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DISCRETE CONTROL AND ESTIMATION OF TWO TIME-SCALE SYSTEMS

by

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Abstract

The singular perturbation method is used for designing multirate controllers for two time-scale systems. Two methods are proposed. In the first method, fast and slow controllers are designed based on system decomposition in the continuous-time domain. The slow subsystem is discretized at a relatively low sampling rate and the fast subsystem is discretized at a higher sampling rate. In the second method, the design is based on system decomposition in the discrete time domain. The latter is quite useful in establishing the stability of the complete system (controller+plant).

A partial control for the fast subsystem is also suggested.

Two numerical examples are given to illustrate the proposed methods.

Another result reported in this thesis is a new method for designing lower order Kalman filters for a class of two time-scale systems. Stability of this Kalman filters also proven.

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List of symbols

Scalars:

a, b, c, d	Constant
h	Singular perturbation parameter
J	Cost functional
k	Discrete time
k_1, k_2	Spring constant
l	Discrete time
m	Dimension of control and observation vectors
m_1, m_2	Masses
n	System dimension
n_1, n_2	Slow and fast subsystem dimensions, respectively
q_i, p_j	Eigevalues
λ	Eigenvalue
t	Time in slow time-scale
τ	Time in fast time-scale
T, T_1, T_2	Sampling period

Vectors

g and f	Vector functions
U, U_s, U_f, U_1	Control variables
V, V_s	Measurement noise
$W, W_{1,2}, W_s, W_{sx}$	Process noise
$X_1, X_2, \bar{X}_2, X_s, X_f$	State variable
Y, Y_1	Measurement vectors

Z	Measurement vector

Matrices

A, AA, A_1, $\tilde{A}_{11}$, $\tilde{A}_{12}$, $\tilde{A}_{12}$, $\tilde{A}_{21}$, $\tilde{A}_{22}$, A_s, A', A_{11}, A_{12}, A_{21}, A_{22}	System matrices
B, $\tilde{B}_1$, $\tilde{B}_2$, $\tilde{B}_2$, $\tilde{B}_{11}$, $\tilde{B}_{12}$, $\tilde{B}_{21}$, $\tilde{B}_{22}$,	Control matrices
C, C'	System matrices
C_1, C_2, C_s, C_1'	Measurement matrices
D, D'	System and uncorrelating matrices
F_1, F_2	System matrices after diagonalizing
G_1, G_2	Disturbance matrices
K_1, K_2	Feedback gains
L, M	Matrices
P	System transformation matrx
Q	Control problem--state weighting matrix Estimation --process noise covariance
R	Control problem--control weighting matrix Estimation problem--measurement noise covariance
T	Transformation matrix

Chapter 1

Introduction

1.1 Background

Two time-scale systems often occur in nature, due to the presence of small parasitic parameters. Many systems in practical applications have two time-scale property. e.g., electrical circuits(Chow, 1982), power systems(Avramovic, 1980), (Sastry, 1980, 1981), (Chow et al, 1983), (Cori et al, 1984), nuclear reactor systems (Asatani et al,1977), scheduling systems(Delebecque et al,1978), (Tenetzis,1980) (Stewart, 1983), chemical kinetics(Bobisud et al, 1980), (Brauner, 1978), economic models(Peponides et al, 1983), and population biology models(Lakin et al, 1981)

This thesis considers the control of such systems. A multirate discrete-time control strategy is proposed. Two design methods are given. It will be shown that the multirate control of two time-scale systems has two main advantages over other traditional controller design techniques such as pole placement by means of state feedback. They are

1. *The dimensionality of the control problem is decreased.*
2. *The practical implementation of the controller is simplified*

In treating this topic, the singular perturbation method is found to be quite useful. This method provides a mechanism for designing lower order controllers for systems possessing two time-scale property.

The singular perturbation method has matured over the past two decades. It is well documented in two survey papers (Kokotovic et al, 1976) and (V.R. Saksena et al, 1984).

The research on multirate systems has become very popular in recent years. See (Amit, 1980), and (Glasson, 1980, 1981, and 1984) for a detailed discussion and relevant references. Very often, systems are described by high order models which include phenomena covering a wide range of characteristic frequencies. The two time-scale system is one typical example. A multirate controller structure allows the designer to implement required control strategies for such systems without excessive computational burden. For instance, in aerospace aircraft applications onboard computational compacity is often a limiting factor. The basic idea of multirate technique is to control the 'fast' phenomena at a 'fast' sampling rate and to control the 'slow' phenomena at a 'slow' sampling rate.

By examining the motivation for the multirate control method and the singular perturbation method, one finds that these two methods have something in common. They are both motivated by the diversity of time-scales in practical control systems. The difference is that while multirate control is valid for any kind of system, the singular perturbation method is valid only for systems with two time-scale property. Interestingly enough, when the multirate control method is applied to the systems with two time-scale property, the design procedure is significantly simplified.

The singular perturbation method has been applied by other researchers to the control of two time-scale systems successfully. Attempts have also been made to apply the singular perturbation method to the estimation of states in two time-scale systems. But so far applications have been limited because of the requirement that one of the subsystems should be quite fast so that the 'fast' subsystem will converge much more quickly than the 'slow' subsystem (Haddad, 1976; Mahmoud, 1982a).

1.2 Objectives of thesis

As discussed earlier, one of the objectives of this thesis is to develop techniques for designing a multirate controller for two time-scale systems. Another objective in this thesis is to develop a procedure for designing slow and fast filters for two time-scale discrete systems, in which the usual asymptotic stability condition will not be required. The technique available at present time for designing fast and slow filters for two time-scale systems requires asymptotic stablilty of the fast subsystems and is only for continuous systems.

This thesis is organized as follows: In chapter 2, a brief review of singular perturbation method for continuous and discrete multitime-scale systems is given. In chapter

3, a discretization procedure as well as controller design methodology for systems with two-time scale property is discussed. To back up the theoretical results, some quantitative investigation is carried out in Chapter 4. The construction of slow and fast filters is discussed in chapter 5. Summary and conclusion appear in chapter 6. A list of references is also included.

Chapter 2

The singular perturbation method for two time-scale systems

2.1 Introduction

In this chapter, we will first present some basic ideas of two time-scale continuous and discrete-time systems. Then some major decomposition methods will be reviewed. Properties of such systems will be discussed next. We will finally discuss the composite control and estimation of two time-scale systems. This review follows closely the excellent survey paper by Saksena et al(1984). Some later development is also included.

2.2 The singular perturbation method

The singular perturbation theory, a traditional tool of fluid dynamics and nonlinear mechanics, embraces a wide variety of dynamic phenomena possessing slow and fast modes. Its assimilation in control theory is recent and rapidly developing.

The theory of singular perturbation for initial and boundary value problems and for stability determination was established in the 1960s, when it became a means for simplifying computation of optimal trajectories. It was soon discovered that singular perturbations are present in most classical and modern control systems which are based on reduced order models since these models disregard high frequency "parasitics". This led to research with applications of time-scale methods to control systems.

More recently, the singular perturbation method has also been used for modeling and control of dynamic networks and certain types of large-scale systems. This versatility of singular perturbation methods is due to their use of time-scale properties that are common to both linear and nonlinear dynamic systems.

2.2.1 Two Time-Scale Systems

2.2.1.1 Continuous Systems

Many multitime scale systems can be modeled by the set of nonlinear differential equations

$$\dot{X}_1=f(X_1,X_2,u,t) \qquad 2\text{-}1$$

$$\dot{X}_2=G(X_1,X_2,u,t) \qquad 2\text{-}2$$

where the n_1-dimensional vector X_1 is predominantly 'slow' and the n_2-dimensional vector X_2 is predominantly 'fast'. The fast transients are superimposed on a slowly varying 'quasi-steady state,' that is $||\dot{X}_1||<<||\dot{X}_2||$. One way of modeling such systems is to let g=hG. The small parameter h is a speed ratio of the slow and fast phenomena. Equation(2-1) and (2-2) become

$$\dot{X}_1=f(X_1,X_2,u,t) \tag{2-3}$$

$$h\dot{X}_2=g(X_1,X_2,u,t) \tag{2-4}$$

This is a generally accepted mathematical model and is used extensively in studies involving singular perturbation methods. The linear system corresponding to (2-1) and (2-2), obtained by linearization, is

$$\begin{bmatrix}\dot{X}_1\\ \dot{X}_2\end{bmatrix}=\begin{bmatrix}A & B\\ C' & D'\end{bmatrix}\begin{bmatrix}X_1\\ X_2\end{bmatrix}+\begin{bmatrix}F\\ G'\end{bmatrix}U \tag{2-5}$$

It can be rescaled to the form of (2-3) and (2-4) by letting $C=hC'$, $D=hD'$, and $G=hG'$.

$$\begin{bmatrix}\dot{X}_1\\ h\dot{X}_2\end{bmatrix}=\begin{bmatrix}A & B\\ C & D\end{bmatrix}\begin{bmatrix}X_1\\ X_2\end{bmatrix}+\begin{bmatrix}F\\ G\end{bmatrix}U \tag{2-6}$$

The choice of the value of small parameter h and the determination of slow and fast states need some insights into the systems to be modeled. This is the challenge which faces the person carrying out the modeling task. The basic principles useful in modeling of singularly perturbed systems have been discussed by Kokotovic(1981 and 1982). They will not be discussed here since such a discussion is beyond the scope of this brief review.

2.2.1.2 Linear Systems

Time-scale properties of time-invariant systems are decided by their eigenvalues. A definition of two time-scale linear systems is as follows. The system(2-5) is said to be a linear two time-scale system if it can be transformed into an upper triangular form

$$\begin{bmatrix}\dot{Y}_1\\ \dot{Y}_2\end{bmatrix}=\begin{bmatrix}F_1 & B\\ 0 & F_2\end{bmatrix}\begin{bmatrix}Y_1\\ Y_2\end{bmatrix} \qquad 2\text{-}7$$

through a linear transformation

$$X=TY \qquad 2\text{-}8$$

where $X^T=(X_1{}^T,X_2{}^T)$ and $Y^T=(Y_1{}^T,Y_2^T)$ and the following condition is satisfied

$$\mathrm{Max}|\lambda(F_1)|<<\mathrm{Min}|\lambda(F_2)| \qquad 2\text{-}9$$

where $\lambda(F_1)$ are the eigenvalues of the matrix F_1 and similarly $\lambda(F_2)$. Eq.(2-9) implies that the largest eigenvalue of matrix F_1 is much smaller than the smallest eigenvalue of matrix F_2. If the above condition is satisfied, T can be found by using a transformation (Narasimhamurthy,1977; Anderson, 1978; Avramovic, 1979; O'Malley, 1982; and Phillips, 1983)

$$Z=X_2+LX_1 \qquad 2\text{-}9a$$

where L is $n_2 \times n_1$ dimensional matrix. This changes eq.(2-5) to

$$\begin{bmatrix} \dot{X}_1 \\ \dot{Z} \end{bmatrix} = \begin{bmatrix} A-BL & B \\ LA-D'L+LBL-C' & D'-LB \end{bmatrix} \begin{bmatrix} X_1 \\ Z \end{bmatrix} \qquad 2\text{-}9b$$

If L satisfies the algebraic Riccati equation

$$-D'L+LA+LBL-C'=0 \qquad 2\text{-}10$$

we then get

$$T=\begin{bmatrix} I & 0 \\ -L & I \end{bmatrix}$$

To completely separate the slow and fast subsystems we let

$$Y=X_1+MZ \qquad 2\text{-}11$$

which yieldes

$$\begin{bmatrix} \dot{Y} \\ \dot{Z} \end{bmatrix} = \begin{bmatrix} A-BL & (A-BL)M-M(D'-BL)+B \\ 0 & D'-LB \end{bmatrix} \begin{bmatrix} Y \\ Z \end{bmatrix} \qquad 2\text{-}11a$$

If M satisfies

$$(A-BL)M-M(D'-LB)+B=0 \qquad 2\text{-}12$$

complete separation is achieved and we get

$$\begin{bmatrix} \dot{Y} \\ \dot{Z} \end{bmatrix} = \begin{bmatrix} A-BL & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} Y \\ Z \end{bmatrix}$$

where D=D'-LB. This complete and exact decomposition is a very useful tool for modeling two time-scale systems. It does not however provide a useful format for the control problem due to the fact that it does not separate controls that contain both fast and slow states of the system. In other words, two subsystems are still coupled. although it is claimed(Phillips, 1983) that this decomposition is accurate for both the control and the modeling problem.

The following approximate decomposition method provides a clearer and more meaningful relation between original state variables and new state variables(Othman et al, 1985). Formally letting h=0 in eq.(2-6) provides(dropping the control)

$$\dot{X}_1=(A-BD^{-1}C)X_1 \qquad 2-13$$

if matrix D is non-singular. Define

$$\bar{X}_2=-D^{-1}CX_1 \qquad 2-14$$

$$X_f=X_2-\bar{X}_2 \qquad 2-14a$$

where $\bar{X}_2$ is called quasi-steady state of X_2 and X_f is called the boundary layer(Kokotovic 1976) which satisfies equation

$$h\dot{X}_f=DX_f \qquad 2-15$$

In this decomposition, $L=D^{-1}C$, and $M=O(h)$. If h is very small, they are the first order solutions of eq.(2-12) and

(2-10).

2.2.2 Discrete Systems

In recent years, considerable progress has been made in the analysis, modeling and control of discrete two time-scale systems. After some difficulties in the initial stages, some convenient and general forms of discrete two time-scale systems have been developed.

The first model which does not define the explicit singular perturbation parameter h is given by Mahmoud(1982a,b)

$$\begin{bmatrix} X_1(k+1) \\ X_2(k+1) \end{bmatrix} = \begin{bmatrix} A & B' \\ C' & D' \end{bmatrix} \begin{bmatrix} X_1(k) \\ X_2(k) \end{bmatrix} \qquad 2\text{-}16$$

If eq.(2-16) is transformed into diagonal form, we get

$$\begin{bmatrix} \eta(k+1) \\ \zeta(k+1) \end{bmatrix} = \begin{bmatrix} F_1 & 0 \\ 0 & F_2 \end{bmatrix} \begin{bmatrix} \eta(k) \\ \zeta(k) \end{bmatrix} \qquad 2\text{-}17$$

where the eigenvalues of matrix F_1 are located near the unity and the eigenvalues of matrix F_2 are located near the origin of the z-plane. This model is inconvenient to use because it does not define the singular perturbation parameter explicitly. Phillips(1980) used scaling by means of $h^{1-j}B=B'$, $h^jC=C'$ and $hD=D'$, where $0<j<1$. Note that h is explicit here. This model is conservative but easier to use. Some specific cases of the model proposed in(Phillips, 1980) have been studied by Rajagopalan et al(1981), Kando et al(1983), Naidu et al(1982) and Syrcos et al(1983). Nevertheless, such models are defined in terms of convergence and their actual use is limited.

The formulation which seems to be most suitable for control applications is that in which the discretization interval is chosen to be compatible with fastest time-scale of the continuous system. Consider the system (2-6) which is customarily called slow time-scale version of singularly perturbed system; the corresponding fast time-scale version is obtained by scaling $\tau=t/h$, then

$$\begin{bmatrix}\dot{X}_1(\tau)\\ \dot{X}_2(\tau)\end{bmatrix}=\begin{bmatrix}hA & hB\\ C & D\end{bmatrix}\begin{bmatrix}X_1(\tau)\\ X_2(\tau)\end{bmatrix} \qquad 2\text{-}18$$

This continuous model is discretized at a fast sampling rate compatible with the fast time-scale. We get the discrete model(Blankenship, 1981; Litkouhi et al,1982)

$$\begin{bmatrix}X_1(k+1)\\ X_2(k+1)\end{bmatrix}=\begin{bmatrix}(I_1+hA_1) & hB_1\\ C_1 & D_1\end{bmatrix}\begin{bmatrix}X_1(k)\\ X_2(k)\end{bmatrix} \qquad 2\text{-}19$$

where I_1 is an $n_1 \times n_1$ identity matrix. This does not require an asymptotically stable fast subsystem because the slow subsystem exhibits the slowness explicitly as a result of the presence of small parameter h.

