The Bandwidth Minimization Problem is NP-complete for lobsters and k-polygon graphs

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Abstract

Assmann et al. [SIAM J. Alg. Disc. Meth., 2 (1981), 387-393] have shown that the bandwidth of caterpillars on n vertices with hairs of length at most two can be found in $O(n \log n)$ time and Monien [SIAM J. Alg. Disc. Meth., 7 (1986), 505-512] has shown that Bandwidth Minimization remains NP-complete when restricted to caterpillars with hair length at most three. In this work it is shown that Bandwidth Minimization remains NP-complete when restricted to lobsters, despite the existing polynomial algorithms for this problem on their modules and prime decompositions. Additionally, we show the problem to be NP-complete on k-polygon graphs, for all $k \geq 3$.

1 Introduction

In 1981, Assmann et al. [1] showed that the bandwidth of caterpillars on n vertices with hairs of length at most two can be determined in $O(n \log n)$ time. Five years later, Monien [8] showed that determining the bandwidth of caterpillars of hair length at most three is NP-complete. Monien's work illustrated that the Bandwidth Minimization problem for trees becomes more difficult as one increases the distance of vertices from the longest path, also known as the backbone. In this work we will show that Bandwidth Minimization remains NP-complete for lobsters, which are trees whose vertices are at most distance two from their backbones. In terms of distance from the backbone, this result tightens the threshold established by Monien. Moreover, it provides an example of a problem which is polynomially solvable on a class of graphs, namely those found in [1], yet NP-complete when vertices are replaced by independent sets, which are the maximal modules of lobsters. Additionally, we show that Bandwidth Minimization remains NP-complete on k-polygon graphs, for all $k \geq 3$.

Let us begin by establishing the definition of the bandwidth of a graph.

Definition 1 *A labelling of a graph (simple, undirected) is an injective mapping from its vertex set to $\mathbb{Z}$.*

Definition 2 *Let σ be a labelling of a graph G with edge set E_G. The width of σ is defined as* $\max_{\{u,v\}\in E_G} |\sigma(u) - \sigma(v)|$ *and denoted by $w(\sigma)$.*

Definition 3 *Let $\mathcal{F}$ be the family of all labellings of a graph G. The bandwidth of G is defined as* $\min_{\sigma\in\mathcal{F}} w(\sigma)$ *and denoted by $b(G)$. A labelling σ of G for which $w(\sigma) = b(G)$ is called an optimal labelling or a bandwidth labelling.*

The reader should observe that for any labelling σ of a graph G on n vertices, there exists a labelling σ' of G using the labels $0, \ldots, n-1$, such that $w(\sigma') \leq w(\sigma)$. Henceforth, we will consider all labellings to be reduced in this manner. Having defined the bandwidth of a graph we present the Bandwidth Minimization problem defined by Garey and Johnson [6].

Bandwidth Minimization

Instance: Graph G and natural number k.

Question: Does G have a labelling whose width is at most k?

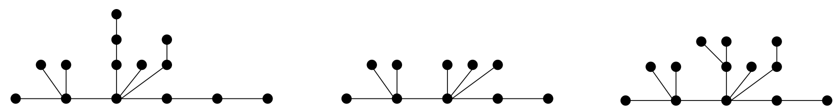
It should be noted that because the bandwidth of a graph is bounded by the number of vertices, the Bandwidth Minimization problem can be phrased as "Given a graph, determine its bandwidth".

As previously mentioned, the works of Assmann et al. [1] and Monien [8] deal with caterpillars having various hair lengths. By hairs, these authors are referring to paths attached to the backbone of the caterpillar. The reader should be cautious as many references, such as [11], define a caterpillar to be a tree whose vertices are a distance at most one from the backbone; that is, the hairs are at most length one. Based upon this definition, Bermond [2] defines a lobster to be a tree whose vertices are a distance at most two from its backbone. Lobsters have also been referred to as 2-distant trees in [9]. Examples of these graph classes are given in Figure 1.

In [8], Monien's proof of the NP-completeness of caterpillars with hairs of length at most three involves a reduction from the strong NP-complete Multiprocessor Scheduling problem. Garey and Johnson [6] state this problem as follows.

