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PIECE-WISE CONGRUENT REGIONS, THEIR AREA MEASURE STRUCTURE, AND GEOMETRIC THINKING PROCESSES

by

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A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES AND RESEARCH IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

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Date March 27, 1981

DEDICATION

To my beloved father

Hishmat Rahim

and

my beloved mother

Hadhiyah Shehab AL-Hemed,

all of blessed memory,

who sent me to school.

To

my wife Sonia, son Leith,

and daughter Sally without

whose understanding and

sacrifice, this study would

not have been possible

ABSTRACT

The purpose of this study was to investigate and identify some of the geometric thinking processes used by eighth grade students in attempting to solve certain spatial problems on polygonal regions and to inquire into how the students justified and explained their performance. In particular, the interest was to explain how grade eight students understand the area concept using piece-wise congruency. Two polygonal regions are said to be piece-wise congruent if one of them can be cut into a finite number of subregions such that they can be rearranged to cover the other region completely. In the inquiry special emphasis was placed on: (1) the effectiveness of physical manipulation on students' performances in certain spatial problem-solving situations encountered; (2) the kind of intuitive reasonings students appeared to exhibit; (3) transfer of learned properties into new situations; (4) the type of errors students appeared to exhibit in justifying their responses; (5) patterns students seemed to form and use in attempting certain spatial problems; (6) the possible utilization of the motions of slide, turn, and flip throughout these problems; (7) the possible adaptation of the unit (Appendix A) as a teaching material through which learning of basic concepts in geometry and in elementary mathematical analysis might take place; and (8) whether the unit and the evaluation tasks (see page 58-61) utilized in the study have any potential value for classroom teachers' use as a tool for diagnosing students' misunderstanding of certain geometric concepts.

Based on pilot testings, an instructional unit was constructed by the experimenter, employing the piece-wise congruency between polygonal regions; teaching aids materials including teachers' manual; and eleven evaluation tasks for interviews.

A sample of 58 grade eight students in two classes was utilized. The classes differed considerably on the final grade 7 mathematics test. From the school's point of view, one of the classes was known as the 'best' and the other was known as the 'poorest' among the grade 8 classes at the school. Twelve students, six from each class, were interviewed prior to the instruction of the unit; the eleven evaluation tasks were used in each interview. The performances of the students on each evaluation task were analyzed and performance categories were developed on the basis of the type of patterns students appeared to exhibit. As such, the processes students appeared to follow in each of the evaluation tasks were characterized by these response-category patterns.

The responses of the students to each of the Geometry Tests were examined and response categories for each item were developed. Accordingly, a score distribution scheme by category for each test was designed which was used to analyze statistically the students' responses and to describe the kind of achievement exhibited.

Qualitatively, the results of the study indicated that: (1) throughout each of the evaluation tasks on piece-wise congruency students seemed to be looking for pairs of congruent edges, angles, or both (learned properties of the linear and angular measure systems were used in the area measure system); (2) a piece-wise congruency of two polygonal regions of equivalent areas could be attained only when a linear congruency, angular congruency, or both were attained; (3) students exhibited surprising patterns, through independent discoveries, in dealing with Pythagorean theorem in particular and with the investigative activities on decomposition of polygonal regions in general; (4) throughout the evaluation tasks, students' processes were characterized by a sequence of manual-mental acts: Superposing one of the regions on the other with two corresponding edges being superposed along each other; cutting one of the regions along the edges of the other; carrying over the resulting subregions and patching them on the other region; repeating the last two steps until either a successful completion of the task or a failure; and (5) students seemed to recognize that two polygonal regions of equal areas were piece-wise congruent only when one of them was decomposed into subregions and superposed to completely cover the other region simultaneously. If the subregions were reassembled into their original region, the students would indicate that the two regions were no longer congruent by pieces. That is, students at the grade eight level appeared to be unable or unwilling to think sequentially, utilizing only static mental representation.

Quantitatively, the results of the study indicated that: (1) there was no significant difference in the students attitude toward geometry and mathematics over the period of the study; (2) students' achievement on the Geometry Tests were significantly different on (a) differentiation and identification of polygonal regions; (b) equidecomposition of polygonal regions of equal areas; and (c) comparison of rational numbers via geometric representations.

Consequently, the piece-wise congruency approach enhanced students' understanding of (a) the meaning of the term 'polygonal'; (b) interrelationships among various polygonal regions; (c) fractions which belong to the same equivalence classes; and (d) the importance of a chosen unit in any visual comparison between rational numbers; and (3) analysis contrasting the two classes on the pretest and post-test indicated that although the 'best' class showed considerable gains, the piece-wise congruency approach was more beneficial for the 'poorest' class.

Finally, the piece-wise congruency unit appeared to be successful on three counts: (1) the student achievement; (2) the students' eliciting of a variety of interesting and surprising patterns through hand-mind activities; and (3) the participating teachers' acceptance and adaptation of the unit.

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CHAPTER I

INTRODUCTION

A. Background to the Problem

Geometry is a subject rich in history and broad in scope; it took its rise from practical activities and from the problems of daily life. The properties of geometric concepts as well as the concepts themselves, have been abstracted from the world around us. It was necessary for people to draw many straight lines before they developed the axiom that a straight line can be drawn through any two arbitrary distinct points. They had to move various plane figures about and apply them to one another on many occasions before they could come to generalize their experiences to the notion of superposition of geometric figures and employ this notion for the proof of theorems. This was done in the famous theorems about the congruence of triangles.

The earliest existing evidences of human activity in the field of geometry are some baked clay tablets unearthed in Mesopotamia and believed to date about 3000 B.C. The great pyramid of Gizeh in Egypt was erected about 2900 B.C. (Eves, 1963). What is known as the Pythagorean Theorem was also known to the Babylonians as far back as approximately 2000 B.C. (Tian-Se, 1978).

The content of geometry and the approach taken to it have remained almost unchanged for a long time. The "Elements" of Euclid was the basic reference for geometry up to the time of Janos Bolyai (Hungarian mathmatician, 1802-1860) and Nikolai Lobachevsky (Russian geometer, 1793-1856). It was then that non-Euclidean geometry was introduced by replacing Euclid's fifth postulate with a contradictory postulate. This is often classified as one of the most significant events of mathematics, after more than two thousand years of unchanging acceptance of the "Elements" of Euclid (Aleksandrov et al., 1969).

A firm foundation of knowledge in spatial concepts is most important for study in geometry. This requires an extension of our knowledge about the child's conception of space and geometry and makes it of immediate importance. Martin (1976a) presented much information about how children, especially those in early grades, perceived geometrical and spatial concepts. Martin stated that, despite the wealth of information available, we know little about the child's conception of space. Robinson (1976) indicated that although geometry has not always been viewed as necessary as arithmetic in every day life, it constitutes an honorable branch of mathematics and deserves a place in the education of children. Considering the many different approaches available to geometry, Robinson regards Euclidean space as a cornerstone in high school geometry programming. She stated,

> At the present time, the high school geometry course is under attack, with heated debates among the proponents of vector approach, a transformation approach, and electic aporoach, and those who favor the traditional course. Yet regardless of how the matter is resolved, Euclidean space will most likely be the center core. (p. 26)

Since the 1960s there have been many questions raised and many statements made about geometry concerning: what geometry should be adopted in the curriculum; why it should be there; if it should be there; when it should be there and how it should be taught (Martin, 1976a). Consequently many conferences took place and many reports were made. Major reports among those are Goals for School Mathematics, The Report of the Cambridge Conference on School Mathematics (1963) in the U.S.A., Mathematics in the Primary School (1965) in England, and Geometry, Kindergarten to Grade 13, Report of the (K-13) Geometry Committee (1967) in Canada. They urged that the approach to geometry, especially during the elementary and junior high years, be highly intuitive with intensive use of concrete materials. Each of these reports stressed that more geometry should be embedded in the school mathematics program throughout the years of elementary and secondary school levels. Similary recommendations can be found in the 36th yearbookd of the National Council of Teachers of Mathematics, Geometry in the Mathematics Curriculum (Henderson, 1973).

One side of the geometry controversy was perhaps well expressed by Gearhart (1975). He reported that teachers of high school tend to reaffirm the importance of geometry in the secondary curriculum and that, in the teachers' opinion, many high school students do not enjoy the subject and do not experience success with it. A similar statement, but for earlier levels, was made when Williford (1972) stated that research indicate that a majority of very young children can show a variety of geometric abilities with a wide range of geometric problems. Nevertheless, Williford added, teachers and students alike, in both levels of schooling, elementary and secondary, experience difficulties with geometry, dislike of geometry, or both.

Although a variety of approaches is used at the present time, in the instruction of geometry, Euclidean properties of the geometric objects studied are the basic common part for almost all of them. Thus, the emergence of the ability to operate in Euclidean space is of great importance to the study of geometry throughout the school mathematics program. And hence, success for an approach to geometry depends, in considerable part, on the experience of the elementary and high school students in Euclidean space. How to meet this crucial need successfully is, in fact, one of the major problems in designing a program in geometry.

Wittenberg (1963) brought up some important issues in his criticism of the treatment of one fundamental topic in elementary geometry, namely the area of plane figures, as given in the revised edition of the Geometry text developed by the School Mathematics Study Group (SMSG) for use in high school. He stated that the treatment was unsatisfactory on at least three counts:

1. as an elementary introduction to geometry and geometrical thinking.
2. as an introduction to logically precise mathematical thinking, and
3. as an introduction to the modern conception of a deductive system.

Wittenberg described it as unsatisfactory in that

a. the text misleads the student, and
b. the text is "definitely poorer than it easily could be at the same level of difficulty."

Wittenberg was in disagreement particularly on the approach adopted by the text on polygonal regions and their areas and rejected the immediate entry of the notion of area as a real-valued function. It appears that Wittenberg was in favor of a prerequisite unit on polygonal regions and their areas before the real-valued function approach. Thus the immediate use of the real-valued function approach to the notion of area as it emerges in the SMSG text was rejected and described as "blind manipulation of formulas...as misleading as it is unsatisfactory."

In Wittenberg's opinion, the textbook proof for 'two triangles with equal altitudes and equal bases have equal areas', is an example supporting the above assertion. The textbook states: "the proof of this is clear because the formula A = 1/2 bh gives the same answer in each case." To which Wittenberg offers the following comment:

> Thus, "having the same area" never takes on any other meaning than this: the formula happens to yield the same value. But this blind manipulating of formulas is as misleading as it is unsatisfactory. For polygonal regions "having the same area" can easily be given a meaning that is at once intuitive and precise: it simply means that one region can be cut up into pieces with which we can build up the other. (Wittenberg, 1963, p. 453)

Obviously, Wittenberg is referring to the idea of equidecomposable figures. These two triangular regions with equal bases and equal altitudes, even if one looks long and thin and the other wide and small (Fig. 1a and Fig. 1b), have equal areas.

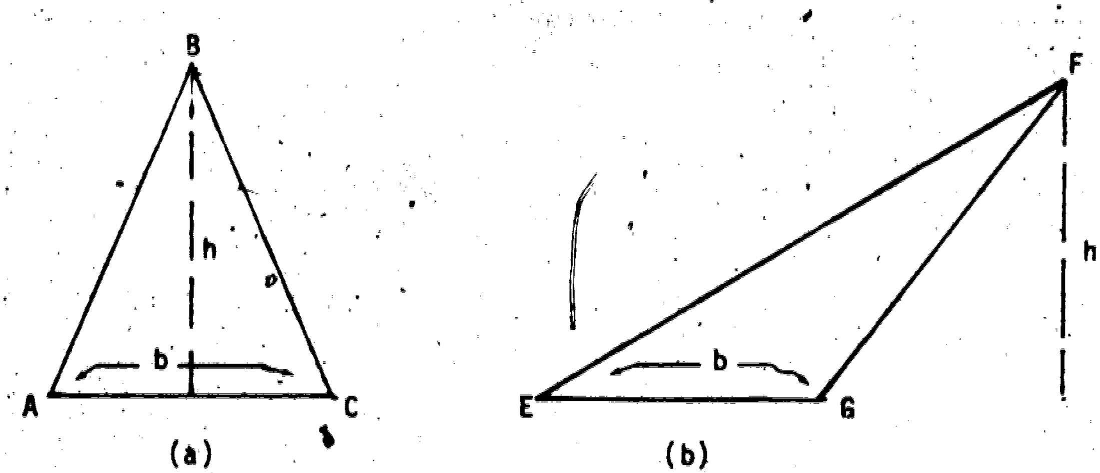


Figure 1

This follows from the fact that both are equivalent to the same rectangular region, that is, each of them can be decomposed into the same rectangular region of the same base but half the altitude of the triangular regions (Fig. 2).

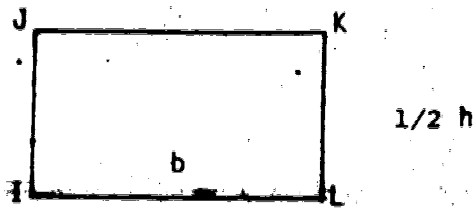


Figure 2

In other words, both triangular region ABC and triangular region EFG are equidecomposable with the rectangular region IJKL (Fig. 3a and Fig. 3b).

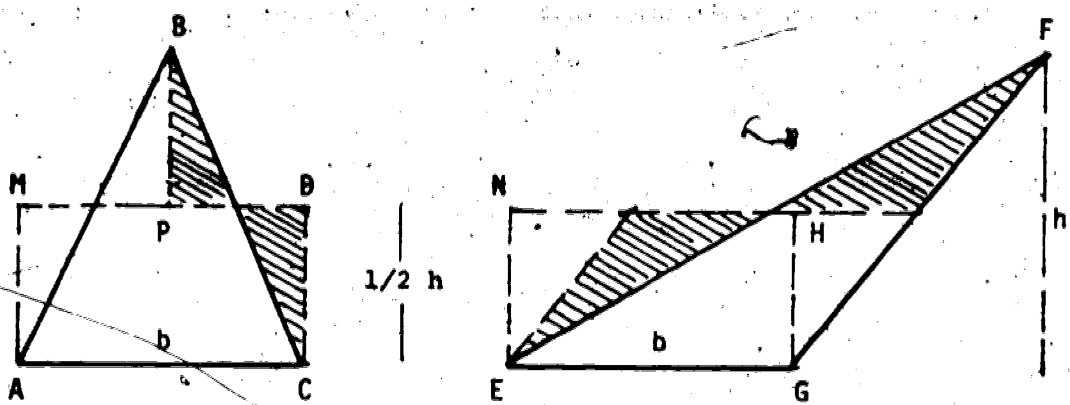


Figure 3

To Wittenberg, many facts about polygonal regions are easily reached by "simple and direct geometric reasoning without any reference to measurement of areas". And that "once a student has a firm grasp of this simple situation, it may be interesting for him to learn that things are not so simple in space.".

In defense of the SMSG approach, Moise (1963) proposed that (1) "the synthetic" concept of area or "the equal-area" concept is not going to be important to the student in the years to come, (2) lengths, areas and volumes are measured in real numbers, and (3) "in the context of modern mathematics and science, the ideas of equal-length and equal-area...simply do not deserve to be regarded as central concepts." Therefore, Moise appears to believe that these reasons are sufficient for presenting area as a real-valued function immediately.

In Moise's words:

> Obviously, if the students are going to get numerical answers to area problems, then we must, at some stage, introduce area as a real-valued function. And granted that we are going to do this eventually, we should do it soon, so that the full force of the theory can be used for things that it is good for. It leads to an easy proof of the Pythagorean Theorem. (Moise, 1963, p. 463)

Finally, Moise indicates two additional reasons concerning his decision to approach area and volume as real-valued functions:

> This is the prevailing practice, for two good reasons: it is a concept that young students can learn and use, and it is the concept that is needed, in applications and in further study of mathematics. (Moise, 1963, p. 466)

B. The Problem

One of the most interesting questions related to the child's mathematical development is how children perceive and conceptualize spatial figures. We do not know precisely how children think geometrically, and in particular, there exists no clear explanation of how children learn the area concept.

Mathematics educators and developmental psychologists have been focusing on the child's recognition and discrimination applied to plane figures. Piaget and Inhelder [illegible] and 1971) hypothesize that children are initially able to deal only with topological properties of figures and only at later stages develop the necessary cognitive structures for perceiving Euclidean properties. Attempts have been made by researchers to verify this hypothesis with contradictory results (Jahoda, Deregowski, and Simbia (1974), and Martin (1976c)). Martin (1976b) stated:

> As a developmental psychologist, Piaget does not always use mathematical language as precisely as the mathematician might desire. But although making implication for mathematics education is not a primary objective of Piaget, mathematics educators do make inferences from his research. One purpose of this paper has been to point out some of these difficulties involved in making such inferences. Another purpose has been to demonstrate that Piaget's evidence that topological representation precedes Euclidean and projective representation, though often tantalizing, is not unequivocal. (Martin, 1976b, p. 24).

There have been continuous changes in the mathematics curriculum for elementary and secondary levels throughout the 1950s and 1960s due to the efforts and concerns of mathematicians and mathematics educators (DeVault _et_ _al_., 1968, p. 31). This need to update school mathematics programs continued over the 1970s (NACOME, 1975) and there is little reason to expect that the extent of innovation will decrease in the near future (Prime-80, 1978 and Davis and McKnight, 1979). An emphasis has been placed on the structural nature of mathematics. This is, perhaps, due to the fact that, from the axiomatic point of view, mathematics appears to be a storehouse of mathematical structures (Bourbaki, 1950). Thus, many approaches to geometry have been introduced, such as analytic, transformation, vector or synthetic with agreement reached neither on a particular approach nor on the general purpose of geometry in the curriculum. However, mathematics ed cators have agreed that more geometry should be included and should begin with earlier stages (Martin, 1976b).

Regardless of the disagreement about which approach to geometry should be adopted in the instruction, they all depend almost entirely on Euclidean properties of geometric figures. Therefore, the ability to operate in Euclidean space is crucial to the study of geometry. Moreover, as is mentioned above, success in geometry, regardless of the approach taken, depends on the experience of the elementary and high school students in Euclidean space.

Thus, research is needed to provide information on possible approaches and information on the way children learn geometry. The problem under consideration in this study falls into this category of research.

The major purposes of this study are:

1. To provide an analysis of the way students in grade eight learn the area and the area measure concepts, especially those for polygonal regions.

2. To observe closely how the students act on the testing instrument before taking the unit, how they act throughout the period of instruction, and how they experience the consequences of their acts.

3. To investigate whether or not the student constructs his own explanations independent of the context.

4. To investigate the relationship, if any, between the ability to equidecomposing polygonal regions and understanding the area and area measure concepts.

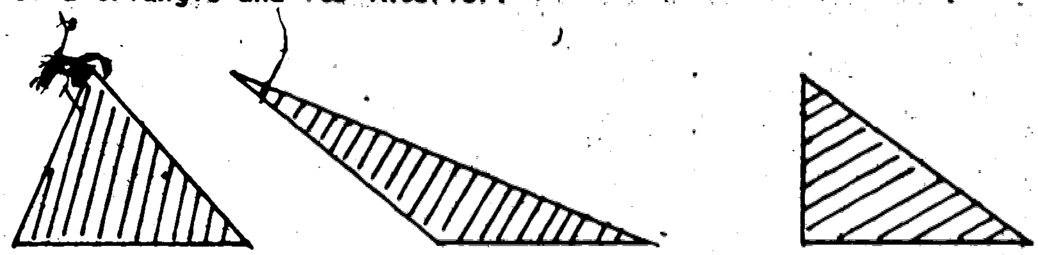
It is a mistake to restrict the study to the properties of the range space of a measure alone and to state that the study is in fact about the child's conception of a measure system.

> The child cannot be said to understand a measure system until the operational definitions of properties of the domain space, the corresponding properties of the range space, and the characterizing function that unites them into the measure system are exhibited. (Osborne, 1976, p. 25)

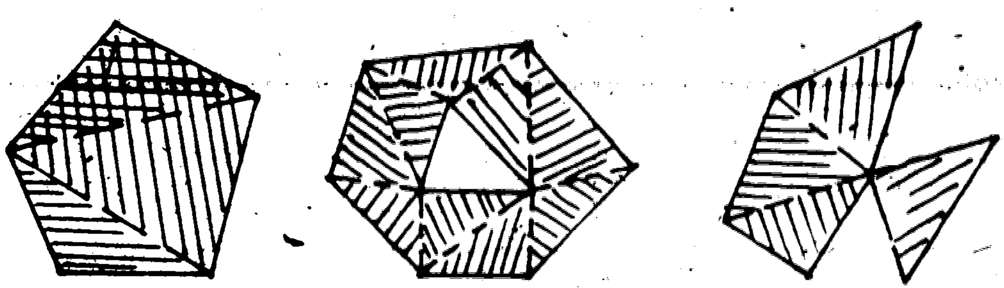
C. Definitions

For the purpose of constructing the unit on polygonal regions and their area measure, the basic undefined terms are as follows: point, line, and plane. Below are definitions for some related and used terms in the study.

Triangular region. A triangular region is a plane region consisting of a triangle and its interior.



Polygonal region. A polygonal region is a plane region such that it can be divided into a finite number of triangular regions that have, at most, a finite number of points and segments in common.



Congruent regions. Two plane regions are said to be congruent if one covers the other completely.

Piece-wise congruency. Two polygonal regions are said to be piece-wise congruent (or equidecomposable or congrument by addition, Boltyanskii, 1963; Eves, 1972) whenever one of them can be cut into a finite number of polygonal subregions in such a way that they can be rearranged to cover the second region completely.

Cut-and-cover. Cut-and-cover refers to the procedure mentioned in the preceding definition.

Space. A space is referred to as the set of all points.

Function. A function in a relation ties two spaces, the domain space and the range space, such that there exists no element in the domain space that is related to more than one element in the range space. (This relation is also called mapping, transformation, operation, correspondence, application.)

Measure system. A measure system is referred to as a structure of ideas consisting of three components: a domain space, the non-negative real numbers (range space), and a function defined on the domain space to the non-negative real numbers (Fig. 4).

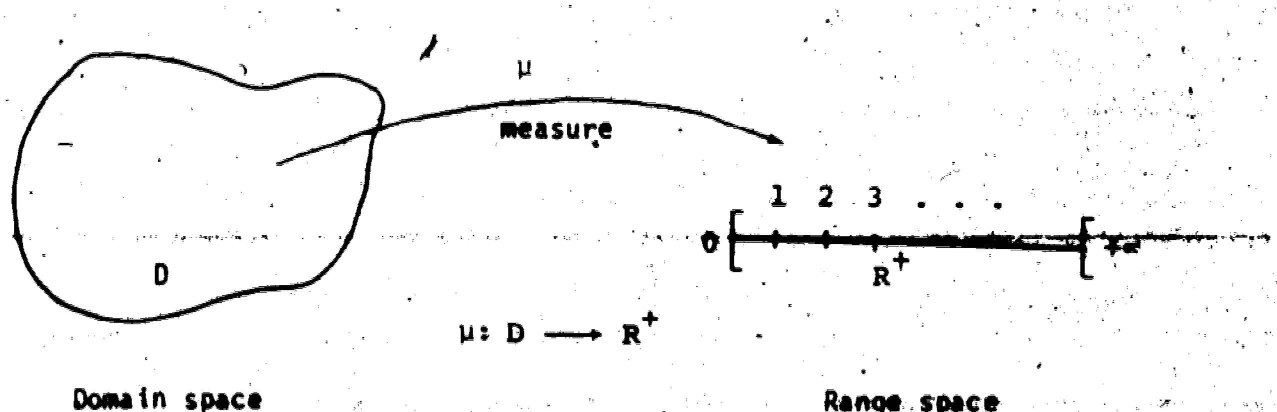
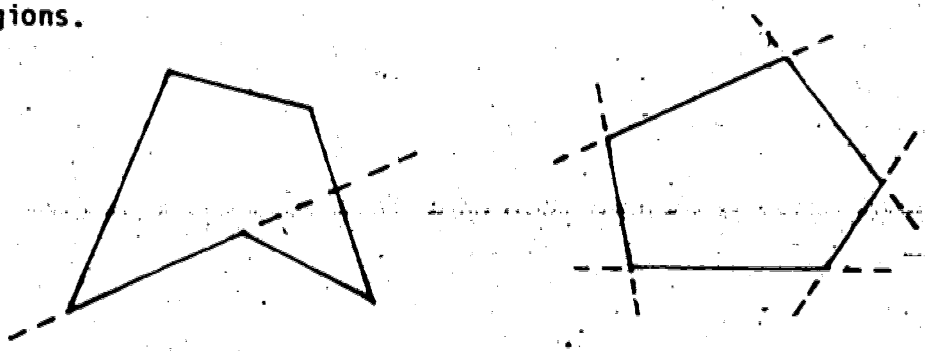


Figure 4

<u>Convex Polygonal Region.</u> A polygonal region is convex if none of its sides, whenever produced, dissects the polygonal region into sub-regions.



non-convex convex

<u>Regular Polygonal Region.</u> A polygonal region is regular if it satisfies the following conditions: (a) it is convex; (b) all of its sides are congruent, and (c) all of its angles are congruent.

<u>Remarks</u>

1. If the area measure for polygonal regions is to be considered, then the domain space would be the set of all polygonal regions.
2. There are always analogous operations, relations and characters between the two spaces; e.g., in area measure

Domain Space			Range Space
Union operation	$\cup$	analogous to	+ additive operation
Set inclusion relation	$\subseteq$	analogous to	$\leq$ order relation
Unit measure		analogous to	1
'Void region'		analogous to	0
Congruency		analogous to	=
Difference		analogous to	- (with some restriction)

D. Need for the Study

The Conference Board of the Mathematical Science of the United States nominated a National Advisory Committee on Mathematics Education (NACOME) in May, 1974. Their report, Overview and Analysis of School Mathematics K-12, states that "Though geometry is mentioned as being part of texts, objectives, and testing, 78% of the teachers report spending fewer than 15 class periods per year on geometry topics." (78% of the elementary school teachers; page 13). They came to the conclusion that the acceptance of geometry in elementary programs has been slow indeed. On the junior and senior high school levels, the picture was not much different (page 5-10). Similar conclusions but on the students' aspect of the problem can be reached from the interpretations of the results of the National Assessment of Educational Progress (NAEP) reports (Carpenter, Coburn, Reys, and Wilson, 1975a and b). For example, about 50% of the 17-year-olds could recall the name of the cube, cylinder, and sphere; while only 25% of the 13-year-olds could recall the names of these figures. The results for younger children were lower. In the conclusions of these two reports it is noted that basic concepts of length, area, and volume are not well aquired by students in both levels. Therefore, a need does exist to provide more information on abilities prerequisite to the learning of geometry and measure systems assuming that the above data can be related to North America as a whole.

Finally, the investigator believes that the need for this study is well expressed in the following statement offered by Coxford (1978):

> the research dealing with geometric concepts needs to progress beyond a description of what children understand at various age levels; it needs to progress beyond repli-

> cation and refinement of Piaget's theory of intellectual development; it needs to begin to relate geometric understanding to the type of geometric experiences learners gain in and out of school; it needs to begin to relate geometric understanding to the nature of the instruction used by teachers. (Coxford, 1978, p. 323)

E. Delimitations

The following delimitations are to be considered:

1. The study is delimited to grade eight students.
2. The study is restricted to classes of Edmonton Public School System.
3. The study is confined to polygonal regions and area measure.

F. Limitations

In interpretation of the data, the following points will ought to be considered:

1. The results of the study will be confined to groups of students similar to those involved in the study.
2. The school in the study was not randomly selected.
3. It is not possible to confirm that all of the students did not consider the instrument as a checking of their knowledge; this in turn could have had an effect on the type of their responses. However, every effort was made to ensure that students did not regard the instrument as a test of their knowledge.

G. Outline of the Report

Chapter II contains a review of some selected relevant literature. A detailed explanation of the design of the study, the pilot study, the testing procedures, interviews, the research questions, the test instruments employed, and the hypotheses tested is reported in Chapter III. Chapter IV reports the results of the study, both quantitatively and qualitatively. Chapter V, the final chapter, includes a summary and discussion of the findings with respect to the questions raised. A discussion of some of the educational implications of the findings and suggestions for further research are also contained in this chapter.

CHAPTER II

REVIEW OF RELATED LITERATURE

A. Introduction

The aim of this chapter is to establish a base for the research questions. Below is a discussion of recommendations made by major commissions and committees, formalized results of research studies on the notion of area, piece-wise congruency and area measure structure, together with some views on perception and thought development.

B. Recommendations of Some Leading Reports

Geometry and the instruction of geometry at elementary and secondary levels have been under intensive study and review over the last two decades. Many conferences and committees were formed and many reports have resulted focusing on geometry programs and the many instrutional patterns. Some of the major reports on mathematics are as follows: The Commission on Mathematics of the College Entrance Board (1959) in the U.S.A., Goals for School Mathematics, The Report of the Cambridge Conference on School Mathematics (1963) in the U.S.A., Mathematics in Primary School (1965) in England, Geometry, Kindergarten to Grade Thirteen, Report of the (K-13) Geometry Committee (1967) in Canada, and Overview and Analysis of School Mathematics Education (1975) in the U.S.A.

These reports suggested major changes in the mathematics programs for elementary and secondary schools, both in content and instruction. These studies not only recommended that a greater emphasis be placed upon geometry throughout the elementary and secondary mathematics programs but should in fact be started as early as kindergarten.

The National Advisory Committee on Mathematics Education (NACOME) produced a technical report, Overview and Analysis of School Mathematics Grades K-13, in an attempt to give a comprehensive overview and analysis of the current situation of mathematics education, with respect to its objectives, current practices, and attainment. The main purpose of this survey was to seek answers to the following questions:

- What have been the goals and rationale for recent curriculum developments?
- How accurately and broadly are these goals realized in current school programs?
- What potent forces and promising innovations should shape mathematics curricula in the years immediately ahead? (p.1)

The committee gave extensive attention to those efforts that have been directed toward the development of school mathematics programs over the past twenty years, 1955-1975. There have been enormous efforts toward developing school mathematics focused on the content of instruction -- new mathematical topics, new organization and grade placements of traditional topics. And, naturally, there have been resulting debates between professionals and the public on the values of these new ideas and techniques and whether or not a mathematics curriculum reform is needed. Even among professionals themselves there are heated debates on what particular technique should be adopted, especially at high school levels.

The extent of use of new programs specifically designed to teach geometry was investigated by the committee. The committee found no immediate survey data that indicate relative emphasis of new and classical junior high school topics. As such, indirect evidence from analysis of National Assessment Educational Progress (NAEP) was adopted by the committee (two surveys were commissioned by the National Council of Teachers of Mathematic (NCTM) - Carpenter, et al., 1975a and 1975b - and a brief account of each was made available to NACOME at that time). A sample of 3000 elementary school teachers was given a questionnaire which contained a list of content topics in mathematics. Among other things, they were asked to judge instructional time for each content topic. As a survey result, geometry was given fewer class periods relative to the mathematics content area by a majority of responding teachers (p. 12). Accordingly, the committee concluded that geometry had a slow acceptance in elementary programs. On secondary school levels, geometry was not a high priority either and the committee, therefore, urged that teachers pay increased attention to geometry as one of the fundamental subjects in mathematics and science.

The report revealed that there is a controversy over the content and organization of high school geometry. The committee stated that attempts to include intuitive geometry in elementary and junior high programs are admirable, yet slow acceptance of it in new programs does exist.

Most of the students search in mathematics for a collection of well-established concepts and methods that are applicable to problems elsewhere. The committee was concerned that the lack of clear explanation of how to group these concepts and methods (into logical structure) can be an obstacle to the learning process. However, they also warn that an over-emphasis exists on the logical structure approach in some recent curricula and instructional practices. In sum, although the committee did not call for overall emphasis of logical reasoning in school mathematics, logical reasoning is regarded as one of the basal attributes of mathematical thought. It is also an essential component of problem solving. Thus, they warned that ignoring logical reasoning in school mathematics would have negative consequences on the learning process. The committee stated that mathematics curricula can be improved by using methods which ensure interaction of concrete experience and abstract thoughts. For this reason, a balanced approach using both the absolute structural approach and complete intuitive approach is recommended.

> Popular reports of this debate suggest inevitable and bitter polarization of mathematics community on the issues:
>
> old or new
> skills or concepts
> concrete or abstract
> intuitive or formal
> inductive or deductive
>
> This dichotomization of curricular issues does not accurately convey the intentions or the accomplishments of recent innovations. (p. 21)

Thus, these phases of mathematical education ought to complement each other: "We are convinced that this is a false antithesis."

Reviewing the National Assessment of Education Progress (NAEP) findings (Carpenter et al., a and b) on geometry and measurement, the Committee found reasons for concern about the level of understanding children are attaining with current educational practices. For instance, performance in applying geometric relationships was poor for 9-year-old students and relatively better for 13-year-old students, though far from being adequate. Only 36% of the 9-year-olds and 60% of the 13-year-olds could calculate the distance between centres of two adjacent squares of the same size. In view of these results the committee came to the conclusion that evidently, at all age levels, students do not understand basic concepts of length, area, and volume.

> While 82% of the 9-year-olds could accurately measure a 7-inch segment, only 48% could measure a 15-inch segment...longer than the foot ruler they were given. Only 7% of the 13-year-olds could calculate the area of square with perimeter 12 inches. Older respondents also had difficulty with area and volume problems. (p. 117)

The committee, therefore, called for thoughtful application of concepts and formulae and more enriched geometry programs for K-12 levels, emphasizing comprehension.

The NACOME report as well as other reports mentioned above urge that the approach to geometry, especially during the elementary and junior high years be highly intuitive with intensive use of concrete materials. They expressed the importance of logical reasoning for later high school levels, but at the same time, they disclosed their fears of over-emphasizing it. Each of them stressed that more geometry should be embedded in the school mathematics curriculum throughout the years of elementary and secondary school levels. All of them emphasized the importance of geometry in everyday life and viewed it as a useful area of mathematics that deserves a special place in the child's education. Although a variety of routes have been recommended for use in the instruction of geometry, the Euclidean properties of the geometric objects are the basic elements for all of them.

It seems, however, that the problem of what, how, when and why we shall teach geometry in elementary and secondary school levels is, at best, partially solved. Differences still exist on the parts "why", "what" and "how" of the problem. Mathematics teachers in different countries have been aware of the deficits of geometry courses based on a condensed version of the "Elements" (Adler, 1968). Due to this growing awareness, different approaches have been advocated. However, the selection of a particular approach to geometry, especially at the high school level, is far from being resolved. Heated debates occur among supporters of the transformation approach, the vector approach, the analytic approach, and the synthetic approach (Robinson, 1976).

In sum, many approaches to geometry have been introduced and different lists of objectives have been recommended with agreement neither on a particular approach nor on the general purpose of geometry in the school curriculum. Mathematics educators, however, have agreed that more geometry should be included in school mathematics curriculum and should begin with earlier stages (Martin, 1976b).

The resolution of the controversy about purpose and method in teaching geometry may be dependent upon increased knowledge over the child's mathematical development -- how children perceive and conceptualize spatial figures. We do not know precisely how children think geometrically; there exists, for example, no clear explanation of how children learn the area concept.

The proposed study, in the author's view, is a response to the NACOME's recommendations.

C. Related Research

Investigations concerning how the child conceives certain properties of area were carried out by Piaget and his associates more than three decades ago. In this chapter, Piaget's theory of intellectual development of a child as it relates to the child's conception of space and geometry will be discussed. Also, the Van Hiele theory describing levels of development in geometry will be outlined.

A large part of our knowledge about the child's conception of space and geometry can be credited either directly or indirectly to Piaget's work (Kidder, 1977). In the Piaget and Inhelder work The Child's Conception of Space (1963), there are three major themes. First, they assert that the earliest spatial concepts the child can attain are topological in nature and the child's projective and Euclidean concepts are attained as extensions of the topological ones. Second, they believe that the child's spatial abilities are built up through the organization of mental actions performed on objects in space. Third, they claim that there exists an important distinction between two levels of the child's knowledge - the perceptual level and the representational level.

The first main Piagetian conclusion concerning the child conception of space and geometry is that ideas which are topological in nature develop first in the child, followed by the development of ideas of projective and Euclidean space. This contradicts the historical and logical sequence of geometry. For, "geometry primers are almost unanimous in presenting the fundamental ideas of space as resting upon Euclidean concepts such as straight lines, angles, squares,

> class of comparable shapes not present to perception. (Piaget, 1963, p. 17).

Most recent research oriented to the study of geometric and spatial concepts has had its roots in Piaget's work. For instance, Kidder (1977) stated that the perceptual level is based on sensory impressions such as manipulating, feeling and seeing. On the other hand, representational space both extends and benefits from perception in such a way that the child is progressively able to perform mental operations not only on physical objects but on objects whose presence can only be imagined. Montangero (1976) stated that mental imagery is particularly fitting for representation of spatial phases of reality and has therefore a fundamental role in spatial knowledge. Kidder (1976) has put forward the main themes of Piaget in space and geometry as a psychological background for his study. The child is able to accomplish mental operations on only objects that can be imagined and objects that are present as well. The final representation, however, results from a long and complex developmental construction that depends more on action than on perception alone (Flavell, 1963). This developmental aspect of imagination implies that the representation of children is different from that for adults as a result of the fact that adults have more experience in manipulating the spatial surrounding (Martin, 1976c).

Van Hiele Levels of Thought Development in Geometry

Radical changes and extensive innovations have been introduced in the new Soviet geometry curriculum due to Russian research motivated by Piaget and van Hiele. The Russians have accepted the Piagetan discovery -- the development of geometric operations in children and the sequence for the development: from the topological notions to the projective and Euclidean notions (Wirszup, 1976).

The van Hielé report, in 1957, described five levels in the child's development of geometric thinking. These five levels were discussed in the P.M. van Hiele article La pensee de l'enfant et la geometrie (The Thought of the Child and Geometry) appearing in 1959.

Van Hiele stated that at the first level young children are perceptual. They perceive geometric figures in their totality and judge them according to their appearance as entities. The parts of a figure or shape are not considered nor are the relationships between figures. At the second level children start to distinguish the components of figures. They recognize figures by their properties and establish relationships between individual figures. These properties, however, are not yet connected with one another. For example, children know that in both a rectangle and a parallelogram opposite sides are equal, but still they are not able to conclude that a rectangle is in fact a parallelogram. However, at the third level this connection is reached and the children are able to relate properties of a figure with each other as well as the figures themselves. A logical ordering can be established for the properties of a figure and of classes of figures as well with the possibility of deducing one property from another. But the child still does not comprehend in full the role of deductive reasoning. Deductive reasoning appears in conjunction with experimentation. At level four, the significance of deduction emerges and the student understands the role and the meaning of definitions, postulates, propositions, structures of proofs and the logical connection between concepts and statements. At the fifth level abstraction is well developed and a theory can be developed without the need of any concrete interpretation.

Van Hiele noticed discontinuities in the learning graph and stated that these jumps reveal the presence of these levels. Students often appear in between levels when little progress is achieved. Van Hiele stated that passage from one stage to another is not a spontaneous process attendent with the student's biological growth and age. Progress to higher geometric level proceeds in van Hiele's view essentially under the influence of learning and hence depends on content and method of instruction. Meanwhile, Piaget's theory asserts that progress from one stage to another can occur in the absence of direct instruction, but not in the absence of learning. The learning happens within an experiential environment and not necessarily from direct instruction. Unlike Piaget's theory, the skipping of levels does not exist in van Hiel's theory. Van Hiele stated that the progress from one level to another requires a certain amount of time and that various methods of instruction allow the regulation of this time. Also it is possible that particular teaching methods do not allow the attainment of higher stages and so the modes of thinking concerning these stages remain inaccessible to the student.

Piece-Wise Congruent Regions in Piaget's Tasks

Piaget (1960) conducted a series of investigations on polygonal regions and their area measure. The investigations were concerned with the development of an understanding of certain properties of area and their connection with area measure.

The first task was "the cow in the field". It is oriented to the study of the Euclidean axiom: "if two equal parts are taken from two equal wholes, the remainders will be equal". This concept is applied to congruent regions. The purpose was to observe the stage at which children understand the idea that by subtracting smaller congruent regions from larger ones one obtains equal remainders. This task was designed to study the area conservation problem , despite the immediate questions about adding or subtracting congruent regions -- to study area conservation of both wholes and parts. In Piaget's words:

> The ability to analyse a whole in this way is a prerequisite to measurement because when measuring an area we assume, as we do for all measurement, that the partial units are conserved and can be composed in a variety of ways to form invariant wholes. (Piaget, 1960, p. 262)

In this context, the capability to understand the Euclid axiom is an immediate necessity to measurement. It was found that children act successfully in the experiment. They recognized that the above Euclid's axiom holds.

> At stage III, however, even at level IIIA (usually at 7 1/2 but sometimes as early as 6 1/2 - 7 years), children recognize that the remainders are always equal... (Piaget, 1960, p. 264)

The task consisted of two identical rectangular regions of green cardboard as meadowlands, two tiny wooden cows and a number of identical wooden houses. All children asked were able to say at once that the cow in the field with a house had less grass to eat than that in the other field without a house. When another house was placed on the field that had no house, the children all agreed that each cow had the same quantity of grass. Younger children were deceived by the arrangement, not observing that two houses occupy the same space whether they are near each other or not. Some children maintained the equivalence of the green area only up to a small number of houses, meanwhile, older children maintained it all the way through, relying on an operational handling of the problem which convinced them of the necessity of their reasoning.

> There is conservation of area as soon as there is operational grouping in the addition and subtraction of areas, but measurement of areas becomes effective only when these processes are fused with change of position. (Piaget, 1960, p. 273)

The second task in Piaget's series of polygonal regions tasks was to investigate what kind of ideas little children might have about conservation of areas. This task was particularly concerned with area of a polygonal region as a stable characteristic independent of alterations of shapes. Essentially, the purpose was to investigate whether or not the child understands that a plane region remains constant under decomposition into a finite number of sub-regions and different rearrangements of these sub-regions. Piaget and his associates presented an area divided into several separate regions and modified the arrangement of these parts to see whether or not the subject concluded that the whole remained invariant. The task consisted of two identical rectangles each of which was made up of six equal squares such that each rectangle was three squares high and two squares wide. The top right-hand square on one of them was transferred by the investigator to the bottom right-hand corner. Children were asked whether the resulting region had the same area - "the same amount of room" as the other rectangle.

A second task was used for the same purpose. Two identical rectangular regions were used. One of them was cut either diagonally and rearranged into a triangular region or randomly into a finite number of pieces and rearranged into an irregular polygonal region. In either case the question thereafter was whether the resulting region and the second rectangular region still had the same size.

Both tasks showed that at the Piagetian third stage, a whole is conceived of as an invariant entity independent of the disposition of its parts.

The third task was related to area conservation outside a closed curve. Two fields and two cows similar to those in the first experiment were used plus two identical square cards for potato patches. One of these square regions was cut into a number of sub-regions. These sub-regions were separated in one field and the second square region left intact on the other field. The children were asked whether there was the same room for potatoes on both fields and the same grass for each of the two cows. Children conserved the interior area (area inside a closed curve) at the early level of the third stage, yet they could not conserve the complementary area at this level. However, at a later level of the third stage both comlementary and interior conservations were attained.

The fourth task was related to measurement by superposition. Children were asked to compare the sizes of a large right angle triangular region and a larger irregular polygonal region by using a number of triangular, square and rectangular cards enough or nearly enough to cover each of the regions. The use of a common unit and its significance in comparison were reached at the third stage. However, at an early level of stage three the need for identical units of measurement was not realized. It was only at later levels that children understood the notion of a unit and its significance in the comparison and thus took the size of measuring cards into account.

Another pattern was used in the study of the measurement of area by superposition, namely measurement by unit iteration. The child was given only one unit of measurement and asked to compare the size of different polygonal regions. One part of the task consisted of three polygonal regions which were equal in area (each of them could be composed of nine smaller squares) and different in shapes (Fig. 5).

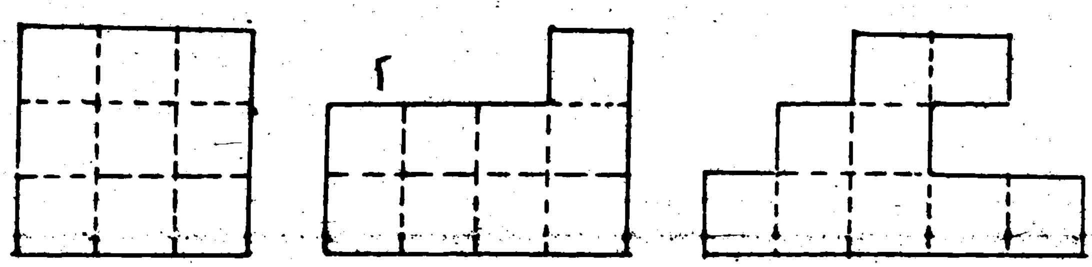


Figure 5

To measure each of them, the child was given a pencil and a cardboard square representing one unit.

The second part of the technique used in the task consisted of different polygonal regions that were not equal in size and three types of measuring cards: a square card equivalent to one-quarter of the polygonal region (a) (Fig. 6); a rectangular card equal to two such squares and a triangular right angle card equivalent to one-half of the square card cut diagonally.

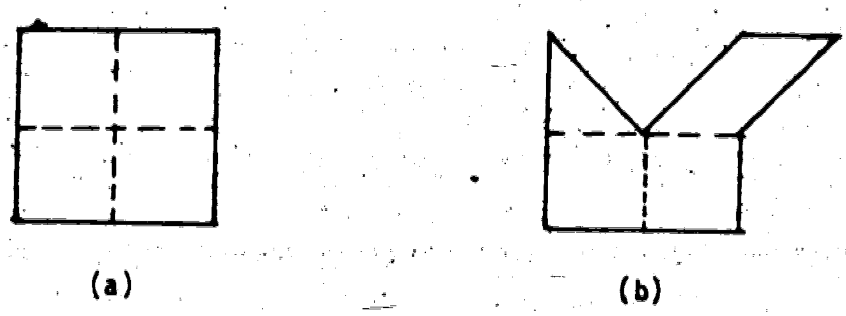


Figure 6

Piaget and his group found that improvement in the composition of sub-regions and in changing their positions is not fully coordinated until children reach the concrete operational stage--III. These two articulations and the coordination of them are attained at two sub-stages -- IIIA and IIIB within the concrete operational stage. Children at the early level (IIIA) of the stage show first the above coordination but still they all count the square and triangular regions as equivalent units unless their error is pointed out. Children were able to measure area by successive application of a unit area within a large area only at the later level of the concrete operational stage -- IIIB.

The fifth Piagetian task was one further step in investigating the beginning of decomposing a polygonal and a circular region, the construction of part-whole relations and the development of fractional notions. The task consisted of a number of clay "cakes" of different shapes. First the children (from 4 to around 7 years) were asked to share these cakes equally among dolls in an increasing number starting with two up to six dolls. Because many children experienced difficulty in dividing the clay, they were supplied with pencils, paper and scissors and asked to cut different paper rectangles, squares, and circles. Second, the children were asked if the pieces they cut could be rearranged together into a whole cake -- "Supposing we stuck all these bits together again, how many whole cakes would we have." (Piaget, 1960, p. 303).

Piaget stated that results of this task suggested that the notions of fractions, even halves, assume a qualitative substructure. Any parts under consideration should initially be grasped as integral parts of a whole that can be both decomposed and rearranged back together, before they can be compared with each other and then transformed into fractions.

> Once the notion of part has been constructed it is comparatively easy to equate the several parts. Therefore, while the elaboration of operations of subdivision is a lengthy process, the concept of a fraction follows closely on that of a part. For parts which are subordinated to the whole can also be related to one another, and when this has been achieved, the notion of a fraction is complete. (Piaget, 1960, p. 335).

Children at the concrete operational stage attained a deductive understanding of the relationship between the fraction to be realized and the original whole. Thus there has been an operational conservation of the whole -- its equality with the sum of its parts was reached as a necessary relation. At the earlier level of the concrete operations stage, the trisection (division into thirds) was understood deductively and at a later level of the stage the anticipatory schema extends to division into fifths or sixths.

The final Piagetian investigation in area was the doubling of an area task. The purpose behind this work was to seek an explanation of how children establish relations between length of the sides and area, and how this lead to mathematical multiplication which replaces merely logical handling of the relations involved. Children were supplied with pencils, paper and rulers. Then, they were asked first to draw a line twice as long as a given line. Second, they were asked to draw a square as large as a given one (Piaget, 1960, p. 337). Piaget stated that the task is much harder than the previous tasks and for the first stage, success in the experiment is ruled out entirely. At the second stage (about 4 - 7½ years), doubling was a matter of an arbitrary small increase in size. At the early level of the third stage (IIIA), children tried different ways to relate the length of sides and the area of a polygonal region with no success. At the later level of the third stage (IIIB), children expressed difficulties in moving from one dimensional space to two dimensional space -- from linear measure to area measure. The actual relation between lengths and areas was reached only at the formal operations stage where the child starts to comprehend that the calculation of an area involves mathematical multiplication as a shortened technique of addition (Piaget, 1960, p. 338-353).

The previous Piagetan studies of some basic properties of area of polygonal regions are related, in a way, to measure theory - the essence of modern mathematical analysis. For, the concept of the measure (A) of a set A in the domain space D (Fig. 4, p. 11) is, in fact, a natural generalization of concepts such as

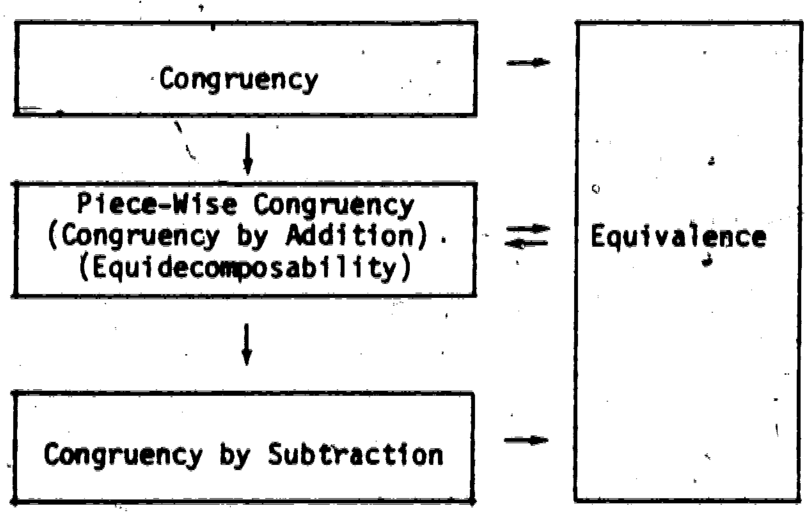
1. The length L(S) of a line segment S;
2. the area A(P) of a plane region P;
3. The volume V(T) of a region T in 3-dimensional space;
4. the increment f(b) - f(a) of a non-decreasing function f over a half open interval [a,b); and
5. the integral of a non-negative function over a set on the real line or over a region in the plane or in 3-dimensional space (Kolmogorov, 1970).

Historically, the notion of measure originated in the theory of functions of real variables and utilized broadly in probability theory, functional anlysis, the theory of dynamic systems and many other branches of mathematics (Boyer, 1968).

Polygonal regions have an essential role in the development of measure theory. For, in almost every text on measure theory and its development, rectangular region is the starting point for developing the general notion of measure in a plane set.

However, the above Piagetan studies, although concentrated on the development of area concepts, are not concerned immediately with congruence and additive properties of area. The doubling an area task essentially deals with square regions and the unit measure tasks are related to the use of particular units -square, rectangular, and right-angle triangular cards and the association of the corresponding sizes.

In the present study, the notion of congruency is of most importance as are its extensions -- piece-wise congruency (congruency by addition or equidecomposability) and congruency by subtraction. From the mathematics point of view, the notion of congruency induces the idea of piece-wise congruency (congruency by addition) and the congruency by addition in turn implies the idea of congruency by subtraction, and each of these ideas implies equivalence.



Moise (1963) stated that since students are going to get numerical answers to area problems, area as a real-valued function should be introduced soon so that the full force of the theory can be used -- "it leads to an easy proof of the Pythagorean Theorem" (p. 463). Yet, the author has developed a new simple procedure in showing Pythagorean Theorem based on the notions of piece-wise congruency and congruency by subtraction (Rahim, October, 1979; see Appendix K).

Replications of Piaget's tasks have been carried out in order to verify the findings of his studies. Lovell, Healey and Rowland (1962) conducted replications of the cow in the field task, the unit measure task and the subdivision of polygonal regions task. Their results support the Piaget findings. Beilin and Franklin (1962) replicated the unit measurement task. The resulting data supports the domination of the child's level of cognitive development on the learning process. They also concluded that length, area, and volume measurement are achieved in this order. Beilin (1964) directed a replication of the second task above - the conservation of area (the second part). Beilin altered the way the task was carried out. In the Piagetian setting, the subject was given two congruent rectangular regions, and then one of them was decomposed in front of him into a polygonal region differing only in shape with the original. Beilin did not use the demonstration of decomposing one of the regions. He focused on the ability of the subject to recognize that two polygonal regions, different in shape only, are of equal areas. Of the sample from grade four, 50% managed to perform correctly. Children with high IQ scores also performed better than those of low scores. Goodnow (1968) studied the conservation of area as related to the cow in the field task. She used a slightly different setting of the task in taking the sample of 10 to 13 year old boys from both Chinese and Western backgrounds with education from full schooling to reading and writing only. She concluded that social status, schooling and nationality have no essential effect on success in the task.

Piece-Wise Congruent Regions in Wagman's Tasks

Wagman (1975) conducted an investigation using several tasks concentrated on one aspect of geometry: "the development of the concept of the area of a polygonal region" (p. 71). She asserted that studies concerned with the study of area concepts overlook some important aspects of area measure, especially when they are viewed from a mathematical standpoint. Furthermore, Wagman stated that the Piagetian experiments are not concerned with the notion of area as a real-valued function with an arbitrary unit nor with how area changes according to the choice of unit.

Wagman's criterion tasks were devised according to the four axioms of the area of polygonal regions as they appeared in the School Mathematics Study Group (1963).

1. The area axiom:

   If R is any given polygonal region, there is a correspondence which associates to each polygonal region in space a unique positive number such that the number assigned to the given polygonal region R is one.

2. The additivity axiom:
   Suppose that the polygonal region R is the union of two polygonal regions R1 and R2 such that the intersection of R1 and R2 is contained in a union of a finite number of segments. Then relative to a given unit area, the area of R is the sum of the areas of R1 and R2.

3. The congruence axiom:
   If two triangles are congruent, then the respective triangular regions consisting of the triangles and their interiors have the same area relative to any given unit area.

4. The unit axiom:
Given a unit pair for measuring distance, the area of a rectangle relative to a unit square is the product of the measures (relative to a given unit pair) of any two consecutive sides of the rectangle. (p. 744-748)

Children included in the sample had to pass two preliminary tasks - a vocabulary task and a conservation of area task. The children were 8, 10 and 11 year olds.

For the congruence axiom task, two congruent isosceles right triangular regions were presented to the child. The child was allowed to handle them. He was asked whether they were the same size and shape. Then the experimenter presented a small right isosceles triangular region and asked the child to find out by tiling, how many small triangular regions were needed to cover one of the large regions. If the child answered the question correctly, then he was asked to explain how he did come to his conclusion.

The collected data from the congruence axiom task was analyzed to test the hypothesis that subjects who conserve area tend to pass the congruence axiom task and are more likely to do so than others.

The unit area task consists of three polygonal regions all different in shapes with the first two having the same area and the third one having a larger area. There was an adequate number of smaller square regions to cover all the three regions and a large number of triangular regions each of which was equal to one half the square region. For the first stage of the task, the child was asked to find out by tiling how many square regions are needed to cover each region perfectly.

The child was asked to pile the square regions needed to cover each region beside that region. In the second part of the task the child was asked again to find out how many of the triangular regions are needed to cover each of the polygonal regions. He was also permitted to tile if it was needed.

Of the eight year old children in the sample, 80% responded correctly in both parts of the task. For the second part, some children reached the correct responses by comparing the square and triangular regions directly without retiling, meanwhile, others reached their conclusions by a rule developed after retiling (and therefore reexamining) one of the equivalent polygonal regions.

Wagman concluded that the high percentage indicated that a large majority of the eight year group could apply the unit area axiom in a concrete setting and, therefore, she labeled them either in substage IIIB or transitional to it (IIIA-B) in Piaget's sense. Children who performed without retiling in the second part of the experiment were categorized at substage IIIB. This included 20% of the eight year group, 25% of the ten year group and 42% of the eleven year group within the sample (p. 91).

The third Wagman task on polygonal regions was constructed on the area additivity axiom. Due to mathematical difficulties, two approaches were adopted. The first approach was designed to investigate the ability to perform the rearrangement part of the problem of piece-wise congruencing of two polygonal regions different in shape and equal in area. The subject was given two polygonal regions, and a finite number of polygonal pieces that actually represented a decomposition of a copy of one of them. Then the subject was asked to find a rearrangement of these polygonal pieces to cover one of the polygonal regions and another rearrangement to cover the second region. While in the second approach, the subject was required to discover a solution to make the two polygonal regions piece-wisely congruent. That is, first, to seek for and to determine certain cuts in one of the given regions dividing it into sub-regions and, second, to discover one rearrangement of these sub-regions to cover the second region. In sum the first approach for performance, given the construction, and the second approach for discovery of a solution of equidecomposing two polygonal regions (University of Chicago Mathematics Staff, 1956).

The subject, having rearranged the sub-regions, was asked whether or not the two regions have equal space and why. To insure an accurate response, there were two pairs of polygonal regions involved, a pair of equal areas and the other of different areas. This procedure prevented the child from assuming equivalence due to past experience.

Wagman found that correct responses for both pairs fall into the first level of the concrete operational stage. This agrees with Piaget's result that operational conservation of area and use of transitivity occur at level IIIA or bigher (Piaget, 1960, p. 294-295). The collected data revealed that about 10% of the eleven year old group failed the task. Out of 94 subjects, of the sixteen children who failed the task, the number failing at differing age levels follows: three failed out of 29 eleven year olds, three failed out of 29 ten year olds, and ten [illegible]led out of 36 eight year olds.

The second approach adopted by Wagman for the area additivity axiom "the additivity without decomposition task" consisted of three different settings.

First Setting. A parallelogram region constructed of a trace paper and an oaktag rectangular region of equal areas were presented to the subject. Then a question was raised as to whether or not the regions had the same area. Superposing one region on the other was demonstrated to the subject when he was unable to respond correctly. If the subject was still not able to see the decomposition required, the investigator had to do the decomposition and the subject was then questioned on the area of the regions.

Second Setting. Copies of the same rectangular region, and the parallelogram region as for the first setting were used together with two more parallelogram regions. These four regions were chosen and arranged such that the rectangular region, the first and the third parallelogram regions were of equal areas, whereas the second parallelogram region was of a different area from the rectangular region. The inclusion of the unequal area parallelogram region was to prevent the child from assuming equality due to previous experience. The first setting's procedure was repeated with no offer of help in superposing unless independent initiation in decomposing was started.

Third setting. The same regions as for the second setting were used, but the third parallelogram region was arranged to have its altitude exterior to the region. The above procedure for the second setting was repeated.

Although the task was administered to conservers, and they should belong to IIIA according to Piaget's findings, nearly 33% of the eight year old conservers were below IIIA stage. That is, they seemed to perceive that different shapes imply different areas which is contrary to conservation of area (Table 9, p. 101). Furthermore, the subjects in this task, in order to respond correctly, have to be aware of the space inside as well as outside the region. In this connection, this task is an analogue of Piaget's conservation of area outside a closed curve. Meanwhile, the performance of the eleven year old group was lower than that expected from Piaget's findings. However, the difference was not that large if subjects at stage IIIB and those in the transitional stage were added together.

The fourth of Wagman's tasks -- the unit axiom task - consisted of two parts:

a. The experimenter presented to the subject two rectangular regions and a number of square regions enough to cover each of the two rectangular regions perfectly. The two rectangular regions were taken so that "for the first, the area is greater than the perimeter; for the second, the perimeter is greater." In addition, the subject was given a number of matchsticks to measure the sides of the unit square and the rectangular regions.

b. The subject was given a parallelogram region and a number of rhombus regions enough to cover the parallelogram region. The corresponding angles of the parallelogram and any rhombus were equal.

By using the matchsticks only, the investigator's aim was to see if the subject could predict the number of square regions needed to cover the rectangular regions and how many rhombus regions were needed to cover the parallelogram region.

It was found that performance on this task increases with age. More than 50% of the eleven year old children were able to respond correctly for all the three regions involved in both parts. They merely used linear measurement of the side of the rectangular regions and the multiplication operation.

Wagman stated that the axioms for area measure define the concept on the basis that all other properties of area are logical consequences of them. With regard to the unit axiom, the findings show that there exists a gap in the child's development between the attainment of a logical basis as described by the axiom and the child's understanding of different logical consequences. Moreover, about a third of the ten year and eleven year groups in the sample failed to apply at least one of the axioms of area, congruency and additivity even in perceptually easy cases. Furthermore, nearly one-third of the subjects did confuse area with perimeter and hence experienced a lack of clear discrimination between these two basic ideas in geometry. Finally, children appeared to be motivated and excited by the use of concrete materials particularly in early levels.

> All through this experiment the children voiced their enthusiasm for the materials. Some refused to believe that this was mathematics! A twelve-year-old girl who was not a subject in this study told the investigator about her confusion in regarding the additivity axiom

> as it was presented in school. She was given the additivity-without-decomposition task to do and, having done it, exclaimed, "So that's what my teacher was talking about!" (Wagman, 1975, p. 109-110).

Reformulations of many key concepts such as area, function and limit have been made by mathematicians after a hard struggle over many centuries. They developed different forms of these notions free from the previous naive explanations that can be found in older textbooks and monographs. Kilpatrick (1975) described Wagman's study above as relatively modern in terms of adopting those new reforms of the area concept. However, he did express his fear as to whether or not Wagman's evidence supports the underlying assumption that these new reforms are closely related to child cognitive development. Kilpatrick offers the following comment on Wagman's studies and others:

> The argument is that these sophisticated reformulations may be more closely related to the child's developing understanding of a concept than are the intuitive notin of an earlier time. It is not clear how far the evidence these researchers have gathered supports the argument, but their analyses and tasks add substantially to our resources for investigating the learning of the concepts. (Kilpatrick, 1975, p. 210)

<u>Olson's Theory of Cognitive Development</u>

Olson (1970) postulated a theory on cognitive development in which he proposes that the basic cognitive process is a perceptual one and that cognitive development results from the elaboration of perceptual knowledge about the surrounding world - the elaboration that occurs in the context of such performatory activities as drawing, making, and speaking.

Olson was strongly influenced by the works of Gibson (1969) and Garner (1966). Gibson's feature theory of perception suggests that objects are perceived by means of certain features or cues which distinguish that object from all other objects. Objects viewed over time are perceived by means of deducted constant features. Perceptual development is basically a matter of perceiving objects and events in terms of larger and higher order sets of features. In any particular situation, the beginning of a search for distinguishing features that are sought will decrease the state of uncertainty, and consequently the search will ultimately be terminated (p. 144).

Olson was not completely satisfied with Gibson's feature theory of perception. He argues that Gibson's theory does not give an explanation as to why children experience difficulties in discriminating two oblique lines nor does it explain why it is easier for them to discriminate the horizontal from the vertical line. In Olson's words:

> Even with this theory we soon run into serious obstacles. For example, since features are primitive elements of perception, and since oppositely-oriented lines differ in at least one distinguishing feature, why are the later more difficult to discriminate than horizontal and vertical lines? (Olson, 1970, p. 1974)

If the answer is natural tendency of the nervous system, i.e., that there are more cells related to horizontal and vertical than those concerned with oblique orientation, then Olson argues that there are also reasons to reject the nervous system tendency as the basis for distinguishing lines orientation.

Olson suggests that if a spatial perception takes place by means of cues and features, it seems important to argue that these cues and features are not elementary in terms of bits of angles and bits of lines. It is rather to suggest that these first features are more general and can be classified as topological. Therefore, viewers (through Olson's tasks) appeared to search for the most general cues that would distinguish the choices with which they were faced.

Garner (1966) indicated that what a person perceives in a stimulus is a function of the perceived and deduced set of alternatives, that is, the contrast set. For example, a single stimulus such as three dots in a vertical setting would be perceived as a vertical array if it was included in a set of choices containing a horizontal and an oblique array. Also it could be perceived as "three" if it was embedded in a set of alternatives containing arrays of one and two dots. Thus, Garner concluded that the feature that one detects is always a function of the set of alternatives. He stated that how the single stimulus is perceived is, in fact, a function not for the stimulus itself, but rather a function for what the total set of alternatives is including of particular subsets. So, in Garner's opinion, we cannot understand the recognition of a single stimulus without understanding the properties of the sets within which it is contained.

Olson appears to accept Garner's hypothesis indicating that the set of alternatives is compatible with a context and has a decisive role in the choice of a feature. It makes specific the fact that a feature which is distinguishing in one context can be less distinguishing, or completely irrelevant, to some other context. Therefore, cues chosen are not those which discriminate objects in the environment, but rather those which differentiate alternatives in the context of performatory acts. That is, cues are not selected simply because they are objective differences between stimuli, but because they furnish information to guide a performatory act such as drawing or reading.

The most crucial aspect in Olson's theory of perception is perhaps well expressed in the following statements.

> It is the elaboration of one's perceptual world under the influence of various media that accounts individually for man's intelligence, and collectively for man's culture, ...it is the experience with the cultural media that gives mind its characteristic properties. These properties develop, not through internalizing the medium in the form of inner speech or mental pictures, or internalized activity, but through requiring additional information to guide the performatory acts in those media. (Olson, 1970, p. 195-196)

In this study the investigator views the piece-wise congruencing process as a vital medium that influences the elaboration of the student's perceptual ideas in two-dimensional Euclidean space. It is the experience in this medium that will provide conditions for better geometric understanding of the area notion and pave the passage for clear perceiving of the area measure structure.

Finally, Olson's notion of 'medium' and its influence in the mental and cultural developments of man seems analogous to Vygotsky's notion of 'mediation' and its role in the behavioral transformations of man throughout the history of his individual development.

Vygotsky's Zone of Proximal Development

Vygotsky (1978) stated that learning should be matched in some manner with the child's developmental levels. We cannot confine ourselves merely to determining developmental levels if we wish to discover the actual relations of the developmental process to learning capabilities. Through a series of research studies, Vygotsky suggested two such developmental levels based essentially on the learning capability of the child. They are: (1) The actual developmental level and (2) the potential developmental level. The actual developmental level is referred to as the level of a child's mental functions that can be identified by the use of standardized problem solving tests. The potential developmental level is referred to as the level of a child's mental functions that can be determined also through problem solving situations but under adult guidance and collaboration. The potential mental functions have not yet matured but are in the process of maturation (p. 85-86). Vygotsky indicated that there is a distance between these two levels which he called the zone of proximal development.

> *It is the distance between the actual developmental level as determined by independent problem solving and the level of potential development as determined through problem solving under adult guidance or in collaboration with more capable peers.* (Vygotsky, 1978, p. 86).

Vygotsky summarized these two developmental levels of the child; "the actual developmental level characterizes mental developmental functions retrospectively, whereas the zone of proximal development characterizes mental developmental functions prospectively" (p. 86-87).

Moreover, Vygotsky extended the notion of mediation and was regarded as the first to attempt to relate it to psychological questions. Traditionally, the notion of mediation is referred to as the historical changes and events in society and material life that produce changes in human nature. Based on the concept of human labor and tool use, Vygotsky modified the notion of mediation (environment interaction) to the use of signs as well as tools. Thus for Vygotsky, sign systems such as language, writing and number systems, like tool system, are created by societies over the years of human history, change with the form of society and change with the level of its cultural development. Vygotsky concluded that the internalization of culturally produced sign systems causes behavioral transformations and establishes the bridge between early and later forms of individual development. Hence for Vygotsky, the mechanism of individual developmental change is deeply rooted in society and culture (Cole and Scribner, p. 7, in Vygotsky, 1978).

> An essential mechanism of the reconstructive processes that take place during a child's development is the creation and use of a number of artificial stimuli. These play an auxiliary role that permits human beings to master their own behavior, at first by external means and later by more complex inner operations. (Vygotsky, 1978, p. 73)

Thus, in the study of cognitive functioning, Vygotsky applied the notion of mediation and as such a new approach was introduced. He rejected the stimulus-response as a useful technique in conducting interviews. Vygotsky did not limit his approach to the method which offers the subject simple stimuli to which he might have a direct response. Rather, the stimulus may be complex and auxiliary stimuli in the forms of leading questions, collaborations or both offered by the experimenter can be employed. These auxiliary stimuli are vital components of the mediation (the new-created context of a given task). By considering the auxiliary stimuli more information can be obtained on the method in which the child deals with a task.

> Although the stimulus-response methodology makes it extremely easy to ascertain subjects' responses, ss it proves useless when our objective is to discover the means and methods that subjects use to organize their own behavior. (Vygotsky, 1978, p. 74).

In this connection, the piece-wise congruency operation can be considered as a mediation or tool through which children can elaborate their perceptions in one of those basic topics in elementary mathematics - polygonal regions. It is by the use and the influence of such medium that the properties of the polygonal regions in the domain space of area measure system, their interrelationships, the area notion and area measure structure and analogous operations, relations, and properties in the range space of area measure can be illustrated. As a consequence, transformations in children thought is believed to occur.

> Man, if you will, is shaped by tools and instruments that he comes to use, and neither the mind nor the hand alone can amount to much. (Bruner, Introduction, in Vygotsky, 1962).

Freudenthal's 'Task'

In Freudenthal's 'Task' (1973), and on pages 401 - 511, Freudenthal postulated his views on geometry, the instruction of geometry, and the role of geometry at the early levels of schooling. Among other interesting views, he stated that geometry is one of those best opportunities to mathematize reality and to learn how to mathematize reality. Geometry, he added, is meaningful only when it utilizes the relation of geometry to the experienced real surrounding.

> It is an opportunity to make discoveries ... numbers are also a realm open to investigation, thinking can be learned by computing, but discoveries made by one's own eyes and hands are more convincing and surprising. Until they can somehow be dispensed with, the shapes in space are indespensable guide to investigation and discovery. (Freudenthal, 1973, p. 407).

CHAPTER III

RESEARCH PROCEDURES

A. Introduction

In order to secure a more efficient and effective procedure for gathering the required information, a pilot study was conducted in May 1979. The undertaking of the pilot study was to examine (1) whether the proposed laboratory unit would be successful in a typical classroom situation (the feasibility of the unit, i.e., whether it is teachable or not) and (2) the amount of time required to cover the unit.

Based on the findings, it was concluded that a more extensive study seeking answers to the proposed research questions could be undertaken.

B. The Framework

Classroom situations and interviews formed the basis for this study. The work comprised five components:

1. constructing a unit,
2. preparing evaluation tasks,
3. classroom teaching of two classes,
4. research questions, and
5. instrumentation.

<u>Construction of a Unit</u>

One of the major aims of the study was to describe and identify some of the methods of geometric problem solving processes used by eighth grade students. The problems were presented within the two dimensional Euclidean space. Hence, a means of presenting a variety of geometrical problems to which the subjects would respond and which could reveal responses exhibiting some of their geometric thinking processes was undertaken. An investigative laboratory unit on polygonal regions and their areas (Appendix A) was constructed by the researcher for grade eight. It was built on the following operations:

a. decomposition,

b. plane transformations, and

c. piece-wise congruency

<u>Preparing Evaluation Tasks</u>

A package employing colored cardboards consisting of eleven tasks was constructed (see pages 58-61). The core of many of the tasks was the idea of decomposing polygonal regions into rectangular regions and the converse. Briefly, in each task a geometric problem was posed which required each student to attempt a solution using the methods of piece-wise congruency (cut-and-cover) and the motions of slide, turn, and flip (translation, rotation, and reflection).

<u>The Sample and Classroom Teaching</u>

Two classes, class A and class B, of grade eight students in an Edmonton public school were selected for the study. There were 58 students with 29 in each class. From the school's historical point of view, the mathematical achivement was quite different for the two classes. On the final mathematics achievement test of the previous year, the mean scores for class A and class B were 71.57 and 62.25 respectively.

A teacher's manual (Appendix B) containing some possible solutions for each investigative activity was provided to both teacher A and teacher B.

Since the unit was designed with laboratory tasks involving piece-wise cutting and covering, a series of geometric activities was constructed and designed in such a way that they could be administered to students on a class basis during regular classroom hours. A basic concept about polygonal regions was embedded in each activity. The student could respond to each activity by tracing, cutting, and covering. At the first class, a copy of a booklet, the student's booklet (Appendix A), containing the unit was provided to each student. During each classroom period the students were supplied with a pair of scissors, tracing paper (colored), and rulers. Instructions not to use measurement were given throughout each investigative activity. At the beginning of each session the teachers requested the students to read the following notice which appeared at the beginning of the booklets:

Notice

THIS IS NOT A TEST!...THERE ARE NO WRONG ANSWERS!

HOWEVER, WE WOULD LIKE TO SEE THE APPROACH YOU USE IN ARRIVING AT YOUR SOLUTION TO EACH TASK. THEREFORE, IT IS ESSENTIAL TO MAKE YOUR ANSWERS AS CLEAR AS POSSIBLE.

In addition, recurring statements were made to the students not to consider the activities as checkpoints for their knowledge nor to have them believe that the kind of responses they would offer could affect their final marks in the ongoing mathematics course. It was assumed that this emphasis would increase the probability that the students would respond spontaneously and that their responses would exhibit their problem-solving behavior.

Based on the information obtained in the pilot study, projectors and magnetic boards were used in each class. A set of 22 transparencies was prepared by the researcher and provided to each teacher. The transparencies contained some possible solutions to each investigative activity. Magnetic boards were used to facilitate the display of the segments of the unit where area formulae for polygonal regions were derived. Thus, decomposable colored cardboard models for each polygonal region involved in the derivation of the area formulae were prepared. The models were designed in such a way that a possible decomposition and the motions involved could be demonstrated on the magnetic board and the desired area formula could be deduced.

Research Questions

Based upon the review of the literature and the purposes stated in Chapter I, the following research questions were posed:

Question 1. How does the piece-wise congruency approach affect the students?

Question 2. What will the student's reaction be to the piece-wise congruency process?

Question 3. What will the student's reaction be to the use of plane transformations throughout the piece-wise congruency process?

Question 4. We do not know what students learn about area; we do not know what they learn about measure systems; we do know that they do not know very much! Can this approach of equidecomposability contribute more to our knowledge?

Question 5. Is there a cognitive structure in the students' thinking similar to that of the content structure or do they think in a different manner?

Question 6. In using the cut-and-cover procedure, can students understand the notion of equidecomposability as an extension of congruency?

Question 7. Do students recognize properties that are invariant under decomposition?

Question 8. Can students successfully perform the practical procedure of equidecomposing two polygonal regions?

Question 9. Does the approach of piece-wise congruency facilitate students' perception of

a. interrelations between polygonal regions?
b. properties which are invariant under transformations?
c. congruency?

Moreover, does this approach increase students' geometric vocabulary?

Question 10. Are there methodological difficulties in using the piece-wise congruency approach? If so, what are they?

Question 11. Does the approach of piece-wise congruency of polygonal regions contribute to the field of algebra?

Question 12. What contribution can the piece-wise congruency approach offer to the arithmetic operations of addition and multiplication?

Question 13. Does the piece-wise congruency approach simplify the reflexive, symmetric, and transitive operational properties?

Question 14. Does the concept of piece-wise congruency facilitate the idea of equivalence relation?

Question 15. What contribution can the piece-wise congruency approach offer to the study of rational numbers?

Question 16. What contribution can the piece-wise congruency approach offer to the concept of ordering on the set of polygonal regions?

Question 17. What contribution can the piece-wise congruency approach offer to the concept of area measure?

Question 18. Can the piece-wise congruency approach simplify some basic theorems in geometry such as those of Apollonius and Pythagoras?

Question 19. What are the differences between the students in class A and those in class B with respect to exploring and perceiving the requirements of the course?

Question 20. What is the attitude of the participating teachers toward the use of the piece-wise congruency approach in their grade eight mathematics classes?

Instrumentation

There were five instrumentation components utilized:

a. interviews,

b. tests,

c. video tape recording,

d. audio tape recording, and

e. observations.

Interviews

Twelve students of the two grade eight classes, class A and class B, with six students from each class were selected for interviews. It was decided that students chosen for interviews should be representative of their classroom with respect to sex and achievement in mathematics. The school records for the final mathematics marks for the previous year of schooling were used in identifying students as high or low-achievers (see Table IV).

Twelve private interviews were videotaped. Each interview lasted approximately 35 minutes. The student in each interview was provided with a pair of scissors, a ruler, and a pencil. Each student was told repeatedly that there would be no wrong answers. The statement

THERE ARE NO WRONG OR RIGHT ANSWERS...WE ARE INTERESTED IN SEEING YOUR APPROACH IN EACH OF THEM.

was written on a plaque and displayed on the table in front of the student. Also each student was instructed that these activities should not be regarded as a test of knowledge, and the results would not affect the final marks in the ongoing mathematics course.

Throughout each of eight investigative activities, a different pair of polygonal regions (made of cardboards of different colors) was provided to the student. One of the regions in each pair was a rectangular region equal in area to the other region. The other region was one of the following:

right angle triangular region,
acute angle triangular region,
parallelogram region,
right angle trapezoid region,
non-right angle non-isosceles trapezoid region,
quadrilateral region with no special condition,
rhombus region, or
regular pentagon region.

The student was asked to demonstrate and explain size or area differences between the members of each pair. Measurement was not allowed.

The remaining three tasks in the package were: the congruency task, the ordering of rectangular regions task, and the ordering of non-rectangular regions task. The congruency task consisted of two different sets of cardboard regions. There were three congruent regions of different colors in each set; they were triangular in one set and rectangular in the other. Each set was displayed at random on the table and the student was asked to show whether or not the members of each set were congruent. The ordering of rectangular regions task consisted of three rectangular regions A, B, and C. Regions A and B were of equal bases but slightly differed in height while region C was of a base narrower than that of B but with a height equal to that of B (Fig. 7).

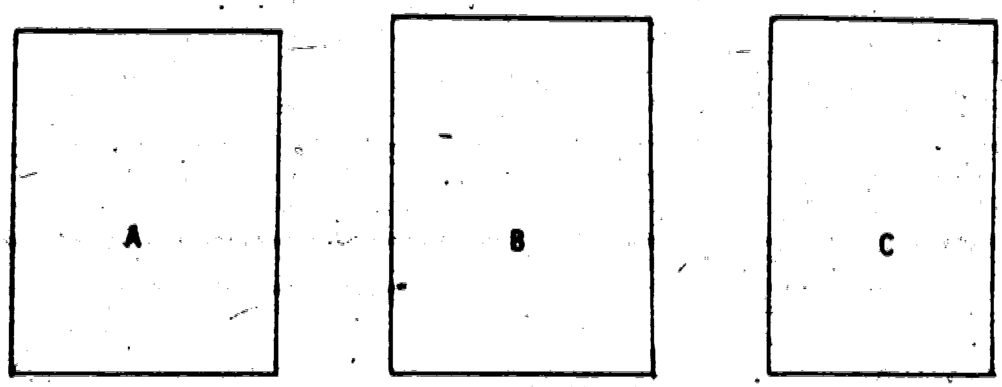


Figure 7

Each student was asked to order the regions A, B, and C with respect to their sizes and again measurement was not allowed. The ordering of non-rectangular regions task comprised three polygonal regions: a right angle triangular region, a non-right angle non-isosceles trapezoid region, and a parallelogram region. The regions were of different sizes: the heights of the first two were equal while the height of the parallelogram region was half of that for the other two (Fig. 8). Each student was asked to explain whether or not it is possible to order these three regions.

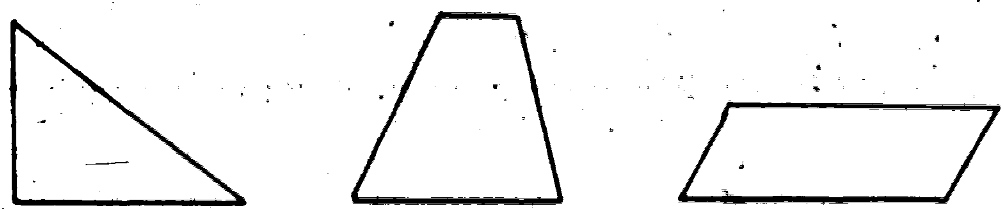


Figure 8

Finally an interview with each of the two participating teachers was conducted by the investigator after the completion of the project. The teachers were asked to give their opinion on the feasibility of the piece-wise congruency approach, its future, and its possible use and adoption for school use. The interview lasted 30 minutes.

Tests

Several tests, based on the research questions stated previously, were administered to students in class A and class B. The tests used in the study were of the three following categories:

1. Opinion Assessment Tests,
2. Geometry Tests,
3. Progress Checking Tests.

1. Opinion Assessment Tests

There were three types of opinion assessment tests involved in the study:

a. Student's Opinion Assessment Test on Geometry.

b. Student's Opinion Assessment Test on Mathematics, and

c. Teacher's Opinion Assessment Test on the Unit.

a. Student's Opinion Assessment Test on Geometry. An opinion assessment test consisting of 15 items (Appendix C) was constructed to check the student's attitude toward geometry. The test was administered to all students both before and after the teaching of the unit. These pretests and posttests were used to determine whether or not there was a change in attitude.

b. Student's Opinion Assessment Test on Mathematics. A similar opinion assessment test consisting of 15 items (Appendix D) was also constructed to check the student's attitude to mathematics. This test was almost identical to the previous one except the word "mathematics" replaced the word "geometry" and the order of the items was reversed. Again, the test was given to all students both before and after the teaching of the unit.

c. Teacher's Opinion Assessment Test on the Unit. A questionnaire of 15 items (Appendix E) was prepared to determine the participant teacher's general opinion on the unit and the use of the piece-wise congruency approach. The teachers were given three days to consider the questions and make their responses.

2. Geometry Tests

There were six geometry tests prepared for the study. The tests were administered as pretests and posttests for all students. The tests (Appendix F) were as follows:

i) Vocabulary Test on Polygonal Regions,

ii) Polygonal Region Differentiation Test,

iii) Polygonal Region Area Formulae Test,

iv) Piece-Wise Congruency - Three Polygonal Regions Test,

v) Piece-Wise Congruency - Two Polygonal Regions Test, and

vi) Piece-Wise Congruency - Rational Numbers Test.

3. Progress Checking Tests

Three progress checking tests (Appendix G) were administered for both class A and class B. They were given over the last 10 minutes of the second classroom session, the last 20 minutes of the fourth classroom session, and the last 15 minutes of the sixth classroom session respectively. These tests were designed to provide feedback as to whether or not some indication of progress was evident in the student's performance as well as to obtain information about the student's interaction with the unit.

Video Tape Recordings

Videotapes were used to record the interviews of the twelve students. The interviews were carried out two weeks prior to the starting of the teaching and lasted seven hours. Two hours of classroom time were then videotaped, an hour for each class, when the teachers started teaching the final section of the unit. This section was devoted to the derivations of area formulae for triangular, convex quadrilateral, and regular n-gon regions (n = 5, 6, 7, ...). Magnetic boards were used throughout this section for displying the magnetized polygonal models of the regions involved.

Audio Tape Recording

Classroom instructions for each class and for all classroom sessions were tape recorded. Classroom dialogues were then transcribed and a written record of all verbal conversations was prepared and analyzed with respect to the research questions.

Observations

During the regular mathematics periods, the investigator attended all these classroom sessions when the unit was being taught. Throughout these class periods, the investigator recorded observations which focused on the students' behavior during various problem solving situations involving decomposition and piece-wise congruency of polygonal regions.

C. Hypotheses Tested

Statistical analysis was used to examine the differences between the response scores of the students throughout the pretest and the posttest for each of the tests described above.

A panel of three people to mark the responses of the six geometry sub-tests described above was used. A response categorization and scores distribution scheme for each item in each of the subtests was then devised. The scheme was not pre-designed but determined on the basis of the different kinds of responses students actually made for each of the items. The categorization and scores distribution scheme was considered appropriate by the three members of the marking panel. However, the fact remains that the categories were developed from the prospective of the panel's members which may be completely differernt from that of the students who made these responses. The categories and scores distribution were as follows:

The six geometry subtests are divided into two groups of tests.

1. Group one. This group consisted of the subtests (i) and (iii). For each item of these tests, the students' responses were categorized into three classes: null, low, and high and apart from the null category, scores with proportional weight ratio of 1:2 were assigned to the other two categories respectively.

2. Group two. This group consisted of the subtests (ii), (iv), (v), and (vi). Each item in these tests consisted of two parts which require the student to respond for the first part and to justify his response for the second part. If part one is wrong, then the whole item would be wrong. Otherwise, the students' responses on part two

would be considered. They were categorized into three classes, that is, their reasonings were categorized as low, medium, and high. With the two categories for part one as wrong (null), and right, the whole category scheme would be: null, right, low, medium, and high. The scores for the last four categories were given proportional weight ratio of 2 : 3 : 4 : 5. With these scores distributions in mind, the members of the marking panel were free to choose whatever scaling unit that might suit them. Below is a summary of the categorization and scores distribution scheme used in the five geometry subtests.

TEST	CATEGORY					NUMBER OF ITEMS
	Null	Right	Low	Medium	High	
i	0	-	1/2	-	1	14
ii	0	2	3	4	5	11
iii	0	-	1/2	-	1	8
iv	0	1	1½	2	2½	4
v	0	1	1½	2	2½	4
vi	0	2	3	4	5	6

The scores for these tests were used to test the following hypotheses:

Hypothesis 1 There is no significant change in the attitude of the classes over the pretest and the posttest as measured by the opinion assessment test toward geometry.

Hypothesis 1A There is no significant change in the mean scores on the pretest and the posttest of the classes over each item of the opinion assessment test toward geometry.

Hypothesis 2 There is no significant change in the attitude of the classes over the pretest and the posttest as measured by the opinion assessment test toward mathematics.

Hypothesis 2A There is no significant change in the mean scores on the pretest and the posttest of the classes over each item of the opinion assessment test toward mathematics.

Hypothesis 3 There is no significant change in the achievement of the classes over the pretest and the posttest of the geometry tests collectively.

Hypothesis 3A There is no significant change in the mean scores on the pretest and the posttest of the classes over each of the geometry tests.

Hypothesis 4 There is no significant difference in the attitude of the classes A and B on the pretest of the opinion assessment test toward geometry.

Hypothesis 4A There is no significant difference between the mean scores of the classes A and B over each item in the pretest of the opinion assessment test toward geometry.

Hypothesis 5 There is no significant difference in the attitude of the classes A and B on the posttest of the opinion assessment test toward geometry.

Hypothesis 5A There is no significant difference between the mean scores of the classes A and B over each item in the posttest of the opinion assessment test toward geometry.

Hypothesis 6 There is no significant difference in the attitude of the classes A and B on the pretest of the opinion assessment test toward mathematics.

Hypothesis 6A There is no significant difference between the mean scores of the classes A and B over each item in the pretest of the opinion assessment test toward mathematics.

Hypothesis 7 There is no significant difference in the attitude of the classes A and B on the posttest of the opinion assessment test toward mathematics.

Hypothesis 7A There is no significant difference between the mean scores of the classes A and B over each item in the posttest of the opinion assessment test toward mathematics.

Hypothesis 8 There is no significant difference in the achievement of the classes A and B on the pretest of the geometry tests collectively.

Hypothesis 8A There is no significant difference in the achievement of the classes A and B over each individual test in the pretest of the geometry tests.

Hypothesis 9 There is no significant difference in the achievement of the classes A and B on the posttest of the geometry tests collectively.

Hypothesis 9A There is no significant difference in the achievement of the classes A and B over each individual test in the posttest of the geometry tests.

Quantitatively, multivariate analysis of variance (Morrison, 1976) was utilized and as such Hotelling T^2 test for both single sample with repeated measures and two independent samples was performed in testing the hypotheses. MULV 07 and MULV 08 programmes were used at the Division of Educational Research Services, University of Alberta.

Qualitatively, Vygotsky's approach based on the notion of mediation was used throughout the interviews. In a given task, leading questions, suggestions, and sometimes collaboration were offered by the experimenter. In this connection, the rationale of Vygotsky's approach is perhaps well illustrated in the following:

> By using this approach, we do not limit ourselves to the usual method of offering the subject simple stimuli to which we expect a direct response. Rather, we simultaneously offer a <u>second series of stimuli</u> that have a special function. In this way, we are able to study the <u>process of accomplishing a task by the aid of specific auxiliary means</u>; thus we are also able to discover the inner structure and development of higher psychological processes. (Vygotsky, 1978, P. 74).