Decomposition techniques are also available for discrete two time-scale systems discussed above. However, since they are similar to those for continuous systems, they will not be reviewed here.

2.3 Stability

In order to check the stability of original system (2-5) or (2-6), it is sufficient if the stability of decoupled systems (2-13) and (2-15) is examined. If system (2-13) and (2-15) are asymptotically stable, then there exists a $h^+>0$ such that for all $h<h^+$, system (2-5) or (2-6) is asymptotically stable. (Kokotovic et al, 1976). A similar result is also available or nonlinear systems (Chow, 1978; Saberi et al, 1981). A similar result was also obtained by (Phillips, 1980) for discrete systems.

2.4 Control Law Designs

The decomposition of a two time-scale system into fast and slow subsystems has made it possible to design two independent controllers for the two subsystems. Control laws can be designed by either using pole placement or optimal control techniques. The fast and slow controls are then combined into a composite control law. Numerous algorithms have been developed for both continuous and discrete systems, e.g., Suzuki et al(1976), Kokotovic et al(1976), Chow et al(1976,a,b), Porter(1976, 1978), Phillips(1980, 1981), Mahmoud(1982b), Othman et al(1985), and Litkouhi et al(1983, 1985). It is a vast topic and therefore only an outline of some basic ideas is presented here.

By means of decomposition, two time-scale systems can be decomposed into a slow and fast subsystems, given by

$$\dot{X}_s(t)=A_sX_s(t)+B_0U(t) \quad 2\text{-}20$$

$$\dot{X}_f(t)=A_fX_f(t)+B_2U(t) \quad 2\text{-}21$$

Suppose that the controls are given by

$$U_s(t)=K_1X_s(t) \quad 2\text{-}22$$

$$U_f(t)=K_2X_f(t) \quad 2\text{-}23$$

Then,

$$U_c=U_s(t)+U_f(t) =K_1X_s(t)+K_2X_f(t) \quad 2\text{-}24$$

is the composite control that affects both slow and fast subsystems. The closed-loop system becomes

$$\begin{bmatrix}\dot{X}_s\\ X_f\end{bmatrix}=\begin{bmatrix}A_s+B_0K_1 & B_0K_2\\ B_2K_1 & A_f+B_2K_2\end{bmatrix}\begin{bmatrix}X_s\\ X_f\end{bmatrix} \quad 2\text{-}25$$

However, this system is expressed in terms of X_s and X_f. Replace them by original state variables X_1 and X_2 using the relation

$$X_s=X_1 \text{ and } X_f=X_2-D^{-1}[C+B_2K_1]X_1$$

which is obtained by using approximate decomposition in the closed-loop system. We then have a realizable composite control given by

$$U_C=K_1X_1+K_2[X_2-D^{-1}(C+B_2K_1)X_1] \qquad 2\text{-}26$$

This results in a closed-loop system. It can be shown that the the closed-loop system can be decomposed into two subsystems(slow and fast) whose system matrices are

$$A_s+B_0K_0,\ [D+B_2K_2+O(h)]/h.$$

respectively. The proof can be found in (Suzuki et al, 1976). Consequently, K_1 and K_2 can be used/for separate slow and fast eigenvalue placement, stabilization and optimal control. This approach was first proposed by Suzuki et al(1976) and Chow et al(1976a).

If matrix D is asymptotically stable, a reduced order system can be formed and a lower order controller can be designed for the system. For discrete systems, the approach is similar.

2.5 Linear filtering of two time-scale linear systems

Linear filtering of two time-scale linear systems has received some attention during the past ten years. Here, we outline the approach given by Haddad(1976) for continuous two time-scale systems.

Consider the linear system

$$\dot{X}_1(t)=A_{11}X_1(t)+A_{12}X_2(t)+G_1W(t) \qquad 2\text{-}27$$

$$h\dot{X}_2(t)=A_{21}X_1(t)+A_{22}X_2(t)+G_2W(t) \qquad 2\text{-}28$$

where W(t) white process noise. To find the reduced order system (2-27) and (2-28), we formally let h=0 in eq.(2-28), and we have

$$\bar{X}_2=-A_{22}^{-1}[A_{21}X_1+G_2W] \qquad 2\text{-}29$$

Here we suppose that the matrix A_{22} is non-singular and asymptotically stable. If A_{22} is not asymptotically stable, $\bar{X}_2$ will not represent X_2 in the mean square sense since the covariance matrix of X_2 will be unbounded.

Suppose the measurement is given by

$$y(t)=C_1X_1(t)+C_2X_2(t)+V(t) \qquad 2\text{-}30$$

where V(t) is white noise. Substitute X_2 by $\bar{X}_2$ in eq.(2-30) and (2-27) and we have

$$\dot{X}_1(t)=A_sX_1(t)+W_s(t) \qquad 2\text{-}31$$

$$y(t)=C_sX_1(t)+V_s(t) \qquad 2\text{-}32$$

where

$$A_s=A_{11}-A_{12}A_{22}^{-1}A_{21} \qquad 2\text{-}33$$

$$C_s=C_1-C_2A_{22}^{-1}A_{21} \qquad 2\text{-}34$$

The covariance of W_s and V_s can also be obtained. As a result of the decomposition, the process and measurement noise become correlated. This difficulty can be resolved by adjoining the output equation to the state equation (2-31) through a adjoining matrix (Bryson et al, 1975). The problem then becomes a standard Kalman filter problem. The fast state variable can be estimated by

$$\bar{X}_2 = -A_{22}^{-1} A_{21} \bar{X}_1 \qquad 2\text{-}35$$

where $\bar{X}_1$ is the estimate of state variable X_1.

Chapter 3

Multirate Control of Two-Time-Scale Systems

3.1 Introduction

One of the two objectives of this thesis, namely to develop a multirate control strategy for two time-scale systems, will be pursued in this chapter. In this task, it is assumed that the fast subsystem can be discretized and controlled at a fast sampling rate, and the slow subsystem can be discretized and controlled at a comparatively low sampling rate. Such an assumption is consistent with Shannon's Sampling Theorem if the original system can be viewed as having two time-scale property in terms of its natural frequencies. It is also consistent with the concept of 'roughness' of digitally controlled systems(Katz, 1974, 1981; Franklin et al, 1980).

Roughness Function(RF) is defined as the weighted sum of the squares of the abrupt changes in the state derivatives or in the control inputs. When continuous time plants are controlled by digital controllers, a Zero-order Hold(ZOH) is used to reconstruct a piecewise continuous signal. The abrupt action of the ZOH at high sampling rate is reduced and smoothed out by the inherent filtering properties of the various electromechanical actuators. However the tendency on the part of designers to shorten the actuator time constants to satisfy various time response criteria diminishes the effectiveness of the actuators to act as filters and consequently, even at a high sampling rate, the action of the control is likely to be abrupt. On the other hand, the designers have to compromise between high sampling rate and computational capacity in hand.

The selection of the sampling rate for a digital control system is a compromise among many factors. The basic motivation in lowering the sampling rate is cost. A decrease in sampling rate means more time available for the control calculations, hence a small computer should be adequate for a given control function. Another way of stating this is: more control capability is available for a given computer. These economic arguments indicate that a suitable engineering choice is to chose the lowest sampling rate possible that meets all performance specifications.

Shannon's Sampling Theorem and the consideration of roughness establish the lower limit for the sampling rate which should be as low as possible from economic point of view. By decomposing the system into a slow and fast subsystems, one can use lower sampling rate for part of the system. Consequently, a multirate control strategy is a better choice.

Two approaches will be used in developing the control law. In the first approach, the decomposition of the given continuous-time two time-scale system is carried out first. The discretization is then performed on the slow and fast subsystems. Separate controllers are designed for the two subsystems. In the second approach, the given system is first discretized and then decomposed into slow and fast discrete time subsystems. The controller design follows this step. The second method is useful in establishing the stability of the entire system (plant+controller).

3.2 Controller Design Based on Continuous Time-Decomposition

3.2.1 System Decomposition

Consider a two-time-scale, continuous-time invariant(also called singularly perturbed) linear system:

$$\dot{X}_1(t)=\tilde{A}_{11}X_1(t)+\tilde{A}_{12}X_2+\tilde{B}_1U(t) \qquad 3\text{-}1$$

$$h\dot{X}_2(t)=\tilde{A}_{21}X_1(t)+\tilde{A}_{22}X_2(t)+\tilde{B}_2U(t) \qquad 3\text{-}2$$

where

X_1 and X_2 are n_1, n_2 dimensional state vectors for slow and fast subsystems respectively, $n_1+n_2=n$ where n is the dimension of the system. U(t) is an m-dimensional control vector, and $h>0$ is a small singular perturbation parameter.

The system (3-1) and (3-2) satisfies the following conditions.

1. $\tilde{A}_{22}$ is non-singular.
2. $\tilde{A}_{11}$, $\tilde{A}_{12}$, $\tilde{A}_{21}$, and $\tilde{A}_{22}$ are bounded.
3. $Max[|eig(\tilde{A}_{11}-\tilde{A}_{12}\tilde{A}_{22}^{-1}\tilde{A}_{21})|]<<Min[|eig()|]$

4. If $Re[eig(\tilde{A}_{22})] \geq 0$,

then $Max\ |[Re(eig(\tilde{A}_{22}))]| / Min|[Im(eig(\tilde{A}_{22}))]| << 1.0$.

These conditions are imposed for practical reasons. Condition 1) is required to decompose the system into two subsystems; Condition 2) is due to the fact that all matrices are dependent on the parameter h theoretically(Kokotovic at el, 1976); Condition 3) ensures that the fast and slow subsystems are grouped into (3-1) and (3-2) explicitly; and condition 4) somewhat weakens condition 3), but is useful from a practical point of view.

Partition the control vector U(t) into two subvectors U_1 and U_2 where U_1 and U_2 have dimension m_1 and m_2 respectively and $m_1+m_2=m$. Several approaches are possible. One choice is to treat U_1 and U_2 as the fast and slow controls, respectively and use them to make up the composite control for the system. Another choice is to use both U_1 and U_2 for slow subsystem and use only one of them for the fast subsystem. Also by assigning different values to m_1 and m_2, different control structures can be generated.

As shown later, the fast subsystem does not contribute very much to the over all cost if regulator design is used. This means that it is not necessary to employ all control variables to the fast subsystem.

It is also true that different controls play different roles in different parts of the system. No matter which choice is made, let us assume that U_f stands for fast control and U_s stands for slow control. Then the system

(3-1) and (3-2) become

$$\dot{X}_1=\tilde{A}_{11}X_1+\tilde{A}_{12}X_2+\tilde{B}_{11}U_s+\tilde{B}_{12}U_f \qquad 3\text{-}3$$

$$h\dot{X}_2=\tilde{A}_{21}X_1+\tilde{A}_{22}X_2+\tilde{B}_{21}U_s+\tilde{B}_{22}U_f \qquad 3\text{-}4$$

where

U_f is an m(if all controls are used for the fast subsystem) or m_2(if only part of controls are used for the fast subsystem) dimensional control vector, U_s is an m dimensional control vector. $\tilde{B}_{11}$, $\tilde{B}_{12}$, $\tilde{B}_{21}$, and $\tilde{B}_{22}$ are matrices with appropriate dimensions. The controls are chosen such that

$$U_f(t)=U_f(kT_2) \qquad \text{if } kT_2\le t<(k+1)T_2 \qquad 3\text{-}5$$

$$U_s(t)=U_s(kT_1) \qquad \text{if } kT_1\le t<(k+1)T_1 \qquad 3\text{-}6$$

where $T_2=hT_1$.

In studying the system (3-3) and (3-4), we find that the corresponding discrete-time model is not available in the slow time-scale if the matrix $\tilde{A}_{22}$ is not an asymptotically stable matrix. If we discretize the system(3-3) and (3-4) with a large sampling period, the discrete model loses its two time-scale property explicitly if the matrix $\tilde{A}_{22}$ is not an asymptotically stable matrix. This causes difficulty in studying the slow version of the system. However, it will been shown later that the slow discrete version of (3-1) and (3-2) can be obtained by considering the closed-loop fast subsystem. As a result, it is more convenient to decompose the system in continuous-time domain, and discretize it at a slow sampling rate.

We assume that X_2 has reached steady state(called quasi-steady state by Chow,1974) and U_f has vanished when considering the slow subsystem (3-3). This assumption is not valid if the two time-scale system is defined in terms of high and low frequencies. However, the fast subsystem does possess 'fastness' in terms of convergence if control effort is applied to the fast subsystem. This can be explained as follows: Consider two independent systems. One has two imaginary modes with period T_1 and the other has two imaginary modes with period T_2. Controllers for these two systems are designed using the same cost function specified in continuous-time domain. The speeds of convergence of two systems will be proportional to the natural frequencies of two systems if two systems are controlled continuously. We justify this argument by considering two separate systems:

$$\dot{x}(t)=Ax(t)+Bu(t) \qquad 3\text{-}7$$

$$h\dot{y}(t)=Ay(t)+Bu(t) \quad 3\text{-}8$$

where x and y are n-dimensional vectors and u is a m-dimensional control vector. We use the cost function

$$J(Z)=1/2\int_0^\infty (Z^TQZ+U^TRU)dt \quad 3\text{-}9$$

where Z can be either x or y.

In solving the problem (3-7) and (3-9), we can obtain a state feedback gain matrix K_1 if the pair (A,B) is stabilizable. For the problem (3-8) and (3-9), a feedback gain K_2 can similarly be obtained. To find the relation between K_1 and K_2, we define a streched time-scale (Tikhonov,1952, Kokotovic,1968) $\tau=t/h$ and substitute it into eq.(3-8) and eq.(3-9). We obtain

$$\dot{y}(\tau)=Ay(\tau)+Bu(\tau) \quad 3\text{-}10$$

$$J(Z)=h/2\int_0^\infty (Z^TQZ+U^TRU)d\tau \quad 3\text{-}11$$

This leads us to the conclusion that the optimal control problem using (3-8) and (3-9) in streched time-scale τ is identical to the optimal problem (3-7) and (3-9). It means that $K_1=K_2$.

In the closed-loop configuration, the two systems have the same time ratio h as they have in the open-loop configuration. We also notice in this that the cost of system (y) is h times of that of system (x). This is used in the subsequent sections.

Applying the transformations in eq.(2-9a) and (2-11) (with appropriate changes in notations)

$$\eta = X_2 + LX_1 \qquad 3\text{-}12$$

where L is chosen such that

$$\tilde{A}_{22}L - hL\tilde{A}_{11} + hL\tilde{A}_{12}L - \tilde{A}_{21} = 0 \qquad 3\text{-}13$$

and

$$\xi = X_1 + M\eta \qquad 3\text{-}14$$

where M is chosen such that

$$(h\tilde{A}_{11} - h\tilde{A}_{12}L)M - M(\tilde{A}_{22} + Lh\tilde{A}_{12}) + h\tilde{A}_{12} = 0 \qquad 3\text{-}15$$

to eq.(3-1) and (3-2), we get(dropping U_1 and U_2)

$$\dot{\xi} = (\tilde{A}_{11} - \tilde{A}_{12}L)\xi \qquad 3\text{-}16$$

h

$$\dot{\eta} = (\tilde{A}_{22} + hL\tilde{A}_{12})\eta \qquad 3\text{-}17$$

This transformation converges if the norm condition(Kokotovic, 1976) is satisfied. It has been stated that the above transformation can achieve any accuracy required both for control and modelling problem. As matter of fact, it is not quite true. In the control problem, the two subsystems are coupled not only by state variables, but also by control variables. The transformation only separates state variables, not the control variables.