Multiprocessor Scheduling

Instance: Set T of tasks, number $m \in \mathbb{N}$ of processors, length $l(t) \in \mathbb{N}$ for each $t \in T$, and a deadline $D \in \mathbb{N}$.



(a) A caterpillar (Assmann et al., Monien).

(b) A caterpillar (West).

(c) A lobster.

Figure 1: Examples of caterpillars and lobsters.

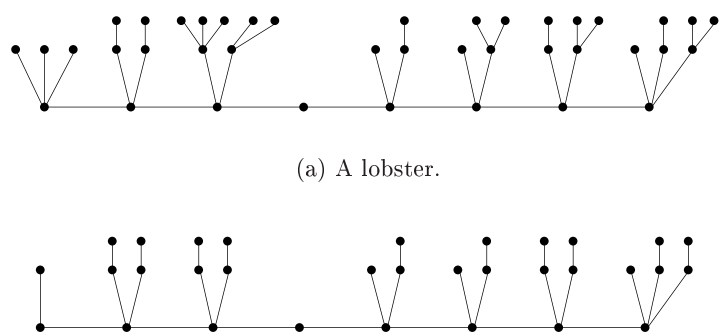
Question: Is there an m-processor schedule for T that meets the overall deadline D, i.e. a function $f : T \longrightarrow \mathbb{N} \cup \{0\}$ such that, for all $u \geq 0$, the number of tasks $t \in T$ for which $f(t) \leq u < f(t) + l(t)$ is no more than m and such that, for all $t \in T$, $f(t) + l(t) \leq D$?

The particular version of Multiprocessor Scheduling used by Monien requires a "perfect fit", that is, $\sum_{t \in T} l(t) = Dm$. We too shall use this version of Multiprocessor Scheduling in our proof of the NP-completeness of Bandwidth Minimization for lobsters, which mirrors that of Monien. As this version remains strong NP-complete, we may assume that the values $l(t)$ are polynomial in n. This ensures that our transformation from Multiprocessor Scheduling to Bandwidth Minimization is polynomial.

The NP-completeness of Bandwidth Minimization on lobsters is particularly interesting when one considers their prime decomposition by maximal proper modules. A module of a graph G is defined as a set M of vertices such that for $v_1, v_2 \in M$ and $w \in V_G \setminus M$, $\{wv_1\} \in E_G \Leftrightarrow \{wv_2\} \in E_G$. By a prime decomposition of a graph by modules we mean a contraction of the proper modules until no further contractions can be made. For further reading on modules and prime decompositions see [3]. Prime decompositions are commonly used in algorithms to solve problems on graphs by first solving the problem on both the prime decomposition and the modules, then using a technique to combine these solutions. As an example, prime decompositions have been used to solve Bandwidth Minimization on cographs in [7].

In contrast, Bandwidth Minimization on lobsters is an example of a problem which has a solution on both the prime decomposition and the modules, but is NP-complete on the class itself. The maximal modules are independent sets of pendant vertices, which have trivial bandwidth labellings. The prime decomposition of a lobster is exactly a caterpillar with hairs of length at most two, as per the definition of Assmann et al. [1], who have shown that Bandwidth Minimization can be solved in $O(n \log n)$ time on that class. An example of a prime decomposition of a lobster is given in Figure 2.

In addition to proving the NP-completeness of Bandwidth Minimization for lobsters we also prove the NP-completeness of this problem on 3-polygon graphs as the lobster generated in the reduction proof is also a 3-polygon graph. Elmallah and Stewart [5] define a graph to be a k-polygon graph if it is the intersection graph of chords in a k-gon, where each chord must have its endpoints on different sides of the k-gon. From this definition we see that the result holds for $k \geq 3$. It should be noted that the proof of Monien [8] does not give the NP-completeness of Bandwidth Minimization on 3-polygon graphs as the caterpillar used in the reduction proof is not a 3-polygon graph.



(a) A lobster.

(b) Its prime decomposition.

Figure 2: The prime decomposition of a lobster results in a caterpillar with hairs of length at most two.

2 NP-completeness

Our proof of the NP-completeness of Bandwidth Minimization on lobsters relies upon two particular graph structures. Monien [8] defines the turning point of height p, denoted by T_p, to be the graph shown in Figure 3. As well, he defines the barrier of height p, denoted by B_p, to be the graph shown in Figure 4.

