Several Studies on Framelets Derived From Compactly Supported Refinable Vector Functions

by

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Abstract

Generalizing wavelets by adding desired redundancy and flexibility, framelets (a.k.a. wavelet frames) are of interest and importance in many applications such as image processing and numerical algorithms. Several key properties of framelets are high vanishing moments for sparse multi-scale representation, fast framelet transforms for numerical efficiency, and redundancy for robustness. The theory and applications of scalar framelets have been extensively studied in the literature. However, vector framelets, or equivalently multiframelets, are far from being well understood. This thesis provides a theoretical investigation of multiframelets.

Framelets are often derived from refinable vector functions via the popular oblique extension principle (OEP), and such framelets are called OEP-based framelets. Constructing OEP-based tight multiframelets with several desired features is a well known challenging problem. We circumvent this issue by considering quasi-tight multiframelets, which are special dual multiframelets but behave almost identical as tight multiframelets. We will show in Chapter 2 that from any compactly supported univariate refinable vector function with at least two entries, one can always obtain a quasi-tight multiframelet such that: (1) its associated discrete framelet transform is compact and has the highest possible balancing order; (2) all compactly supported framelet generators have the highest possible order of vanishing moments. Several illustrative examples will be provided. The results of this chapter are summarized as [43], which has been published in *Applied and Computational Harmonic Analysis*.

In Chapter 3, we extend the theory of univariate quasi-tight multiframelets in Chapter 2 to arbitrary dimensions. The generalization is not straight forward. Several new challenges and elements are involved. The work of this chapter is summarized as [44], which has been published in *Science China Mathematics*.

In Chapter 4, we will discuss the more general question on how to construct multivariate dual multiframelets satisfying all desired properties from any pair of compactly supported refinable vector functions. Our study on constructing OEP-based multiframelets relies on a newly developed normal form of matrix-valued filters, which is of independent interest and importance for greatly reducing the difficulty of studying refinable vector functions and multiframelets. This chapter's work is summarized as [57], which is submitted and is under review.

In Chapter 5, we introduce framelets with mixed dilation factors. Unlike a traditional framelet system which only involves a single dilation factor, we consider framelet systems involves different dilation factors. Recent advances on constructing tight framelets with low redundancy and good directionality give us a taste of framelets with mixed dilation factors, and demonstrate the interest and importance of establishing the corresponding theory. In this thesis, we develop the basic theory of framelets with mixed dilation factors.

Preface

Some of the research conducted for this thesis forms part of research collaboration with Dr. Bin Han at the University of Alberta. I was the key investigator of all research projects in Chapters 2 and 3. The research conducted in Chapters 4 and 5 is my original work.

Chapter 2 of this thesis has been published as B. Han and R. Lu, "Compactly supported quasi-tight multiframelets with high balancing orders and compact framelet transform," *Appl. Comput. Harmon. Anal.*, **51** (2021), 295-332. I was responsible for the theoretical analysis, example constructions as well as manuscript composition. Dr. Han was involved with concept/project formation, theoretical analysis including the key result Theorem 2.3.1 and manuscript improvement.

Chapter 3 of this thesis has been published as B. Han and R. Lu, "Multivariate quasi-tight framelets with high balancing orders derived from any compactly supported refinable vector functions," *Sci. China Math.* **64** (2021), https://doi.org/10.1007/s11425-020-1786-9. I was responsible for the theoretical analysis and manuscript composition. Dr. Han was involved with the proof of the crucial result Theorem 3.2.1 and manuscript improvement.

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Contents

List of Figures **viii**

1 Introduction **1**

1.1 Backgrounds on Framelets 1

1.2 The Major Shortcoming of OEP 6

1.3 Advantages and Difficulties of Multiframelets 9

1.4 Framelets with Mixed Dilation Factors 12

1.5 Contributions 14

1.6 Thesis Structure 15

2 One-dimensional Quasi-tight Multiframelets with High Balancing Orders **17**

2.1 The Perfect Reconstruction Property of a Multi-level Discrete Framelet Transform 18

2.2 The Balancing Property of a Multi-level Discrete Framelet Transform . . . 28

2.3 A Newly Developed Normal Form of a Matrix-valued Filter 29

2.4 The Main Theorem 41

2.5 Guidelines for Constructing Balanced OEP-based Quasi-tight Multiframelets 52

2.6 Some Examples of Spline Quasi-tight Framelets with High Balancing Orders 59

2.7 Summary of the Chapter 70

2.8 References 71

3 Balanced Quasi-tight Multiframelets in Arbitrary Dimensions **74**

3.1 Properties of a Discrete Framelet Transform 76

3.2 A Matrix-valued Filter Normal Form for an Arbitrary Dimension d 82

3.3 Multivariate Quasi-tight Multiframelets with High Vanishing Moments and High Balancing Orders 88

3.4 The Structure of OEP-based Balanced Multivariate Quasi-tight Multiframelets 100

3.5 Summary of the Chapter 111

3.6 References 111

4 Multivariate Dual Multiframelets Derived from Arbitrary Compactly Supported Refinable Vector Functions **116**

4.1 The Main Theorem on OEP-based Dual Multiframelets 117

4.2 Structural Properties of OEP-Based Balanced Dual Multiframelets 124

4.3 Summary of the Chapter 131

5 Framelets with Mixed Dilation Factors **132**

5.1 Discrete Framelet Transforms with Mixed Dilation Factors 133

5.2 Discrete Affine Systems in $l_2(\mathbb{Z}^d)$ 145

5.3 Framelets and Wavelets in $L_2(\mathbb{R}^d)$ with Mixed dilation Factors 154

5.4 Summary of the Chapter 171

6 Summary and Future Work **172**

Bibliography **175**

List of Figures

2.1 Graphs of $\phi = [B_2(\cdot-1), 0]^\mathsf{T}$ and a new refinable vector function $\mathring{\phi}$, together with graphs of $\psi^1, , \ldots, \psi^4$ constructed from ϕ in Example 2.3. A graph with a solid (resp. dash) line denotes the first (resp. second) component of a vector function. $\{\mathring{\phi}; (\psi^1, \ldots, \psi^4)^\mathsf{T}\}_{(1,1,1,-1)}$ is a compactly supported quasi-tight 2-framelet in $L_2(\mathbb{R})$ with balanced vanishing moments 2. . . . 63

2.2 Graphs of $\phi = [\varphi(2\cdot), \varphi(2\cdot -1)]^\mathsf{T}$ and the new refinable vector function $\mathring{\phi}$, together with graphs of $\psi^1, , \ldots, \psi^4$ constructed from ϕ in Example 2.4. A graph with a solid (resp. dash) line denotes the first (resp. second) component of a function vector. $\{\mathring{\phi}; (\psi^1, \ldots, \psi^4)^\mathsf{T}\}$ is a compactly supported tight 2-framelet in $L_2(\mathbb{R})$ with balanced vanishing moments 2. 66

2.3 Graphs of ϕ and the new refinable vector function $\mathring{\phi}$, together with graphs of $\psi^1, , \ldots, \psi^6$ constructed from the Hermite cubic splines ϕ defined as (2.6.3) in Example 2.5. $\{\mathring{\phi}; [\psi^1, \ldots, \psi^6]^\mathsf{T}\}_{(-1,-1,1,1,1,1)}$ is a compactly supported quasi-tight 2-framelet in $L_2(\mathbb{R})$ with balanced vanishing moments 2. . . . 70

3.1 The entries of the $M_{\sqrt{2}}$-refinable vector function $\phi = (\phi_1, \phi_2)^\mathsf{T}$ in Example 3.1.105

Chapter 1

Introduction

1.1 Backgrounds on Framelets

Generalizing wavelets by adding desired redundancy and flexibility, framelets (a.k.a. wavelet frames) are of interest and importance in many applications such as image processing and numerical algorithms. Several key properties of framelets are high vanishing moments for sparse multiscale representation, fast framelet transforms for numerical efficiency, and redundancy for robustness. The theory and applications of framelets have been extensively studied over the past decades, see e.g. [8, 11–15, 18, 19, 21, 22, 24, 28, 30, 40, 41, 50, 51, 62] and many references therein.

To better explain the motivation of our work, we recall some basic concepts and notations on framelets. Throughout the thesis, by M we always denote a **dilation matrix**, which is a $d \times d$ integer matrix whose eigenvalues are greater than one in modulus, or equivalently, $\lim_{j\to\infty} \mathsf{M}^{-j} = 0$. Moreover, we define

$$d_{\mathsf{M}} := |\det(\mathsf{M})|. \tag{1.1.1}$$

By $f \in (L_2(\mathbb{R}^d))^{r\times s}$ we mean that f is an $r \times s$ matrix of square integrable functions in

$L_2(\mathbb{R}^d)$. In particular, $(L_2(\mathbb{R}^d))^r := (L_2(\mathbb{R}^d))^{r\times 1}$. Define the inner product by

$$\langle f, g\rangle := \int_{\mathbb{R}^d} f(x)\overline{g(x)}^{\mathsf{T}} dx, \qquad f \in (L_2(\mathbb{R}^d))^{r\times s}, \quad g \in (L_2(\mathbb{R}^d))^{t\times s}. \tag{1.1.2}$$

Let $\mathring{\phi} = (\mathring{\phi}_1, \dots, \mathring{\phi}_r)^{\mathsf{T}} \in (L_2(\mathbb{R}^d))^r$ and $\psi = (\psi_1, \dots, \psi_s)^{\mathsf{T}} \in (L_2(\mathbb{R}^d))^s$. We say that $\{\mathring{\phi}; \psi\}$ is **an M-framelet** in $L_2(\mathbb{R}^d)$ if there exist positive constants C_1 and C_2 such that

$$C_1\|f\|^2_{L_2(\mathbb{R}^d)} \leqslant \sum_{k\in\mathbb{Z}^d} |\langle f, \mathring{\phi}(\cdot - k)\rangle|^2 + \sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^d} |\langle f, \psi_{\mathsf{M}^j;k}\rangle|^2 \leqslant C_2\|f\|^2_{L_2(\mathbb{R}^d)}, \qquad \forall\, f \in L_2(\mathbb{R}^d), \tag{1.1.3}$$

where

$$|\langle f, \psi_{\mathsf{M}^j;k}\rangle|^2 := \|\langle f, \psi_{\mathsf{M}^j;k}\rangle\|^2_{l_2} = \sum_{\ell=1}^{s} |\langle f, \psi_{\ell,\mathsf{M}^j;k}\rangle|^2, \tag{1.1.4}$$

and

$$\psi_{\mathsf{M}^j;k} := d_{\mathsf{M}}^{j/2}\psi(\mathsf{M}^j \cdot - k), \qquad \psi_{\ell,\mathsf{M}^j;k} := d_{\mathsf{M}}^{j/2}\psi_\ell(\mathsf{M}^j \cdot - k), \quad \ell = 1, \dots, s. \tag{1.1.5}$$

The number r is called **the multiplicity** of the framelet. If $r = 1$, we often call $\{\mathring{\phi}; \psi\}$ **a scalar framelet**. If $r > 1$, then $\{\mathring{\phi}; \psi\}$ is often called **a multiframelet** or **a vector framelet**. For simplicity, we refer both of them framelet.

For $\mathring{\phi}, \tilde{\mathring{\phi}} \in (L_2(\mathbb{R}^d))^r$ and $\psi, \tilde{\psi} \in (L_2(\mathbb{R}^d))^r$, we say that $(\{\mathring{\phi}; \psi\}, \{\tilde{\mathring{\phi}}; \tilde{\psi}\})$ is **a dual M-framelet** in $L_2(\mathbb{R}^d)$ if both $\{\mathring{\phi}; \psi\}$ and $\{\tilde{\mathring{\phi}}; \tilde{\psi}\}$ are M-framelets in $L_2(\mathbb{R}^d)$ and satisfy

$$f = \sum_{k\in\mathbb{Z}^d} \langle f, \mathring{\phi}(\cdot - k)\rangle \tilde{\mathring{\phi}}(\cdot - k) + \sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^d} \langle f, \psi_{\mathsf{M}^j;k}\rangle \tilde{\psi}_{\mathsf{M}^j;k}, \quad \forall f \in L_2(\mathbb{R}^d), \tag{1.1.6}$$

with the above series converging unconditionally in $L_2(\mathbb{R}^d)$. We say that $\{\mathring{\phi}; \psi\}_{(\epsilon_1,\dots,\epsilon_s)}$ is **a quasi-tight M-framelet** in $L_2(\mathbb{R}^d)$ if $\{\mathring{\phi}; \psi\}$ is an M-framelet in $L_2(\mathbb{R}^d)$ and satisfies

$$f = \sum_{k\in\mathbb{Z}^d} \langle f, \mathring{\phi}(\cdot - k)\rangle \mathring{\phi}(\cdot - k) + \sum_{j=0}^{\infty}\sum_{\ell=1}^{s}\sum_{k\in\mathbb{Z}^d} \epsilon_\ell \langle f, \psi_{\ell,\mathsf{M}^j;k}\rangle \psi_{\ell,\mathsf{M}^j;k} \qquad \forall\, f \in L_2(\mathbb{R}^d) \tag{1.1.7}$$

for some $\epsilon_1, \dots, \epsilon_s \in \{\pm 1\}$, with the above series converging unconditionally in $L_2(\mathbb{R}^d)$. We say that $\{\mathring{\phi}; \psi\}$ is **a tight framelet** in $L_2(\mathbb{R}^d)$ if (1.1.3) holds with $C_1 = C_2 = 1$, or equivalently, $\{\mathring{\phi}; \psi\}_{(\epsilon_1,\dots,\epsilon_s)}$ is a quasi-tight M-framelet with $\epsilon_1 = \dots = \epsilon_s = 1$. From the above definitions, we see that a tight framelet is a special case of a quasi-tight framelet, and a quasi-tight framelet is a special case of a dual framelet.

Framelet is a special type of of frames in $L_2(\mathbb{R}^d)$. For $\mathring{\phi} = (\mathring{\phi}_1, \dots, \mathring{\phi}_r)^\mathsf{T} \in (L_2(\mathbb{R}^d))^r$ and $\psi = (\psi_1, \dots, \psi_s)^\mathsf{T} \in (L_2(\mathbb{R}^d))^s$, we define the **$\mathsf{M}$-affine system** generated by $\{\mathring{\phi}; \psi\}$ via

$$\begin{aligned} \mathrm{AS}(\{\mathring{\phi}; \psi\}) :=& \{\mathring{\phi}_\ell(\cdot - k) : \ell = 1, \dots, r, \quad k \in \mathbb{Z}^d\} \\ & \cup \{\psi_{\ell, \mathsf{M}^j; k} : \ell = 1, \dots, s, \quad k \in \mathbb{Z}^d, \quad j \in \mathbb{N}_0\}. \end{aligned} \tag{1.1.8}$$

For $\mathring{\phi}, \tilde{\mathring{\phi}} \in (L_2(\mathbb{R}^d))^r$ and $\psi, \tilde{\psi} \in (L_2(\mathbb{R}^d))^r$, it is trivial that $(\{\mathring{\phi}; \psi\}, \{\tilde{\mathring{\phi}}; \tilde{\psi}\})$ is a dual M-framelet in $L_2(\mathbb{R}^d)$ if and only if $(\mathrm{AS}(\{\mathring{\phi}; \psi\}), \mathrm{AS}(\{\tilde{\mathring{\phi}}; \tilde{\psi}\}))$ is a pair of dual frames of $L_2(\mathbb{R}^d)$. Similarly, $\{\mathring{\phi}; \psi\}_{(\epsilon_1,\dots,\epsilon_s)}$ is a quasi-tight M-framelet in $L_2(\mathbb{R}^d)$ if and only if $(\mathrm{AS}(\{\mathring{\phi}; \psi\}), \mathrm{AS}(\{\mathring{\phi}; \mathrm{Diag}(\epsilon_1, \dots, \epsilon_s)\tilde{\psi}\}))$ is a pair of dual frames of $L_2(\mathbb{R}^d)$, and $\{\mathring{\phi}; \psi\}$ is a tight framelet in $L_2(\mathbb{R}^d)$ if and only if $\mathrm{AS}(\{\mathring{\phi}; \psi\})$ is a tight frame of $L_2(\mathbb{R}^d)$.

Most known framelets are constructed from **refinable vector functions**. By $(l_0(\mathbb{Z}^d))^{r \times s}$ we denote the space of all $r \times s$ matrix-valued finitely supported sequences $u = \{u(k)\}_{k \in \mathbb{Z}^d} : \mathbb{Z}^d \to \mathbb{C}^{r \times s}$ such that $\{k \in \mathbb{Z}^d \ : \ u(k) \neq 0\}$ is a finite set. For a vector function $\phi \in (L_2(\mathbb{R}^d))^r$, we say that ϕ is **an M-refinable vector function** with **a refinement filter/mask** $a \in (l_0(\mathbb{Z}^d))^{r \times r}$ if

$$\phi = |\det(\mathsf{M})| \sum_{k \in \mathbb{Z}^d} a(k) \phi(\mathsf{M} \cdot -k). \tag{1.1.9}$$

The integer r is the multiplicity of ϕ. If $r = 1$, then we simply say that ϕ is an M-refinable (scalar) function. For $f \in L_1(\mathbb{R}^d)$, let $\widehat{f}(\xi) := \int_{\mathbb{R}^d} f(x) e^{-ix \cdot \xi} dx$ for $\xi \in \mathbb{R}^d$ be its Fourier transform. The definition of the Fourier transform can be naturally extended to $L_2(\mathbb{R}^d)$ functions and tempered distributions. For any finitely supported filter $u \in (l_0(\mathbb{Z}^d))^{s \times r}$, define its Fourier series via

$$\widehat{u}(\xi) := \sum_{k\in\mathbb{Z}^d} u(k)e^{-ik\cdot\xi}, \qquad \xi \in \mathbb{R}^d. \tag{1.1.10}$$

It is easy to see that the refinable equation (1.1.9) is equivalent to

$$\widehat{\phi}(\mathsf{M}^{\mathsf{T}}\xi) = \widehat{a}(\xi)\widehat{\phi}(\xi), \qquad \xi \in \mathbb{R}^d, \tag{1.1.11}$$

where $\widehat{\phi}$ is the $r \times 1$ vector obtained by taking entry-wise Fourier transform on ϕ. To construct framelets from refinable vector functions, an **oblique extension principle (OEP)** was introduced. The scalar framelet version of the OEP was introduced in [8] and [15]. The univariate multiframelet version of the OEP was studied in [33, 45] (also see [41, Theorem 6.4.1]). Its corresponding multivariate version is as follows:

Theorem 1.1.1 (*Oblique Extension Principle (OEP)*)**.** *Let* M *be a* $d\times d$ *dilation matrix. Let* $\theta, \tilde{\theta}, a, \tilde{a} \in (l_0(\mathbb{Z}^d))^{r\times r}$ *and* $\phi, \tilde{\phi} \in (L_2(\mathbb{R}^d))^r$ *be compactly supported* M*-refinable vector functions with refinement filters* $a, \tilde{a} \in (l_0(\mathbb{Z}^d))^{r\times r}$*, respectively. For matrix-valued filters* $b, \tilde{b} \in (l_0(\mathbb{Z}^d))^{s\times r}$*, define* $\mathring{\phi}, \psi, \mathring{\tilde{\phi}}, \tilde{\psi}$ *as*

$$\widehat{\mathring{\phi}}(\xi) := \widehat{\theta}(\xi)\widehat{\phi}(\xi), \qquad \widehat{\psi}(\xi) := \widehat{b}(\mathsf{M}^{-\mathsf{T}}\xi)\widehat{\phi}(\mathsf{M}^{-\mathsf{T}}\xi), \tag{1.1.12}$$

$$\widehat{\mathring{\tilde{\phi}}}(\xi) := \widehat{\tilde{\theta}}(\xi)\widehat{\tilde{\phi}}(\xi), \quad \widehat{\tilde{\psi}}(\xi) := \widehat{\tilde{b}}(\mathsf{M}^{-\mathsf{T}}\xi)\widehat{\tilde{\phi}}(\mathsf{M}^{-\mathsf{T}}\xi). \tag{1.1.13}$$

Then $(\{\mathring{\phi}; \psi\}, \{\mathring{\tilde{\phi}}; \tilde{\psi}\})$ *is a dual* M*-framelet in* $L_2(\mathbb{R}^d)$ *if the following conditions are satisfied:*

(1) $\overline{\widehat{\phi}(0)}^{\mathsf{T}}\widehat{\Theta}(0)\widehat{\tilde{\phi}}(0) = 1$ *with* $\widehat{\Theta}(\xi) := \overline{\widehat{\theta}(\xi)}^{\mathsf{T}}\widehat{\tilde{\theta}}(\xi)$*;*

(2) all entries in ψ *and* $\tilde{\psi}$ *have at least one vanishing moment, i.e.,* $\widehat{\psi}(0) = \widehat{\tilde{\psi}}(0) = 0$.

(3) $(\{a; b\}, \{\tilde{a}; \tilde{b}\})_{\Theta}$ *forms* ***an OEP-based dual*** M***-framelet filter bank****, i.e.,*

$$\overline{\widehat{a}(\xi)}^{\mathsf{T}}\widehat{\Theta}(\mathsf{M}^{\mathsf{T}}\xi)\widehat{\tilde{a}}(\xi + 2\pi\omega) + \overline{\widehat{b}(\xi)}^{\mathsf{T}}\widehat{\tilde{b}}(\xi + 2\pi\omega) = \boldsymbol{\delta}(\omega)\widehat{\Theta}(\xi), \tag{1.1.14}$$

for all $\xi \in \mathbb{R}^d$ *and* $\omega \in \Omega_{\mathsf{M}}$*, where*

$$\boldsymbol{\delta}(0) := 1 \quad \textit{and} \quad \boldsymbol{\delta}(x) := 0, \qquad \forall\, x \neq 0 \tag{1.1.15}$$

and Ω_{M} *is a particular choice of the representatives of cosets in* $[\mathsf{M}^{-\mathsf{T}}\mathbb{Z}^d]/\mathbb{Z}^d$ *given by*

$$\Omega_{\mathsf{M}} := \{\omega_1, \ldots, \omega_{d_{\mathsf{M}}}\} := (\mathsf{M}^{-\mathsf{T}}\mathbb{Z}^d) \cap [0,1)^d \quad \textit{with} \quad \omega_1 := 0. \tag{1.1.16}$$

Framelets derived from refinable vector functions via OEP are called **OEP-based framelets**.

The most important feature of the multiscale representation in (1.1.6) is its sparsity, which is highly desired for processing multidimensional data. By $\mathbb{P}_{m-1}$ we denote the space of all d-variate polynomials of degree less than m. The sparsity of the multiscale representation in (1.1.6) comes from the vanishing moments of ψ. We say that a function ψ has **order** m **vanishing moments** if

$$\langle \mathsf{p}, \psi \rangle = 0, \qquad \forall\, \mathsf{p} \in \mathbb{P}_{m-1}, \quad \text{or equivalently,} \quad \widehat{\psi}(\xi) = \mathscr{O}(\|\xi\|^m), \qquad \xi \to 0,$$

where the notation $f(\xi) = g(\xi) + \mathscr{O}(\|\xi\|^m)$ as $\xi \to 0$ simply means $\partial^\mu f(0) = \partial^\mu g(0)$ for all $\mu = (\mu_1, \ldots, \mu_d)^{\mathsf{T}} \in \mathbb{N}_0^d$ with $|\mu| := \mu_1 + \cdots + \mu_d < m$. We define $\mathrm{vm}(\psi) := m$ with m being the largest such integer. It is easy to deduce from (1.1.6) that a necessary condition for all framelet generators ψ_ℓ, $\ell = 1, \ldots, s$ to have order m vanishing moments is the following polynomial preservation property:

$$\sum_{k \in \mathbb{Z}^d} \langle \mathsf{p}, \mathring{\phi}(\cdot - k) \rangle \mathring{\tilde{\phi}}(\cdot - k) := \sum_{\ell=1}^{r} \sum_{k \in \mathbb{Z}^d} \langle \mathsf{p}, \mathring{\phi}_\ell(\cdot - k) \rangle \mathring{\tilde{\phi}}_\ell(\cdot - k) = \mathsf{p}, \qquad \forall\, \mathsf{p} \in \mathbb{P}_{m-1}, \tag{1.1.17}$$

which plays a crucial role in approximation theory and numerical analysis for the convergence rate of the associated approximation/numerical scheme. Using the Fourier transform, it is well known in approximation theory (see e.g. [41, Proposition 5.5.2]) that if

$\mathrm{vm}(\psi) = m$ and $\mathrm{vm}(\tilde{\psi}) = \tilde{m}$, then we necessarily have

$$\overline{\widehat{\mathring{\phi}}(\xi)}^{\mathsf{T}} \widehat{\mathring{\tilde{\phi}}}(\xi + 2\pi k) = \mathscr{O}(\|\xi\|^m), \qquad \overline{\widehat{\mathring{\phi}}(\xi + 2\pi k)}^{\mathsf{T}} \widehat{\mathring{\tilde{\phi}}}(\xi) = \mathscr{O}(\|\xi\|^{\tilde{m}}), \qquad \xi \to 0, \quad k \in \mathbb{Z}^d \backslash \{0\}, \tag{1.1.18}$$

and

$$\overline{\widehat{\mathring{\phi}}(\xi)}^{\mathsf{T}} \widehat{\mathring{\tilde{\phi}}}(\xi) = 1 + \mathscr{O}(\|\xi\|^{\tilde{m}+m}), \quad \xi \to 0. \tag{1.1.19}$$

For an OEP-based dual M-framelet $(\{\mathring{\phi}; \psi\}, \{\mathring{\tilde{\phi}}; \tilde{\psi}\})$, the vanishing moments on the framelet generators ψ and $\tilde{\psi}$ is closely related to the orders of sum rules of the refinement masks a and $\tilde{a}$ associated with the refinement vector functions ϕ and $\tilde{\phi}$. We say that a filter $a \in (l_0(\mathbb{Z}^d))^{r\times r}$ has **order m sum rules with respect to M** with a matching filter $\upsilon \in (l_0(\mathbb{Z}^d))^{1\times r}$ if $\widehat{\upsilon}(0) \neq 0$ and

$$\widehat{\upsilon}(\mathsf{M}^{\mathsf{T}}\xi)\widehat{a}(\xi + 2\pi\omega) = \boldsymbol{\delta}(\omega)\widehat{\upsilon}(\xi) + \mathscr{O}(\|\xi\|^m), \quad \xi \to 0, \quad \forall\, \omega \in \Omega_{\mathsf{M}}. \tag{1.1.20}$$

In particular, we define $\mathrm{sr}(a, \mathsf{M}) := m$ with m being the largest possible integer in (1.1.20). It can be easily deduced from (1.1.14) that $\mathrm{vm}(\psi) \leqslant \mathrm{sr}(\tilde{a}, \mathsf{M})$ and $\mathrm{vm}(\tilde{\psi}) \leqslant \mathrm{sr}(a, \mathsf{M})$ always hold no matter how we choose θ and $\tilde{\theta}$. The purpose of OEP is to improve the orders of vanishing moments of the framelet generators ψ and $\tilde{\psi}$ by properly constructing $\theta, \tilde{\theta} \in (l_0(\mathbb{Z}^d))^{r\times r}$ such that (1.1.14), (1.1.18) and (1.1.19) are satisfied with m and $\tilde{m}$ being as large as possible.

1.2 The Major Shortcoming of OEP

With OEP, a lot of compactly supported scalar tight or dual framelets with the highest possible vanishing moments have been constructed in the literature, to mention only a few, see [2, 3, 8, 14–17, 30, 33, 41, 45, 46, 52, 53, 58, 62, 66] and many references therein. In particular, see Chapter 3 of [41] for comprehensive study and references on scalar tight or dual framelets. Though OEP appears perfect for improving the vanishing moments of framelet generators, it has a serious shortcoming. To properly address this issue, we need to briefly recall the discrete framelet transform employing an OEP-based filter bank.

By $(l(\mathbb{Z}^d))^{s\times r}$ we denote the linear space of all sequences $v:\mathbb{Z}^d\to\mathbb{C}^{s\times r}$. We call every element $v\in(l(\mathbb{Z}^d))^{s\times r}$ **a matrix-valued filter**. For a filter $a\in(l_0(\mathbb{Z}^d))^{r\times r}$, we define the filter $a^\star$ via $\widehat{a^\star}(\xi):=\overline{\widehat{a}(\xi)}^{\mathsf{T}}$, or equivalently, $a^\star(k):=\overline{a(-k)}^{\mathsf{T}}$ for all $k\in\mathbb{Z}^d$. We define the *convolution* of two filters via

$$[v*a](n):=\sum_{k\in\mathbb{Z}} v(k)a(n-k),\quad n\in\mathbb{Z}^d,\quad v\in(l(\mathbb{Z}^d))^{s\times r},\quad a\in(l_0(\mathbb{Z}^d))^{r\times r}.$$

Let M be a $d\times d$ dilation matrix, define the **upsampling operator** $\uparrow\mathsf{M}:(l(\mathbb{Z}^d))^{s\times r}\to(l(\mathbb{Z}^d))^{s\times r}$ and the **downsampling operator** $\downarrow\mathsf{M}:(l(\mathbb{Z}^d))^{s\times r}\to(l(\mathbb{Z}^d))^{s\times r}$ as

$$[v\uparrow\mathsf{M}](k):=\begin{cases}v(\mathsf{M}^{-1}k), & \text{if } k\in\mathsf{M}\mathbb{Z}^d,\\ 0, & \text{elsewhere,}\end{cases},\qquad [v\downarrow\mathsf{M}](k):=v(\mathsf{M}k),\qquad \forall k\in\mathbb{Z}^d,\quad (1.2.1)$$

for all $v\in(l(\mathbb{Z}^d))^{s\times r}$.

We introduce the following operators acting on matrix-valued sequence spaces:

- For $u\in(l_0(\mathbb{Z}^d))^{r\times t}$, the **subdivision operator** $\mathcal{S}_{u,\mathsf{M}}$ is defined via

$$\mathcal{S}_{u,\mathsf{M}}v=|\det(\mathsf{M})|^{\frac{1}{2}}[v\uparrow\mathsf{M}]*a=|\det(\mathsf{M})|^{\frac{1}{2}}\sum_{k\in\mathbb{Z}^d}v(k)u(\cdot-\mathsf{M}k),\qquad (1.2.2)$$

  for all $v\in(l(\mathbb{Z}^d))^{s\times r}$.

- For $u\in(l_0(\mathbb{Z}^d))^{t\times r}$, the **transition operator** $\mathcal{T}_{u,\mathsf{M}}$ is defined via

$$\mathcal{T}_{u,\mathsf{M}}v=|\det(\mathsf{M})|^{\frac{1}{2}}[v*u^\star]\downarrow\mathsf{M}=|\det(\mathsf{M})|^{\frac{1}{2}}\sum_{k\in\mathbb{Z}^d}v(k)\overline{u(k-\mathsf{M}\cdot)}^{\mathsf{T}},\qquad (1.2.3)$$

  for all $v\in(l(\mathbb{Z}^d))^{s\times r}$.

Let $a, \tilde{a}, \theta, \tilde{\theta} \in (l_0(\mathbb{Z}))^{r\times r}$ and $b, \tilde{b} \in (l_0(\mathbb{Z}))^{s\times r}$ be finitely supported filters. Define a filter $\Theta := \theta^\star * \tilde{\theta}$, i.e., $\widehat{\Theta}(\xi) := \overline{\widehat{\theta}(\xi)}^{\mathsf{T}}\widehat{\tilde{\theta}}(\xi)$. We now state the discrete multiframelet transform using these finitely supported filters. For any $J \in \mathbb{N}$ and any (vector-valued) input signal/data $v_0 \in (l(\mathbb{Z}^d))^{1\times r}$, the J-level discrete framelet transform using a filter bank $(\{a; b\}, \{\tilde{a}; \tilde{b}\})_\Theta$ is as follows:

(S1) *Decomposition:* Recursively compute the **framelet coefficients** $v_j, w_j, j = 1, \ldots, J$ by

$$v_j := \mathcal{T}_{a,\mathsf{M}} v_{j-1} = \mathcal{T}_{a,\mathsf{M}}^{j} v_0, \quad w_j := \mathcal{T}_{b,\mathsf{M}} v_{j-1} = \mathcal{T}_{b,\mathsf{M}} \mathcal{T}_{a,\mathsf{M}}^{j-1} v_0, \qquad j = 1, \ldots, J.$$

(S2) *Reconstruction:* Compute $\tilde{v}_J := v_J * \Theta$ and recursively compute $\tilde{v}_{j-1}, j = J, \ldots, 1$ by

$$\tilde{v}_{j-1} := \mathcal{S}_{\tilde{a},\mathsf{M}} \tilde{v}_j + \mathcal{S}_{\tilde{b},\mathsf{M}} w_j, \qquad j = J, \ldots, 1.$$

(S3) Recover $\mathring{v}_0$ from $\tilde{v}_0$ through the deconvolution $\mathring{v}_0 * \Theta = \tilde{v}_0$.

The deconvolution step (S3) is where the trouble arises. If $(\{a; b\}, \{\tilde{a}; \tilde{b}\})_\Theta$ is an OEP-based dual M-framelet filter bank satisfying (1.1.14), then the original input data v_0 is guaranteed to be a solution of the deconvolution problem $\mathring{v}_0 * \Theta = \tilde{v}_0$. However, the deconvolution is inefficient and non-stable, that is, there could be multiple solutions of the deconvolution problem. Thus we cannot expect that the input data can be exactly retrieved by implementing the transform. As we shall see later in Chapter 3, one necessary condition to avoid this issue is that Θ is **strongly invertible**.

Definition 1.2.1. *Let $\Theta \in (l_0(\mathbb{Z}^d))^{r\times r}$ be a finitely supported filter. We say that $\widehat{\Theta}$ (or simply Θ) is strongly invertible if there exists $\Theta^{-1} \in (l_0(\mathbb{Z}^d))^{r\times r}$ such that $\widehat{\Theta^{-1}} = \widehat{\Theta}^{-1}$, or equivalently all entries of $\widehat{\Theta}^{-1}$ are $2\pi\mathbb{Z}^d$-periodic trigonometric polynomials.*

For a scalar filter Θ (i.e., $r = 1$), it is strongly invertible if and only if $\widehat{\Theta}$ is **a non-zero monomial**, i.e., $\widehat{\Theta}(\xi) = ce^{-ik\cdot\xi}$ for some $c \in \mathbb{C} \setminus \{0\}$ and $k \in \mathbb{Z}^d$. When Θ is strongly invertible, the discrete framelet transform is said to be **compact**, i.e., the transform is implemented by convolution/deconvolution with finitely supported filters only. Note that the strong invertibility of Θ forces both $\widehat{\theta}$ and $\widehat{\tilde{\theta}}$ to be monomials, and we lose the main advantage of OEP of improving the vanishing moments of framelet generators by choosing such filters θ and $\tilde{\theta}$. For instance, one of the most important examples of refinable scalar functions are B-splines. For $m \in \mathbb{N}$, the **B-spline function** B_m **of order** m is defined by

$$B_1 := \chi_{[0,1]} \quad \text{and} \quad B_m := B_{m-1} * B_1 = \int_0^1 B_{m-1}(\cdot - t)dt. \tag{1.2.4}$$

The B-spline function B_m is a piecewise polynomial function, belongs to $C^{m-1}(\mathbb{R})$ with support $[0, m]$, and is M-refinable: $\widehat{B_m}(\mathsf{M}\xi) = \widehat{a^B_{m,\mathsf{M}}}(\xi)\widehat{B_m}(\xi)$ with

$$\widehat{a^B_{m,\mathsf{M}}}(\xi) := \mathsf{M}^{-m}(1 + e^{-i\xi} + \cdots + e^{-i(\mathsf{M}-1)\xi})^m. \tag{1.2.5}$$

We see that $\operatorname{sr}(a^B_{m,\mathsf{M}}, \mathsf{M}) = m$ and $1 - |\widehat{a^B_{m,\mathsf{M}}}(\xi)|^2 = \mathscr{O}(|\xi|^2)$ as $\xi \to 0$. Thus any OEP-based tight M-framelet derived from B_m with the trivial choice $\widehat{\Theta} \equiv 1$ has at most 1 vanishing moments on the framelet generators, even though m can be arbitrarily large.

The situation becomes even more complicated for multiframelets (i.e., $r > 1$). In this case, Θ is strongly invertible if and only if $\det(\widehat{\Theta})$ is a non-zero monomial, which is in general too much to expect. Nevertheless, we will see in Chapters 2, 3 and 4 that this condition can be satisfied without sacrificing other good properties of an OEP-based multiframelet, which demonstrates great advantages of multiframelets over scalar framelets.

1.3 Advantages and Difficulties of Multiframelets

The previously mentioned shortcoming of OEP motivates us to consider multiframelets, that is, framelets with multiplicity $r > 1$. Multiframelets have certain advantages over scalar framelets and have been initially studied in [27, 29] and references therein. In sharp contrast to the extensively studied OEP-based scalar framelets, constructing multiframelets through OEP is much more difficult and is much less studied. To our best knowledge, we are only aware of [58, Chapter 2] for studying OEP-based tight multiframelets, and [33, 45] for investigating OEP-based dual multiframelets with vanishing moments, which both focus only on the one-dimensional setting (i.e., $d = 1$). In this section, we briefly explain the difficulties involved in studying multiframelets.

We see from Theorem 1.1.1 that the most important step of constructing OEP-based framelets is choosing the appopriate filters $\theta, \tilde{\theta} \in (l_0(\mathbb{Z}^d))^{r\times r}$. In many situations, this is not easy. Except for the examples in [33], all constructed OEP-based dual framelets with non-trivial Θ (where $\Theta := \theta^\star * \tilde{\theta}$) do not have a compact underlying discrete framelet transform, i.e., Θ is not strongly invertible. We will present some concrete examples in Chapter 2. Choosing suitable filters θ and $\tilde{\theta}$ becomes even harder when constructing OEP-based tight framelets, in which case we require that $\theta = \tilde{\theta}$. Moreover, define

$$P_{u;\mathsf{M}}(\xi) := [\widehat{u}(\xi + 2\pi\omega_1), \ldots, \widehat{u}(\xi + 2\pi\omega_{d_\mathsf{M}})], \qquad \xi \in \mathbb{R}^d. \tag{1.3.1}$$

for every matrix-valued filter u, where $\omega_1, \ldots, \omega_{d_\mathsf{M}}$ are defined as in (1.1.16). A tight M-framelet filter bank $\{a; b\}_\Theta$ satisfies (1.1.14) with $a = \tilde{a}$ and $b = \tilde{b}$, which is further equivalent to

$$\mathcal{M}_{a,\Theta}(\xi) = \overline{P_{b;\mathsf{M}}(\xi)}^\mathsf{T} P_{b;\mathsf{M}}(\xi), \tag{1.3.2}$$

where

$$\mathcal{M}_{a,\Theta}(\xi) := \begin{bmatrix} \widehat{\Theta}(\xi + 2\pi\omega_1) & & \\ & \ddots & \\ & & \widehat{\Theta}(\xi + 2\pi\omega_{d_\mathsf{M}}) \end{bmatrix} - \overline{P_{a;\mathsf{M}}(\xi)}^\mathsf{T} \widehat{\Theta}(\mathsf{M}^\mathsf{T}\xi) P_{a;\mathsf{M}}(\xi). \tag{1.3.3}$$

Note that (1.3.2) forces $\mathcal{M}_{a,\Theta}(\xi) \geqslant 0$ (i.e., $\mathcal{M}_{a,\Theta}(\xi)$ is positive semi-definite) for all $\xi \in \mathbb{R}^d$. This extra requirement makes the construction of Θ much harder, even in the simplest case $d = 1$. Even though one can obtain Θ such that $\mathcal{M}_{a,\Theta} \geqslant 0$, we still have the problem of obtaining a spectral factorization of $\mathcal{M}_{a,\Theta}$ as in (1.3.2). If $d=1$, the Fejér-Riesz lemma guarantees the existence of a spectral factorization. However, the spectral factorization of a trigonometric polynomial matrix is much harder when $d \geqslant 2$, due to its intrinsic connections to factorization and syzygy modules of multivariate polynomial matrices (see [4, 5]).

On the other hand, the sparsity of a discrete framelet transform is another issue which needs to be worried about when the multiplicity $r>1$. First we look at the scalar case $r=1$. Let $(\{\mathring{\phi};\psi\},\{\mathring{\tilde{\phi}};\tilde{\psi}\})$ be an OEP-based dual M-framelet obtained through Theorem 1.1.1 with an underlying OEP-based dual M-framelet filter bank $(\{a;b\},\{\tilde{a};\tilde{b}\})_\Theta$. Suppose that $\mathrm{vm}(\psi)=m$. Then the framelet representation (1.1.6) has sparsity in the sense that the polynomial preservation property (1.1.17) holds. Moreover, item (1) of Theorem 1.1.1 yields $\widehat{\phi}(0)\neq 0$. Thus it follows from $\widehat{\psi}:=\widehat{b}(\mathsf{M}^{-\mathsf{T}}\cdot)\widehat{\phi}(\mathsf{M}^{-\mathsf{T}}\cdot)$ that $\widehat{b}(\xi)=\mathscr{O}(\|\xi\|^m)$ as $\xi\to 0$. For any polynomial $\mathsf{p}\in\mathbb{P}_{m-1}$, using Taylor expansion yields $\mathsf{p}(x-k)=\sum_{\alpha\in\mathbb{N}_0^d}\frac{(-k)^\alpha}{\alpha!}\partial^\alpha\mathsf{p}(x)$ for all $x,k\in\mathbb{R}^d$. Thus for any finitely supported sequence $u\in l_0(\mathbb{Z}^d)$, we have

$$[\mathsf{p}*u](x)=\sum_{k\in\mathbb{Z}^d}\mathsf{p}(x-k)u(k)=\sum_{\alpha\in\mathbb{N}_0^d}[\partial^\alpha\mathsf{p}](x)\left(\sum_{k\in\mathbb{Z}^d}\frac{(-k)^\alpha}{\alpha!}u(k)\right)=\sum_{\alpha\in\mathbb{N}_0^d}\frac{(-i)^{|\alpha|}}{\alpha!}[\partial^\alpha\mathsf{p}](x)[\partial^\alpha\widehat{u}](0),$$

which is a polynomial whose degree is no bigger than the degree of p, i.e., $\mathsf{p}*u\in\mathbb{P}_{m-1}$. We now input a polynomial sequence data $\mathsf{p}\in\mathbb{P}_{m-1}$ and implement the J-level discrete framelet transform with the filter bank $(\{a;b\},\{\tilde{a};\tilde{b}\})_\Theta$. Observe that the framelet coefficient v_1 satisfies $v_1=\mathcal{T}_{a,\mathsf{M}}\mathsf{p}=|\det(\mathsf{M})|^{\frac{1}{2}}[\mathsf{p}*a^\star](\mathsf{M}\cdot)\in\mathbb{P}_{m-1}$, and by induction we conclude that $v_j\in\mathbb{P}_{m-1}$ for all $j=1,2,\dots,J$. It follows that the framelet coefficients w_j satisfy

$$w_j=\mathcal{T}_{b,\mathsf{M}}v_{j-1}=|\det(\mathsf{M})|^{\frac{1}{2}}[v_{j-1}*b^\star](\mathsf{M}\cdot)=|\det(\mathsf{M})|^{\frac{1}{2}}\sum_{\alpha\in\mathbb{N}_0^d}\frac{(-i)^{|\alpha|}}{\alpha!}[\partial^\alpha v_{j-1}](\cdot)[\partial^\alpha\widehat{b}](0)=0,$$

for all $j=1,\dots,J$. where the last step follows from $\widehat{b}(\xi)=\mathscr{O}(\|\xi\|^m)$ as $\xi\to 0$ and $v_j\in\mathbb{P}_{m-1}$. Consequently, all framelet coefficients w_j vanish. This means that the sparsity of the framelet expansion (1.1.6) automatically guarantees the sparsity of the underlying multi-level discrete framelet transform. Unfortunately this is in general not the case when $r > 1$, simply due to the fact that $\widehat{\psi}(\xi) = \widehat{b}(\mathsf{M}^{-\mathsf{T}}\xi)\widehat{\phi}(\mathsf{M}^{-\mathsf{T}}\xi) = \mathscr{O}(\|\xi\|^m)$ does not imply any moment property of $\widehat{b}(\xi)$ at $\xi = 0$. This issue is known as the **balancing property** of a framelet in the literature ([9, 10, 33, 35, 41, 56, 65]). For the case $r > 1$, the sparsity of a discrete framelet transform is measured by its **balancing order** (see the definition in Chapter 2 for $d = 1$ and Chapter 3 for $d > 1$). For an OEP-based dual framelet $(\{\mathring{\phi}; \psi\}, \{\mathring{\tilde{\phi}}; \tilde{\psi}\})$ has the highest possible orders of vanishing moments on framelet generators, and the balancing order underlying discrete framelet transform is the same as $\operatorname{vm}(\psi)$, then we say that the dual framelet is **balanced**. We will leave further discussion of this issue to Chapters 2 and 3.

1.4 Framelets with Mixed Dilation Factors

The redundancy of framelet systems not only offers flexibility in their constructions but also improves the performance when handling multi-dimensional data ([1,15,30,36,38,47–49,60–62,67]). The degree of redundancy of a dual framelet is measured by its *redundancy rate*. Here let us roughly explain the redundancy rate of a framelet transform.

Let M be a $d \times d$ dilation matrix. For simplicity let us consider a J-level discrete framelet transform employing the OEP-based filter bank $(\{a; b\}, \{\tilde{a}; \tilde{b}\})_{\boldsymbol{\delta} I_r}$ for some $a, \tilde{a} \in (l_0(\mathbb{Z}^d))^{r\times r}$ and $b, \tilde{b} \in (l_0(\mathbb{Z}^d))^{s\times r}$. In many applications, we consider an initial data $v_0 \in (l_0(\mathbb{Z}^d))^{1\times r}$ which has finite support. Suppose that v_0 contains n real numbers, i.e. $\#\{\operatorname{Re}(v_0(k)_l), \operatorname{Im}(v_0(k)_l) : k \in \mathbb{Z}^d; l = 1, \dots, r\} = n$. One first extend v_0 to a periodic sequence $v_0^{per} \in (l(\mathbb{Z}^d))^{1\times r}$, and use v_0^{per} as the input data of the discrete framelet transform. Noting that $v_0^{per} * a^\star$ and $v_0^{per} * b^\star$ are both periodic sequences which have the same period as v_0^{per}, both sequences contain only finitely many real numbers. Thus the framelet coefficients v_j and w_j contain finitely many real numbers for all $j = 1, \dots, J$, and say all framelet coefficients contain a total of N real numbers. Then the ratio $\frac{N}{n}$ measures the redundancy of the J-level discrete framelet transform. For a J-level discrete framelet transform employing a dual M-framelet filter bank $(\{a; b\}, \{\tilde{a}; \tilde{b}\})_{\boldsymbol{\delta} I_r}$, its redundancy rate only depends on the level J, the number s and the dilation factor M.

Most dual framelets which offer good performance in practice have relatively high redundacy rates (see e.g. [1, 15, 30, 36, 60–62, 67]). Though the high redundancy rate of a dual framelet improves performance in practice, the computational cost will also increase as dimension increases. Recently, a version of tensor product complex tight framelets (TP-$\mathbb{C}$TF) was introduced in [49], which not only offers good performance on image processing, but also has a significantly lower redundancy rate compared with existing systems in the literature. This newly developed TP-$\mathbb{C}$TF is a particular case of **framelets with mixed dilation factors**. That is, instead of using one single dilation factor M, different dilation factors are used in the framelet system. To be specific, let $\psi^0 = (\psi^0_1, \dots, \psi^0_r)^{\mathsf{T}} \in (L_2(\mathbb{R}^d))^r, \psi^1, \dots, \psi^s \in L_2(\mathbb{R}^d)$ and $\mathsf{M}_0, \dots, \mathsf{M}_s$ be $d \times d$ dilation matrices. Define **an affine system with mixed dilation factors** via

$$\begin{aligned}\mathrm{AS}(\{\psi^l ! \mathsf{M}_l\}_{l=0}^s) :=& \{\psi^0_q(\cdot - k) : q = 1, \dots, r, \quad k \in \mathbb{Z}^d\} \\ & \cup \{|\det(\mathsf{M}_0^{-1}\mathsf{M}_l)|^{\frac{1}{2}} \psi^l_{\mathsf{M}_0^j; \mathsf{M}_0^{-1}\mathsf{M}_l k} : l = 1, \dots, s, \quad j \in \mathbb{N}_0, \quad k \in \mathbb{Z}^d\}.\end{aligned}$$

We say that $(\{\psi^l ! \mathsf{M}_l\}_{l=0}^s, \{\tilde{\psi}^l ! \mathsf{M}_l\}_{l=0}^s)$ is **a dual framelet in $L_2(\mathbb{R}^d)$ with mixed dilation factors** if both $(\mathrm{AS}(\{\psi^l ! \mathsf{M}_l\}_{l=0}^s)$ and $\mathrm{AS}(\{\tilde{\psi}^l ! \mathsf{M}_l\}_{l=0}^s))$ are frames of $L_2(\mathbb{R}^d)$ and satisfy

$$\begin{aligned}\langle f, g\rangle =& \sum_{k \in \mathbb{Z}^d} \langle f, \psi^0(\cdot - k)\rangle \langle \tilde{\psi}^0(\cdot - k), g\rangle \\ & + \sum_{l=1}^s \sum_{j=0}^\infty \sum_{k \in \mathbb{Z}^d} |\det(\mathsf{M}_0^{-1}\mathsf{M}_l)| \langle f, \psi^l_{\mathsf{M}_0^j; \mathsf{M}_0^{-1}\mathsf{M}_l k}\rangle \langle \tilde{\psi}^l_{\mathsf{M}_0^j; \mathsf{M}_0^{-1}\mathsf{M}_l k}, g\rangle\end{aligned} \tag{1.4.1}$$

for all $f, g \in L_2(\mathbb{R}^d)$, with the above series converging absolutely.

Allowing mixed dilation factors creates more flexibility in constructing framelets, and makes it possible to achieve a low redundancy rate while all desired properties for good performance are being kept. There is a small amount of work in the literature on lowering redundancy rates of wavelet/framelet systems by considering mixed dilation factors (e.g. [68, 70]). To our best knowledge, the only existing work in the literature discussing framelets with mixed dilation factors theoretically is [49], which only addresses the case of tight framelets with multiplicity $r = 1$.

1.5 Contributions

The majority of this thesis is devoted to the investigation on how to avoid the previously mentioned shortcomings and difficulties on OEP-based framelets. We mainly focus on multiframelets. For $\mathring{\phi} \in (L_2(\mathbb{R}^d))^r$ and $\psi \in (L_2(\mathbb{R}^d))^s$ and $\epsilon_1, \dots, \epsilon_s \in \{\pm 1\}$, recall that $\{\mathring{\phi}; \psi\}_{(\epsilon_1,\dots,\epsilon_s)}$ is a quasi-tight M-framelet in $L_2(\mathbb{R}^d)$ if $(\{\mathring{\phi}; \psi\}, \{\mathring{\phi}; \mathrm{Diag}(\epsilon_1, \dots, \epsilon_s)\psi\})$ is a dual M-framelet in $L_2(\mathbb{R}^d)$. Quasi-tight framelets are special dual framelets, but behave almost identically as tight framelets. From any compactly supported refinable vector function with multiplicity $r > 1$, we show that one can always construct a quasi-tight multiframelet through OEP such that (1) all framelet generators have the highest possible vanishing moments; (2) its underlying discrete framelet transform is compact; (3) its underlying discrete framelet transform has the highest possible balancing order, which makes the transform sparse. Several illustrative examples will be provided for the case $d = 1$. Our result on quasi-tight multiframelets can be further generalized to dual multiframelets. We prove that from any pair of compactly supported refinable vector functions, an OEP-based dual framelet satisfying all desired properties can be obtained. This part of the work demonstrates great advantages of OEP on multiframelets over scalar framelets. The key ingredient of our study is a newly developed normal form of matrix-valued filters, which is of independent interest and importance for greatly reducing the difficulty of studying refinable vector functions and multiframelets. We also study discrete multiframelet transforms employing OEP-based multiframelet filter banks in detail.

On the other hand, one chapter will be devoted to framelets with mixed dilation factors. The basics of the theory of framelets with mixed dilation factors and with arbitrary multiplicity will be developed. We will study properties of framelet filter banks with mixed dilation factors, and then make connections with framelets in $L_2(\mathbb{R}^d)$.

1.6 Thesis Structure

Chapter 2 deals with one dimensional quasi-tight multiframelets. First, we develop the normal form of a matrix-valued filter and demonstrate its power. Next, we study the discrete framelet transform employing an OEP-based dual framelet filter bank. We shall discuss various properties including the compactness and the balancing property of the framelet transform. Then we show how to construct an OEP-based quasi-tight multiframelet from any compactly supported refinable vector function, and provide some examples to illustrate our result.

Chapter 3 is the multivariate counterpart of Chapter 2. However, we will see that the multivariate case is much harder than the univariate case. There are new challenges that we have to deal with, and several new elements will be involved in our study. We shall also provide a structural characterization of OEP-based quasi-tight multiframelets with high vanishing moments and high balancing orders.

In Chapter 4, we focus on OEP-based dual multiframelets. We prove that from any pairs of refinable vector functions, one can always obtain a dual multiframelet through OEP, with all desired properties being satisfied. We comment that the univariate dual multiframelet case has been covered in [33]. Our main result in this chapter is a non-trivial generalization of the corresponding result on the case $d = 1$ to the case $d > 1$.

In Chapter 5, we deal with framelets with mixed dilation factors and with arbitrary multiplicity. The notion of a dual framelet filter bank with mixed dilation factors will be encountered and its various properties will be studied. We will also link the theory in discrete settings to function settings, by making connections between filter banks and framelet systems in $L_2(\mathbb{R}^d)$.

Finally in Chapter 6, we provide a summary of the thesis, and discuss some potential future research topics.

Chapter 2

One-dimensional Quasi-tight Multiframelets with High Balancing Orders

In this chapter, we investigate the construction of one-dimensional OEP-based quasi-tight multiframelets with high vanishing moments and high balancing orders. The work of this chapter is summarized as [43], which has been published in *Applied and Computational Harmonic Analysis*. Since we work on the case when the dimension is $d = 1$, the dilation factor M is an integer with $|\mathsf{M}| \geqslant 2$. Without loss of generality and for simplicity of discussion, throughout this chapter, we assume $\mathsf{M} \geqslant 2$, in which case the set Ω_{M} defined as (1.1.16) simply becomes

$$\Omega_{\mathsf{M}} = \left\{0, \frac{1}{\mathsf{M}}, \dots, \frac{\mathsf{M}-1}{\mathsf{M}}\right\},$$

with $\omega_j = \frac{j-1}{\mathsf{M}}$ for $j = 1, \dots, \mathsf{M}$.

As discussed in Chapter 1, the key is to construct a desired filter Θ as in Theorem 1.1.1, which maximizes the vanishing moments of all framelet generators, and makes the underlying discrete framelet transform compact and balanced. We start with this chapter by discussing various properties of a discrete framelet transform employing an OEP-based filter bank. Next, we move on to develop the theory of the normal form of a matrix-valued filter, which plays a key role in our study of OEP-based multiframelets. Then we show that from any compactly supported refinable vector function, one can always obtain an OEP-based quasi-tight multiframelet satisfying all desired properties. Several examples will be provided at the end of the chapter to illustrate our main result.

2.1 The Perfect Reconstruction Property of a Multi-level Discrete Framelet Transform

We shall address several important issues on a multi-level discrete framelet transform such as the perfect reconstruct property and the balancing property. We have mentioned in Section 1.2 that the deconvolution problem in (S3) of a multi-level discrete framelet reconstruction may have infinitely many solutions, which cause nonstability and inaccuracy for reconstruction. We say that a multi-level discrete framelet transform has **the generalized perfect reconstruction property** if any original input signal v_0 can be reconstructed as one of the solutions $\mathring{v}_0$ of the deconvolution problem $\tilde{v}_0 = \mathring{v}_0 * \Theta$.

Let $a, \tilde{a}, \theta, \tilde{\theta} \in (l_0(\mathbb{Z}))^{r\times r}$ and $b, \tilde{b} \in (l_0(\mathbb{Z}))^{s\times r}$ be finitely supported filter, and let $J \in \mathbb{N}$. To analyze a J-level discrete framelet transform using the filter bank $(\{a; b\}, \{\tilde{a}; \tilde{b}\})_\Theta$ (where $\Theta := \theta^\star * \tilde{\theta}$), we define:

- The ***J*-level discrete framelet analysis/decomposition operator**:

$$\mathcal{W}_J : (l(\mathbb{Z}))^{1\times r} \to (l(\mathbb{Z}))^{1\times(sJ+r)}, \quad \mathcal{W}_J(v) = (\mathcal{T}_{b,\mathsf{M}} v, \mathcal{T}_{b,\mathsf{M}}\mathcal{T}_{a,\mathsf{M}} v, \ldots, \mathcal{T}_{b,\mathsf{M}}\mathcal{T}_{a,\mathsf{M}}^{J-1} v, \mathcal{T}_{a,\mathsf{M}}^{J} v).$$

  Define $\mathcal{W} := \mathcal{W}_1$ as the one-level analysis/decomposition operator.

- The J-**level discrete framelet synthesis/reconstruction operator**:

$$\tilde{\mathcal{V}}_J : (l(\mathbb{Z}))^{1\times(sJ+r)} \to (l(\mathbb{Z}))^{1\times r}, \quad \tilde{\mathcal{V}}_J(\mathring{w}_1, \mathring{w}_2, \dots, \mathring{w}_J, \mathring{v}_J) = \tilde{v}_0,$$

  for all $\mathring{w}_j \in (l(\mathbb{Z}))^{1\times s}$ and $\mathring{v}_J \in (l(\mathbb{Z}))^{1\times r}$, where $\tilde{v}_{j-1}, j = J, \dots, 1$ are recursively computed via

$$\tilde{v}_{j-1} := \mathcal{S}_{\tilde{a},\mathsf{M}}\tilde{v}_j + \mathcal{S}_{\tilde{b},\mathsf{M}}\mathring{w}_j, \qquad j = J, \dots, 1,$$

  with $\tilde{v}_J := \mathring{v}_J$. Define $\tilde{\mathcal{V}} := \tilde{\mathcal{V}}_1$ as the one-level synthesis/reconstruction operator.

- The J-**level convolution operator**:

$$C_{\Theta;J} : (l(\mathbb{Z}))^{1\times(sJ+r)} \to (l(\mathbb{Z}))^{1\times(sJ+r)}, \quad C_{\Theta;J}(\mathring{w}_1, \mathring{w}_2, \dots, \mathring{w}_J, \mathring{v}_J) = (\mathring{w}_1, \mathring{w}_2, \dots, \mathring{w}_J, \mathring{v}_J * \Theta),$$

  for all $\mathring{w}_j \in (l(\mathbb{Z}))^{1\times s}$ and $\mathring{v}_J \in (l(\mathbb{Z}))^{1\times r}$.

We observe that the J-level discrete framelet transform using the filter bank $(\{a; b\}, \{\tilde{a}; \tilde{b}\})_\Theta$ has the generalized perfect reconstruction property for an input signal $v \in (l(\mathbb{Z}))^{1\times r}$ if and only if

$$\tilde{\mathcal{V}}_J C_{\Theta;J} \mathcal{W}_J(v) = C_\Theta(v), \tag{2.1.1}$$

where C_Θ is the **convolution operator** $C_\Theta(v) = v * \Theta$. Moreover, by

$$\mathcal{W}_J = (\mathrm{Id}_{(l(\mathbb{Z}))^{1\times s(J-1)}} \otimes \mathcal{W}) \cdots (\mathrm{Id}_{(l(\mathbb{Z}))^{1\times s}} \otimes \mathcal{W})\mathcal{W}$$

and

$$\tilde{\mathcal{V}}_J = \tilde{\mathcal{V}}(\mathrm{Id}_{(l(\mathbb{Z}))^{1\times s}} \otimes \tilde{\mathcal{V}}) \cdots (\mathrm{Id}_{(l(\mathbb{Z}))^{1\times s(J-1)}} \otimes \tilde{\mathcal{V}}),$$

we see that the J-level discrete framelet transform has the generalized perfect reconstruction property for v if and only if the one-level discrete framelet transform does, that is,

$$\mathcal{S}_{\tilde{a},\mathsf{M}}([\mathcal{T}_{a,\mathsf{M}} v] * \Theta) + \mathcal{S}_{\tilde{b},\mathsf{M}}(\mathcal{T}_{b,\mathsf{M}} v) = v * \Theta. \tag{2.1.2}$$

Following the approach in [37, 41] for scalar framelets, we now provide the necessary and sufficient conditions for the generalized perfect reconstruction property of a discrete framelet transform as follows:

Theorem 2.1.1. *Let $a, \tilde{a}, \theta, \tilde{\theta} \in (l_0(\mathbb{Z}))^{r\times r}$ and $b, \tilde{b} \in (l_0(\mathbb{Z}))^{s\times r}$ be finitely supported filters. Define $\Theta := \theta^\star * \tilde{\theta}$. The following statements are equivalent to each other:*

(i) For any $J \in \mathbb{N}$, the J-level discrete framelet transform using the filter bank $(\{a; b\}, \{\tilde{a}; \tilde{b}\})_\Theta$ has the generalized perfect reconstruction property for any input signal $v \in (l(\mathbb{Z}))^{1\times r}$ (or for any input signal $v \in (l_0(\mathbb{Z}))^{1\times r}$).

(ii) $(\{a; b\}, \{\tilde{a}; \tilde{b}\})_\Theta$ is an OEP-based dual M-framelet filter bank satisfying (1.1.14)*.*

Proof. (i) $\Rightarrow$ (ii): The generalized perfect reconstruction property of the one-level discrete multiframelet transform for $v \in (l(\mathbb{Z}))^{1\times r}$ is equivalent to (2.1.2). For $v \in (l_0(\mathbb{Z}))^{1\times r}$, we observe that

$$\widehat{\mathcal{S}_{a,\mathsf{M}} v}(\xi) = \mathsf{M}^{1/2}\widehat{v}(\mathsf{M}\xi)\widehat{a}(\xi) \quad \text{and} \quad \widehat{\mathcal{T}_{a,\mathsf{M}} v}(\mathsf{M}\xi) = \mathsf{M}^{-1/2}\sum_{\gamma=0}^{\mathsf{M}-1} \widehat{v}(\xi + \tfrac{2\pi\gamma}{\mathsf{M}})\overline{\widehat{a}(\xi + \tfrac{2\pi\gamma}{\mathsf{M}})}^{\mathsf{T}}.$$

Therefore, in the frequency domain, (2.1.2) is equivalent to

$$\sum_{\gamma=0}^{\mathsf{M}-1} \widehat{v}(\xi + \tfrac{2\pi\gamma}{\mathsf{M}}) \left[\overline{\widehat{a}(\xi + \tfrac{2\pi\gamma}{\mathsf{M}})}^{\mathsf{T}} \widehat{\Theta}(\mathsf{M}\xi)\widehat{\tilde{a}}(\xi) + \overline{\widehat{b}(\xi + \tfrac{2\pi\gamma}{\mathsf{M}})}^{\mathsf{T}} \widehat{\tilde{b}}(\xi)\right] = \widehat{v}(\xi)\widehat{\Theta}(\xi). \tag{2.1.3}$$

We now use the same argument as in [37, Theorem 2.1] and [41, Theorem 1.1.1] by selecting v as Dirac sequences in (2.1.3) to prove (1.1.14). Observe that (2.1.3) actually holds for all $v \in (l_0(\mathbb{Z}))^{r\times r}$. For each $j = 0, \dots, \mathsf{M}-1$, plug $v := \boldsymbol{\delta}(\cdot - j)I_r$ into (2.1.3) and noting that $\widehat{v}(\xi) = e^{-ij\xi}I_r$, we deduce from (2.1.3) that

$$\sum_{\gamma=0}^{\mathsf{M}-1} e^{-ij\cdot\frac{2\pi\gamma}{\mathsf{M}}} \left[\overline{\widehat{a}(\xi + \tfrac{2\pi\gamma}{\mathsf{M}})}^{\mathsf{T}} \widehat{\Theta}(\mathsf{M}\xi)\widehat{\tilde{a}}(\xi) + \overline{\widehat{b}(\xi + \tfrac{2\pi\gamma}{\mathsf{M}})}^{\mathsf{T}} \widehat{\tilde{b}}(\xi)\right] = \widehat{\Theta}(\xi), \tag{2.1.4}$$

for all $j = 0, \dots, \mathsf{M}-1$. Define the $(r\mathsf{M}) \times (r\mathsf{M})$ matrix $\mathsf{F} := \left(e^{-ij\cdot\frac{2\pi\gamma}{\mathsf{M}}} I_r\right)_{0\leqslant j,\gamma\leqslant \mathsf{M}-1}$. For each $\gamma = 0, \ldots, \mathsf{M}-1$, define

$$\widehat{u_\gamma}(\xi) := \overline{\widehat{a}(\xi + \tfrac{2\pi\gamma}{\mathsf{M}})}^{\mathsf{T}} \widehat{\Theta}(\mathsf{M}\xi)\widehat{\tilde{a}}(\xi) + \overline{\widehat{b}(\xi + \tfrac{2\pi\gamma}{\mathsf{M}})}^{\mathsf{T}} \widehat{\tilde{b}}(\xi), \qquad \xi \in \mathbb{R}.$$

Note that (2.1.4) yields

$$\mathsf{F} \begin{bmatrix} \widehat{u_0}(\xi) \\ \widehat{u_1}(\xi) \\ \vdots \\ \widehat{u_{\mathsf{M}-1}}(\xi) \end{bmatrix} = \begin{bmatrix} I_r \\ I_r \\ \vdots \\ I_r \end{bmatrix} \widehat{\Theta}(\xi). \tag{2.1.5}$$

It is easy to verify that $\overline{\mathsf{F}}^{\mathsf{T}}\, \mathsf{F} = \mathsf{M} I_{r\mathsf{M}}$. Thus (2.1.5) implies

$$\mathsf{M} \begin{bmatrix} \widehat{u_0}(\xi) \\ \widehat{u_1}(\xi) \\ \vdots \\ \widehat{u_{\mathsf{M}-1}}(\xi) \end{bmatrix} = \overline{\mathsf{F}}^{\mathsf{T}} \begin{bmatrix} I_r \\ I_r \\ \vdots \\ I_r \end{bmatrix} \widehat{\Theta}(\xi) = \begin{bmatrix} \mathsf{M} I_r \\ \mathbf{0}_{r\times r} \\ \vdots \\ \mathbf{0}_{r\times r} \end{bmatrix} \widehat{\Theta}(\xi), \tag{2.1.6}$$

where $\mathbf{0}_{r\times r}$ denotes the $r \times r$ zero matrix. Hence we have $\widehat{u_\gamma}(\xi) = \boldsymbol{\delta}(\gamma)\widehat{\Theta}(\xi)$, that is, $(\{a; b\}, \{\tilde{a}; \tilde{b}\})_\Theta$ is an OEP-based dual M-framelet filter bank. This proves item (ii).

(ii) $\Rightarrow$ (i): Suppose that $(\{a; b\}, \{\tilde{a}; \tilde{b}\})_\Theta$ is an OEP-based dual M-framelet filter bank satisfying (1.1.14). Then (2.1.3) must hold for all $v \in (l_0(\mathbb{Z}))^{1\times r}$. In the time domain, (2.1.3) is equivalent to (2.1.2), which further implies (2.1.1). This proves the generalized perfect reconstruction property for any $v \in (l_0(\mathbb{Z}))^{1\times r}$. Now for arbitrary $v \in (l(\mathbb{Z}))^{1\times r}$, we use the locality of the subdivision and transition operators (see the proof of [37, Theorem 2.1] and [41, Theorem 1.1.1]) to prove that (2.1.3) holds for all $v \in (l(\mathbb{Z}))^{1\times r}$. Since all filters $a, \tilde{a}, b, \tilde{b}, \Theta$ are all finitely supported, there exists $N \in \mathbb{N}$ such that all these filters vanish outside $[-N, N]$. Fix $n \in \mathbb{Z}$. For any $v \in (l(\mathbb{Z}))^{1\times r}$, define $v_n \in (l_0(\mathbb{Z}))^{1\times r}$

via

$$v_n(k) := \begin{cases} v(k), & \text{if } n-(\mathsf{M}+2)N \leqslant k \leqslant n+(\mathsf{M}+2)N \\ 0, & \text{elsewhere.} \end{cases}$$

For $k \in \mathbb{Z} \cap \left[\frac{n-N}{\mathsf{M}}, \frac{n+N}{\mathsf{M}}\right]$, we have

$$\begin{aligned}
([\mathcal{T}_{a,\mathsf{M}} v] * \Theta)(k) &= \sum_{t\in\mathbb{Z}} \left(\sum_{q\in\mathbb{Z}} v(q)\overline{a(q-\mathsf{M}t)}^{\mathsf{T}} \right) \Theta(k-t) \\
&= \sum_{t\in\mathbb{Z}} \left(\sum_{q\in\mathbb{Z}} v(q-\mathsf{M}t)\overline{a(q-\mathsf{M}k)}^{\mathsf{T}} \right) \Theta(t) \\
&= \sum_{t=-N}^{N} \left(\sum_{q=n-2N}^{n+2N} v(q-\mathsf{M}t)\overline{a(q-\mathsf{M}k)}^{\mathsf{T}} \right) \Theta(t) \\
&= \sum_{t=-N}^{N} \left(\sum_{q=n-2N}^{n+2N} v_n(q-\mathsf{M}t)\overline{a(q-\mathsf{M}k)}^{\mathsf{T}} \right) \Theta(t) \\
&= \sum_{t\in\mathbb{Z}} \left(\sum_{q\in\mathbb{Z}} v_n(q-\mathsf{M}t)\overline{a(q-\mathsf{M}k)}^{\mathsf{T}} \right) \Theta(t) \\
&= ([\mathcal{T}_{a,\mathsf{M}} v_n] * \Theta)(k).
\end{aligned}$$

Similarly we deduce that $(\mathcal{T}_{b,\mathsf{M}} v)(k) = (\mathcal{T}_{b,\mathsf{M}} v_n)(k)$ for all $k \in \mathbb{Z} \cap \left[\frac{n-N}{\mathsf{M}}, \frac{n+N}{\mathsf{M}}\right]$. By direct calculation, we have

$$\begin{aligned}
&[\mathcal{S}_{\tilde{a},\mathsf{M}}([\mathcal{T}_{a,\mathsf{M}} v] * \Theta)](n) + [\mathcal{S}_{\tilde{b},\mathsf{M}}(\mathcal{T}_{b,\mathsf{M}} v)](n) \\
=&\mathsf{M}^{1/2} \sum_{k\in\mathbb{Z}\cap\left[\frac{n-N}{\mathsf{M}}, \frac{n+N}{\mathsf{M}}\right]} \Big(([\mathcal{T}_{a,\mathsf{M}} v] * \Theta)(k)\tilde{a}(n-\mathsf{M}k) + \mathcal{T}_{b,\mathsf{M}} v(k)\tilde{b}(n-\mathsf{M}k) \Big) \\
=&\mathsf{M}^{1/2} \sum_{k\in\mathbb{Z}\cap\left[\frac{n-N}{\mathsf{M}}, \frac{n+N}{\mathsf{M}}\right]} \Big(([\mathcal{T}_{a,\mathsf{M}} v_n] * \Theta)(k)\tilde{a}(n-\mathsf{M}k) + \mathcal{T}_{b,\mathsf{M}} v_n(k)\tilde{b}(n-\mathsf{M}k) \Big) \\
=&[\mathcal{S}_{\tilde{a},\mathsf{M}}([\mathcal{T}_{a,\mathsf{M}} v_n] * \Theta)](n) + [\mathcal{S}_{\tilde{b},\mathsf{M}}(\mathcal{T}_{b,\mathsf{M}} v_n)](n) \\
=&(v_n * \Theta)(n),
\end{aligned}$$

where the last identity follows from the generalized perfect reconstruction property of the transform for all $v \in (l_0(\mathbb{Z}))^{1\times r}$. Note that

$$(v_n*\Theta)(n) = \sum_{k\in\mathbb{Z}} v_n(k)\Theta(n-k) = \sum_{k=n-N}^{n+N} v_n(k)\Theta(n-k) = \sum_{k=n-N}^{n+N} v(k)\Theta(n-k) = (v*\Theta)(n).$$

Since $n \in \mathbb{Z}$ is arbitrary, this proves (2.1.2) for all $v \in (l_0(\mathbb{Z}))^{1\times r}$. The proof is now complete. $\square$

If the deconvolution in (S3) of the J-level discrete framelet reconstruction has infinitely many solutions or no solution at all, without any extra information on the input signal, then one cannot exactly recover the original input signal v_0 from the transform. Hence, we say that a discrete framelet transform has **the perfect reconstruction property** if any original input signal v_0 can be reconstructed as the unique solution $\mathring{v}_0$ of $\tilde{v}_0 = \mathring{v}_0*\Theta$ in (S3).

To study the perfect reconstruction property of a discrete framelet transform, we need the following auxiliary result.

Lemma 2.1.2. *Let $\Theta \in (l_0(\mathbb{Z}))^{r\times r}$ be a finitely supported filter. Define the convolution operator C_Θ by $C_\Theta(v) := v * \Theta$ for any sequences $v \in (l(\mathbb{Z}))^{1\times r}$. Then*

(1) The mapping $C_\Theta : (l_\infty(\mathbb{Z}))^{1\times r} \to (l_\infty(\mathbb{Z}))^{1\times r}$ is injective (or bijective) if and only if $\det(\widehat{\Theta}(\xi)) \neq 0$ for all $\xi \in \mathbb{R}$.

(2) The mapping $C_\Theta : (l_{si}(\mathbb{Z}))^{1\times r} \to (l_{si}(\mathbb{Z}))^{1\times r}$ is injective (or bijective) if and only if $\det(\widehat{\Theta}(\xi)) \neq 0$ for all $\xi \in \mathbb{R}$, where $l_{si}(\mathbb{Z})$ denotes the space of all slowly increasing sequences, i.e., $v \in l_{si}(\mathbb{Z})$ if $(1+|\cdot|^2)^{-m}v \in l_\infty(\mathbb{Z})$ for some $m \in \mathbb{N}$.

(3) The mapping $C_\Theta : (l(\mathbb{Z}))^{1\times r} \to (l(\mathbb{Z}))^{1\times r}$ is injective (or bijective) if and only if $\det(\widehat{\Theta}(\xi))$ is a nontrivial monomial (i.e., $\det(\widehat{\Theta}(\xi)) = ce^{-im\xi}$ for some $m \in \mathbb{Z}$ and $c \in \mathbb{C}\backslash\{0\}$).

Proof. We first prove items (1) and (2) simultaneously. Let V_0 be either $(l_\infty(\mathbb{Z}))^{1\times r}$ or $(l_{si}(\mathbb{Z}))^{1\times r}$. Suppose that $C_\Theta : V_0 \to V_0$ is injective, but $\det(\widehat{\Theta}(\xi_0)) = 0$ for some $\xi_0 \in \mathbb{R}$.

We start with the case $r = 1$. In this case, we have $0 = \widehat{\Theta}(\xi_0) = \sum_{k\in\mathbb{Z}} \Theta(k)e^{-ik\xi_0}$. Let $v \in l_\infty(\mathbb{Z})$ be defined by

$$v(k) = e^{-ik\xi_0}, \quad \forall k \in \mathbb{Z}. \tag{2.1.7}$$

By definition, we have $(v * \Theta)(n) = e^{-in\xi_0}\widehat{\Theta}(\xi_0) = 0$ for all $n \in \mathbb{Z}$. So we find a non-zero sequence v with $v * \Theta = 0$. Hence C_Θ is not injective, which is a contradiction. So we must have $\widehat{\Theta}(\xi) \neq 0$ for all $\xi \in \mathbb{R}$.

Now we consider $r > 1$. As $\det(\widehat{\Theta}(\xi_0)) = 0$, we can find an invertible $r \times r$ matrix Q such that all elements in the first row of $Q\widehat{\Theta}(\xi_0)$ are zero. Let $v \in l_\infty(\mathbb{Z})$ be defined as (2.1.7), and let $u \in (l_\infty(\mathbb{Z}))^{1\times r}$ be defined by $u(k) = (v(k), 0, \ldots, 0)Q$ for all $k \in \mathbb{Z}$. It follows immediately that $u * \Theta = 0$, which again contradicts our assumption that C_Θ is injective. Hence, $\det(\widehat{\Theta}(\xi)) \neq 0$ for all $\xi \in \mathbb{R}$.