Instead of the 'exact' decomposition discussed above, a more meaningful and clearer decomposition is adopted in this thesis. In studying the slow subsystem, we assume that the fast subsystem driven by slow state variables and controls has reached its steady state and the fast control has vanished. This steady state is called quasi-steady state. Then eq.(3-4) becomes

$$\tilde{A}_{21}X_1+\tilde{A}_{22}X_2+\tilde{B}_{21}U_s=0 \qquad 3\text{-}18$$

or

$$\bar{X}_2 =-\tilde{A}_{22}^{-1}\tilde{A}_{21}X_1-\tilde{A}_{22}^{-1}\tilde{B}_{22}U_s \qquad 3\text{-}19$$

Substituting $\bar{X}_2$ for X_2 into eq.(3-3), we get

$$\dot{X}_1=[\tilde{A}_{11}-\tilde{A}_{12}\tilde{A}_{22}^{-1}\tilde{A}_{21}]X_1 +[\tilde{B}_{11}-\tilde{B}_{12}\tilde{A}_{22}^{-1}\tilde{B}_{21}]U_s \qquad 3\text{-}20$$

In the short term, it is assumed that $X_1(t)$ and $U_s(t)$ are constant and we define $X_f=X_2-\bar{X}_2$; then, we have

$$h\dot{X}_f=\tilde{A}_{22}X_f+\tilde{B}_{22}U_f \tag{3-21}$$

Equation (3-21) is called boundary layer equation(O'Mally, 1969; Chang, 1972). In this approximate decomposition method, we replace matrix L and M by $L=\tilde{A}_{22}^{-1}\tilde{A}_{21}$ and M=0, it is accurate to the degree O(h). As discussed earlier, using accurate L and M will not result in any better decomposition. It only shows that small eigenvalues of the system are close to the eigenvalues of matrix $(\tilde{A}_{11}-\tilde{A}_{12}\tilde{A}_{22}^{-1}\tilde{A}_{21})$ and large eigenvalues are close to those of $(\tilde{A}_{22})$. The discrete form of eq.(3-20) is

$$X_1\{(k+1)T_1\}=A_1X_1(kT_1)+B_1U_s(kT_1) \tag{3-22}$$

where

$$A_1=e^{\{\tilde{A}_{11}-\tilde{A}_{12}\tilde{A}_{22}^{-1}\tilde{A}_{21}\}T_1} \tag{3-23}$$

$$B_1=\int_0^{T_1} e^{\{\tilde{A}_{11}-\tilde{A}_{12}\tilde{A}_{22}^{-1}\tilde{A}_{21}\}t}\{\tilde{B}_{11}-\tilde{B}_{12}\tilde{A}_{22}^{-1}\tilde{B}_{21}\}dt \tag{3-24}$$

The discrete version of (3-21) is

$$X_f\{(l+1)T_2\}=A_2X_f(lT_2)+B_2U_f(lT_2) \tag{3-25}$$

where

$$A_2=e^{\tilde{A}_{22}/hT_2} \quad 3\text{-}26$$

$$B_2=\int_0^{T_2} e^{\tilde{A}_{22}/ht}\tilde{B}_{22}/h\,dt \quad 3\text{-}27$$

and

$$X_f=X_2-\bar{X}_2 \quad 3\text{-}28$$

3.2.2 Multirate controller design

In the preceding subsection, a given continuous system is decomposed first and the subsystems have been discretized. The next step in the design problem is to design separate controllers for the slow and fast subsystems. The controllers are designed using either pole placement or optimal control technique.

Considering eq(3-22), if the controller for the slow subsystem is designed using optimal technique, the pair (A_1,B_1) has to be stabilizable. This means the that unstable modes of matrix A_1 are controllable. The cost function to be minimized is

$$J=1/2\sum_{i=0}^{\infty}[X_1^T(iT_1)QX_1(iT_1)+U_s^T(iT_1)RU_s(iT_1)] \quad 3\text{-}29$$

where Q is an $n_1 \times n_1$ symmetric, positive semi-definite matrix, and R an m×m symmetric, positive definite matrix.

If pole placement technique is used, the modes of A_1 that are to be relocated must be controllable. In this case, the feedback gain C_1 is chosen such that

$$\mathrm{Spec}(A_1+B_1C_1)=(p_1,p_2,\ldots,p_{n1}) \qquad 3\text{-}30$$

where $p_1,p_2,\ldots,p_{n1}$ are desired locations of eigenvalues of the closed loop slow subsystem. These eigenvalues are at the slow time scale.

Whichever design technique is used, the controller has the form

$$U_s(kT_1)=C_1X_1(kT_1) \qquad 3\text{-}31$$

As far as the fast subsystem is concerned, the design procedure is exactly the same as the slow controller design. If regulator design technique is used, the controller has the form:

$$U_f(kT_2)=C_2X_2(kT_2)+C_2\tilde{A}_{22}^{-1}\tilde{A}_{21}X_1(kT_2)+ C_2\tilde{A}_{22}^{-1}\tilde{B}_{21}U_s(kT_2) \qquad 3\text{-}32$$

where C_2 is chosen such

that

$$J_f=1/2\sum_{k=0}^{\infty} [X_f^T(kT_2)Q_1X_f(kT_2)+U_f^TR_fU_f(kT_2)] \qquad 3\text{-}33$$

is minimized. Q_1 is an $n_2 \times n_2$ symmetric, positive semidefinie matrix and R_f is m×m symmetric, positive definite matrix.

If the pole placement is used, the feedback gain C_2 is chosen such that

$$\mathrm{Spec}(A_2+B_2C_2)=(q_1, q_2, \ldots, q_{n2}) \qquad 3\text{-}34$$

Where q_1, q_2, ... q_{n2} are deesired locations of the eigenvalues of closed-loop fast subsystem.

Equation (3-32) can be implemented in two different ways:

Method 1: Direct implementation of eq.(3-32). *Here it is necessary to measure $X_1(t)$ at the fast sampling rate. This may be a disadvantage of this method since it measures the slow variable at a fast sampling rate.*

Method 2: Another method of implementing the fast subsystem controller is to measure $X_1(t)$ at the slow sampling rate. *The error caused by this may be tolerable because $X_1(t)$ changes very slowly compared to $X_2(t)$. This is also consistent with the* *assumption that $X_1(t)$ is constant when the fast subsystem is considered. In this method, a great deal of on-line computational time will be saved if the order of slow subsystem is high.*

A simpler form of eq.(3-32) results

$$U_f(t)=C_2X_2(lT_2)-C_2\tilde{A}_{22}^{-1}\tilde{A}_{21}X_1'(lT_2)+C_2\tilde{A}_{22}^{-1}B$$
$$\tilde{\ }_{21}C_1X_1'(lT_2) = C_{22}X_2+C_{21}X_1'(lT_2) \qquad 3\text{-}35$$

where

$$X_1'(lT_2)=X_1(kT_1) \quad \text{if} \quad kT_1 \le lT_2 < (k+1)T_1 \qquad 3\text{-}36$$

In the selection of the cost function J when the regulator design approach is used or in the selection of eigenvalues of closed-loop system when pole placement technique is used, it must be ensured that the closed-loop system possesses two-time scale property besides other requirements, in terms of time separation etc.

3.3 Controller design using discrete time domain decomposition

3.3.1 Discrete time decomposition in fast time-scale

While continuous time domain decomposition discussed in previous section is straightforward, it is difficult to prove the stability of resulting control system. In order to overcome this difficulty, another method is proposed in this section.

It is difficult to obtain an explicitly expressed slow time-scale discrete analog of system (3-1) and (3-2) if the fast subsystem is not asymptotically stable. In this section, it is proposed to first design a controller for fast subsystem that will stabilize it and then transform the fast version of the system into a slow one.

Notice that eq.(3-1) and (3-2) describe the system in a slow time scale. By defining $\tau=t/h$, we get the modified form of eq.(3-1) and (3-2) as given below

$$\dot{X}_1(\tau)=h\tilde{A}_{11}X_1(\tau)+h\tilde{A}_{12}X_2+ h\tilde{B}_1U(\tau) \qquad 3\text{-}37$$

$$\dot{X}_2(\tau)=\tilde{A}_{21}X_1(\tau) + \tilde{A}_{22}X_2(\tau) + \tilde{B}_2U(\tau) \qquad 3\text{-}38$$

This time-scale transformation does not affect the original singularly perturbed nature of the system. The discrete analog of (3-37) and (3-38) is obtained by sampling (3-37) and (3-38) at $\tau=0,T,2T,\ldots$, or $t=0,hT,2hT,\ldots$

$$X_1(n+1)=(I_1+hA_{11})X_1(n)+hA_{12}X_2(n)+hB_1U(n) \qquad 3\text{-}39$$

$$X_2(n+1)=A_{21}X_2(n)+A_{22}X_2(n)+B_2U(n) \qquad 3\text{-}40$$

This is true whether it is obtained by exactly calculating the matrix exponential or by using approximation(Blakenship, 1981). The physical meaning of this discrete model is that the slow eigenvalues of the system are located near the unity while the the fast eigenvalues are located elsewhere in the Z-plane.

The following assumptions are useful:

1. Only partial controls are used for the fast subsystem. This means that some components of U_f are forced to be zero.
2. $U(n)=U_s(n)+U_f(n)$

For convenience, we express (3-39) and and (3-40) as

$$X_1(n+1)\doteq(I_1+hA_{11})X_1(n)+hA_{12}X_2(n)+hB_{11}U_s(n)+ hB_{12}U_f(n) \qquad 3\text{-}41$$

$$X_2(n+1)=A_{21}X_1(n)+A_{22}X_2(n)+B_{21}U_s(n)+ B_{22}U_f(n) \qquad 3\text{-}42$$

Following the technique used for continuous systems, the decomposition transformation(Kokotovic, 1975)

$$y=Px \qquad 3\text{-}43$$

where

$$P=\begin{bmatrix} I-hML & -hM \\ L & I \end{bmatrix} \text{ and } P^{-1}=\begin{bmatrix} I & hM \\ -L & I-hLM \end{bmatrix}$$

is applied to eq.(3-41) and (3-42), where L and M are $n_2 \times n_1$ and $n_1 \times n_2$ matrices satisfying the conditions

$$A_{21}+L-A_{22}L+hL(A_{11}-A_{12}L)=0 \tag{3-44}$$

$$A_{12}+M-MA_{22}+h[A_{11}-A_{12}L]M-hMLA_{12}=0 \tag{3-45}$$

M and L exist if h is small and norm condition(Kokotovic,1975) is satisfied. However, There is no great advantage in using (3-44) and (3-45) compared to its first order approximation given by

$$M=-A_{12}(I_2-A_{22})^{-1} \tag{3-46}$$

$$L=-(I_2-A_{22})^{-1}A_{21} \tag{3-47}$$

Here eq.(3-46) and (3-47) will be used, which implies that we are assuming that the slow phenomena remain constant while the fast phenomena are being considered and the fast transient vanishes by the time the slow transient is considered. This transformation can achieve the same accuracy as the transformation (3-44) and (3-45). The resulting decomposed system is given by

$$X_1(n+1)=(I_1+hA_s)X_1(n)+O(h)X_f(n)+hB_sU_s(n) \tag{3-48}$$

$$X_f(n+1)=A_{22}X_f(n)+B_{22}U_f(n) \tag{3-49}$$

It should be noticed that U_f may not have the same dimension as U_s. Also note that

$$A_s=A_{11}-A_{12}(I_2-A_{22})^{-1}A_{21} \quad \text{3-49a}$$

$$B_s=B_{11}-A_{12}(I_2-A_{22})^{-1}B_{21} \quad \text{3-49b}$$

$$X_f=X_2-\bar{X}_2 \quad \text{3-49c}$$

3.3.2 Fast controller design

Since our objective is to design a multirate controller(two rate controller) for the system, eq.(3-48) and (3-49) can be used to design a fast controller for the fast subsystem, that is to obtain K_f such that $U_f=K_fX_f$. This can be done if the pair(A_{22},B_{22}) is stabilizable. K_f is chosen such that the matrix $(A_{22}+B_{22}K_f)$ is an asymptotically stable matrix. i.e.,

$$|\lambda(A_{22}+B_{22}K_f)|<1.0 \quad \text{3-50}$$

U_f can be expressed as

$$U_f(n)=F_1X_1(n)+F_2X_2(n)+F_3U_s(n) \quad \text{3-51}$$

where F_1, F_2, and F_3 can be obtained.

Substitute (3-51) into eq.(3-48) and (3-49) to get

$$X_1(n+1) = (I_1 + hA_1)X_1(n) + hA_2X_2(n) + hB_1U_s(n) \qquad 3\text{-}52$$

$$X_2(n+1) = A_3X_1(n) + A_4X_2(n) + B_2U_s(n) \qquad 3\text{-}53$$

where A_1, A_2, A_3, A_4, B_1, and B_2 can be calculated. Eq. (3-52) and (3-53) still retain the structure of a two-time-scale system. Notice that A_4 is an asymptotically stable matrix due to the presence of fast control effort. This is a very useful result.

3.3.3 Discrete time decomposition in slow time-scale

One model used by Phillips(1980) and Rao et al(1981) for discrete two-time scale systems is —

$$X_1(k+1) = AX_1(k) + hBX_2(k) + B_1U(k) \qquad 3\text{-}54$$

$$X_2(k+1) = CX_1(k) + hDX_2(k) + B_2U(k) \qquad 3\text{-}55$$

It is reported in the literature--a Ph.D thesis by Litkouhi(1983), this is the analog of slow version of singularly perturbed system (3-1) and (3-2) if matrix $\tilde{A}_{22}$ is an asymptotically stable matrix. Inspecting eq.(3-54) and (3-55), a very interesting feature can be observed, i.e., it is already in a decomposed form. It means that the fast state variable $X_2(k)$ has little affect on the behavior of slow subsystem. The discretization process has also yielded a decomposed model of the system. It has been shown that (3-52) and (3-53) can be transformed into the form (3-54) and (3-55) by propagating (3-52) and (3-53) and assuming that $U_s(n)$ remains constant during the propagating interval, i.e.,

$$U_s(n)=U_s(kl) \text{ if } kl \le n < k(l+1) \qquad 3\text{-}56$$

where k=[1/h] and [x] is defined as the largest integer that satisfies [x]≤x.

We have achieved a slow version of system(3-1) and (3-2), but without requiring that the matrix A_{22} be an asymptotically stable matrix by stabilizing it. In other words, we design a fast controller first for the fast subsystem in the fast time-scale. Consequently, we are able to study two-time-scale discrete systems in the slow-time-scale in an explicit form.