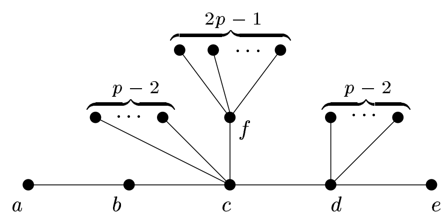


Figure 3: The turning point of height p, denoted by T_p.

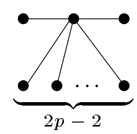


Figure 4: The barrier of height p, denoted by B_p.

In establishing necessary results on turning points and barriers, the following lemma from Chvátal [4] proves quite useful.

Lemma 4 (Chvátal) *For any graph G on n vertices having diameter d,* $b(G) \geq \left\lceil \frac{n-1}{d} \right\rceil$.

Now consider the following results based upon Lemma 4.

Lemma 5 $b(T_p) = p$.

Proof. From Lemma 4 we obtain $b(T_p) \geq p$. If P_c, P_d, and P_f are the sets of pendant vertices, excluding e, adjacent to c, d, and f respectively, then the labelling σ defined by $\sigma(a) = 1$, $\sigma(b) = p + 1$, $\sigma(c) = 2p$, $\sigma(d) = p$, $\sigma(e) = 0$, $\sigma(f) = 3p$, $\sigma[P_c] = \{p + 2 \ldots, 2p - 1\}$, $\sigma[P_d] = \{2 \ldots, p - 1\}$, and $\sigma[P_f] = \{2p + 1 \ldots, 3p - 1, 3p + 1, \ldots, 4p\}$, gives that $b(T_p) = p$. □

Lemma 6 *In any optimal labelling of T_p, c is labelled $2p$.*

Proof. Let σ be an optimal labelling of T_p and let v_i be the vertex for which $\sigma(v_i) = ip$, $0 \leq i \leq 4$. Since $w(\sigma) = p$ and $\sigma(v_4) - \sigma(v_0) = 4p$, the vertices v_0 and v_4 must be a distance at least four from each other. Yet T_p is a tree of diameter four, so v_0 and v_4 are distance four apart, and the unique shortest path between them is v_0, v_1, v_2, v_3, v_4. Since c is the central vertex of all paths of length four in T_p, $\sigma(c) = 2p$. □

Observation 7 *Consider a tree T containing a subtree T' of diameter d on $d \cdot b(T) + 1$ vertices. In any optimal labelling of T, T' must receive $d \cdot b(T) + 1$ consecutive labels. In particular, in any optimal labelling of T_p, the induced subgraph on $N[f]$, the closed neighbourhood of f, must receive $2p + 1$ consecutive labels.*

Lemma 8 *If σ is an optimal labelling of T_p, then either $\sigma(c) \geq \sigma(v)$, or $\sigma(c) \leq \sigma(v)$, for all $v \in N[f]$.*

Proof. Since $|N[f]| = 2p + 1$, Lemma 6 and Observation 7 give that either some vertex in $N[f] \setminus \{c\}$ has label p, or some vertex in $N[f] \setminus \{c\}$ has label $3p$. Recall from the proof of Lemma 6 that the vertices labelled ip, $0 \leq i \leq 4$, form a path of length four. Since f is the only vertex in $N[f]$ adjacent to c, f must be labelled either p or $3p$. If $\sigma(f) = p$, then $\sigma(c) \geq \sigma(v)$ for all $v \in N[f]$. Otherwise, $\sigma(c) \leq \sigma(v)$ for all $v \in N[f]$. □

Corollary 9 *If σ is an optimal labelling of T_p, then either $\sigma(a), \sigma(e) < \sigma(c)$, or $\sigma(a), \sigma(e) > \sigma(c)$.*