CHAPTER IV

RESULTS OF THE STUDY

A. Introduction

Based on the observations made, scrutinizing the transcripts and the video tape records, each of the research questions is answered. Excerpts from the transcripts of the interviews are identified by sequential numbers. Students are coded by two digit numbers followed by the letter M (male) or F (female) for sex identification. An excerpt attributed to a particular student is expressed by the number of the excerpt followed by the student's code; for example Excerpt 2 (32F) identifies the second excerpt attributed to a female student coded 32F. Also, the letters I and S are used within each excerpt to denote the investigator and the student, respectively.

B. Research Questions

Question 1. How does the piece-wise congruency approach affect the students?

The attitude of the students toward geometry and mathematics was examined twice, once early in the project and then at the end, using the questionnaires: Student's Opinion Assessment Test on Geometry and Mathematics (see Appendixes C and D). The mean scores of class A and class B on the pretest and on the posttest in each questionnaire are shown in Table I and Table II.

TABLE I

MEAN SCORES OF CLASS A AND CLASS B ON THE PRETEST AND THE POSTTEST - STUDENT'S OPINION ASSESSMENT TEST ON GEOMETRY: FIFTEEN ITEMS

TEST ITEM	CLASS A		CLASS B	
	PRE	POST	PRE	POST
1	3.154	3.538	3.000	3.300
2	4.346	4.731	3.900	3.900
3	2.462	2.731	2.200	2.500
4	2.500	2.962	2.500	2.950
5	4.308	4.462	3.900	3.900
6	2.808	2.962	3.650	3.650
7	3.538	3.385	3.200	3.450
8	3.692	3.769	3.850	3.800
9	2.500	3.423	2.350	2.600
10	3.000	3.115	2.950	2.950
11	3.885	3.885	3.800	3.900
12	3.769	3.462	3.600	3.400
13	3.154	3.345	2.350	3.050
14	2.923	2.962	2.650	2.750
15	3.462	3.346	3.200	3.350

TABLE II

MEAN SCORES OF CLASS A AND CLASS B ON THE PRETEST AND THE POSTTEST - STUDENT'S OPINION ASSESSMENT TEST ON MATHEMATICS: FIFTEEN ITEMS

TEST ITEM	CLASS A		CLASS B	
	PRE	POST	PRE	POST
1	3.960	3.880	3.500	3.500
2	4.520	4.640	4.200	4.100
3	3.120	3.560	2.750	2.350
4	3.360	3.640	2.800	2.550
5	3.800	4.200	3.800	3.950
6	3.880	3.920	3.400	3.050
7	3.920	4.160	3.850	3.600
8	4.720	4.563	4.100	4.150
9	3.200	3.680	3.500	3.300
10	3.960	3.840	3.500	3.300
11	4.480	4.440	4.200	4.000
12	3.880	4.120	3.900	3.950
13	3.560	3.920	2.950	3.050
14	3.800	4.160	3.450	3.050
15	3.840	3.920	3.850	3.850

There were five categories of responses scored 1, 2, 3, 4, and 5 on each item; they were strongly disagree, disagree, indifferent, agree, and strongly agree, respectively.

These scores were considered as repeated measures and Hotelling T^2 test for single sample with repeated measures was then used. The results indicated that hypotheses (see Chapter III) 1, 1A, 2 and 2A ($p > 0.05$) not be rejected. Hence, dramatic changes in the attitude of the students toward geomtry and mathematics over the period of the project was not apparent.

However, the following information in Table III part A, B, and C show some notable changes in the attitude of the students toward geometry.

TABLE IIIA

FREQUENCIES OF THE FIVE CATEGORIES OF RESPONSES FOR THE OPINION ASSESSMENT TEST - GEOMETRY OF CLASS A AND CLASS B ON THE FOLLOWING ITEMS:

1. Geometry is a pleasant subject.
2. Geometry makes me embarrased
3. I like to do geometric problems in other subjects.
4. I like working on my geometry homework.
5. When I have to do geometry I get nervous.

Item	Test	CLASS A							CLASS B						
		Negative Response			Indif-ferent	Positive Response			Negative Response			Indif-ferent	Positive Response		
		(total)	S.D	D	I	(total)	A	S.A	(total)	S.D	D	I	(total)	S	S.A
1	pr	5	3	2	13	10	7	3	8	2	6	2	10	10	0
	po	3	0	3	10	15	12	3	5	0	5	6	9	7	2
2	pr	27	14	13	0	1	0	1	14	7	7	4	2	1	1
	po	28	21	7	0	0	0	0	17	3	14	1	2	2	0
3	pr	15	7	8	7	6	5	1	14	3	11	5	1	1	0
	po	13	3	10	8	7	5	2	11	2	9	6	3	3	0
4	pr	16	7	9	6	6	6	0	12	3	9	3	5	5	0
	po	12	4	8	6	10	9	1	9	1	8	5	6	5	1
5	pr	24	15	9	3	1	1	0	17	3	14	1	2	2	0
	po	27	15	12	1	0	0	0	16	3	13	3	1	1	0

(*) S.D= Strongly Disagree; D= Disagree; I= Indifferent; S.A = Strongly Agree; A = Agree; Pr= Pretest and Po = Posttest.

TABLE IIIB

FREQUENCIES OF THE FIVE CATEGORIES OF RESPONSES FOR THE OPINION ASSESSMENT TEST ON GEOMETRY OF CLASS A AND CLASS B ON THE FOLLOWING ITEMS:

6. I have always like geometry in school.
7. Geometry is a boring subject.
8. Geometry is not a useful subject.
9. I can see geometry everywhere.
10. I am interested in studying more geometry in coming years of my schooling.

Item	Test	Class A Negative Response (total)	Class A S.D	Class A D	Class A Indifferent I	Class A Positive Response (total)	Class A A	Class A S.A	Class B Negative Response (total)	Class B S.D	Class B D	Class B Indifferent I	Class B Positive Response (total)	Class B A	Class B S.A
6	Pr	12	5	7	4	12	10	2	9	2	7	7	4	4	0
	Po	9	1	8	10	9	6	3	12	1	11	3	5	5	0
7	Pr	16	6	10	8	4	3	1	9	2	7	7	4	1	3
	Po	15	4	11	8	5	4	1	12	3	9	6	2	2	0
8	Pr	20	7	13	4	4	3	1	16	3	13	2	2	2	0
	Po	18	7	11	7	3	2	1	14	5	9	3	3	3	0
9	Pr	14	6	8	6	8	7	1	13	4	9	3	4	4	0
	Po	8	4	4	2	18	13	5	11	2	9	5	4	3	1
10*	Pr	10	3	7	5	11	11	0	4	1	3	11	5	5	0
	Po	9	0	9	8	9	8	1	8	2	6	5	7	5	2

(*) Number of subjects with repeated scores in class A dropped to 26.

TABLE IIIC

FREQUENCIES OF THE FIVE CATEGORIES OF RESPONSES FOR THE OPINION ASSESSMENT TEST ON GEOMETRY OF CLASS A AND CLASS B ON THE FOLLOWING ITEMS:

11. Geometry is my most hated subject.
12. Geometry is not a practical subject.
13. I enjoy trying difficult problems in geometry.
14. I would like to take more geometry when I have the opportunity.
15. I like to help others in geometry.

		CLASS A							CLASS B						
		Negative Response			Indif-ferent	Positive Response			Negative Response			Indif-ferent	Positive Response		
Item	Test	Total	S.D	D	I	Total	A	S.A	Total	S.D	D	I	Total	A	S.A
11	Pr	17	7	10	6	3	3	0	16	3	13	2	2	1	1
	Po	19	6	13	5	2	2	0	15	5	10	4	1	0	1
12	Pr	17	3	14	6	3	2	1	11	1	10	5	3	3	0
	Po	11	4	7	13	2	1	1	10	0	10	8	2	1	1
13	Pr	8	3	5	7	11	7	4	12	5	7	4	4	4	0
	Po	6	1	5	8	12	8	4	7	1	6	5	8	5	3
14	Pr	7	2	5	12	7	7	0	9	2	7	7	4	4	0
	Po	10	0	10	10	6	5	1	10	3	7	3	7	6	1
15	Pr	5	2	3	8	13	7	6	7	0	7	4	9	7	2
	Po	6	1	5	6	14	12	2	6	2	4	3	11	7	4

To contrast the responses of the students in class A with those of class B on the pretest and on the posttest, Hotelling T^2 test for two independent samples was performed on each of the opinion assessment tests. The results indicated that hypothesis 4, 4A, 6, and 6A were not rejected. In other words, the responses of both class A and class B over the pretest for each of the opinion assessment tests were almost identical. A similar lack of significant difference held over the posttests when hypotheses 5, 5A, 7, and 7A were not rejected either. The two classes, however, were remarkably different on the grade 7 final mathematics achievement scores (see Chapter III).

An examination of actual scores of the students who were interviewed early in the project illustrates a shift in the attitude for individual students in this group (Table IV). With respect to the scoring scheme, the minimum total score on the 15 items was 15 points and the total maximum score was 75 points.

Table IV indicates that the most prominent change in scores occurred for 30F, 26F, and 12M on geometry and 19F and 30F on mathematics. Student 30F, the <u>lowest</u> <u>aohiever</u> <u>in</u> <u>class</u> <u>B</u>, seemed to increase in motivation throughout the project. She had the <u>highest</u> <u>gain</u> in her scores on both geometry and mathematics. Student 12M, the <u>highest</u> <u>achiever</u> <u>in</u> <u>class</u> <u>A</u>, whose responses were indicative of a high positive change on geometry, showed a smaller increment in his scores on mathematics. Student 26F, a <u>below</u> <u>average</u> <u>achiever</u> in class A, appeared to have a large positive change on geometry while maintaining a stable attitude on mathematics. Students 19F, a below

TABLE IV

STUDENTS' TOTAL SCORES ON THE OPINION ASSESSMENT TEST: GEOMETRY AND MATHEMATICS: TWELVE STUDENTS WHO WERE INTERVIEWED.

STUDENT	GEOMETRY		MATHEMATICS		FINAL GRADE 7 SCHOOL MATHEMATICS MARK (%)
(by code)	Pre	Post	Pre	Post	
03M	46	48	34	35	41
05F	65	66	71	75	89
12M	44	52	61	64	94
19F	45	48	39	48	54
26F	29	40	46	48	58
27M	63	63	56	60	87
30F	52	71	54	65	36
34F	55	56	66	62	83
41F	43	42	48	41	63
44F	54	56	58	61	94
46M	45	50	58	63	61
48M	43	42	54	52	42

average achiever in class A, whose scores showed a small gain on geometry, showed a high positive change toward mathematics. Another positive change on both geometry and mathematics was shown in 46M's scores although it was not as large a gain as that of 30F. However, student 41F, a below average achiever, appeared to have a negative change of one point on geometry and a drop of seven points on mathematics. Another negative change was in student 48M's scores on both geometry and mathematics by one and two points respectively. Finally, student 34F showed a positive change of one point on geometry and a negative change of four points on mathematics.

In general, on geometry, three students had large positive changes, six with slight positive changes, one with no change, and two with negative changes. With respect to mathematics, there were two students with high positive changes, seven with slight positive changes, two with slight negative change, and one with a large negative change.

In summary, the above information, quantitatively and qualitatively, indicate that students' attitude towards geometry and mathematics did not considerably change throughout the period between the pretest and the posttest (about 3 weeks). Only a few individual changes in the attitude could be regarded as notably high. As well, the two classes, class A vs class B, showed no significant difference in attitude either on the pretest or on the posttest on each of opinion assessment tests.

Question 2. What will the students' reaction be to the piece-wise congruency process?

This question is designed to give a general overview of the way grade eight students perceive the cut-and-cover process. That is, how well do they cope with it?

Throughout the evaluation tasks (see pages 58-61), it was observed that students were persistent in attempting those tasks on piece-wise congruency. In many cases, experiencing failure on the first attempt of a task did not discourage the students. For example, student 48M did not concede when the chosen strategy was unsuccessful on the first attempt of task 1 - equidecomposing a right triangular region and a rectangular region (Fig. 9).

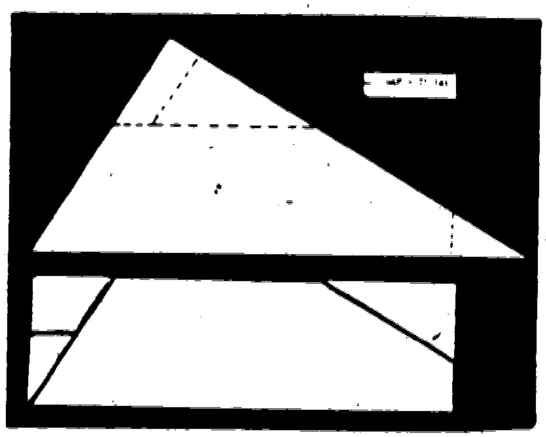


Figure 9

~~(A copy of the 48M's original 1st attempt)~~

The student then asked for another rectangular region and a correct solution was later produced (Fig. 10)

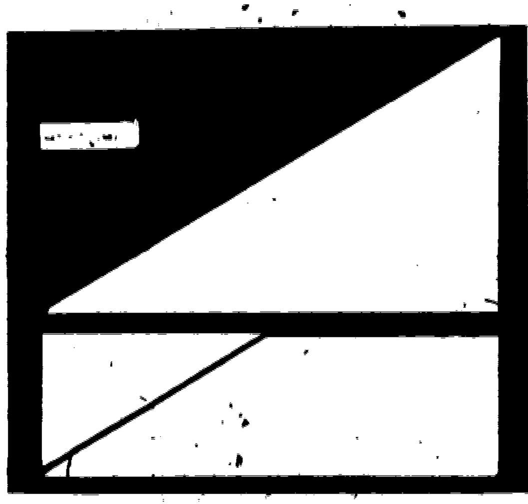


Figure 10
(A copy of the 48M's original 2nd attempt)

Another student, 27M, met with failure on the first attempt (Fig. 11) but was successful later (Fig. 12) on task 2 - equidecomposing a triangular region and a rectangular region.

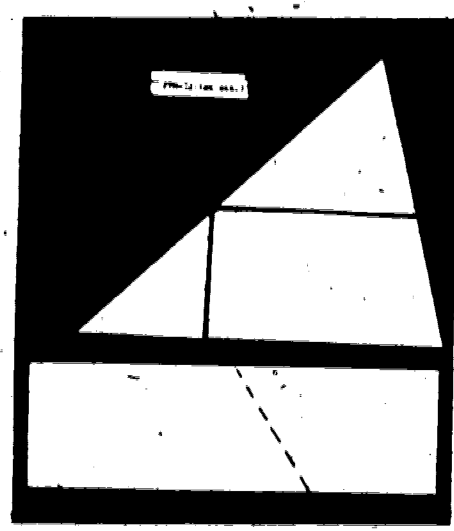


Figure 11

(A copy of the 27M's original 1st attempt)

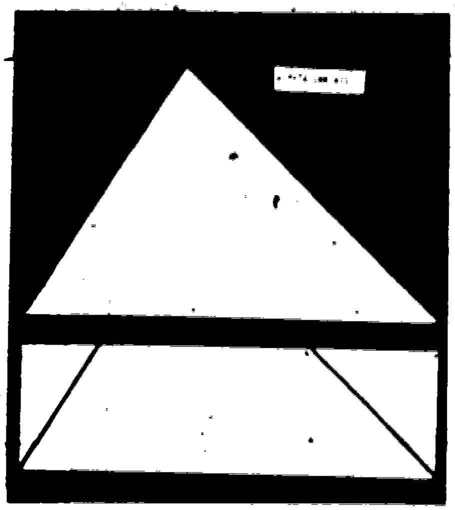


Figure 12

(A copy of the 27M's original 2nd attempt)

As well, student 05F performed task 1 successfully on the second attempt while she was successful neither on the first nor on the second attempt of task 2. However, another student, 03M, after a second unsuccessful attempt on task 2 gave up: "I can't do it" he said (see Appendix H). Student 48M persued task 5 and eventually succeeded on the second attempt (Fig. 13 and Fig. 14).

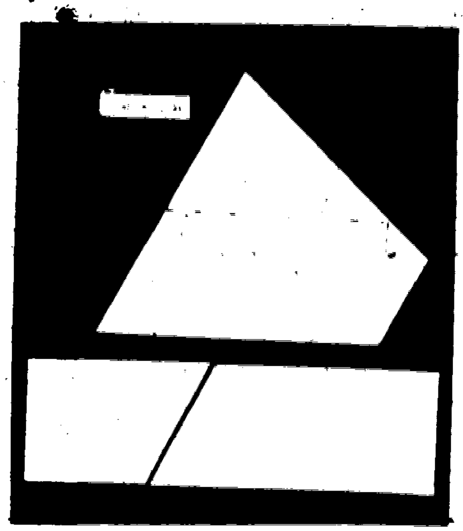


Figure 13

(A copy of the 48M's original 1st attempt)

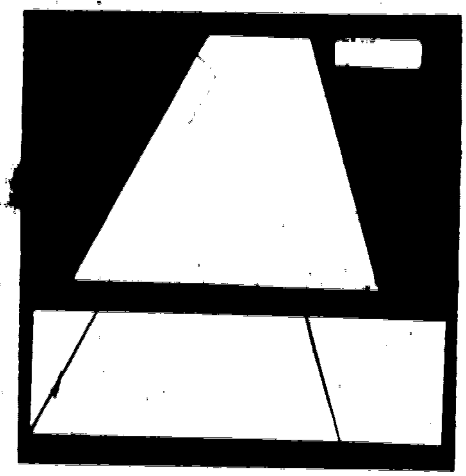


Figure 14

(A copy of the 48M's original 2nd attempt)

Tasks 6 and 7 (equidecomposing a quadrilateral region with a rectangular region; and a rhombus region with a rectangular region) appeared to be the most difficult tasks presented in the interviews. All students failed both of them. There was, however, [redacted] approximate covering performed on task 6 by the highest achiever in class A (12M). This approximation was achieved after a failure on the first attempt (Fig. 15 and Fig. 16).

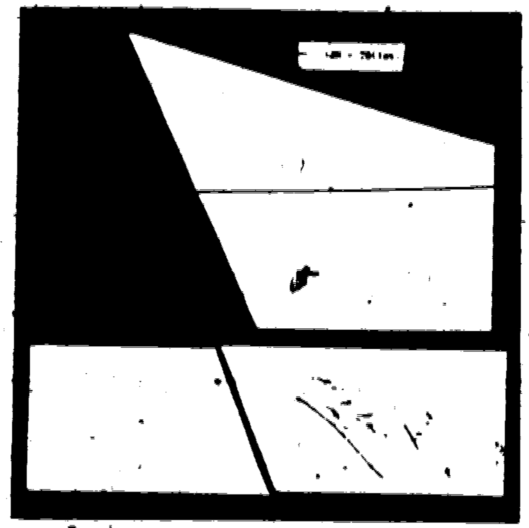


Figure 15

(A copy of the 12M's original 1st attempt)

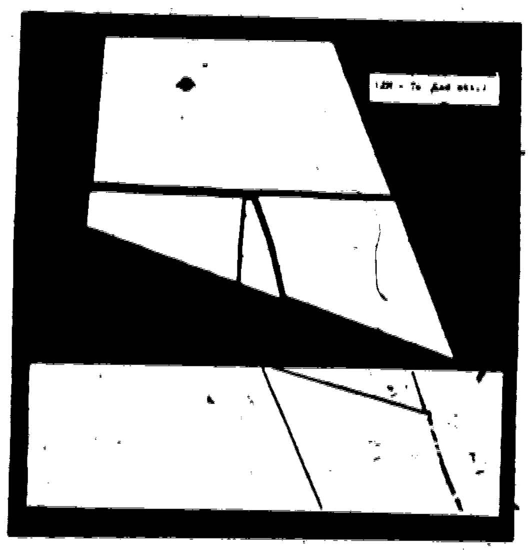


Figure 16

(A copy of the 12M's original 2nd attempt)

On task 7, the performance was worse. No student achieved success, nor an approximate solution either. In fact, four of the students did not attempt any cut on any of the rhombus or the rectangular regions. In view of the content structure of tasks 6 and 7, one of the key ideas is to draw one of the diagonals first for each the quadrilateral and the rhombus regions. It seems that students are less familiar with diagonal lines than with horizontal and vertical lines. This was evident on the performance of the students on task 7 where horizontal and vertical cuts were used repetitively without success (See Appendix H, task 7). Also, it was noticed that, when two regions were superposed, and a vertical or a horizontal cut was found for one region it generates its dual the oblique cut on the other region and vice-versa. This was apparent throughout all the cuttings at tasks 1, 2, 3, 4, 5, 6, and 7 especially where the dual lines for the cutting lines are illustrated (see Appendix H). It was a matter of which region was facing the student that determined the cutting along its edges of the other region. In task 1 (equi-decomposing a triangular region and a rectangular region), for example, if the triangular region was superposed on the rectangular region, an oblique cut on the rectangular region would more likely be used by the student and vice versa. As well, the 12M's performmance was interesting on task 7 though his approximate covering was unsuccessful (Fig. 17 and Fig. 18).

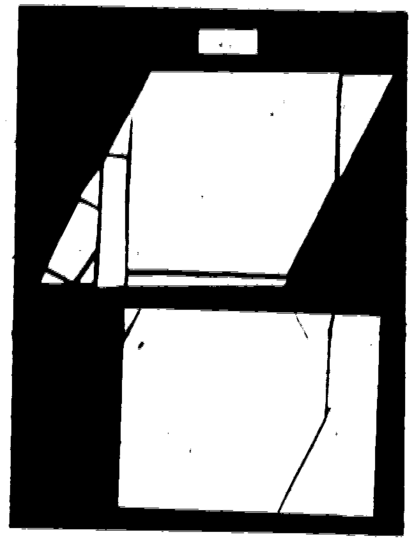


Figure 17

(A copy of the 12M's original cuttings)

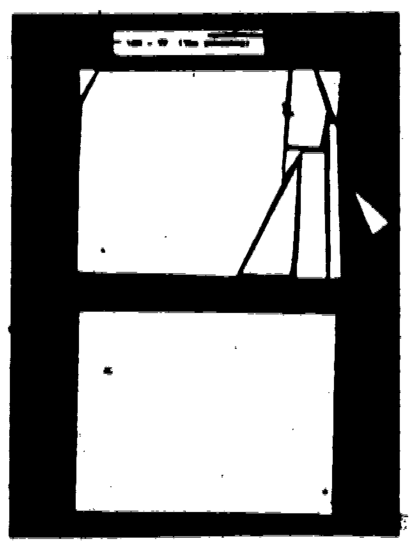


Figure 18

(A copy of the 12M's original rearrangement)

His procedure ends in an infinite cutting and patching process which ultimately took him nowhere.

One of the main purposes was to investigate the extent to which the piece-wise congruency approach affects the students' achievement on the Geometry Tests when instruction is designed for the use of such an approach. Thus, the two classes, A and B, were first pretested and the Geometry Tests were among the instruments used in this testing. Both Teacher A and Teacher B confirmed that polygonal regions and their areas, as part of the grade eight mathematics programme, were covered prior to the starting of the project. After the completion of the piece-wise congruency unit, the students were then retested. Table V shows the mean scores of the two classes jointly on the pretest and the posttest for the Geometry Tests (i), (ii), (iii), (iv), (v), and (vi). The testing scores were considered as repeated measures and Table VI gives a summary of the Hotelling T^2 test for single sample with repeated measures. The pretest achievement scores were contrasted with the corresponding posttest achievement scores both overall and over each of the tests simultaneously.

Table VI shows enough evidence for significant changes in the students' achievement on the Geometry Tests. In particular, there were significant changes in:

1. The overall achievement of the classes over the Geometry Tests collectively, $F = 27.586$, $p < 0.001$.

2. The achievement of the classes over the Polygonal Region

TABLE V

MEAN SCORES OF THE STUDENTS ON THE PRETEST AND THE POSTTEST OF THE GEOMETRY TESTS: SIX SUBTESTS.

GEOMETRY TEST	PRETEST	POSTTEST
i	9.745	10.585
ii	14.404	48.447
iii	3.777	4.947
iv	4.255	7.894
v	4.894	8.117
vi	14.085	20.925

TABLE VI

HOTELLING T^2 CONTRAST BETWEEN ACHIEVEMENT SCORES ON THE PRETEST AND THE POSTTEST OF THE TWO CLASSES JOINTLY: SIX SUBTESTS.

TEST	T^2	DF_1	DF_2	F-RATIO	PROBABILITY
i	14.003	11	36	0.996	0.468
ii	360.616	11	36	25.656	0.000
iii	14.468	11	36	1.029	0.442
iv	34.363	11	36	2.445	0.021
v	57.029	11	36	4.057	0.001
vi	32.002	11	36	2.277	0.031
ALL	387.743	11	36	27.586	0.000

Differentiation Test; $F = 25.656$, $p < 0.001$.

3. The achievement of the classes over the Piece-Wise Congruency - Three Polygonal Regions Test; $F = 2.445$, $p < 0.05$.

4. The achievement of the classes over the Piece-Wise Congruency - Two Regions Test; $F = 4.057$, $p < 0.05$.

5. The achievement of the classes over the Piece-Wise Congruency - Rational Numbers Test; $F = 2.277$, $p < 0.05$.

On the other hand, there was no significant change in the achievement of the classes over the Vocabulary Test on polygonal regions ($F = 0.996$, $p > 0.05$) nor over the Area Formulae Test ($F = 1.029$, $p > 0.05$).

Based on the above information, hypothesis 3 was rejected, while hypothesis 3A was rejected over the geometry subtests (ii), (iv), (v), and (vi). Hypothesis 3A, however, was not rejected over subtests (i) and (iii).

The means which were included in Table V are plotted in Fig. 19.

The previous quantitative analysis indicates that the piece-wise congruency approach enhanced the students' perception of (a) interrelationships among various polygonal regions, (b) the meaning of the term 'polygonal', (c) fractions which belong to the same equivalence classes, and (d) the importance of the role of a chosen unit in any visual comparison between rational numbers.

Qualitatively, the piece-wise congruency approach introduced problem solving situations in which students appeared interested and persistent. That is, they were doing more than listening.

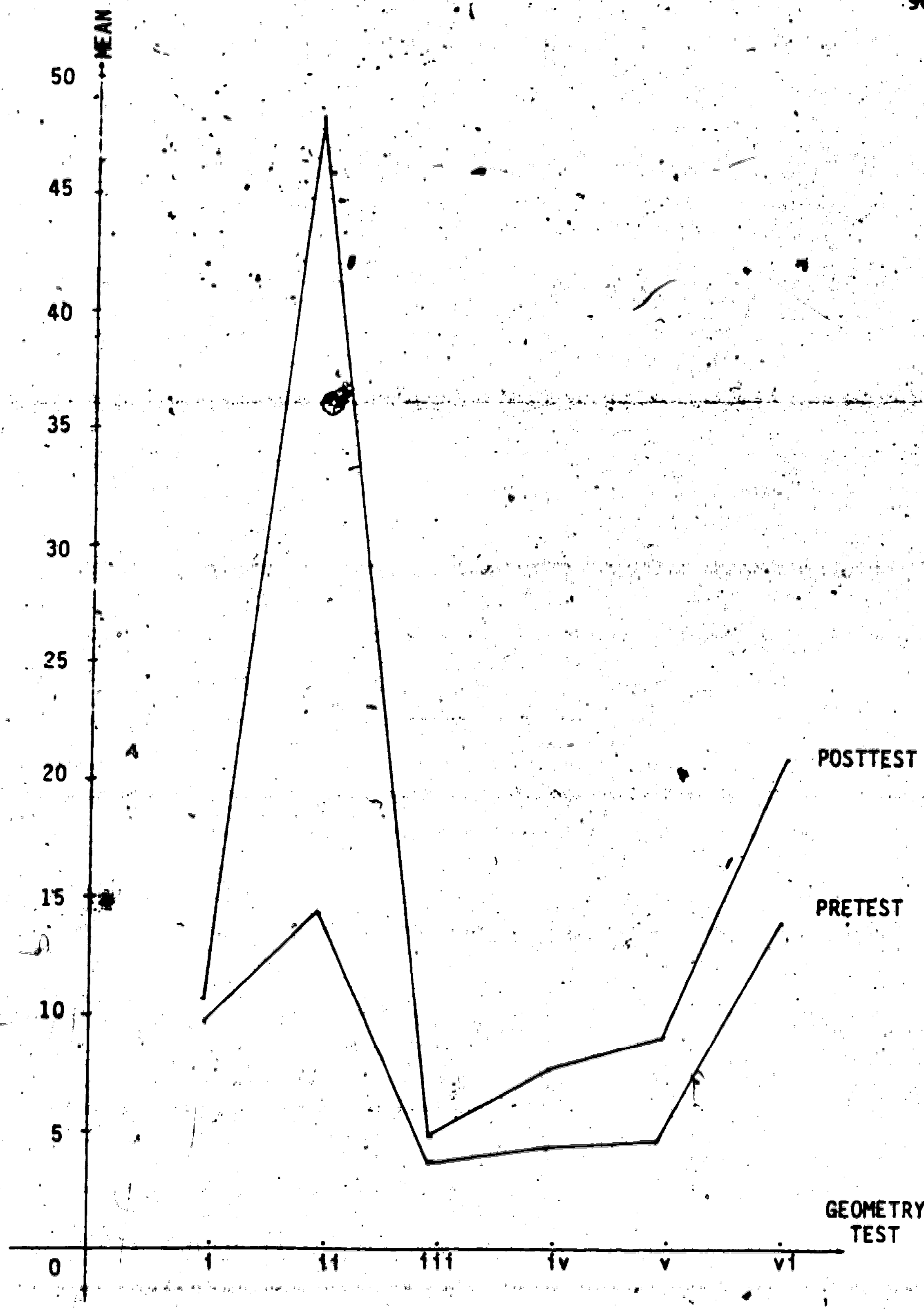

MEAN
50
45
40
35
30
25
20
15
10
5
0
POSTTEST
PRETEST
GEOMETRY TEST
i
ii
iii
iv
v
vi

Figure 19

Question 3. What will the student's reaction be to the use of plane transformations throughout the piece-wise congruency process?

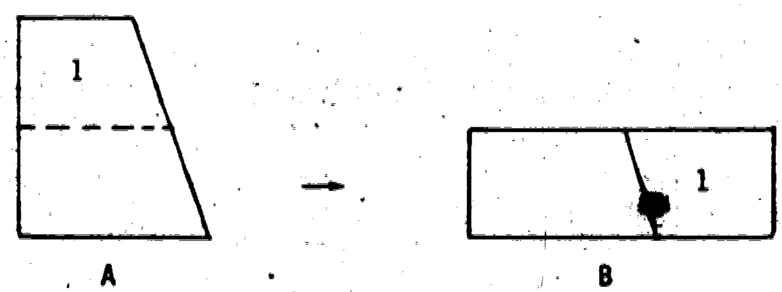
This question is primarily designed to provide an indication of how well the students employ the motions of slide, turn, and flip in the cut-and-cover process.

In each of the evaluation tasks on equidecomposability after cutting a polygonal region into pieces, students did not use motions (transformations) in rearranging the pieces to cover the other region. Instead, they picked up the pieces and simply placed them on the other region. Their procedures resembled those of jigsaw puzzle fittings rather than motion geometry operations. However, when the investigator brought their attention to the possibility of using a slide, turn, or flip, their performance on the rearrangement of the subregions changed. The jigsaw puzzle style disappeared and motion geometry operations were used. Another peculiar behavior was that some students, after cutting a region into pieces, slid the pieces through curvelinear paths, often with a rotational component. A third observation, and perhaps a more interesting one was that some students turned their hands a half-turn spontaneously, and then placed the pieces on the second region. However, there were few cases in which students did use the motions of slide, turn, and flip properly; some students failed to identify these motions and yet used them.

Students who placed the pieces of a region on the other region experienced difficulties when they were asked to return the pieces or reassemble them to their original shape. On the other hand, those who used motions properly, did the reassembling easily.

The following excerpts will illustrate a variety of different situations in which the use of the motions of slide, turn, and flip were encountered; the tasks involved were successfully performed.

Excerpt 1 (05F) - a high achiever:
(This is related to the evaluation task number 4 - equidecomposing a right trapezoid region and a rectangular region).



(The student decomposed region A into region B)

I: What kind of motion did you use? (The regions were superposed on each other).

S: Amm, things like this? (She starts returning region 1 back).

I: Yes, it was here wasn't it? (Pointing to subregion 1 of A.)

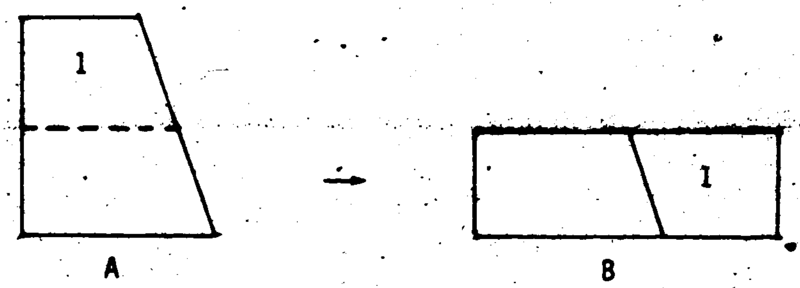
S: Yes, a, turn.

I: Do you know how many right angles you turned this piece? (Pointing to piece 1).

S: One.

Thus, this student, 05F, performed the required motion alright and yet failed to identify that a half turn (or two right angles) was actually used. The following excerpt notes the procedure of another student:

Excerpt 2 (34F) - a high achiever.
(This is related to task 4 as well. The student decomposed the trapezoid region into the rectangular region).



I: Could you return this piece to where it was before? (Pointing to piece 1 while the regions A and B were superposed on each other). (The student starts returning piece 1 back to assemble region A.)

I: What motion would you use to cover this bit? (Pointing to piece 1 of the rectangular region left uncovered).

S: It would be a turn.

I: Do it, use your finger as a point of turn.

S: I...(the student starts turning)

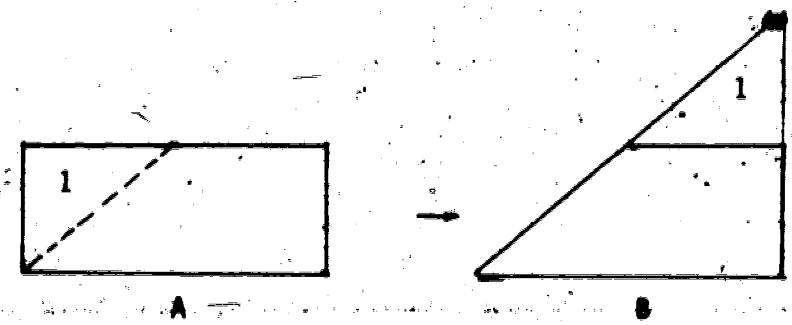
I: So, a half turn?

S: No, three-quarters.

As well, this student performed the required motion when she was asked but failed to recognize that a half turn was really used. The following excerpt further illustrates the lack of perception in recognizing the three motions:

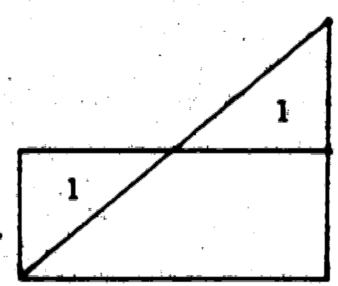
Excerpt 3 (20F) - a below average achiever:
(This is related to the evaluation task number 1. The student

decomposed the rectangular region into the right triangular region).



(While the two regions were superposed on each other the investigator returned region 1 back and reassembled the rectangular region.)

I: This was the situation before the cut, OK? (The two regions A and B, were displayed as in the figure below.)



S: Yes.

I: What motion have you used on this piece (Pointing to piece 1 of region A) to cover this piece? (Pointing to piece 1 of region B).

S: aaa

I: Do you know what motion?

S: Yes, flip

I: You said flip?

S: Yes.

I: Is it? (starts to turn a piece of cardboard on the desk in front of the student.)

S: Alright.

I: This motion, is it a flip?

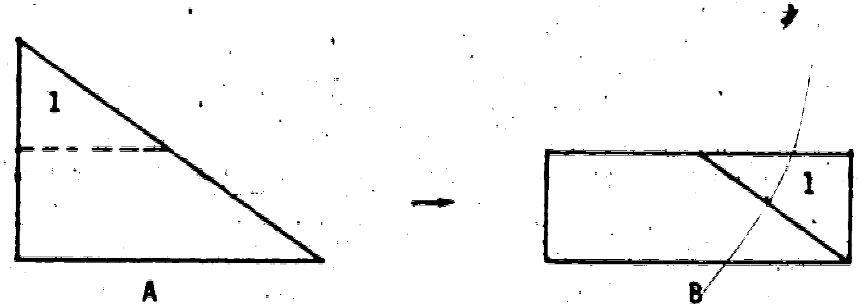
S: No.

I: No, what is it then?

S: It is a turn.

The following excerpt notes what occurred with the highest achiever of class A.

Excerpt 4 (12M): the highest achiever in class A.
(This is related to task 1 as well. The student decomposed the right triangular region into the rectangular region.)



I: What motion have you used?

S: Turn.

I: Do you know how many right angles you have used?

S: Amm.

I: Turn it. (The student starts turning piece 1 of the triangular region counter-clockwise and clockwise alternately while regions A and B were superposed on each other.)

I: Yes, how many right angles have you turned?

S: Amm, one.

I: One?

S: Or you can say two, those together.

I: OK, one right angle or two?

S: One.

From the above excerpts, it would appear that students had little difficulty in performing a particular motion whenever they were asked to do so. On the other hand, they showed uncertainty in identifying the motions that were used. In particular, recognizing whether or not a half-turn (two right angles) was used seemed not easy to perceive even for the highest achieving student. This, in turn, indicates that a lack of manipulative experiences on this aspect is perhaps one of the reasons for these confusions and uncertainty.

> Question 4. We do not know what students learn about area; we do not know what they learn about measure systems; we do know that they do not know very much! Can this approach of equidecomposability contribute more to our knowledge?

The students' actions were observed on several occasions - in class periods, through interviews, and testing. Some parts of their booklets (Student's booklets) were examined where homework was indicated. The video tape recordings for the evaluation tasks were viewed several times. As a result of this scrutiny of students' performance the following points were made:

1. Throughout each task involving cut-and-cover, the students

seemed to be looking first for pairs of congruent edges. That is, a linear congruency was sought at the first instant.. Moreover, it was noticed in most cases the student who failed to find a linear congruency tended to make a linear superposition first and then a linear congruency with respect to an edge of one of the regions involved (Fig. 20). For other cases, see Appendix H.

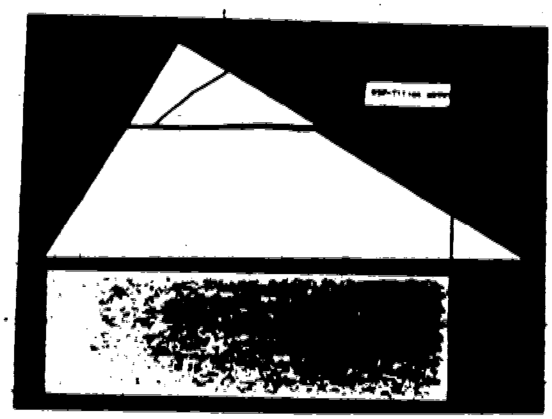


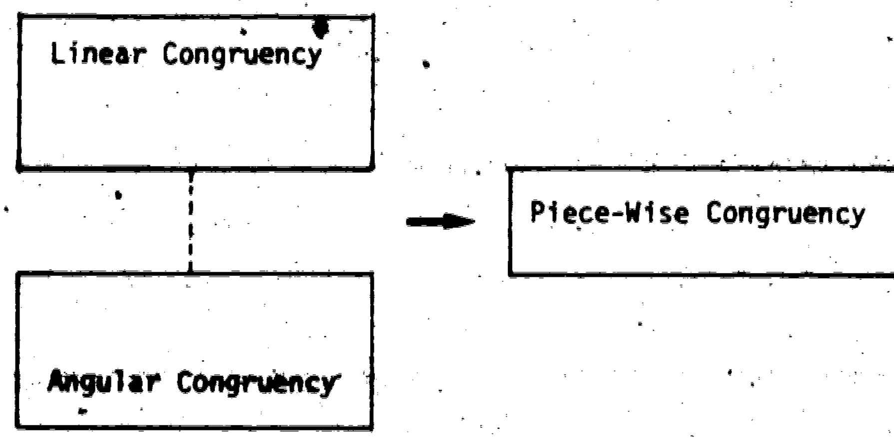
Figure 20

(A copy of the 05F's original 1st attempt - task 1)

2. The students, after searching for a linear congruency, then searched for a pair of congruent angles. That is, an angular congruency was sought.

3. The students, after searching for congruent pairs of edges and angles, continued looking for congruent pairs of subregions. That is, a piece-wise congruency was sought later.

4. A piece-wise congruency could be attained only if a linear congruency, or angular congruency, or both was attained.



There were enough evidences supporting these points. For, almost all of the students who passed tasks 1, 2, 3, 4, and 5 in the interviews attained congruent pairs of sides, angles, or both. In task 8, however, despite the existence of a congruent pair of sides, seven students failed the task. The reasons for this peculiar performance were (a) task 8 requires partitioning of the pentagon region into five congruent triangular subregions (in case of decomposing the pentagon into the rectangular region); and (b) recalling and utilizing task 2's strategy in rectangulating each triangular subregion.

Failing to find a linear congruency, an angular congruency, or both was the primary reason for the student not attaining a successful piece-wise congruency on tasks 6 and 7. On the other hand, there existed a linear congruency in both task 6 and task 7; it was between a diagonal in each of the non-rectangular regions and a side of the corresponding rectangular region. The diagonals in both regions were not drawn however.

The students, both high achievers as well as low achievers, were using learned properties of the domain space for the linear measure system (the set of all segments in a plane) in approaching a piece-wise congruency in the domain space of the area measure system as defined in Chapter I (the set of all polygonal regions in a plane). As well, a similar situation was exhibited when students used properties learned earlier of the domain space for the angular measure system (the set of all angles in a plane) in arriving at a piece-wise congruency. Thus, a typical student did transfer into another new measure system learned properties of measure systems learned previously.

In addition, students' performance of the piece-wise congruency operation was characterized by no number involvement and no measurement performed. Regions were compared. Students seemed, therefore, to perceive area through the cut-and-cover medium. Whenever students performed a task successfully they responded that the two regions were:

'equal', 'congruent', 'the same'.

"Why are they equal? Congruent? The same?"

"Because they fit."

Thus, area through the students' actions, appears as a notion of fitting, after cutting, a region on the other.

Therefore, two polygonal regions are equal in area because one of them can be cut into pieces such that they can be rearranged to cover the other region completely and not only because their area formulae happen to give the same number!

Moreoever, performances described above suggest that some plane quantities can be compared apart from numbers involvement. The comparisons in the tasks could be considered as examples supporting Bateson's (1979) assertion that quantity and number can be separated (p. 49).

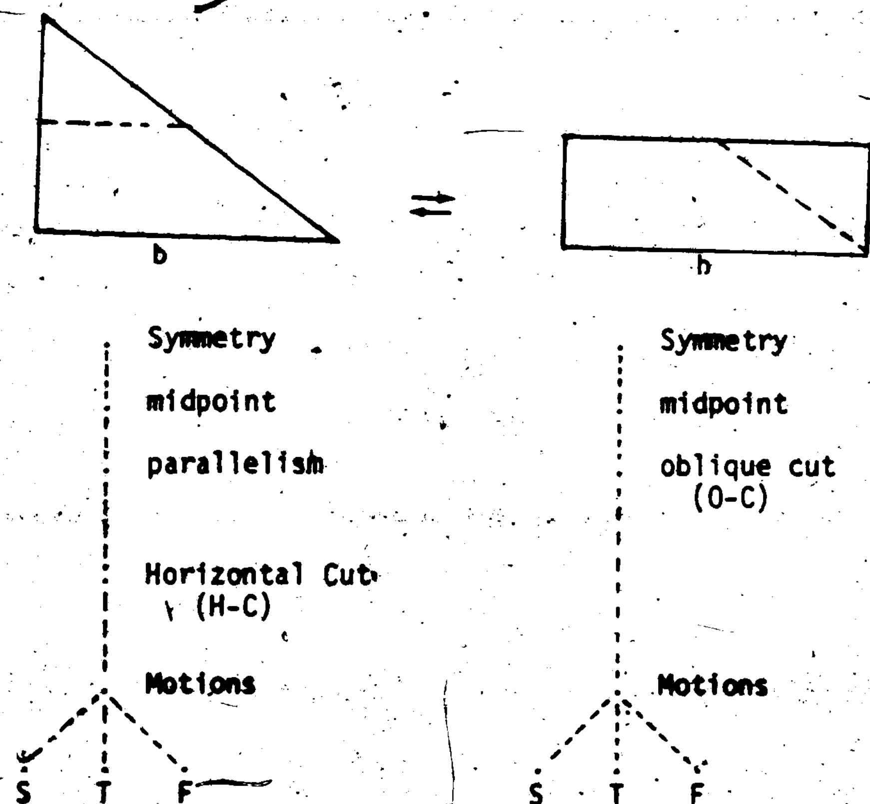
Question 5. Is there a cognitive structure in the students' thinking similar to that of the content structure or do they think in a different manner?

The main purpose of this question is to provide a general overview of the students' performance with respect to those key ideas and their priority-ordering (sequence) involved in each of the tasks based on the piece-wise congruency concept presented throughout the interviews.

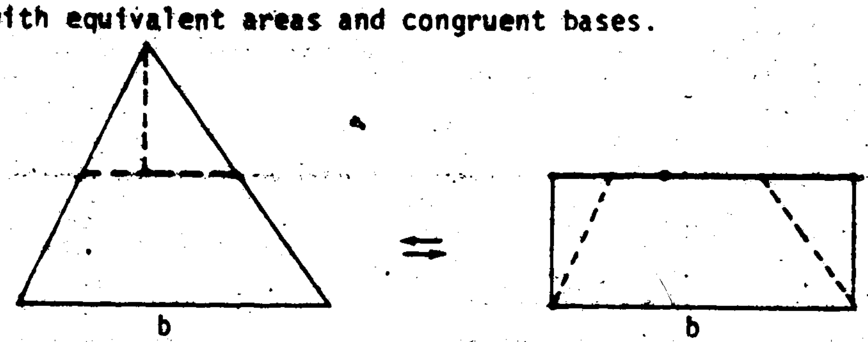
Clearly, there exist some key ideas in each of the evaluation tasks on piece-wise congruency. Obviously, each key idea has a priority with respect to the other key ideas. Hence, these key ideas, in a given task, are in a special priority-ordering. Thus, following these key ideas in their priority ordering might lead to a possible solution for the task concerned. This <u>priority-ordering of key ideas</u> in a task is referred to as its <u>content structure</u>. While the <u>observed priority-ordering</u>, the student assigns to whatever key ideas he might consider for a given task is referred to as the <u>cognitive structure</u> in his thinking.

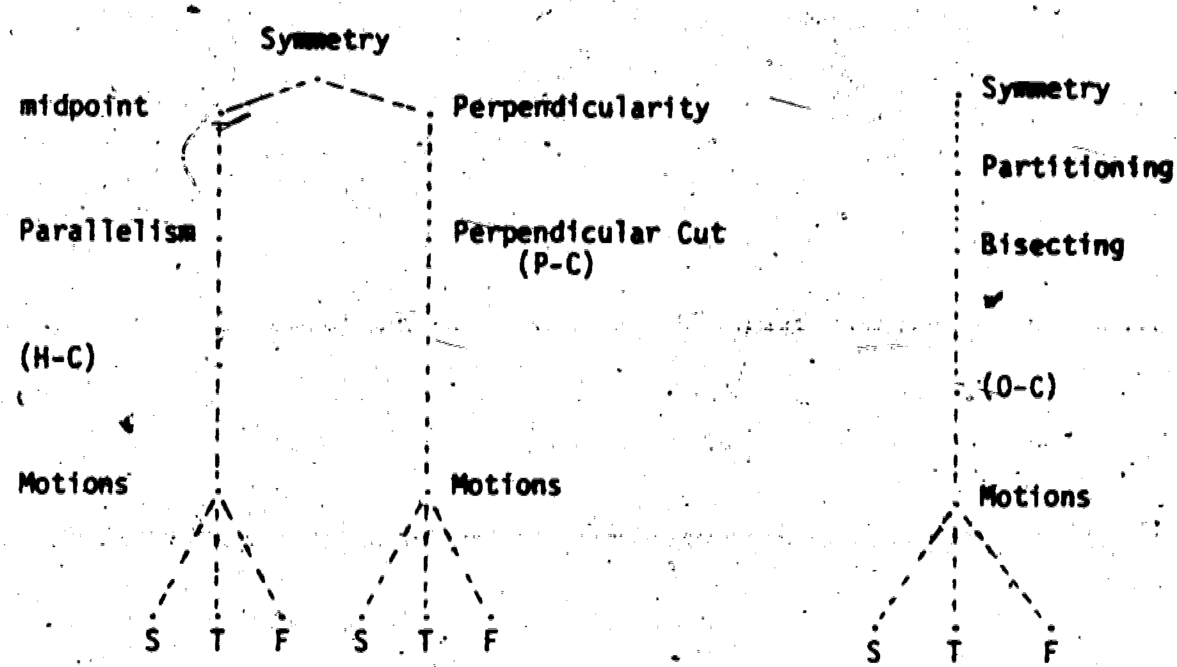
To illustrate the above statements, the following are felt to be the most probable priority orderings of those key ideas involved in the eight evaluation tasks on the piece-wise congruency operation.

Task 1: Piece-wise congruencing a right angle triangular region and a rectangular region of equivalent areas and congruent bases.

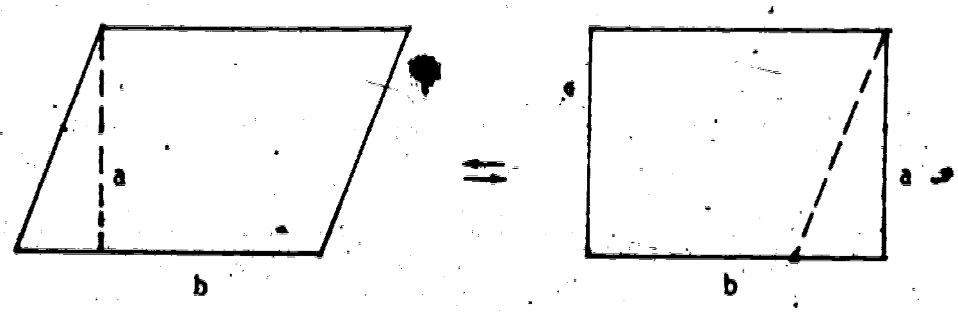


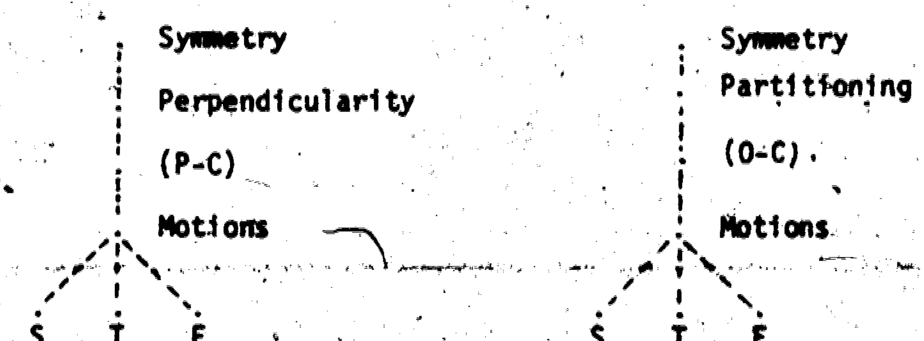
Task 2: Equidecomposing an acute-triangular region and a rectangular region with equivalent areas and congruent bases.



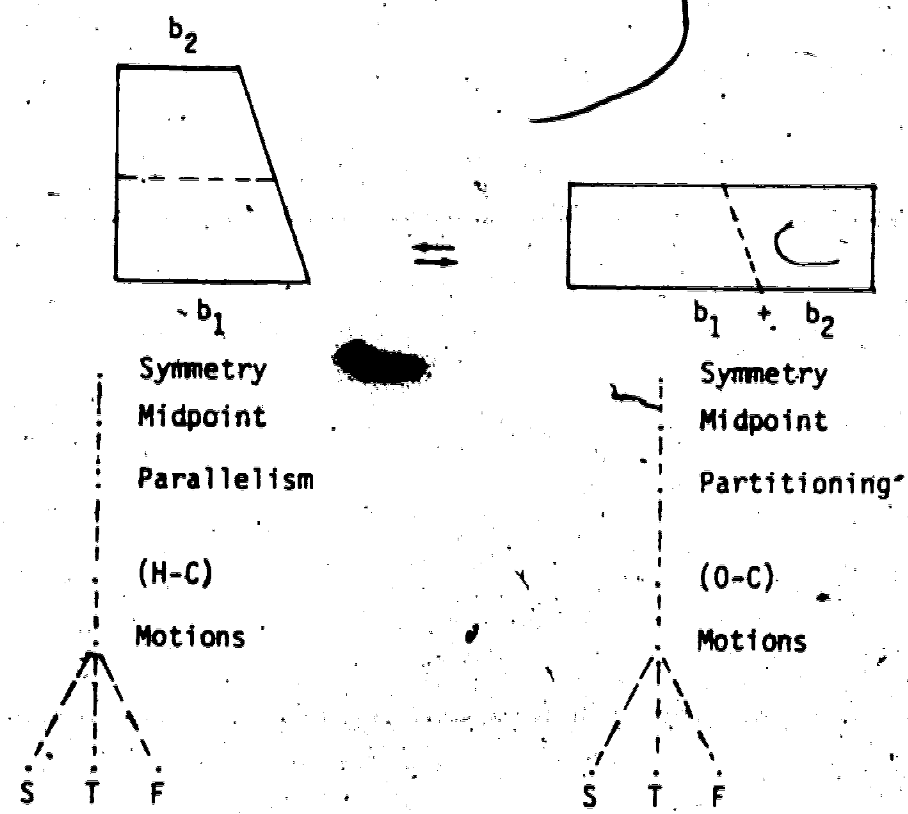


Task 3: Equidecompsoing a parallelogram region and a rectangular region of equivalent areas and congruent hases (or altitudes).

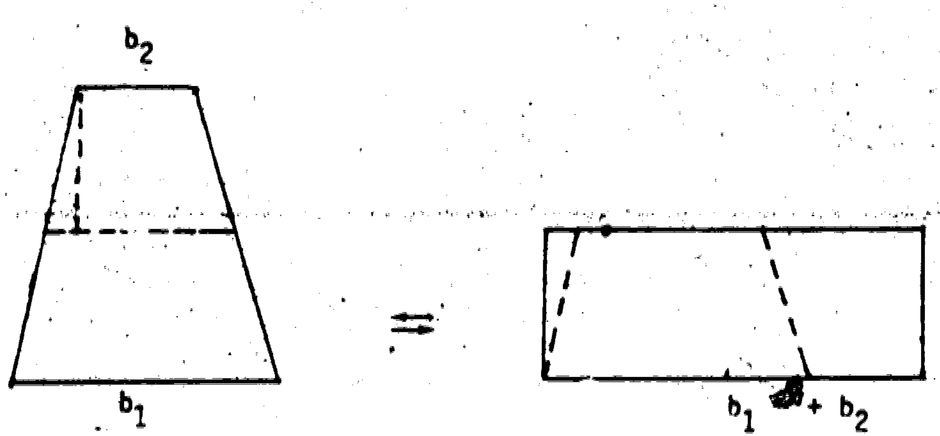


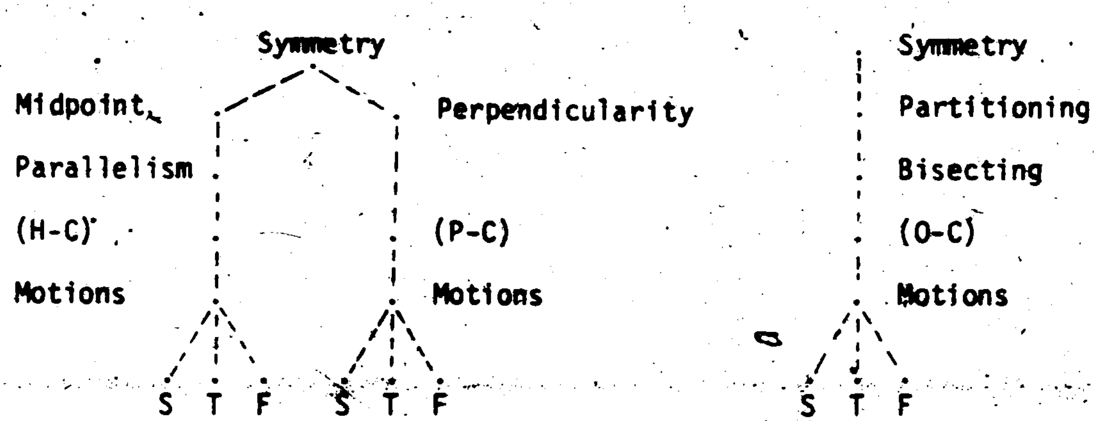


Task 4: Piece-wise congruencing a right trapezoid region and a rectangular region with equivalent areas and the base of the second region equals the sum of the bases of the first region.

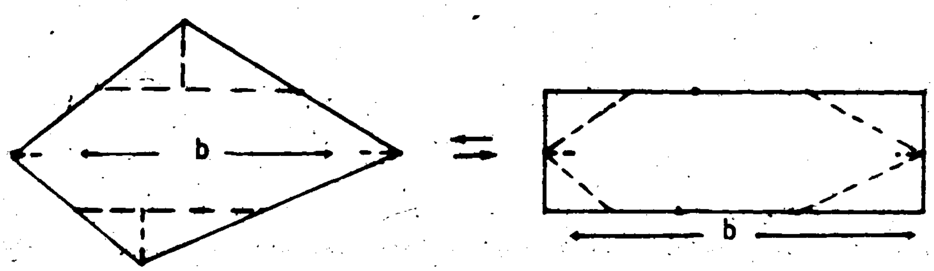


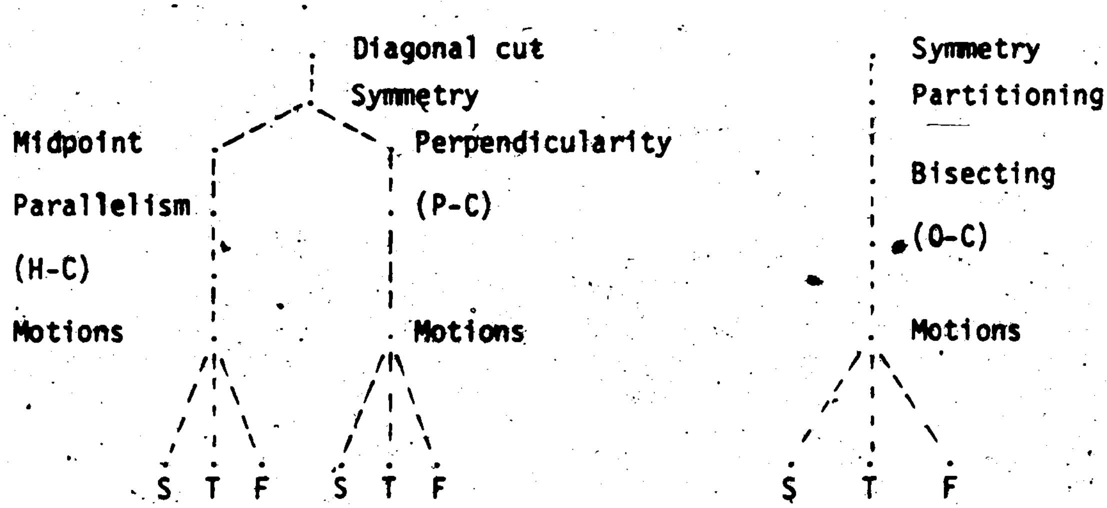
Task 5: Piece-wise congruencing a trapezoid region and a rectangular region with equivalent areas and the base of the second region equals the sum of the bases of the first region.



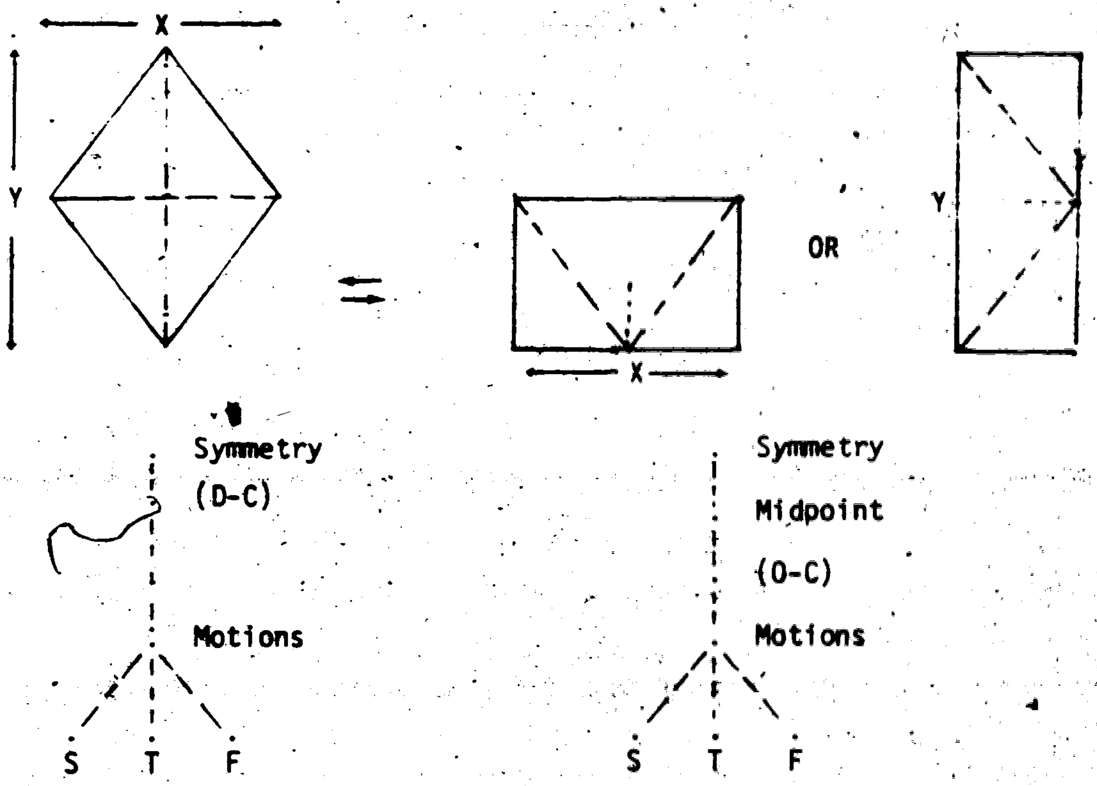


Task 6: Piece-wise congruencing a convex quadrilateral region and a rectangular region with equivalent areas and the base of the second region is congruent with a diagonal of the first region.

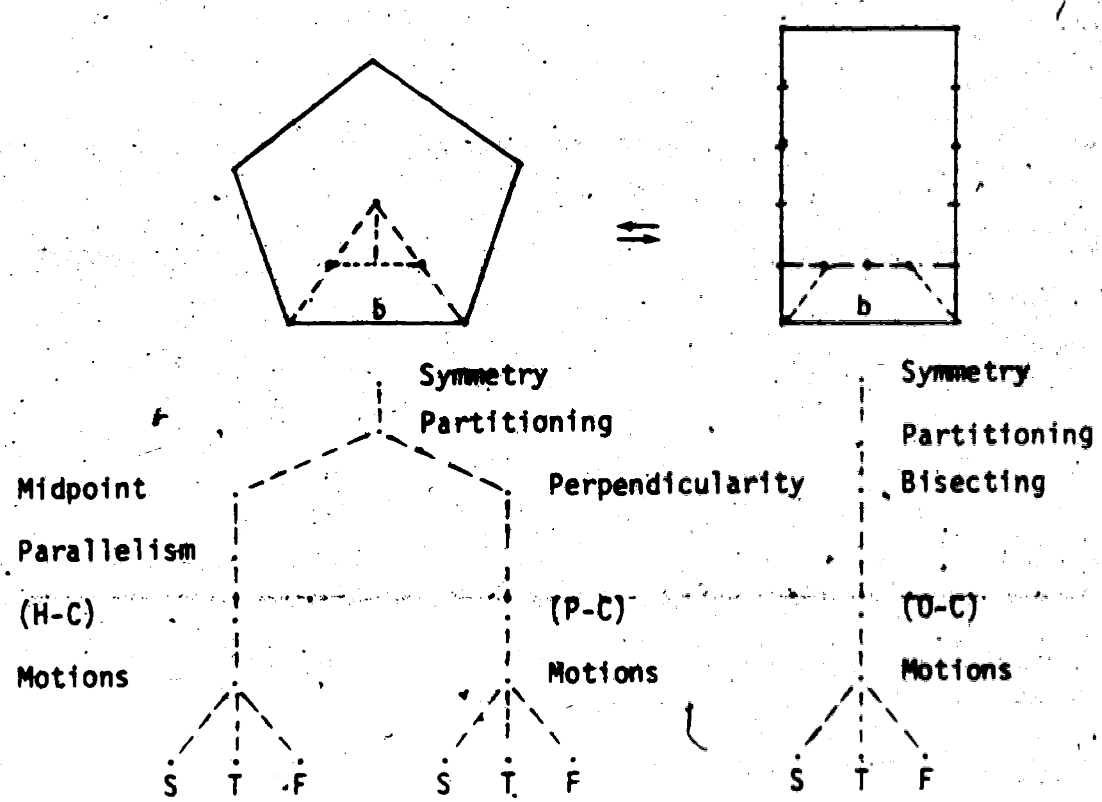




Task 7: Piece-wise congruencing a rhombus region and a rectangular region of equivalent areas and the base (or the altitude) of the second region is congruent with a diagonal of the first region.



Task 8: Piece-wise congruencing a regular pentagon region and a rectangular region of equivalent areas and the base of the second region is congruent with a side of the first region.



From tasks 1, 2, 3, ..., and 8, [illegible]lows that every right triangular region, acute triangular region, convex quadrilateral region, or regular n-gon region (n = 5,6,7,...) is decomposable into a rectangular region of equivalent area. Furthermore, the rectangular region in each task is characterized as one of its sides depends on a side, a diagonal, or sides of the corresponding polygonal region. In other words, beside the area equivalency condition of the two polygonal regions, there is another condition. In each of the tasks 1, 2, 3, and 8, the base, say, of the rectangular region equals a side of the non-rectangular region. In tasks 4 and 5, the base of the rectangular region equals the sum of the bases of the trapezoid region. In task 6, the base of the rectangular region equals a diagonal of the quadrilateral region whereas in task 7, the base of the rectangular region equals either a half diagonal or a diagonal of the rhombus region. Thus, under the two conditions of area equivalency and the existing relation between a side of the rectangular region and a side, a diagonal, or sides of the corresponding non-rectangular regions, the two polygonal regions are piece-wise congruent. That is, the rectangular region of equivalent area in each of the eight tasks is not arbitrary. As such, the rectangulation operation (the process of decomposing a polygonal region into a rectangular region of equivalent area) and the derectangulation operation (the converse of the former) for each of the above eight tasks hold only under these two conditions: area equivalency and a relation between corresponding segments of the two polygonal regions involved. The removal of the second condition- the linear relation between corresponding segments of each pair of the polygonal regions- is possible however.

It should be noticed however that the manipulative nature of each task called for an immediate attainment of some of the key ideas involved. In task 1, for example, as explained in Question 4, while students were searching for a piece-wise congruency they searched for a linear congruency, or angular congruency, or both. Once they found two congruent edges (the bases in task 1), midpoints (symmetry), parallelism (with respect to the base of the triangular region), horizontal cut, and oblique cut were attained. This automatic attainment was also apparent throughout tasks 2, 3, 4, and 5.

In fact, the use in the tasks of the cardboard models for polygonal regions created the following situations: Whenever the two polygonal regions in tasks 1, 2, 3, 4, 5, 6, or 7 are superposed such that a linear congruency, or angular congruency or both are attained, the rectangular region in each task generates a horizontal or a perpendicular cut on the other region. The non-rectangular region in return generates an oblique cut on the rectangular region (for examples, see Appendix H). That is, for a horizontal or a perpendicular cut on a region there exists a dual cut on the other region. This duality existed in the student's performance on these tasks which facilitated the achievement of the converse decomposition of a successfully performed task. This could be one of the main reasons that caused some students to perform oblique cuts on the rectangular regions. Table VII shows the number of students whose initial cuts were on the non-rectangular region, rectangular regions, or both. Photocopies of the students' original cuttings on each of the eight evaluation tasks are

TABLE VII

FREQUENCIES OBSERVED: INITIAL CUTTING

EVALUATION TASK	CUTTING: NON-RECTANGULAR REGION	RECTANGULAR REGION	BOTH	NONE	TOTAL
1	6	5	1	-	12
2	7	5	-	-	12
3	6	6	-	-	12
4	9	3	-	-	12
5	6	6	-	-	12
6	5	7	-	-	12
7	4	4	-	4	12
8	12	-	-	-	12
TOTAL	55	36	1	4	96

presented in Appendix H.

Students, in attempting a cut-and-cover task, went through three stages, (a) searching for corresponding congruent edges, angles, and then pieces (subregions), (b) cutting, where Table VII shows the diversity of the students' initial cuttings, and (c) rearranging the subregions of one region to cover the second region completely so that a piece-wise congruency is attained.

It was a matter of the way two regions were superposed on each other that determined which line to be drawn and cut first. That is, if the non-rectangular region was upward, oblique lines were drawn along its edges. Otherwise, a horizontal or perpendicular line was drawn along the edges of the rectangular region. Some students, however, drew lines on both regions while others did not. They simply cut along the edges of the upper region while the two regions were superposed on each other.

Based on these illustrations and the scrutiny of the students' performance described in Question 4, it would appear that there was not enough evidence to justify whether or not students were aware of the key ideas of midpoints, parallelism, perpendicularity, partitioning, and motions. It was evident however that most of the students did not use the motions of slide, turn, and flip spontaneously (Question 3).

With respect to those priority-orderings suggested for tasks 1, 2, 3, 4, 5, 6, and 7, the students can not be said to have used the same priority-orderings either. This is not to say that the students were unable to recognize and use those key ideas in a particular order rather the manipulative nature of these tasks had drastically influenced the outcomes.

It would be more informative if each task was redesigned so that only one polygon was available at a time. Otherwise, the manipulative influence of one region on the other is unavoidable as in our case.

Task 8, however, is an example of having the manipulative influence of one region on the other diminished though there are two regions involved simultaneously. Beside partitioning, it requires the student recall and utilize task 2 technique (see p. 101). The following is a full description of the twelve students' performances on this task with respect to the key ideas and their priority ordering involved:

1. Seven students failed the task; their performances were classified into two categories:

a. The first category consisted of two students who failed earlier in the task when they did not partition the regular pentagon region into five congruent triangular regions. Accordingly, there were none of the key ideas exhibited at the first place. In a sense, their actions appeared drastically different from any pattern that might be expected - see copies of their original performances in Apppendix H. These students were an upper high achiever (05F) and a below average achiever (19F) - Table IV.

b. The second category consisted of the other five students; they were 34F (a high achiever), 41F (a low achiever), 44F (the highest achiever in class B), 46M (a low achiever), and 48M (a low achiever). The performances of these students were almost identical. All of them failed to exhibit the key idea 'midpoint' and hence appeared similarly unsuccessful. In addition, 46M's performance did not exhibit the key idea 'perpendicularity'. Their cuttings are shown in Fig. 21.

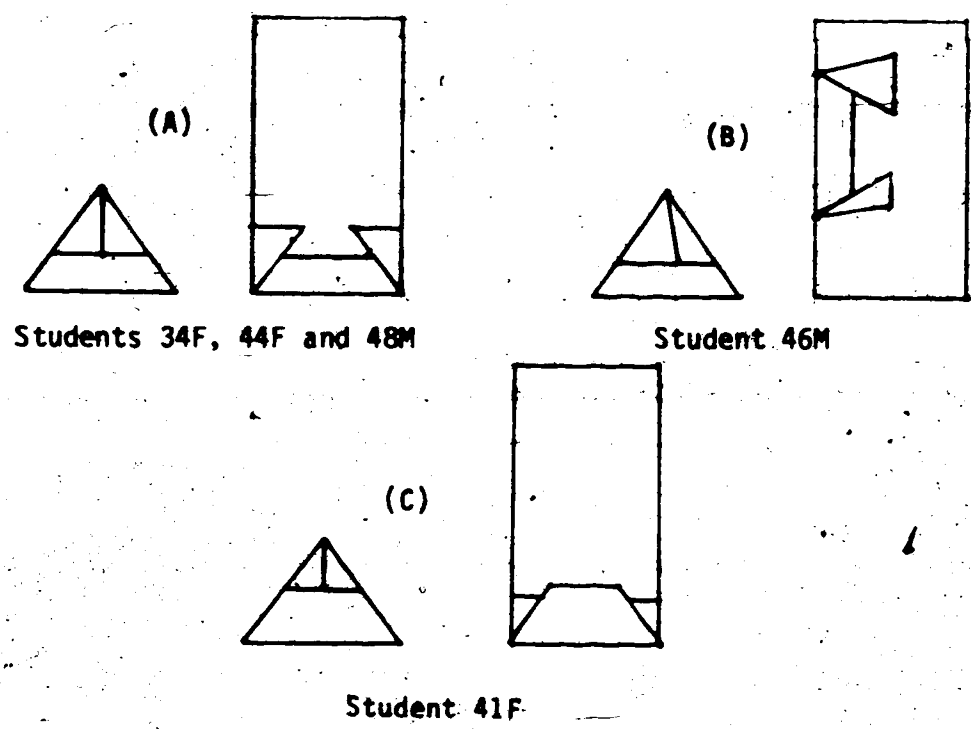


Figure 21

The key ideas of parallelism, perpendicularity (in A and C), horizontal cut, and perpendicular cut were apparent in each of the students' actions. Experiencing the motion operations throughout the previous tasks, the motion of turn was also exhibited. Thus, the actions of the students in this category were nearly the same as the content structure of the task.

2. Five students successfully performed the task; their performances were divided into two categories:

a. The first category represented by two identical performances related to the students 12M (the highest achiever in class A) and 03M (a low achiever). These performances are shown in Figure 22.

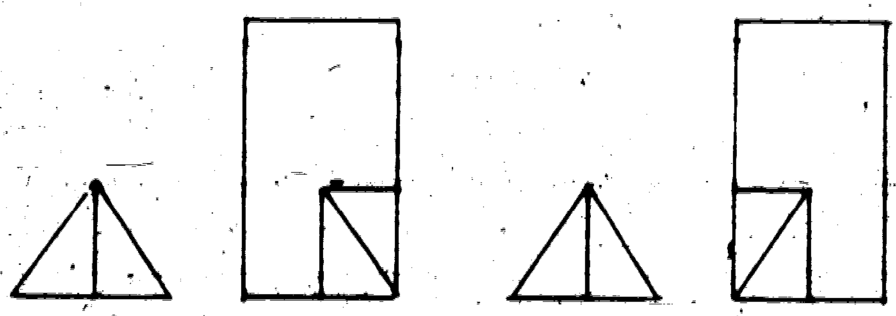


Figure 22

The key ideas of symmetry, perpendicularity, and the motions of turn and flip were exhibited. these are the only key ideas needed should this pattern of cutting be followed (which is correct provided that the fifth generated sub-rectangular region would be partitioned into two halves to cover the upper part of the rectangular region as in the Figure 23 below).

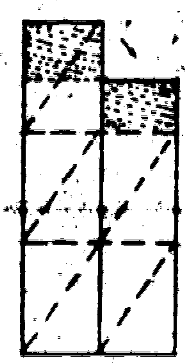


Figure 23

In a sense, initiation of this type of performance suggests a divergence from the content structure. It can be regarded as a purely individual discovery.

b. The second category consisted of the remaining three performances which seemed to be identical to the content structure. They are related to the students 27M (a high achiever), 20F (a low achiever), and 30F (the lowest achiever in class B). Their performances are shown in Figure 24.

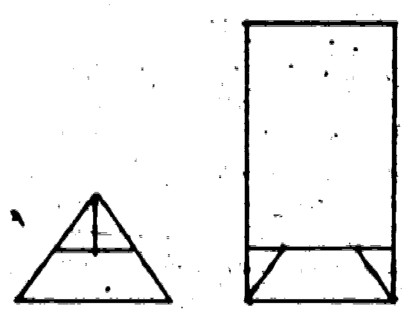


Figure 24

The content structure is exhibited in each of the performances and hence a cognitive structure in the students' thinking similar to the content structure exists for these three students.

In general, category (b) of the unsuccessful performances together with category (b) of the successful performances could be regarded as one group of performances; they are characterized by a cognitive structure that is almost similar to the content structure. Category (a) for both successful and unsuccessful actions together are not analogous to the content structure.

Question 6: In using the cut-and-cover procedure, can students understand the notion of equidecomposability as an extension of congruency?

In each of the evaluation tasks involving the piece-wise congruency operation, students were asked to explain the relationship between the sizes of the two regions involved. The responses were different; some of the students performed accurate cuts and after successful equidecompositions they indicated that the regions were 'the same', 'congruent', or 'equal'. Others who lacked accurate cuttings suggested that the two regions were 'almost' or 'approximately' the same. A third group of them who performed neither a complete nor an approximate covering suggested that regions involved were not 'equal', 'the same', or 'congruent'.

Throughout the cases in which students attained successful cuttings and coverings, it was noticed that students did not acquire the piece-wise congruency concept as an extension of congruency. They simply seemed to accept the piece-wise congruency idea only if the two regions were being superposed on each other so that one shape for both regions was apparent. It is very interesting to notice that when the pieces (subregions) were reassembled into their original region the students reversed themselves and indicated that the two regions were no longer piece-wise congruent. The following excerpts (taken from the transcription) will illustrate difficulties encountered.

All the tasks mentioned in the following excerpts were successfully performed by the student performing them.

Excerpt 5 (12M) - the highest achiever in class A:
(The excerpt is related to the evaluation task number 4 - equidecomposing a right trapezoid region and a rectangular region.)

I: OK, and they are now of the same? (After the completion of the task by the student.)

S: Yes.

I: Of the same what?

S: Area.

I: Good.

S: And the same shape.

I: Are they congruent?...The two regions?

S: Now they are.

I: They were different in shape and they are now...?

S: Congruent.

I: Of equal area?

S: Yes. (The trapezoid region was decomposed into two pieces which were superposed on the rectangular region and covered it completely.)

I: Now, (the pieces were reassembled by the investigator into the trapezoid region), can you say they are congruent now? You had done it.

S: You can't say because they are not the same shape.

I: OK, but you cut this into pieces and rearranged them to cover the second one didn't you?

S: Yes.

I: Well, do you agree with me if I say they are congruent piecewisely?

S: Yes, if you cut them into pieces and put them together they are congruent.

I: Piece-wisely?

S: Yes.

Note the persistence of the student in rejecting the two regions of being piece-wisely congruent without having the same shape. The following excerpt further illustrates the rejection of the notion of piece-wise congruency of two regions because of different shapes:

Excerpt 6 (27M) - a high achiever:
(This excerpt is related to the evaluation task number 2 - equidecomposing a triangular region and a rectangular region.)

I: Now, you may say something about the area of these two regions. Are they equal? (The cut-and-cover was completed).

S: Yes, the area is equal.

I: Are they congruent?

S: These two?

I: Yes, these two.

S: You mean the two different shapes?

I: Yes. You covered one by the other; you said they are of equal area. Now, are they congruent?

S: Yes they are, I think they are congruent.

I: Do you agree with me if I call this congruency as congruency by pieces. This piece is congruent with the one beneath it and this one with the other one. Do you agree with me?

S: Yes.

I: Even though they have different shapes but equal area, we can call them congruent piecewisely, I mean piece to piece. Do you agree with me?

S: No.

The uncertainty is clear as to whether the two regions were piece-wise congruent even though he already made the required cutting and covering. The condition of the same shape for two polygonal regions to be piece-wisely congruent was prevailing over the other aspects of the problem.

The following occurred with another student:

Excerpt 7 (34F) - a high achiever:
(The excerpt is related to the evaluation task number 1 - equidecomposing a right angle triangular region and a rectangular region.)

I: OK, what do you say now? They are...?

S: They are congruent.

I: Do you agree if I say they are congruent piece-wisely that is, piece to piece?

S: Like

I: You had them in different shapes and you changed this into the rectangular region, so in fact these two regions are congruent but piece to piece.

S: No.

I: No, well.

S: If I cut them then they are.

As well, this upper high achiever seems to reject the idea of piece to piece congruency without the two regions being superposed on each other and taking one shape. In connection with the same situation, the following took place with another student:

Excerpt 8 (41F) - a below achiever:
(The excerpt is related to the evaluation task number 1 - equidecomposing a right angle triangular region and a rectangular region.)

I: What do you say about their areas? (The student already performed the task.)

S: They are about the same.

I: Do you agree if I say they are congruent?

S: Yes.

I: Even though one of them is a triangular region and the other is a rectangular region?

S: Yes they are congruent, the same.

I: Do you agree if I say they are congruent piece to piece?

S: Piece to piece means as I cut them.

I: Yes.

S: No.

I: Well, that is what you did.

S: Oh! They weren't congruent, this was a triangle but it is congruent when I cut it and put it on.

I: Now they are congruent or they were congruent?

S: They now...they're congruent.

I: What happens if I put this...(reassembled the pieces into the triangular region) are they congruent now?

S: No.

I: Can you say now they are congruent?

S: No.

I: Why?

S: Because this is a different shape than this.

Therefore, two polygonal regions of equal area and different shapes are piece-wisely congruent only if they are equidecomposed into one shape and superposed on each other simultaneously. If the pieces of one region are reassembled into the original region, the two regions would not be piece-wisely congruent even if the pieces previously covered the other region completely.

Question 7: Do students recognize properties that are invariant under equidecomposition?

Throughout classroom observations and the evaluation tasks, students experienced no difficulty in recognizing that segments, angles, and areas of subregions (resulting from cutting) are unchanged under equidecompositions. Also, apart from the obvious partitioning made over the main polygonal region, other features of segments, angles, and the whole area of the original region remained conserved. It was trivial to the students either to perceive that the original region and the other region which was covered by rearranging the subregions were of the same area. There was no difficulty in performing the reverse equidecomposition nor in recognizing those unchanged properties of the other region. In sum, the change in the characteristics of the decomposed region and the conservation of segment, angle, and area were easily achieved. The following excerpt is a typical sample of what was occurring during the interviews on this matter:

Excerpt 9 (03M) - a low-achiever:
(This excerpt is related to the evaluation task number 4 - equidecomposing a right angle trapezoid and rectangular region.)

I: Is this angle going to be changed if you move this piece? (At this stage, the student has already performed the cutting and covering.)

S: No.

I: And neither this side?

S: No.

I: Do you think this side would be changed? (Pointing to another side of a subregion resulting from the student's cutting.)

S: No.

I: This angle?

S: No.

I: And this side?

S: No.

I: So, nothing would be changed.

S: Right.

I: And the area won't be changed.

S: Yes.

I: Have we lost something?

S: No.

I: Has the area been reduced?

S: No.

Since the rest of the students who participated in the evaluation tasks offered similar responses, it would seem that a positive answer to the question exists.

Question 8: Can students successfully perform the practical procedure of equidecomposing two polygonal regions?

Initially, there was a fear that a class of about 30 students each of whom was supplied with a pair of scissors, tracing paper, and a ruler would create a chaotic situation which might negatively effect the ongoing project. Practically, the picture was absolutely different. The students were relaxed, deeply engaged, and the classes were impressively disciplined. Teachers A and B were asked through the Teacher's Opinion Assessment questionnaire on the unit the following question:

> Do you think that the use of the laboratory approach that involved actual tracing, cutting, and covering of polygonal regions creates a chaotic situation in the classroom?

Both teachers responded that the classes went fine and the students got a lot of enjoyment and fun through discovering solutions for various geometric tasks. Teacher A, in addition, offers the following statement in defending the nature of the unit:

> Students of Junior High age enjoy and are generally adept at manipulating objects and trying to devise different possible solutions.

Some of the students were observed not willing to use the ruler at all; they were drawing lines simply by hand and hence their cuts were inaccurate (see Fig. 25 and Fig. 26 for example). Others were consistent in utilizing whatever tools they were supplied with and therefore their cuts were neat. The former group of students were hesitant to make a decision as to whether or not the two regions involved were precisely piece-wise congruent. They tended to use 'almost', 'nearly', or 'approximately' within their syntax of responses. Figures 24 and 25 are copies of the original cuttings related to the students 03M and 34F respectively.

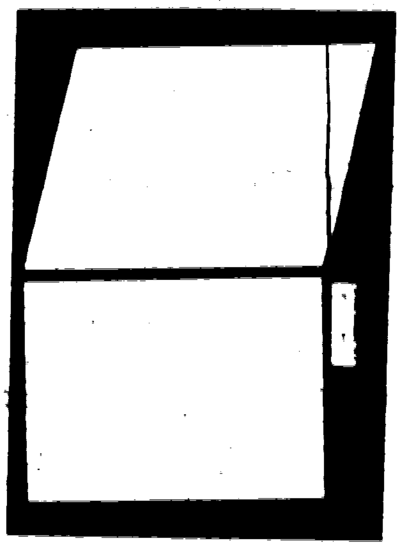


Figure 25
(A copy of the 03M's original 2nd attempt-task 3)

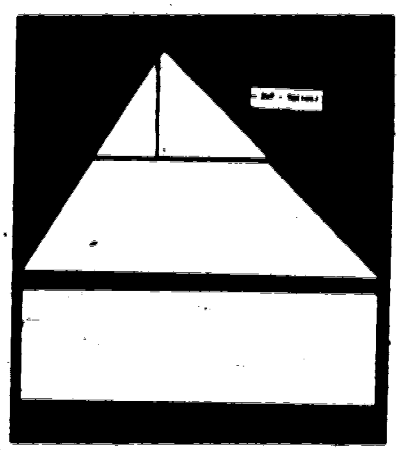


Figure 26
(A copy of the 34F's original 1st attempt-task 2)

Question 9: Does the approach of piece-wise congruency facilitate students' perception of

a. interrelations between polygonal regions?

b. properties which are invariant under transformations?

c. congruency?

Moreover, does this approach increase students' geometric vocabulary?

In general students in junior high schools do not pocess a clear understanding on how polygonal regions are really interrelated. There is no text that shows in a <u>perceptual</u> <u>setting</u> the precise systematic interrelationships among polygonal regions. The unit presented in Appendix A is an attempt to fill this vacuum. Physical manipulation is the core of it and relations between polygonal regions are intuitively produced. Students' results on the first testing showed vague understanding of what is meant by a region in a plane being polygonal (Table V, subtest ii). On interrelating polygonal regions with each other Geometry Tests iv, v and vi were designed. The information in Table V shows the effect the piece-wise congruency concept had on the students' achievement. In Question 2, the statistical analysis shows that the change was significant in the achievement scores on the subtests iv, v, and vi. Hence, the piece-wise congruency approach did facilitate the students' perception of the kind of interrelationships between the polygonal regions involved.

Properties that are invariant under transformations were already recognized by the students and it is not possible to assign an effect to the approach regarding this aspect. The approach, however, appeared to be a medium in which the conservation of segments, angles, and areas were exhibited (Question 7).

On the congruency aspect, it was hoped that the piece-wise congruency would be looked at as an extension of the notion of congruency when the 'same shape' condition is removed. The actual situation however was different. As discussed in Question 6, for two polygonal regions of equal areas but different in shapes to be piece-wise congruent, the pieces of one region must superpose and cover the other region completely. If the pieces are reassembled into their original region, the two regions are no longer piece-wise congruent. Thus, the approach revealed how the condition of 'same shape' ruled the students' perception and caused them not to view piece-wise congruency as an extension of the congruency notion.

On the geometric vocabulary aspect of the question, although Table V shows some positive change, Table VI indicates that the change in the students' achievement on the vocabulary test (subtest i) was not significant; $F = 0.996$, $P > 0.05$.