To capture the separation property of eq.(3-54) and (3-55), we introduce the linear transformation(P) (Kando et al, 1983; Syrcos et al, 1983; Naidu et al, 1982; Phillips, 1980) Z(k)=PX(k), where

$$Z^T(k)=[X^T_s(k),X_f^T(k)]^T \qquad 3\text{-}56b$$

and

$$P=\begin{bmatrix} I+KL & K \\ K & I \end{bmatrix} \text{ and } P^{-1}=\begin{bmatrix} I & -K \\ -L & I+LK \end{bmatrix}$$

where L and K satisfy

$$hDL-LA+hLBL-B=0 \tag{3-57}$$

$$hK(D+LB)-(A-hBL)K+hB=0 \tag{3-58}$$

If A is non-singular and matrix A, B, C, and D are bounded, and h is small, then we have the following first order approximation

$$L=-BA^{-1}+O(h) \tag{3-59}$$

$$K=hA^{-1}B+O(h^2) \tag{3-60}$$

Since the fast subsystem is already a well damped system, we focus our attention only to the slow subsystem in subsequent discussions. Using the first order solution of (3-57) and (3-58), (3-59) and (3-60), results in

$$X_s(k+1)=A_sX_s(k)+B_sU_s(k) \tag{3-61}$$

where

$$A_s=A+hBCA^{-1}+O(h^2) \tag{3-62}$$

$$B_s=(I-hCB)B_1+hA^{-1}BB_2+O(h^2) \quad 3\text{-}63$$

and

$$X_1(k)=X_s(k)-hA^{-1}BX_f \,. \quad 3\text{-}64$$

Furthering our approximation, we formally let $h=0$ in (3-62), (3-63) and (3-64). It proves that $\lambda(A)$ approximates the slow eigenvalues of original system to the degree $O(h)$. This formally shows that system(3-55) and (3-54) is already in decomposed form. It is also shown that we do not have to design a fast controller first. Instead, we only need to find a stabilizing feedback gain for the fast subsystem. The choice of this stabilizing feedback gain has no significant effect on the slow controller design.

3.3.4 Slow controller design

If the pair(A_s,B_s) is stabilizable, then the slow controller can be designed such that

$$U_s(k)=K_sX_1(k) \quad 3\text{-}65$$

and the matrix $A'=(A+B_1K_s)$ is an asymptotically stable matrix. K_s can be determined by using either pole placement or optimal control technique.

To investigate stability of the control system, we substitute $U_s(k)$ into (3-54) and (3-55)

$$X_1(k+1)=A'X_1(k)+hBX_2(k) \qquad 3\text{-}66$$

$$X_2(k+1)=C'X_1(k)+hDX_2(k) \qquad 3\text{-}67$$

Matrix A' and C' can be obtained. The stability of (3-66) and (3-67) is guaranteed by the following theorem.

Theorem: *If matrix* $A'=(A+\tilde{B}_1K_s)$ **is an asymptotically stable matrix, there exists $h^+>0$, such that for all $0<h<h^+$, the system (3-66) and (3-67) is an symptotically stable system and**

$$X_1(k)=A'X_1(k)+O(h) \qquad 3\text{-}68$$

Proof: The theorem can be proved simply by reapplying the transformation (P).

3.4 Conclusion

In this chapter, the modeling and multirate control of two time-scale systems have been discussed. The relationship of two models currently used for discrete time systems has been given. The discrete models are derived for systems described by a group of differential equations. Two different discretization and decomposition methods for controller design are proposed. In the first method, the system is decomposed in continuous time domain; the fast and slow subsystems are then discretized at different sampling rates. This method is straightforward and simple. In the second method, the given system is discretized at a fast sampling rate and then the fast controller is designed in the fast time-scale. The system is transformed into the slow time-scale. The controller design procedure has been put into a theoretical framework and the stability of overall system has been proven. A partial control strategy is proposed for the fast subsystem.

Chapter 4

Quantitative Investigation of Multirate Control of Two Time-Scale Systems

4.1 Introduction

The results of some quantitative investigation are reported in this chapter.

The primary objective is to demonstrate the theoretical results discussed in chpater 3. In chapter 3, two controller design methods were presented, one using decomposition in continuous time domain and the other using decomposition directly in discrete time domain. In this chapter, we will give two examples to illustrate the two methods.

It should be emphasized that singular perturbation method is an approximation method; consequently, some degradation in performance can be expected to traditional control methods such as optimal control method. The relationship between sampling rate and performance degradation will be examined.

In chapter 3, we have given a theorem which states that there exists a small parameter h^+ such that for all $h<h^+$, the control system designed is asymptotically stable. It is believed that the small parameter h^+ is dependent on the system and related to the pole locations as well. We will illustrate this relation through an example.

4.2 Cost function transformation

A linear system and cost function are often given as

$$\dot{X}=AX+BU \qquad 4\text{-}1$$

$$J=1/2\int_0^\infty [X^TQX+U^TRU]dt \qquad 4\text{-}2$$

where Q is an n×n symmetric, positive semi-definite matrix, R is an m×m symmetric, positive definite matrix, A is an n×n matrix, B is an n×m matrix, X is n-dimensional state vector and U is m-dimensional control vector. It is assumed that system (4-1) can be partitioned into the form (3-1) and (3-2).

In practical application, control and state weighting matrices in cost function are often chosen to be diagonal. Without losing generality, we assume that the cost function have the following form

$$J=1/2\int_0^\infty [X_1^TQ_1X_1+U^TRU+X_2^TQ_2X_2]dt \qquad 4\text{-}2a$$

where Q_1 and Q_2 are $n_1 \times n_1$ and $n_2 \times n_2$ positive, semidefinite and symetric matrices, and R is m×m positive definite and symetric control weighting matrix. $X^T=[X_1^T,X_2^T]^T$ and $U=U_s+U_f$ are state variables and control, respectively.

The cost function J is transformed into discrete form for the following three different control structures.

4.2.1 Standard LQR design

If this control method is used, the cost function is transformed as follows

$$J_T=1/2\sum_{i=0}^{\infty}[X^T(iT)\tilde{Q}X(iT)+2X^T(iT)\tilde{M}U(iT)+U^T(iT)\tilde{R}U(iT)] \tag{4-2c}$$

where

$$\tilde{Q}=\int_0^T[e^{A(T-t)}]^T Q e^{A(T-t)}dt \tag{4-3}$$

$$\tilde{M}=\int_0^T[[e^{A(T-t)}]^T Q\int_0^t e^{A(T-q)}Bdq]dt \tag{4-4}$$

$$\tilde{R}=\int_0^T[[\int_0^t e^{A(T-q)}Bdq]^T Q\int_0^t e^{A(T-p)}Bdp]dt+R \tag{4-5}$$

and

$$Q=\begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix}, \quad A=\begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix},$$

$$B=\begin{bmatrix} \tilde{B}_1 & 0 \\ \tilde{B}_2 & 0 \end{bmatrix}$$

and T is sampling period.

4.2.2 Singular perturbation method, LQR design with all controls used for the fast subsystem

The singular perturbation method provides a very flexible control structure decided in accordance with practical applications and system dynamics for digitally controlled systems. Sometime, it is not neccesary to apply all control variables to the fast subsytems.

As given in chapter 3, $X_2(t)$ can be replaced by its quasi-steady state

$$\bar{X}_2(t)=-\tilde{A}_{22}^{-1}\tilde{A}_{21}X_1(t)-\tilde{A}_{22}^{-1}\tilde{B}_{21}U_s(t) \qquad 4\text{-}6$$

If the closed-loop system has two time-scale property, the above assumption is reasonalbly accurate.

In designing the slow subcontrol system, we let $U=U_f+U_s$ and

$U_f=0$ formally, and substitute X_2 in eq.(4-2a), by $\bar{X}_2$ from eq.(4-6), then we have

$$J_s=1/2\int_0^\infty [X_1^T Q_s X_1+U_s^T R_s U_s+2X_1^T M_s U_s] \qquad 4\text{-}7$$

where

$$Q_s=Q_1+[\tilde{A}_{22}^{-1}\tilde{A}_{21}]^T Q_2\tilde{A}_{22}^{-1}\tilde{A}_{21} \qquad 4\text{-}8$$

$$M_s=[\tilde{A}_{22}^{-1}\tilde{A}_{21}]Q_2\tilde{A}_{22}^{-1}\tilde{B}_{21} \qquad 4\text{-}9$$

and

$$R_s=[\tilde{A}_{22}^{-1}\tilde{B}_{21}]^T Q_2\tilde{A}_{22}^{-1}\tilde{B}_{21}+R \qquad 4\text{-}10$$

Define $\underline{X}_s=[X_1^T,\ U_s^T]^T$, then we have

$$\dot{\underline{X}}_s=\underline{A}_s\underline{X}_s \qquad 4\text{-}11$$

where

$$\underline{A}_s=\begin{bmatrix}\tilde{A}_{11}-\tilde{A}_{12}\tilde{A}_{22}^{-1}\tilde{A}_{21} & \tilde{B}_{11}-\tilde{A}_{12}\tilde{A}_{22}^{-1}\tilde{B}_{21}\\ 0 & 0\end{bmatrix} \qquad 4\text{-}12$$

The cost function can be rewritten as

$$J_s = 1/2\int_0^{\infty} \underline{X}_s^T Q_{ss} \underline{X}_s dt \qquad 4\text{-}13$$

where

$$Q_{ss} = \begin{bmatrix} Q_s & M_s \\ M_s^T & R_s \end{bmatrix}$$

The discrete form of the system and the cost function sampled at the sampling rate $1/T_1$ are

$$X_1(k+1) = A_1 X_1(k) + B_1 U_s(k) \qquad 4\text{-}14$$

$$J_{T_1} = 1/2\sum_{k=0}^{\infty} [X_1^T(k) Q_{T_1} X_1(k) + 2X_1^T(k) M_{T_1} U_s(k) + U_s^T(k) R_{T_1} U_s(k)] \qquad 4\text{-}15$$

where A_1, B_1, Q_{T_1}, M_{T_1}, and R_{T_1} can be determined by the following identities

$$\begin{bmatrix} A_1 & B_1 \\ 0 & I \end{bmatrix} = e^{\underline{A}_s T_1} \qquad 4\text{-}16$$

$$\begin{bmatrix} Q_{T_1} & M_{T_1} \\ M_{T_1}^T & R_{T_1} \end{bmatrix} = \int_0^{T_1} [e^{A_s[T_1-t]}] Q_s e^{A_s[T_1-t]} dt \qquad 4\text{-}17$$

In short term run, the cost contributed by the fast subsystem is very little compared to the slow subsystem as given in chapter 3. For simplification, we assume that the slow state variables and controls be zero in considering the fast subsystem. Then we have the cost function for the fast subsytem design as

$$J_f=1/2\int_0^\infty [X_f^T Q_2 X_f+U_f^T RU_f]dt \qquad 4\text{-}18$$

and the fast subsystem

$$\dot{X}_f=\tilde{A}_{22}X_f+\tilde{B}_{22}U_f \qquad 4\text{-}19$$

Then J_s can be transformed into a discrete form in exactly the same way as for the slow subsystem.

4.2.3 Singular perturbation method, only part of control variables are used for the fast subsystem

As discussed, different controls play different roles for different parts of systems to be controlled.

In this formulation, it is somewhat the same as for the case in which all controls are used to transform the cost function for slow and fast subsystems. However, differences occur in the matrix $\tilde{B}_{22}$ and control weighting matrix R. In the fast cost function transformation, $\tilde{B}_{22}$ is replaced by its columns and R is replaced by its sub-diagonal matrix.

4.2.4 Example 1

To illustrate the theoretical results, it is better to have a physical system in hand. In this simulation, we use a spring-mass system given in fig. 4.1.

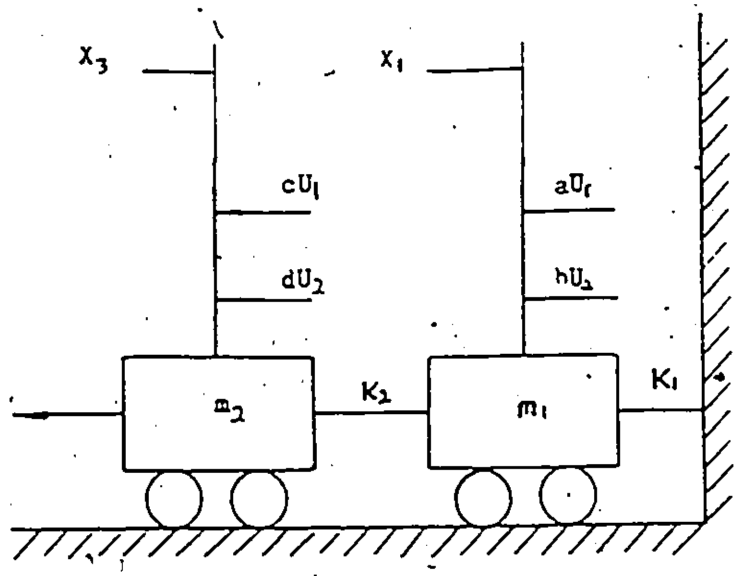

Fig. 4.1 A mass-spring system

If we define $X_2=\dot{X}_1$ and $X_4=\dot{X}_3$, we have the state space equation

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{pmatrix}=\begin{pmatrix}0 & 1 & 0 & 0\\ -\frac{k_2}{m_1}-\frac{k_1}{m_1} & 0 & \frac{k_2}{m_1} & 0\\ 0 & 0 & 0 & 1\\ \frac{-k_2}{m_2}-\frac{k_2}{m_1}-\frac{k_1}{m_1} & 0 & -\frac{k_2}{m_2}-\frac{k_2}{m_1} & 0\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\\ x_4\end{pmatrix}+\begin{pmatrix}0 & 0\\ \frac{a}{m_1} & \frac{b}{m_1}\\ 0 & 0\\ \frac{c}{m_1}-\frac{a}{m_2} & \frac{b}{m_1}-\frac{d}{m_2}\end{pmatrix}\begin{pmatrix}U_1\\ U_2\end{pmatrix}$$

Examining the system, we find that the system has two pair imaginary eigenvalues. Through the control or parameters k_1, k_2, m_1 and m_2, we can have the system in two timee-scale form. In this simulation, we let m_1=1kg., m_2=0.05kg., k_1=0.5nt/m, k_2=1nt./m., and a=b=d=1 and c=0. The time-scale separation is about h=1/6. As suggested in (Powell et al, 1980), we choose the fast sampling rate as seven times the highest frequencies and the slow sampling rate to be nine times the slow frequencies of the open-loop system for the multirate control.

4.3 Result and discussion

Simulation results are shown in fig. 4.2-11. Fig. 4.2 shows the placement of two masses; fig. 4.3 gives the control history when all controls are used for the fast subsystem. Fig. 4.4 illustrates the placement of two masses and fig. 4.5 shows the control varables when only U_2 is used for the fast subsystem. Fig. 4.6 and fig. 4.7 show the placement and the control when the standard LQR control is used. It is observed that degradation exists in the singular perturbation method compared to the standard LQR design since it is only an approximation method. If partial control strategy is applied, the computational time is significantly reduced. If the computational compacity is fixed, we can use faster sampling rate for the system. In term of cost, we have listed the costs for different control strategies in the following tables.

TABLE 1.

METHOD	Partial cont Sing. pert. method, way1	Full cont. Sing. pert. method way1	Standard LQR	Partial cont sing. pert. method, way2	Full contro sing. pert. method,way2
Fast samp rate	0.2sec.	0.2sec.	0.2sec.	0.2sec.	0.2sec.
Slow samp rate	1.0sec.	1.0sec.	***	1.0sec.	1.0sec.
Cost	13.410	14.231	7.801	14.691	17.523

TABLE 2.