Given an instance $\Upsilon(\{l(t_1), \ldots, l(t_n)\}, D, m)$ of Multiprocessor Scheduling, for which $\sum_{i=1}^{n} l(t_i) = Dm$, we construct the lobster $\mathcal{L}$ shown in Figure 5 in order to prove the NP-completeness of Bandwidth Minimization on lobsters. We define the "ground line" of $\mathcal{L}$ to be the part of the backbone from the outermost point of the barrier of height β up to, and including, the closest vertex of the turning point. Similarly, the "sweeping line" of $\mathcal{L}$ is defined as the remainder of the backbone, less the vertices of the turning point. The "i^{th} block", $1 \leq i \leq n$ is defined to be the induced subgraph on the vertices in the chain of length t_i in the sweeping line and the pendant vertices attached to them. The reader should observe the correspondence between blocks and tasks, and as such we will denote both by t_i. The terms ground line, sweeping line, and i^{th} block will be used in the proofs of Lemmas 10 and 11 which show that, for appropriate choice of p, Υ has a solution if and only if $\mathcal{L}$ has bandwidth $\beta = p + 2n + 1$.

Lemma 10 *If Υ has a solution then $\mathcal{L}$ has bandwidth $\beta = p + 2n + 1$.*

Proof. Let $\lambda = m(D + 2) + 1$ and label the vertices of the ground line as $i\beta, 0 \leq i \leq \lambda - 1$ beginning at the barrier of height β and working towards the turning point. Then label the remaining vertices of the barriers as shown in Figures 6 and 7. By the argument found in the proof of Lemma 5 we can label the turning point with the labels $\lambda\beta + j, 0 \leq j \leq 4\beta$, where in

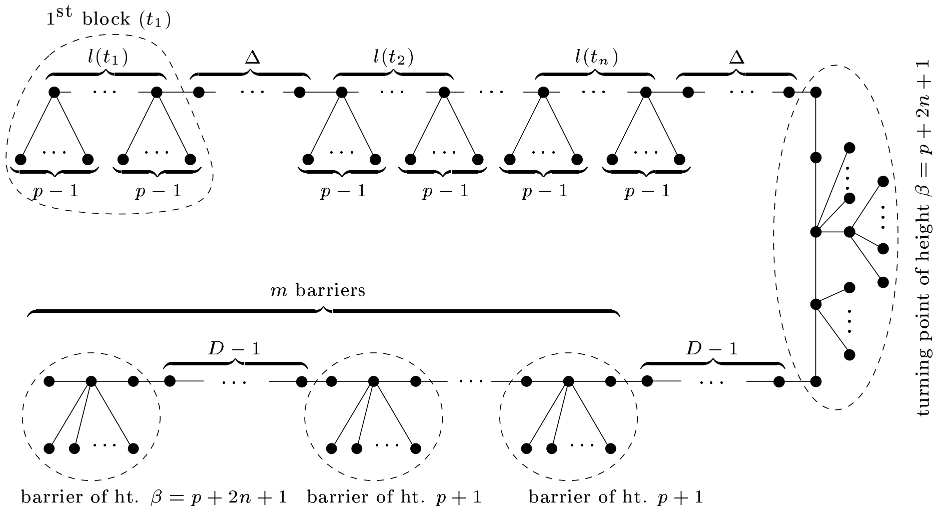


Figure 5: The lobster $\mathcal{L}$ constructed from the instance $\Upsilon = (\{t_1, \ldots, t_n\}, D, m)$ of Multiprocessor Scheduling, where $\Delta = 2(m(D+2)-2)$ and $p > 0$.

particular e and a are labelled $\lambda\beta$ and $\lambda\beta + 1$, respectively. All remaining nodes (sweeping line and its hairs) will get labels smaller than $\lambda\beta$.

Since Υ has a solution there exists $I_j, 0 \leq j \leq m-1$ such that $\bigcup_{j=0}^{m-1} I_j = \{1, \ldots, n\}$ and $\sum_{i \in I_j} l(t_i) = D$ for all $j \in \{0, \ldots, m-1\}$. For each $i \in \{1, \ldots, n\}$, if $i \in I_j$ then the vertices of the i^{th} block should be labelled so as to lie between the j^{th} and $j+1^{\text{st}}$ barriers as shown in Figure 8. This leaves $2n$ labels between each pair of members, $i\beta$ and $(i+1)\beta$, of the ground line, except those in the barrier of height β. That is, there are $n\Delta$ labels left for the $n\Delta$ vertices in the n chains of length Δ.