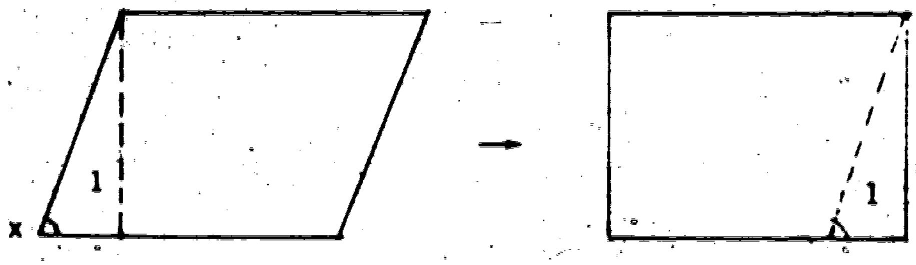
The following excerpts will illustrate, on the other hand, how students perceived properties that are invariant under transformations:

Excerpt 10 (34F) - a high achiever:
(This is related to the evaluation task number 3).

I: So, you can change any parallelogram region into a rectangular region?

S: Yes.

I: Do you think this angle will be changed if you translate this piece there? (Pointing to angle x in subregion 1.)



S: Amm.

I: Translate it and see. (The student starts translating subregion 1.)

S: No.

I: OK, put it back please. Now, could you tell me if the length of this side would be changed if you translate this piece there? (The student starts translating piece 1 again.)

S: No.

I: OK, put it back please, what about this side will its length be changed?

S: No.

I: What about this? (Pointing to another side.)

S: No.

I: So, in fact, you won't have anything changed would you?

S: Yes.

The role of the cut-and-cover approach above is clear; it is a medium in which practicing motions and perceiving invariant properties were experienced. The following excerpt will further exemplify provisions presented by the use of the approach:

Excerpt 11 (46M) - a below achiever:
(This is related to the evaluation task number 2)

I: So, you are going both ways, from the rectangular region to the triangular region and vice-versa.

S: Yes.

I: Now, could you tell me if the length of this side will be the same or it will be changed if you turn this piece?

S: Amm, (starts turning), it will be the same.

I: OK, what about this angle.

S: Will be the same.

I: And this angle?

S: The same.

I: This side?

S: The same.

I: So, there won't be any change?

S: Right.

Finally Teacher A and Teacher B indicated that they favor the use of the piece-wise congruency approach in geometry instruction. What follows are comments from the teachers.

> Good way to introduce area of different polygonal regions by relating them to rectangles.
>
> Teacher B

> An aid to developing the concept of area as the interior of a polygonal region.
>
> Teacher A

Question 10. Are there methodological difficulties in using the piece-wise congruency approach? If so, what are they?

Prior to the start of the project two joint meetings were made with the teachers. At the first meeting, main steps for implementing the unit were outlined and a schedule was arranged. A copy of each of the Teacher's manual and Student's booklet were provided to the teachers. They were asked to revise them and make notes for the next meeting. Both of the teachers suggested that a session of at least one hour should be provided to prepare teachers should the unit be used in schools.

Other difficulties were: (1) obtaining about 60 end-rounded pairs of scissors (for safety reasons), (2) obtaining tracing paper as cheaply as possible (a pink and white onionskin fine paper was eventually used), (3) preparing a magnetic board for each class, (4) preparing a set of magnetized cardboard models for the polygonal regions involved in the derivation of the area formulae, and (5) preparing overhead transparencies for all activities for subsequent discussions. These were the practical difficulties encountered. Overhead projectors and rulers were not obstacles; they were available at each class.

Question 11. Can the approach of piece-wise congruency of polygonal regions contribute to the field of algebra?

This question was designed to present a general overview of the unit, especially part three of the unit which deals with the derivation of the area formulae for triangular, convex quadrilateral, and regular n-gon regions (n = 5, 6, 7, ...); see Appendix A - Part Three.

The area postulate of a rectangular region was the only statement to be assumed. The rest of the formulae were deduced through piece-wise congruency. Each formula was deduced in an intuitive manner consisting of few steps in most of the cases. This physically simple manner of derivation could not have been made without the piece-wise congruency concept.

> ...It is also interesting to the students to be able to manipulate things to find area.
>
> Teacher B

> I would not hesitate to use this method again in developing the concept of area or in obtaining area formulae.
>
> Teacher A

On the other hand, the use of cardboard models proved useful. It increased the speed of the derivations, and motivation through students' participation. They enjoyed moving things around on the magnetic boards and changing regions from one to another. It seemed to be a pleasant experience for them.

However, things were not as encouraging in the testing aspect of the investigation. Students' achievement was not significantly changed and hence no essential progress can be credited to the approach. As discussed in Question 2, the students' achievement scores on the Area Formulae subtest (of the Geometry Tests) over the pretest and the posttest were considered as repeated measures. The Hotelling T^2 test for single sample with repeated measure yielded no significant change and hypothesis 3A on this subtest was not rejected; $F = 1.029$, $p > 0.05$ (Table VI).

In sum, it would appear that although the piece-wise congruency approach seemed, through the class sessions, a useful means for classroom practice, it had no significant effect on the students' achievement in topics related to the field of algebra. Accordingly, a contribution to the field of algebra was not in evidence in this study. The area formulae were taught (to each class) through one classroom period however.

Question 12. What contribution can the piece-wise congruency approach offer to the arithmetic operations of addition and multiplication?

Both teacher A and teacher B while they were discussing activity number 2, displayed the overhead transparency and represented the corresponding subregions of the parallelogram and the rectangular regions by letters a and b (Fig. 26). Each teacher showed first that the two regions are congruent piecewisely. Then he asked the students to make

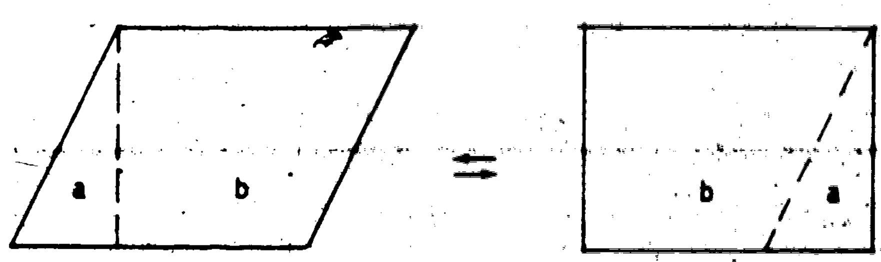


Figure 26

a guess as to what they were perceiving. The answer came from some students:

$a + b = b + a.$

The teacher after assigning positive values to a and b raised the question as what positive numbers then a and b could be? The answer was that a and b could be some positive numbers. The teacher finally inquired on whether or not a and b could be negative numbers and the answer was made by many students. The Teacher then made the following statement:

> The addition operation is commutative over the positive numbers; that is, $a + b = b + a$, a and b are positive numbers.

On the multiplication operation, the teachers, using a rectangular cardboard on the magnetic board, the commutative property of the multiplication operation was easily demonstrated through the area formula of a rectangular region and the turn operation. Many students easily perceived that $A = b \times a$ would be $A = a \times b$ if one fourth turn was performed on the rectangular cardboard. The teacher similarly made the following statement:

> The multiplication operation is commutative over the positive numbers; that is, $b \times a = a \times b$, a and b are positive numbers.

The commutativity of the addition and multiplication over the positive real numbers was induced through intuitive and manipulative opeartions in the domain space of the area measure and the length measure systems. That is, the commutativity of the union operation of non-overlaping polygonal regions in the domain space of the area measure system was directly analogized to its counterpart, the addition operation in the range space. Similar comparison was made for the multiplication operation inside the length measure system.

In sum, the piece-wise congruency approach would appear as a medium in which such type of analogies inside some measure systems could be illustrated in an intuitive manner. Whether or not students understood such ideas within measure systems as described above was not checked in this study.

Question T3. Can the piece-wise congruency approach simplify the reflexive, symmetric and transitive properties?

Among the twelve students interviewed over the evaluation tasks, there was no one who could not perform the converse of a successfully achieved task. That is, the piece-wise congruency operation once initiated on one region, facilitates an immediate induction of the converse (its conjugate). In this connection therefore the piece-wise congruency operation is symmetric on the set of polygonal regions and as such a manipulative medium is presented in which the symmetric property is physically experienced.

The transitivity property of the cut-and-cover operation was brought to light both during the interviews and the class sessions. The following excerpts will exemplify some situations encountered:

Excerpt 12 (44F) - the highest achiever in class B:
(This is related to the evaluation tasks number 2 and 3.)

I: (Presenting copies of both the triangular region of task 2 and the parallelogram region of task 3 together with a copy of the rectangular region used in both tasks.)

You have dissected this and this into this pink region haven't you? (Pointing to the triangular, the parallelogram, and the rectangular regions respectively.)

S: Yes

I: So, you changed this one and covered this OK? And this one as well, you cut and covered this. (Pointing to the triangular and the rectangular and then to the parallelogram and the rectangular regions.) OK, what can you deduce then about these two regions? (Pointing to the triangular region and the parallelogram region.)

S: Aaa. Can you repeat the last words?

I: OK, what can you say about these two? What is the relation between the area of these two?

S: Oh! They're both congruent to the area of this (pointing to the rectangular region.)

I: And therefore?

S: Amm, the rectangle is congruent to the area of these.

I: Yes, and what about them?

S: O! They are congruent.

The student used the word 'congruent' repeatedly possibly as a crude substitute for 'equal'. The following excerpt shows what occurred with another student:

Excerpt 13 (48M) - a low-achiever:
(This is related to the evaluation tasks number 2 and 3).

I: (Presenting copies of the non-rectangular regions and the rectangular region used in both tasks.)
You have had these two regions before (pointing to triangular and the parallelogram regions)
and you, in fact, decomposed each of them into this region didn't you?
(Pointing to the rectangular region.)

S: Yes.

I: So, what do you call them? (Pointing to the triangular and the rectangular regions.)

S: Congruent.

I: and these two? (Pointing to the parallelogram and the rectangular regions.)

S: Congruent.

I: Piece-wisely?

S: Yes.

I: OK, what about these two regions? (Pointing to the triangular and the parallelogram regions.)
Can you say they are congruent?

S: No.

I: Why not?

S: Because they aren't the same,---or--- (the student did not elaborate).

However, this student appeared to view that piece-wise congruency holds only when the same shape condition holds for the regions involved; he, therefore, overlooked the transitivity. The following further illustrates how transitivity was perceived through the cut-and-cover operation:

Excerpt 14 (41F) - a below achiever:
(This is related to the evaluation tasks number 1 and 2.)

I: (Presenting copies of the right triangular region in task 1, the acute triangular region of task 2 and a copy of the rectangular region used in both tasks.)

Now, you have this and you had this before. (Pointing to the acute and right triangular regions.)

You decomposed this into this. (Pointing to the right triangular region and the rectangular region.)

S: Yes.

I: And now you decomposed this into this region. (Pointing to the acute triangular region and the rectangular region.)

S: Yes.

I: And you said these are congruent in pieces or piece-wisely. (Pointing to task 1 regions.)

S: Yes.

I: So are these? (Pointing to task 2 regions.)

S: Yes.

I: Now, what do you say about these two regions? (Pointing to the right and acute triangular regions.)

S: They are congruent, they are congruent.

I: Yes.

S: And piece-wisely. (The term 'piece-wisely' was used prior to this excerpt.)

I: Why?

S: Well if you cut them up and put them on here (pointing to the rectangular region.)

I: Yes.

S: They'll fit.

I: Yes.

S: Without any loss of pieces any where.

I: Yes.

S: And it would fit all the things and there wouldn't be any of the rectangle left off.

This student would seem to have acquired the transitivity property of the cut-and-cover process without the 'same shape' condition. He was the only student that showed such acquirement among those who were interviewed.

Finally, the third Progress Checking Test was designed to give some indications on how well the students can approach the transitivity property through manipulative means (Fig. 27). See Appendix G for details.

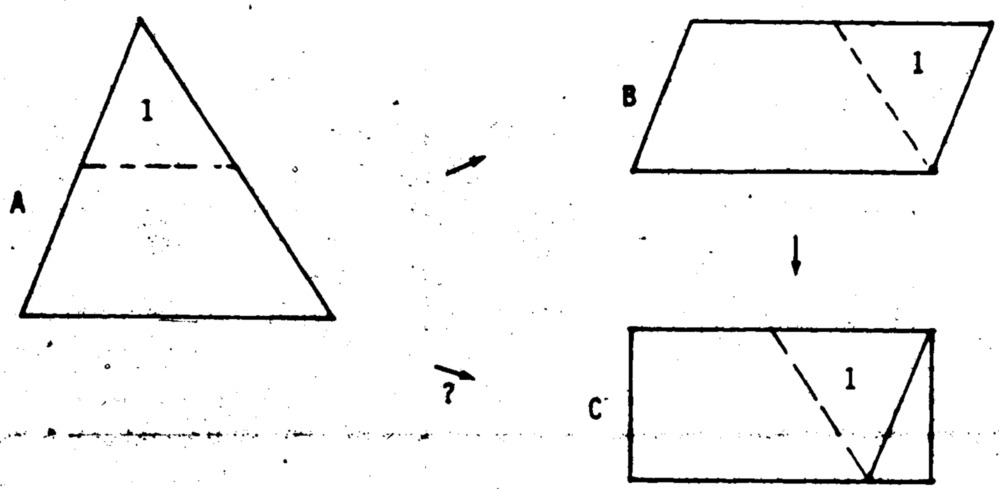


Figure 27

TABLE VIII

FREQUENCIES OBSERVED: TRANSITIVITY

	PASS	FAIL	TOTAL
CLASS A	13	11	24
CLASS B	19	7	26
TOTAL	32	18	50

The results of the classes on the test are shown in the contingency table - Table VIII. There was no significant difference in the performance between class A and class B; $\chi^2 = 1.94 < \chi^2_{.05} = 3.84$, df = 1.

From the above information it would seem that students had successfully perceived the symmetric and the transitivity properties for the piece-wise congruency operation. In other words, the piece-wise congruency process appeared to be one of those advantageous media for presenting such vital operational properties. Given the fact that each polygonal region is congruent with itself, the reflexivity was not considered.

Question 14. Does the concept of piece-wise congruency facilitate the idea of equivalence relation?

An equivalence relation is defined as follows: If X and Y be two sets of points (or objects) such that they can be put into one-to-one correspondence, then X is said to be equivalent to Y, and written $X \sim Y$. The relation $\sim$ just defined has the following properties:

a. $X \sim X$; (it is reflexive).

b. If $X \sim Y$ then $Y \sim X$; (it is symmetric).

c. If $X \sim Y$ and $Y \sim Z$ then $X \sim Z$; (it is transitive).

Any relation which satisfies these three properties is called an equivalence relation (Rudin, 1976).

In Question 13, it was suggested that the properties of symmetry and transitivity of the piece-wise congruency operation were approached through physical manipulative actions. As a result, it would appear that an adoption of such approach in schools could offer provisions in which students can learn not only basic concepts in plane geometry but also some other basic concepts in elementary mathematical analysis simultaneously.

For example, in any of the investigative activities (Appendix A) the symmetric property can easily be illustrated. An exercise similar to the one included in the Progress Checking Test #3 can be employed and the transitivity property could be physically manipulated. Adding to these the obvious congruency of a region with itself, the cut-and-cover operation therefore would appear as an equivalence relation on the set of all polygonal regions. This equivalence relation in the domain space of area measure is analogous to the equivalence relation, equality, in the range space, of the positive real numbers. In this connection this analogy together with others may serve to improve understanding of the structure of the area measure system. For the Progress Checking Test #3, see Appendix G.

Question 15. What contribution can the piece-wise congruency approach offer to the study of rational numbers?

The Piece-wise Congruency Rational Numbers Test was designed to examine the role of the piece-wise congruency concept in recognizing and understanding some aspects of rational numbers. In particular the test was constructed to disclose whether or not grade eight students are aware of the necessity of defining a unit whenever rational numbers are visually encountered for comparison. For this purpose, each of the six items in the test was constructed to contain three square regions two of which being of equal size; the three regions were partitioned differently into sub-regions of square, triangular, or rectangular shapes. The second purpose of the test was to investigate the effect of instructing the piece-wise congruency unit on the students' performance. The overall purpose was to examine the relationship between visual (area representations) comparisons of rational numbers some of which belong to the same equivalence class in the presence of different area units. A secondary purpose was to examine the effect, if any, of the piece-wise congruency concept on such relationships. That is, to investigate and determine first, errors students might exhibit in comparing visually rational numbers where different area unit are presented; and second, what improvement in the students' performances can be induced by instruction in the piece-wise congruency unit. For the Rational Numbers Test, see Appendix F - subtest vi.

Hypothesis 3A (Chapter III) is translated below solely for the rational numbers subtest (vi) of the Geometry Tests:

Hypothesis 3A (vi) There is no significant change in the mean scores on the pretest and the posttest of the classes over the Piece-wise Congruency-Rational Numbers Test.

Table VI shows that the effect of the piece-wise congruency concept was significant and hypothesis 3A (vi) was rejected, $F = 2.277$, $p < 0.05$.

Appendix I, on the other hand, shows copies of the original performances of two students on the Piece-Wise Congruency-Rational Numbers Test both as a pretest and posttest. They were a high achiever (07F) and a low achiever (51F). An examination of these performances would show how effectively the piece-wise congruency approach shaped these students' achievement. Interestingly, on the pretest -item 4-, student 07F showed that regions A and B are of equal areas by complementation!

Question 16. What contribution can the piece-wise congruency approach offer to the concept of ordering on the set of polygonal regions?

Among the evaluation tasks there were tasks 9 and 10, dealing with ordering of rectangular regions and non-rectangular regions respectively (outlined in Chapter III, p. 60-61).

Task 9 consisted of three rectangular regions A, B, and C of different sizes. A and B have congruent bases but slightly different altitudes while B and C have congruent altitudes with slightly different bases (Figure 7, p. 60). Task 10 comprised a right triangular region, a trapezoid region, and a parallelogram region. The three regions were of different sizes (Figure 8, p. 61).

The performance of the students over these tasks indicates the following:

1. All of the students passed task 9 through visual comparisons. They juxtaposed the three regions with respect to their edges (looking

for linear congruencies) and superposed one over the other repeatedly prior to the ordering of the regions. It is interesting that all of the students performed the task successfully and yet an upper high achiever in class A, student 27M, suggested that the ordering "is not that easy":

2. Students were less sure on task 10 and seemed surprised by it. However, some of them did perform the task and explained that if they rectangulate each of the three regions, they might order them as they did in task 9.

The following excerpts will illustrate some related situations encountered:

Excerpt 15 (41F) - a below achiever:

I: Can you order this set of rectangles?

S: OK, can I cut them?

I: No, just order them. (The student starts juxtaposing and superposing.)

I: Which one is the largest?

S: This. (Correct).

I: Which is the smallest?

S: This one. (Correct answer and the student ordered them on the desk.)

I: Now, can you order these in the same way? (Presenting the other set of task 10 - triangular, trapezoid, and parallelogram regions.)

S: aa...

I: As you did these? (Pointing to the rectangular regions of task 9.)

S: No.

I: Why?

S: Because they are of different shapes and I can't tell which one is bigger.

I: OK, now you tell me if you have an idea?

S: Amm...you cut them and fit them on each other.

I: You cut them into what?

S: Into shapes to cover whatever one you want.

I: What shapes?

S: These shapes. (She pointed to the rectangular regions of the previous task.)

I: Do you agree if I say you cut each into a rectangular region?

S: Yes.

I: You did this, and this (pointing to the non-rectangular regions) so each of these you can change into---?

S: Rectangle.

I: And afterwards you would have this situation (pointing to the rectangular regions set of the previous task.)

S: Yes.

I: Could you tell me how can you order them?

S: You put them all into rectangles and then get them like this (pointing to the set of rectangular regions) and you could get them into biggest - smallest.

It would appear that this student did use the concept of piece-wise congruency (rectangulation) over the non-rectangular region using ideas learned through the previous tasks.

The following excerpt further illustrates the role of the piecewise congruency concept in ordering polygonal regions of different shapes:

Excerpt 16 (48M) - a low achiever:
(After ordering the rectangular regions in task 9.)

I: Can you arrange these (presenting the regions of task 10) in the same way? I mean is it as easy as in the previous set?

S: No.

I: Do you think the set of rectangular regions is easy to order?

S: Yes.

I: But not this? (Pointing to the regions of task 10.)

S: No.

I: Is it possible to decompose these into rectangular regions? You had done them.

S: Yes.

I: So, tell me what you have to do here first?

S: Change these into rectangular regions.

I: Yes.

S: Then put them in order.

I: Why do you have to change them? (Short interval)

I: Can you order them now?

S: No.

I: So, why do you have to change them?

S: To be easier to rearrange.

Thus "to be easier to rearrange" might be interpreted as being one consequence the rectangulating process has produced. This student found it easy to order a set of non-rectangular regions whenever they were decomposed into rectangular regions. It is interesting to notice that ordering rectangular regions was "not that easy" in 27M's view and rectangulating non-rectangular regions was necessary for these regions "to be easier to rearrange" in 48M's view. The following excerpt will elaborate more on the effective and facilitative role of the piece-wise congruency approach in producing and defining an ordering on the set of polygonal regions:

Excerpt 17 (46M) - a low achiever:

(Through juxtaposing, superposing, and making linear comparisons between edges this student ordered the set of the rectangular regions.)

S: Small, medium and large (pointing to the rectangular regions.)

I: Good. This is neat; you have rectangular regions and you ordered them easily.

S: Yes.

I: Now, what happens if you have different figures like these? (Presenting the non-rectangular regions.) Is it as easy in that case to order them?

S: No; maybe you have to measure them and find out their areas.

I: Yes; or...?

S: Or, you can do it the way I did before in cutting them up.

I: And...?

S: And place them to see which one has a bit more of the other.

I: OK, you mean cut each of them into...?

S: Rectangle.

I: Good, then you will have the same situation as there. (Pointing to the regions of task 9).

S: Yes.

From the above information, it would seem appropriate to suggest that the piece-wise congruency approach did offer situations in which the notion of ordering was introduced on merely plane regions with no involvement of the real numbers. This regions-ordering in the domain space of the area measure may be looked at as an analogy to the numbers-ordering in the range space.

Question 17. What contribution can the piece-wise congruency approach offer to learning of measure systems?

This question was designed to present a general overview of the notion of transfer and the possible influence of instructing the piece-wise congruency approach on attaining it. The following situations were considered:

a. Internal Transfer - in the area measure system.

b. Between Transfer - of some measure systems.

c. External Transfer - out of the area measure system.

Transfer is referred to as the utilization of learned properties of a space or system in learning similar properties and analogies of another space or system. The three types of transfer above are identical to 'within transfer', 'across transfer', and 'outside transfer' described by Osborne (1976, p. 19-20). He defines the three types of transfer as follows:

Within Transfer: In a given system of measure, a within transfer is referred to as the use of the structure of one space for support and guidence in learning the structure of the other space and in attaining a sense of the function relating the two structures.

Across Transfer: An across transfer is referred to as the learning of one system of measure in terms of the related previously learned measure system.

Outside Transfer: In a given system of measure, outside transfer is referred to as the use of the measure system context in learning some ideas that belong to topics in, say, algebra or geometry.

a. Internal Transfer

1. Throughout the unit, triangular regions, convex quadrilateral regions, and regular n-gon regions (n = 5, 6, 7, ...) were decomposed into rectangular regions. As a results, the rectangular region could be considered as the basic region in the domain space of the area measure defined on the set of all polygonal regions. Moreover, the converse decomposition of a rectangular region into the shape of each of the polygonal regions illustrates the interrelationships between these regions. Each of these regions is decomposable into any of the other polygonal regions (of equal size) through rectangulation and derectangulation processes. This is not the only way to interrelate polygonal regions; it is more consistent for this study however.

2. Through the process of piece-wise congruency, it was shown that all regions which are decomposable into a given rectangular region constitute an equivalence class (see Question 15).

3. The reflexive, symmetric, and transitive properties of the piece-wise congruency operation, as an equivalence relation, were discussed in Question 14.

4. The ordering of polygonal regions in the domain space of the area measure system was exemplified in Question 16.

The properties discussed above are considered to have analogies in the range space of the area measure system. First of all, a rectangular region that might be chosen as a unit with respect to which other polygonal regions could be compared is analogous to the number one of the range space. The set of rectangular regions within the set of all polygonal regions is as vital as the set of rational numbers within the set of the real numbers.

An equivalence class of polygonal regions with respect to a given rectangular or square region is analogous to that in the real numbers with respect to a given positive number. In the domain space the classes are visual whereas in the range space they are symbolic.

The idea of equivalence relation is physically approached and the piece-wise congruency is shown as a manipulative equivalence relation - an analogy to the equivalence relation, equality, in the range space of the area measure system.

The ordering concept is introduced through direct comparisons between polygonal regions. The piece-wise congruency concept through rectangulation did extend the possibility of comparing only rectangular regions to all polygonal regions (with the exception of non-regular n-gon regions, $n \geq 5$; n is an integer).

Internal transfer within the area measure system was attained. The Hotelling T^2 test shows enough evidence that hypothesis 3A be rejected over the Piece-wise Congruency-Rational Numbers Test. That is a significant change in the student performances was apparent; $F = 2.277$, $p < 0.05$ (Table VI). This means:

1. The awareness of the role of choosing and fixing a unit, whenever comparisons among rational numbers were encountered, was enhanced significantly and hence transfer was evident.

2. The awareness of relating rational numbers through geometric representation to some equivalence classes was enhanced significantly and therefore transfer was evident.

No attempt was made to examine the other analogous properties and their mutual effects on each other within the learning process.

b. Between Transfer

Transfer between systems of linear measure, angular measure, and area measure was investigated. There was no testing procedure employed however.

Students' performances throughout the evaluation tasks and the investigative activities in the classroom were scrutinized. Students in searching for a piece-wise congruency between two regions looked first for linear congruency between corresponding edges, then for angular congruency between corresponding angles in each of the tasks.

Some students searched for angular congruency first and then linear congruency. In both cases, students appeared to use early learned properties of the domain spaces of the linear and angular measure systems in attempting to attain analogous properties in the domain space of a new measure system - the area measure system (Question 14).

Based on the above observations, it is safe to suggest that, what students might learn about properties of the domain space of the area measure system through the piece-wise congruency process, could similarly be used in learning analogies in the volume measure defined on the set of all prisms. Equidecomposability operation on the set of all prisms can be introduced; it is an immediate extension of the piece-wise congruency operation. That a transfer between the previous measure systems and this volume measure system may take place yet needs empirical justification. Similarly, as is the case in the area measure system, there could be internal transfer situations as well as external ones. Unfortunately the extension is restricted to the set of all prisms (Boltyanskii, 1963, p. 37).

c. External Transfer

Students' performances on the Polygonal Regions - Area Formulae Test were not significantly changed. Hypothesis 3A related to the Area Formulae Test was not rejected: $F = 1.029$, $p > 0.05$ (Table VI, subtest iii). On the assumption that a study of rational numbers is part of studying concepts of number, and that area formulae and their derivations are more related to the study of concepts of algebra the following situations were evident:

1. External transfer to the study of concepts of numbers was attained: Hypothesis 3A (p. 67), related to the Rational Numbers test, was rejected ($F = 2.277$, $p < 0.05$, Table VI, subtest vi, page 88).

2. External transfer to the study of algebra (in terms of the area formulae test) was not apparent. Hypothesis 3A, related to this test, was not rejected ($F = 1.029$, $p > 0.05$, Table VI, subtest iii).

It should be noticed, however, that there was only one class period devoted to part three of the unit - the area formulae and their derivations. Both teachers, at the end of the project, expressed their concern about the short time spent on this section.

> Much time should be spent on formulae development (part III) using decomposition models and problems. This was not well done during the project and the students did not relate well to the "cut-and-cover" used to develop formulae.
>
> (Teacher A, Teachers' Opinion Test)

Teacher B recommended that one to three class periods would be necessary to have part three of the unit taught well.

Question 18. Can the piece-wise congruency approach simplify some basic theorems in geometry such as those of Apollonius and Pythagoras?

The Progress Checking Test #2 (Appendix G) comprised two cases of the Pythagorean Theorem - a special case when the right triangle is isosceles, and a general case. The two classes were given the test over the last 20 minutes period of the fourth classroom session. The two cases were given to the students simultaneously.

In case 1, the special case, there were no failures in class A and only one failure in class B. The students' performances fall in four categories:

1. Cutting each of the square regions A and B into two triangular regions through a diagonal and the square region C of the hypotenus is covered (Fig. 28).

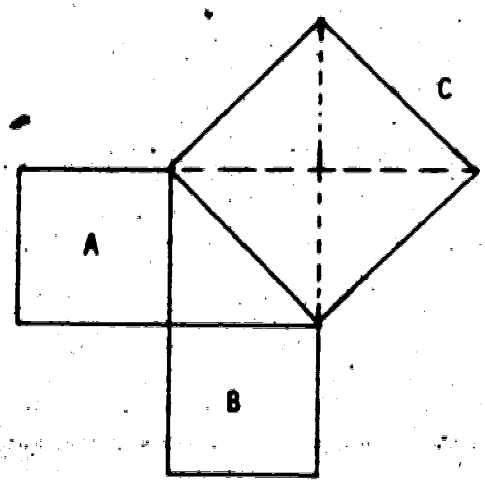


Figure 28

2. Cutting one of the square regions, either region A or region B, diagonally into four triangular regions and the square region C of the hypotenuse is covered (Fig. 29).

3. Cutting one of the square regions, either region A or region B, vertically or horizontally into three rectangular regions such that the proportional ratio of their sizes is 1 : 1 : 1/2. The thinner rectangular region then was divided into two halves (Fig. 30). The five cases which represent this category are included in Appendix J; they are related to students 04M, 13F, 20F, 21M, and 24M of class A. No such type of pattern was exhibited at class B however.

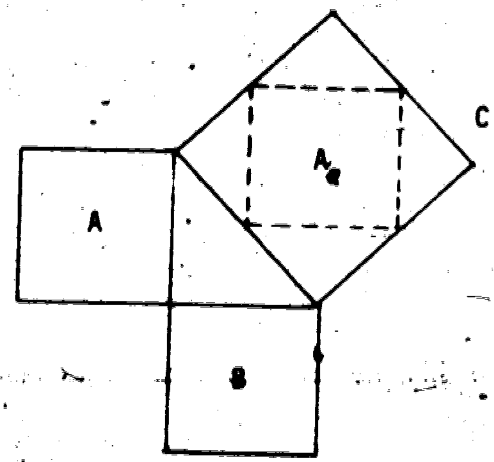


Figure 29

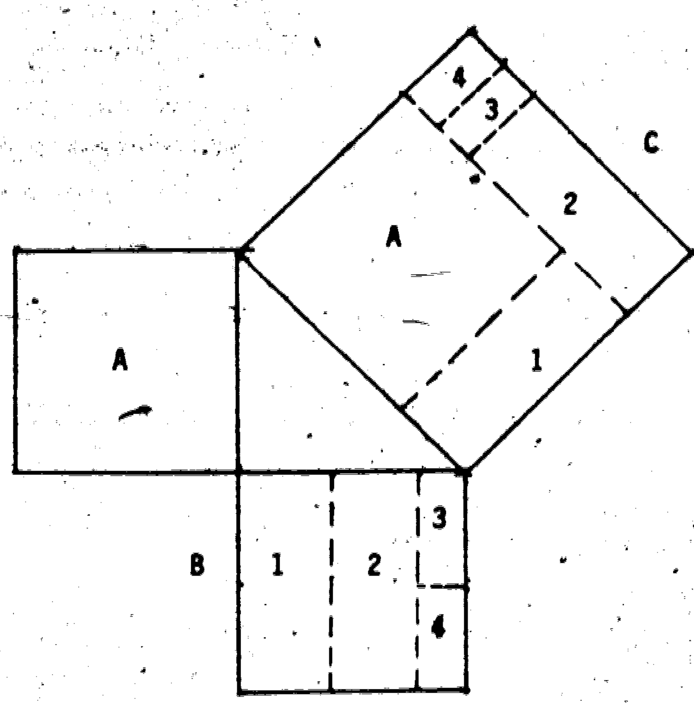


Figure 30

4. The cuttings in this category were different from case to case: There were no common features that united them. They were interesting as surprising cuttings; they were performed by the students 34F, 35F, 54F, and 55M of class B; see Appendix J.

By an examination of the students' performances in Figures 28, 29 and 30 and Appendix J one can see the kind of innovative cutting and covering students have achieved, and how an intuitive proof of the theorem was then made.

In case 2, the general case, only the rearrangement of the five regions 1, 2, 3, 4, and 5 to cover the square region C was required (Appendix G). The students' performances of the two classes are shown in the contingency Table - Table IX.

There was no association between performance level and class membership as indicated by the χ^2 test ($\chi^2 = .152 < \chi^2_{.05} = 3.84$, df = 1).

All of the students except one did prove the special case of the Pythagorean Theorem. On the general case, Table IX shows that 64.2% of the students did pass the covering exercise of the general case of the theorem. It would appear that the students did very well on the two cases through the cut-and-cover operation. Thus the proof of the Pythagorean Theorem was simplified and attained in an intuitive manner.

Question 19. What are the differences between the students in class A and those in class B with respect to exploring and perceiving the requirements of the course?

The Geometry Tests (Appendix F) were used to measure the dif-

TABLE IX

FREQUENCIES OBSERVED: PYTHAGOREAN THEOREM

	PASS	FAIL	TOTAL
CLASS A	18	9	27
CLASS B	16	10	26
TOTAL	34	19	53

ferences in achievement for class A and class B. The Hotelling T^2 for independent samples was performed on the scores of the classes in the pretest and again in the posttest to test hypotheses 8, 8A, 9 and 9A.

Table X gives a summary of the mean scores of the two classes in each of the Geometry Tests on the pretest.

Table XI shows a summary of the multivariate analysis of variance in contrasting the two classes over their scores on the pretest.

There were 27 students in class A and 20 in class B with both pretest and posttest scores.

Table XI gives enough evidence to reject hypothesis 8; $F = 7.735$, $P < 0.001$. On each of the individual tests, hypothesis 8A was rejected on:

Test (i); $F = 3.771$, $P < 0.05$,

Test (ii); $F = 3.305$, $P < 0.05$, and

Test (iii); $F = 2.984$, $P < 0.05$.

Thus, the overall performances on the pretest of class A and class B over the Geometry tests were significantly different. Class A did better over the whole test.

Moreover, class A scores on the Vocabulary Test, Polygonal Regions Differentiation Test, and Area Formulae Test were significantly better than the corresponding scores for class B.

The two classes, however, were almost identical in their performance on the Piece-wise Congruency Tests, (iv), (v), and (vi); hypothesis 8A was not rejected over each of them.

Table XII contains a summary of the mean scores in each of the Geometry Tests on the posttest.

TABLE X

MEAN SCORES OF CLASS A AND CLASS B ON THE GEOMETRY TEST - SIX SUBTESTS: PRETEST

GEOMETRY TEST (PRE)	CLASS A	CLASS B
i	11.426	7.475
ii	17.148	10.700
iii	4.907	2.250
iv	4.667	3.700
v	5.407	4.200
vi	14.704	13.250

TABLE XI

HOTELLING T^2 CONTRAST BETWEEN ACHIEVEMENT OF CLASS A AND CLASS B ON THE GEOMETRY TEST - SIX SUBTESTS: PRETEST

TEST	T^2	DF_1	DF_2	F-RATIO	PROBABILITY
i	25.452	6	40	3.771	0.005
ii	22.309	6	40	3.305	0.010
iii	20.142	6	40	2.984	0.017
iv	0.710	6	40	0.105	0.995
v	2.837	6	40	0.420	0.861
vi	0.300	6	40	0.045	1.000
ALL	52.213	6	40	7.735	0.000

Table XIII shows a summary of the multivariate analysis of variance in contrasting the two classes over their scores on the posttest.

Table XIII presents enough evidence that hypothesis 9 be rejected; $F = 5.962$, $P < 0.001$. On the individual subtests, hypothesis 9A (page 68) was rejected on test (i) only; $F = 3.558$, $p < 0.05$. Therefore, the overall performance on the posttest of class A and class B over the Geometry Tests was significantly different.

Furthermore, class A scores on the Vocabulary Test remained significantly better than those for class B. Hence, the situation over the posttest was almost identical to that over the pretest apart from a slight gain for both classes.

But, class B scores over the Polygonal Regions Differentiation Test, and the Area Formulae Test were significantly improved and hence hypothesis 9A was not rejected; $F = 0.220$, $p > 0.05$, and $F = 0.337$, $p > 0.05$ respectively. This means that class B bridged the gaps which existed over the pretest scores of these two tests (Table XI, tests ii and iii).

On the other hand, hypothesis 9A was not rejected on the achievement scores over each of the Piece-wise Congruency Tests, (iv), (v), and (vi).

Based on these results, it seems evident that although class A showed considerable gains (Table X and XII) the piece-wise congruency approach was more beneficial to Class B - firstly on the aspect of differentiating polygonal regions from non-polygonal regions and secondly on the aspect of area formulae of polygonal regions included in

TABLE XII

MEAN SCORES OF CLASS A AND CLASS B ON THE GEOMETRY TEST - SIX SUBTESTS: POSTTEST

GEOMETRY TEST (POST)	CLASS A	CLASS B
i	12.019	8.650
ii	46.741	50.750
iii	5.333	4.4[illegible]5
iv	8.630	6.900
v	8.667	7.375
vi	21.463	20.200

TABLE XIII

HOTELLING T^2 CONTRAST BETWEEN ACHIEVEMENT OF CLASS A AND CLASS B ON THE GEOMETRY TEST - SIX SUBTESTS: POSTTEST

TEST	T^2	DF_1	DF_2	F-RATIO	PROBABILITY
i	24.015	6	40	3.558	0.006
ii	1.484	6	40	0.220	0.968
iii	2.273	6	40	0.337	0.913
iv	2.535	6	40	0.376	0.890
v	2.910	6	40	0.431	0.854
vi	0.172	6	40	0.026	1.000
ALL	40.240	6	40	5.962	0.000

the unit. Class B initially fell behind class A over these aspects, but later recovered in narrowing the gaps significantly. It is probably useful to recall that the means for grade 7 mathematics final marks were 71.57 and 62.26 for class A and class B respectively.

> Question 20. What is the attitude of the participating teachers toward the use of the piece-wise congruency approach in their grade eight mathematics classes?

The Teachers' Opinion Assessment Test (Appendix E) was conducted and both teacher A and teacher B showed positive responses. The teachers expressed their willingness and enthusiasm for the project after the first meeting. They were able and highly cooperative teachers.

Some of the teachers' responses have been included throughout the discussion of the foregoing research questions. Below is a fuller account of their stance toward the piece-wise congruency approach:

1. Both teacher A and teacher B were in favor of the use of the approach in geometry instruction.

2. In supporting the above response the teachers offered the following:

> As an aid to developing the concept of area as the interior of a polygonal region, this method is helpful.
>
> Teacher A

> Good way to introduce area of different polygons by relating them to rectangles.
> It is interesting also to the students to be able to manipulate things to find area.
>
> Teacher B

3. Both teachers indicated that the use of approach which involved actual tracing, cutting, and covering of polygonal regions did not create chaotic situations in the classrooms. Teacher A supported his indication in the following statement:

> Students of Junior high age enjoy and are generally adept at manipulating objects and trying to devise different possible solutions.

4. Both teachers were unable to determine the extent of the possible effect of the approach regarding students' performance in attempting spatial problems on polygonal regions.

5. On the effectiveness of the approach for the students' performance in mathematics, Teacher B was "unable to determine" while Teacher A offered the following elaboration:

> Students were more aware of spatial relations between sides and angles of polygonal shapes when solving problems involving geometric concepts.

6. On the possible change in the students' attitude, teacher B was unable to determine while teacher A made the following response:

> I feel there was positive attitude development.

7. In responding to whether or not the approach can positively change the students' self-confidence in performing spacial tasks, both teachers positively responded. In addition, Teacher A made the following remark:

> The fact that most students were able to find their own and completely different solutions, improved student self-confidence.

8. Teacher B was unable to determine whether or not the use of the approach can change the students' attitude toward mathematics. Teacher A, however, offered the following comment:

> Increased awareness of geometric concepts and especially area. It increased self-confidence of students. It also showed students that math can be enjoyable as well as a learning experience.

9. On whether or not the Edmonton Public School System should adopt the use of the piece-wise congruency approach in grade eight classrooms, both teachers responded positively. Teacher B suggested that "end of grade 7, and in grade 8" are the stages in which the approach should be utilized. While Teacher A suggested that:

> Grade 8 and/or Grade 9. Perhaps Grade 6 or 7 if formulae are not included.

10. Both teachers indicated that the Edmonton Public School System should provide instructional materials to its students in the use of the piece-wise congruency approach.

11. The teachers made the following additional comment on their own:

> I have found the project to be an innovative and thought provoking experience. The students were, at first, skeptical about the unknown situation. They soon became very involved with "cut-and-cover".
>
> The decomposition of regular polygonal regions appeared to become tedious. Perhaps heptagon and dodecagon should be omitted.
>
> I would not hesitate to use this method again in developing the concept of area or in obtaining area formulae.
>
> Teacher A.

> If the school board is going to implement this program, an inservice would be beneficial.
>
> Teacher B.

CHAPTER V

SUMMARY, CONCLUSIONS, DISCUSSION, IMPLICATION AND RECOMMENDATIONS

I. SUMMARY OF THE INVESTIGATION

A. Introduction

The purpose of this study was to investigate and identify some of the geometric thinking processes used by eighth grade students in attempting to solve spatial problems on polygonal regions presented to them, and to justify and explain their performances. The major purpose of this enterprise was to attempt to explain how students really understand area, with special emphasis on (a) the effectiveness of physical manipulations on students' performances in spatial problem solving situations; (b) the kind of inuitive reasonings students appear to exhibit in those situations; (c) transfer of learned properties into new situations; (d) the type of errors students appear to exhibit when attempting to justify their actions; (e) patterns students appear to form and use when problem solving situations are encountered; (f) the possible utilization of the motions of slide, turn, and flip throughout the students' responses in those situations; (g) the possible use by the classroom teachers of the piece-wise congruency unit as a medium through which better learning of some geometric concepts, measure system structures, and some basic concepts in elementary mathematical analaysis might take place; and (h) whether or not the unit and the evaluation tasks have any potential for classroom teachers' use as a tool for diagnosing students' misunderstanding of some fundamental geometric concepts.

Based on the piece-wise congruency operation, an instructional unit was constructed consisting of a sequence of geometric theorems each of which was formulated in an investigative activity. The mathematical structure of the unit was interpreted through intuitive and physical operations such that 'learning by doing' might take place. The theorems are summarized in the following:

> Each triangular region (all kinds), quadrilateral regions (all kinds) and regular n-gon regions ; $n \geq 5$, n being an integer are decomposable into a rectangular region.

The interrelationships shown in Fig. 31 illustrates one of the unit's aims.

B. The sample

A sample of 58 grade eight students in two classes of an Edmonton public school was utilized for analysis purposes. The two selected classes, class A and class B, differed considerably on the final grade 7 mathematics achievement test. The mean scores for class A and class B were 71.57 and 62.25 respectively. From the school's point of view, class A was known as the 'best' grade eight class while class B was the 'poorest' among the eight grade eight classes in the school.

C. Procedures

Investigative activities based on the piece-wise congruency operation were developed covering most of the polygonal regions (except the non-regular n-gon region, $n \geq 5$ and n is an integer) and

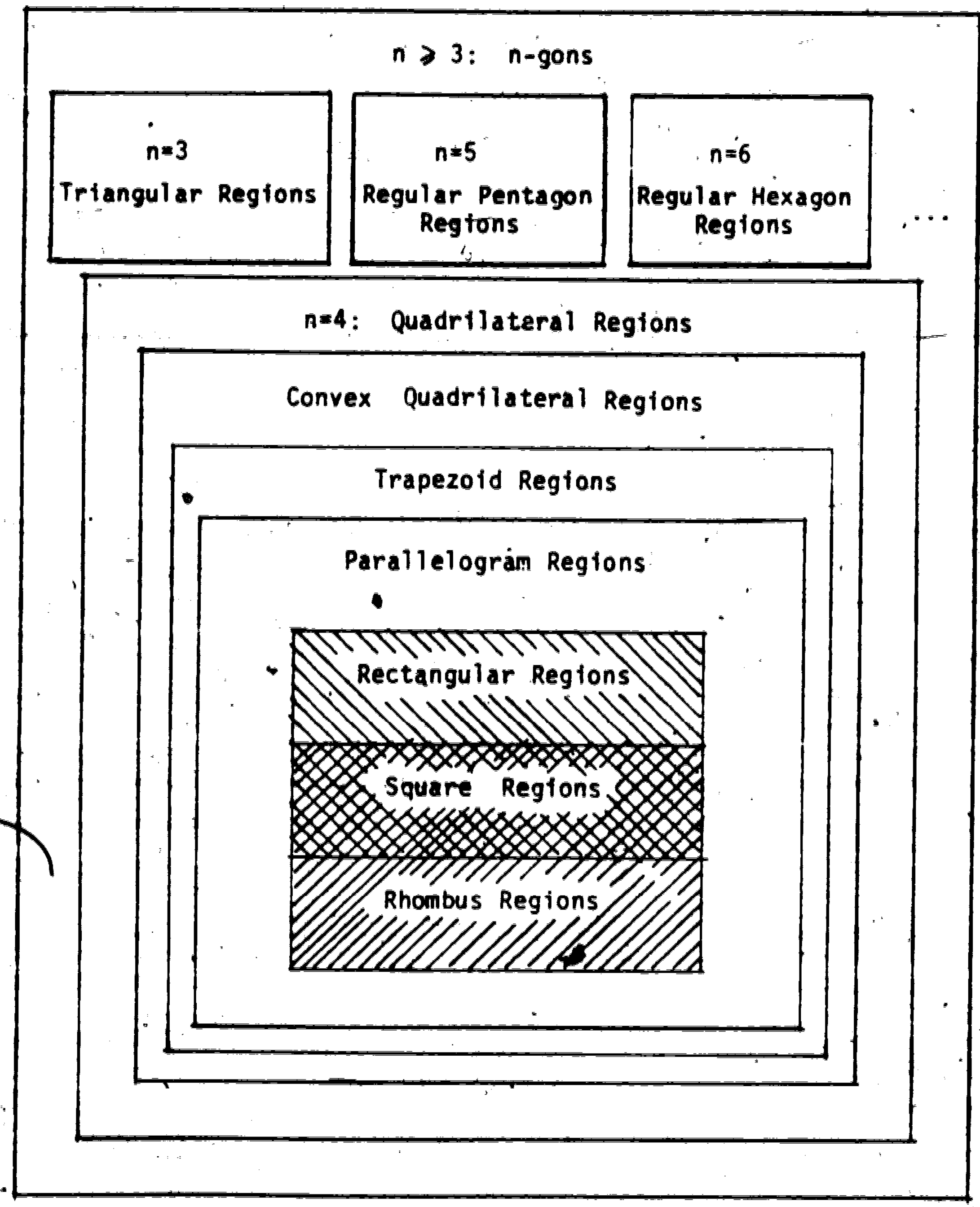


Figure 31

Interrelationships Among the Polygonal Regions some basic properties of the area measure defined on them. A section on the area formulae and their derivations, also based on the piece-wise congruency operation, was developed. These instructional materials were put together, as a unit, and included both in a booklet (Student's Booklet) and a teacher's manual. The unit covered the part on polygonal regions of the grade eight mathematics programme; and it was used in both classes.

The students' responses on the pretests and the posttests were checked by three people. But first, the students' responses were carefully examined on each item. As a result a weighting factor for each item was then designed for scoring purposes (see page 66, Ch. III). The scores were then used in teting the null hypotheses.

A package of cardboard models for polygonal regions, which comprised most of the investigative activities in the unit was developed (evaluation tasks, see p. 58-61). It consisted of eleven tasks which in turn were used for interviews.

Magnetic boards, decomposable magnetized cardboard models for each polygonal region involved in the investigative activities of the unit, and a complete set of transparencies (Appendix B - transparencies' masters) containing possible solutions for each task were prepared. They were used as teaching aids.

The investigation began in March-April of the school year 1979-1980 and lasted about one month.

D. The Instrument

A Student's Opinion Assessment Test consisting of 15 items both on geometry and mathematics was utilized to determine if a change in the students' attitude was exhibited. The two tests were administered to all students both before and after the teaching of the unit.

A Geometry Test consisting of six subtests was employed to determine whether or not there was a change in the students' achievement with respect to geometric vocabulary, differentiating polygonal and non-polygonal regions, area formulae, and interrelating polygonal regions to each other. These tests were given to all students before and after the instruction of the unit. The tests were confined to polygonal regions, some of the area measure properties, and the piecewise congruency concept. For all these instruments, the pretesting was teacher administered while the posttesting was investigator-teacher administered.

A set of three Progress Checking Tests were used to provide information about the student's interaction with the unit. These tests were investigator administered to each of the classes over the last 10, 20, and 15 minutes of the second, the fourth, and the sixth classroom sessions respectively.

All classroom sessions for each class were tape recorded. A single class period for each class when teachers began teaching the final section of the unit (derivations of area formulae) was videotaped. The investigator attended all the classroom sessions for both classes and recorded observations focussing on the students' behavior when problem situations involving cut-and-cover were encountered.

Six students from each class were interviewed. The school records for the final grade 7 mathematics marks were used in identifying students as low-or-high-achievers for this interview. The Evaluation Tasks' package was used in each interview. The interviews were carried out approximately two weeks prior to the start of the teaching. Each interview lasted about 35 minutes and all were videotaped. The transcripts of the interviews and the video tapes were used to glean information concerning students' preception of area, interrelationships between polygonal regions, properties that are invariant under decomposition, and students' pattern formation in equidecomposing regions of different shapes to each other. Throughout the interviews, Vygotsky's approach was used. Leading questions, suggestions, or collaboration were offered by the investigator. In this connection, Vygotsky offered the following statement:

> In studies of children's mental development it is generally assumed that only those things that children can do on their own are indicative of mental abilities.--On the other hand, if we offer leading questions or show how the problem is to be solved and the child then solves it, or if the teacher initiates the solution and the child completes it or solves it in collaboration with other children - in short, if the child barely misses an independent solution of the problem - the solution is not regarded as indicative of his mental development. This "truth" was familiar and reinforced by common sense. Over a decade even the profoundest thinkers never questioned the assumption; they never entertained the notion that what children can do with the assistance of others might be in some sense even more indicative of their mental development than what they can do alone.
> (Vygotsky, 1978, p. 85)

The student's Opinion Assessment data both on geometry and mathematics was analyzed using multivariate analysis of variance. The

Hotelling T^2 test for single sample with repeated measures was used to determine an overall contrast between the pretest and the posttest scores as well as contrasts on each item individually. Also, the Hotelling T^2 test for independent samples was utilized to contrast the two classes' scores both on the pretest and the posttest for each of the Opinion Assessment Tests. As well, the contrasting was both overall and itemwise over each of the pretest and the posttest. Similar analysis was used over the Geometry Tests data. For the Progress Checking data however, the Chi-square test was used to determine performance differences between the classes.

II. CONCLUSIONS

A summary of the findings will be presented as follows: Quantitatively, on the basis of testing the hypotheses; and qualitatively, on the basis of scrutinizing the transcripts and the video tape records of the students who were interviewed, the classroom observations, and the participating teachers' reactions.

A. Conclusions and Results (Quantitative)

Analysis of student attitudes toward both geometry and mathematics suggested that there was no significant attitude change overall or on each individual item of the two questionnaires. Accordingly hypotheses 1, 1A, 2, and 2A (see Chapter III, P.67) were not rejected. As well, analysis of contrasting class A and class B on each of the two testing periods, the pretest and the posttest, yielded no significant difference in the attitude of class A vs class B overall or on each individual item. Subsequently, hypotheses 4, 4A, 5, 5A, 6, 6A, 7, and 7A were not rejected.

Analysis of students' geometry achievement scores on the two testing periods, the pretesting and the posttesting, yielded significant changes ($p < 0.05$) overall and on four of the six subtests of the Geometry Tests. These significant changes occurred on the following subtests: the differentiation of polygonal and non-polygonal regions subtest; the piece-wise congruency related to three polygonal regions subtest; the two polygonal regions subtest; and the rational numbers subtest. On the basis of these results, hypothesis 3 was rejected while hypothesis 3A was rejected only on the subtests (ii), (iv), (v), and (vi). Also, analysis of contrasting class A and class B on each of the pretest and the posttest yielded significant changes ($p < 0.05$) overall and on three of the six subtests of the Geometry Tests. These significant differences occurred on Vocabulary, Differentiation, and area formulae subtests. Accordingly, hypothesis 8 was rejected while hypothesis 8A was rejected only on the subtest (i), (ii), and (iii). On the posttest scores, analysis of contrasting class A with class B indicated significant differences ($p < 0.05$) overall and on the Vocabulary subtest. On the basis of these results, hypothesis 9 was rejected while hypothesis 9A was rejected only on the subtest (i). The Chi-square test on Progress Checking Test data over test #2 and #3 indicated that the classes' responses were not significantly different on each of these checkings. On the Progress Checking Test #1 the mean scores for classes A and B were

11.482 and 10.875 respectively, and hence were taken to be approximately the same.

B. Conclusions and Results (Qualitative)

The following conclusions were drawn after scrutiny of interview transcripts, video tapes, and the students' original cutting and covering, together with observations recorded during the classroom sessions. Teachers' reactions were also examined.

a. Throughout each evaluation task on piece-wise congruency, students seemed to be looking first for pairs of congruent edges. In other words, at the first instant, a linear congruency was sought. The students then searched for pairs of congruent angles; that is, an angular congruency was sought. Most of the students looked first for a linear congruency. There were however, a few cases where students looked first for angular congruency. Finally, searching for corresponding pairs of congruent edges and angles was the initial action for all of the students in their search for congruent pairs of subregions. That is, a piece-wise congruency was sought later (Ch. IV. Question 4). This sequencing in the students' action would appear to be due to the fact that a linear congruency is simpler to identify and check than an angular congruency. In a linear congruency, the student deals with simple comparisons of one dimensional figures in a plane whereas in an angular congruency, comparisons would be less simple and would require an extra condition to be fulfilled; he turns to deal with two dimensional figures. The situation would seem not obvious where a region-wise congruency in a plane is required. This is due to the fact that more conditions must be met so that the required congruency can be made. Moreover, students' experiences with comparisons in a plane started with the linear one, the angular, and later with the regional and hence were learned in this sequence.

b. There was enough evidence to suggest that a piece-wise congruency could be attained only if a linear congruency, or angular congruency, or both was attained (Question 4). On the congruency task for example (task O of the evaluation tasks) the students' responses were interesting. As mentioned in Chapter III, the task consisted of two different sets of cardboard regions. There were three congruent regions in each set; triangular in one set and rectangular in the other. Each set was displayed at random on the desk and the student was asked to show whether or not the regions of each set were congruent. Some of the students quickly responded indicating that the regions of each set were not congruent. The investigator then asked for justification and suggested holding the regions of each set. Accordingly each of these students reversed their responses. The point here is that even in the case of having already congruent regions, students seemed to search first for linear congruency, or angular congruency, or both in attempting the task.

c. Most of the students exhibited the kind of behavior described in points (a) and (b) above. They spontaneously introduced into a new measure system, learned properties of measure systems experienced previously (Question 4 and 17). This kind of behavior contradicts what Osborne (1976) asserted earlier.

> Given the extensive experience of children and adolescents with different systems of measure, a remarkable feature of their performances is that it proceeds so inefficiently. That is to say, the typical child does not use the learning of properties for one characterizing function to advantage in learning about other measure system.
> (Osborne, 1976, p. 19-20)

The notions of congruency and comparison between both segments and angles were advantageously used in comparing and congruencing polygonal regions in a plane.

d. In some cases, students had little difficulty in performing a particular motion whenever they were asked to do so. In other instances, they experienced uncertainty in identifying the motions that were used (Question 3).

e. Few individual changes in attitude could be regarded as positively high (Question 1). Student 30F, the lowest achiever in class B, had the highest positive change in the opinion assessment test both on geometry and mathematics. As well, the highest achiever in class A, student 12M showed a high positive change on geometry and a less positive change on mathematics (Table IV). Incidently, student 12M and student 30F were the highest and the lowest achievers in their respective classes. Tables IIIA, IIIB, and IIIC show notable itemwise changes in attitude toward geometry. An examination of Table I and Table II would indicate that students in general, tended to a positive change in attitude toward geometry while tending to a negative change toward mathematics. This opposite tendency might suggest that students looked at geometry as a different subject independent of mathematics.

f. In many cases, students were persistent in attempting tasks on piece-wise congruency. They requested additional materials to continue when initial strategies were abandoned (Question 2).

g. With respect to the context of each of the evaluation tasks, it would appear that students exhibited key ideas that are analogous to the content structure in each of the evaluation tasks 1, 2, 3, 4, and 5. On tasks 6 and 7, there was no similarity between the students' exhibited key ideas and the content structure for both tasks. On task 8, actions of only three students were characterized as indicating cognitive structures that are similar to the content structure. Actions for the remaining seven students were marked as exhibiting cognitive structures that are not similar to the content structure of the task despite some successful actions (Question 5).

h. Students seemed to recognize that two polygonal regions of equal area and different shapes were piece-wisely congruent only if one of the regions was decomposed into subregions and superposed to completely cover the other regions simultaneously. If the subregions were reassembled into their original region, the students would reverse themselves and indicate that the two polygonal regions were no longer congruent by pieces. They (the students) appeared to suggest that as long as the two regions possess different shapes they were not congruent even if the subregions of one of them already covered the other region completely (Question 6). In sum, students in grade eight level appeared to be not able or willing to think sequentially but only with static mental representations.

i. The change of the properties of decomposed regions and the conservation of segments, angles and areas under decomposition were easily achieved (Question 7).

j. The practical aspect of the use of the piece-wise congruency approach in a laboratory setting was evident. The students were at ease and totally engaged in the activities. No discipline problems were apparent, (Question 8).

k. The cut-and-cover approach seemed to be a medium through which invariant properties under the motions of slide, turn, and flip were perceived and exhibited (Question 9).

l. An in-service session of one to three hours was seen as necessary to prepare teachers should the unit be adopted for use in schools (Question 10).

m. The algebraic aspect of the piece-wise congruency approach as well as the whole approach in general were favorably viewed by the participating teachers. The two teachers showed a strong acceptance of the approach and indicated their firm willingness to use the unit in their future instruction (Question 11).

n. The equidecomposability approach seemed to be a medium through which the commutative property of the arithmetic operations of addition and multiplication was physically demonstrated in the domain space of the area and length measure systems. It was immediately analogized with its counterpart in the range space, the positive real numbers,

o. Students intuitively perceived the symmetric and transitive properties of the piece-wise congruency as an operation defined on the set of all polygonal regions into itself (Question 13).

p. If a polygonal region is considered congruent to itself, then the piece-wise congruency operation would be characterized as a reflexive operation and hence an equivalence relation (Question 14).

q. The piece-wise congruency approach created problem solving situations in which the notion of ordering was introduced on polygonal regions. This was based merely on rectangulating each of them apart from real number considerations (Question 16).

r. Students did very well proving the Pythagorean Theorem through piece-wise congruency processing. They proved the special case of the theorem (isosceles right triangular region) with only one failure in both classes. On the general case of the theorem, about two thirds of the sample proved it. Appendix J shows some of the patterns students used in proving the theorem (Question 18).

s. The participating teachers viewed the piece-wise congruency approach as "an innovative and thought-provoking experience" as a "good way to introduce area of different polygons by relating them to rectangles". The teachers indicated that they would not hesitate to use the approach again in obtaining the area formulae and developing the area concept (Question 19).

t. It seems likely that the students' cuttings in task 6 were influenced by the pattern used in previous tasks such as tasks 2 and 4. As well, the cuttings in task 7 appear to be influenced by the pattern used in task 3. An examination of the photographs of the students' original cuttings (Appendix H) for tasks 2, 4 and 6 and tasks 3 and 7 supports this conclusion.

u. Analysis of the students' performance on the evaluation tasks: The congruency task (task 0), the eight piece-wise congruency tasks (tasks 1, 2, 3, ..., and 8), and the ordering tasks (tasks 9 and 10) for rectangular regions and non-rectangular regions respectively. For details see pages 58-61 and pages 101-105.

Task 0. Students' performances, with respect to their initial action, fell into two categories.

Category 1: The performances in this category were characterized by (a) an immediate reaction to each of the two sets of congruent triangular regions and congruent rectangular regions; (b) justification of response by visual judgement only; and (c) unsuccessful completion of the task. They processed the task by simply looking at each set and quickly stating that the regions in each of them were not congruent.

Category 2: The performances in this category were characterized by (a) the use of visual means; (b) the use of manual means; and (c) successful completion of the task. The students processed the task by: picking up the regions of each set; superposing them on each other; and then a visual judgement was used to conclude a correct judgement. That is, they processed the task by a collaboration of visual and manual means.

Interestingly, the students in category 1, when later asked by the experimenter to pick up the pieces of each set by hand and see what could happen (collaboration - suggestion in Vygotsky's terms, 1978), they reversed themselves and responded that the polygonal regions in each set were congruent. They discovered through hand-mind actions, the correct answer as well as their mistakes (Freudenthal, 1973, p. 407). Their performances thereafter, when an auxiliary tool was introduced, fell into category 2.

The second type of performance was potential (zone of proximal development, Vygotsky, 1978): it was not exhibited until some sort of collaborative suggestions were offered.

Task 1. The students' performances in this task were classified into three categories.

Category 1: The performances were characterized by (a) the discovery of the linear congruency and the angular congruency that existed between the right triangular region and the rectangular region; (b) the dissection of one of the regions into two subregions; and (c) successful completion of the task. The process used in this type of performances can be described by the following hand-mind sequence of actions: superposing one of the regions on the other with the congruent sides and the congruent angles being superposed on each other; cutting one of the regions (or each) along the edges of the other region; carrying over the subregions and patching them on the other region, and concluding that the two regions were 'equal' 'the same' or 'congruent'. For this type of performances, see Appendix H, task 1, photographs 03M - 1st and 2nd attempts, 05F - 2nd attempt, 12M, 19F-1st and 2nd attempts, 20F - 1st and 2nd attempts, 27M, 30F, 41F, 44F - 1st and 2nd attempts, and 48M - 2nd attempt.

Category 2: The performances in this category have everything in common with those in category 1 except the number of subregions used were more than two (see Appendix H, task 1, photograph 34F). The process student 34F used was identical to that used by students in category 1 except the actions of cutting, carrying over, and patching were repeated once more

(Fig. 32). Note that the motion of flip (slide or turn) was implicitly used through the patching.

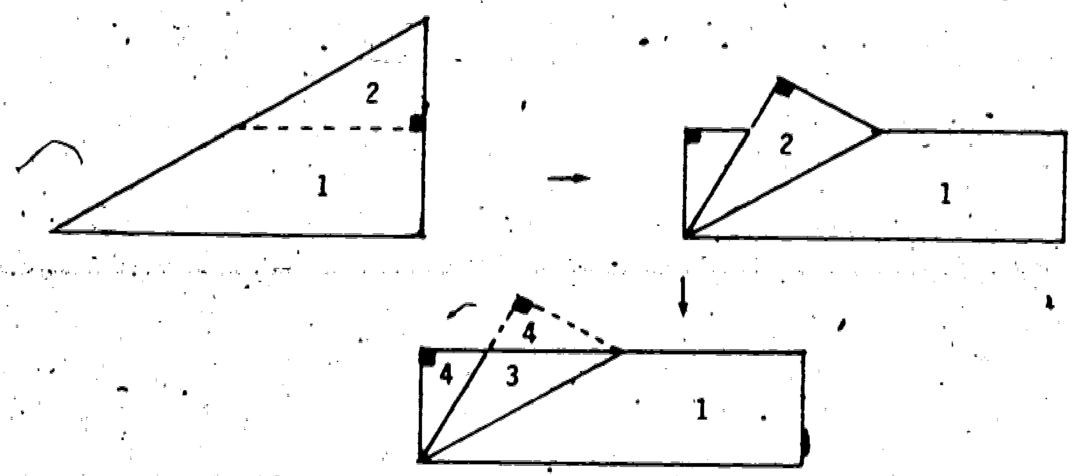


Figure 33

The 34F's Process on task 1 via Rectangulation

Category 3: The students' performances were characterized by (a) failing to discover the linear and angular congruencies between the two regions; (b) the dissection of one of the regions into more than two subregions (in the existing cases the number of the subregions were more than three); and (c) unsuccessful completion of the task. The process used through these performances can be described in the following hand-mind actions: superposing one of the regions on the other with two sides being superposed along each other; cutting one of the regions along the edges of the other; carrying over and patching the resulting subregions on the other regions; repeating the actions of cutting, carrying over, and patching one or more times; realizing that the pattern used was leading to a continuous cutting and patching in a cyclic fashion; and the process was terminated. For this type of process, see Appendix H, task 1, photographs 05F - 1st attempt, 46M, and 48M - 1st attempt.

Task 2. The students' performances in this task were classified into four categories.

Category 1: The performances in this category were characterized by (a) the discovery of the linear congruency between the acute triangular region and the rectangular region; (b) the dissection of one of the regions into three subregions - two right triangular regions and a trapezoid region - through horizontal and perpendicular cuttings of the triangular region or oblique cuttings of the rectangular region; and (c) successful completion of the task. The process used in the category can be described by the following sequence of hand-mind actions: superposing one of the regions on the other with the two congruent sides being superposed on each other; cutting one (sometimes each) of the regions along the edges of the other; carrying over the patching the subregions on the other region; repeating the actions of cutting, carrying over, and patching once more; with a successful equidecomposition attained. For this type of performances, see Appendix H, task 2, photographs 12M, 19F, 20F - 1st and 2nd attempts, 27M - 2nd attempt, 30F, 34F - 1st and 2nd attempts, 41F, 46M - 1st and 2nd attempts, and 48M.

Category 2: The performances in the cases included in this category were characterized by (a) the discovery of the linear congruency between the two polygonal regions involved; (b) the dissection of one of the regions into more than three subregions (in the existing cases there were four subregions - a trapezoid region, a quadrilateral region, and two triangular regions); and (c) successful completion of the task. The process employed for this type of performances can be described by the following chain of actions: superposing one of the regions on the other with the two congruent sides being superposed on each other; cutting the triangular region - through a horizontal cut along the edges of the rectangular region - into two subregions (a trapezoid region and a triangular region) or cutting the rectangular region - through oblique cuts along the sides of the triangular region - into three subregions (a trapezoid region and two right triangular regions); carrying over and patching the pieces on the other region; repeating the actions of cutting, carrying over, and patching on some of the resulting subregions; with a successful equidecomposition exhibited. For this type of performances, see Appendix H, task 2, photographs 44F - 1st and 2nd attempts. These two performances are elaborated in Figure 33 and Figure 34 (via the video tape recordings).

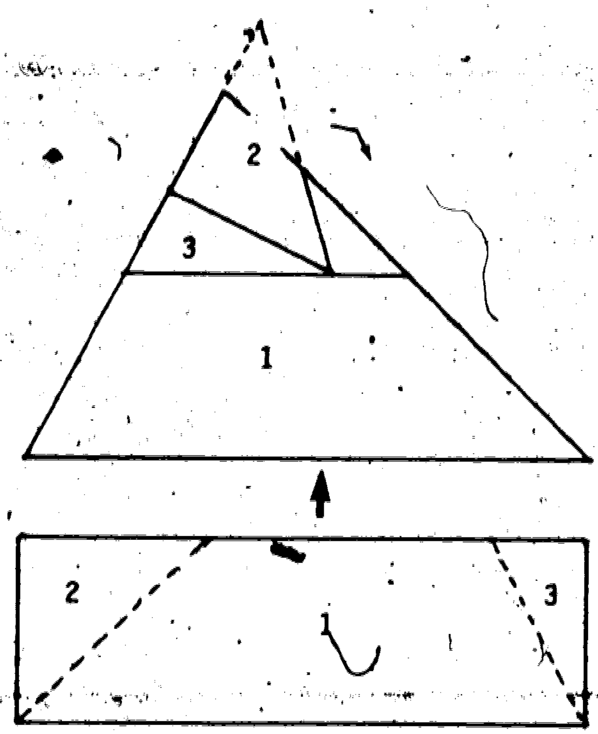


Figure 33
The 44F's Process on task 2 - 1st Successful Attempt via Derectangulation

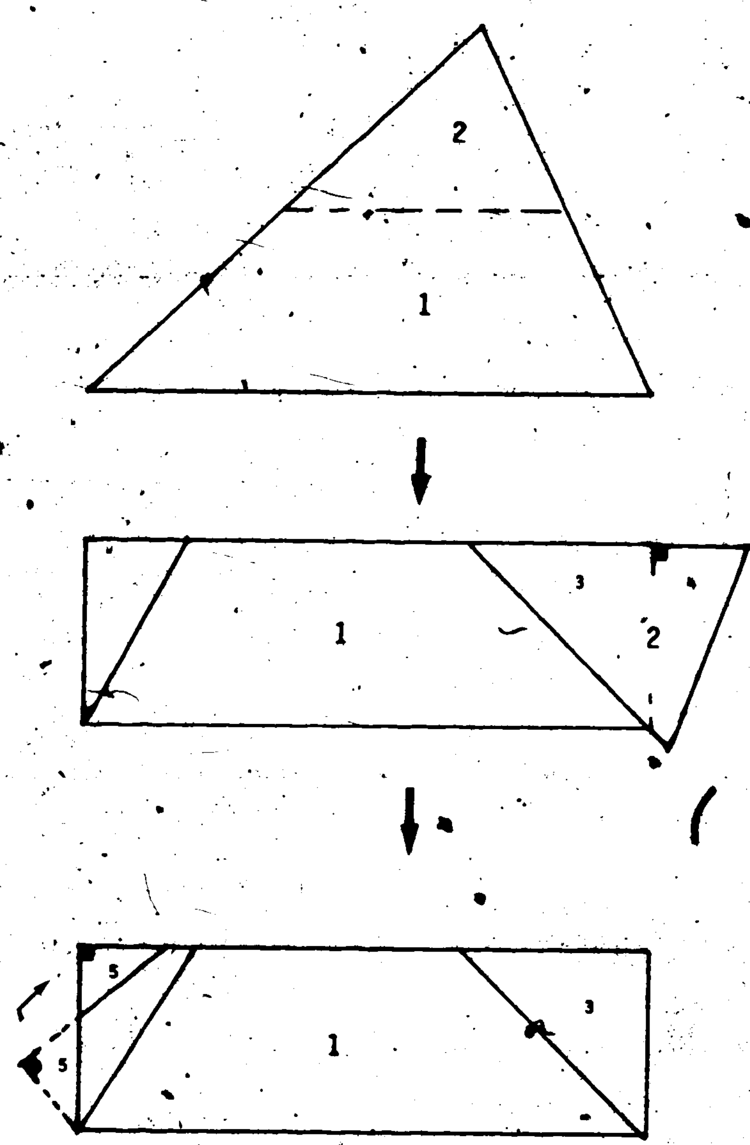


Figure 34

The 44F's Process on task 2 - 2nd Successful Attempt via Rectangulation

The patterns exhibited in the Figures 33, 34, as well as 32 show the kind of innovative performances students can demonstrate when manual and mental actions are incorporated. Students 34F and 44F are high achievers of class B; in fact, 44F is the highest achiever of class B (see Table IV, p. 77).

Category 3: The performances in this category were characterized by (a) the discovery of the existing linear congruency between the two polygonal regions; (b) the dissection of one of the regions into two or more subregions; and (c) unsuccessful completion of the task. The process used in these performances can be described by the following sequence of hand-mind actions: superposing one of the regions on the other with the two congruent sides being superposed on each other; cutting one of the regions along the edges of the other region (in each of the existing cases, the triangular region was cut along the edges of the rectangular region); carrying over and patching the pieces on the other region; repeating the actions of cutting, carrying over, and patching on the resulting subregions (in one of the cases: 03M - 1st attempt, the performance exhibited no further actions of this kind); terminating the attempt which indicates that the pattern used would lead nowhere or the two regions were not 'the same', or 'equal'. For this type of performance see Appendix H, task 2, photographs 03M - 1st attempt, 05F - 1st and 2nd attempts.

Category 4: The performances in this category were characterized by (a) failing to discover the existed linear congruency between the two regions; (b) dissecting one or each of the regions along the edges of the other regions; and (c) unsuccessful completion of the task. The process used in these performances can be described in the following hand-mind sequence of actions: superposing one of the regions on the other with two sides being superposed along each other; cutting one (or each) region along the edges of the other; carrying over and patching the subregions on the other region; finally, terminating the attempt with no apparent success. For this type of performance, see Appendix H, task 2, photographs 03M - 2nd attempt and 27M - 1st attempt.

<u>Task 3</u>. The performances on this task were classified into one category based on (a) the discovery of the existing linear congruency between two sides of the parallelogram region and the rectangular region; (b) the dissection of one of the regions into two subregions - a trapezoid and a triangular subregion -; and (c) successful completion of the task. The process used can be described in the following sequence of hand-mind actions: superposing one of the regions on the other with the two congruent sides being superposed on each other; cutting one of the regions along the edges of the other; carrying over and patching the pieces on the other region; and attaining a piece-wise congruency. For this type of performances, see Appendix H, all task 3's photographs.

<u>Task 4</u>. The students' performances on this task were classified into two categories.

Category 1: The performances in this category were characterized on the basis of (a) the discovery of the existing angular congruency between two

angles of the trapezoid region and the rectangular region; (b) the dissection of one of the regions into two right trapezoid subregions; and (c) successful completion of the task. The students processed the task in the following chain of hand-mind actions: superposing one of the regions on the other with two congruent angles being superposed on each other; cutting one of the regions along the edges of the other; carrying over and patching the pieces on the other region; with an equidecomposition attained. For this type of performance, see Appendix H, task 4, photographs 03M, 05F, 12M - 1st and 2nd attempts, 19F, 20F, 27M, 30F, 34F, 41F - 1st and 2nd attempts, and 44F.

Category 2: The performances in this category were characterized on the following basis: (a) failing to discover (or not utilizing) the existing angular congruency; (b) the dissection of the rectangular region into three subregions - a right trapezoid region, a rectangular region, and a triangular region -; and (c) successful completion of the task. The process used can be described in the following sequence of hand-mind actions: superposing one of the regions on the other with two bases being superposed along each other; cutting the rectangular region along the edges of the trapezoid region; carrying over and patching the pieces on the other region; with a piece-wise congruency exhibited (see Appendix H, task 5, photographs 46M and 48M).

Note that, not utilizing or missing the angular congruency caused students to have three subregions for the equidecomposition process which could be two subregions otherwise (category 1).

Task 5. The students' performances on this task fell into four categories.

Category 1: The performances of this category were characterized by (a) utilizing the parallelism property between the bases of the trapezoid region throughout the superposition act, that is, one of the trapezoid bases is used for linear superposition; (b) the dissection of one of the regions into three subregions - a right trapezoid, a trapezoid, and a right triangular subregions -; and (c) successful completion of the task. The students' process can be outlined in the following sequence of hand-mind actions: superposing one of the regions on the other with two corresponding bases being superposed along each other; cutting one of the regions along the edges of the other; carrying over and patching the pieces on the other region; repeating the actions of cutting, carrying over, and patching - in case of cutting the trapezoid region -; and finally a congruency by addition exhibited. For examples of this type of performances, see Appendix H, task 5, photographs 03M, 05F, 12M, 20F, 27M, 30F, 34F, and 48M - 2nd attempt.

Category 2: The performances in this category were characterized on the same basis as category 1 except the three resulting subregions were two right trapezoid and a trapezoid subregions. The students processed the task through a chain of hand-mind actions identical to that of category 1. For this type of performances, see Appendix H, task 5, photographs 19F, 41F, and 44F.

Category 3: The characterization for performances included in this category is made on the following basis: (a) utilization of parallelism property for the bases of the trapezoid region; that is, one of the bases is used for linear superposition; (b) the dissection of the trapezoidal regions into more than three subregions; and (c) unsuccessful completion of the task. The process employed in this category can be outlined in the following chain of hand-mind actions: superposing one of the regions on the other with two corresponding bases being superposed along each other; cutting one of the regions (the trapezoid region in the present case) along the edges of the other region; carrying over and patching the pieces on the other region; repeating the actions of cutting, carrying over, and patching on some of the resulted subregions once, twice, ...; terminating the attempt which indicated that the pattern used was leading to a continuous cutting and patching in a cyclic fashion or the two polygonal regions were not 'equal' or 'the same' (see Appendix H, task 5, photograph 46M).

Category 4: For this category, the performances were characterized on the following basis: (a) no utilization of the parallelism of the trapezoid region's bases, that is, none of them were used for linear superposition; (b) the dissection of one of the regions (the rectangular in the present case) into two (or more) subregions; and (c) unsuccessful completion of the task. The process used in this type can be outlined in the following sequence of manual-mental actions: superposing one of the regions on the other with one of the oblique sides of the trapezoid being superposed along the base of the rectangular region; cutting one of the regions (the rectangular region in the present case) along the edges of the other; carrying over and patching the pieces on the other region; terminating the process with no further cutting, carrying over, or patching indicating that the pattern used wouldn't lead to an equidecomposition or the two polygonal regions were not 'equal' or 'the same' (see Appendix H, task 5, photograph 48M - 1st attempt).

Task 6. The students' performances on this task were classified into one category. This category is characterized on the basis of (a) failing to discover the existing linear congruency between a diagonal of the quadrilateral region and a side of the rectangular region; (b) the dissection of one of the regions along the edges of the other; and (c) unsuccessful completion of the task. The process used in this type of performances can be described by the following sequence of (hand-mind) actions: superposing one of the regions on the other with one of the sides of the quadrilateral region and the base of the rectangular region being superposed along each other; cutting one of the regions along the edges of the other; carrying over and patching the resulting pieces on the other region; repeating (no repetition was exhibited in some cases) the actions of cutting, carrying over, and patching on some of the resulted subregions; terminating the attempt which indicated that the pattern used was leading to a continuous cutting and patching in a cyclic fashion or the two polygonal regions were not equidecomposable (see Appendix H, task 6, all the photographs).

Task 7. The performances in this task fell into four categories.

Category 1: The performances in this category are characterized by (a) failing to discover the linear congruency between a diagonal of the rhombus region and a side of the rectangular region; (b) termination of cutting. The students attempted the task through superposing one region on the other with no further action, such as cutting one of the regions along the edges of the other, or performing other cuts. They, after a series of superposing and re-superposing, terminated their actions. This sort of act was performed by students 03M, 20F, 30F, and 48M (see Appendix H, task 7's photographs).

Category 2: The performances in this category were characterized by (a) failing to discover the linear congruency mentioned in category 1; (b) the dissection (which was inaccurate) of one of the regions along the edges of the other into two subregions - a triangular and a right trapezoid subregions; (c) unsuccessful completion of the task. The process used in this type of performances can be described by the following series of hand-mind actions: superposing one of the regions on the other with two corresponding sides being superposed along each other; cutting one region along the edges of the other region; carrying over the pieces and patching them on the other region; terminating the actions, with no further cutting and patching, concluding that the two regions were not 'equal' or 'the same' (see Appendix H, photographs 05F and 19F).

Category 3: The performances in this category were characterized by (a) failing to discover the linear congruency mentioned in category 1; (b) the dissection (which was more accurate than that described in category 2) of the rhombus region into three subregions (a triangular region, a right trapezoid region, and a quadrilateral strip; see Appendix H, photographs 27M and 41F) or the dissection of the rectangular region into two subregions (two right trapezoid regions or a triangular region and a right trapezoid region; see Appendix H, photographs 19F, 34F, and 44F); and (c) unsuccessful completion of the task. The process used in this type of performances is identical to that of category 2.

Category 4: The performances in this category were characterized by: (a) failing to discover the linear congruency mentioned in category 1; (b) the dissection of the rhombus region into more than three subregions in a peculiar pattern (the initial cutting yielded three subregions - a triangular, pentagon, and quadrilateral subregions); and (c) unsuccessful completion of the task. The process used in this type of performances was identical to the process of category 2 except the actions of cutting and patching were repeatedly performed. This lead to continuous actions of cuttings and patchings until the student realized the actions were fruitless, and the performances were terminated (see Appendix H, photographs 12M - Initial cuts, 12M - Final cuts, and 12M - The patching).

Task 8. The performances on this task were classified into four categories.

Category 1: The performances in this category were characterized by (a) failing to utilize the regularity property of the pentagon region, and hence failing to partition it into five congruent triangular regions; (b) the dissection of one of the regions along the edges of the other; and (c) unsuccessful completion of the task. The processes used in these performances can be described by the following sequence of hand-mind actions: superposing one region on the other with two sides being superposed along each othe; cutting one of the regions (the pentagon region in this case) along the edges of the other region; carrying over the pieces and patching them on the other region; repeating the cutting and patching actions (as in the case of student 05F); terminating the performance by concluding that the two regions were not 'equal' or 'the same' (see Appendix H, photographs 05F anf 19F).

Category 2: The performances were characterized by (a) utilizing the regularity property (partitioning the pentagon region into five congruent triangular regions); (b) utilizing the pattern of task 2 on one of the five triangular regions but failing to utilize the midpoint idea in drawing a horizontal line (student 46M failed to utilize properly the perpendicularity idea); (c) unsuccessful completion of the task. The process used in these performances can be described by the following actions: partitioning the pentagon region into five congruent triangular regions; superposing one of the triangular regions on the rectangular region with the two congruent sides being superposed on each other (except 46M); utilizing the pattern of task 2 on the triangular region; carrying over and patching the pieces to cover a rectangular strip; terminating the actions and concluding that the regions were not of equal area, 'equal' or 'the same' (see Appendix H, photographs 34F, 41F, 44F, 46M, and 48M).

Category 3: The performances were characterized by (a) partitioning the pentagon region into five triangular regions; (b) utilizing task 2's pattern using the ideas of midpoint, parallelism, perpendicularity in dissecting one of triangular region into a rectangular region; (c) successful completion of the task. The process used here is identical to the process of category 2 except that students concluded that their performances were leading to a successful equidecomposition (see Appendix H, photographs 20F, 27M, and 30F).

Category 4: The performances were characterized by (a) partitioning of the pentagon region into five congruent triangular regions; (b) the decomposition of one of the triangular regions into a rectangular region by a perpendicular cut; and (c) successful completion of the task. The process used can be described in the following sequence of actions: partitioning the pentagon region into five triangular regions, superposing one of the triangular regions on the rectangular region with the two congruent sides being superposed on each other; decomposing the rectangular region through a perpendicular cut into a rectangular region; carrying over and patching the pieces to cover a sub-rectangular region; concluding that their pattern was leading to a possible equidecomposition (see Appendix H, photographs 03M and 12M).

Task 9. The students' performances fell into one category. They were characterized by (a) the use of manual means; (b) the use of visual means; and (c) successful completion of the task. The students processed the task by: juxtaposing the three rectangular regions; superposing the regions pairwisely on each other with two sides being superposed along each other; repeating these two actions several times, with a correct ordering exhibited.

Task 10. The students' performances were classified into two categories.

Category 1: The performances in this category were characterized by (a) recalling the pattern of task 9; and (b) suggesting that if the three polygonal regions were to be compared, each region then should be changed into a rectangular region. There were no actual cuttings; the performances were considered (potentially) successful however.

Category 2: The performances in this category were characterized by (a) failing to recall the pattern of task 9 for possible utilization; (b) suggesting that an ordering was not possible; and (c) unsuccessful completion of the task.

III. DISCUSSION AND IMPLICATIONS OF THE RESULTS

Taylor (1980) states:

> There is only one way to create a strong and lasting interest in mathematics and that is to make the subject (material) belong to the student. We are interested in that which is ours. This is the ultimate and the only truly natural source of motivation. (p. 51)

On the basis of this study, the students at the junior high school grade eight level appeared highly motivated in manipulative problem solving situations. They seemed relaxed, fully engaged and enthusiastic throughout tasks that were based on more intuitive and less rigorous reasonings.

a. All of the students who were interviewed passed tasks 0, 3, 4, and 9 while there was one failure only on each of the tasks 1 and 5, and two on task 2. On the other hand, none of the students passed tasks 6 and 7 (except an approximate solution for task 6 performed by the student 12M, the highest achiever in class A). Only five students passed task 8 and six passed task 10 (full description of these tasks are included in Chapter III).

Olson (1970) concluded that:

> Certain orientations of line are more difficult to discriminate than others...The discrimination of lines which are oppositely oriented obliques are the most difficult of those considered, followed by those differing in left-right orientation then up-down orientation, with horizontal-vertical alternatives being somewhat easier. (p. 173)

In view of this conclusion, the failure performances on tasks 6 and 7 would seem more likely to result from the fact that in each of tasks

6 and 7, a diagonal of the non-rectangular region should be discriminated as congruent to one of the sides of the rectangular regions involved (see p. 53-56). Thus, the difficulty in 'the child's acquisition of diagonality' would seem to have an analogy in these grade eight students performances.

b. The students' achievement in the sample changed significantly over the geometry subtests (ii), (iv), (v), and (vi). That is, students achieved significant improvement in their perception of interrelating polygonal regions to each other, differentiating polygonal and non-polygonal regions, fractions which belong to the same equivalence classes, and the importance of the role of a chosen and fixed unit in visual comparisons between rational numbers. The findings of no significant change in the students' achievement on the Vocabulary and Area Formulae subtests could be related to the following: (1) the size of the sample was relatively small; it was 58 students, dropped to only 47 students who took both of the two testings, and (2) the last section of the unit which dealt with the area formulae was instructed over one class period - only 50 minutes. In the joint interview, the teachers expressed their concern that one class period was not sufficient and that at least three class periods should have to be reserved for the area formulae section.

c. The students' performances in utilizing the piece-wise congruency operation to prove the Pythagorean Theorem and to show the transitivity property of the operation were impressively successful. The kinds of creative proofs, the students invented were purely theirs.

for a theorem that wasn't known by them, and hence the subject materials once more could be taken as theirs. All the students (except one) proved the special case of the theorem while about two thirds of the students passed the general case. On the transitivity property of the piece-wise congruency operation, out of the 50 students who took the test, 32 of them passed with proofs that were absolutely theirs (Progress Checking Test 2 and 3).

d. The students' overall performance on the unit impressed their teachers who accordingly expressed their willingness to adopt and use the approach in the future. In this respect, the thinking of these junior high school students would seem more productive in situations where more intuitive and less rigorous reasonings are required. Furthermore, students in class B, the 'poor' class, gained more than those in class A, the 'best' class, over the two testing periods on the Geometry Tests. Class A was significantly better on the pretest over the geometry subtests (i), (ii), and (iii). While on the posttest, class A was significantly better only over the subtest (i). Class A's scores increased over the two testing periods however (see Tables X and XII).

The findings of no significant change in the attitude of the students was not a surprise since the period between the two testings was relatively short (less than four weeks).

Wagman (1975) concluded that:

> All through this experiment the children voiced their enthusiasm for the materials. Some refuse to believe that this was mathematics! (p. 109)

It was evident from this study that this conclusion is valid but only for geometry. Although the students' attitude toward geometry was not significantly changed, it still showed a positive tendency, while regarding mathematics, a negative tendency was evident, especially in class B (Table I and II). It would seem, therefore that student did not actually believe that geometry is really mathematics!

The following implications can be drawn from the findings:

1. Manipulative materials of the kind similar to the piece-wise congruency unit not only would increase motivation and interest for geometry but through physical comparisons of polygonal regions based on cut-and-cover operation would be a natural way to introduce area concept. It would then seem appropriate to introduce area as a real-valued function later.

2. An examination of the photographs of the students' original performances, contained in Appendix H and the related findings would suggest the following: The evaluation tasks package could be used, partially or totally depending on Piagetian children's cognitive stages, as a tool to detect the formation of rules which the child exhibits, and which generate a pattern to be followed, in responding to a set of stimuli (Milligan, 1979).

> A major problem in the study of cognitive development is the use of verbal materials. Cognitive abilities such as the acquisition of concepts using verbal materials are confounded with verbal fluency. Non-verbal materials or procedures are required. (Milligan, P. 204)

3. Teaching aids such as magnetic boards and decomposable mag-

netized cardboard models, in teaching the derivations of area formulae seemed highly motivating. The students appeared enthusiastic in using the boards and changing regions into others.

4. On the basis of the findings, inservice sessions covering subject materials similar to that included in the unit would seem to be necessary. Both, teacher A and teacher B stressed this point.

5. Paper folding seemed to be used by many students throughout their attempts of the activities in the classroom. Thus, a paper folding material similar to Olson's (1975) work would seem to be a useful prerequisite and complementary to subject materials similar to those included in the unit.

6. Based on the findings of this study, it would seem that items of the type used in the unit can be employed by teachers for (a) transfer of learned properties of one measure system to another measure system (length, angular, area, and volume measure systems); (b) utilizing the motions of slide, turn, and flip; (c) diagnosing the type of errors in students' thinking in relation to key ideas and theorems in geometry; and (d) introducing some elementary mathematical analysis concepts in concrete settings for later abstraction.