METHOD	Partial cont Sing. pert. method, way1	Full cont. Sing. pert. method way1	Standard LQR	Partial cont sing. pert. method, way2	Full contro sing. pert. method,way2
Fast samp rate	0.1sec.	***	0.4sec	***	***
Slow samp rate	0.5sec.	***	0.4sec	***	***
Cost	13.443	***	20.384	***	***

As shown in table 1 and 2, degradation occurs when two control methods(Partial fast control and full fast control using the singular perturbation method) are used, compared to the standard LQR design method. One of the reasons for this is that the cost functions used for the different control methods are not exactly the same since we have used approximation in transformation of the cost function for the singular perturbation method. In this transformation, we assumed that the fast subsystem does not contribute too much to the over all cost. In fact, it is underestimated since the changes in the slow control will disturb the fast subsystem continuously untill the slow subsystem is converged. Another reason for the degradation is that the singular perturbation method is based on the assumption that the singular perturbation parameter h is very small. In this example it is not as small as it should be. However, the degradation is not very large.

As calculated, the computational time if partial control is used is almost half of that if LQR control is used. If we increase the sampling rate for the singular pertubation method and only partial control used, the overall cost does not improve very much. In contrast, if we slow down the sampling rate in LQR design, the cost increases significantly. It implies that the singular perturbation method is a useful method to design near optimal controllers for two time-scale systems if the computational capacity is limited.