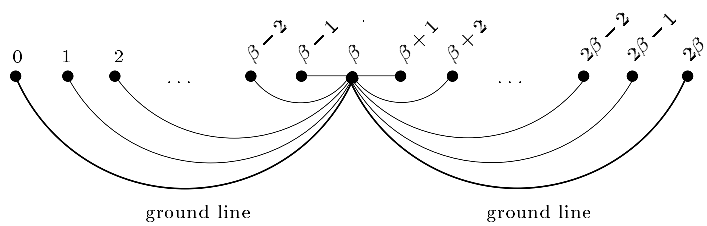


Figure 6: Labelling of the barrier of height β

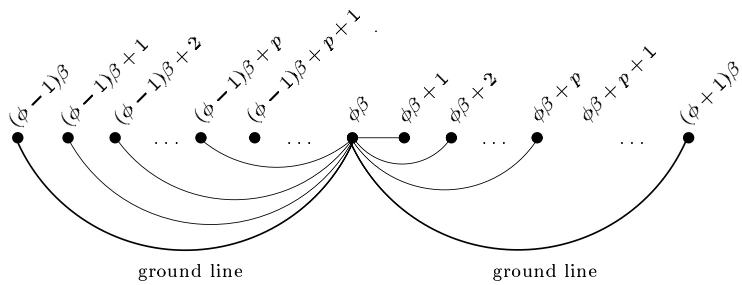


Figure 7: Labelling of a barrier of height $p+1$, where $\phi = (i(D+2)+1)$, $1 \leq i \leq m-1$.

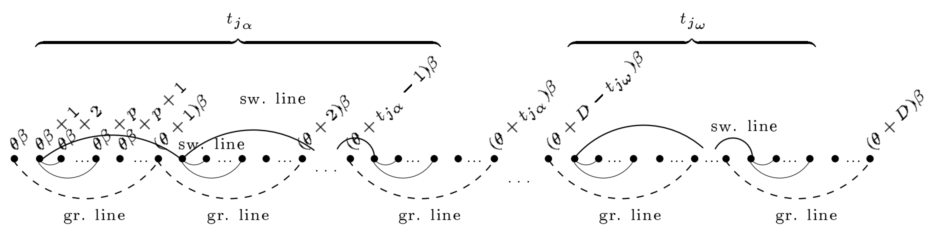


Figure 8: Placement of the blocks belonging to processor j, where $\theta = j(D+2)+2$, $0 \leq j \leq m-1$. Note that t_{j_α} refers to the arbitrary block which receives the lowest labels and t_{j_ω} refers to the arbitrary block which receives the highest labels.

We label the vertices of the chains of length Δ such that two vertices of each chain lie between each pair of the vertices of the ground line. Consider a chain C connecting t_i and t_{i+1} where, without loss of generality, $i \in I_{j_1}$ and $i+1 \in I_{j_2}$, where $j_1 \leq j_2$. We label this chain as shown in Figure 9. Having labelled each of the chains in this manner we have completed a labelling of $\mathcal{L}$ that is of width β. By Lemma 4 we have that $b(\mathcal{L}) = \beta$. $\square$

Lemma 11 *If $p > 2n(D+4)$ and if $\mathcal{L}$ has bandwidth $\beta = p + 2n + 1$ then $\Upsilon(\{l(t_1), \ldots, l(t_n)\}, D, m)$ has a solution.*

Proof. Let σ be an optimal labelling of $\mathcal{L}$ such that $w(\sigma) = \beta$. First we show that σ uniquely labels the ground line, up to symmetry. Recall that from Observation 7 the turning point must be labelled with $4\beta+1$ consecutive numbers. By Corollary 9 we know that the labels of the vertices of the ground line and sweeping line are either all less than or all greater than