7. It would seem appropriate to pay more attention to the statement of Osborne (1976):

> Measure is ubiquitous. Measure concepts surround the learner. The learner uses an understanding of measure to quantify and interpret his or her world. This understanding provides the base for instruction for many new mathematical and scientific concepts. As a consequence, the acquisition of an understanding of measure is of fundamental importance to the learner. (p. 33)

8. The findings of this study are highly supportive to the conviction indicated in the following statement made by Vance and Kieren (1971):

> The concrete materials serve not only to create interest and motivate learning but to provide a real-world setting for the problem to be solved or the concept to be investigated. (P. 586)

IV. RECOMMENDATIONS FOR FURTHER RESEARCH

Primarily, the present study investigated the way in which students in grade eight understand the concept of area, the effect of concrete operational activities introduced through the unit and the evaluation tasks on the students' performance, and patterns students might elicit through spatial problem solving situations.

Many studies are suggested as a result of this investigation. It is recommended that a study similar to this one be conducted such that a larger sample be used and that more class periods be given to the section of the unit on area formulae and their derivations. Moreover, it would seem that studies of this type could be undertaken with students at all levels, from Elementary School to Junior High School to Senior High School and perhaps even to beginning University. The main objective of such 'stretched enterprise' could be to determine (a) properties of stages through which the students' perception of area evolves, (b) the development of students' perception of properties of polygonal regions and their interrelationships in view of Van Hiele Levels of Thought Development in Geometry discussed in Chapter III, (c) the role of the piece-wise congruency operation in revealing how students appear to use transfer over various stages of cognitive development (in Piaget's terms), (d) what effect the instruction of a unit similar to the piece-wise congruency unit (Appendix A) can cause on the Van Hiele's levels of thought development in geometry, and (e) the students' patterns formation in responding to various stimuli (piece-wise congruency tasks) as described by Milligan (1979).

A unit on prisms based on the present unit and Cavalieri's Principle could be constructed such that the operation of equidecomposing prisms of all kinds to rectangular parallelopipes (right rectangular prisms) and hence to each other would be an extension of the equidecomposability operation. This extension, however, is restricted to the prisms alone when it comes to the problem of whether or not two polyhedra of equal volumes are equidecomposable (Boltyanskii, 1963, p. 37). Accordingly, an analogous study for Senior and perhaps late Junior high school levels could be conducted with the possibility of determining relatively similar main objectives. However, there would be some difficulties in preparing the materials.

EPILOGUE

For the benefit of those who are interested in continuing the course of research reported in this study, it would be useful (perhaps) to suggest certain possible organizations of the tasks used in this study. Then, further potential information and improvements to this study might have produced more valid and reliable results should the proposed organizations be used.

The investigative activities number 1, 2, 3, 4, 5, and 6, that are included in the unit -Appendix A-, can be organized in the following six settings: One way to introduce them is by preparing a pair of cardboard models for the two regions in each of the activeties above; provide each pair of the cardboard models to the students on an individual basis (or in small groups); and ask the student to seek for a decomposition (transform or change), if any, of one of the two cardboard regions into the other using scissors and straightedges. A second way is to present the cardboard model for the non-rectangular region in each of the pairs and ask the student to investigate whether or not the cardboard region is decomposable into a rectangular region. One can starts in the converse order (a third way) of the second setting to present the rectangular cardboard model and ask the student to try to transform it into the corresponding non-rectangular region. A fourth way is to provide each student with tracing paper, scissors, and a sheet of paper on which the two regions of one of the investigative activities are drawn; then ask the student to trace and cut out copies of the two regions and to try to seek for a decomposition of one copy into the other via cutting and covering. Similarly, a fifth way analogous to the second way can be utilized with a single figure of each of the non-rectangular region is drawn on a single sheet of paper. And finally, a sixth way is the converse of the fifth one where the rectangular region in each pair is drawn on a single sheet of paper. The procedures for the last two methods are identical to those of the second and the third methods respectively. In the last three settings, folding paper would be a helpful aid especially in dividing a rectangular strip into equal parts.

A combination of two (or more) of the six methods can be adopted (as is the case in this study). However, the choice of a particular method depends on the level of the child's cognitive development (in Piaget's terms).

Courant and Robbins (1958) have stated that:

> Fortunately, creative minds forget dogmatic philosophical beliefs whenever adherence to them would impede constructive achievement. For scholars and layman alike it is not philosophy but active experience in mathematics itself that alone can answer the question: What is mathematics?

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APPENDICES

APPENDIX A

(STUDENT'S BOOKLET)

THE UNIT

LABORATORY INVESTIGATIONS

IN

GEOMETRY

A PIECE - WISE CONGRUENCY APPROACH*

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PART ONE

DEFINITIONS

Definition

A triangular region is a plane region consisting of a triangle and its interior (Fig. 1).

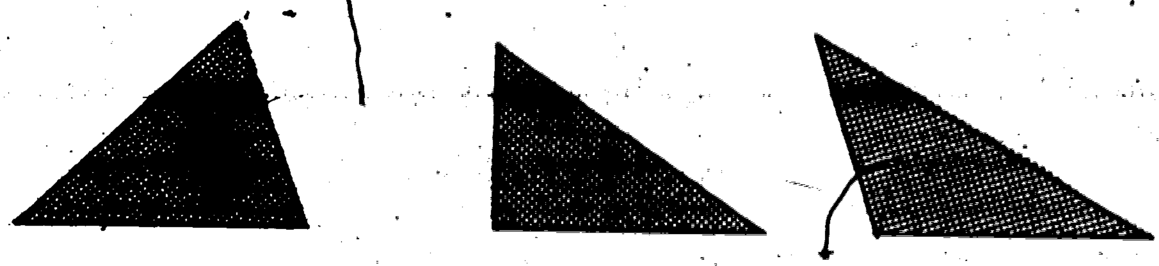
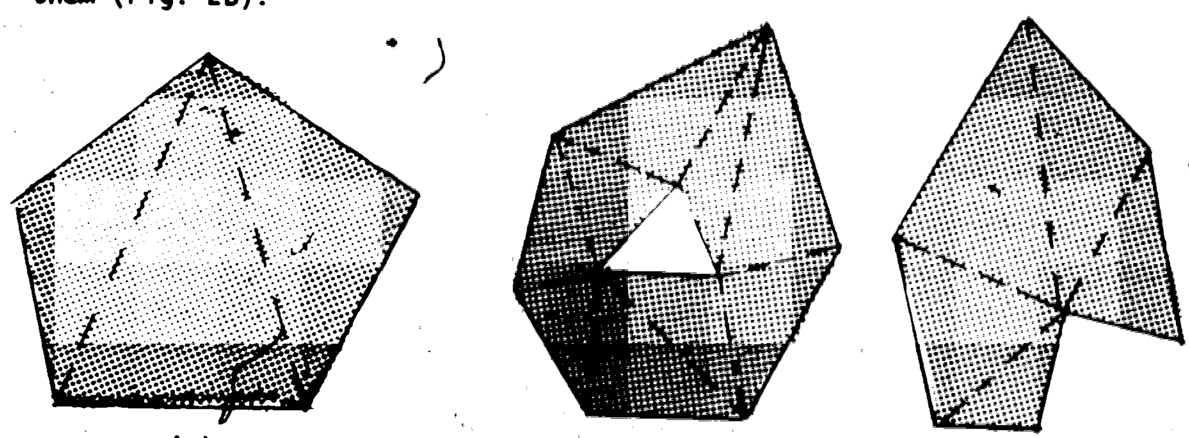


Fig. 1

Definition

A polygonal region is any region in a plane such that it can be decomposed into a finite number of triangular regions (Fig. 2).

Note that figures examplifying polygonal regions may have holes in them (Fig. 2b).



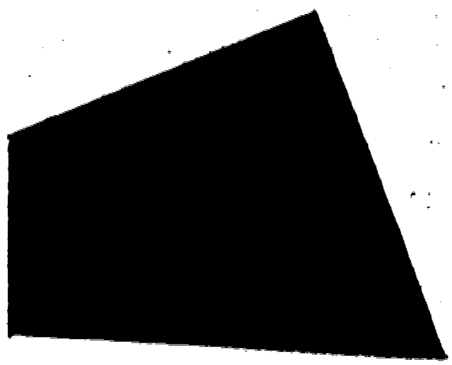
(a) (b) (c)

Fig. 2

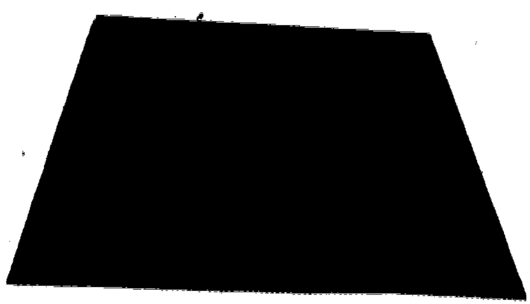
Exercise Set 1

Show that each region below is polygonal by triangulating each one using the definition of a polygonal region.

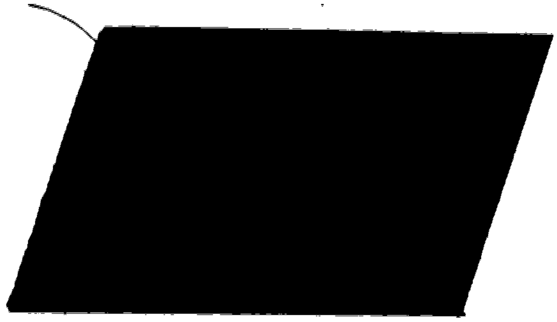
(1)



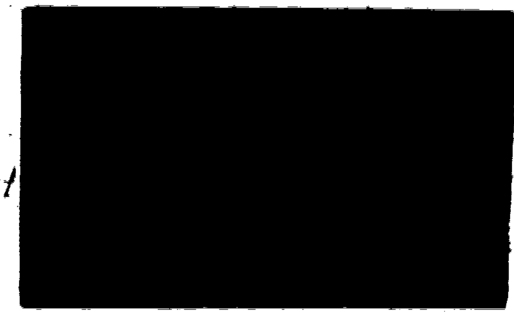
(2)



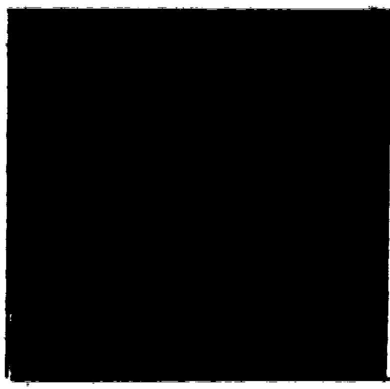
(3)



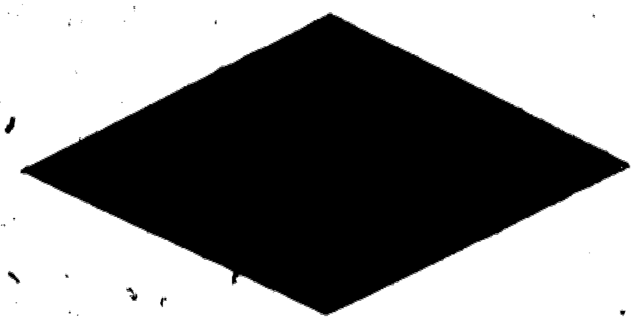
(4)



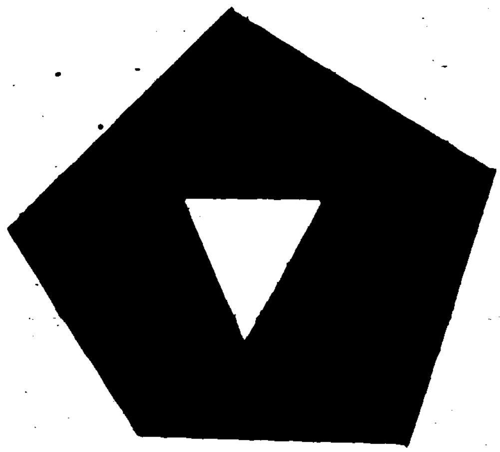
(5)



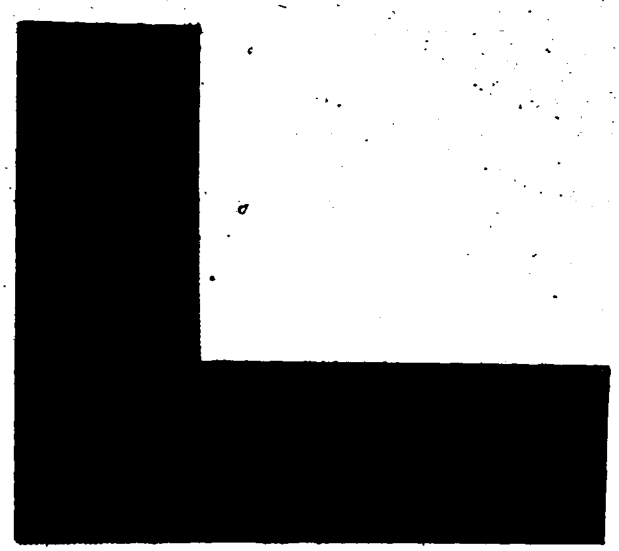
(6)



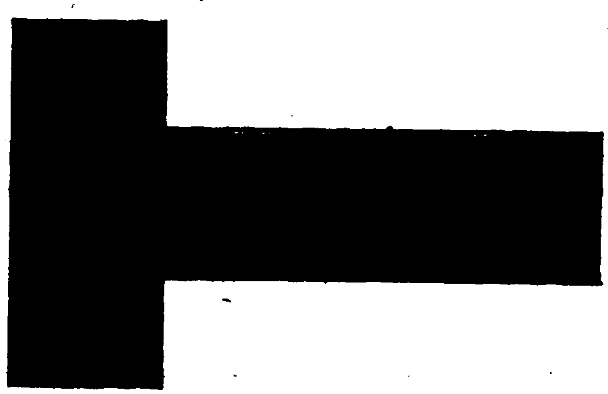
(7)



(8)



(9)



HOMEWORKE

Exercise Set 2

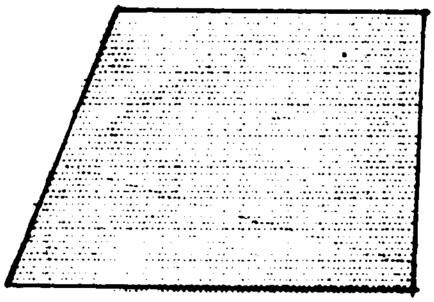
The following is a series of regions in a plane. Show whether or not each region is polygonal. Try to use as less number of triangular regions as possible in each case of decomposition. As well, give reasons for your responses.

Notice in particular that a polygonal region may have one, two, or more holes in it.

N.B.

p.r. stands for polygonal region.

(1)

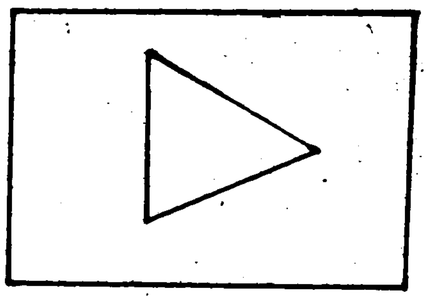


1) p.r.

2) Not p.r.

Why?

(2)

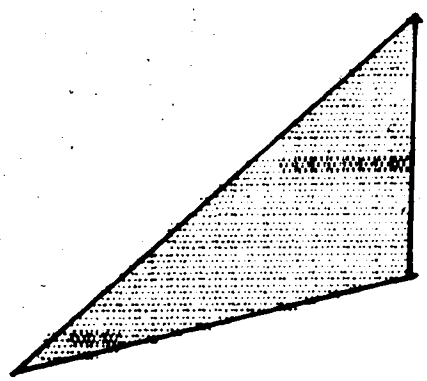


1) p.r.

2) Not p.r.

Why?

(3)

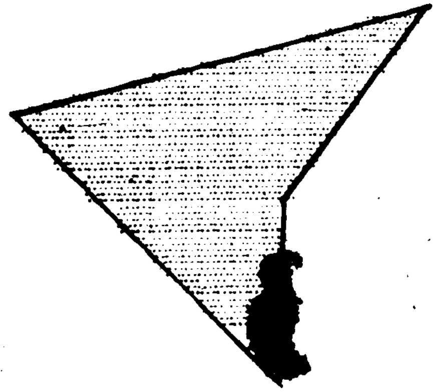


1) p.r.

2) Not p.r.

Why?

(4)

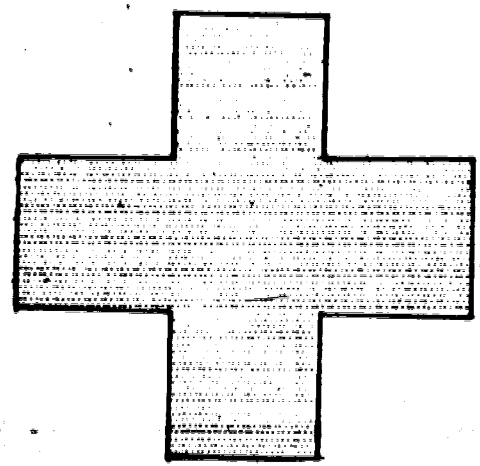


1) p.r.

2) Not p.r.

Why?

(5)

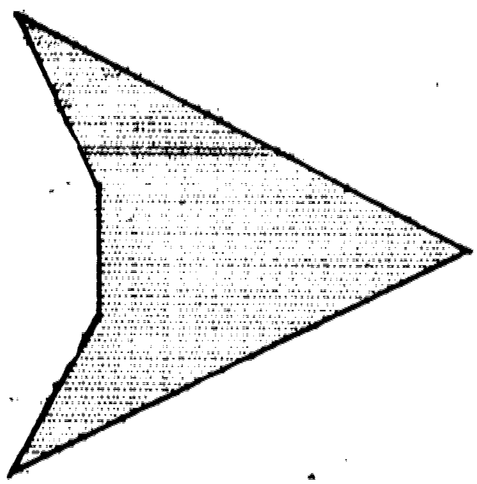


1) p.r.

2) Not p.r.

Why?

(6)

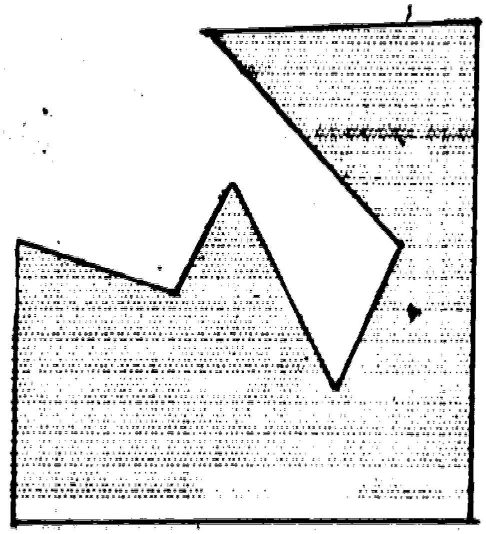


1) p.r.

2) Not p.r.

Why?

(7)

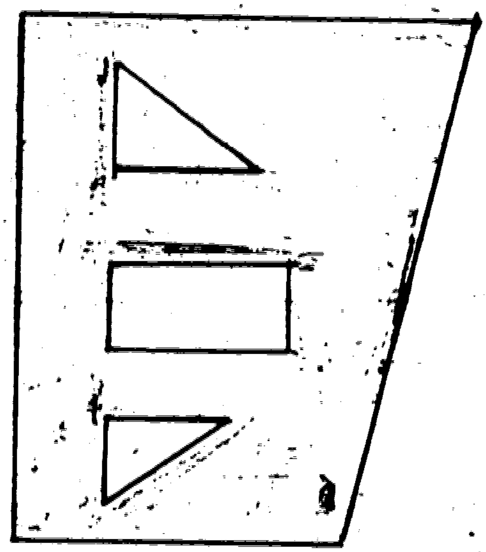


1) p.r.

2) Not p.r.

Why?

(8)

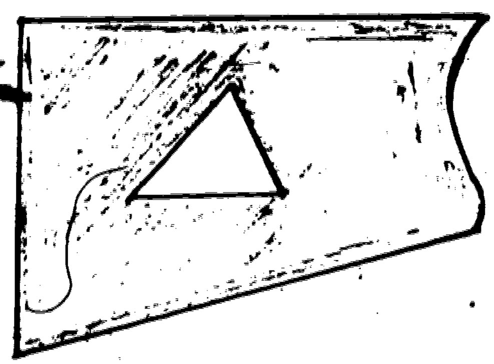


1) p.r.

2) Not p.r.

Why?

(9)

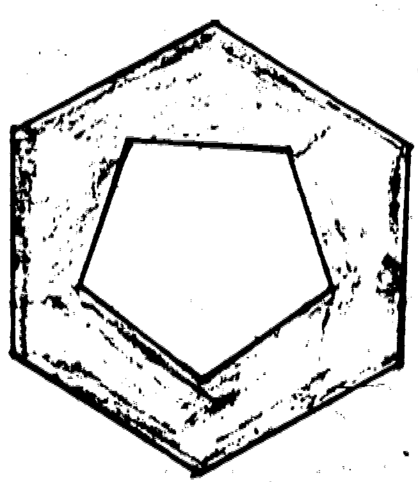


1) p.r.

2) Not p.r.

Why?

(10)



1) p.r.

2) Not p.r.

Why?

PART TWO

LABORATORY INVESTIGATIONS

NOTICE

THIS IS NOT A TEST! ... THERE ARE NO WRONG ANSWERS! HOWEVER, WE WOULD LIKE TO SEE THE APPROACH YOU USE IN ARRIVING AT YOUR SOLUTION TO EACH TASK. THEREFORE, IT IS ESSENTIAL TO MAKE YOUR ANSWERS AS CLEAR AS POSSIBLE.

TRIANGULAR REGIONS

Investigation 1a:

Below (Figure 1) are two polygonal regions, a triangular region A and a rectangular region B.

Trace and cut out a copy for each of them.
Now, you are required to find an answer to the following question:

ARE REGION A AND REGION B OF EQUAL SIZE? HOW DO YOU KNOW?

(you may cut out as many copies as you may need).

Try to make a conclusion out of this activity.

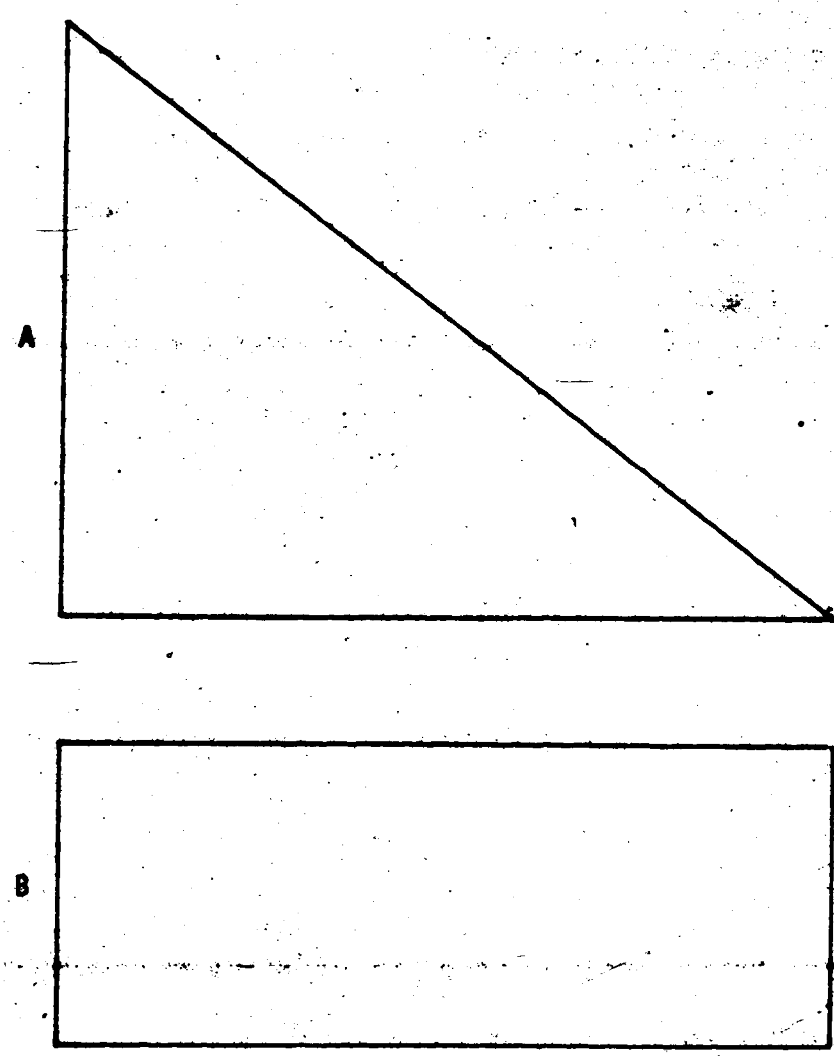


Fig. 1

Investigation 1b

Below are two polygonal regions, a triangular region A and a rectangular region B (Figure 2).

Trace and cut out a copy for each of them.

You are required now to look for an answer to the following question:

ARE REGION A AND REGION B OF EQUAL SIZE? HOW DO YOU KNOW?

Try to make a conclusion out of this activity.

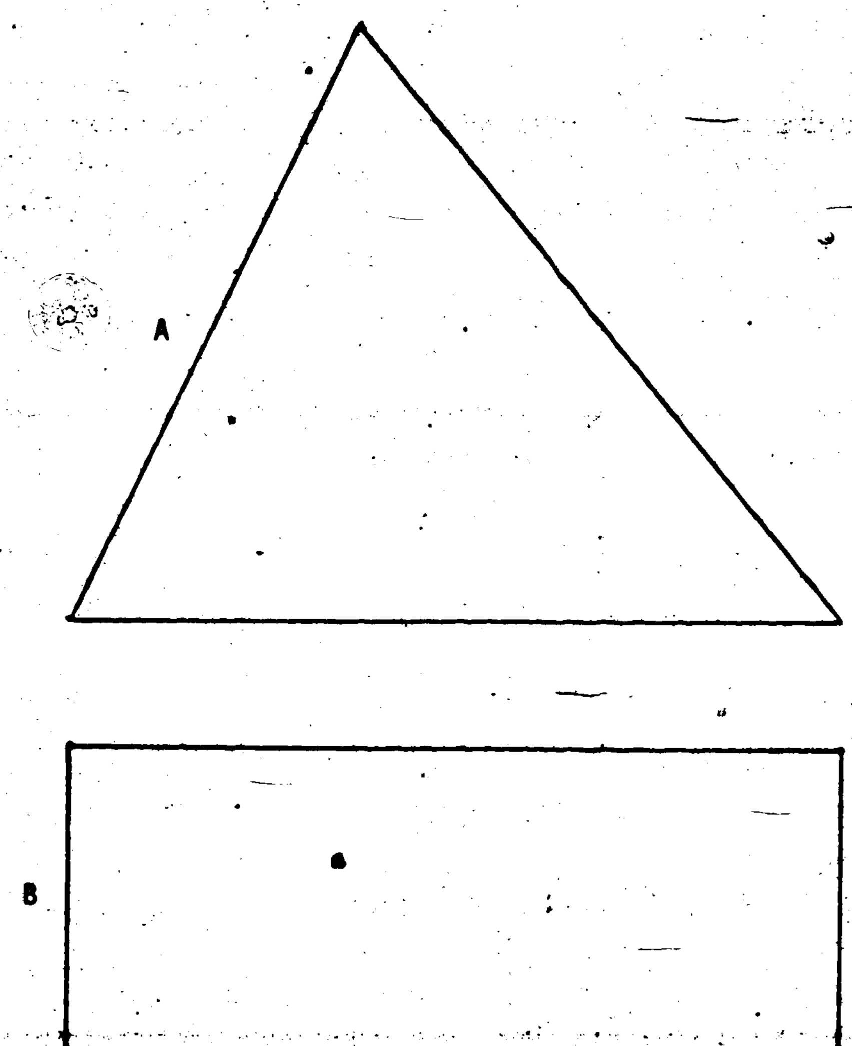


Fig. 2

Investigation 1c:

Given a triangular region A and a rectangular region B (Figure 3).

Trace and cut out a copy for each of them.

You are now required to find an answer to the following question:

ARE REGION A AND REGION B OF EQUAL SIZE? HOW DO YOU KNOW?

Try to make a conclusion out of this activity.

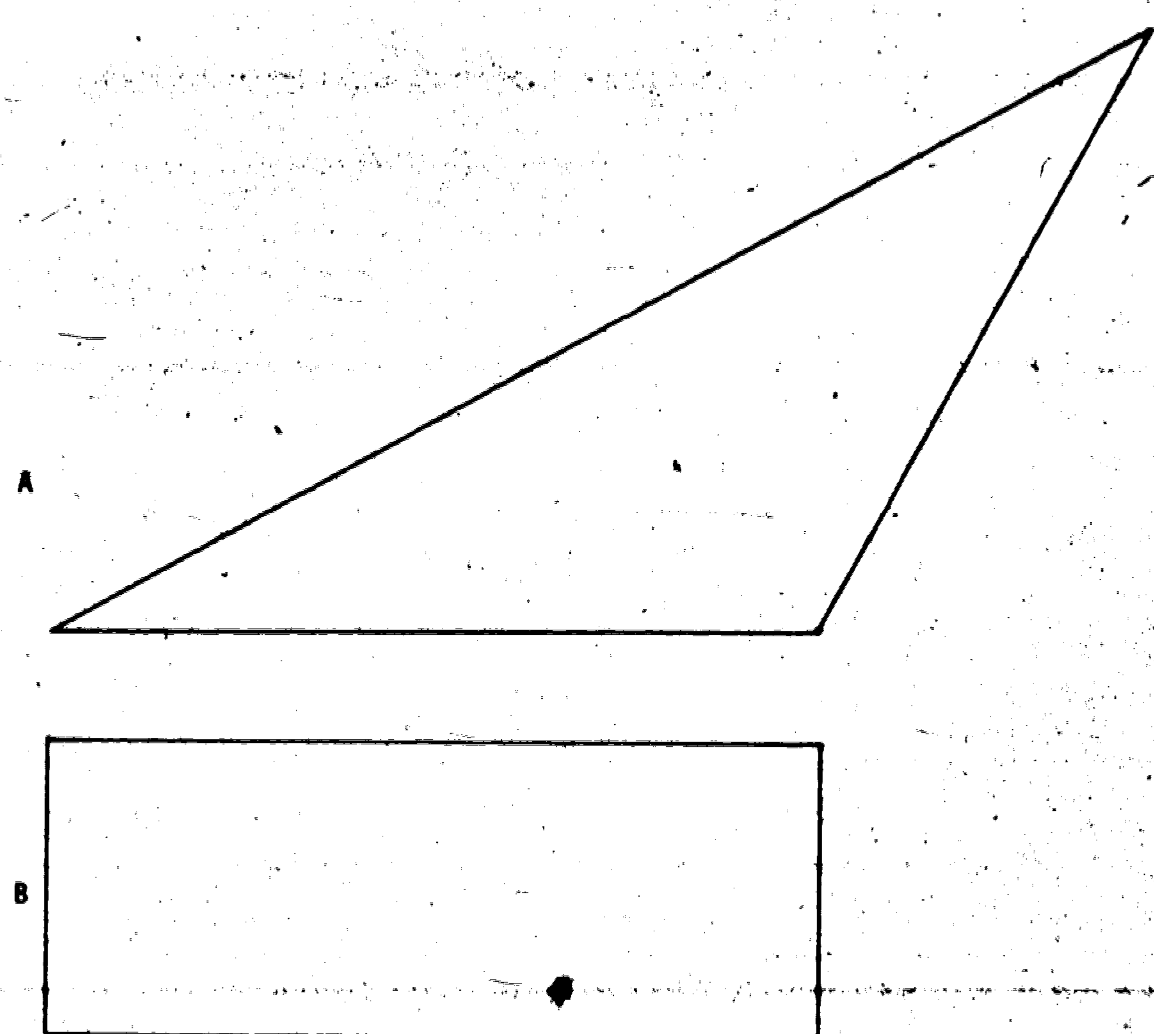


Fig. 3

Investigation 1d:

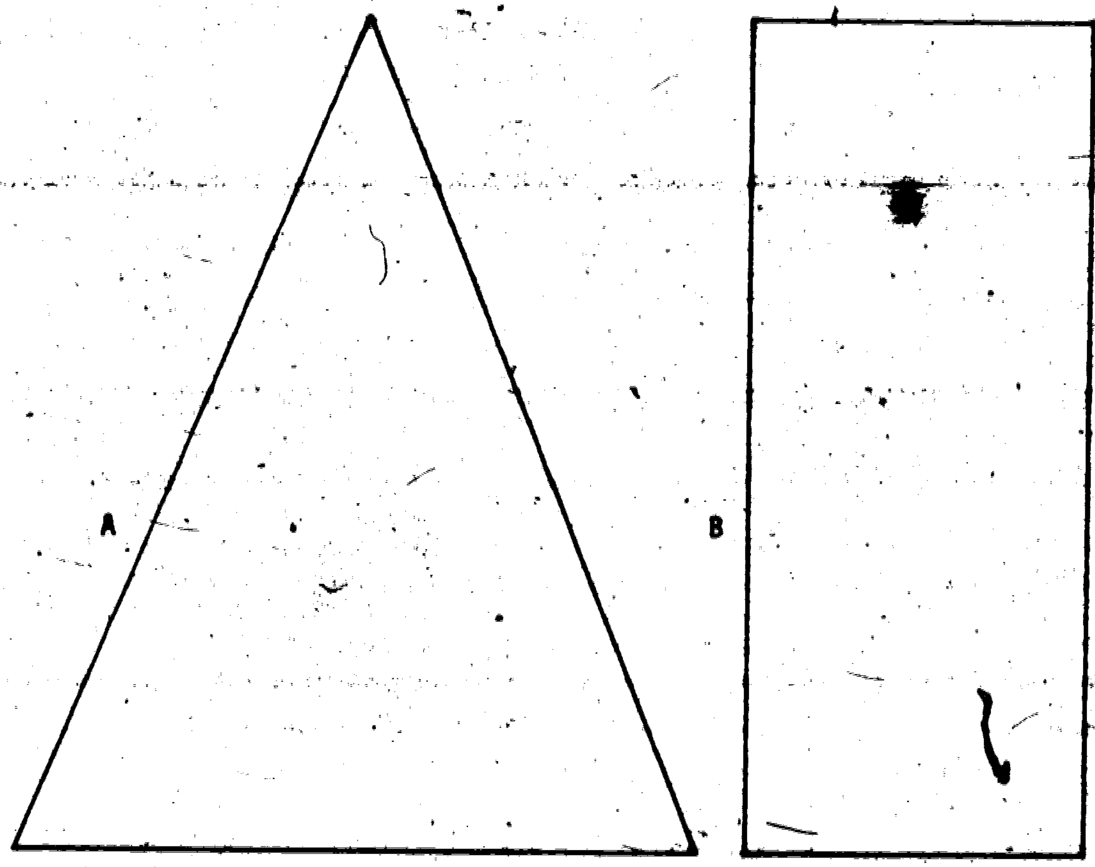


Fig. 4

Above, in Figure 4, are two polygonal regions, a triangular region A and a rectangular region B.

Trace and cut out a copy for each of them.

Now, you are required to look for an answer to the following question:

ARE REGION A AND REGION B OF EQUAL SIZE? HOW DO YOU KNOW?

Try to make a conclusion out of this activity.

<u>Investigation 1e</u>:

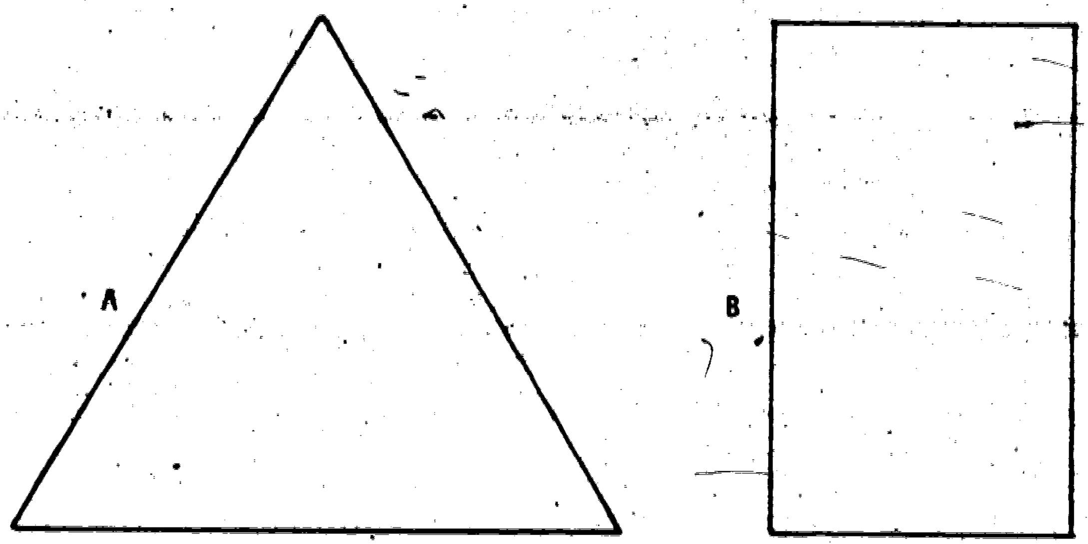


Fig. 5

In Figure 5 above are two polygonal regions, a triangular region A and a rectangular region B.

Trace and cut out a copy for each of them.

You are required to find an answer to the following question:

ARE REGION A AND REGION B OF EQUAL SIZE? HOW DO YOU KNOW?

Try to make a conclusion based on your findings.

QUADRILATERAL REGIONS: GENERAL CASE

Investigation 2:

In Figure 6 below are two polygonal regions, a quadrilateral region A (with no parallel opposite sides) and a rectangular region B.

Trace and cut out a copy for each of them.

You are now required to look for an answer to the following question:

ARE REGION A AND REGION B OF EQUAL SIZE? HOW DO YOU KNOW?

Try to make a conclusion out of this activity.

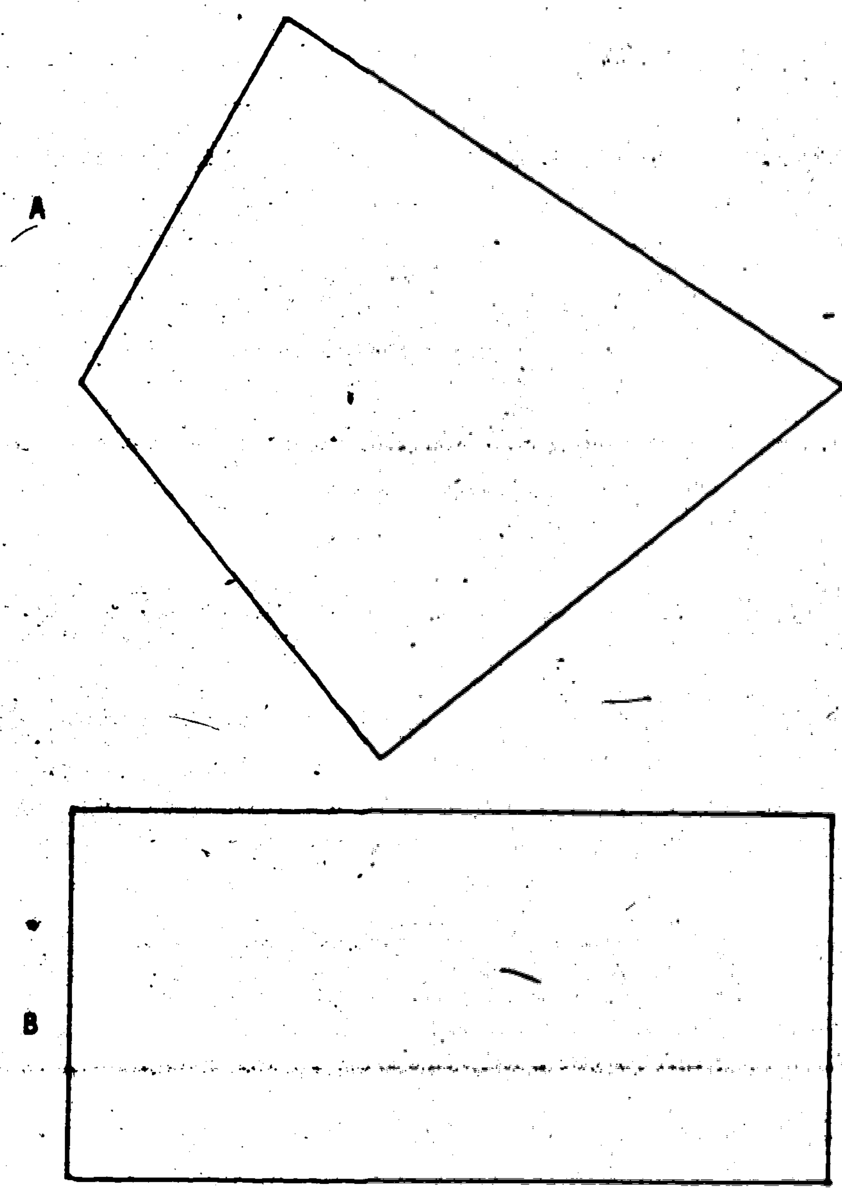


Fig. 6

QUADRILATERAL REGIONS: SPECIAL CASES

TRAPEZOID REGIONS

Investigation 3a:

Below are two polygonal regions, a trapezoid region A and a rectangular region B (Figure 7).

Trace and cut out a copy for each of them.

Now, you are required to find an answer to the following question:

ARE REGION A AND REGION B OF EQUAL SIZE? HOW DO YOU KNOW?

Try to make a conclusion out of this activity.

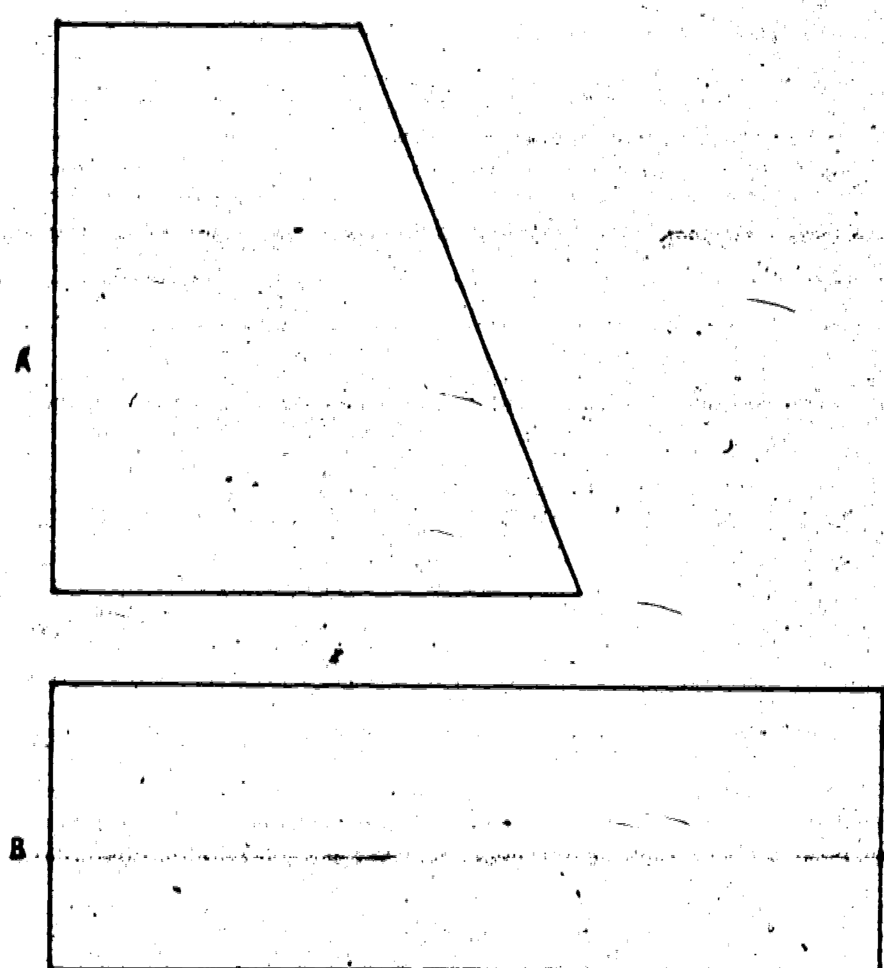


Fig. 7

Investigation 3b:

Below are two polygonal regions, a trapezoid region A and a rectangular region B (Figure 8).

Trace and cut out a copy for each of them.

Now, you are required to find an answer to the following question:

ARE REGION A AND REGION B OF EQUAL SIZE? HOW DO YOU KNOW?

Try to make a conclusion out of this activity.

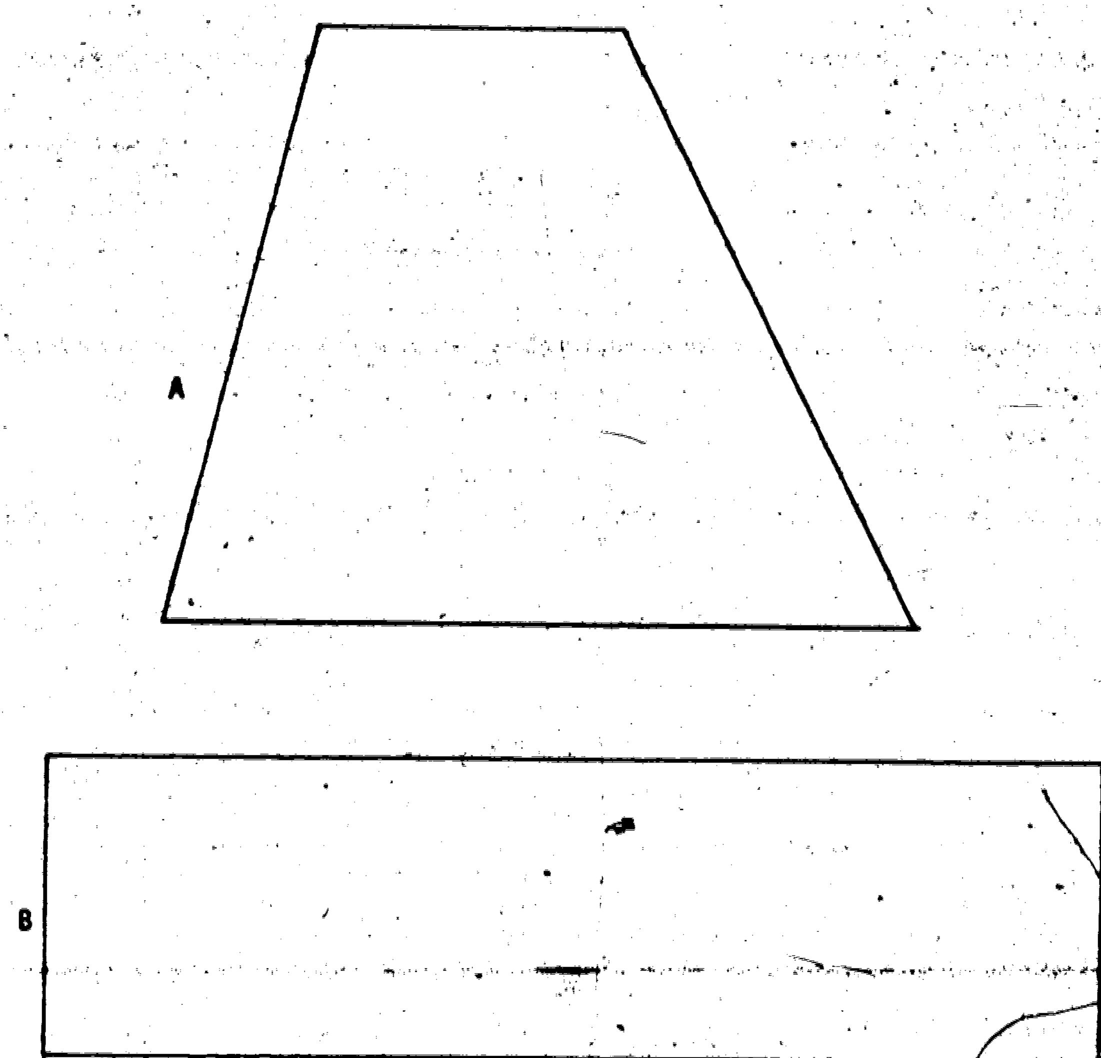


Fig. 8

Investigation 3c:

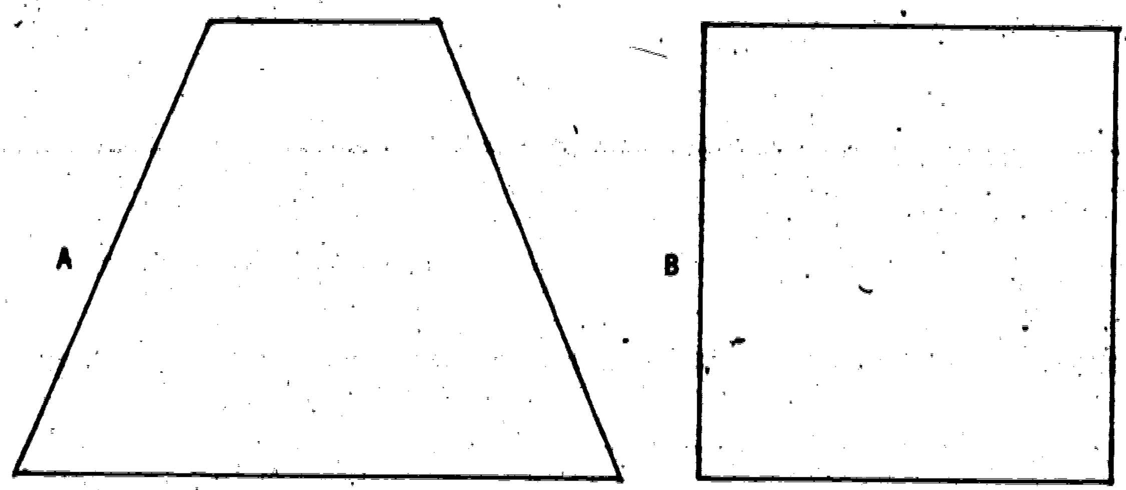


Fig. 9

Above (Figure 9) are two polygonal regions, a trapezoid region A and a rectangular region B.

Trace and cut out a copy for each of them.

You are required now to find an answer to the following question:

ARE REGION A AND REGION B OF EQUAL SIZE? HOW DO YOU KNOW?

Try to make a conclusion out of this activity.

PARALLELOGRAM REGIONS

Investigation 4:

Below are two polygonal regions, a parallelogram region A and a rectangular region B (Figure 10).

Trace and cut out a copy for each of them.

Now, you are required to look for an answer to the following question:

ARE REGION A AND REGION B OF EQUAL AREA? HOW DO YOU KNOW?

Try to make a conclusion out of this activity.

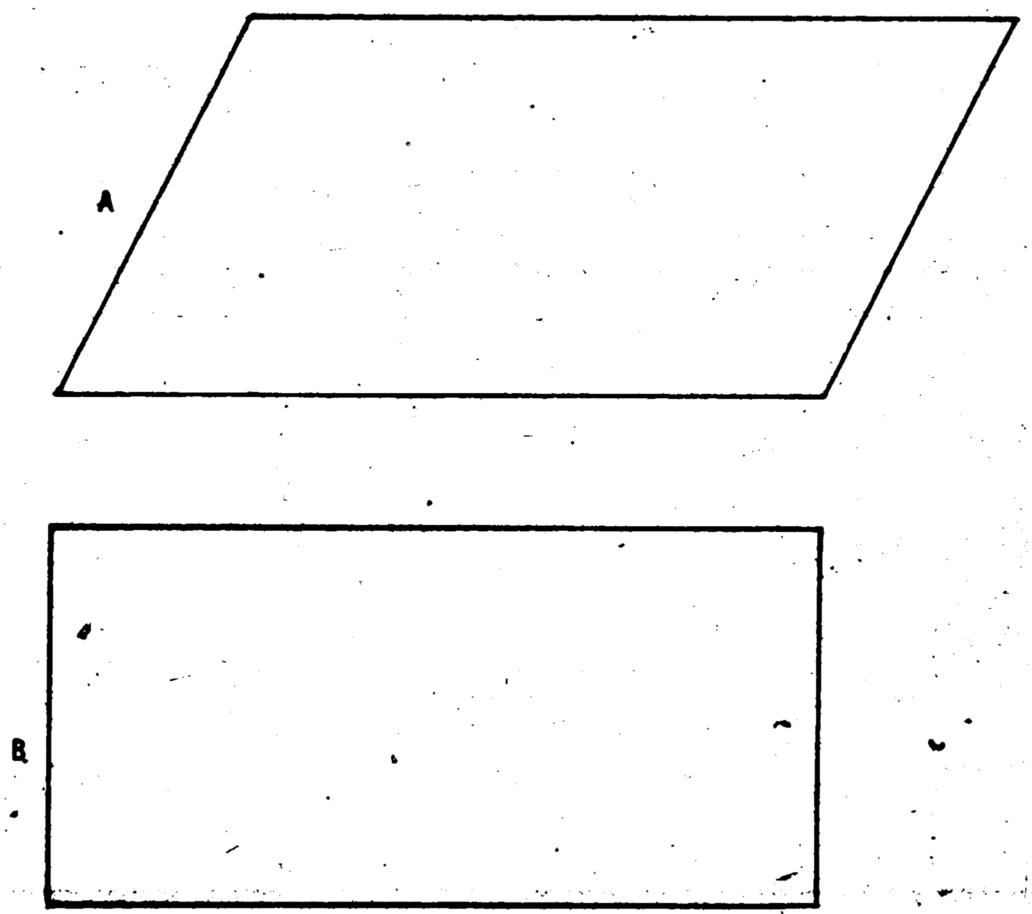


Fig. 10

RHOMBUS REGIONS

Investigation 5:

In Figure 11 below are two polygonal regions, a rhombus region A and a rectangular region B.

Trace and cut out a copy for each of them.

Now, you are required to look for an answer to the following question:

ARE REGION A AND REGION B OF EQUAL AREA? HOW DO YOU KNOW?

Try to make a conclusion out of this activity.

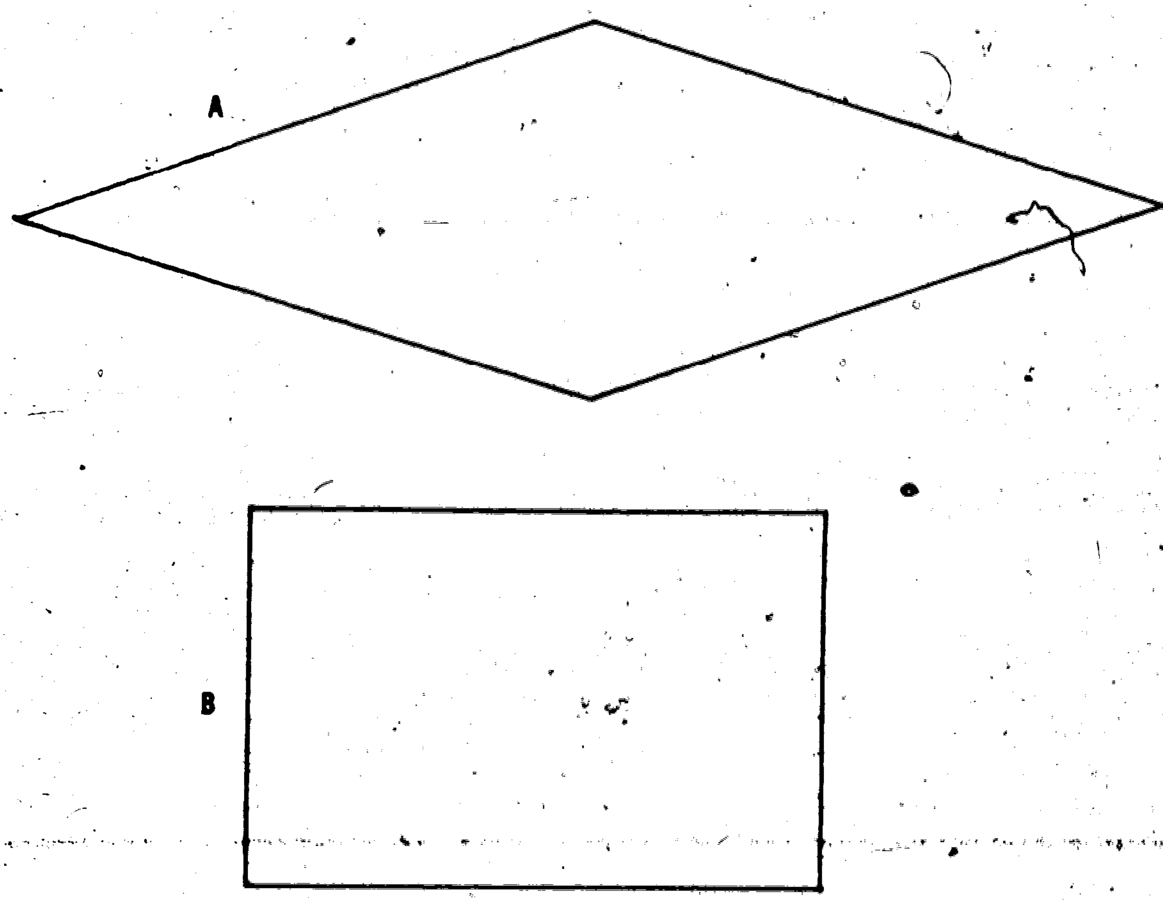


Fig. 11

N-GON REGIONS: N $\geq$ 5

REGULAR PENTAGON REGIONS

Investigation 6:

In Figures 12a, 12b, and 12c below are three polygonal regions, a regular pentagon region A and two rectangular regions B and C.

Trace and cut out a copy for each of them.
Now, try to find an answer to the following questions:

ARE REGION A AND REGION B OF EQUAL SIZE? HOW DO YOU KNOW?
ARE REGION A AND REGION C OF EQUAL SIZE? HOW DO YOU KNOW?

Try to make a conclusion out of this activity.

A

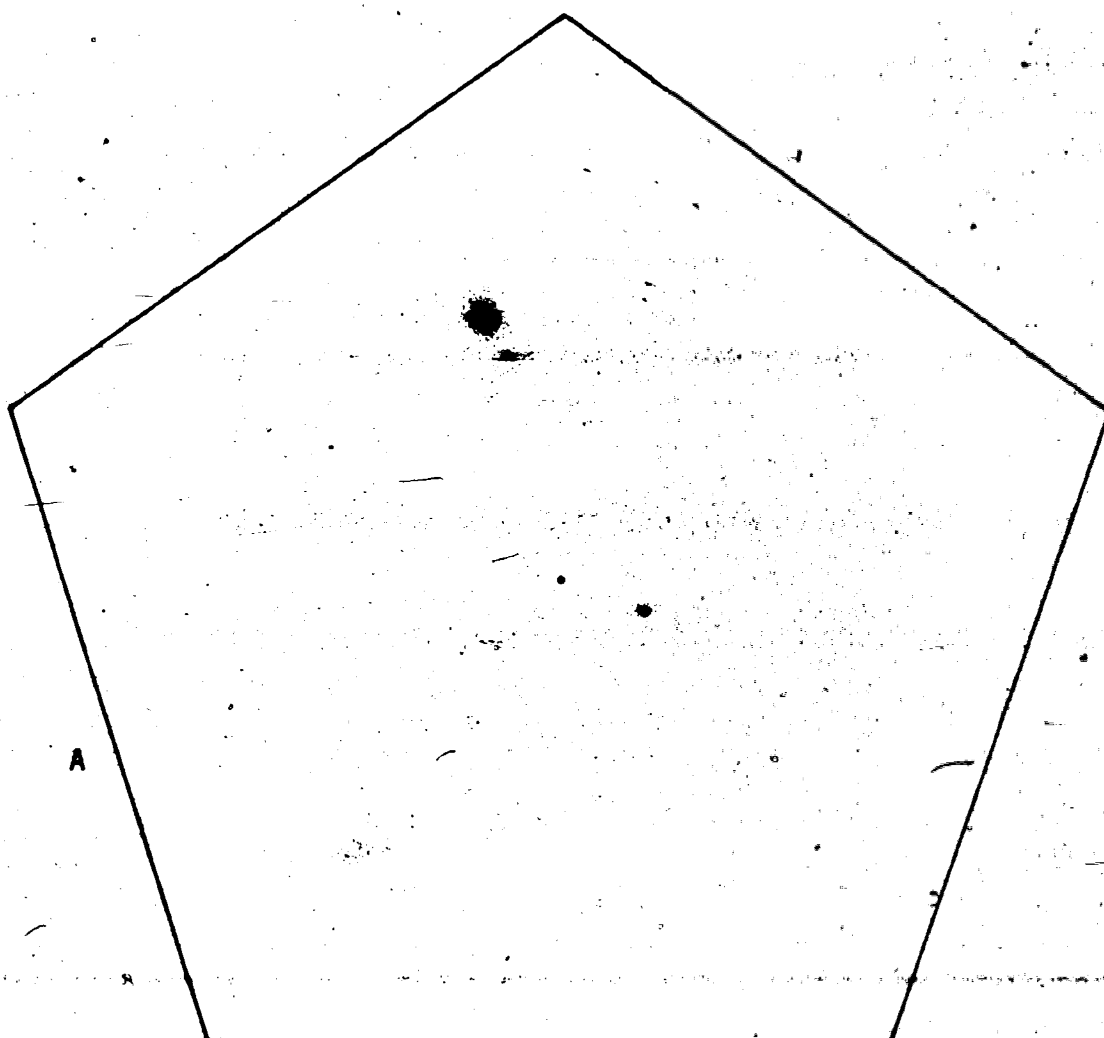


Fig. 12a

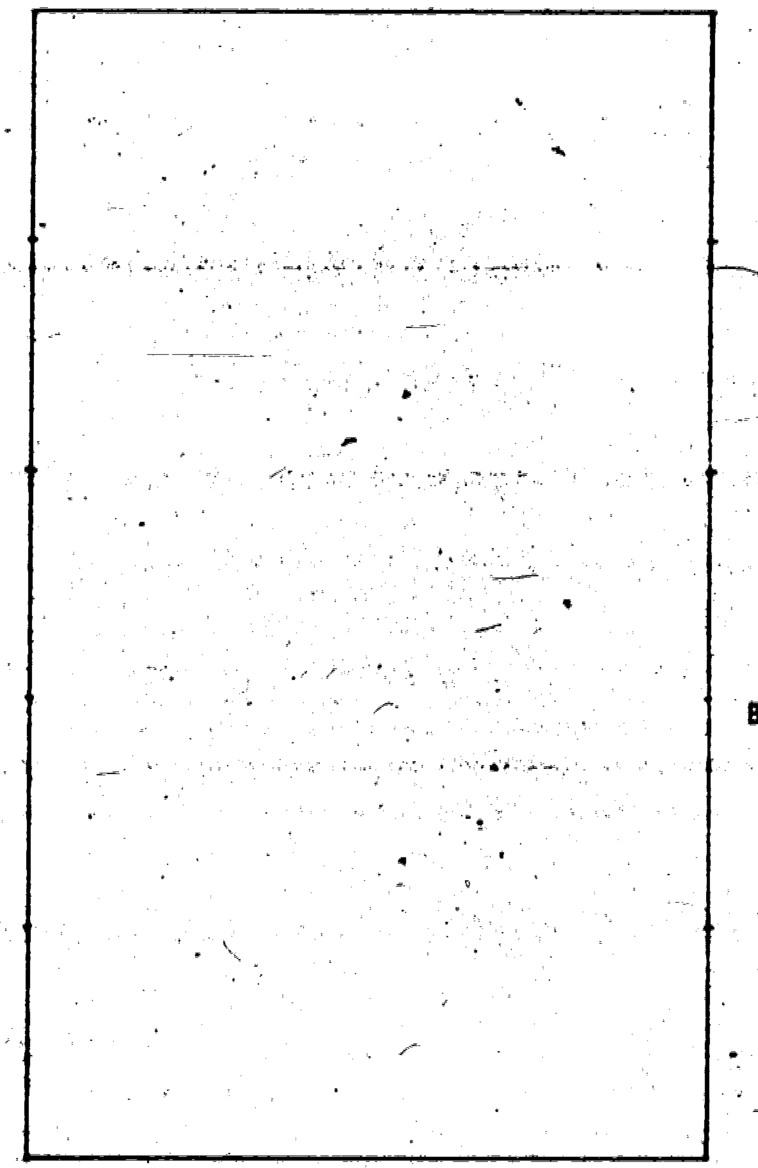


Fig. 12b

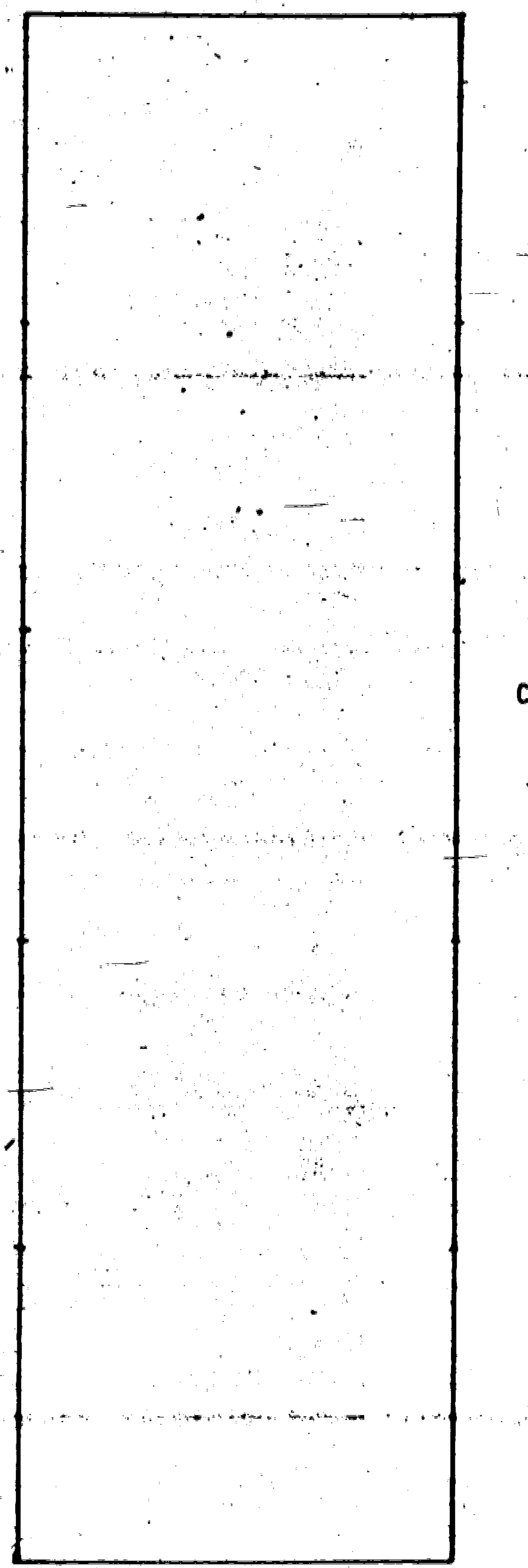


Fig. 12c

REGULAR HEXAGON REGIONS

Investigation 7:

In Figures 13a, 13b, and 13c below are three polygonal regions, a regular hexagon region A and two rectangular regions B and C.

Trace and cut out a copy for each of them.

Try now to find an answer to each of the following questions:

ARE REGION A AND REGION B OF EQUAL SIZE? HOW DO YOU KNOW?
ARE REGION A AND REGION C OF EQUAL SIZE? HOW DO YOU KNOW?

Try to make a conclusion out of this activity.

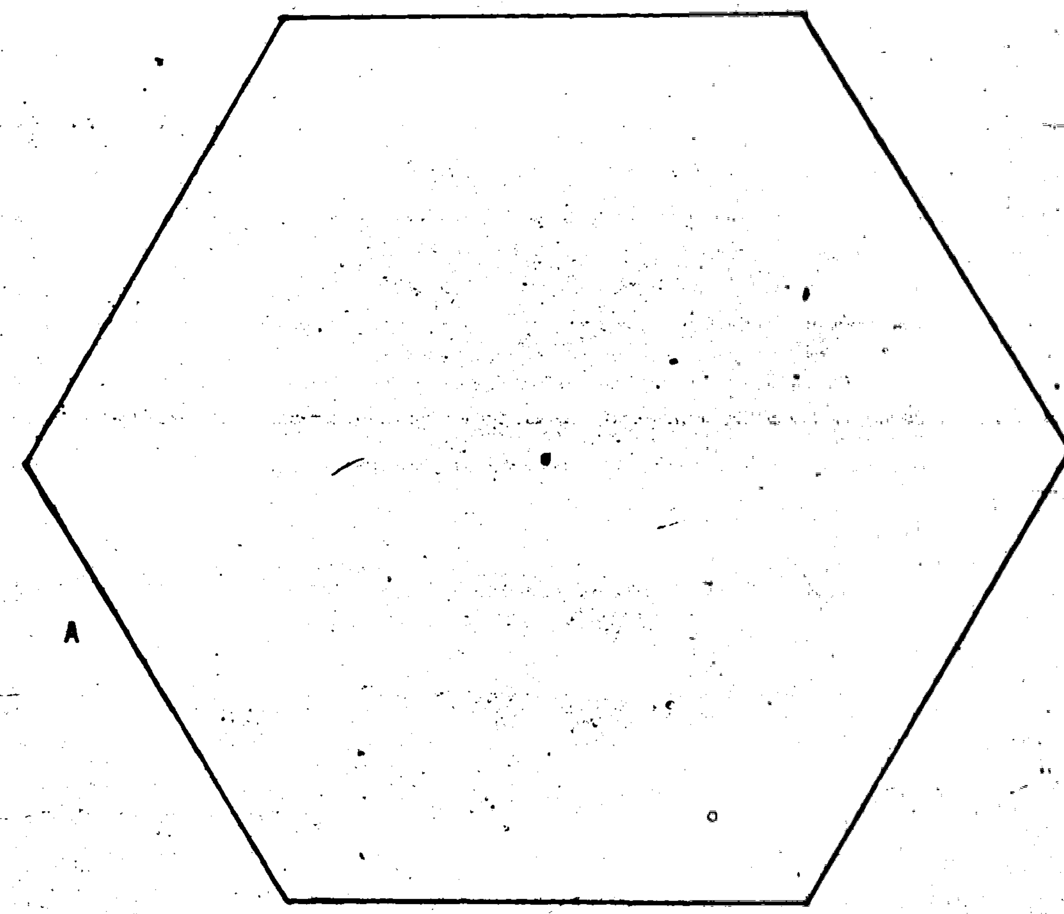


Fig. 13a

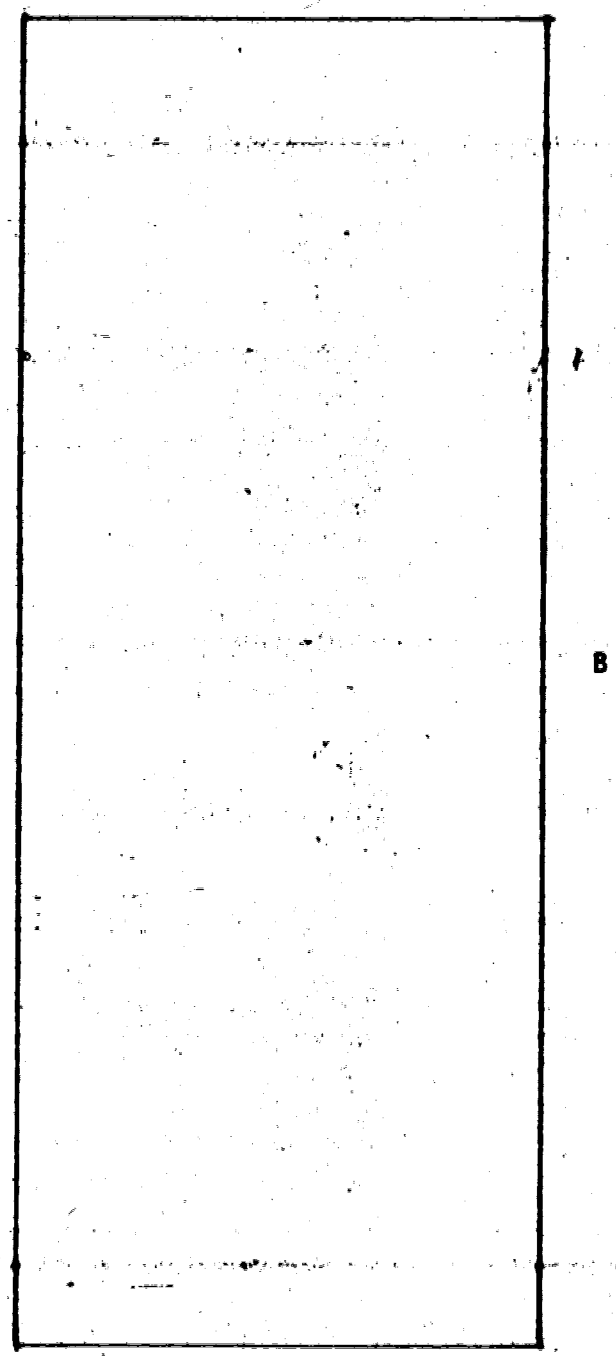


Fig. 13b

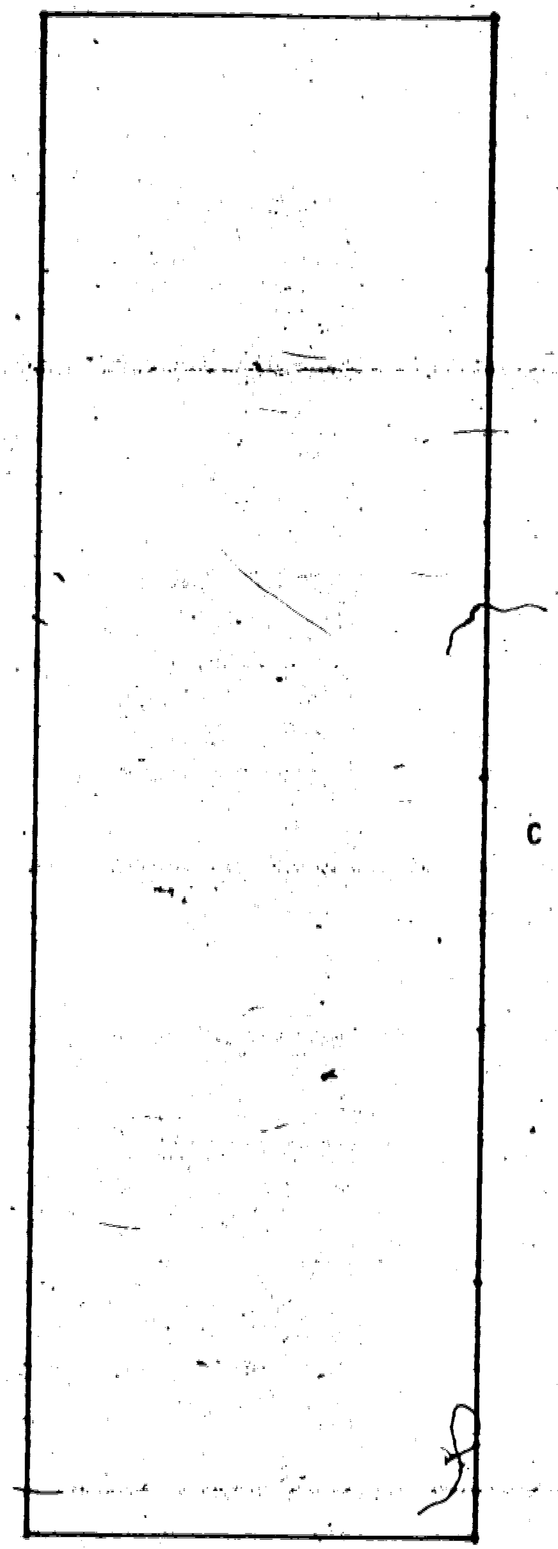


Fig. 13c

REGULAR HEPTAGON REGIONS

Investigation 8:

In Figures 14a, 14b, and 14c below are three polygonal regions, a regular heptagon region A and two rectangular regions B and C.

Trace and cut out a copy for each of them.

You are required to try to find an answer to each of the following questions:

ARE REGION A AND REGION B OF EQUAL SIZE? HOW DO YOU KNOW?

ARE REGION A AND REGION C OF EQUAL SIZE? HOW DO YOU KNOW?

Try to make a conclusion out of this activity.

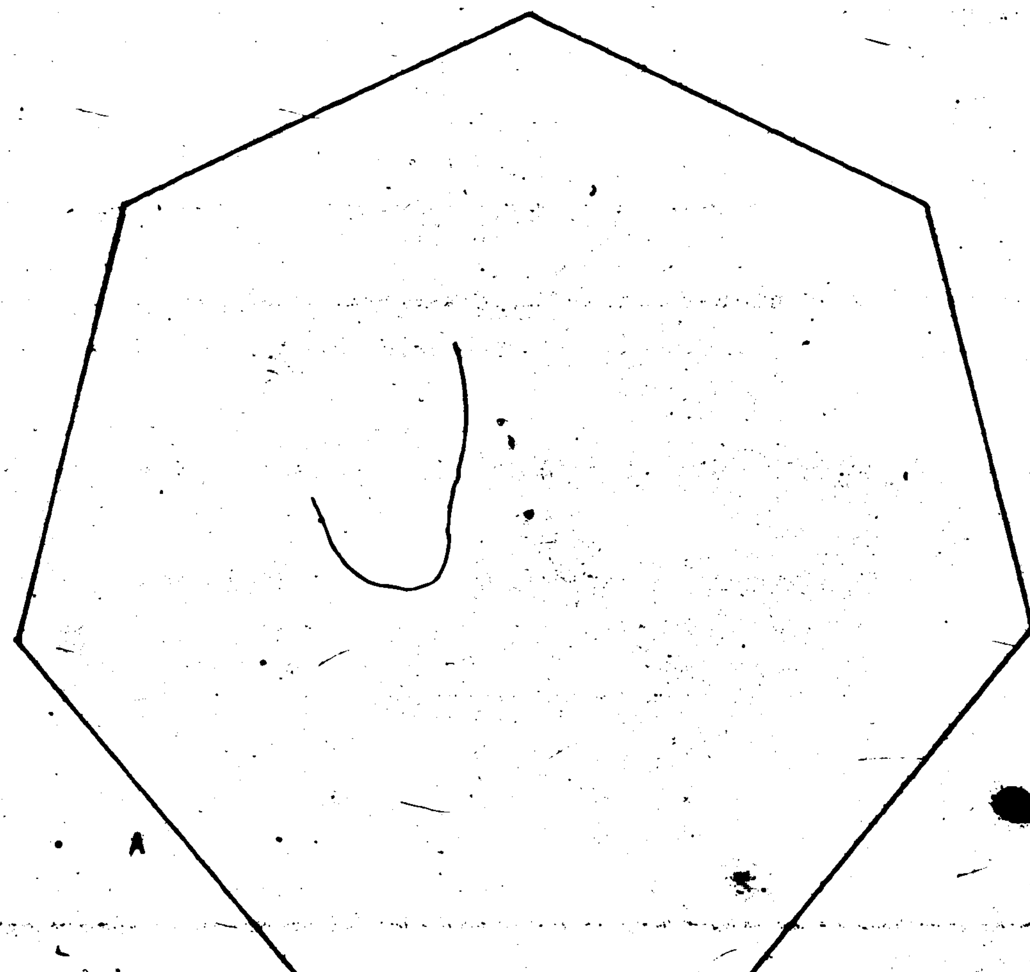
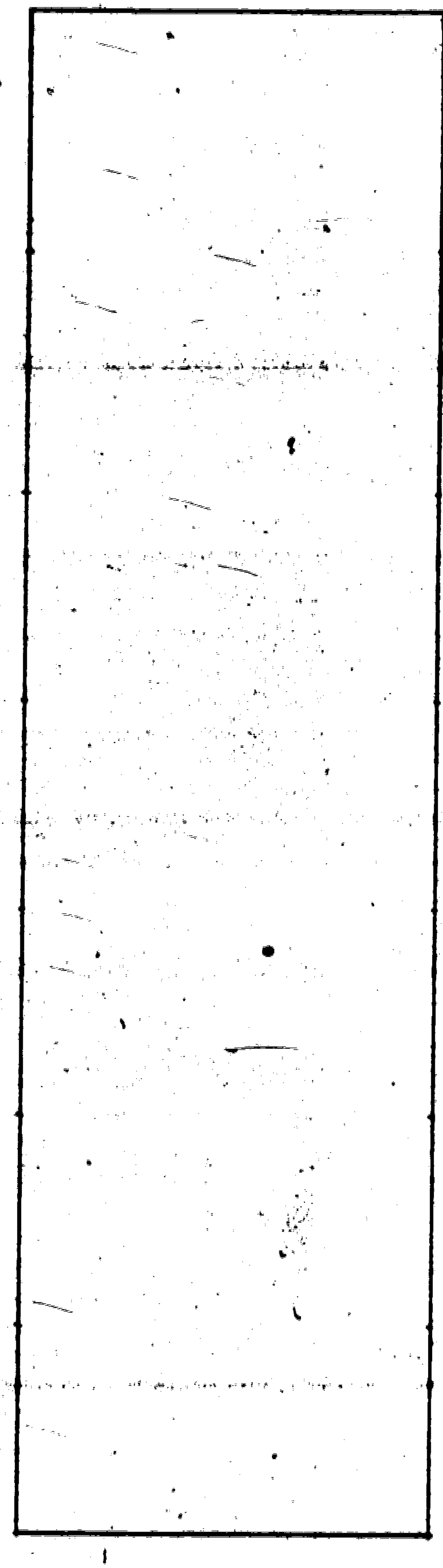


Fig. 14a



B

Fig. 14b

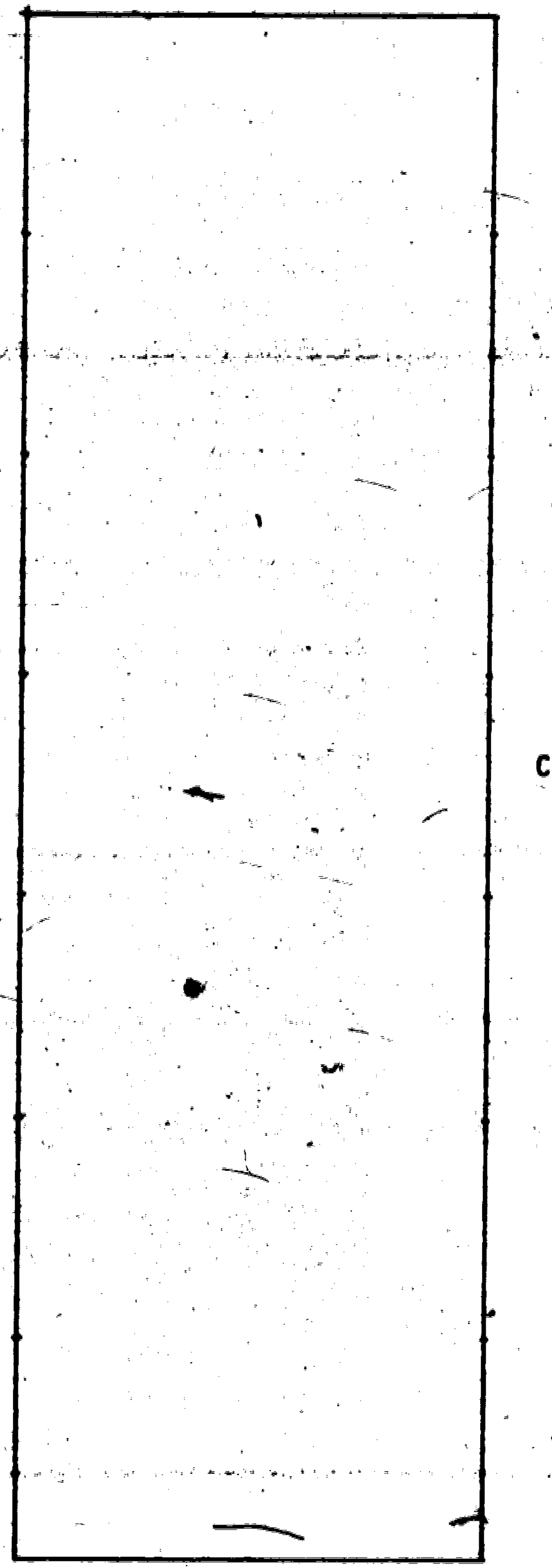


Fig. 14c

REGULAR OCTAGON REGIONS

Investigation 9:

In Figures 15a, 15b, and 15c below are three polygonal regions, a regular octagon region A and two rectangular regions B and C Trace and cut out a copy for each of them.

Try now to find an answer to each of the follwing questions:

ARE REGION A AND REGION B OF EQUAL SIZE? HOW DO YOU KNOW?

ARE REGION A AND REGION C OF EQUAL SIZE? HOW DO YOU KNOW?

Try to make a conclusion out of this activity.