Conversely, suppose that $\det(\widehat{\Theta}(\xi)) \neq 0$ for all $\xi \in \mathbb{R}$. Then the filter Θ^{-1}, which is defined by $\widehat{\Theta^{-1}}(\xi) := \widehat{\Theta}(\xi)^{-1}$, must be well defined and has exponential decaying coefficients. Consequently, we can deduce that

$$(v * \Theta) * \Theta^{-1} = v * (\Theta * \Theta^{-1}) = v = v * (\Theta^{-1} * \Theta) = (v * \Theta^{-1}) * \Theta \tag{2.1.8}$$

and $v * \Theta^{-1} \in V_0$ for all $v \in V_0$. Hence, C_Θ must be bijective. This proves items (1) and (2).

Finally, we prove item (3). Suppose that $C_\Theta : (l(\mathbb{Z}))^{1\times r} \to (l(\mathbb{Z}))^{1\times r}$ is injective, but $\det(\widehat{\Theta}(\xi))$ is not a non-trivial monomial. Then there exist $\xi_0 \in \mathbb{C}$ such that $\det(\widehat{\Theta}(\xi_0)) = 0$. Then by applying the same argument as in the proof of item (1), we conclude that C_Θ is not injective, which is a contradiction.

Conversely, if $\det(\widehat{\Theta}(\xi))$ is a non-trivial monomial, then Θ is strongly invertible and $\Theta^{-1} \in (l_0(\mathbb{Z}))^{r\times r}$. Consequently, (2.1.8) must hold for all $v \in (l(\mathbb{Z}))^{1\times r}$. Hence, C_Θ is bijective. $\square$

Now Lemma 2.1.2 yields the following characterization on the perfect reconstruction property of a discrete framelet transform employing an OEP-based dual framelet filter bank.

Theorem 2.1.3. *Let $a, \tilde{a}, \theta, \tilde{\theta} \in (l_0(\mathbb{Z}))^{r\times r}$ and $b, \tilde{b} \in (l_0(\mathbb{Z}))^{s\times r}$ be finitely supported filters. Define $\Theta := \theta^\star * \tilde{\theta}$. Let $V_0 = (l_{si}(\mathbb{Z}))^{1\times r}$ (or respectively, $V_0 = (l(\mathbb{Z}))^{1\times r}$). For any $J \in \mathbb{N}$, the J-level discrete framelet transform using the filter bank $(\{a; b\}, \{\tilde{a}; \tilde{b}\})_\Theta$ has the perfect reconstruction property for any input signal from V_0 if and only if*

(i) $(\{a; b\}, \{\tilde{a}; \tilde{b}\})_\Theta$ *is an OEP-based dual* M*-framelet filter bank satisfying* (1.1.14)*;*

(ii) $\det(\widehat{\Theta}(\xi)) \neq 0$ *for all* $\xi \in \mathbb{R}$ *(or respectively,* $\det(\widehat{\Theta}(\xi))$ *is a non-trivial monomial), where* $\widehat{\Theta}(\xi) := \overline{\widehat{\theta}(\xi)}^{\mathsf{T}}\widehat{\tilde{\theta}}(\xi)$.

Except for the examples in [33], all constructed OEP-based dual framelet filter banks with non-trivial Θ do not satisfy item (ii) of Theorem 2.1.3. For the convenience of the reader, we now present two concrete examples of tight framelet filter banks such that item (ii) of Theorem 2.1.3 fails.

Example 2.1. *Let $\mathsf{M} = 2$ and consider the B-spline filter $a^B_{2,2} \in l_0(\mathbb{Z})$:*

$$\widehat{a^B_{2,2}}(\xi) = \frac{1}{4}(1 + e^{-i\xi})^2, \quad \xi \in \mathbb{R}.$$

It is well known that $a^B_{2,2}$ is the mask associated to the refinable function $\widehat{B_2}(\xi) = \left(\frac{1-e^{-i\xi}}{i\xi}\right)^2$. That is, $\widehat{B_2}(2\xi) = \widehat{a^B_{2,2}}(\xi)\widehat{B_2}(\xi)$ for all $\xi \in \mathbb{R}$. With

$$\widehat{\theta}(\xi) = (1 + e^{-i\xi})/2, \quad \widehat{\Theta}(\xi) = \overline{\widehat{\theta}(\xi)}\widehat{\theta}(\xi) = (2 + e^{-i\xi} + e^{i\xi})/4,$$

one can construct a tight 2-framelet filter bank $\{a^B_{2,2}; b\}_\Theta$ satisfying

$$\overline{\widehat{a^B_{2,2}}(\xi)}\widehat{\Theta}(2\xi)\widehat{a^B_{2,2}}(\xi) + \overline{\widehat{b}(\xi)}^{\mathsf{T}}\widehat{b}(\xi) = \widehat{\Theta}(\xi), \quad \overline{\widehat{a^B_{2,2}}(\xi)}\widehat{\Theta}(2\xi)\widehat{a^B_{2,2}}(\xi + \pi) + \overline{\widehat{b}(\xi)}^{\mathsf{T}}\widehat{b}(\xi + \pi) = 0$$

for all $\xi \in \mathbb{R}$, where $b := [b_1, b_2]^{\mathsf{T}} \in (l_0(\mathbb{Z}))^{2\times 1}$ is given by

$$\widehat{b_1}(\xi) = \frac{\sqrt{2}}{8}(e^{-i\xi} - 1)(e^{-i\xi} + 1)^3, \quad \widehat{b_2}(\xi) = \frac{\sqrt{2}}{4}(e^{-2i\xi} - 1), \quad \xi \in \mathbb{R}.$$

Define $\widehat{\mathring{\phi}}(\xi) := \widehat{\theta}(\xi)\widehat{B_2}(\xi)$ *and* $\widehat{\psi}(\xi) := \widehat{b}(\xi/2)\widehat{B_2}(\xi/2)$ *for all* $\xi \in \mathbb{R}$*. Note that* $\widehat{\mathring{\phi}}(0) = \widehat{\theta}(0)\widehat{B_2}(0) = 1$ *and* ψ *has one vanishing moment. By Theorem 1.1.1,* $\{\mathring{\phi}; \psi\}$ *forms a compactly supported tight* 2*-framelet in* $L_2(\mathbb{R})$*. However, because* $\widehat{\Theta}(\pi) = 0$*, Theorem 2.1.3 tells us that the tight* 2*-framelet filter bank* $\{a^B_{2,2}; b\}_\Theta$ *cannot have the perfect reconstruction property for certain input signals.*

Example 2.2. *Let* $\phi_1(x) = B_2(\cdot - 1) = \max(1 - |x|, 0)$ *for all* $x \in \mathbb{R}$*. Then* $\phi := [\phi_1, 0]^\mathsf{T}$ *is a* 2*-refinable vector of compactly supported functions satisfying* $\widehat{\phi}(2\xi) = \widehat{a}(\xi)\widehat{\phi}(\xi)$ *with*

$$\widehat{a}(\xi) = \frac{1}{4}\begin{bmatrix} e^{-i\xi} + 2 + e^{i\xi} & 0 \\ 0 & 1 \end{bmatrix}, \quad \xi \in \mathbb{R}.$$

With

$$\widehat{\theta}(\xi) = \begin{bmatrix} (1 + e^{-i\xi})/2 & 0 \\ 0 & 0 \end{bmatrix}, \quad \widehat{\Theta}(\xi) = \overline{\widehat{\theta}(\xi)}^\mathsf{T}\widehat{\theta}(\xi), \quad \xi \in \mathbb{R},$$

we can construct a tight 2*-multiframelet filter bank* $\{a; b\}_\Theta$ *satisfying*

$$\overline{\widehat{a}(\xi)}^\mathsf{T}\widehat{\Theta}(2\xi)\widehat{a}(\xi) + \overline{\widehat{b}(\xi)}^\mathsf{T}\widehat{b}(\xi) = \widehat{\Theta}(\xi), \quad \overline{\widehat{a}(\xi)}^\mathsf{T}\widehat{\Theta}(2\xi)\widehat{a}(\xi+\pi) + \overline{\widehat{b}(\xi)}^\mathsf{T}\widehat{b}(\xi+\pi) = 0$$

for all $\xi \in \mathbb{R}$*, where* $b \in (l_0(\mathbb{Z}))^{2\times 2}$ *is given by*

$$\widehat{b}(\xi) = \frac{\sqrt{2}}{8}\begin{bmatrix} (1 - e^{-i\xi})(1 + e^{-i\xi})^3 & 0 \\ 2e^{-i\xi}(e^{-2i\xi} - 1) & 0 \end{bmatrix}, \quad \xi \in \mathbb{R}.$$

Define $\widehat{\mathring{\phi}}(\xi) := \widehat{\theta}(\xi)\widehat{\phi}(\xi)$ *and* $\widehat{\psi}(\xi) = \widehat{b}(\xi/2)\widehat{\phi}(\xi/2)$*. Note that* $\|\widehat{\mathring{\phi}}(0)\|^2 = \overline{\widehat{\phi}(0)}^\mathsf{T}\widehat{\Theta}(0)\widehat{\phi}(0) = 1$ *and* ψ *has one vanishing moment. By Theorem 1.1.1,* $\{\mathring{\phi}; \psi\}$ *is a tight* 2*-multiframelet in* $L_2(\mathbb{R})$*. However,* $\det(\widehat{\Theta})$ *is identically zero, which clearly does not satisfy item (ii) of Theorem 2.1.3. As a consequence, the tight* 2*-multiframelet filter bank* $\{a; b\}_\Theta$ *does not have the perfect reconstruction property.*

The essence of OEP is to replace the original pair of refinable vector functions ϕ and

$\tilde{\phi}$ by another desired pair of refinable vector function $\mathring{\phi}$ and $\mathring{\tilde{\phi}}$ satisfying

$$\widehat{\mathring{\phi}}(\mathsf{M}\xi) = \widehat{\mathring{a}}(\xi)\widehat{\mathring{\phi}}(\xi), \quad \widehat{\mathring{\tilde{\phi}}}(\mathsf{M}\xi) = \widehat{\mathring{\tilde{a}}}(\xi)\widehat{\mathring{\tilde{\phi}}}(\xi),$$

with

$$\widehat{\mathring{a}}(\xi) := \widehat{\theta}(\mathsf{M}\xi)\widehat{a}(\xi)\widehat{\theta}(\xi)^{-1}, \quad \widehat{\mathring{\tilde{a}}}(\xi) := \widehat{\tilde{\theta}}(\mathsf{M}\xi)\widehat{\tilde{a}}(\xi)\widehat{\tilde{\theta}}(\xi)^{-1}.$$

When item (ii) of Theorem 2.1.3 fails, we see that $\Theta := \theta^\star * \tilde{\theta}$ is not strongly invertible, and consequently so is at least one of θ and $\tilde{\theta}$. In this case, at least one of the two refinement masks/filters $\mathring{a}$ and $\mathring{\tilde{a}}$ has infinite support, even though both $\mathring{\phi}$ and $\mathring{\tilde{\phi}}$ have compact support. In the multiframelet case ($r > 1$), the determinant of $\widehat{\Theta}$ could even be identically zero and therefore, the solution of the deconvolution problem is not unique at all. Hence, it appears impossible for framelets constructed through OEP to achieve both high vanishing moments and an efficient framelet transform simultaneously. The first breakthrough to knock down this dead-end for OEP is probably [33] showing the real advantage of OEP for $r > 1$. If Θ is strongly invertible, then the solution $\mathring{v}_0$ to the deconvolution problem $\tilde{v}_0 = \mathring{v}_0 * \Theta$ is simply given by $\mathring{v}_0 = \tilde{v}_0 * \Theta^{-1}$ (here $\Theta^{-1} \in (l_0(\mathbb{Z}))^{r\times r}$ is the filter with $\widehat{\Theta^{-1}} = \widehat{\Theta}^{-1}$) and the trouble of deconvolution is completely gone. Indeed, as proved in [33, Theorem 1.2], if $r > 1$, then one can always construct a dual M-framelet through OEP in Theorem 1.1.1 from any pair of matrix-valued filters such that the dual framelet has the highest possible vanishing moments and both θ and $\tilde{\theta}$ are strongly invertible (consequently, Θ is strongly invertible). It is the purpose of this chapter to show that we can always construct quasi-tight multiframelets, which are much stronger than simply dual multiframelets, with all desired properties being kept.

2.2 The Balancing Property of a Multi-level Discrete Framelet Transform

Next, we discuss the discrete vanishing moments and the balancing property of a discrete multiframelet transform. "Smooth" signals are often modelled by polynomial sequences. For $m \in \mathbb{N}$, by $\mathbb{P}_{m-1}$ we denote the space of all polynomial sequences of degree less than m. The sparsity of a discrete multiframelet transform is described by its ability to have zero framelet coefficients w_j for polynomial input data. The input to a discrete multiframelet transform is a vector sequence in $(l(\mathbb{Z}))^{1\times r}$, while most data in applications are scalar-valued, i.e., in $l(\mathbb{Z})$. Hence, we have to convert a scalar sequence into a vector sequence by using the *standard vector conversion operator*

$$\mathring{E} : l(\mathbb{Z}) \to (l(\mathbb{Z}))^{1\times r} \quad \text{with} \quad \mathring{E}v := [v(r\cdot), v(r\cdot+1), \ldots, v(r\cdot+(r-1))]. \tag{2.2.1}$$

Note that $\mathring{E}$ is a linear bijective mapping. Let $(\{a; b\}, \{\tilde{a}; \tilde{b}\})_\Theta$ be an OEP-based dual M-framelet filter bank and $(\{\mathring{\phi}; \psi\}, \{\mathring{\tilde{\phi}}; \tilde{\psi}\})$ be its corresponding dual M-framelet. Define $m := \operatorname{sr}(\tilde{a}, \mathsf{M})$ be its sum rule order. Ideally, since the multi-level discrete framelet transform is recursive, to have sparsity of a framelet transform, we hope that

$$\mathcal{T}_{b,\mathsf{M}}\mathring{E}(\mathsf{p}) = 0, \qquad \forall\, \mathsf{p} \in \mathbb{P}_{m-1} \tag{2.2.2}$$

and

$$\mathcal{T}_{a,\mathsf{M}}\mathring{E}(\mathsf{p}) \in \mathring{E}(\mathbb{P}_{m-1}), \qquad \forall\, \mathsf{p} \in \mathbb{P}_{m-1}. \tag{2.2.3}$$

The condition in (2.2.3) guarantees that the output signal $\mathcal{T}_{a,\mathsf{M}}\mathring{E}(\mathsf{p})$ is still in the space $\mathring{E}(\mathbb{P}_{m-1})$ for any input data $\mathsf{p} \in \mathring{E}(\mathbb{P}_{m-1})$. The condition in (2.2.2) preserves sparsity for all levels, that is, the framelet coefficients $w_j := \mathcal{T}_{b,\mathsf{M}}\mathcal{T}_{a,\mathsf{M}}^{j-1}\mathring{E}(\mathsf{p}) = 0$ for all $\mathsf{p} \in \mathbb{P}_{m-1}$ and $j \in \mathbb{N}$. Hence, we say that a filter b has m **balanced vanishing moments** if (2.2.2) holds. Moreover, we define $\operatorname{bvm}(b, \mathsf{M}) := m$ with m being the largest possible integer such that (2.2.2) holds. Similarly, we say that a discrete multiframelet transform (using the filter bank $(\{a;b\},\{\tilde{a};\tilde{b}\})_\Theta$) or a filter bank $\{a;b\}$ has m **balancing order** with respect to the dilation factor M if both (2.2.2) and (2.2.3) hold. In particular, we define $\operatorname{bo}(\{a;b\},\mathsf{M}) := m$ with m being the largest such integer satisfying both (2.2.2) and (2.2.3). For the case $r=1$, we always have $\operatorname{bo}(\{a;b\},\mathsf{M}) = \operatorname{bvm}(b,\mathsf{M}) = \operatorname{vm}(\psi)$. But for $r>1$, it was first observed in [56] that $\operatorname{bo}(\{a,b\},\mathsf{M}) < \operatorname{vm}(\psi)$ often happens. This reduced sparsity hurdles the applications of multiwavelets and multiframelets. How to remedy this shortcoming has been extensively studied in the function setting in [9, 65] and in the setting of discrete multiframelet transforms in [33, 35, 41].

The following result on characterizing the balanced vanishing moments and the balancing property is known (e.g., see [33, Theorem 4.4] or [41, Lemma 7.6.3]).

Theorem 2.2.1. *Let $\mathsf{M} \geqslant 2$ be a positive integer. Let $a \in (l_0(\mathbb{Z}))^{r\times r}$ and $b \in (l_0(\mathbb{Z}))^{s\times r}$. Define*

$$\widehat{\Upsilon}(\xi) := \left(1, e^{i\xi/r}, \ldots, e^{i(r-1)\xi/r}\right). \tag{2.2.4}$$

Then

(1) The filter b has m balanced vanishing moments satisfying (2.2.2) *if and only if*

$$\widehat{\Upsilon}(\xi)\overline{\widehat{b}(\xi)}^{\mathsf{T}} = \mathscr{O}(|\xi|^m), \qquad \xi \to 0. \tag{2.2.5}$$

(2) The filter bank $\{a;b\}$ has m balancing order satisfying both (2.2.2) *and* (2.2.3) *if and only if* (2.2.5) *holds and there exists $c \in l_0(\mathbb{Z})$ with $\widehat{c}(0) \neq 0$ such that*

$$\widehat{c}(\xi)\widehat{\Upsilon}(\xi)\overline{\widehat{a}(\xi)}^{\mathsf{T}} = \widehat{\Upsilon}(\mathsf{M}\xi) + \mathscr{O}(|\xi|^m), \quad \xi \to 0. \tag{2.2.6}$$

2.3 A Newly Developed Normal Form of a Matrix-valued Filter

In this section, we study the normal form of a matrix-valued filter for the case $d=1$. We say that a function f is **smooth near the origin** if all the derivatives of f at the origin exist. The main results of this section are the following two theorems.

Theorem 2.3.1. *Let* $\mathsf{M} \geqslant 2$ *be an integer and* $a \in (l_0(\mathbb{Z}))^{r\times r}$ *be a finitely supported matrix-valued filter. Let* ϕ *be an* $r\times 1$ *vector of compactly supported distributions satisfying* $\widehat{\phi}(\mathsf{M}\xi) = \widehat{a}(\xi)\widehat{\phi}(\xi)$ *with* $\widehat{\phi}(0) \neq 0$. *Suppose that the filter* a *has* m *sum rules with respect to* M *satisfying* (1.1.20) *with a matching filter* $\upsilon \in (l_0(\mathbb{Z}))^{1\times r}$ *such that* $\widehat{\upsilon}(0)\widehat{\phi}(0) = 1$. *Let* $\widehat{\mathring{\upsilon}}$ *be a* $1 \times r$ *row vector and* $\widehat{u_\phi}$ *be an* $r \times 1$ *column vector such that all the entries of* $\widehat{\mathring{\upsilon}}$ *and* $\widehat{u_\phi}$ *are functions which are smooth near the origin and*

$$\widehat{\mathring{\upsilon}}(\xi)\widehat{u_\phi}(\xi) = 1 + \mathscr{O}(|\xi|^m), \qquad \xi \to 0. \tag{2.3.1}$$

If the multiplicity $r \geqslant 2$, *then for any positive integer* $n \in \mathbb{N}$, *there exists a strongly invertible* $r \times r$ *matrix* $\widehat{U}$ *of* 2π*-periodic trigonometric polynomials such that*

$$\widehat{\upsilon}(\xi)\widehat{U}(\xi)^{-1} = \widehat{\mathring{\upsilon}}(\xi) + \mathscr{O}(|\xi|^m) \quad \text{and} \quad \widehat{U}(\xi)\widehat{\phi}(\xi) = \widehat{u_\phi}(\xi) + \mathscr{O}(|\xi|^n), \qquad \xi \to 0. \tag{2.3.2}$$

Define $\widehat{\mathring{\phi}}(\xi) := \widehat{U}(\xi)\widehat{\phi}(\xi)$ *and* $\widehat{\mathring{a}}(\xi) := \widehat{U}(\mathsf{M}\xi)\widehat{a}(\xi)\widehat{U}(\xi)^{-1}$. *Then the following statements hold:*

(i) The new vector function $\mathring{\phi}$ *is a vector of compactly supported distributions satisfying the refinement equation* $\widehat{\mathring{\phi}}(\mathsf{M}\xi) = \widehat{\mathring{a}}(\xi)\widehat{\mathring{\phi}}(\xi)$ *for all* $\xi \in \mathbb{R}$ *and* $\widehat{\mathring{\phi}}(\xi) = \widehat{u_\phi}(\xi) + \mathscr{O}(|\xi|^n)$ *as* $\xi \to 0$.

(ii) The new finitely supported matrix filter/mask $\mathring{a}$ *has* m *sum rules with respect to* M *with the matching filter* $\mathring{\upsilon}$ *such that* $\widehat{\mathring{\upsilon}}(0)\widehat{\mathring{\phi}}(0) = 1$, *i.e.,* (1.1.20) *holds with* a *and* υ *being replaced by* $\mathring{a}$ *and* $\mathring{\upsilon}$, *respectively.*

As a special case of Theorem 2.3.1, we have the following result.

Theorem 2.3.2. *Let* $\mathsf{M} \geqslant 2$ *be an integer and* $a \in (l_0(\mathbb{Z}))^{r\times r}$ *be a finitely supported matrix-valued filter. Let* ϕ *be an* $r\times 1$ *vector of compactly supported distributions satisfying* $\widehat{\phi}(\mathsf{M}\xi) = \widehat{a}(\xi)\widehat{\phi}(\xi)$ *with* $\widehat{\phi}(0) \neq 0$. *Suppose that the filter* a *has* m *sum rules with respect to* M *satisfying* (1.1.20) *with a matching filter* $\upsilon \in (l_0(\mathbb{Z}))^{1\times r}$ *and* $\widehat{\upsilon}(0)\widehat{\phi}(0) = 1$. *If the* *multiplicity $r \geqslant 2$, then for any positive integer $n \in \mathbb{N}$, there exists a strongly invertible $r \times r$ matrix $\widehat{U}$ of 2π-periodic trigonometric polynomials such that the following properties hold:*

(i) $\widehat{\mathring{a}}(\xi) := \widehat{U}(\mathsf{M}\xi)\widehat{a}(\xi)\widehat{U}(\xi)^{-1}$ *takes **the ideal (m, n)-normal form**, i.e.,*

$$\begin{bmatrix} (1+e^{-i\xi}+\cdots+e^{-i(\mathsf{M}-1)\xi})^m P_{1,1}(\xi) & (1-e^{-i\mathsf{M}\xi})^m P_{1,2}(\xi) \\ (1-e^{-i\xi})^n P_{2,1}(\xi) & P_{2,2}(\xi) \end{bmatrix}, \tag{2.3.3}$$

with

$$\widehat{\mathring{a}_{1,1}}(\xi) := (1+e^{-i\xi}+\cdots+e^{-i(\mathsf{M}-1)\xi})^m P_{1,1}(\xi) = 1 + \mathscr{O}(|\xi|^n), \quad \xi \to 0, \tag{2.3.4}$$

where $P_{1,1}, P_{1,2}, P_{2,1}$ and $P_{2,2}$ are some 1×1, $1\times(r-1)$, $(r-1)\times 1$ and $(r-1)\times(r-1)$ matrices of 2π-periodic trigonometric polynomials. Moreover, define

$$\widehat{\mathring{v}}(\xi) := \left(\widehat{\mathring{v}}_1(\xi), \dots, \widehat{\mathring{v}}_r(\xi)\right) := \widehat{v}(\xi)\widehat{U}(\xi)^{-1}, \tag{2.3.5}$$

$$\widehat{\mathring{\phi}}(\xi) := \left(\widehat{\mathring{\phi}}_1(\xi), \dots, \widehat{\mathring{\phi}}_r(\xi)\right)^{\mathsf{T}} := \widehat{U}(\xi)\widehat{\phi}(\xi), \tag{2.3.6}$$

we have $\widehat{\mathring{\phi}}(\mathsf{M}\xi) = \widehat{\mathring{a}}(\xi)\widehat{\mathring{\phi}}(\xi)$ with

$$\widehat{\mathring{\phi}}_1(\xi) = 1 + \mathscr{O}(|\xi|^n) \quad \text{and} \quad \widehat{\mathring{\phi}}_\ell(\xi) = \mathscr{O}(|\xi|^n), \quad \xi \to 0, \quad \ell = 2, \dots, r, \tag{2.3.7}$$

and $\mathring{a}$ has m sum rules with respect to M with the matching filter $\mathring{v}$ satisfying

$$\widehat{\mathring{v}}_1(\xi) = 1 + \mathscr{O}(|\xi|^m) \quad \text{and} \quad \widehat{\mathring{v}}_\ell(\xi) = \mathscr{O}(|\xi|^m), \quad \xi \to 0, \quad \ell = 2, \dots, r. \tag{2.3.8}$$

(ii) If in addition

$$\widehat{v}(\xi) = \frac{\overline{\widehat{\phi}(\xi)}^{\mathsf{T}}}{\|\widehat{\phi}(\xi)\|^2} + \mathscr{O}(|\xi|^m), \quad \xi \to 0, \tag{2.3.9}$$

then $\widehat{U}$ in item (i) can be chosen such that the following "almost orthogonal" struc-

ture holds:

$$\overline{\widehat{U}(\xi)}^{-\mathsf{T}}\widehat{U}(\xi)^{-1} = Diag\left(\|\widehat{\phi}(\xi)\|^2, \|\widehat{u_2}(\xi)\|^2, \ldots, \|\widehat{u_r}(\xi)\|^2\right) + \mathscr{O}(|\xi|^{\max(m,n)}), \quad \xi \to 0, \tag{2.3.10}$$

where $\widehat{u_j}$ is the j-th column of $\widehat{U}^{-1}$ for $j = 2, \ldots, r$.

Conversely, if there exists $\widehat{U}$ such that item (i) and (2.3.10) *hold, then* (2.3.9) *must hold.*

We make some comments on the importance of the normal form of a matrix-valued filter in the study of refinable vector functions and multiwavelets/multiframelets. The normal form (also called the canonical form) of a matrix-valued filter was initially introduced in [45, Theorem 2.2] for dimension one and was further developed in [32, Proposition 2.4] for high dimensions to study multivariate vector subdivision schemes and multivariate refinable vector functions. In the scalar case (i.e., $r = 1$), recall that a scalar filter a has m sum rules with respect to the dilation factor M if and only if $(1 + e^{-i\xi} + \cdots + e^{-i(\mathsf{M}-1)\xi})^m \mid \widehat{a}(\xi)$. That is, $\widehat{a}(\xi) = (1 - e^{-i\mathsf{M}\xi})^m A(\xi)(1 - e^{-i\xi})^{-m} = (1 + e^{-i\xi} + \cdots + e^{-i(\mathsf{M}-1)\xi})^m A(\xi)$ for a unique 2π-periodic trigonometric polynomial $A(\xi)$. Now consider the case $r > 1$. If a filter $\mathring{a}$ takes the ideal normal form (2.3.3) in item (i) of Theorem 2.3.2, we can factorize $\mathring{a}$ as $\widehat{\mathring{a}}(\xi) = B(\mathsf{M}\xi)A(\xi)B(\xi)^{-1}$ with

$$B(\xi) := \widehat{U}(\xi)^{-1}\begin{bmatrix} (1 - e^{-i\xi})^m & \\ & I_{r-1} \end{bmatrix}, \quad A(\xi) := \begin{bmatrix} P_{1,1}(\xi) & P_{1,2}(\xi) \\ (1 - e^{-i\xi})^{m+n}P_{2,1}(\xi) & P_{2,2}(\xi) \end{bmatrix}.$$

The above factorization of a matrix-valued filter allows us to theoretically study and construct multiwavelets/multiframelets with high vanishing moments from refinable vector functions, in almost the same way as the scalar case using the popular factorization technique in the scalar case (i.e., $r = 1$). On the other hand, as we will see later in the construction of OEP-based quasi-tight multiframelets, the almost orthogonal structure introduced in item (ii) of Theorem 2.3.2 is the key to achieve the balancing property of the associated discrete multiframelet transform.

To prove Theorems 2.3.1 and 2.3.2, we need a few auxiliary results.

Lemma 2.3.3. *Let $\widehat{v} = (\widehat{v_1}, \dots, \widehat{v_r})$ and $\widehat{u} = (\widehat{u_1}, \dots, \widehat{u_r})$ be $1 \times r$ vectors of functions which are smooth near the origin such that $\widehat{v}(0) \neq 0$ and $\widehat{u}(0) \neq 0$. If $r \geqslant 2$, then for any positive integer $n \in \mathbb{N}$, there exists a strongly invertible $r \times r$ matrix $\widehat{U}$ of 2π-periodic trigonometric polynomials such that*

$$\widehat{u}(\xi) = \widehat{v}(\xi)\widehat{U}(\xi) + \mathscr{O}(|\xi|^n), \qquad \xi \to 0. \tag{2.3.11}$$

Proof. We first prove the claim for the special case $\widehat{u}(\xi) = (1, 0, \dots, 0) + \mathscr{O}(|\xi|^n)$ as $\xi \to 0$. Since $\widehat{v}(0) \neq 0$, by permuting the entries of $\widehat{v}$, we can assume that $\widehat{v_1}(0) \neq 0$. Moreover, since $\widehat{v}$ is smooth near the origin, we can find a $1 \times r$ vector $\widehat{\mathring{v}}$ of 2π-periodic trigonometric polynomials such that $\widehat{v}(\xi) = \widehat{\mathring{v}}(\xi) + \mathscr{O}(|\xi|^n)$ as $\xi \to 0$. Hence, without loss of generality, we assume that $\widehat{v}$ is a vector of 2π-periodic trigonometric polynomials. Since $\widehat{v_1}(0) \neq 0$, there exist 2π-periodic trigonometric polynomials $\widehat{w_j}(\xi), j = 2, \dots, r$ such that

$$\widehat{w_j}(\xi) = -\widehat{v_j}(\xi)/\widehat{v_1}(\xi) + \mathscr{O}(|\xi|^n), \qquad \xi \to 0, \quad j = 2, \dots, r.$$

Define

$$\widehat{U_1}(\xi) := \begin{bmatrix} 1 & \widehat{w_2}(\xi) & \cdots & \widehat{w_r}(\xi) \\ 0 & 1 & \cdots & 0 \\ & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}.$$

Since $\det(\widehat{U_1}(\xi)) = 1$, $\widehat{U_1}$ is strongly invertible and

$$\widehat{v}(\xi)\widehat{U_1}(\xi) = (\widehat{v_1}(\xi), 0, \dots, 0) + \mathscr{O}(|\xi|^n), \qquad \xi \to 0. \tag{2.3.12}$$

Note that $\widehat{v_1}$ is a 2π-periodic trigonometric polynomial with $\widehat{v_1}(0) \neq 0$. We now adopt an idea in the proof of [33, Theorem 2.1] to prove the claim. Choose $c \in l_0(\mathbb{Z})$ such that

$$\widehat{c}(\xi) := \frac{1}{\widehat{v_1}(\xi)} + \mathscr{O}(|\xi|^{2n}), \qquad \xi \to 0, \tag{2.3.13}$$

and define $d \in l_0(\mathbb{Z})$ via

$$\widehat{d}(\xi) = \sum_{k=1}^{2n} (-1)^{k-1} \binom{2n}{k} \frac{[\widehat{c}(\xi)]^{k-1}}{[\widehat{c}(0)]^k}. \tag{2.3.14}$$

Note that

$$(1 - \widehat{c}(\xi)/\widehat{c}(0))^{2n} = 1 - \widehat{c}(\xi)\widehat{d}(\xi). \tag{2.3.15}$$

Due to our assumption $r \geqslant 2$, we can define

$$\widehat{U_2}(\xi) = \begin{bmatrix} \widehat{c}(\xi) & -(1-\widehat{c}(\xi)/\widehat{c}(0))^n & 0 \\ (1-\widehat{c}(\xi)/\widehat{c}(0))^n & \widehat{d}(\xi) & 0 \\ 0 & 0 & I_{r-2} \end{bmatrix}.$$

Using (2.3.12) and (2.3.15), we trivially conclude that

$$\widehat{v}(\xi)\widehat{U_1}(\xi)\widehat{U_2}(\xi) = (\widehat{v_1}(\xi), 0, \ldots, 0)\widehat{U_2}(\xi) + \mathscr{O}(|\xi|^n) = (\widehat{v_1}(\xi)\widehat{c}(\xi), 0, \ldots, 0) + \mathscr{O}(|\xi|^n)$$

as $\xi \to 0$. Due to (2.3.13) and (2.3.15), we have $\widehat{v_1}(\xi)\widehat{c}(\xi) = 1 + \mathscr{O}(|\xi|^n)$ as $\xi \to 0$ and $\det(\widehat{U_2}(\xi)) = 1$. Hence, $\widehat{U_2}$ is strongly invertible and $\widehat{v}(\xi)\widehat{U_1}(\xi)\widehat{U_2}(\xi) = (1, 0, \ldots, 0) + \mathscr{O}(|\xi|^n)$ as $\xi \to 0$. The proof is completed for the special case of $\widehat{u}$ by taking $\widehat{U}(\xi) := \widehat{U_1}(\xi)\widehat{U_2}(\xi)$.

Generally, by what has been proved, there exist strongly invertible matrices $\widehat{U_v}$ and $\widehat{U_u}$ such that

$$\widehat{v}(\xi)\widehat{U_v}(\xi) = (1, 0, \ldots, 0) + \mathscr{O}(|\xi|^n), \qquad \widehat{u}(\xi)\widehat{U_u}(\xi) = (1, 0, \ldots, 0) + \mathscr{O}(|\xi|^n), \qquad \xi \to 0.$$

Define $\widehat{U}(\xi) := \widehat{U_v}(\xi)\widehat{U_u}(\xi)^{-1}$. Then $\widehat{U}$ is strongly invertible and (2.3.11) holds. $\square$

Note that Lemma 2.3.3 often fails for $r = 1$, since (2.3.11) holds for $r = 1$ if and only if $\widehat{u}(\xi)/\widehat{v}(\xi) = ce^{-ik\xi} + \mathscr{O}(|\xi|^n)$ as $\xi \to 0$ for some $c \neq 0$ and $k \in \mathbb{Z}$.

The following result shows that the condition in (2.3.1) of Theorem 2.3.1 is also a necessary condition.

Lemma 2.3.4. *Let $\mathsf{M} \geqslant 2$ be an integer and $a \in (l_0(\mathbb{Z}))^{r\times r}$ be a finitely supported matrix-valued filter. Let ϕ be an $r \times 1$ vector of compactly supported distributions satisfying $\widehat{\phi}(\mathsf{M}\xi) = \widehat{a}(\xi)\widehat{\phi}(\xi)$ with $\widehat{\phi}(0) \neq 0$. Suppose that the filter a has m sum rules with respect to M satisfying (1.1.20) with a matching filter $\upsilon \in (l_0(\mathbb{Z}))^{1\times r}$ such that $\widehat{\upsilon}(0)\widehat{\phi}(0) = 1$. Then*

$$\widehat{\upsilon}(\xi)\widehat{\phi}(\xi) = 1 + \mathscr{O}(|\xi|^m), \quad \xi \to 0. \tag{2.3.16}$$

Proof. The claim is essentially known in [32, Proposition 3.2]. Here we provide a simple proof. Since the filter a satisfies (1.1.20), we have $\widehat{\upsilon}(\mathsf{M}\xi)\widehat{a}(\xi) = \widehat{\upsilon}(\xi) + \mathscr{O}(|\xi|^m)$ as $\xi \to 0$. Now we deduce from $\widehat{\phi}(\mathsf{M}\xi) = \widehat{a}(\xi)\widehat{\phi}(\xi)$ that

$$\widehat{\upsilon}(\mathsf{M}\xi)\widehat{\phi}(\mathsf{M}\xi) = \widehat{\upsilon}(\mathsf{M}\xi)\widehat{a}(\xi)\widehat{\phi}(\xi) = \widehat{\upsilon}(\xi)\widehat{\phi}(\xi) + \mathscr{O}(|\xi|^m), \quad \xi \to 0. \tag{2.3.17}$$

Considering the derivatives of the function $\widehat{\upsilon}(\xi)\widehat{\phi}(\xi)$ at $\xi = 0$, (2.3.17) yields

$$\mathsf{M}^j[\widehat{\upsilon}\widehat{\phi}]^{(j)}(0) = [\widehat{\upsilon}\widehat{\phi}]^{(j)}(0), \qquad \forall j = 0, \dots, m-1.$$

Since we assumed $\widehat{\upsilon}(0)\widehat{\phi}(0) = 1$ and $\mathsf{M} \geqslant 2$, we can straightforwardly deduce that (2.3.16) must hold. $\square$

Lemma 2.3.5. *Let $m \in \mathbb{N}$ be a positive integer. Let $\widehat{\upsilon}$ be a $1 \times r$ row vector and $\widehat{u}$ be an $r \times 1$ column vector such that all the entries of $\widehat{\upsilon}$ and $\widehat{u}$ are functions which are smooth near the origin such that*

$$\widehat{\upsilon}(\xi)\widehat{u}(\xi) = 1 + \mathscr{O}(|\xi|^m), \quad \xi \to 0. \tag{2.3.18}$$

For any positive integer n, there must exist $1 \times r$ vector $\widehat{\mathring{\upsilon}}$ of functions which are smooth *near the origin such that*

$$\widehat{\mathring{v}}(\xi)=\widehat{v}(\xi)+\mathscr{O}(|\xi|^m) \quad \textit{and} \quad \widehat{\mathring{v}}(\xi)\widehat{u}(\xi)=1+\mathscr{O}(|\xi|^n), \quad \xi\to 0. \tag{2.3.19}$$

Proof. If $n\leqslant m$, then we can simply take $\widehat{\mathring{v}}:=\widehat{v}$ and it follows directly from our assumption in (2.3.18) that (2.3.19) trivially holds. So, we assume $n>m$. We consider two cases $r=1$ and $r>1$. If $r=1$, then $\widehat{u}(0)\neq 0$. Taking $\widehat{\mathring{v}}(\xi):=1/\widehat{u}(\xi)$, we see that (2.3.19) is satisfied.

Suppose that $r>1$. By Lemma 2.3.3, there exists a strongly invertible $r\times r$ matrix $\widehat{U}$ such that $\widehat{\breve{u}}(\xi):=\widehat{U}(\xi)\widehat{u}(\xi)=(1,0,\ldots,0)^{\mathsf{T}}+\mathscr{O}(|\xi|^n)$ as $\xi\to 0$. We define $\widehat{\breve{v}}(\xi)=(\widehat{\breve{v}}_1(\xi),\ldots,\widehat{\breve{v}}_r(\xi)):=\widehat{v}(\xi)\widehat{U}(\xi)^{-1}$. Then it follows from (2.3.18) that

$$\widehat{\breve{v}}_1(\xi)=\widehat{\breve{v}}(\xi)\widehat{\breve{u}}(\xi)+\mathscr{O}(|\xi|^n)=\widehat{v}(\xi)\widehat{U}(\xi)^{-1}\widehat{U}(\xi)\widehat{u}(\xi)+\mathscr{O}(|\xi|^n)=\widehat{v}(\xi)\widehat{u}(\xi)=1+\mathscr{O}(|\xi|^m),$$

as $\xi\to 0$. We define $\widehat{\mathring{v}}(\xi):=(1,\widehat{\breve{v}}_2(\xi),\ldots,\widehat{\breve{v}}_r(\xi))\widehat{U}(\xi)$. Then

$$\widehat{\mathring{v}}(\xi)\widehat{u}(\xi)=(1,\widehat{\breve{v}}_2(\xi),\ldots,\widehat{\breve{v}}_r(\xi))\widehat{U}(\xi)\widehat{u}(\xi)=(1,\widehat{\breve{v}}_2(\xi),\ldots,\widehat{\breve{v}}_r(\xi))\widehat{\breve{u}}(\xi)=1+\mathscr{O}(|\xi|^n)$$

as $\xi\to 0$. By $\widehat{\breve{v}}_1(\xi)=1+\mathscr{O}(|\xi|^m)$ as $\xi\to 0$ and noting $\widehat{\breve{v}}(\xi)=\widehat{v}(\xi)\widehat{U}(\xi)^{-1}$, we have

$$\widehat{\mathring{v}}(\xi)=\widehat{\breve{v}}(\xi)\widehat{U}(\xi)+\mathscr{O}(|\xi|^m)=\widehat{v}(\xi)+\mathscr{O}(|\xi|^m), \quad \xi\to 0.$$

This completes the proof. $\square$

We are now ready to prove Theorem 2.3.1, which includes all the results on the normal form of a matrix-valued filter in [32,33,41,45] as special cases for dimension one. Following the lines of our proof for Theorem 2.3.1 below, we also point out that Theorem 2.3.1 can be generalized without much difficulty to multidimensional matrix-valued filters.

Proof of Theorem 2.3.1. Obviously, it suffices to prove the claims for $n\geqslant m$. By Lemma 2.3.4, we see that (2.3.16) holds. By our assumption in (2.3.1) and the fact that

$\widehat{\phi}$ is smooth at every $\xi \in \mathbb{R}$ (because ϕ is a vector of compactly supported distributions), using Lemma 2.3.5, without loss of generality we can assume that

$$\widehat{\upsilon}(\xi)\widehat{\phi}(\xi) = 1 + \mathscr{O}(|\xi|^n) \quad \text{and} \quad \widehat{\mathring{\upsilon}}(\xi)\widehat{u_\phi}(\xi) = 1 + \mathscr{O}(|\xi|^n), \qquad \xi \to 0. \tag{2.3.20}$$

Define $\widehat{\breve{\upsilon}}(\xi) := (1, 0, \ldots, 0)$. Since $\widehat{\mathring{\upsilon}}(0) \neq 0$ and $\widehat{\upsilon}(0) \neq 0$, by Lemma 2.3.3, there exist strongly invertible $r \times r$ matrices $\widehat{U_1}$ and $\widehat{U_2}$ of 2π-periodic trigonometric polynomials such that

$$\widehat{\breve{\upsilon}}(\xi) = \widehat{\mathring{\upsilon}}(\xi)\widehat{U_1}(\xi) + \mathscr{O}(|\xi|^n) \quad \text{and} \quad \widehat{\upsilon}(\xi) = \widehat{\breve{\upsilon}}(\xi)\widehat{U_2}(\xi) + \mathscr{O}(|\xi|^n), \quad \xi \to 0. \tag{2.3.21}$$

Define

$$\widehat{\breve{u}_\phi}(\xi) := \widehat{U_1}(\xi)^{-1}\widehat{u_\phi}(\xi), \qquad \widehat{\breve{\phi}}(\xi) := \widehat{U_2}(\xi)\widehat{\phi}(\xi), \qquad \text{and} \qquad \widehat{\breve{a}}(\xi) := \widehat{U_2}(\mathsf{M}\xi)\widehat{a}(\xi)\widehat{U_2}(\xi)^{-1}.$$

Then it is trivial to check that $\widehat{\breve{\phi}}(\mathsf{M}\xi) = \widehat{\breve{a}}(\xi)\widehat{\breve{\phi}}(\xi)$ and $\breve{a}$ has m sum rules with the matching filter $\breve{\upsilon}$. Write $\breve{u}_\phi = (\breve{u}_1, \ldots, \breve{u}_r)^\mathsf{T}$. Using (2.3.20) and (2.3.21) as well as $\widehat{\breve{\upsilon}}(\xi) = (1, 0, \ldots, 0)$, we observe that

$$\widehat{\breve{u}_1}(\xi) = \widehat{\breve{\upsilon}}(\xi)\widehat{\breve{u}_\phi}(\xi) = \widehat{\mathring{\upsilon}}(\xi)\widehat{U_1}(\xi)\widehat{U_1}(\xi)^{-1}\widehat{u_\phi}(\xi) + \mathscr{O}(|\xi|^n) = \widehat{\mathring{\upsilon}}(\xi)\widehat{u_\phi}(\xi) = 1 + \mathscr{O}(|\xi|^n), \quad \xi \to 0.$$

Write $\breve{\phi} = (\breve{\phi}_1, \ldots, \breve{\phi}_r)^\mathsf{T}$. Since $\widehat{\breve{\upsilon}}(\xi) = (1, 0, \ldots, 0)$, we deduce from (2.3.20) and (2.3.21) that

$$\widehat{\breve{\phi}_1}(\xi) = \widehat{\breve{\upsilon}}(\xi)\widehat{\breve{\phi}}(\xi) = \widehat{\upsilon}(\xi)\widehat{U_2}(\xi)^{-1}\widehat{U_2}(\xi)\widehat{\phi}(\xi) = \widehat{\upsilon}(\xi)\widehat{\phi}(\xi) = 1 + \mathscr{O}(|\xi|^n), \quad \xi \to 0.$$

There exist 2π-periodic trigonometric polynomials $\widehat{w_\ell}$ for all $\ell = 2, \ldots, r$ such that

$$\widehat{w_\ell}(\xi) = \widehat{\breve{u}_\ell}(\xi) - \widehat{\breve{\phi}_\ell}(\xi) + \mathscr{O}(|\xi|^n), \quad \xi \to 0, \ell = 2, \ldots, r.$$

Define

$$\widehat{U_3}(\xi) := \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \widehat{w_2}(\xi) & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \widehat{w_r}(\xi) & 0 & \cdots & 1 \end{bmatrix}.$$

Since $\det(\widehat{U_3}(\xi)) = 1$, the matrix $\widehat{U_3}$ is strongly invertible. Moreover, by the definition of $\widehat{w_\ell}$, we have

$$\widehat{U_3}(\xi)\widehat{\breve{\phi}}(\xi) = \widehat{\breve{u}_\phi}(\xi) + \mathscr{O}(|\xi|^n), \qquad \xi \to 0, \tag{2.3.22}$$

where we also used $\widehat{\breve{u}_1}(\xi) = 1 + \mathscr{O}(|\xi|^n)$ and $\widehat{\breve{\phi}_1}(\xi) = 1 + \mathscr{O}(|\xi|^n)$ as $\xi \to 0$.

Define $\widehat{U}(\xi) := \widehat{U_1}(\xi)\widehat{U_3}(\xi)\widehat{U_2}(\xi)$. Then $\widehat{U}$ is strongly invertible and we now prove that all the claims in Theorem 2.3.1 are satisfied. We first check (2.3.1). Using (2.3.21) and $n \geqslant m$, we have

$$\begin{aligned}\widehat{v}(\xi)\widehat{U}(\xi)^{-1} &= \widehat{v}(\xi)\widehat{U_2}(\xi)^{-1}\widehat{U_3}(\xi)^{-1}\widehat{U_1}(\xi)^{-1} = \widehat{\breve{v}}(\xi)\widehat{U_3}(\xi)^{-1}\widehat{U_1}(\xi)^{-1} + \mathscr{O}(|\xi|^n)\\ &= \widehat{\breve{v}}(\xi)\widehat{U_1}(\xi)^{-1} + \mathscr{O}(|\xi|^n) = \widehat{\mathring{v}}(\xi) + \mathscr{O}(|\xi|^n) = \widehat{\mathring{v}}(\xi) + \mathscr{O}(|\xi|^m),\end{aligned}$$

as $\xi \to 0$, since the first row of $\widehat{U_3}(\xi)^{-1}$ is $[1, 0, \dots, 0]$ and $\widehat{\breve{v}}(\xi) = (1, 0, \dots, 0)$. Similarly, by $\widehat{\breve{\phi}}(\xi) = \widehat{U_2}(\xi)\widehat{\phi}(\xi)$ and using (2.3.22), as $\xi \to 0$, we have

$$\widehat{U}(\xi)\widehat{\phi}(\xi) = \widehat{U_1}(\xi)\widehat{U_3}(\xi)\widehat{U_2}(\xi)\widehat{\phi}(\xi) = \widehat{U_1}(\xi)\widehat{U_3}(\xi)\widehat{\breve{\phi}}(\xi) = \widehat{U_1}(\xi)\widehat{\breve{u}_\phi}(\xi) + \mathscr{O}(|\xi|^n) = \widehat{u_\phi}(\xi) + \mathscr{O}(|\xi|^n),$$

where in the last identity we used the definition $\widehat{\breve{u}_\phi}(\xi) = \widehat{U_1}(\xi)^{-1}\widehat{u_\phi}(\xi)$. This proves (2.3.2).

We now check items (i) and (ii). By $\widehat{\phi}(\mathsf{M}\xi) = \widehat{a}(\xi)\widehat{\phi}(\xi)$, we obviously have

$$\widehat{\mathring{\phi}}(\mathsf{M}\xi) = \widehat{U}(\mathsf{M}\xi)\widehat{\phi}(\mathsf{M}\xi) = \widehat{U}(\mathsf{M}\xi)\widehat{a}(\xi)\widehat{\phi}(\xi) = \widehat{\mathring{a}}(\xi)\widehat{\mathring{\phi}}(\xi).$$

Now by (2.3.1), we have $\widehat{\mathring{\phi}}(\xi) = \widehat{U}(\xi)\widehat{\phi}(\xi) = \widehat{u_\phi}(\xi) + \mathscr{O}(|\xi|^n)$ as $\xi \to 0$. This proves item (i).

Since $\widehat{U}$ is strongly invertible, the filter $\mathring{a}$ must be finitely supported. Since a satisfies

(1.1.20) and (2.3.1) holds, for $\gamma = 0, \dots, \mathsf{M}-1$, we have

$$\begin{aligned}
&\widehat{\mathring{v}}(\mathsf{M}\xi)\widehat{\mathring{a}}(\xi + \tfrac{2\pi\gamma}{\mathsf{M}}) = \widehat{v}(\mathsf{M}\xi)\widehat{U}(\mathsf{M}\xi)^{-1}\widehat{U}(\mathsf{M}\xi)\widehat{a}(\xi + \tfrac{2\pi\gamma}{\mathsf{M}})\widehat{U}(\xi + \tfrac{2\pi\gamma}{\mathsf{M}})^{-1}\\
=&\widehat{v}(\mathsf{M}\xi)\widehat{a}(\xi + \tfrac{2\pi\gamma}{\mathsf{M}})\widehat{U}(\xi + \tfrac{2\pi\gamma}{\mathsf{M}})^{-1} = \boldsymbol{\delta}(\gamma)\widehat{v}(\xi)\widehat{U}(\xi)^{-1} + \mathscr{O}(|\xi|^m) = \boldsymbol{\delta}(\gamma)\widehat{\mathring{v}}(\xi) + \mathscr{O}(|\xi|^m),,
\end{aligned}$$

as $\xi \to 0$, which proves item (ii). $\square$

Finally, we prove Theorem 2.3.2.

Proof of Theorem 2.3.2. We first prove item (i). By Theorem 2.3.1, there exists a strongly invertible $r \times r$ matrix $\widehat{U}$ of 2π-periodic trigonometric polynomials such that all the claims of Theorem 2.3.1 hold with $\widehat{\mathring{v}}(\xi) = (1, 0, \dots, 0)$ and $\widehat{u_\phi}(\xi) = (1, 0, \dots, 0)^\mathsf{T}$. Now by item (ii) of Theorem 2.3.1, we conclude that

$$\widehat{\mathring{a}_{1,1}}(\xi + \tfrac{2\pi\gamma}{\mathsf{M}}) = \mathscr{O}(|\xi|^m), \qquad \xi \to 0, \quad \gamma = 1, \dots, \mathsf{M}-1 \tag{2.3.23}$$

and

$$\widehat{\mathring{a}_{1,2}}(\xi + \tfrac{2\pi\gamma}{\mathsf{M}}) = \mathscr{O}(|\xi|^m), \qquad \xi \to 0, \quad \gamma = 0, \dots, \mathsf{M}-1. \tag{2.3.24}$$

(2.3.23) is equivalent to $(1 + e^{-i\xi} + \dots + e^{-i(\mathsf{M}-1)\xi})^m \mid \widehat{\mathring{a}_{1,1}}(\xi)$, and (2.3.24) is equivalent to $(1 - e^{-i\mathsf{M}\xi})^m \mid \widehat{\mathring{a}_{1,2}}(\xi)$. On the other hand, we have $\widehat{\mathring{\phi}}(\xi) = \widehat{U}(\xi)\widehat{\phi}(\xi) = \widehat{u_\phi}(\xi) = (1, 0, \dots, 0)^\mathsf{T} + \mathscr{O}(|\xi|^n)$ as $\xi \to 0$, which is simply (2.3.7). Observing that $\widehat{\mathring{\phi}}(\mathsf{M}\xi) = \widehat{\mathring{a}}(\xi)\widehat{\mathring{\phi}}(\xi)$, we conclude from (2.3.7) that $\widehat{\mathring{a}_{1,1}}(\xi) = 1 + \mathscr{O}(|\xi|^n)$ and $\widehat{\mathring{a}_{2,1}}(\xi) = \mathscr{O}(|\xi|^n)$ as $\xi \to 0$. Thus (2.3.3) and (2.3.4) hold, and this proves item (i).

Next, we prove item (ii). By Theorem 2.3.1, there exists a strongly invertible filter $V \in (l_0(\mathbb{Z}))^{r\times r}$ such that

$$\widehat{v}(\xi)\widehat{V}(\xi) = \widehat{\mathring{v}}(\xi) = (1, 0, \dots, 0) + \mathscr{O}(|\xi|^m), \quad \widehat{V}(\xi)^{-1}\widehat{\phi}(\xi) = \widehat{\mathring{\phi}}(\xi) = (1, 0, \dots, 0)^\mathsf{T} + \mathscr{O}(|\xi|^{\tilde{n}}),$$

as $\xi \to 0$, where $\tilde{n} = \max(m, n)$. It follows from (2.3.9) and the above identities that

$$(1, 0, \dots, 0)\widehat{V}^{-1}(\xi) = \frac{\overline{\widehat{\phi}(\xi)}^{\mathsf{T}}}{\|\widehat{\phi}(\xi)\|^2} + \mathscr{O}(|\xi|^m), \quad (1, 0, \dots, 0)\overline{\widehat{V}(\xi)}^{\mathsf{T}} = \overline{\widehat{\phi}(\xi)}^{\mathsf{T}} + \mathscr{O}(|\xi|^{\tilde{n}}), \quad (2.3.25)$$

as $\xi \to 0$. For $j = 1, \dots, r$, denote $\widehat{V_j}$ the j-th column of $\widehat{V}$. It is easy to see from (2.3.25) that $\widehat{V_1}(\xi) = \widehat{\phi}(\xi) + \mathscr{O}(|\xi|^{\tilde{n}})$ as $\xi \to 0$. Set $\widehat{u_1}(\xi) := \widehat{V_1}(\xi)$ and choose $g_1 \in l_0(\mathbb{Z})$ such that $\widehat{g_1}(\xi) = \frac{1}{\|\widehat{\phi}(\xi)\|^2} + \mathscr{O}(|\xi|^{\tilde{n}})$ as $\xi \to 0$. For $j = 2, \dots, r$, define $u_j \in (l_0(\mathbb{Z}))^{r\times 1}$ and choose $g_j \in l_0(\mathbb{Z})$ recursively via

$$\widehat{u_j}(\xi) = \widehat{V_j}(\xi) - \sum_{l=1}^{j-1} \widehat{V_j}(\xi)^{\mathsf{T}}\overline{\widehat{u_l}(\xi)}\widehat{g_l}(\xi)\widehat{u_l}(\xi), \quad (2.3.26)$$

$$\widehat{g_j}(\xi) = \frac{1}{\|\widehat{u_j}(\xi)\|^2} + \mathscr{O}(|\xi|^{\tilde{n}}), \quad \xi \to 0. \quad (2.3.27)$$

Define

$$\widehat{U}(\xi)^{-1} := [\widehat{u_1}(\xi), \widehat{u_2}(\xi), \dots, \widehat{u_r}(\xi)] = [\widehat{\phi}(\xi) + \mathscr{O}(|\xi|^{\tilde{n}}), \widehat{u_2}(\xi), \dots, \widehat{u_r}(\xi)], \quad \xi \to 0. \quad (2.3.28)$$

By our construction, it is not hard to verify that all claims of item (i) hold. Moreover, we have $\det(\widehat{U}^{-1}) = \det(\widehat{V})$, which implies that U is strongly invertible.

For $j = 1, \dots, r$, we have

$$\widehat{u_j}(\xi) = \left(\widehat{V_j}(\xi) - \sum_{l=1}^{j-1} \widehat{V_j}(\xi)^{\mathsf{T}}\overline{\widehat{u_l}(\xi)}\frac{\widehat{u_l}(\xi)}{\|\widehat{u_l}(\xi)\|^2}\right) + \mathscr{O}(|\xi|^{\tilde{n}}), \quad \xi \to 0.$$

This means whenever $j \neq k$, we have

$$\overline{\widehat{u_j}(\xi)}^{\mathsf{T}}\widehat{u_k}(\xi) = \mathscr{O}(|\xi|^{\tilde{n}}), \quad \xi \to 0. \quad (2.3.29)$$

Note that the first column of $\widehat{U}^{-1}$ is $\widehat{V_1}$. It follows that

$$\overline{\widehat{U}(\xi)}^{-\mathsf{T}}\widehat{U}(\xi)^{-1} = \mathrm{Diag}\left(\|\widehat{\phi}(\xi)\|^2, \|\widehat{u_2}(\xi)\|^2, \dots, \|\widehat{u_r}(\xi)\|^2\right) + \mathscr{O}(|\xi|^{\tilde{n}}), \quad \xi \to 0.$$

Hence (2.3.10) holds since $\tilde{n} = \max(m, n)$. Hence item (ii) is proved.

Conversely, suppose that item (i) and (2.3.10) hold. As $\widehat{U}$ is strongly invertible, we see from (2.3.10) that $\|\widehat{\phi}(0)\|^2 \neq 0$. Now use item (i), (2.3.10) and $\max(m, n) \geqslant m$, we have

$$\widehat{\upsilon}(\xi) = (1, 0, \dots, 0)\widehat{U}(\xi) + \mathscr{O}(|\xi|^m) = \frac{1}{\|\widehat{\phi}(\xi)\|^2}(1, 0, \dots, 0)\overline{\widehat{U}(\xi)}^{-\mathsf{T}}\widehat{U}(\xi)^{-1}\widehat{U}(\xi) + \mathscr{O}(|\xi|^m)$$

$$= \frac{1}{\|\widehat{\phi}(\xi)\|^2}(1, 0, \dots, 0)\overline{\widehat{U}(\xi)}^{-\mathsf{T}} + \mathscr{O}(|\xi|^m) = \frac{\overline{\widehat{\phi}(\xi)}^{\mathsf{T}}}{\|\widehat{\phi}(\xi)\|^2} + \mathscr{O}(|\xi|^m), \quad \xi \to 0.$$

This proves (2.3.9). This completes the proof of Theorem 2.3.2. $\square$

2.4 The Main Theorem

In this section, we establish our main result on one-dimensional quasi-tight multiframelets with high balancing orders. The main theorem is stated as the following.

Theorem 2.4.1. *Let $\mathsf{M} \geqslant 2$ be an integer and $\phi \in (L_2(\mathbb{R}))^r$ be a compactly supported M-refinable vector function with a matrix-valued refinement filter/mask $a \in (l_0(\mathbb{Z}))^{r\times r}$ satisfying $\widehat{\phi}(\mathsf{M}\xi) = \widehat{a}(\xi)\widehat{\phi}(\xi)$. Suppose that the filter a has m sum rules with respect to the dilation factor M satisfying (1.1.20) with a matching filter $\upsilon \in (l_0(\mathbb{Z}))^{1\times r}$ such that $\widehat{\upsilon}(0)\widehat{\phi}(0) = 1$. If the multiplicity $r \geqslant 2$, then there exist filters $\theta \in (l_0(\mathbb{Z}))^{r\times r}$, $b \in (l_0(\mathbb{Z}))^{s\times r}$ and $\epsilon_1, \dots, \epsilon_s \in \{\pm 1\}$ such that*

(1) $\{\mathring{\phi}; \psi\}_{(\epsilon_1,\dots,\epsilon_s)}$ is a compactly supported quasi-tight M-framelet in $L_2(\mathbb{R})$, where $\widehat{\mathring{\phi}}(\xi) := \widehat{\theta}(\xi)\widehat{\phi}(\xi)$ and $\widehat{\psi}(\xi) := \widehat{b}(\xi/\mathsf{M})\widehat{\phi}(\xi/\mathsf{M})$. Furthermore, ψ has m vanishing moments.

(2) θ is strongly invertible. Moreover, the filter bank $\{\mathring{a}; \mathring{b}\}_{(\epsilon_1,\dots,\epsilon_s)}$ is a finitely supported quasi-tight M-framelet filter bank, i.e.,

$$\overline{\widehat{\mathring{a}}(\xi)}^{\mathsf{T}}\widehat{\mathring{a}}(\xi) + \overline{\widehat{\mathring{b}}(\xi)}^{\mathsf{T}} Diag(\epsilon_1, \dots, \epsilon_s)\widehat{\mathring{b}}(\xi) = I_r, \tag{2.4.1}$$

$$\overline{\widehat{\mathring{a}}(\xi)}^{\mathsf{T}}\widehat{\mathring{a}}(\xi + \tfrac{2\pi\gamma}{\mathsf{M}}) + \overline{\widehat{\mathring{b}}(\xi)}^{\mathsf{T}} Diag(\epsilon_1, \dots, \epsilon_s)\widehat{\mathring{b}}(\xi + \tfrac{2\pi\gamma}{\mathsf{M}}) = 0, \tag{2.4.2}$$

for all $\gamma = 1, \dots, \mathsf{M}-1$ *and for all* $\xi \in \mathbb{R}$*, where the finitely supported matrix-valued filters* $\mathring{a} \in (l_0(\mathbb{Z}))^{r\times r}$ *and* $\mathring{b} \in (l_0(\mathbb{Z}))^{s\times r}$ *are defined by*

$$\widehat{\mathring{a}}(\xi) := \widehat{\theta}(\mathsf{M}\xi)\widehat{a}(\xi)\widehat{\theta}(\xi)^{-1} \quad \text{and} \quad \widehat{\mathring{b}}(\xi) := \widehat{b}(\xi)\widehat{\theta}(\xi)^{-1}. \tag{2.4.3}$$

(3) The filter $\mathring{b}$ *has* m *balanced vanishing moments.*

(4) The associated discrete multiframelet transform employing the quasi-tight M*-framelet filter bank* $\{\mathring{a}; \mathring{b}\}_{(\epsilon_1,\dots,\epsilon_s)}$ *is compact and has* m *balancing order with respect to* M*, i.e.,* $\operatorname{bo}(\{\mathring{a}; \mathring{b}\}, \mathsf{M}) = m$.

Moreover, the compactly supported vector functions $\mathring{\phi}$ *and* ψ *satisfy*

$$\widehat{\mathring{\phi}}(\mathsf{M}\xi) = \widehat{\mathring{a}}(\xi)\widehat{\mathring{\phi}}(\xi) \quad \text{and} \quad \widehat{\psi}(\mathsf{M}\xi) = \widehat{\mathring{b}}(\xi)\widehat{\mathring{\phi}}(\xi). \tag{2.4.4}$$

Theorem 2.4.1 demonstrates that we can construct quasi-tight multiframelets from any compactly supported refinable vector functions. This is not like existing works in the literature that study tight framelets, which often require that the refinable vector function ϕ should have stable integer shifts. This condition guarantees the existence of $\Theta \in (l_0(\mathbb{Z}))^{r\times r}$ (which is often not strongly invertible at all) such that $\mathcal{M}_{a,\Theta}$ is positive semi-definite, where $\mathcal{M}_{a,\Theta}$ is defined in (1.3.3) (see [59, Proposition 3.4 and Theorem 4.3]). The positive semi-definiteness of $\mathcal{M}_{a,\Theta}$ is a necessary condition for the existence of tight framelets.

On the other hand, Theorem 2.4.1 also demonstrates great advantages of OEP for multiframelets. In the scalar case ($r = 1$), OEP can increase the order of vanishing moments on framelet generators, but quite often it is inevitable to sacrifice the compactness of the associated discrete framelet transform. For example, [33, Theorem 1.3] proves that for any scalar dual framelet constructed through OEP from any pair of scalar spline refinable functions, if it has a compact framelet transform, then it can have at most one vanishing moment. Besides, most of the multiframelets constructed in existing literatures lack the balancing property, which reduces sparsity when implementing a multi-level discrete multiframelet transform. Theorem 2.4.1 guarantees the existence of quasi-tight multiframelets with all desired properties: (i) high order vanishing moments on framelet generators; (ii) a compact and balanced associated discrete multiframelet transform.

To prove Theorem 2.4.1 and for the convenience of later presentation, we need the following notations:

(1) For $\gamma \in \mathbb{Z}$ and $u \in (l(\mathbb{Z}))^{s\times r}$, the γ-*coset sequence* of u with respect to the dilation factor M is the sequence $u^{[\gamma;\mathsf{M}]} \in (l(\mathbb{Z}))^{s\times r}$ given by

$$u^{[\gamma;\mathsf{M}]}(k) = u(\gamma + \mathsf{M}k), \quad k \in \mathbb{Z}.$$

It is straightforward to check that

$$\widehat{u}(\xi) = \sum_{\gamma=0}^{\mathsf{M}-1} \widehat{u^{[\gamma;\mathsf{M}]}}(\mathsf{M}\xi) e^{-i\gamma\xi}, \quad \forall u \in (l_0(\mathbb{Z}))^{s\times r}, \quad \xi \in \mathbb{R}. \tag{2.4.5}$$

Thus by letting $\mathsf{F}_{r;\mathsf{M}}(\xi)$ to be the $(\mathsf{M}r) \times (\mathsf{M}r)$ matrix defined via

$$\mathsf{F}_{r;\mathsf{M}}(\xi) := \left(e^{-i(l-1)(\xi+2\pi\frac{k-1}{\mathsf{M}})} I_r\right)_{1\leqslant l,k\leqslant \mathsf{M}}, \tag{2.4.6}$$

we have

$$\begin{aligned}&\left[\widehat{u}(\xi), \widehat{u}(\xi + \tfrac{2\pi}{\mathsf{M}}), \ldots, \widehat{u}(\xi + \tfrac{2\pi(\mathsf{M}-1)}{\mathsf{M}})\right] \\ =& \left[\widehat{u^{[0;\mathsf{M}]}}(\mathsf{M}\xi), \widehat{u^{[1;\mathsf{M}]}}(\mathsf{M}\xi), \ldots, \widehat{u^{[\mathsf{M}-1;\mathsf{M}]}}(\mathsf{M}\xi)\right] \mathsf{F}_{r;\mathsf{M}}(\xi).\end{aligned} \tag{2.4.7}$$

Observe that $\overline{\mathsf{F}_{r;\mathsf{M}}(\xi)}^{\mathsf{T}} \mathsf{F}_{r;\mathsf{M}}(\xi) = \mathsf{M} I_{\mathsf{M}r}$, (2.4.7) is equivalent to

$$\begin{aligned}&\left[\widehat{u}(\xi), \widehat{u}(\xi + \tfrac{2\pi}{\mathsf{M}}), \ldots, \widehat{u}(\xi + \tfrac{2\pi(\mathsf{M}-1)}{\mathsf{M}})\right] \overline{\mathsf{F}_{r;\mathsf{M}}(\xi)}^{\mathsf{T}} \\ =&\mathsf{M} \left[\widehat{u^{[0;\mathsf{M}]}}(\mathsf{M}\xi), \widehat{u^{[1;\mathsf{M}]}}(\mathsf{M}\xi), \ldots, \widehat{u^{[\mathsf{M}-1;\mathsf{M}]}}(\mathsf{M}\xi)\right].\end{aligned} \tag{2.4.8}$$

(2) For $j \in \{1, \ldots, \mathsf{M}\}$ and $u \in (l_0(\mathbb{Z}))^{r\times r}$, let $D_{u;\mathsf{M}}(\xi)$ be the $(\mathsf{M}r)\times(\mathsf{M}r)$ block diagonal

matrix defined via

$$D_{u;\mathsf{M}}(\xi) := \begin{bmatrix} \widehat{u}(\xi) & & & \\ & \widehat{u}(\xi+\frac{2\pi}{\mathsf{M}}) & & \\ & & \ddots & \\ & & & \widehat{u}(\xi+\frac{2\pi(\mathsf{M}-1)}{\mathsf{M}}) \end{bmatrix}, \tag{2.4.9}$$

and let $E_{u;\mathsf{M}}(\xi)$ be the $(\mathsf{M}r) \times (\mathsf{M}r)$ block matrix, whose (l,k)-th $r \times r$ block is

$$(E_{u;\mathsf{M}}(\xi))_{l,k} := \widehat{u^{[k-l;\mathsf{M}]}}(\xi), \tag{2.4.10}$$

for $1 \leqslant l, k \leqslant \mathsf{M}$. Then direct calculation yields

$$\mathsf{F}_{r;\mathsf{M}}(\xi) D_{u;\mathsf{M}}(\xi) \overline{\mathsf{F}_{r;\mathsf{M}}(\xi)}^{\mathsf{T}} = \mathsf{M} E_{u;\mathsf{M}}(\mathsf{M}\xi). \tag{2.4.11}$$

The following theorem plays a key role in the proof of Theorem 3.3.1.

Theorem 2.4.2. *Let $\mathsf{M} \geqslant 2$ and $r \geqslant 2$ be positive integers and let $a \in (l_0(\mathbb{Z}))^{r\times r}$. Suppose that a has m sum rules with respect to M with a matching filter $\upsilon \in (l_0(\mathbb{Z}))^{1\times r}$ satisfying $\widehat{\upsilon}(\xi) = (1,0,\dots,0) + \mathscr{O}(|\xi|^m)$ as $\xi \to 0$. Further suppose that ϕ is an $r \times 1$ vector of compactly supported functions in $L_2(\mathbb{R})$ satisfying $\widehat{\phi}(\mathsf{M}\xi) = \widehat{a}(\xi)\widehat{\phi}(\xi)$ and $\widehat{\phi}(\xi) = (1,0,\dots,0)^{\mathsf{T}} + \mathscr{O}(|\xi|^n)$ as $\xi \to 0$ for some $n \geqslant 2m$. Then for any strongly invertible $U \in (l_0(\mathbb{Z}))^{r\times r}$ satisfying (2.3.10), there exist $b \in (l_0(\mathbb{Z}))^{s\times r}$ and $\epsilon_1, \dots \epsilon_s \in \{\pm 1\}$ for some $s \in \mathbb{N}$ such that*

(i) $\{a; b\}_{\mathbf{U};(\epsilon_1,\dots,\epsilon_s)}$ is an OEP-based quasi-tight M-framelet filter bank, i.e.,

$$\overline{\widehat{a}(\xi)}^{\mathsf{T}} \widehat{\mathbf{U}}(\mathsf{M}\xi)\widehat{a}(\xi+2\pi\tfrac{\gamma}{\mathsf{M}}) + \overline{\widehat{b}(\xi)}^{\mathsf{T}} Diag(\epsilon_1,\dots,\epsilon_s)\widehat{b}(\xi+2\pi\tfrac{\gamma}{\mathsf{M}}) = \boldsymbol{\delta}(\omega)\widehat{\mathbf{U}}(\xi), \tag{2.4.12}$$

for all $\gamma = 0, \dots, \mathsf{M}-1$, where $\boldsymbol{\delta}$ is defined as in (1.1.15) and $\widehat{\mathbf{U}}(\xi) = \overline{\widehat{U}(\xi)}^{-\mathsf{T}} \widehat{U}(\xi)^{-1}$ for all $\xi \in \mathbb{R}$.

(ii) $\{\eta;\psi\}_{(\epsilon_1,\ldots,\epsilon_s)}$ *is a compactly supported quasi-tight* M*-multiframelet in* $L_2(\mathbb{R})$ *such that all the entries of* ψ *have* m *vanishing moments, where*

$$\widehat{\eta}(\xi) = \widehat{U}(\xi)^{-1}\widehat{\phi}(\xi), \quad \widehat{\psi}(\xi) = \widehat{b}(\xi/\mathsf{M})\widehat{\phi}(\xi/\mathsf{M}), \quad \xi \in \mathbb{R}. \tag{2.4.13}$$

Proof. By our assumptions on $\widehat{v}$ and $\widehat{\phi}$, we see that $\widehat{a}$ must take the form in (2.3.3), i.e.,

$$\widehat{a}(\xi) = \begin{bmatrix} \widehat{a_{1,1}}(\xi) & \widehat{a_{1,2}}(\xi) \\ \widehat{a_{2,1}}(\xi) & P_{2,2}(\xi) \end{bmatrix},$$

with

$$\widehat{a_{1,1}}(\xi) = (1+e^{-i\xi}+\cdots+e^{-i(\mathsf{M}-1)\xi})^m P_{1,1}(\xi) = 1+\mathscr{O}(|\xi|^n), \quad \xi \to 0,$$
$$\widehat{a_{1,2}}(\xi) = (1-e^{-i\mathsf{M}\xi})^m P_{1,2}(\xi), \quad \widehat{a_{2,1}}(\xi) = (1-e^{-i\xi})^n P_{2,1}(\xi),$$

where $P_{1,1}, P_{1,2}, P_{2,1}$ and $P_{2,2}$ are some 1×1, $1\times(r-1)$, $(r-1)\times 1$ and $(r-1)\times(r-1)$ matrices of 2π-periodic trigonometric polynomials. Define

$$\widehat{a_1}(\xi) := \widehat{\mathbf{U}}(\xi) - \overline{\widehat{a}(\xi)}^{\mathsf{T}}\widehat{\mathbf{U}}(\mathsf{M}\xi)\widehat{a}(\xi),$$

$$\widehat{a_j}(\xi) := -\overline{\widehat{a}(\xi)}^{\mathsf{T}}\widehat{\mathbf{U}}(\mathsf{M}\xi)\widehat{a}(\xi+\tfrac{2\pi(j-1)}{\mathsf{M}}), \quad j=2,\ldots,\mathsf{M}.$$

For $j=1$, using (2.3.10) and the fact that $\|\widehat{\phi}(\xi)\|^2 = 1+\mathscr{O}(|\xi|^n)$ as $\xi\to 0$, we have

$$\begin{bmatrix} p_1(\xi) & p_2(\xi) \\ p_3(\xi) & p_4(\xi) \end{bmatrix} := \widehat{a_1}(\xi) = \begin{bmatrix} 1 & \\ & \widehat{C}(\xi) \end{bmatrix} - \overline{\widehat{a}(\xi)}^{\mathsf{T}} \begin{bmatrix} 1 & \\ & \widehat{C}(\mathsf{M}\xi) \end{bmatrix} \widehat{a}(\xi) + \mathscr{O}(|\xi|^n), \quad \xi \to 0,$$

where $\widehat{C}(\xi) = \mathrm{Diag}\,(\|\widehat{u_2}(\xi)\|^2,\ldots,\|\widehat{u_r}(\xi)\|^2)$ and $\widehat{u_j}$ denote the j-th column of $\widehat{U}^{-1}$ for all $j=1,\ldots,r$. Then

- p_1 is a 2π-periodic trigonometric polynomial satisfying

$$p_1(\xi) = 1 - \left(| \underbrace{\widehat{a_{1,1}}(\xi)}_{=1+\mathscr{O}(|\xi|^n)} |^2 + \underbrace{\overline{\widehat{a_{2,1}}(\xi)}^{\mathsf{T}}}_{=\mathscr{O}(|\xi|^n)} \widehat{C}(\mathsf{M}\xi)\widehat{a_{2,1}}(\xi) \right) + \mathscr{O}(|\xi|^n) = \mathscr{O}(|\xi|^n), \quad \xi \to 0.$$

- p_2 is a $1 \times (r-1)$ vector of 2π-periodic trigonometric polynomials satisfying

$$p_2(\xi) = -\overline{\widehat{a_{1,1}}(\xi)} \underbrace{\widehat{a_{1,2}}(\xi)}_{=\mathscr{O}(|\xi|^m)} - \underbrace{\overline{\widehat{a_{2,1}}(\xi)}^{\mathsf{T}}}_{=\mathscr{O}(|\xi|^n)} \widehat{C}(\mathsf{M}\xi) P_{2,2}(\xi) + \mathscr{O}(|\xi|^n) = \mathscr{O}(|\xi|^m), \quad \xi \to 0.$$

- p_3 and p_4 are $(r-1) \times 1$ and $(r-1) \times (r-1)$ matrices of 2π-periodic trigonometric polynomials satisfying $p_3(\xi) = \overline{p_2(\xi)}^{\mathsf{T}} = \mathscr{O}(|\xi|^m), \quad \xi \to 0.$

Since $n \geqslant 2m$, we conclude that $\widehat{a_1}$ admits the following factorization:

$$\widehat{a_1}(\xi) = \overline{\begin{bmatrix} (1-e^{-i\xi})^m & \\ & I_{r-1} \end{bmatrix}}^{\mathsf{T}} \widehat{C_1}(\xi) \begin{bmatrix} (1-e^{-i\xi})^m & \\ & I_{r-1} \end{bmatrix}, \tag{2.4.14}$$

for some $C_1 \in (l_0(\mathbb{Z}))^{r \times r}$.