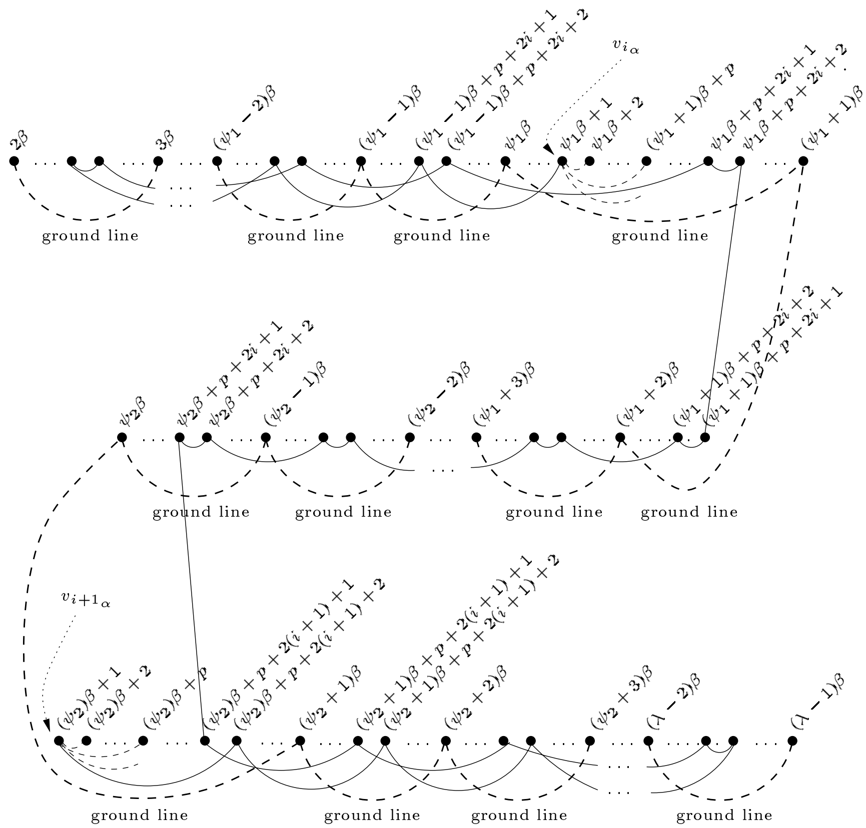


Figure 9: The placement of the chain of length Δ which joins the i^{th} block to the $i+1^{\text{st}}$ block. Note that v_{i_α} and v_{i+1_α} refer to the vertices from the i^{th} and $i+1^{\text{st}}$ blocks, respectively, which receive the smallest labels.

those of the turning point. Without loss of generality assume that they are all less than those of the turning point and that the vertices of the turning point receive the labels $m(D+2)\beta+j$, $0 \leq j \leq 4\beta$. Recall from Observation 7 that the barrier of height β must receive $2\beta+1$ consecutive labels. Since no edge can cross this barrier it receives the labels $0, \ldots, 2\beta$. Between the turning point and the barrier of height β there exist $m(D+2)-3$ vertices in the ground line which must be labelled. The turning point and the barrier of height β are connected by a path of length $m(D+2)-2$ in the ground line where the labels range from 2β to $m(D+2)\beta$. Thereby, the members of the ground line must receive the labels $i\beta$ for $0 \leq i \leq m(D+2)$. That is, every labelling of width β numbers the ground line and the turning point in the same way, up to symmetry.

We now need to show that if $\mathcal{L}$ has bandwidth β, an optimal labelling of $\mathcal{L}$ describes a solution of Υ. Observe that in an optimal labelling σ, the centers of the barriers have the labels $Z_j = (j(D+2)+1)\beta$ for $0 \leq j \leq m-1$. In Υ, let task i belong to the j^{th} interval if and only if $Z_{j-1} < \sigma(u) < Z_j$ holds for some node u in the sweeping line of the i^{th} block.

First we show that a task cannot belong to two different intervals. Assume task i belongs to two different intervals. Then there exist adjacent vertices u, v in the chain of length t_i of the sweeping line such that $\sigma(u) < Z_j < \sigma(v)$ for some j. Let w be the vertex for which $\sigma(w) = Z_j$. Let $\delta_1 = \sigma(u) - (Z_j - \beta)$ and $\delta_2 = Z_j + \beta - \sigma(v)$. Since u is adjacent to v, $\sigma(v) - \sigma(u) \leq \beta$, so $\delta_1 + \delta_2 \geq \beta$. This scenario is depicted in Figure 10. There are $p-1$ pendant vertices adjacent to each of u and v as well as $2p$ pendant vertices adjacent to w. At most $\beta - \delta_1$ pendant vertices adjacent to u can get labels less than $Z_j - \beta$, and at most $\beta - \delta_2$ pendant vertices adjacent to v can get labels greater than $Z_j + \beta$. Thereby, $5 + 2p + 2(p-1) - (\beta - \delta_1) - (\beta - \delta_2) \geq 3 + 4p - \beta = 2 + 3p - 2n$ nodes have to use the $2\beta + 1$ labels from $Z_j - \beta$ to $Z_j + \beta$. But $n \geq 1$ and $p \geq 8n$, so $2 + 3p - 2n \geq 2 + 2p + 6n \geq 4 + 2p + 4n > 3 + 2p + 4n = 2\beta + 1$, that is, there are too many vertices for the $2\beta + 1$ labels. Thereby, every task belongs to exactly one interval.