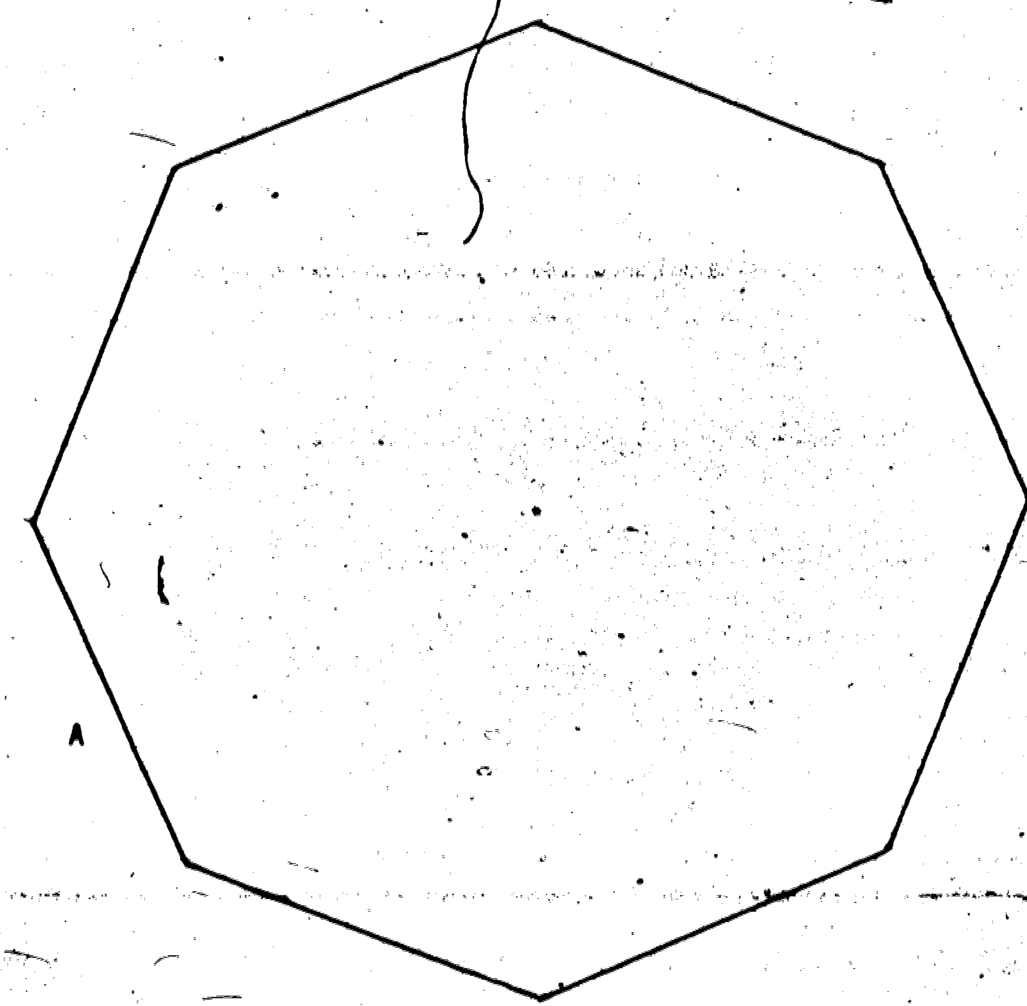


Fig. 15a

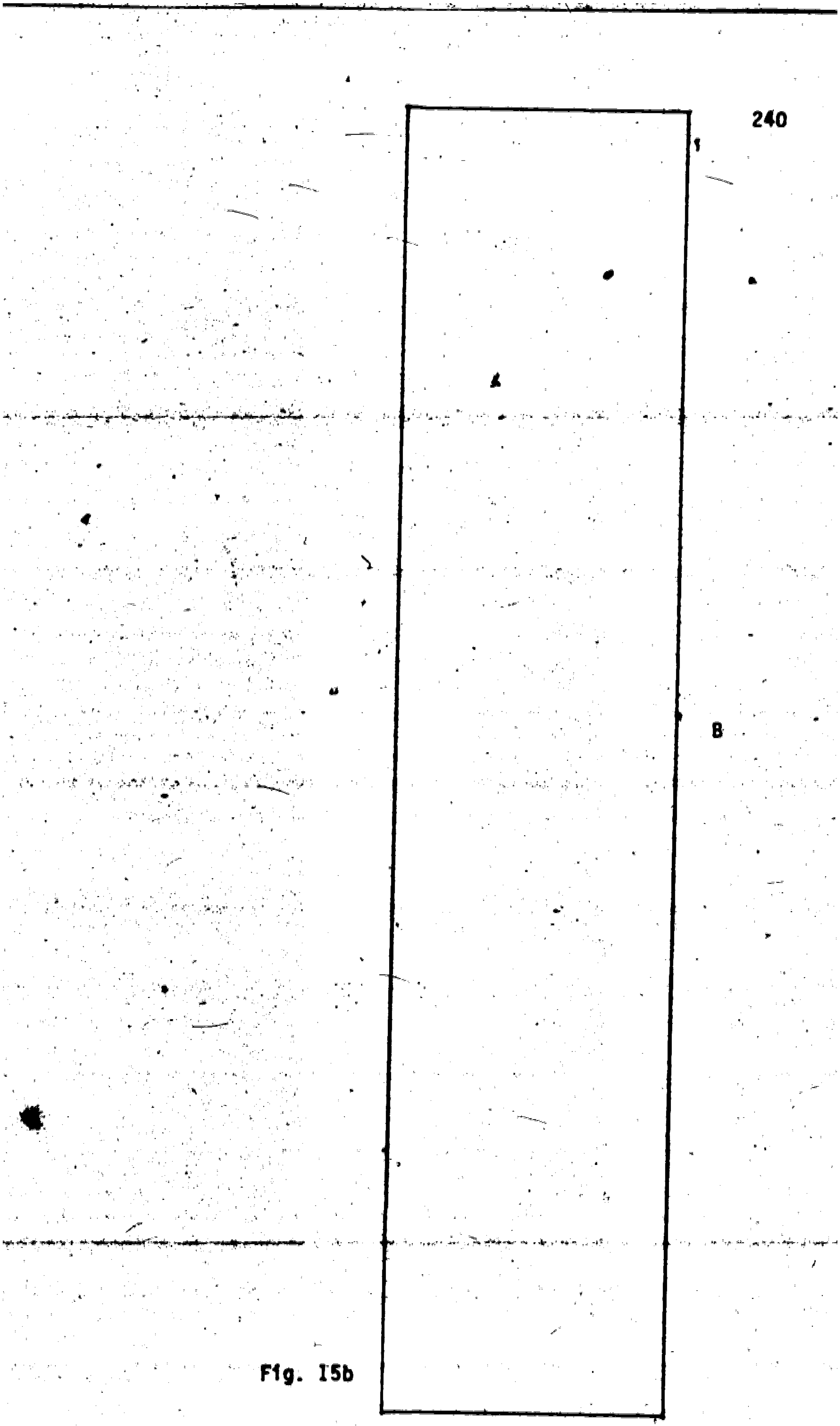


Fig. I5b

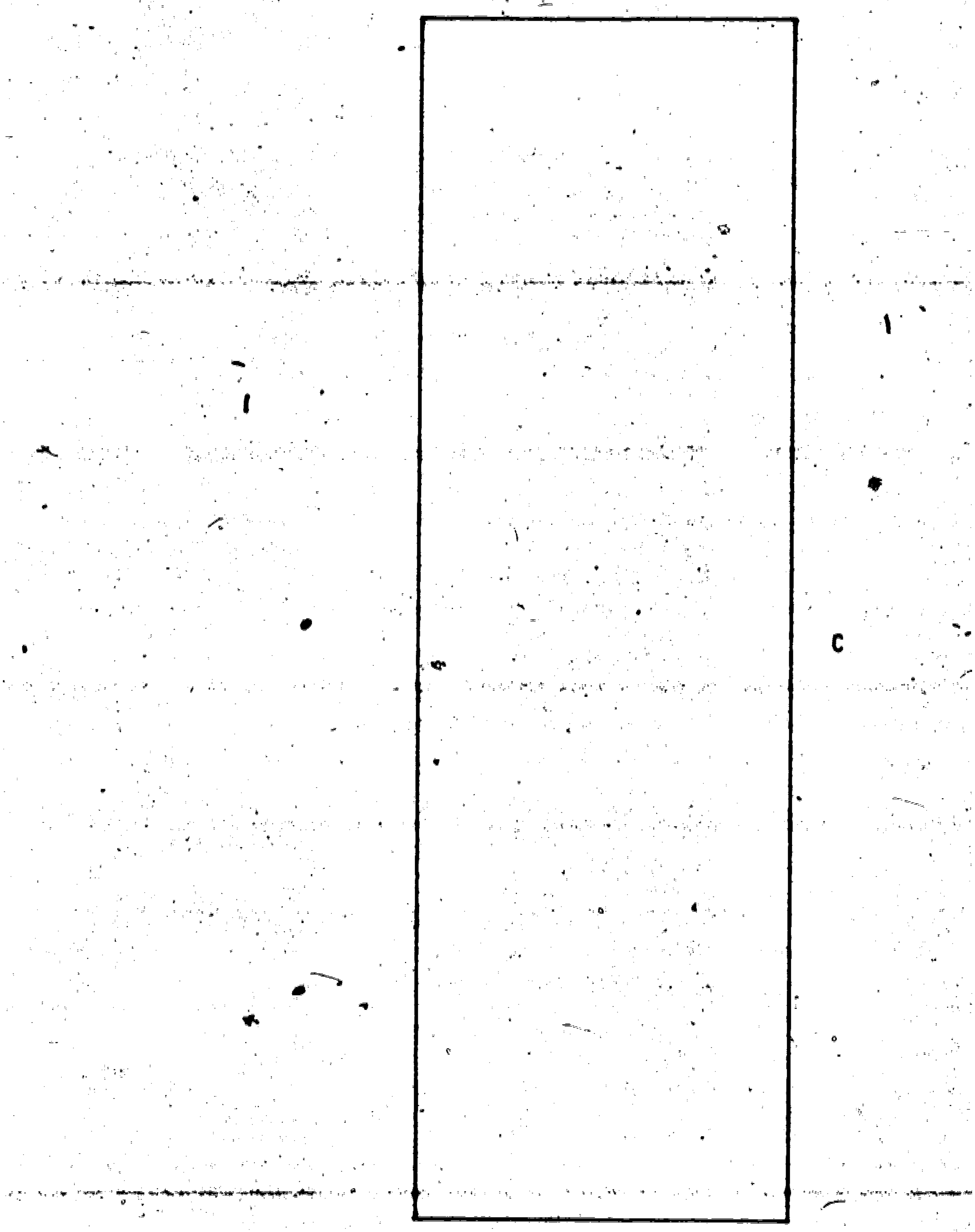


Fig. 15c

REGULAR NONAGON REGIONS

Investigation 10:

In Figures 16a, 16b, and 16c below are three polygonal regions, a regular nonagon region A and two rectangular regions B and C.

Trace and cut out a copy for each of them.

Try now to find an answer to each of the following questions:

ARE REGION A AND REGION B OF EQUAL SIZE? HOW DO YOU KNOW?

ARE REGION A AND REGION C OF EQUAL SIZE? HOW DO YOU KNOW?

Try to make a conclusion out of this activity.

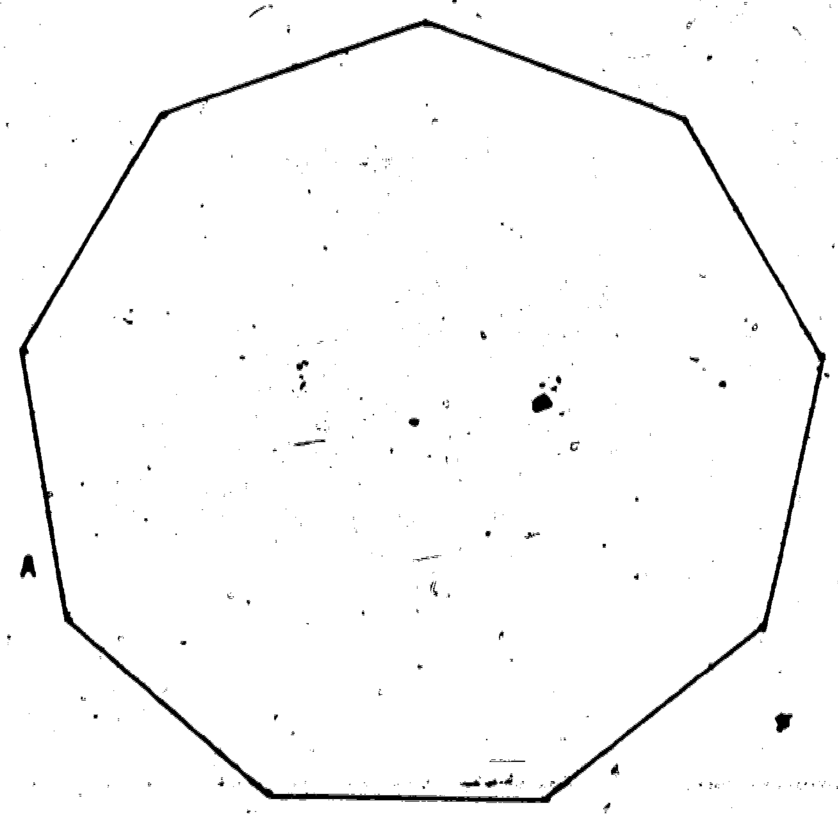


Fig. 16a

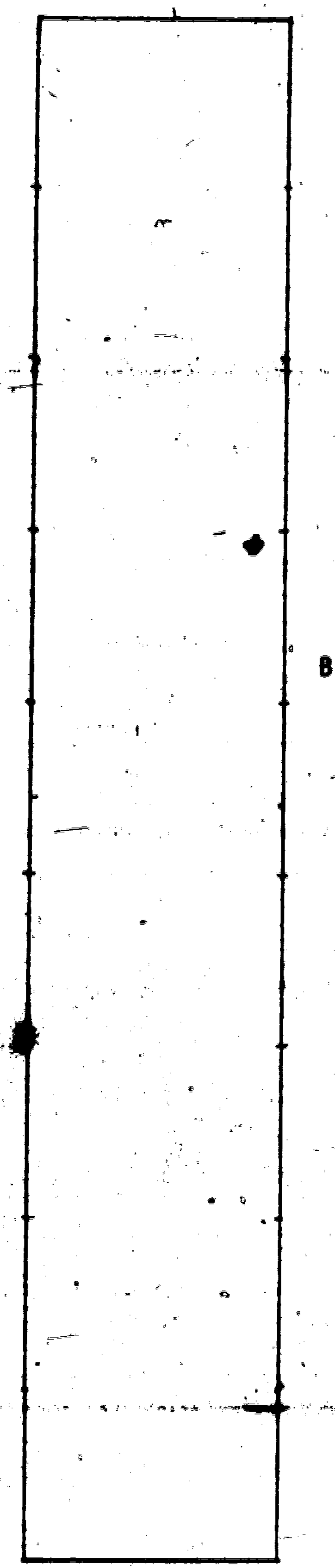


Fig. 16b

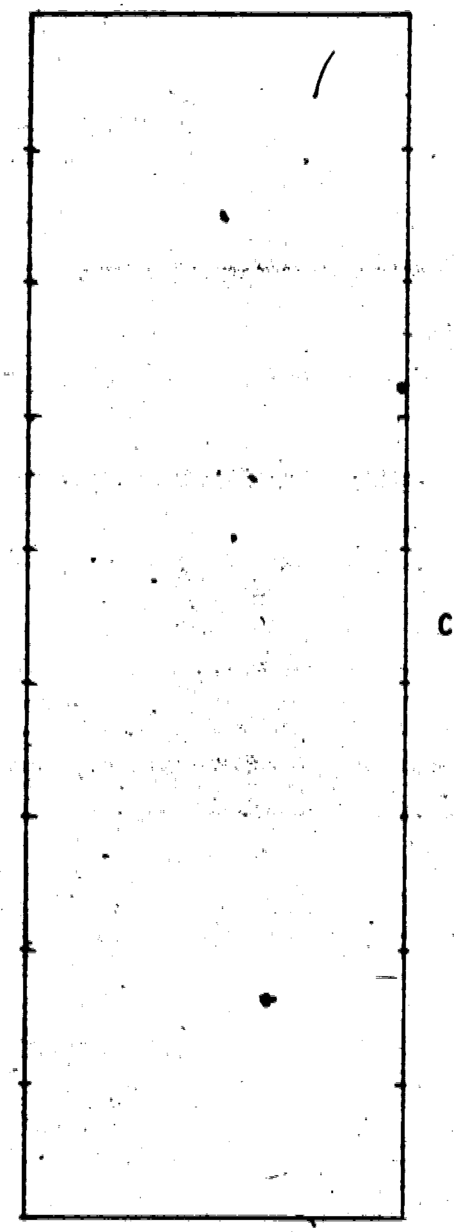


Fig. 16c

REGULAR DECAGON REGIONS

Investigation 11:

In Figures 17a, 17b, and 17c below are three polygonal regions, a regular decagon region A and two rectangular regions B and C.

Trace and cut out a copy for each of them.

Try now to find an answer to each of the following questions:

ARE REGION A AND REGION B OF EQUAL SIZE? HOW DO YOU KNOW?

ARE REGION A AND REGION C OF EQUAL SIZE? HOW DO YOU KNOW?

Try to make a conclusion out of this activity.

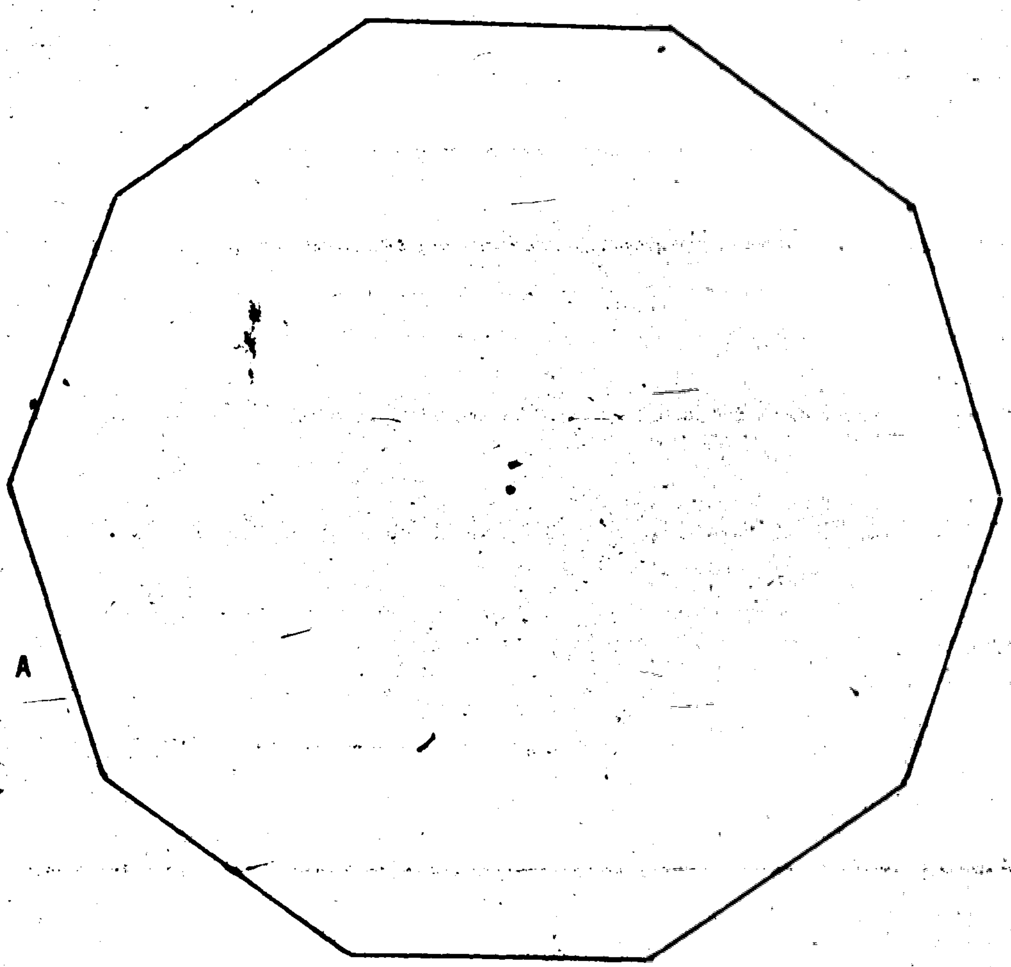


Fig. 17a

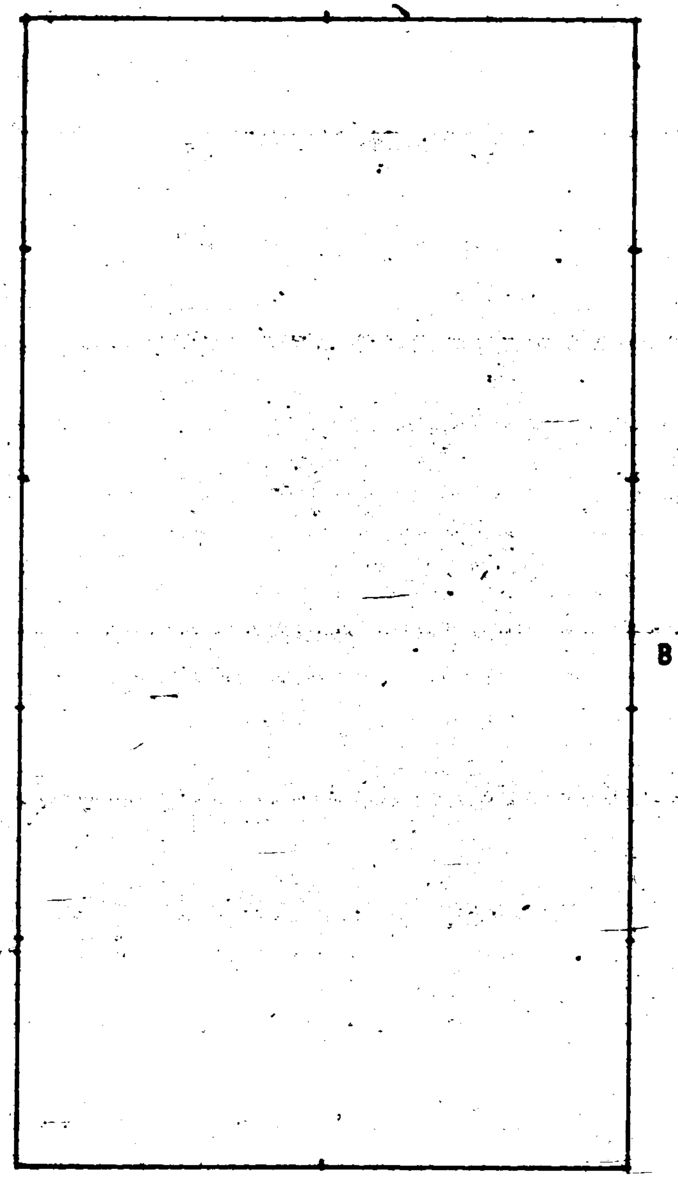


Fig. 17b

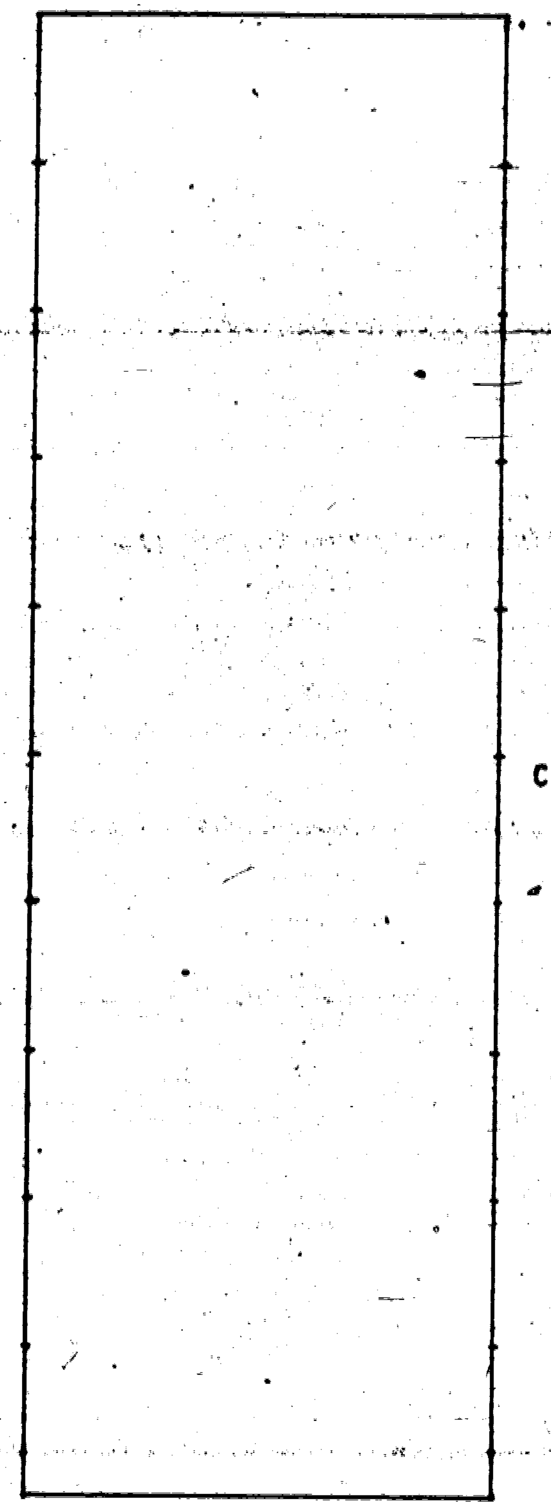


Fig. 17c

REGULAR DODECAGON REGIONS

Investigation 12:

In Figures 18a, 18b, and 18c below are three polygonal regions, a regular dodecagon region A and two rectangular regions B and C.

Trace and cut out a copy for each of them.

Try now to find an answer to each of the follwing questions:

ARE REGION A AND REGION B OF EQUAL SIZE? HOW DO YOU KNOW?

ARE REGION A AND REGION C OF EQUAL SIZE? HOW DO YOU KNOW?

Try to make a conclusion out of this activity.

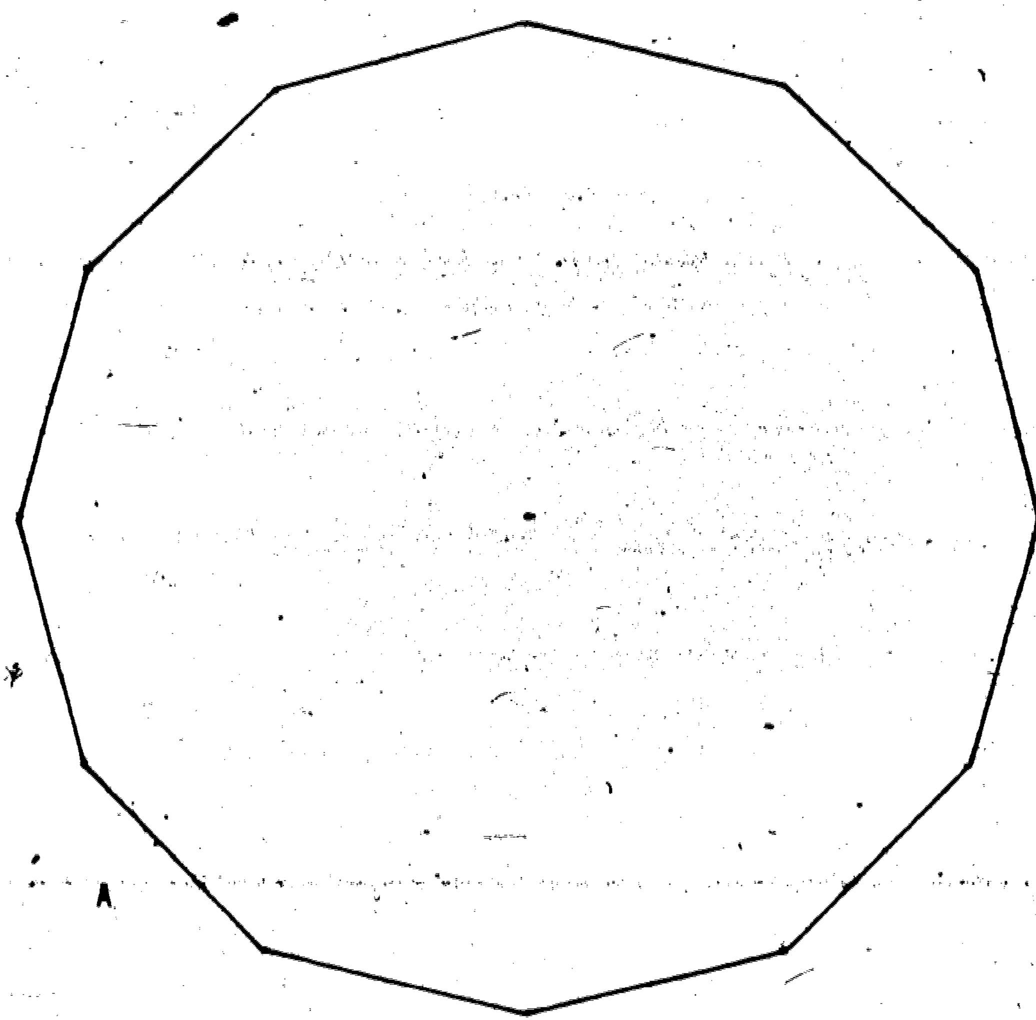


Fig. 18a

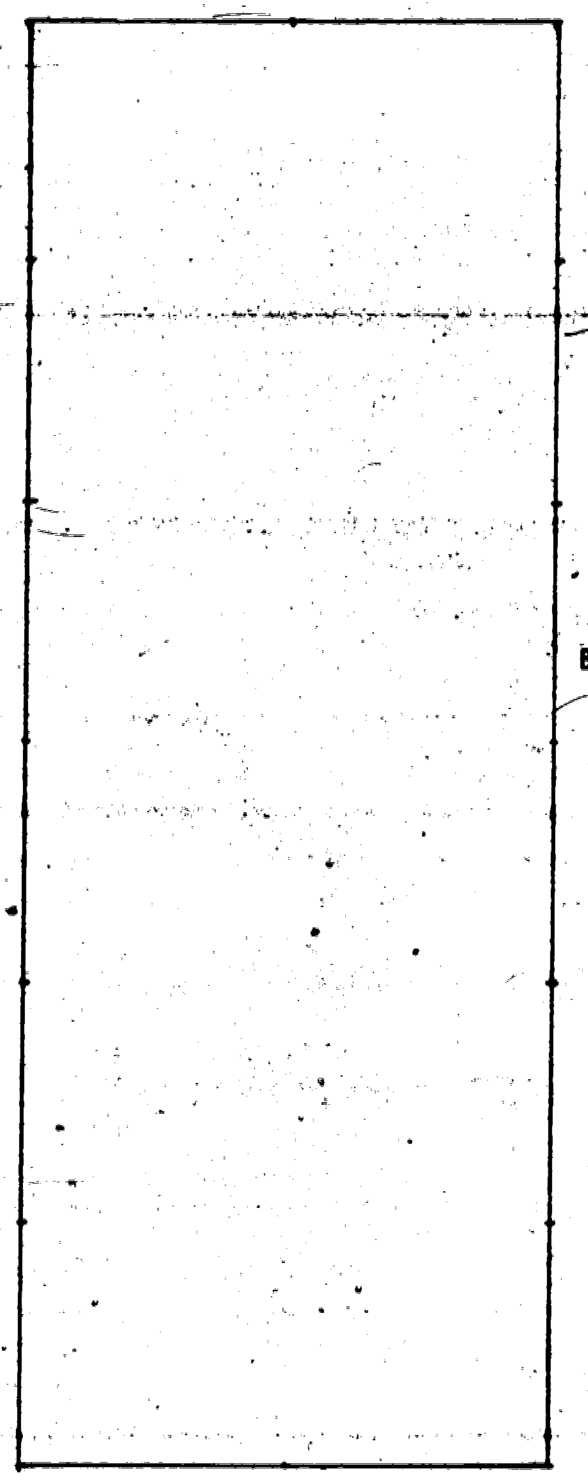


Fig. 18b

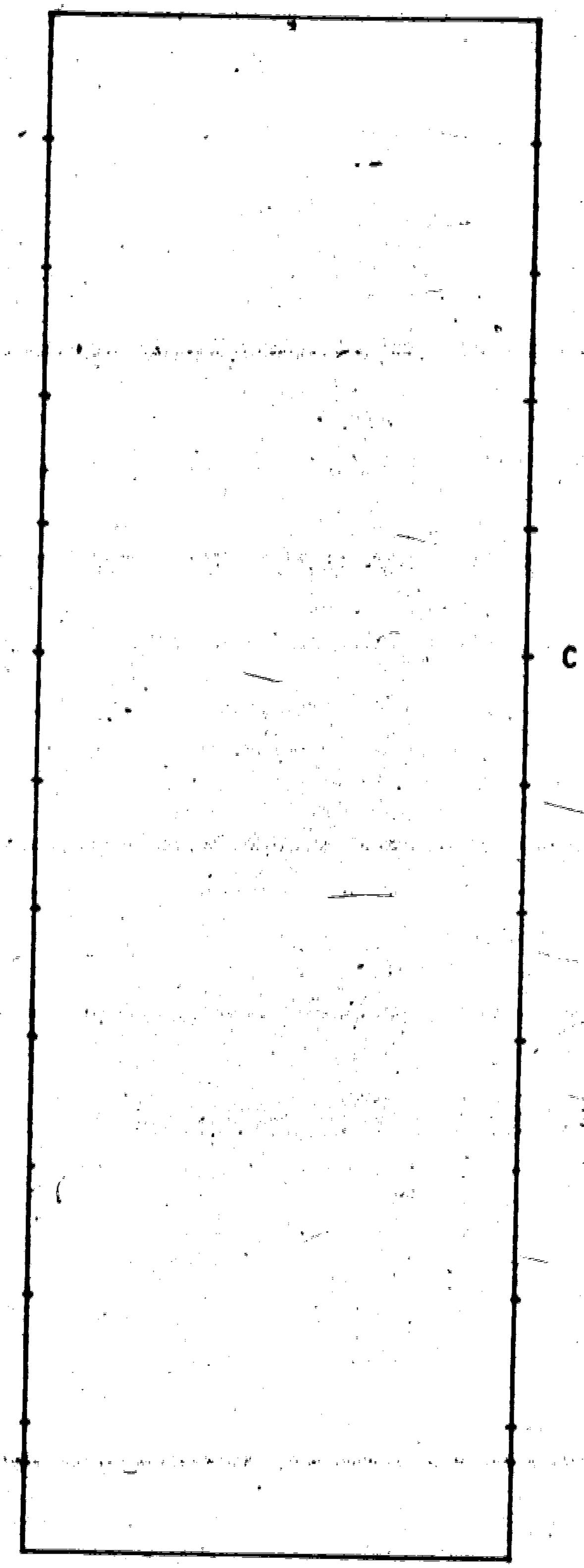


Fig. 18c

PART THREE

AREA FORMULAE

It is shown above (Part Two), through the previous activities, that each triangular, quadrilateral, and regular n-gon region, where n = 5, 6, 7, ..., can be decomposed into a rectangular region. Now, we are in a position to set up the following postulate for the area of a rectangular region and use the Piece-Wise Congruency notion (cut-and-cover) to deduce the area formulae for triangular, quadrilatral, and regular n-gon regions.

The Area Postulate for a Rectangular Region

> The area of a rectangular region is the product of its base and its altitude (Fig. 19).

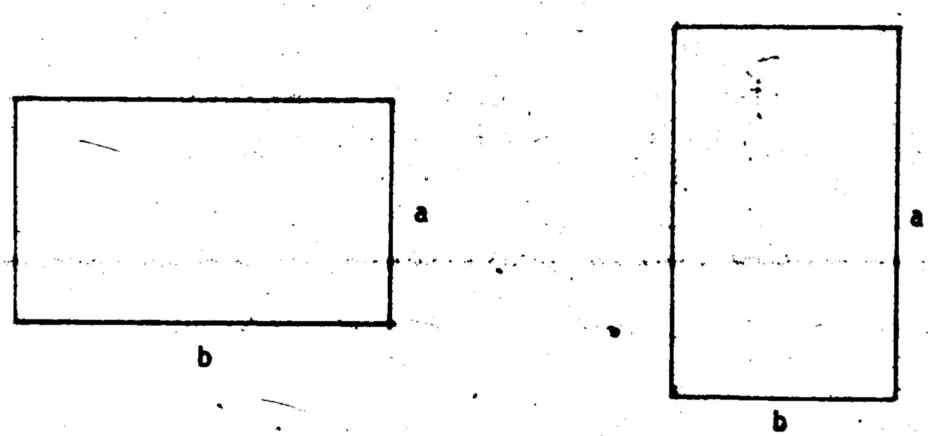


Fig. 19

Area = Base x altitude

$$A = b \cdot a \qquad \ldots (1)$$

Remark

In a rectangular region, two consecutive sides would be a base and an altitude. Therefore, the area of a rectangular region can be rephrased as:

The product of any two cosecutive sides.

The Area Formula of a Parallelogram Region

Given a parallelogram region BCDE of base = b and altitude = a, it is piece-wise congruent with the rectangular region FGHI (Fig. 20).

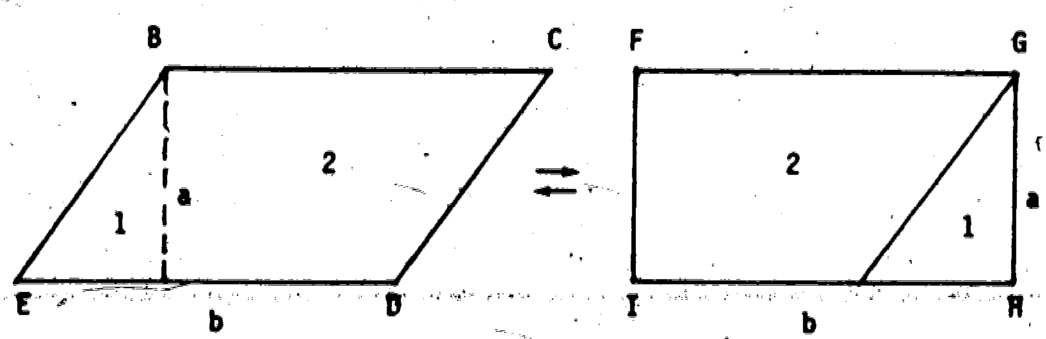


Fig. 20

Therefore,

Area of BCDE = Area of FGHI.

Using relation (1) above, we have

Area of BCDE = b . a	... (2)

That is,

> the area of a parallelogram region is the product of its base and its corresponding altitude.

The Area Formula of a Triangular Region

Let BCD be a triangular region of base = b and altitude = a. It is piece-wise congruent with a parallelogram region of base = b and altitude = ½a (Fig. 21). Also, it is piece-wise congruent with a rectangular region of base = b and altitude = ½a (Fig. 22).

Case I:

The Triangular region BCD is taken as right , acute, or obtuse angle in both Figure 21 and Figure 22.

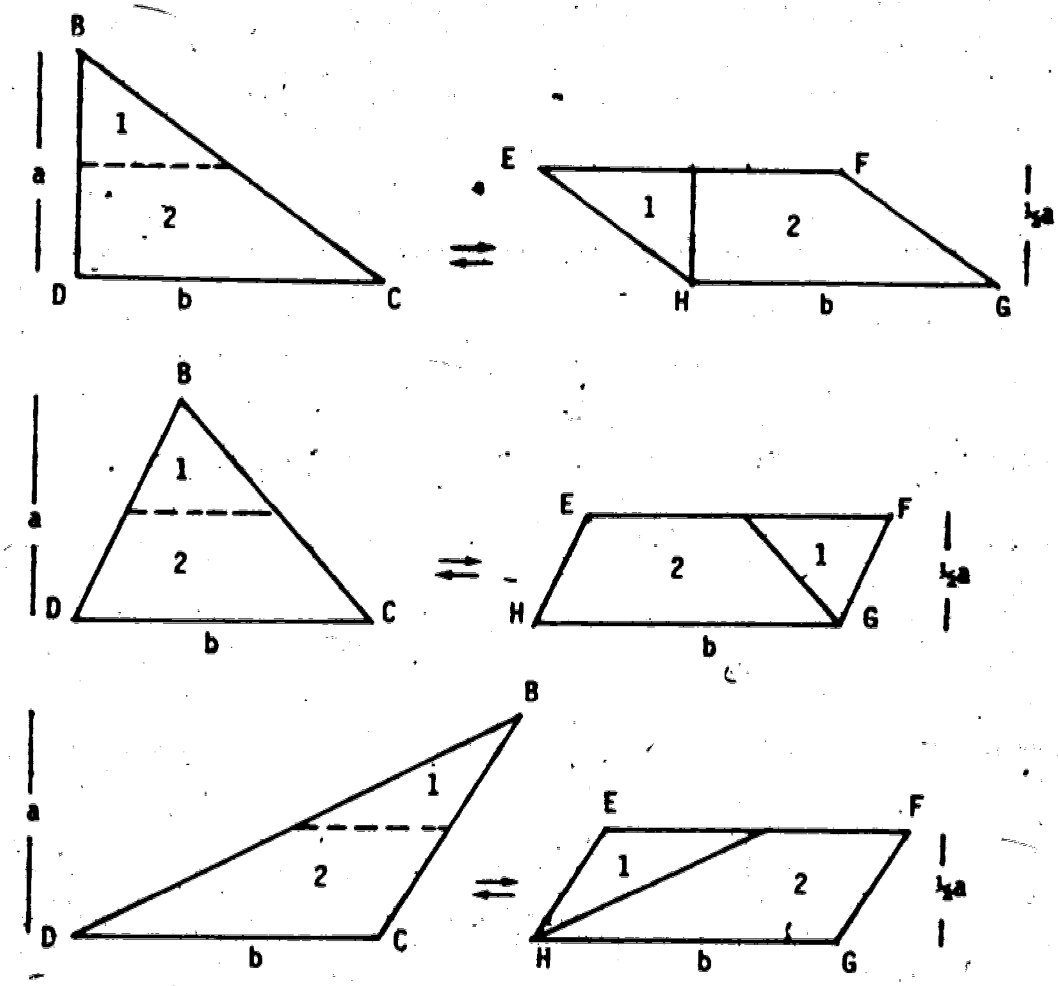


Fig. 21

The piece-wise congruency implies,

Area of BCD = Area of EFGH.

And using relation (2), it follows that

$$\text{Area of BCD} = \tfrac{1}{2}b \cdot a \qquad \ldots (3).$$

That is,

> **the area of a triangular region is half the product of its base and its corresponding altitude.**

<u>Case II</u>:

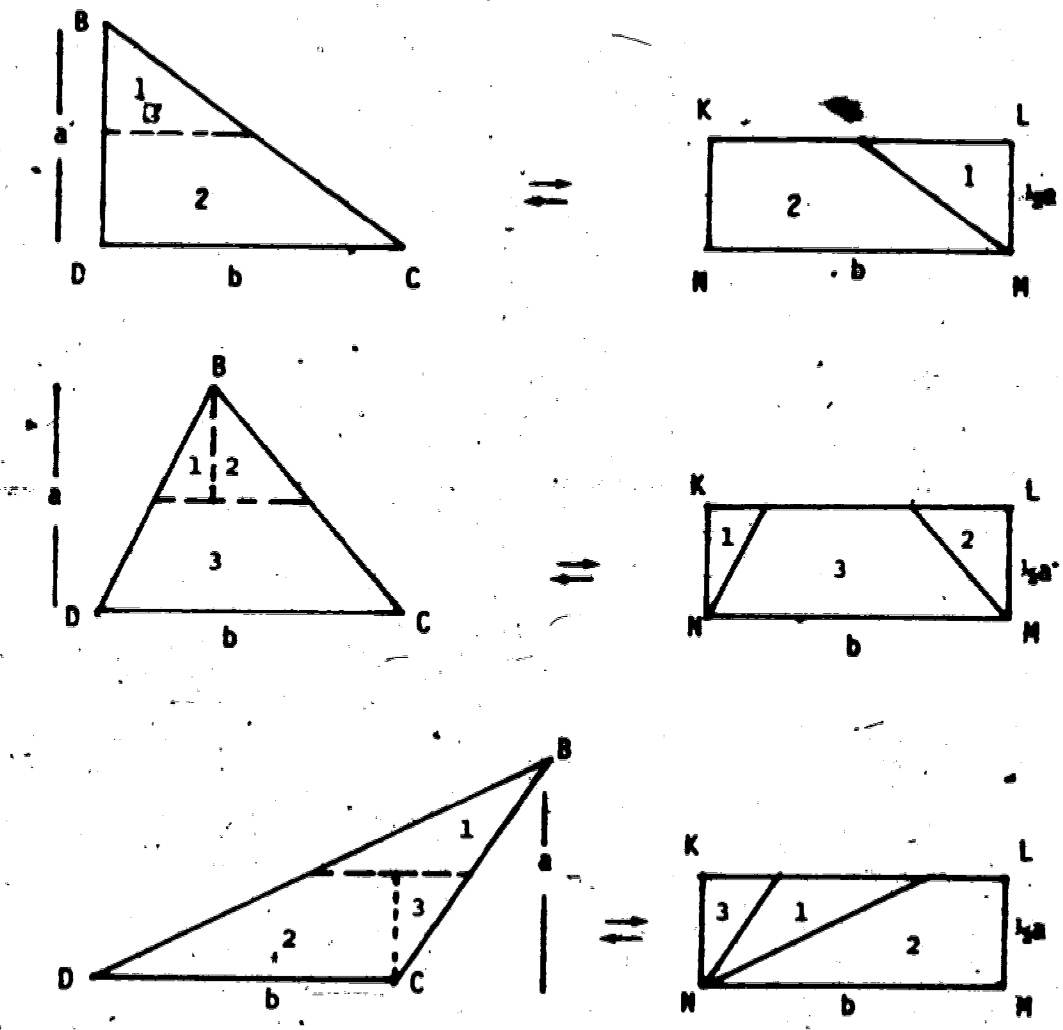


Fig. 22

The piece-wise congruency implies that

Area of BCD = Area of KLMN.

Using relation (1), it follows that

> **Area of BCD = $\frac{1}{2}b \cdot a$**

The Area Formula of a Trapezoid Region

Given a trapezoid region BCDE of lower base = b_1, upper base = b_2, and altitude = a, it is piece-wise congruent with the triangular region FGH of base = $b_1 + b_2$, and altitude = a (Fig. 23).

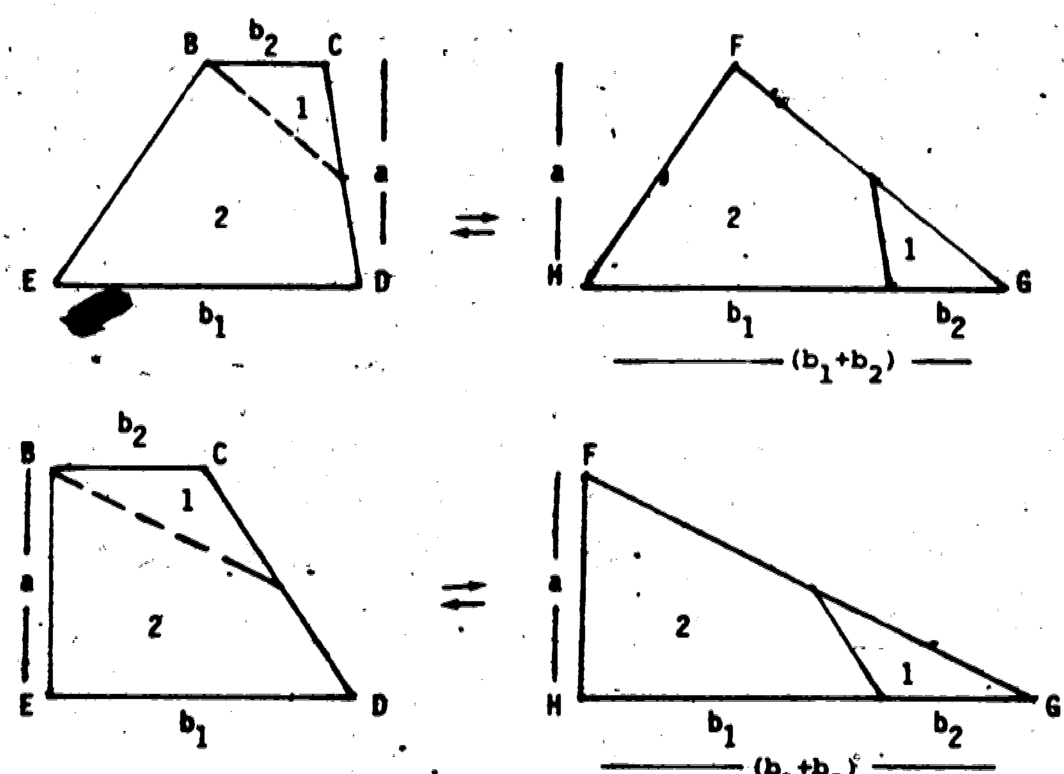


Fig. 23

Therefore,

$$\text{Area of BCDE} = \text{Area of FGH}.$$

Using relation (3), we have

$$\text{Area of } FGH = \tfrac{1}{2}(b_1 + b_2) \cdot a \quad \text{and hence}$$

$$\text{Area of } BCDE = \tfrac{1}{2}a(b_1 + b_2) \qquad \ldots (4).$$

That is,

> the area of a trapezoid region is half the product of its altitude and the sum of its bases.

The Area Formula of a Square Region

Given a square region, it is a special case of the rectangular region and therefore its area can be taken as the product of any two cosecutive sides which yields its area as the square of the length of its side. However, the following is another way of producing the area formula for a square region.

Let BCDE be a square region with the length of its side = X. It is piece-wise congruent with a triangular region FGH of base = 2X and altitude = X (Fig. 24).

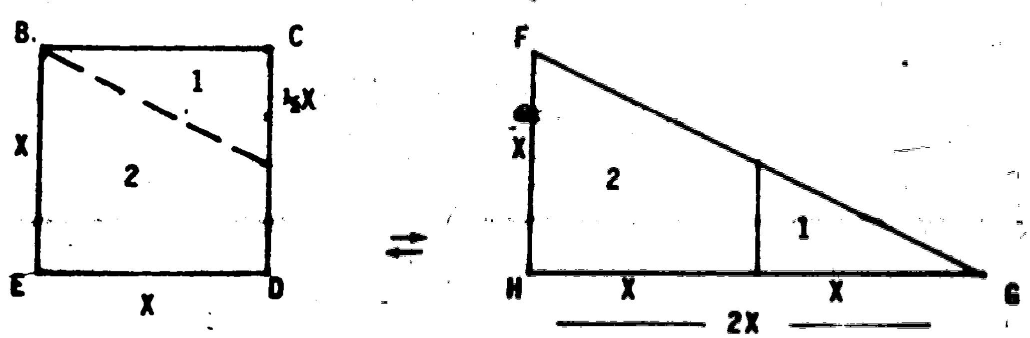


Fig. 24

Thus,

Area of BCDE = Area of FGH.

Since the area of FGH = $\frac{1}{2}(2X) \cdot X$ ... by (3), it follows that

Area of BCDE = X^2 ... (5).

That is,

the area of a square region is the square of the length of its side.

The Area Formula of a Rhombus Region

Given a rhombus region BCDE of major diagonal BD = X and minor diagonal CE = Y, it is piece-wise congruent with the rectangular region FGHI of two consecutive sides equal either $\frac{1}{2}X$ and Y or $\frac{1}{2}Y$ and X respectively (Fig. 25a & b).

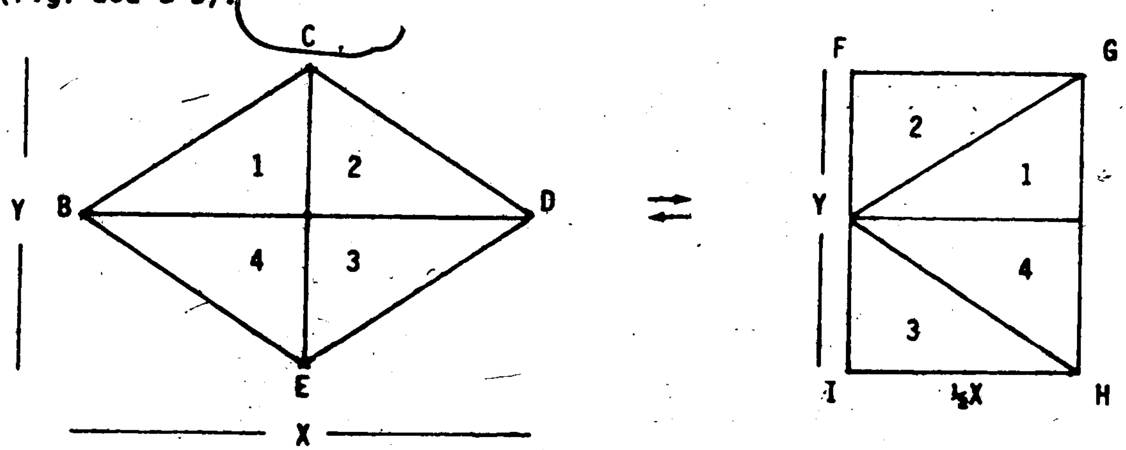


Fig. 25a

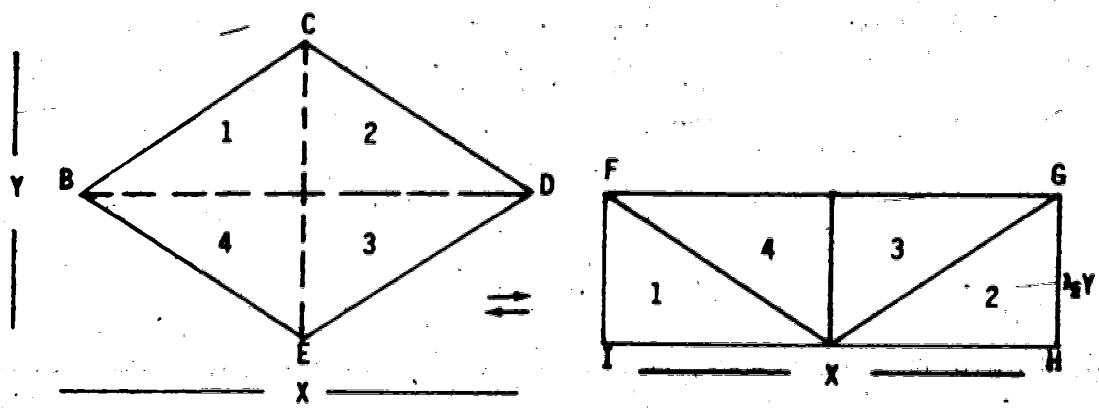


Fig. 25b.

It follows that

Area of BCDE = Area of FGHI.

Since the area of FGHI = ½ X . Y (or ½ Y . X) ... by (1),

Area of BCDE = ½ X . Y ... (6).

That is,

the area of a rhombus region is half the product of its diagonals.

The Area formula for a Regular Pentagon Region

Let BCDEF be a regular pentagon region and let the point O be its center. Join the center O with the verteqies B, C, D, E, and F and let its side = b and the altitude of one of the five congruent triangles = a (Fig. 26). It is then piece-wise congruent with the rectangular region KLMN of base = b and altitude = 5(½a).

Therefore,

Area of BCDEF = Area of KLMN.

And,

Area of BCDEF = (5/2) b . a ... (7).

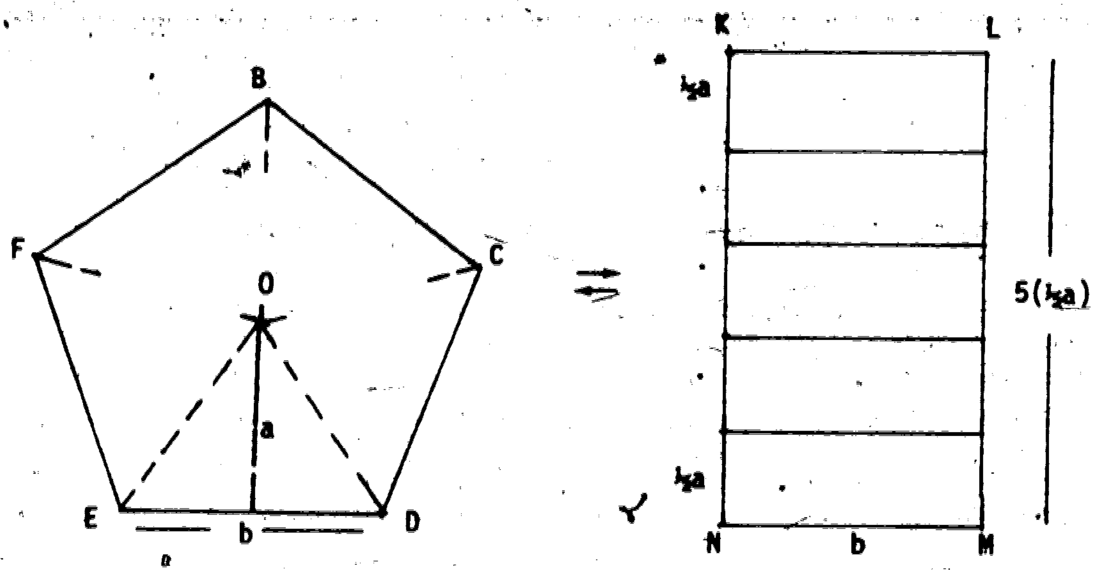


Fig. 26

Similar figures to Figure 26 will show that for regular n-gons with n = 6, 7, 8, ... formula (7) holds.

In general: Given a regular n-gon region, $n \in Z+$, $n \geq 5$, with the length of its side = b and the altitude of one of the n triangular regions equals a, it is piece-wise congruent with a rectangular region of two consecutive sides equal b and n(½a) respectively (Fig. 27).

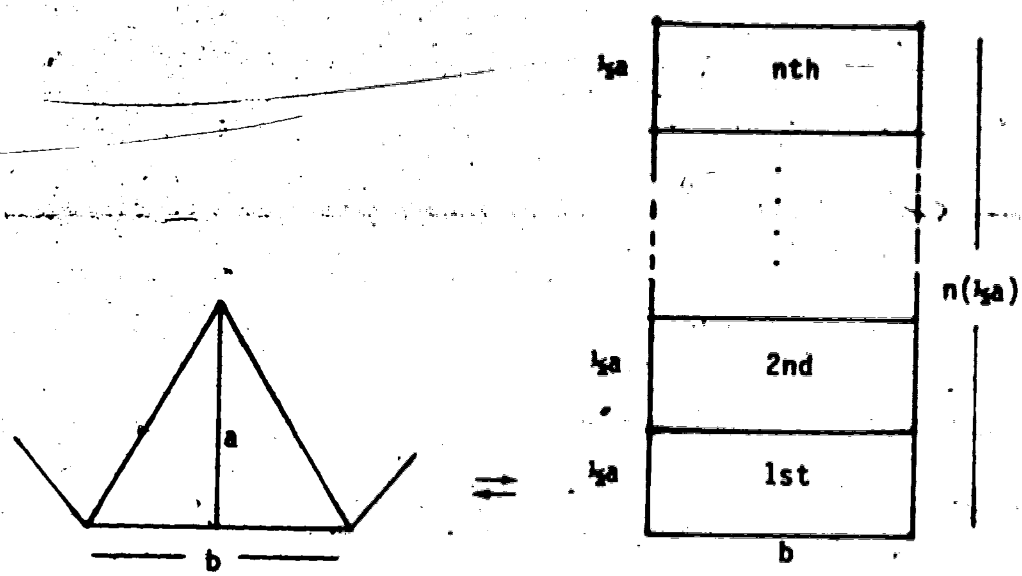


Fig. 27

Therefore,

Area of a regular n-gon region = (n/2) b . a ... (8),

where n = the number of the sides,

b = the length of each side, and

a = the altitude of one of the n congruent triangular regions.

****

APPENDIX B

THE TEACHERS' MANUAL FOR THE UNIT

The included remarks in the TEACHERS' MANUAL are inteded to present some suggestions and practical help in teaching the UNIT (App. A). They include some POSSIBLE solutions to problems raised in the UNIT such that some of the teachers' time would be saved. However, this MANUAL is complete only in its essentials; some details are omitted.

Two types of teaching aids have been prepared for the UNIT.

1. Transparancies: Appendix B contains their masters. For each activity or part of it there is a transparency that contains a possible solution and an intended conclusion.

2. Magnetized card board models: For each polygonal region involved in the derivations of the area formulae (part three of the UNIT), a magnetized and already decomposed card board model is prepared. Thus, in deriving a particular area formula, the related model would be displayed on a magnetic board, the required decomposition and motions would be demonstrated, and the wanted area formula could be derived.

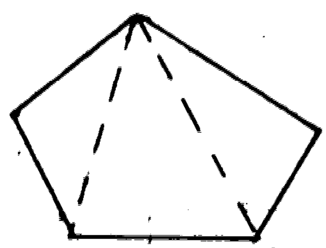
PART ONE

DEFINITIONS

For the two sets of exercises, Exercise Set 1 and 2, although there are many ways for triangulating polygonal regions the following two types are felt appropriate:

1. Vertex - Vertex Triangulating.

   For example, dissect through the diagonals.



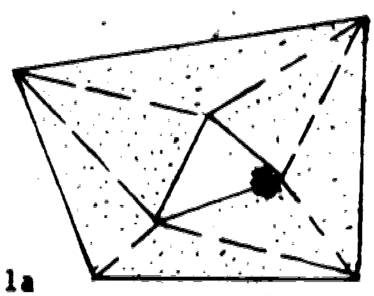


Fig. 1a

2. Inner Point - Vertex Triangulating.

   For example, dissect through lines joining the center and the vertices.

The point C in the Figure below can be taken anywhere inside the region.

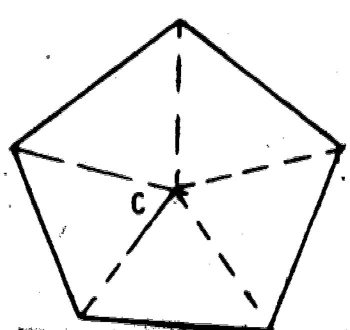


Fig. 1b

PART TWO

LABORATORY INVESTIGATIONS

TRIANGULAR REGIONS

Investigation 1a:

Bisect the two sides of the triangular region; join the two points; cut through this line; and a half turn will give a possible solution. You may use transparency #1.

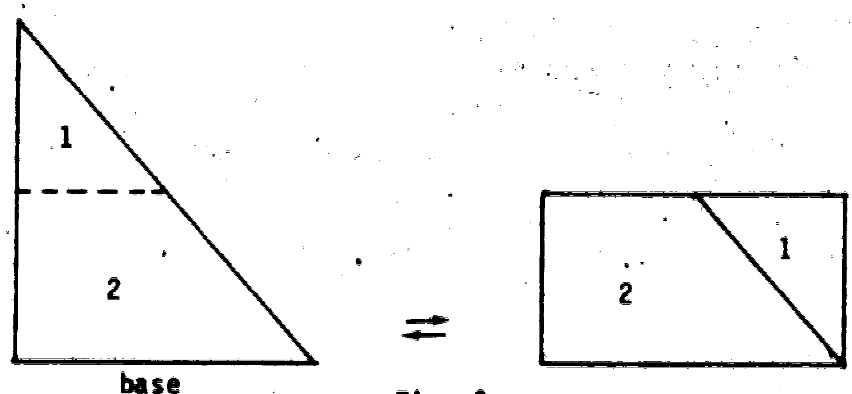


Fig. 2

Investigation 1b:

Bisect the two sides; join the two points of bisection; drop a perpendicular on this line (the bisection line); cut through these two lines; and a half turn counter-clockwise and clockwise will give a possible solution. You may use transparency #2.

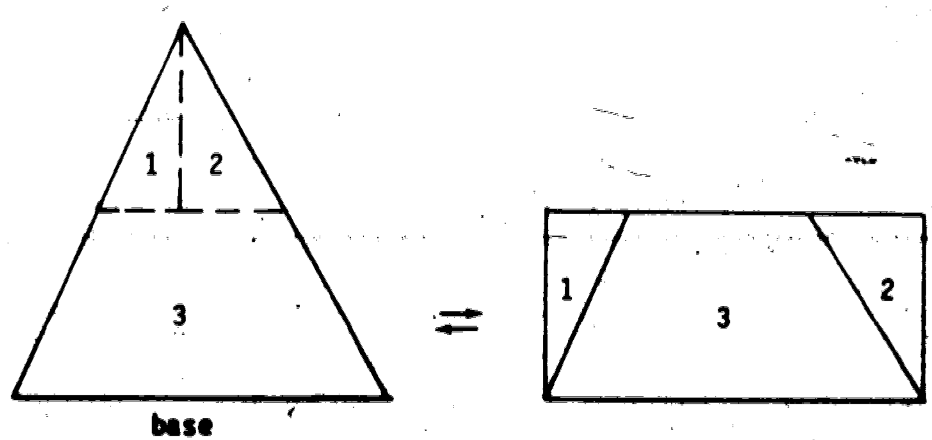


Fig. 3

Investigation 1c:

Bisect the two sides; join the points of bisection; draw a perpendicular on this line from the lower right vertex; cut through these two lines; and the motions of a half turn and a slide will give a possible solution. You may use tranparency #3.

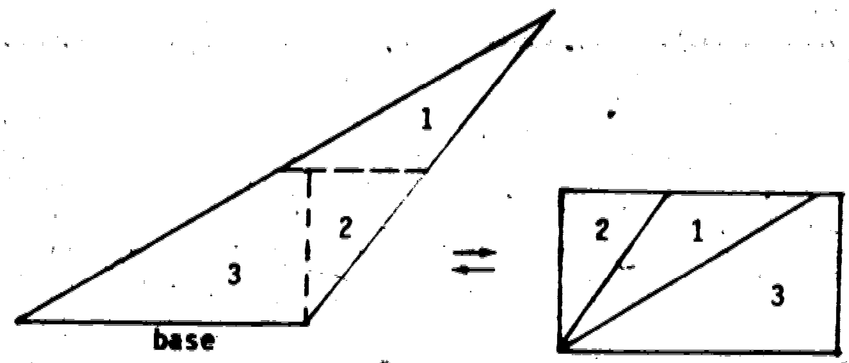


Fig. 4

Investigation 1d:

Drop a perpendicular line from the upper vertex on the base; cut through this line; and the motions of a flip and a half turn will provide a possible solution. You may use transparency #4.

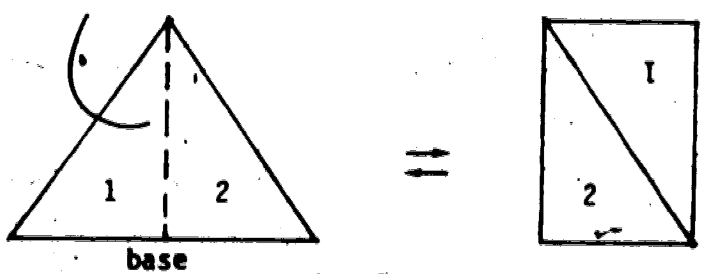


Fig. 5

Investigation 1e:

Follow the same procedure of Investigation 1d above. You could use transparency #5.

QUADRILATERAL REGIONS: GENERAL CASE

Investigation 2:

Choose a diagonal; draw it and you have two triangular regions; apply Investigation 1b's method on each of them and you will have a possible solution. You could use transparency #6.

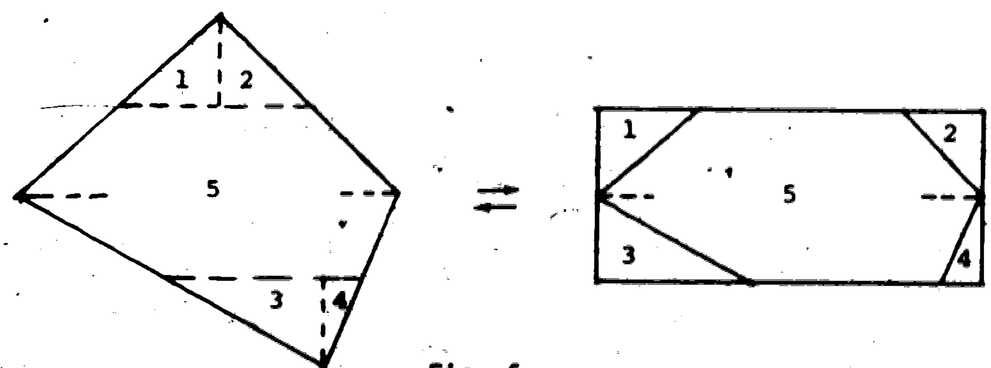


Fig. 6

QUADRILATERAL REGIONS: SPECIAL CASES

TRAPEZOID REGIONS

Investigation 3a:

Bisect the two sides; join the points of bisection; cut through this line; and a half turn will provide a possible solution. Use transparency #7. As you notice, this pattern is identical to that for Investigation 1a above.

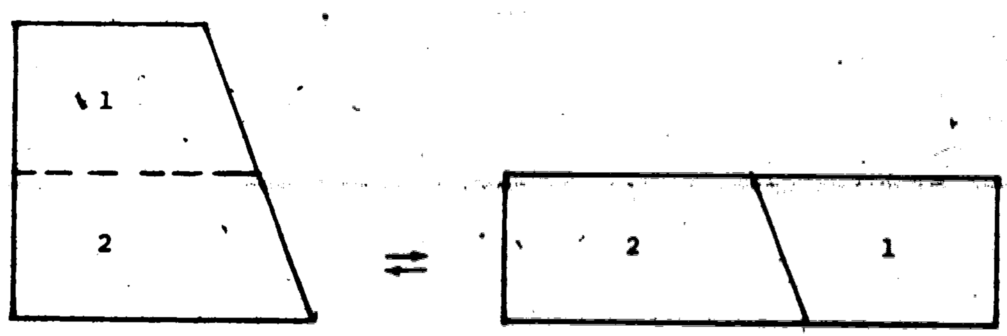


Fig. 7

Investigation 3b:

Bisect the two sides; join the points of bisection; drop a perpendicular on this line from any point you may choose on the upper base; cut through these two lines; and a half turn counter-clockwise and clockwise will provide a possible solution. Notice that the pattern here is almost identical to the Investigation 1b's pattern. You may use transparencies #8, #9, and #10.

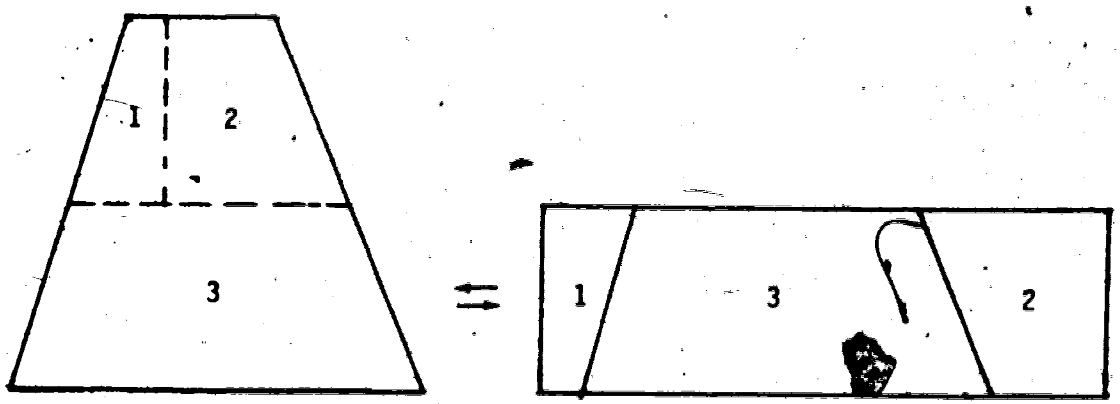


Fig. 8

Investigation 3c:

Choose a point on the upper base; drop a perpendicular line from this point on the lower base; cut through this line; and the motions of flip and a half turn will give a possible solution. Note that the pattern here is almost identical to the Invetigation 1d's pattern. You may use transparency #11.

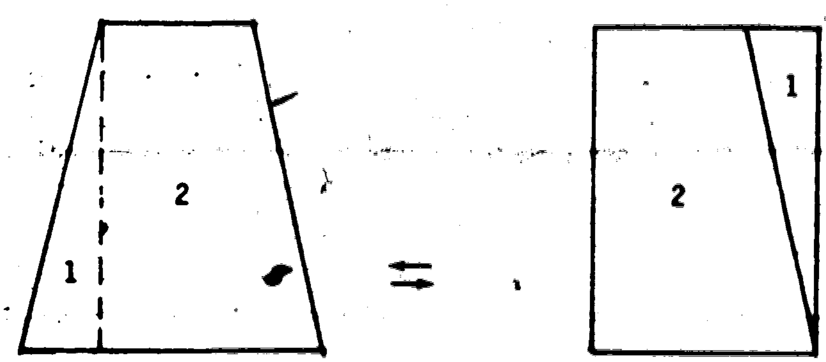


Fig. 9

PARALLELOGRAM REGIONS

Invetigation 4:

Choose a point on the upper base; drop a perpendicular line from this point on the lower base; cut through this line; and the motion of slide will provide a possible solution. Note that the pattern for this activity is identical to the Investigation 3c's pattern. You may use transparency #12.

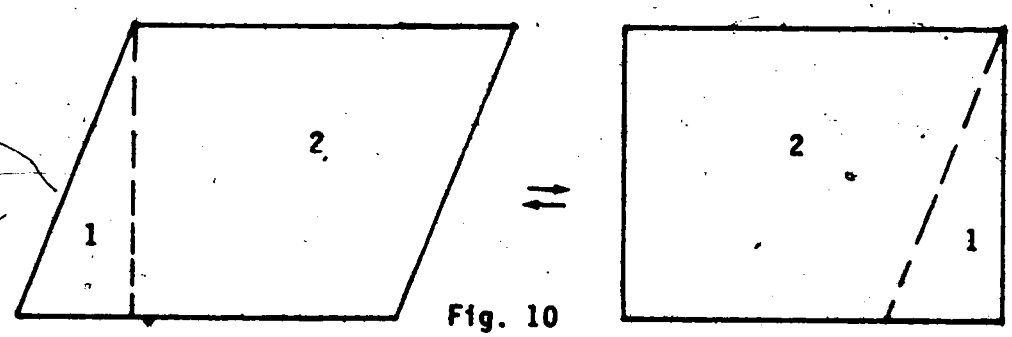


Fig. 10

RHOMBUS REGIONS

Investigation 5:

Draw either the major or the minor diagonal; drop a perpendicular line on the diagonal from one of the other two vertices; cut through these two lines; and the motions of flip and a half turn will provide a possible solution. You may cut through both diagonals. Use transparency #13.

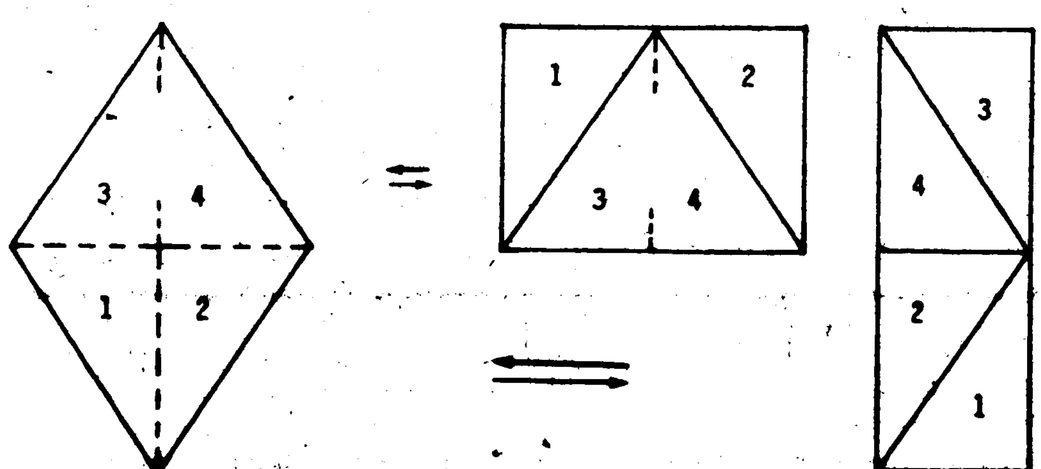


Fig. 11

N-GON REGIONS: $N \geq 5$

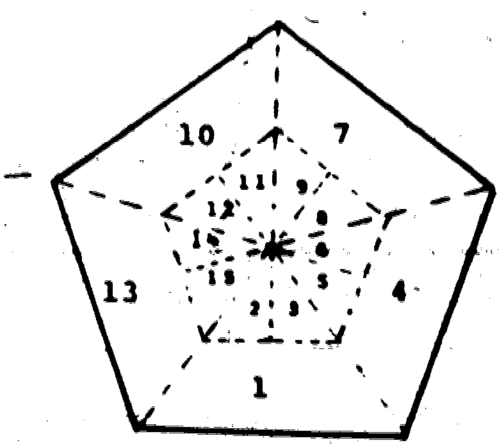
REGULAR PENTAGON REGIONS

Investigation 6:

CASE 1. For the first question concerning regions A and B, the following procedure may be followed: Partition the regular pentagon region (region A) into five congruent triangular subregions; apply Invetigation 1b's pattern on one of the triangular regions; repeat the procedure on each of the remainig triangular subregions; re-arrange the pieces to cover region B completely and a possible sol-ution is provided (Fig. 12). You may use transparency #14.

CASE 2. For the second question concerning regions A and C, proceed as follows: Partition the regular pentagon region (region A) as in case 1 into five congruent triangular subregions; apply Investigation 1d's pattern on one of these subregions; repeat the procedure on the rest of the subregions; rearrange the pieces to cover region C completely and a possible solution is made (Fig. 13).

You may use transparency #15.



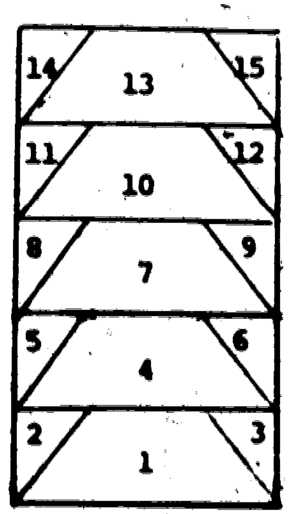


Fig. 12

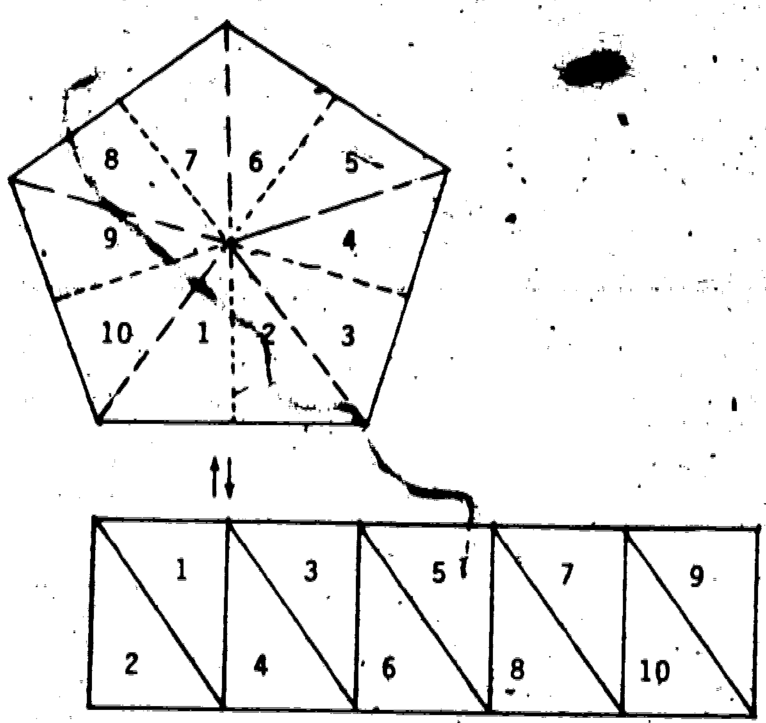


Fig. 13

REGULAR HEXAGON REGIONS

Investigation 7:

For the first question on regions A and B and the second question on regions A and C repeat the Investigation 6's procedure Case 1 and Case 2 respectively and a possible solution is made. Use transparency #16.

REGULAR HEPTAGON REGIONS

Investigation 8:

Repeat the Investigation 6's procedure. Use transparency #17.

REGULAR OCTAGON REGIONS

Investigation 9:

Follow the Investigation 6's procedure. Use tra[illegible]ncy #18.

REGULAR NONAGON REGIONS

Investigation 10:

Repeat the procedure for investigation 6. Use transparency #19.

REGULAR DECAGON REGIONS

Investigation 11:

Repeat the Investigation 6's procedure. Use transparency #20.

REGULAR DODECAGON REGIONS

Investigation 12:

Follow the Investigation 6's procedure. Use transparency #21.

PART THREE

AREA FORMULAE

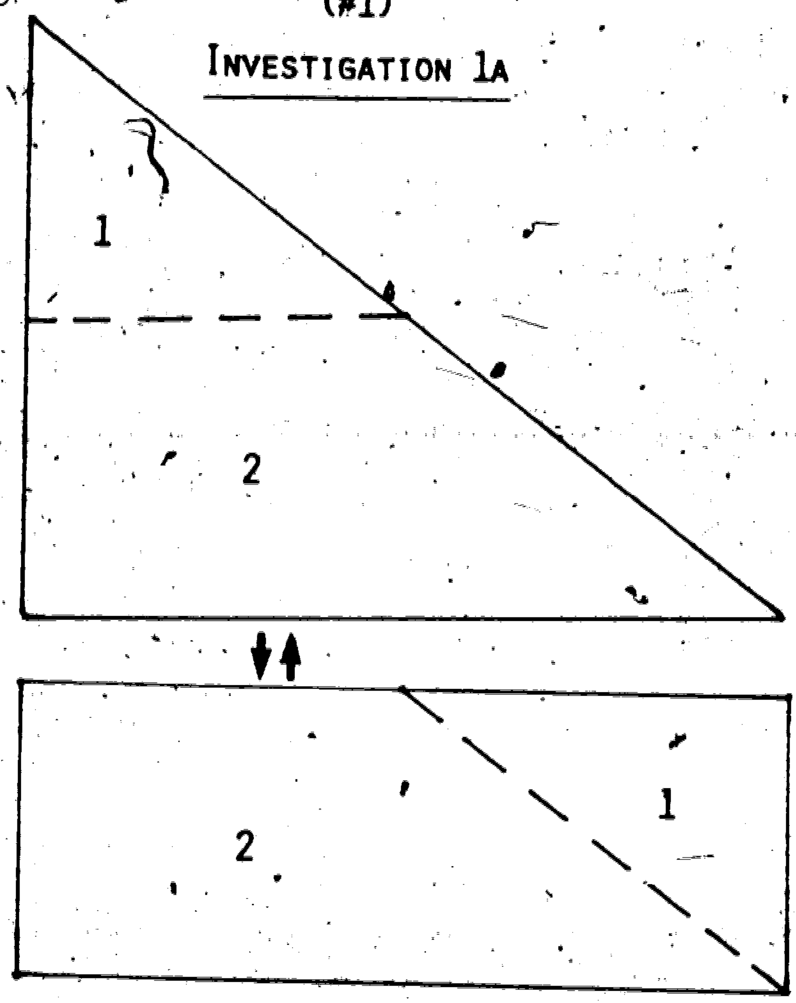
This section is part three of the UNIT repeated here. In addition, a package of magnetized card board models is prepared as a teaching aid for the derivations of tha formulae. It contains 14 modele representing the 14 figures included in part three. Each model is decomposed so that the related rearrangement can be demonstrated as well as the motions involved and that the related area formula can be deduced. The students may participate in demonstrating the motions and the rearrangements throughout this section.

TRANSPARENCIES' MASTERS

(#1)

INVESTIGATION 1A

A



B

FIG. 1

CONCLUSION:

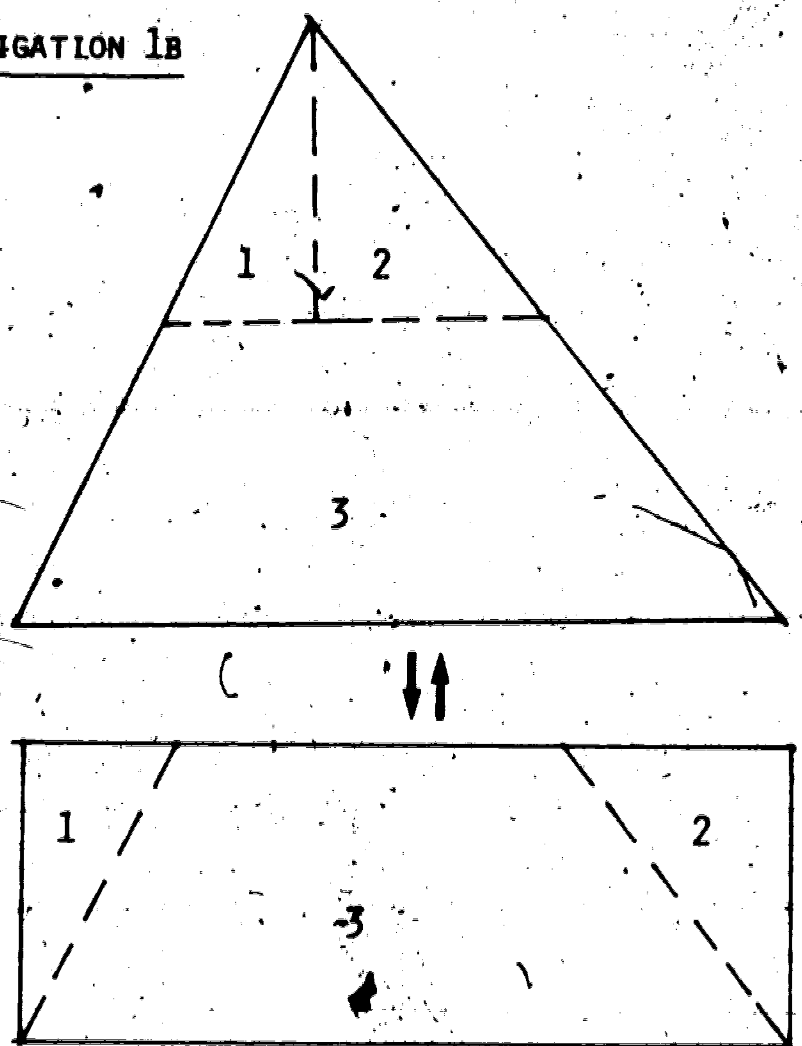
EACH RIGHT ANGLE TRIANGULAR REGION CAN BE DECOMPOSED INTO A RECTANGULAR REGION.

THE TWO REGIONS ARE PIECE - WISE CONGRUENT, THAT IS, THEY ARE OF EQUAL AREA.

IN OTHER WORDS: ONE REGION CAN BE CUT UP INTO PIECES WITH WHICH WE CAN COVER THE OTHER REGION COMPLETELY.

INVESTIGATION 1B

A



B

FIG. 2

CONCLUSION:

EACH ACUTE ANGLE TRIANGULAR REGION CAN BE DECOMPOSED INTO A RECTANGULAR REGION.

THE TWO REGIONS ARE PIECE-WICE CONGRUENT, THAT IS, THEY ARE OF EQUAL AREA.

IN OTHER WORDS: ONE REGION (ANY ONE) CAN BE CUT UP INTO PIECES SUCH THAT WE CAN COVER THE OTHER REGION COMPLETELY.

INVESTIGATION 1C

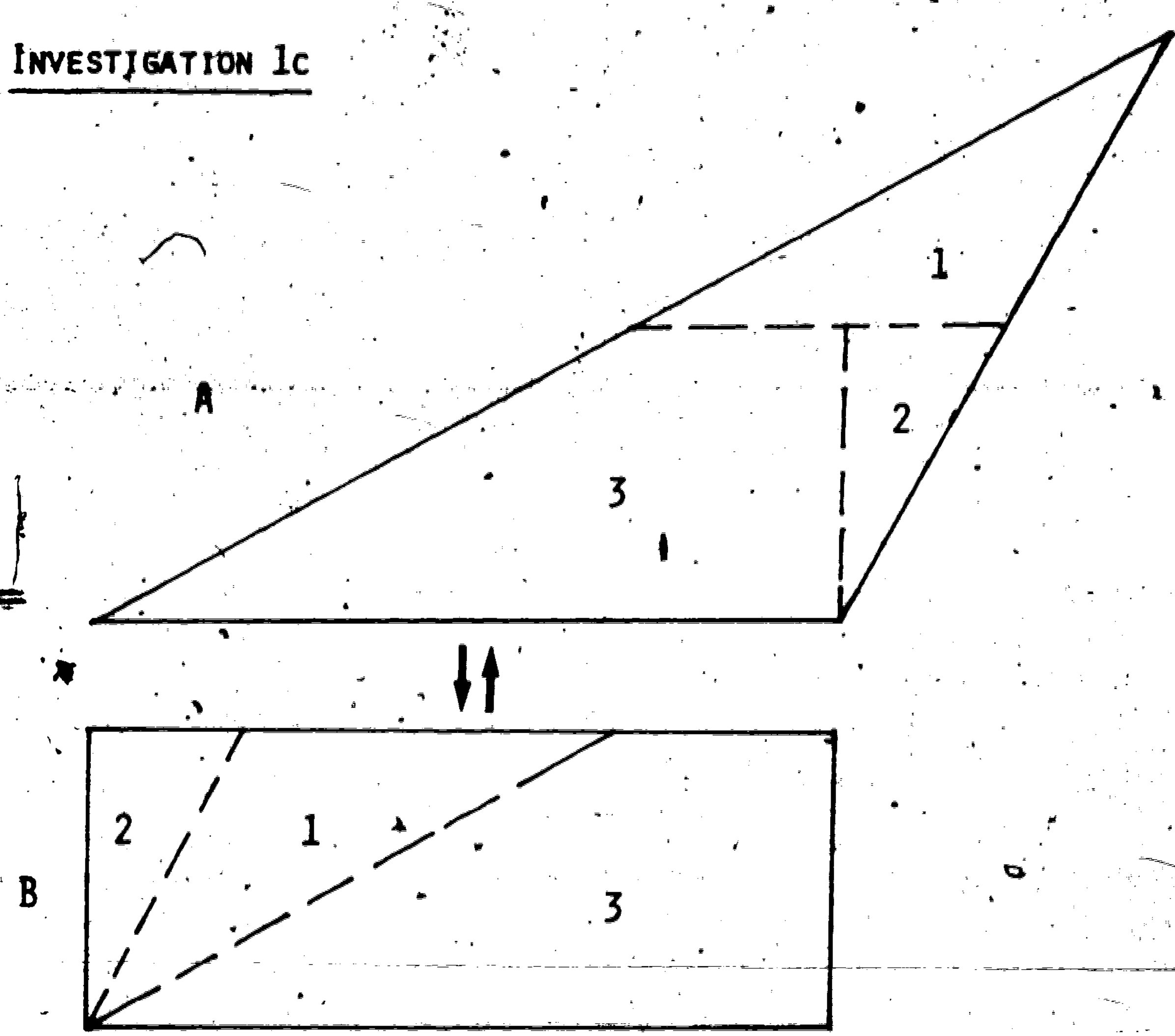


FIG. 3

CONCLUSION:

EACH OBTUSE TRIANGULAR REGION CAN BE DECOMPOSED INTO A RECTANGULAR REGION.

THE TWO REGIONS ARE PIECE-WISE CONGRUENT, THAT IS, THEY ARE OF EQUAL AREA.

IN OTHER WORDS: ONE REGION CAN BE CUT UP INTO PIECES SUCH THAT WE CAN COVER THE OTHER REGION COMPLETELY.