For $j = 2, \dots, \mathsf{M}$, we have

$$\begin{bmatrix} p_{j,1}(\xi) & p_{j,2}(\xi) \\ p_{j,3}(\xi) & p_{j,4}(\xi) \end{bmatrix} := \widehat{a_j}(\xi) = -\overline{\widehat{a}(\xi)}^{\mathsf{T}} \begin{bmatrix} 1 & \\ & \widehat{C}(\mathsf{M}\xi) \end{bmatrix} \widehat{a}(\xi + 2\pi\tfrac{(j-1)}{\mathsf{M}}) + \mathscr{O}(|\mathsf{M}\xi|^n), \quad \xi \to 0.$$

By $n \geqslant 2m$, we observe that

- $p_{j,1}$ is a 2π-periodic trigonometric polynomial satisfying

$$\begin{aligned} & p_{j,1}(\xi) \\ = & - \left(\underbrace{\overline{\widehat{a_{1,1}}(\xi)}}_{=\mathscr{O}(|\xi+2\pi(j-1)/\mathsf{M}|^m)} \underbrace{\widehat{a_{1,1}}(\xi + 2\pi\tfrac{(j-1)}{\mathsf{M}})}_{=\mathscr{O}(|\xi|^m)} + \underbrace{\overline{\widehat{a_{2,1}}(\xi)}^{\mathsf{T}}}_{=\mathscr{O}(|\xi|^n)} \widehat{C}(\mathsf{M}\xi) \underbrace{\widehat{a_{2,1}}(\xi + 2\pi\tfrac{(j-1)}{\mathsf{M}})}_{=\mathscr{O}(|\xi+2\pi(j-1)/\mathsf{M}|^n)} \right) + \mathscr{O}(|\mathsf{M}\xi|^n) \\ = & (1-e^{i\xi})^m (1-e^{-i(\xi+2\pi\frac{(j-1)}{\mathsf{M}})})^m \widehat{F_j}(\xi), \end{aligned}$$

where $\widehat{F_j}(\xi)$ denotes some 2π-periodic trigonometric polynomial.

- $p_{j,2}$ is a $1\times(r-1)$ vector of 2π-periodic trigonometric polynomials satisfying

$$p_{j,2}(\xi) = -\overline{\widehat{a_{1,1}}(\xi)}\,\underbrace{\widehat{a_{1,2}}(\xi+2\pi\tfrac{(j-1)}{\mathsf{M}})}_{=\mathscr{O}(|\xi|^m)} - \underbrace{\overline{\widehat{a_{2,1}}(\xi)}^{\mathsf{T}}}_{=\mathscr{O}(|\xi|^n)}\widehat{C}(\mathsf{M}\xi)P_{2,2}(\xi+2\pi\tfrac{(j-1)}{\mathsf{M}}) + \mathscr{O}(|\mathsf{M}\xi|^n)$$

$$= \mathscr{O}(|\xi|^m), \quad \xi\to 0.$$

- $p_{j,3}$ is a $(r-1)\times 1$ vector of 2π-periodic trigonometric polynomials satisfying

$$p_{j,3}(\xi) = -\underbrace{\overline{\widehat{a_{1,2}}(\xi)}}_{=\mathscr{O}(|\xi+2\pi(j-1)/\mathsf{M}|^m)}\widehat{a_{1,1}}(\xi+2\pi\tfrac{(j-1)}{\mathsf{M}}) - \overline{P_{2,2}(\xi)}^{\mathsf{T}}\widehat{C}(\mathsf{M}\xi)\underbrace{\widehat{a_{2,1}}(\xi+2\pi\tfrac{(j-1)}{\mathsf{M}})}_{=\mathscr{O}(|\xi+2\pi(j-1)/\mathsf{M}|^m)} + \mathscr{O}(|\mathsf{M}\xi|^n)$$

$$= \mathscr{O}(|\xi+2\pi\tfrac{(j-1)}{\mathsf{M}}|^m), \quad \xi\to 0.$$

- $p_{j,4}$ is some $(r-1)\times(r-1)$ matrix of 2π-periodic trigonometric polynomials.

Thus $\widehat{a_j}$ admits the following factorization:

$$\widehat{a_j}(\xi) = \overline{\begin{bmatrix}(1-e^{-i\xi})^m & \\ & I_{r-1}\end{bmatrix}}^{\mathsf{T}}\widehat{C_j}(\xi)\begin{bmatrix}(1-e^{-i(\xi+2\pi\frac{(j-1)}{\mathsf{M}})})^m & \\ & I_{r-1}\end{bmatrix}, \qquad (2.4.15)$$

for some $C_j \in (l_0(\mathbb{Z}))^{r\times r}$ for all $j=2,\ldots,\mathsf{M}$. Hence by letting

$$\widehat{\Delta_m}(\xi) := \begin{bmatrix}(1-e^{-i\xi})^m & 0\\ 0 & I_{r-1}\end{bmatrix}, \qquad (2.4.16)$$

and

$$
\begin{aligned}
\mathcal{M}_{a,\mathsf{U}}(\xi) :=& \begin{bmatrix} \widehat{\mathsf{U}}(\xi) & & \\ & \ddots & \\ & & \widehat{\mathsf{U}}(\xi+\frac{2\pi(\mathsf{M}-1)}{\mathsf{M}}) \end{bmatrix} \\
&- \begin{bmatrix} \overline{\widehat{a}(\xi)}^{\mathsf{T}} \\ \vdots \\ \overline{\widehat{a}(\xi+\frac{2\pi(\mathsf{M}-1)}{\mathsf{M}})}^{\mathsf{T}} \end{bmatrix} \widehat{\mathsf{U}}(\mathsf{M}\xi) \left[\widehat{a}(\xi), \quad \ldots, \quad \widehat{a}(\xi+\frac{2\pi(\mathsf{M}-1)}{\mathsf{M}})\right],
\end{aligned} \tag{2.4.17}
$$

it follows from (2.4.14) and (2.4.15) that

$$
\mathcal{M}_{a,\mathsf{U}}(\xi) = \overline{D_{\Delta_m;\mathsf{M}}(\xi)}^{\mathsf{T}} \mathcal{M}(\xi) D_{\Delta_m;\mathsf{M}}(\xi), \tag{2.4.18}
$$

where $\mathcal{M}$ is some $(\mathsf{M}r)\times(\mathsf{M}r)$ Hermitian matrix of 2π-periodic trigonometric polynomials, and $D_{\Delta_m;\mathsf{M}}$ is defined via (2.4.9) with $u = \Delta_m$.

It follows from (2.4.11) and (2.4.18) that

$$
\begin{aligned}
&\mathsf{M}^{-2}\, \mathsf{F}_{r;\mathsf{M}}(\xi)\mathcal{M}_{a,\mathsf{U}}(\xi)\overline{\mathsf{F}_{r;\mathsf{M}}(\xi)}^{\mathsf{T}} \\
=&\mathsf{M}^{-4}\left(\mathsf{F}_{r;\mathsf{M}}(\xi)\overline{D_{\Delta_m;\mathsf{M}}(\xi)}^{\mathsf{T}}\overline{\mathsf{F}_{r;\mathsf{M}}(\xi)}^{\mathsf{T}}\right)\left(\mathsf{F}_{r;\mathsf{M}}(\xi)\mathcal{M}(\xi)\overline{\mathsf{F}_{r;\mathsf{M}}(\xi)}^{\mathsf{T}}\right)\left(\mathsf{F}_{r;\mathsf{M}}(\xi)D_{\Delta_m;\mathsf{M}}(\xi)\overline{\mathsf{F}_{r;\mathsf{M}}(\xi)}^{\mathsf{T}}\right) \\
=&\overline{E_{\Delta_m;\mathsf{M}}(\mathsf{M}\xi)}^{\mathsf{T}}\tilde{\mathcal{M}}(\xi)E_{\Delta_m;\mathsf{M}}(\mathsf{M}\xi),
\end{aligned}
$$

where $\tilde{\mathcal{M}}(\xi) = \mathsf{M}^{-2}\,\mathsf{F}_{r;\mathsf{M}}(\xi)\mathcal{M}(\xi)\overline{\mathsf{F}_{r;\mathsf{M}}(\xi)}^{\mathsf{T}}$ and $E_{\Delta_m;\mathsf{M}}$ is defined as in (2.4.10) with $u = \Delta_m$.

On the other hand, using(2.4.8) and (2.4.11), we see that

$$
\begin{aligned}
&\mathsf{M}^{-2}\,\mathsf{F}_{r;\mathsf{M}}(\xi)\mathcal{M}_{a,\mathsf{U}}(\xi)\overline{\mathsf{F}_{r;\mathsf{M}}(\xi)}^{\mathsf{T}} \\
=&\mathsf{M}^{-1}E_{\mathsf{U};\mathsf{M}}(\mathsf{M}\xi) - \begin{bmatrix} \overline{\widehat{a^{[0;\mathsf{M}]}}(\mathsf{M}\xi)}^{\mathsf{T}} \\ \vdots \\ \overline{\widehat{a^{[\mathsf{M}-1;\mathsf{M}]}}(\mathsf{M}\xi)}^{\mathsf{T}} \end{bmatrix} \widehat{\mathsf{U}}(\mathsf{M}\xi)\left[\widehat{a^{[0;\mathsf{M}]}}(\mathsf{M}\xi), \quad \ldots, \quad \widehat{a^{[\mathsf{M}-1;\mathsf{M}]}}(\mathsf{M}\xi)\right].
\end{aligned} \tag{2.4.19}
$$

Hence $\tilde{\mathcal{M}}(\xi)$ only depends on $\mathsf{M}\xi$, say $\tilde{\mathcal{M}}(\xi) = \mathring{\mathcal{M}}(\mathsf{M}\xi)$, where $\mathring{\mathcal{M}}$ is some $(\mathsf{M}r)\times(\mathsf{M}r)$

Hermitian matrix of 2π-periodic trigonometric polynomials. We now claim that $\mathring{\mathcal{M}}$ can be factorized in the following way:

$$\mathring{\mathcal{M}}(\xi) = \overline{\tilde{U}(\xi)}^{\mathsf{T}} \operatorname{Diag}(I_{s_1}, -I_{s_2}) \tilde{U}(\xi), \tag{2.4.20}$$

for some $(s_1 + s_2) \times (\mathsf{M}r)$ matrix of 2π-periodic trigonometric polynomials $\tilde{U}(\xi)$. In fact, there always exist $(\mathsf{M}r) \times (\mathsf{M}r)$ matrices of 2π-periodic trigonometric polynomials $\mathring{\mathcal{M}}_1(\xi)$ and $\mathring{\mathcal{M}}_2(\xi)$ such that

$$\mathring{\mathcal{M}}(\xi) = \overline{\mathring{\mathcal{M}}_1(\xi)}^{\mathsf{T}} \mathring{\mathcal{M}}_1(\xi) - \overline{\mathring{\mathcal{M}}_2(\xi)}^{\mathsf{T}} \mathring{\mathcal{M}}_2(\xi).$$

For example, take $\mathring{\mathcal{M}}_1(\xi) = I_{\mathsf{M}r} + \frac{1}{4}\mathring{\mathcal{M}}(\xi)$ and $\mathring{\mathcal{M}}_2(\xi) = I_{\mathsf{M}r} - \frac{1}{4}\mathring{\mathcal{M}}(\xi)$. Then simply choose $\tilde{U} = [\mathring{\mathcal{M}}_1^{\mathsf{T}}, \mathring{\mathcal{M}}_2^{\mathsf{T}}]^{\mathsf{T}}$, we see that (2.4.20) holds with $s_1 = s_2 = \mathsf{M}r$. Once we have factorized $\mathring{\mathcal{M}}$ as in (2.4.20), define $b \in (l_0(\mathbb{Z}))^{(s_1+s_2)\times r}$ and $\epsilon_1, \dots, \epsilon_{s_1+s_2} \in \{\pm 1\}$ via

$$\widehat{b}(\xi) := \tilde{U}(\mathsf{M}\xi)\, \mathsf{F}_{r;\mathsf{M}}(\xi) \begin{bmatrix} I_r \\ \mathbf{0}_{(\mathsf{M}-1)r\times r} \end{bmatrix} \widehat{\Delta_m}(\xi), \tag{2.4.21}$$

$$\epsilon_1 = \dots = \epsilon_{s_1} = 1, \quad \epsilon_{s_1+1} = \dots = \epsilon_{s_1+s_2} = -1, \tag{2.4.22}$$

where $\mathbf{0}_{q\times t}$ denotes the $q \times t$ zero matrix. Using (2.4.21) and (2.4.22), we have

$$\begin{aligned}
&\begin{bmatrix} \overline{\widehat{b}(\xi)}^{\mathsf{T}} \\ \vdots \\ \overline{\widehat{b}(\xi + 2\pi\frac{\mathsf{M}-1}{\mathsf{M}})}^{\mathsf{T}} \end{bmatrix} \operatorname{Diag}(\epsilon_1, \dots, \epsilon_{s_1+s_2}) \begin{bmatrix} \widehat{b}(\xi) & \dots & \widehat{b}(\xi + 2\pi\frac{\mathsf{M}-1}{\mathsf{M}}) \end{bmatrix} \\
=&\overline{D_{\Delta_m;\mathsf{M}}(\xi)}^{\mathsf{T}} \overline{\mathsf{F}_{r;\mathsf{M}}(\xi)}^{\mathsf{T}} \overline{\tilde{U}(\mathsf{M}\xi)}^{\mathsf{T}} \operatorname{Diag}(I_{s_1}, -I_{s_2}) \tilde{U}(\mathsf{M}\xi)\, \mathsf{F}_{r;\mathsf{M}}(\xi) D_{\Delta_m;\mathsf{M}}(\xi) \qquad (2.4.23) \\
=&\overline{D_{\Delta_m;\mathsf{M}}(\xi)}^{\mathsf{T}} \overline{\mathsf{F}_{r;\mathsf{M}}(\xi)}^{\mathsf{T}} \mathring{\mathcal{M}}(\mathsf{M}\xi)\, \mathsf{F}_{r;\mathsf{M}}(\xi) D_{\Delta_m;\mathsf{M}}(\xi) \\
=&\overline{\mathsf{F}_{r;\mathsf{M}}(\xi)}^{\mathsf{T}} \overline{E_{\Delta_m;\mathsf{M}}(\mathsf{M}\xi)}^{\mathsf{T}} \mathring{\mathcal{M}}(\mathsf{M}\xi) E_{\Delta_m;\mathsf{M}}(\mathsf{M}\xi)\, \mathsf{F}_{r;\mathsf{M}}(\xi) \\
=&\mathcal{M}_{a,\mathbf{u}}(\xi).
\end{aligned}$$

Note that (2.4.23) is equivalent to say that (2.4.12) holds for all $\gamma = 0, \dots, \mathsf{M} - 1$. This proves item (i).

Define η and ψ as in (2.4.13). By $\widehat{\phi}(\xi) = (1, 0, \dots, 0)^{\mathsf{T}} + \mathscr{O}(|\xi|^n)$ as $\xi \to 0$ and $n \geqslant 2m$, we have

$$\widehat{\psi}(\mathsf{M}\xi) = \widehat{b}(\xi)\widehat{\phi}(\xi) = \tilde{U}(\mathsf{M}\xi)\,\mathsf{F}_{r;\mathsf{M}}(\xi)\begin{bmatrix} I_r \\ \mathbf{0}_{(\mathsf{M}-1)r\times r}\end{bmatrix}\underbrace{\widehat{\Delta_m}(\xi)\widehat{\phi}(\xi)}_{=\mathscr{O}(|\xi|^m)} = \mathscr{O}(|\xi|^m), \quad \xi \to 0,$$

which means that all the entries of ψ have m vanishing moments. Note that $\overline{\widehat{\phi}(0)}^{\mathsf{T}}\widehat{\mathbf{U}}(0)\widehat{\phi}(0) = 1$. Now by Theorem 1.1.1, $\{\eta; \psi\}_{(\epsilon_1,\dots,\epsilon_s)}$ is a quasi-tight M-framelet in $L_2(\mathbb{R})$. This proves item (ii). □

Now we are ready to prove Theorem 2.4.1.

Proof of Theorem 2.4.1. Let $n \geqslant 2m$ be a positive integer. By Theorem 2.3.1, there exists a strongly invertible filter $\theta \in (l_0(\mathbb{Z}))^{r\times r}$ such that

$$\widehat{\mathring{v}}(\xi) := \widehat{v}(\xi)\widehat{\theta}(\xi)^{-1} = r^{-\frac{1}{2}}\widehat{\Upsilon}(\xi) + \mathscr{O}(|\xi|^m), \quad \widehat{\mathring{\phi}}(\xi) := \widehat{\theta}(\xi)\widehat{\phi}(\xi) = r^{-\frac{1}{2}}\overline{\widehat{\Upsilon}(\xi)}^{\mathsf{T}} + \mathscr{O}(|\xi|^n), \quad \xi \to 0,$$

where $\widehat{\Upsilon}(\xi) := (1, e^{i\xi/r}, \dots, e^{i(r-1)\xi/r})$ as in (2.2.4)). Now by Theorem 2.3.2, there exists a strongly invertible $U \in (l_0(\mathbb{Z}))^{r\times r}$ such that

$$\begin{aligned}\overline{\widehat{U}(\xi)}^{-\mathsf{T}}\widehat{U}(\xi)^{-1} &= \operatorname{Diag}\left(\|\widehat{\mathring{\phi}}(\xi)\|^2, \|\widehat{u_2}\|^2, \dots, \|\widehat{u_r}\|^2\right) + \mathscr{O}(|\xi|^n) \\ &= \operatorname{Diag}\left(1, \|\widehat{u_2}\|^2, \dots, \|\widehat{u_r}\|^2\right) + \mathscr{O}(|\xi|^n), \quad \xi \to 0,\end{aligned}$$

where $\widehat{u_j}$ denotes the j-th column of $\widehat{U}^{-1}$, and

$$\widehat{\breve{v}}(\xi) := \widehat{\mathring{v}}(\xi)\widehat{U}(\xi)^{-1} = (1, 0, \dots, 0) + \mathscr{O}(|\xi|^m), \quad \widehat{\breve{\phi}}(\xi) := \widehat{U}(\xi)\widehat{\mathring{\phi}}(\xi) = (1, 0, \dots, 0)^{\mathsf{T}} + \mathscr{O}(|\xi|^n),$$

as $\xi \to 0$. Define $\mathring{a}, \breve{a} \in (l_0(\mathbb{Z}))^{r\times r}$ via

$$\widehat{\mathring{a}}(\xi) = \widehat{\theta}(\mathsf{M}\xi)\widehat{a}(\xi)\widehat{\theta}(\xi)^{-1}, \quad \widehat{\breve{a}}(\xi) = \widehat{U}(\mathsf{M}\xi)\widehat{\mathring{a}}(\xi)\widehat{U}(\xi)^{-1}.$$

It is trivial that $\breve{a}$ has m sum rules with the matching filter $\breve{v}$ and $\widehat{\breve{\phi}}(\mathsf{M}\xi) = \widehat{\breve{a}}(\xi)\widehat{\breve{\phi}}(\xi)$ holds for all $\xi \in \mathbb{R}$. Thus by item (i) of Theorem 2.4.2, there exist $\breve{b} \in (l_0(\mathbb{Z}))^{s\times r}$ and $\epsilon_1, \dots, \epsilon_s \in \{\pm 1\}$ such that (2.4.12) holds for all $\gamma = 0, \dots, \mathsf{M}-1$ with a and b being replaced by $\breve{a}$ and $\breve{b}$ respectively. Hence by defining $b, \mathring{b} \in (l_0(\mathbb{Z}))^{s\times r}$ via

$$\widehat{b}(\xi) = \widehat{\breve{b}}(\xi)\widehat{U}(\xi)\widehat{\theta}(\xi), \quad \widehat{\mathring{b}}(\xi) = \widehat{b}(\xi)\widehat{\theta}(\xi)^{-1},$$

we see that item (2) follows right away.

Next, define

$$\widehat{\psi}(\xi) := \widehat{b}(\xi/\mathsf{M})\widehat{\phi}(\xi/\mathsf{M}) = \widehat{\mathring{b}}(\xi/\mathsf{M})\widehat{\mathring{\phi}}(\xi/\mathsf{M}) = \widehat{\breve{b}}(\xi/\mathsf{M})\widehat{\breve{\phi}}(\xi/\mathsf{M}),$$

for all $\xi \in \mathbb{R}$. By item (ii) of Theorem 2.4.2, $\{\mathring{\phi}; \psi\}_{(\epsilon_1,\dots,\epsilon_s)}$ is a quasi-tight M-framelet in $L_2(\mathbb{R})$ with $\widehat{\psi}(\xi) = \mathscr{O}(|\xi|^m)$ as $\xi \to 0$. This proves item (1). Moreover, (2.4.4) holds trivially.

Finally, we have

$$\widehat{\Upsilon}(\xi)\overline{\widehat{\mathring{b}}(\xi)}^{\mathsf{T}} = \widehat{\Upsilon}(\xi)\overline{\widehat{\theta}(\xi)}^{-\mathsf{T}}\overline{\widehat{b}(\xi)}^{\mathsf{T}} = r^{\frac{1}{2}}\overline{\widehat{\phi}(\xi)}^{\mathsf{T}}\overline{\widehat{b}(\xi)}^{\mathsf{T}} + \mathscr{O}(|\xi|^n) = r^{\frac{1}{2}}\overline{\widehat{\psi}(\mathsf{M}\xi)}^{\mathsf{T}} + \mathscr{O}(|\xi|^n) = \mathscr{O}(|\xi|^m),$$

and

$$\begin{aligned}\widehat{\Upsilon}(\xi)\overline{\widehat{\mathring{a}}(\xi)}^{\mathsf{T}} &= \widehat{\Upsilon}(\xi)\overline{\widehat{\theta}(\xi)}^{-\mathsf{T}}\overline{\widehat{a}(\xi)}^{\mathsf{T}}\overline{\widehat{\theta}(\mathsf{M}\xi)}^{\mathsf{T}} = r^{\frac{1}{2}}\overline{\widehat{\phi}(\xi)}^{\mathsf{T}}\overline{\widehat{a}(\xi)}^{\mathsf{T}}\overline{\widehat{\theta}(\mathsf{M}\xi)}^{\mathsf{T}} + \mathscr{O}(|\xi|^n)\\ &= r^{\frac{1}{2}}\overline{\widehat{\phi}(\mathsf{M}\xi)}^{\mathsf{T}}\overline{\widehat{\theta}(\mathsf{M}\xi)}^{\mathsf{T}} + \mathscr{O}(|\xi|^n) = \widehat{\Upsilon}(\mathsf{M}\xi) + \mathscr{O}(|\xi|^m),\end{aligned}$$

as $\xi \to 0$. Hence by Theorem 2.2.1, items (3) and (4) must hold. The proof is now complete. $\square$

2.5 Guidelines for Constructing Balanced OEP-based Quasi-tight Multiframelets

To construct a quasi-tight framelet in Theorem 2.4.1 from a matrix-valued filter having multiplicity greater than one, a desired filter θ in Theorem 2.4.1 plays a key role and is guaranteed to exist by Theorem 2.4.1. In this section, we study properties of θ which allow us to construct quasi-tight framelet filter banks and quasi-tight framelets in Theorem 2.4.1 having all the desired properties. Our theoretical investigation enable us to develop an algorithm for construction.

For $m \in \mathbb{N}$, we define a sequence $\nabla^m \boldsymbol{\delta} \in l_0(\mathbb{Z})$ through $\widehat{\nabla^m \boldsymbol{\delta}}(\xi) := (1 - e^{-i\xi})^m$. Before proceeding further, we need the following technical lemma, which provides an equivalent way of interpreting the balanced vanishing moments condition.

Lemma 2.5.1. *Let $r \geqslant 2$ and $s \in \mathbb{N}$ be positive integers. For any $m \in \mathbb{N}$, a filter $b \in (l_0(\mathbb{Z}))^{s\times r}$ has m balanced vanishing moments (i.e.,* (2.2.5) *holds) if and only if*

$$\widehat{b}(\xi) = \left[\widehat{q^{[0;r]}}(\xi), \widehat{q^{[1;r]}}(\xi), \ldots, \widehat{q^{[r-1;r]}}(\xi)\right] E_{\nabla^m \boldsymbol{\delta};r}(\xi), \quad \xi \in \mathbb{R}, \qquad (2.5.1)$$

for some $q \in (l_0(\mathbb{Z}))^{s\times 1}$, where $E_{\nabla^m \boldsymbol{\delta};r}$ is defined in (2.4.10) *with $\mathsf{M} = r$ and $u = \nabla^m \boldsymbol{\delta}$.*

Proof. Suppose that b has m balanced vanishing moments, i.e., (2.2.5) holds. We deduce that

$$\widehat{b_1}(r\xi) + e^{-i\xi}\widehat{b_2}(r\xi) + \cdots + e^{-i\xi(r-1)}\widehat{b_r}(r\xi) = \widehat{\nabla^m \boldsymbol{\delta}}(\xi)\widehat{q}(\xi), \qquad (2.5.2)$$

for some $q \in (l_0(\mathbb{Z}))^{s\times 1}$, where $\widehat{b_j}$ denotes the j-th column of $\widehat{b}$. Since $\widehat{u}(\xi) = \sum_{\gamma=0}^{\mathsf{M}-1} \widehat{u^{[\gamma;\mathsf{M}]}}(\mathsf{M}\xi)e^{-i\gamma\xi}$ in (2.4.5), we have

$$\begin{aligned}
&\sum_{l=1}^{r} e^{-i(l-1)\xi}\widehat{b_l}(r\xi) = \left(\sum_{j=0}^{r-1} e^{-ij\xi}\widehat{\nabla^m \boldsymbol{\delta}^{[j;r]}}(r\xi)\right)\left(\sum_{k=0}^{r-1} e^{-ik\xi}\widehat{q^{[k;r]}}(r\xi)\right)\\
&= \sum_{j=0}^{r-1} e^{-ij\xi} \sum_{k=0}^{j} \widehat{\nabla^m \boldsymbol{\delta}^{[j-k;r]}}(r\xi)\widehat{q^{[k;r]}}(r\xi) + \sum_{j=0}^{r-2} e^{-ij\xi}e^{-ir\xi} \sum_{k=j+1}^{r-1} \widehat{\nabla^m \boldsymbol{\delta}^{[j+r-k;r]}}(r\xi)\widehat{q^{[k;r]}}(r\xi).
\end{aligned}$$

Hence

$$\widehat{b_r}(\xi) = \sum_{k=0}^{r-1} \widehat{\nabla^m \boldsymbol{\delta}^{[r-1-k;r]}}(\xi)\widehat{q^{[k;r]}}(\xi), \tag{2.5.3}$$

and

$$\begin{aligned}\widehat{b_{j+1}}(\xi) &= \sum_{k=0}^{j} \widehat{\nabla^m \boldsymbol{\delta}^{[j-k;r]}}(\xi)\widehat{q^{[k;r]}}(\xi) + e^{-i\xi} \sum_{k=j+1}^{r-1} \widehat{\nabla^m \boldsymbol{\delta}^{[j+r-k;r]}}(\xi)\widehat{q^{[k;r]}}(\xi) \\ &= \sum_{k=0}^{r-1} \widehat{\nabla^m \boldsymbol{\delta}^{[j-k;r]}}(\xi)\widehat{q^{[k;r]}}(\xi), \quad j = 0, \ldots, r-2,\end{aligned} \tag{2.5.4}$$

for all $\xi \in \mathbb{R}$, where the last line of (2.5.4) follows from the fact that

$$e^{-i\xi}\widehat{u^{[r-l;r]}}(\xi) = \widehat{u^{[-l;r]}}(\xi), \quad \forall u \in (l_0(\mathbb{Z}))^{t\times r}, \quad l \in \mathbb{Z}. \tag{2.5.5}$$

Thus (2.5.1) follows right away from (2.5.3) and (2.5.4).

Conversely, suppose that (2.5.1) holds. Then (2.5.3) and (2.5.4) must hold. Thus we deduce that (2.5.2) hold. Now (2.2.5) follows trivially. $\square$

The following result provides a characterization for all the desired filters θ in Theorem 2.4.1.

Theorem 2.5.2. *Let $\mathsf{M} \geqslant 2$ and $r \geqslant 2$ be integers and $a \in (l_0(\mathbb{Z}))^{r\times r}$ be a finitely supported matrix-valued filter. Let $\phi \in (L_2(\mathbb{R}))^r$ be a compactly supported vector function satisfying $\widehat{\phi}(\mathsf{M}\xi) = \widehat{a}(\xi)\widehat{\phi}(\xi)$. Assume that a has m sum rules with respect to M with a matching filter $\upsilon \in (l_0(\mathbb{Z}))^{1\times r}$ such that $\widehat{\upsilon}(0)\widehat{\phi}(0) = 1$. Define $\widehat{\Upsilon}(\xi) := (1, e^{i\xi/r}, \ldots, e^{i(r-1)\xi/r})$ as in (2.2.4). Let $\theta \in (l_0(\mathbb{Z}))^{r\times r}$ be a strongly invertible filter. If the filter θ satisfies the following two conditions:*

(i) There exist 2π-periodic trigonometric polynomials $\widehat{c}$ and $\widehat{d}$ with $|\widehat{c}(0)| = |\widehat{d}(0)| = \|\widehat{\Upsilon}(0)\|^{-1} = r^{-\frac{1}{2}}$ such that

$$\widehat{\mathring{\upsilon}}(\xi) := \widehat{\upsilon}(\xi)\widehat{\theta}(\xi)^{-1} = \widehat{c}(\xi)\widehat{\Upsilon}(\xi) + \mathscr{O}(|\xi|^m), \quad \xi \to 0, \tag{2.5.6}$$

$$\widehat{\mathring{\phi}}(\xi) := \widehat{\theta}(\xi)\widehat{\phi}(\xi) = \widehat{d}(\xi)\overline{\widehat{\Upsilon}(\xi)}^{\mathsf{T}} + \mathscr{O}(|\xi|^m), \quad \xi \to 0; \tag{2.5.7}$$

(ii) All the entries of the following two matrices are 2π-periodic trigonometric polynomials:

$$M_0(\xi) := \overline{E_{\nabla^m \boldsymbol{\delta};r}(\xi)}^{-\mathsf{T}} \left(I_r - \overline{\widehat{\mathring{a}}(\xi)}^{\mathsf{T}} \widehat{\mathring{a}}(\xi) \right) E_{\nabla^m \boldsymbol{\delta};r}(\xi)^{-1}, \tag{2.5.8}$$

$$M_j(\xi) := -\overline{E_{\nabla^m \boldsymbol{\delta};r}(\xi)}^{-\mathsf{T}} \overline{\widehat{\mathring{a}}(\xi)}^{\mathsf{T}} \widehat{\mathring{a}}(\xi + \tfrac{2\pi j}{\mathsf{M}}) E_{\nabla^m \boldsymbol{\delta};r}(\xi + \tfrac{2\pi j}{\mathsf{M}})^{-1}, \quad j = 1, \ldots, \mathsf{M}-1, \tag{2.5.9}$$

where $\widehat{\mathring{a}}(\xi) = \widehat{\theta}(\mathsf{M}\xi)\widehat{a}(\xi)\widehat{\theta}(\xi)^{-1}$ *and* $E_{\nabla^m \boldsymbol{\delta};r}$ *is defined in* (2.4.10) *with* $\mathsf{M} = r$ *and* $u = \nabla^m \boldsymbol{\delta}$*,*

then there must exist $b \in (l_0(\mathbb{Z}))^{s \times r}$ *and* $\epsilon_1, \ldots, \epsilon_s \in \{\pm 1\}$ *such that all the items (1)–(4) of Theorem 2.4.1 are satisfied. Conversely, if all the items (1)–(4) of Theorem 2.4.1 are satisfied for some* $b \in (l_0(\mathbb{Z}))^{s \times r}$ *and* $\epsilon_1, \ldots, \epsilon_s \in \{\pm 1\}$*, then the filter* θ *must satisfy item (ii) above and if additionally*

$$\begin{aligned} &1 \textit{ is a simple eigenvalue of } \widehat{a}(0) \\ &\textit{and } \det(\mathsf{M}^j I_r - \widehat{a}(0)) \neq 0 \textit{ for all } j \in \mathbb{Z} \setminus \{0\} \textit{ with } |j| \leqslant \mathsf{M} - 1, \end{aligned} \tag{2.5.10}$$

and

$$\widehat{\mathring{c}}(\xi) \widehat{\Upsilon}(\xi) \overline{\widehat{\mathring{a}}(\xi)}^{\mathsf{T}} = \widehat{\Upsilon}(\mathsf{M}\xi) + \mathscr{O}(|\xi|^m), \quad \xi \to 0, \textit{ for some } \mathring{c} \in l_0(\mathbb{Z}) \textit{ with } \widehat{\mathring{c}}(0) = 1, \tag{2.5.11}$$

then θ *must also satisfy item (i) above.*

Proof. For simplicity of presentation, we define $\widehat{E_{m;r}}(\xi) := E_{\nabla^m \boldsymbol{\delta};r}(\xi)$. Then $E_{m;r}$ is a finitely supported matrix-valued filter. First observe from (2.4.11) and the fact $\mathsf{F}_{r;r}(\xi)\overline{\mathsf{F}_{r;r}}^{\mathsf{T}} = rI_{r^2}$ that

$$\det(\widehat{E_{m;r}}(\xi)) = \det(E_{\nabla^m \boldsymbol{\delta};r}(\xi)) = \det(D_{\nabla^m \boldsymbol{\delta};r}(\xi/r)) = \prod_{j=0}^{r-1} (1 - e^{-i(\xi + 2\pi j)/r})^m = (1 - e^{-i\xi})^m, \quad \xi \in \mathbb{R}.$$

Therefore, $\widehat{E_{m;r}}(\xi)$ is invertible for all $\xi \in \mathbb{R} \backslash 2\pi\mathbb{Z}$, and thus all the matrices $M_0, M_1, \ldots, M_{\mathsf{M}-1}$ in item (ii) are well defined for all $\xi \in \mathbb{R} \setminus 2\pi\mathbb{Z}$.

Suppose that items (i) and (ii) hold. Define

$$\mathcal{M}_{\mathring{a}}(\xi) := I_{\mathsf{M}r} - \begin{bmatrix} \overline{\widehat{\mathring{a}}(\xi)}^{\mathsf{T}} \\ \vdots \\ \overline{\widehat{\mathring{a}}(\xi + \frac{2\pi(\mathsf{M}-1)}{\mathsf{M}})}^{\mathsf{T}} \end{bmatrix} \left[\widehat{\mathring{a}}(\xi), \ldots, \widehat{\mathring{a}}(\xi + \tfrac{2\pi(\mathsf{M}-1)}{\mathsf{M}})\right].$$

By item (ii), $\mathcal{M}_{\mathring{a}}$ admits the following factorization:

$$\mathcal{M}_{\mathring{a}}(\xi) = \overline{D_{E_{m;r};\mathsf{M}}(\xi)}^{\mathsf{T}} M(\xi) D_{E_{m;r};\mathsf{M}}(\xi), \quad \xi \in \mathbb{R}, \tag{2.5.12}$$

where $M(\xi)$ is some $(\mathsf{M}r) \times (\mathsf{M}r)$ matrix of 2π-periodic trigonometric polynomials. Applying the same argument as in the proof of Theorem 2.4.2, we have

$$\frac{1}{\mathsf{M}^2} \mathsf{F}_{r;\mathsf{M}}(\xi) \mathcal{M}_{\mathring{a}}(\xi) \overline{\mathsf{F}_{r;\mathsf{M}}(\xi)}^{\mathsf{T}} = \overline{E_{E_{m;r};\mathsf{M}}(\mathsf{M}\xi)}^{\mathsf{T}} \tilde{M}(\mathsf{M}\xi) E_{E_{m;r};\mathsf{M}}(\mathsf{M}\xi)^{\mathsf{T}},$$

where $\tilde{M}(\xi)$ is some $\mathsf{M}r \times \mathsf{M}r$ Hermitian matrix of 2π-periodic trigonometric polynomials. Thus there exists an $s \times r$ matrix $\tilde{U}(\xi)$ of 2π-periodic trigonometric polynomials such that

$$\tilde{M}(\xi) = \overline{\tilde{U}(\xi)}^{\mathsf{T}} \operatorname{Diag}(I_{s_1}, -I_{s_2}) \tilde{U}(\xi), \quad \xi \in \mathbb{R},$$

for some $s_1, s_2 \in \mathbb{N}_0$ satisfying $s_1 + s_2 = s$. Define $\mathring{b}, b \in (l_0(\mathbb{Z}))^{s \times r}$ and $\epsilon_1, \ldots, \epsilon_s \in \{\pm 1\}$ via

$$\widehat{\mathring{b}}(\xi) := \tilde{U}(\mathsf{M}\xi)\, \mathsf{F}_{r;\mathsf{M}}(\xi) \begin{bmatrix} I_r \\ \mathbf{0}_{r(\mathsf{M}-1)\times r} \end{bmatrix} \widehat{E_{m;r}}(\xi), \quad \widehat{b}(\xi) := \widehat{\mathring{b}}(\xi)\widehat{\theta}(\xi), \tag{2.5.13}$$

$$\epsilon_1 = \cdots = \epsilon_{s_1} = 1, \quad \epsilon_{s_1+1} = \cdots = \epsilon_s = -1. \tag{2.5.14}$$

Using (2.5.12), (2.5.13) and (2.5.14), it is straightforward to check that item (2) of Theorem 2.4.1 holds. Next, by letting $q \in (l_0(\mathbb{Z}))^{s \times 1}$ be such that

$$\left[\widehat{q^{[0;r]}}(\xi), \widehat{q^{[1;r]}}(\xi), \ldots, \widehat{q^{[r-1;r]}}(\xi)\right] = \tilde{U}(\mathsf{M}\xi)\, \mathsf{F}_{r;\mathsf{M}}(\xi) \begin{bmatrix} I_r \\ \mathbf{0}_{r(\mathsf{M}-1)\times r} \end{bmatrix},$$

we see that items (3) and (4) of Theorem 2.4.1 follow immediately from Lemma 2.5.1. On the other hand, since item (i) holds, it follows that $\|\widehat{\mathring{\phi}}(0)\|^2 = 1$ and thus Theorem 1.1.1 yields that $\{\mathring{\phi}; \psi\}_{(\epsilon_1,\ldots,\epsilon_s)}$ is a quasi-tight M-framelet in $L_2(\mathbb{R})$, where $\widehat{\psi}(\xi) = \widehat{b}(\xi/\mathsf{M})\widehat{\phi}(\xi/\mathsf{M})$. Moreover, (2.5.7) and item (3) of Theorem 2.4.1 guarantee that ψ has m vanishing moments. This proves item (1) of Theorem 2.4.1. Hence all the claims of Theorem 2.4.1 hold.

Conversely, suppose that θ is a strongly invertible filter and all the claims in Theorem 2.4.1 holds. By item (3) of Theorem 2.4.1, (2.2.5) holds. Thus by Lemma 2.5.1, there exists $q \in (l_0(\mathbb{Z}))^{s\times 1}$ such that

$$\widehat{\mathring{b}}(\xi) = \left[\widehat{q^{[0;r]}}(\xi), \widehat{q^{[1;r]}}(\xi), \ldots, \widehat{q^{[r-1;r]}}(\xi)\right] \widehat{E_{m;r}}(\xi), \quad \xi \in \mathbb{R}. \tag{2.5.15}$$

By (2.4.1) and (2.4.2), we have

$$I_r - \overline{\widehat{\mathring{a}}(\xi)}^{\mathsf{T}} \widehat{\mathring{a}}(\xi) = \overline{\widehat{E_{m;r}}(\xi)}^{\mathsf{T}} \begin{bmatrix} \overline{\widehat{q^{[0;r]}}(\xi)}^{\mathsf{T}} \\ \vdots \\ \overline{\widehat{q^{[r-1;r]}}(\xi)}^{\mathsf{T}} \end{bmatrix} \operatorname{Diag}(\epsilon_1, \ldots, \epsilon_s) \left[\widehat{q^{[0;r]}}(\xi), \ldots \widehat{q^{[r-1;r]}}(\xi)\right] \widehat{E_{m;r}}(\xi),$$

and

$$\begin{aligned}
&- \overline{\widehat{\mathring{a}}(\xi)}^{\mathsf{T}} \widehat{\mathring{a}}(\xi + \tfrac{2\pi j}{\mathsf{M}}) \\
=& \overline{\widehat{E_{m;r}}(\xi)}^{\mathsf{T}} \begin{bmatrix} \overline{\widehat{q^{[0;r]}}(\xi)}^{\mathsf{T}} \\ \vdots \\ \overline{\widehat{q^{[r-1;r]}}(\xi)}^{\mathsf{T}} \end{bmatrix} \operatorname{Diag}(\epsilon_1, \ldots, \epsilon_s) \left[\widehat{q^{[0;r]}}(\xi + \tfrac{2\pi j}{\mathsf{M}}), \ldots \widehat{q^{[r-1;r]}}(\xi + \tfrac{2\pi j}{\mathsf{M}})\right] \widehat{E_{m;r}}(\xi + \tfrac{2\pi j}{\mathsf{M}}),
\end{aligned}$$

for all $\xi \in \mathbb{R}$ and $j = 1, \ldots, \mathsf{M} - 1$. By letting M_0 as (2.5.8) and M_j as (2.5.9) for $j = 1, \ldots, \mathsf{M} - 1$, item (ii) follows immediately from the above two identities.

Finally, assume in addition that (2.5.10) and (2.5.11) hold. Since $\{\mathring{a}, \mathring{b}\}_{(\epsilon_1,\ldots,\epsilon_s)}$ is a quasi-tight M-framelet filter bank, (2.4.3) holds. Multiplying $\widehat{\mathring{c}}\widehat{\Upsilon}$ on the left to both sides of (2.4.3) and using (2.5.11) and the fact that $\widehat{\Upsilon}(\xi)\overline{\widehat{\mathring{b}}(\xi)}^{\mathsf{T}} = \mathscr{O}(|\xi|^m)$ as $\xi \to 0$, we conclude that

$$\widehat{\Upsilon}(\mathsf{M}\xi)\widehat{\mathring{a}}(\xi) = \widehat{\mathring{c}}(\xi)\widehat{\Upsilon}(\xi) + \mathscr{O}(|\xi|^m), \qquad \xi \to 0. \tag{2.5.16}$$

Choose $c \in l_0(\mathbb{Z})$ such that $\widehat{c}(\xi) := \prod_{j=0}^{\infty}\widehat{\mathring{c}}(\mathsf{M}^{-j}\xi) + \mathscr{O}(|\xi|^m)$ as $\xi \to 0$ (the infinite product is well-defined as $\widehat{\mathring{c}}(0) = 1$, see e.g. [41, Lemma 4.1.8]). It follows from (2.5.16) that

$$\widehat{c}(\mathsf{M}\xi)\widehat{\Upsilon}(\mathsf{M}\xi)\widehat{\mathring{a}}(\xi) = \widehat{c}(\xi)\widehat{\Upsilon}(\xi) + \mathscr{O}(|\xi|^m), \qquad \xi \to 0. \tag{2.5.17}$$

Moreover, (2.5.11) yields

$$\overline{\widehat{c}(\mathsf{M}\xi)\widehat{\Upsilon}(\mathsf{M}\xi)}^{\mathsf{T}} = \widehat{\mathring{a}}(\xi)\overline{\widehat{c}(\xi)\widehat{\Upsilon}(\xi)}^{\mathsf{T}} + \mathscr{O}(|\xi|^m), \qquad \xi \to 0. \tag{2.5.18}$$

As $\mathring{a}$ has order m sum rules with the matching filter $\mathring{\upsilon}$ with $\widehat{\mathring{\upsilon}} := \widehat{\upsilon}\widehat{\theta}^{-1}$, the condition (2.5.10) implies that $\widehat{\mathring{\upsilon}}^{j}(0)$ are uniquely determined by $\widehat{\mathring{\upsilon}}(\mathsf{M}\xi)\widehat{\mathring{a}}(\xi) = \widehat{\mathring{\upsilon}}(\xi) + \mathscr{O}(|\xi|^m)$ as $\xi \to 0$ via the recurrence relation: $\widehat{\mathring{\upsilon}}(0)\widehat{\mathring{a}}(0) = \widehat{\mathring{\upsilon}}(0)$ and

$$\widehat{\mathring{\upsilon}}^{(j)}(0) = \left(I_r - \mathsf{M}^j\widehat{\mathring{a}}(0)\right)^{-1}\sum_{k=0}^{j-1}\binom{j}{k}\mathsf{M}^k\widehat{\mathring{\upsilon}}^{k}(0)\widehat{\mathring{a}}^{(j-k)}(0), \qquad j = 1, \dots, m-1. \tag{2.5.19}$$

Thus (2.5.6) follows right away from (2.5.17). Similarly, by letting $\widehat{\mathring{\phi}} := \widehat{\phi}\widehat{\theta}$, (2.5.10) implies that $\widehat{\mathring{\phi}}^{(j)}$ are uniquely determined by $\widehat{\mathring{\phi}}(\mathsf{M}\xi) = \widehat{\mathring{a}}(\xi)\widehat{\mathring{\phi}}(\xi)$ via the recurrence relation: $\widehat{\mathring{\phi}}(0) = \widehat{\mathring{a}}(0)\widehat{\mathring{\phi}}(0)$ and

$$\widehat{\mathring{\phi}}^{(j)}(0) = \left(\mathsf{M}^j I_r - \widehat{\mathring{a}}(0)\right)^{-1}\sum_{k=0}^{j-1}\binom{j}{k}\widehat{\mathring{a}}^{(j-k)}(0)\widehat{\mathring{\phi}}^{k}(0), \qquad j = 1, \dots, m-1. \tag{2.5.20}$$

Thus by (2.5.18), we have (2.5.7) holds with $\widehat{d} = \bar{\mathring{c}}$. Note that $\|\widehat{\mathring{\phi}}(0)\|^2 = 1$, which immediately implies that $|\widehat{c}(0)| = r^{-\frac{1}{2}}$. This proves item (i). $\square$

The following corollary is an immediate consequence of Theorem 2.5.2.

Corollary 2.5.3. *Let* $\mathsf{M} \geqslant 2$ *and* $r \geqslant 2$ *be integers and* $a \in (l_0(\mathbb{Z}))^{r\times r}$ *be a finitely sup-*

ported matrix-valued filter. Let $\phi \in (L_2(\mathbb{R}))^r$ be a compactly supported vector function satisfying $\widehat{\phi}(\mathsf{M}\xi) = \widehat{a}(\xi)\widehat{\phi}(\xi)$ and define $\widehat{\Upsilon}(\xi) := (1, e^{i\xi/r}, \ldots, e^{i(r-1)\xi/r})$ as in (2.2.4). Assume that a has m sum rules with respect to M with a matching filter $\upsilon \in (l_0(\mathbb{Z}))^{1\times r}$ such that $\widehat{\upsilon}(0)\widehat{\phi}(0) = 1$. If $\theta \in (l_0(\mathbb{Z}))^{r\times r}$ is a strongly invertible filter such that item (ii) of Theorem 2.5.2 holds and

$$\widehat{\mathring{\phi}}(\xi) := \widehat{\theta}(\xi)\widehat{\phi}(\xi) = \widehat{c}(\xi)\overline{\widehat{\Upsilon}(\xi)}^{\mathsf{T}} + \mathscr{O}(|\xi|^m), \quad \xi \to 0, \tag{2.5.21}$$

for some 2π-periodic trigonometric polynomial $\widehat{c}$ with $\widehat{c}(0) \neq 0$, then

$$\|\widehat{\mathring{\phi}}(\xi)\|^2 = \|\widehat{\mathring{\phi}}(0)\|^2 + \mathscr{O}(|\xi|^{2m}), \quad \xi \to 0. \tag{2.5.22}$$

Proof. Let $\theta \in (l_0(\mathbb{Z}))^{r\times r}$ be a strongly invertible filter such that all above assumptions are satisfied. From the proof of Theorem 2.5.2, we deduce that there exist $b \in (l_0(\mathbb{Z}))^{s\times r}$ and $\epsilon_1, \ldots, \epsilon_s \in \{\pm 1\}$ such that item (2) of Theorem 2.4.1 and the following condition hold:

$$\widehat{\mathring{\phi}}(\mathsf{M}\xi) = \widehat{\mathring{a}}(\xi)\widehat{\mathring{\phi}}(\xi) \quad \text{and} \quad \widehat{\psi}(\xi) := \widehat{\mathring{b}}(\xi/\mathsf{M})\widehat{\mathring{\phi}}(\xi/\mathsf{M}) = \mathscr{O}(|\xi|^m) \quad \text{as} \quad \xi \to 0, \tag{2.5.23}$$

where $\widehat{\mathring{\phi}} := \widehat{\theta}\widehat{\phi}$, $\widehat{\mathring{a}} := \widehat{\theta}(\mathsf{M}\cdot)\widehat{a}\widehat{\theta}^{-1}$ and $\widehat{\mathring{b}} := \widehat{b}\widehat{\theta}^{-1}$. By multiplying $\overline{\widehat{\mathring{\phi}}(\xi)}^{\mathsf{T}}$ to the left and $\widehat{\mathring{\phi}}(\xi)$ to the right to both sides of (2.4.1) and using (2.5.23), we have

$$\|\widehat{\mathring{\phi}}(\mathsf{M}\xi)\|^2 + \overline{\widehat{\psi}(\mathsf{M}\xi)}^{\mathsf{T}} \operatorname{Diag}(\epsilon_1, \ldots, \epsilon_s)\widehat{\psi}(\mathsf{M}\xi) = \|\widehat{\mathring{\phi}}(\xi)\|^2, \tag{2.5.24}$$

which yields

$$\|\widehat{\mathring{\phi}}(\mathsf{M}\xi)\|^2 = \|\widehat{\mathring{\phi}}(\xi)\|^2 + \mathscr{O}(|\xi|^{2m}), \quad \xi \to 0. \tag{2.5.25}$$

Hence (2.5.22) follows from (2.5.25) and $\mathsf{M} \geqslant 2$. $\square$

The condition in (2.5.22) a key for vanishing moments of derived framelets ψ from $\mathring{\phi}$. Based on our previous investigation, we now present the general procedure of constructing quasi-tight framelets with all desired properties in Theorem 2.4.1. Let $\mathsf{M} \geqslant 2$ and $r \geqslant 2$ be integers and $a \in (l_0(\mathbb{Z}))^{r\times r}$ be a finitely supported matrix-valued filter. Let $\phi \in (L_2(\mathbb{R}))^r$ be a compactly supported refinable vector function satisfying $\widehat{\phi}(\mathsf{M}\xi) = \widehat{a}(\xi)\widehat{\phi}(\xi)$. Assume that a has m sum rules with respect to M with a matching filter $\upsilon \in (l_0(\mathbb{Z}))^{1\times r}$ such that $\widehat{\upsilon}(0)\widehat{\phi}(0) = 1$. Let $\widehat{\Upsilon}(\xi) := (1, e^{i\xi/r}, \dots, e^{i(r-1)\xi/r})$ as in (2.2.4). The general construction steps are as follows:

(1) Construct a strongly invertible filter $\theta \in (l_0(\mathbb{Z}))^{r\times r}$ with short support satisfying items (i) and (ii) of Theorem 2.5.2.

(2) Construct a filter $b \in (l_0(\mathbb{Z}))^{s\times r}$ such that (2.4.1) and (2.4.2) are satisfied (where $\mathring{a}$ and $\mathring{b}$ are given by (2.4.4)) for some $\epsilon_1, \dots, \epsilon_s \in \{\pm 1\}$, and $\widehat{b}(\xi)\widehat{\phi}(\xi) = \mathscr{O}(|\xi|^m)$ as $\xi \to 0$. The existence of such $b \in (l_0(\mathbb{Z}))^{s\times r}$ and $\epsilon_1, \dots, \epsilon_s \in \{\pm 1\}$ is guaranteed by Theorem 2.5.2.

Define $\widehat{\psi}(\xi) = \widehat{b}(\xi/\mathsf{M})\widehat{\phi}(\xi/\mathsf{M})$. Then $\{\mathring{\phi}; \psi\}_{(\epsilon_1,\dots,\epsilon_s)}$ is a compactly supported quasi-tight M-framelet in $L_2(\mathbb{R})$ satisfying all the desired properties in Theorem 2.4.1.

2.6 Some Examples of Spline Quasi-tight Framelets with High Balancing Orders

In this section, we present some examples to illustrate our main result Theorem 2.4.1.

Example 2.3. *Let $\mathsf{M} = r = 2$, and consider $\phi = (B_2(\cdot - 1), 0)^{\mathsf{T}}$, where B_2 is the B-spline of order 2 in (1.2.4). Then ϕ satisfies $\widehat{\phi}(2\xi) = \widehat{a}(\xi)\widehat{\phi}(\xi)$ with a filter $a \in (l_0(\mathbb{Z}))^{2\times 2}$ being given by*

$$\widehat{a}(\xi) = \begin{bmatrix} \widehat{A}(\xi) & 0 \\ 0 & p(\xi) \end{bmatrix},$$

with

$$\widehat{A}(\xi) = \frac{1}{4}(e^{i\xi} + 2 + e^{-i\xi}), \tag{2.6.1}$$

and $p(\xi)$ *is any* 2π*-periodic trigonometric polynomial. Note that* $\mathrm{sr}(a,2) = 2$ *with any matching filter* $\upsilon \in (l_0(\mathbb{Z}))^{1\times 2}$ *satisfying* $\widehat{\upsilon}(\xi) = (1,0) + \mathscr{O}(|\xi|^2)$ *as* $\xi \to 0$*. We obtain a strongly invertible filter* $\theta \in (l_0(\mathbb{Z}))^{2\times 2}$ *satisfying items (i) and (ii) of Theorem 2.5.2 as follows:*

$$\widehat{\theta}(\xi) = \frac{1}{6\sqrt{2}(1+\sqrt{3})}\begin{bmatrix} -(3+\sqrt{3})e^{-i\xi} + 8\sqrt{3} + 9 - \sqrt{3}e^{i\xi} & (7+13\sqrt{3})e^{-i\xi} - 11 + (4+3\sqrt{3})e^{i\xi} \\ (3\sqrt{3}-1)e^{-i\xi} + 3\sqrt{3} + 8 - e^{i\xi} & -\left(\frac{23\sqrt{3}}{3}+9\right)e^{-i\xi} - \frac{11\sqrt{3}}{3} + \left(\frac{4\sqrt{3}}{3}+3\right)e^{i\xi} \end{bmatrix}.$$

Direct computation shows that (2.5.6) *and* (2.5.7) *hold with* $m = 2$ *and*

$$\widehat{c}(\xi) = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}(\sqrt{3}-1)i}{8}\xi + \mathscr{O}(|\xi|^2), \quad \widehat{d}(\xi) = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}(\sqrt{3}-1)i}{8}\xi + \mathscr{O}(|\xi|^2), \quad \xi \to 0.$$

Here we have the freedom to choose $p(\xi)$ *such that the degree of* $\widehat{\mathring{a}}(\xi) := \widehat{\theta}(2\xi)\widehat{a}(\xi)\widehat{\theta}(\xi)^{-1}$ *is as small as possible for simple presentation. By choosing* $p(\xi) = \frac{1}{4}e^{-i\xi} + \frac{7}{2} + \frac{1}{4}e^{i\xi}$*, we obtain* $b \in (l_0(\mathbb{Z}))^{4\times 2}$ *such that* $\{\mathring{a}; \mathring{b}\}_{(\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4)}$*, with*

$$\widehat{\mathring{a}}(\xi) := \widehat{\theta}(\mathsf{M}\xi)\widehat{a}(\xi)\widehat{\theta}(\xi)^{-1} \quad \textit{and} \quad \widehat{\mathring{b}}(\xi) := \widehat{b}(\xi)\widehat{\theta}(\xi)^{-1}$$

and $\epsilon_1 = \epsilon_2 = \epsilon_3 = 1$ *and* $\epsilon_4 = -1$*, is a finitely supported quasi-tight* 2*-multiframelet filter bank with* 2 *balancing orders, where*

$$\widehat{b}(\xi) := e^{i\xi} D\left(U_1 + \sqrt{3}U_2\right)\widehat{U_3}(2\xi)\widehat{F}(\xi)\widehat{\theta}(\xi),$$

with

- $D = \frac{1}{9+4\sqrt{3}} Diag\left(d_1\lambda_1, d_2\sqrt{46}\sqrt{\lambda_2+\lambda_3\sqrt{3}}, d_3\sqrt{927889}\sqrt{\lambda_4+\lambda_5\sqrt{3}}, d_4\sqrt{\lambda_6+\lambda_7\sqrt{3}}\right)$ *where*

$$d_1 = -\frac{1}{25180085734704776}, \quad d_2 = \frac{1}{170664162417838019952956},$$

$$d_3 = \frac{1}{2817007141763662549987911271714008626532826712614384057821861523676 77},$$

$$d_4 = \frac{1}{500906215504938895435066189905155695167066 8};$$

$$\lambda_1 = \sqrt{5437900978131564 + 3120526414024498\sqrt{3}},$$

$$\lambda_2 = 3719615046635084853, \quad \lambda_3 = 2147522348686212558,$$

λ_4

$=22703059042075680129405086247429130016139847446609946732846213394218921936 11324,$

λ_5

$=13107617249370361168619925170215855005204294209258139770128911228607260540 86295,$

$$\lambda_6 = 37853329481006809861894104277104413571 2181,$$

$$\lambda_7 = 21854629965582943055398476074583481906 2292.$$

- *U_1 and U_2 are the 4×4 constant matrices given by*

$$U_1 = \begin{bmatrix} 3392119873 & -2778324120 & -4025874315 & 8672724540 \\ 0 & -8288658986045588 & -10141655898429575 & 6150856039101563 4 \\ 0 & 0 & 0 & \lambda_8 \\ 0 & 0 & \lambda_9 & \lambda_{10} \end{bmatrix},$$

$$U2 = \begin{bmatrix} 0 & 670619101 & 457342305 & -11821258848 \\ 0 & 481466912421912 & 5873639680107160 & -35445004131731402 \\ 0 & 0 & 0 & \lambda_{11} \\ 0 & 0 & \lambda_{12} & \lambda_{13} \end{bmatrix},$$

where

$$\lambda_8 = 65563889896748829166195428223750445718 6876,$$

$$\lambda_9 = 117383088836173699817238498607 8,$$

$$\lambda_{10} = -1055722690591344263872331100,$$

$$\lambda_{11} = -3785332948100680986189410427710441357 12181$$

$$\lambda_{12} = -6777110743808945446298137 29391,$$

$$\lambda_{13} = 609520590739704131515104420.$$

- $\widehat{U_3}$ *is the* 4×4 *matrix of* 2π*-periodic trigonometric polynomials given by* $\widehat{U_3} = \widehat{U_{3,1}}\widehat{U_{3,2}}$ *where*

$$\widehat{U_{3,1}}(\xi) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \left(\frac{782849\sqrt{3}}{61240674} + \frac{1521092}{10206779}\right)e^{-i\xi} & \frac{699997 - 272171\sqrt{3}}{10206779}e^{-i\xi} & 1 & 0 \\ \left(\frac{17348101\sqrt{3}}{122481348} - \frac{9216904}{30620337}\right)e^{-i\xi} & \left(\frac{4881421\sqrt{3}}{61240674} - \frac{5507049}{40827116}\right)e^{-i\xi} & 0 & 1 \end{bmatrix},$$

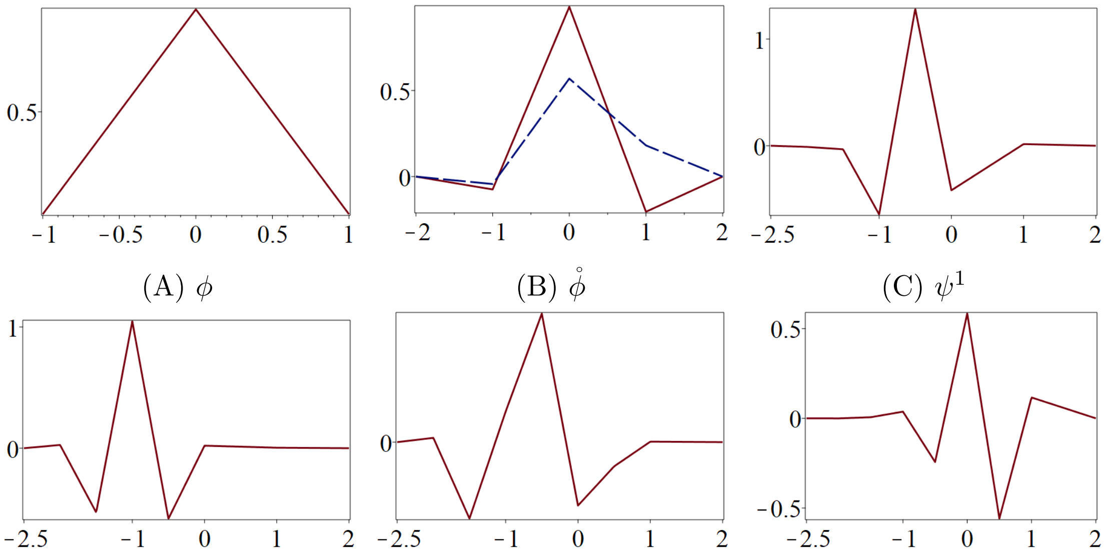
$$\widehat{U_{3,2}}(\xi) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \left(\frac{\sqrt{3}}{6} + \frac{1}{2}\right)e^{-i\xi} & 1 & -\frac{\sqrt{3}}{3} & \left(\frac{2\sqrt{3}}{3} - 3\right)e^{i\xi} + \frac{5\sqrt{3}}{6} + \frac{3}{2} \\ 0 & 0 & 0 & \frac{1}{2} \\ -\frac{1}{6}e^{-i\xi} & 0 & \frac{1}{3} & -\frac{5}{6} - \frac{2}{3}e^{i\xi} \end{bmatrix}.$$

- $\widehat{F}(\xi)$ *is the* 4×2 *matrix of* 2π*-periodic trigonometric polynomials given by*

$$\widehat{F}(\xi) = \begin{bmatrix} 2 & -1 - e^{i\xi} \\ -1 - e^{-i\xi} & 2 \\ 2e^{-i\xi} & -1 - e^{-i\xi} \\ -e^{-2i\xi} - e^{-i\xi} & 2e^{-i\xi} \end{bmatrix}. \qquad (2.6.2)$$

The filter b *is supported on* $[-4,3]$*. Define* $\psi = [\psi^1, \psi^2, \psi^3, \psi^4]^\mathsf{T}$ *via* $\widehat{\psi}(\xi) = \widehat{b}(\xi/2)\widehat{\phi}(\xi/2)$*. Define a new refinable vector function* $\widehat{\mathring{\phi}}(\xi) := \widehat{\theta}(\xi)\widehat{\phi}(\xi)$*. Then* $\|\widehat{\mathring{\phi}}(\xi)\|^2 = 1 + \mathscr{O}(|\xi|^4)$ *as*

$\xi \to 0$ and $\{\mathring{\phi}; \psi\}_{(\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4)}$ is a compactly supported quasi-tight 2-framelet in $L_2(\mathbb{R})$ such that all the items (1)–(4) of Theorem 2.4.1 are satisfied with $m = 2$. Note that ψ has 2 vanishing moments. See Figure 2.1 for graphs of $\phi, \mathring{\phi}, \psi^1, \ldots, \psi^4$.



(A) ϕ (B) $\mathring{\phi}$ (C) ψ^1

(D) ψ^2 (E) ψ^3 (F) ψ^4

Figure 2.1: Graphs of $\phi = [B_2(\cdot - 1), 0]^\mathsf{T}$ and a new refinable vector function $\mathring{\phi}$, together with graphs of $\psi^1, , \ldots, \psi^4$ constructed from ϕ in Example 2.3. A graph with a solid (resp. dash) line denotes the first (resp. second) component of a vector function. $\{\mathring{\phi}; (\psi^1, \ldots, \psi^4)^\mathsf{T}\}_{(1,1,1,-1)}$ is a compactly supported quasi-tight 2-framelet in $L_2(\mathbb{R})$ with balanced vanishing moments 2.

Example 2.4. *Let $\mathsf{M} = r = 2$, and $\varphi := B_2(\cdot - 1)$ where B_2 is the B-spline of order 2 in (1.2.4). Then φ satsfies $\widehat{\varphi}(2\xi) = \widehat{A}(\xi)\widehat{\varphi}(\xi)$ where $\widehat{A}$ is given by (2.6.1). Note that* $\operatorname{sr}(A, 2) = 2$. *Define $\phi := (\varphi(2\cdot), \varphi(2\cdot -1))^\mathsf{T}$. By [33, Proposition 6.2], ϕ satisfies $\widehat{\phi}(2\xi) = \widehat{a}(\xi)\widehat{\phi}(\xi)$, where*

$$\widehat{a}(\xi) = \frac{1}{4}\begin{bmatrix} 2 & 1 + e^{i\xi} \\ 2e^{-i\xi} & 1 + e^{-i\xi} \end{bmatrix}, \quad \xi \in \mathbb{R}.$$

Moreover, $\operatorname{sr}(a, 2) = 2$ with a matching filter $\upsilon \in (l_0(\mathbb{Z}))^{1\times 2}$ satisfying $\widehat{\upsilon}(\xi) = (1, e^{i\xi/2}) + \mathscr{O}(|\xi|^2)$ as $\xi \to 0$. Now applying the general construction steps presented above, we obtain a desired strongly invertible filter $\theta \in (l_0(\mathbb{Z}))^{2\times 2}$ satisfying items (i) and (ii) of Theo-

rem 2.5.2:

$$\widehat{\theta}(\xi) = \frac{\sqrt{2}}{24}\begin{bmatrix} 1 & -e^{i\xi}+2-e^{-i\xi} \\ 0 & 1 \end{bmatrix}.$$

Direct computation shows that (2.5.6) *and* (2.5.7) *hold with* $m = 2$ *and*

$$\widehat{c}(\xi) = \widehat{d}(\xi) = \frac{\sqrt{2}}{2}.$$

We obtain $b \in (l_0(\mathbb{Z}))^{4\times 2}$ *such that* $\{\mathring{a}; \mathring{b}\}$*, with* $\mathring{a}$ *and* $\mathring{b}$ *being defined in* (2.4.3)*, is a finitely supported tight* 2*-framelet filter bank with* 2 *balancing orders. For simplicity of presentation, we write*

$$\widehat{b}(\xi) = \widehat{D}(2\xi)\widehat{E}(2\xi)\widehat{F}(\xi),$$

with

- $\widehat{D}(\xi)$ *is the* 4×4 *diagonal matrix of* 2π*-periodic trigonometric polynomials given by* $\widehat{D}(\xi) = Diag\,(d_1(\xi), d_2, d_3, d_4)$ *with*

$$d_1(\xi) = \frac{\sqrt{70}}{43680\lambda_1}\left(-52\sqrt{5249}+364\sqrt{105}+\left(364\sqrt{105}+52\sqrt{5249}\right)e^{i\xi}\right),$$

$$d_2 = -\frac{1}{96\lambda_1\lambda_2}, \quad d_3 = -\frac{\sqrt{6}}{576\lambda_3\lambda_4}, \quad d_4 = -\frac{\sqrt{2}}{48\lambda_3},$$

where

$$\lambda_1 = \sqrt{1542994447951920545029009}, \quad \lambda_2 = \sqrt{392716620870},$$

$$\lambda_3 = \sqrt{28366141}, \quad \lambda_4 = \sqrt{1870079147}.$$

- $\widehat{E}(\xi)$ *is the* 4×4 *matrix of* 2π*-periodic trigonometric polynomials given by* $\widehat{E}(\xi) =$

$E_{-1}e^{i\xi} + E_0 + E_1 e^{-i\xi}$ *with*

$$E_{-1} = \begin{bmatrix} 0 & 0 & -116055812828 & -116541733616 \\ 0 & 0 & 31811040936954791 & 514403328290664092 \\ 0 & 0 & -24580769 & -98323076 \\ 0 & 0 & 449 & 1796 \end{bmatrix},$$

$$E_0^{\mathsf{T}} = \begin{bmatrix} 2591577536 & -2573825532553116272 & 393292304 & -7184 \\ 18434692032 & -13507206835285209804 & 2064784596 & -37716 \\ 6760552105 & -7105187917736625576 & 6589364832 & -81521 \\ 0 & -3908121799627104752 & -16304613472 & -117450 \end{bmatrix},$$

$$E_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 86906212475244992 & 618190750766215104 & 226709010062624185 & 0 \\ -205796480 & -398928912 & 239683385 & 0 \\ -128302 & -76394 & -20406 & 0 \end{bmatrix}.$$

- $\widehat{F}$ *is the* 4×2 *matrix of* 2π*-periodic trigonometric polynomials given by* (2.6.2).

The filter b *is supported on* $[-4, 3]$*, i.e.,* $b(k) = 0$ *whenever* $k \notin \mathbb{Z} \cap [-4, 3]$*. Define* $\psi = [\psi^1, \psi^2, \psi^3, \psi^4]^{\mathsf{T}}$ *via* $\widehat{\psi}(\xi) = \widehat{b}(\xi/2)\widehat{\phi}(\xi/2)$*. Define a new refinable vector function* $\widehat{\mathring{\phi}}(\xi) := \widehat{\theta}(\xi)\widehat{\phi}(\xi)$*. Then* $\|\widehat{\mathring{\phi}}(\xi)\|^2 = 1 + \mathscr{O}(|\xi|^4)$ *as* $\xi \to 0$ *and* $\{\mathring{\phi}; \psi\}$ *is a compactly supported tight 2-framelet in* $L_2(\mathbb{R})$ *such that all the desired properties in items (1)–(4) of Theorem 2.4.1 are satisfied. Note that* ψ *has* 2 *vanishing moments. See Figure 2.2 for* *graphs of* $\phi, \mathring{\phi}, \psi^1, \psi^2, \psi^3, \psi^4$.

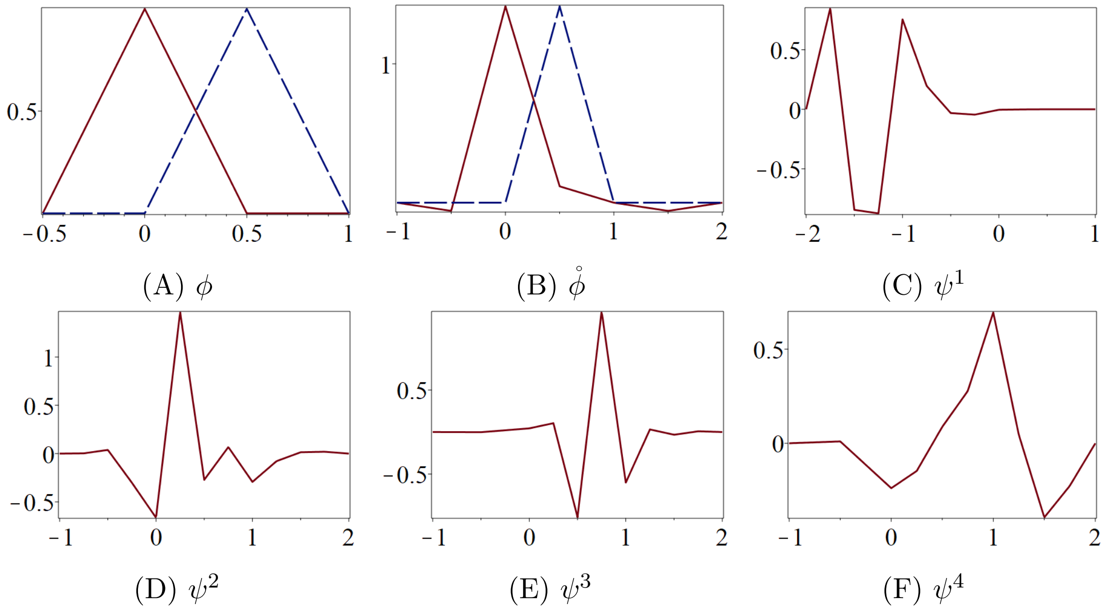


Figure 2.2: Graphs of $\phi = [\varphi(2\cdot), \varphi(2\cdot -1)]^\mathsf{T}$ and the new refinable vector function $\mathring{\phi}$, together with graphs of $\psi^1, , \dots, \psi^4$ constructed from ϕ in Example 2.4. A graph with a solid (resp. dash) line denotes the first (resp. second) component of a function vector. $\{\mathring{\phi}; (\psi^1, \dots, \psi^4)^\mathsf{T}\}$ is a compactly supported tight 2-framelet in $L_2(\mathbb{R})$ with balanced vanishing moments 2.

As we discussed in item (3) of Lemma 2.1.2, a 2π-periodic trigonometric polynomial $\widehat{\Theta}$ is strongly invertible (i.e., $1/\widehat{\Theta}$ is also a 2π-periodic trigonometric polynomial) if and only if $\widehat{\Theta}(\xi) = ce^{-im\xi}$ for some $m \in \mathbb{Z}$ and $c \in \mathbb{C}\backslash\{0\}$. Thus for framelets constructed from scalar refinable functions to have high vanishing moments, usually it is inevitable to sacrifice the compactness of the associated discrete (scalar) framelet transform, because a non-trivial scalar filter Θ is not strongly invertible. Examples 2.3 and 2.4 demonstrate that this difficulty can be easily resolved by simply vectorizing the scalar refinable function, and do the constructions by using the new refinable vector function.