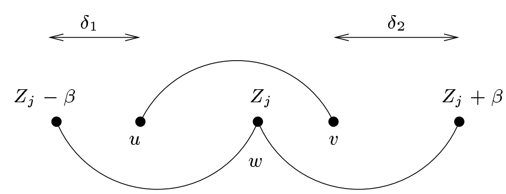


Figure 10: Two vertices u and v in the sweeping line corresponding to a task belonging to two intervals.

It remains to show that for all j, $\sum_{i \in I_j} t_i \leq D$. The labelling σ gives the $(D+4)\beta+1$ labels from $Z_{j-1} - \beta$ to $Z_j + \beta$ to all the vertices associated with task j (of which there are $p \sum_{i \in I_j} t_i$) as well as to the $4p$ pendant vertices of the two barriers and the $D+5$ vertices of the ground line. Therefore,

$$
\begin{aligned}
p\sum_{i\in I_j} t_i + 4p + D + 5 &\le (D+4)\beta + 1 \\
&= (D+4)(p+2n+1)+1 \\
&= pD + 2nD + D + 4p + 8n + 5,
\end{aligned}
$$

giving

$$
\begin{aligned}
\sum_{i\in I_j} t_i &\le \frac{pD + 2nD + 8n}{p} \\
&= D + \frac{2nD + 8n}{p} \\
&< D + 1.
\end{aligned}
$$

Yet $\sum_{i\in I_j} t_i$ is an integer so $\sum_{i\in I_j} t_i \le D$, and the result holds. □

Using Lemmas 10 and 11 we get our main result.

Theorem 12 *BANDWIDTH MINIMIZATION remains NP-complete when restricted to lobsters and k-polygon graphs, for all $k \ge 3$.*

Proof. Given a natural number k and a lobster or 3-polygon graph G with labelling σ, it can be determined in $O(|E_G|)$ time if the width of σ is at most k, so Bandwidth Minimization is in NP for lobsters and 3-polygon graphs. Given an instance $\Upsilon(\{l(t_1), \ldots, l(t_n)\}, D, m)$ of the strong NP-complete "perfect fit" Multiprocessor Scheduling we construct the lobster $\mathcal{L}$ in polynomial time, and by Lemmas 10 and 11 we obtain that Bandwidth Minimization is NP-complete on lobsters. As shown in Figure 11, $\mathcal{L}$ is also a 3-polygon graph, so the problem is also NP-complete for 3-polygon graphs. By the definition of k-polygon graphs, Bandwidth Minimization is NP-complete on k-polygon graphs, for all $k \ge 3$. □

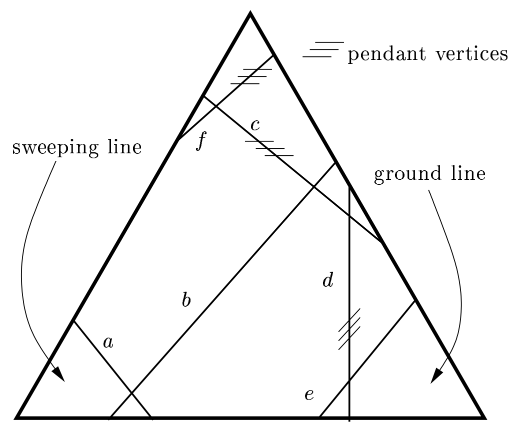


Figure 11: The lobster $\mathcal{L}$ as a 3-polygon graph.

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