INVESTIGATION 1D:

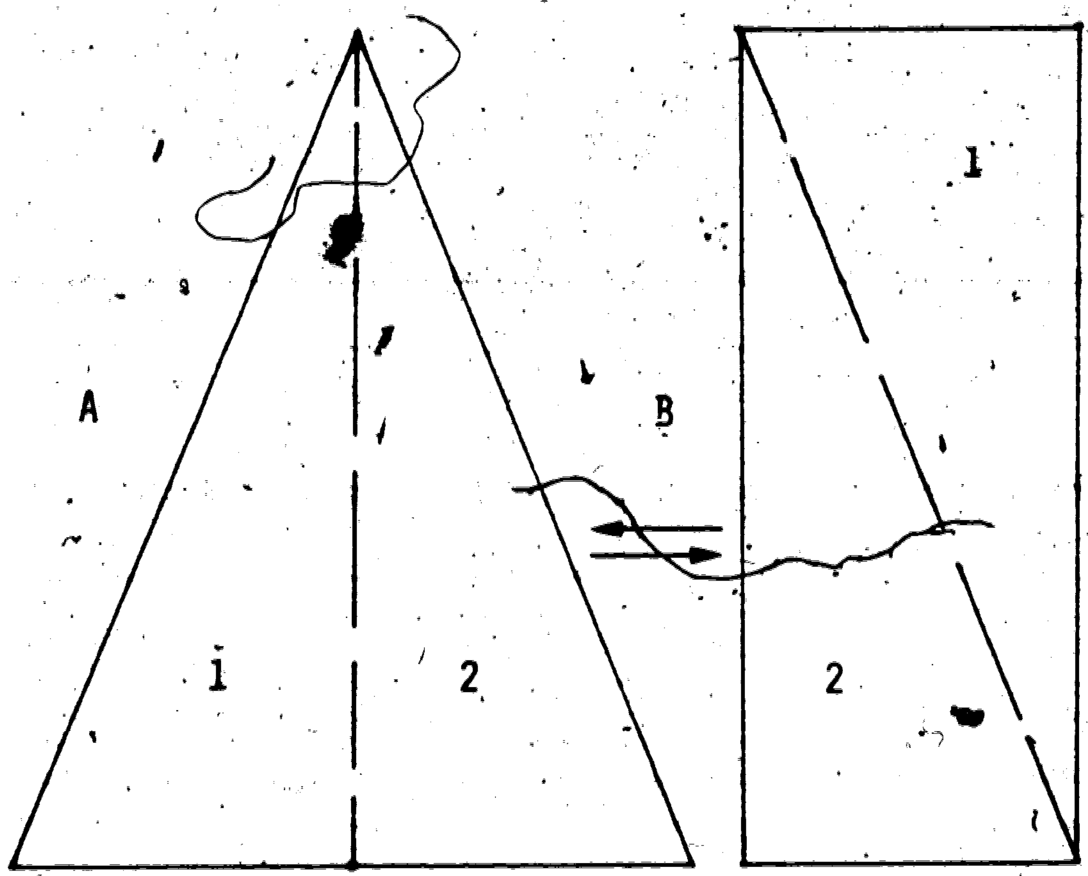


FIG. 4

CONCLUSION:

EACH ISOSCELES TRIANGULAR REGION CAN BE DECOMPOSED INTO A RECTANGULAR REGION.

THE TWO REGIONS ARE PIECE-WISE CONGRUENT, THAT IS, THEY ARE OF EQUAL AREA.

IN OTHER WORD : ONE OF THE TWO REGIONS (ANY ONE) CAN BE CUT UP INTO PIECES SUCH THAT THEY CAN BE REARRANGED TO COVER THE OTHER REGION COMPLETELY.

(# 5)

INVESTIGATION 1E

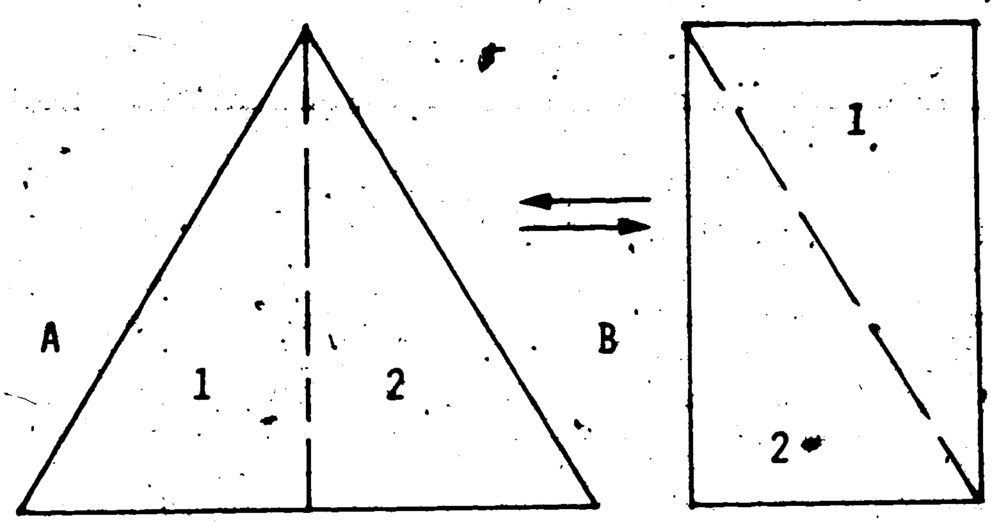


FIG. 5

CONCLUSION:

EACH EQUILATERAL TRIANGULAR REGION CAN BE DECOMPOSED INTO A RECTANGULAR REGION.

****

IN GENERAL, FROM THE FIVE PREVIOUS ACTIVITIES WE HAVE:

ANY TRIANGULAR REGION WHATSOEVER CAN BE DECOMPOSED INTO A RECTANGULAR REGION

INVESTIGATION 2 (# 6)

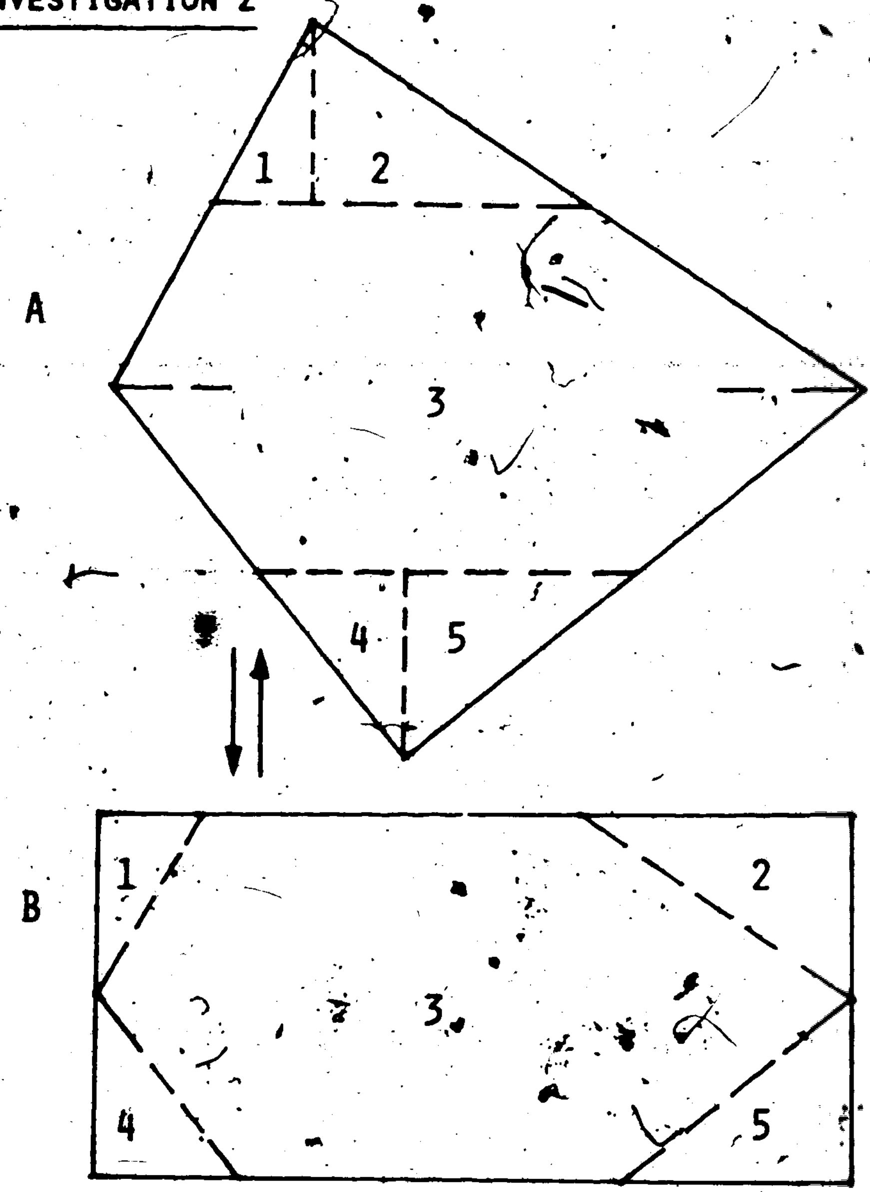


FIG. 6

CONCLUSION:

EACH CONVEX QUADRILATERAL REGION CAN BE DECOMPOSED INTO A RECTANGULAR REGION.

THE TWO REGIONS ARE PIECE-WISE CONGRUENT. I.E. ONE OF THEM CAN BE CUT INTO PIECES SUCH THAT THEY CAN BE REARRANGED TO COVER THE OTHER REGION COMPLETELY.

INVESTIGATION 3A

(# 7)

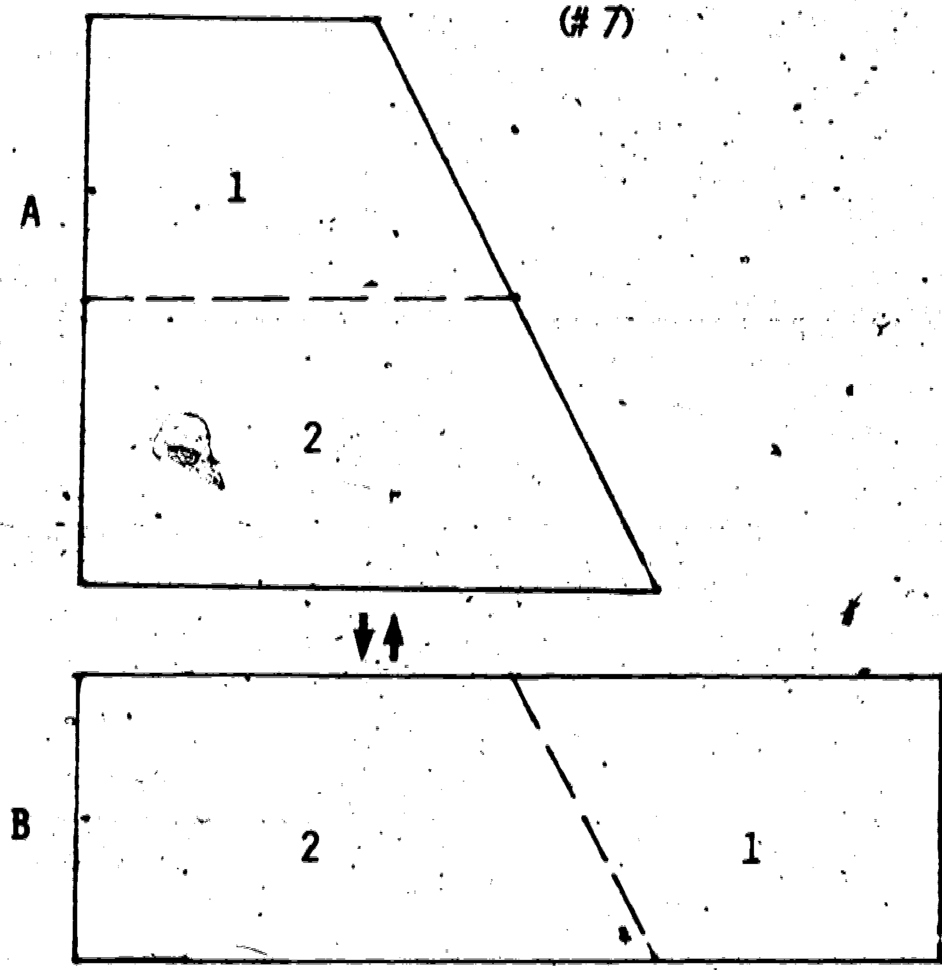


FIG. 7

CONCLUSION:

EACH RIGHT ANGLE TRAPEZOID REGION CAN BE DECOMPOSED INTO A RECTANGULAR REGION.

IN OTHER WORDS, THE TWO REGIONS A & B ARE PIECE-WISE CONGRUENT ...THAT IS, ONE REGION CAN BE CUT INTO PIECES BY WHICH WE CAN COVER THE OTHER REGION COMPLETELY.

INVESTIGATION 3B: THERE ARE MANY POSSIBLE DECOMPOSITIONS FOR THIS INVESTIGATION. THIS IS ONE OF THEM.

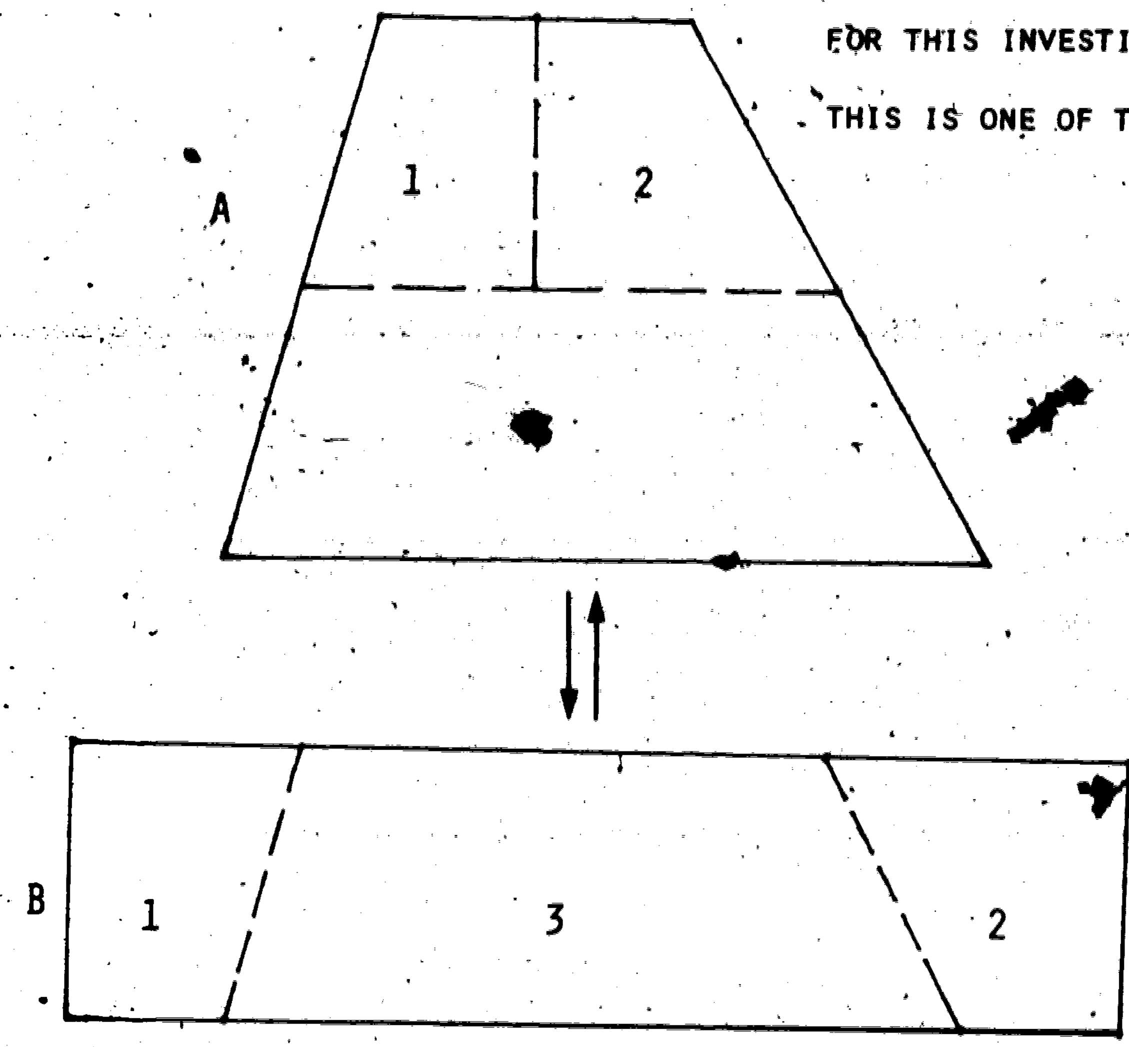


FIG. 8A

CONCLUSION:

EACH NON-REGULAR TRAPEZOID REGION CAN BE DECOMPOSED INTO A RECTANGULAR REGION.

IN OTHER WORDS, THE TWO REGIONS A AND B ARE PIECE-WISE CONGRUENT ... THAT IS, ONE REGION CAN BE CUT UP INTO PIECES WITH WHICH WE CAN COVER THE OTHER REGION COMPLETELY.

INVESTIGATION 3B - A SECOND POSSIBLE DECOMPOSITION.

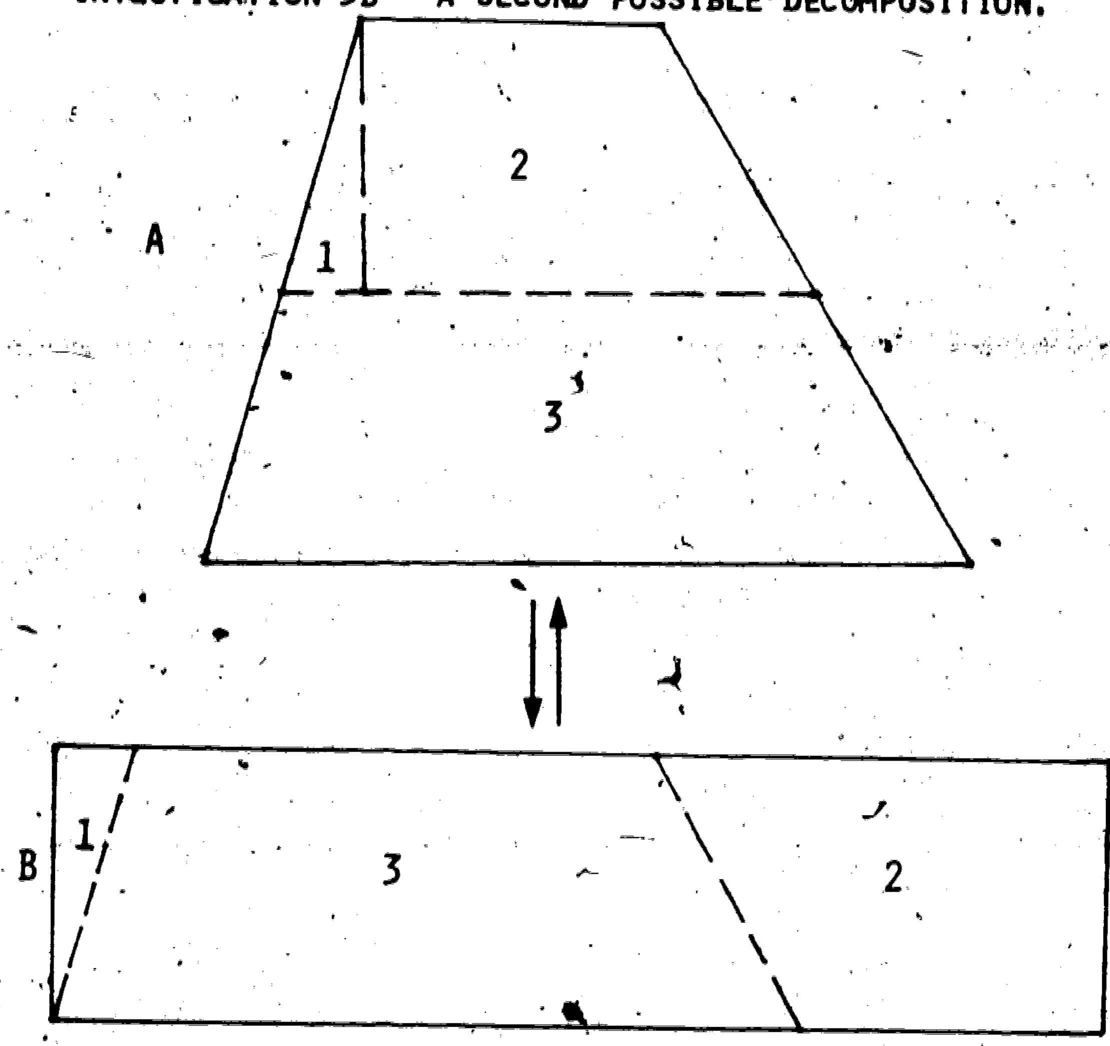


FIG. 8B
(2ND DECOMPOSITION)

CONCLUSION:

EACH NON-REGULAR TRAPEZOID REGION CAN BE DECOMPOSED INTO A RECTANGULAR REGION.

THE CONVERSE HOLDS.

IN OTHER WORDS, THE TWO REGIONS A & B ARE PIECE-WISE CONGRUENT ... THAT IS, ONE REGION CAN BE CUT UP INTO PIECES WITH WHICH WE CAN COVER THE OTHER REGION COMPLETELY.

INVESTIGATION 3B - A THIRD POSSIBLE DECOMPOSITION.

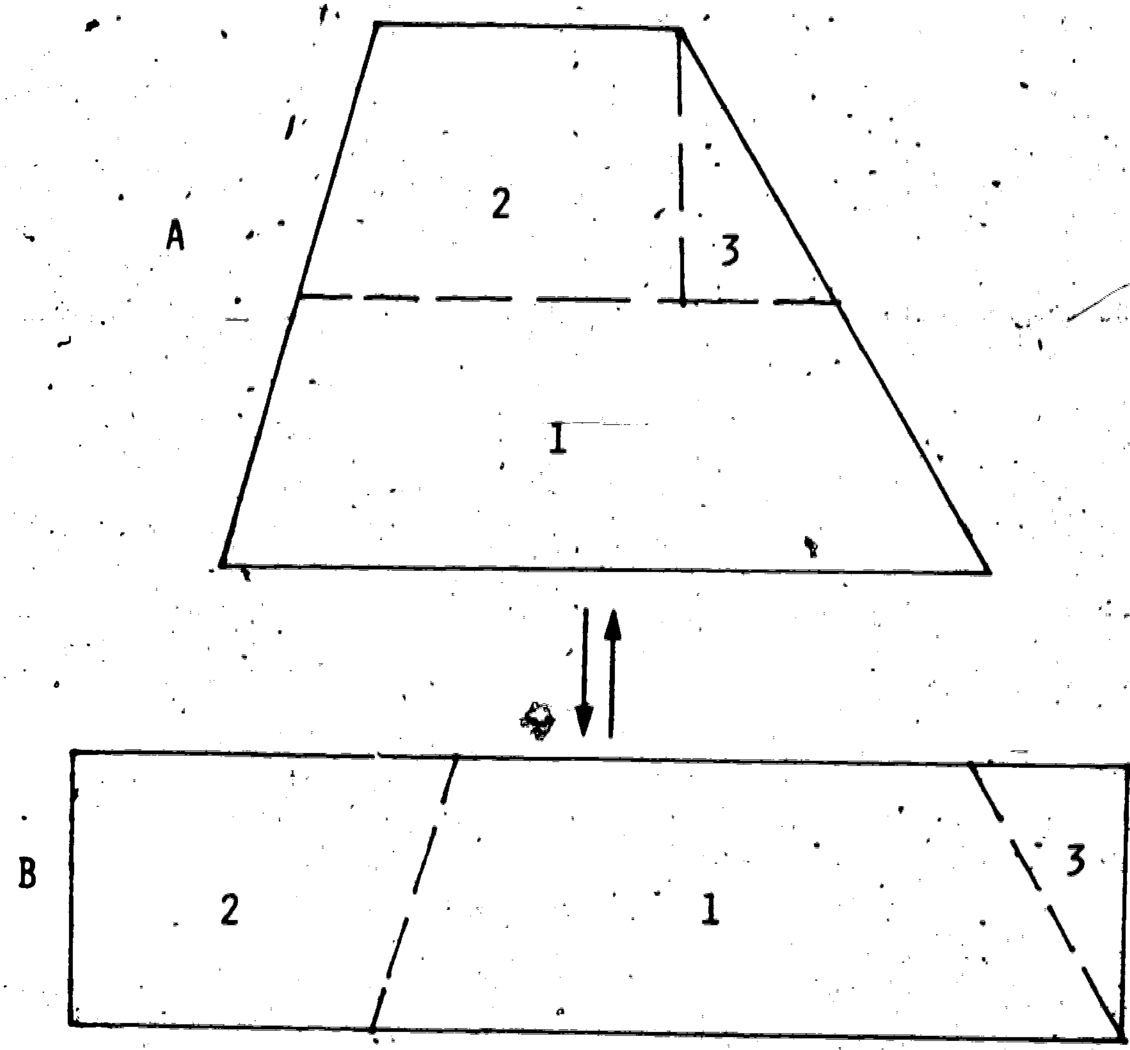


FIG. 8C
(3RD DECOMPOSITION)

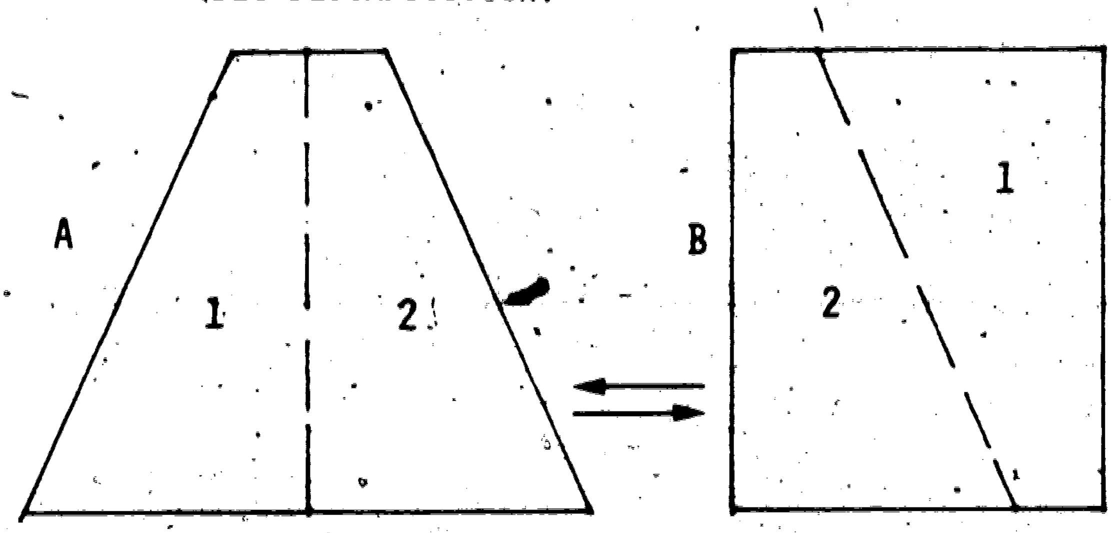
CONCLUSION:

EACH NON-REGULAR TRAPEZOID REGION CAN BE DECOMPOSED INTO A RECTANGULAR REGION.

IN OTHER WORDS, THE TWO REGIONS A & B ARE PIECE-WISE CONGRUENT ... THAT IS, ONE REGION CAN BE CUT UP INTO PIECES SUCH THAT THEY CAN BE REARRANGED TO COVER THE OTHER REGION COMPLETELY.

INVESTIGATION 3C:

ONE POSSIBLE DECOMPOSITION:



SECOND POSSIBLE DECOMPOSITION:

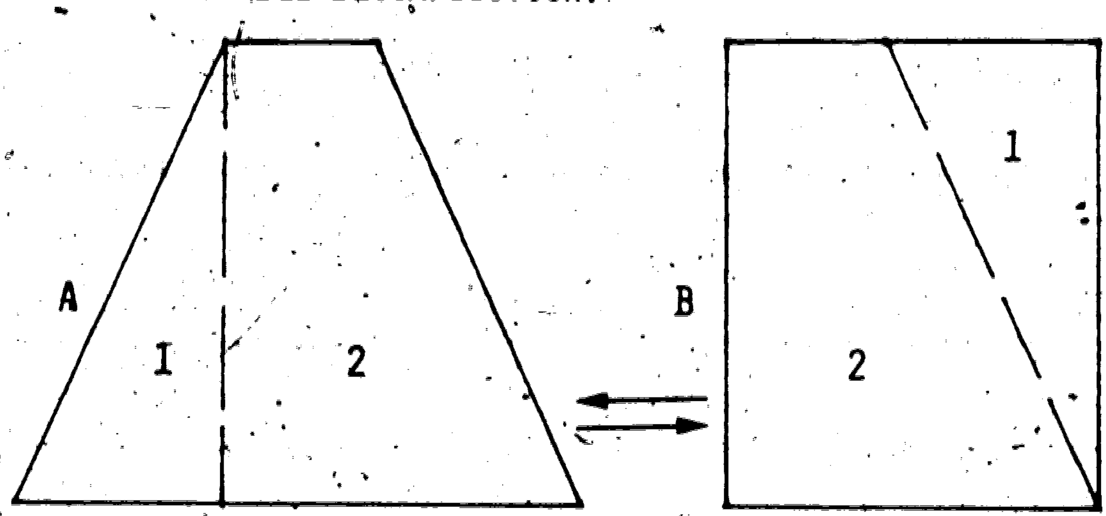


FIG. 9

CONCLUSION:

EACH REGULAR TRAPEZOID REGION CAN BE DECOMPOSED INTO A RECTANGULAR REGION.
IN OTHER WORDS, THE TWO REGIONS A & B ARE PIECE-WISE CONGRUENT ... THAT IS, ONE REGION CAN BE CUT UP INTO PIECES WITH WHICH WE CAN COVER THE OTHER REGION COMPLETLY.

FROM THE INVESTIGATIONS 3A, 3B, & 3C, WE CONCLUDE THAT ANY TRAPEZOID REGION IS DECOMPOSABLE INTO A RECTANGULAR REGION.

(# 12)

INVESTIGATION 4:

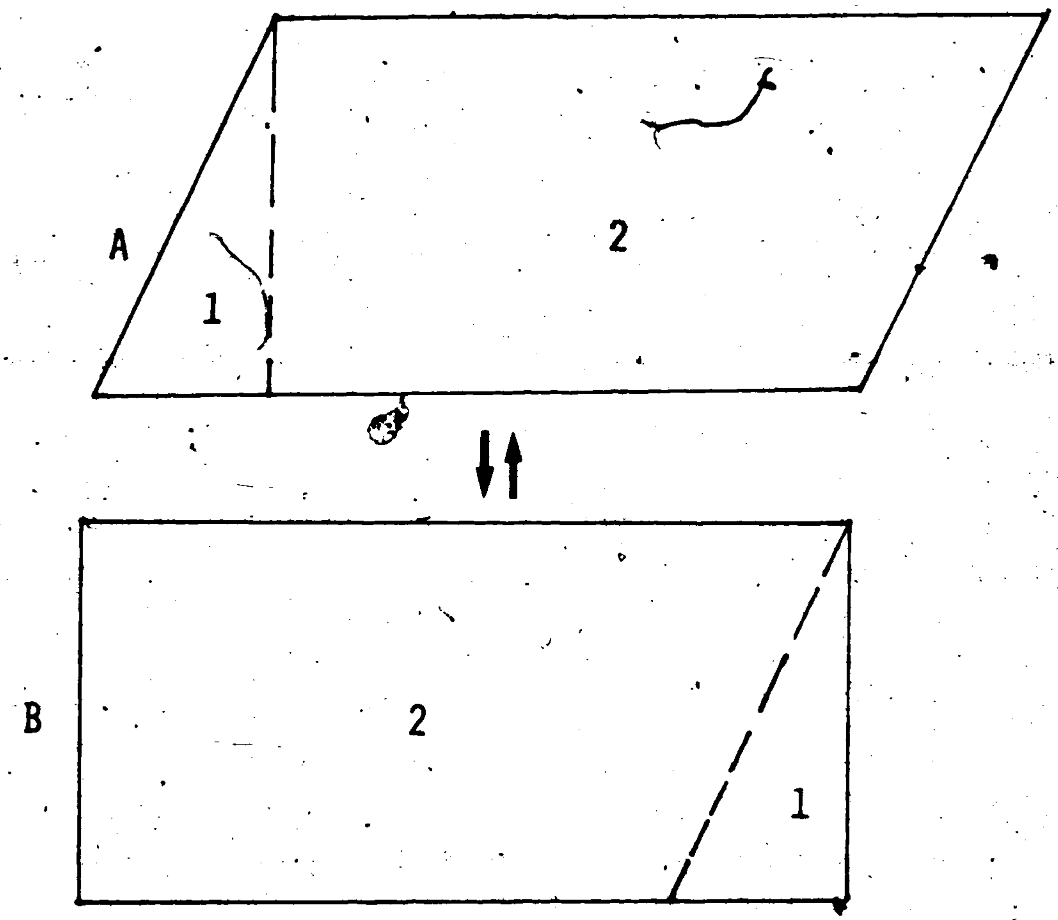


FIG. 10

CONCLUSION:

EACH PARALLELOGRAM REGION CAN BE DECOMPOSED INTO A RECTANGULAR REGION.

IN OTHER WORDS, THE TWO REGIONS A & B ARE PIECE-WISE CONGRUENT ... THAT IS, ONE REGION CAN BE CUT UP INTO PIECES WITH WHICH WE CAN COVER THE OTHER REGION COMPLETELY.

INVESTIGATION 5:

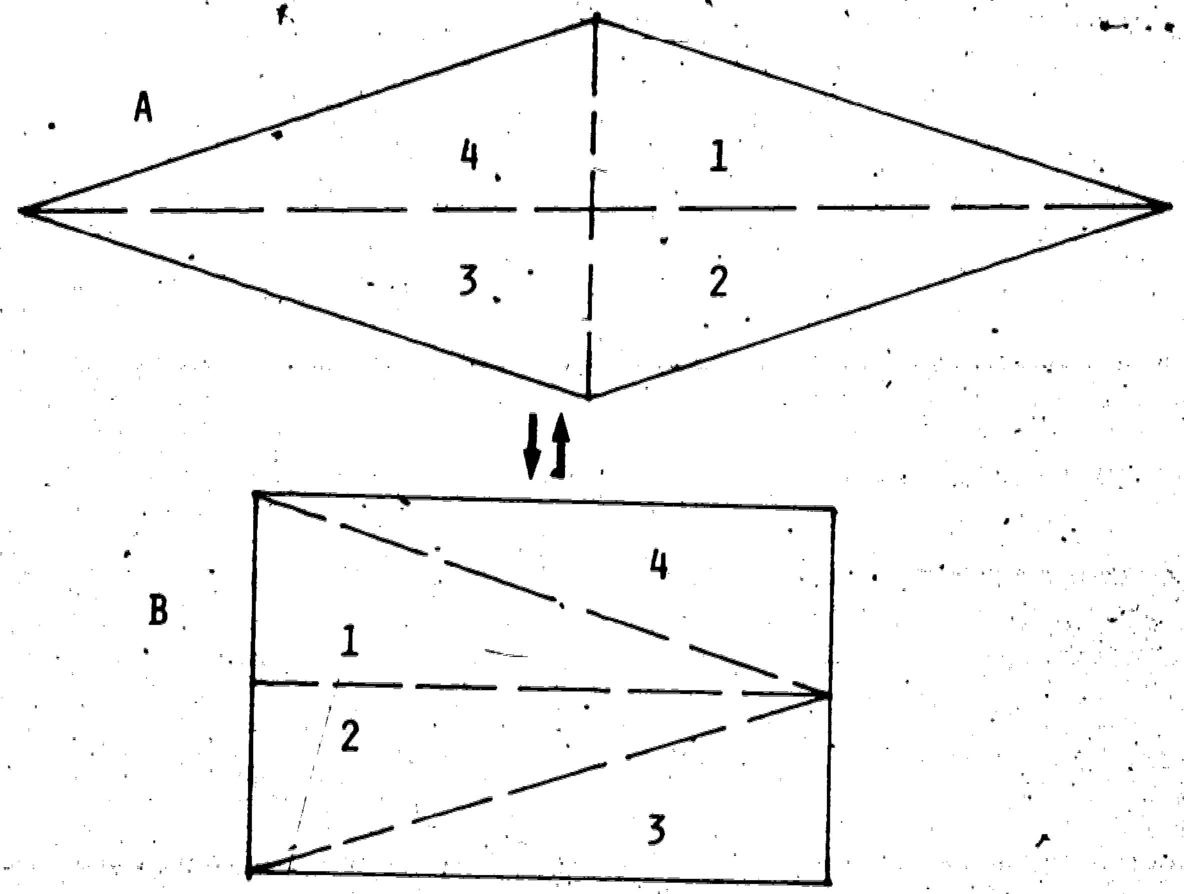


FIG. 11

CONCLUSION:

ANY RHOMBUS REGION CAN BE DECOMPOSED INTO A RECTANGULAR REGION.

IN OTHER WORDS, THE TWO REGIONS ARE PIECE-WISE CONGRUENT ... THAT IS ONE OF THE REGIONS A & B CAN BE CUT UP INTO PIECES SUCH THAT THEY CAN BE REARRANGED TO COVER THE OTHER REGION COMPLETELY.

INVESTIGATION 6: (# 14)

FIRST POSSIBLE DECOMPOSITION:

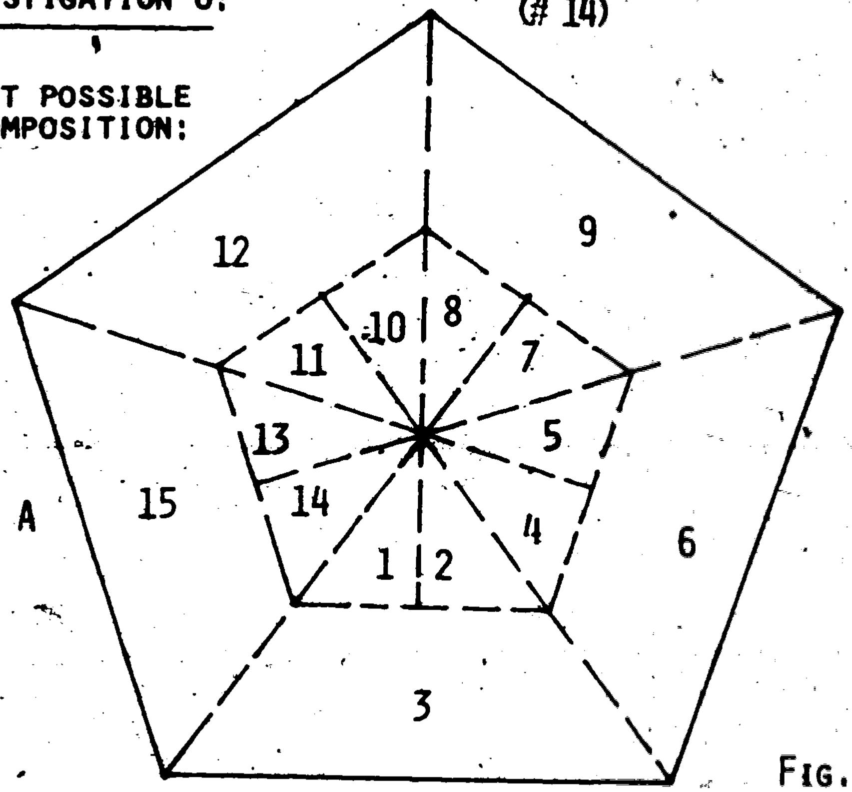
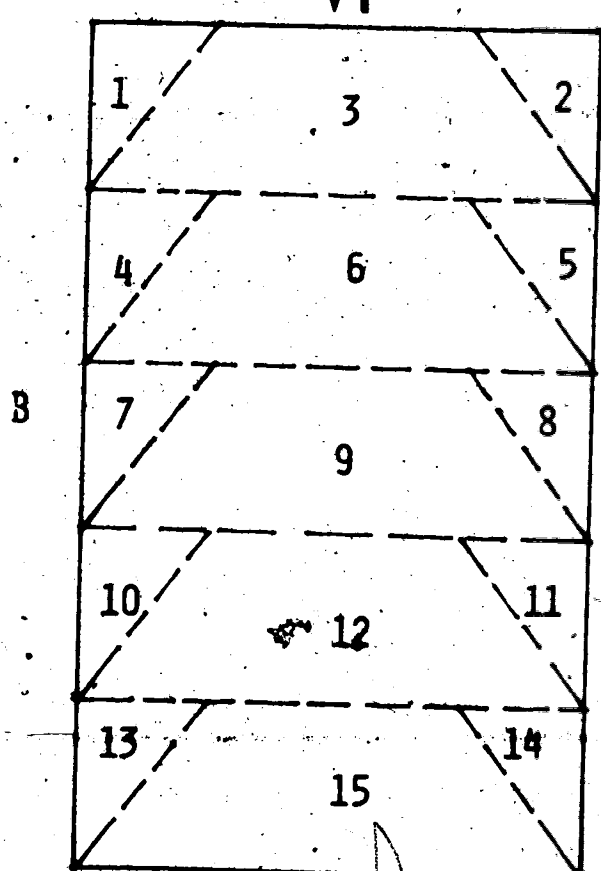


FIG. 12A



CONCLUSION:

ANY REGULAR PENTAGON REGION CAN BE DECOMPOSED INTO A RECTANGULAR REGION.

IN OTHER WORDS, THE TWO REGIONS A & B ARE PIECE-WICE CONGRUENT... THAT IS, ONE OF THE REGIONS CAN BE CUT UP INTO PIECES TO COVER THE OTHER REGION COMPLETELY.

FIG. 12B

INVESTIGATION 6: (# 15)

SECOND POSSIBLE DECOMPOSITION :

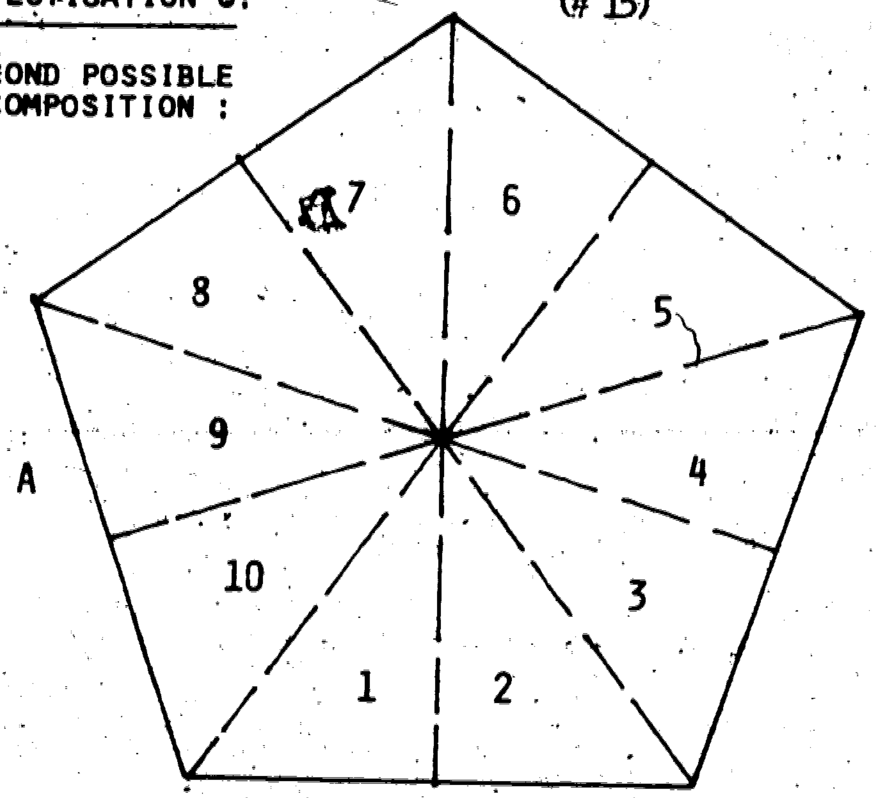


FIG. 12A

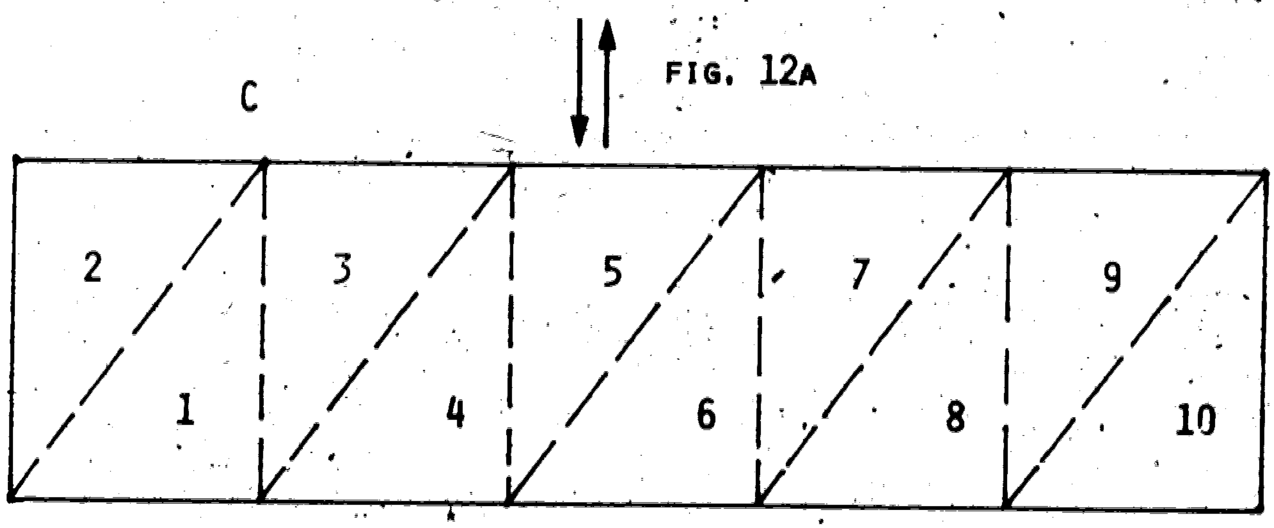


FIG. 12C

CONCLUSION:

ANY REGULAR PENTAGON REGION CAN BE DECOMPOSED INTO A RECTANGULAR REGION.

IN OTHER WORDS, THE TWO REGIONS A & C ARE PIECE-WISE CONGRUENT ... THAT IS, ONE REGION CAN BE CUT UP INTO PIECES WITH WHICH WE CAN COVER THE OTHER REGION COMPLETELY.

(# 16)

INVESTIGATION 7: FIRST POSSIBLE DECOMPOSITION:

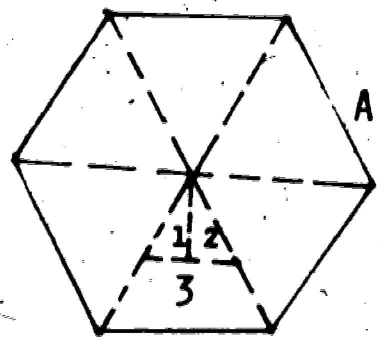


FIG. 13A

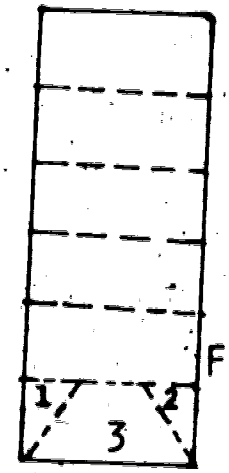


FIG. 13B

SECOND POSSIBLE DECOMPOSITION :

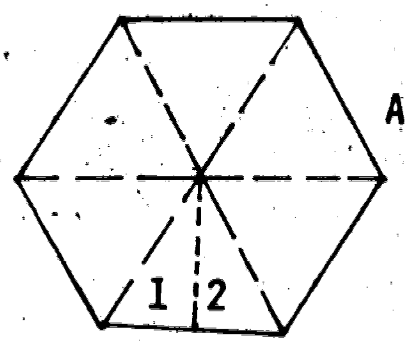


FIG. 13A

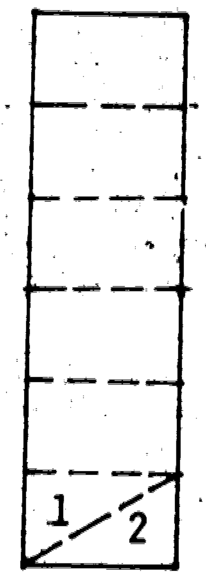


FIG. 13C

CONCLUSION: ANY REGULAR HEXAGON REGION CAN BE DECOMPOSED INTO A RECTANGULAR REGION.

IN OTHER WORDS, THE TWO REGIONS A & B ARE PIECE-WISE CONGRUENT ... THAT IS, ONE REGION CAN BE CUT UP INTO PIECES SUCH THAT THEY CAN BE REARRANGED TO COVER THE OTHER REGION COMPLETELY.
(SIMILARLY FOR REGIONS A & C).

INVESTIGATION 8:

FIRST POSSIBLE DECOMPOSITION :

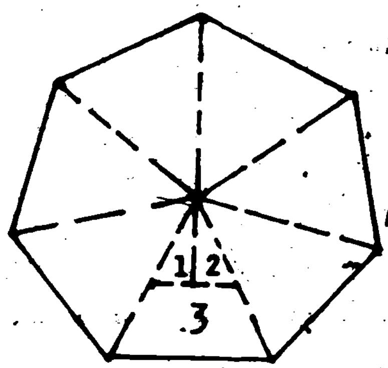
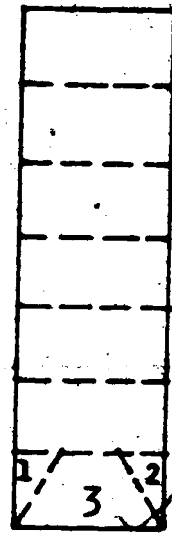


FIG. 14A



B

FIG. 14B

SECOND POSSIBLE DECOMPOSITION :

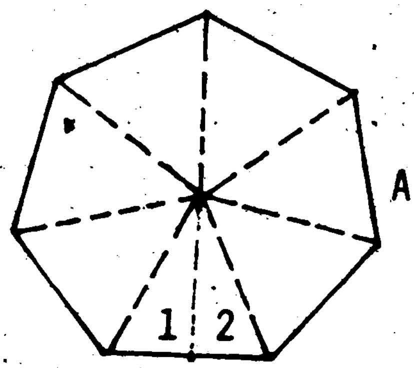
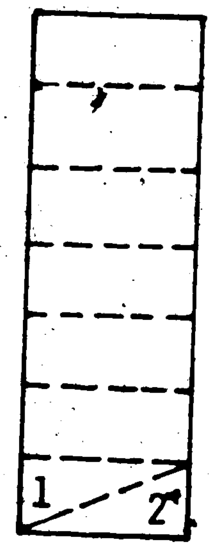


FIG. 14A



C

FIG. 14C

CONCLUSION: ANY REGULAR HEPTAGON REGION CAN BE DECOMPOSED INTO A RECTANGULAR REGION.

IN OTHER WORDS, THE TWO REGIONS A & B ARE PIECE-WISE CONGRUENT ... THAT IS, ONE REGION CAN BE CUT UP INTO PIECES SUCH THAT THEY CAN BE REARRANGED TO COVER THE OTHER REGION COMPLETELY.
(SIMILARLY FOR REGIONS A & C).

INVESTIGATION 9: (# 18)

FIRST POSSIBLE DECOMPOSITION:

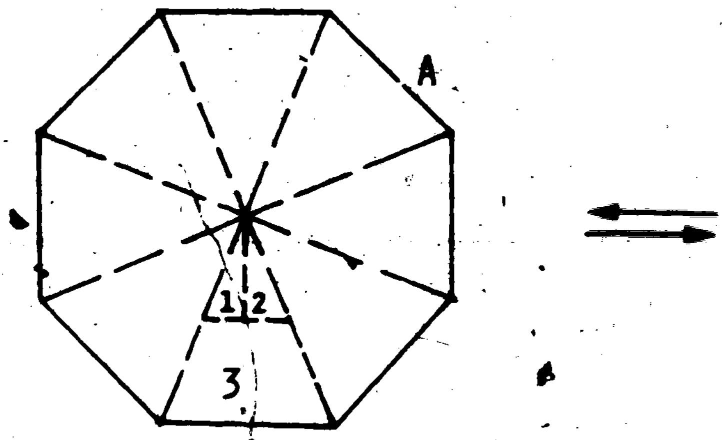


FIG. 15A

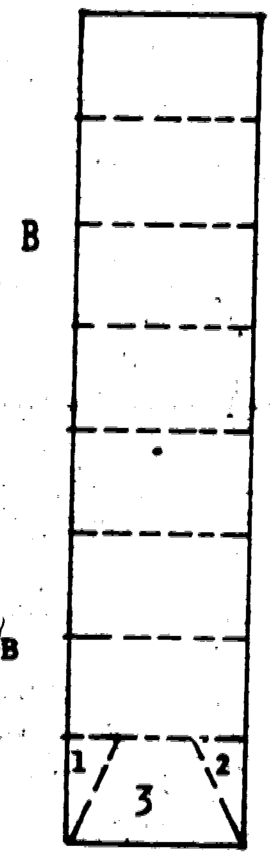


FIG. 15B

SECOND POSSIBLE DECOMPOSITION :

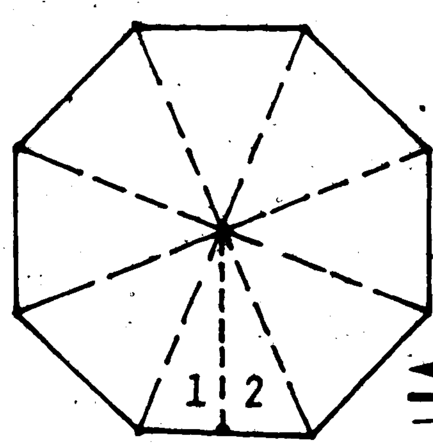


FIG. 15A

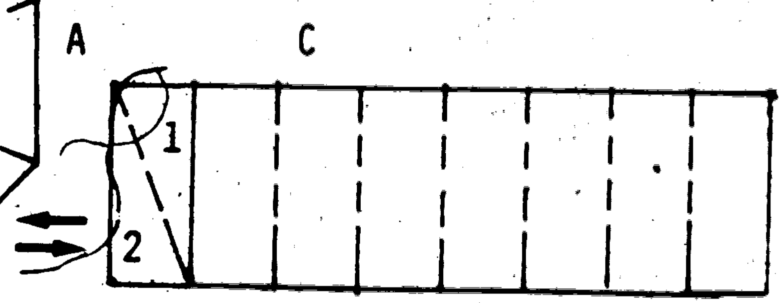


FIG. 15C

CONCLUSION: ANY REGULAR OCTAGON REGION CAN BE DECOMPOSED INTO A RECTANGULAR REGION.

THAT IS, ONE REGION CAN BE CUT UP INTO PIECES WITH WHICH WE CAN COVER THE OTHER REGION COMPLETELY ... IN OTHER WORDS, THE TWO REGIONS A AND B ARE PIECE-WISE CONGRUENT.
(SIMILARLY FOR REGIONS A & C).

INVESTIGATION 10:

FIRST POSSIBLE DECOMPOSITION:

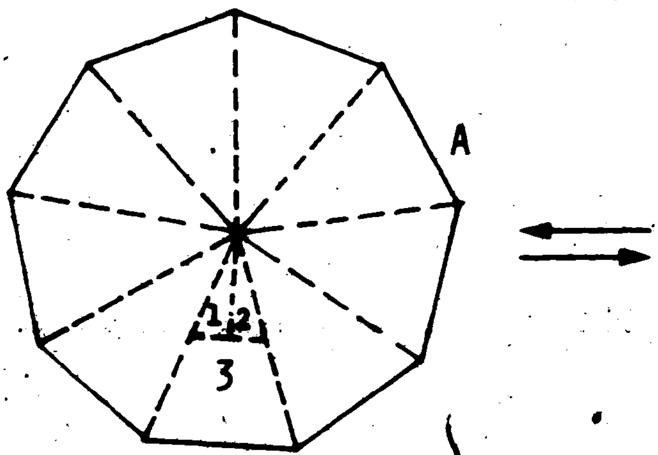


FIG. 16A

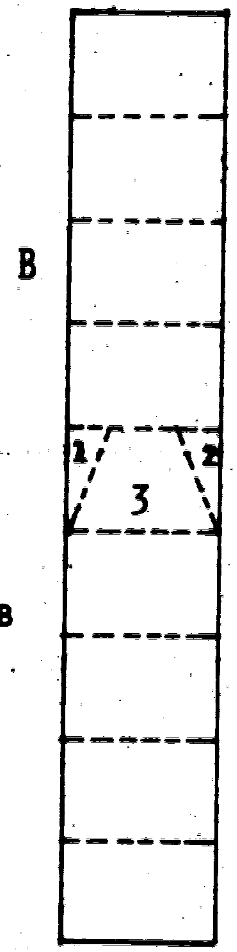


FIG. 16B

SECOND POSSIBLE DECOMPOSITION :

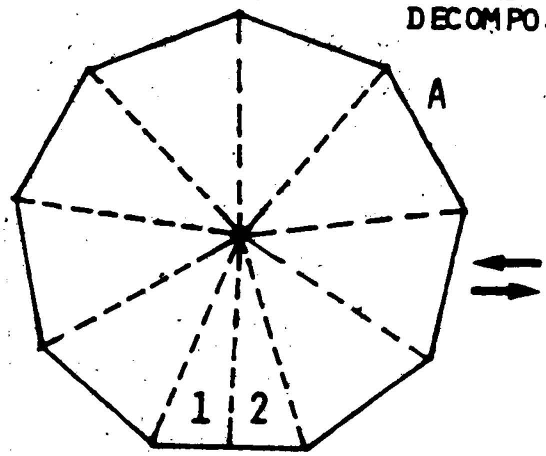


FIG 16A.

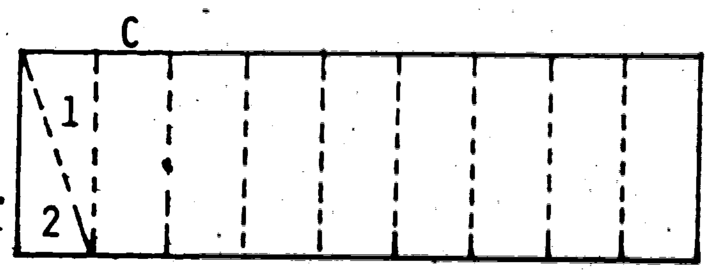


FIG. 16C

CONCLUSION: ANY REGULAR NONAGON REGION CAN BE DECOMPOSED INTO A RECTANGULAR REGION.

IN OTHER WORDS, THE TWO REGIONS A & B ARE PIECE-WISE CONGRUENT THAT IS, ONE REGION CAN BE CUT INTO PIECES WITH WHICH WE CAN COVER THE OTHER REGION COMPLETELY.

(SIMILARLY FOR REGIONS A & C).

INVESTIGATION 11: FIRST POSSIBLE DECOMPOSITION: (# 20)

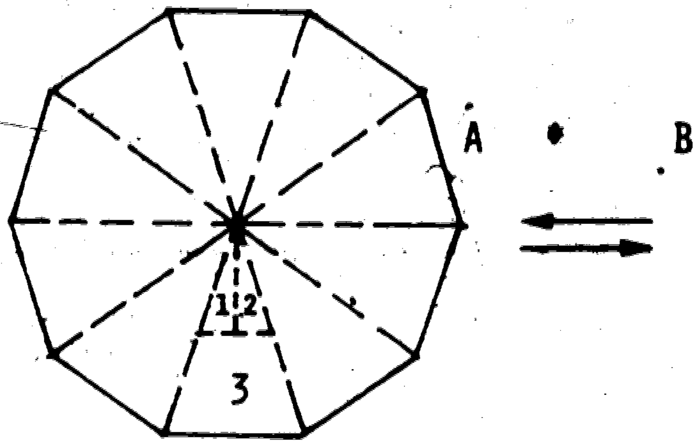


FIG. 17A

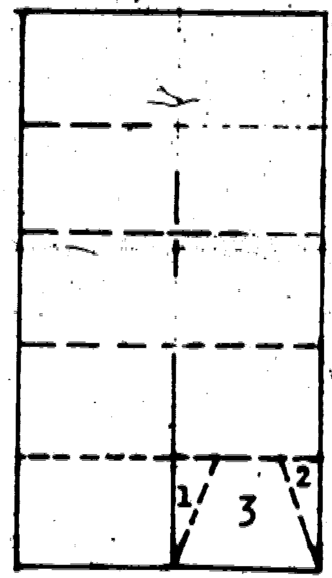


FIG. 17B

SECOND POSSIBLE DECOMPOSITION :

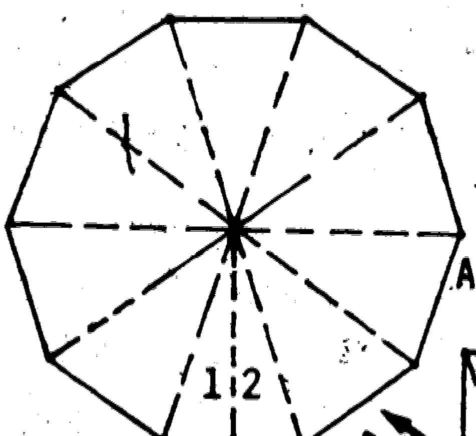


FIG. 17A

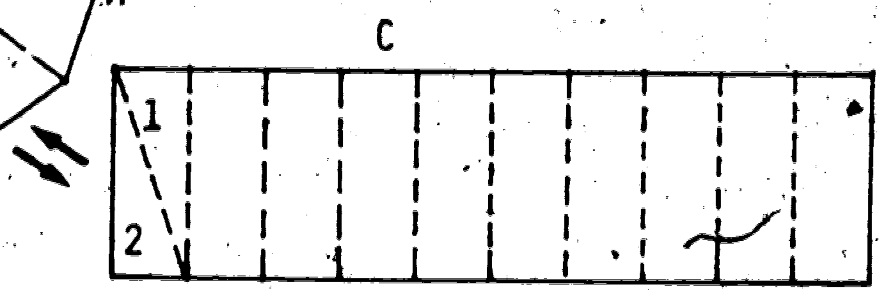


FIG. 17C

CONCLUSION:

ANY REGULAR DECAGON REGION CAN BE DECOMPOSED INTO A RECTANGULAR REGION.

IN OTHER WORDS THE TWO REGIONS A & B ARE PIECE-WISE CONGRUENT THAT IS, ONE REGION CAN BE CUT UP INTO PIECES WITH WHICH WE CAN COVER THE OTHER REGION COMPLETELY.
(SIMILARLY FOR REGIONS A & C).

INVESTIGATION 12: FIRST POSSIBLE DECOMPOSITION:

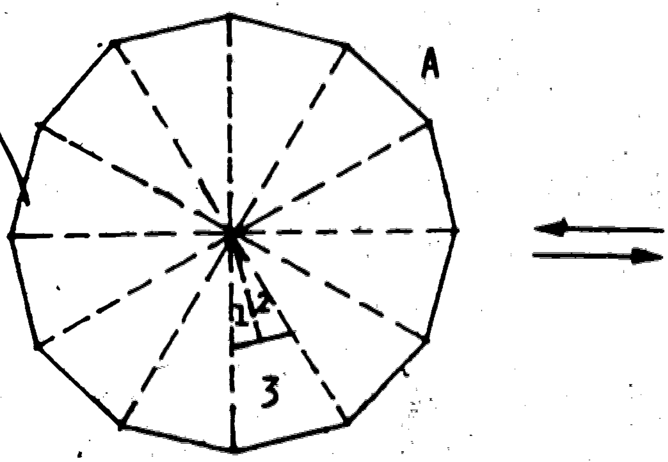


FIG. 18A

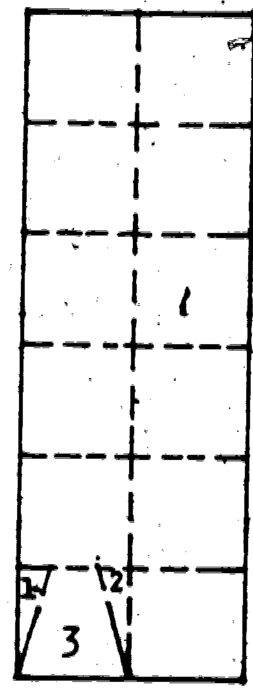


FIG. 18B

SECOND POSSIBLE DECOMPOSITION :

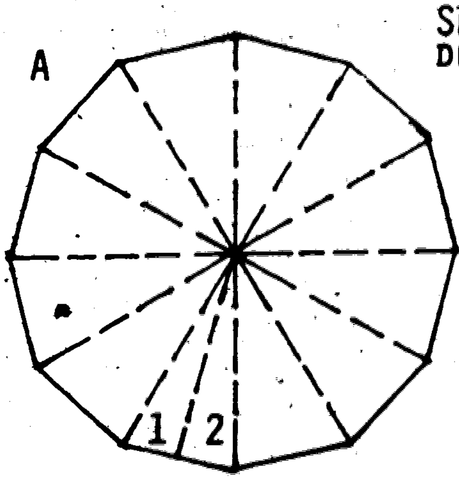


FIG. 18A

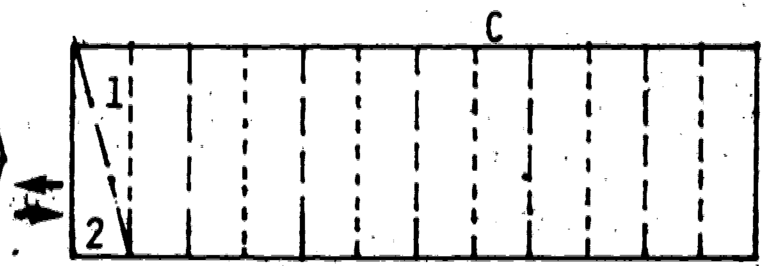


FIG. 18C

CONCLUSION: ANY REGULAR DODECAGON REGION CAN BE DECOMPOSED INTO A RECTANGULAR REGION.

IN OTHER WORDS THE TWO REGIONS A & B ARE PIECE-WISE CONGRUENT ... THAT IS, ONE REGION CAN BE DECOMPOSED INTO PIECES WITH WHICH WE CAN COVER THE OTHER REGION COMPLETELY. (SIMILARLY FOR REGIONS A & C).

$$$$$$$$$

APPENDIX C

OPINION ASSESSMENT ON GEOMETRY

NAME ____________________ SCHOOL ____________________

GRADE ____________________ DATE ____________________

In each of the fifteen items below there are five categories of responding. Circle the one that you believe best describes your opinion. Note that there are no right or wrong answers since your responses depend simply on your opinion. Just give your honest feeling to each of the items below. Notice that the meaning of SD, D, I, A, and SA are as follows:

SD -- means Strongly Disagree

D -- means Disagree

I -- means Indifferent

A -- means Agree

SA -- means Strongly Agree

1.	Geometry is a pleasant subject.	SD	D	I	A	SA
2.	Geometry makes me embarrased.	SD	D	I	A	SA
3.	I like to do geometric problems in other subjects.	SD	D	I	A	SA
4.	I like working on my geometry homework.	SD	D	I	A	SA
5.	When I have to do geometry I get nervous.	SD	D	I	A	SA
6.	I have always liked geometry in school.	SD	D	I	A	SA
7.	Geometry is a boring subject.	SD	D	I	A	SA
8.	Geometry is not a useful subject.	SD	D	I	A	SA
9.	I can see geometry everywhere.	SD	D	I	A	SA

10.	I am interested in studying more geometry in coming years of my schooling.	SD	D	I	A	SA
11.	Geometry is my most hated subject.	SD	D	I	A	SA
12.	Geometry is not a practical subject.	SD	D	I	A	SA
13.	I enjoy trying difficult problems in geometry.	SD	D	I	A	SA
14.	I would like to take more geometry when I have the opportunity.	SD	D	I	A	SA
15.	I like to help others with geometry.	SD	D	I	A	SA

****

APPENDIX D

OPINION ASSESSMENT ON MATHEMATICS

NAME ____________________ SCHOOL ____________________

GRADE ____________________ DATE ____________________

In each of the fifteen items below there are five categories of responding. Circle the one that you believe best describes your opinion. Note that there are no right or wrong answers since your responses depend simply on your opinion. Just give your honest feeling to each of the items below. Notice that the meaning of SD, D, I, A, and SA are as follows:

SD -- means Strongly Disagree

D -- means Disagree

I -- means Indifferent

A -- means Agree

SA -- means Strongly Agree

1. I like to help others with mathematics.	SD D I A SA
2. I would like to take more mathematics when I have the opportunity.	SD D I A SA
3. I enjoy trying difficult problems in mathematics.	SD D I A SA
4. Mathematics is not a practical subject.	SD D I A SA
5. Mathematics is my most hated subject.	SD D I A SA
6. I am interested in studying more mathematics in coming years of my schooling.	SD D I A SA
7. I can see mathematics everywhere.	SD D I A SA
8. Mathematics is not a useful subject.	SD D I A SA

9. Mathematics is a boring subject.	SD	D	I	A	SA
10. I have always liked mathematics in school.	SD	D	I	A	SA
11. When I have to do mathematics I get nervous.	SD	D	I	A	SA
12. I like working on my mathematics homework.	SD	D	I	A	SA
13. I like to do mathematical problems in other subjects.	SD	D	I	A	SA
14. Mathematics makes me embarrased.	SD	D	I	A	SA
15. Mathematics is a pleasant subject.	SD	D	I	A	SA

****

APPENDIX E

TEACHERS' OPINION CHECKING ON THE UNIT

This survey is an attempt to obtain your present opinion on the use of the Piece-Wise Congruency operation in teaching polygonal regions in a plane. You are being asked these questions because of your participation in the project involving the use of the Piece-Wise Congruency operation in teaching polygonal regions, their interrelationships, and their area formulae. Your opinion will be of assistance in forming an overall assessment on the status of geometry and the use of a manipulative (concrete) technique in a laboratory setting for geometry instruction based on learning by doing.

Names of students and teachers will not appear in any final statement that may be made as a result of the project.

1. In general, do you favor the use of the Piece-Wise Congruency operation in geometry instruction?

   Yes _____ No _____

2. If you answered "Yes" to question 1, at what grade do you think the notion of Piece-Wise Congruency would be most appropriate?

   Below grade 7 _____ Grade 7 _____ Grade 8 _____

   Grade 9 _____ Grade 10 _____ Grade 11 _____

   Grade 12 _____

3. If you answered "Yes" to question 1, please elaborate on why you favor the use of the Piece-Wise Congruency notion.

   ____

   ____

   ____

   ____

   ____

   ____

   ____

4. Do you think that the use of the laboratory approach that involved tracing, cutting, and covering of polygonal regions creates chaotic situations in the classroom?

Yes _____ No _____

5. If you answered "Yes" to question 4, please give your reasons.

____

____

____

____

____

____

6. If you answered "No" to question 4, please give your reasons.

____

____

____

____

____

____

7. In what way did the use of the Cut-and-Cover technique in grade 8 classroom affect your students' performance in solving spatial two-dimension problems?

____

____

____

____

____

8. In what way did the use of th[illegible]Wise Congruency notion in grade 8 classroom affect [illegible]udents' performance in mathematics?

9. In what way did the use of the decomposition technique in grade 8 classroom affect your students' attitude towards geometry?

10. Will the use of the Piece-Wise Congruency operation in grade 8 make your students more self-confedence in performing spatial tasks?

11, In what way did the use of Cut-and-Cover technique in grade 8 classroom affect your students' attitude towards mathematics?

____

____

____

____

____

12. In your opinion, should the Edmonton Public School System adopt the use of the Piece-Wise Congruency approach in grade 8 classroom?

Yes _____ No _____ No opinion _____

13. If you answered "Yes" to quetion 12, in what level do you think the Piece-Wise Congruency might be appropriate?

____

____

____

14. In your opinion, should the Edmonton Public School System provide instructional materials to its students in the use of the Piece-Wise Congruency approach?

Yes _____ No _____ No opinion _____

15. If you wish, make any comment you like about the use of the Piece-Wise Congruency approach in school.

Your comment:

(You may use the other side)

APPENDIX F

GEOMETRY TESTS*
-SIX SUBTESTS-

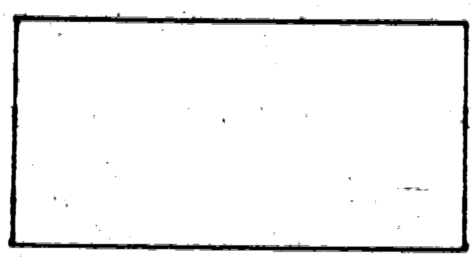
(*) The first polygonal region of item number 1 in both, subtest iv and subtest v are taken from MATHEMATICS CALENDER 79 of Springer-Varlag, 1978.

POLYGONAL REGIONS
VOCABULARY TEST

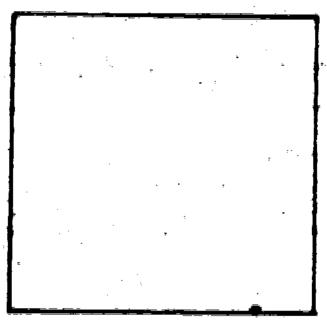
NAME ____________________ SCHOOL ____________________

GRADE ____________________ DATE ____________________

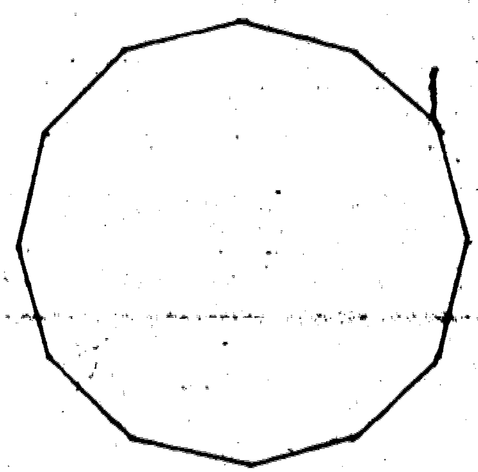
BELOW IS A COLLECTION OF FIGURES IN A PLANE. NAME EACH OF THEM.

1. 

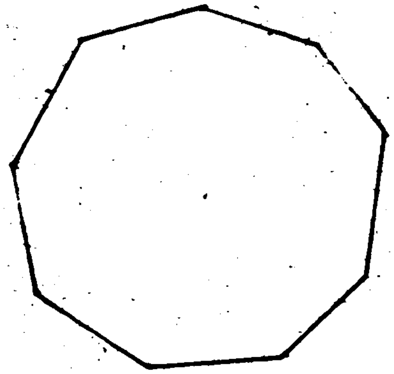
Name: ____________________

2. 

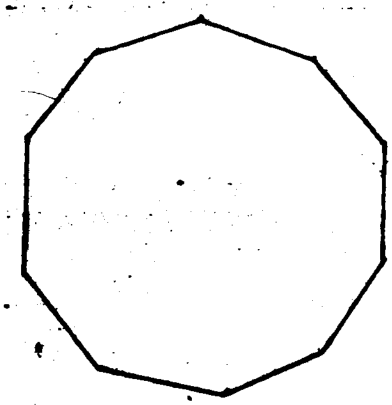
name: ____________________

3. 

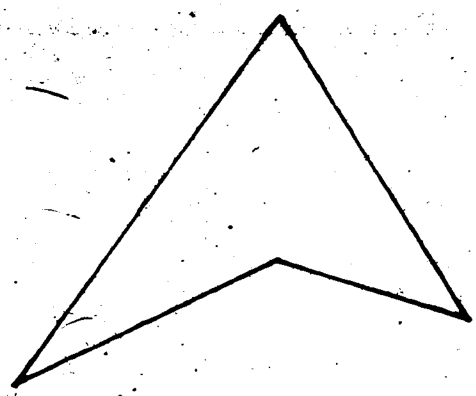
Name: ____________________

4. 

Name: ______________

5. 

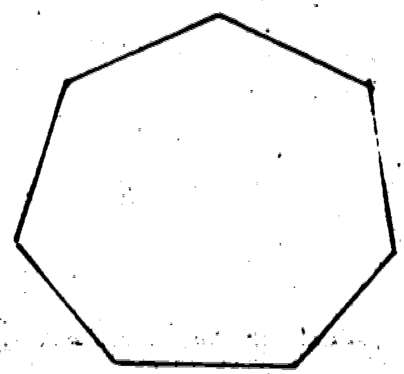
Name: ______________

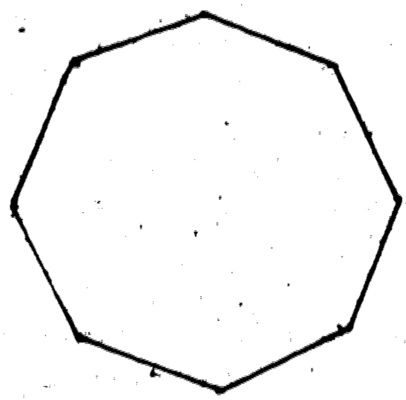
6. 

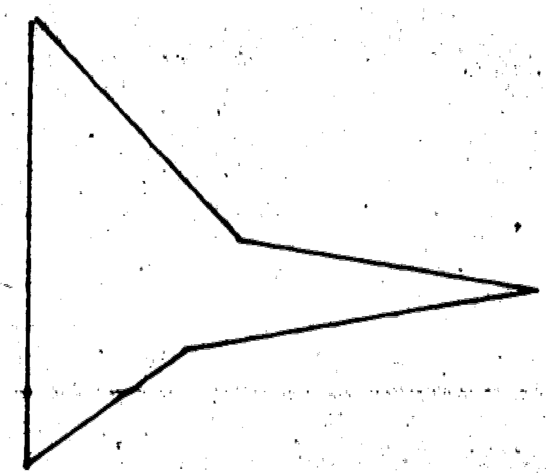
Name: ______________

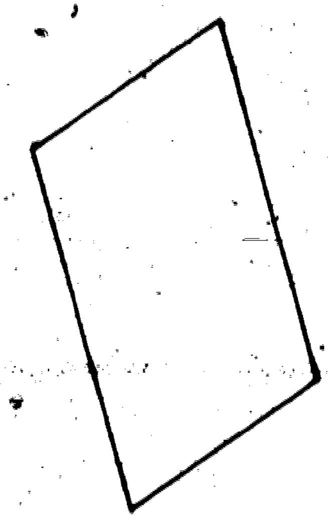
7. 

Name: ______________

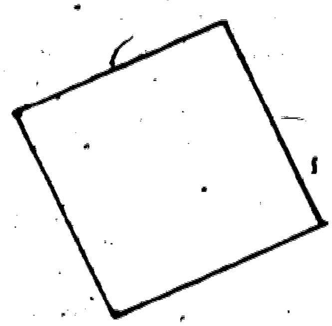
8.  Name: ____________

9.  Name: ____________

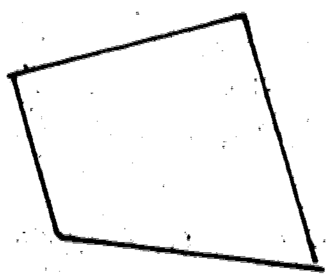
10.  Name: ____________

11. 

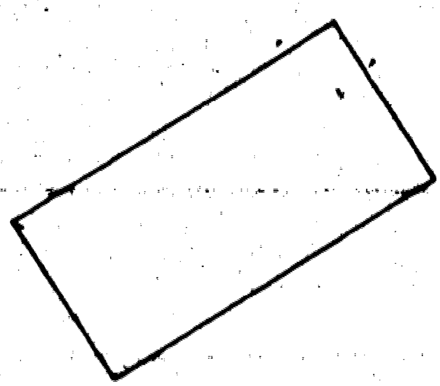
Name: ________________

12. 

Name: ________________

13. 

Name: ________________

14. 

Name: ________________

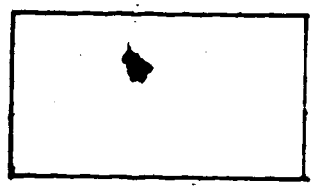
(ii)

POLYGONAL REGIONS DIFFERENTIATION TEST

NAME ____________________ SCHOOL ____________________

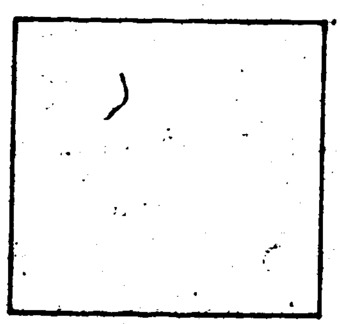
GRADE ____________________ DATE ____________________

The following is a series of regions in a plane. Show whether or not each region is a **POLYGONAL REGION**. Give reasons for your responses. Note that some of the regions below have one, two or more holes in them. The abbreviation (p.r.) stands for (Polygonal Region).

1. 

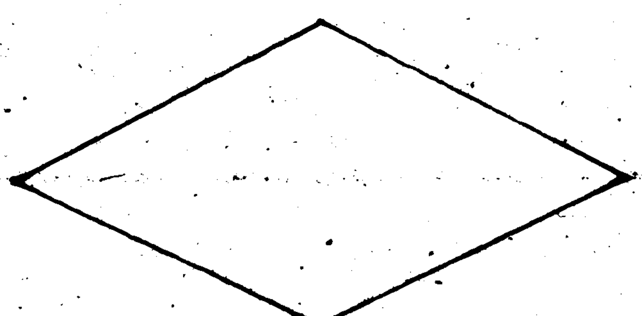
1) p.r.
2) Not p.r.

Why?

2. 

1) p.r.
2) Not p.r.

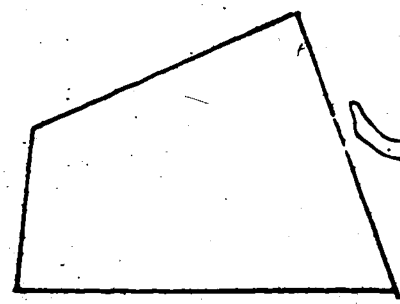
Why?

3. 

1) p.r.
2) Not p.r.

Why?

4.

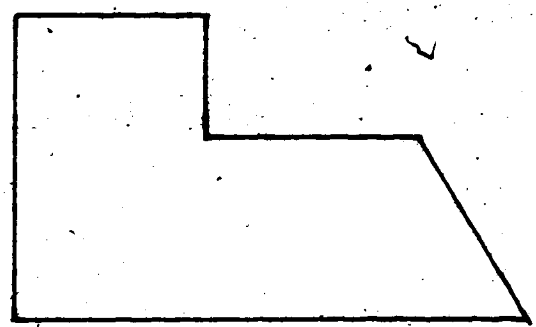


1) p.r.

2) Not p.r.

Why?

5.



1) p.r.

2) Not p.r.

Why?

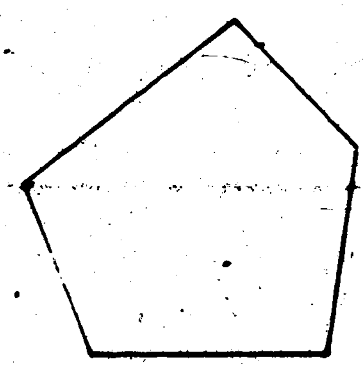
6.

1) p.r.

2) Not p.r.

Why?

7.

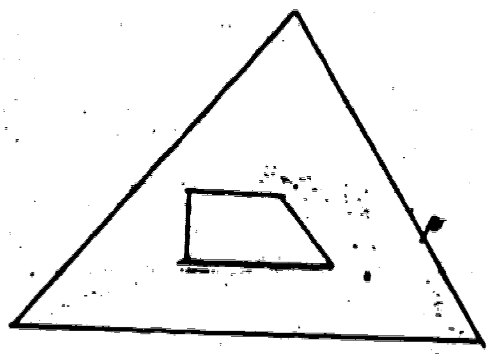


1) p.r.

2) Not p.r.

Why?

8.

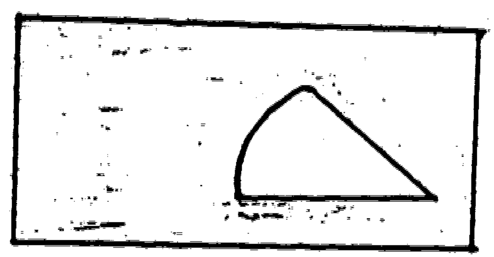


1) p.r.

2) Not p.r.

Why?

9.

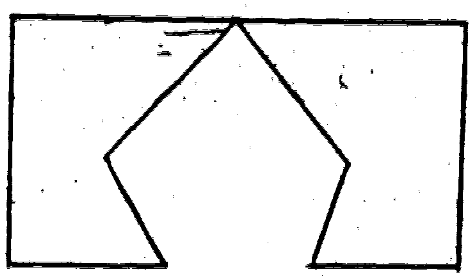


1) p.r.

2) Not p.r.

Why?

10.

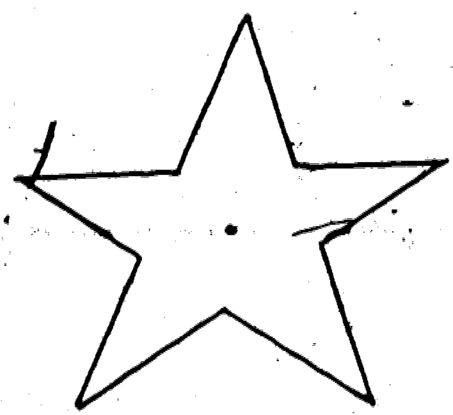


1) p.r.

2) Not p.r.

Why?

11.



1) p.r.

2) Not p.r.

Why?

(iii)

POLYGONAL REGIONS

AREA FORMULA TEST

NAME ______________________ SCHOOL ______________________

GRADE ______________________ DATE ______________________

ANSWER EACH OF THE FOLLOWING QUESTIONS.

YOU MAY DRAW A FIGURE FOR EACH OF THEM.

QUESTION 1.

What is the area formula for a triangular region?

QUESTION 2.

What is the area formula for a trapezoid region?

QUESTION 3.

What is the area formula for a parallelogram region?

QUESTION 4.

What is the area formula for a rhombus region?

QUESTION 5.

What is the area formula for a square region?

QUESTION 6.

What is the area formula for a regular pentagon region?

QUESTION 7.

What is the area formula for a rectangular region?

QUESTION 8.

Which one of these area formulae you just wrote above do you think you should learn first? Why?

(iv)

PIECE-WISE CONGRUENCY

THREE POLYGONAL REGIONS TEST

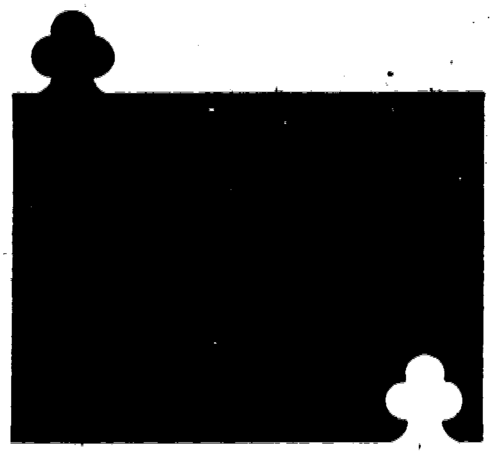
NAME ____________________ SCHOOL ____________________

GRADE ____________________ DATE ____________________

Below are three regions in a plane: A, B, and C. Do you think some or all of these regions have the same size? Why?

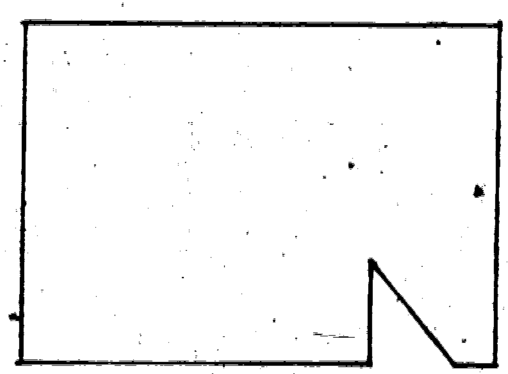
1.

A

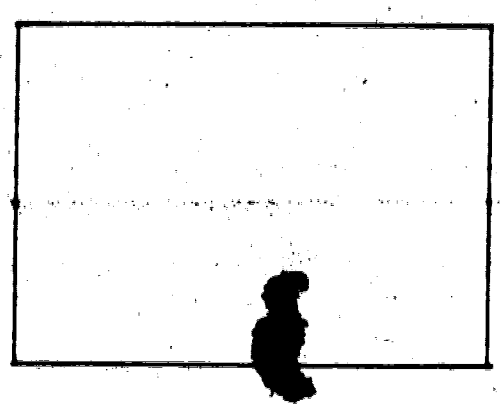


Answer:

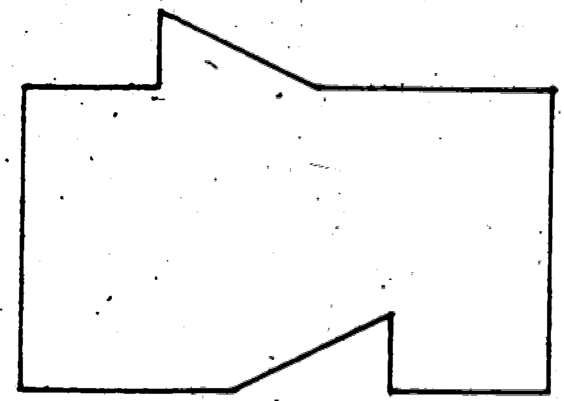
B



C

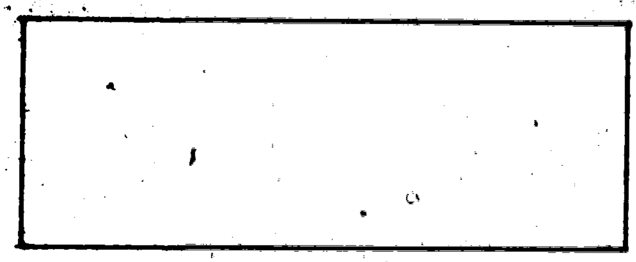


2.