Example 2.5. *Let* $\phi = (\phi_1, \phi_2)^\mathsf{T}$ *be* the Hermite cubic splines *as*

$$\phi_1(x) = \begin{cases} (1-x)^2(1+2x), & x \in [0,1] \\ (1+x)^2(1-2x), & x \in [-1,0) \\ 0, & \textit{otherwise}, \end{cases} \quad \phi_2(x) = \begin{cases} (1-x)^2 x, & x \in [0,1] \\ (1+x)^2 x, & x \in [-1,0) \\ 0, & \textit{otherwise}. \end{cases} \tag{2.6.3}$$

We have $\widehat{\phi}(2\cdot) = \widehat{a}\widehat{\phi}$*, where* $a \in (l_0(\mathbb{Z}))^{2\times 2}$ *is* the Hermite interpolatory filter*:*

$$\widehat{a}(\xi) = \frac{1}{16}\begin{bmatrix} 4e^{i\xi}+8+4e^{-i\xi} & 6(e^{i\xi}-e^{-i\xi}) \\ -(e^{i\xi}-e^{-i\xi}) & -e^{i\xi}+4-e^{-i\xi} \end{bmatrix}, \quad \xi \in \mathbb{R}. \tag{2.6.4}$$

. We have $\operatorname{sr}(a,2) = 4$ *with a matching filter* $v \in (l_0(\mathbb{Z}))^{1\times 2}$ *satisfying* $\widehat{v}(\xi) = (1, i\xi) + \mathscr{O}(|\xi|^4)$ *as* $\xi \to 0$*. Thus there exist quasi-tight* 2*-framelets derived from* ϕ *which satisfy all claims of Theorem 2.4.1, with the maximum possible choice* $m = 4$ *for balanced vanishing moments. For simplicity of presentation, here we present an example of quasi-tight framelets with* $m = 2$ *instead. Following the construction guidelines, we first construct a desired strongly invertible filter* $\theta \in (l_0(\mathbb{Z}))^{2\times 2}$ *as follows:*

$$\widehat{\theta}(\xi) = \frac{\sqrt{2}}{256}\begin{bmatrix} 96e^{i\xi}+32 & -77e^{-i\xi}+506+51e^{i\xi} \\ 160e^{i\xi}-32 & -385e^{-i\xi}-78-17e^{i\xi} \end{bmatrix}.$$

Direct computation shows that (2.5.6) *and* (2.5.7) *hold with* $m = 2$ *and*

$$\widehat{c}(\xi) = \frac{\sqrt{2}}{2} - \frac{i\sqrt{2}}{4}\xi + \mathscr{O}(|\xi|^2), \quad \widehat{d}(\xi) = \frac{\sqrt{2}}{2} + \frac{i\sqrt{2}}{4}\xi + \mathscr{O}(|\xi|^2), \quad \xi \to 0.$$

We obtain $b \in (l_0(\mathbb{Z}))^{6\times 2}$ *such that* $\{a; b\}_{\Theta;(\epsilon_1,\dots,\epsilon_6)}$ *is a finitely supported quasi-tight* 2*-multiframelet filter bank with* $\widehat{\Theta}(\xi) = \overline{\widehat{\theta}(\xi)}^{\mathsf{T}}\widehat{\theta}(\xi)$*,* $\epsilon_1 = \epsilon_2 = -1$*, and* $\epsilon_3 = \cdots = \epsilon_6 = 1$*. For simplicity of presentation, we write*

$$\widehat{b}(\xi) = \begin{bmatrix} Diag(-1,1) - \frac{1}{4}Diag(-1,1)N(2\xi) & \mathbf{0}_{2\times 2} \\ Diag(-1,1) + \frac{1}{4}Diag(-1,1)N(2\xi) & \mathbf{0}_{2\times 2} \\ \mathbf{0}_{2\times 2} & De^{2i\xi} \end{bmatrix} \tilde{D}\widehat{E}(2\xi)\widehat{F}(\xi)\widehat{\theta}(\xi),$$

where $\mathbf{0}_{q\times t}$ *denotes the* $q \times t$ *zero matrix and*

- *$N(\xi)$ is the 2×2 matrix of 2π-periodic trigonometric polynomials given by*

$$N(\xi) = \left(N_{-1}e^{i\xi} + N_0 + N_1 e^{-i\xi}\right) \mathit{Diag}\left(\frac{28831}{93273477387000584 6016}, \frac{104831}{70129776762814464}\right),$$

where

$$N_{-1} = \mathit{Diag}\left(\frac{2219987}{92160}, \frac{8071987}{74502}\right) \begin{bmatrix} -280650717637 & 32961105478501 \\ 1406309548267 & -205511772233035 \end{bmatrix} \mathit{Diag}\left(47, \frac{5}{53759}\right),$$

$$N_0 = \mathit{Diag}\left(\frac{28831}{230400}, \frac{104831}{37251}\right) \begin{bmatrix} 45834164001503531 & -3709367687537217 \\ -3709367687537217 & 2370098256094979 \end{bmatrix} \mathit{Diag}\left(1, \frac{25}{53759}\right),$$

$$N_1 = \mathit{Diag}\left(\frac{104339389}{18432}, \frac{8071987}{74502}\right) \begin{bmatrix} -280650717637 & 1406309548267 \\ 164805527392505 & -1027558861165175 \end{bmatrix} \mathit{Diag}\left(\frac{1}{5}, \frac{1}{53759}\right).$$

- *$D = \mathit{Diag}(d_1, d_2)$ where*

$$d_1 = \frac{1761312\sqrt{13212226268199396309514273}}{51195503191172527}, \quad d_2 = \frac{\sqrt{547545488675642}}{37192694}.$$

- *$\tilde{D} = \mathit{Diag}(d_3, d_4, d_5, d_6)$ where*

$$d_3 = \frac{1}{16174191}, \quad d_4 = \frac{1}{1505540889600},$$

$$d_5 = \frac{1}{240649943852714023924531 20}, \quad d_6 = \frac{1}{269129558593851555840}.$$

- *$\widehat{E}$ is the 4×4 matrix of 2π-periodic trigonometric polynomials given by*

$$\widehat{E}(\xi) = D_{-1}E_{-1}e^{i\xi} + D_0E_0 + D_1E_1e^{-i\xi} + D_2E_2e^{-2i\xi} + E_3e^{-3i\xi},$$

where

$$D_{-1} = \mathit{Diag}\left(187, 1505540889600, 263278430792912227, 370729762969181\right),$$

$$D_0 = Diag\,(1, 1, 187, 18596347)\,,$$

$$D_1 = Diag\,(318703, 318703, 14399, 18596347)\,,$$

$$D_2 = Diag\,(16174191, 4589004497, 1108723, 1431918719)\,,$$

$$E_{-1} = \begin{bmatrix} 0 & -3713 & -18565 & 5417 \\ 0 & 0 & 0 & 0 \\ 0 & 3713 & 18565 & -5417 \\ 0 & 3713 & 18565 & -5417 \end{bmatrix}$$

$$E_0^{\mathsf{T}} = \begin{bmatrix} 16174191 & 0 & 192032829613853577201 & -1724291840139 \\ 7619899 & -204477398651 & -210030256538769846979 & -4565648413949 \\ 4457703 & -262538115175 & 192032829613853577201 & -9050015332017 \\ 956109 & -230405377741 & 427912364665047103611 & -535702395579 \end{bmatrix},$$

$$E_1^{\mathsf{T}} = \begin{bmatrix} -1 & -505023 & 51107253271088914 43 & -824326243735 \\ 0 & -344267 & 1450280624007721927 & -1270122003803 \\ 0 & -201399 & -383860742942300317 & --1307289140551 \\ 0 & -43197 & -1583445964329957343 & -967745336173 \end{bmatrix},$$

$$E_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -19768729169137143 & -12286216971251881 & -7187537033154957 & -1541616578141871 \\ -8001370677 & -5625792235 & -3291134295 & -705897885 \end{bmatrix},$$

$$E_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 569741919122396546911 & 0 & 0 & 0 \\ 336929465078003105 & 0 & 0 & 0 \end{bmatrix}.$$

- $\widehat{F}$ *is the* 4×2 *matrix of* 2π*-periodic trigonometric polynomials given by* (2.6.2).

The filter b is supported on $[-3,5]$. Define $\psi = [\psi^1,\psi^2,\psi^3,\psi^4,\psi^5,\psi^6]^\mathsf{T}$ via $\widehat{\psi}(\xi) = \widehat{b}(\xi/2)\widehat{\phi}(\xi/2)$. Define a new refinable vector function $\widehat{\mathring{\phi}}(\xi) := \widehat{\theta}(\xi)\widehat{\phi}(\xi)$. Then $\|\widehat{\mathring{\phi}}(\xi)\|^2 = 1+\mathscr{O}(|\xi|^4)$ as $\xi \to 0$ and $\{\mathring{\phi};\psi\}_{(\epsilon_1,\ldots,\epsilon_6)}$ is a compactly supported quasi-tight 2-framelet in $L_2(\mathbb{R})$ such that all the desired properties in items (1)–(4) of Theorem 2.4.1 are satisfied with $m=2$. Note that ψ has 2 vanishing moments. See Figure 2.3 for graphs of $\phi, \mathring{\phi}, \psi^1,\ldots,\psi^6$.

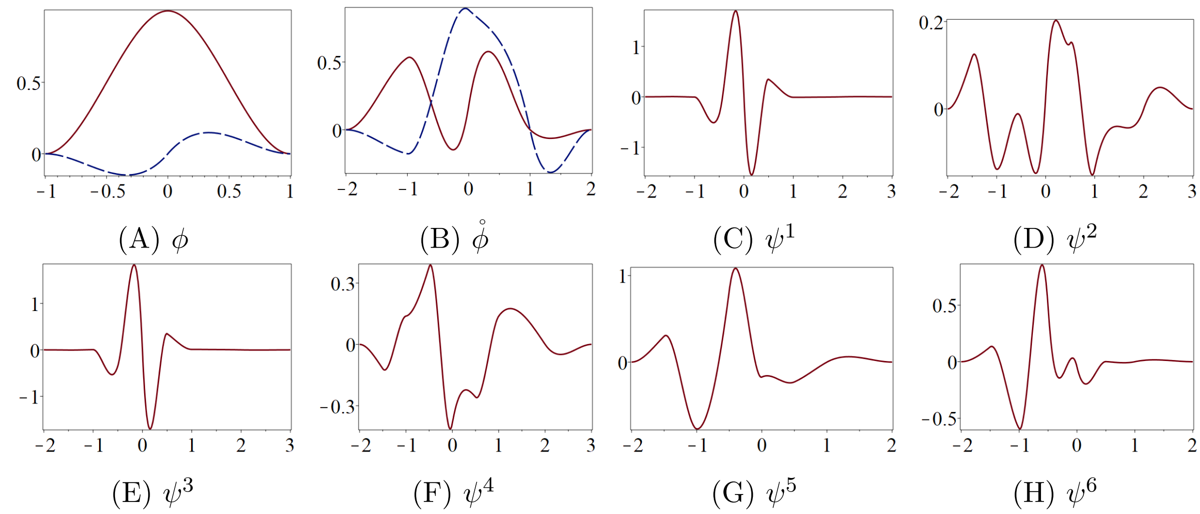


Figure 2.3: Graphs of ϕ and the new refinable vector function $\mathring{\phi}$, together with graphs of $\psi^1,,\ldots,\psi^6$ constructed from the Hermite cubic splines ϕ defined as (2.6.3) in Example 2.5. $\{\mathring{\phi};[\psi^1,\ldots,\psi^6]^\mathsf{T}\}_{(-1,-1,1,1,1,1)}$ is a compactly supported quasi-tight 2-framelet in $L_2(\mathbb{R})$ with balanced vanishing moments 2.

2.7 Summary of the Chapter

From arbitrary univariate compactly supported refinable vector function with multiplicity $r>1$, we proved in this chapter that we can always obtain a compactly supported quasi-tight multiframelet such that its associated discrete framelet transform is compact and has the highest order of balancing orders and vanishing moments. Moreover, in order to prove our main result, we further developed the normal form of a matrix-valued masks, which is of interest in itself for studying refinable vector functions and multiframelets.

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Chapter 3

Balanced Quasi-tight Multiframelets in Arbitrary Dimensions

In this chapter, we study quasi-tight multiframelets derived from any compactly supported refinable vector functions in arbitrary dimensions. The results in this chapter are summarized in [44], which has been published in *Science China Mathematics*.

The theory of multivariate framelets and wavelets are of interest in both theory and applications. One way to obtain multivariate framelets is by tensor product from one-dimensional tight framelets, and these are what we called separable framelets. However, it is often important but challenging to construct non-separable framelets through OEP, mainly due to their intrinsic connections to the factorization of multivariate polynomial matrices. The problem becomes even more challenging if we consider multiframelets with certain desired properties (e.g. high order of vanishing moments), mainly due to the difficulty of constructing a suitable filter Θ in OEP. Due to the existing difficulties on multivariate framelet, most papers in the literature (e.g., see [4–7,9,16,25,42,53–55,63,64] and references therein) study OEP-based multivariate framelets only for the particular case $r = 1, \Theta = \boldsymbol{\delta}$ and special choices of ϕ and $\tilde{\phi}$, in which case we have i.e., $\widehat{\Theta} = 1$, $\mathring{\phi} = \phi$ and $\mathring{\tilde{\phi}} = \tilde{\phi}$, which is called unitary extension principle in [62]. Indeed, many known refinable scalar functions such as spline refinable functions satisfy (1.1.18) with $\Theta = \boldsymbol{\delta}$

for a large positive integer m, which guarantees that the integer shifts of ϕ provide m approximation order for approximating functions. But (1.1.19) with $\Theta = \boldsymbol{\delta}$ can only hold with $m = 1$ for most known refinable (scalar) functions including all spline functions. Hence, it is not surprising that most known multivariate tight framelets including those derived from all spline functions can have only one vanishing moment.

We circumvent all above-mentioned difficulties by considering multivariate quasi-tight multiframelets. From an arbitrary compactly supported refinable vector function ϕ with multiplicity greater than one ($r > 1$), we prove that we can always derive from ϕ a compactly supported multivariate quasi-tight framelet such that

(i) all the framelet generators have the highest possible order of vanishing moments;

(ii) its associated discrete framelet transform is compact with the highest balancing order.

For a refinable scalar function ϕ (i.e., $r = 1$), the above item (ii) often cannot be achieved intrinsically but we show that we can always construct a compactly supported OEP-based multivariate quasi-tight framelet derived from ϕ satisfying item (i).

The work of this chapter generalizes theory presented in Chapter 2 for the case $d = 1$. We will develop the normal form of a matrix-valued filter, study the properties of a discrete framelet transform employing an OEP-based filter bank, and establish the main theorem on constructing OEP-based quasi-tight multiframelets, all in arbitrary dimensions. Furthermore, a structural characterization of multivariate OEP-based quasi-tight multiframelets will be given. However, due to the aforementioned difficulties on multivariate multiframelets, the generalization is not trivial. Several new challenges and difficulties are involved in the study of multivariate multiframelets.

3.1 Properties of a Discrete Framelet Transform

In this section, we discuss a discrete multiframelet transform employing an OEP-based dual framelet filter bank in an arbitrary dimension d.

Let $a, \tilde{a}, \theta, \tilde{\theta} \in (l_0(\mathbb{Z}^d))^{r\times r}$ and $b, \tilde{b} \in (l_0(\mathbb{Z}^d))^{s\times r}$ be finitely supported filters, and let $\Theta := \theta^\star * \tilde{\theta}$. For $J \in \mathbb{N}$, we say that the J-level discrete framelet transform employing the filter bank $(\{a; b\}, \{\tilde{a}; \tilde{b}\})_\Theta$ has **the perfect reconstruction property** if any original input signal v_0 can be exactly recovered through the above J-level discrete framelet reconstruction steps in (S1)–(S3) (recall from Section 1.2).

Define the *convolution operator* $C_\Theta : (l(\mathbb{Z}^d))^{1\times r} \to (l(\mathbb{Z}^d))^{1\times r}$ by

$$C_\Theta(v) := v * \Theta, \qquad \forall v \in (l(\mathbb{Z}^d))^{1\times r}. \tag{3.1.1}$$

Observe that a J-level discrete framelet transform employing the filter bank $(\{a; b\}, \{\tilde{a}; \tilde{b}\})_\Theta$ has the perfect reconstruction property if and only if

$$\mathcal{S}_{\tilde{a},\mathsf{M}}([\mathcal{T}_{a,\mathsf{M}}v] * \Theta) + \mathcal{S}_{\tilde{b},\mathsf{M}}(\mathcal{T}_{b,\mathsf{M}}v) = v * \Theta \tag{3.1.2}$$

holds for all $v \in (l(\mathbb{Z}^d))^{1\times r}$ and the convolution operator C_Θ in (3.1.1) is bijective.

Lemma 3.1.1. *For $\Theta \in (l_0(\mathbb{Z}^d))^{r\times r}$, the mapping $C_\Theta : (l(\mathbb{Z}^d))^{1\times r} \to (l(\mathbb{Z}^d))^{1\times r}$ is bijective if and only if Θ is strongly invertible, that is, $\widehat{\Theta}^{-1}$ is an $r \times r$ matrix of $2\pi\mathbb{Z}^d$-periodic trigonometric polynomials.*

Proof. Suppose that C_Θ is bijective, but Θ is not strongly invertible. This means that $\det(\widehat{\Theta})$ is not a non-trivial monomial. Here a non-trivial monomial is of the form $ce^{ik\cdot\xi}$ for some $c \in \mathbb{C} \setminus \{0\}$ and $k \in \mathbb{Z}^d$. Thus, $\det(\widehat{\Theta}(\xi_0)) = 0$ for some $\xi_0 \in \mathbb{C}^d$. We start with the case $r = 1$. In this case, we have $0 = \widehat{\Theta}(\xi_0) = \sum_{k\in\mathbb{Z}^d} \Theta(k)e^{-ik\cdot\xi_0}$. Define $v \in l(\mathbb{Z}^d)$ by

$$v(k) = e^{ik\cdot\xi_0}, \quad k \in \mathbb{Z}^d. \tag{3.1.3}$$

It is easy to see that $(v * \Theta)(n) = e^{in\cdot\xi_0}\widehat{\Theta}(\xi_0) = 0$ for all $n \in \mathbb{Z}^d$, which contradicts the injectivity of C_Θ. For $r > 1$, as $\det(\widehat{\Theta}(\xi_0)) = 0$, we can find an invertible $r \times r$ matrix Q such that all elements in the first row of $Q\widehat{\Theta}(\xi_0)$ are zero. Let $v \in l(\mathbb{Z}^d)$ be defined as in (3.1.3). Define $u := (v, 0, \dots, 0)Q \in (l(\mathbb{Z}^d))^{1\times r}$. It follows that $u * \Theta = 0$, which again contradicts the assumption that C_Θ is injective. Therefore, Θ must be strongly invertible.

Conversely, if Θ is strongly invertible, then $\Theta^{-1} \in (l_0(\mathbb{Z}^d))^{r\times r}$ with $\widehat{\Theta^{-1}} := [\widehat{\Theta}(\xi)]^{-1}$. Consequently, we have $v = (v * \Theta) * \Theta^{-1} = (v * \Theta^{-1}) * \Theta$ for $v \in (l(\mathbb{Z}^d))^{1\times r}$. Hence, C_Θ is bijective. $\square$

We now characterize the perfect reconstruction property of a J-level discrete framelet transform. The following theorem generalizes Theorem 2.1.3 for the case $d = 1$ to arbitrary dimensions.

Theorem 3.1.2. *Let $a, \tilde{a}, \theta, \tilde{\theta} \in (l_0(\mathbb{Z}^d))^{r\times r}$ and $b, \tilde{b} \in (l_0(\mathbb{Z}^d))^{s\times r}$ be finitely supported filters. Define $\Theta := \theta^\star * \tilde{\theta}$. Then the following statements are equivalent to each other:*

(1) For any $J \in \mathbb{N}$, the J-level discrete framelet transform employing the filter bank $(\{a; b\}; \{\tilde{a}; \tilde{b}\})_\Theta$ has the perfect reconstruction property.

(2) Both filters θ and $\tilde{\theta}$ are strongly invertible and $(\{a; b\}, \{\tilde{a}; \tilde{b}\})_\Theta$ is an OEP-based dual M-framelet filter bank satisfying (1.1.14).

Proof. (1) $\Rightarrow$ (2). Suppose that item (1) holds. By Lemma 3.1.1, Θ is strongly invertible, and thus implies that both θ and $\tilde{\theta}$ are strongly invertible. On the other hand, observe that

$$\widehat{\mathcal{S}_{a,\mathsf{M}} v}(\xi) = |\det(\mathsf{M})|^{1/2}\widehat{v}(\mathsf{M}^\mathsf{T}\xi)\widehat{a}(\xi), \tag{3.1.4}$$

$$\widehat{\mathcal{T}_{a,\mathsf{M}} v}(\xi) = |\det(\mathsf{M})|^{-1/2}\sum_{\omega\in\Omega_\mathsf{M}} \widehat{v}(\mathsf{M}^{-\mathsf{T}}\xi + 2\pi\omega)\overline{\widehat{a}(\mathsf{M}^{-\mathsf{T}}\xi + 2\pi\omega)}^\mathsf{T}, \tag{3.1.5}$$

for all $v \in (l_0(\mathbb{Z}^d))^{1\times r}$, where Ω_M is defined as (1.1.16). Therefore, (3.1.2) yields that for all $v \in (l_0(\mathbb{Z}^d))^{1\times r}$,

$$\sum_{\omega\in\Omega_{\mathsf{M}}} \widehat{v}(\xi+2\pi\omega)\left[\overline{\widehat{a}(\xi+2\pi\omega)}^{\mathsf{T}}\widehat{\Theta}(\mathsf{M}^{\mathsf{T}}\xi)\widehat{\tilde{a}}(\xi)+\overline{\widehat{b}(\xi+2\pi\omega)}^{\mathsf{T}}\widehat{\tilde{b}}(\xi)\right]=\widehat{v}(\xi)\widehat{\Theta}(\xi). \qquad (3.1.6)$$

Let Γ_{M} be a complete set of canonical representatives of the quotient group $\mathbb{Z}^d/[\mathsf{M}\mathbb{Z}^d]$ given by

$$\Gamma_{\mathsf{M}} := \{\gamma_1,\ldots,\gamma_{d_{\mathsf{M}}}\} =: [\mathsf{M}[0,1)^d]\cap\mathbb{Z}^d \quad \text{with} \quad \gamma_1 := 0. \qquad (3.1.7)$$

Note that (3.1.6) holds for all $v \in (l_0(\mathbb{Z}^d))^{r\times r}$. Plugging $\widehat{v_\gamma}(\xi) = e^{-i\gamma\cdot\xi}I_r$ with $\gamma\in\Gamma_{\mathsf{M}}$ into (3.1.6) and using the same argument as in the proof of Theorem 2.1.3, we deduce from (3.1.6) that (1.1.14) must hold. This proves (1) $\Rightarrow$ (2).

(2) $\Rightarrow$ (1). Suppose item (2) holds. Then (1.1.14) implies that (3.1.2) must hold for all $v \in (l_0(\mathbb{Z}^d))^{1\times r}$. As all filters $a, \tilde{a}, b, \tilde{b}$ and Θ are finitely supported, using the locality of the subdivision and transition operators (see the proof of Theorem 2.1.3 or [37, Theorem 2.1]), we can prove that (3.1.2) holds for all $v \in (l(\mathbb{Z}^d))^{1\times r}$. Noting that Θ is strongly invertible, we conclude that the J-level discrete framelet transform employing the filter bank $(\{a;b\};\{\tilde{a};\tilde{b}\})_\Theta$ has the perfect reconstruction property for every $J\in\mathbb{N}$. This proves (2) $\Rightarrow$ (1). $\square$

If both θ and $\tilde{\theta}$ are strongly invertible, then we see that the following filters are finitely supported:

$$\widehat{\mathring{a}}(\xi) := \widehat{\theta}(\mathsf{M}^{\mathsf{T}}\xi)\widehat{a}(\xi)\widehat{\theta}(\xi)^{-1}, \quad \widehat{\mathring{\tilde{a}}}(\xi) := \widehat{\tilde{\theta}}(\mathsf{M}^{\mathsf{T}}\xi)\widehat{\tilde{a}}(\xi)\widehat{\tilde{\theta}}(\xi)^{-1}, \qquad (3.1.8)$$

$$\widehat{\mathring{b}}(\xi) := \widehat{b}(\xi)\widehat{\theta}(\xi)^{-1}, \quad \widehat{\mathring{\tilde{b}}}(\xi) := \widehat{\tilde{b}}(\xi)\widehat{\tilde{\theta}}(\xi)^{-1}. \qquad (3.1.9)$$

Therefore, instead of using the dual framelet filter bank $(\{a;b\},\{\tilde{a};\tilde{b}\})_\Theta$, we can implement a J-level discrete framelet transform using the new dual framelet filter bank $(\{\mathring{a};\mathring{b}\},\{\mathring{\tilde{a}};\mathring{\tilde{b}}\})_{\boldsymbol{\delta} I_r}$ as follows:

(S1') The J-level discrete framelet decomposition: recursively compute

$$\mathring{v}_j := \mathcal{T}_{\mathring{a},\mathsf{M}}\mathring{v}_{j-1}, \quad \mathring{w}_j := \mathcal{T}_{\mathring{b},\mathsf{M}}\mathring{v}_{j-1}, \qquad j=1,\ldots,J,$$

for an input data $\mathring{v}_0 \in (l(\mathbb{Z}^d))^{1\times r}$.

(S2') The J-level discrete framelet reconstruction: recursively compute $\tilde{\mathring{v}}_j, j = J, \ldots, 1$ by

$$\tilde{\mathring{v}}_{j-1} := \mathcal{S}_{\tilde{\mathring{a}},\mathsf{M}} \tilde{\mathring{v}}_j + \mathcal{S}_{\tilde{\mathring{b}},\mathsf{M}} \mathring{w}_j, \qquad j = J, \ldots, 1.$$

We see that the deconvolution step disappears with the new filter bank $(\{\mathring{a}; \mathring{b}\}, \{\tilde{\mathring{a}}; \tilde{\mathring{b}}\})_{\boldsymbol{\delta} I_r}$, which greatly increases the efficiency of the discrete framelet transform.

Next, we discuss the balancing property of a discrete framelet transform in an arbitrary dimension d. Like what we have for the case $d = 1$, vectorizing a scalar data $v \in l(\mathbb{Z}^d)$ so that the input data is a vector sequence in $(l(\mathbb{Z}^d))^{1\times r}$ is needed to implement a discrete framelet transform employing matrix-valued filters. Let N be a $d \times d$ integer matrix with $|\det(\mathsf{N})| = r$, and let Γ_{N} be a complete set of canonical representatives of the quotient group $\mathbb{Z}^d/[\mathsf{N}\mathbb{Z}^d]$ given by

$$\Gamma_{\mathsf{N}} := \{\mathring{\gamma}_1, \ldots, \mathring{\gamma}_r\} =: [\mathsf{N}[0,1)^d] \cap \mathbb{Z}^d \quad \text{with} \quad \mathring{\gamma}_1 := 0. \tag{3.1.10}$$

We define **the standard vector conversion operator associated with** N via

$$[E_{\mathsf{N}} v](k) := (v(\mathsf{N}k + \mathring{\gamma}_1), v(\mathsf{N}k + \mathring{\gamma}_2), \ldots, v(\mathsf{N}k + \mathring{\gamma}_r)), \qquad k \in \mathbb{Z}^d, \quad v \in l(\mathbb{Z}^d). \tag{3.1.11}$$

It is obvious that E_{N} is a linear bijective mapping. For the case $d = 1$, we have a natural choice $\mathsf{N} = r$ so that $\Gamma_{\mathsf{N}} = \{0, 1, \ldots, r-1\}$ and E_{N} is simply the one dimensional standard vector conversion operator $\mathring{E}$ defined as in (2.2.1).

Let $(\{a; b\}, \{\tilde{a}; \tilde{b}\})_{\Theta}$ be an OEP-based dual M-framelet filter bank and $(\{\mathring{\phi}; \psi\}, \{\tilde{\mathring{\phi}}; \tilde{\psi}\})$ be its corresponding dual M-framelet. Let $\mathbb{P}_{m-1}$ be the space of all d-variate polynomial sequences of degree less than m. The sparsity of a multiframelet transform is described by its ability to annihilate framelet coefficients w_j for polynomial input data, which is measured by the following properties:

(1) The operator $\mathcal{T}_{a,\mathsf{M}}$ is invariant on $E_{\mathsf{N}}(\mathbb{P}_{m-1})$, that is,

$$\mathcal{T}_{a,\mathsf{M}} E_{\mathsf{N}}(\mathsf{p}) \in E_{\mathsf{N}}(\mathbb{P}_{m-1}), \qquad \forall\, \mathsf{p} \in \mathbb{P}_{m-1}. \tag{3.1.12}$$

(2) The high-pass filter b has **order m E_{N}-balanced vanishing moments**, that is,

$$\mathcal{T}_{b,\mathsf{M}} E_{\mathsf{N}}(\mathsf{p}) = 0, \qquad \forall\, \mathsf{p} \in \mathbb{P}_{m-1}. \tag{3.1.13}$$

Items (1) and (2) preserve sparsity for all levels when implementing a multi-level discrete framelet transform, because the framelet coefficients $w_j := \mathcal{T}_{b,\mathsf{M}} \mathcal{T}_{a,\mathsf{M}}^{j-1} E_{\mathsf{N}}(\mathsf{p}) = 0$ for all $\mathsf{p} \in \mathbb{P}_{m-1}$ and $j \in \mathbb{N}$. We define $\operatorname{bvm}(b, \mathsf{M}, \mathsf{N}) := \sup\{m \in \mathbb{N}_0 : \text{(3.1.13) holds}\}$. A discrete framelet transform or a filter bank $\{a; b\}$ has m **E_{N}-balancing order** if both (3.1.12) and (3.1.13) hold. In particular, we define $\operatorname{bo}(\{a; b\}, \mathsf{M}, \mathsf{N}) := \sup\{m \in \mathbb{N}_0 : \text{(3.1.12) and (3.1.13) hold}\}$. The balancing property for multiwavelets has been studied in [9, 33, 41, 56, 65] and references therein.

Let $a, \tilde{a}, \theta, \tilde{\theta} \in (l_0(\mathbb{Z}^d))^{r\times r}$ and $b, \tilde{b} \in (l_0(\mathbb{Z}^d))^{s\times r}$ such that $(\{a; b\}, \{\tilde{a}; \tilde{b}\})_\Theta$ is an OEP-based dual M-multiframelet filter bank, where $\Theta = \theta^\star * \tilde{\theta}$. Suppose that $\phi, \tilde{\phi} \in (L_2(\mathbb{R}^d))^r$ are compactly supported M-refinable vector functions in $L_2(\mathbb{R}^d)$ satisfying $\widehat{\phi}(\mathsf{M}^\mathsf{T}\xi) = \widehat{a}(\xi)\widehat{\phi}(\xi)$ and $\widehat{\tilde{\phi}}(\mathsf{M}^\mathsf{T}\xi) = \widehat{\tilde{a}}(\xi)\widehat{\tilde{\phi}}(\xi)$. Define $\mathring{\phi}, \psi, \mathring{\tilde{\phi}}, \tilde{\psi}$ as in (1.1.12) and (1.1.13). If $\overline{\widehat{\phi}(0)}^\mathsf{T} \widehat{\Theta}(0)\widehat{\tilde{\phi}}(0) = 1$ and $\widehat{\psi}(0) = \widehat{\tilde{\psi}}(0) = 0$, then Theorem 1.1.1 tells us that $(\{\mathring{\phi}; \psi\}, \{\mathring{\tilde{\phi}}; \tilde{\psi}\})$ is a dual M-framelet in $L_2(\mathbb{R}^d)$. With $m := \operatorname{sr}(\tilde{a}, \mathsf{M})$, we observe that $\operatorname{vm}(\psi) \leqslant m$, $\operatorname{bvm}(b, \mathsf{M}, \mathsf{N}) \leqslant m$ and $\operatorname{bo}(\{a; b\}, \mathsf{M}, \mathsf{N}) \leqslant \operatorname{bvm}(b, \mathsf{M}, \mathsf{N})$. If $\operatorname{bo}(\{a; b\}, \mathsf{M}, \mathsf{N}) = \operatorname{bvm}(b, \mathsf{M}, \mathsf{N}) = \operatorname{vm}(\psi) = m$, then we say that the discrete framelet transform or the filter bank $\{a; b\}$ is **order m E_{N}-balanced**. For $r > 1$, $\operatorname{bo}(\{a, b\}, \mathsf{M}, \mathsf{N}) < \operatorname{vm}(\psi)$ often happens. Hence, having high vanishing moments on framelet generators does not guarantee the balancing property and thus significantly reduces the sparsity of the associated discrete multiframelet transform. How to overcome this shortcoming has been extensively studied in the setting of functions in [9, 56, 65] and in the setting of discrete framelet transforms in [33, 35, 41].

The following result on properties of the subdivision and the transition operators that are related to the standard vector conversion operator were investigated, we refer the reader to [35] for detailed discussions and proofs of the following result.

Theorem 3.1.3. *Let* M *be a* $d \times d$ *dilation matrix,* $s \in \mathbb{N}$ *and* $r \geqslant 2$ *be positive integers. Let* N *be a* $d \times d$ *integer matrix with* $|\det(\mathsf{N})| = r$ *and* E_{N} *be the standard vector conversion operator associated with* N *in* (3.1.11)*. Define* $\{\mathring{\gamma}_1, \ldots, \mathring{\gamma}_r\} := \Gamma_{\mathsf{N}}$ *as in* (3.1.10) *and*

$$\widehat{\Upsilon_{\mathsf{N}}}(\xi) := \left(e^{i\mathsf{N}^{-1}\mathring{\gamma}_1 \cdot \xi}, \ldots, e^{i\mathsf{N}^{-1}\mathring{\gamma}_r \cdot \xi}\right), \qquad \xi \in \mathbb{R}^d. \tag{3.1.14}$$

Define $\mathbb{P}_{m,y} := \{\mathsf{p} * y : \mathsf{p} \in \mathbb{P}_m\}$ *for* $y \in (l_0(\mathbb{Z}^d))^{1\times r}$*. Then the following statements hold:*

(1) $E_{\mathsf{N}}(\mathbb{P}_m) = \mathbb{P}_{m,y} \subseteq (\mathbb{P}_m)^{1\times r}$ *with* $y \in (l_0(\mathbb{Z}^d))^{1\times r}$ *if and only if*

$$\widehat{y}(\xi) = \widehat{c}(\xi)\widehat{\Upsilon_{\mathsf{N}}}(\xi) + \mathscr{O}(\|\xi\|^{m+1}) \text{ as } \xi \to 0 \text{ for some } c \in l_0(\mathbb{Z}^d) \text{ with } \widehat{c}(0) \neq 0.$$

(2) For $u \in (l_0(\mathbb{Z}^d))^{r\times r}$ *and* $y \in (l_0(\mathbb{Z}^d))^{1\times r}$*,* $\mathcal{T}_{u,\mathsf{M}}\mathbb{P}_{m,y} = \mathbb{P}_{m,y}$ *if and only if*

$$\widehat{c}(\xi)\widehat{y}(\mathsf{M}^{\mathsf{T}}\xi) = \widehat{y}(\xi)\overline{\widehat{u}(\xi)}^{\mathsf{T}} + \mathscr{O}(\|\xi\|^{m+1}) \text{ as } \xi \to 0 \text{ for some } c \in l_0(\mathbb{Z}^d) \text{ with } \widehat{c}(0) \neq 0.$$

(3) For $u \in (l_0(\mathbb{Z}^d))^{r\times r}$ *and* $y \in (l_0(\mathbb{Z}^d))^{1\times r}$*,* $\mathcal{S}_{u,\mathsf{M}}\mathbb{P}_{m,y} \subseteq (\mathbb{P}_m)^{1\times r}$ *if and only if*

$$\widehat{y}(\mathsf{M}^{\mathsf{T}}\xi)\widehat{u}(\xi + 2\pi\omega) = \mathscr{O}(\|\xi\|^{m+1}), \qquad \xi \to 0, \quad \omega \in \Omega_{\mathsf{M}} \setminus \{0\}.$$

(4) For $u, \tilde{u} \in (l_0(\mathbb{Z}^d))^{r\times r}$ *and* $y \in (l_0(\mathbb{Z}^d))^{1\times r}$*,* $\mathcal{S}_{u,\mathsf{M}}\mathcal{T}_{\tilde{u},\mathsf{M}}(v) = v$ *for all* $v \in \mathbb{P}_{m,y}$ *if and only if*

$$\widehat{y}(\xi)\overline{\widehat{\tilde{u}}(\xi)}^{\mathsf{T}}\widehat{u}(\xi + 2\pi\omega) = \boldsymbol{\delta}(\omega)\widehat{y}(\xi) + \mathscr{O}(\|\xi\|^{m+1}), \qquad \xi \to 0, \quad \omega \in \Omega_{\mathsf{M}}.$$

The following result is known (see [35, Proposition 3.1, Theorem 4.1]), which characterizes the balancing property of a discrete framelet transform in any dimension.

Theorem 3.1.4. *Let* M *be a* $d \times d$ *dilation matrix and* $r \geqslant 2$ *be a positive integer. Let* $a \in (l_0(\mathbb{Z}^d))^{r\times r}$ *and* $b \in (l_0(\mathbb{Z}^d))^{s\times r}$ *for some* $s \in \mathbb{N}$*. Let* N *be a* $d \times d$ *integer matrix with* $|\det(\mathsf{N})| = r$ *and* E_{N} *in* (3.1.11)*. Define* $\widehat{\Upsilon_{\mathsf{N}}}$ *as in* (3.1.14)*. Then the following statements hold:*

(1) The filter b *has order* m E_{N}*-balanced vanishing moments satisfying* (3.1.13) *if and only if*

$$\widehat{\Upsilon_{\mathsf{N}}}(\xi)\overline{\widehat{b}(\xi)}^{\mathsf{T}} = \mathscr{O}(\|\xi\|^m), \qquad \xi \to 0. \tag{3.1.15}$$

(2) The filter bank $\{a; b\}$ *is order* m E_{N}*-balanced satisfying both* (3.1.12) *and* (3.1.13) *if and only if* (3.1.15) *holds and*

$$\begin{aligned}\widehat{\Upsilon_{\mathsf{N}}}(\xi)\overline{\widehat{a}(\xi)}^{\mathsf{T}} =& \widehat{c}(\xi)\widehat{\Upsilon_{\mathsf{N}}}(\mathsf{M}^{\mathsf{T}}\xi) + \mathscr{O}(\|\xi\|^m), \quad \xi \to 0,\\ &\text{for some } c \in l_0(\mathbb{Z}^d) \text{ with } \widehat{c}(0) \neq 0.\end{aligned} \tag{3.1.16}$$

3.2 A Matrix-valued Filter Normal Form for an Arbitrary Dimension d

In this section, we extend the normal form of a matrix valued-filter developed in Chapter 2 for the case $d = 1$ to arbitrary dimentions. Some ideas from the case $d = 1$ can be borrowed, but several new elements and challenges are involved for the high dimensional case. First, we state the main result of this section as the following.

Theorem 3.2.1. *Let* M *be a* $d\times d$ *dilation matrix and* ϕ *be a vector of compactly supported distributions satisfying* $\widehat{\phi}(\mathsf{M}^{\mathsf{T}}\xi) = \widehat{a}(\xi)\widehat{\phi}(\xi)$ *with* $\widehat{\phi}(0) \neq 0$ *and a finitely supported matrix-valued filter* $a \in (l_0(\mathbb{Z}^d))^{r\times r}$*. Suppose that the filter* a *has order* m *sum rules with respect to* M *satisfying* (1.1.20) *with a matching filter* $\upsilon \in (l_0(\mathbb{Z}^d))^{1\times r}$ *such that* $\widehat{\upsilon}(0)\widehat{\phi}(0) = 1$*. If* $r \geqslant 2$*, then for any positive integer* $n \in \mathbb{N}$*, there exists a strongly invertible* $r \times r$ *matrix* $\widehat{U}$ *of* $2\pi\mathbb{Z}^d$*-periodic trigonometric polynomials such that the following properties hold:*

(1) Define $\widehat{\mathring{\upsilon}}(\xi) := (\widehat{\mathring{\upsilon}}_1(\xi), \dots, \widehat{\mathring{\upsilon}}_r(\xi)) := \widehat{\upsilon}(\xi)\widehat{U}(\xi)^{-1}$ *and* $\widehat{\mathring{\phi}}(\xi) := (\widehat{\mathring{\phi}}_1(\xi), \dots, \widehat{\mathring{\phi}}_r(\xi))^{\mathsf{T}} :=$

$\widehat{U}(\xi)\widehat{\phi}(\xi)$. *Then*

$$\widehat{\mathring{\phi}_1}(\xi) = 1 + \mathscr{O}(\|\xi\|^n) \quad \text{and} \quad \widehat{\mathring{\phi}_\ell}(\xi) = \mathscr{O}(\|\xi\|^n), \quad \xi \to 0, \quad \ell = 2, \dots, r, \qquad (3.2.1)$$

$$\widehat{\mathring{v}_1}(\xi) = 1 + \mathscr{O}(\|\xi\|^m) \quad \text{and} \quad \widehat{\mathring{v}_\ell}(\xi) = \mathscr{O}(\|\xi\|^m), \quad \xi \to 0, \quad \ell = 2, \dots, r. \qquad (3.2.2)$$

(2) Define a finitely supported matrix-valued filter $\mathring{a} \in (l_0(\mathbb{Z}^d))^{r\times r}$ *by* $\widehat{\mathring{a}}(\xi) := \widehat{U}(\mathsf{M}^\mathsf{T}\xi)\widehat{a}(\xi)\widehat{U}(\xi)^{-1}$. *Then the filter* $\mathring{a}$ *takes the* ***ideal*** (m,n)***-normal form****, i.e.,*

$$\widehat{\mathring{a}}(\xi) = \begin{bmatrix} \widehat{\mathring{a}_{1,1}}(\xi) & \widehat{\mathring{a}_{1,2}}(\xi) \\ \widehat{\mathring{a}_{2,1}}(\xi) & \widehat{\mathring{a}_{2,2}}(\xi) \end{bmatrix}, \qquad (3.2.3)$$

where $\widehat{\mathring{a}_{1,1}}, \widehat{\mathring{a}_{1,2}}, \widehat{\mathring{a}_{2,1}}$ *and* $\widehat{\mathring{a}_{2,2}}$ *are* 1×1, $1\times(r-1)$, $(r-1)\times 1$ *and* $(r-1)\times(r-1)$ *matrices of* $2\pi\mathbb{Z}^d$*-periodic trigonometric polynomials such that the following moment conditions hold as* $\xi \to 0$*:*

$$\widehat{\mathring{a}_{1,1}}(\xi) = 1 + \mathscr{O}(\|\xi\|^n), \quad \widehat{\mathring{a}_{1,1}}(\xi + 2\pi\omega) = \mathscr{O}(\|\xi\|^m), \quad \forall \omega \in \Omega_\mathsf{M} \setminus \{0\}, \qquad (3.2.4)$$

$$\widehat{\mathring{a}_{1,2}}(\xi + 2\pi\omega) = \mathscr{O}(\|\xi\|^m), \quad \forall \omega \in \Omega_\mathsf{M}, \qquad (3.2.5)$$

$$\widehat{\mathring{a}_{2,1}}(\xi) = \mathscr{O}(\|\xi\|^n), \qquad (3.2.6)$$

where Ω_M *is defined as* (1.1.16). *Moreover,* $\widehat{\mathring{\phi}}(\mathsf{M}^\mathsf{T}\xi) = \widehat{\mathring{a}}(\xi)\widehat{\mathring{\phi}}(\xi)$ *and the new filter* $\mathring{a}$ *has order* m *sum rules with respect to* M *with the matching filter* $\mathring{v} \in (l_0(\mathbb{Z}^d))^{1\times r}$.

(3) Define $\|\widehat{\phi}(\xi)\|^2 := \|\widehat{\phi}(\xi)\|_{l_2}^2 := \overline{\widehat{\phi}(\xi)}^\mathsf{T}\widehat{\phi}(\xi)$. *If in addition*

$$\widehat{v}(\xi) = \|\widehat{\phi}(\xi)\|^{-2}\overline{\widehat{\phi}(\xi)}^\mathsf{T} + \mathscr{O}(\|\xi\|^m), \quad \xi \to 0, \qquad (3.2.7)$$

then the strongly invertible $\widehat{U}$ *can satisfy the following additional "almost orthogonality" moment condition as* $\xi \to 0$*:*

$$\overline{\widehat{U}(\xi)}^{-\mathsf{T}}\widehat{U}(\xi)^{-1} = \mathit{Diag}\left(\|\widehat{\phi}(\xi)\|^2, \|\widehat{u_2}(\xi)\|^2, \dots, \|\widehat{u_r}(\xi)\|^2\right) + \mathscr{O}(\|\xi\|^{\tilde{n}}), \qquad (3.2.8)$$

where $\widehat{u_j}$ is the j-th column of the matrix $\widehat{U}^{-1}$ for $j = 2, \dots, r$ and $\tilde{n} := \max(m, n)$. Conversely, if there exists a strongly invertible matrix $\widehat{U}$ of $2\pi\mathbb{Z}^d$-periodic trigonometric polynomials such that items (1) and (2) and (3.2.8) *hold with $n \geqslant m$, then* (3.2.7) *must hold.*

If $d = 1$ and without loss of generality assume that M is positive, the three moment conditions (3.2.4), (3.2.5) and (3.2.6) further yield

$$\widehat{\mathring{a}_{1,1}}(\xi) = (1 + e^{-i\xi} + \cdots + e^{-i(\mathsf{M}-1)\xi})^m P_{1,1}(\xi) = 1 + \mathscr{O}(|\xi|^n), \qquad \xi \to 0,$$

$$\widehat{\mathring{a}_{1,2}}(\xi) = (1 - e^{-i\mathsf{M}\xi})^m P_{1,2}(\xi), \qquad \widehat{\mathring{a}_{2,1}}(\xi) = (1 - e^{-i\xi})^n P_{2,1}(\xi),$$

where $P_{1,1}, P_{1,2}$ and $P_{2,1}$ are some $1 \times 1, 1 \times (r-1)$ and $(r-1) \times 1$ matrices of 2π-periodic trigonometric polynomials. So the filter $\mathring{a}$ takes the form (2.3.3), which demonstrates that Thereom 2.3.1 on the one-dimensional normal form is indeed a special case of Thereom 3.2.1. Unfortunately for $d \geqslant 2$, there are no corresponding factors for $(1 + e^{-i\xi} + \cdots + e^{-i(\mathsf{M}-1)\xi})^m$ and $(1 - e^{-i\xi})^m$. This means the factorization technique that we had for the case $d = 1$ is no longer available, which illustrates that the investigation is more difficult for $d > 1$.

To prove Theorem 3.2.1, several auxiliary results are needed. We start with the following result, which is a straightforward generalization of Lemma 2.3.3 (see also [35, Lemma 2.3]).

Lemma 3.2.2. *Let $\widehat{v} = (\widehat{v_1}, \dots, \widehat{v_r})$ and $\widehat{u} = (\widehat{u_1}, \dots, \widehat{u_r})$ be $1 \times r$ vectors of functions which are infinitely differentiable at 0 with $\widehat{v}(0) \neq 0$ and $\widehat{u}(0) \neq 0$. If $r \geqslant 2$, then for any positive integer $n \in \mathbb{N}$, there exists a strongly invertible $U \in (l_0(\mathbb{Z}^d))^{r \times r}$ such that $\widehat{u}(\xi) = \widehat{v}(\xi)\widehat{U}(\xi) + \mathscr{O}(\|\xi\|^n)$ as $\xi \to 0$.*

Next, we establish the following lemma on the moment conditions for vectors of smooth functions, which is a simple extension of Lemma 2.3.5 for the case $d = 1$ to an arbitrary dimention d

Lemma 3.2.3. *Let $m \in \mathbb{N}$. Let $\widehat{\upsilon}$ be a $1 \times r$ row vector and $\widehat{u}$ be an $r \times 1$ column vector such that all the entries of $\widehat{\upsilon}$ and $\widehat{u}$ are functions which are infinitely differentiable at the origin such that*

$$\widehat{\upsilon}(\xi)\widehat{u}(\xi) = 1 + \mathscr{O}(\|\xi\|^m), \quad \xi \to 0. \tag{3.2.9}$$

Then for any positive integer n, there exists an $1 \times r$ vector $\widehat{\mathring{\upsilon}}$ of $2\pi\mathbb{Z}^d$-periodic trigonometric polynomials such that

$$\widehat{\mathring{\upsilon}}(\xi) = \widehat{\upsilon}(\xi) + \mathscr{O}(\|\xi\|^m) \quad \text{and} \quad \widehat{\mathring{\upsilon}}(\xi)\widehat{u}(\xi) = 1 + \mathscr{O}(\|\xi\|^n), \quad \xi \to 0. \tag{3.2.10}$$

To prove Theorem 3.2.1, we also need the following result linking a refinable vector function ϕ with the matching filter υ for the associated matrix-valued filter of ϕ.

Lemma 3.2.4. *Let M be a dilation matrix and $a \in (l_0(\mathbb{Z}^d))^{r\times r}$. Let ϕ be an $r \times 1$ vector of compactly supported distributions satisfying $\widehat{\phi}(\mathsf{M}^\mathsf{T}\xi) = \widehat{a}(\xi)\widehat{\phi}(\xi)$ with $\widehat{\phi}(0) \neq 0$. If a has order m sum rules with respect to M satisfying* (1.1.20) *with a matching filter $\upsilon \in (l_0(\mathbb{Z}^d))^{1\times r}$ and $\widehat{\upsilon}(0)\widehat{\phi}(0) = 1$, then*

$$\widehat{\upsilon}(\xi)\widehat{\phi}(\xi) = 1 + \mathscr{O}(\|\xi\|^m), \quad \xi \to 0. \tag{3.2.11}$$

Proof. This is the multivariate version of Lemma 2.3.4, but the proof does not follow trivially.

By our assumption on a, using $\widehat{\upsilon}(\mathsf{M}^\mathsf{T}\xi)\widehat{a}(\xi) = \widehat{\upsilon}(\xi) + \mathscr{O}(\|\xi\|^m)$ as $\xi \to 0$ and $\widehat{\phi}(\mathsf{M}^\mathsf{T}\xi) = \widehat{a}(\xi)\widehat{\phi}(\xi)$, we deduce that

$$\widehat{\upsilon}(\mathsf{M}^\mathsf{T}\xi)\widehat{\phi}(\mathsf{M}^\mathsf{T}\xi) = \widehat{\upsilon}(\mathsf{M}^\mathsf{T}\xi)\widehat{a}(\xi)\widehat{\phi}(\xi) = \widehat{\upsilon}(\xi)\widehat{\phi}(\xi) + \mathscr{O}(\|\xi\|^m), \quad \xi \to 0. \tag{3.2.12}$$

We now prove that (3.2.12) yields (3.2.11) using [32, Proposition 2.1]. For a $p \times q$ matrix $A = (a_{kj})_{1\leqslant k\leqslant p, 1\leqslant j\leqslant q}$ and an $s \times t$ matrix B, their Kronecker product $A \otimes B$ is the $(ps) \times (qt)$

block matrix given by

$$A \otimes B = \begin{bmatrix} a_{11}B & \dots & a_{1q}B \\ \vdots & \ddots & \vdots \\ a_{p1}B & \dots & a_{pq}B \end{bmatrix}.$$

For any $n \in \mathbb{N}$, define $\otimes^n A := A \otimes \cdots \otimes A$ with n copies of A. Recall that if A, B, C and E are matrices of sizes such that one can perform the matrix products AC and BE, then we have $(A \otimes B)(C \otimes E) = (AC) \otimes (BE)$. Thus by induction, we have $(\otimes^n (AC)) \otimes (BE) = [(\otimes^n A) \otimes B][(\otimes^n C) \otimes E]$.

Define the $1 \times d$ vector of differential operators $D := (\partial_1, \dots, \partial_d)$, where $\partial_j := \frac{\partial}{\partial \xi_j}$ for $j = 1, \dots, d$. For simplicity, we define $\widehat{g}(\xi) := \widehat{\upsilon}(\xi)\widehat{\phi}(\xi)$. Direct calculation yields $D \otimes [\widehat{g}(\mathsf{M}^\mathsf{T} \cdot)] = [(D\mathsf{M}^\mathsf{T}) \otimes \widehat{g}](\mathsf{M}^\mathsf{T} \cdot)$. Here $D\mathsf{M}^\mathsf{T} := \left(\sum_{j=1}^d \mathsf{M}_{1j}\partial_j, \dots, \sum_{j=1}^d \mathsf{M}_{dj}\partial_j\right)$ is a $1 \times d$ vector of differential operators where $\mathsf{M} := (\mathsf{M}_{jk})_{1 \leqslant j,k \leqslant d}$. By induction, for $j \in \mathbb{N}$, we have

$$[\otimes^j D] \otimes [\widehat{g}(\mathsf{M}^\mathsf{T} \cdot)] = [(\otimes^j (D\mathsf{M}^\mathsf{T})) \otimes \widehat{g}](\mathsf{M}^\mathsf{T} \cdot) = \left([(\otimes^j D) \otimes \widehat{g}](\mathsf{M}^\mathsf{T} \cdot)\right)(\otimes^j (\mathsf{M}^\mathsf{T})). \quad (3.2.13)$$

It follows from (3.2.12) and (3.2.13) that

$$\left([(\otimes^j D) \otimes \widehat{g}](0)\right)(\otimes^j (\mathsf{M}^\mathsf{T})) = [(\otimes^j D) \otimes \widehat{g}](0), \qquad j = 1, \dots, m-1.$$

Since all the eigenvalues of M are greater than 1 in modulus, so are the eigenvalues of $\otimes^j (\mathsf{M}^\mathsf{T})$ for every $j \in \mathbb{N}$. This forces the above linear system to have only the trivial solution $[(\otimes^j D) \otimes \widehat{g}](0) = \mathbf{0}_{1 \times dj}$ for $j = 1, \dots, m-1$. Hence we conclude that $\partial^\mu \widehat{g}(0) = 0$ for all $\mu \in \mathbb{N}_0^d$ with $1 \leqslant |\mu| \leqslant m-1$. By $g(0) = \widehat{\upsilon}(0)\widehat{\phi}(0) = 1$, we proved $g(\xi) = 1 + \mathscr{O}(\|\xi\|^m)$ as $\xi \to 0$, which is just (3.2.11). $\square$

We now prove the following theorem, which generalizes all results on the standard normal form of a matrix-valued filter in [32, 33, 35, 41, 43, 45] but under much weaker conditions.

Theorem 3.2.5. *Let* M *be a* $d \times d$ *dilation matrix and* $a \in (l_0(\mathbb{Z}^d))^{r \times r}$ *be a matrix-*

valued filter. Let ϕ be an $r \times 1$ vector of compactly supported distributions satisfying $\widehat{\phi}(\mathsf{M}^{\mathsf{T}}\xi) = \widehat{a}(\xi)\widehat{\phi}(\xi)$ with $\widehat{\phi}(0) \neq 0$. Suppose the filter a has order m sum rules with respect to M satisfying (1.1.20) with a matching filter $\upsilon \in (l_0(\mathbb{Z}^d))^{1\times r}$ such that $\widehat{\upsilon}(0)\widehat{\phi}(0) = 1$. Let $\widehat{\mathring{\upsilon}}$ be a $1 \times r$ row vector and $\widehat{u_\phi}$ be an $r \times 1$ column vector such that all the entries of $\widehat{\mathring{\upsilon}}$ and $\widehat{u_\phi}$ are functions which are infinitely differentiable at 0 and

$$\widehat{\mathring{\upsilon}}(\xi)\widehat{u_\phi}(\xi) = 1 + \mathscr{O}(\|\xi\|^m), \qquad \xi \to 0. \tag{3.2.14}$$

If $r \geqslant 2$, then for any positive integer $n \in \mathbb{N}$, there exists a strongly invertible $r \times r$ matrix $\widehat{U}$ of $2\pi\mathbb{Z}^d$-periodic trigonometric polynomials such that

$$\widehat{\upsilon}(\xi)\widehat{U}(\xi)^{-1} = \widehat{\mathring{\upsilon}}(\xi) + \mathscr{O}(\|\xi\|^m) \quad \textit{and} \quad \widehat{U}(\xi)\widehat{\phi}(\xi) = \widehat{u_\phi}(\xi) + \mathscr{O}(\|\xi\|^n), \qquad \xi \to 0. \tag{3.2.15}$$

Define $\widehat{\mathring{\phi}}(\xi) := \widehat{U}(\xi)\widehat{\phi}(\xi)$ and $\widehat{\mathring{a}}(\xi) := \widehat{U}(\mathsf{M}^{\mathsf{T}}\xi)\widehat{a}(\xi)\widehat{U}(\xi)^{-1}$. Then the following statements hold:

(i) The new vector function $\mathring{\phi}$ is a vector of compactly supported distributions satisfying $\widehat{\mathring{\phi}}(\mathsf{M}^{\mathsf{T}}\xi) = \widehat{\mathring{a}}(\xi)\widehat{\mathring{\phi}}(\xi)$ for all $\xi \in \mathbb{R}$ and $\widehat{\mathring{\phi}}(\xi) = \widehat{u_\phi}(\xi) + \mathscr{O}(\|\xi\|^n)$ as $\xi \to 0$.

(ii) The new finitely supported filter $\mathring{a}$ has order m sum rules with respect to M with the matching filter $\mathring{\upsilon}$ satisfying $\widehat{\mathring{\upsilon}}(0)\widehat{\mathring{\phi}}(0) = 1$ and (1.1.20) with a and υ being replaced by $\mathring{a}$ and $\mathring{\upsilon}$, respectively.

Proof. It suffices to prove the claims for $n \geqslant m$. By Lemma 3.2.4, we see that (3.2.11) holds. Note that $\widehat{\phi}$ is smooth at every $\xi \in \mathbb{R}^d$, which follows from the Paley-Wiener theorem. Thus by (3.2.14) and Lemma 3.2.3, without loss of generality we may assume that

$$\widehat{\upsilon}(\xi)\widehat{\phi}(\xi) = 1 + \mathscr{O}(\|\xi\|^n) \quad \text{and} \quad \widehat{\mathring{\upsilon}}(\xi)\widehat{u_\phi}(\xi) = 1 + \mathscr{O}(\|\xi\|^n), \qquad \xi \to 0. \tag{3.2.16}$$

Now by applying the argument as in the proof of Theorem 2.3.1, all claims can be proved. □

We are now ready to prove Theorem 3.2.1.

Proof of Theorem 3.2.1. Choose a strongly invertible $r \times r$ matrix $\widehat{U}$ of $2\pi\mathbb{Z}^d$-periodic trigonometric polynomials such that all claims of Theorem 3.2.5 hold with $\widehat{\mathring{v}}(\xi) = (1, 0, \dots, 0)$ and $\widehat{u_\phi}(\xi) = (1, 0, \dots, 0)^{\mathsf{T}}$. Then we immediately observe that item (1) holds.

Next, we prove item (2). By Theorem 3.2.5, we see that $\widehat{\mathring{\phi}}(\mathsf{M}^{\mathsf{T}}\xi) = \widehat{\mathring{a}}(\xi)\widehat{\mathring{\phi}}(\xi)$ and $\mathring{a}$ has order m sum rules with respect to M with the matching filter $\mathring{v}$. Moreover,

$$(1, 0, \dots, 0)\widehat{\mathring{a}}(\xi + 2\pi\omega) = \boldsymbol{\delta}(\omega)(1, 0, \dots, 0) + \mathscr{O}(\|\xi\|^m), \quad \xi \to 0, \quad \forall \omega \in \Omega_{\mathsf{M}}.$$

It follows that $\widehat{\mathring{a}_{1,1}}(\xi + 2\pi\omega) = \mathscr{O}(\|\xi\|^m)$ as $\xi \to 0$ for all $\omega \in \Omega_{\mathsf{M}} \setminus \{0\}$, and $\widehat{\mathring{a}_{1,2}}(\xi + 2\pi\omega) = \mathscr{O}(\|\xi\|^m)$ as $\xi \to 0$ for all $\omega \in \Omega_{\mathsf{M}}$. This proves the second moment condition in (3.2.4) and (3.2.5). On the other hand, by (3.2.1) and $\widehat{\mathring{\phi}}(\mathsf{M}^{\mathsf{T}}\xi) = \widehat{\mathring{a}}(\xi)\widehat{\mathring{\phi}}(\xi)$, it follows immediately that $\widehat{\mathring{a}_{1,1}}(\xi) = 1 + \mathscr{O}(\|\xi\|^n)$ and $\widehat{\mathring{a}_{2,1}}(\xi) = \mathscr{O}(\|\xi\|^n)$. This proves the first identity in (3.2.4) and (3.2.6). Hence item (2) is proved.

Finally, item (3) can be proved by applying the same argument as in the proof of item (ii) in Theorem 2.3.2.

□

3.3 Multivariate Quasi-tight Multiframelets with High Vanishing Moments and High Balancing Orders

In this section, we study OEP-based quasi-tight multiframelets with balancing property and compact discrete multiframelet transforms. The main result is the following theorem, which is a generalization of Theorem 2.4.1 to arbitrary dimensions.

Theorem 3.3.1. *Let M be a $d \times d$ dilation matrix and $\phi \in (L_2(\mathbb{R}^d))^r$ be a compactly supported M-refinable vector function satisfying $\widehat{\phi}(\mathsf{M}^{\mathsf{T}}\xi) = \widehat{a}(\xi)\widehat{\phi}(\xi)$ with $\widehat{\phi}(0) \neq 0$ and a matrix-valued filter $a \in (l_0(\mathbb{Z}^d))^{r \times r}$. Suppose that the filter a has order m sum rules with respect to M satisfying* (1.1.20) *with a matching filter $\upsilon \in (l_0(\mathbb{Z}^d))^{1 \times r}$ such that*

$\widehat{v}(0)\widehat{\phi}(0) = 1$. *Let* N *be a* $d \times d$ *integer matrix with* $|\det(\mathsf{N})| = r$. *If* $r \geqslant 2$, *then there exist filters* $b \in (l_0(\mathbb{Z}^d))^{s\times r}$, $\theta \in (l_0(\mathbb{Z}^d))^{r\times r}$ *and* $\epsilon_1, \dots, \epsilon_s \in \{\pm 1\}$ *such that*

(1) $\{\mathring{\phi}; \psi\}_{(\epsilon_1,\dots,\epsilon_s)}$ *is a compactly supported quasi-tight* M*-framelet in* $L_2(\mathbb{R}^d)$ *such that* ψ *has order* m *vanishing moments, where* $\mathring{\phi}$ *and* ψ *are defined in* (1.1.12). *Moreover,* $\mathring{\phi}$ *and* ψ *satisfy the refinable structure*

$$\widehat{\mathring{\phi}}(\mathsf{M}^{\mathsf{T}}\xi) = \widehat{\mathring{a}}(\xi)\widehat{\mathring{\phi}}(\xi), \qquad \widehat{\psi}(\mathsf{M}^{\mathsf{T}}\xi) = \widehat{\mathring{b}}(\xi)\widehat{\mathring{\phi}}(\xi), \qquad \xi \in \mathbb{R}^d, \tag{3.3.1}$$

with the filters $\mathring{a}, \mathring{b}$ *being defined as*

$$\widehat{\mathring{a}}(\xi) := \widehat{\theta}(\mathsf{M}^{\mathsf{T}}\xi)\widehat{a}(\xi)\widehat{\theta}(\xi)^{-1} \qquad \text{and} \qquad \widehat{\mathring{b}}(\xi) := \widehat{b}(\xi)\widehat{\theta}(\xi)^{-1}. \tag{3.3.2}$$

(2) $\widehat{\theta}$ *is* strongly invertible.

(3) $\{a; b\}_{\Theta,(\epsilon_1,\dots,\epsilon_s)}$ *and* $\{\mathring{a}; \mathring{b}\}_{\boldsymbol{\delta} I_r,(\epsilon_1,\dots,\epsilon_s)}$ *are finitely supported quasi-tight* M*-framelet filter banks, satisfying*

$$\overline{\widehat{a}(\xi)}^{\mathsf{T}}\widehat{\Theta}(\mathsf{M}^{\mathsf{T}}\xi)\widehat{a}(\xi + 2\pi\omega) + \overline{\widehat{b}(\xi)}^{\mathsf{T}} Diag(\epsilon_1, \dots, \epsilon_s)\widehat{b}(\xi + 2\pi\omega) = \boldsymbol{\delta}(\omega)\widehat{\Theta}(\xi) \tag{3.3.3}$$

and

$$\overline{\widehat{\mathring{a}}(\xi)}^{\mathsf{T}}\widehat{\mathring{a}}(\xi + 2\pi\omega) + \overline{\widehat{\mathring{b}}(\xi)}^{\mathsf{T}} Diag(\epsilon_1, \dots, \epsilon_s)\widehat{\mathring{b}}(\xi + 2\pi\omega) = \boldsymbol{\delta}(\omega)I_r, \tag{3.3.4}$$

for all $\xi \in \mathbb{R}^d, \omega \in \Omega_{\mathsf{M}}$, *where* $\boldsymbol{\delta}$ *and* Ω_{M} *are defined as in* (1.1.15) *and* (1.1.16), *respectively.*

(4) The associated discrete multiframelet transform employing $\{\mathring{a}; \mathring{b}\}_{\boldsymbol{\delta} I_r,(\epsilon_1,\dots,\epsilon_s)}$ *is compact and order* m E_{N}*-balanced, where* E_{N} *is the vector conversion operator in* (3.1.11).

To prove Theorem 3.3.1, let us first recall some notations. For $k \in \mathbb{Z}^d$, *the difference* *operator* ∇_k is defined via

$$\nabla_k u(n) = u(n) - u(n-k), \qquad \forall n \in \mathbb{Z}^d, \quad u \in (l(\mathbb{Z}^d))^{t\times r}.$$

For any multi-index $\beta = (\beta_1, \dots, \beta_d)^\mathsf{T} \in \mathbb{N}_0^d$, define $\nabla^\beta := \nabla_{e_1}^{\beta_1}\nabla_{e_2}^{\beta_2}\dots\nabla_{e_d}^{\beta_d}$, where $\{e_1, \dots, e_d\}$ is the standard basis of $\mathbb{R}^d$. For $u \in (l_0(\mathbb{Z}^d))^{r\times r}$, we have

$$\widehat{\nabla^\beta u}(\xi) = \widehat{\nabla^\beta \boldsymbol{\delta}}(\xi)\widehat{u}(\xi) = (1-e^{-i\xi_1})^{\beta_1}(1-e^{-i\xi_2})^{\beta_2}\cdots(1-e^{-i\xi_d})^{\beta_d}\widehat{u}(\xi), \quad \xi = (\xi_1, \dots, \xi_d)^\mathsf{T} \in \mathbb{R}^d.$$

For $x = (x_1, \dots, x_d)$ and $y = (y_1, \dots, y_d)$, we say $x \prec y$ if there exists $l \in \{1, \dots, d\}$ such that $x_j = y_j$ for all $j < l$ and $x_l < y_l$. By $x \preceq y$ we mean that $x \prec y$ or $x = y$.

By Theorem 1.1.1, the key step to obtain an OEP-based dual framelet is the construction of an OEP-based dual framelet filter bank $(\{a; b\}, \{\tilde{a}; \tilde{b}\})_\Theta$ satisfying (1.1.14) of Theorem 1.1.1. Let us now rewrite (1.1.14) into a matrix form below. For $\gamma \in \mathbb{Z}^d$ and $u \in (l(\mathbb{Z}^d))^{s\times r}$, the γ-*coset sequence* of u with respect to M is the sequence $u^{[\gamma;\mathsf{M}]} \in (l(\mathbb{Z}^d))^{s\times r}$ given by

$$u^{[\gamma;\mathsf{M}]}(k) = u(\gamma + \mathsf{M}k), \quad k \in \mathbb{Z}^d.$$

Trivially, $\widehat{u}(\xi) = \sum_{\gamma\in\Gamma_\mathsf{M}} \widehat{u^{[\gamma;\mathsf{M}]}}(\mathsf{M}^\mathsf{T}\xi)e^{-i\gamma\cdot\xi}$, where Γ_M is defined as (3.1.7). Let Ω_M be defined in (1.1.16). Define $\mathsf{F}_{r;\mathsf{M}}(\xi)$ to be the $(rd_\mathsf{M}) \times (rd_\mathsf{M})$ matrix below

$$\mathsf{F}_{r;\mathsf{M}}(\xi) := \left(e^{-i\gamma_l\cdot(\xi+2\pi\omega_k)}I_r\right)_{1\leqslant l,k\leqslant d_\mathsf{M}}. \tag{3.3.5}$$

For $\omega \in \Omega_\mathsf{M}$ and $u \in (l_0(\mathbb{Z}^d))^{r\times r}$, let $D_{u,\omega;\mathsf{M}}(\xi)$ and $E_{u,\omega;\mathsf{M}}(\xi)$ be the $(rd_\mathsf{M}) \times (rd_\mathsf{M})$ block matrices, whose (l, k)-th $r \times r$ blocks are given by

$$(D_{u,\omega;\mathsf{M}}(\xi))_{l,k} := \begin{cases} \widehat{u}(\xi + 2\pi\omega_l), & \text{if } \omega_l + \omega - \omega_k \in \mathbb{Z}^d \\ 0, & \text{otherwise,} \end{cases} \tag{3.3.6}$$

and

$$(E_{u,\omega;\mathsf{M}}(\xi))_{l,k} := \widehat{u^{[\gamma_k-\gamma_l;\mathsf{M}]}}(\xi)e^{-i\gamma_k\cdot(2\pi\omega)}. \tag{3.3.7}$$

Following the lines of the proof of [16, Lemma 7], we have

$$\mathsf{F}_{r;\mathsf{M}}(\xi)D_{u,\omega;\mathsf{M}}(\xi)\overline{\mathsf{F}_{r;\mathsf{M}}(\xi)}^{\mathsf{T}} = d_{\mathsf{M}}E_{u,\omega;\mathsf{M}}(\mathsf{M}^{\mathsf{T}}\xi), \qquad \xi \in \mathbb{R}^d, \omega \in \Omega_{\mathsf{M}}. \tag{3.3.8}$$

Recall that $P_{u;\mathsf{M}}(\xi) := \left[\widehat{u}(\xi + 2\pi\omega_1), \widehat{u}(\xi + 2\pi\omega_2), \dots, \widehat{u}(\xi + 2\pi\omega_{d_{\mathsf{M}}})\right]$ in (1.3.1). It is straightforward to check that $P_{u;\mathsf{M}}(\xi) = Q_{u;\mathsf{M}}(\mathsf{M}^{\mathsf{T}}\xi)\,\mathsf{F}_{r;\mathsf{M}}(\xi)$, where

$$Q_{u;\mathsf{M}}(\xi) := \left[\widehat{u^{[\gamma_1;\mathsf{M}]}}(\xi), \widehat{u^{[\gamma_2;\mathsf{M}]}}(\xi), \dots, \widehat{u^{[\gamma_{d_{\mathsf{M}}};\mathsf{M}]}}(\xi)\right]. \tag{3.3.9}$$

Since $\overline{\mathsf{F}_{r;\mathsf{M}}(\xi)}^{\mathsf{T}}\,\mathsf{F}_{r;\mathsf{M}}(\xi) = d_{\mathsf{M}}I_{d_{\mathsf{M}}r}$, it is trivial to observe that $P_{u;\mathsf{M}}(\xi)\overline{\mathsf{F}_{r;\mathsf{M}}(\xi)}^{\mathsf{T}} = d_{\mathsf{M}}Q_{u;\mathsf{M}}(\mathsf{M}^{\mathsf{T}}\xi)$. Now by (3.3.8), it is clear that (1.1.14) is equivalent to

$$\mathcal{N}_{a,\tilde{a},\Theta}(\xi) = \overline{Q_{b;\mathsf{M}}(\xi)}^{\mathsf{T}}Q_{\tilde{b};\mathsf{M}}(\xi), \tag{3.3.10}$$

with

$$\mathcal{N}_{a,\tilde{a},\Theta}(\xi) := d_{\mathsf{M}}^{-1}E_{\Theta,0;\mathsf{M}}(\xi) - \overline{Q_{a;\mathsf{M}}(\xi)}^{\mathsf{T}}\widehat{\Theta}(\xi)Q_{\tilde{a};\mathsf{M}}(\xi). \tag{3.3.11}$$

Note that $\{a;b\}_{\Theta,(\epsilon_1,\dots,\epsilon_s)}$ is a quasi-tight M-framelet filter bank if and only if $(\{a;b\},\{a;\tilde{b}\})_{\Theta}$ is a dual M-framelet filter bank with $\tilde{b} := \mathrm{Diag}(\epsilon_1,\dots,\epsilon_s)b$. In this case, (3.3.10) yields

$$\overline{Q_{b;\mathsf{M}}(\xi)}^{\mathsf{T}}\mathrm{Diag}(\epsilon_1,\dots,\epsilon_s)Q_{b;\mathsf{M}}(\xi) = \mathcal{N}_{a,a,\Theta}(\xi). \tag{3.3.12}$$

For $d = 1$, recall that a 2π-periodic trigonometric polynomial $\widehat{u}$ satisfies $\widehat{u}(\xi) = \mathscr{O}(\|\xi\|^m)$ as $\xi \to 0$ if and only if $(1 - e^{-i\xi})^m \mid \widehat{u}(\xi)$. When $d \geqslant 2$, we can no longer separate out a moment factor that plays the role of $(1 - e^{-i\xi})^m$. Nevertheless, the following result is known in [16, Lemma 5] and [32, Theorem 3.6], which characterizes the moment condition for arbitrary dimensions.

Lemma 3.3.2. *Let $m \in \mathbb{N}$ and $v \in l_0(\mathbb{Z}^d)$. Then $\widehat{v}(\xi) = \mathscr{O}(\|\xi\|^m)$ as $\xi \to 0$ if and only if*

$\widehat{v}(\xi) = \sum_{\beta \in \mathbb{N}_{0;m}^d} \widehat{\nabla^\beta \boldsymbol{\delta}}(\xi)\widehat{u_\beta}(\xi)$ *for some* $u_\beta \in l_0(\mathbb{Z}^d)$ *for all* $\beta \in \mathbb{N}_{0;m}^d$*, where* $\mathbb{N}_{0;m}^d := \{\beta \in \mathbb{N}_0^d : |\beta| = m\}$*.*

For simplicity of later presentation, we introduce the following definition.

Definition 3.3.3. *Let* M *be a* $d \times d$ *dilation matrix and* N *be a* $d \times d$ *integer matrix with* $|\det(\mathsf{N})| = r \geqslant 2$*. Let* E_{N} *be the vector conversion operator in* (3.1.11) *and* $\widehat{\Upsilon_{\mathsf{N}}}$ *in* (3.1.14)*.*

(a) For $\mathring{a} \in (l_0(\mathbb{Z}^d))^{r\times r}$*, we say that* $\mathring{a}$ *has* order m E_{N}-balanced sum rules with respect to M *if* a *has order* m *sum rules with respect to* M *with a matching filter* $\mathring{v} \in (l_0(\mathbb{Z}^d))^{1\times r}$ *satisfying*

$$\widehat{\mathring{v}}(\xi) = \widehat{c}(\xi)\widehat{\Upsilon_{\mathsf{N}}}(\xi) + \mathscr{O}(\|\xi\|^m), \qquad \xi \to 0, \text{ for some } c \in l_0(\mathbb{Z}^d) \text{ satisfying } \widehat{c}(0) \neq 0.$$

Such a filter $\mathring{v}$ *is called* an E_{N}-balanced matching filter *for* $\mathring{a}$*. We define* $\operatorname{bsr}(\mathring{a}, \mathsf{M}, \mathsf{N}) := m$ *with* m *being the largest such integer.*

(b) For $n \in \mathbb{N}$*, we say that* $\mathring{a}$ *is* an order n E_{N}-balanced refinement filter *associated to an* $r \times 1$ *vector* $\mathring{\phi}$ *of compactly supported distributions if* $\widehat{\mathring{\phi}}(\mathsf{M}^{\mathsf{T}}\xi) = \widehat{\mathring{a}}(\xi)\widehat{\mathring{\phi}}(\xi)$ *for all* $\xi \in \mathbb{R}^d$ *and*

$$\widehat{\mathring{\phi}}(\xi) = \widehat{d}(\xi)\overline{\widehat{\Upsilon_{\mathsf{N}}}(\xi)}^{\mathsf{T}} + \mathscr{O}(\|\xi\|^n), \quad \xi \to 0, \text{ for some } d \in l_0(\mathbb{Z}^d) \text{ satisfying } \widehat{d}(0) \neq 0.$$

We first prove a special case of Theorem 3.3.1, which states that certain balanced filters can be used to construct quasi-tight framelets with high order vanishing moments. This result plays a key role in our proof of Theorem 3.3.1 on multivariate quasi-tight framelets.

Theorem 3.3.4. *Let* M *be a* $d \times d$ *dilation matrix and* N *be a* $d \times d$ *integer matrix with* $|\det(\mathsf{N})| = r \geqslant 2$*. Define* E_{N} *and* $\widehat{\Upsilon_{\mathsf{N}}}$ *as in* (3.1.11) *and* (3.1.14)*, respectively. Suppose that* $\mathring{a} \in (l_0(\mathbb{Z}))^{r\times r}$ *is an order* m E_{N}*-balanced refinement filter associated to an* $r \times 1$

vector $\mathring{\phi}$ of compactly supported functions in $L_2(\mathbb{R}^d)$, and $\mathring{a}$ has order m E_{N}-balanced sum rules with respect to M with an E_{N}-balanced matching filter $\mathring{v} \in (l_0(\mathbb{Z}^d))^{1\times r}$. If

$$\widehat{\mathring{v}}(\xi) = \|\widehat{\mathring{\phi}}(\xi)\|^{-2}\overline{\widehat{\mathring{\phi}}(\xi)}^{\mathsf{T}} + \mathscr{O}(\|\xi\|^m) = \widehat{g}(\xi)\widehat{\Upsilon_{\mathsf{N}}}(\xi) + \mathscr{O}(\|\xi\|^m) \text{ as } \xi \to 0 \quad \text{for some } g \in l_0(\mathbb{Z}^d) \text{ with } \widehat{g}(0) \neq 0, \tag{3.3.13}$$

and

$$\|\widehat{\mathring{\phi}}(\xi)\|^2 = 1 + \mathscr{O}(\|\xi\|^n), \qquad \xi \to 0, \tag{3.3.14}$$

for some $n \geqslant 2m$, then there exist $\mathring{b} \in (l_0(\mathbb{Z}))^{s\times r}$ and $\epsilon_1, \ldots, \epsilon_s \in \{\pm 1\}$ for some $s \in \mathbb{N}$ such that

(i) $\{\mathring{a}; \mathring{b}\}_{\boldsymbol{\delta} I_r,(\epsilon_1,\ldots,\epsilon_s)}$ is a quasi-tight M-multiframelet filter bank satisfying (3.3.4).

(ii) $\{\mathring{\phi}; \psi\}_{(\epsilon_1,\ldots,\epsilon_s)}$ is a quasi-tight M-framelet in $L_2(\mathbb{R}^d)$ and ψ has order m vanishing moments, where $\widehat{\psi}(\xi) := \widehat{\mathring{b}}(\mathsf{M}^{-\mathsf{T}}\xi)\widehat{\mathring{\phi}}(\mathsf{M}^{-\mathsf{T}}\xi)$ for $\xi \in \mathbb{R}^d$.