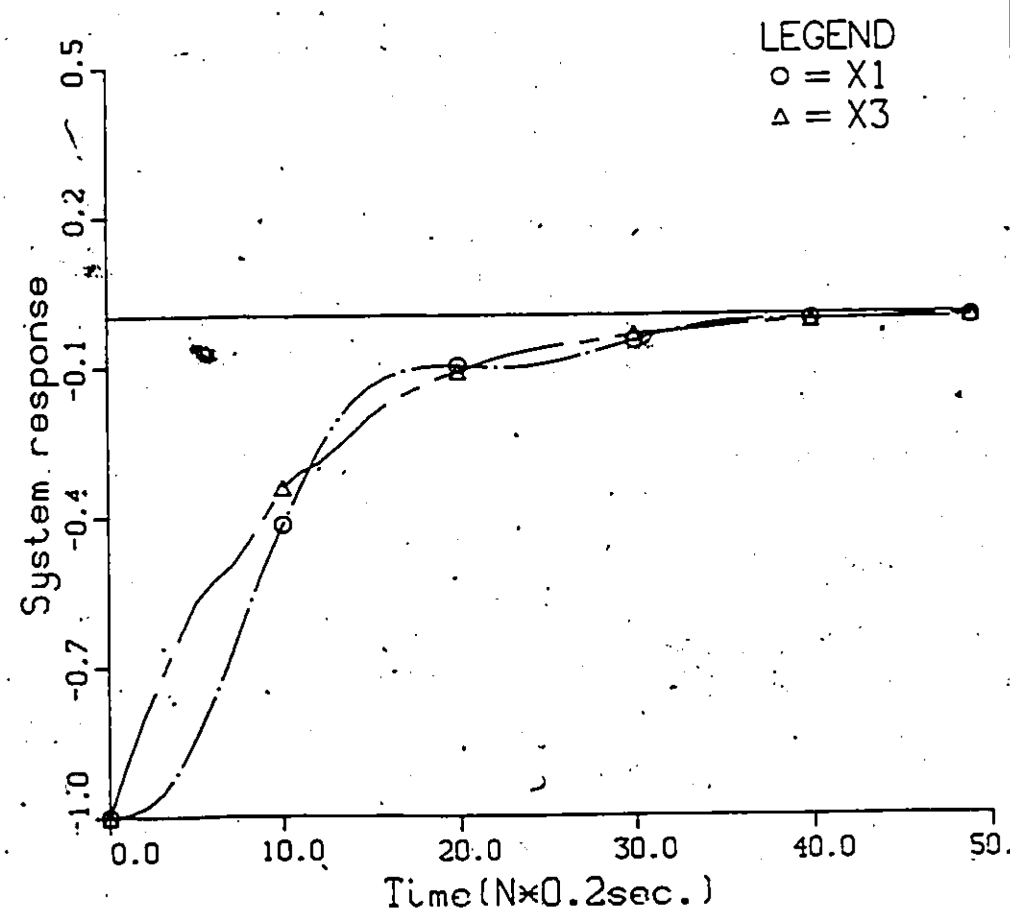

Fig. 4.2 System response via partial control, singular perturbation method

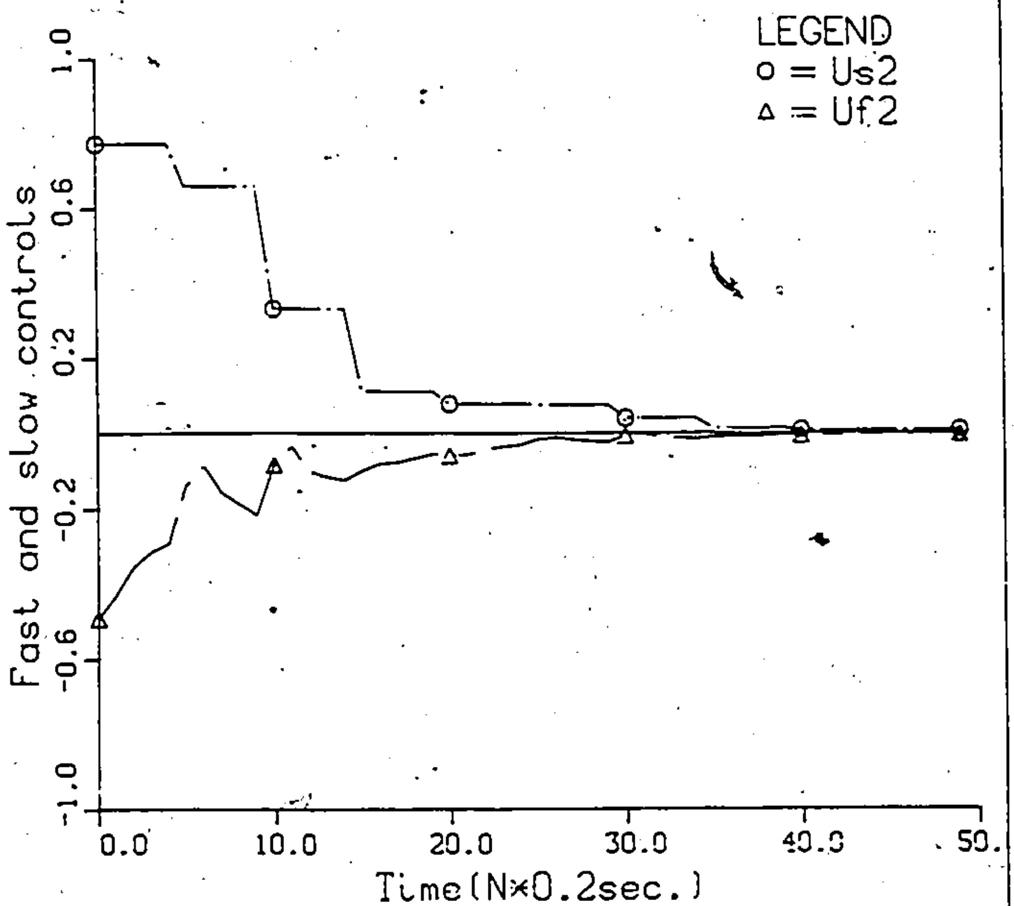

Fig. 4.3 Control via partial control, singular perturbation method

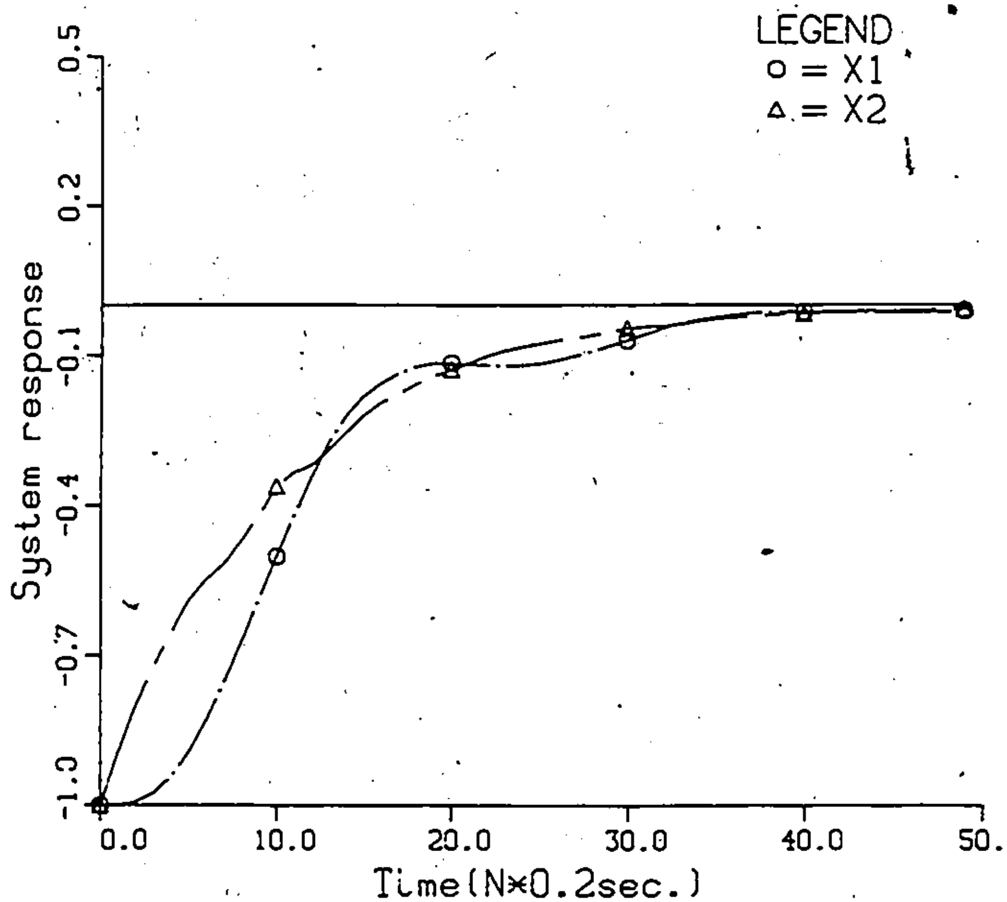

Fig. 4.4 System response via full control, singular perturbation method

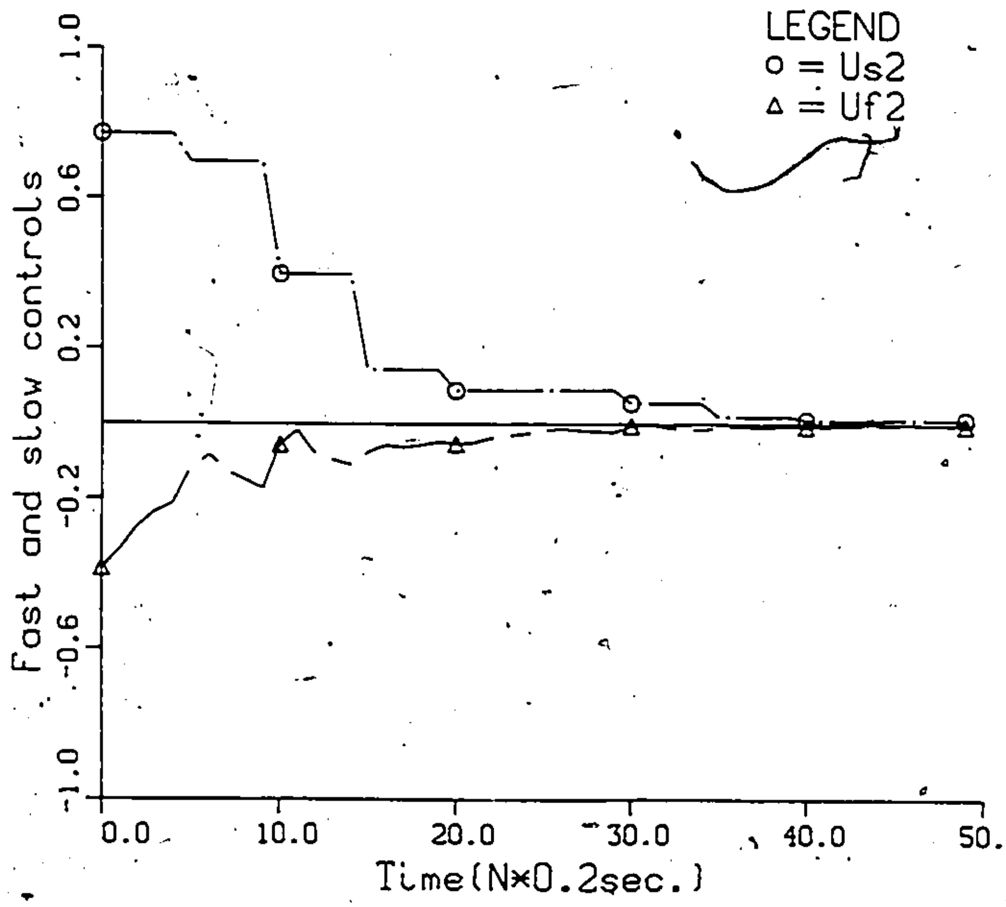

Fig. 4.5 Control via full control, singular perturbation method

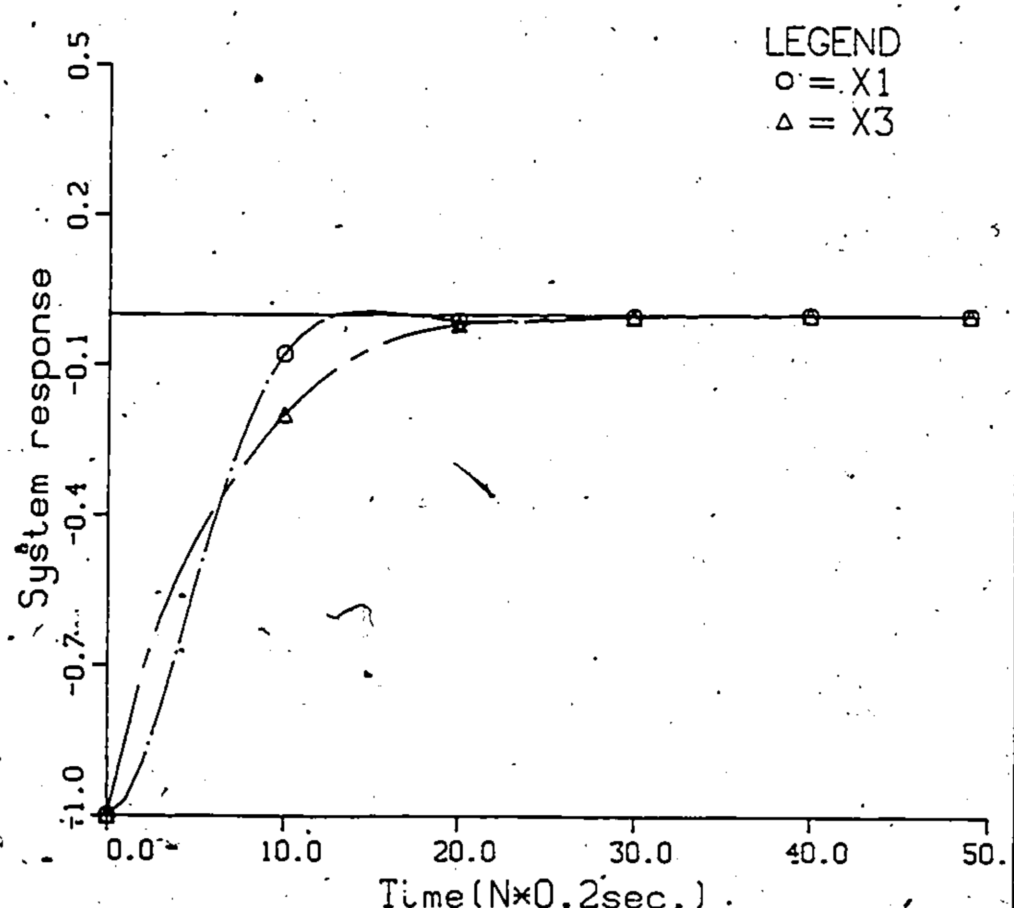

Fig. 4.6 System response via standard LQR design

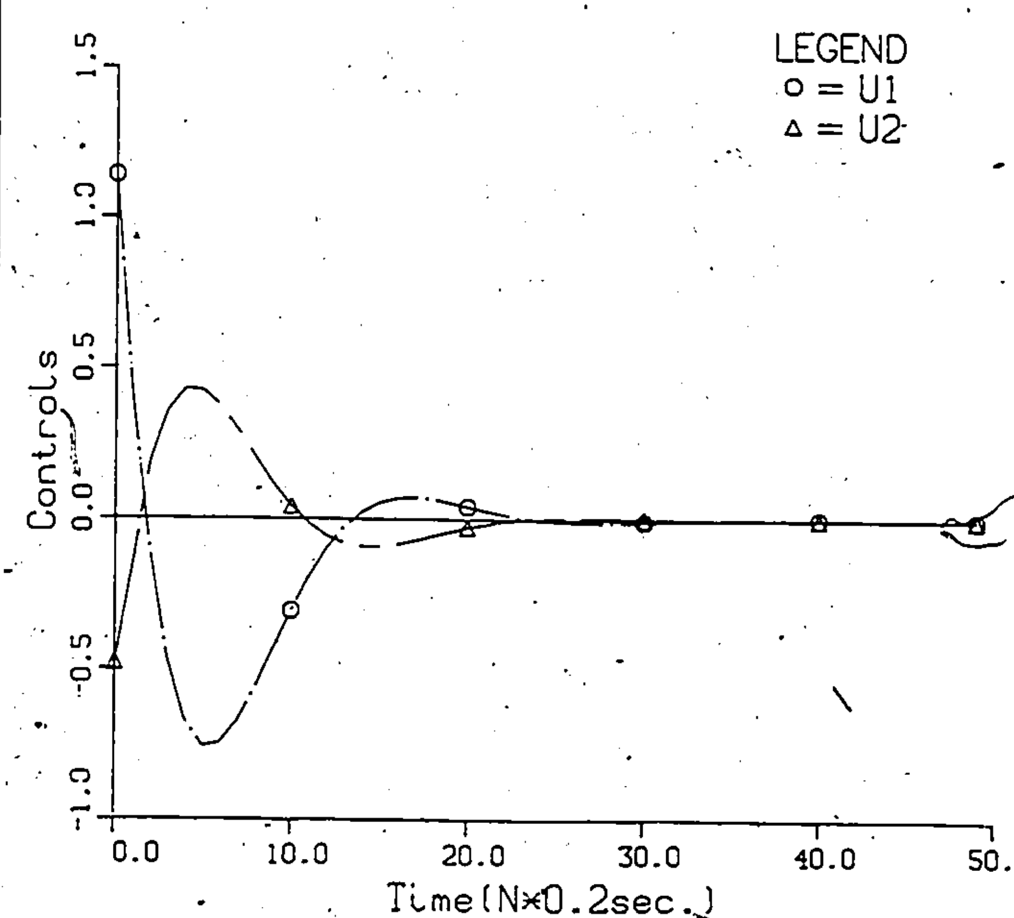

Fig. 4.7 Control via standard LQR design

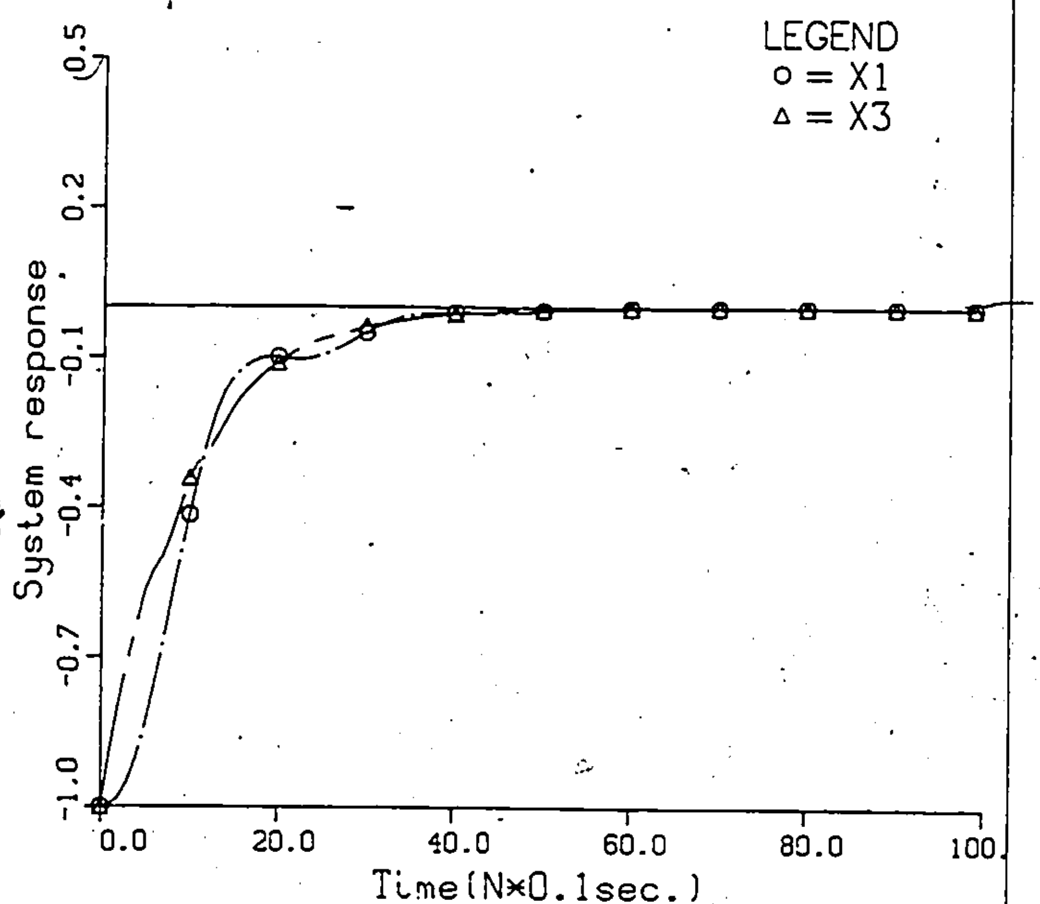

Fig. 4.8 System response via partial control singular perturbation method

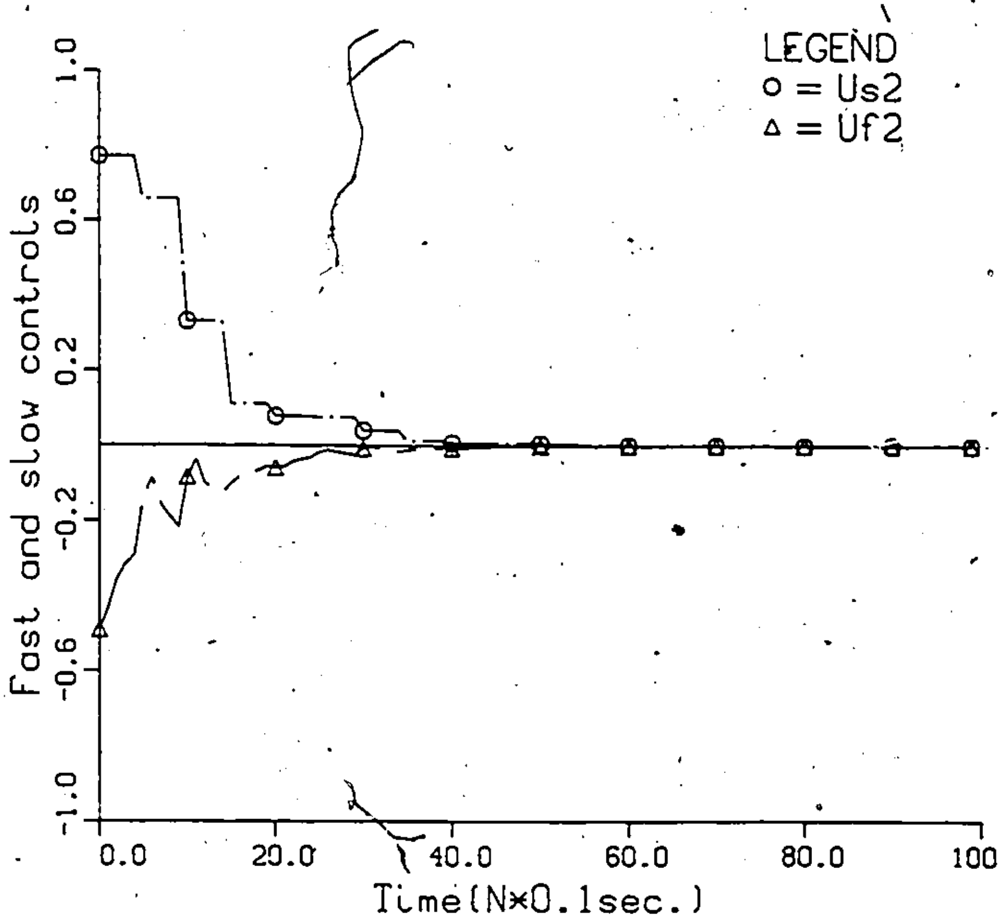

Fig. 4.9 Control via partial control, singular perturbation method

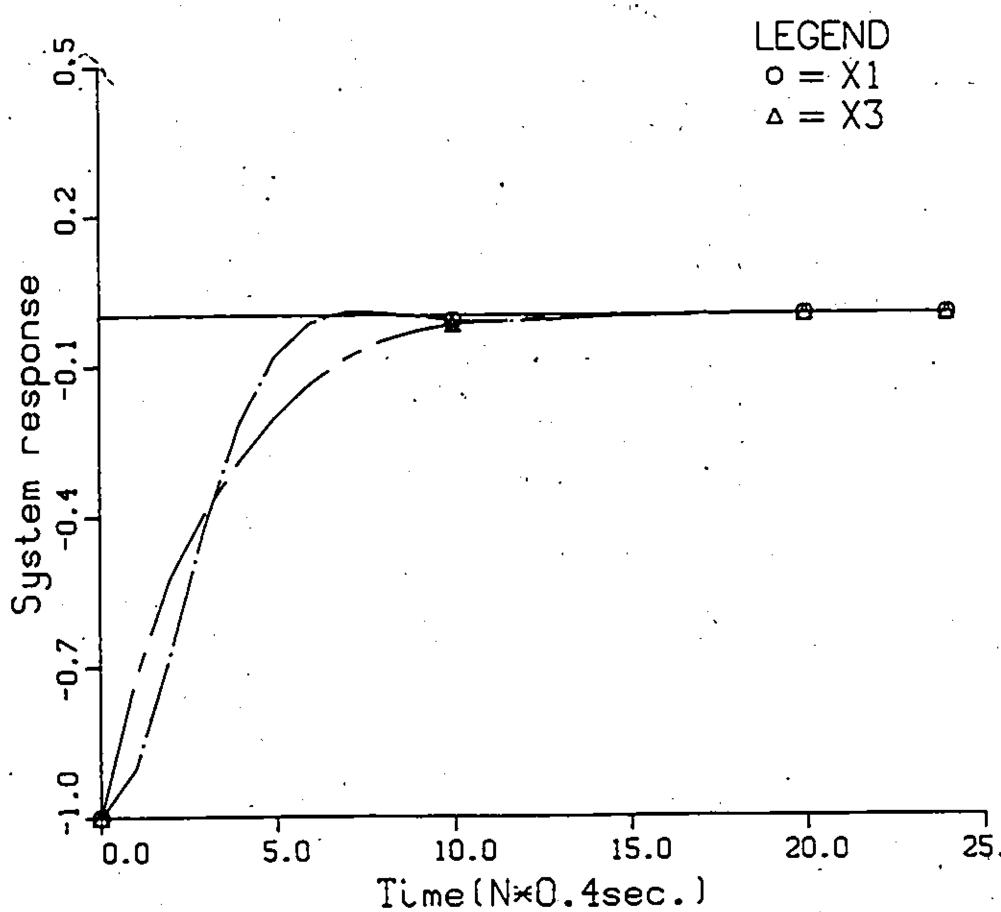

Fig. 4.10 System response via standard LQR design

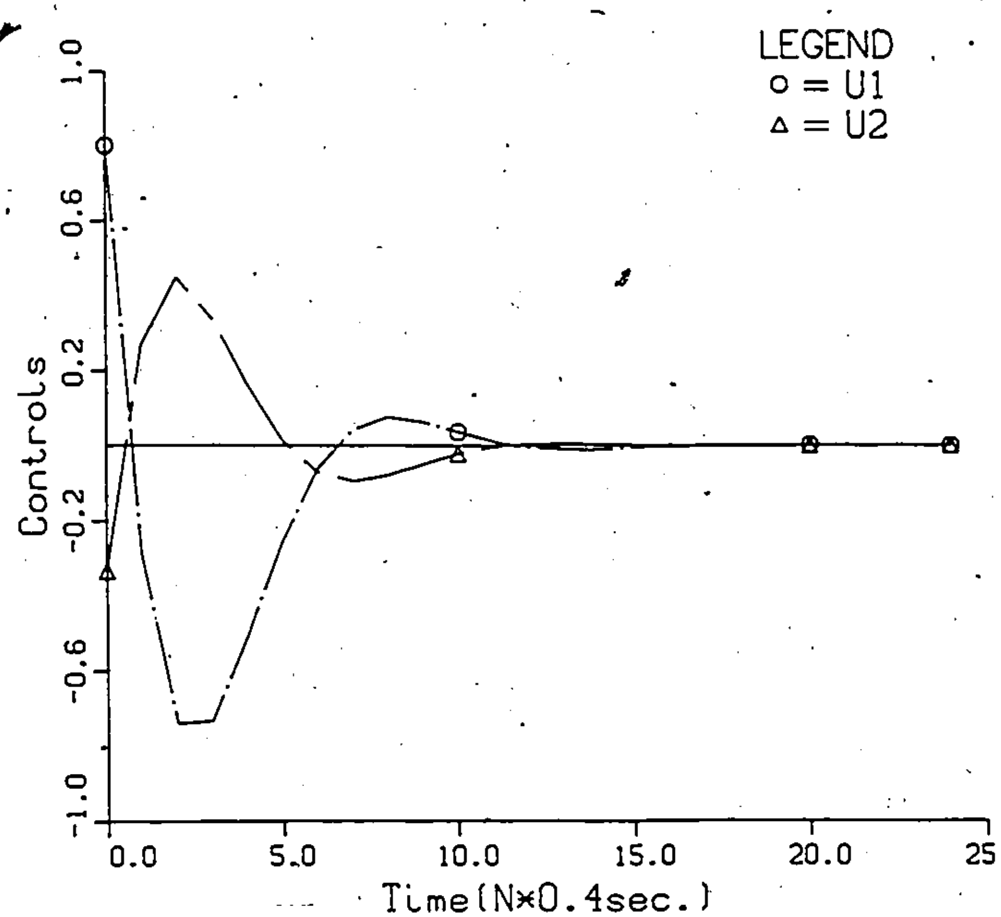

Fig. 4.11 Control via standard LQR design

4.4 Example 2

This section is designed to illustrate the second multirate controller design method for singularly perturbed systems. With a numerical example and design method in hand, we can show validity of the method and design procedure. We will also investigate the relationship between pole location and time-scale ratio that gives a marginally stable closed-loop system.