A 

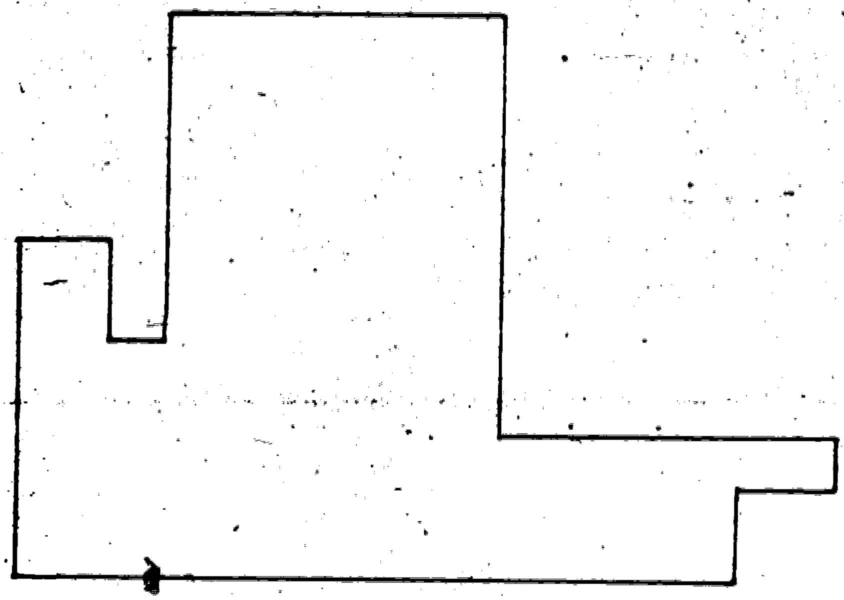
Answer: ____________

B 

C 

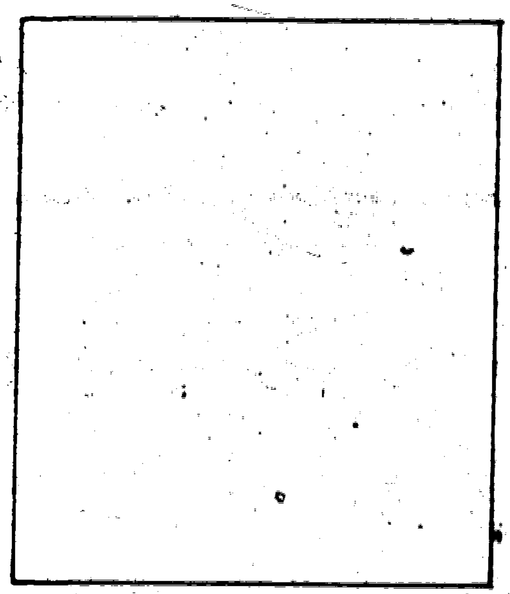
3.

A

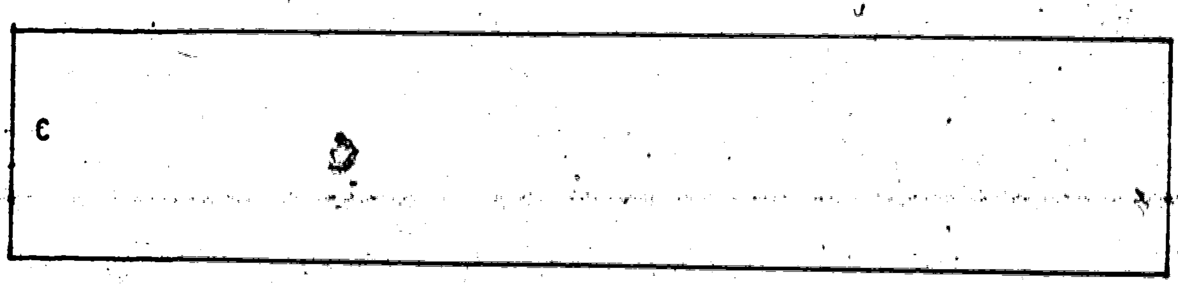


Answer:

B

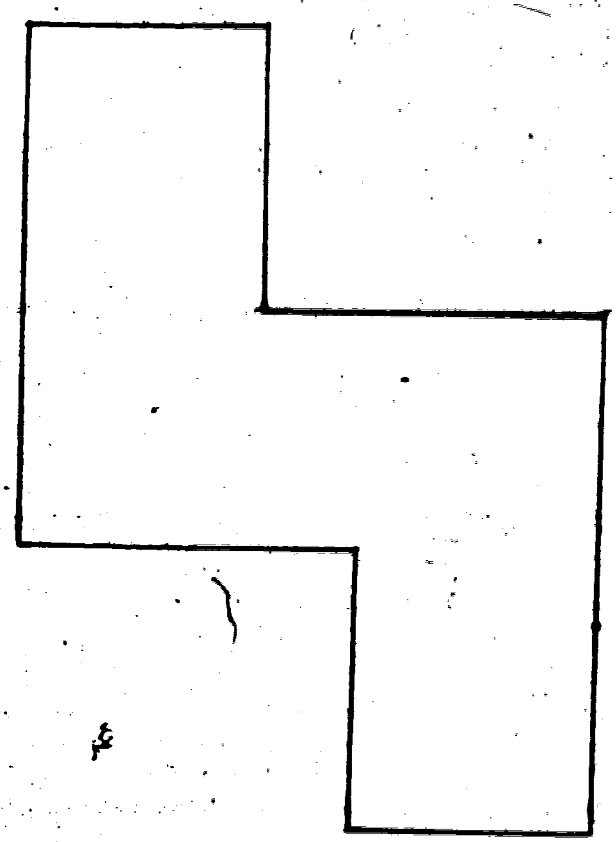


C

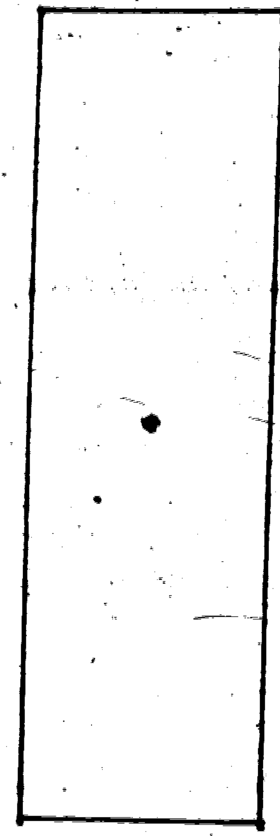


4.

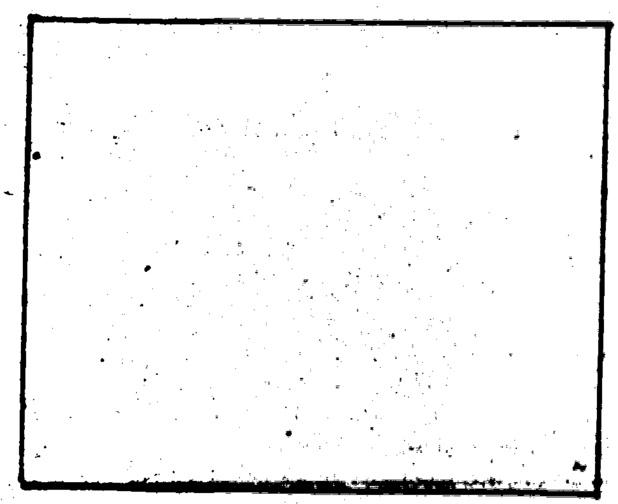
A



B



C



Answer:

(v)

PIECE-WISE CONGRUENCY

TWO POLYGONAL REGIONS TEST

NAME ____________________ SCHOOL ____________________

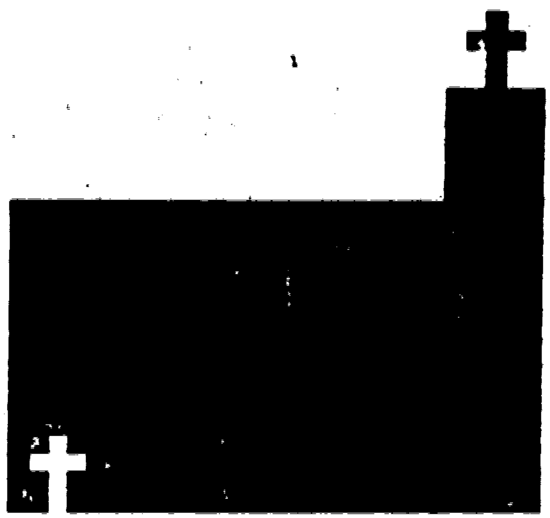
GRADE ____________________ DATE ____________________

Below are two polygonal regions in a plane, A and B (you may think of them as two gardens or yards). Which region, A or B has bigger size or should they be of the same size? Why?

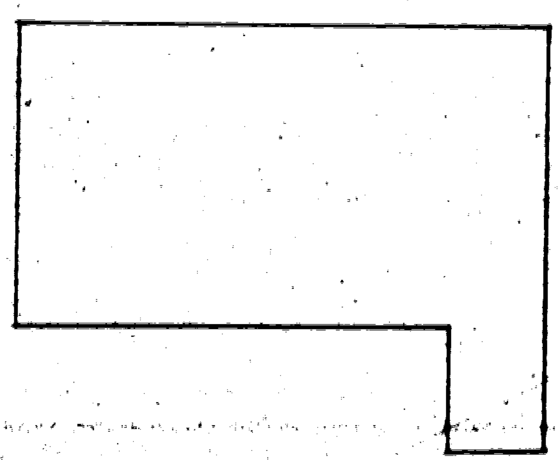
1.

Answer:

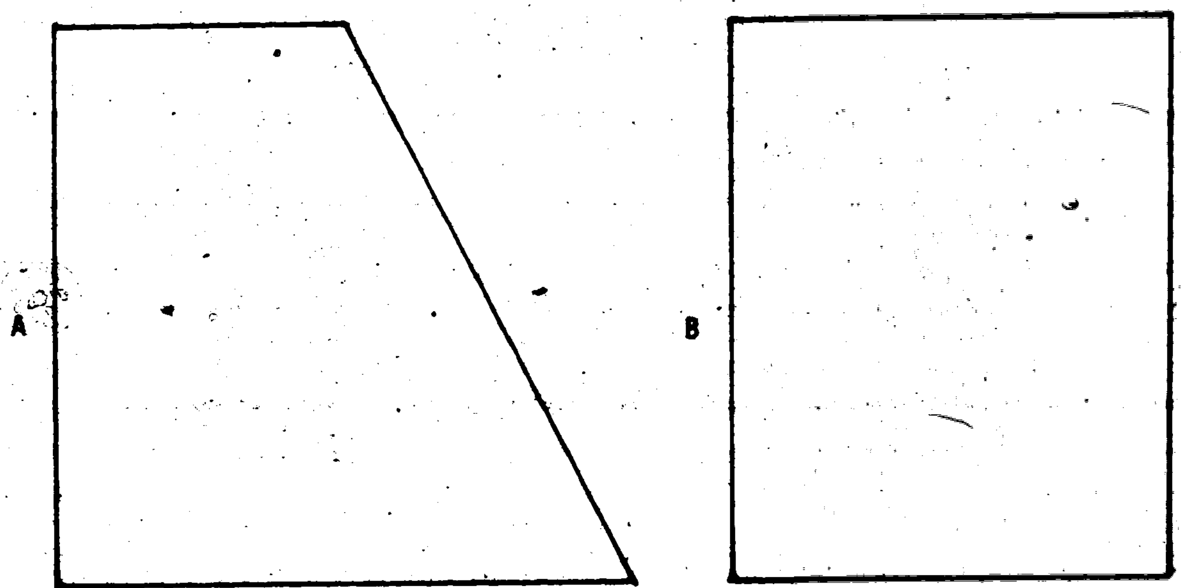
A



B

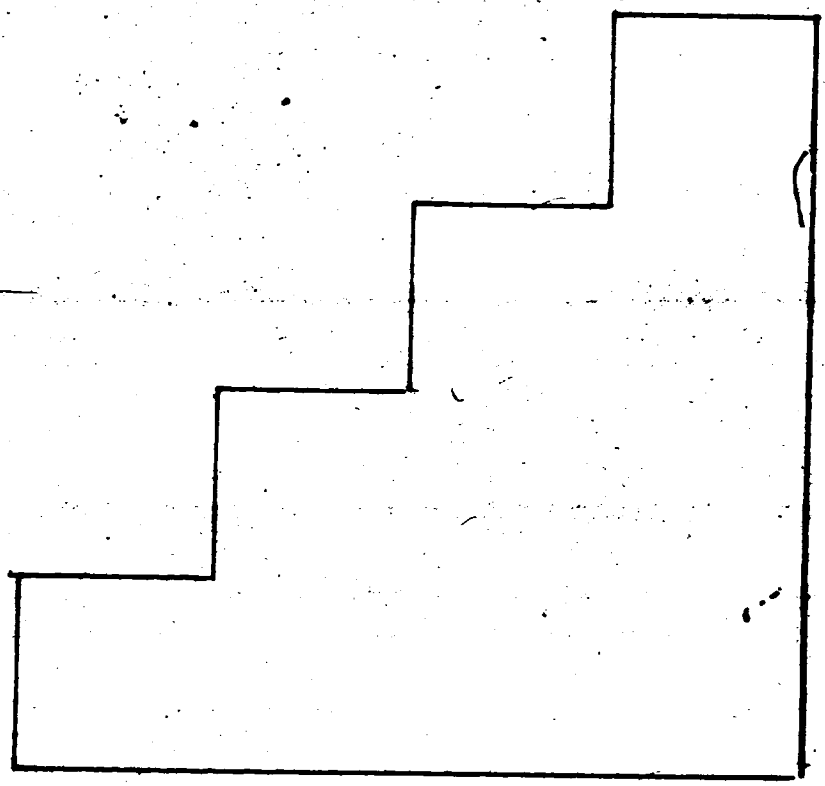


2.

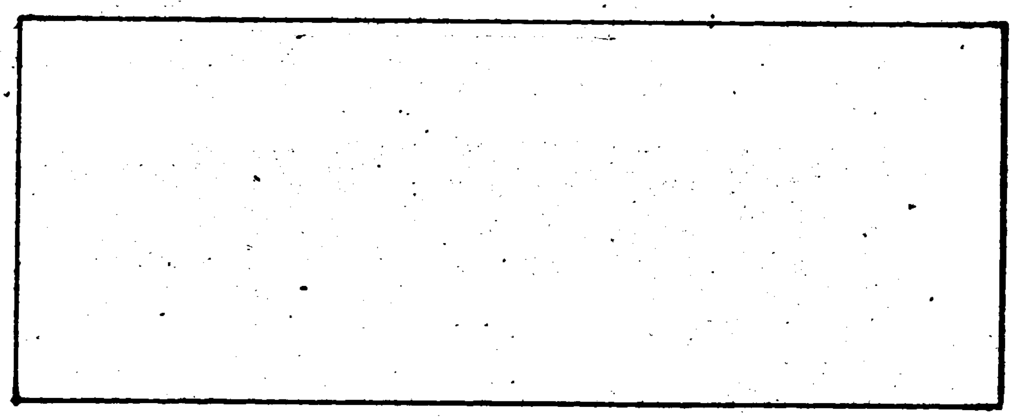


Answer:

3.



A

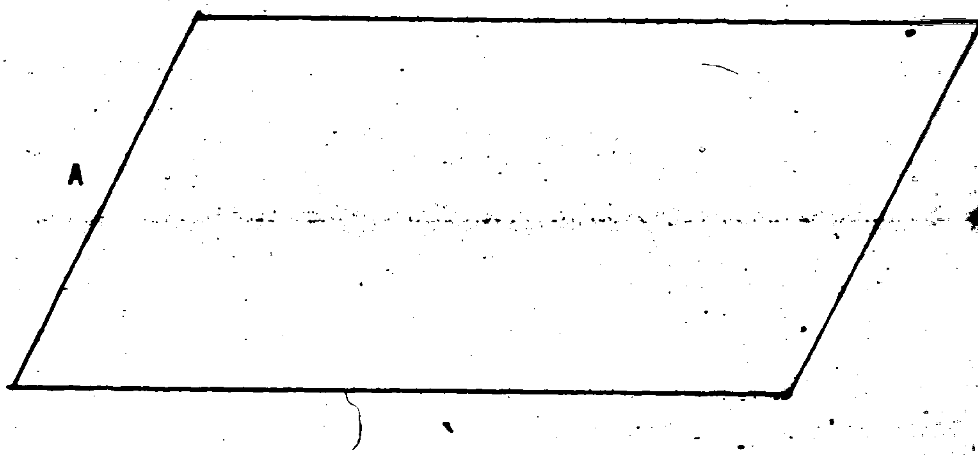


B

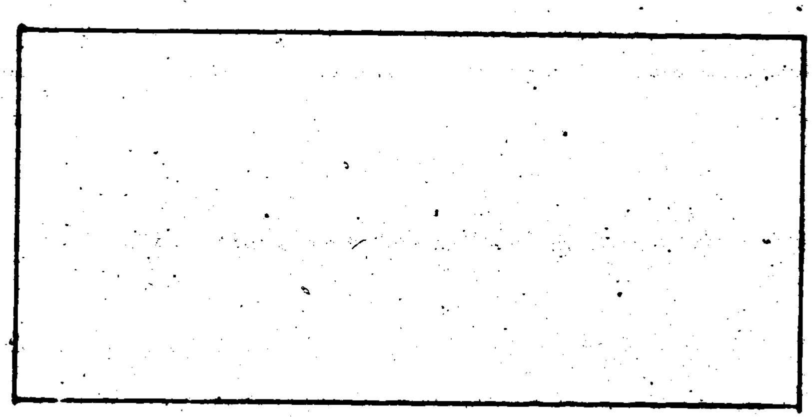
Answer:

4.

A



B



Answer:

(vi)

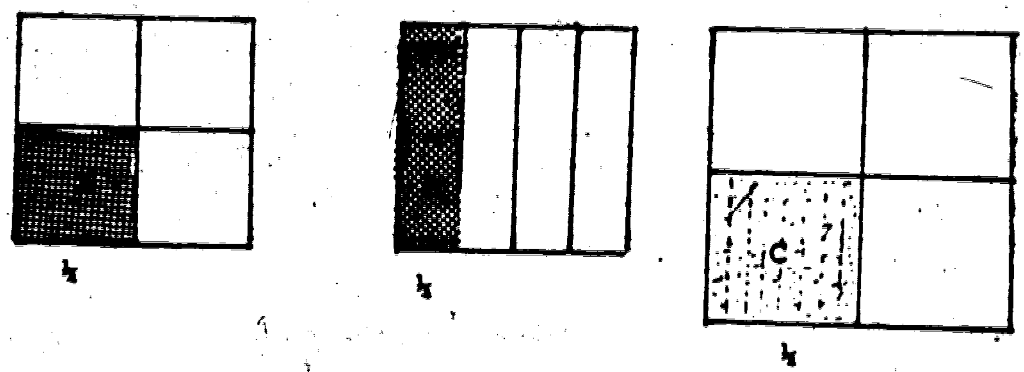
PIECE-WISE CONGRUENCY RATIONAL NUMBERS TEST

NAME ______________________ SCHOOL ______________________

GRADE ______________________ DATE ______________________

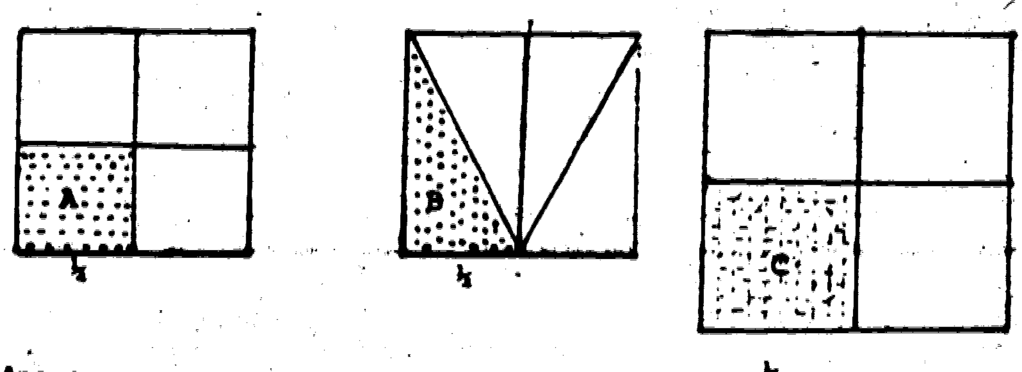
Below are six groups of figures each of which has THREE figures. The three regions in each group have shaded subregions A, B, and C. Now, which of these subregions A, B, and C do you think are equal? Why?

1.



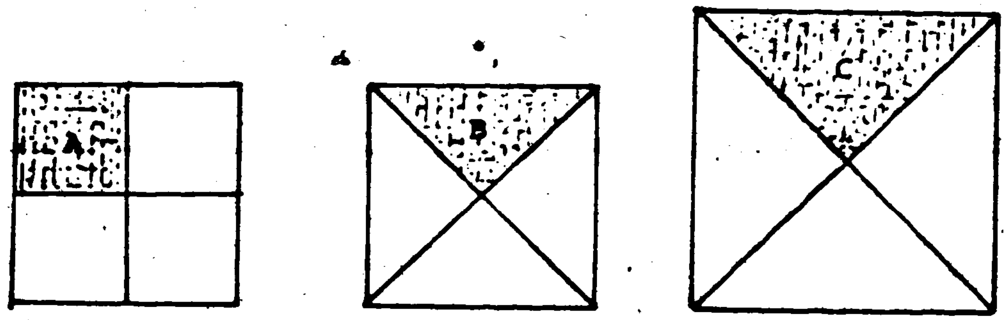
Answer:

2.



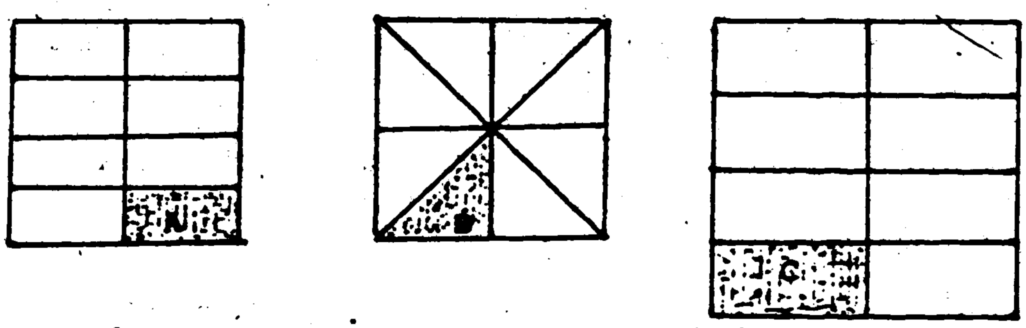
Answer:

3.



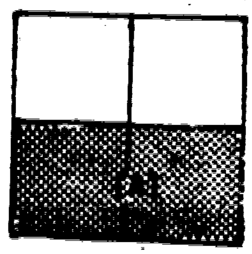
Answer:

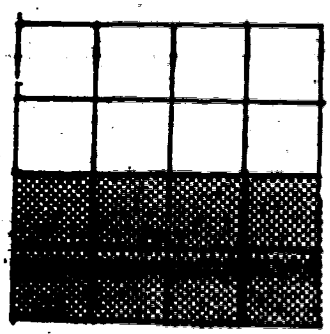
4.

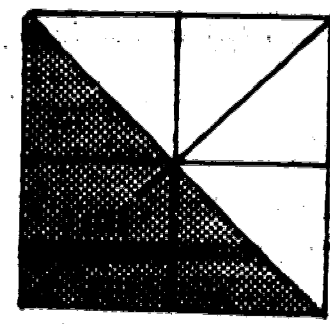


Answer:

5.

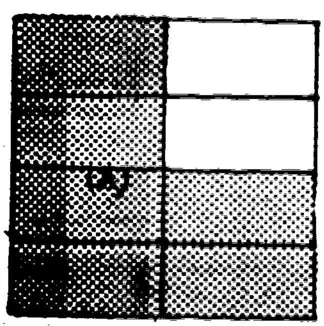


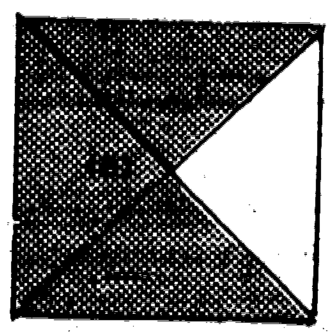




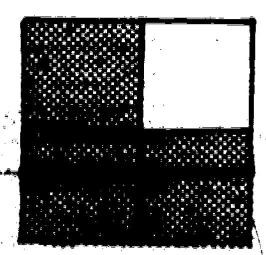
Answer:

6.





Answer:



APPENDIX G

PROGRESS CHECKING TESTS

CHECKING ONE

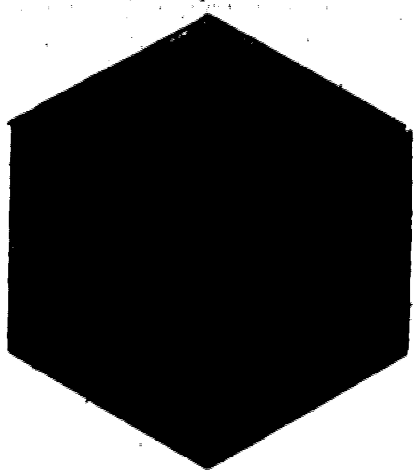
NAME______________

DATE______________

SCHOOL____________

Show whether or not each region below is polygonal according to the definition of polygonal region.

(a)

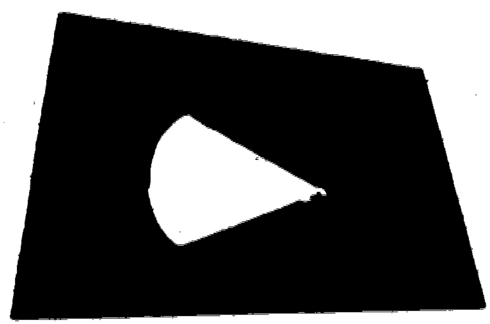


1) Polygonal region (p.r.).

2) Not p.r.

Why?

(b)

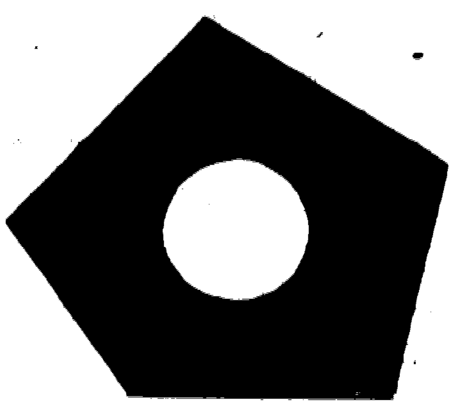


1) p.r.

2) Not p.r.

Why?

(c)

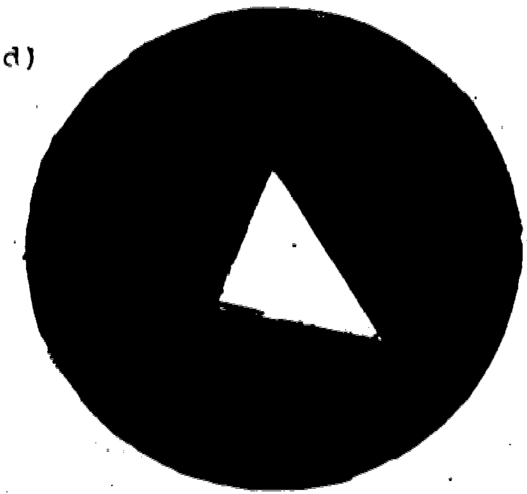


1) p.r.

2) Not p.r.

Why?

(d)

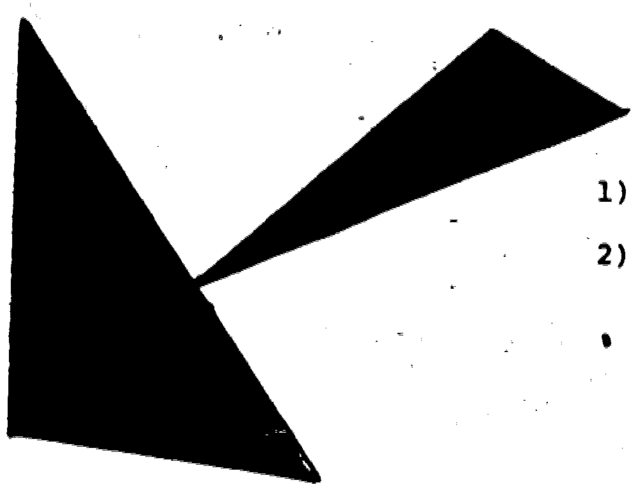


1) p.r.

2) Not p.r.

Why?

(e)



1) p.r.

2) Not p.r.

Why?

(f)

1) p.r.

2) Not p.r.

Why?

CHECKING TWO

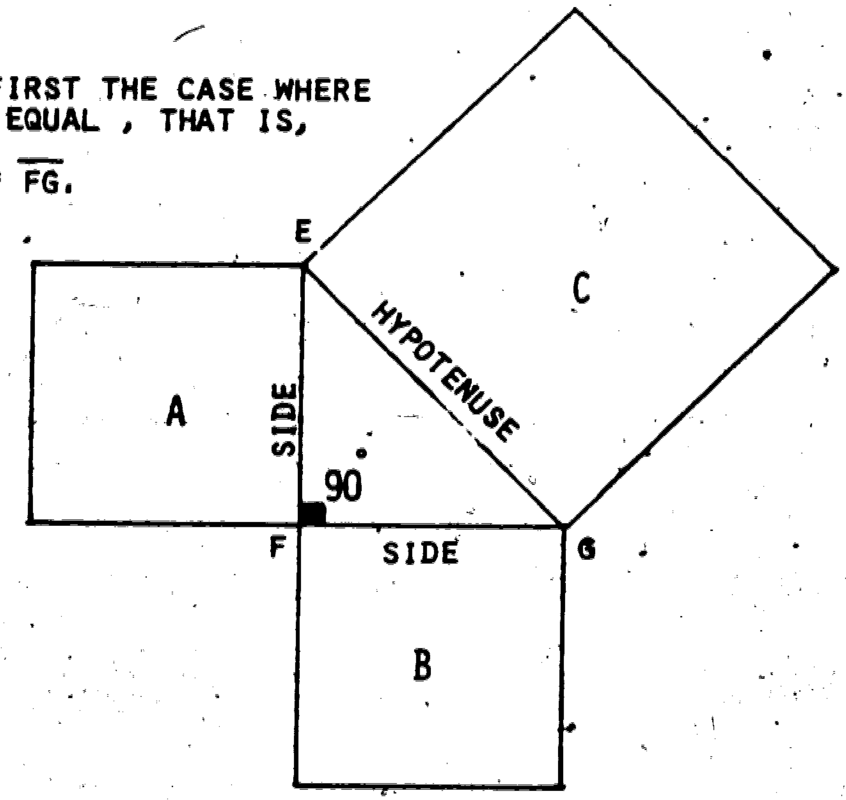
PYTHAGOREAN THEOREM: - CASE 1 -

PYTHAGOREAN THEOREM IS ONE OF THE MOST IMPORTANT THEOREMS, NOT ONLY IN GEOMETRY AND ELEMENTARY MATHEMATICS, BUT IN ADVANCED MATHEMATICS AS WELL. THE STATEMENT OF THIS THEOREM IS THE FOLLOWING:

> IN A RIGHT ANGLE TRIANGLE, THE SQUARE OF THE HYPOTENUSE IS EQUAL TO THE SUM OF THE SQUARES OF THE OTHER TWO SIDES.

LET US TAKE FIRST THE CASE WHERE THE SIDES ARE EQUAL, THAT IS,

$\overline{EF} = \overline{FG}$.



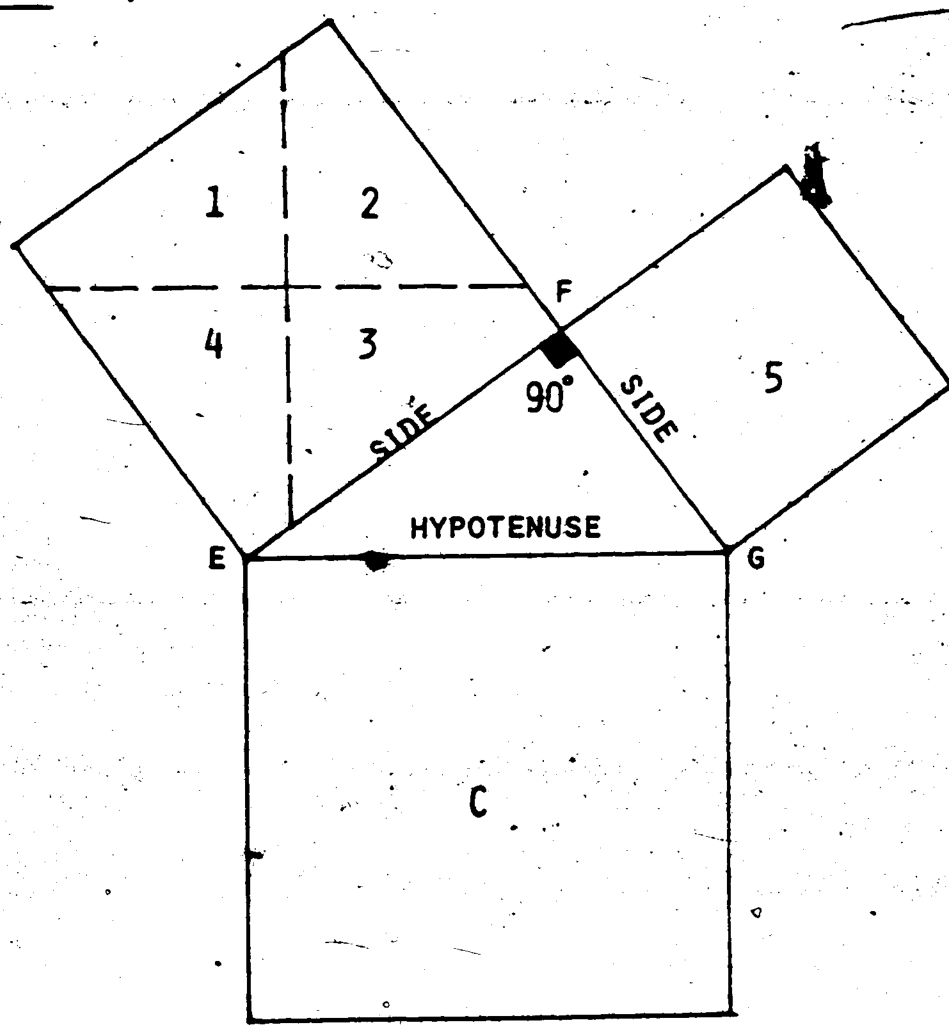
NOW,

- TRACE AND CUT OUT A COPY OF EACH OF THE SQUARES A & B. TRY TO COVER THE SQUARE C USING THE COPIES OF THE SQUARES A AND B.

(*) SUCCEEDED IN THIS EXCERCISE, YOU HAVE, THEN, ALREADY PROVED THIS IMPORTANT THEOREM!

PYTHAGOREAN THEOREM:- CASE 2 -

THIS IS THE GENERAL CASE OF THIS THEOREM WHERE THE SIDES ARE NOT EQUAL, THAT IS, $\overline{EF} \neq \overline{FG}$.



TRACE AND CUT OUT A COPY FOR EACH OF THE SQUARES ON THE SIDES $\overline{EF}$ AND $\overline{FG}$. THE SQUARE ON THE SIDE $\overline{EF}$ IS DECOMPOSED FOR YOU INTO THE REGIONS 1, 2, 3, & 4. THE SQUARE ON THE SIDE $\overline{FG}$ HAS NUMBER 5. NOW, TRY TO COVER THE SQUARE C BY REARRANGING THESE FIVE REGIONS.

*) SHOULD BE SUCCESSFUL, THE GENERAL CASE OF THE THEOREM IS PROVED.

NAME :____________________

CHECKING THREE

SCHOOL:____________________

DATE: :____________________

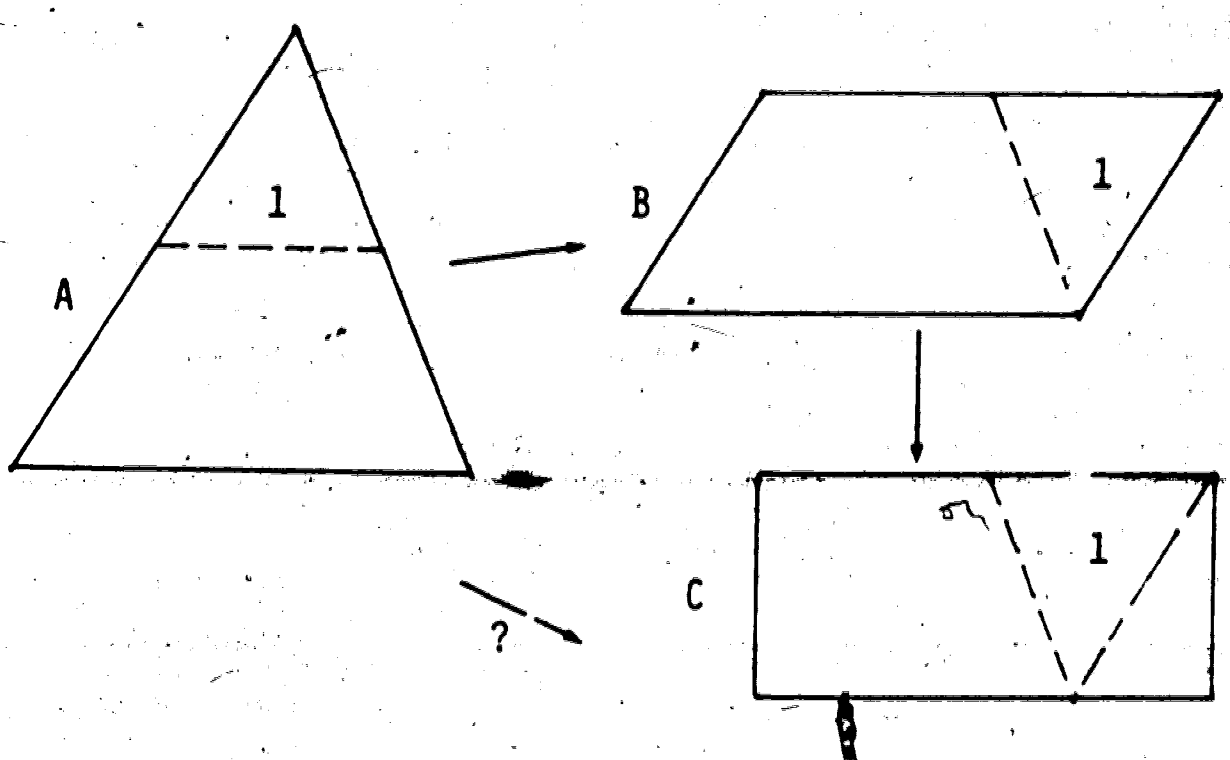
EXCERCISE:

IN THE FIGURE BELOW ARE THREE REGIONS, REGION A, REGION B, AND REGION C. AS IT IS SHOWN IN THE FIGURE BELOW, REGION A IS DECOMPOSED INTO REGION B, AND REGION B IS DECOMPOSED INTO REGION C. THAT IS,

REGION A —decomposed to→ REGION B —decomposed to→ REGION C.

?

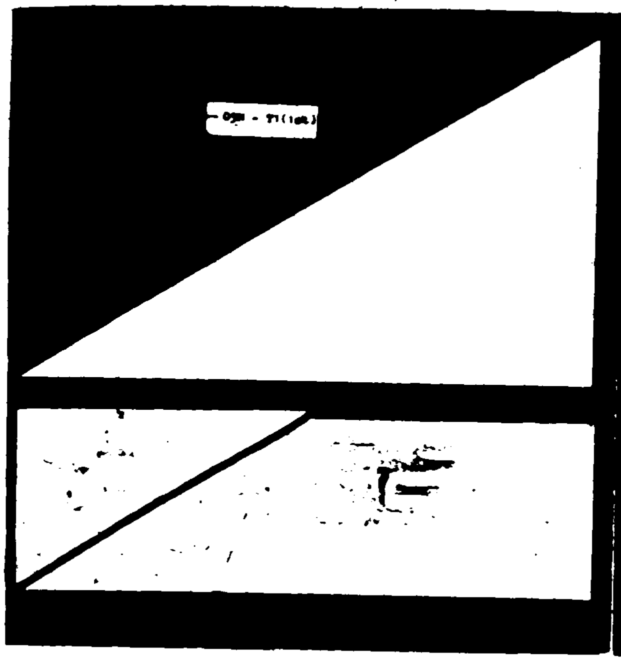
CAN REGION A, THEREFORE, BE DECOMPOSED INTO REGION C?
IF SO, SHOW THE DECOMPOSITIONS REQUIRED FOR REGION A, REGION B, AND REGION C BY DRAWING **BROKEN LINES** ON EACH OF THEM.