Proof. As (3.3.13) holds, by Theorem 3.2.1, there exists a strongly invertible $U \in (l_0(\mathbb{Z}^d))^{r\times r}$ such that

$$\begin{aligned}
\widehat{\phi}(\xi) &:= \widehat{U}(\xi)\widehat{\mathring{\phi}}(\xi) = (1, 0, \ldots, 0)^{\mathsf{T}} + \mathscr{O}(\|\xi\|^n), \qquad \xi \to 0,\\
\widehat{v}(\xi) &:= \widehat{\mathring{v}}(\xi)\widehat{U}(\xi)^{-1} = (1, 0, \ldots, 0) + \mathscr{O}(\|\xi\|^m), \qquad \xi \to 0,
\end{aligned}$$

and (3.2.8) holds with ϕ being replaced by $\mathring{\phi}$. Moreover, by letting $\widehat{a}(\xi) = \widehat{U}(\mathsf{M}^{\mathsf{T}}\xi)\widehat{\mathring{a}}(\xi)\widehat{U}(\xi)^{-1}$, we see that $\widehat{a}$ takes the ideal (m, n)-normal form in item (2) of Theorem 3.2.1 with $\mathring{a}$ being replaced by a. Enumerate Ω_{M} as in (1.1.16). Define $\widehat{\mathbf{U}} := \overline{\widehat{U}}^{-\mathsf{T}}\widehat{U}^{-1}$ and

$$\widehat{a_1}(\xi) := \widehat{\mathbf{U}}(\xi) - \overline{\widehat{a}(\xi)}^{\mathsf{T}}\widehat{\mathbf{U}}(\mathsf{M}^{\mathsf{T}}\xi)\widehat{a}(\xi) \quad \text{and} \quad \widehat{a_j}(\xi) := -\overline{\widehat{a}(\xi)}^{\mathsf{T}}\widehat{\mathbf{U}}(\mathsf{M}^{\mathsf{T}}\xi)\widehat{a}(\xi + 2\pi\omega_j), \quad j = 2, \ldots, d_{\mathsf{M}}.$$

For $j = 1$, using (3.2.8) and (3.3.14), we have

$$\widehat{a_1}(\xi) = \begin{bmatrix} 1 & \\ & \widehat{\tilde{U}}(\xi) \end{bmatrix} - \overline{\widehat{a}(\xi)}^{\mathsf{T}} \begin{bmatrix} 1 & \\ & \widehat{\tilde{U}}(\mathsf{M}^{\mathsf{T}}\xi) \end{bmatrix} \widehat{a}(\xi) + \mathscr{O}(\|\xi\|^n) = \begin{bmatrix} p_1(\xi) & p_2(\xi) \\ p_3(\xi) & p_4(\xi) \end{bmatrix}, \quad \xi \to 0.$$

where $\widehat{\widetilde{U}}(\xi) = \mathrm{Diag}\,(\|\widehat{u_2}(\xi)\|^2, \ldots, \|\widehat{u_r}(\xi)\|^2)$ and $\widehat{u_j}$ denotes the j-th column of $\widehat{U}^{-1}$. Here p_1, p_2, p_3, p_4 are 1×1, $1\times(r-1)$, $(r-1)\times 1$ and $(r-1)\times(r-1)$ matrices of $2\pi\mathbb{Z}^d$-periodic trigonometric polynomials. Using (3.2.4), (3.2.5) and (3.2.6) with $\mathring{a}$ being replaced by a, we deduce the following moment conditions as $\xi \to 0$:

$$p_1(\xi) = 1 - \left(|\widehat{a_{1,1}}(\xi)|^2 + \overline{\widehat{a_{2,1}}(\xi)}^{\mathsf{T}}\widehat{\widetilde{U}}(\mathsf{M}^{\mathsf{T}}\xi)\widehat{a_{2,1}}(\xi)\right) + \mathscr{O}(\|\xi\|^n) = \mathscr{O}(\|\xi\|^n), \tag{3.3.15}$$

$$p_2(\xi) = -\overline{\widehat{a_{1,1}}(\xi)}\widehat{a_{1,2}}(\xi) - \overline{\widehat{a_{2,1}}(\xi)}^{\mathsf{T}}\widehat{\widetilde{U}}(\mathsf{M}^{\mathsf{T}}\xi)\widehat{a_{2,2}}(\xi) + \mathscr{O}(\|\xi\|^n) = \mathscr{O}(\|\xi\|^m), \tag{3.3.16}$$

$$p_3(\xi) = \overline{p_2(\xi)}^{\mathsf{T}} = \mathscr{O}(\|\xi\|^m). \tag{3.3.17}$$

For every $\beta \in \mathbb{N}_0^d$ with $|\beta| = m$, define $\Delta_\beta := \mathrm{Diag}(\nabla^\beta \boldsymbol{\delta}, \boldsymbol{\delta} I_{r-1}) \in (l_0(\mathbb{Z}^d))^{r\times r}$. Using (3.3.15),(3.3.16),(3.3.17), Lemma 3.3.2 and $n \geqslant 2m$, we see that there exist $B_{1,\alpha,\beta} \in (l_0(\mathbb{Z}^d))^{r\times r}$ for all $\alpha, \beta \in \mathbb{N}_{0;m}^d$ such that

$$\widehat{a_1}(\xi) = \sum_{\alpha,\beta\in\mathbb{N}_{0;m}^d} \overline{\widehat{\Delta_\alpha}(\xi)}^{\mathsf{T}}\widehat{B_{1,\alpha,\beta}}(\xi)\widehat{\Delta_\beta}(\xi), \tag{3.3.18}$$

where $\mathbb{N}_{0;m}^d := \{\beta \in \mathbb{N}_0^d : |\beta| = m\}$. For $j = 2, \ldots, d_{\mathsf{M}}$, we have

$$\widehat{a_j}(\xi) = -\overline{\widehat{a}(\xi)}^{\mathsf{T}}\begin{bmatrix} 1 & \\ & \widehat{\widetilde{U}}(\mathsf{M}^{\mathsf{T}}\xi)\end{bmatrix}\widehat{a}(\xi + 2\pi\omega_j) + \mathscr{O}(\|\mathsf{M}^{\mathsf{T}}\xi\|^n) = \begin{bmatrix} p_{j,1}(\xi) & p_{j,2}(\xi) \\ p_{j,3}(\xi) & p_{j,4}(\xi)\end{bmatrix}.$$

Here $p_{j,1}, p_{j,2}, p_{j,3}, p_{j,4}$ are 1×1, $1\times(r-1)$, $(r-1)\times 1$ and $(r-1)\times(r-1)$ matrices of $2\pi\mathbb{Z}^d$-periodic trigonometric polynomials. It follows from the (3.2.4), (3.2.5), (3.2.6) with $\mathring{a}$ being replaced by a and $n \geqslant 2m$ that as $\xi \to 0$,

$$p_{j,1}(\xi) = -\left(\overline{\widehat{a_{1,1}}(\xi)}\widehat{a_{1,1}}(\xi+2\pi\omega_j) + \overline{\widehat{a_{2,1}}(\xi)}^{\mathsf{T}}\widehat{\widetilde{U}}(\mathsf{M}^{\mathsf{T}}\xi)\widehat{a_{2,1}}(\xi+2\pi\omega_j)\right) + \mathscr{O}(\|\mathsf{M}^{\mathsf{T}}\xi\|^n) = \mathscr{O}(\|\xi\|^m),$$

$$p_{j,2}(\xi) = -\overline{\widehat{a_{1,1}}(\xi)}\widehat{a_{1,2}}(\xi+2\pi\omega_j) - \overline{\widehat{a_{2,1}}(\xi)}^{\mathsf{T}}\widehat{\widetilde{U}}(\mathsf{M}^{\mathsf{T}}\xi)P_{2,2}(\xi+2\pi\omega_j) + \mathscr{O}(\|\mathsf{M}^{\mathsf{T}}\xi\|^n) = \mathscr{O}(\|\xi\|^m),$$

$$p_{j,3}(\xi) = -\overline{\widehat{a_{1,2}}(\xi)}\widehat{a_{1,1}}(\xi+2\pi\omega_j) - \overline{\widehat{a_{2,2}}(\xi)}^{\mathsf{T}}\widehat{\widetilde{U}}(\mathsf{M}^{\mathsf{T}}\xi)\widehat{a_{2,1}}(\xi+2\pi\omega_j) + \mathscr{O}(\|\mathsf{M}^{\mathsf{T}}\xi\|^n) = \mathscr{O}(\|\xi\|^m),$$

and using symmetry and the same argument, we further have

$$p_{j,1}(\xi+2\pi\omega_j)=\mathscr{O}(\|\xi\|^m) \quad \text{and} \quad p_{j,3}(\xi+2\pi\omega_j)=\mathscr{O}(\|\xi\|^m), \quad \xi\to 0, j=2,\dots,d_{\mathsf{M}}.$$

Hence the above identities and Lemma 3.3.2 yield

$$\widehat{a_j}(\xi)=\sum_{\alpha,\beta\in\mathbb{N}_{0;m}^d}\overline{\widehat{\Delta_\alpha}(\xi)}^{\mathsf{T}}\widehat{B_{j,\alpha,\beta}}(\xi)\widehat{\Delta_\beta}(\xi+2\pi\omega_j), \tag{3.3.19}$$

for some $B_{j,\alpha,\beta}\in(l_0(\mathbb{Z}^d))^{r\times r}$ for all $\alpha,\beta\in\mathbb{N}_{0;m}^d$ and all $j=2,\dots,d_{\mathsf{M}}$. Recall that $P_{a;\mathsf{M}}(\xi):=[\widehat{a}(\xi+\omega_1),\dots,\widehat{a}(\xi+2\pi\omega_{d_{\mathsf{M}}})]$ as in (1.3.1). It follows from (3.3.18) and (3.3.19) that

$$\begin{aligned}\mathcal{M}_{a,\mathbf{U}}(\xi):=&\operatorname{Diag}\left(\widehat{\mathbf{U}}(\xi+\omega_1),\dots,\widehat{\mathbf{U}}(\xi+2\pi\omega_{d_{\mathsf{M}}})\right)-\overline{P_{a;\mathsf{M}}(\xi)}^{\mathsf{T}}\widehat{\mathbf{U}}(\mathsf{M}^{\mathsf{T}}\xi)P_{a;\mathsf{M}}(\xi)\\ =&\sum_{j=1}^{d_{\mathsf{M}}}D_{a_j,\omega_j}(\xi)=\sum_{j=1}^{d_{\mathsf{M}}}\sum_{\alpha,\beta\in\mathbb{N}_{0;m}^d}\overline{D_{\Delta_\alpha,0}(\xi)}^{\mathsf{T}}D_{B_{j,\alpha,\beta},\omega_j}(\xi)D_{\Delta_\beta,0}(\xi),\end{aligned} \tag{3.3.20}$$

where $D_{u,\omega}:=D_{u,\omega;\mathsf{M}}$ is defined via (3.3.6) for every $u\in(l_0(\mathbb{Z}^d))^{r\times r}$ and $\omega\in\Omega_{\mathsf{M}}$ with the subscript M being dropped for simplicity. It follows from (3.3.8) and (3.3.20) that

$$\begin{aligned}&d_{\mathsf{M}}^{-2}\,\mathsf{F}_{r;\mathsf{M}}(\xi)\mathcal{M}_{a,\mathbf{U}}(\xi)\overline{\mathsf{F}_{r;\mathsf{M}}(\xi)}^{\mathsf{T}}\\ =&d_{\mathsf{M}}^{-4}\sum_{\alpha,\beta\in\mathbb{N}_{0;m}^d}\left(\mathsf{F}_{r;\mathsf{M}}(\xi)\overline{D_{\Delta_\alpha,0}(\xi)}^{\mathsf{T}}\overline{\mathsf{F}_{r;\mathsf{M}}(\xi)}^{\mathsf{T}}\right)\left(\sum_{j=1}^{d_{\mathsf{M}}}\mathsf{F}_{r;\mathsf{M}}(\xi)D_{B_{j,\alpha,\beta},\omega_j}(\xi)\overline{\mathsf{F}_{r;\mathsf{M}}(\xi)}^{\mathsf{T}}\right)\left(\mathsf{F}_{r;\mathsf{M}}(\xi)D_{\Delta_\beta,0}(\xi)\overline{\mathsf{F}_{r;\mathsf{M}}(\xi)}^{\mathsf{T}}\right)\\ =&\sum_{\alpha,\beta\in\mathbb{N}_{0;m}^d}\overline{E_{\Delta_\alpha,0}(\mathsf{M}^{\mathsf{T}}\xi)}^{\mathsf{T}}\left(d_{\mathsf{M}}^{-1}\sum_{j=1}^{d_{\mathsf{M}}}E_{B_{j,\alpha,\beta},\omega_j}(\mathsf{M}^{\mathsf{T}}\xi)\right)E_{\Delta_\beta,0}(\mathsf{M}^{\mathsf{T}}\xi),\end{aligned}$$

where $E_{u,\omega}:=E_{u,\omega;\mathsf{M}}$ is defined via (3.3.7) for every $u\in(l_0(\mathbb{Z}^d))^{r\times r}$ and $\omega\in\Omega_{\mathsf{M}}$ by dropping the subscript M. Define

$$\mathring{E}_{\alpha,\beta}(\xi):=\frac{d_{\mathsf{M}}^{-1}}{2}\sum_{j=1}^{d_{\mathsf{M}}}\left(E_{B_{j,\alpha,\beta},\omega_j}(\xi)+\overline{E_{B_{j,\beta,\alpha},\omega_j}(\xi)}^{\mathsf{T}}\right).$$

It is straightforward to see that $\overline{\mathring{E}_{\alpha,\alpha}}^{\mathsf{T}} = \mathring{E}_{\alpha,\alpha}$ for all $\alpha \in \mathbb{N}_{0;m}^d$. It follows that

$$
\begin{aligned}
& d_{\mathsf{M}}^{-2}\, \mathsf{F}_{r;\mathsf{M}}(\xi)\mathcal{M}_{a,\mathbf{u}}(\xi)\overline{\mathsf{F}_{r;\mathsf{M}}(\xi)}^{\mathsf{T}} \\
= & \sum_{\alpha,\beta\in\mathbb{N}_{0;m}^d,\alpha\prec\beta} \Big(\overline{E_{\Delta_\alpha,0}(\mathsf{M}^{\mathsf{T}}\xi)}^{\mathsf{T}}\mathring{E}_{\alpha,\beta}(\mathsf{M}^{\mathsf{T}}\xi)E_{\Delta_\beta,0}(\mathsf{M}^{\mathsf{T}}\xi) + \overline{E_{\Delta_\beta,0}(\mathsf{M}^{\mathsf{T}}\xi)}^{\mathsf{T}}\overline{\mathring{E}_{\alpha,\beta}(\mathsf{M}^{\mathsf{T}}\xi)}^{\mathsf{T}}E_{\Delta_\alpha,0}(\mathsf{M}^{\mathsf{T}}\xi)\Big) \\
& + \sum_{\alpha\in\mathbb{N}_{0;m}^d} \overline{E_{\Delta_\alpha,0}(\mathsf{M}^{\mathsf{T}}\xi)}^{\mathsf{T}}\mathring{E}_{\alpha,\alpha}(\mathsf{M}^{\mathsf{T}}\xi)E_{\Delta_\alpha,0}(\mathsf{M}^{\mathsf{T}}\xi).
\end{aligned}
$$

For $\alpha, \beta \in \mathbb{N}_{0;m}^d$ and $\alpha \prec \beta$, we take any factorization $\mathring{E}_{\alpha,\beta}(\xi) = \overline{E_{\alpha,\beta,1}(\xi)}^{\mathsf{T}}E_{\alpha,\beta,2}(\xi)$ such that $E_{\alpha,\beta,1}$ and $E_{\alpha,\beta,2}$ are $r \times r$ matrices of $2\pi\mathbb{Z}^d$-periodic trigonometric polynomials. By calculation:

$$
\begin{aligned}
& \sum_{\alpha,\beta\in\mathbb{N}_{0;m}^d,\alpha\prec\beta} \overline{\left(E_{\alpha,\beta,1}(\xi)E_{\Delta_\alpha,0}(\xi) + E_{\alpha,\beta,2}(\xi)E_{\Delta_\beta,0}(\xi)\right)}^{\mathsf{T}}\left(E_{\alpha,\beta,1}(\xi)E_{\Delta_\alpha,0}(\xi) + E_{\alpha,\beta,2}(\xi)E_{\Delta_\beta,0}(\xi)\right) \\
= & \sum_{\alpha,\beta\in\mathbb{N}_{0;m}^d,\alpha\prec\beta} \Big(\overline{E_{\Delta_\alpha,0}(\xi)}^{\mathsf{T}}\mathring{E}_{\alpha,\beta}(\xi)E_{\Delta_\beta,0}(\xi) + \overline{E_{\Delta_\beta,0}(\xi)}^{\mathsf{T}}\overline{\mathring{E}_{\alpha,\beta}(\xi)}^{\mathsf{T}}E_{\Delta_\alpha,0}(\xi)\Big) \\
& + \sum_{\alpha,\beta\in\mathbb{N}_{0;m}^d,\alpha\prec\beta} \Big(\overline{E_{\Delta_\alpha,0}(\xi)}^{\mathsf{T}}\overline{E_{\alpha,\beta,1}(\xi)}^{\mathsf{T}}E_{\alpha,\beta,1}(\xi)E_{\Delta_\alpha,0}(\xi) + \overline{E_{\Delta_\beta,0}(\xi)}^{\mathsf{T}}\overline{E_{\alpha,\beta,2}(\xi)}^{\mathsf{T}}E_{\alpha,\beta,2}(\xi)E_{\Delta_\beta,0}(\xi)\Big) \\
= & \sum_{\alpha,\beta\in\mathbb{N}_{0;m}^d,\alpha\prec\beta} \Big(\overline{E_{\Delta_\alpha,0}(\xi)}^{\mathsf{T}}\mathring{E}_{\alpha,\beta}(\xi)E_{\Delta_\beta,0}(\xi) + \overline{E_{\Delta_\beta,0}(\xi)}^{\mathsf{T}}\overline{\mathring{E}_{\alpha,\beta}(\xi)}^{\mathsf{T}}E_{\Delta_\alpha,0}(\xi)\Big) \\
& + \sum_{\alpha\in\mathbb{N}_{0;m}^d} \overline{E_{\Delta_\alpha,0}(\xi)}^{\mathsf{T}}E_\alpha(\xi)E_{\Delta_\alpha,0}(\xi),
\end{aligned}
$$

where E_α is a Hermitian $d_{\mathsf{M}}r \times d_{\mathsf{M}}r$ matrix of $2\pi\mathbb{Z}^d$-periodic trigonometric polynomials for every $\alpha \in \mathbb{N}_{0;m}^d$. Define $\epsilon_{\alpha,\beta,k} := 1$ and $b_{\alpha,\beta,k} \in (l_0(\mathbb{Z}^d))^{1\times r}$ for $k = 1, \dots, rd_{\mathsf{M}}$ via

$$
\widehat{b_{\alpha,\beta}}(\xi) := \begin{bmatrix} \widehat{b_{\alpha,\beta,1}}(\xi) \\ \vdots \\ \widehat{b_{\alpha,\beta,rd_{\mathsf{M}}}}(\xi) \end{bmatrix} := E_{\alpha,\beta,1}(\mathsf{M}^{\mathsf{T}}\xi)\,\mathsf{F}_{r;\mathsf{M}}(\xi)\begin{bmatrix} \widehat{\Delta_\alpha}(\xi) \\ \mathbf{0}_{(d_{\mathsf{M}}-1)r\times r} \end{bmatrix} + E_{\alpha,\beta,2}(\mathsf{M}^{\mathsf{T}}\xi)\,\mathsf{F}_{r;\mathsf{M}}(\xi)\begin{bmatrix} \widehat{\Delta_\beta}(\xi) \\ \mathbf{0}_{(d_{\mathsf{M}}-1)r\times r} \end{bmatrix},
$$

for all $\alpha, \beta \in \mathbb{N}_{0;m}^d$ with $\alpha \prec \beta$, where $\mathbf{0}_{q\times t}$ denotes the $q \times t$ zero matrix. Define $P_{b;\mathsf{M}}$ via (1.3.1) for all matrix-valued filter b. By (3.3.8) and $\overline{\mathsf{F}_{r;\mathsf{M}}(\xi)}^{\mathsf{T}}\mathsf{F}_{r;\mathsf{M}}(\xi) = d_{\mathsf{M}}I_{d_{\mathsf{M}}r}$, it follows that

$$\begin{aligned} P_{b_{\alpha,\beta};\mathsf{M}}(\xi) =&E_{\alpha,\beta,1}(\mathsf{M}^\mathsf{T}\xi)\,\mathsf{F}_{r;\mathsf{M}}(\xi)D_{\Delta_\alpha,0}(\xi)+E_{\alpha,\beta,2}(\mathsf{M}^\mathsf{T}\xi)\,\mathsf{F}_{r;\mathsf{M}}(\xi)D_{\Delta_\beta,0}(\xi)\\ =&E_{\alpha,\beta,1}(\mathsf{M}^\mathsf{T}\xi)E_{\Delta_\alpha,0}(\mathsf{M}^\mathsf{T}\xi)\,\mathsf{F}_{r;\mathsf{M}}(\xi)+E_{\alpha,\beta,2}(\mathsf{M}^\mathsf{T}\xi)E_{\Delta_\beta,0}(\mathsf{M}^\mathsf{T}\xi)\,\mathsf{F}_{r;\mathsf{M}}(\xi), \end{aligned} \tag{3.3.21}$$

Similarly, for $\ell \in \{1,2\}$, we define $\epsilon_{\ell;\alpha,k} := (-1)^{\ell+1}$ and $b_{\ell;\alpha,k}$ by

$$\widehat{b_{\ell;\alpha}}(\xi) := \begin{bmatrix} \widehat{b_{\ell;\alpha,1}}(\xi)\\ \vdots \\ \widehat{b_{\ell;\alpha,d_\mathsf{M}r}}(\xi)\end{bmatrix} := \Big(pI_r-(-1)^\ell q(\mathring{E}_{\alpha,\alpha}(\mathsf{M}^\mathsf{T}\xi)-E_\alpha(\mathsf{M}^\mathsf{T}\xi))\Big)\,\mathsf{F}_{r;\mathsf{M}}(\xi)\begin{bmatrix}\widehat{\Delta_\alpha}(\xi)\\ \mathbf{0}_{(d_\mathsf{M}-1)r\times r}\end{bmatrix},$$

for $\alpha \in \mathbb{N}_{0;m}^d$ and $k=1,\dots,d_\mathsf{M}r$, where $p,q\in\mathbb{R}$ satisfy $p+q=\frac{1}{4}$. We conclude that

$$P_{b_{\ell;\alpha};\mathsf{M}}(\xi) = \Big(pI_r-(-1)^\ell q(\mathring{E}_{\alpha,\alpha}(\mathsf{M}^\mathsf{T}\xi)-E_\alpha(\mathsf{M}^\mathsf{T}\xi))\Big)\,E_{\Delta_\alpha,0}(\mathsf{M}^\mathsf{T}\xi)\,\mathsf{F}_{r;\mathsf{M}}(\xi), \tag{3.3.22}$$

for $\ell\in\{1,2\}$. Define

$$\begin{aligned} \{(b_\ell,\epsilon_\ell):\ell=1,\dots,s\} :=&\{(b_{\alpha,\beta,k},\epsilon_{\alpha,\beta,k}):\alpha,\beta\in\mathbb{N}_{0;m}^d \text{ with } \alpha\prec\beta,\quad k=1,\dots,d_\mathsf{M}r\}\\ &\cup\{(b_{\ell;\alpha,k},\epsilon_{\ell;\alpha,k}):\alpha\in\mathbb{N}_{0;m}^d,\quad k=1,\dots,d_\mathsf{M}r,\quad \ell=1,2\}, \end{aligned}$$

and let $b := [b_1^\mathsf{T},\dots,b_s^\mathsf{T}]^\mathsf{T}$. We claim that $\{a;b\}_{\mathbf{U};(\epsilon_1,\dots,\epsilon_s)}$ is an OEP-based quasi-tight

M-framelet filter bank. Indeed, by (3.3.21) and (3.3.22), we have

$$
\begin{aligned}
&\overline{P_{b;\mathsf{M}}(\xi)}^{\mathsf{T}}\operatorname{Diag}(\epsilon_1,\ldots,\epsilon_s)P_{b;\mathsf{M}}(\xi)\\
=&\sum_{\alpha,\beta\in\mathbb{N}_{0;m}^d,\alpha\prec\beta}\overline{P_{b_{\alpha,\beta};\mathsf{M}}(\xi)}^{\mathsf{T}}P_{b_{\alpha,\beta};\mathsf{M}}(\xi)+\sum_{\alpha\in\mathbb{N}_{0;m}^d}\left(\overline{P_{b_{1;\alpha};\mathsf{M}}(\xi)}^{\mathsf{T}}P_{b_{1;\alpha};\mathsf{M}}(\xi)-\overline{P_{b_{2;\alpha};\mathsf{M}}(\xi)}^{\mathsf{T}}P_{b_{2;\alpha};\mathsf{M}}(\xi)\right)\\
=&\overline{\mathsf{F}_{r;\mathsf{M}}(\xi)}^{\mathsf{T}}\left(\sum_{\alpha,\beta\in\mathbb{N}_{0;m}^d,\alpha\prec\beta}\left(\overline{E_{\Delta_\alpha,0}(\mathsf{M}^{\mathsf{T}}\xi)}^{\mathsf{T}}\mathring{E}_{\alpha,\beta}(\mathsf{M}^{\mathsf{T}}\xi)E_{\Delta_\beta,0}(\mathsf{M}^{\mathsf{T}}\xi)+\overline{E_{\Delta_\beta,0}(\mathsf{M}^{\mathsf{T}}\xi)}^{\mathsf{T}}\overline{\mathring{E}_{\alpha,\beta}(\mathsf{M}^{\mathsf{T}}\xi)}^{\mathsf{T}}E_{\Delta_\alpha,0}(\mathsf{M}^{\mathsf{T}}\xi)\right)\right.\\
&\left.+\sum_{\alpha\in\mathbb{N}_{0;m}^d}\overline{E_{\Delta_\alpha,0}(\mathsf{M}^{\mathsf{T}}\xi)}^{\mathsf{T}}\mathring{E}_{\alpha,\alpha}(\mathsf{M}^{\mathsf{T}}\xi)E_{\Delta_\alpha,0}(\mathsf{M}^{\mathsf{T}}\xi)\right)\mathsf{F}_{r;\mathsf{M}}(\xi)\\
=&\overline{\mathsf{F}_{r;\mathsf{M}}(\xi)}^{\mathsf{T}}\left(d_{\mathsf{M}}^{-2}\,\mathsf{F}_{r;\mathsf{M}}(\xi)\mathcal{M}_{a,\mathbf{U}}(\xi)\overline{\mathsf{F}_{r;\mathsf{M}}(\xi)}^{\mathsf{T}}\right)\mathsf{F}_{r;\mathsf{M}}(\xi)\\
=&\mathcal{M}_{a,\mathbf{U}}(\xi).
\end{aligned}
$$

This proves the claim. Define $\mathring{b}\in(l_0(\mathbb{Z}^d))^{s\times r}$ via $\widehat{\mathring{b}}=\widehat{b}\widehat{U}$. The above identity is equivalent to saying that $\{\mathring{a};\mathring{b}\}_{\boldsymbol{\delta}I_r;(\epsilon_1,\ldots,\epsilon_s)}$ is a quasi-tight M-framelet filter bank. This proves item (i).

By definition in item (ii), $\widehat{\psi}(\mathsf{M}^{\mathsf{T}}\xi)=\widehat{\mathring{b}}(\xi)\widehat{\mathring{\phi}}(\xi)$. Note that $\overline{\widehat{\phi}(0)}^{\mathsf{T}}\widehat{\mathbf{U}}(0)\widehat{\phi}(0)=\|\widehat{\mathring{\phi}}(0)\|^2=1$. Thus by Theorem 1.1.1, $\{\mathring{\phi};\psi\}_{(\epsilon_1,\ldots,\epsilon_s)}$ is a quasi-tight M-framelet in $L_2(\mathbb{R}^d)$. Moreover, note that

$$
\begin{bmatrix}\widehat{\Delta_\alpha}(\xi)\\ \mathbf{0}_{(d_{\mathsf{M}}-1)r\times r}\end{bmatrix}\widehat{\phi}(\xi)=\begin{bmatrix}\widehat{\Delta_\alpha}(\xi)\\ \mathbf{0}_{(d_{\mathsf{M}}-1)r\times r}\end{bmatrix}(1,0,\ldots,0)^{\mathsf{T}}+\mathscr{O}(\|\xi\|^n)=\mathscr{O}(\|\xi\|^m),\qquad \xi\to0,\quad\forall\alpha\in\mathbb{N}_{0;m}^d.
$$

Thus it follows that $\widehat{\psi}(\mathsf{M}^{\mathsf{T}}\xi)=\widehat{\mathring{b}}(\xi)\widehat{\mathring{\phi}}(\xi)=\widehat{b}(\xi)\widehat{\phi}(\xi)=\mathscr{O}(\|\xi\|^m)$ as $\xi\to0$. This proves item (ii). $\square$

We are now ready to prove the main result Theorem 3.3.1 on multivariate quasi-tight framelets.

Proof of Theorem 3.3.1. By Theorem 3.2.5, there exists a strongly invertible $\theta\in$

$(l_0(\mathbb{Z}^d))^{r\times r}$ such that

$$\widehat{\mathring{v}}(\xi) := \widehat{v}(\xi)\widehat{\theta}(\xi)^{-1} = \frac{1}{\sqrt{r}}\widehat{\Upsilon_{\mathsf{N}}}(\xi) + \mathscr{O}(\|\xi\|^m), \qquad \xi \to 0, \tag{3.3.23}$$

$$\widehat{\mathring{\phi}}(\xi) := \widehat{\theta}(\xi)\widehat{\phi}(\xi) = \frac{1}{\sqrt{r}}\overline{\widehat{\Upsilon_{\mathsf{N}}}(\xi)}^{\mathsf{T}} + \mathscr{O}(\|\xi\|^n), \qquad \xi \to 0, \tag{3.3.24}$$

for some $n \geqslant 2m$, where $\widehat{\Upsilon_{\mathsf{N}}}$ is defined in (3.1.14). In fact, the proof works as long as (3.3.13) and (3.3.14) hold with $n \geqslant 2m$.

By the choice of θ, item (2) trivially holds. Let $\widehat{\mathring{a}}(\xi) := \widehat{\theta}(\mathsf{M}^{\mathsf{T}}\xi)\widehat{a}(\xi)\widehat{\theta}(\xi)^{-1}$. Then $\mathring{a}$ is an order m E_{N}-balanced refinement filter associated to the refinement vector function $\mathring{\phi}$, with the E_{N}-balanced matching filter $\mathring{v} \in (l_0(\mathbb{Z}^d))^{1\times r}$. Moreover, (3.3.13) and (3.3.14) hold. Thus by Theorem 3.3.4, there exist $b \in (l_0(\mathbb{Z}^d))^{s\times r}$ and $\epsilon_1,\dots,\epsilon_s \in \{\pm 1\}$ such that items (1) and (3) hold.

On the other hand, using (3.3.23) and (3.3.24), we have

$$\widehat{\Upsilon_{\mathsf{N}}}(\xi)\overline{\widehat{\mathring{a}}(\xi)}^{\mathsf{T}} = \sqrt{r}\overline{\widehat{\mathring{\phi}}(\xi)}^{\mathsf{T}}\overline{\widehat{\mathring{a}}(\xi)}^{\mathsf{T}} + \mathscr{O}(\|\xi\|^n) = \sqrt{r}\overline{\widehat{\mathring{\phi}}(\mathsf{M}^{\mathsf{T}}\xi)}^{\mathsf{T}} + \mathscr{O}(\|\xi\|^n) = \widehat{\Upsilon_{\mathsf{N}}}(\mathsf{M}^{\mathsf{T}}\xi) + \mathscr{O}(\|\xi\|^m),$$

and

$$\widehat{\Upsilon_{\mathsf{N}}}(\xi)\overline{\widehat{\mathring{b}}(\xi)}^{\mathsf{T}} = \sqrt{r}\overline{\widehat{\mathring{\phi}}(\xi)}^{\mathsf{T}}\overline{\widehat{\mathring{b}}(\xi)}^{\mathsf{T}} + \mathscr{O}(\|\xi\|^n) = \sqrt{r}\overline{\widehat{\psi}(\mathsf{M}^{\mathsf{T}}\xi)}^{\mathsf{T}} + \mathscr{O}(\|\xi\|^n) = \mathscr{O}(\|\xi\|^m), \quad \xi \to 0.$$

as $\xi \to 0$. Hence item (4) follows from Theorem 3.1.4. The proof is now complete. $\square$

Though Theorem 3.3.1 is for multiplicity $r \geqslant 2$, one can easily obtain a similar but weaker result for $r = 1$. For $r = 1$, the notion of balancing property will not come into play since we no longer need the vectorization of scalar data. On the other hand, a scalar filter $\theta \in l_0(\mathbb{Z}^d)$ is strongly invertible if and only if $\widehat{\theta}(\xi) = ce^{-ik\cdot\xi}$ for some $c \in \mathbb{C}\setminus\{0\}$ and $k \in \mathbb{Z}^d$. Thus it is too much to expect the strong invertibility of θ when $r = 1$ (see [33] for details about the case $d = 1$). By applying the Hermitian matrix decomposition technique as presented in the proof of Theorem 3.3.4, one can still achieve high vanishing moments on framelet generators. We have the following corollary of Theorem 3.3.4.

Corollary 3.3.5. *Let* M *be a* $d \times d$ *dilation matrix and let* $\phi \in L_2(\mathbb{R}^d)$ *be a compactly supported refinable function satisfying* $\widehat{\phi}(\mathsf{M}^{\mathsf{T}}\xi) = \widehat{a}(\xi)\widehat{\phi}(\xi)$ *with* $\widehat{\phi}(0) \neq 0$, *where* $a \in l_0(\mathbb{Z}^d)$ *has order* m *sum rules with respect to* M *satisfying* (1.1.20) *with a matching filter* $\upsilon \in l_0(\mathbb{Z}^d)$ *such that* $\widehat{\upsilon}(\xi) = 1/\widehat{\phi}(\xi) + \mathscr{O}(\|\xi\|^m)$ *as* $\xi \to 0$. *Then there exist* $b \in (l_0(\mathbb{Z}^d))^{s\times 1}$, $\epsilon_1, \dots, \epsilon_s \in \{\pm 1\}$ *and* $\theta \in l_0(\mathbb{Z}^d)$ *such that*

1. $\{a; b\}_{\Theta;(\epsilon_1,\dots,\epsilon_s)}$ *forms an OEP-based quasi-tight* M*-framelet filter bank satisfying* (3.3.3).
2. $\{\mathring{\phi}; \psi\}_{(\epsilon_1,\dots,\epsilon_s)}$ *is a compactly supported quasi-tight* M*-framelet in* $L_2(\mathbb{R}^d)$ *and* ψ *has order* m *vanishing moments, where* $\mathring{\phi}$ *and* ψ *are defined in* (3.3.1).

3.4 The Structure of OEP-based Balanced Multivariate Quasi-tight Multiframelets

In this section, we investigate the structure of OEP-based balanced quasi-tight multiframelets. To derive a balanced quasi-tight multiframelet through OEP, the filter θ in Theorem 3.3.1 plays a key role in our investigation. Hence it is important for us to understand the underlying structure that θ must satisfy. The proof of Theorem 3.3.1 reveals some ideas on which strongly invertible $\theta \in (l_0(\mathbb{Z}^d))^{r\times r}$ serves as a desired filter for constructing balanced quasi-tight framelets.

For simplicity of later discussion, we need the following definition.

Definition 3.4.1. *Let* N *be a* $d \times d$ *integer matrix with* $|\det(\mathsf{N})| = r \geqslant 2$ *and* E_{N} *be the vector conversion operator defined as* (3.1.11). *Let* M *be a* $d \times d$ *dilation matrix and* $a \in (l_0(\mathbb{Z}^d))^{r\times r}$ *be a filter associated to a compactly supported refinable vector function* $\phi \in (L_2(\mathbb{R}^d))^r$ *satisfying* $\widehat{\phi}(\mathsf{M}^{\mathsf{T}}\xi) = \widehat{a}(\xi)\widehat{\phi}(\xi)$. *Suppose that the filter* a *has order* m *sum rules with respect to* M *satisfying* (1.1.20) *with a matching filter* $\upsilon \in (l_0(\mathbb{Z}^d))^{1\times r}$ *such that* $\widehat{\upsilon}(0)\widehat{\phi}(0) = 1$. *We say that* $\theta \in (l_0(\mathbb{Z}^d))^{r\times r}$ *is* an order m E_{N}-balanced moment correction filter *associated to the filter* a *if* θ *is strongly invertible and there exist* $b \in (l_0(\mathbb{Z}^d))^{s\times r}$, $\epsilon_1, \dots, \epsilon_s \in \{\pm 1\}$ *such that all claims in Theorem 3.3.1 hold.*

A concrete characterization of balanced moment correction filters is given by the following theorem.

Theorem 3.4.2. *Let* N *be a* $d \times d$ *integer matrix with* $|\det(\mathsf{N})| = r \geqslant 2$ *and define* E_{N} *and* $\widehat{\Upsilon_{\mathsf{N}}}$ *as in* (3.1.11) *and* (3.1.14), *respectively. Let* M *be a* $d \times d$ *dilation matrix and* $\phi \in (L_2(\mathbb{R}^d))^r$ *be a compactly supported* M*-refinable vector function satisfying* $\widehat{\phi}(\mathsf{M}^{\mathsf{T}}\xi) = \widehat{a}(\xi)\widehat{\phi}(\xi)$ *such that* $a \in (l_0(\mathbb{Z}^d))^{r\times r}$ *has order* m *sum rules with respect to* M *with a matching filter* $\upsilon \in (l_0(\mathbb{Z}^d))^{1\times r}$ *and* $\widehat{\upsilon}(0)\widehat{\phi}(0) \neq 0$. *Then the following statements hold:*

(i) If $\theta \in (l_0(\mathbb{Z}^d))^{r\times r}$ *is strongly invertible and if* (3.3.13) *and* (3.3.14) *hold with* $n = 2m$ *and*

$$\widehat{\mathring{a}}(\xi) := \widehat{\theta}(\mathsf{M}^{\mathsf{T}}\xi)\widehat{a}(\xi)\widehat{\theta}(\xi)^{-1}, \quad \widehat{\mathring{\upsilon}}(\xi) := \widehat{\upsilon}(\xi)\widehat{\theta}(\xi)^{-1} \quad \text{and} \quad \widehat{\mathring{\phi}}(\xi) := \widehat{\theta}(\xi)\widehat{\phi}(\xi),$$

then θ *is an order* m E_{N}*-balanced moment correction filter associated to* ϕ *and* a.

(ii) If $\theta \in (l_0(\mathbb{Z}^d))^{r\times r}$ *is an order* m E_{N}*-balanced moment correction filter associated to* ϕ *and* a, *then* θ *is strongly invertible and* (3.3.14) *must hold with* $n = 2m$. *If in addition*

$$\begin{aligned} &1 \text{ is a simple eigenvalue of } \widehat{a}(0), \\ &\text{and } \det(\lambda^{\pm\beta} I_r - \widehat{a}(0)) \neq 0 \text{ for all } \beta \in \mathbb{N}_0^d \text{ with } 0 < |\beta| < m, \end{aligned} \tag{3.4.1}$$

where $\lambda = (\lambda_1, \ldots, \lambda_d)$ *is the vector of all the eigenvalues of* M, *and if*

$$\widehat{p}(\mathsf{M}^{\mathsf{T}}\xi)\widehat{\Upsilon_{\mathsf{N}}}(\mathsf{M}^{\mathsf{T}}\xi)\widehat{\mathring{a}}(\xi) = \widehat{p}(\xi)\widehat{\Upsilon_{\mathsf{N}}}(\xi) + \mathscr{O}(\|\xi\|^m) \text{ as } \xi \to 0, \tag{3.4.2}$$

for some $p \in l_0(\mathbb{Z}^d)$ *with* $\widehat{p}(0) \neq 0$, *then* (3.3.13) *must hold.*

Proof. Following the lines of the proof of Theorem 3.3.1, if θ is strongly invertible such that (3.3.13) and (3.3.14) hold with $n = 2m$, then one can obtain $b \in (l_0(\mathbb{Z}^d))^{s\times r}$ and $\epsilon_1, \ldots, \epsilon_s \in \{\pm 1\}$ such that all claims of Theorem 3.3.1 hold, which implies that θ must be an order m E_{N}-balanced moment correction filter associated to ϕ and a. This proves item (i).

Conversely, if θ is an order m E_{N}-balanced moment correction filter associated to a, then there exist $b \in (l_0(\mathbb{Z}^d))^{s\times r}$ and $\epsilon_1, \dots, \epsilon_s \in \{\pm 1\}$ such that all claims of Theorem 3.3.1 hold. In particular,

$$\overline{\widehat{\mathring{a}}(\xi)}^{\mathsf{T}}\widehat{\mathring{a}}(\xi) + \overline{\widehat{\mathring{b}}(\xi)}^{\mathsf{T}}\operatorname{Diag}(\epsilon_1, \dots, \epsilon_s)\widehat{\mathring{b}}(\xi) = I_r, \tag{3.4.3}$$

where $\widehat{\mathring{b}}(\xi) := \widehat{b}(\xi)[\widehat{\theta}(\xi)]^{-1}$. By multiplying $\overline{\widehat{\mathring{\phi}}(\xi)}^{\mathsf{T}}$ to the left and $\widehat{\mathring{\phi}}(\xi)$ to the right on both sides of (3.4.3), and using item (1) of Theorem 3.3.1, we deduce that (3.3.14) holds with $n = 2m$.

By item (4) of Theorem 3.3.1, we see that (3.1.13) holds with b being replaced by $\mathring{b}$ respectively. Consequently, we deduce from (3.1.4), (3.1.5) and (3.3.4) that for all $u \in E_{\mathsf{N}}(\mathbb{P}_{m-1})$,

$$\begin{aligned}\widehat{u}(\xi) &= \sum_{j=1}^{d_{\mathsf{M}}} \widehat{u}(\xi + 2\pi\omega_j)\overline{\widehat{\mathring{a}}(\xi + 2\pi\omega_j)}^{\mathsf{T}}\widehat{\mathring{a}}(\xi) + \sum_{j=1}^{d_{\mathsf{M}}} \widehat{u}(\xi + 2\pi\omega_j)\overline{\widehat{\mathring{b}}(\xi + 2\pi\omega_j)}^{\mathsf{T}}\operatorname{Diag}(\epsilon_1, \dots, \epsilon_s)\widehat{\mathring{b}}(\xi)\\ &= d_{\mathsf{M}}^{\frac{1}{2}}\widehat{\mathcal{T}_{\mathring{a},\mathsf{M}}u}(\mathsf{M}^{\mathsf{T}}\xi)\widehat{\mathring{a}}(\xi) + d_{\mathsf{M}}^{\frac{1}{2}}\widehat{\mathcal{T}_{\mathring{b},\mathsf{M}}u}(\mathsf{M}^{\mathsf{T}}\xi)\operatorname{Diag}(\epsilon_1, \dots, \epsilon_s)\widehat{\mathring{b}}(\xi) = \widehat{\mathcal{S}_{\mathring{a},\mathsf{M}}\mathcal{T}_{\mathring{a},\mathsf{M}}u}(\xi).\end{aligned}$$

Suppose in addition that (3.4.1) and (3.4.2) hold. Let $y \in (l_0(\mathbb{Z}^d))^{1\times r}$ be such that

$$\widehat{y}(\xi) = \widehat{p}(\xi)\widehat{\Upsilon_{\mathsf{N}}}(\xi) + \mathscr{O}(\|\xi\|^m) \text{ where } p \text{ is the same as in (3.4.2).} \tag{3.4.4}$$

By item (1) of Theorem 3.1.3, we have $E_{\mathsf{N}}(\mathbb{P}_{m-1}) = \mathbb{P}_{m-1,y} \subseteq (\mathbb{P}_{m-1})^{1\times r}$. Thus by item (2) of Theorem 3.1.3, we have

$$\mathcal{S}_{\mathring{a},\mathsf{M}}(\mathbb{P}_{m-1,y}) = \mathcal{S}_{\mathring{a},\mathsf{M}}\mathcal{T}_{\mathring{a},\mathsf{M}}E_{\mathsf{N}}(\mathbb{P}_{m-1}) = E_{\mathsf{N}}(\mathbb{P}_{m-1}) \subseteq (\mathbb{P}_{m-1})^{1\times r}.$$

Hence by item (3) of Theorem 3.1.3 and (3.4.2), $\mathring{a}$ has order m sum rules with respect to M, with a matching filter $y \in (l_0(\mathbb{Z}^d))^{1\times r}$ satisfying (3.4.4). On the other hand, since $\mathring{a}$

has order m sum rules with a matching filter $\mathring{v}$ with $\widehat{\mathring{v}} := \widehat{v}\widehat{\theta}^{-1}$, we have

$$\widehat{\mathring{v}}(\mathsf{M}^\mathsf{T}\xi)\widehat{\mathring{a}}(\xi) = \widehat{\mathring{v}}(\xi) + \mathscr{O}(\|\xi\|^m), \quad \xi \to 0.$$

Now the condition in (3.4.1) will force $\widehat{\mathring{v}}(\xi) = \widehat{y}(\xi) + \mathscr{O}(\|\xi\|^m)$ as $\xi \to 0$.

By our assumption in item (ii) on θ, item (4) of Theorem 3.3.1 holds. Hence, (3.1.15) and (3.1.16) of Theorem 3.1.4 hold with $a = \mathring{a}$ and $b = \mathring{b}$. Multiplying $\widehat{\Upsilon_\mathsf{N}}(\xi)$ from the left-hand side of (3.3.4) with $\omega = 0$, we deduce from (3.1.15) and (3.1.16) that

$$\widehat{\Upsilon_\mathsf{N}}(\xi) = \widehat{\Upsilon_\mathsf{N}}(\xi)\overline{\widehat{\mathring{a}}(\xi)}^\mathsf{T}\widehat{\mathring{a}}(\xi) + \mathscr{O}(\|\xi\|^m) = \widehat{c}(\xi)\widehat{\Upsilon_\mathsf{N}}(\mathsf{M}^\mathsf{T}\xi)\widehat{\mathring{a}}(\xi) + \mathscr{O}(\|\xi\|^m) = \widehat{c}(\xi)\frac{\widehat{p}(\xi)}{\widehat{p}(\mathsf{M}^\mathsf{T}\xi)}\widehat{\Upsilon_\mathsf{N}}(\xi) + \mathscr{O}(\|\xi\|^m)$$

as $\xi \to 0$. Since $\widehat{p}(0) \neq 0$ and $\widehat{\Upsilon}(0) \neq 0$, we conclude from the above identity that $\widehat{c}(0) = 1$. Since $\widehat{\mathring{\phi}}(\mathsf{M}^\mathsf{T}\xi) = \widehat{\mathring{a}}(\xi)\widehat{\mathring{\phi}}(\xi)$ and (3.1.16) holds with $\widehat{c}(0) = 1$ and $a = \mathring{a}$, (3.4.1) will force

$$\widehat{\mathring{\phi}}(\xi) = \widehat{f}(\xi)\overline{\widehat{\Upsilon_\mathsf{N}}(\xi)}^\mathsf{T} + \mathscr{O}(\|\xi\|^m), \quad \xi \to 0$$

with $\widehat{f}(\xi) := \prod_{j=1}^\infty \overline{\widehat{c}((\mathsf{M}^\mathsf{T})^{-j}\xi)}$. Note that $\widehat{f}(0) = 1$, $\widehat{\mathring{v}}(\xi)\widehat{\mathring{\phi}}(\xi) = 1 + \mathscr{O}(\|\xi\|^m)$ and $\|\widehat{\mathring{\phi}}(\xi)\|^2 = r\widehat{f}(\xi)\overline{\widehat{f}(\xi)} + \mathscr{O}(\|\xi\|^m)$ as $\xi \to 0$. It follows that $\widehat{p}(\xi) = \frac{1}{r\widehat{f}(\xi)} + \mathscr{O}(\|\xi\|^m)$ as $\xi \to 0$. Thus

$$\widehat{\mathring{v}}(\xi) = [r\widehat{f}(\xi)]^{-1}\widehat{\Upsilon_\mathsf{N}}(\xi) + \mathscr{O}(\|\xi\|^m) = \|\widehat{\mathring{\phi}}(\xi)\|^{-2}\overline{\widehat{\mathring{\phi}}(\xi)}^\mathsf{T} + \mathscr{O}(\|\xi\|^m), \qquad \xi \to 0.$$

Therefore, (3.3.13) holds. This proves item (ii). $\square$

Here we give an example to illustrate such an E_N-balanced moment correction filter θ.

Example 3.1. *Consider a compactly supported $M_{\sqrt{2}}$-refinable vector function $\phi = (\phi_1, \phi_2)^\mathsf{T}$ given in [26] (see Figure 3.1 for details) with its refinement matrix filter $a \in (l_0(\mathbb{Z}^2))^{2\times 2}$ being given by*

$$\widehat{a}(\xi_1, \xi_2) := \frac{1}{4}\begin{bmatrix} 2 & 1 + e^{i\xi_1} + e^{i\xi_2} + e^{i(\xi_1+\xi_2)} \\ 2e^{-i\xi_1} & 0 \end{bmatrix} \quad \textit{and} \quad M_{\sqrt{2}} := \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}. \qquad (3.4.5)$$

The filter a has order 2 sum rules with respect to $M_{\sqrt{2}}$, with a matching filter $\upsilon \in (l_0(\mathbb{Z}^2))^{1\times 2}$ satisfying

$$\widehat{\upsilon}(\xi) = \left(1, 1 + \frac{i}{2}(\xi_1+\xi_2)\right) + \mathscr{O}(\|\xi\|^2), \qquad \xi = (\xi_1, \xi_2) \to (0,0).$$

Let $\mathsf{N} := M_{\sqrt{2}}$. One can obtain an order 2 E_{N}-balanced moment correction filter θ given by

$$\widehat{\theta}(\xi) := \begin{bmatrix} p_1(\xi) & p_2(\xi) \\ p_3(\xi) & p_4(\xi) \end{bmatrix}, \qquad \xi \in \mathbb{R}^2,$$

where p_1, p_2, p_3, p_4 are the following $2\pi\mathbb{Z}^2$-periodic bivariate trigonometric polynomials:

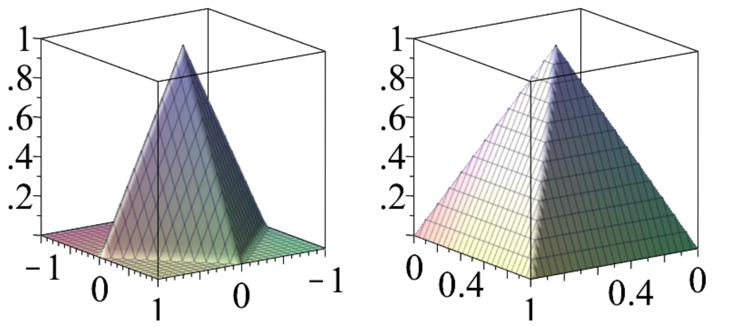
$$\begin{aligned}
p_1(\xi) &:= \frac{\sqrt{2}}{272}\big(542001 - 3225e^{-2i\xi_1} - 7740e^{-i(\xi_1+\xi_2)} - 265735e^{-i\xi_1} + 12267e^{i\xi_2} \\
&\qquad -4522e^{i(\xi_1-\xi_2)} - 273258e^{i\xi_1}\big),\\
p_2(\xi) &:= -\frac{\sqrt{2}}{17}e^{-i(\xi_1+\xi_2)}\left(645e^{-i\xi_1} - 646\right),\\
p_3(\xi) &:= \frac{\sqrt{2}}{3264}e^{i(\xi_1+\xi_2)}\big[(7740e^{-2i\xi_1} - 12267e^{-i\xi_1} + 4522)e^{-2i\xi_2} - 1075e^{-2i\xi_1} - 89655e^{-i\xi_1} + 90873 \\
&\qquad +(3225e^{-3i\xi_1} + 265735e^{-2i\xi_1} - 544581e^{-i\xi_1} + 274763)e^{-i\xi_2}\big],\\
p_4(\xi) &:= \frac{\sqrt{2}}{204}e^{-i\xi_1}(645e^{-i(\xi_1+\xi_2)} - 646e^{-i\xi_2} - 215).
\end{aligned}$$

Define $\widehat{\mathring{\upsilon}}(\xi) := \widehat{\upsilon}(\xi)\widehat{\theta}(\xi)^{-1}$ and $\widehat{\mathring{\phi}}(\xi) := \widehat{\theta}(\xi)\widehat{\phi}(\xi)$, we have $\|\widehat{\mathring{\phi}}(\xi)\|^2 = 1 + \mathscr{O}(\|\xi\|^4)$ as $\xi \to (0,0)$ and

$$\widehat{\mathring{\upsilon}}(\xi) = \overline{\widehat{\mathring{\phi}}(\xi)}^{\mathsf{T}} = -\frac{\sqrt{2}}{24}\left(12 + 429i\xi_1 - i\xi_2, 12 + 435i\xi_1 + 5\xi_2\right) + \mathscr{O}(\|\xi\|^2), \qquad \xi \to (0,0).$$

By Theorem 3.4.2, there exist $b \in (l_0(\mathbb{Z}^d))^{s\times r}$ and $\epsilon_1, \dots, \epsilon_s \in \{\pm 1\}$ such that all the claims in Theorem 3.3.1 hold with $m = 2$. For simplicity of presentation, we skip details about filters $b \in (l_0(\mathbb{Z}^d))^{s\times r}$.

The characterization of balanced moment correction filters in Theorem 3.4.2 motivates us to establish an algorithm for constructing quasi-tight multiframelets with high



(A) ϕ_1 (B) ϕ_2

Figure 3.1: The entries of the $M_{\sqrt{2}}$-refinable vector function $\phi = (\phi_1, \phi_2)^\mathsf{T}$ in Example 3.1.

balancing orders.

Lemma 3.4.3. *Let $r \geqslant 2$ and $s \in \mathbb{N}$ be positive integers. Let N be a $d \times d$ integer matrix with $|\det(\mathsf{N})| = r$ and define E_N as in* (3.1.11). *Then $b \in (l_0(\mathbb{Z}^d))^{s\times r}$ has order m E_N-balanced vanishing moments if and only if there exist $q_\beta \in (l_0(\mathbb{Z}^d))^{s\times 1}$ for all $\beta \in \mathbb{N}^d_{0;m}$ such that*

$$\widehat{b}(\xi) = \sum_{\beta\in\mathbb{N}^d_{0;m}} Q_{q_\beta;\mathsf{N}}(\xi)\widehat{E_{\beta;\mathsf{N}}}(\xi) \qquad \text{with} \quad \widehat{E_{\beta;\mathsf{N}}} := E_{\nabla^\beta\boldsymbol{\delta},0;\mathsf{N}}, \tag{3.4.6}$$

where $\mathbb{N}^d_{0;m} := \{\beta \in \mathbb{N}^d_0 : |\beta| = m\}$, $E_{\nabla^\beta\boldsymbol{\delta},0;\mathsf{N}}$ is defined as in (3.3.7) *with u and M being replaced by $\nabla^\beta\boldsymbol{\delta}$ and N, respectively, and $Q_{q_\beta;\mathsf{N}}$ is defined in* (3.3.9) *with u and M being replaced by q_β and N, respectively.*

Proof. Suppose that b has order m E_N-balanced vanishing moments, i.e., (3.1.15) holds. Recall that $\{\mathring{\gamma}_1, \dots, \mathring{\gamma}_r\} = \Gamma_\mathsf{N}$ as in (3.1.10). For simplicity, we define $u^{[\mathring{\gamma}_j]} := u^{[\mathring{\gamma}_j;\mathsf{N}]}$ for the $\mathring{\gamma}_j$-coset of u with respect to N. Let $\widehat{b_j}$ be the j-th column of $\widehat{b}$ for $j = 1, \dots, r$. By

Lemma 3.3.2, there exist $q_\beta \in (l_0(\mathbb{Z}^d))^{s\times 1}$ for all $\beta \in \mathbb{N}_{0;m}^d$ such that

$$
\begin{aligned}
\sum_{j=1}^{r} e^{-i\mathring{\gamma}_j\cdot\xi}\widehat{b_j}(\mathsf{N}^\mathsf{T}\xi) &= \sum_{\beta\in\mathbb{N}_{0;m}^d} \widehat{\nabla^\beta \boldsymbol{\delta}}(\xi)\widehat{q_\beta}(\xi) \\
&= \sum_{\beta\in\mathbb{N}_{0;m}^d} \left(\sum_{k=1}^{r} \widehat{\nabla^\beta \boldsymbol{\delta}^{[\mathring{\gamma}_k]}}(\mathsf{N}^\mathsf{T}\xi)e^{-i\mathring{\gamma}_k\cdot\xi}\right)\left(\sum_{l=1}^{r} \widehat{q_\beta^{[\mathring{\gamma}_l]}}(\mathsf{N}^\mathsf{T}\xi)e^{-i\mathring{\gamma}_l\cdot\xi}\right) \qquad (3.4.7)\\
&= \sum_{\beta\in\mathbb{N}_{0;m}^d}\sum_{k=1}^{r}\sum_{l=1}^{r} \widehat{\nabla^\beta \boldsymbol{\delta}^{[\mathring{\gamma}_k]}}(\mathsf{N}^\mathsf{T}\xi)\widehat{q_\beta^{[\mathring{\gamma}_l]}}(\mathsf{N}^\mathsf{T}\xi)e^{-i(\mathring{\gamma}_k+\mathring{\gamma}_l)\cdot\xi}.
\end{aligned}
$$

For every pair of indices $k, l \in \{1, \dots, r\}$, there exist unique $\mathring{\gamma}_{k,l} \in \Gamma_\mathsf{N}$ and $p_{k,l} \in \mathbb{Z}^d$ such that $\mathring{\gamma}_k + \mathring{\gamma}_l - \mathring{\gamma}_{k,l} = \mathsf{N}p_{k,l}$. Note that $\widehat{u^{[\mathring{\gamma}]}}(\xi)e^{ip\cdot\xi} = \widehat{u^{[\mathring{\gamma}+\mathsf{N}p]}}(\xi)$ for all $\mathring{\gamma}, p \in \mathbb{Z}^d$. It follows that

$$
\begin{aligned}
&\widehat{\nabla^\beta \boldsymbol{\delta}^{[\mathring{\gamma}_k]}}(\mathsf{N}^\mathsf{T}\xi)\widehat{q_\beta^{[\mathring{\gamma}_l]}}(\mathsf{N}^\mathsf{T}\xi)e^{-i(\mathring{\gamma}_k+\mathring{\gamma}_l)\cdot\xi} = \widehat{\nabla^\beta \boldsymbol{\delta}^{[\mathring{\gamma}_k]}}(\mathsf{N}^\mathsf{T}\xi)\widehat{q_\beta^{[\mathring{\gamma}_l]}}(\mathsf{N}^\mathsf{T}\xi)e^{-i\mathring{\gamma}_{k,l}\cdot\xi}e^{-ip_{k,l}\cdot\mathsf{N}^\mathsf{T}\xi}\\
&\quad=\widehat{\nabla^\beta \boldsymbol{\delta}^{[\mathring{\gamma}_k-\mathsf{N}p_{k,l}]}}(\mathsf{N}^\mathsf{T}\xi)\widehat{q_\beta^{[\mathring{\gamma}_l]}}(\mathsf{N}^\mathsf{T}\xi)e^{-i\mathring{\gamma}_{k,l}\cdot\xi} = \widehat{\nabla^\beta \boldsymbol{\delta}^{[\mathring{\gamma}_{k,l}-\mathring{\gamma}_k]}}(\mathsf{N}^\mathsf{T}\xi)\widehat{q_\beta^{[\mathring{\gamma}_l]}}(\mathsf{N}^\mathsf{T}\xi)e^{-i\mathring{\gamma}_{k,l}\cdot\xi}.
\end{aligned}
$$

It follows from (3.4.7) and the above identities that

$$
\sum_{k=1}^{r} e^{-i\mathring{\gamma}_k\cdot\xi}\widehat{b_k}(\mathsf{N}^\mathsf{T}\xi) = \sum_{k=1}^{r}\sum_{\beta\in\mathbb{N}_{0;m}^d}\sum_{l=1}^{r} \widehat{\nabla^\beta \boldsymbol{\delta}^{[\mathring{\gamma}_k-\mathring{\gamma}_l]}}(\mathsf{N}^\mathsf{T}\xi)\widehat{q_\beta^{[\mathring{\gamma}_l]}}(\mathsf{N}^\mathsf{T}\xi)e^{-i\mathring{\gamma}_k\cdot\xi}. \qquad (3.4.8)
$$

Hence, $\widehat{b_k}(\xi) = \sum_{\beta\in\mathbb{N}_{0;m}^d}\sum_{l=1}^{r} \widehat{\nabla^\beta \boldsymbol{\delta}^{[\mathring{\gamma}_k-\mathring{\gamma}_l]}}(\xi)\widehat{q_\beta^{[\mathring{\gamma}_l]}}(\xi)$ for all $k = 1, \dots, r$ and (3.4.6) follows immediately.

Conversely, if (3.4.6) holds, then (3.4.8) must hold. By working out the above calculation backwards, we obtain (3.4.7), which precisely means that b has order m E_N-balanced vanishing moments. $\square$

The following result is an immediate consequence of Lemma 3.4.3.

Proposition 3.4.4. *Let M be a $d\times d$ dilation matrix and N be a $d\times d$ integer matrix with $|\det(\mathsf{N})| = r \geqslant 2$ and define E_N as in (3.1.11). Let $\phi \in (L_2(\mathbb{R}^d))^r$ be a compactly supported vector function satisfying $\widehat{\phi}(\mathsf{M}^\mathsf{T}\xi) = \widehat{a}(\xi)\widehat{\phi}(\xi)$ with $a \in (l_0(\mathbb{Z}^d))^{r\times r}$. Suppose a has order m sum rules with respect to M satisfying (1.1.20) with a matching filter $\upsilon \in (l_0(\mathbb{Z}^d))^{1\times r}$*

such that $\widehat{\upsilon}(0)\widehat{\phi}(0) = 1$. *If* $\theta \in (l_0(\mathbb{Z}^d))^{r\times r}$ *is an order* m E_{N}*-balanced moment correction filter associated to* a*, by letting* $\widehat{\mathring{a}}(\xi) = \widehat{\theta}(\mathsf{M}^{\mathsf{T}}\xi)\widehat{a}(\xi)\widehat{\theta}(\xi)^{-1}$ *and*

$$\mathcal{M}_{\mathring{a}}(\xi) := I_{d_{\mathsf{M}}r} - \overline{P_{\mathring{a};\mathsf{M}}(\xi)}^{\mathsf{T}} P_{\mathring{a};\mathsf{M}}(\xi), \tag{3.4.9}$$

then there exist $A_{\alpha,\beta} \in (l_0(\mathbb{Z}^d))^{d_{\mathsf{M}}r\times d_{\mathsf{M}}r}$ *for all* $\alpha, \beta \in \mathbb{N}_{0;m}^d$ *with* $\alpha \preceq \beta$ *such that*

$$\overline{\widehat{A_{\alpha,\alpha}}(\xi)}^{\mathsf{T}} = \widehat{A_{\alpha,\alpha}}(\xi) \tag{3.4.10}$$

and

$$\begin{aligned}
d_{\mathsf{M}}^{-2}\,\mathsf{F}_{r;\mathsf{M}}(\xi)\mathcal{M}_{\mathring{a}}(\xi)\overline{\mathsf{F}_{r;\mathsf{M}}(\xi)}^{\mathsf{T}} &= \sum_{\alpha\in\mathbb{N}_{0;m}^d} \overline{E_{E_{\alpha;\mathsf{N}},0}(\mathsf{M}^{\mathsf{T}}\xi)}^{\mathsf{T}}\widehat{A_{\alpha,\alpha}}(\mathsf{M}^{\mathsf{T}}\xi)E_{E_{\alpha;\mathsf{N}},0}(\mathsf{M}^{\mathsf{T}}\xi)+\\
\sum_{\alpha,\beta\in\mathbb{N}_{0;m}^d,\alpha\prec\beta} &\Big[\overline{E_{E_{\alpha;\mathsf{N}},0}(\mathsf{M}^{\mathsf{T}}\xi)}^{\mathsf{T}}\widehat{A_{\alpha,\beta}}(\mathsf{M}^{\mathsf{T}}\xi)E_{E_{\beta;\mathsf{N}},0}(\mathsf{M}^{\mathsf{T}}\xi) \\
&+\overline{E_{E_{\beta;\mathsf{M}},0}(\mathsf{M}^{\mathsf{T}}\xi)}^{\mathsf{T}}\overline{\widehat{A_{\alpha,\beta}(\mathsf{M}^{\mathsf{T}}\xi)}}^{\mathsf{T}}E_{E_{\alpha;\mathsf{N}},0}(\mathsf{M}^{\mathsf{T}}\xi)\Big],
\end{aligned} \tag{3.4.11}$$

where $F_{r;\mathsf{M}}$ *is defined in* (3.3.5)*,* $E_{\beta;\mathsf{N}}$ *is defined via* (3.4.6)*, and* $E_{E_{\beta;\mathsf{N}},0} := E_{E_{\beta;\mathsf{N}},0;\mathsf{M}}$ *is defined via* (3.3.7) *with* u *and* ω *being replaced by* $E_{\beta;\mathsf{N}}$ *and* 0 *respectively.*

Proof. Since θ is an order m E_{N}-balanced moment correction filter associated to ϕ and a, there exist $b \in (l_0(\mathbb{Z}^d))^{s\times r}$ and $\epsilon_1, \dots, \epsilon_s \in \{\pm 1\}$ such that all the claims of Theorem 3.3.1 hold. In particular, by letting $\widehat{\mathring{b}} := \widehat{b}\widehat{\theta}$, $\{\mathring{a}; \mathring{b}\}_{(\epsilon_1,\dots,\epsilon_s)}$ is a quasi-tight M-framelet filter bank satisfying

$$\mathcal{M}_{\mathring{a}}(\xi) = \overline{P_{\mathring{b};\mathsf{M}}(\xi)}^{\mathsf{T}}\mathrm{Diag}(\epsilon_1, \dots, \epsilon_s)P_{\mathring{b};\mathsf{M}}(\xi).$$

Moreover, the filter $\mathring{b}$ has order m E_{N}-balanced vanishing moments. So by Lemma 3.4.3, there exist $q_\beta \in (l_0(\mathbb{Z}^d))^{s\times 1}$ for all $\beta \in \mathbb{N}_{0;m}^d$ such that (3.4.6) holds with b being replaced by $\mathring{b}$. Define

$$\widehat{\mathring{b}}_j(\xi) := \overline{\widehat{\mathring{b}}(\xi)}^{\mathsf{T}}\mathrm{Diag}(\epsilon_1, \dots, \epsilon_s)\widehat{\mathring{b}}(\xi + 2\pi\omega_j), \qquad j = 1, \dots, d_{\mathsf{M}}.$$

It follows that for $j=1,\dots,d_{\mathsf{M}}$,

$$\widehat{\mathring{b}_j}(\xi)=\sum_{\alpha,\beta\in\mathbb{N}_{0;m}^d}\overline{\widehat{E_{\alpha;\mathsf{N}}}(\xi)}^{\mathsf{T}}\overline{Q_{q_\alpha;\mathsf{N}}(\xi)}^{\mathsf{T}}\mathrm{Diag}(\epsilon_1,\dots,\epsilon_s)Q_{q_\beta;\mathsf{N}}(\xi+2\pi\omega_j)\widehat{E_{\beta;\mathsf{N}}}(\xi+2\pi\omega_j).$$

For $j=1,\dots,d_{\mathsf{M}}$, letting

$$\widehat{q_{\alpha,\beta,j}}(\xi):=\overline{Q_{q_\alpha;\mathsf{N}}(\xi)}^{\mathsf{T}}\mathrm{Diag}(\epsilon_1,\dots,\epsilon_s)Q_{q_\beta;\mathsf{N}}(\xi+2\pi\omega_j),\qquad \alpha,\beta\in\mathbb{N}_{0;m}^d,$$

we have

$$\mathcal{M}_{\mathring{a}}(\xi)=\sum_{j=1}^{d_{\mathsf{M}}}D_{\mathring{b}_j,\omega_j;\mathsf{M}}(\xi)=\sum_{j=1}^{d_{\mathsf{M}}}\sum_{\alpha,\beta\in\mathbb{N}_{0;m}^d}\overline{D_{E_{\alpha;\mathsf{N}},0;\mathsf{M}}(\xi)}^{\mathsf{T}}D_{q_{\alpha,\beta,j},\omega_j;\mathsf{M}}(\xi)D_{E_{\beta;\mathsf{N}},0;\mathsf{M}}(\xi).\quad(3.4.12)$$

Note that the decomposition in (3.4.12) is similar to the one in (3.3.20). Thus by applying the same idea as in the proof of Theorem 3.3.4, one can obtain (3.4.11). $\square$

We now provide an algorithm for constructing balanced multivariate quasi-tight framelets. This offers an alternative constructive proof to Theorem 3.3.1 on multivariate quasi-tight framelets.

Theorem 3.4.5. *Let M be a $d\times d$ dilation matrix and $\phi\in(L_2(\mathbb{R}^d))^r$ be a compactly supported vector function satisfying $\widehat{\phi}(\mathsf{M}^{\mathsf{T}}\xi)=\widehat{a}(\xi)\widehat{\phi}(\xi)$ with $a\in(l_0(\mathbb{Z}^d))^{r\times r}$. Suppose that the filter a has order m sum rules with respect to M satisfying* (1.1.20) *with a matching filter $\upsilon\in(l_0(\mathbb{Z}^d))^{1\times r}$ such that $\widehat{\upsilon}(0)\widehat{\phi}(0)=1$. If N is an $d\times d$ integer matrix with $|\det(\mathsf{N})|=r\geqslant 2$, then one can obtain $b\in(l_0(\mathbb{Z}^d))^{s\times r}$, $\epsilon_1,\dots,\epsilon_s\in\{\pm1\}$ and $\theta\in(l_0(\mathbb{Z}^d))^{r\times r}$ such that all claims of Theorem 3.3.1 hold by implementing the following steps:*

(Step 1) Construct a strongly invertible $\theta\in(l_0(\mathbb{Z}^d))^{r\times r}$ such that (3.3.13) *and* (3.3.14) *hold with $n=2m$, where $\widehat{\mathring{\upsilon}}(\xi):=\widehat{\upsilon}(\xi)\widehat{\theta}^{-1}(\xi)$ and $\widehat{\mathring{\phi}}(\xi)=\widehat{\theta}(\xi)\widehat{\phi}(\xi)$.*

(Step 2) Define $\widehat{\mathring{a}}(\xi):=\widehat{\theta}(\mathsf{M}^{\mathsf{T}}\xi)\widehat{a}(\xi)\widehat{\theta}(\xi)^{-1}$ and $\mathcal{M}_{\mathring{a}}(\xi)$ as in (3.4.9)*. Apply Proposition 3.4.4 to find $A_{\alpha,\beta}\in(l_0(\mathbb{Z}^d))^{rd_{\mathsf{M}}\times rd_{\mathsf{M}}}$ for all $\alpha,\beta\in\mathbb{N}_{0;m}^d$ with $\alpha\preceq\beta$ such that* (3.4.10) *and* (3.4.11) *hold.*

(Step 3) For all $\alpha, \beta \in \mathbb{N}_{0;m}^d$ with $\alpha \prec \beta$, factorize $\widehat{A_{\alpha,\beta}}(\xi) = \overline{\widehat{A_{\alpha,\beta,1}}(\xi)}^{\mathsf{T}} \widehat{A_{\alpha,\beta,2}}(\xi)$ with $A_{\alpha,\beta,1}, A_{\alpha,\beta,2} \in (l_0(\mathbb{Z}^d))^{d_{\mathsf{M}}r \times d_{\mathsf{M}}r}$. Find $B_\alpha \in (l_0(\mathbb{Z}^d))^{rd_{\mathsf{M}} \times rd_{\mathsf{M}}}$ for every $\alpha \in \mathbb{N}_{0;m}^d$ such that $\overline{\widehat{B_\alpha}(\xi)}^{\mathsf{T}} = \widehat{B_\alpha}(\xi)$ and

$$\sum_{\alpha,\beta\in\mathbb{N}_{0;m}^d,\alpha\prec\beta} \overline{\left(\widehat{A_{\alpha,\beta,1}}(\xi)E_{E_{\alpha;\mathsf{N}},0}(\xi) + \widehat{A_{\alpha,\beta,2}}(\xi)E_{E_{\beta;\mathsf{N}},0}(\xi)\right)}^{\mathsf{T}} \left(\widehat{A_{\alpha,\beta,1}}(\xi)E_{E_{\alpha;\mathsf{N}},0}(\xi) + \widehat{A_{\alpha,\beta,2}}(\xi)E_{E_{\beta;\mathsf{N}},0}(\xi)\right)$$
$$+ \sum_{\alpha\in\mathbb{N}_{0;m}^d} \overline{E_{E_{\alpha;\mathsf{N}},0}(\xi)}^{\mathsf{T}} \widehat{B_\alpha}(\xi) E_{E_{\alpha;\mathsf{N}},0}(\xi) = \mathcal{N}(\xi),$$

where $\mathcal{N}(\mathsf{M}^{\mathsf{T}}\xi) := d_{\mathsf{M}}^{-2}\, \mathsf{F}_{r;\mathsf{M}}(\xi)\mathcal{M}_{\mathring{a}}(\xi)\overline{\mathsf{F}_{r;\mathsf{M}}(\xi)}^{\mathsf{T}}$ and $E_{E_{\beta;\mathsf{N}},0} := E_{E_{\beta;\mathsf{N}},0;\mathsf{M}}$ is defined via (3.3.7) *with u and ω being replaced by $E_{\beta;\mathsf{N}}$ and 0, respectively.*

(Step 4) Define $\epsilon_{\alpha,\beta,k} = 1$ and $\mathring{b}_{\alpha,\beta,k} \in (l_0(\mathbb{Z}^d))^{1\times r}$ for $k = 1, \dots, d_{\mathsf{M}}r$ and $\alpha, \beta \in \mathbb{N}_{0;m}^d$ with $\alpha \prec \beta$ via

$$\widehat{\mathring{b}_{\alpha,\beta}}(\xi) := \begin{bmatrix} \widehat{\mathring{b}_{\alpha,\beta,1}}(\xi) \\ \vdots \\ \widehat{\mathring{b}_{\alpha,\beta,d_{\mathsf{M}}r}}(\xi) \end{bmatrix} := \widehat{A_{\alpha,\beta,1}}(\mathsf{M}^{\mathsf{T}}\xi)\, \mathsf{F}_{r;\mathsf{M}}(\xi) \begin{bmatrix} \widehat{E_{\alpha;\mathsf{N}}}(\xi) \\ \mathbf{0}_{(d_{\mathsf{M}}-1)r\times r} \end{bmatrix} + \widehat{A_{\alpha,\beta,2}}(\mathsf{M}^{\mathsf{T}}\xi)\, \mathsf{F}_{r;\mathsf{M}}(\xi) \begin{bmatrix} \widehat{E_{\beta;\mathsf{N}}}(\xi) \\ \mathbf{0}_{(d_{\mathsf{M}}-1)r\times r\cdot} \end{bmatrix},$$

for $\ell \in \{1, 2\}$ and $k = 1, \dots, d_{\mathsf{M}}r$, define $\epsilon_{\ell;\alpha,k} = (-1)^{\ell+1}$ and $\mathring{b}_{\ell;\alpha,k} \in (l_0(\mathbb{Z}^d))^{1\times r}$ by

$$\widehat{\mathring{b}_{\ell;\alpha}}(\xi) := \begin{bmatrix} \widehat{\mathring{b}_{\ell;\alpha,1}}(\xi) \\ \vdots \\ \widehat{\mathring{b}_{\ell;\alpha,d_{\mathsf{M}}r}}(\xi) \end{bmatrix} := \left(pI_r - (-1)^\ell q(\widehat{A_{\alpha,\alpha}}(\mathsf{M}^{\mathsf{T}}\xi) - \widehat{B_\alpha}(\mathsf{M}^{\mathsf{T}}\xi))\right) \mathsf{F}_{r;\mathsf{M}}(\xi) \begin{bmatrix} \widehat{E_{\alpha;\mathsf{N}}}(\xi) \\ \mathbf{0}_{(d_{\mathsf{M}}-1)r\times r} \end{bmatrix}$$

for $\alpha \in \mathbb{N}_{0;m}^d$, where $p, q \in \mathbb{R}$ satisfy $p + q = \frac{1}{4}$. Define

$$\begin{aligned}\{(\mathring{b}_\ell, \epsilon_\ell) : \ell = 1, \dots, s\} :=& \{(\mathring{b}_{\alpha,\beta,k}, \epsilon_{\alpha,\beta,k}) : \alpha, \beta \in \mathbb{N}_{0;m}^d \text{ with } \alpha \prec \beta, \quad k = 1, \dots, d_{\mathsf{M}}r\} \\ & \cup \{(\mathring{b}_{\ell;\alpha,k}, \epsilon_{\ell;\alpha,k}) : \alpha \in \mathbb{N}_{0;m}^d, \quad k = 1, \dots, d_{\mathsf{M}}r, \quad \ell = 1, 2\}.\end{aligned}$$

Let $\mathring{b} := [\mathring{b}_1^{\mathsf{T}}, \dots, \mathring{b}_s^{\mathsf{T}}]^{\mathsf{T}}$ and $b = \mathring{b} * \theta$. Then $\{\mathring{a}; \mathring{b}\}_{\boldsymbol{\delta} I_r,(\epsilon_1,\dots,\epsilon_r)}$ is a finitely supported quasi-tight M-framelet filter bank such that all the claims of Theorem 3.3.1 hold.