4.4.1 System

The system served as an example is given as

$$\dot{Y}=AX+BU \qquad 4\text{-}21$$

where $\dot{Y}=[\dot{X}_1{}^T, h\dot{X}_2{}^T]^T$, $X=[X_1{}^T, X_2{}^T]^T$ and

$$A=\begin{bmatrix} 0.4 & -0.3 & 0.4 & 0.10 \\ -0.2 & -0.5 & 0.4 & 0.60 \\ 0.4 & 0.2 & 0.0 & -0.62 \\ 0.5 & 0.3 & 0.62 & 0.0 \end{bmatrix} \qquad B=\begin{bmatrix} 1.00 & 0.50 \\ 0.60 & 0.70 \\ 0.80 & 0.60 \\ 0.40 & 0.90 \end{bmatrix}$$

These four poles locate at [0.17,j5.8], [0.17,-j5.8], [-0.677], and [0.24] in the s-plane with h=0.1.

4.4.2 Results and Discussion

Since the open-loop system has a pair of bending modes that have frequecy f=1Hz., we sample the system in fast time-scale at T=0.1sec(i.e., ten times of the highest natural frequency) and propagating interval is n=10. The discrete analog of (4-21) (that is in fast time-scale)

$$X(n+1)=AX(n)+BU(n) \tag{4-22}$$

where

$$A=\begin{bmatrix} 1.0497 & -0.026 & 0.0407 & -0.003 \\ 0.0026 & 0.9635 & 0.0543 & 0.0432 \\ 0.2313 & 0.0909 & 0.8231 & -0.578 \\ 0.6010 & 0.3267 & 0.6008 & 0.8224 \end{bmatrix}$$

$$B=\begin{bmatrix} 0.1181 & 0.0630 \\ 0.0884 & 0.1056 \\ 0.6512 & 0.3072 \\ 0.6596 & 0.0558 \end{bmatrix}$$

It is clear that eq.(4-22) can be easily scaled to the form of (3-39) and (3-40). The small parameter h=0.1. To illustrate the accuracy of the transformation (3-43), we obtain the eigenvalues of the discrete system (4-22 as $p_{1,2}=(0.8507, \pm j0.5586)$(fast modes) and $q_1=0.110226$ and $q_2=0.93453$. As discussed in Chapter 3 the fast eigenvalues can be approximated by the eigenvalues of matrix A_{22}. The eigenvalues of matrix A_{22} are

$$\lambda_{1,2}=(0.82279, \pm j0.55865)$$

It can be seen that $p_{1,2}$ are very close to $\lambda_{1,2}$.

In designing the fast subsystem, we specify the control index as $Q_2=\text{diag}(1.0,\ 1.0)$ and $R_2=\text{diag}(1.0,\ 1.0)$. The ideal eigenvalues of the fast subsystem are

$$\eta=(0.36546, \pm j0.20463)$$

The magnitude of the eigenvalues is very small compared to the magnitude of eigenvalues of the slow subsystem. In fact, difference exists between 'ideal' and 'real'. The real fast modes are

$$\eta_2=(0.36182, \pm j0.24765)$$

It is an very good approximation of η. To show the effect of fast controls on the slow subsystem, we substitute the fast controls into the system, the closed-loop system have two slow modes

$$\bar{q}_1=1.0224 \text{ and } \bar{q}_2=0.9445$$

From these data, we conclude that the fast controls have very little effect on the slow subsystem.

The new system after substituting the fast controls into eq.(4-21)

$$X(n+1)=A_1X(n)+B_1U_s(n) \qquad 4\text{-}23$$

where

$$A_1=\begin{bmatrix} 1.0093 & -0.050 & -0.026 & -0.025 \\ -0.014 & 0.9503 & -0.008 & -0.009 \\ -0.000 & -0.046 & 0.4623 & -0.671 \\ 0.5357 & 0.2520 & 0.0764 & 0.2686 \end{bmatrix} \text{ and}$$

$$B_1=\begin{bmatrix}0.0998 & -0.014\\ 0.1143 & 0.0648\\ 0.5228 & -0.131\\ 1.0349 & 0.8294\end{bmatrix}$$

The slow version of eq.(4-23) after propogating at an interval K=10

$$X(k+1)=A_s X(k)+B_s U_s(k) \qquad 4\text{-}24$$

where

$$A_s\begin{bmatrix}1.1135 & -0.367 & -0.046 & 0.0096\\ -0.112 & 0.6581 & -0.006 & 0.0017\\ -0.817 & -0.119 & 0.0411 & -0.007\\ 0.6771 & -0.014 & 0.032 & 0.0072\end{bmatrix} \text{ and}$$

$$B_{ss}=\begin{bmatrix}0.5716 & -0.188\\ 0.7581 & 0.4988\\ -1.290 & -1.530\\ 1.8826 & 1.009\end{bmatrix}$$

As is seen, eq.(4-24) is in the form of eq.(3-54) and (3-55).

Inspecting the eigenvalues of the system in slow time-scale

$$q_{s1}=1.2209,\ q_{s2}=0.5984, \text{ and}$$

$$p_{s3,4}=(0.00038, \pm j0.00046)$$

we say that the fast subsystem is dead beat and, thus has no influence on the ~~sl~~ow subsystem.

In designing the slow subsystem, the control index is given as Q_1=diag(1.0,1.0) and R_2=diag(1.0,1.0). The over all closed-loop system is

$$X(k+1)=A_{ss}X(k) \tag{4-25}$$

where

$$A_{ss}=\begin{bmatrix} 0.6119 & -0.225 & -0.046 & 0.0096 \\ -0.046 & 0.2949 & -0.006 & 0.0017 \\ -1.596 & 1.0018 & 0.0411 & -0.007 \\ 0.6168 & -0.746 & -0.032 & 0.0072 \end{bmatrix}$$

and the eigenvalues of A_{ss} are

$$p_{1,2,3,4}=0.7470,\ 0.2631,\ -0.0563,\ 0.0015$$

The closed system is an asymptotically stable one, and the fast subsystem is almost deadbeat in the slow time-scale.

4.5 Stablity, Separation Ratio and Pole Locations

As given in Chapter 3, there exists h^+ such that the overall control system (3-66) and (3-67) is asymptotically stable for all $h<h^+$. The algorithm for finding h^+ is not available. However, it has been shown that it has some relation with pole locations of control systems. For convenience, we will use the fast version (3-39) and (3-40) of singularly perturbed systems and fix the magnitude of fast subsystem and change it along a circle centered at the origin. As matter of fact, we can not adjust the fast modes exactly; therefore we adjust the modes of matix A_{22}

instead since they represent the fast modes very closely if h is very small. The result is shown in Table. 3.

Table 3.

Real(p)	6.2	6.0	5.5	5.0	4.0	3.0
largest stable h	0.08	0.09	0.11	0.15	0.2	0.5

As is shown in table 3, the larger the unstable fast eigenvalues are(real parts), the smaller the parameter h is required for a stablizing control. It is common requirement for system have fast response, less overshot, and less settling time; therefore, the speed separation ratio must be much smaller than h^+. It is quite unrealistic to have such system designed using the singular perturbation method. Consequently, we require that the fast eigenvalues have large imaginary part.

4.6 Summary

In this chapter, we have demonstrated the two controller design methods and investigated the relation between pole locations and the parameter h^+.

Chapter 5

State estimation of two time-scale discrete systems

5.1 Introduction

As discussed in chapter 2, considerable progress has been made recently in the development of the singular perturbation method in control theory. The development has focused on the control of two time-scale systems such as optimal control and eigenvalue placement through state feedback. However, state variables of systems are not always directly accesible for feedback in practical applications. In the standard control system, an observer or Kalman filter is constructed in order to obtain the estimated state variables. Estimation of states in two time-scale systems, can be achieved by constructing two low order observers or Kalman filters for the slow and fast subsystems(Haddad,1976; and Altshuler et al, 1978). The work by Haddad has laid the foundation for the solution of the filtering problem in two time-scale systems.

However, it is required that the fast subsystems should be asymptotically stable in the approach, i.e., fast subsystem converge more quickly than the slow subsystem. In other words, the n eigenvalues of the system should be capable of being grouped into two groups: n_1 are located near origin(for continuous systems) and near the unity(for discrete systems) which represent the slow modes; Similarly, n_2 are located at the left of the s-plane(for continuous systems) or near the origin on z-plane(for discrete systems). In practice, it is more common to have systems that do not have a fast convergent fast subsystem. The main objective of this chapter is to propose a method of designing lower dimension filters in such cases.

State estimation in discrete-time systems will be considered in this chapter. Extension of the method to continuous time systems is straightforward. We will follow the same sequence as was adopted in chpter 3. System decomposition will be discussed first and followed by the technique for filter design.

5.2 Modeling and decomposition of two time scale systems for filter design

As in the case of controller design, there are two ways of modeling two time-scale discrete systems, namely, fast and slow versions. We adopt the fast version of the two time-scale system since it does not require that the fast subsystem should be asymptotically stable. Consider the system

$$X_1(k+1)=(I_1+hA_{11})X_1(k)+hA_{12}X_2(k)+hW_1(k) \qquad 5\text{-}1$$

$$X_2(k+1)=A_{21}X_1(k)+A_{22}X_2(k)+W_2(k) \qquad 5\text{-}2$$

and

$$y(k)=C_1X_1(k)+C_2X_2(k) \qquad 5\text{-}3$$

where $X_1(k)$, $X_2(k)$, and $y(k)$ are n_1, n_2, and m slow, fast state vectors and measurement, respectively; A_{11}, A_{12}, A_{21}, A_{22}, C_1, and C_2 are constant matrices with appropriate dimensions; $I_i (i=1,2,...)$ stands for $n_i \times n_i$ dimensional identity matrix.

$$E[W_1(k)W_1^T(l)]=Q_{11}\delta(k-l) \tag{5-4}$$

$$E[W_2(k)W_2^T(l)]=Q_{22}\delta(k-l) \tag{5-5}$$

$$E[W_1(k)W_2^T(l)]=Q_{12}\delta(k-l) \tag{5-6}$$

In studying the system (5-1), we can reasonably assume that $X_1(k)$ is constant. And let

$$X_2(k)=\eta(k)+GX_1(k) \tag{5-7}$$

where

$$\eta(k+1)=A_{22}\eta(k)+W_2(k) \tag{5-8}$$

and

$$G=(I_2-A_{22})^{-1}A_{21} \tag{5-8}$$

The only approximation involved is that we assume that $X_1(k)$ is constant, which quite accurate if h is sufficiently small. If matrix A_{22} is not an asymptotically stable, $\eta(k)$ does not converge at all. In control problems, $\eta(k)$ does converge if fast control effort has been applied. In this case, we can ignore the definite effect of $\eta(k)$ and, instead, replace it by $(I_2-A_{22})^{-1}W_2(k)$. However in estimation problem, $\eta(k)$ can not be ignored if the matrix A_{22} is not asymptotically stable. If this is the case, two subfilters can not be designed independently.

Substitute $X_2(k)$ and $\eta(k)$ into eq.(5-1) and (5-2), we have

$$X_1(k+1)=(I_1+hA_S)X_1(k)+hA_{12}\eta(k)+hW_1(k) \qquad 5\text{-}9$$

$$\eta(k)=A_{22}\eta(k)+W_2(k) \qquad 5\text{-}10$$

and

$$y(k)=C'_1X_1(k)+C'_2\eta(k)+V(k) \qquad 5\text{-}11$$

where

$$A_S=A_{11}+A_{12}(I_2-A_{22})^{-1}A_{21} \qquad 5\text{-}12$$

$$C'_1=C_1+C_2(I_2-A_{22})^{-1}A_{21} \text{ and } C'_2=C_2$$

In studying the covariance of state variables of fast and slow subsystems, we can model them as hP_1 and P_2, respectively, which can be deduced by assuming that the matrix A_{22} is asymptotically stable. In considering the estimation error covariance, this assumption can be relaxed further.

It can be easily proven that

$$X_1(k+1)=[I+hA_s]X_1(k)+hA_{12}\eta(k)+hW_1(k)+O(h^2) \qquad \text{5-12a}$$

and

$$\eta(k+1)=A_{22}\eta(k)+O(h) \qquad \text{5-12b}$$

$$X_2(k)=-(I-A_{22})^{-1}A_{21}X_1(k)+A_{22}\eta(k)+O(h) \qquad \text{5-12c}$$

The decomposition is a partial decomposition discussed in chapter 3.

5.3 Filter Design

In the design of filters for slow and fast subsystems, it is observed that the slow filter can not be designed prior to the designing of the filter for the fast subsystem due to the presence of fast state variables in both state and output equations. Although the slow state variable $X_1(k)$ is also present in the measurement equation, the estimated value can be used. It is feasible to design the fast filter first for fast subsystems. Since the process noise input to the slow subsystems is h times of the noise input to the fast subsystem, we can model the state covariance of slow subsystems as hP_1.

5.3.1 Fast filter design

Rewrite the decomposed system(5-9), (5-10) and (5-11)

$$X_1(k+1)=(I_1+hA_s)X_1(k)+hA_{12}\eta(k)+hW_1(k) \quad 5\text{-}9$$

$$\eta(k)=A_{22}\eta(k)+W_2(k) \quad 5\text{-}10$$

and

$$y(k)=C'_1X_1(k)+C'_2\eta(k) + V(k) \quad 5\text{-}11$$

Since $Cov[X_1(k)]=hP_1$, the covariance of estimated error of slow subsystems will behave similarly. Therefore we substitute the estimated value of $X_1(k)$(this including the estimation of $X_1(k)$ before or after measurement) as its real value in the estimation of $\eta(k)$ without considering the uncertainty involved. The estimated value of $X_1(k)$ is not yet known. However, as will be seen, this will not present any difficulties.

Define

$$y'(k)=y(k)-C'_1X_1(k) \quad 5\text{-}13$$

If the estimated or predicted value of $X_1(k)$ is used, $y'(k)$ is a known value. A Kalman filter can therefore be generated if the pair (A_{22},C_2) is detectable. The filter equations are

$$\tilde{\eta}(k)=\bar{\eta}(k)+K_2(y'(k)-C_2\bar{\eta}(k)) \quad 5\text{-}13$$

$$\bar{\eta}(k)=A_{22}\tilde{\eta}(k) \qquad 5\text{-}14$$

$$K_2=M_2C_2^T(C_2M_2C_2^T+R)^{-1} \qquad 5\text{-}15$$

$$P_2=M_2-M_2C_2^T(C_2M_2C_2^T+R)^{-1}C_2M_2 \qquad 5\text{-}16$$

$$M_2=A_{22}PA_{22}^T+Q_{22} \qquad 5\text{-}17$$

where P_2 is steady state covariance of estimated fast state variable $\eta(k)$ after measurement and M_2 is steady state covariance of estimated $\eta(k)$ before the measurement. If the detectability condition is satisfied, the matrix

$$A_2=A_{22} - K_2C_2A_{22} \qquad 5\text{-}18$$

is asymptotically stable. The estimation error is then given by

$$e_2(k+1)=-K_2C_1e_1(k)+(A_{22}-K_2C_2A_{22})e_2(k)+W_2(k) -K_2V(k) \qquad 5\text{-}19$$

Although $||e_1(k)||$ is very small compared to $||e_2(k)||$, $e_2(k)$ contains a componenent that is driven by $e_1(k)$, that could affect the slow filter design significantly.