Proof. The existence of $\theta \in (l_0(\mathbb{Z}^d))^{r\times r}$ satisfying all the conditions in (Step 1) is guaranteed by Theorem 3.2.1 (e.g. choose $\theta \in (l_0(\mathbb{Z}^d))^{r\times r}$ such that (3.3.23) and (3.3.24) hold). So it is straightforward to see that item (2) of Theorem 3.3.1 holds. Moreover, θ is an order m E_{N}-balanced moment correction filter associated to ϕ and a. Thus, (Step 2) is justified by Proposition 3.4.4. The filters B_α satisfying the identity in (Step 3) can be obtained by using the same idea as in the proof of Theorem 3.3.4. Now define $\mathring{b}$ as in (Step 4). By (3.3.8) and the identity $\overline{\mathsf{F}_{r;\mathsf{M}}(\xi)}^{\mathsf{T}}\,\mathsf{F}_{r;\mathsf{M}}(\xi) = d_{\mathsf{M}} I_{d_{\mathsf{M}}r}$, we deduce that

$$P_{\mathring{b}_{\alpha,\beta};\mathsf{M}}(\xi) = \widehat{A_{\alpha,\beta,1}}(\mathsf{M}^{\mathsf{T}}\xi)E_{E_{\alpha;\mathsf{N}},0}(\mathsf{M}^{\mathsf{T}}\xi)\,\mathsf{F}_{r;\mathsf{M}}(\xi) + \widehat{A_{\alpha,\beta,2}}(\mathsf{M}^{\mathsf{T}}\xi)E_{E_{\beta;\mathsf{N}},0}(\mathsf{M}^{\mathsf{T}}\xi)\,\mathsf{F}_{r;\mathsf{M}}(\xi), \tag{3.4.13}$$

$$P_{\mathring{b}_{\ell;\alpha};\mathsf{M}}(\xi) = \left(pI_r - (-1)^{\ell}q(\widehat{A_{\alpha,\alpha}}(\mathsf{M}^{\mathsf{T}}\xi) - \widehat{B_\alpha}(\mathsf{M}^{\mathsf{T}}\xi))\right) E_{E_{\alpha;\mathsf{N}},0}(\mathsf{M}^{\mathsf{T}}\xi)\,\mathsf{F}_{r;\mathsf{M}}(\xi), \tag{3.4.14}$$

for $\ell \in \{1,2\}$. By (3.4.13) and (3.4.14), item (3) of Theorem 3.3.1 can be verified by direct calculation.

Define $q_{\alpha,\beta,l}, q_{l;\alpha} \in (l_0(\mathbb{Z}^d))^{d_{\mathsf{M}}r\times 1}$ for $\ell = 1,2$ and for all $\alpha,\beta \in \mathbb{N}_{0;m}^d$ with $\alpha \prec \beta$ such that

$$Q_{q_{\alpha,\beta,\ell};\mathsf{N}}(\xi) := \widehat{A_{\alpha,\beta,\ell}}(\mathsf{M}^{\mathsf{T}}\xi)\,\mathsf{F}_{r;\mathsf{M}}(\xi)\begin{bmatrix} I_r \\ \mathbf{0}_{(d_{\mathsf{M}}-1)r\times r}\end{bmatrix},$$

$$Q_{q_{\ell;\alpha};\mathsf{N}}(\xi) := \left(pI_r - (-1)^{\ell}q(\widehat{A_{\alpha,\alpha}}(\mathsf{M}^{\mathsf{T}}\xi) - \widehat{B_\alpha}(\mathsf{M}^{\mathsf{T}}\xi))\right)\mathsf{F}_{r;\mathsf{M}}(\xi)\begin{bmatrix} I_r \\ \mathbf{0}_{(d_{\mathsf{M}}-1)r\times r}\end{bmatrix}$$

for $\ell \in \{1,2\}$. We see that

$$\widehat{\mathring{b}_{\alpha,\beta}}(\xi) = Q_{q_{\alpha,\beta,1};\mathsf{N}}(\xi)\widehat{E_{\alpha;\mathsf{N}}}(\xi) + Q_{q_{\alpha,\beta,2};\mathsf{N}}(\xi)\widehat{E_{\beta;\mathsf{N}}}(\xi)$$

for all $\alpha,\beta \in \mathbb{N}_{0;m}^d$ with $\alpha \prec \beta$, and $\widehat{\mathring{b}_{\ell;\alpha}}(\xi) = Q_{q_{\ell;\alpha};\mathsf{N}}(\xi)\widehat{E_{\alpha;\mathsf{N}}}(\xi)$ for all $\alpha \in \mathbb{N}_{0;m}^d$ and $\ell = 1,2$. Hence Lemma 3.4.3 implies that $\mathring{b}$ has order m E_{N}-balanced vanishing moments. Combining this fact with (3.3.14), we conclude that items (1) and (4) of Theorem 3.3.1 follow right away. □

3.5 Summary of the Chapter

In sharp contrast to univariate quasi-tight framelets, multivariate quasi-tight framelets are much harder to study and construct. This is because constructing framelet filter banks is related to the factorization of polynomial matrices, which is challenging when the dimension $d > 1$. Furthermore, it is more difficult to achieve high order of vanishing moments on the framelet generators when $d > 1$, as the technique of separate out the special factors $(1 - e^{-i\xi})^m$ in the case $d = 1$ is no longer valid. Nevertheless, we were able to generalize the theory of quasi-tight multiframelets to arbitrary dimentions. We developed the multivariate version of a matrix-valued filter normal form as a tool to study multiframelets. Consequently, we proved the existence of OEP-based multivariate quasi-tight multiframelets, and performed analysis on their structural properties.

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Chapter 4

Multivariate Dual Multiframelets Derived from Arbitrary Compactly Supported Refinable Vector Functions

In Chapters 2 and 3, we studied how to obtain a quasi-tight multiframelet with several desired properties from any given compactly supported refinable vector functions. In this chapter, we consider the more general question, on how to derive a dual multiframelet with high vanishing moments and high balancing orders, from any given pair of compactly supported refinable vector functions. The work of this chapter is summarized as [57].

Comparing with univariate framelets, the main challenge involved in studying multivariate framelets is that we have to deal with the highly non-trivial problem of factorizing multivariate polynomial matrices (see e.g. [4, 5]). As a consequence, multivariate framelets are much less studied than univariate framelets in the literature. Among those existing works on multivariate framelets, most of them are for the scalar case (see e.g. [16, 20, 23, 31, 36, 39, 55, 63, 69]). We are only aware of the following papers which discuss multivariate dual multiframelets from the theoretical aspect: (1) [35] studies the balancing property and the structure of OEP-based multivariate dual multiframelets; (2) [44]

studies multivariate OEP-based quasi-tight framelets, which we have seen these work in Chapter 3. How one can obtain a desired OEP-based multivariate dual multiframelet from refinable vector functions remains unsolved, and we will answer this question in this chapter. The work of this chapter generalizes the results on univariate dual multiframelets in [33].

4.1 The Main Theorem on OEP-based Dual Multiframelets

In this section, we prove the following theorem on OEP-based balanced dual multiframelets.

Theorem 4.1.1. *Let* M *be a* $d \times d$ *dilation matrix and* $r \geqslant 2$ *be an integer. Let* $\phi, \tilde{\phi} \in (L_2(\mathbb{R}^d))^r$ *be compactly supported* M *refinable vector functions associated with refinement masks* $a, \tilde{a} \in (l_0(\mathbb{Z}^d))^{r\times r}$. *Suppose that* $\operatorname{sr}(a, \mathsf{M}) = \tilde{m}$ *and* $\operatorname{sr}(\tilde{a}, \mathsf{M}) = m$ *with matching filters* $\upsilon, \tilde{\upsilon} \in (l_0(\mathbb{Z}^d))^{1\times r}$ *respectively such that* $\widehat{\upsilon}(0)\widehat{\phi}(0) \neq 0$ *and* $\widehat{\tilde{\upsilon}}(0)\widehat{\tilde{\phi}}(0) \neq 0$. *Let* N *be a* $d \times d$ *integer matrix with* $|\det(\mathsf{N})| = r$. *Then there exist* $\theta, \tilde{\theta} \in (l_0(\mathbb{Z}^d))^{r\times r}$ *and* $b, \tilde{b} \in (l_0(\mathbb{Z}^d))^{s\times r}$ *for some* $s \in \mathbb{N}$ *such that*

(1) θ *and* $\tilde{\theta}$ *are both strongly invertible.*

(2) Define finitely supported filters $\mathring{a}, \mathring{b}, \mathring{\tilde{a}}, \mathring{\tilde{b}}$ *via* (3.1.8) *and* (3.1.9). *Then* $(\{\mathring{a}; \mathring{b}\}, \{\mathring{\tilde{a}}; \mathring{\tilde{b}}\})_{\boldsymbol{\delta} I_r}$ *is an OEP-based dual* M*-framelet filter bank. Moreover,* $\operatorname{bo}(\{\mathring{a}; \mathring{b}\}, \mathsf{M}, \mathsf{N}) = \operatorname{sr}(\mathring{\tilde{a}}, \mathsf{M}) = m$.

(3) $(\{\mathring{\phi}; \psi\}, \{\mathring{\tilde{\phi}}; \tilde{\psi}\})$ *is a compactly supported dual* M*-framelet in* $L_2(\mathbb{R}^d)$ *with* $\operatorname{vm}(\psi) = m$ *and* $\operatorname{vm}(\tilde{\psi}) = \tilde{m}$, *where* $\mathring{\phi}, \psi, \mathring{\tilde{\phi}}, \tilde{\psi}$ *are vector-valued functions defined as in* (1.1.12) *and* (1.1.13).

Proof. Let $\widehat{\Upsilon_{\mathsf{N}}}$ be defined as in (3.1.14), and let $n := m + \tilde{m}$. By Theorem 3.2.5, there exist strongly invertible filters $\theta, \tilde{\theta} \in (l_0(\mathbb{Z}^d))^{r\times r}$ such that

$$\widehat{\mathring{\upsilon}}(\xi) = r^{-1/2}\widehat{\Upsilon_{\mathsf{N}}}(\xi) + \mathscr{O}(\|\xi\|^{\tilde{m}}), \quad \widehat{\mathring{\phi}}(\xi) = r^{-1/2}\overline{\widehat{\Upsilon_{\mathsf{N}}}(\xi)}^{\mathsf{T}} + \mathscr{O}(\|\xi\|^{n}), \tag{4.1.1}$$

$$\widehat{\mathring{\tilde{\upsilon}}}(\xi) = r^{-1/2}\widehat{\Upsilon_{\mathsf{N}}}(\xi) + \mathscr{O}(\|\xi\|^{m}), \quad \widehat{\mathring{\tilde{\phi}}}(\xi) = r^{-1/2}\overline{\widehat{\Upsilon_{\mathsf{N}}}(\xi)}^{\mathsf{T}} + \mathscr{O}(\|\xi\|^{n}), \tag{4.1.2}$$

as $\xi \to 0$, where $\widehat{\mathring{\upsilon}} := \widehat{\upsilon}\widehat{\theta}^{-1}, \widehat{\mathring{\phi}} := \widehat{\theta}\widehat{\phi}, \widehat{\mathring{\tilde{\upsilon}}} := \widehat{\tilde{\upsilon}}\widehat{\tilde{\theta}}^{-1}$ and $\widehat{\mathring{\tilde{\phi}}} := \widehat{\tilde{\theta}}\widehat{\tilde{\phi}}$. In particular, item (1) holds. Define $\mathring{a}, \mathring{\tilde{a}} \in (l_0(\mathbb{Z}^d))^{r\times r}$ as in (3.1.8). It is trivial that $\mathring{a}$ (resp. $\mathring{\tilde{a}}$) has $\tilde{m}$ (resp. m) sum rules with respect to M with a matching filter $\mathring{\upsilon}$ (resp. $\mathring{\tilde{\upsilon}}$), and satisfies the refinement relation $\widehat{\mathring{\phi}}(\mathsf{M}^{\mathsf{T}}\cdot) = \widehat{\mathring{a}}\widehat{\mathring{\phi}}$ (resp. $\widehat{\mathring{\tilde{\phi}}}(\mathsf{M}^{\mathsf{T}}\cdot) = \widehat{\mathring{\tilde{a}}}\widehat{\mathring{\tilde{\phi}}}$).

It remains to show the existence of finitely supported filters $b, \tilde{b} \in (l_0(\mathbb{Z}^d))^{s\times r}$ such that items (2) and (3) hold. By Theorems 3.2.1, there exists a strongly invertible filter $U \in (l_0(\mathbb{Z}^d))^{r\times r}$ such that

$$r^{-1/2}\widehat{\Upsilon_{\mathsf{N}}}(\xi)\widehat{U}(\xi)^{-1} = (1,0,\dots,0)+\mathscr{O}(\|\xi\|^{\max(m,\tilde{m})}), \quad r^{-1/2}\widehat{U}(\xi)\overline{\widehat{\Upsilon_{\mathsf{N}}}(\xi)}^{\mathsf{T}} = (1,0,\dots,0)^{\mathsf{T}}+\mathscr{O}(\|\xi\|^{n}),$$

as $\xi \to 0$. Moreover, $\widehat{U}$ can be chosen such that the following relation holds:

$$\begin{aligned}\widehat{\mathcal{U}}(\xi) := \overline{\widehat{U}(\xi)}^{-\mathsf{T}}\widehat{U}(\xi)^{-1} &= \operatorname{Diag}\left(\|r^{-1/2}\widehat{\Upsilon_{\mathsf{N}}}(\xi)\|^2, \|\widehat{U_2}\|^2, \dots, \|\widehat{U_r}(\xi)\|^2\right) + \mathscr{O}(\|\xi\|^{n}) \\ &= \operatorname{Diag}\left(1, \widehat{\mathcal{U}}(\xi)\right) + \mathscr{O}(\|\xi\|^{n}), \quad \xi \to 0,\end{aligned}$$

where U_j denotes the j-th column of U^{-1} for all $j = 2, \dots, r$ and $\widehat{\mathcal{U}} := \operatorname{Diag}(\|\widehat{U_2}\|^2, \dots, \|\widehat{U_r}\|^2)$.

Define $\breve{a}, \breve{\tilde{a}} \in (l_0(\mathbb{Z}^d))^{r\times r}$ via $\widehat{\breve{a}} := \widehat{U}(\mathsf{M}^{\mathsf{T}}\cdot)\widehat{\mathring{a}}\widehat{U}^{-1}$ and $\widehat{\breve{\tilde{a}}} := \widehat{U}(\mathsf{M}^{\mathsf{T}}\cdot)\widehat{\mathring{\tilde{a}}}\widehat{U}^{-1}$. It is not hard to see that $\breve{a} = \begin{bmatrix} \breve{a}_{1,1} & \breve{a}_{1,2} \\ \breve{a}_{2,1} & \breve{a}_{2,2} \end{bmatrix}$ takes the ideal $(\tilde{m}, n)$-normal form, and $\breve{\tilde{a}} = \begin{bmatrix} \breve{\tilde{a}}_{1,1} & \breve{\tilde{a}}_{1,2} \\ \breve{\tilde{a}}_{2,1} & \breve{\tilde{a}}_{2,2} \end{bmatrix}$ takes the ideal (m, n)-normal form, that is, $\widehat{\breve{a}_{1,1}}, \widehat{\breve{a}_{1,2}}, \widehat{\breve{a}_{2,1}}$ and $\widehat{\breve{a}_{2,2}}$ (resp. $\widehat{\breve{\tilde{a}}_{1,1}}, \widehat{\breve{\tilde{a}}_{1,2}}, \widehat{\breve{\tilde{a}}_{2,1}}$ and $\widehat{\breve{\tilde{a}}_{2,2}}$) are $1\times 1, 1\times(r-1), (r-1)\times 1$ and $(r-1)\times(r-1)$ matrices of $2\pi\mathbb{Z}^d$-periodic trigonometric polynomials such that

- $\widehat{\breve{a}_{1,1}}(\xi) = 1 + \mathscr{O}(\|\xi\|^{n})$ (resp. $\widehat{\breve{\tilde{a}}_{1,1}}(\xi) = 1 + \mathscr{O}(\|\xi\|^{n})$) as $\xi \to 0$, and $\widehat{\breve{a}_{1,1}}(\xi + 2\pi\omega) = \mathscr{O}(\|\xi\|^{\tilde{m}})$ (resp. $\widehat{\breve{\tilde{a}}_{1,1}}(\xi + 2\pi\omega) = \mathscr{O}(\|\xi\|^{m})$) as $\xi \to 0$ for all $\omega \in \Omega_{\mathsf{M}} \setminus \{0\}$.

- $\widehat{\breve{a}_{1,2}}(\xi+2\pi\omega) = \mathscr{O}(\|\xi\|^{\tilde{m}})$ (resp. $\widehat{\breve{\tilde{a}}_{1,2}}(\xi+2\pi\omega) = \mathscr{O}(\|\xi\|^{m})$) as $\xi \to 0$ for all $\omega \in \Omega_{\mathsf{M}}$.

- $\widehat{\breve{a}_{2,1}}(\xi) = \mathscr{O}(\|\xi\|^{n})$ (resp. $\widehat{\breve{\tilde{a}}_{2,1}}(\xi) = \mathscr{O}(\|\xi\|^{n})$) as $\xi \to 0$.

Define $\mathcal{M}_{\breve{a},\breve{\tilde{a}},\tilde{U}}$ as

$$\mathcal{M}_{\breve{a},\breve{\tilde{a}},\tilde{U}}(\xi) := \begin{bmatrix} \widehat{\tilde{U}}(\xi+2\pi\omega_1) & & \\ & \ddots & \\ & & \widehat{\tilde{U}}(\xi+2\pi\omega_{d_{\mathsf{M}}}) \end{bmatrix} - \overline{P_{\breve{a};\mathsf{M}}(\xi)}^{\mathsf{T}}\widehat{\tilde{U}}(\mathsf{M}^{\mathsf{T}}\xi)P_{\breve{\tilde{a}};\mathsf{M}}(\xi), \qquad (4.1.3)$$

where $P_{u;\mathsf{M}}$ is defined as in (1.3.1) for any matrix valued filter u. Note that $\mathcal{M}_{\breve{a},\breve{\tilde{a}},\tilde{U}} = \sum_{j=1}^{d_{\mathsf{M}}} D_{A_j,\omega_j;\mathsf{M}}$ where $A_j \in (l_0(\mathbb{Z}^d))^{r\times r}, j = 1,\dots,d_{\mathsf{M}}$ are defined via

$$\widehat{A_j}(\xi) := \boldsymbol{\delta}(\omega_j)\widehat{\tilde{U}}(\xi) - \overline{\widehat{\breve{a}}(\xi)}^{\mathsf{T}}\widehat{\tilde{U}}(\mathsf{M}^{\mathsf{T}}\xi)\widehat{\breve{\tilde{a}}}(\xi+2\pi\omega_j), \qquad j = 1,\dots,d_{\mathsf{M}}, \qquad (4.1.4)$$

and $D_{A_j,\omega_j;\mathsf{M}}$ is defined as in (3.3.6) with u and ω being replaced by A_j and ω_j respectively. We now perform structural analysis on each $\widehat{A_j}$. First consider $j = 1$, we have

$$\widehat{A_1}(\xi) = \begin{bmatrix} 1 & \\ & \widehat{\breve{U}(\xi)} \end{bmatrix} - \overline{\widehat{\breve{a}}(\xi)}^{\mathsf{T}} \begin{bmatrix} 1 & \\ & \widehat{\breve{U}}(\mathsf{M}^{\mathsf{T}}\xi) \end{bmatrix} \widehat{\breve{\tilde{a}}}(\xi) + \mathscr{O}(\|\xi\|^n) = \begin{bmatrix} \widehat{A_{1;1}} & \widehat{A_{1;2}} \\ \widehat{A_{1;3}} & \widehat{A_{1;4}} \end{bmatrix}, \quad \xi \to 0,$$

where $\widehat{A_{1;1}}, \widehat{A_{1;2}}, \widehat{A_{1;3}}$ and $\widehat{A_{1;4}}$ are $1\times 1, 1\times(r-1), (r-1)\times 1$ and $(r-1)\times(r-1)$ matrices of $2\pi\mathbb{Z}^d$-periodic trigonometric polynomials, satisfying the following moment conditions as $\xi \to 0$:

$$\widehat{A_{1;1}}(\xi) = 1 - \left(\overline{\widehat{\breve{a}_{1,1}}(\xi)}\widehat{\breve{\tilde{a}}_{1,1}}(\xi) + \overline{\widehat{\breve{a}_{2,1}}(\xi)}^{\mathsf{T}}\widehat{\breve{U}}(\mathsf{M}^{\mathsf{T}}\xi)\widehat{\breve{\tilde{a}}_{2,1}}(\xi)\right) + \mathscr{O}(\|\xi\|^n) = \mathscr{O}(\|\xi\|^n), \quad (4.1.5)$$

$$\widehat{A_{1;2}}(\xi) = -\overline{\widehat{\breve{a}_{1,1}}(\xi)}\widehat{\breve{\tilde{a}}_{1,2}}(\xi) - \overline{\widehat{\breve{a}_{2,1}}(\xi)}^{\mathsf{T}}\widehat{\breve{U}}(\mathsf{M}^{\mathsf{T}}\xi)\widehat{\breve{\tilde{a}}_{2,2}}(\xi) + \mathscr{O}(\|\xi\|^n) = \mathscr{O}(\|\xi\|^m), \qquad (4.1.6)$$

$$\widehat{A_{1;3}}(\xi) = -\overline{\widehat{\breve{a}_{1,2}}(\xi)}^{\mathsf{T}}\widehat{\breve{\tilde{a}}_{1,1}}(\xi) - \overline{\widehat{\breve{a}_{2,2}}(\xi)}^{\mathsf{T}}\widehat{\breve{U}}(\mathsf{M}^{\mathsf{T}}\xi)\widehat{\breve{\tilde{a}}_{2,1}}(\xi) + \mathscr{O}(\|\xi\|^n) = \mathscr{O}(\|\xi\|^{\tilde{m}}). \qquad (4.1.7)$$

Recall that $\Delta_\alpha := \operatorname{Diag}(\nabla^\alpha\boldsymbol{\delta}, \boldsymbol{\delta}I_{r-1})$ for all $\alpha \in \mathbb{N}_0^d$ and $\mathbb{N}_{0;m}^d := \{\beta \in \mathbb{N}_0^d : |\beta| = m\}$ for all $m \in \mathbb{N}_0$. It follows from (4.1.5),(4.1.6),(4.1.7) and Lemma 3.3.2 that

$$\widehat{A_1}(\xi) = \sum_{\alpha\in\mathbb{N}_{0;m}^d, \beta\in\mathbb{N}_{0;\tilde{m}}^d} \overline{\widehat{\Delta_\alpha}(\xi)}^{\mathsf{T}} \widehat{A_{1,\alpha,\beta}}(\xi)\widehat{\Delta_\beta}(\xi), \tag{4.1.8}$$

where $A_{1,\alpha,\beta} \in (l_0(\mathbb{Z}^d))^{r\times r}$ for all $\alpha \in \mathbb{N}_{0;m}^d, \beta \in \mathbb{N}_{0;\tilde{m}}^d$.

For $j = 2, \dots, d_{\mathsf{M}}$, we have

$$\widehat{A_j}(\xi) = -\overline{\widehat{\breve{\tilde{a}}}(\xi)}^{\mathsf{T}} \begin{bmatrix} 1 & \\ & \widehat{\breve{U}}(\mathsf{M}^{\mathsf{T}}\xi) \end{bmatrix} \widehat{\breve{\tilde{a}}}(\xi + 2\pi\omega_j) + \mathscr{O}(\|\mathsf{M}^{\mathsf{T}}\xi\|^n) = \begin{bmatrix} \widehat{A_{j;1}} & \widehat{A_{j;2}} \\ \widehat{A_{j;3}} & \widehat{A_{j;4}} \end{bmatrix},$$

as $\xi \to 0$, where $\widehat{A_{j;1}}, \widehat{A_{j;2}}, \widehat{A_{j;3}}$ and $\widehat{A_{j;4}}$ are $1\times1, 1\times(r-1), (r-1)\times1$ and $(r-1)\times(r-1)$ matrices of $2\pi\mathbb{Z}^d$-periodic trigonometric polynomials for each j, satisfying the following moment conditions as $\xi \to 0$:

$$\begin{aligned}
\widehat{A_{j;1}}(\xi) &= -\left(\overline{\widehat{\breve{a}_{1,1}}(\xi)}\widehat{\breve{\tilde{a}}_{1,1}}(\xi + 2\pi\omega_j) + \overline{\widehat{\breve{a}_{2,1}}(\xi)}^{\mathsf{T}}\widehat{\breve{U}}(\mathsf{M}^{\mathsf{T}}\xi)\widehat{\breve{\tilde{a}}_{2,1}}(\xi + 2\pi\omega_j)\right) + \mathscr{O}(\|\mathsf{M}^{\mathsf{T}}\xi\|^n) = \mathscr{O}(\|\xi\|^m),\\
\widehat{A_{j;1}}(\xi - 2\pi\omega_j) &= -\left(\overline{\widehat{\breve{a}_{1,1}}(\xi - 2\pi\omega_j)}\widehat{\breve{\tilde{a}}_{1,1}}(\xi) + \overline{\widehat{\breve{a}_{2,1}}(\xi - 2\pi\omega_j)}^{\mathsf{T}}\widehat{\breve{U}}(\mathsf{M}^{\mathsf{T}}\xi)\widehat{\breve{\tilde{a}}_{2,1}}(\xi)\right) + \mathscr{O}(\|\mathsf{M}^{\mathsf{T}}\xi\|^n)\\
&= \mathscr{O}(\|\xi\|^{\tilde{m}}),\\
\widehat{A_{j;2}}(\xi) &= -\overline{\widehat{\breve{a}_{1,1}}(\xi)}\widehat{\breve{\tilde{a}}_{1,2}}(\xi + 2\pi\omega_j) - \overline{\widehat{\breve{a}_{2,1}}(\xi)}^{\mathsf{T}}\widehat{\breve{U}}(\mathsf{M}^{\mathsf{T}}\xi)\widehat{\breve{\tilde{a}}_{2,2}}(\xi + 2\pi\omega_j) + \mathscr{O}(\|\mathsf{M}^{\mathsf{T}}\xi\|^n) = \mathscr{O}(\|\xi\|^m),\\
\widehat{A_{j;3}}(\xi - 2\pi\omega_j) &= -\overline{\widehat{\breve{a}_{1,2}}(\xi - 2\pi\omega_j)}\widehat{\breve{\tilde{a}}_{1,1}}(\xi) - \overline{\widehat{\breve{a}_{2,2}}(\xi - 2\pi\omega_j)}^{\mathsf{T}}\widehat{\breve{U}}(\mathsf{M}^{\mathsf{T}}\xi)\widehat{\breve{\tilde{a}}_{2,1}}(\xi) + \mathscr{O}(\|\mathsf{M}^{\mathsf{T}}\xi\|^n)\\
&= \mathscr{O}(\|\xi\|^{\tilde{m}}).
\end{aligned}$$

Hence one can conclude that

$$\widehat{A_j}(\xi) = \sum_{\alpha\in\mathbb{N}_{0;m}^d, \beta\in\mathbb{N}_{0;\tilde{m}}^d} \overline{\widehat{\Delta_\alpha}(\xi)}^{\mathsf{T}} \widehat{A_{j,\alpha,\beta}}(\xi)\widehat{\Delta_\beta}(\xi + 2\pi\omega_j), \tag{4.1.9}$$

where $A_{j,\alpha,\beta} \in (l_0(\mathbb{Z}^d))^{r\times r}$ for all $\alpha \in \mathbb{N}_{0;m}^d, \beta \in \mathbb{N}_{0;\tilde{m}}^d$ and all $j = 2, \dots, d_{\mathsf{M}}$. It follows from

(4.1.8) and (4.1.9) that

$$\mathcal{M}_{\breve{a},\breve{\tilde{a}},\tilde{U}}(\xi)=\sum_{j=1}^{d_{\mathsf{M}}} D_{A_j,\omega_j;\mathsf{M}}(\xi)=\sum_{j=1}^{d_{\mathsf{M}}}\sum_{\alpha\in\mathbb{N}_{0;m}^d,\beta\in\mathbb{N}_{0;\tilde{m}}^d}\overline{D_{\Delta_\alpha,0;\mathsf{M}}(\xi)}^{\mathsf{T}} D_{A_{j,\alpha,\beta},\omega_j;\mathsf{M}}(\xi) D_{\Delta_\beta,0;\mathsf{M}}(\xi).$$

Define $\mathcal{N}_{\breve{a},\breve{\tilde{a}},\tilde{U}}$ as in (3.3.11) with $a,\tilde{a}$ and Θ being replaced by $\breve{a},\breve{\tilde{a}}$ and $\tilde{U}$ respectively. Define $E_{u,\omega;\mathsf{M}}$ as in (3.3.7) for any $u\in(l_0(\mathbb{Z}^d))^{r\times r}$ and $\omega\in\Omega_{\mathsf{M}}$. By (3.3.8) and $\mathsf{F}_{r;\mathsf{M}}\overline{\mathsf{F}_{r;\mathsf{M}}}^{\mathsf{T}}=d_{\mathsf{M}}I_{d_{\mathsf{M}}r}$ with $\mathsf{F}_{r;\mathsf{M}}$ being defined as in (3.3.5), we have

$$\begin{aligned}&\mathcal{N}_{\breve{a},\breve{\tilde{a}},\tilde{U}}(\mathsf{M}^{\mathsf{T}}\xi)=d_{\mathsf{M}}^{-2}\,\mathsf{F}_{r;\mathsf{M}}(\xi)\mathcal{M}_{\breve{a},\breve{\tilde{a}},\tilde{U}}(\xi)\overline{\mathsf{F}_{r;\mathsf{M}}(\xi)}^{\mathsf{T}}\\ =&d_{\mathsf{M}}^{-1}\sum_{j=1}^{d_{\mathsf{M}}}\sum_{\alpha\in\mathbb{N}_{0;m}^d,\beta\in\mathbb{N}_{0;\tilde{m}}^d}\overline{E_{\Delta_\alpha,0;\mathsf{M}}(\mathsf{M}^{\mathsf{T}}\xi)}^{\mathsf{T}}E_{A_{j,\alpha,\beta},\omega_j;\mathsf{M}}(\mathsf{M}^{\mathsf{T}}\xi)E_{\Delta_\beta,0;\mathsf{M}}(\mathsf{M}^{\mathsf{T}}\xi).\end{aligned}\tag{4.1.10}$$

By letting

$$E_{\alpha,\beta}(\xi):=d_{\mathsf{M}}^{-1}\sum_{j=1}^{d_{\mathsf{M}}}E_{A_{j,\alpha,\beta},\omega_j;\mathsf{M}}(\xi),\qquad \xi\in\mathbb{R}^d,\quad \alpha\in\mathbb{N}_{0;m}^d,\quad \beta\in\mathbb{N}_{0;\tilde{m}}^d,\tag{4.1.11}$$

we have

$$\mathcal{N}_{\breve{a},\breve{\tilde{a}},\tilde{U}}(\xi)=\sum_{\alpha\in\mathbb{N}_{0;m}^d,\beta\in\mathbb{N}_{0;\tilde{m}}^d}\overline{E_{\Delta_\alpha,0;\mathsf{M}}(\xi)}^{\mathsf{T}}E_{\alpha,\beta}(\xi)E_{\Delta_\beta,0;\mathsf{M}}(\xi).\tag{4.1.12}$$

For every $\alpha\in\mathbb{N}_{0;\tilde{m}}^d$ and $\beta\in\mathbb{N}_{0;m}^d$, choose $E_{\alpha,\beta,1}$ and $E_{\alpha,\beta,1}$ which are $d_{\mathsf{M}}r\times d_{\mathsf{M}}r$ matrices of $2\pi\mathbb{Z}^d$-periodic trigonometric polynomials such that $E_{\alpha,\beta}=\overline{E_{\alpha,\beta,1}}^{\mathsf{T}}E_{\alpha,\beta,2}$. Define $\breve{b}_{\alpha,\beta,k},\breve{\tilde{b}}_{\alpha,\beta,k}\in(l_0(\mathbb{Z}^d))^{1\times r}$ for $k=1,\dots,d_{\mathsf{M}}r$ and all $\alpha\in\mathbb{N}_{0;m}^d,\beta\in\mathbb{N}_{0;\tilde{m}}^d$ via

$$\widehat{\breve{b}_{\alpha,\beta}}(\xi):=\begin{bmatrix}\widehat{\breve{b}_{\alpha,\beta,1}}(\xi)\\ \vdots\\ \widehat{\breve{b}_{\alpha,\beta,d_{\mathsf{M}}r}}(\xi)\end{bmatrix}:=E_{\alpha,\beta,1}(\mathsf{M}^{\mathsf{T}}\xi)\,\mathsf{F}_{r;\mathsf{M}}(\xi)\begin{bmatrix}\widehat{\Delta_\alpha}(\xi)\\ \mathbf{0}_{d_{\mathsf{M}}(r-1)\times r}\end{bmatrix},\tag{4.1.13}$$

$$\widehat{\breve{\tilde{b}}_{\alpha,\beta}}(\xi):=\begin{bmatrix}\widehat{\breve{\tilde{b}}_{\alpha,\beta,1}}(\xi)\\ \vdots\\ \widehat{\breve{\tilde{b}}_{\alpha,\beta,d_{\mathsf{M}}r}}(\xi)\end{bmatrix}:=E_{\alpha,\beta,2}(\mathsf{M}^{\mathsf{T}}\xi)\,\mathsf{F}_{r;\mathsf{M}}(\xi)\begin{bmatrix}\widehat{\Delta_\beta}(\xi)\\ \mathbf{0}_{d_{\mathsf{M}}(r-1)\times r}\end{bmatrix},\tag{4.1.14}$$

where $\mathbf{0}_{t\times q}$ denotes the $t\times q$ zero matrix. Recall that $P_{u;\mathsf{M}}(\xi) = [\widehat{u}(\xi+2\pi\omega_1),\dots,\widehat{u}(\xi+2\pi\omega_{d_\mathsf{M}})]$ as in (1.3.1) for all matrix-valued filter u. It follows from (3.3.8) and $\mathsf{F}_{r;\mathsf{M}}\overline{\mathsf{F}_{r;\mathsf{M}}}^\mathsf{T}$ that

$$P_{\breve{b}_{\alpha,\beta};\mathsf{M}}(\xi) = E_{\alpha,\beta,1}(\mathsf{M}^\mathsf{T}\xi)E_{\Delta_\alpha,0;\mathsf{M}}(\mathsf{M}^\mathsf{T}\xi)\overline{\mathsf{F}_{r;\mathsf{M}}(\xi)}^\mathsf{T}. \tag{4.1.15}$$

Similarly,

$$P_{\breve{\tilde{b}}_{\alpha,\beta};\mathsf{M}}(\xi) = E_{\alpha,\beta,2}(\mathsf{M}^\mathsf{T}\xi)E_{\Delta_\beta,0;\mathsf{M}}(\mathsf{M}^\mathsf{T}\xi)\overline{\mathsf{F}_{r;\mathsf{M}}(\xi)}^\mathsf{T}. \tag{4.1.16}$$

It follows from (4.1.10), (4.1.12), (4.1.15) and (4.1.16) that

$$\mathcal{M}_{\breve{a},\breve{\tilde{a}},\tilde{U}}(\xi) = \overline{\mathsf{F}_{r;\mathsf{M}}(\xi)}^\mathsf{T}\mathcal{N}_{\breve{a},\breve{\tilde{a}},\tilde{U}}(\mathsf{M}^\mathsf{T}\xi)\,\mathsf{F}_{r;\mathsf{M}}(\xi) = \sum_{\alpha\in\mathbb{N}_{0;m}^d,\beta\in\mathbb{N}_{0;\tilde{m}}^d}\overline{P_{\breve{b}_{\alpha,\beta};\mathsf{M}}(\xi)}^\mathsf{T}P_{\breve{\tilde{b}}_{\alpha,\beta};\mathsf{M}}(\xi). \tag{4.1.17}$$

Define

$$\{\breve{b}_\ell : \ell = 1,\dots,s\} := \{\breve{b}_{\alpha,\beta} : \alpha\in\mathbb{N}_{0;m}^d,\quad \beta\in\mathbb{N}_{0;\tilde{m}}^d\}, \tag{4.1.18}$$
$$\{\breve{\tilde{b}}_\ell : \ell = 1,\dots,s\} := \{\breve{\tilde{b}}_{\alpha,\beta} : \alpha\in\mathbb{N}_{0;m}^d,\quad \beta\in\mathbb{N}_{0;\tilde{m}}^d\}, \tag{4.1.19}$$

and let $\breve{b} := [\breve{b}_1^\mathsf{T},\dots,\breve{b}_s^\mathsf{T}]^\mathsf{T}, \breve{\tilde{b}} := [\breve{\tilde{b}}_1^\mathsf{T},\dots,\breve{\tilde{b}}_s^\mathsf{T}]^\mathsf{T}$. We see that (4.1.17) becomes

$$\mathcal{M}_{\breve{a},\breve{\tilde{a}},\tilde{U}}(\xi) = \overline{P_{\breve{b};\mathsf{M}}(\xi)}^\mathsf{T}P_{\breve{\tilde{b}};\mathsf{M}}(\xi), \tag{4.1.20}$$

which is equivalent to say that $(\{\breve{a};\breve{b}\},\{\breve{\tilde{a}};\breve{\tilde{b}}\})_{\tilde{U}}$ is an OEP-based dual M-framelet filter bank satisfying

$$\overline{\widehat{\breve{a}}(\xi)}^\mathsf{T}\widehat{\tilde{U}}(\mathsf{M}^\mathsf{T}\xi)\widehat{\breve{\tilde{a}}}(\xi+2\pi\omega) + \overline{\widehat{\breve{b}}(\xi)}^\mathsf{T}\widehat{\breve{\tilde{b}}}(\xi+2\pi\omega) = \boldsymbol{\delta}(\omega)\widehat{\tilde{U}}(\xi),\qquad \xi\in\mathbb{R}^d,\omega\in\Omega_\mathsf{M}. \tag{4.1.21}$$

Now define $\mathring{b},\mathring{\tilde{b}},b,\tilde{b}\in(l_0(\mathbb{Z}^d))^{s\times r}$ via

$$\widehat{\mathring{b}}(\xi) := \widehat{\breve{b}}(\xi)\widehat{U}(\xi),\quad \widehat{\mathring{\tilde{b}}}(\xi) := \widehat{\breve{\tilde{b}}}(\xi)\widehat{U}(\xi),\quad \widehat{b}(\xi) := \widehat{\mathring{b}}(\xi)\widehat{\theta}(\xi)^{-1},\quad \widehat{\tilde{b}}(\xi) := \widehat{\mathring{\tilde{b}}}(\xi)\widehat{\tilde{\theta}}(\xi)^{-1}.$$

It follows from (4.1.21) that $(\{\mathring{a};\mathring{b}\},\{\mathring{\tilde{a}};\mathring{\tilde{b}}\})_{I_r}$ is an OEP-based dual M-framelet filter bank satisfying

$$\overline{\widehat{\mathring{a}}(\xi)}^{\mathsf{T}}\widehat{\mathring{\tilde{a}}}(\xi+2\pi\omega)+\overline{\widehat{\mathring{b}}(\xi)}^{\mathsf{T}}\widehat{\mathring{\tilde{b}}}(\xi+2\pi\omega)=\boldsymbol{\delta}(\omega)I_r, \qquad \xi\in\mathbb{R}^d,\omega\in\Omega_{\mathsf{M}}, \tag{4.1.22}$$

and $(\{a;b\},\{\tilde{a};\tilde{b}\})_{\Theta}$ (where $\Theta:=\theta^\star*\tilde{\theta}$) is an OEP-based dual M-framelet filter bank satisfying (1.1.14). Note that

$$\widehat{\Upsilon_{\mathsf{N}}}(\xi)\overline{\widehat{\mathring{b}}(\xi)}^{\mathsf{T}}=\widehat{\Upsilon_{\mathsf{N}}}(\xi)\overline{\widehat{U}(\xi)}^{\mathsf{T}}\overline{\widehat{\breve{b}}(\xi)}^{\mathsf{T}}=r^{1/2}(1,0,\ldots,0)\overline{\widehat{\breve{b}}(\xi)}^{\mathsf{T}}+\mathscr{O}(\|\xi\|^n), \quad \xi\to 0. \tag{4.1.23}$$

Moreover, it is trivial that $(1,0,\ldots,0)\left[\overline{\widehat{\Delta_\alpha}(\xi)}^{\mathsf{T}},\mathbf{0}_{r\times d_{\mathsf{M}}(r-1)}\right]=\mathscr{O}(\|\xi\|^m)$ as $\xi\to 0$ for all $\alpha\in\mathbb{N}_{0;m}^d$. Thus by (4.1.23) and the way we define $\breve{b}$, we conclude that

$$\widehat{\Upsilon_{\mathsf{N}}}(\xi)\overline{\widehat{\mathring{b}}(\xi)}^{\mathsf{T}}=\mathscr{O}(\|\xi\|^m), \qquad \xi\to 0. \tag{4.1.24}$$

Similarly, we deduce that

$$\widehat{\Upsilon_{\mathsf{N}}}(\xi)\overline{\widehat{\mathring{\tilde{b}}}(\xi)}^{\mathsf{T}}=\mathscr{O}(\|\xi\|^{\tilde{m}}), \qquad \xi\to 0. \tag{4.1.25}$$

On the other hand, it follows immediately from the definition of $\mathring{\phi}$ in (4.1.1) and the refinement structure $\widehat{\mathring{\phi}}(\mathsf{M}^{\mathsf{T}}\cdot)=\widehat{\mathring{a}}\widehat{\mathring{\phi}}$ that

$$\widehat{\Upsilon_{\mathsf{N}}}(\xi)\overline{\widehat{\mathring{a}}(\xi)}^{\mathsf{T}}=\widehat{\Upsilon_{\mathsf{N}}}(\mathsf{M}^{\mathsf{T}}\xi)+\mathscr{O}(\|\xi\|^m), \qquad \xi\to 0. \tag{4.1.26}$$

Hence by Theorem 3.1.4, we have $\operatorname{bo}(\{\mathring{a};\mathring{b}\},\mathsf{M},\mathsf{N})=m=\operatorname{sr}(\mathring{a};\mathsf{M})$. This proves item (2).

Now define vector functions $\mathring{\phi},\psi,\mathring{\tilde{\phi}},\psi$ as in (1.1.12) and (1.1.13). We have

$$\widehat{\psi}(\mathsf{M}^{\mathsf{T}}\xi)=\widehat{b}(\xi)\widehat{\phi}(\xi)=\widehat{\mathring{b}}(\xi)\widehat{\mathring{\phi}}(\xi)=r^{-1/2}\widehat{\mathring{b}}(\xi)\widehat{\Upsilon_{\mathsf{N}}}(\xi)+\mathscr{O}(\|\xi\|^n)=\mathscr{O}(\|\xi\|^m), \quad \xi\to 0,$$

where the last identity follows from (4.1.24). Thus $\operatorname{vm}(\psi)=m$. Similarly, (4.1.25) yields

$\mathrm{vm}(\tilde{\psi}) = \tilde{m}$. Further note that

$$\overline{\widehat{\phi}(0)}^{\mathsf{T}}\widehat{\Theta}(0)\widehat{\tilde{\phi}}(0) = \overline{\widehat{\mathring{\phi}}(0)}^{\mathsf{T}}\widehat{\mathring{\tilde{\phi}}}(0) = 1.$$

It follows from Theorem 1.1.1 that $(\{\mathring{\phi}; \psi\}, \{\mathring{\tilde{\phi}}; \tilde{\psi}\})$ is a dual M-framelet in $L_2(\mathbb{R}^d)$. This proves item (3). $\square$

Theorem 4.1.1 is valid for the case $r > 1$. For the case $r = 1$, we have to sacrifice the strong invertibility of θ and $\tilde{\theta}$ to improve the orders of vanishing moments of the framelet generators. Neverthe less, the matrix decomposition technique in the proof of Theorem 4.1.1 can be applied to deduce the following result for the case $r = 1$.

Corollary 4.1.2. *Let M be a $d \times d$ dilation matrix and let $\phi, \tilde{\phi} \in L_2(\mathbb{R}^d)$ be compactly supported refinable functions satisfying $\widehat{\phi}(\mathsf{M}^{\mathsf{T}}\xi) = \widehat{a}(\xi)\widehat{\phi}(\xi)$ and $\widehat{\tilde{\phi}}(\mathsf{M}^{\mathsf{T}}\xi) = \widehat{\tilde{a}}(\xi)\widehat{\tilde{\phi}}(\xi)$, where $a, \tilde{a} \in l_0(\mathbb{Z}^d)$ have order $\tilde{m}$ and m sum rules with respect to M with matching filters $\upsilon, \tilde{\upsilon} \in l_0(\mathbb{Z}^d)$, respectively. Suppose that $\widehat{\upsilon}(0)\widehat{\phi}(0) = \widehat{\tilde{\upsilon}}(0)\widehat{\tilde{\phi}}(0) = 1$. Then there exist $b, \tilde{b} \in (l_0(\mathbb{Z}^d))^{s\times 1}$ and $\theta, \tilde{\theta} \in l_0(\mathbb{Z}^d)$ such that*

1. *$(\{a; b\}, \{\tilde{a}; \tilde{b}\})_{\theta^\star * \tilde{\theta}}$ forms an OEP-based dual M-framelet filter bank.*
2. *$(\{\mathring{\phi}; \psi\}, \{\mathring{\tilde{\phi}}; \tilde{\psi}\})$ is a compactly supported dual M-framelet in $L_2(\mathbb{R}^d)$, where $\mathring{\phi}, \psi, \mathring{\tilde{\phi}}$ and $\tilde{\psi}$ are defined as in* (1.1.12) *and* (1.1.13). *Moreover,* $\mathrm{vm}(\psi) = m$ *and* $\mathrm{vm}(\tilde{\psi}) = \tilde{m}$.

4.2 Structural Properties of OEP-Based Balanced Dual Multiframelets

In this section, we investigate properties of the filters $\theta, \tilde{\theta} \in (l_0(\mathbb{Z}^d))^{r\times r}$ which allow us to construct dual multiframelets satisfying all claims of Theorem 4.1.1. The following theorem states the sufficient conditions on θ and $\tilde{\theta}$.

Theorem 4.2.1. *Let N be a $d \times d$ integer matrix with $|\det(\mathsf{N})| = r \geqslant 2$ and define E_{N} and $\widehat{\Upsilon_{\mathsf{N}}}$ in (3.1.11) and (3.1.14), respectively. Let M be a $d \times d$ dilation matrix and $\phi, \tilde{\phi} \in (L_2(\mathbb{R}^d))^r$ be compactly supported M-refinable vector functions satisfying $\widehat{\phi}(\mathsf{M}^{\mathsf{T}}\xi) = \widehat{a}(\xi)\widehat{\phi}(\xi)$ and $\widehat{\tilde{\phi}}(\mathsf{M}^{\mathsf{T}}\xi) = \widehat{\tilde{a}}(\xi)\widehat{\tilde{\phi}}(\xi)$ for some $a, \tilde{a} \in (l_0(\mathbb{Z}^d))^{r\times r}$. Suppose a has order $\tilde{m}$ sum rules with respect to M with a matching filter $\upsilon \in (l_0(\mathbb{Z}^d))^{1\times r}$, and $\tilde{a}$ has order m sum rules with respect to M with a matching filter $\tilde{\upsilon} \in (l_0(\mathbb{Z}^d))^{1\times r}$, such that $\widehat{\upsilon}(0)\widehat{\phi}(0) \neq 0$ and $\widehat{\tilde{\upsilon}}(0)\widehat{\tilde{\phi}}(0) \neq 0$. If $\theta, \tilde{\theta} \in (l_0(\mathbb{Z}^d))^{r\times r}$ are strongly invertible filters such that the following moment conditions hold as $\xi \to 0$:*

$$\widehat{\mathring{\upsilon}}(\xi) = C\overline{\widehat{\mathring{\tilde{\phi}}}(\xi)}^{\mathsf{T}} + \mathscr{O}(\|\xi\|^{\tilde{m}}) = \widehat{c}(\xi)\widehat{\Upsilon_{\mathsf{N}}}(\xi) + \mathscr{O}(\|\xi\|^{\tilde{m}}), \tag{4.2.1}$$

$$\widehat{\mathring{\tilde{\upsilon}}}(\xi) = \tilde{C}\overline{\widehat{\mathring{\phi}}(\xi)}^{\mathsf{T}} + \mathscr{O}(\|\xi\|^{m}) = \widehat{d}(\xi)\widehat{\Upsilon_{\mathsf{N}}}(\xi) + \mathscr{O}(\|\xi\|^{m}), \tag{4.2.2}$$

$$\overline{\widehat{\mathring{\phi}}(\xi)}^{\mathsf{T}}\widehat{\mathring{\tilde{\phi}}}(\xi) = 1 + \mathscr{O}(\|\xi\|^{m+\tilde{m}}), \tag{4.2.3}$$

for some $c, d \in l_0(\mathbb{Z}^d)$ with $\widehat{c}(0) \neq 0$ and $\widehat{d}(0) \neq 0$, and some $C, \tilde{C} \in \mathbb{C} \setminus \{0\}$, where $\widehat{\mathring{\upsilon}} := \widehat{\upsilon}\widehat{\theta}^{-1}, \widehat{\mathring{\phi}} := \widehat{\theta}\widehat{\phi}, \widehat{\mathring{\tilde{\upsilon}}} := \widehat{\tilde{\upsilon}}\widehat{\tilde{\theta}}^{-1}$ and $\widehat{\mathring{\tilde{\phi}}} := \widehat{\tilde{\theta}}\widehat{\tilde{\phi}}$. Then there exist $b, \tilde{b} \in (l_0(\mathbb{Z}^d))^{s\times r}$ for some $s \in \mathbb{N}$ such that all claims of Theorem 4.1.1 hold.

Proof. Define $\mathring{a}, \mathring{\tilde{a}} \in (l_0(\mathbb{Z}^d))^{r\times r}$ as in (3.1.8). We have $\widehat{\mathring{\phi}}(\mathsf{M}^{\mathsf{T}}\cdot) = \widehat{\mathring{a}}\widehat{\mathring{\phi}}$, and $\widehat{\mathring{\tilde{\phi}}}(\mathsf{M}^{\mathsf{T}}\cdot) = \widehat{\mathring{\tilde{a}}}\widehat{\mathring{\tilde{\phi}}}$. Furthermore, $\mathring{a}$ (resp. $\mathring{\tilde{a}}$) has order $\tilde{m}$ (resp. m) sum rules with respect to M with a matching filter $\mathring{\upsilon}$ (resp. $\mathring{\tilde{\upsilon}}$).

By Theorem 3.2.5, there exists a strongly invertible $U \in (l_0(\mathbb{Z}^d))^{r\times r}$ such that

$$\widehat{\breve{\phi}}(\xi) := \widehat{U}(\xi)\widehat{\mathring{\phi}}(\xi) = (1, 0, \ldots, 0)^{\mathsf{T}} + \mathscr{O}(\|\xi\|^{\tilde{m}+m}), \quad \widehat{\breve{\upsilon}}(\xi) := \widehat{\mathring{\upsilon}}(\xi)\widehat{U}(\xi)^{-1} = (1, 0, \ldots, 0) + \mathscr{O}(\|\xi\|^{\tilde{m}}),$$

as $\xi \to 0$. Thus by letting $\widehat{\breve{a}} := \widehat{U}(\mathsf{M}^{\mathsf{T}}\cdot)\widehat{\mathring{a}}\widehat{U}^{-1}$ and $n := \tilde{m} + m$, we see that $\breve{a}$ takes the ideal $(\tilde{m}, n)$-normal form, that is,

$$\widehat{\breve{a}}(\xi) = \begin{bmatrix} \widehat{\breve{a}_{1,1}}(\xi) & \widehat{\breve{a}_{1,2}}(\xi) \\ \widehat{\breve{a}_{2,1}}(\xi) & \widehat{\breve{a}_{2,2}}(\xi) \end{bmatrix}, \tag{4.2.4}$$

where $\widehat{\breve{a}_{1,1}}, \widehat{\breve{a}_{1,2}}, \widehat{\breve{a}_{2,1}}$ and $\widehat{\breve{a}_{2,2}}$ are $1 \times 1, 1 \times (r-1), (r-1) \times 1$ and $(r-1) \times (r-1)$ matrices of $2\pi\mathbb{Z}^d$-periodic trigonometric polynomials such that

$$\widehat{\breve{a}_{1,1}}(\xi) = 1 + \mathscr{O}(\|\xi\|^n), \quad \widehat{\breve{a}_{1,1}}(\xi + 2\pi\omega) = \mathscr{O}(\|\xi\|^{\tilde{m}}), \quad \xi \to 0, \quad \forall \omega \in \Omega_{\mathsf{M}} \setminus \{0\}, \tag{4.2.5}$$

$$\widehat{\breve{a}_{1,2}}(\xi + 2\pi\omega) = \mathscr{O}(\|\xi\|^{\tilde{m}}), \quad \xi \to 0, \quad \forall \omega \in \Omega_{\mathsf{M}}, \tag{4.2.6}$$

$$\widehat{\breve{a}_{2,1}}(\xi) = \mathscr{O}(\|\xi\|^n), \quad \xi \to 0, \tag{4.2.7}$$

as $\xi \to 0$.

On the other hand, we have

$$\widehat{\breve{\tilde{v}}}(\xi) := \widehat{\mathring{\tilde{v}}}(\xi)\overline{\widehat{U}(\xi)}^{\mathsf{T}} = \overline{\widehat{\mathring{\phi}}(\xi)}^{\mathsf{T}}\overline{\widehat{U}(\xi)}^{\mathsf{T}} + \mathscr{O}(\|\xi\|^m) = (1, 0, \dots, 0) + \mathscr{O}(\|\xi\|^m), \tag{4.2.8}$$

$$\widehat{\breve{\tilde{\phi}}}(\xi) := \overline{\widehat{U}(\xi)}^{-\mathsf{T}}\widehat{\mathring{\tilde{\phi}}}(\xi) = \overline{\widehat{U}(\xi)}^{-\mathsf{T}}\overline{\widehat{\mathring{v}}(\xi)}^{\mathsf{T}} + \mathscr{O}(\|\xi\|^{\tilde{m}}) = (1, 0, \dots, 0)^{\mathsf{T}} + \mathscr{O}(\|\xi\|^{\tilde{m}}), \tag{4.2.9}$$

as $\xi \to 0$. Moreover, the condition (4.2.3) implies that

$$\widehat{\breve{\tilde{\phi}}_1}(\xi) = 1 + \mathscr{O}(\|\xi\|^n), \qquad \xi \to 0, \tag{4.2.10}$$

where $\breve{\tilde{\phi}}_1$ is the first coordinate of $\breve{\tilde{\phi}}$. Thus by letting $\widehat{\breve{\tilde{a}}} := \overline{\widehat{U}(\mathsf{M}^{\mathsf{T}}\cdot)}^{-\mathsf{T}}\widehat{\mathring{\tilde{a}}}\overline{\widehat{U}}^{\mathsf{T}}$, we see that $\widehat{\breve{\tilde{\phi}}}(\mathsf{M}^{\mathsf{T}}\cdot) = \widehat{\breve{\tilde{a}}}\widehat{\breve{\tilde{\phi}}}$ and $\breve{\tilde{a}}$ has order m sum rules with respect to M with a matching filter $\breve{\tilde{v}}$. We have

$$\widehat{\breve{\tilde{a}}}(\xi) = \begin{bmatrix} \widehat{\breve{\tilde{a}}_{1,1}}(\xi) & \widehat{\breve{\tilde{a}}_{1,2}}(\xi) \\ \widehat{\breve{\tilde{a}}_{2,1}}(\xi) & \widehat{\breve{\tilde{a}}_{2,2}}(\xi) \end{bmatrix}, \tag{4.2.11}$$

where $\widehat{\breve{\tilde{a}}_{1,1}}, \widehat{\breve{\tilde{a}}_{1,2}}, \widehat{\breve{\tilde{a}}_{2,1}}$ and $\widehat{\breve{\tilde{a}}_{2,2}}$ are $1 \times 1, 1 \times (r-1), (r-1) \times 1$ and $(r-1) \times (r-1)$ matrices of $2\pi\mathbb{Z}^d$-periodic trigonometric polynomials such that

$$\widehat{\breve{\tilde{a}}_{1,1}}(\xi) = 1 + \mathscr{O}(\|\xi\|^n), \quad \widehat{\breve{\tilde{a}}_{1,1}}(\xi + 2\pi\omega) = \mathscr{O}(\|\xi\|^m), \quad \xi \to 0, \quad \forall \omega \in \Omega_{\mathsf{M}} \setminus \{0\}, \tag{4.2.12}$$

$$\widehat{\breve{\tilde{a}}_{1,2}}(\xi + 2\pi\omega) = \mathscr{O}(\|\xi\|^m), \quad \xi \to 0, \quad \forall \omega \in \Omega_{\mathsf{M}}, \tag{4.2.13}$$

$$\widehat{\breve{\tilde{a}}_{2,1}}(\xi) = \mathscr{O}(\|\xi\|^{\tilde{m}}), \quad \xi \to 0, \tag{4.2.14}$$

as $\xi \to 0$.

For $j = 1, \dots, d_{\mathsf{M}}$, define

$$\widehat{\breve{A}_j}(\xi) := \boldsymbol{\delta}(\omega_j) I_r - \overline{\widehat{\breve{a}}(\xi)}^{\mathsf{T}} \widehat{\breve{\tilde{a}}}(\xi + 2\pi\omega_j). \tag{4.2.15}$$

we have

$$\widehat{\breve{A}_1}(\xi) = I_r - \overline{\widehat{\breve{a}}(\xi)}^{\mathsf{T}} \widehat{\breve{\tilde{a}}}(\xi) = \begin{bmatrix} \widehat{\breve{A}_{1;1}}(\xi) & \widehat{\breve{A}_{1;2}}(\xi) \\ \widehat{\breve{A}_{1;3}}(\xi) & \widehat{\breve{A}_{1;4}}(\xi) \end{bmatrix}, \tag{4.2.16}$$

where $\widehat{\breve{A}_{1;1}}, \widehat{\breve{A}_{1;2}}, \widehat{\breve{A}_{1;3}}$ and $\widehat{\breve{A}_{1;4}}$ are $1 \times 1, 1 \times (r-1), (r-1) \times 1$ and $(r-1) \times (r-1)$ matrices of $2\pi\mathbb{Z}^d$-periodic trigonometric polynomials, satisfying the following moment conditions as $\xi \to 0$:

$$\widehat{\breve{A}_{1;1}}(\xi) = 1 - \left(\overline{\widehat{\breve{a}_{1,1}}(\xi)} \widehat{\breve{\tilde{a}}_{1,1}}(\xi) + \overline{\widehat{\breve{a}_{2,1}}(\xi)}^{\mathsf{T}} \widehat{\breve{\tilde{a}}_{2,1}}(\xi) \right) = \mathscr{O}(\|\xi\|^n), \tag{4.2.17}$$

$$\widehat{\breve{A}_{1;2}}(\xi) = -\overline{\widehat{\breve{a}_{1,1}}(\xi)} \widehat{\breve{\tilde{a}}_{1,2}}(\xi) - \overline{\widehat{\breve{a}_{2,1}}(\xi)}^{\mathsf{T}} \widehat{\breve{\tilde{a}}_{2,2}}(\xi) = \mathscr{O}(\|\xi\|^m), \tag{4.2.18}$$

$$\widehat{\breve{A}_{1;3}}(\xi) = -\overline{\widehat{\breve{a}_{1,2}}(\xi)}^{\mathsf{T}} \widehat{\breve{\tilde{a}}_{1,1}}(\xi) - \overline{\widehat{\breve{a}_{2,2}}(\xi)}^{\mathsf{T}} \widehat{\breve{\tilde{a}}_{2,1}}(\xi) = \mathscr{O}(\|\xi\|^{\tilde{m}}). \tag{4.2.19}$$

For $j = 2, \dots, d_{\mathsf{M}}$, we have

$$\widehat{\breve{A}_j}(\xi) = -\overline{\widehat{\breve{a}}(\xi)}^{\mathsf{T}} \widehat{\breve{\tilde{a}}}(\xi + 2\pi\omega_j) = \begin{bmatrix} \widehat{\breve{A}_{j;1}}(\xi) & \widehat{\breve{A}_{j;2}}(\xi) \\ \widehat{\breve{A}_{j;3}}(\xi) & \widehat{\breve{A}_{j;4}}(\xi) \end{bmatrix}, \tag{4.2.20}$$

where $\widehat{\breve{A}_{j;1}}, \widehat{\breve{A}_{j;2}}, \widehat{\breve{A}_{j;3}}$ and $\widehat{\breve{A}_{j;4}}$ are $1 \times 1, 1 \times (r-1), (r-1) \times 1$ and $(r-1) \times (r-1)$ matrices of $2\pi\mathbb{Z}^d$-periodic trigonometric polynomials for each j, satisfying the following moment conditions as $\xi \to 0$:

$$
\begin{aligned}
\widehat{\breve{A}_{j;1}}(\xi) &= -\left(\overline{\widehat{\breve{a}_{1,1}}(\xi)}\widehat{\breve{\tilde{a}}_{1,1}}(\xi+2\pi\omega_j) + \overline{\widehat{\breve{a}_{2,1}}(\xi)}^{\mathsf{T}}\widehat{\breve{\tilde{a}}_{2,1}}(\xi+2\pi\omega_j)\right) = \mathscr{O}(\|\xi\|^m),\\
\widehat{\breve{A}_{j;1}}(\xi-2\pi\omega_j) &= -\left(\overline{\widehat{\breve{a}_{1,1}}(\xi-2\pi\omega_j)}\widehat{\breve{\tilde{a}}_{1,1}}(\xi) + \overline{\widehat{\breve{a}_{2,1}}(\xi-2\pi\omega_j)}^{\mathsf{T}}\widehat{\breve{\tilde{a}}_{2,1}}(\xi)\right) = \mathscr{O}(\|\xi\|^{\tilde{m}}),\\
\widehat{\breve{A}_{j;2}}(\xi) &= -\overline{\widehat{\breve{a}_{1,1}}(\xi)}\widehat{\breve{\tilde{a}}_{1,2}}(\xi+2\pi\omega_j) - \overline{\widehat{\breve{a}_{2,1}}(\xi)}^{\mathsf{T}}\widehat{\breve{\tilde{a}}_{2,2}}(\xi+2\pi\omega_j) = \mathscr{O}(\|\xi\|^m),\\
\widehat{\breve{A}_{j;3}}(\xi-2\pi\omega_j) &= -\overline{\widehat{\breve{a}_{1,2}}(\xi-2\pi\omega_j)}\widehat{\breve{\tilde{a}}_{1,1}}(\xi) - \overline{\widehat{\breve{a}_{2,2}}(\xi-2\pi\omega_j)}^{\mathsf{T}}\widehat{\breve{\tilde{a}}_{2,1}}(\xi) = \mathscr{O}(\|\xi\|^{\tilde{m}}).
\end{aligned}
$$

Hence one can conclude that

$$
\widehat{\breve{A}_j}(\xi) = \sum_{\alpha\in\mathbb{N}_{0;m}^d,\beta\in\mathbb{N}_{0;\tilde{m}}^d} \overline{\widehat{\Delta_\alpha}(\xi)}^{\mathsf{T}}\widehat{\breve{A}_{j,\alpha,\beta}}(\xi)\widehat{\Delta_\beta}(\xi+2\pi\omega_j), \tag{4.2.21}
$$

for some $\breve{A}_{j,\alpha,\beta} \in (l_0(\mathbb{Z}^d))^{r\times r}$ for all $\alpha \in \mathbb{N}_{0;m}^d, \beta \in \mathbb{N}_{0;\tilde{m}}^d$ and all $j = 1,\ldots,d_{\mathsf{M}}$.

Note that the factorization in (4.2.21) takes the same form as the one in (4.1.9). Hence by applying the same argument as in the proof of Theorem 4.1.1, there exist $\breve{b}, \breve{\tilde{b}} \in (l_0(\mathbb{Z}^d))^{s\times r}$ such that $(\{\breve{a};\breve{b}\},\{\breve{\tilde{b}};\breve{\tilde{b}}\})_{\boldsymbol{\delta}I_r}$ is an OEP-based dual M-framelet filter bank, and

$$
\widehat{\psi}(\xi) := \widehat{\breve{b}}(\mathsf{M}^{-\mathsf{T}}\xi)\widehat{\breve{\phi}}(\mathsf{M}^{-\mathsf{T}}\xi) = \mathscr{O}(\|\xi\|^m), \quad \widehat{\tilde{\psi}}(\xi) := \widehat{\breve{\tilde{b}}}(\mathsf{M}^{-\mathsf{T}}\xi)\widehat{\breve{\tilde{\phi}}}(\mathsf{M}^{-\mathsf{T}}\xi) = \mathscr{O}(\|\xi\|^{\tilde{m}}), \quad \xi \to 0.
$$

Finally, define $b, \tilde{b}, \mathring{b}, \mathring{\tilde{b}} \in (l_0(\mathbb{Z}^d))^{r\times r}$ via

$$
\widehat{\mathring{b}} := \widehat{\breve{b}}\widehat{U}^{-1}, \quad \widehat{\mathring{\tilde{b}}} := \widehat{\breve{\tilde{b}}}\overline{\widehat{U}}^{\mathsf{T}}, \quad \widehat{b} := \widehat{\mathring{b}}\widehat{\theta}^{-1}, \quad \widehat{\tilde{b}} := \widehat{\mathring{\tilde{b}}}\widehat{\tilde{\theta}}^{-1}.
$$

It is straight forward to check that all claims of Theorem 4.1.1 hold. This completes the proof. $\square$

By imposing some mild conditions on the filters $a, \tilde{a} \in (l_0(\mathbb{Z}^d))^{r\times r}$, we obtain the following converse result of Theorem 4.2.1.

Theorem 4.2.2. *Let* N *be a* $d \times d$ *integer matrix with* $|\det(\mathsf{N})| = r \geqslant 2$ *and define* E_{N} *and* $\widehat{\Upsilon_{\mathsf{N}}}$ *in* (3.1.11) *and* (3.1.14)*, respectively. Let* M *be a* $d \times d$ *dilation matrix and* $\phi, \tilde{\phi} \in (L_2(\mathbb{R}^d))^r$ *be compactly supported* M*-refinable vector functions satisfying* $\widehat{\phi}(\mathsf{M}^{\mathsf{T}}\xi) = \widehat{a}(\xi)\widehat{\phi}(\xi)$ *and* $\widehat{\tilde{\phi}}(\mathsf{M}^{\mathsf{T}}\xi) = \widehat{\tilde{a}}(\xi)\widehat{\tilde{\phi}}(\xi)$ *for some* $a, \tilde{a} \in (l_0(\mathbb{Z}^d))^{r\times r}$*. Suppose* a *has order* $\tilde{m}$ *sum rules with respect to* M *with a matching filter* $\upsilon \in (l_0(\mathbb{Z}^d))^{1\times r}$*, and* $\tilde{a}$ *has order* m *sum rules with respect to* M *with a matching filter* $\tilde{\upsilon} \in (l_0(\mathbb{Z}^d))^{1\times r}$*, such that* $\widehat{\upsilon}(0)\widehat{\phi}(0) \neq 0$ *and* $\widehat{\tilde{\upsilon}}(0)\widehat{\tilde{\phi}}(0) \neq 0$.

Suppose $\theta, \tilde{\theta} \in (l_0(\mathbb{Z}^d))^{r\times r}$ *and* $b, \tilde{b} \in (l_0(\mathbb{Z}^d))^{s\times r}$ *are finitely supported filters such that all claims of Theorem 4.1.1 hold. Further assume*

(i) 1 *is a simple eigenvalue of* $\widehat{a}(0)$ *and* $\widehat{\mathring{\tilde{a}}}(0)$*. Moreover,*

$$\lambda^{\alpha} I_r - \widehat{a}(0), \quad I_r - \lambda^{\beta}\widehat{a}(0), \quad I_r - \lambda^{\alpha}\widehat{\tilde{a}}(0), \quad \lambda^{\beta} I_r - \widehat{\tilde{a}}(0)$$

are invertible matrices for all $\alpha, \beta \in \mathbb{N}_0^d$ *with* $0 < |\alpha| < \tilde{m}$ *and* $0 < |\beta| < m$*, where* $\lambda := (\lambda_1, \ldots, \lambda_d)$ *is the vector of the eigenvalues of* M.

(ii) $\widehat{p}(\mathsf{M}^{\mathsf{T}}\xi)\widehat{\Upsilon_{\mathsf{N}}}(\mathsf{M}^{\mathsf{T}}\xi)\widehat{\mathring{\tilde{a}}}(\xi) = \widehat{p}(\xi)\widehat{\Upsilon_{\mathsf{N}}}(\xi) + \mathscr{O}(\|\xi\|^m)$ *as* $\xi \to 0$ *for some* $p \in l_0(\mathbb{Z}^d)$ *with* $\widehat{p}(0) \neq 0$*, where* $\widehat{\mathring{\tilde{a}}} := \widehat{\tilde{\theta}}(\mathsf{M}^{\mathsf{T}}\cdot)\widehat{\tilde{a}}\widehat{\tilde{\theta}}^{-1}$.

(iii) $\widehat{q}(\xi)\widehat{\mathring{a}}(\xi)\overline{\widehat{\Upsilon_{\mathsf{N}}}(\xi)}^{\mathsf{T}} = \widehat{q}(\mathsf{M}^{\mathsf{T}}\xi)\overline{\widehat{\Upsilon_{\mathsf{N}}}(\mathsf{M}^{\mathsf{T}}\xi)}^{\mathsf{T}} + \mathscr{O}(\|\xi\|^{\tilde{m}})$ *as* $\xi \to 0$ *for some* $q \in l_0(\mathbb{Z}^d)$ *with* $\widehat{q}(0) \neq 0$.

Then the moment conditions (4.2.1) — (4.2.3) *must hold as* $\xi \to 0$ *for some* $c, d \in l_0(\mathbb{Z}^d)$ *with* $\widehat{c}(0) \neq 0$ *and* $\widehat{d}(0) \neq 0$ *and some* $C, \tilde{C} \in \mathbb{C} \setminus \{0\}$.

Proof. Define $\mathring{a} \in (l_0(\mathbb{Z}^d))^{r\times r}$ as in (3.1.8) and define $\mathring{b}, \mathring{\tilde{b}} \in (l_0(\mathbb{Z}^d))^{s\times r}$ as in (3.1.9). By item (2) of Theorem 4.1.1, we have

$$\overline{\widehat{\mathring{a}}(\xi)}^{\mathsf{T}}\widehat{\mathring{\tilde{a}}}(\xi) + \overline{\widehat{\mathring{b}}(\xi)}^{\mathsf{T}}\widehat{\mathring{\tilde{b}}}(\xi) = I_r, \tag{4.2.22}$$

and $\operatorname{bo}(\{\mathring{a};\mathring{b}\},\mathsf{M},\mathsf{N})=m$. By Theorem 3.1.4, we have

$$\widehat{\Upsilon_{\mathsf{N}}}(\xi)\overline{\widehat{\mathring{b}}(\xi)}^{\mathsf{T}}=\mathscr{O}(\|\xi\|^m),\quad \widehat{\Upsilon_{\mathsf{N}}}(\xi)\overline{\widehat{\mathring{a}}(\xi)}^{\mathsf{T}}=\widehat{\mathring{c}}(\xi)\widehat{\Upsilon_{\mathsf{N}}}(\mathsf{M}^{\mathsf{T}}\xi)+\mathscr{O}(\|\xi\|^m),\quad \xi\to 0,\tag{4.2.23}$$

for some $\mathring{c}\in l_0(\mathbb{Z}^d)$ with $\widehat{\mathring{c}}(0)\neq 0$.

Assume in addition that items (iii) - (v) hold.

By left multiplying $\widehat{\Upsilon_{\mathsf{N}}}$ on both sides of (4.2.22) and using item (iv), we have

$$\widehat{\Upsilon_{\mathsf{N}}}(\xi)=\widehat{\mathring{c}}(\xi)\widehat{\Upsilon_{\mathsf{N}}}(\mathsf{M}^{\mathsf{T}}\xi)\widehat{\mathring{\tilde{a}}}(\xi)+\mathscr{O}(\|\xi\|^m)=\widehat{\mathring{c}}(\xi)\frac{\widehat{p}(\xi)}{\widehat{p}(\mathsf{M}^{\mathsf{T}}\xi)}\widehat{\Upsilon_{\mathsf{N}}}(\xi)+\mathscr{O}(\|\xi\|^m),\quad \xi\to 0.$$

From the above relation we conclude that $\widehat{\mathring{c}}(0)=1$, and thus

$$\widehat{\mathring{d}}(\mathsf{M}^{\mathsf{T}}\xi)\widehat{\Upsilon_{\mathsf{N}}}(\mathsf{M}^{\mathsf{T}}\xi)\widehat{\mathring{\tilde{a}}}(\xi)=\widehat{\mathring{d}}(\xi)\widehat{\Upsilon_{\mathsf{N}}}(\xi)+\mathscr{O}(\|\xi\|^m),\quad \xi\to 0,\tag{4.2.24}$$

where $\mathring{d}\in l_0(\mathbb{Z}^d)$ satisfies

$$\widehat{\mathring{d}}(\xi)=\prod_{j=1}^{\infty}\widehat{\mathring{c}}((\mathsf{M}^{\mathsf{T}})^{-j}\xi)+\mathscr{O}(\|\xi\|^m),\qquad \xi\to 0.\tag{4.2.25}$$

Moreover, it is easy to see from the second relation in (4.2.23) that

$$\overline{\widehat{\mathring{d}}(\mathsf{M}^{\mathsf{T}}\xi)\widehat{\Upsilon_{\mathsf{N}}}(\mathsf{M}^{\mathsf{T}}\xi)}^{\mathsf{T}}=\widehat{\mathring{a}}(\xi)\overline{\widehat{\mathring{d}}(\xi)\widehat{\Upsilon_{\mathsf{N}}}(\xi)}^{\mathsf{T}}+\mathscr{O}(\|\xi\|^m),\quad \xi\to 0.\tag{4.2.26}$$

Since $\mathring{\tilde{a}}$ has m sum rules with a matching filter $\mathring{\tilde{\upsilon}}$ with $\widehat{\mathring{\tilde{\upsilon}}}:=\widehat{\tilde{\upsilon}}\widehat{\tilde{\theta}}^{-1}$, we have $\widehat{\mathring{\tilde{\upsilon}}}(\mathsf{M}^{\mathsf{T}}\xi)\widehat{\mathring{\tilde{a}}}(\xi)=\widehat{\mathring{\tilde{\upsilon}}}(\xi)+\mathscr{O}(\|\xi\|^m)$ as $\xi\to 0$. Furthermore, note that the refinement relation $\widehat{\mathring{\phi}}(\mathsf{M}^{\mathsf{T}}\cdot)=\widehat{\mathring{a}}\widehat{\mathring{\phi}}$ (where $\widehat{\mathring{\phi}}=\widehat{\theta}\widehat{\phi}$) holds. Hence by the conditions in item (i) on $\mathring{a}$ and $\mathring{\tilde{a}}$, we conclude from (4.2.24) and (4.2.26) that (4.2.2) holds for some $\tilde{C}\in\mathbb{C}\setminus\{0\}$, with $d\in l_0(\mathbb{Z}^d)$ being a non-zero scalar multiple of $\mathring{d}$.

On the other hand, the condition on $\mathring{\tilde{a}}$ in item (iii) and item (v) together yield

$$\widehat{\mathring{\tilde{\phi}}}(\xi) = K\widehat{q}(\xi)\overline{\widehat{\Upsilon_{\mathsf{N}}}(\xi)}^{\mathsf{T}} + \mathscr{O}(\|\xi\|^{\tilde{m}}), \qquad \xi \to 0, \tag{4.2.27}$$

for some non-zero constant K. As item (ii) holds, then in particular item (3) of Theorem 4.1.1 holds. Then $\mathrm{vm}(\tilde{\psi}) = \tilde{m}$ and (4.2.27) imply that $\widehat{\mathring{\tilde{b}}}(\xi)\overline{\widehat{\Upsilon_{\mathsf{N}}}(\xi)}^{\mathsf{T}} = \mathscr{O}(\|\xi\|^{\tilde{m}})$ as $\xi \to 0$. Now right multiplying $\overline{\widehat{q}\widehat{\Upsilon_{\mathsf{N}}}}^{\mathsf{T}}$ to both sides of (4.2.22) yields

$$\overline{\widehat{q}(\mathsf{M}^{\mathsf{T}}\xi)}\widehat{\Upsilon_{\mathsf{N}}}(\mathsf{M}^{\mathsf{T}}\xi)\widehat{\mathring{a}}(\xi) = \overline{\widehat{q}(\xi)}\widehat{\Upsilon_{\mathsf{N}}}(\xi) + \mathscr{O}(\|\xi\|^{\tilde{m}}), \quad \xi \to 0. \tag{4.2.28}$$

Since $\mathring{a}$ has $\tilde{m}$ sum rules with a matching filter $\mathring{\upsilon}$ with $\widehat{\mathring{\upsilon}} := \widehat{\upsilon}\widehat{\theta}^{-1}$, we have $\widehat{\mathring{\upsilon}}(\mathsf{M}^{\mathsf{T}}\xi)\widehat{\mathring{a}}(\xi) = \widehat{\mathring{\upsilon}}(\xi) + \mathscr{O}(\|\xi\|^{\tilde{m}})$ as $\xi \to 0$. Moreover, $\mathring{\tilde{a}}$ satisfies the refinement equation $\widehat{\mathring{\tilde{\phi}}}(\mathsf{M}^{\mathsf{T}}\cdot) = \widehat{\mathring{\tilde{a}}}\widehat{\mathring{\tilde{\phi}}}$. By the condition in item (iii) on $\mathring{a}$, we conclude from (4.2.27) and (4.2.28) that (4.2.1) must hold for some $C \in \mathbb{C} \setminus \{0\}$, with $c \in l_0(\mathbb{Z}^d)$ being a non-zero scalar multiple of q.