5.3.2 Slow filter design

Since it is quite inconvenient to study the slow filter in terms of natural state variables, we will study the slow filter in terms of error variables.

Suppose we have a slow filter .

$$\hat{X}_1(k+1)=(I_1+hA_s)+hA_{12}\tilde{\eta}(k) + hK_1[y(k+1)-C_1'\hat{X}_1(k+1)- C_2\tilde{\eta}(k+1)] =(I_1+hA_s)\hat{X}_1(k)+hA_{12}\tilde{\eta}(k) +hK_1[C_1'e_1(k)+ C_2A_{22}e_2(k) + V(k+1)] \quad 5\text{-}20$$

with error $e_1(k)=X_1(k)-\hat{X}_1(k)$, that satisfies

$$e_1(k+1)=(I+hA_s)e_1(k)+hA_{12}e_2(k)+hK_1[C_1'e_1(k)+ C_2A_{22}e_2(k)+V(k)] \quad 5\text{-}21$$

and

$$e_2(k+1)=-K_2C_1'e_1(k)+(A_{22}-K_2C_2A_{22})e_2(k)+W_2(k)- K_2V(k+1) \quad 5\text{-}22$$

Since the matrix $[A_{22}-K_2C_2A_{22}]$ is asymptotically stable, and the small parmameter is also present in eq.(5-21), the systems (5-21) and (5-22) still represent a two time-scale system. Consequently, we can, as in the case of continuous systems which approximates the fast state variable by its quasi-steady state(Haddad, 1976), approximate $e_2(k)$ by its steady state and some noise terms.

$$\bar{e}_2(k)=-(I-A_{22}+K_2C_2A_{22})^{-1}K_2C_1'e_1(k)(I-A_{22}+ K_2C_2A_{22})^{-1}[W_2(k-1)- K_2V(k)] =G_1e_1(k)+G_2W_2(k-1)+G_3V(k) \quad 5\text{-}23$$

If we substitute $e_2(k)$ by $\bar{e}_2(k)$ in (5-21), we have

$$\begin{aligned} e_1(k+1)=&[I+h(A_s+A_{12}G_1)]e_1(k)+ hA_{12}G_2W_2(k-1)+ \\ &hA_{12}V(k)+ \\ &hW_1(k)-hK_1\{(C_1'+A_{22}C_2G_1)e_1(k)+C_2G_2W_2(k-1)+ \\ &C_2G_3V(k)+ V(k+1)\} \end{aligned} \quad 5\text{-}24$$

This is equivalent to the estimation problem

$$Z(k+1)=(I+hA_1)Z(k) + hW_{sx}(k) \quad 5\text{-}25$$

with measurement

$$U(k)=C_sZ(k) + V_s(k) \quad 5\text{-}26$$

where

$$W_{sx}=A_{12}G_2W_2(k-1) + W_1(k) + A_{12}G_3V(k) \quad 5\text{-}27$$

$$V_s=C_2G_2W_2(k-2) + C_2G_3V(k-1) + V(k) \quad 5\text{-}28$$

and

$$A_1=A_s + A_{12}G_1 \text{ and } C_s=C_1'+A_{22}C_2G_1$$

As given in eq.(5-27) and (5-28), the noise term $W_{sx}(k)$ and $V_s(k)$ are not white at all. Nevertheless, we can assume that they are white since the correlation time is only one sampling period which is very short compared to the time-scale of slow subsystem. To design a Kalman filter, we need the following variables

$$E[W_{sx}(k)V^T_s(k)]=G_3R=R_c \quad \text{5-29}$$

$$E[W_{sx}(k)]=0, \text{ and } E[V_s(k)]=0$$

$$\begin{aligned}Cov[W_{sx}(k)]&=A_{12}G_2Q_{22}(A_{12}G_2)^T + Q_{11} + A_{12}G_3R(A_{12}G_3)^T\\ &=Q_1\end{aligned} \quad \text{5-30}$$

$$\begin{aligned}Cov[V_s(k)]&=C_2G_2Q_{22}(C_2G_2)^T + R+C_2G_3R(C_2G_3)^T\\ &=R_s\end{aligned} \quad \text{5-31}$$

Introducing the uncorrelating procedure in(Bryson et al, 1975)

$$\begin{aligned}Z(k+1)&=(I+hA_1)Z(k)+hW_{sx}(k)+hD[U(k)-C_sZ(k)-V_s(k)]\\ &=(I+A_{ss})Z(k)+hW_s(k)+hU(k)\end{aligned} \quad \text{5-33}$$

where

$$A_{ss}=A_1-DC_s \quad \text{5-34}$$

$$Cov[W_s(k)=Q_s=Q_1 + DR_sD^{-1} - G_3RD^T-DRG_3^T \quad \text{5-35}$$

To find the matrix D, Let

$$E[W_s(k)V^T{}_s(K)]=0 \qquad 3\text{-}35a$$

We then have

$$D=-R_s^{-1}R_c \qquad 3\text{-}35b$$

Two appproaches are possible to design the slow filter for the slow substystem. We could follow the procedure similar to that proposed for controller design(Hoppensteadt et al, 1977) and to associate eq.(5-33) with a differential equation. Alternatively, we use a slow time scale n=kh. We will adopt the first approach here. We shall associate a differential equation which characterizes the asymptotical behavior of incremental motion of slow variables with the respect to the small parameter h and sampling period. The solution of (5-33) is sought in the time-scale t=hk and this equation can be rewritten as

$$Z(t+h) - Z(t)=hA_{ss}Z(t) + hW_s(t) \qquad 5\text{-}35c$$

Divide both sides of equation (5-35c) by h and taking the limit h-->0.0, yields

$$\frac{dZ(t)}{dt}=A_{ss}Z(t)+W_s(t) \qquad 5\text{-}36a$$

with the measurement equation

$$U(t)=C_sZ(t) + V_s(t) \qquad 5\text{-}36b$$

The following condition has to be satisfied.

The pair (A_{ss}, C_s) must be detectable in continuous sense.

If this condition is satisfied, the slow filter can be devised as

$$\frac{d\hat{Z}(t)}{dt} = A_{ss}\hat{Z}(t) + K_1[U(t) - C_s\hat{Z}(t)] \qquad 5\text{-}37$$

where

$$K_1 = PC_s^T R_s^{-1} \qquad 5\text{-}38$$

$$A_{ss}P_1 + P_1A_{ss}^T + Q_s - P_1C_s^T R_s^{-1} C_s P_1 = 0 \qquad 5\text{-}39$$

As is given by eq.(5-20), we have the slow filter in discrete-time domain, in natural state

$$\hat{X}_1(k+1) = (I + hA_s)\hat{X}_1(k) + hA_{12}\tilde{\eta}(k) + hK_1[Y(k) - C_1\hat{X}_2(k) - C_2A_{22}\tilde{\eta}(k)] \qquad 5\text{-}40$$

If the detectability condition is satisfied, the matrix

$$A_1 - K_1C_s = A_{11} + A_{12}(I - A_{22})^{-1}A_{21} - K_1\{C_1 + C_2(I - A_{22})^{-1}A_{21} - C_2A_{22}(I - A_{22} + K_2C_2A_{22})^{-1}K_2[C_1 + C_2(I - A_{22})^{-1}A_{21}]\} = A_c \qquad 5\text{-}41$$

is asymptotically stable in continuous sense. And for small h, $hA_c + I$ is asymptotically stable in discrete sense. In this filter design, we see that the slow filter is dependent of the fast filter. The fast filter is independent of slow filter.

It is noted(not proof) that the term of $\tilde{\eta}(k)$ can be neglected in LQG or stochastic problem in eq.(5-40) since $E[\tilde{\eta}(k)]$ will vanish very quickly compared to the slow state variable $X_1(k)$. It means it has little affect on the estimation of state varaible $X_1(k)$.

In this case, the method devised for designing lower order filters does not only result in the lower order design, but also gives some simplification in practical implementation of control systems. Some computational time can be saved as well..

5.4 Stability

In filter or controller design, system stability is a basic basic requirement. In this section, our main purpose is to prove that the filters designed in the preceding section is stable for some small h.

Define

$$\bar{X}_2(k)=(I-A_{22})^{-1}A_{21}\bar{X}_1(k)+\tilde{\eta}(k) \qquad 5\text{-}42$$

In subsequent discussions, we use

$$e_2(k)=X_2(k)-\bar{X}_2(k) \qquad 5\text{-}43$$

instead of $\eta(k)-\tilde{\eta}(k)$ since it is more convenient. To find the law which the estimation error follows, we need the following in terms of the estimation error.

$$y(k+1)-C_1'\bar{X}_1-C_2\bar{\eta}(k+1)=[C_1'-C_2A_{22}(I-A_{22})^{-1}A_{21}]e_1(k) + C_2A_{22}e_2(k) + O(h) \qquad 5\text{-}44$$

where $\bar{X}_1(k)$ is defined as

$$\bar{X}_1(k+1)=(I+hA_s)\bar{X}_1(k)+hA_{12}\tilde{\eta}(k) \qquad 5\text{-}45$$

and $\tilde{\eta}(k)$, $\eta(k)$, and $\tilde{X}_1(k)$ are given in eq.(5-13), (5-14), and (5-40), respectively. In order to prove stability of the filter problem, we drop all noise terms as we procede because they do not affect the stability of the over all problem. Then, the error $e_1(k)=X_1(k)-\tilde{X}_1(k)$ and $e_2(k)=X_2(k)-\tilde{X}_2(k)$ are governed by

$$e_1(k+1)=\{I+h[A_{11}-K_1C_1'+K_1C_2A_{22}(I-A_{22})^{-1}A_{21}]\}e_2(k)+ h[A_{12}-K_1C_2A_{22}]e_2(k) \qquad 5\text{-}46$$

$$e_2(k+1)=[A_{21}-K_2C_1'+K_2C_2A_{22}(I-A_{22})^{-1}A_{21}]e_1(k) + (A_{22}-K_2C_2A_{22})e_2(k) \qquad 5\text{-}47$$

Rewrite eq.(5-[illegible]) as

$$e_2(k+1)=[(A_{22}-K_2C_2A_{22})(I-A_{22})^{-1}A_{21}-K_2C_1']e_1(k) + (A_{22}-K_2C_2A_{22})e_2(k) \qquad 5\text{-}48$$

To ensure the stability, we apply the decomposition technique since eq.(5-46) and (5-48) are still in two time-scale form. If the decomposed system is stable, the original system will be stable for some small h. Since the matrix $(A_{22}-K_2C_2A_{22})$ is asymptotically stable, we can, in long term run, approximate $e_2(k)$ by $\bar{e}_2(k)$, where

$$\bar{e}_2(k)=[-(I-A_{22}+K_2C_2A_{22})^{-1}K_2C_1'+(I-A_{22})^{-1}A_{21}]e_1(k) \qquad 5\text{-}49$$

Substitute $e_2(k)$ in eq.(5-46) by $\bar{e}_2(k)$, we have

$$e_1(k+1)=(I+hA_c)e_1(k) \qquad 5\text{-}50$$

If we define $e_2'(k)=e_2(k)-\bar{e}_2(k)$, we have

$$e_2'(k+1)=(A_{22}-K_2C_2A_{22})e_2'(k) \qquad 5\text{-}51$$

Due to the fact that matrix AA is asymptotically stable in the continuous sense and matrix $(A_{22}-K_2C_2A_{22})$ is asymptotically stable in discrete sense, we have the following theorem:

Theorem 2. If the pair(A_{ss},C_s) is detectable in the continous sense and the pair(A_{22},C_2) is detectable in the discrete sense, there exists a h^+ such that for all $0<h<h^+$, the slow filter (5-40) and the fast filter (5-13) and (5-14) are asymptotically stable.

5.5 Reduced order filtering of two time-scale systems

In the study of the linear filtering of two time-scale systems, we do not require matrix A_{22} to be asymptotically stable. In some practical applications, matrix A_{22} is asymptotically stable. In this case, the filter design can be further simplified. In this type of system, the fast subsystem converge much more quickly than the slow subsystem. If matrix A_{22} is asymptotically stable, $X_2(k)$ can be replaced, in long term run, by its quasi-steady state

$$\bar{X}_2=(I-A_{22})^{-1}A_{21}X_1(k)+(I-A_{22})^{-1}W_2(k-1) \qquad 5\text{-}52$$

In this expression, we use $W_2(k-1)$ instead of $W_2(k)$ because that $X_2(k)$ does not depend on $W_2(k)$ at all. In the continuous case, we do not have this problem. Substitute eq.(5-52) into eq.(5-1) and (5-3), we have the reduced order filtering problem

$$X_1(k+1)=\{I+h[A_{11}+A_{12}(I-A_{22})^{-1}A_{21}]\}X_1(k) + A_{12}(I-A_{22})^{-1}W_2(k-1)+W_1(k) \qquad 5\text{-}53$$

$$y(k)=[C_1+C_2(I-A_{22})^{-1}]X_1(k)+ (I-A_{22})^{-1}W_2(k-1)+V(k) \qquad 5\text{-}54$$

Consequently, a reduced order filter can be designed by applying uncorrelating technique and considering the 'slowness' of the slow subsystem. By using the expression (5-52), X_2 can be estimated. However this can not be used to alternate the fast dynamics of the system since it only reflects the slow part of the system.

5.6 Summary

In this chapter, we have studied linear filtering of two time-scale discrete systems. The requirement that the fast subsystem be asymptotically stable has been relaxed.

Chapter 6

Conclusion

The contributions of this thesis have been mainly twofold.

First, two multirate controller design techniques for two time-scale systems are presented, 1)multirate controller design based on system decompositon in continuous-time domain in which systems considered are continuous, linear, and two time-scale; systems are decomposed into a fast and a slow subsytems; the fast subsystem is discretized at a high sampling rate and the slow subsytem is discretized at a low sampling rate; and two controllers are independently designed for the slow and fast subsystems, respectively; and 2)multirate controller design based on system decomposition in the discrete time domain in which the system considered is discrete, linear, two time-scale obtained by discretizing the continuous system at the sampling rate which is compatible with the fast time-scale; fast controller is designed at the fast time-scale; then a slow the system in slow time-scale is obtained by system propogating and the slow controller is designed in the slow time-scale. The second method is used to prove the stability of the control systems.

Partial control for the fast subsystem is also suggested.

Second, a technique for designing lower order filters for discrete two time-scale systems is also investigated.

Two lower dimension filters are described for the system. The stability of the filters is also proven. The singular perturbation method used in treating the topics mentioned above is an approximation method. Degradation exists in the control system designed using the method given. Two numerical examples are given to illustrate the method and the degradation. It is shown that the degradation is tolerable. The designer has to make his or her own choice between the degradation and computational capacity available.

In the multirate control design proposed by Glasson(1980), a periodical Riccatti equation has to be solved, and in the multirate control design proposed by Amit(1980), equivalent single rate discrete system and cost function have to be established, which is a tedious task to be performed in terms of programming. The methods proposed in this thesis are relatively simpler. The drawback however is the requirement that the system should possess two time-scale property.

Further research can be done on the stochastic control of two time-scale or multiple time scale systems, to deveelop a method, which does not require an asymptotically stable fast subsystem.

Chapter 7

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