Finally, by left multiplying $\overline{\widehat{\mathring{\phi}}}^{\mathsf{T}}$ and right multiplying $\widehat{\mathring{\tilde{\phi}}}$ (where $\widehat{\mathring{\tilde{\phi}}} := \widehat{\tilde{\theta}}\widehat{\tilde{\phi}}$) to (4.2.22), we have

$$\overline{\widehat{\mathring{\phi}}(\mathsf{M}^{\mathsf{T}}\xi)}^{\mathsf{T}}\widehat{\mathring{\tilde{\phi}}}(\mathsf{M}^{\mathsf{T}}\xi) = \overline{\widehat{\mathring{\phi}}(\xi)}^{\mathsf{T}}\widehat{\mathring{\tilde{\phi}}}(\xi) + \mathscr{O}(\|\xi\|^{\tilde{m}+m}), \quad \xi \to 0.$$

By applying the same argument as in the proof of Lemma 3.2.4, (4.2.3) follows from the above identity. The proof is now complete. $\square$

4.3 Summary of the Chapter

In this chapter, we answered the question on how to derive a balanced dual multiframelet from a pair of multivariate compactly supported refinable vector functions. We only performed a theoretical approach to investigate this question, and we did not address any construction guidelines. It is of practical interest to develop an effective algorithm for constructing OEP-based multivariate dual multiframelets.

Chapter 5

Framelets with Mixed Dilation Factors

In this chapter, we develop the basic theory of framelets with mixed dilation factors. As discussed in Chapter 1, the main reason for considering framelets with mixed dilation factors is to achieve a relatively low redundancy rate on the framelet system without sacrificing its desired properties such as directionality. To our best knowledge, the first and the only paper which investigates the theory of framelets with mixed dilation factors is [49], which addresses basic concepts and properties of scalar tight framelets with mixed dilation factors. In this chapter, we will systematically investigate framelets with mixed dilation factors, with arbitrary multiplicity in arbitrary dimensions. In Section 5.1, we will first study the discrete framelet transform employing a filter bank with mixed dilation factors, and discuss its various properties. Next, the notion of a discrete affine system in $l_2(\mathbb{Z}^d)$ will be introduced in Section 5.2, which greatly facilitates our study of discrete framelet transforms with mixed dilation factors. Finally, we will discuss framelets and wavelets with mixed dilation factors in Section 5.3.

5.1 Discrete Framelet Transforms with Mixed Dilation Factors

To discuss discrete framelet transforms with mixed dilation factors, we need to recall some preoperties of the subdivision and transition operators.

For $u \in (l_2(\mathbb{Z}^d))^{t\times r}$ and $v \in (l_2(\mathbb{Z}^d))^{s\times r}$, we define:

$$\langle u, v\rangle_{l_2(\mathbb{Z}^d)} := \sum_{k\in\mathbb{Z}^d} u(k)\overline{v(k)}^{\mathsf{T}}. \tag{5.1.1}$$

Note that $\langle u, v\rangle_{l_2(\mathbb{Z}^d)} \in \mathbb{C}^{t\times s}$ is a matrix of complex numbers.

Recall the subdivision operator $\mathcal{S}_{u;\mathsf{M}}$ and the transition operator $\mathcal{T}_{u;\mathsf{M}}$ defined as in (1.2.2) and (1.2.3), respectively. The following two lemmas are the matrix-valued filter versions of [37, Lemma 2.3 and Lemma 4.3], which are important properties on the subdivision and transition operators.

Lemma 5.1.1. *Let $a \in (l_0(\mathbb{Z}^d))^{r\times q}$ be a finitely supported matrix-valued filter and M be a $d\times d$ dilation matrix. Then the following operators are well-defined:*

$$\mathcal{S}_{a,\mathsf{M}} : (l_2(\mathbb{Z}^d))^{s\times r} \to (l_2(\mathbb{Z}^d))^{s\times q}, \quad \mathcal{T}_{a,\mathsf{M}} : (l_2(\mathbb{Z}^d))^{s\times q} \to (l_2(\mathbb{Z}^d))^{s\times r}.$$

Moreover, we have $\mathcal{S}_{a,\mathsf{M}} = \mathcal{T}^{\star}_{a,\mathsf{M}}$. That is

$$\langle \mathcal{S}_{a,\mathsf{M}}v, w\rangle_{l_2(\mathbb{Z}^d)} = \langle v, \mathcal{T}_{a,\mathsf{M}}w\rangle_{l_2(\mathbb{Z}^d)}, \qquad v \in (l_2(\mathbb{Z}^d))^{s\times r}, w \in (l_2(\mathbb{Z}^d))^{s\times q}. \tag{5.1.2}$$

Lemma 5.1.2. *Let $u_1 \in (l_0(\mathbb{Z}^d))^{q\times t}, u_2 \in (l_0(\mathbb{Z}^d))^{r\times q}$ be finitely supported matrix-valued filters and let $\mathsf{M}_1, \mathsf{M}_2$ be $d\times d$ dilation matrices. Then*

$$\mathcal{S}_{u_1,\mathsf{M}_1}\mathcal{S}_{u_2,\mathsf{M}_2}v = \mathcal{S}_{(u_2\uparrow\mathsf{M}_1)*u_1,\mathsf{M}_1\mathsf{M}_2}v, \qquad \mathcal{T}_{u_2,\mathsf{M}_2}\mathcal{T}_{u_1,\mathsf{M}_1}w = \mathcal{T}_{(u_2\uparrow\mathsf{M}_1)*u_1,\mathsf{M}_1\mathsf{M}_2}w, \tag{5.1.3}$$

for all $v \in (l(\mathbb{Z}^d))^{s\times r}$ and $w \in (l(\mathbb{Z}^d))^{s\times t}$.

Suppose we have finitely supported matrix-valued filters $b_0, \tilde{b}_0 \in (l_0(\mathbb{Z}^d))^{r\times r}$ and $b_1, \dots, b_s, \tilde{b}_1, \dots, \tilde{b}_s \in (l_0(\mathbb{Z}^d))^{1\times r}$, and $d \times d$ dilation matrices $\mathsf{M}_0, \mathsf{M}_1, \dots, \mathsf{M}_s$. We now state the discrete framelet transform employing the filter bank $(\{b_l!\mathsf{M}_l\}_{l=0}^s, \{\tilde{b}_l!\mathsf{M}_l\}_{l=0}^s)$ with mixed dilation factors. For $J \in \mathbb{N}$, the J-level discrete framelet transform can be stated as the following:

Step 1. For any input signal $v_{0,0} \in (l(\mathbb{Z}^d))^{1\times r}$, perform the J-level discrete framelet decomposition procedure:

$$v_{l,j} = \mathcal{T}_{b_l,\mathsf{M}_l} v_{0,j-1}, \qquad j = 1, \dots, J, \quad l = 0, \dots, s. \tag{5.1.4}$$

Step 2. Set $\tilde{v}_{l,J} := v_{l,J}$ for all $l = 0, \dots, s$. Recursively compute $\tilde{v}_{j-1}, j = J, \dots, 1$ via

$$\tilde{v}_{j-1} := \sum_{l=0}^{s} \mathcal{S}_{\tilde{b}_l,\mathsf{M}_l} \tilde{v}_{l,j}, \qquad j = J, \dots, 1. \tag{5.1.5}$$

We say that a J-level discrete framelet transform has **the perfect reconstruction (PR) property** if the original input data v_0 is equal to the output data $\tilde{v}_0$. To study the J-level discrete framelet transform employing the filter bank $(\{b_l!\mathsf{M}_l\}_{l=0}^s, \{\tilde{b}_l!\mathsf{M}_l\}_{l=0}^s)$, we need to define the following linear operators:

(1) *The J-level discrete framelet analysis operator* $\mathcal{W}_J : (l(\mathbb{Z}^d))^{1\times r} \to (l(\mathbb{Z}^d))^{1\times(sJ+r)}$ employing the filter bank $\{b_l!\mathsf{M}_l\}_{l=0}^s$:

$$\mathcal{W}_J v := (v_{1,1}, \dots, v_{s,1}, \dots, v_{1,J}, \dots, v_{s,J}, v_{0,J}), \tag{5.1.6}$$

for all $v \in l(\mathbb{Z}^d)$, where $v_{l,j}, l = 0, \dots, s, j = 1, \dots, J$ are defined as (5.1.4). Define $\mathcal{W} := \mathcal{W}_1$.

(2) *The J-level discrete framelet synthesis operator* $\tilde{\mathcal{V}}_J : (l(\mathbb{Z}^d))^{1\times(sJ+r)} \to (l(\mathbb{Z}^d))^{1\times r}$ employing the filter bank $\{\tilde{b}_1!\mathsf{M}_l\}_{l=0}^s$:

$$\tilde{\mathcal{V}}_J(\tilde{v}_{1,1}, \dots, \tilde{v}_{s,1}, \dots, \tilde{v}_{1,J}, \dots, \tilde{v}_{s,J}, \tilde{v}_{0,J}) := \tilde{v}_0, \tag{5.1.7}$$

for all $\tilde{v}_{1,1},\dots,\tilde{v}_{s,1},\dots,\tilde{v}_{1,J},\dots,\tilde{v}_{s,J}\in l(\mathbb{Z}^d),\tilde{v}_{0,J}\in(l(\mathbb{Z}^d))^{1\times r}$, where $\tilde{v}_0$ is calculated via (5.1.5). Define $\tilde{\mathcal{V}}:=\tilde{\mathcal{V}}_1$.

It follows immediately that the J-level discrete framelet transform has the PR property if and only if

$$\tilde{\mathcal{V}}_J\mathcal{W}_J=\mathrm{Id}_{(l(\mathbb{Z}^d))^{1\times r}}. \tag{5.1.8}$$

Moreover, by noting that

$$\mathcal{W}_J=(\mathrm{Id}_{(l(\mathbb{Z}^d))^{1\times[s(J-1)]}}\otimes\mathcal{W})\dots(\mathrm{Id}_{(l(\mathbb{Z}^d))^{1\times s}}\otimes\mathcal{W})\mathcal{W}, \tag{5.1.9}$$

and

$$\tilde{\mathcal{V}}_J=\tilde{\mathcal{V}}(\mathrm{Id}_{(l(\mathbb{Z}^d))^{1\times s}}\otimes\tilde{\mathcal{V}})\dots(\mathrm{Id}_{(l(\mathbb{Z}^d))^{1\times[s(J-1)]}}\otimes\tilde{\mathcal{V}}), \tag{5.1.10}$$

it is obvious that (5.1.8) is equivalent to

$$\tilde{\mathcal{V}}\mathcal{W}=\mathrm{Id}_{(l(\mathbb{Z}^d))^{1\times r}}. \tag{5.1.11}$$

Here we provide a characterization of the PR property of a multi-level discrete framelet transform with mixed dilation factors.

Theorem 5.1.3. *Let $b_0,\tilde{b}_0\in(l_0(\mathbb{Z}^d))^{r\times r}, b_1,\dots,b_s,\tilde{b}_1,\dots,\tilde{b}_s\in(l_0(\mathbb{Z}^d))^{1\times r}$ be finitely supported matrix-valued filters and $\mathsf{M}_0,\dots,\mathsf{M}_s$ be $d\times d$ dilation matrices. The following statements are equivalent:*

(i) For every $J\in\mathbb{N}$, the J-level discrete framelet transform employing the filter bank $(\{b_l!\mathsf{M}_l\}_{l=0}^s,\{\tilde{b}_l!\mathsf{M}_l\}_{l=0}^s)$ has the PR property.

(ii) $\tilde{\mathcal{V}}\mathcal{W}v=v$ for all $v\in(l_0(\mathbb{Z}^d))^{1\times r}$, where $\mathcal{W}$ and $\tilde{\mathcal{V}}$ are the 1-level discrete analysis and synthesis operators respectively.

(iii) $(\{b_l!\mathsf{M}_l\}_{l=0}^s,\{\tilde{b}_l!\mathsf{M}_l\}_{l=0}^s)$ is a dual framelet filter bank with mixed dilation factors, i.e.,

$$\sum_{l=0}^{s}\chi_{\Omega_l}(\omega)\overline{\widehat{b_l}(\xi)}^{\mathsf{T}}\widehat{\tilde{b}_l}(\xi+2\pi\omega)=\boldsymbol{\delta}(\omega)I_r, \tag{5.1.12}$$

for almost every $\xi \in \mathbb{R}^d$ *and every* $\omega \in \bigcup_{l=0}^{s} \Omega_l$*, where*

$$\Omega_l := [\mathsf{M}_l^{-\mathsf{T}}\mathbb{Z}^d] \cap [0,1)^d, \qquad l = 0, \dots, s, \tag{5.1.13}$$

and χ_E *is the indicator function of the subset* $E \subseteq \mathbb{R}^d$*.*

Proof. (i) ⇔ (ii): (i) ⇒ (ii) is trivial. To prove (ii) ⇒ (i), one uses the locality of the subdivision and transition operators (see the proof of Theorem 2.1.1, or [37, Theorem 2.1] and [41, Theorem 1.1.1]).

(i) ⇒ (iii): By definition we have

$$\tilde{\mathcal{V}}\mathcal{W}v = \sum_{l=0}^{s} \mathcal{S}_{\tilde{b}_l,\mathsf{M}_l}\mathcal{T}_{b_l,\mathsf{M}_l}v, \qquad v \in (l(\mathbb{Z}^d))^{1\times r}. \tag{5.1.14}$$

By item (i) and taking Fourier series on both sides of (5.1.14), we have

$$\widehat{v}(\xi) = \sum_{l=0}^{s} \sum_{\omega_l \in \Omega_l} \widehat{v}(\xi + 2\pi\omega_l)\overline{\widehat{b_l}(\xi + 2\pi\omega_l)}^{\mathsf{T}}\widehat{\tilde{b}_l}(\xi), \qquad \xi \in \mathbb{R}^d. \tag{5.1.15}$$

For simplicity of presentation, define $\Omega := \bigcup_{l=0}^{s} \Omega_l$. For every $\omega \in \Omega$, define

$$\widehat{u_\omega}(\xi) := \left(\sum_{l=0}^{s} \chi_{\Omega_l}(\omega)\overline{\widehat{b_l}(\xi + 2\pi\omega)}^{\mathsf{T}}\widehat{\tilde{b}_l}(\xi)\right) - \boldsymbol{\delta}(\omega)I_r, \qquad \xi \in \mathbb{R}^d. \tag{5.1.16}$$

It follows that (5.1.15) is equivalent to

$$\widehat{v}(\xi) = \sum_{\omega \in \Omega} \widehat{v}(\xi + 2\pi\omega)\widehat{u_\omega}(\xi), \qquad \xi \in \mathbb{R}^d. \tag{5.1.17}$$

Define

$$\epsilon_1 := \inf_{x \in \Omega\setminus\{0\}} \|x\|_\infty, \quad \epsilon_2 := \inf_{x \in \Omega\setminus\{0\}, y \in \{0,1\}^d} \|x - y\|_\infty. \tag{5.1.18}$$

Let $x_0 \in (-\pi,\pi)^d$ be fixed. For every $\epsilon \in (0, \min(\epsilon_1, \epsilon_2))$, define $E_{x_0,\epsilon} := x_0 + [-\frac{\epsilon}{\pi}, \frac{\epsilon}{\pi}]^d$.

Let $v \in (l(\mathbb{Z}^d))^{1\times r}$ be such that

$$(\operatorname{supp}(\widehat{v}) \cap [-\pi,\pi)^d) \subseteq E_{x_0,\epsilon} \subseteq (-\pi,\pi)^d. \tag{5.1.19}$$

Note that for any $\xi \in E_{x_0,\epsilon}$ and $\omega \in \Omega \setminus \{0\} \subseteq [0,1)^d$, the point $\xi + 2\pi\omega$ lies in $[-\pi,\pi)^d + 2\pi y$ for some (unique) $y \in \{0,1\}^d$. Moreover, for any $\xi \in E_{x_0,\epsilon}$, $\omega \in \Omega \setminus \{0\}$ and $y \in \{0,1\}^d$, we have

$$\|\xi + 2\pi\omega - (x_0 + 2\pi y)\|_\infty \geqslant 2\pi\|\omega - y\|_\infty - \|\xi - x_0\|_\infty > 2\pi\epsilon - \frac{\epsilon}{\pi} > \frac{\epsilon}{\pi}.$$

It follows that $\widehat{v}(\xi+2\pi\omega) = \mathbf{0}_{1\times r}$ for all $\xi \in E_{x_0,\epsilon}$ and $\omega \in \Omega\setminus\{0\}$. By (5.1.17), $\widehat{v}(\xi)\widehat{u_0}(\xi) = \mathbf{0}_{1\times r}$ for all $\xi \in E_{x_0,\epsilon}$, and thus $\widehat{u_0}(\xi) = \mathbf{0}_{1\times r}$ for all $\xi \in E_{x_0,\epsilon}$ as $v \in (l(\mathbb{Z}^d))^{1\times r}$ satisfying (5.1.19) is arbitrary. Since $x_0 \in (-\pi,\pi)^d$ is arbitrary and $E_{x_0,\epsilon}$ is a neighbourhood of x_0, we conclude that $\widehat{u_0}(\xi) = \mathbf{0}_{r\times r}$ for all $\xi \in (-\pi,\pi)^d$. Since $\widehat{u_0}$ is $2\pi\mathbb{Z}^d$-periodic, it follows that $\widehat{u_0} = \mathbf{0}_{r\times r}$ for a.e. $\xi \in \mathbb{R}^d$. For $\mathring{\omega} \in \Omega \setminus \{0\}$, note that (5.1.17) is equivalent to

$$\sum_{\omega\in\Omega} \widehat{v}(\xi + 2\pi(\omega - \mathring{\omega}))\widehat{u_\omega}(\xi - 2\pi\mathring{\omega}) = \mathbf{0}_{1\times r}, \quad \forall \xi \in \mathbb{R}^d. \tag{5.1.20}$$

By applying the same argument as above, we have $\widehat{u_{\mathring{\omega}}} = \mathbf{0}_{r\times r}$ for $\xi \in \mathbb{R}^d$. This proves item (iii).

(iii) $\Rightarrow$ (i): This follows immediately from item (iii) and the fact that (5.1.17) holds for all $v \in (l_0(\mathbb{Z}^d))^{1\times r}$. $\square$

Remark 5.1.4. *It is not hard to observe that* $(\{b_l!\mathsf{M}_l\}_{l=0}^s, \{\tilde{b}_l!\mathsf{M}_l\}_{l=0}^s)$ *is a dual framelet filter bank if and only if* $(\{\tilde{b}_l!\mathsf{M}_l\}_{l=0}^s, \{b_l!\mathsf{M}_l\}_{l=0}^s)$ *is a dual framelet filter bank.*

Next, we discuss the stability of a discrete framelet transform with mixed dilation factors.

Definition 5.1.5. *A filter bank* $\{b_l!\mathsf{M}_l\}_{l=0}^s$ ***has stability*** *in* $l_2(\mathbb{Z}^d)$ *if there exist* $C_1, C_2 > 0$

such that

$$C_1\|v\|^2_{(l_2(\mathbb{Z}^d))^{1\times r}} \leqslant \|\mathcal{W}_J v\|^2_{(l_2(\mathbb{Z}^d))^{1\times[(sJ+r)]}} \leqslant C_2\|v\|^2_{(l_2(\mathbb{Z}^d))^{1\times r}} \tag{5.1.21}$$

for all $J \in \mathbb{N}$ *and* $v \in (l_2(\mathbb{Z}^d))^{1\times r}$, *where* $\mathcal{W}_J$ *is the* J*-level discrete analysis operator employing the filter bank* $\{b_l!\mathsf{M}_l\}_{l=0}^s$. *In this case,* $\{b_l!\mathsf{M}_l\}_{l=0}^s$ *is called* ***a framelet filter bank*** *with mixed dilation factors.*

The following result provides several equivalent ways to interpret the stability of a discrete filter bank.

Theorem 5.1.6. *Let* $(\{b_l!\mathsf{M}_l\}_{l=0}^s, \{\tilde{b}_l!\mathsf{M}_l\}_{l=0}^s)$ *be a dual framelet filter bank with mixed dilation factors, and let* $0 < C_1 \le C_2 < \infty$. *For each* $J \in \mathbb{N}$, *let* $\mathcal{W}_J$ *and* $\mathcal{V}_J$ *(resp.* $\tilde{\mathcal{W}}_J$ *and* $\tilde{\mathcal{V}}_J$*) be the* J*-level discrete framelet analysis and synthesis operators employing the filter bank* $\{b_l!\mathsf{M}_l\}_{l=0}^s$ *(resp.* $\{\tilde{b}_l!\mathsf{M}_l\}_{l=0}^s$*). The following statements are equivalent.*

(1) $\{b_l!\mathsf{M}_l\}_{l=0}^s$ *and* $\{\tilde{b}_l!\mathsf{M}_l\}_{l=0}^s$ *have stability in* $l_2(\mathbb{Z}^d)$ *with* (5.1.21) *and*

$$C_2^{-1}\|v\|^2_{(l_2(\mathbb{Z}^d))^{1\times r}} \leqslant \|\tilde{W}_J v\|^2_{(l_2(\mathbb{Z}^d))^{1\times[(sJ+r)]}} \leqslant C_1^{-1}\|v\|^2_{(l_2(\mathbb{Z}^d))^{1\times r}} \tag{5.1.22}$$

hold for all $J \in \mathbb{N}$ *and* $v \in (l_2(\mathbb{Z}^d))^{1\times r}$.

(2) $\|\mathcal{W}_J\|^2 \leqslant C_2$ *and* $\|\tilde{\mathcal{W}}_J\|^2 \leqslant C_1^{-1}$ *for all* $J \in \mathbb{N}$.

(3) $\|\mathcal{V}_J\|^2 \leqslant C_2$ *and* $\|\tilde{\mathcal{V}}\|^2 \leqslant C_1^{-1}$ *for all* $J \in \mathbb{N}$.

(4) $\|\mathcal{V}_J\|^2 \leqslant C_2$ *and* $\|\tilde{\mathcal{W}}_J\|^2 \leqslant C_1^{-1}$ *for all* $J \in \mathbb{N}$.

(5) $\|\mathcal{W}_J\|^2 \leqslant C_2$ *and* $\|\tilde{\mathcal{V}}_J\|^2 \leqslant C_1^{-1}$ *for all* $J \in \mathbb{N}$.

Proof. By Lemma 5.1.1, we have

$$\langle \mathcal{V}_J w, v\rangle_{l_2(\mathbb{Z}^d)} = \langle w, \mathcal{W}_J v\rangle_{l_2(\mathbb{Z}^d)}, \quad \langle \tilde{\mathcal{V}}_J w, v\rangle_{l_2(\mathbb{Z}^d)} = \langle w, \tilde{\mathcal{W}}_J v\rangle_{l_2(\mathbb{Z}^d)} \tag{5.1.23}$$

for all $J \in \mathbb{N}$, $w \in (l_2(\mathbb{Z}^d))^{1\times[(sJ+r)]}$ and $v \in (l_2(\mathbb{Z}^d))^{1\times r}$. Thus we have (2)$\Leftrightarrow$(3)$\Leftrightarrow$(4)$\Leftrightarrow$(5). We finish the proof by proving (1)$\Leftrightarrow$(2). It's trivial to see that (1)$\Rightarrow$(2). To prove the converse, note that by Remark 5.1.4, we have $\mathcal{V}_J\tilde{\mathcal{W}}_Jv = \tilde{\mathcal{V}}_J\mathcal{W}_Jv = v$ for all $v \in l_2(\mathbb{Z}^d)$ and $J \in \mathbb{N}$. Moreover, by Lemma 5.1.1, we have $\|\mathcal{W}_J\| = \|\mathcal{V}_J\|$ and $\|\tilde{\mathcal{W}}_J\| = \|\tilde{\mathcal{V}}_J\|$ for all $J \in \mathbb{N}$. Thus item (2) yields

$$\|\mathcal{V}_J\tilde{\mathcal{W}}_Jv\|^2_{(l_2(\mathbb{Z}^d))^{1\times r}} = \|v\|^2_{l_2(\mathbb{Z}^d)} \leqslant C_2\|\tilde{\mathcal{W}}_Jv\|^2_{(l_2(\mathbb{Z}^d))^{1\times[(sJ+r)]}}, \tag{5.1.24}$$

$$\|\tilde{\mathcal{V}}_J\mathcal{W}_Jv\|^2_{(l_2(\mathbb{Z}^d))^{1\times r}} = \|v\|^2_{(l_2(\mathbb{Z}^d))^{1\times r}} \leqslant C_1^{-1}\|\mathcal{W}_Jv\|^2_{(l_2(\mathbb{Z}^d))^{1\times[(sJ+r)]}}, \tag{5.1.25}$$

for all $J \in \mathbb{N}$ and $v \in (l_2(\mathbb{Z}^d))^{1\times r}$. Thus

$$\|\mathcal{W}_J\|^2 = \|\mathcal{V}_J\|^2 \leqslant C_2, \quad \|\tilde{\mathcal{W}}_J\|^2 = \|\tilde{\mathcal{V}}_J\|^2 \leqslant C_1^{-1}. \tag{5.1.26}$$

This proves (2)⇒(1), and the proof is now complete. □

We now introduce the notion of a wavelet filter bank with mixed dilation factors. Briefly speaking, wavelet filter banks are special framelet filter banks which are non-redundant (or critically sampled). To do this, we need the following proposition on the analysis and synthesis operators.

Proposition 5.1.7. *Let $b_0, \tilde{b}_0 \in (l_0(\mathbb{Z}^d))^{r\times r}, b_1, \dots, b_s, \tilde{b}_1, \dots, \tilde{b}_s \in (l_0(\mathbb{Z}^d))^{1\times r}$ be finitely supported filters and $\mathsf{M}_0, \dots, \mathsf{M}_s$ be $d\times d$ dilation matrices such that $(\{b_l!\mathsf{M}_l\}_{l=0}^s, \{\tilde{b}_l!\mathsf{M}_l\}_{l=0}^s)$ is a dual framelet filter bank with mixed dilation factors. Let $\mathcal{W}$ and $\tilde{\mathcal{V}}$ be the discrete framelet analysis and synthesis operators employing the filter banks $\{b_l!\mathsf{M}_l\}_{l=0}^s$ and $\{\tilde{b}_1!\mathsf{M}_l\}_{l=0}^s$. The following statements are equivalent.*

(i) $\mathcal{W}$ is surjective.

(ii) $\tilde{\mathcal{V}}$ is injective.

(iii) $\tilde{\mathcal{V}}\mathcal{W} = Id_{(l(\mathbb{Z}^d))^{1\times r}}$ and $\mathcal{W}\tilde{\mathcal{V}} = Id_{(l(\mathbb{Z}^d))^{1\times(s+r)}}$.

Proof. (iii) ⇒ (i) and (iii) ⇒ (ii) are trivial.

We now prove that either (i) or (ii) implies (iii). The condition $\tilde{\mathcal{V}}\mathcal{W} = \mathrm{Id}_{(l(\mathbb{Z}^d))^{1\times r}}$ follows immediately from Theorem 5.1.3. It remains to show that either (i) or (ii) implies that $\mathcal{W}\tilde{\mathcal{V}} = \mathrm{Id}_{(l(\mathbb{Z}^d))^{1\times(s+r)}}$. First assume item (i) holds. For every $v \in (l(\mathbb{Z}^d))^{1\times r}$ we have

$$\mathcal{W}v = \mathcal{W}(\tilde{\mathcal{V}}\mathcal{W})v = \mathcal{W}\tilde{\mathcal{V}}(\mathcal{W}v). \tag{5.1.27}$$

By the surjectivity of $\mathcal{W}$, we conclude that $\mathcal{W}\tilde{\mathcal{V}} = \mathrm{Id}_{(l(\mathbb{Z}^d))^{1\times(s+r)}}$. Now assume item (ii) holds. For every $w \in (l(\mathbb{Z}^d))^{1\times(s+r)}$, we have

$$\tilde{\mathcal{V}}(\mathcal{W}\tilde{\mathcal{V}})w = (\tilde{\mathcal{V}}\mathcal{W})\tilde{\mathcal{V}}w = \tilde{\mathcal{V}}w. \tag{5.1.28}$$

Since $\tilde{\mathcal{V}}$ is injective, we see that $(\mathcal{W}\tilde{\mathcal{V}})w = w$ for all $w \in (l(\mathbb{Z}^d))^{1\times(s+r)}$. $\square$

Definition 5.1.8. *A dual framelet filter bank $(\{b_l!\mathsf{M}_l\}_{l=0}^s, \{\tilde{b}_l!\mathsf{M}_l\}_{l=0}^s)$ is called a biorthogonal wavelet filter bank with mixed dilation factors if any one of the equivalent conditions in Proposition 5.1.7 holds.*

The following theoreom characterizes the biorthogonal wavelet filter banks with mixed dilation factors.

Theorem 5.1.9. *Let $b_0, \tilde{b}_0 \in (l_0(\mathbb{Z}^d))^{r\times r}, b_1, \dots, b_s, \tilde{b}_1, \dots, \tilde{b}_s \in (l_0(\mathbb{Z}^d))^{1\times r}$ be finitely supported filters and let $\mathsf{M}_0, \dots, \mathsf{M}_s$ be $d \times d$ dilation matrices. The following are equivalent:*

(i) $(\{b_l!\mathsf{M}_l\}_{l=0}^s, \{\tilde{b}_l!\mathsf{M}_l\}_{l=0}^s)$ is a biothogonal wavelet filter bank with mixed dilation factors.

(ii) $(\{b_l!\mathsf{M}_l\}_{l=0}^s, \{\tilde{b}_l!\mathsf{M}_l\}_{l=0}^s)$ is a dual framelet filter bank with mixed dilation factors, and satisfies

$$\sum_{\omega\in\Omega_k\cap\Omega_l} \widehat{b_k}(\xi+2\pi\omega)\overline{\widehat{\tilde{b}_l}(\xi+2\pi\omega)}^{\mathsf{T}} = \begin{cases} I_r, & k=l=0 \\ \boldsymbol{\delta}(k-l), & k,l\neq 0 \\ \mathbf{0}_{1\times r}, & k\neq 0, l=0 \\ \mathbf{0}_{r\times 1}, & k=0, l\neq 0, \end{cases} \qquad \xi\in\mathbb{R}^d,$$

for all $k, l = 0, \dots, s$, where Ω_l is defined via (5.1.13).

Proof. Let $\mathcal{W}$ and $\tilde{\mathcal{V}}$ be the discrete framelet analysis and synthesis operators employing the filter bank $(\{b_l!\mathsf{M}_l\}_{l=0}^s, \{\tilde{b}_l!\mathsf{M}_l\}_{l=0}^s)$.

(i) $\Rightarrow$ (ii): If item (i) holds, then by item (iii) of Proposition 5.1.7, we have $\tilde{\mathcal{V}}\mathcal{W} = \mathrm{Id}_{l(\mathbb{Z}^d)}$. This means that $(\{b_l!\mathsf{M}_l\}_{l=0}^s, \{\tilde{b}_l!\mathsf{M}_l\}_{l=0}^s)$ must be a dual framelet filter bank with mixed dilation factors.

On the other hand, let $w := (w_1, \dots, w_s, w_0)$ with $w_1, \dots, w_s \in l_1(\mathbb{Z}^d)$ and $w_0 \in (l_1(\mathbb{Z}^d))^{1\times r}$. By $\mathcal{W}\tilde{\mathcal{V}} = \mathrm{Id}_{(l(\mathbb{Z}^d))^{1\times(s+r)}}$, we have

$$w_k = (\mathcal{W}\tilde{\mathcal{V}}w)_k = \sum_{l=0}^{s} \mathcal{T}_{b_k,\mathsf{M}_k}\mathcal{S}_{\tilde{b}_l,\mathsf{M}_l} w_l, \qquad k = 0, \dots, s. \tag{5.1.29}$$

By taking Fourier series of both sides of (5.1.29), we have

$$\begin{aligned}
\widehat{w_k}(\xi) &= \sum_{l=0}^{s} |\det(\mathsf{M}_k)|^{-\frac{1}{2}} \sum_{\omega\in\Omega_k} \widehat{\mathcal{S}_{\tilde{b}_l,\mathsf{M}_l} w_l}(\mathsf{M}_k^{-\mathsf{T}}\xi + 2\pi\omega)\overline{\widehat{b_k}(\mathsf{M}_k^{-\mathsf{T}} + 2\pi\omega)}^{\mathsf{T}} \\
&= \sum_{l=0}^{s} |\det(\mathsf{M}_k)|^{-\frac{1}{2}} |\det(\mathsf{M}_l)|^{\frac{1}{2}} \sum_{\omega\in\Omega_k} \widehat{w_l}(\mathsf{M}_l^{\mathsf{T}}(\mathsf{M}_k^{-\mathsf{T}}\xi + 2\pi\omega))\widehat{\tilde{b}_l}(\mathsf{M}_k^{-\mathsf{T}}\xi + 2\pi\omega)\overline{\widehat{b_k}(\mathsf{M}_k^{-\mathsf{T}} + 2\pi\omega)}^{\mathsf{T}} \\
&= \sum_{l=0}^{s} \sum_{\omega\in\Omega_k} \widehat{w_l}(\mathsf{M}_l^{\mathsf{T}}(\mathsf{M}_k^{-\mathsf{T}}\xi + 2\pi\omega))\widehat{u_{l,k}}(\mathsf{M}_k^{-\mathsf{T}}\xi + 2\pi\omega)
\end{aligned}$$

where

$$\widehat{u_{l,k}}(\xi) := |\det(\mathsf{M}_k)|^{-\frac{1}{2}} |\det(\mathsf{M}_l)|^{\frac{1}{2}} \widehat{\tilde{b}_l}(\xi)\overline{\widehat{b_k}(\xi)}^{\mathsf{T}}, \qquad \xi \in \mathbb{R}^d. \tag{5.1.30}$$

As $(\mathsf{M}_l^{-\mathsf{T}}\mathbb{Z}^d)\cap(\mathsf{M}_k^{-\mathsf{T}}\mathbb{Z}^d)$ is a sublattice of $\mathsf{M}_k^{-\mathsf{T}}\mathbb{Z}^d$ which contians $\mathbb{Z}^d$, there exist $\omega_{k,1}, \dots, \omega_{k,L_{k,l}} \in \Omega_k$ with $\omega_{k,1} = 0$ and $L_{k,l} := \frac{\#\Omega_k}{\#(\Omega_k\cap\Omega_l)}$ such that

$$\mathsf{M}_k^{-\mathsf{T}}\mathbb{Z}^d = \bigsqcup_{j_{k,l}=1}^{L_{k,l}} \left(\omega_{k,j_{k,l}} + [(\mathsf{M}_l^{-\mathsf{T}}\mathbb{Z}^d) \cap (\mathsf{M}_k^{-\mathsf{T}}\mathbb{Z}^d)]\right). \tag{5.1.31}$$

It follows that

$$\begin{aligned}&\sum_{\omega\in\Omega_k}\widehat{w_l}(\mathsf{M}_l^{\mathsf{T}}(\mathsf{M}_k^{-\mathsf{T}}\xi+2\pi\omega))\widehat{u_{l,k}}(\mathsf{M}_k^{-\mathsf{T}}\xi+2\pi\omega)\\ =&\sum_{j_{k,l}=1}^{L_{k,l}}\widehat{w_l}(\mathsf{M}_l^{\mathsf{T}}\mathsf{M}_k^{-\mathsf{T}}\xi+2\pi\mathsf{M}_l^{\mathsf{T}}\omega_{k,j_{k,l}}))\sum_{\omega\in\Omega_k\cap\Omega_l}\widehat{u_{l,k}}(\mathsf{M}_k^{-\mathsf{T}}\xi+2\pi\mathsf{M}_l^{\mathsf{T}}\omega_{k,j_{k,l}}+2\pi\omega).\end{aligned}$$

Consequently, we have

$$\widehat{w_k}(\xi)=\sum_{l=0}^{s}\sum_{j_{k,l}=1}^{L_{k,l}}\widehat{w_l}(\mathsf{M}_l^{\mathsf{T}}\mathsf{M}_k^{-\mathsf{T}}\xi+2\pi\mathsf{M}_l^{\mathsf{T}}\omega_{k,j_{k,l}}))\sum_{\omega\in\Omega_k\cap\Omega_l}\widehat{u_{l,k}}(\mathsf{M}_k^{-\mathsf{T}}\xi+2\pi\mathsf{M}_l^{\mathsf{T}}\omega_{k,j_{k,l}}+2\pi\omega),\qquad \xi\in\mathbb{R}^d,$$

for all $w\in(l(\mathbb{Z}^d))^{1\times(s+1)}$. Denote $\{e_1^r,\ldots,e_r^r\}$ the standard basis of $\mathbb{R}^r$. Choose $w_0:=\boldsymbol{\delta}e_j^r, j=1,\ldots,r$ and $w_1=\cdots=w_s:=0$, we conclude that the special case of item (ii) with $k=l=0$ holds. Similarly choose $w_0:=\mathbf{0}_{1\times r}$, $w_k:=1$ for some $k\in\{1,\ldots,s\}$ and $w_l:=0$ for all $l\neq k$, we can prove the special case of item (ii) with $k=l$ and $k,l\neq 0$.

Next, we consider the case when $k\neq l$. First fix $k\in\{1,\ldots,s\}$. Choose any $w=(w_1,\ldots,w_s,w_0)\in(l_1(\mathbb{Z}^d))^{1\times(s+r)}$ such that $w_1=\cdots=w_s=0$ and $w_0\neq\mathbf{0}_{1\times r}$. By (5.1.29),

$$0=\widehat{w_k}(\mathsf{M}_k^{\mathsf{T}}\xi)=\sum_{j_{k,0}=1}^{L_{k,0}}\widehat{w_0}(\mathsf{M}_0^{\mathsf{T}}\xi+2\pi\mathsf{M}_0^{\mathsf{T}}\omega_{k,j_{k,0}}))\sum_{\omega\in\Omega_0\cap\Omega_k}\widehat{u_{0,k}}(\xi+2\pi\omega_{k,j_{k,0}}+2\pi\omega),\quad(5.1.32)$$

for all $\xi\in\mathbb{R}^d$. Note that if $\Omega_k\subseteq\Omega_0$, then $\mathsf{M}_0^{\mathsf{T}}\omega_{k,j_{k,0}}\in\mathbb{Z}^d$ and $L_{k,0}=1$. Thus (5.1.32) reduces to

$$0=\widehat{w_0}(\mathsf{M}_0^{\mathsf{T}}\xi)\sum_{\omega\in\Omega_k}\widehat{u_{0,k}}(\xi+2\pi\omega),\qquad\xi\in\mathbb{R}^d.\qquad(5.1.33)$$

As $w_0\in(l_1(\mathbb{Z}^d))^{1\times r}\setminus\{\mathbf{0}_{1\times r}\}$ is arbitrary, it follows immediately from (5.1.33) that

$$\sum_{\omega\in\Omega_0\cap\Omega_k}\widehat{u_{0,k}}(\xi+2\pi\omega)=\mathbf{0}_{r\times 1}\qquad(5.1.34)$$

for all $\xi \in \mathbb{R}^d$. If $\Omega_k \nsubseteq \Omega_0$, then $\mathsf{M}_0^{\mathsf{T}}\omega \notin \mathbb{Z}^d$ for all $\omega \in \Omega_k \setminus \Omega_0$. Thus

$$\delta := \inf_{\omega\in\Omega_k\setminus\Omega_0, y\in\mathbb{Z}^d} 2\pi\|\mathsf{M}_0^{\mathsf{T}}\omega - y\| > 0. \tag{5.1.35}$$

Fix $x_0 \in (-\pi,\pi)^d$ and set $y_0 := \mathsf{M}_l^{\mathsf{T}}x_0$. Choose w_0 such that $\epsilon \in (0,\delta/2)$ and $\widehat{w_0} \in (C^\infty(\mathbb{T}^d))^{1\times r}$ such that the following hold:

1. $\widehat{w_0}(\xi) = (e_j^r)^{\mathsf{T}}$ for all $\xi \in B_\epsilon(y_0)$ with $j = 1,\dots,r$;
2. $\widehat{w_0}(\xi) = \mathbf{0}_{1\times r}$ for all $\xi \in B_\epsilon(y_0) + 2\pi\mathsf{M}_0^{\mathsf{T}}\omega$ with $\omega \in \Omega_k \setminus \Omega_0$;
3. $\mathsf{M}_0^{-\mathsf{T}}B_\epsilon(y_0) \subseteq (-\pi,\pi)^d$;
4. $\mathrm{supp}(\widehat{w_0}) \cap [y_0 + (-\pi,\pi^d)] \subseteq B_{2\epsilon}(y_0)$.

By our choice of w_l, it follows from (5.1.33) that

$$\sum_{\omega\in\Omega_0\cap\Omega_k} \widehat{u_{0,k}}(\xi + 2\pi\omega) = \mathbf{0}_{r\times 1}, \qquad \xi \in \mathsf{M}_0^{\mathsf{T}}B_\epsilon(y_0). \tag{5.1.36}$$

As $x_0 \in (-\pi,\pi)^d$ is arbitrary and $\widehat{u_{0,k}}$ is $2\pi\mathbb{Z}^d$-periodic, we conclude that (5.1.34) holds for a.e. $\xi \in \mathbb{R}^d$. Similarly, one can prove that

$$\sum_{\omega\in\Omega_l\cap\Omega_0} \widehat{u_{l,0}}(\xi + 2\pi\omega) = \mathbf{0}_{1\times r}, \qquad \sum_{\omega\in\Omega_l\cap\Omega_k} \widehat{u_{l,k}}(\xi + 2\pi\omega) = \boldsymbol{\delta}(k-l), \tag{5.1.37}$$

for a.e. $\xi \in \mathbb{R}^d$ and for all $k,l = 1,\dots,s$. This proves item (ii).

(ii) $\Rightarrow$ (i): Conversely suppose item (ii) holds. Then by Theorem 5.1.3, $\tilde{\mathcal{V}}\mathcal{W} = \mathrm{Id}_{(l(\mathbb{Z}^d))^{1\times r}}$.

It remains to show that $\mathcal{W}\tilde{\mathcal{V}} = \mathrm{Id}_{(l(\mathbb{Z}^d))^{1\times(s+r)}}$. Let $u_{k,l}$ be defined via (5.1.30). For all

$w = (w_1, \dots, w_s, w_0) \in (l_0(\mathbb{Z}^d))^{1\times[r(s+1)]}$, we have

$$\begin{aligned}
(\widehat{\mathcal{W}\tilde{\mathcal{V}}w})_k(\xi) &= \sum_{l=0}^{s} \sum_{\omega\in\Omega_k} \widehat{w_l}(\mathsf{M}_l^{\mathsf{T}}(\mathsf{M}_k^{-\mathsf{T}}\xi + 2\pi\omega))\widehat{u_{l,k}}(\mathsf{M}_k^{-\mathsf{T}}\xi + 2\pi\omega)\\
&= \sum_{j_{k,l}=1}^{L_{k,l}} \widehat{w_l}(\mathsf{M}_l^{\mathsf{T}}\mathsf{M}_k^{-\mathsf{T}}\xi + 2\pi\mathsf{M}_l^{\mathsf{T}}\omega_{k,j_{k,l}})) \sum_{\omega\in\Omega_l\cap\Omega_k} \widehat{u_{l,k}}(\mathsf{M}_k^{-\mathsf{T}}\xi + 2\pi\omega_{k,j_{k,l}} + 2\pi\omega)\\
&= \widehat{w_k}(\xi),
\end{aligned}$$

for a.e. $\xi \in \mathbb{R}^d$ and all $k = 0, 1, \dots, s$. Thus $\mathcal{W}\tilde{\mathcal{V}}w = w$ for all $w \in (l_0(\mathbb{Z}^d))^{1\times(s+r)}$. By using the locality of the subdivision and transition operators, we conclude that $\mathcal{W}\tilde{\mathcal{V}} = \mathrm{Id}_{(l(\mathbb{Z}^d))^{1\times(s+r)}}$. This proves item (i). ☐

The following result states the non-redundant property of a biorthogonal wavelet filter bank.

Lemma 5.1.10. *Let $b_0, \tilde{b}_0 \in (l_0(\mathbb{Z}^d))^{r\times r}, b_1, \dots, b_s, \tilde{b}_1, \dots, \tilde{b}_s \in (l_0(\mathbb{Z}^d))^{1\times r}$ be finitely supported filters and let $\mathsf{M}, \dots, \mathsf{M}_s$ be $d\times d$ dilation matrices. If $(\{b_l!\mathsf{M}_l\}_{l=0}^s, \{\tilde{b}_l!\mathsf{M}_l\}_{l=0}^s)$ is a biorthogonal wavelet filter bank with mixed dilation factors, then*

$$\frac{r}{|\det(\mathsf{M}_0)|} + \sum_{l=1}^{s} \frac{1}{|\det(\mathsf{M}_l)|} = r. \tag{5.1.38}$$

Proof. Let $\mathcal{V}$ and $\mathcal{W}$ (resp. $\tilde{\mathcal{V}}$ and $\tilde{\mathcal{W}}$) be the discrete framelet synthesis and analysis operators employing the filter bank $\{b_l!\mathsf{M}_l\}_{l=0}^s$ (resp. $\{\tilde{b}_l!\mathsf{M}_l\}_{l=0}^s$). By definition of a biorthogonal wavelet filter bank, we have $\mathcal{W}\tilde{\mathcal{V}} = \mathrm{Id}_{(l(\mathbb{Z}^d))^{1\times(s+r)}}$. Fix $k =\in \{1, \dots, s\}$ and define $v := (v_1, \dots, v_s, v_0) \in (l(\mathbb{Z}^d))^{1\times(s+r)}$ via

$$v_0 := \mathbf{0}_{1\times r}, \qquad v_k := \boldsymbol{\delta}, \qquad v_l = 0, \qquad l \in \{1, \dots, s\} \setminus \{k\}. \tag{5.1.39}$$

By Lemma 5.1.1, we have

$$
\begin{aligned}
1 = \langle v, v\rangle_{l_2(\mathbb{Z}^d)} &= \langle v, \mathcal{W}\tilde{\mathcal{V}}v\rangle_{l_2(\mathbb{Z}^d)} = \langle \boldsymbol{\delta}, \mathcal{T}_{b_k,\mathsf{M}_k}\mathcal{S}_{\tilde{b}_k,\mathsf{M}_k}\boldsymbol{\delta}\rangle_{l_2(\mathbb{Z}^d)}\\
=&\langle \mathcal{S}_{b_k,\mathsf{M}_k}\boldsymbol{\delta}, \mathcal{S}_{\tilde{b}_k,\mathsf{M}_k}\boldsymbol{\delta}\rangle_{l_2(\mathbb{Z}^d)} = \langle \widehat{\mathcal{S}_{b_k,\mathsf{M}_k}\boldsymbol{\delta}}, \widehat{\mathcal{S}_{\tilde{b}_k,\mathsf{M}_k}\boldsymbol{\delta}}\rangle_{L_2(\mathbb{T}^d)}\\
=&(2\pi)^{-d}|\det(\mathsf{M}_k)|\int_{[0,2\pi)^d} \widehat{b_k}(\xi)\overline{\widehat{\tilde{b}_k}(\xi)}^{\mathsf{T}}\, d\xi
\end{aligned}
\tag{5.1.40}
$$

Similarly, denote $\{e_1^r, \dots, e_r^r\}$ the standard basis of $\mathbb{R}^r$ and choose $w^j := (w_1, \dots, w_s, w_0^j) \in (l(\mathbb{Z}^d))^{1\times(s+r)}$ with

$$
w_0^j := (e_j^r)^{\mathsf{T}}\boldsymbol{\delta}, \qquad w_l = 0, \qquad l = 1, \dots, s. \tag{5.1.41}
$$

One can conclude that

$$
1 = \langle w^j, w^j\rangle_{l_2(\mathbb{Z}^d)} = (2\pi)^{-d}|\det(\mathsf{M}_0)|\int_{[0,2\pi)^d}\left(\widehat{b_0}(\xi)\overline{\widehat{\tilde{b}_0}(\xi)}^{\mathsf{T}}\right)_{j,j} d\xi, \qquad j = 1, \dots, r. \tag{5.1.42}
$$

It follows from (5.1.12), (5.1.40) and (5.1.42) that

$$
\begin{aligned}
r = \mathrm{Trace}(I_r) &= \sum_{l=0}^{s}(2\pi)^{-d}\int_{[0,2\pi)^d}\mathrm{Trace}\left(\overline{\widehat{\tilde{b}_l}(\xi)}^{\mathsf{T}}\widehat{b_l}(\xi)\right) d\xi\\
&= \sum_{l=0}^{s}(2\pi)^{-d}\int_{[0,2\pi)^d}\mathrm{Trace}\left(\widehat{b_l}(\xi)\overline{\widehat{\tilde{b}_l}(\xi)}^{\mathsf{T}}\right) d\xi\\
&= \frac{r}{|\det(\mathsf{M}_0)|} + \sum_{l=1}^{s}\frac{1}{|\det(\mathsf{M}_l)|}.
\end{aligned}
\tag{5.1.43}
$$

□

5.2 Discrete Affine Systems in $l_2(\mathbb{Z}^d)$

In this section, we further study the discrete framelet transforms by introducing the notion of a discrete affine system in $l_2(\mathbb{Z}^d)$.

Let $b_0, \tilde{b}_0 \in (l_0(\mathbb{Z}^d))^{r\times r}, b_1, \dots, b_s, \tilde{b}_1, \dots, \tilde{b}_s \in (l_0(\mathbb{Z}^d))^{1\times r}$ be finitely supported filters, and let $\mathsf{M}_0, \ldots, \mathsf{M}_s$ be $d \times d$ dilation matrices. Define

$$\widehat{b_{l,j}}(\xi) := \widehat{b_l}((\mathsf{M}_0^\mathsf{T})^{j-1}\xi)\widehat{b_0}((\mathsf{M}_0^\mathsf{T})^{j-2}\xi)\ldots\widehat{b_0}(\mathsf{M}_0^\mathsf{T}\xi)\widehat{b_0}(\xi), \tag{5.2.1}$$

$$\widehat{\tilde{b}_{l,j}}(\xi) := \widehat{\tilde{b}_l}((\mathsf{M}_0^\mathsf{T})^{j-1}\xi)\widehat{\tilde{b}_0}((\mathsf{M}^\mathsf{T})^{j-2}\xi)\ldots\widehat{\tilde{b}_0}(\mathsf{M}_0^\mathsf{T}\xi)\widehat{\tilde{b}_0}(\xi), \tag{5.2.2}$$

for all $j \in \mathbb{N}$, $l = 0, \ldots, s$ and $\xi \in \mathbb{R}^d$, with the convention $b_{0,0} := \boldsymbol{\delta} I_r =: \tilde{b}_{0,0}$, $b_{l,1} := b_l$ and $\tilde{b}_{l,1} := \tilde{b}_l$ for $l = 1, \ldots, s$. In other words:

$$b_{l,j} := (b_l \uparrow \mathsf{M}_0^{j-1}) * (b_0 \uparrow \mathsf{M}_0^{j-2}) * \cdots * (b_0 \uparrow \mathsf{M}_0) * b_0, \tag{5.2.3}$$

$$\tilde{b}_{l,j} := (\tilde{b}_l \uparrow \mathsf{M}_0^{j-1}) * (\tilde{b}_0 \uparrow \mathsf{M}_0^{j-2}) * \cdots * (\tilde{b}_0 \uparrow \mathsf{M}_0) * \tilde{b}_0, \tag{5.2.4}$$

where $\uparrow$ is the upsampling operator defined as in (1.2.1). For all $k \in \mathbb{Z}^d$, $J \in \mathbb{N}$ and $l = 0, \ldots, s$, define

$$b_{l,j;k} := |\det(\mathsf{M}_0)|^{\frac{j-1}{2}} |\det(\mathsf{M}_l)|^{\frac{1}{2}} b_{l,j}(\cdot - \mathsf{M}_0^{j-1}\mathsf{M}_l k), \tag{5.2.5}$$

$$\tilde{b}_{l,j;k} := |\det(\mathsf{M}_0)|^{\frac{j-1}{2}} |\det(\mathsf{M}_l)|^{\frac{1}{2}} \tilde{b}_{l,j}(\cdot - \mathsf{M}_0^{j-1}\mathsf{M}_l k). \tag{5.2.6}$$

The following lemma is an important result on the multi-level discrete analysis and synthesis processes.

Lemma 5.2.1. *Let $b_0 \in (l_0(\mathbb{Z}^d))^{r \times r}, b_1, \ldots, b_s \in (l_0(\mathbb{Z}^d))^{1 \times r}$ be finitely supported filters, and let $\mathsf{M}_0, \ldots, \mathsf{M}_s$ be $d \times d$ dilation matrices.*

(i) For any fixed $v_{0,0} \in (l(\mathbb{Z}^d))^{1 \times r}$, define $v_{l,j}$ as in (5.1.4) for all $l = 0, \ldots, s$ and $j = 1, \ldots, J$. Then we have

$$v_{l,j}(k) = \langle v_{0,0}, b_{l,j;k} \rangle_{l_2(\mathbb{Z}^d)} = \sum_{n \in \mathbb{Z}^d} v_{0,0}(n) \overline{b_{l,j;k}(n)}^\mathsf{T}, \qquad k \in \mathbb{Z}^d. \tag{5.2.7}$$

(ii) For $v_{0,J}, v_{l,j} \in l(\mathbb{Z}^d)$ with $l = 1, \ldots, s$ and $j = 1, \ldots, J$, we have

$$\mathcal{V}_J(0, \ldots, 0, v_{0,J}) = \sum_{k \in \mathbb{Z}^d} v_{0,J}(k) b_{0,J;k} \tag{5.2.8}$$

and

$$\mathcal{V}_J(0, \ldots, 0, v_{l,j}, 0, \ldots, 0) = \sum_{k \in \mathbb{Z}^d} v_{l,j}(k) b_{l,j;k}, \quad l = 1, \ldots, s, \quad j = 1, \ldots, J, \tag{5.2.9}$$

where $\mathcal{V}_J$ is the J-level discrete synthesis operator employing the filter bank $\{b_l!\mathsf{M}_l\}_{l=0}^s$.

Proof. For every $k \in \mathbb{Z}^d$, we have

$$\begin{aligned}
v_{l,j}(k) &= \mathcal{T}_{b_l,\mathsf{M}_l} v_{0,j-1}(k) = \mathcal{T}_{b_l,M_l} \mathcal{T}^{j-1}_{b_0,\mathsf{M}_0} v_{0,0}(k) \\
&= \mathcal{T}_{(b_l \uparrow \mathsf{M}_0^{j-1}) * (b_0 \uparrow \mathsf{M}_0^{j-2}) * \cdots * (b_0 \uparrow \mathsf{M}_0) * b_0, \mathsf{M}_0^{j-1}\mathsf{M}_l} v_{0,0}(k) \quad \text{(by Lemma 5.1.2)} \\
&= \mathcal{T}_{b_{l,j}, \mathsf{M}_0^{j-1}\mathsf{M}_l} v_{0,0}(k) \\
&= |\det(\mathsf{M}_l)|^{\frac{1}{2}} |\det(\mathsf{M}_0)|^{\frac{j-1}{2}} \sum_{n \in \mathbb{Z}^d} v_{0,0}(n) \overline{b_{l,j}(n - \mathsf{M}_0^{j-1}\mathsf{M}_l k)}^{\mathsf{T}} \\
&= \langle v_{0,0}, b_{l,j;k} \rangle_{l_2(\mathbb{Z}^d)}.
\end{aligned}$$

This proves item (i).

To prove item (ii), Define

$$\mathring{v}_{0,j-1} = \mathcal{S}_{b_0,\mathsf{M}_0} \mathring{v}_{l,j}, \qquad j = J, \ldots, 1, \tag{5.2.10}$$

with then convention $\mathring{v}_{0,J} = v_{0,J}$. Then

$$
\begin{aligned}
\mathcal{V}_J(0,\dots,0,v_{0,J}) &= \mathring{v}_{0,0} = \mathcal{S}^J_{b_0,\mathsf{M}_0} v_{0,J} \\
&= \mathcal{S}_{(b_0\uparrow\mathsf{M}_0^{j-1})*\cdots*(b_0\uparrow\mathsf{M}_0)*b_0,\mathsf{M}_0^J} v_{0,J} \quad \text{(by Lemma 5.1.2)} \\
&= \mathcal{S}_{b_{0,J},\mathsf{M}_0^J} v_{0,J} \\
&= |\det(\mathsf{M}_0)|^{\frac{J}{2}} \sum_{k\in\mathbb{Z}^d} v_{0,J}(k) b_{0,J}(\cdot - \mathsf{M}_0^J k) \\
&= \sum_{k\in\mathbb{Z}^d} v_{0,J}(k) b_{0,J;k}.
\end{aligned}
$$

Similarly, one can conclude that

$$
\mathcal{V}_J(0,\dots,0,v_{l,j},0,\dots,0) = \sum_{k\in\mathbb{Z}^d} v_{l,j}(k) b_{l,j;k}, \qquad l = 1,\dots,s, \quad j = 1,\dots,J. \tag{5.2.11}
$$

This proves item (ii). $\square$

In traditional framelet theory, discrete affine systems are introduced to study the frame representation property of discrete framelet systems generated by framelet filter banks (see [37, 41]). By mimicing the way we did in traditional framelet theory, we introduce the definition of a discrete affine system with mixed dilation factors as the following.

Definition 5.2.2. *Let $b_0 \in (l_0(\mathbb{Z}^d))^{r\times r}, b_1,\dots,b_s \in (l_0(\mathbb{Z}^d))^{1\times r}$ be finitely supported filters and let $\mathsf{M}_0,\dots,\mathsf{M}_s$ be $d\times d$ dilation matrices. For every $J\in\mathbb{N}$, the J-**level discrete affine system** associated to the filter bank $\{b_l!\mathsf{M}_l\}_{l=0}^s$ is defined via*

$$
\mathrm{DAS}_J(\{b_l!\mathsf{M}_l\}_{l=0}^s) = \{b_{0,J;k} : k\in\mathbb{Z}^d\} \cup \{b_{l,j;k} : l = 1,\dots,s, \quad j = 1,\dots,J, \quad k\in\mathbb{Z}^d\},
$$

where $b_{l,j;k}$ is defined via (5.2.5) *for all $k\in\mathbb{Z}^d, j\in\mathbb{N}$ and $l = 0,\dots,s$.*

The stability of a filter bank is naturally linked to the frame property of its associated discrete affine systems.

Lemma 5.2.3. *Let $b_0 \in (l_0(\mathbb{Z}^d))^{r\times r}, b_1,\dots,b_s \in (l_0(\mathbb{Z}^d))^{1\times r}$ be finitely supported filters and let $\mathsf{M}_0,\dots,\mathsf{M}_s$ be $d\times d$ dilation matrices. Then the filter bank $\{b_l!\mathsf{M}_l\}_{l=0}^s$ has stability* *in $l_2(\mathbb{Z}^d)$ if and only if there exist $C_1, C_2 > 0$ such that*

$$\begin{aligned} C_1\|v\|^2_{(l_2(\mathbb{Z}^d))^{1\times r}} \leqslant \sum_{k\in\mathbb{Z}^d} \|\langle v, b_{0,J;k}\rangle_{l_2(\mathbb{Z}^d)}\|^2 \\ + \sum_{j=1}^{J}\sum_{l=1}^{s}\sum_{k\in\mathbb{Z}^d} |\langle v, b_{l,j;k}\rangle_{l_2(\mathbb{Z}^d)}|^2 \leqslant C_2\|v\|^2_{(l_2(\mathbb{Z}^d))^{1\times r}} \end{aligned} \tag{5.2.12}$$

holds for all $v \in (l_2(\mathbb{Z}^d))^{1\times r}$ and $J \in \mathbb{N}$.

Proof. Let $\mathcal{W}_J$ be the J-level discrete analysis operator employing the filter bank $\{b_l!\mathsf{M}_l\}_{l=0}^{s}$. By item (i) of Lemma 5.2.1, we have

$$\begin{aligned} \|\mathcal{W}_J v\|^2_{(l_2(\mathbb{Z}^d))^{1\times(sJ+r)}} &= \|v_{0,J}\|^2_{(l_2(\mathbb{Z}^d))^{1\times r}} + \sum_{j=1}^{J}\sum_{l=1}^{s} \|v_{l,j}\|^2_{(l_2(\mathbb{Z}^d))^{1\times r}} \\ &= \sum_{k\in\mathbb{Z}^d} \|\langle v, b_{0,J;k}\rangle_{l_2(\mathbb{Z}^d)}\|^2 + \sum_{j=1}^{J}\sum_{l=1}^{s}\sum_{k\in\mathbb{Z}^d} \|\langle v, b_{l,j;k}\rangle_{l_2(\mathbb{Z}^d)}\|^2, \end{aligned} \tag{5.2.13}$$

for all $v \in (l_2(\mathbb{Z}^d))^{1\times r}$ and $J \in \mathbb{N}$. Hence the result follows immediately. $\square$

The associated discrete affine systems of a dual framelet filter bank have the frame expansion property, which is illustrated by the following result.

Lemma 5.2.4. *Let $b_0, \tilde{b}_0 \in (l_0(\mathbb{Z}^d))^{r\times r}, b_1, \dots, b_s, \tilde{b}_1, \dots, \tilde{b}_s \in (l_0(\mathbb{Z}^d))^{1\times r}$ be finitely supported filters and let $\mathsf{M}_0, \dots, \mathsf{M}_s$ be $d \times d$ dilation matrices. Then $(\{b_l!\mathsf{M}_l\}_{l=0}^{s}, \{\tilde{b}_l!\mathsf{M}_l\}_{l=0}^{s})$ is a dual framelet filter bank with mixed dilation factors if and only if*

$$v = \sum_{k\in\mathbb{Z}^d} \langle v, b_{0,J;k}\rangle_{l_2(\mathbb{Z}^d)} \tilde{b}_{0,J;k} + \sum_{j=1}^{J}\sum_{l=1}^{s}\sum_{k\in\mathbb{Z}^d} \langle v, b_{l,j;k}\rangle_{l_2(\mathbb{Z}^d)} \tilde{b}_{l,j;k} \tag{5.2.14}$$

for all $v \in (l_2(\mathbb{Z}^d))^{1\times r}$ and $J \in \mathbb{N}$, where $b_{l,j;k}$ and $\tilde{b}_{l,j;k}$ are defined as (5.2.5) *and* (5.2.6) *for all $k \in \mathbb{Z}^d, j \in \mathbb{N}$ and $l = 0, \dots, s$.*

Proof. Let $v_{0,0} \in (l_2(\mathbb{Z}^d))^{1\times r}$ and $J \in \mathbb{N}$. Define $v_{l,j}$ as in (5.1.4) for all $j = 1, \dots, J$ and

$l = 0, \dots, s$. Then by Lemma 5.2.1, we have

$$\begin{aligned}
\tilde{\mathcal{V}}_J \mathcal{W}_J v_{0,0} &= \tilde{\mathcal{V}}_J(v_{1,1}, \dots, v_{s,1}, \dots, v_{1,J}, \dots, v_{s,J}, v_{0,J}) \\
&= \sum_{k\in\mathbb{Z}^d} v_{0,J}(k)\tilde{b}_{0,J;k} + \sum_{j=1}^{J}\sum_{l=1}^{s}\sum_{k\in\mathbb{Z}^d} \tilde{v}_{l,j}(k)\tilde{b}_{l,j;k} \\
&= \sum_{k\in\mathbb{Z}^d} \langle v_{0,0}, b_{0,J;k}\rangle_{l_2(\mathbb{Z}^d)}\tilde{b}_{0,J;k} + \sum_{j=1}^{J}\sum_{l=1}^{s}\sum_{k\in\mathbb{Z}^d} \langle v_{0,0}, b_{l,j;k}\rangle_{l_2(\mathbb{Z}^d)}\tilde{b}_{l,j;k}.
\end{aligned} \tag{5.2.15}$$

Hence $(\{b_l!\mathsf{M}_l\}_{l=0}^{s}, \{\tilde{b}_l!\mathsf{M}_l\}_{l=0}^{s})$ is a dual framelet filter bank with mixed dilation factors if and only if (5.2.14) holds for all $v \in (l_2(\mathbb{Z}^d))^{1\times r}$ and $J \in \mathbb{N}$. $\square$

Now we are at the stage to give a complete characterization of a dual framelet filter bank by using its associated discrete affine systems, which is summarized as the following theorem:

Theorem 5.2.5. *Let $b_0, \tilde{b}_0 \in (l_0(\mathbb{Z}^d))^{r\times r}, b_1, \dots, b_s, \tilde{b}_1, \dots, \tilde{b}_s \in (l_0(\mathbb{Z}^d))^{1\times r}$ be finitely supported filters and let $\mathsf{M}_0, \dots, \mathsf{M}_s$ be $d \times d$ dilation matrices. The following statements are equivalent.*

(i) $(\{b_l!\mathsf{M}_l\}_{l=0}^{s}, \{\tilde{b}_l!\mathsf{M}_l\}_{l=0}^{s})$ is a dual framelet filter bank with mixed dilation factors.

(ii) For all $v, w \in (l_2(\mathbb{Z}^d))^{1\times r}$:

$$\langle v, w\rangle_{l_2(\mathbb{Z}^d)} = \sum_{l=0}^{s}\sum_{k\in\mathbb{Z}^d} \langle v, b_{l,1;k}\rangle_{l_2(\mathbb{Z}^d)} \langle \tilde{b}_{l,1;k}, w\rangle_{l_2(\mathbb{Z}^d)}. \tag{5.2.16}$$

(iii) For all $J \in \mathbb{N}$ and $v, w \in (l_2(\mathbb{Z}^d))^{1\times r}$:

$$\begin{aligned}
\langle v, w\rangle_{l_2(\mathbb{Z}^d)} &= \sum_{k\in\mathbb{Z}^d} \langle v, b_{0,J;k}\rangle_{l_2(\mathbb{Z}^d)} \langle \tilde{b}_{0,J;k}, w\rangle_{l_2(\mathbb{Z}^d)} \\
&\quad + \sum_{j=1}^{J}\sum_{l=1}^{s}\sum_{k\in\mathbb{Z}^d} \langle v, b_{l,j;k}\rangle_{l_2(\mathbb{Z}^d)} \langle \tilde{b}_{l,j;k}, w\rangle_{l_2(\mathbb{Z}^d)}.
\end{aligned} \tag{5.2.17}$$

*(iv) (**Cascade structure**) For all $j \in \mathbb{N}$ and $v, w \in (l_2(\mathbb{Z}^d))^{1\times r}$:*

$$\sum_{k\in\mathbb{Z}^d} \langle v, b_{0,j-1;k}\rangle_{l_2(\mathbb{Z}^d)} \langle \tilde{b}_{0,j-1;k}, w\rangle_{l_2(\mathbb{Z}^d)} = \sum_{l=0}^{s} \sum_{k\in\mathbb{Z}^d} \langle v, b_{l,j;k}\rangle_{l_2(\mathbb{Z}^d)} \langle \tilde{b}_{l,j;k}, w\rangle_{l_2(\mathbb{Z}^d)}, \quad (5.2.18)$$

with the convention $\tilde{b}_{0,0} = b_{0,0} := \boldsymbol{\delta} I_r$ and $\tilde{b}_{0,0;k} = b_{0,0;k} := \boldsymbol{\delta}_k I_r$ where $\boldsymbol{\delta}_k := \boldsymbol{\delta}(\cdot - k)$.

If further assume that $\{b_l!M_l\}_{l=0}^{s}$ and $\{\tilde{b}_l!M_l\}_{l=0}^{s}$ have stability in $l_2(\mathbb{Z}^d)$, then each of items (i) - (iv) is equivalent to the following statement:

(v) (5.2.14) holds, and moreover, there exist $C_1, C_2 > 0$ such that (5.2.12) and

$$\begin{aligned} C_2^{-1}\|v\|^2_{(l_2(\mathbb{Z}^d))^{1\times r}} \leqslant \sum_{k\in\mathbb{Z}^d} &\|\langle v, \tilde{b}_{0,J;k}\rangle_{l_2(\mathbb{Z}^d)}\|^2 \\ &+ \sum_{j=1}^{J}\sum_{l=1}^{s}\sum_{k\in\mathbb{Z}^d} |\langle v, \tilde{b}_{l,j;k}\rangle_{l_2(\mathbb{Z}^d)}|^2 \leqslant C_1^{-1}\|v\|^2_{(l_2(\mathbb{Z}^d))^{1\times r}} \end{aligned} \quad (5.2.19)$$

hold for all $J \in \mathbb{N}$ and $v \in (l_2(\mathbb{Z}^d))^{1\times r}$.

Proof. (i)⇔(ii): Follows immediately from Lemma 5.2.4.

(ii)⇔(iv): Suppose that (iv) holds. Recall that $b_{0,0;k} = \tilde{b}_{0,0;k} = \boldsymbol{\delta}_k I_r$. Then we have

$$v = \sum_{k\mathbb{Z}^d} \langle v, b_{0,0;k}\rangle_{l_2(\mathbb{Z}^d)} \tilde{b}_{0,0;k} = \sum_{l=0}^{s}\sum_{k\in\mathbb{Z}^d} \langle v, b_{l,1;k}\rangle_{l_2(\mathbb{Z}^d)} \tilde{b}_{l,1;k} \quad (5.2.20)$$

for all $v \in (l_2(\mathbb{Z}^d))^{1\times r}$. Thus (ii) holds. Conversely, suppose that (ii) holds. Note that for all $l = 0, \ldots, s$ and $j \in \mathbb{N}$, we have

$$b_{l,j} = (b_{l,1} \uparrow \mathsf{M}_0^{j-1}) * b_{0,j-1} = \sum_{k\in\mathbb{Z}^d} b_{l,1}(k) b_{0,j-1}(\cdot - \mathsf{M}_0^{j-1}k). \quad (5.2.21)$$

For all $k \in \mathbb{Z}^d$, $l = 0, \ldots, s$ and $j \in \mathbb{N}$, it follows that

$$\begin{aligned} b_{l,j;k} &= |\det(\mathsf{M}_0)|^{\frac{j-1}{2}} |\det(\mathsf{M}_l)|^{\frac{1}{2}} b_{l,j}(\cdot - \mathsf{M}_0^{j-1}\mathsf{M}_l k) \\ &= |\det(\mathsf{M}_0)|^{\frac{j-1}{2}} |\det(\mathsf{M}_l)|^{\frac{1}{2}} \sum_{m\in\mathbb{Z}^d} b_{l,1}(m) b_{0,j-1}(\cdot - \mathsf{M}_0^{j-1}\mathsf{M}_l k - \mathsf{M}_0^{j-1} m) \\ &= |\det(\mathsf{M}_0)|^{\frac{j-1}{2}} |\det(\mathsf{M}_l)|^{\frac{1}{2}} \sum_{m\in\mathbb{Z}^d} b_{l,1}(m - \mathsf{M}_l k) b_{0,j-1}(\cdot - \mathsf{M}_0^{j-1} m) \\ &= \sum_{m\in\mathbb{Z}^d} b_{l,1;k}(m) b_{0,j-1;m}. \end{aligned} \tag{5.2.22}$$

Thus for all $v \in (l_2(\mathbb{Z}^d))^{1\times r}$, $k \in \mathbb{Z}^d$, $l = 0, \ldots, s$ and $j \in \mathbb{N}$, we have

$$\langle v, b_{l,j;k}\rangle_{l_2(\mathbb{Z}^d)} = \sum_{m\in\mathbb{Z}^d} \langle v, b_{0,j-1;m}\rangle_{l_2(\mathbb{Z}^d)} \overline{b_{l,1;k}(m)}^{\mathsf{T}} = \langle B_{0,j-1;v}, b_{l,1;k}\rangle_{l_2(\mathbb{Z}^d)}, \tag{5.2.23}$$

where

$$B_{0,j-1;v}(m) = \langle v, b_{0,j-1;m}\rangle_{l_2(\mathbb{Z}^d)}, \qquad m \in \mathbb{Z}^d. \tag{5.2.24}$$

It follows from (5.2.22) and(5.2.23) that

$$\begin{aligned} \sum_{l=0}^{s}\sum_{k\in\mathbb{Z}^d} \langle v, b_{l,j;k}\rangle_{l_2(\mathbb{Z}^d)} \tilde{b}_{l,j;k} &= \sum_{l=0}^{s}\sum_{k\in\mathbb{Z}^d} \langle B_{0,j-1;v}, b_{l,1;k}\rangle_{l_2(\mathbb{Z}^d)} \sum_{m\in\mathbb{Z}^d} \tilde{b}_{l,1;k}(m)\tilde{b}_{0,j-1;m} \\ &= \sum_{m\in\mathbb{Z}^d}\left(\sum_{l=0}^{s}\sum_{k\in\mathbb{Z}^d} \langle B_{0,j-1;v}, b_{l,1;k}\rangle_{l_2(\mathbb{Z}^d)} \tilde{b}_{l,1;k}(m)\right)\tilde{b}_{0,j-1;m} \\ &= \sum_{m\in\mathbb{Z}^d} B_{0,j-1;v}\tilde{b}_{0,j-1;m} \\ &= \sum_{m\in\mathbb{Z}^d} \langle v, b_{0,j-1;m}\rangle_{l_2(\mathbb{Z}^d)} \tilde{b}_{0,j-1;m}. \end{aligned} \tag{5.2.25}$$

This proves (iv).

(ii)⇔(iii): (iii) trivially implies (ii). Conversely, (ii) and (iv) are equivalent to each other, and together imply (iii).

If we further assume that both $\{b_l!\mathsf{M}_l\}_{l=0}^s$ and $\{\tilde{b}_l!\mathsf{M}_l\}_{l=0}^s$ have stability in $l_2(\mathbb{Z}^d)$, then by Lemma 5.2.3, each of items (i)-(iv) is equivalent to (v). □

Similarly, we have the following characterization of biorthogonal filter banks using discrete affine systems.

Theorem 5.2.6. *Let $b_0, \tilde{b}_0 \in (l_0(\mathbb{Z}^d))^{r\times r}$ and $b_1, \dots, b_s, \tilde{b}_1, \dots, \tilde{b}_s \in (l_0(\mathbb{Z}^d))^{1\times r}$ be finitely supported filters and let $\mathsf{M}_0, \dots, \mathsf{M}_s$ be $d \times d$ dilation matrices. The following statements are equivalent.*

(i) $(\{b_l!\mathsf{M}_l\}_{l=0}^s, \{\tilde{b}_l!\mathsf{M}_l\}_{l=0}^s)$ *is a biorthogonal wavelet filter bank with mixed dilation factors.*

(ii) For every $J \in \mathbb{N}$, (5.2.14) *holds for all $v \in (l_2(\mathbb{Z}^d))^{1\times r}$, and moreover,* $(\mathrm{DAS}_J(\{b_l!\mathsf{M}_l\}_{l=0}^s), \mathrm{DAS}_J(\{\tilde{b}_l!\mathsf{M}_l\}_{l=0}^s))$ *is a pair of biorthogonal systems in $l_2(\mathbb{Z}^d)$ which satisfies the following biorthogonality relations:*

$$\langle \tilde{b}_{0,J;k'}, b_{0,J;k}\rangle = \boldsymbol{\delta}(k-k')I_r, \qquad \langle \tilde{b}_{l,j;k}, b_{0,J;k'}\rangle = \mathbf{0}_{1\times r}, \tag{5.2.26}$$

for all $l = 1, \dots, s$, $j = 1, \dots, J$, $k, k' \in \mathbb{Z}^d$, and

$$\langle \tilde{b}_{l',j';k'}, b_{l,j;k}\rangle = \boldsymbol{\delta}(l-l')\boldsymbol{\delta}(j-j')\boldsymbol{\delta}(k-k'), \tag{5.2.27}$$

for all $l, l' = 1, \dots, s$, $j, j' = 1, \dots, J$, and $k, k' \in \mathbb{Z}^d$.

If we further assume that both $\{b_l!\mathsf{M}_l\}_{l=0}^s$ and $\{\tilde{b}_l!\mathsf{M}_l\}_{l=0}^s$ have stability in $l_2(\mathbb{Z}^d)$, then each of items (i) and (ii) is equivalent to the following statement:

(iii) (5.2.14) *holds and there exist $C_1, C_2 > 0$ such that* (5.2.12) *and* (5.2.19) *hold for all $v \in l_2(\mathbb{Z}^d)$ and $J \in \mathbb{N}$. Moreover,* $(\mathrm{DAS}_J(\{b_l!\mathsf{M}_l\}_{l=0}^s), \mathrm{DAS}_J(\{\tilde{b}_l!\mathsf{M}_l\}_{l=0}^s))$ *is a pair of biorthogonal systems which satisfies* (5.2.26) *and* (5.2.27).

Proof. (i)⇔(ii): Suppose that (i) holds. Then in particular $(\{b_l!\mathsf{M}_l\}_{l=0}^s, \{\tilde{b}_l!\mathsf{M}_l\}_{l=0}^s)$ is a dual framelet filter bank with mixed dilation factors. Thus (5.2.14) holds for all $J \in \mathbb{N}$

and $v \in (l_2(\mathbb{Z}^d))^{1\times r}$ by Lemma 5.2.4. Moreover, we have $\tilde{\mathcal{V}}_J \mathcal{W}_J = \mathrm{Id}_{(l(\mathbb{Z}^d))^{1\times r}}$ for all $J \in \mathbb{N}$. Now the biorthogonality relations (5.2.26) and (5.2.27) follow straight away from (5.2.15) and the injectivity of $\tilde{\mathcal{V}}_J$.

Conversely suppose that (ii) holds, then (5.2.14) implies that $(\{b_l!\mathsf{M}_l\}_{l=0}^s, \{\tilde{b}_l!\mathsf{M}_l\}_{l=0}^s)$ is a dual framelet filter bank with mixed dilation factors. Moreover the injectivity of $\mathcal{V}_J$ follows from the biorthogonality relations (5.2.26) and (5.2.27) and the fact that

$$\tilde{\mathcal{V}}_J w = \sum_{k\in\mathbb{Z}^d} w_{0,J}(k)\tilde{b}_{0,J;k} + \sum_{j=1}^{J}\sum_{l=1}^{s}\sum_{k\in\mathbb{Z}^d} w_{l,j}(k)\tilde{b}_{l,j;k} \tag{5.2.28}$$

holds for all $w = (w_{1,1}, \dots, w_{1,J}, \dots, w_{s,1}, \dots, w_{s,J}, w_{0,J}) \in (l(\mathbb{Z}^d))^{1\times(sJ+r)}$. This proves item (i).

Finally, by further assuming that both $\{b_l!\mathsf{M}_l\}_{l=0}^s$ and $\{\tilde{b}_l!\mathsf{M}_l\}_{l=0}^s$ have stability in $l_2(\mathbb{Z}^d)$, it follows from Lemma 5.2.3 that each of items (i) and (ii) is equivalent to item (iii). $\square$

5.3 Framelets and Wavelets in $L_2(\mathbb{R}^d)$ with Mixed dilation Factors

In this section, we discuss the connections between framelet filter banks and framelets in $L_2(\mathbb{R}^d)$. First, we briefly review several definitions and results related to refinable functions obtained from finitely supported filters. The following result is well known (see e.g. [41, Theorem 5.1.2]).

Lemma 5.3.1. *Let $b_0 \in (l_0(\mathbb{Z}^d))^{r\times r}$ be a finitely supported filter and let M_0 be a $d\times d$ dilation matrix. Suppose there exist $C_0, C_b, \tau > 0$ and $u \in (l_0(\mathbb{Z}^d))^{r\times 1}$ such that*

$$0 < \left|\left|\left|\widehat{b}(0)\right|\right|\right| \|\mathsf{M}_0\|^{-\tau} < 1, \qquad \|\widehat{b_0}(\xi)\widehat{u}(\xi) - \widehat{u}(\mathsf{M}_0^{\mathsf{T}}\xi)\| \leqslant C_b\|\xi\|^{\tau}, \qquad \xi \in [-\pi,\pi]^d, \tag{5.3.1}$$

where $|||\cdot|||$ *is some fixed sub-multiplicative matrix norm (i.e.,* $|||EF||| \leqslant |||E||| |||F|||$*). Then*

$$\varphi(\xi) := \lim_{n\to\infty} \left(\prod_{j=1}^{n} \widehat{b_0}((\mathsf{M}_0^{-\mathsf{T}})^j \xi) \right) \widehat{u}((\mathsf{M}_0^{-\mathsf{T}})^n \xi), \qquad \xi \in \mathbb{R}^d \tag{5.3.2}$$

is a well-define locally bounded measurable function which satisfies

$$\varphi(\xi) = \widehat{u}(\xi) + \mathscr{O}(\|\xi\|^\tau), \qquad \xi \to 0. \tag{5.3.3}$$

Moreover, there exists a compactly supported vector distribution ψ^0 *such that* $\widehat{\psi^0} = \varphi$*, and satisfies the following refinement equation:*

$$\psi^0(\cdot) = \sum_{k\in\mathbb{Z}^d} b_0(k)\psi^0(\mathsf{M}_0 \cdot -k), \tag{5.3.4}$$

with the above series converging in the sense of tempered distributions.

Definition 5.3.2. *For any finitely supported filter* $b_0 \in (l_0(\mathbb{Z}^d))^{r\times r}$ *satisfying* (5.3.1) *for some* $C_0, C_b, \tau > 0$ *and* $u \in (l_0(\mathbb{Z}^d))^{r\times 1}$*, the function* ϕ *which is defined via* (5.3.2) *is called* ***a standard refinable function*** *associated to the filter* a*.*

The following proposition is an important result for our later study on framelets and wavelets in $L_2(\mathbb{R}^d)$.

Proposition 5.3.3. *Let* $b_0 \in (l_0(\mathbb{Z}^d))^{r\times r}, b_1, \dots, b_s \in (l_0(\mathbb{Z}^d))^{1\times r}$ *be finitely supported filters and let* $\mathsf{M}_0, \dots, \mathsf{M}_s$ *be* $d \times d$ *dilation matrices. Suppose there exist* $C_0, C_b, \tau > 0$ *and* $u \in (l_0(\mathbb{Z}^d))^{r\times 1}$ *such that* (5.3.1) *holds. Define a compactly supported standard refinable vector function* ψ^0 *via* (5.3.2) *and define* $\psi^1, \dots, \psi^s$ *via*

$$\widehat{\psi^l}(\xi) = \widehat{b_l}(\mathsf{M}_0^{-\mathsf{T}}\xi)\widehat{\psi^0}(\mathsf{M}_0^{-\mathsf{T}}\xi), \qquad \xi \in \mathbb{R}^d, \quad l = 1, \dots, s. \tag{5.3.5}$$

If $\{b_l ! \mathsf{M}_l\}_{l=0}^s$ *has stability in* $l_2(\mathbb{Z}^d)$*, then there exists* $C > 0$ *such that*

$$\sum_{k\in\mathbb{Z}^d} \widehat{\psi^0}(\xi + 2\pi k)\overline{\widehat{\psi^0}(\xi + 2\pi k)}^{\mathsf{T}} \leqslant C I_r, \qquad \sum_{l=1}^{s} \sum_{k\in\mathbb{Z}^d} \left| \widehat{\psi^l}(\mathsf{M}_l^{-\mathsf{T}}\mathsf{M}_0^{\mathsf{T}}(\xi + 2\pi k)) \right|^2 \leqslant C \tag{5.3.6}$$

for a.e. $\xi \in \mathbb{R}^d$.

Proof. For $J \in \mathbb{N}$, let $\mathcal{V}_J$ and $\mathcal{W}_J$ be the J-level discrete framelet synthesis and analysis operators employing the filter bank $\{b_l!\mathsf{M}_l\}_{l=0}^s$. By the stability of $\{b_l!\mathsf{M}_l\}_{l=0}^s$ in $l_2(\mathbb{Z}^d)$, there exists $K > 0$ such that

$$\|\mathcal{V}_J w\|^2_{(l_2(\mathbb{Z}^d))^{1\times r}} \leqslant K\|w\|^2_{(l_2(\mathbb{Z}^d))^{1\times(sJ+r)}}, \qquad w \in (l_2(\mathbb{Z}^d))^{1\times(sJ+r)}, \quad J \in \mathbb{N}. \tag{5.3.7}$$

Let $w = (0, \dots, 0, v) \in (l_2(\mathbb{Z}^d))^{1\times(sJ+r)}$ with $v \in (l_2(\mathbb{Z}^d))^{1\times r}$. We have

$$\mathcal{V}_J w = \mathcal{S}^J_{b_0,\mathsf{M}_0} v = \mathcal{S}_{b_{0,J},\mathsf{M}_0^J} v, \tag{5.3.8}$$

where $b_{0,J} = (b_0 \uparrow \mathsf{M}_0^{J-1}) * \cdots * (b_0 \uparrow \mathsf{M}_0) * b_0$. Thus by Parseval's identity, we have

$$\begin{aligned}
&\|\mathcal{S}_{u,\mathsf{M}_0}\mathcal{V}_J w\|^2_{l_2(\mathbb{Z}^d)}\\
=&(2\pi)^{-d}\int_{(\mathsf{M}_0^{\mathsf{T}})^J[0,2\pi)^d} \widehat{v}(\xi)\widehat{b_{0,J}}((\mathsf{M}_0^{-\mathsf{T}})^J\xi)\widehat{u}((\mathsf{M}_0^{-\mathsf{T}})^J\xi)\overline{\widehat{b_{0,J}}((\mathsf{M}_0^{-\mathsf{T}})^J\xi)\widehat{u}((\mathsf{M}_0^{-\mathsf{T}})^J\xi)}^{\mathsf{T}}\,\overline{\widehat{v}(\xi)}^{\mathsf{T}} d\xi.
\end{aligned}$$

It now follows from Fatou's lemma that

$$\begin{aligned}
&(2\pi)^{-d}\int_{\mathbb{R}^d} \widehat{v}(\xi)\widehat{\psi^0}(\xi)\overline{\widehat{\psi^0}(\xi)}^{\mathsf{T}}\,\overline{\widehat{v}(\xi)}^{\mathsf{T}} d\xi \leqslant \liminf_{J\to\infty} \|\mathcal{S}_{u,\mathsf{M}_0}\mathcal{V}_J w\|^2_{l_2(\mathbb{Z}^d)}\\
\leqslant&\liminf_{J\to\infty} |\det(\mathsf{M}_0)|\|\mathcal{V}_J w\|^2_{(l_2(\mathbb{Z}^d))^{1\times(s+r)}}\|u\|^2_{(l_1(\mathbb{Z}^d))^{r\times 1}}\\
\leqslant&K|\det(\mathsf{M}_0)|\|u\|^2_{(l_1(\mathbb{Z}^d))^{r\times 1}}\|v\|^2_{(l_2(\mathbb{Z}^d))^{1\times r}}
\end{aligned} \tag{5.3.9}$$

for all $v \in l_2(\mathbb{Z}^d)$. Note that

$$\int_{\mathbb{R}^d} \widehat{v}(\xi)\widehat{\psi^0}(\xi)\overline{\widehat{\psi^0}(\xi)}^{\mathsf{T}}\,\overline{\widehat{v}(\xi)}^{\mathsf{T}} d\xi = \int_{[0,2\pi)^d} \widehat{v}(\xi)\left(\sum_{k\in\mathbb{Z}^d} \widehat{\psi^0}(\xi+2\pi k)\overline{\widehat{\psi^0}(\xi+2\pi k)}^{\mathsf{T}}\right)\overline{\widehat{v}(\xi)}^{\mathsf{T}} d\xi,$$

for all $v \in (l_2(\mathbb{Z}^d))^{1\times r}$. It follows from (5.3.9) and the periodicity of $\sum_{k\in\mathbb{Z}^d} \widehat{\psi^0}(\xi +$

$2\pi k)\overline{\widehat{\psi^0}(\xi+2\pi k)}^{\mathsf{T}}$ that

$$\sum_{k\in\mathbb{Z}^d}\widehat{\psi^0}(\xi+2\pi k)\overline{\widehat{\psi^0}(\xi+2\pi k)}^{\mathsf{T}}\leqslant K|\det(\mathsf{M}_0)|\|u\|^2_{(l_1(\mathbb{Z}^d))^{r\times 1}}I_r, \tag{5.3.10}$$

for a.e. $\xi\in\mathbb{R}^d$.

Similarly, choose $w=(0,\ldots,w_{l,J},\ldots,0)\in(l_2(\mathbb{Z}^d))^{1\times(sJ+r)}$ with $w_{l,J}\in l_2(\mathbb{Z}^d)$ for some fixed $l\in\{1,\ldots,s\}$, and using the fact that

$$\widehat{\psi^l}(\xi)=\lim_{J\to\infty}\widehat{b_l}(\mathsf{M}_0^{-\mathsf{T}}\xi)\prod_{j=1}^{J-1}\widehat{\psi^0}((\mathsf{M}_0^{-\mathsf{T}})^j\xi),\qquad l=1,\ldots,s,\quad \xi\in\mathbb{R}^d, \tag{5.3.11}$$

one can apply the above same arguments to conclude that there exists $\tilde{K}>0$ such that $\sum_{k\in\mathbb{Z}^d}\left|\widehat{\psi^l}((\mathsf{M}_l^{-\mathsf{T}}\mathsf{M}_0^{\mathsf{T}})(\xi+2\pi k))\right|^2\leqslant\tilde{K}$ for all $l=1,\ldots,s$ and a.e. $\xi\in\mathbb{R}^d$. This completes the proof. $\square$

For a function (distribution) matrix $f:\mathbb{R}^d\to\mathbb{C}^{s\times r}$ and a $d\times d$ invertible real matrix U, define

$$f_{U;k,n}(x):=|\det(U)|^{\frac{1}{2}}e^{-in\cdot Ux}f(Ux-k),\qquad x,k,n\in\mathbb{R}^d. \tag{5.3.12}$$

In particular, define $f_{U;k}:=f_{U;k,0}$. It is straight forward to verity that

$$\widehat{f_{U;k,n}}=\widehat{f}_{U^{-\mathsf{T}};n,k}.$$

Definition 5.3.4. *Let $f:\mathbb{R}^d\to\mathbb{C}^{s\times r}$ and $g:\mathbb{R}^d\to\mathbb{C}^{t\times r}$ are matrices of measurable functions, and U is a $d\times d$ invertible real matrix, define the U**-bracket product** of f and g via:*

$$[f,g]_U(x):=\sum_{k\in\mathbb{Z}^d}f(x+2\pi U^{-1}k)\overline{g(x+2\pi U^{-1}k)}^{\mathsf{T}}, \tag{5.3.13}$$

whenever the series converges absolutely for a.e. $x\in\mathbb{R}^d$. Define $[f,g]:=[f,g]_{I_d}$. Note that $[f,g]_U$ is an $s\times t$ matrix of $2\pi U^{-1}\mathbb{Z}^d$-periodic functions.

We discuss some important properties of the bracket product.

Lemma 5.3.5. *Let $f \in (L_2(\mathbb{R}^d))^{s\times r}, g \in (L_2(\mathbb{R}^d))^{t\times r}$ and let U be a $d\times d$ invertible real matrix. We have $[\widehat{f}, \widehat{g}]_U \in (L_1(U^{-1}\mathbb{T}^d))^{s\times t}$ and its Fourier series is*

$$|\det(U)| \sum_{k\in\mathbb{Z}^d} \langle f, g(\cdot + U^{\mathsf{T}}k)\rangle e^{-ik\cdot Ux}. \tag{5.3.14}$$

Proof. Let $f \in (L_2(\mathbb{R}^d))^{s\times r}$ and $g \in (L_2(\mathbb{R}^d))^{t\times r}$. For $j \in \{1,\dots,s\}$ and $l \in \{1,\dots,t\}$, the (j,l)-th entry of $[\widehat{f}, \widehat{g}]_U$ satisfies

$$\left|\Big([\widehat{f}, \widehat{g}]_U(\xi)\Big)_{j,l}\right| \leqslant \sum_{k=1}^{r} [\widehat{f}_{j,k}, \widehat{f}_{j,k}]_U [\widehat{g}_{k,l}, \widehat{g}_{k,l}]_U < \infty, \qquad \xi \in \mathbb{R}^d. \tag{5.3.15}$$

It follows from (5.3.15) that

$$\int_{U^{-1}[0,2\pi)^d} \left|\Big([\widehat{f}, \widehat{g}]_U(\xi)\Big)_{j,l}\right| d\xi \leqslant \sum_{k=1}^{r} \|\widehat{f}_{j,k}\|_{L_2(\mathbb{R}^d)} \|\widehat{g}_{k,l}\|_{L_2(\mathbb{R}^d)}. \tag{5.3.16}$$

Thus $[\widehat{f}, \widehat{g}]_U \in (L_1(U^{-1}\mathbb{T}^d))^{s\times t}$, and its k-th Fourier coefficient is given by

$$\begin{aligned}
&|\det(U)|(2\pi)^{-d} \int_{U^{-1}[0,2\pi)^d} \sum_{n\in\mathbb{Z}^d} \widehat{f}(\xi + 2\pi U^{-1}n)\overline{\widehat{g}(\xi + 2\pi U^{-1}n)}^{\mathsf{T}} e^{-ik\cdot U\xi} d\xi \\
=&|\det(U)|(2\pi)^{-d} \int_{\mathbb{R}^d} \widehat{f}(\xi)\overline{\widehat{g}(\xi)e^{ik\cdot U\xi}}^{\mathsf{T}} d\xi \\
=&|\det(U)|(2\pi)^{-d} \langle \widehat{f}, \widehat{g}_{I_d;0,-U^{\mathsf{T}}k}\rangle \\
=&|\det(U)|\langle f, g(\cdot + U^{\mathsf{T}}k)\rangle.
\end{aligned}$$

□

The following lemma is a generalization of [41, Lemma 4.1.1] to high dimensions and arbitrary dilation factors. The result can be proved by following the lines of the proof of [41, Lemma 4.1.1].

Lemma 5.3.6. *Let U be an invertible $d\times d$ real matric, and $f, g : \mathbb{R}^d \to \mathbb{C}, h, \tilde{h} : \mathbb{R}^d \to \mathbb{C}^r$*

be such that

$$\int_{\mathbb{R}^d}\sum_{k\in\mathbb{Z}^d}\|f(x)\overline{h(Ux)}^{\mathsf{T}}\|\|f(x+2\pi U^{-1}k)\overline{h(Ux+2\pi k)}^{\mathsf{T}}\|dx<\infty \tag{5.3.17}$$

and

$$\int_{\mathbb{R}^d}\sum_{k\in\mathbb{Z}^d}\|g(x)\overline{\tilde{h}(Ux)}^{\mathsf{T}}\|\|g(x+2\pi U^{-1}k)\overline{\tilde{h}(Ux+2\pi k)}^{\mathsf{T}}\|dx<\infty. \tag{5.3.18}$$

Then

$$\sum_{k\in\mathbb{Z}^d}\langle f,h_{U;0,k}\rangle\langle\tilde{h}_{U;0,k},g\rangle=(2\pi)^d\int_{\mathbb{R}^d}\sum_{k\in\mathbb{Z}^d}f(x)\overline{h(Ux)}^{\mathsf{T}}\tilde{h}(Ux+2\pi k)\overline{g(x+2\pi U^{-1}k)}dx. \tag{5.3.19}$$

Proof. Note that

$$\begin{aligned}\int_{U^{-1}[-\pi,\pi)^d}\|[f,h(U\cdot)]_U(x)\|^2dx&=\int_{\mathbb{R}^d}f(x)\overline{h(Ux)}^{\mathsf{T}}\overline{[f,h(U\cdot)]_U(x)}^{\mathsf{T}}dx\\ \leqslant\int_{\mathbb{R}^d}\sum_{k\in\mathbb{Z}^d}\|f(x)\overline{h(Ux)}^{\mathsf{T}}\|&\|f(x+2\pi U^{-1}k)\overline{h(Ux+2\pi k)}^{\mathsf{T}}\|dx<\infty.\end{aligned} \tag{5.3.20}$$

Thus $[f,h(U\cdot)]_U\in(L_2(U^{-1}\mathbb{T}^d))^{1\times r}$, and by Lemma 5.3.5, its k-th Fourier coefficient is given by:

$$|\det(U)|^{\frac{1}{2}}(2\pi)^{-d}\langle f,h_{U;0,-k}\rangle. \tag{5.3.21}$$

Similarly we have $[g,\tilde{h}(U\cdot)g]_U\in(L_2(U^{-1}\mathbb{T}^d))^{1\times r}$, with its k-th Fourier coefficient equals

$$|\det(U)|^{\frac{1}{2}}(2\pi)^{-d}\langle g,\tilde{h}_{U;0,-k}\rangle. \tag{5.3.22}$$

By Parseval's identity:

$$\begin{aligned}\sum_{k\in\mathbb{Z}^d}\langle f,h_{U;0,k}\rangle\langle\tilde{h}_{U;0,k},g\rangle&=(2\pi)^d\int_{U^{-1}[-\pi,\pi)^d}[f,h(U\cdot)]_U(x)[\tilde{h}(U\cdot),g]_U(x)dx\\ =(2\pi)^d\int_{\mathbb{R}^d}\sum_{k\in\mathbb{Z}^d}f(x)\overline{h(Ux)}^{\mathsf{T}}&\tilde{h}(Ux+2\pi k)\overline{g(x+2\pi U^{-1}k)}dx.\end{aligned} \tag{5.3.23}$$

$\square$

Similar to what we have done in the discrete case, we now introduce the notion of an affine system in $L_2(\mathbb{R}^d)$ to facilitate our study of framelets in function spaces.

Definition 5.3.7. *Let* $\psi^0 = (\psi_1^0, \dots, \psi_r^0)^\mathsf{T} \in (L_2(\mathbb{R}^d))^r, \psi^1, \dots, \psi^s \in L_2(\mathbb{R}^d)$ *and* $\mathsf{M}_0, \dots, \mathsf{M}_s$ *be* $d \times d$ *dilation matrices. Define an affine system via*

$$\begin{aligned}\mathrm{AS}(\{\psi^l ! \mathsf{M}_l\}_{l=0}^s) :=& \{\psi_q^0(\cdot - k) : q = 1, \dots, r; k \in \mathbb{Z}^d\} \\ & \cup \{|\det(\mathsf{M}_0^{-1}\mathsf{M}_l)|^{\frac{1}{2}} \psi^l_{\mathsf{M}_0^j;\mathsf{M}_0^{-1}\mathsf{M}_l k} : l = 1, \dots, s, \quad j \in \mathbb{N}_0, \quad k \in \mathbb{Z}^d\}.\end{aligned}$$

The following result connects discrete affine systems in $l_2(\mathbb{Z}^d)$ with affine systems in $L_2(\mathbb{R}^d)$.

Proposition 5.3.8. *Let* $b_0 \in (l_0(\mathbb{Z}^d))^{r\times r}, b_1, \dots, b_s \in (l_0(\mathbb{Z}^d))^{1\times r}$ *be finitely supported filters and let* $\mathsf{M}_0, \dots, \mathsf{M}_s$ *be* $d \times d$ *dilation matrices. Suppose that* $\psi^0 \in (L_2(\mathbb{R}^d))^r$ *is a refinable vector function associated to* b_0 *satisfying* (5.3.4). *Define* $\psi^1, \dots, \psi^s$ *via* (5.3.5) *for* $l = 1, \dots, s$. *Suppose in addition that* $\psi^0 \in (L_2(\mathbb{R}^d))^r$. *For* $f \in L_2(\mathbb{R}^d)$, *define*

$$w_{l,j}(k) = \langle f, |\det(\mathsf{M}_0^{-1}\mathsf{M}_l)|^{\frac{1}{2}} \psi^l_{\mathsf{M}_0^j;\mathsf{M}_0^{-1}\mathsf{M}_l k}\rangle, \qquad k \in \mathbb{Z}^d, \quad l = 0, \dots, s, \quad j \in \mathbb{N}_0. \quad (5.3.24)$$

Then the following statements hold:

(i) For every $j \in \mathbb{N}_0, l = 0, \dots, s$ *and* $k \in \mathbb{Z}^d$, *we have*

$$w_{l,j}(k) = \mathcal{T}_{b_l \mathsf{M}_l} w_{0,j+1}(k). \quad (5.3.25)$$

(ii) For every $J, j \in \mathbb{N}_0$ *with* $J > j$, $l = 0, \dots, s$ *and* $k \in \mathbb{Z}^d$, *we have*

$$w_{l,J}(k) = \langle w_{0,J}, b_{l,J-j;k}\rangle_{l_2(\mathbb{Z}^d)}, \quad (5.3.26)$$

where $b_{l,j;k}$ *is defined via* (5.2.5).

(iii) If further assume that $\{b_l ! \mathsf{M}_l\}_{l=0}^s$ *has stability in* $l_2(\mathbb{Z}^d)$, *then* $\mathrm{AS}(\{\psi^l ! \mathsf{M}_l\}_{l=0}^s)$ *is a*

Bessel sequence in $L_2(\mathbb{R}^d)$.

Proof. Note that (5.3.5) is equivalent to

$$\psi^l(x) = |\det(\mathsf{M}_l)| \sum_{k\in\mathbb{Z}^d} b_l(k)\psi^0(\mathsf{M}_0 x - k), \qquad x \in \mathbb{R}^d, \quad l = 0, \ldots, s. \tag{5.3.27}$$

Thus we see that $\psi^1, \ldots, \psi^s \in L_2(\mathbb{R}^d)$. By calculation:

$$\begin{aligned}
w_{l,j}(k) &= \int_{\mathbb{R}^d} f(x)|\det(\mathsf{M}_0^{-1}\mathsf{M}_l)|^{\frac{1}{2}} \overline{\psi^l_{\mathsf{M}_0^j;\mathsf{M}_0^{-1}\mathsf{M}_l k}(x)}^{\mathsf{T}} dx \\
&= |\det(\mathsf{M}_0^{j-1}\mathsf{M}_l)|^{\frac{1}{2}} \int_{\mathbb{R}^d} f(x)\overline{\psi^l(\mathsf{M}_0^j x - \mathsf{M}_0^{-1}\mathsf{M}_l k)}^{\mathsf{T}} dx \\
&= \sum_{m\in\mathbb{Z}^d} |\det(\mathsf{M}_0^{j+1}\mathsf{M}_l)|^{\frac{1}{2}} \left(\int_{\mathbb{R}^d} f(x)\overline{\psi^0(\mathsf{M}_0^{j+1}x - \mathsf{M}_l k - m)}^{\mathsf{T}} dx\right) \overline{\widehat{b_l}(m)}^{\mathsf{T}} \\
&= \sum_{m\in\mathbb{Z}^d} |\det(\mathsf{M}_l)|^{\frac{1}{2}} \langle f, \psi^0_{\mathsf{M}_0^{j+1};\mathsf{M}_l k+m}\rangle \overline{\widehat{b_l}(m)}^{\mathsf{T}} \\
&= \sum_{m\in\mathbb{Z}^d} |\det(\mathsf{M}_l)|^{\frac{1}{2}} w_{0,j+1}(\mathsf{M}_l k + m)\overline{\widehat{b_l}(m)}^{\mathsf{T}} \\
&= \mathcal{T}_{b_l,\mathsf{M}_l} w_{0,j+1}(k),
\end{aligned}$$

for all $k \in \mathbb{Z}^d$, $j \in \mathbb{N}_0$ and $l = 0, \ldots, s$. This proves item (i)

Next, define $b_{l,j}$ via (5.2.1) for all $j \in \mathbb{N}$ and $l = 0, \ldots, s$. Use item (i), we have

$$\begin{aligned}
w_{l,j}(k) &= \mathcal{T}_{b_l,\mathsf{M}_l}\mathcal{T}^{J-j-1}_{b_0,\mathsf{M}_0} w_{0,J}(k) \\
&= \mathcal{T}_{(b_l\uparrow\mathsf{M}_0^{J-j-1})*(b_0\uparrow\mathsf{M}_0^{J-j-2})*\cdots*(b_0\uparrow\mathsf{M}_0)*b_0,\mathsf{M}_0^{J-j-1}\mathsf{M}_l} w_{0,J}(k) \\
&= \mathcal{T}_{b_{l,J-j},\mathsf{M}_0^{J-j-1}\mathsf{M}_l} w_{0,J}(k) \\
&= |\det(\mathsf{M}_l)|^{\frac{1}{2}} |\det(\mathsf{M}_0)|^{\frac{J-j-1}{2}} \sum_{m\in\mathbb{Z}^d} w_{0,J}(m)\overline{b_{l,J-j}(m - \mathsf{M}_0^{J-j-1}\mathsf{M}_l k)}^{\mathsf{T}} \\
&= \langle w_{0,J}, b_{l,J-j;k}\rangle_{l_2(\mathbb{Z}^d)},
\end{aligned}$$

whenever $J, j \in \mathbb{N}_0$ and $J > j$. This proves item (ii).

Finally, we prove item (iii). By items (i) and (ii), we have

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}^d}\|\langle f,\psi^0_{I_d;k}\rangle\|^2+\sum_{j=0}^{J-1}\sum_{k\in\mathbb{Z}^d}|\langle f,|\det(\mathsf{M}_0^{-1}\mathsf{M}_l)|^{\frac{1}{2}}\psi^l_{\mathsf{M}_0^j;\mathsf{M}_0^{-1}\mathsf{M}_lk}\rangle|^2\\
&=\sum_{k\in\mathbb{Z}^d}\|w_{0,0}(k)\|^2+\sum_{j=0}^{J-1}\sum_{k\in\mathbb{Z}^d}|w_{l,j}|^2\\
&=\sum_{k\in\mathbb{Z}^d}\|\langle w_{0,J},b_{0,J;k}\rangle_{l_2(\mathbb{Z}^d))}\|^2+\sum_{j=0}^{J-1}\sum_{k\in\mathbb{Z}^d}|\langle w_{0,J},b_{l,J-j;k}\rangle_{l_2(\mathbb{Z}^d)}|^2,
\end{aligned}\tag{5.3.28}$$

for all $J\in\mathbb{N}$. Since $\{b_l!\mathsf{M}_l\}_{l=0}^s$ has stability in $l_2(\mathbb{Z}^d)$, Lemma 5.2.3 yields that there exists $C>0$ such that

$$\sum_{k\in\mathbb{Z}^d}\|\langle w_{0,J},b_{0,J;k}\rangle_{l_2(\mathbb{Z}^d))}\|^2+\sum_{j=0}^{J-1}\sum_{k\in\mathbb{Z}^d}|\langle w_{0,J},b_{l,J-j;k}\rangle_{l_2(\mathbb{Z}^d)}|^2\leqslant C\|w_{0,J}\|^2_{l_2(\mathbb{Z}^d)},\tag{5.3.29}$$

for all $J\in\mathbb{N}$. Moreover, by Proposition 5.3.3, there exists $K>0$ such that $[\widehat{\psi^0}^{\mathsf{T}},\widehat{\psi^0}^{\mathsf{T}}](\xi)\leqslant K$ for a.e. $\xi\in\mathbb{R}^d$, i.e., $[\widehat{\psi^0}^{\mathsf{T}},\widehat{\psi^0}^{\mathsf{T}}]\in L_\infty(\mathbb{T}^d)$. Thus $[\widehat{f},\widehat{\psi^0}]\in L_2(\mathbb{T}^d)$ for all $f\in L_2(\mathbb{R}^d)$. Now use Lemma 5.3.5 and Parseval's identity, we have

$$\begin{aligned}
\|w_{0,J}\|^2_{l_2(\mathbb{Z}^d)}&=\sum_{k\in\mathbb{Z}^d}\|\langle f,\psi^0_{\mathsf{M}_0^J;k}\rangle\|^2=\sum_{k\in\mathbb{Z}^d}\|\langle f_{\mathsf{M}_0^{-J};0},\psi^0(\cdot-k)\rangle\|^2\\
&=(2\pi)^{-d}\int_{[0,2\pi)^d}|[\widehat{f_{(\mathsf{M}_0^{\mathsf{T}})^J;0}},\widehat{\psi^0}](\xi)|^2d\xi\\
&\leqslant(2\pi)^{-d}\int_{[0,2\pi)^d}[\widehat{f_{(\mathsf{M}_0^{\mathsf{T}})^J;0}},\widehat{f_{(\mathsf{M}_0^{\mathsf{T}})^J;0}}](\xi)[\widehat{\psi^0}^{\mathsf{T}},\widehat{\psi^0}^{\mathsf{T}}](\xi)d\xi\\
&\leqslant(2\pi)^{-d}K\int_{[0,2\pi)^d}[\widehat{f_{(\mathsf{M}_0^{\mathsf{T}})^J;0}},\widehat{f_{(\mathsf{M}_0^{\mathsf{T}})^J;0}}](\xi)d\xi\\
&=(2\pi)^{-d}K\|\widehat{f_{(\mathsf{M}_0^{\mathsf{T}})^J;0}}\|^2_{L_2(\mathbb{R}^d)}=K\|f\|^2_{L_2(\mathbb{R}^d)}.
\end{aligned}\tag{5.3.30}$$

It follows from (5.3.28), (5.3.29) and (5.3.30) that

$$\sum_{k\in\mathbb{Z}^d}\|\langle f,\psi^0_{I_d;k}\rangle\|^2+\sum_{j=0}^{J-1}\sum_{k\in\mathbb{Z}^d}|\langle f,|\det(\mathsf{M}_0^{-1}\mathsf{M}_l)|^{\frac{1}{2}}\psi^l_{\mathsf{M}_0^j;\mathsf{M}_0^{-1}\mathsf{M}_lk}\rangle|^2\leqslant CK\|f\|^2_{L_2(\mathbb{R}^d)}\tag{5.3.31}$$

for all $f\in L_2(\mathbb{R}^d)$ and $J\in\mathbb{N}$. By letting $J\to\infty$, we see that $\mathrm{AS}(\{\phi^l!\mathsf{M}_l\}_{l=0}^s)$ is a Bessel sequence in $L_2(\mathbb{R}^d)$. This proves item (iii). $\square$

We are now ready to state the main theorem which connects discrete framelet filter banks and frameltes in $l_2(\mathbb{Z}^d)$.

Theorem 5.3.9. *Let $b_0, \tilde{b}_0 \in (l_0(\mathbb{Z}^d))^{r\times r}, b_1, \ldots, b_s, \tilde{b}_1, \ldots \tilde{b}_s \in (l_0(\mathbb{Z}^d))^{1\times r}$ be finitely supported filters and let $\mathsf{M}_0, \ldots, \mathsf{M}_s$ be $d \times d$ dilation matrices. Suppose $\psi^0, \tilde{\psi}^0 \in (L_2(\mathbb{R}^d))^r$ are compactly supported standard refinable vector functions satisfying*

$$\widehat{\psi^0}(\mathsf{M}_0^{\mathsf{T}}\xi) = \widehat{b_0}(\xi)\widehat{\psi^0}(\xi), \qquad \widehat{\tilde{\psi}^0}(\mathsf{M}_0^{\mathsf{T}}\xi) = \widehat{\tilde{b}_0}(\xi)\widehat{\tilde{\psi}^0}(\xi), \qquad \xi \in \mathbb{R}^d, \tag{5.3.32}$$

and $\overline{\widehat{\psi^0}(0)}^{\mathsf{T}}\widehat{\tilde{\psi}^0}(0) = 1$. Define $\psi^1, \ldots, \psi^s, \tilde{\psi}^1, \ldots, \tilde{\psi}^s \in L_2(\mathbb{R}^d)$ via

$$\widehat{\psi^l}(\xi) = \widehat{b_l}(\mathsf{M}_0^{-\mathsf{T}}\xi)\widehat{\psi^0}(\mathsf{M}_0^{-\mathsf{T}}\xi), \qquad \widehat{\tilde{\psi}^l}(\xi) = \widehat{\tilde{b}_l}(\mathsf{M}_0^{-\mathsf{T}}\xi)\widehat{\tilde{\psi}^0}(\mathsf{M}_0^{-\mathsf{T}}\xi), \qquad \xi \in \mathbb{R}^d. \tag{5.3.33}$$

Then

(i) $(\{b_l!\mathsf{M}_l\}_{l=0}^s, \{\tilde{b}_l!\mathsf{M}_l\}_{l=0}^s)$ is a dual framelet filter bank with mixed dilation factors. Furthermore, both $\{b_l!\mathsf{M}_l\}_{l=0}^s$ and $\{\tilde{b}_l!\mathsf{M}_l\}_{l=0}^s$ have stability in $l_2(\mathbb{Z}^d)$,

implies

(ii) $(\{\psi^l!\mathsf{M}_l\}_{l=0}^s, \{\tilde{\psi}^l!\mathsf{M}_l\}_{l=0}^s)$ is a dual framelet in $L_2(\mathbb{R}^d)$. Furthermore, there exists $C > 0$ such that

$$[\widehat{\psi^0}, \widehat{\psi^0}](\xi) + [\widehat{\tilde{\psi}^0}, \widehat{\tilde{\psi}^0}](\xi) \leqslant C I_r, \qquad a.e.\ \xi \in \mathbb{R}^d. \tag{5.3.34}$$

Conversely, if in addition

$$[\widehat{\psi^0}, \widehat{\psi^0}](\xi) \geqslant C' I_r, \qquad [\widehat{\tilde{\psi}^0}, \widehat{\tilde{\psi}^0}](\xi) \geqslant C' I_r, \qquad a.e.\ \xi \in \mathbb{R}^d, \tag{5.3.35}$$

for some constant $C' > 0$, then item (ii) implies item (i).

Proof. Define

$$D := \{f \in L_2(\mathbb{R}^d) : \widehat{f} \in \mathscr{D}(\mathbb{R}^d)\}, \tag{5.3.36}$$

where $\mathscr{D}(\mathbb{R}^d)$ denotes the space of compactly supported C^∞-functions on $\mathbb{R}^d$. By Lemma 5.3.6, for every $f, g \in D$ and $j \in \mathbb{N}_0$, we have

$$\begin{aligned}&\sum_{k\in\mathbb{Z}^d} \langle \widehat{f}, \widehat{\psi^0}_{((\mathsf{M}_0)^{-\mathsf{T}})^j;0,k}\rangle \langle \widehat{\tilde{\psi}^0}_{((\mathsf{M}_0)^{-\mathsf{T}})^j;0,k}, \widehat{g}\rangle \\ =&(2\pi)^d \int_{\mathbb{R}^d} \sum_{k\in\mathbb{Z}^d} \widehat{f}(\xi)\overline{\widehat{\psi^0}((\mathsf{M}_0^{-\mathsf{T}})^j\xi)}^{\mathsf{T}} \widehat{\tilde{\psi}^0}((\mathsf{M}_0^{-\mathsf{T}})^j\xi + 2\pi k)\overline{\widehat{g}(\xi + 2\pi(\mathsf{M}_0^{\mathsf{T}})^j k)} d\xi.\end{aligned} \tag{5.3.37}$$

Choose j sufficiently large such that $\widehat{f}(\xi)\hat{g}(\xi + 2\pi(\mathsf{M}_0^{\mathsf{T}})^j k) = 0$ for all $\xi \in \mathbb{R}^d$ and $k \in \mathbb{Z}^d \setminus \{0\}$, we have

$$\sum_{k\in\mathbb{Z}^d} \langle \widehat{f}, \widehat{\psi^0}_{((\mathsf{M}_0)^{-\mathsf{T}})^j;0,k}\rangle \langle \widehat{\tilde{\psi}^0}_{((\mathsf{M}_0)^{-\mathsf{T}})^j;0,k}, \widehat{g}\rangle = (2\pi)^d \int_{\mathbb{R}^d} \widehat{f}(\xi)\overline{\widehat{\psi^0}((\mathsf{M}_0^{-\mathsf{T}})^j\xi)}^{\mathsf{T}} \widehat{\tilde{\psi}^0}((\mathsf{M}_0^{-\mathsf{T}})^j\xi)\overline{\widehat{g}(\xi)} d\xi. \tag{5.3.38}$$

By Lemma 5.3.1, $\widehat{\psi^0}$ and $\widehat{\tilde{\psi}^0}$ are vectors of locally bounded measurable functions and

$$\lim_{j\to\infty} \overline{\widehat{\psi^0}((\mathsf{M}_0^{-\mathsf{T}})^j\xi)}^{\mathsf{T}} \widehat{\tilde{\psi}^0}((\mathsf{M}_0^{-\mathsf{T}})^j\xi) = \overline{\widehat{\psi^0}(0)}^{\mathsf{T}} \widehat{\tilde{\psi}^0}(0) = 1, \qquad \xi \in \mathbb{R}^d. \tag{5.3.39}$$

It follows from the Dominated convergence theorem that

$$\lim_{j\to\infty} \sum_{k\in\mathbb{Z}^d} \langle \hat{f}, \widehat{\psi^0}_{((\mathsf{M}_0)^{-\mathsf{T}})^j;0,k}\rangle \langle \widehat{\tilde{\psi}^0}_{((\mathsf{M}_0)^{-\mathsf{T}})^j;0,k}, \widehat{g}\rangle = (2\pi)^d \langle \widehat{f}, \widehat{g}\rangle. \tag{5.3.40}$$

Define

$$\eta^l := \psi^l(\mathsf{M}_0^{-1}\mathsf{M}_l\cdot), \qquad \tilde{\eta}^l = \tilde{\psi}^l(\mathsf{M}_0^{-1}\mathsf{M}_l\cdot), \qquad l = 1, \ldots, s. \tag{5.3.41}$$

We have

$$\psi^l(\cdot - \mathsf{M}_0^{-1}\mathsf{M}_l k) = \eta^l(\mathsf{M}_l^{-1}\mathsf{M}_0 \cdot - k), \qquad \widehat{\eta^l} = |\det(\mathsf{M}_0^{-1}\mathsf{M}_l)|^{-1}\widehat{\psi^l}(\mathsf{M}_0^{\mathsf{T}}\mathsf{M}_l^{-\mathsf{T}}\cdot), \tag{5.3.42}$$

$$\tilde{\psi}^l(\cdot - \mathsf{M}_0^{-1}\mathsf{M}_l k) = \tilde{\eta}^l(\mathsf{M}_l^{-1}\mathsf{M}_0 \cdot - k), \qquad \widehat{\tilde{\eta}^l} = |\det(\mathsf{M}_0^{-1}\mathsf{M}_l)|^{-1}\widehat{\tilde{\psi}^l}(\mathsf{M}_0^{\mathsf{T}}\mathsf{M}_l^{-\mathsf{T}}\cdot), \tag{5.3.43}$$

for all $l = 1, \ldots, s$. For $f, g \in D$, Lemma 5.3.6 yields

$$\begin{aligned}
&|\det(\mathsf{M}_0^{-1}\mathsf{M}_l)|^2 \sum_{k\in\mathbb{Z}^d} \langle \widehat{f}, \widehat{\eta^l}_{\mathsf{M}_l^{\mathsf{T}}\mathsf{M}_0^{-\mathsf{T}};0,k}\rangle \langle \widehat{\tilde{\eta}^l}_{\mathsf{M}_l^{\mathsf{T}}\mathsf{M}_0^{-\mathsf{T}};0,k}, \widehat{g}\rangle \\
=&(2\pi)^d |\det(\mathsf{M}_0^{-1}\mathsf{M}_l)|^2 \int_{\mathbb{R}^d} \sum_{k\in\mathbb{Z}^d} \widehat{f}(\xi)\overline{\widehat{\eta^l}(\mathsf{M}_l^{\mathsf{T}}\mathsf{M}_0^{-\mathsf{T}}\xi)}^{\mathsf{T}} \widehat{\tilde{\eta}^l}(\mathsf{M}_l^{\mathsf{T}}\mathsf{M}_0^{-\mathsf{T}}\xi + 2\pi k)\overline{\widehat{g}(\xi + 2\pi(\mathsf{M}_0^{\mathsf{T}}\mathsf{M}_l^{-\mathsf{T}})k)}d\xi. \\
=&(2\pi)^d \int_{\mathbb{R}^d} \sum_{k\in\mathbb{Z}^d} \widehat{f}(\xi)\overline{\widehat{\psi^l}(\xi)}^{\mathsf{T}} \widehat{\tilde{\psi}^l}(\xi + 2\pi(\mathsf{M}_0^{\mathsf{T}}\mathsf{M}_l^{-\mathsf{T}})k)\overline{\widehat{g}(\xi + 2\pi(\mathsf{M}_0^{\mathsf{T}}\mathsf{M}_l^{-\mathsf{T}})k)}d\xi. \\
=&(2\pi)^d \int_{\mathbb{R}^d} \sum_{k\in\mathbb{Z}^d} \widehat{f}(\xi)\overline{\widehat{b_l}(\mathsf{M}_0^{-\mathsf{T}}\xi)\widehat{\psi^0}(\mathsf{M}_0^{-\mathsf{T}}\xi)}^{\mathsf{T}} \widehat{\tilde{b}_l}(\mathsf{M}_0^{-\mathsf{T}}\xi + 2\pi(\mathsf{M}_l^{-\mathsf{T}})k)\widehat{\tilde{\psi}^0}(\mathsf{M}_0^{-\mathsf{T}}\xi + 2\pi(\mathsf{M}_l^{-\mathsf{T}})k) \\
&\times \overline{\widehat{g}(\xi + 2\pi(\mathsf{M}_0^{\mathsf{T}}\mathsf{M}_l^{-\mathsf{T}})k)}d\xi. \\
=&(2\pi)^d \int_{\mathbb{R}^d} \widehat{f}(\xi)\overline{\widehat{\psi^0}(\mathsf{M}_0^{-\mathsf{T}}\xi)}^{\mathsf{T}} \left(\sum_{\omega\in\Omega_l} \overline{\widehat{b_l}(\mathsf{M}_0^{-\mathsf{T}}\xi)}\widehat{\tilde{b}_l}(\mathsf{M}_0^{-\mathsf{T}}\xi + 2\pi\omega)\right) \\
&\times \sum_{k\in\mathbb{Z}^d} \widehat{\tilde{\psi}^0}(\mathsf{M}_0^{-\mathsf{T}}\xi + 2\pi\omega + 2\pi k)\overline{\widehat{g}(\xi + 2\pi\mathsf{M}_0\omega + 2\pi\mathsf{M}_0 k)}d\xi.
\end{aligned}$$

Similarly:

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}^d} \langle \widehat{f}, \widehat{\psi^0}_{I_d;0,k}\rangle \langle \widehat{\tilde{\psi}^0}_{I_d;0,k}, \widehat{g}\rangle \\
=&(2\pi)^d \int_{\mathbb{R}^d} \widehat{f}(\xi)\overline{\widehat{\psi^0}(\mathsf{M}_0^{-\mathsf{T}}\xi)}^{\mathsf{T}} \left(\sum_{\omega\in\Omega_0} \overline{\widehat{b_0}(\mathsf{M}_0^{-\mathsf{T}}\xi)}\widehat{\tilde{b}_0}(\mathsf{M}_0^{-\mathsf{T}}\xi + 2\pi\omega)\right) \\
&\times \sum_{k\in\mathbb{Z}^d} \widehat{\tilde{\psi}^0}(\mathsf{M}_0^{-\mathsf{T}}\xi + 2\pi\omega + 2\pi k)\overline{\widehat{g}(\xi + 2\pi\mathsf{M}_0\omega + 2\pi\mathsf{M}_0 k)}d\xi.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}^d} \langle \widehat{f}, \widehat{\psi^0}_{I_d;0,k}\rangle \langle \widehat{\tilde{\psi}^0}_{I_d;0,k}, \widehat{g}\rangle + \sum_{l=1}^{s} |\det(\mathsf{M}_0^{-1}\mathsf{M}_l)|^2 \sum_{k\in\mathbb{Z}^d} \langle \widehat{f}, \widehat{\eta^l}_{\mathsf{M}_l^{\mathsf{T}}\mathsf{M}_0^{-\mathsf{T}};0,k}\rangle \langle \widehat{\tilde{\eta}^l}_{\mathsf{M}_l^{\mathsf{T}}\mathsf{M}_0^{-\mathsf{T}};0,k}, \widehat{g}\rangle \\
=&(2\pi)^d \int_{\mathbb{R}^d} \widehat{f}(\xi)\overline{\widehat{\psi^0}(\mathsf{M}_0^{-\mathsf{T}}\xi)}^{\mathsf{T}} \left(\sum_{\omega\in\Omega} \sum_{l=0}^{s} \chi_{\Omega_l}(\omega)\overline{\widehat{b_l}(\mathsf{M}_0^{-\mathsf{T}}\xi)}\widehat{\tilde{b}_l}(\mathsf{M}_0^{\mathsf{T}}\xi + 2\pi\omega)\right) \\
&\times \sum_{k\in\mathbb{Z}^d} \widehat{\tilde{\psi}^0}(\mathsf{M}_0^{-\mathsf{T}}\xi + 2\pi\omega + 2\pi k)\overline{\widehat{g}(\xi + 2\pi\mathsf{M}_0\omega + 2\pi\mathsf{M}_0 k)}d\xi
\end{aligned}$$

for all $f, g \in D$.

Suppose item (i) holds. By (5.1.12) and (5.3.37), we have

$$\sum_{k\in\mathbb{Z}^d}\langle \widehat{f}, \widehat{\psi^0}_{I_d;0,k}\rangle\langle \widehat{\tilde{\psi}^0}_{I_d;0,k}, \widehat{g}\rangle + \sum_{l=1}^{s}|\det(\mathsf{M}_0^{-1}\mathsf{M}_l)|^2\sum_{k\in\mathbb{Z}^d}\langle \widehat{f}, \widehat{\eta}^l{}_{\mathsf{M}_l^{\mathsf{T}}\mathsf{M}_0^{-\mathsf{T}};0,k}\rangle\langle \widehat{\tilde{\eta}}^l{}_{\mathsf{M}_l^{\mathsf{T}}\mathsf{M}_0^{-\mathsf{T}};0,k}, \widehat{g}\rangle$$
$$=(2\pi)^d\int_{\mathbb{R}^d}\widehat{f}(\xi)\overline{\widehat{\psi^0}(\mathsf{M}_0^{-\mathsf{T}}\xi)}^{\mathsf{T}}\sum_{k\in\mathbb{Z}^d}\widehat{\tilde{\psi}^0}(\mathsf{M}_0^{-\mathsf{T}}\xi+2\pi k)\overline{\widehat{g}(\xi+2\pi\mathsf{M}_0 k)}d\xi$$
$$=\sum_{k\in\mathbb{Z}^d}\langle \widehat{f}, \widehat{\psi^0}_{\mathsf{M}_0^{-\mathsf{T}};0,k}\rangle\langle \widehat{\tilde{\psi}^0}_{\mathsf{M}_0^{-\mathsf{T}};0,k}, \widehat{g}\rangle,$$

for all $f, g \in D$. Note that

$$\langle \widehat{f}_{U^{-1};0}, \widehat{g}\rangle = \langle \widehat{f}, \widehat{g}_{U;0}\rangle, \qquad f, g \in L_2(\mathbb{R}^d). \tag{5.3.44}$$

It follows that

$$\sum_{k\in\mathbb{Z}^d}\langle \widehat{f}, \widehat{\psi^0}_{(\mathsf{M}_0^{-\mathsf{T}})^j;0,k}\rangle\langle \widehat{\tilde{\psi}^0}_{(\mathsf{M}_0^{-\mathsf{T}})^j;0,k}, \widehat{g}\rangle + \sum_{l=1}^{s}|\det(\mathsf{M}_0^{-1}\mathsf{M}_l)|^2\sum_{k\in\mathbb{Z}^d}\langle \widehat{f}, \widehat{\eta}^l{}_{\mathsf{M}_l^{\mathsf{T}}(\mathsf{M}_0^{-\mathsf{T}})^j;0,k}\rangle\langle \widehat{\tilde{\eta}}^l{}_{\mathsf{M}_l^{\mathsf{T}}(\mathsf{M}_0^{-\mathsf{T}})^j;0,k}, \widehat{g}\rangle$$
$$=\sum_{k\in\mathbb{Z}^d}\langle \widehat{f}, \widehat{\psi^0}_{(\mathsf{M}_0^{-\mathsf{T}})^{j+1};0,k}\rangle\langle \widehat{\tilde{\psi}^0}_{(\mathsf{M}_0^{-\mathsf{T}})^{j+1};0,k}, \widehat{g}\rangle,$$

for all $f, g \in D$ and $j \in \mathbb{N}_0$. So for $m, n \in \mathbb{N}_0$ with $m > n$, we have

$$\sum_{k\in\mathbb{Z}^d}\langle \widehat{f}, \widehat{\psi^0}_{(\mathsf{M}_0^{-\mathsf{T}})^n;0,k}\rangle\langle \widehat{\tilde{\psi}^0}_{(\mathsf{M}_0^{-\mathsf{T}})^n;0,k}, \widehat{g}\rangle + \sum_{j=n}^{m}\sum_{l=1}^{s}|\det(\mathsf{M}_0^{-1}\mathsf{M}_l)|^2\sum_{k\in\mathbb{Z}^d}\langle \widehat{f}, \widehat{\eta}^l{}_{\mathsf{M}_l^{\mathsf{T}}(\mathsf{M}_0^{-\mathsf{T}})^j;0,k}\rangle\langle \widehat{\tilde{\eta}}^l{}_{\mathsf{M}_l^{\mathsf{T}}(\mathsf{M}_0^{-\mathsf{T}})^j;0,k}, \widehat{g}\rangle$$
$$=\sum_{k\in\mathbb{Z}^d}\langle \widehat{f}, \widehat{\psi^0}_{(\mathsf{M}_0^{-\mathsf{T}})^{m+1};0,k}\rangle\langle \widehat{\tilde{\psi}^0}_{(\mathsf{M}_0^{-\mathsf{T}})^{m+1};0,k}, \widehat{g}\rangle.$$

By letting $m \to \infty$, (5.3.40) yields

$$\sum_{k\in\mathbb{Z}^d}\langle \widehat{f}, \widehat{\psi^0}_{(\mathsf{M}_0^{-\mathsf{T}})^n;0,k}\rangle\langle \widehat{\tilde{\psi}^0}_{(\mathsf{M}_0^{-\mathsf{T}})^n;0,k}, \widehat{g}\rangle + \sum_{j=n}^{\infty}\sum_{l=1}^{s}|\det(\mathsf{M}_0^{-1}\mathsf{M}_l)|^2\sum_{k\in\mathbb{Z}^d}\langle \widehat{f}, \widehat{\eta}^l{}_{\mathsf{M}_l^{\mathsf{T}}(\mathsf{M}_0^{-\mathsf{T}})^j;0,k}\rangle\langle \widehat{\tilde{\eta}}^l{}_{\mathsf{M}_l^{\mathsf{T}}(\mathsf{M}_0^{-\mathsf{T}})^j;0,k}, \widehat{g}\rangle$$
$$=(2\pi)^d\langle \widehat{f}, \widehat{g}\rangle,$$

for all $f, g \in D$ and $n \in \mathbb{N}_0$. By Plancherel's theorem and letting $n = 0$ in the above identity, we see that (1.4.1) holds for all $f, g \in D$. Since D is dense in $L_2(\mathbb{R}^d)$, we conclude that (1.4.1) holds for all $f, g \in L_2(\mathbb{R}^d)$. Next, by item (iii) of Proposition 5.3.8, the stability of $\{b_l!\mathsf{M}_l\}_{l=0}^s$ and $\{\tilde{b}_l!\mathsf{M}_l\}_{l=0}^s$ implies that both $\mathrm{AS}(\{\psi^l!\mathsf{M}_l\}_{l=0}^s)$ and $\mathrm{AS}(\{\tilde{\psi}^l!\mathsf{M}_l\}_{l=0}^s)$ are Bessel sequences. Thus by theory of Hilbert spaces (see e.g. [41, Theorem 4.2.5]), $(\mathrm{AS}(\{\tilde{\psi}^l!\mathsf{M}_l\}_{l=0}^s), \mathrm{AS}(\{\psi^l!\mathsf{M}_l\}_{l=0}^s))$ is a pair of dual frames for $L_2(\mathbb{R}^d)$. Finally, (5.3.34) follows rightaway from item (iii) of Proposition 5.3.3. This proves the direction (i) $\Rightarrow$ (ii).

Conversely, suppose that $(\{\psi^l!\mathsf{M}_l\}_{l=0}^s, \{\tilde{\psi}^l!\mathsf{M}_l\}_{l=0}^s)$ is a dual framelet in $L_2(\mathbb{R}^d)$. It follows from (5.3.37) that

$$\begin{aligned} 0 = \int_{\mathbb{R}^d} & \widehat{f}(\xi)\overline{\widehat{\psi^0}(\mathsf{M}_0^{-\mathsf{T}}\xi)}^{\mathsf{T}} \left(\sum_{\omega\in\Omega}\sum_{l=0}^{s} \chi_{\Omega_l}(\omega)\overline{\widehat{b_l}(\mathsf{M}_0^{-\mathsf{T}}\xi)}\widehat{\tilde{b}_l}(\mathsf{M}_0^{-\mathsf{T}}\xi + 2\pi\omega) - \boldsymbol{\delta}(\omega)I_r\right) \\ &\times \sum_{k\in\mathbb{Z}^d} \widehat{\tilde{\psi}^0}(\mathsf{M}_0^{-\mathsf{T}}\xi + 2\pi\omega + 2\pi k)\overline{\widehat{g}(\xi + 2\pi\mathsf{M}_0\omega + 2\pi\mathsf{M}_0 k)}d\xi, \end{aligned} \tag{5.3.45}$$

for all $f, g \in L_2(\mathbb{R}^d)$. By using the similar argument as in the proof of (ii) $\Rightarrow$ (iii) in Theorem 5.1.3 (or see the proof of [34, Lemma 5]), one can conclude that

$$\overline{\widehat{\psi^0}(\xi)}^{\mathsf{T}} \left(\sum_{l=0}^{s} \chi_{\Omega_l}(\omega)\overline{\widehat{b_l}(\xi)}\widehat{\tilde{b}_l}(\xi + 2\pi\omega) - \boldsymbol{\delta}(\omega)I_r\right) \widehat{\tilde{\psi}^0}(\xi + 2\pi\omega + 2\pi k) = 0 \tag{5.3.46}$$

for a.e. $\xi \in \mathbb{R}^d$. Note that the condition (5.3.35) implies that

$$\operatorname{span}\left\{\widehat{\psi^0}(\xi + 2\pi k) : k \in \mathbb{Z}^d\right\} = \mathbb{C}^r, \quad \operatorname{span}\left\{\widehat{\tilde{\psi}^0}(\xi + 2\pi k) : k \in \mathbb{Z}^d\right\} = \mathbb{C}^r, \tag{5.3.47}$$

for a.e. $\xi \in \mathbb{R}^d$. Thus (5.3.46) yields (5.1.12), and we conclude that $(\{b_l!\mathsf{M}_l\}_{l=0}^s, \{\tilde{b}_l!\mathsf{M}_l\}_{l=0}^s)$ is a dual framelet filter bank with mixed dilation factors. It remains to prove that both $\{b_l!\mathsf{M}_l\}_{l=0}^s$ and $\{\tilde{b}_l!\mathsf{M}_l\}_{l=0}^s$ have stability in $l_2(\mathbb{Z}^d)$ under the additional assumptions that (5.3.34) and (5.3.35) hold. Define the *shift invariant space* generated by

$\psi^0 = (\psi_1^0, \dots, \psi_r^0)^{\mathsf{T}}$ via

$$S(\psi^0) := \overline{\mathrm{span}\{\psi_l^0(\cdot - k) : l = 1, \dots, r, \quad k \in \mathbb{Z}^d\}} \subseteq L_2(\mathbb{R}^d). \tag{5.3.48}$$

By (5.3.34), (5.3.35) and [41, Theorem 4.4.12], $\{\psi_l^0(\cdot - k) : l = 1, \dots, r, \quad k \in \mathbb{Z}^d\}$ is a Riesz basis of $S(\psi^0)$. This means that the map

$$\mathcal{W}_{\psi^0} : S(\psi^0) \to (l_2(\mathbb{Z}^d))^{1\times r}, \qquad \mathcal{W}_{\psi^0}(f) = \{\langle f, \psi^0(\cdot - k)\rangle\}_{k\in\mathbb{Z}^d} \tag{5.3.49}$$

is a well-defined bounded isomorphism. Thus for each $v \in (l_2(\mathbb{Z}^d))^{1\times r}$, there exists a unique $h^v \in S(\psi^0)$ such that $\mathcal{W}_{\psi^0} h^v = v$ and that

$$\|v\|^2_{(l_2(\mathbb{Z}^d))^{1\times r}} = \sum_{k\in\mathbb{Z}^d} |\langle h^v, \psi^0(\cdot - k)\rangle|^2 \geqslant C_2 \|h^v\|^2_{L_2(\mathbb{R}^d)}, \tag{5.3.50}$$

for some constant $C_2 > 0$. On the other hand, let $\mathcal{W}_J$ be the J-level discrete framelet analysis operator employing the filter bank $\{b_l!\mathsf{M}_l\}_{l=0}^s$. By (5.2.13), (5.3.28) and the fact that $\langle h^v_{\mathsf{M}_0^J;0}, \psi^0_{\mathsf{M}_0^J;k}\rangle = \langle h^v, \psi^0(\cdot - k)\rangle = v(k)$, we have

$$\begin{aligned}\|\mathcal{W}_J v\|^2_{(l_2(\mathbb{Z}^d))^{1\times(sJ+r)}} =& \sum_{k\in\mathbb{Z}^d} \|\langle h^v_{\mathsf{M}_0^J}, \psi^0(\cdot - k)\rangle\|^2 + \sum_{j=0}^{J-1}\sum_{k\in\mathbb{Z}^d} |\langle h^v_{\mathsf{M}_0^J}, |\det(\mathsf{M}_0^{-1}\mathsf{M}_l)|^{\frac{1}{2}} \psi^l_{\mathsf{M}_0^j;\mathsf{M}_0^{-1}\mathsf{M}_l k}\rangle|^2 \\ \leqslant& D\|h^v\|^2_{L_2(\mathbb{R}^d)},\end{aligned} \tag{5.3.51}$$

for some constant $D > 0$. Hence (5.3.50) and (5.3.51) together yield

$$\|\mathcal{W}_J v\|^2_{(l_2(\mathbb{Z}^d))^{1\times(sJ+r)}} \leqslant DC_2^{-1}\|v\|^2_{(l_2(\mathbb{Z}^d))^r}, \qquad v \in (l_2(\mathbb{Z}^d))^r, J \in \mathbb{N}. \tag{5.3.52}$$

Similarly we can prove that there exists $D' > 0$ such that

$$\|\tilde{\mathcal{W}}_J v\|^2_{(l_2(\mathbb{Z}^d))^{1\times(sJ+r)}} \leqslant D'C_2^{-1}\|v\|^2_{(l_2(\mathbb{Z}^d))^r}, \qquad v \in (l_2(\mathbb{Z}^d))^r, J \in \mathbb{N}. \tag{5.3.53}$$

Thus by item (ii) of Theorem 5.1.6, we see that both $\{b_l!\mathsf{M}_l\}_{l=0}^s)$ and $\{\tilde{b}_l!\mathsf{M}_l\}_{l=0}^s$ have stability in $l_2(\mathbb{Z}^d)$. This completes the proof. $\square$

A similar result which connects discrete wavelet filter banks in $l_2(\mathbb{Z}^d)$ and biorthogonal wavelets in $L_2(\mathbb{R}^d)$ can be established, which is the following theorem.

Theorem 5.3.10. *Let $b_0, \tilde{b}_0 \in (l_0(\mathbb{Z}^d))^{r\times r}, b_1 \dots, b_s, \tilde{b}_1, \dots, \tilde{b}_s \in (l_0(\mathbb{Z}^d))^{1\times r}$ be finitely supported filters and let $\mathsf{M}_0, \dots, \mathsf{M}_s$ be $d\times d$ dilation matrices. Suppose $\psi^0, \tilde{\psi}^0 \in (L_2(\mathbb{R}^d))^r$ are compactly supported standard refinable vector functions satisfying (5.3.32) and $\overline{\widehat{\psi^0}(0)}^{\mathsf{T}}\,\widehat{\tilde{\psi}^0}(0) = 1$. Define $\psi^1, \dots, \psi^s, \tilde{\psi}^1, \dots, \tilde{\psi}^2 \in L_2(\mathbb{R}^d)$ via (5.3.33). Then $(\{\psi^l!\mathsf{M}_l\}_{l=0}^s, \{\tilde{\psi}^l!\mathsf{M}_l\}_{l=0}^s)$ is a biorthogonal wavelet in $L_2(\mathbb{R}^d)$ if the following conditions are satisfied:*

(i) $(\{b_l!\mathsf{M}_l\}_{l=0}^s, \{\tilde{b}_l!\mathsf{M}_l\}_{l=0}^s)$ is a biorthogonal wavelet filter bank with mixed dilation factors.

(ii) $\{b_l!\mathsf{M}_l\}_{l=0}^s$ and $\{\tilde{b}_l!\mathsf{M}_l\}_{l=0}^s$ have stability in $l_2(\mathbb{Z}^d)$.

(iii) The biorthogonality relation

$$\langle \tilde{\psi}^0, \psi^0(\cdot - k)\rangle = \boldsymbol{\delta}(k) I_r \tag{5.3.54}$$

holds for all $k \in \mathbb{Z}^d$.

Conversely, if $(\{\psi^l!\mathsf{M}_l\}_{l=0}^s, \{\tilde{\psi}^l!\mathsf{M}_l\}_{l=0}^s)$ is a biorthogonal wavelet in $L_2(\mathbb{R}^d)$ and assume in addition that (5.3.35) holds for some constant $C' > 0$, then items (i)-(iii) hold.

Proof. By Theorem 5.3.9, items (i) and (ii) imply that $(\{\psi^l!\mathsf{M}_l\}_{l=0}^s, \{\tilde{\psi}^l!\mathsf{M}_l\}_{l=0}^s)$ is a dual framelet in $L_2(\mathbb{R}^d)$. On the other hand, define $b_{l,j}, \tilde{b}_{l,j}$ as in (5.2.3) and (5.2.4) and define $b_{l,j;k}, \tilde{b}_{l,j;k}$ as in (5.2.5) and (5.2.6) for all $l = 0, \dots, s, j \in \mathbb{N}$ and $k \in \mathbb{Z}^d$. It follows from (5.3.33) that

$$\psi^l(x) = |\det(\mathsf{M}_0)|^j \sum_{k\in\mathbb{Z}^d} b_{l,j}(k)\psi^0(\mathsf{M}_0^j x - k), \quad \tilde{\psi}^l(x) = |\det(\mathsf{M}_0)|^j \sum_{k\in\mathbb{Z}^d} \tilde{b}_{l,j}(k)\tilde{\psi}^0(\mathsf{M}_0^j x - k)$$

for a.e. $\xi \in \mathbb{R}^d$ and for all $l = 0, \dots, s$ and $j \in \mathbb{N}$. By calculation, we have

$$
\begin{aligned}
&\langle |\det(\mathsf{M}_0^{-1}\mathsf{M}_l)|^{\frac{1}{2}}\tilde{\psi}^l_{I_d;\mathsf{M}_0^{-1}\mathsf{M}_l m}, |\det(\mathsf{M}_0^{-1}\mathsf{M}_t)|^{\frac{1}{2}}\psi^t_{\mathsf{M}_0^{j-1};\mathsf{M}_0^{-1}\mathsf{M}_t n}\rangle \\
=&|\det(\mathsf{M}_0^{-2}\mathsf{M}_l\mathsf{M}_t)|^{\frac{1}{2}}|\det(\mathsf{M}_0)|^{\frac{j-1}{2}}|\det(\mathsf{M}_0)|^{j+1} \\
&\times \sum_{p,q\in\mathbb{Z}^d} \tilde{b}_{l,j}(p-\mathsf{M}_0^{j-1}\mathsf{M}_l m)\left(\int_{\mathbb{R}^d}\tilde{\psi}^0(x-p)\overline{\psi^0(x-q)}^{\mathsf{T}}dx\right)\overline{b_t(q-\mathsf{M}_t n)}^{\mathsf{T}} \\
=&|\det(\mathsf{M}_0)|^{\frac{j-1}{2}}|\det(\mathsf{M}_l)|^{\frac{1}{2}}|\det(\mathsf{M}_t)|^{\frac{1}{2}} \\
&\times \sum_{p,q\in\mathbb{Z}^d} \tilde{b}_{l,j}(p-\mathsf{M}_0^{j-1}\mathsf{M}_l m)\left(\int_{\mathbb{R}^d}\tilde{\psi}^0(x-p)\overline{\psi^0(x-q)}^{\mathsf{T}}dx\right)\overline{b_t(q-\mathsf{M}_t n)}^{\mathsf{T}} \qquad (5.3.55)\\
=&\sum_{p,q\in\mathbb{Z}^d}\tilde{b}_{l,j;m}(p)\langle\psi^0(\cdot-p),\tilde{\psi}^0(\cdot-q)\rangle_{L_2(\mathbb{R}^d)}\overline{b_{t,1;n}(q)}^{\mathsf{T}} \\
=&\sum_{p,q\in\mathbb{Z}^d}\boldsymbol{\delta}(p-q)\tilde{b}_{l,j;m}(p)\overline{b_{t,1;n}(q)}^{\mathsf{T}} \\
=&\sum_{p\in\mathbb{Z}^d}\tilde{b}_{l,j;m}(p)\overline{b_{t,1;n}(p)}^{\mathsf{T}} \\
=&\langle\tilde{b}_{l,j;m}, b_{t,1;n}\rangle_{l_2(\mathbb{Z}^d)},
\end{aligned}
$$

for all $l, t = 0, \dots, s$, $j \in \mathbb{N}$ and $m, n \in \mathbb{Z}^d$. Similarly, by a simple scaling technique, one can prove that

$$\langle |\det(\mathsf{M}_0^{-1}\mathsf{M}_l)|^{\frac{1}{2}}\psi^l_{\mathsf{M}_0^{j'};\mathsf{M}_0^{-1}\mathsf{M}_l m}, |\det(\mathsf{M}_0^{-1}\mathsf{M}_l)|^{\frac{1}{2}}\tilde{\psi}^t_{\mathsf{M}_0^{j-1};\mathsf{M}_0^{-1}\mathsf{M}_t n}\rangle = \langle b_{l,j;m}, \tilde{b}_{t,j';n}\rangle_{l_2(\mathbb{Z}^d)} \quad (5.3.56)$$

for all $l, t = 0, \dots, s$, $m, n \in \mathbb{Z}^d$ and $j, j' \in \mathbb{N}$. By Theorem 5.2.6, (5.3.55) and (5.3.56), $(\mathrm{AS}(\{\psi^l!\mathsf{M}_l\}_{l=0}^s), \mathrm{AS}(\{\tilde{\psi}^l!\mathsf{M}_l\}_{l=0}^s))$ is a pair of biorthogonal sequences in $L_2(\mathbb{R}^d)$. Hence $(\{\psi^l!\mathsf{M}_l\}_{l=0}^s, \{\tilde{\psi}^l!\mathsf{M}_l\}_{l=0}^s)$ is a biorthogonal wavelet in $L_2(\mathbb{R}^d)$.

Conversely, suppose that $(\{\psi^l!\mathsf{M}_l\}_{l=0}^s, \{\tilde{\psi}^l!\mathsf{M}_l\}_{l=0}^s)$ is a biorthogonal wavelet in $L_2(\mathbb{R}^d)$. Then item (iii) trivially holds. Next, by the frame property of $\mathrm{AS}\{\psi^l!\mathsf{M}_l\}_{l=0}^s$ and $\mathrm{AS}\{\tilde{\psi}^l!\mathsf{M}_l\}_{l=0}^s$, [41, Proposition 4.4.13] yields that (5.3.34) holds for some constant $C > 0$. If in addition that (5.3.35) holds for some $C' > 0$, then by the proof of Theorem 5.3.9, we conclude that $\{b_l!\mathsf{M}_l\}_{l=0}^s$ and $\{\tilde{b}_l!\mathsf{M}_l\}_{l=0}^s$ have stability in $l_2(\mathbb{Z}^d)$. This proves item (ii).

Finally, using (5.3.55), (5.3.56) and the biorthogonality of $(\mathrm{AS}\{\psi^l!\mathsf{M}_l\}_{l=0}^s, \mathrm{AS}\{\tilde{\psi}^l!\mathsf{M}_l\}_{l=0}^s)$, we see that $(\mathrm{DAS}\{b_l!\mathsf{M}_l\}_{l=0}^s, \mathrm{DAS}\{\tilde{b}_l!\mathsf{M}_l\}_{l=0}^s)$ is a pair of biorthogonal sequences. Hence by Theorem 5.2.6, we conclude that $(\{b_l!\mathsf{M}_l\}_{l=0}^s, \{\tilde{b}_l!\mathsf{M}_l\}_{l=0}^s)$ is a biorthogonal wavelet filter bank with mixed dilation factors. This proves item (i). $\square$

5.4 Summary of the Chapter

In this chapter, we introduced the theory of framelets with mixed dilation factors. We first studied properties of a multi-level discrete framelet transform employing framelet filter banks with mixed dilation factors. The notion of a discrete affine system was introduced to further facilitate our study on the frame property of a discrete framelet system. Morevoer, we made connections between discrete framelet filter banks and framelets in $L_2(\mathbb{R}^d)$.

Chapter 6

Summary and Future Work

In this thesis, we studied framelets derived from refinable vector functions with arbitrary multiplicity in arbitrary dimensions. In Chapter 2, we showed that from any univariate compactly supported refinable vector function with multiplicity greater than 2, one can always use the oblique extension principle (OEP) to construct a quasi-tight multiframelet with the highest possible orders of vanishing moments, and its underlying discrete multiframelet transform is compact and balanced. In Chapter 3, we developed the multivariate counterpart of the work of Chapter 2. Motivated by our discussion of OEP-based quasi-tight multiframelets, we proved in Chapter 4 that from any pair of compactly supported multivariate refinable vector functions with at least two entries, a balanced OEP-based dual multiframelet with a compact underlying discrete multiframelet transform can always be obtained. Not only we have proved the possibility of constructing multiframelets from refinable vector functions, but the structure of balanced multiframelets have been also studied. The key ingredient of our investigation is a newly developed normal form of a matrix-valued filter, which greatly benefits the study of OEP-based multiframelets. Finally, in Chapter 5 we established the basic theory of framelets with mixed dilation factors, which is a topic that is of interest in itself.

Some related questions remain open, which could be research tasks in the future.

The most important question to ask is whether it is possible to construct a OEP-based tight multiframelet with the highest possible order of vanishing moments, and the associated discrete framelet transform is balanced and compact. To out best knowledge, the construction of OEP-based tight multiframelets have been only discussed in [58]. However, the tight framelets constructed in [58] either fail to have a compact associated discrete framelet transform or lack high orders of vanishing moments. It is challenging and interesting to study whether we can obtain an OEP-based tight multiframelet with all desired properties being kept.

Next, we comment on the construction process of OEP-based multiframelets. From Chapters 2, 3 and 4, obtaining suitable filters $\theta, \tilde{\theta}$ and factorizing matrices of trigonometric polynomials are required for construction. The construction algorithms we developed are good for theoretical investigation, but may not be ideal in practice. The examples we have in Chapter 2 of univariate quasi-tight multiframelets are already too complicated to present. The situation is getting even harder for the multivariate case. The main reason that causes the difficulty in construction is that, suitable choices of $\theta, \tilde{\theta}$ often make the matrix way too complicated to factorize. How to find a more feasible algorithm for construction could be a future research topic.

On the other hand, we are interested in whether we can get stronger versions of Theorem 2.4.1, Theorem 3.3.1 and Theorem 4.1.1 on OEP-based multiframelets. Say we are given refinable vector functions $\phi, \tilde{\phi} \in (L_2(\mathbb{R}^d))^r$ whose associated refinement filters $a, \tilde{a} \in (l_0(\mathbb{Z}^d))^{r\times r}$ are *rational* matrix-valued . Can we obtain *rational* matrix-valued filters $\theta, \tilde{\theta} \in (l_0(\mathbb{Z}^d))^{r\times r}$ and $b, \tilde{b} \in (l_0(\mathbb{Z}^d))^{s\times r}$ such that $(\{a; b\}, \{\tilde{a}; \tilde{b}\})_{\theta^\star * \tilde{\theta}}$ is an OEP-based filter bank such that the asscoaited OEP-based multiframelet has all desired properties? More generally, is it possible to obtain a filter banks such that all elements lie in the same arbitrary algebraic number field (e.g., $\mathbb{Q}, \mathbb{Q}\sqrt{2}$, etc.)? We may need some new ideas to get the solutions.

Last but not the least, framelets with mixed dilation factors could be a future research topic. We only developed the basics of the theory of this topic in this thesis. It is clear that the theory is not as comprehensive as the work on traditional framelets. Extending this topic could lead to future research problems.

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