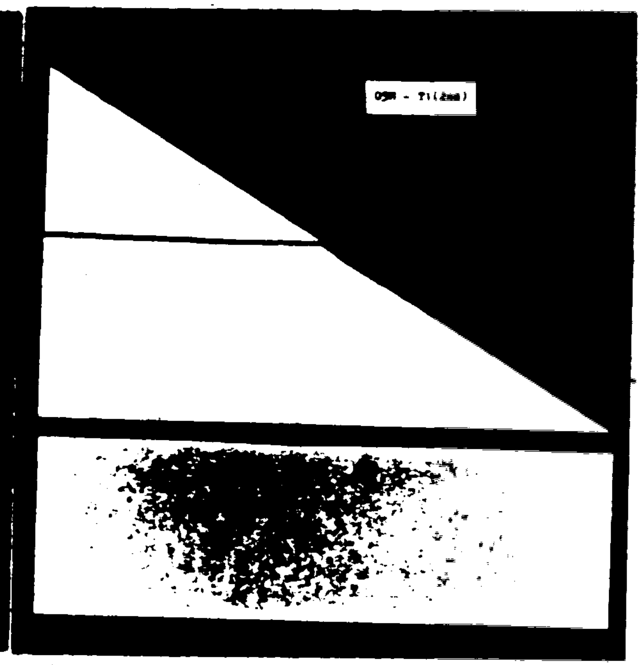
APPENDIX H

PHOTOGRAPHS OF THE STUDENTS' ORIGINAL CUTTINGS

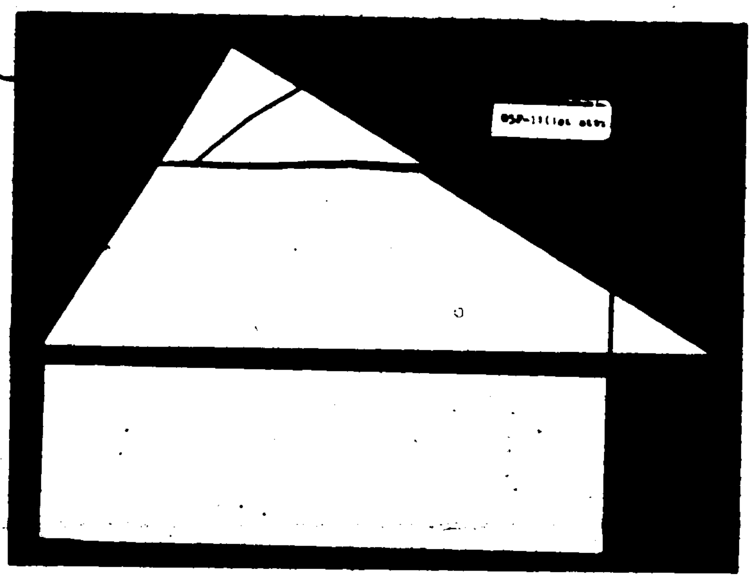
TASK 1



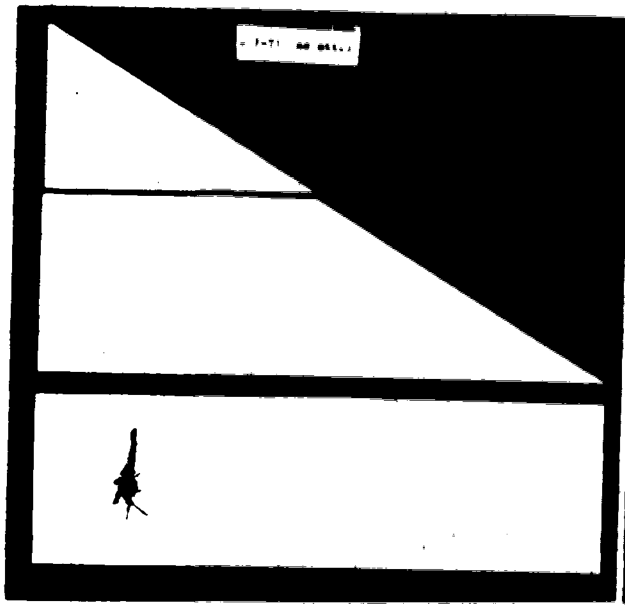
03M-1st attempt



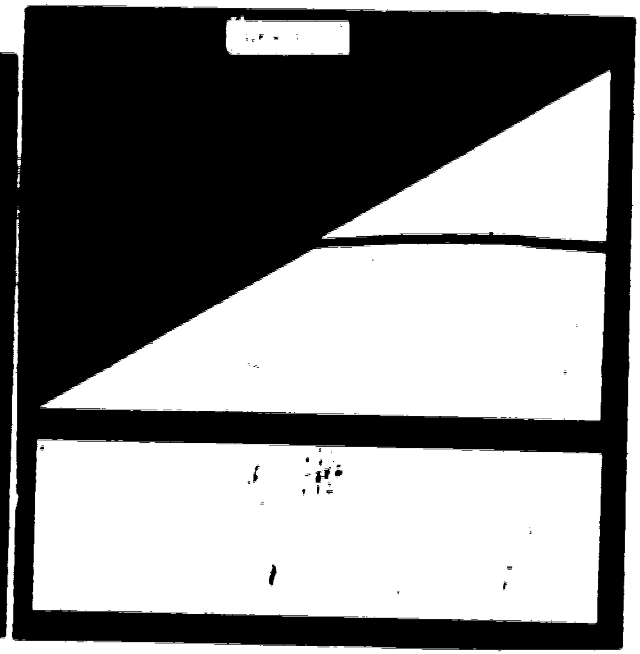
03M-2nd attempt



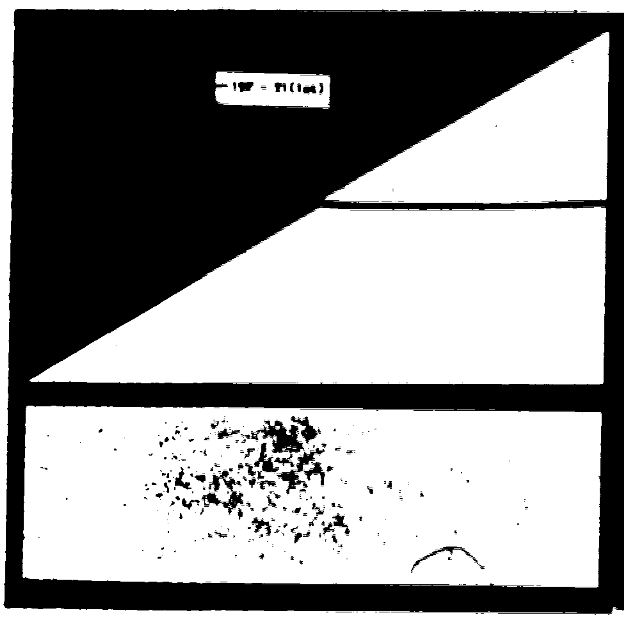
05F-1st attempt



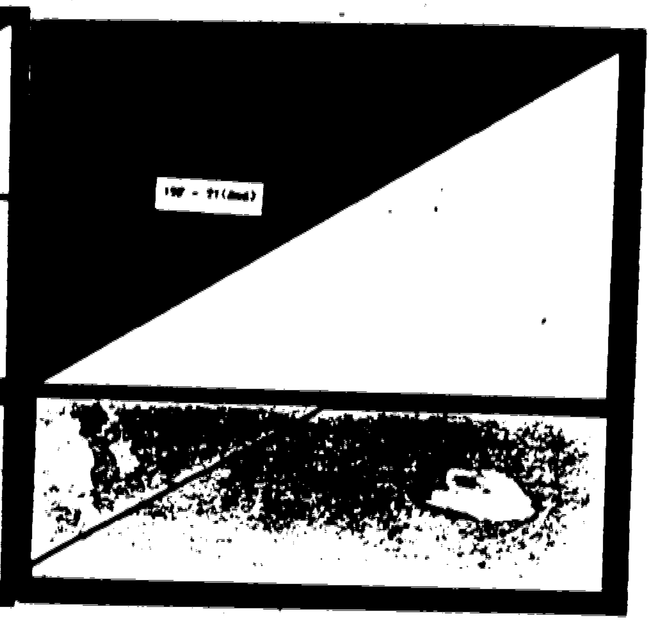
05F-2nd attempt



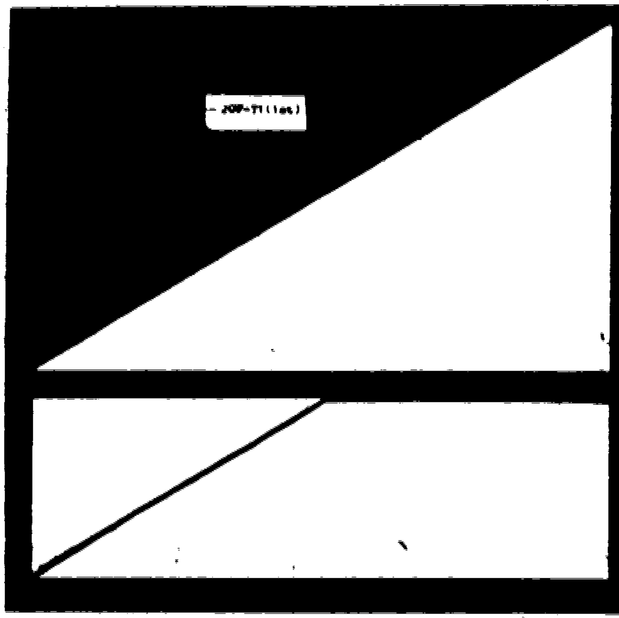
12M



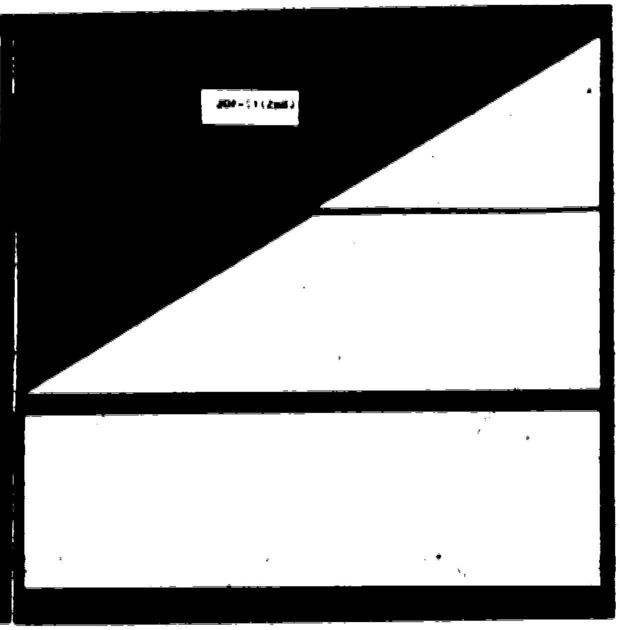
19F-1st attempt



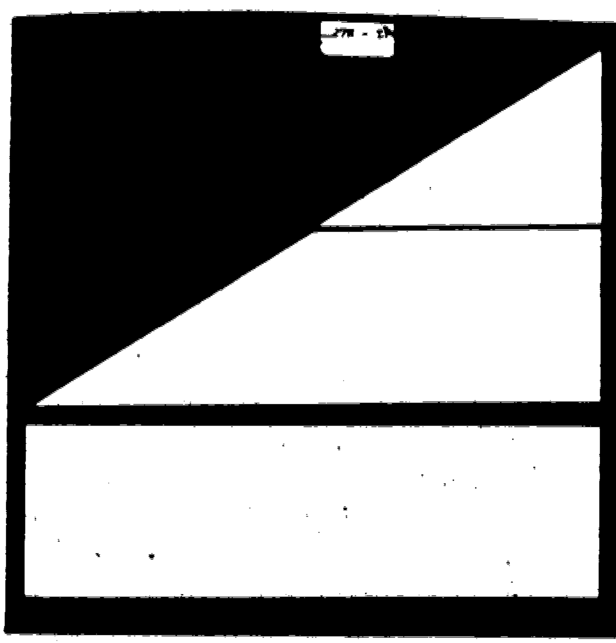
19F-2nd attempt



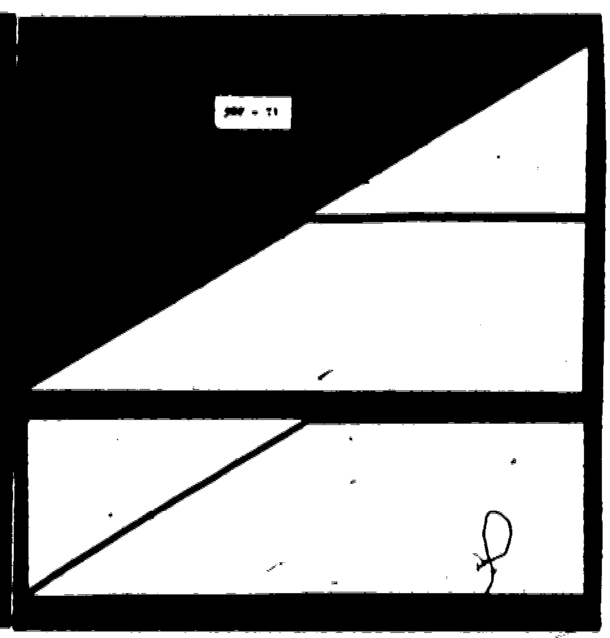
20F-1st attempt



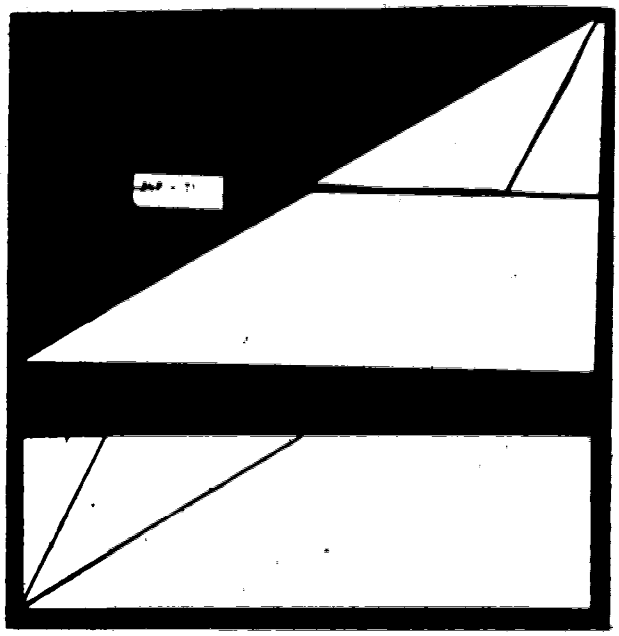
20F-2nd attempt



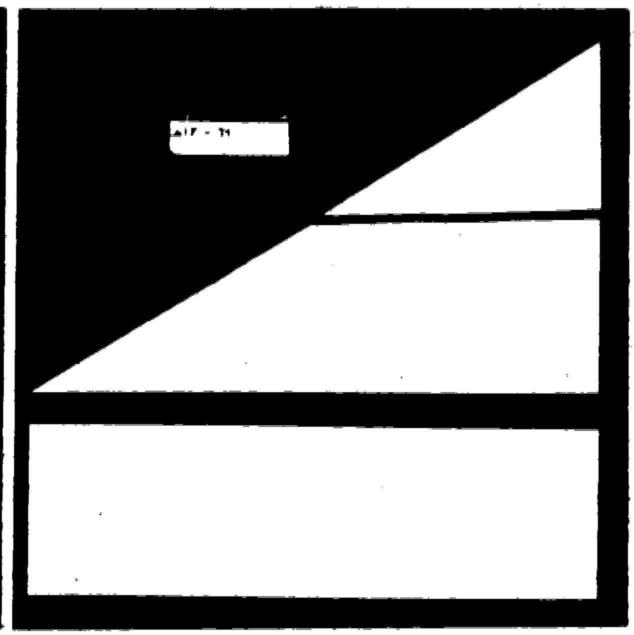
27M



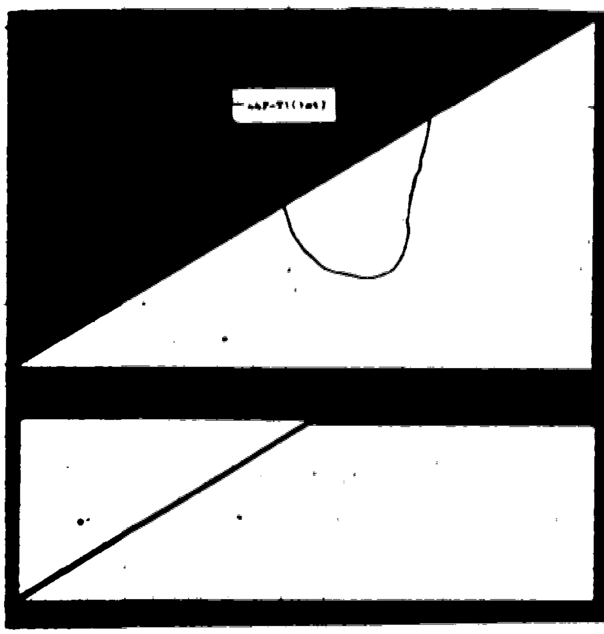
30F



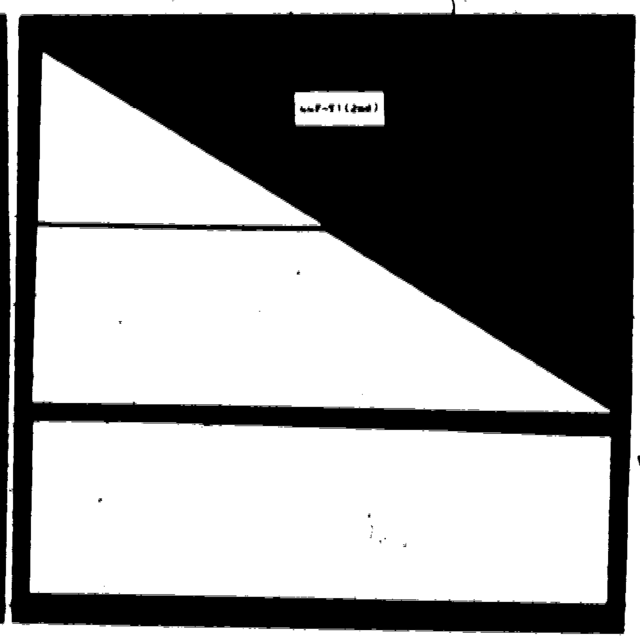
34F



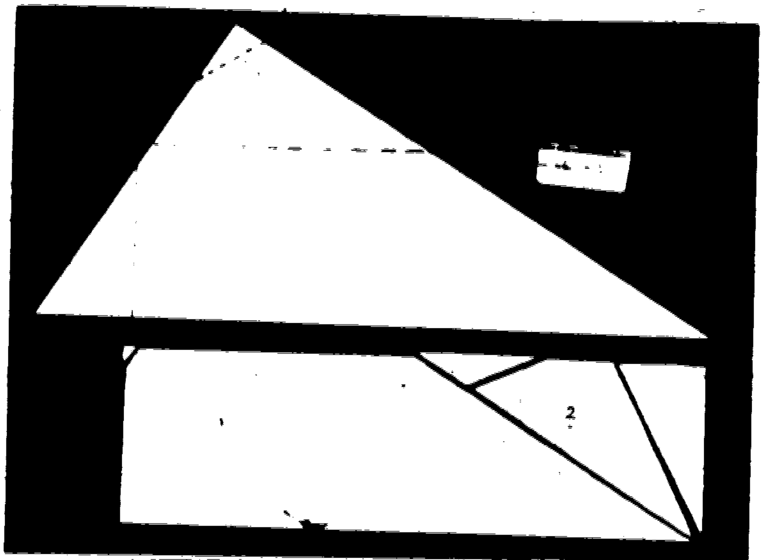
41F



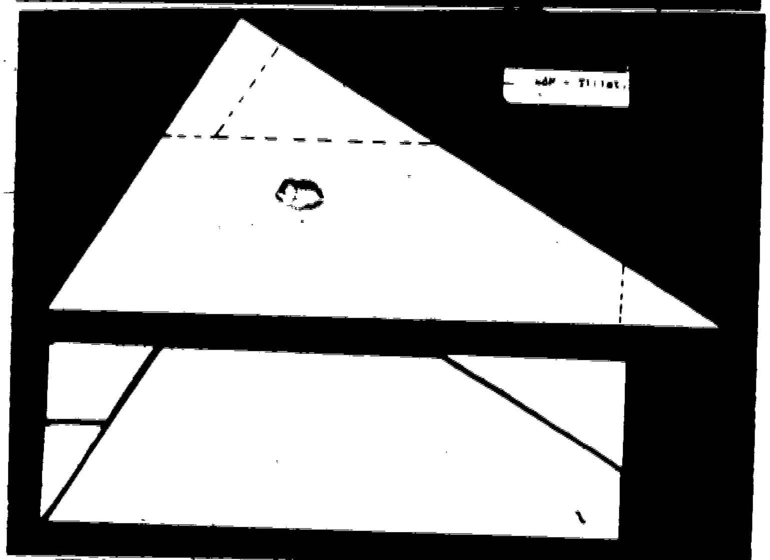
44F-1st attempt



44F-2nd attempt



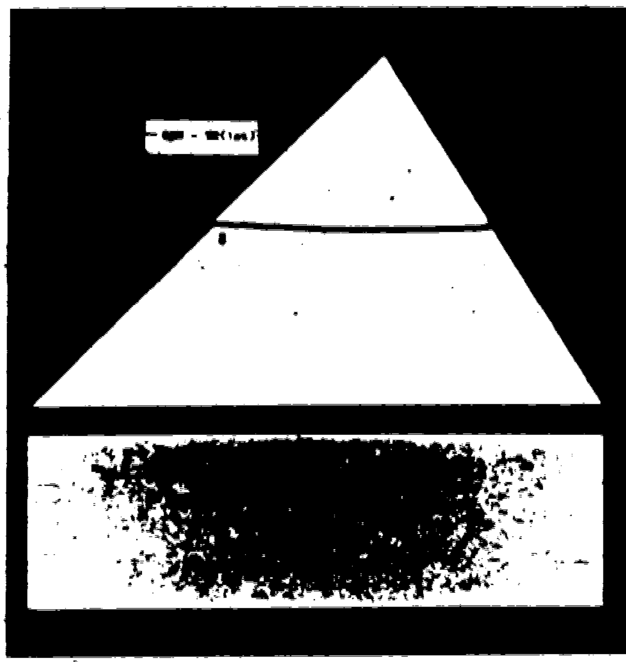
46M



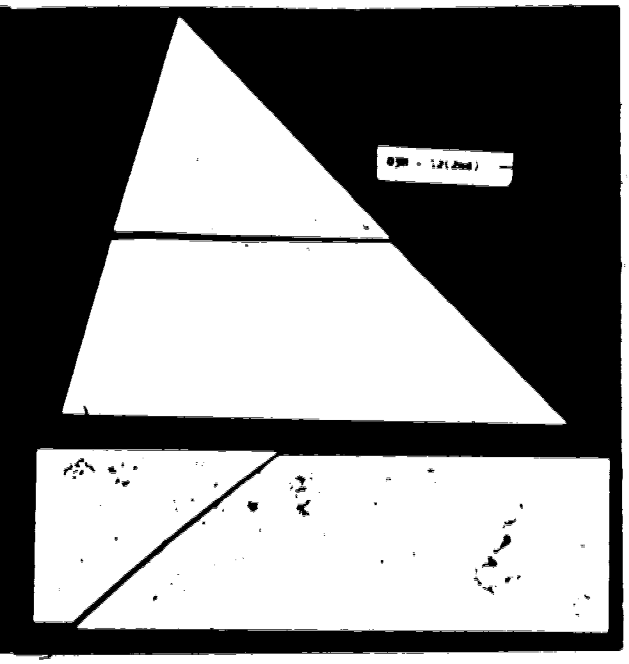
48M-1st attempt

48M-2nd attempt

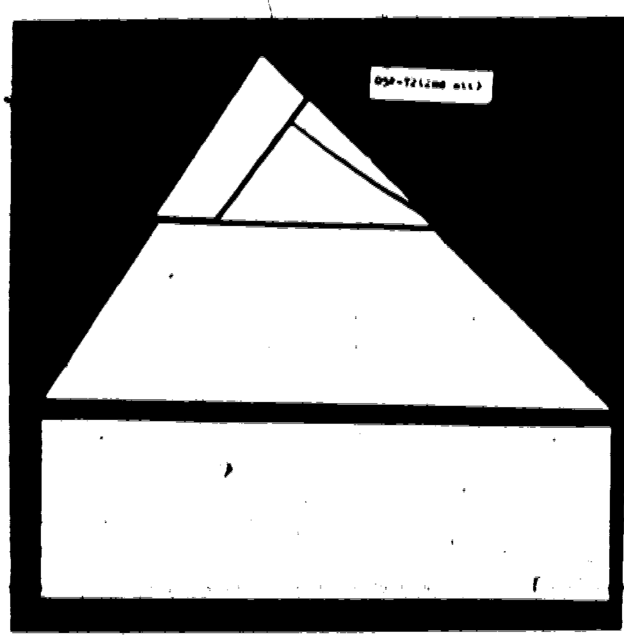
TASK 2



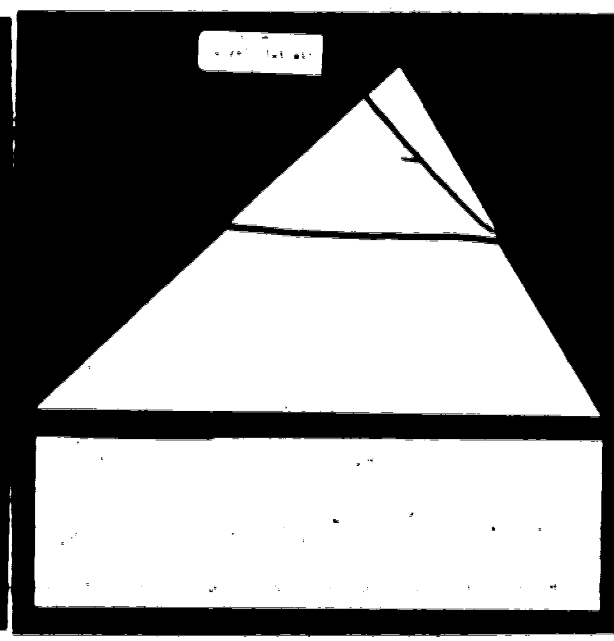
03M-1st attempt



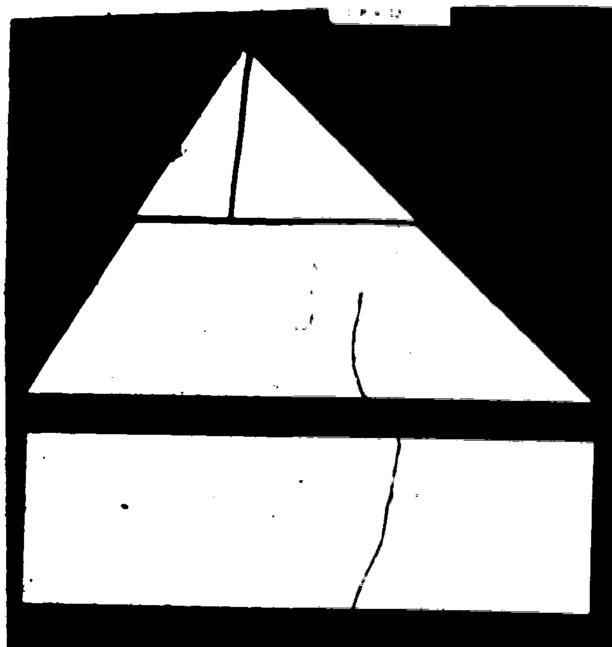
03M-2nd attempt



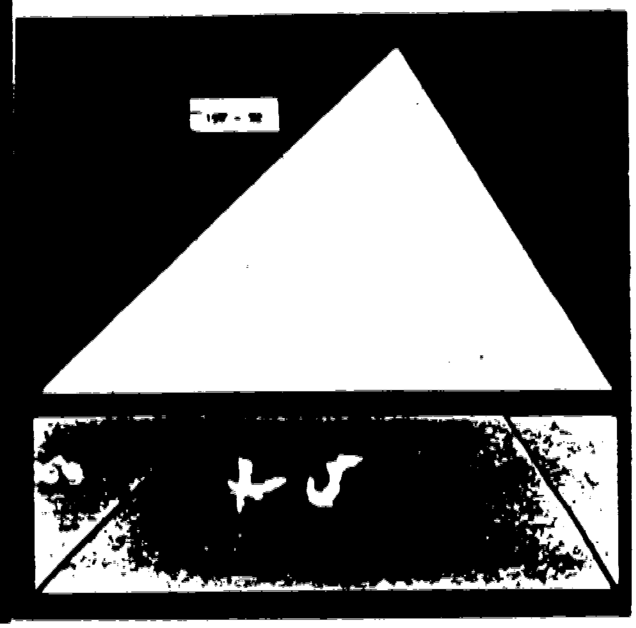
05F-1st attempt



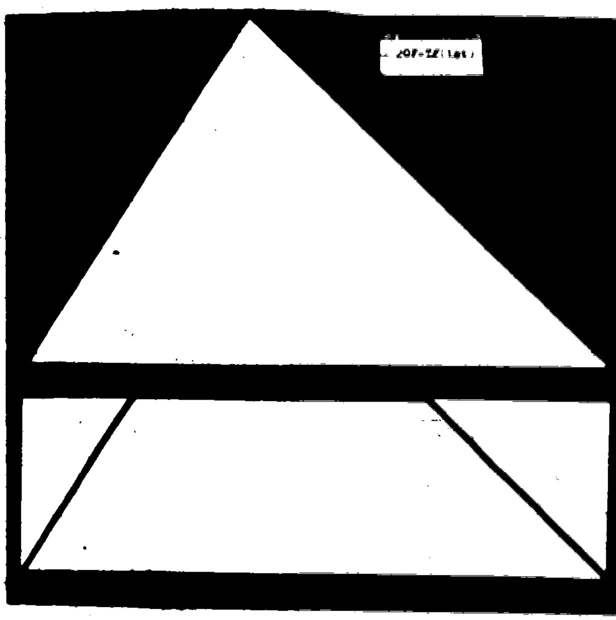
05F-2nd attempt



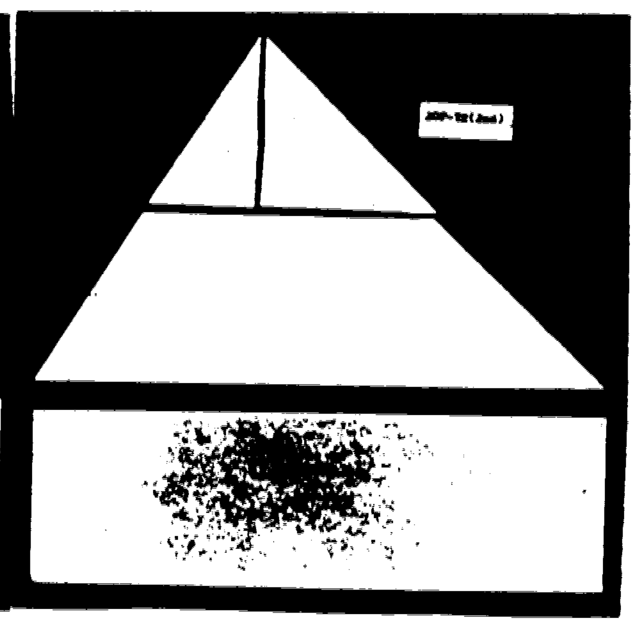
12M



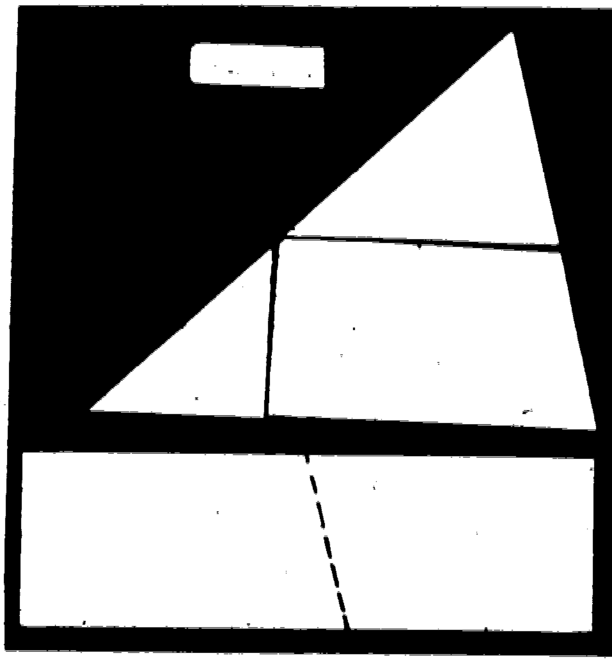
19F



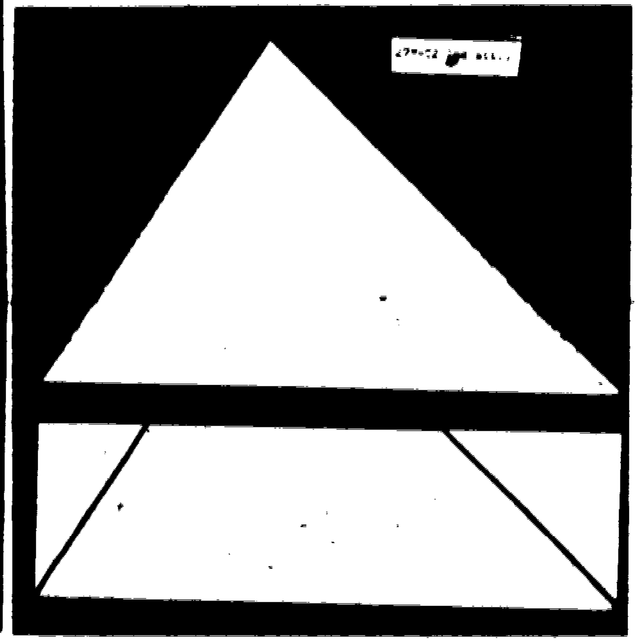
20F-1st attempt



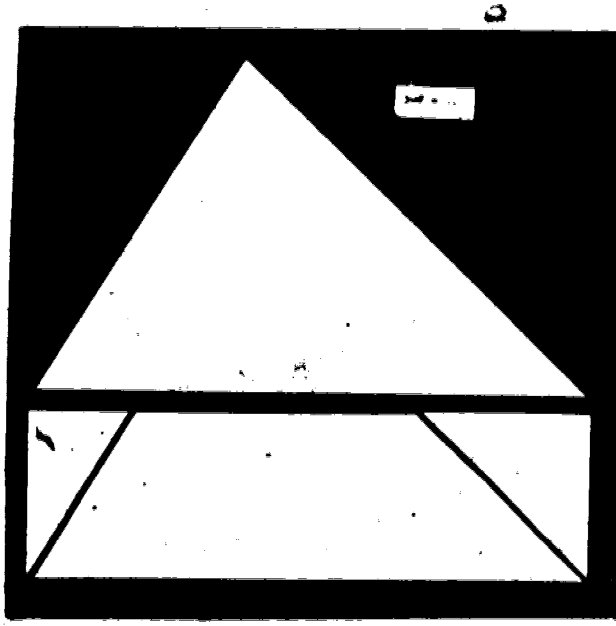
20F-2nd attempt



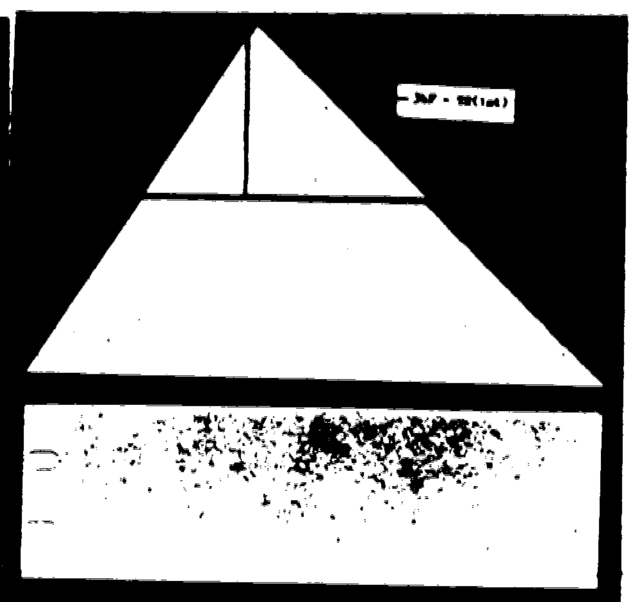
27M-1st attempt



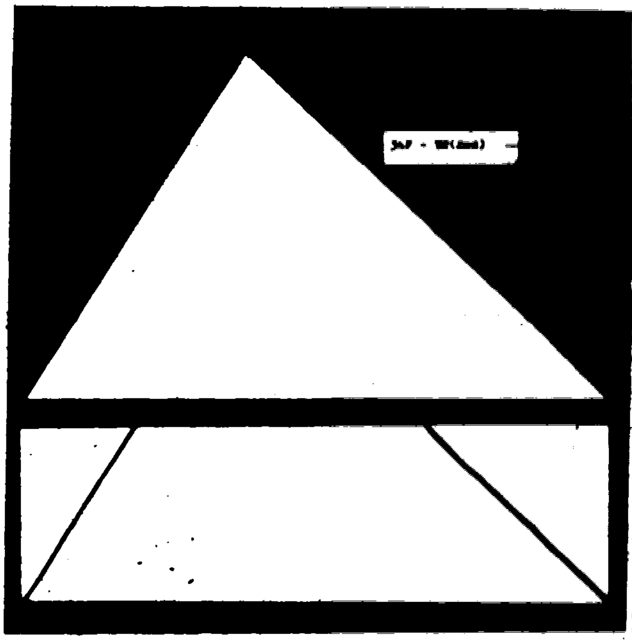
27M-2nd attempt



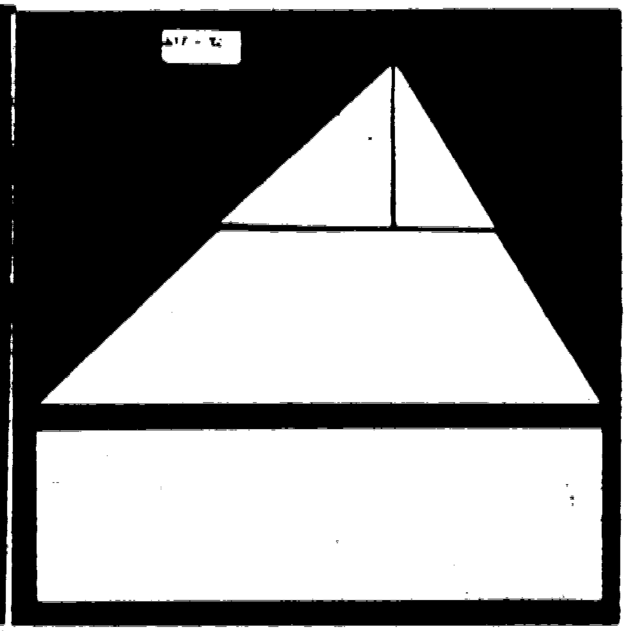
30F



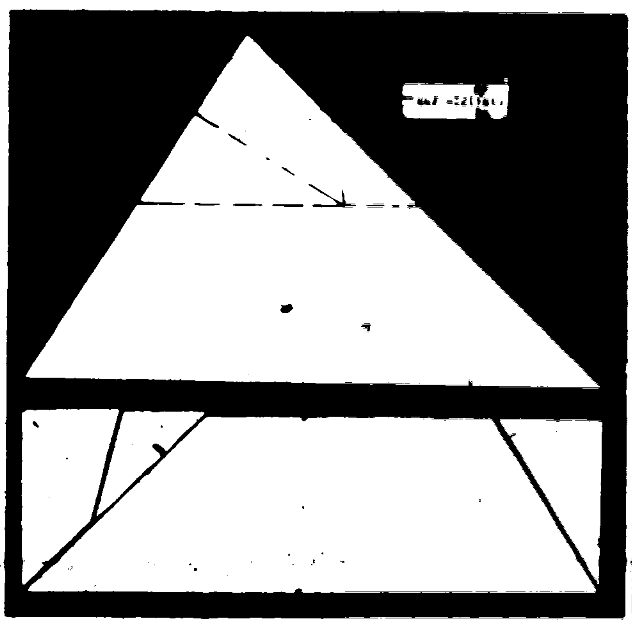
34F-1st attempt



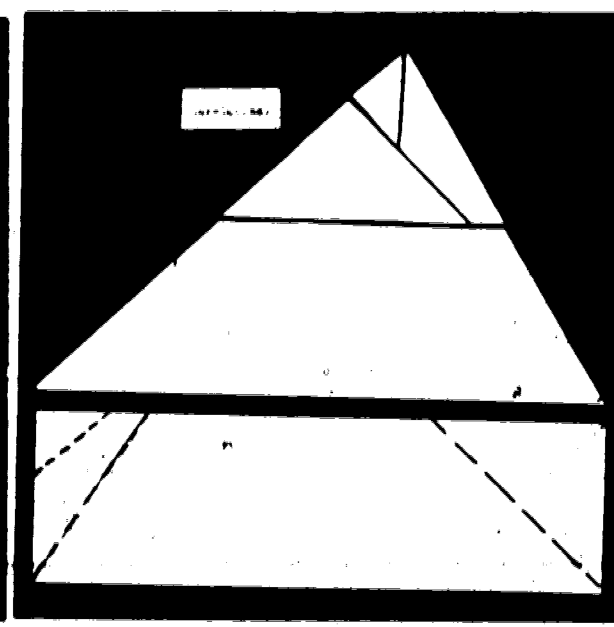
34F-2nd attempt



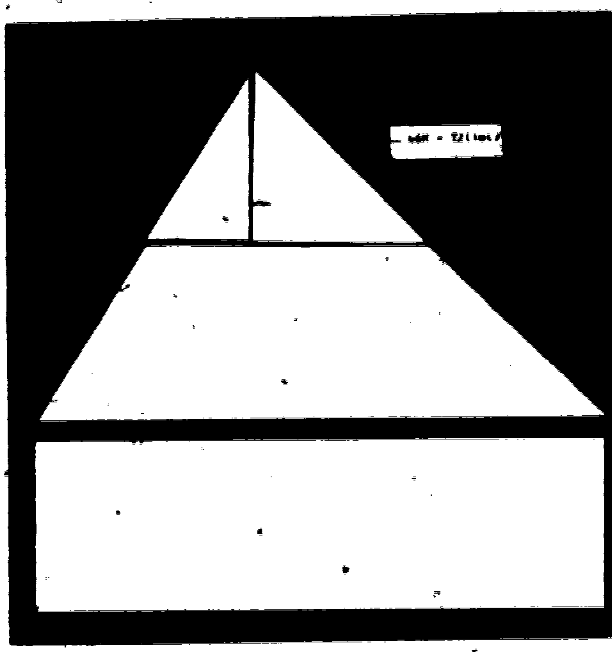
41F



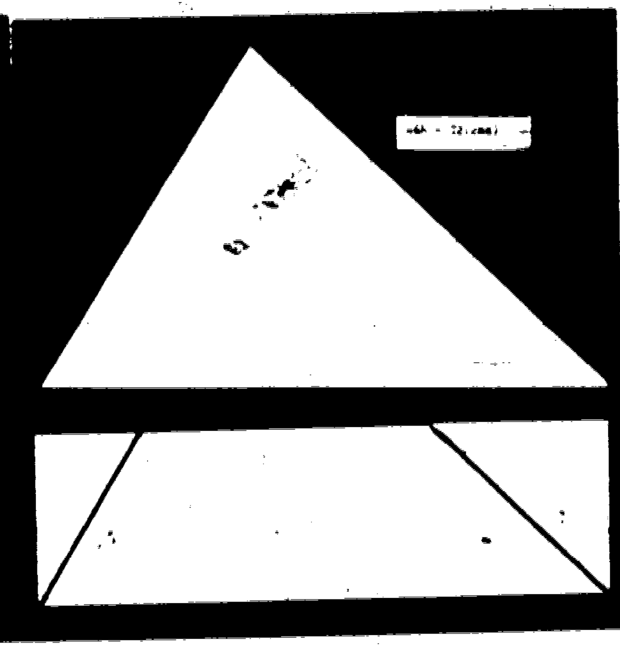
44F-1st attempt



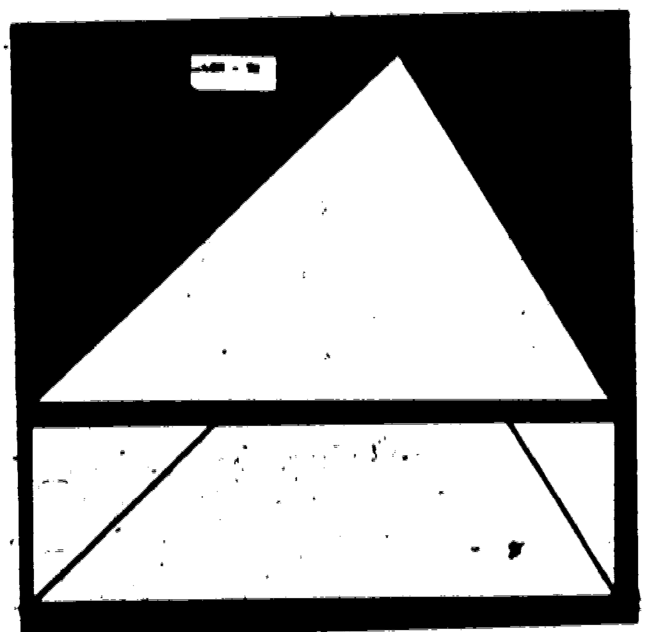
44F-2nd attempt



46M-1st attempt

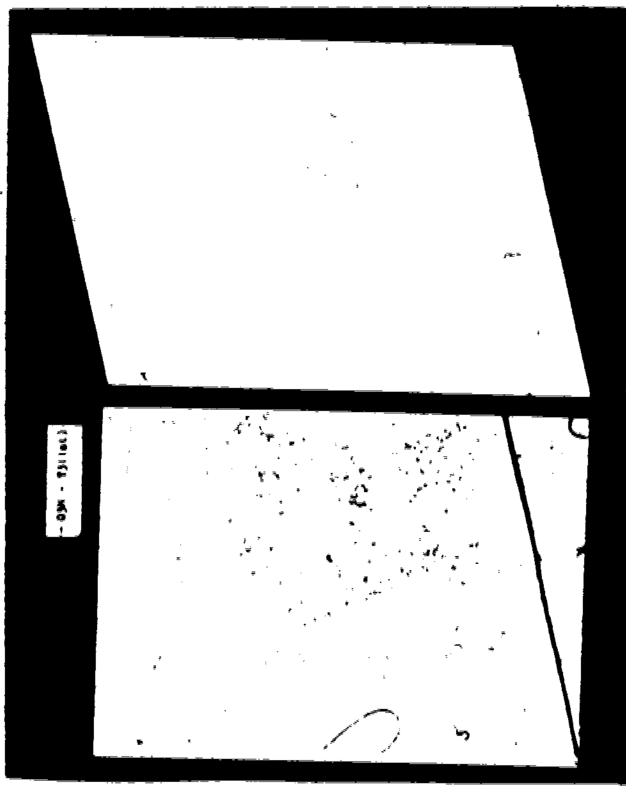


46M-2nd attempt

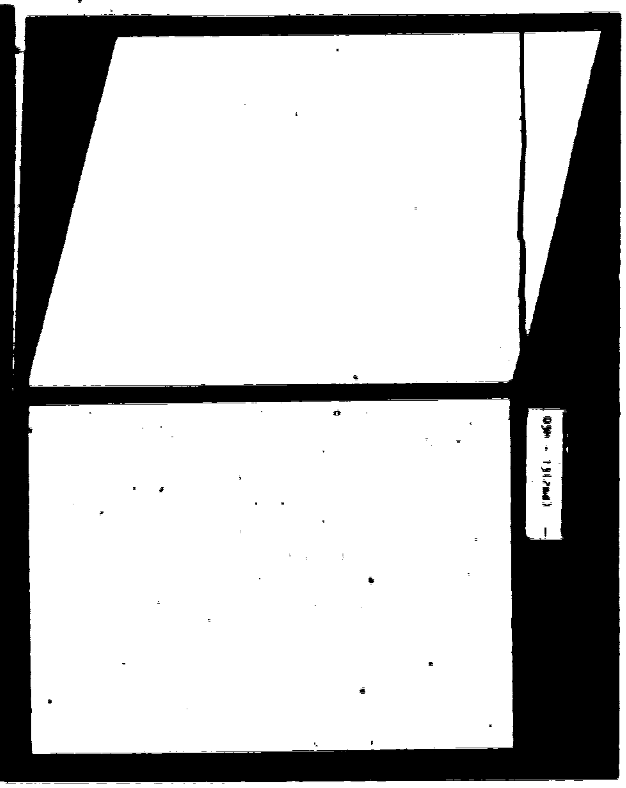


48M

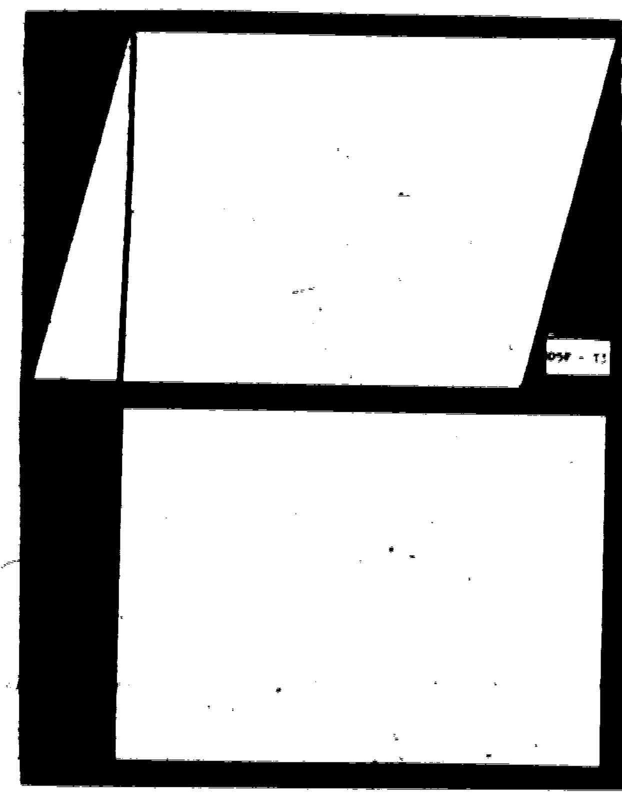
TASK 3



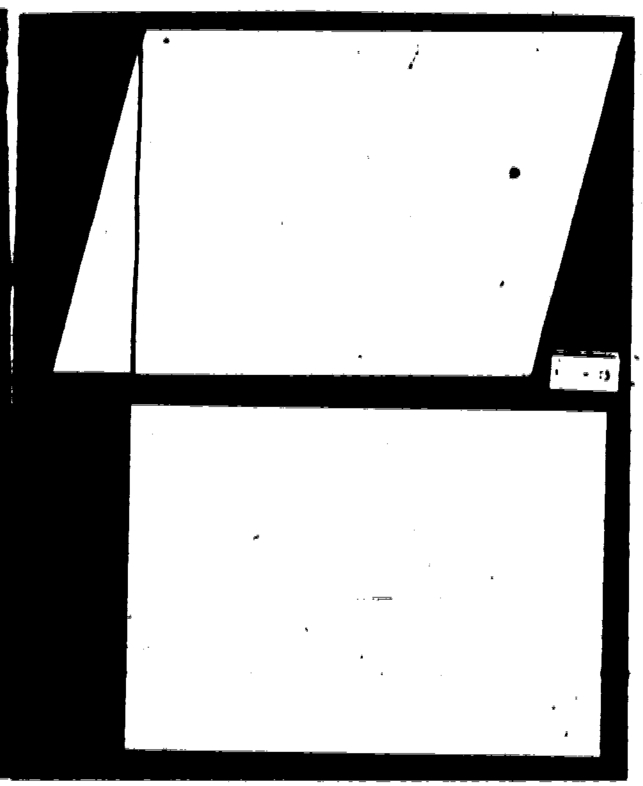
03M-1st attempt



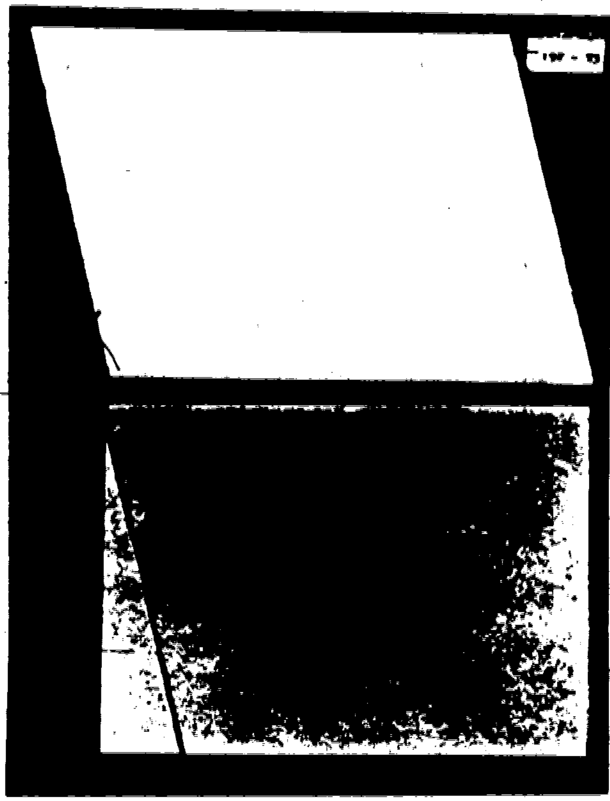
03M-2nd attempt



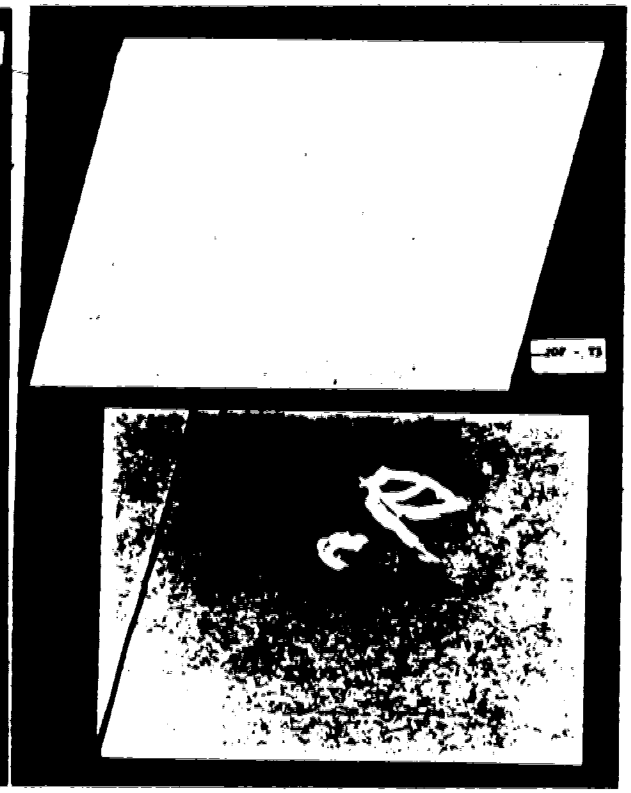
05F



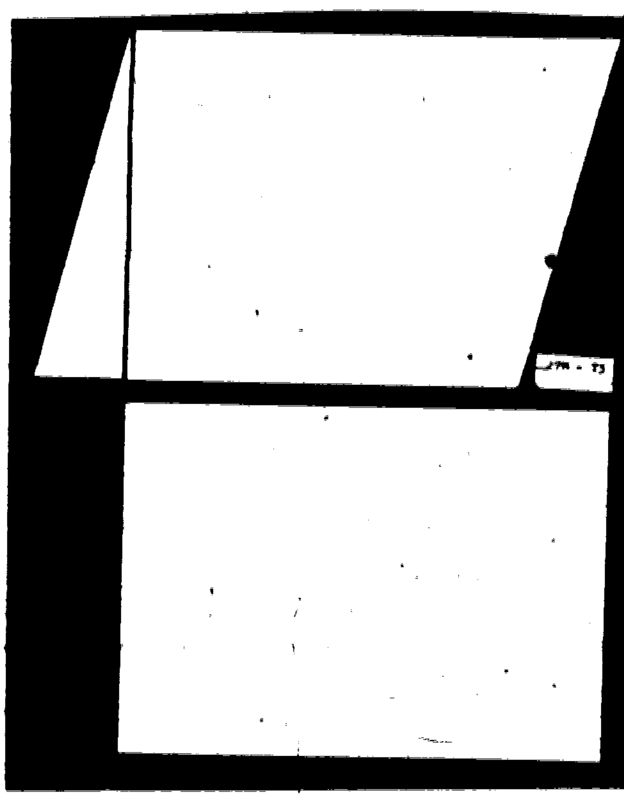
12M



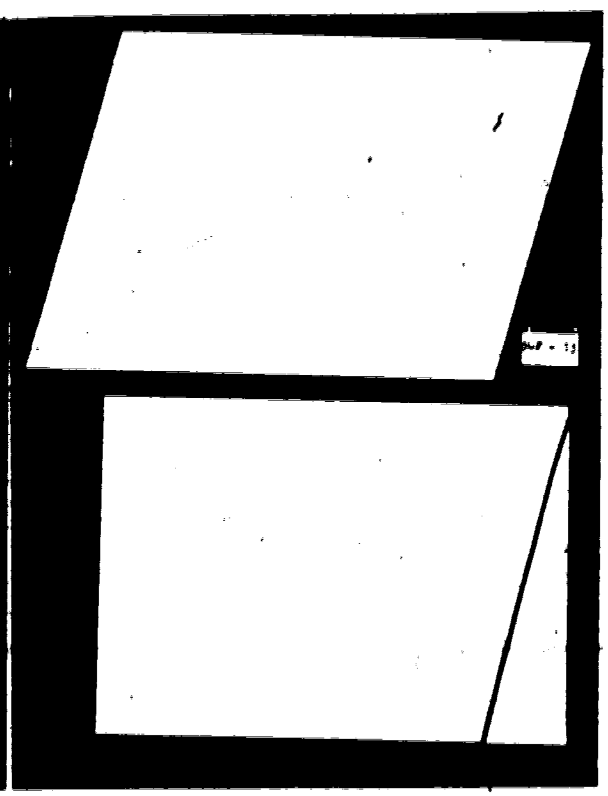
19F



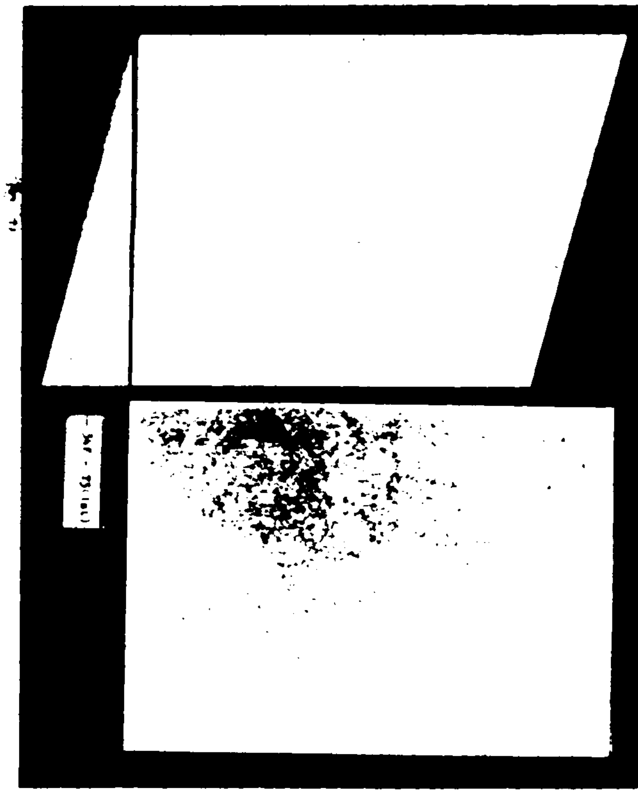
20F



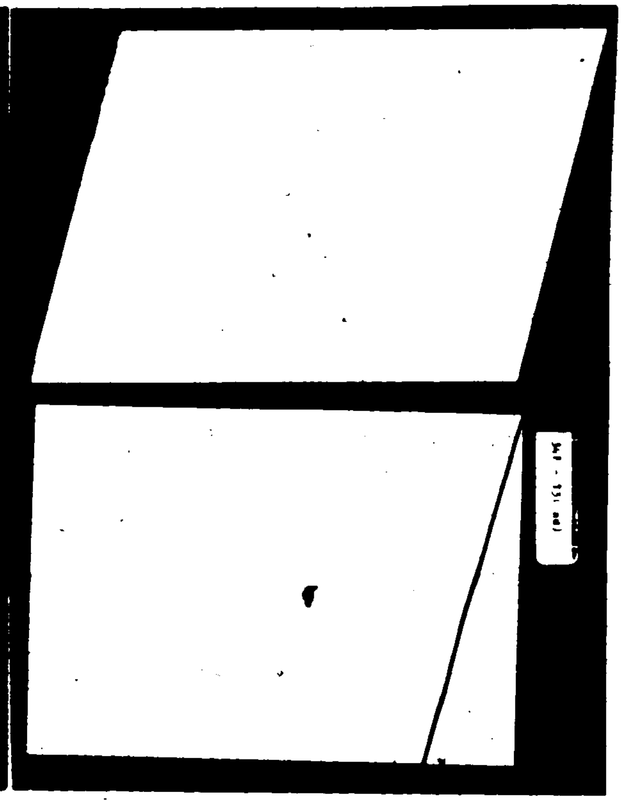
27M



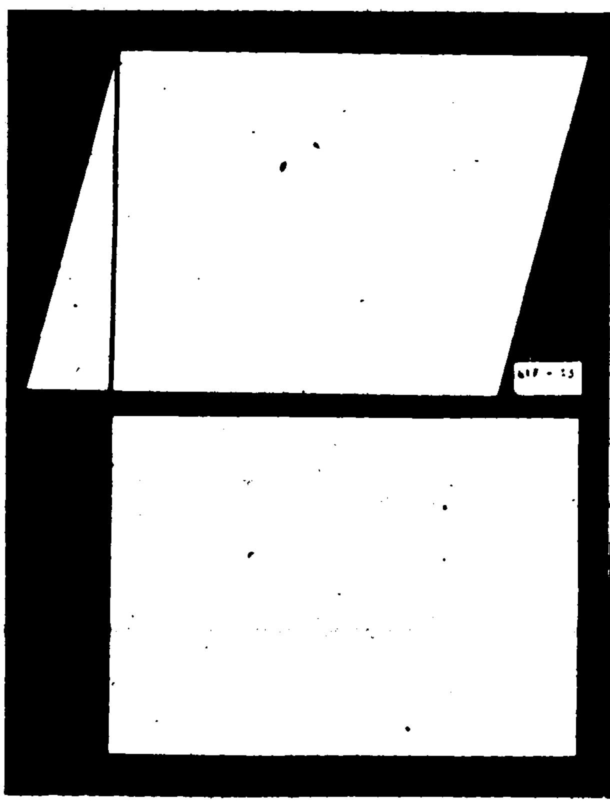
30F



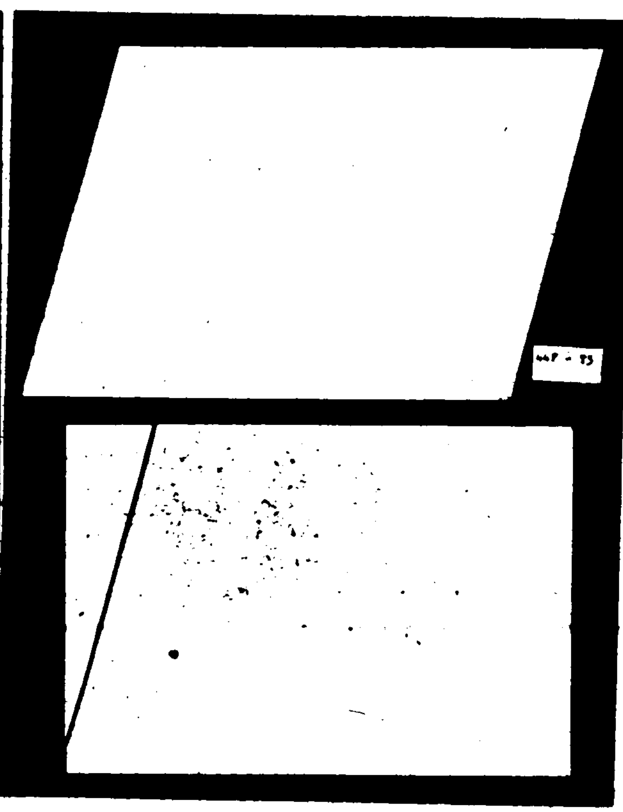
34F-1st attempt



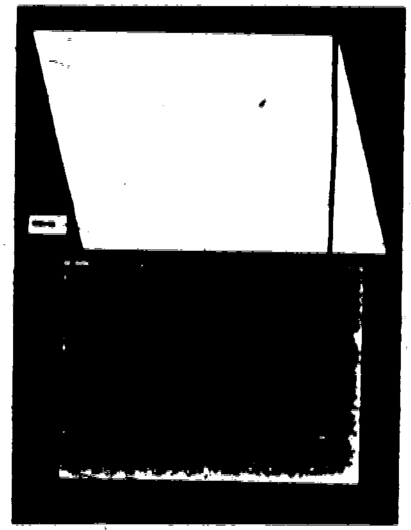
34F-2nd attempt



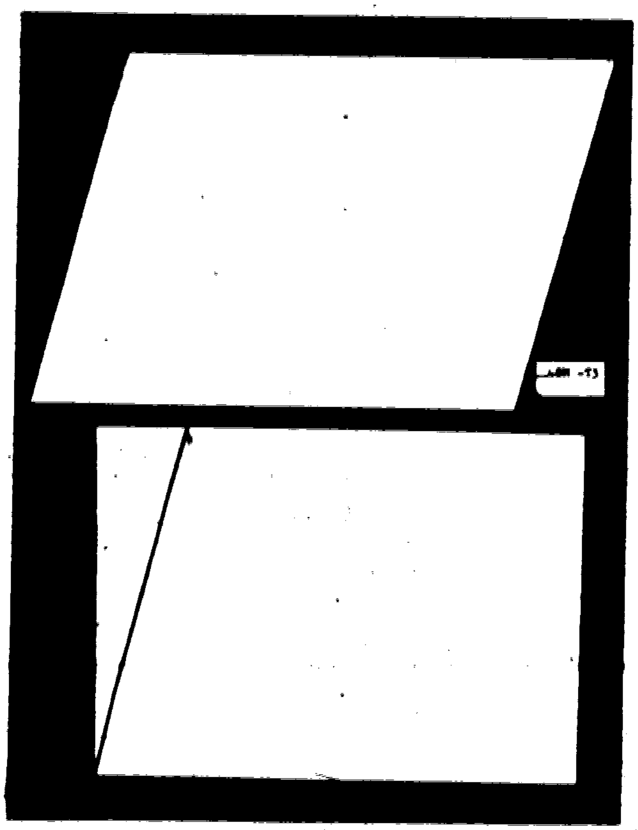
41F



44F

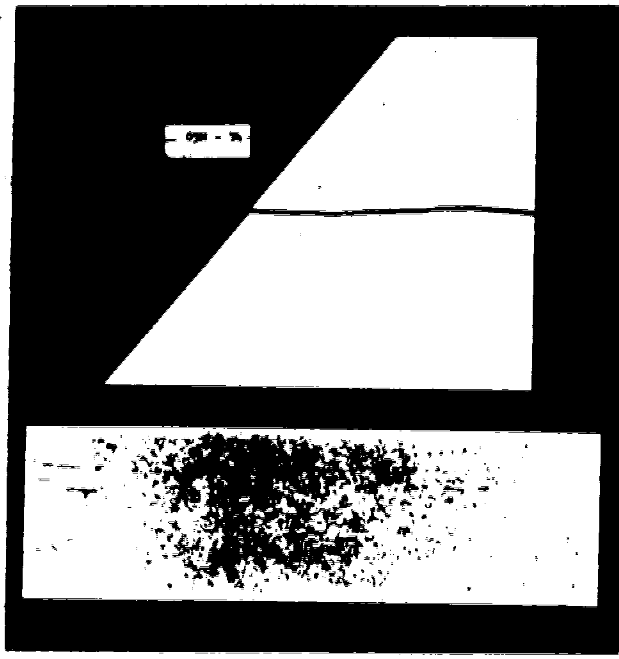


46M

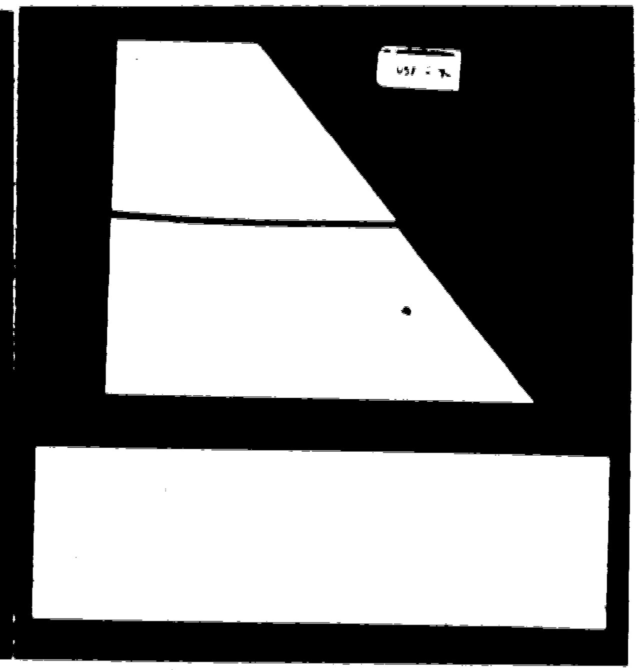


48M

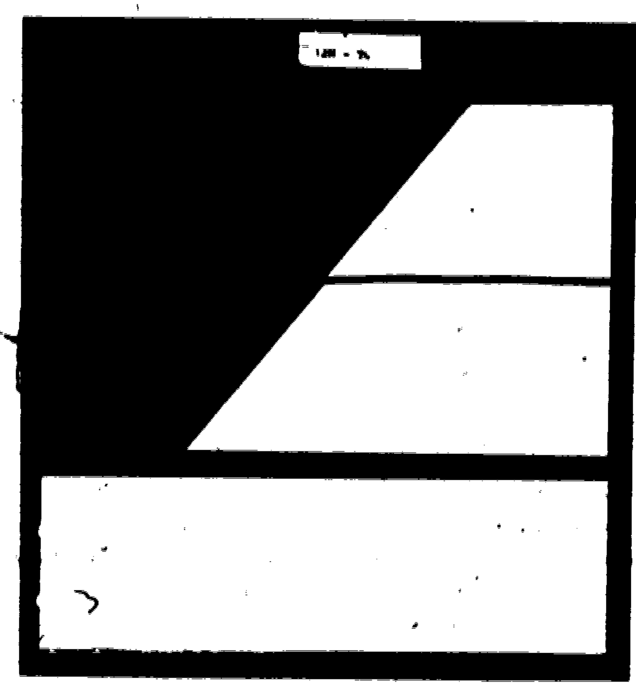
TASK 4



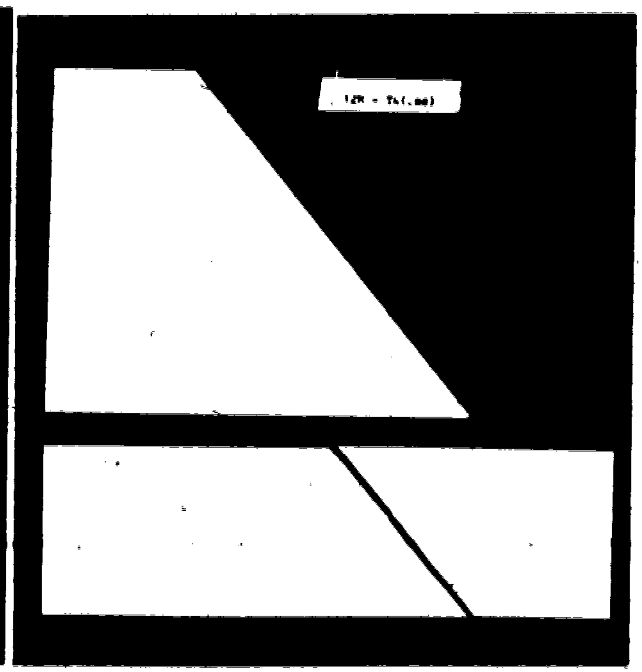
03M



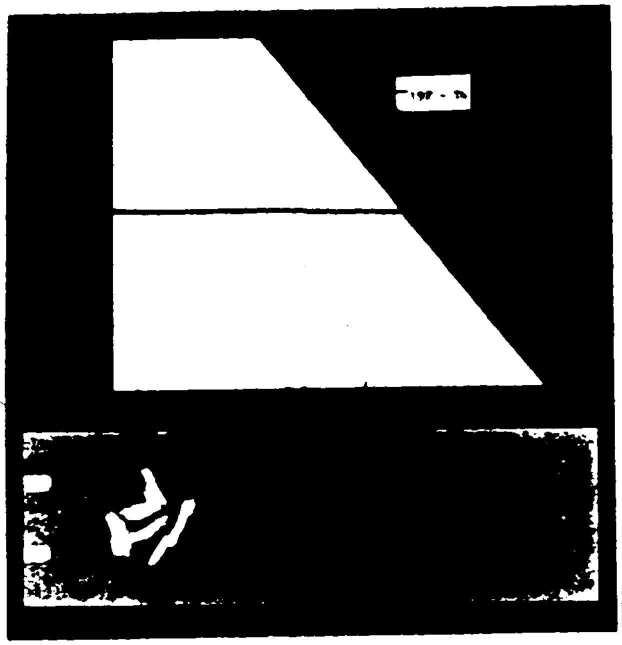
05F



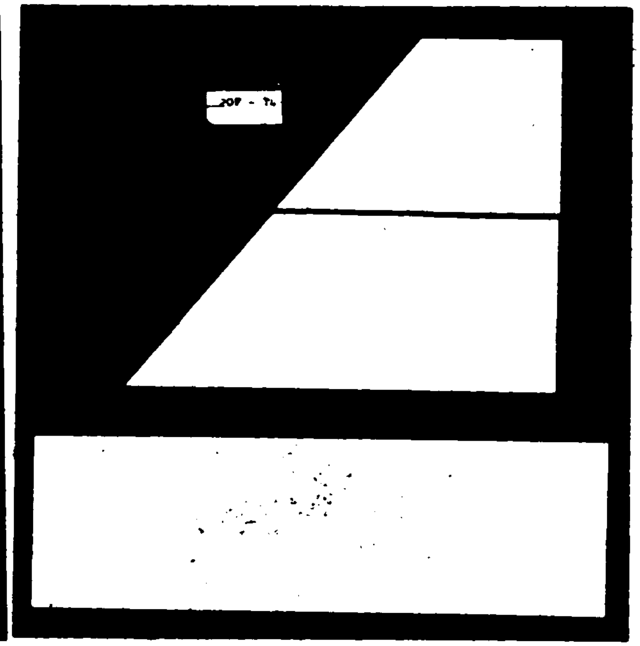
12M-1st attempt



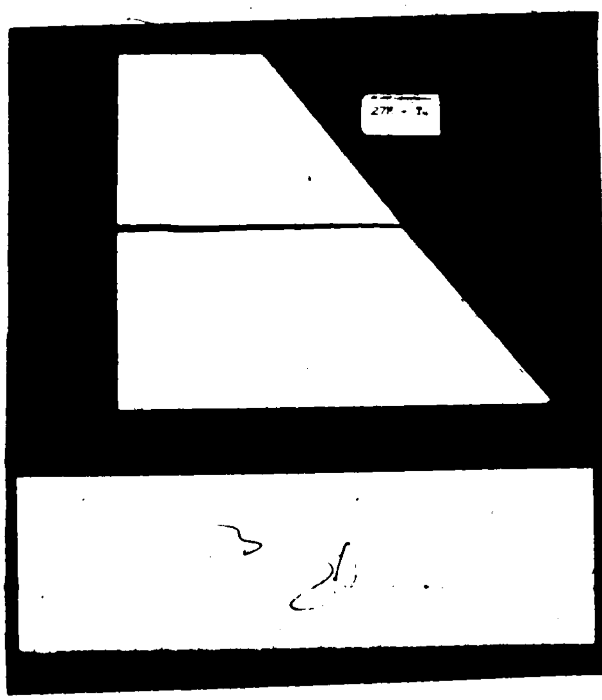
12M-2nd attempt



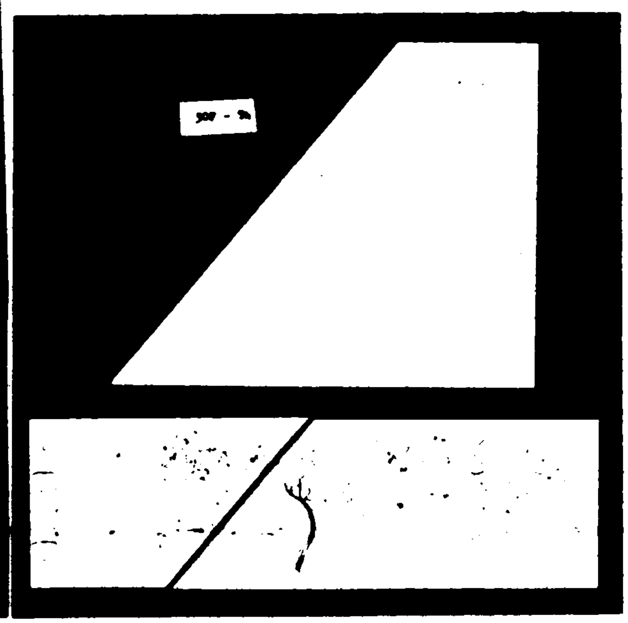
19F



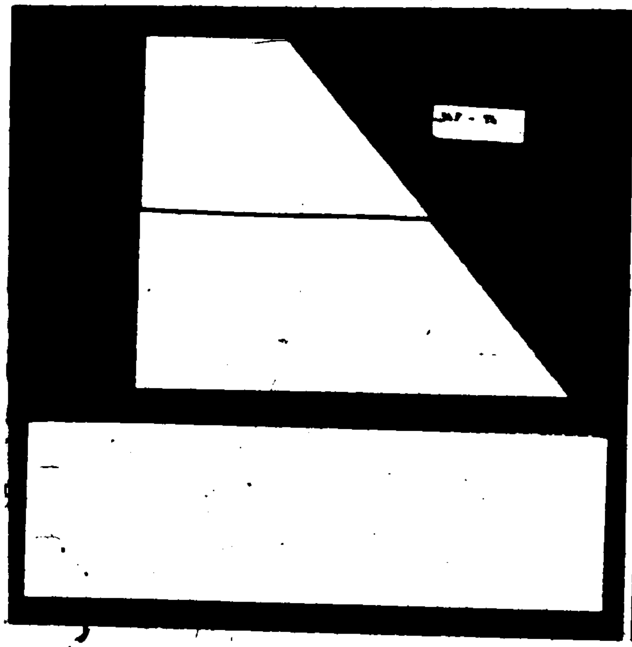
20F



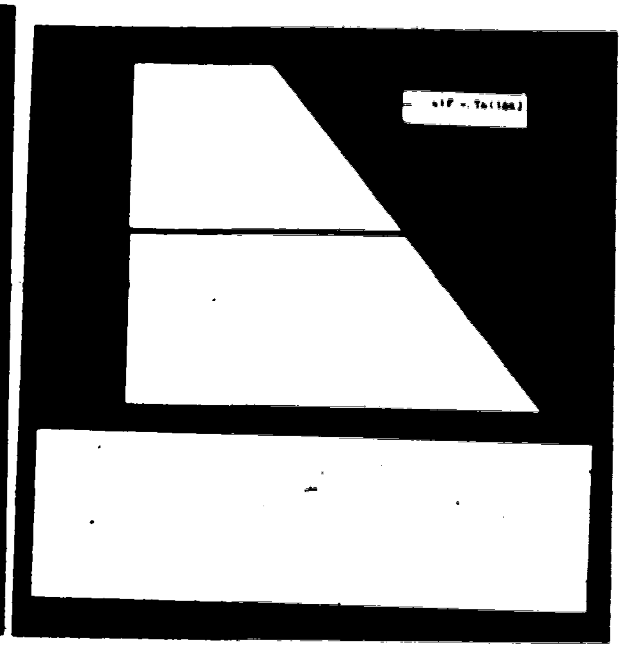
27M



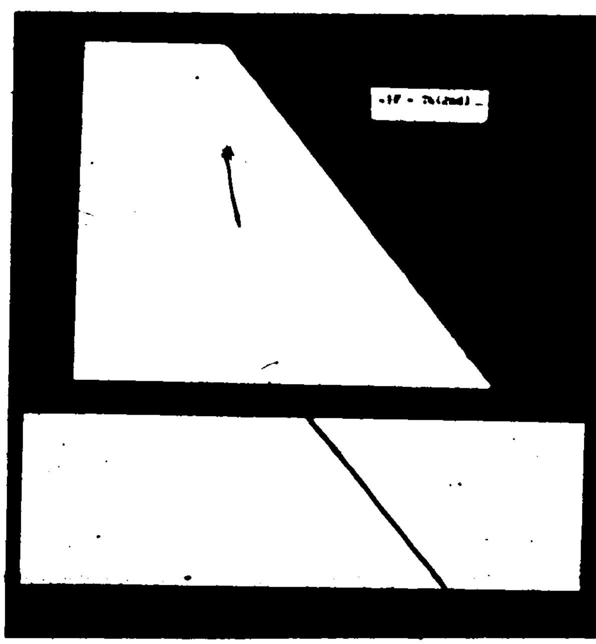
30F



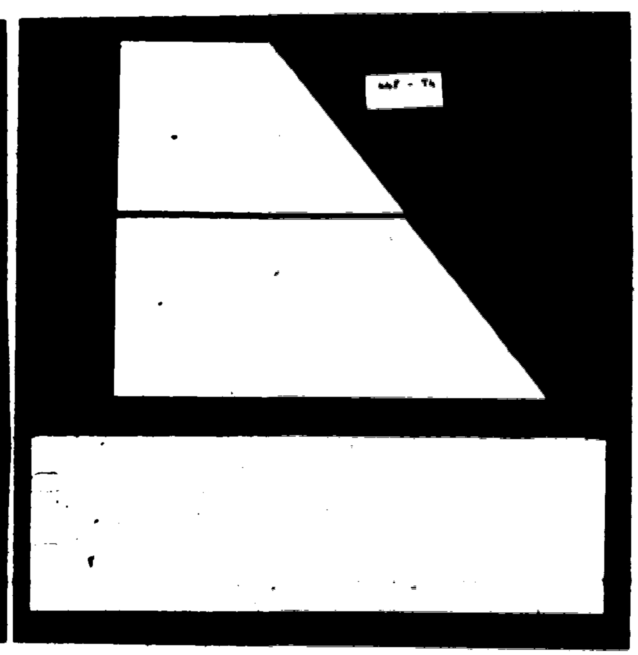
34F



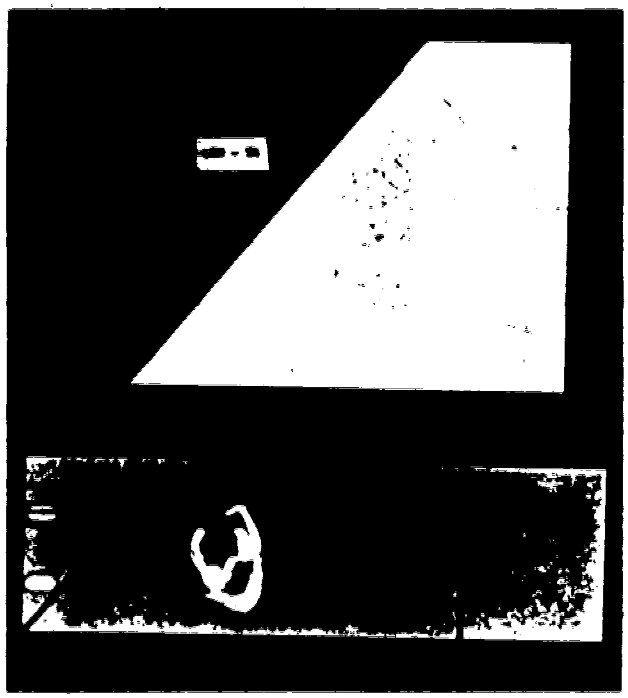
41F-1st attempt



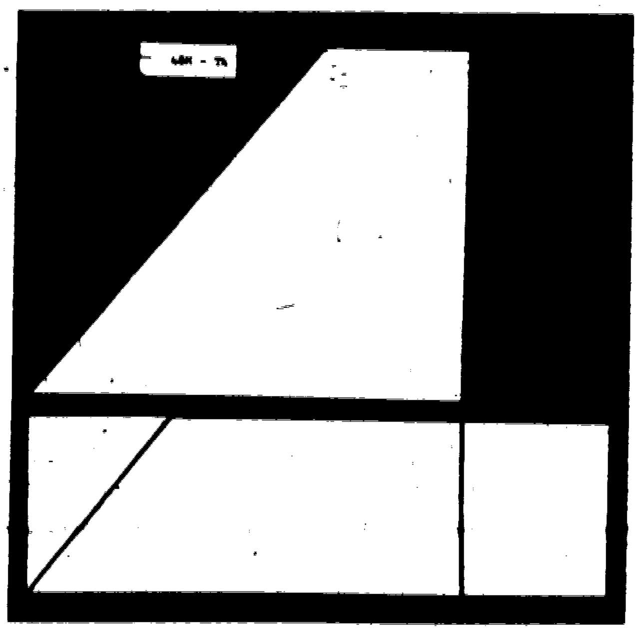
41F-2nd attempt



44F

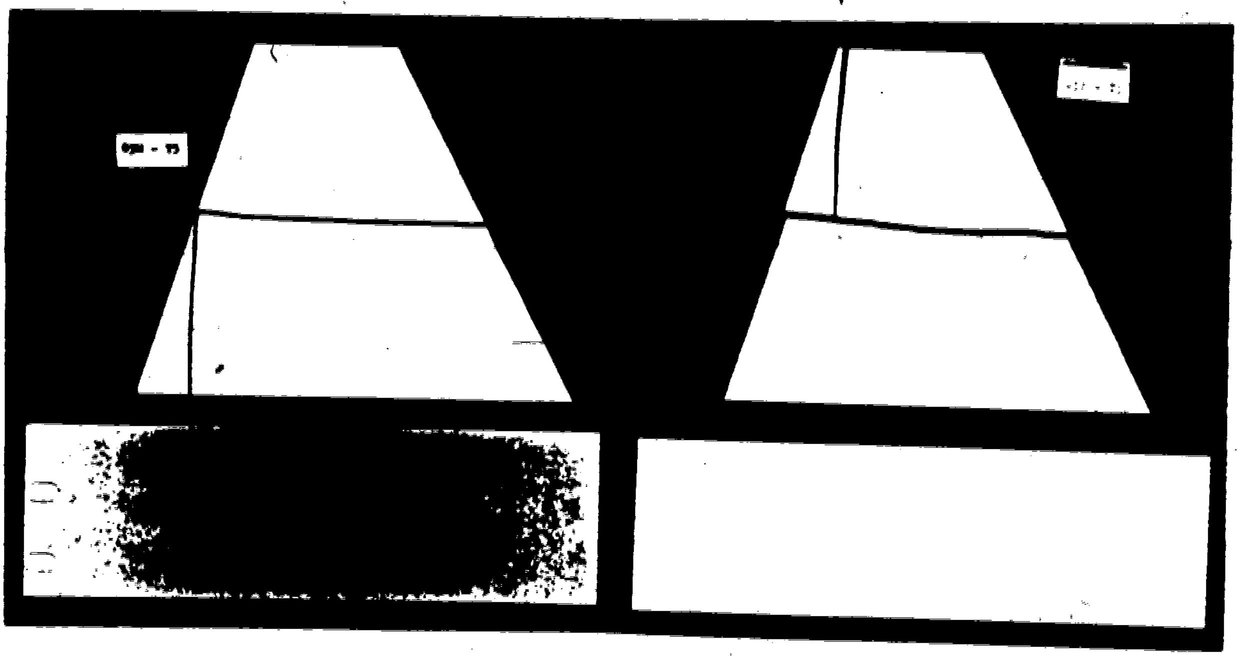


46M

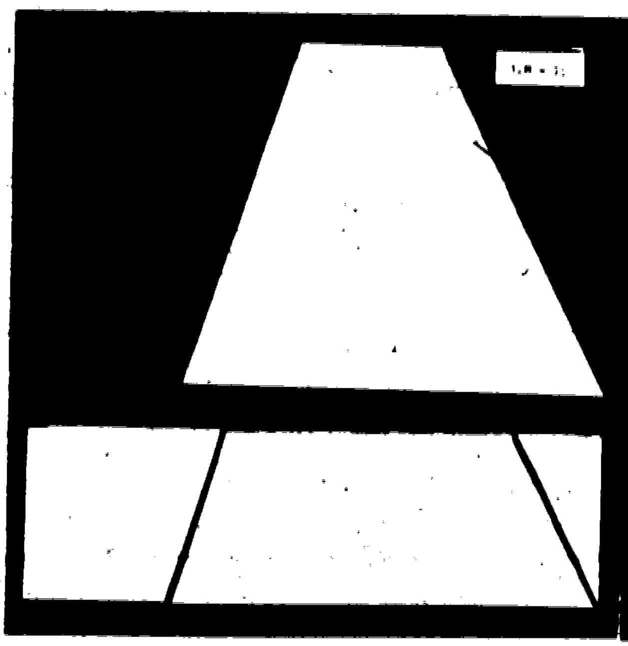


48M

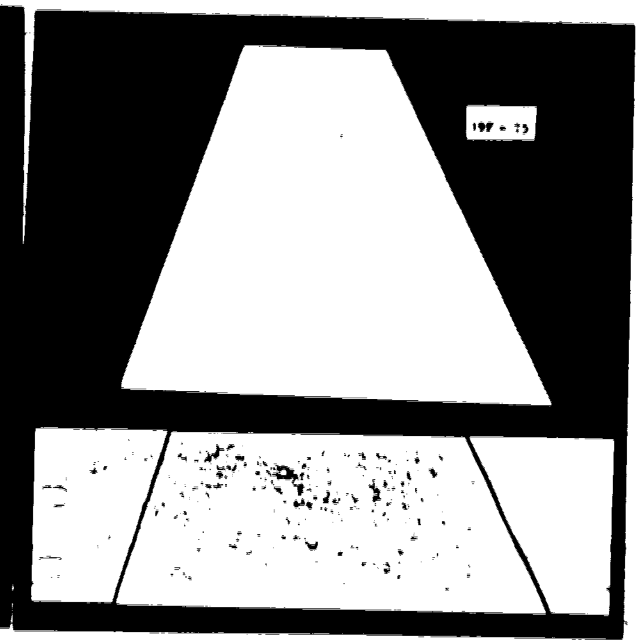
TASK 5



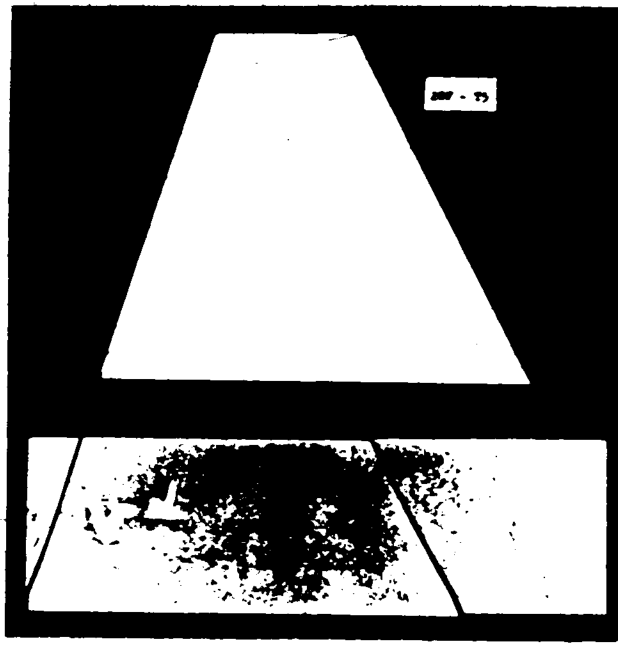
03M 05F



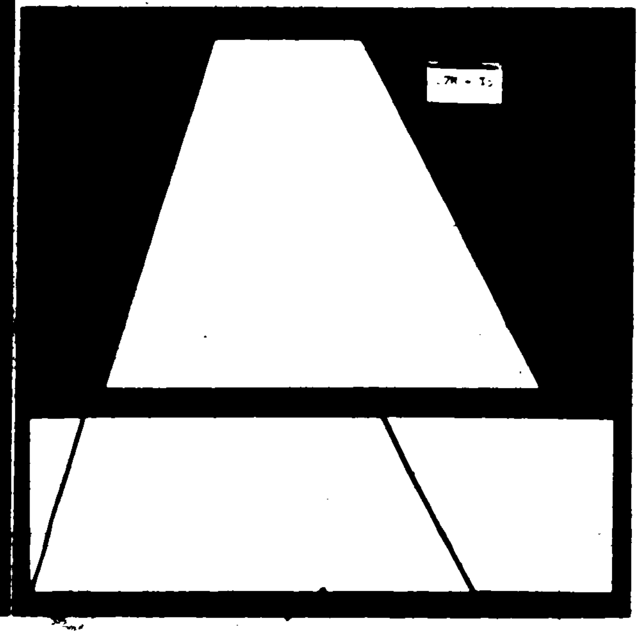
12M



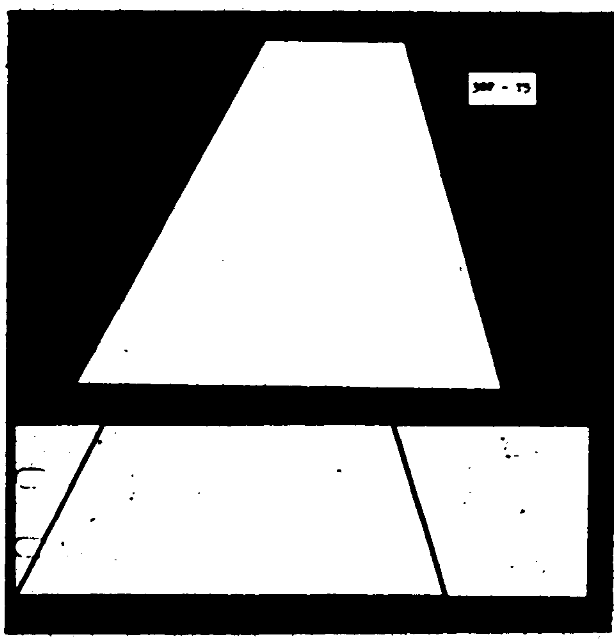
19F



20F



27M



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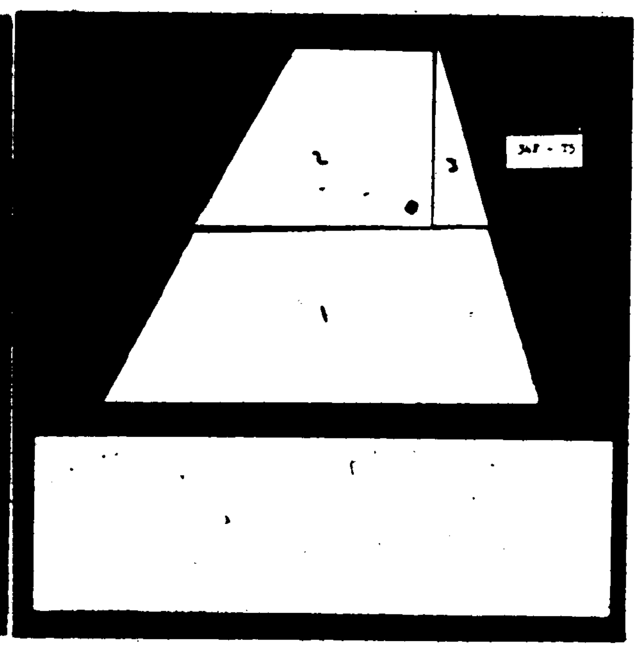
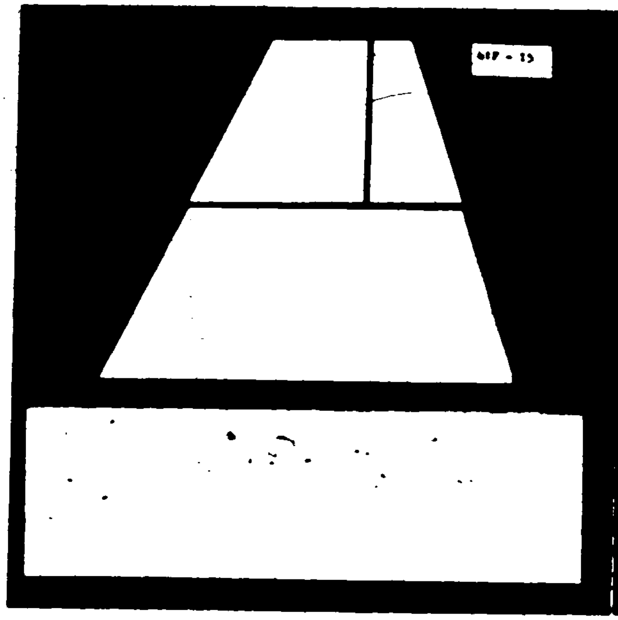
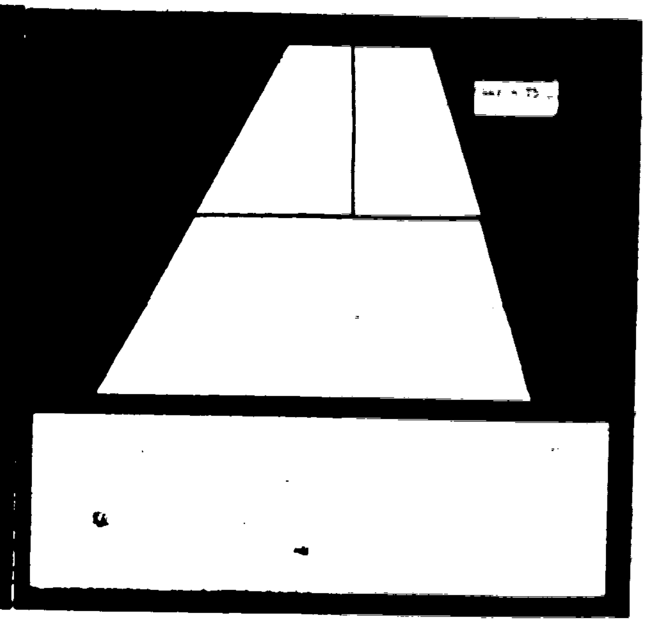


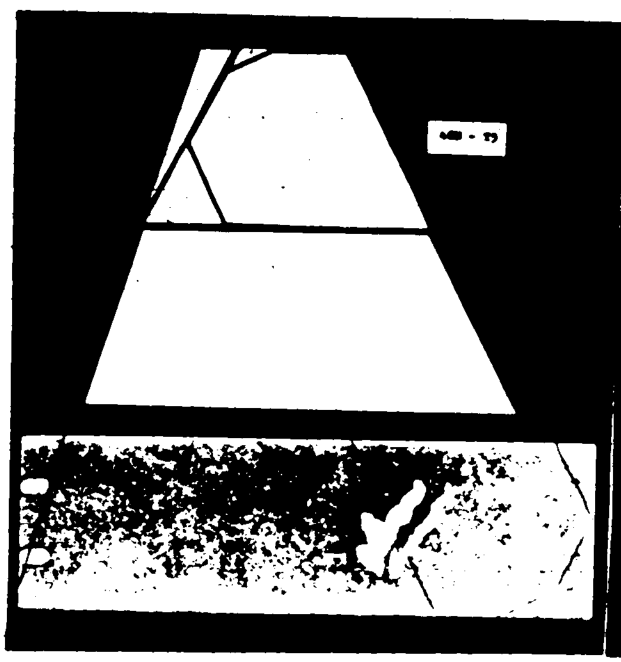
34F



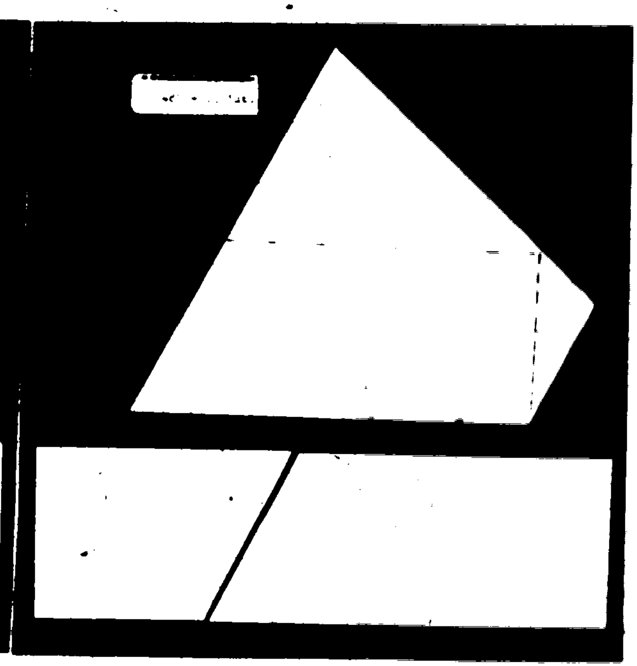
41F



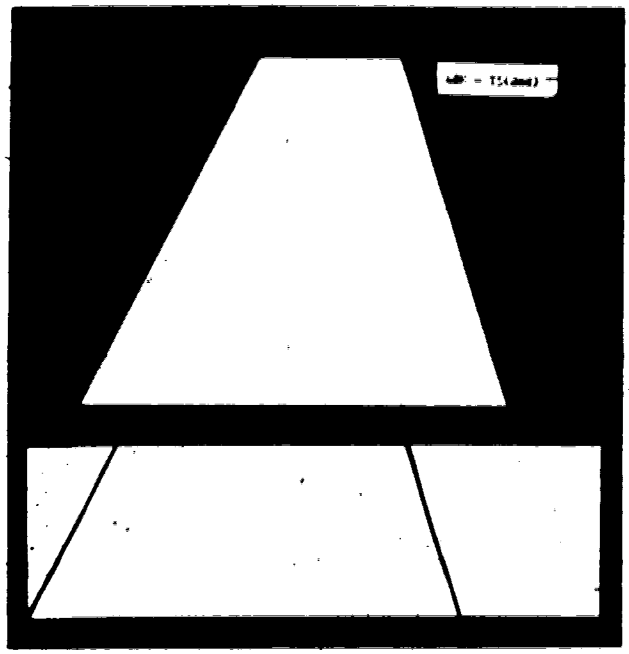
44F



46M

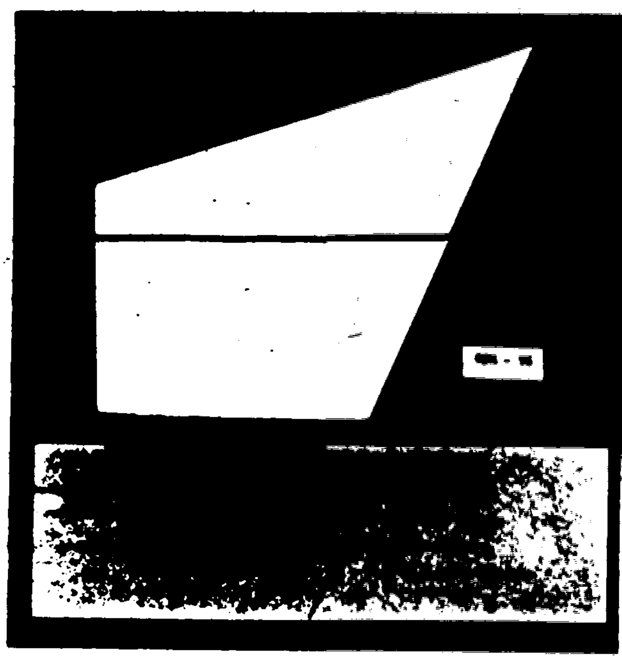


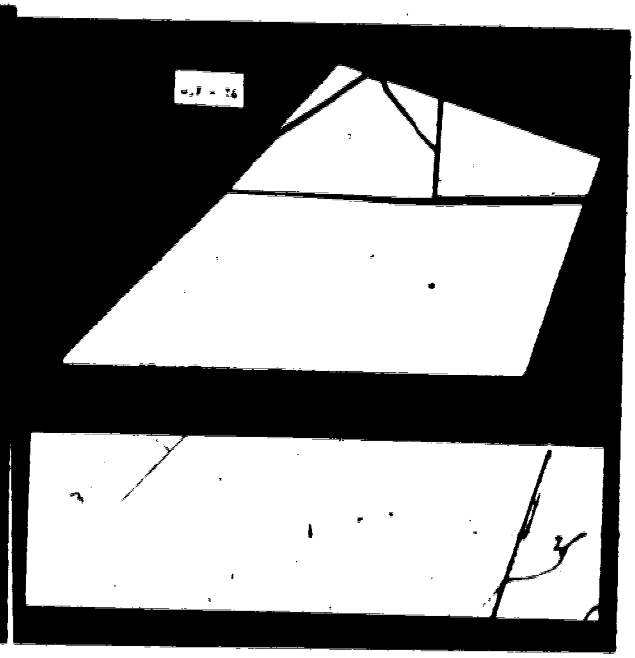
48M-1st attempt



48H-2nd attempt

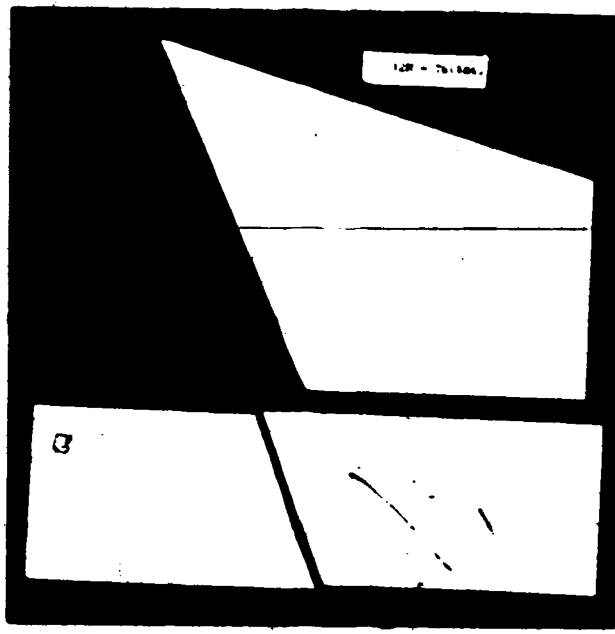
TASK 6



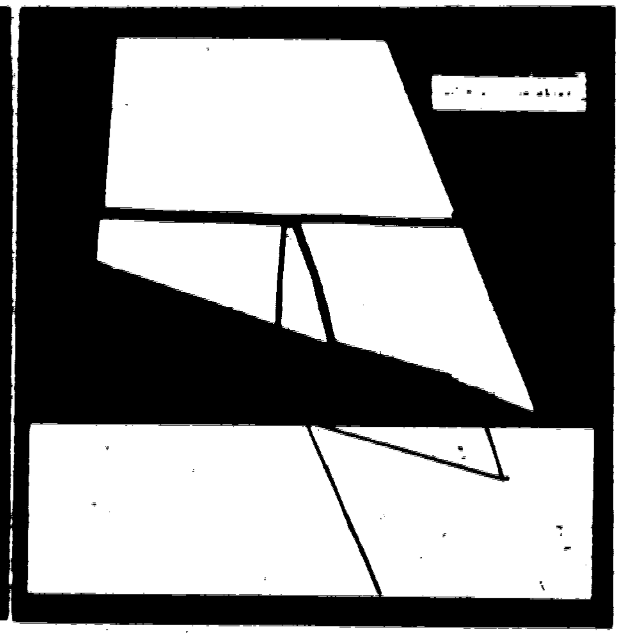


03M

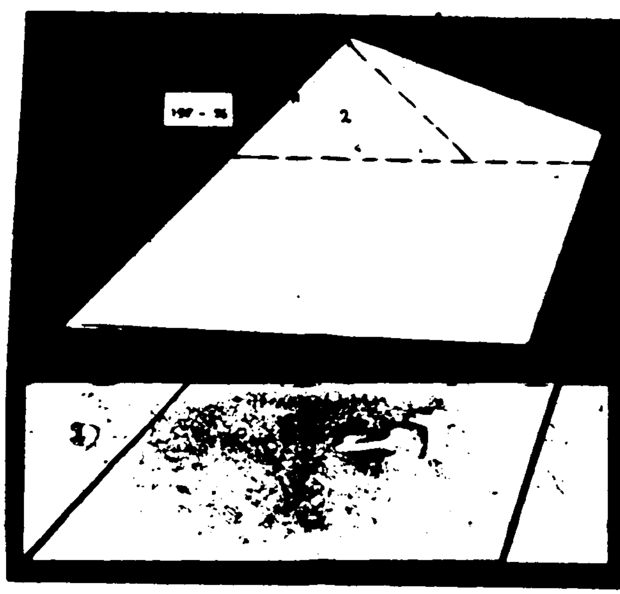
05F



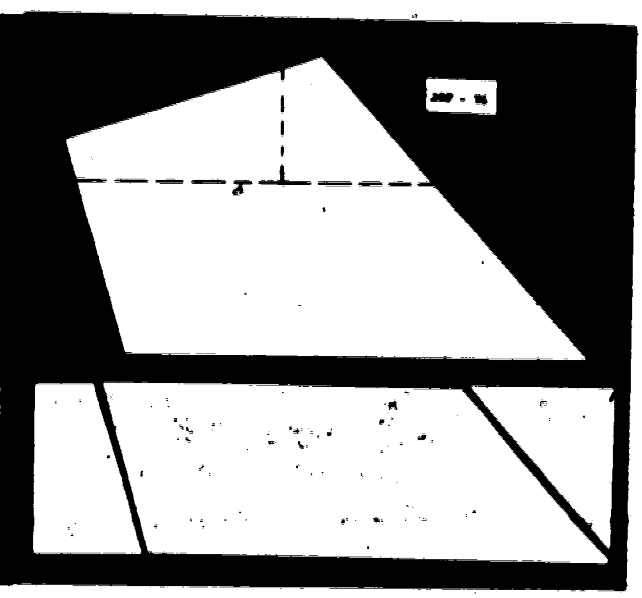
12M-1st attempt



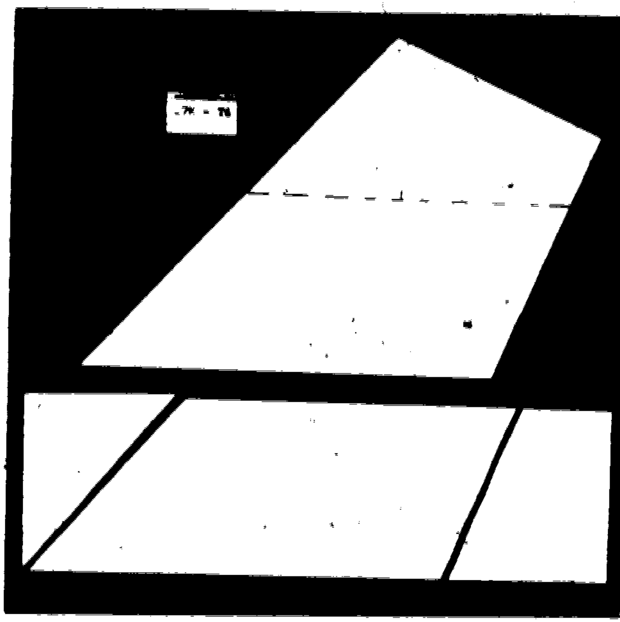
12M-2nd attempt



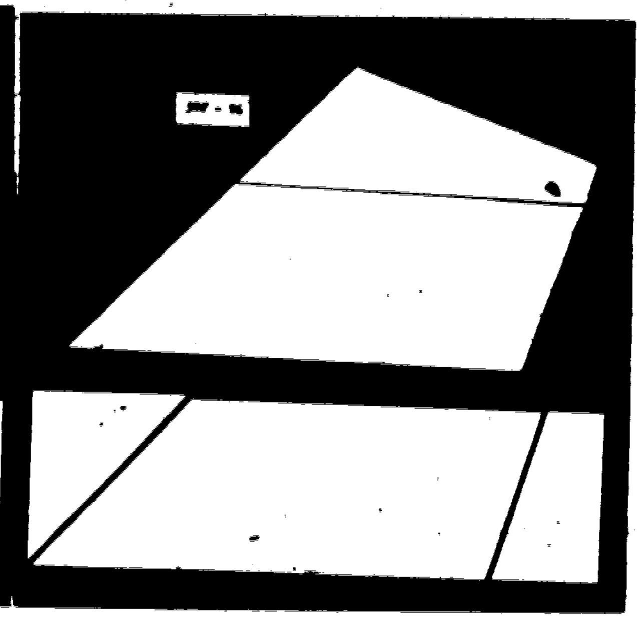
19F



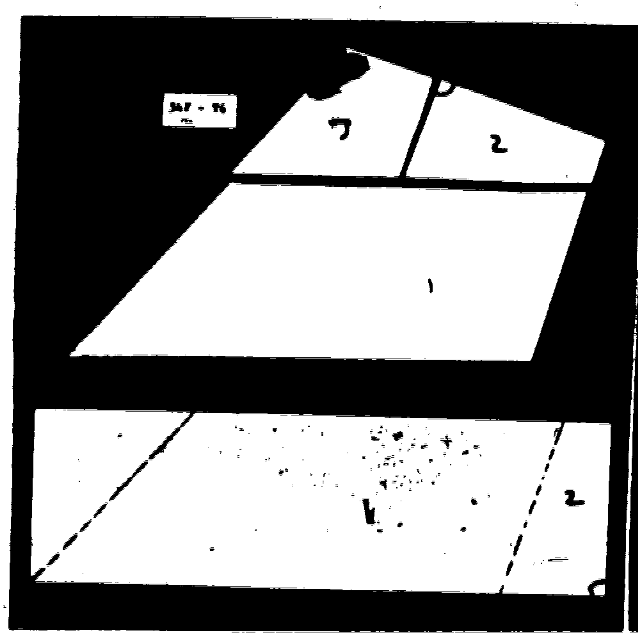
20F



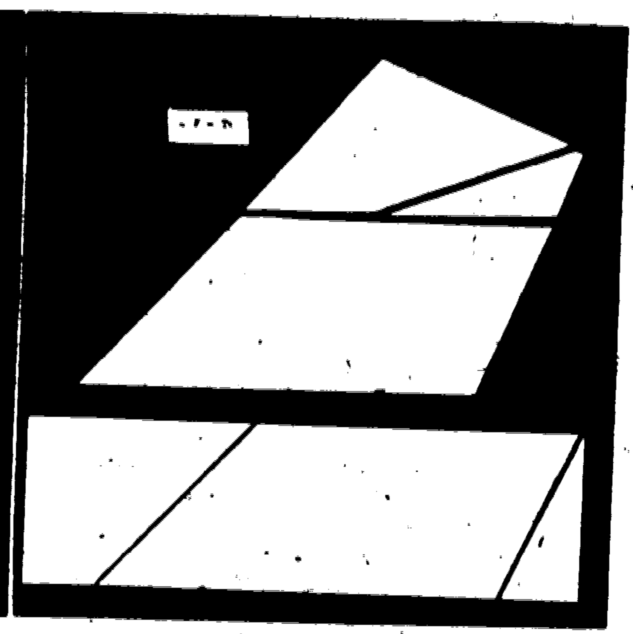
27M



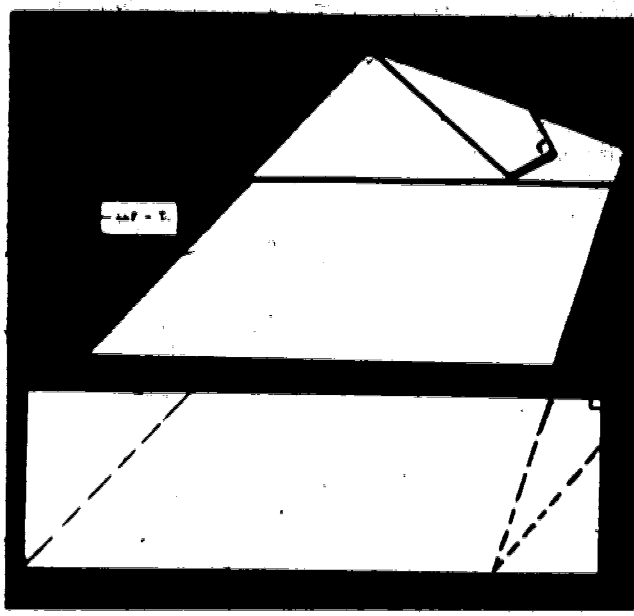
30F



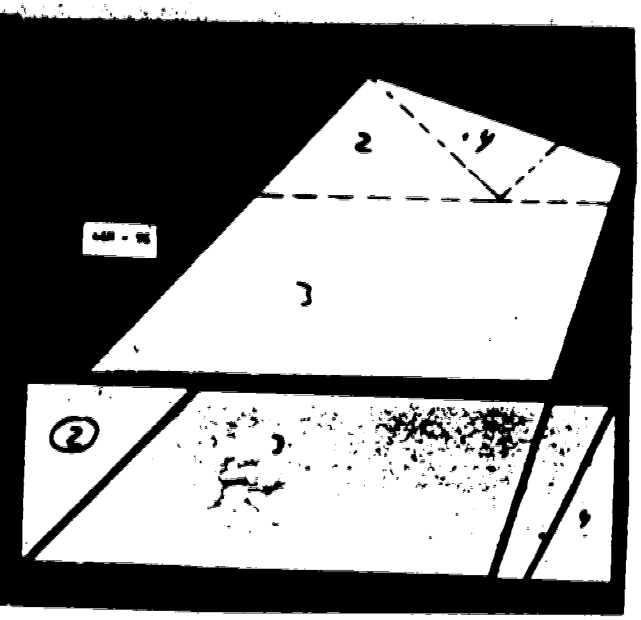
34F



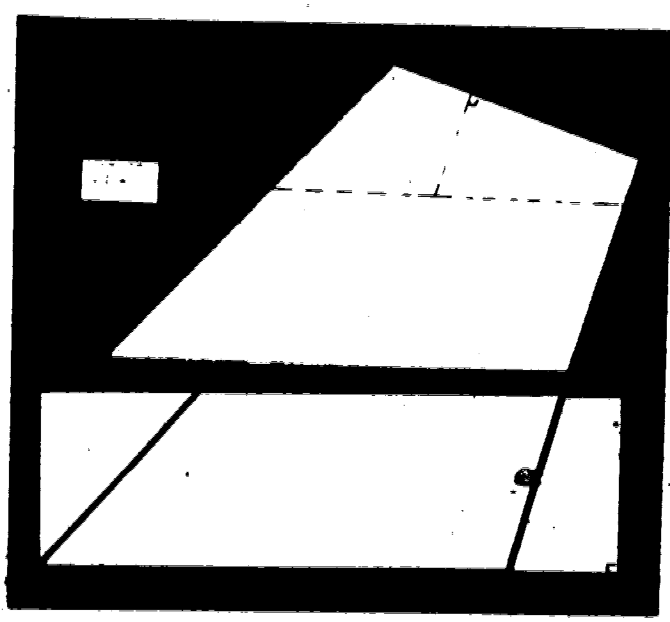
41F



44F

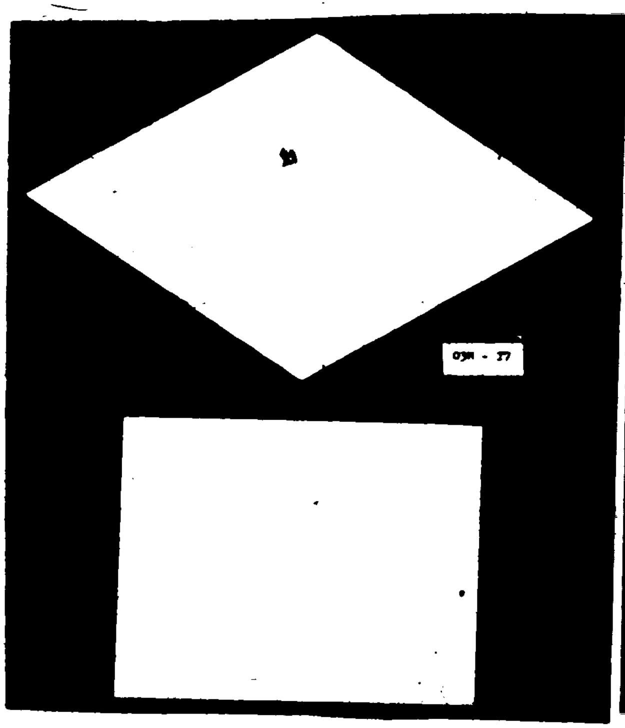


46M

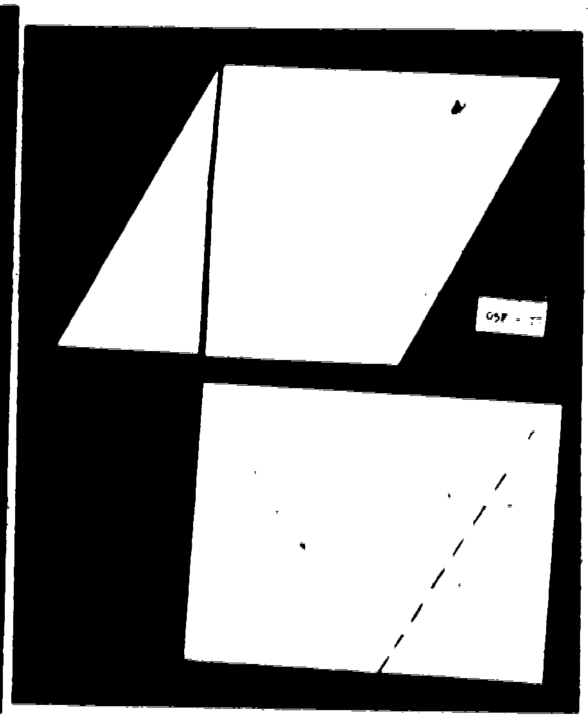


48M

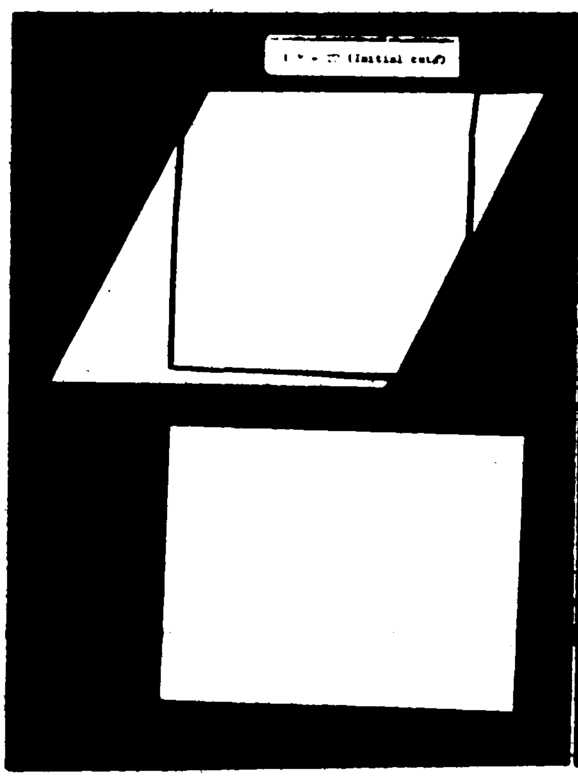
TASK 7



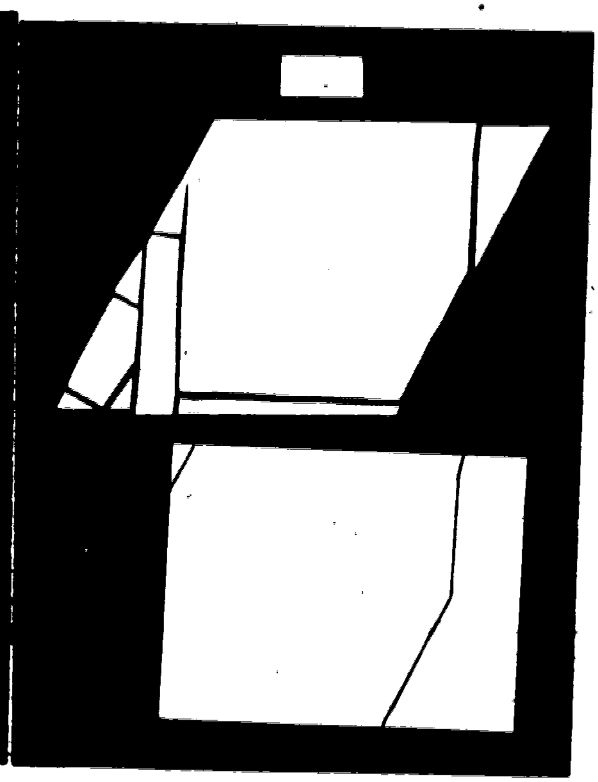
03M



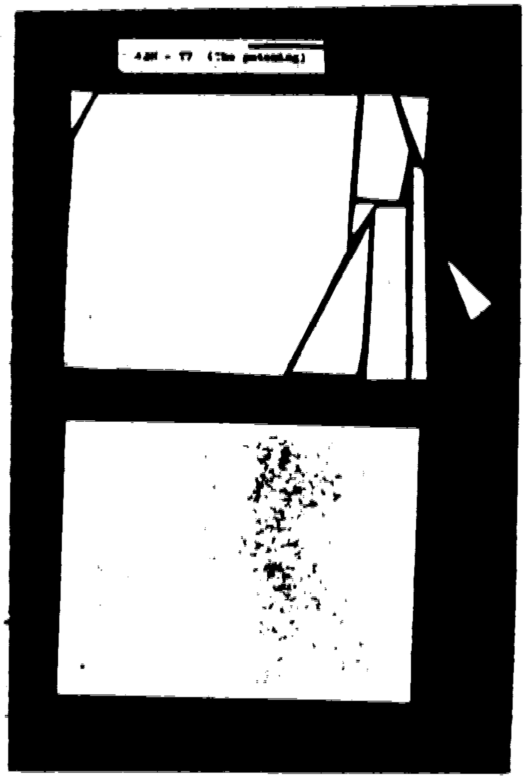
05F



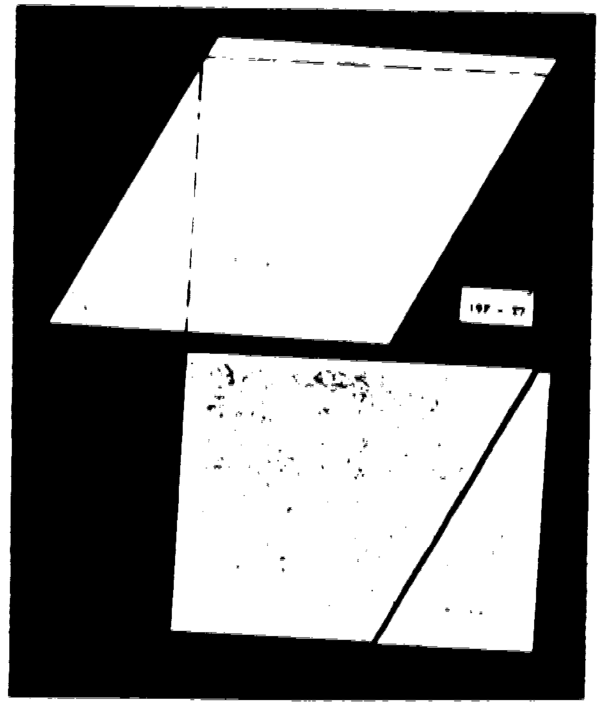
12M-Initial cuts



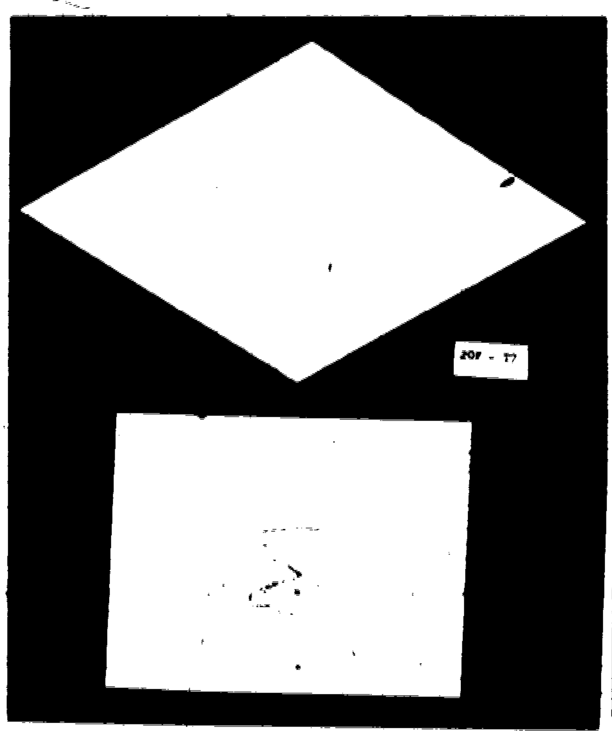
12M-Final cuts



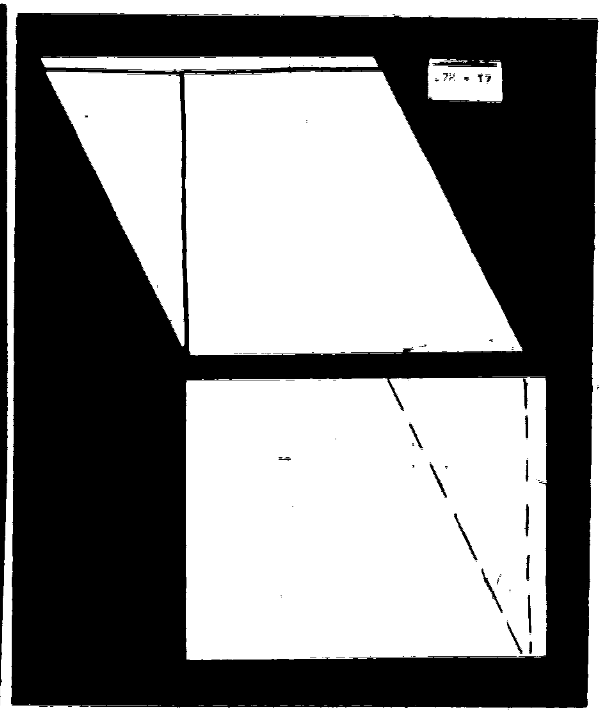
12M-The patching



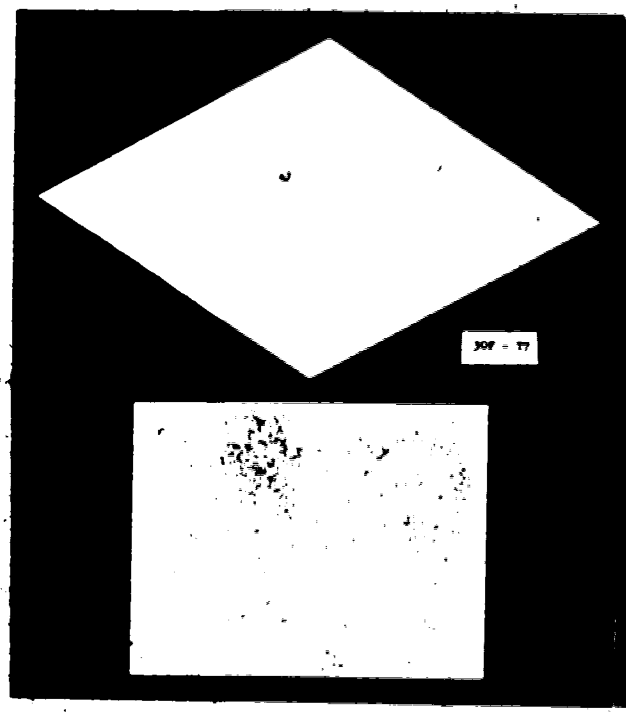
19F



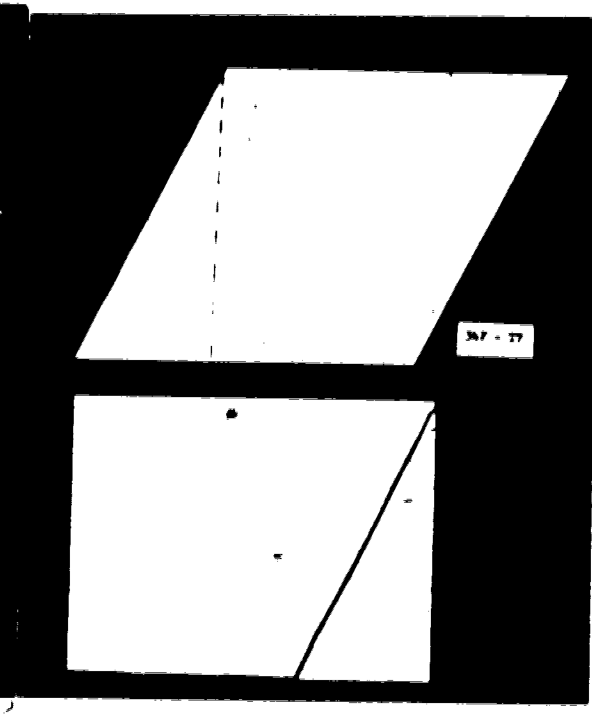
20F



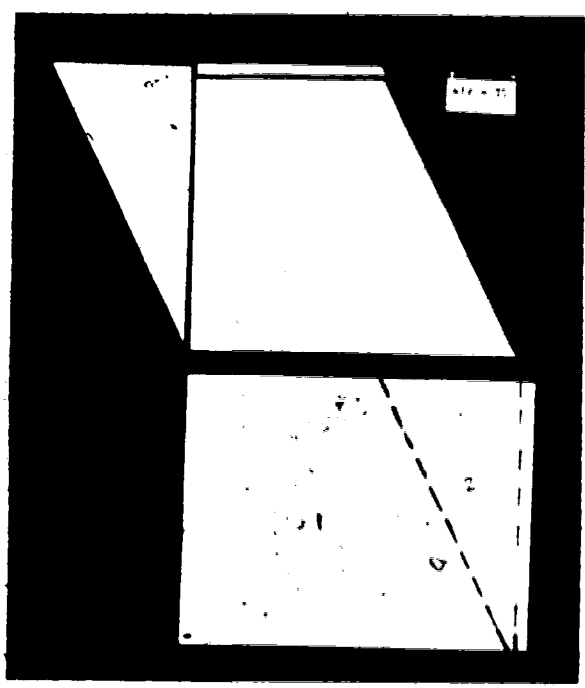
27M



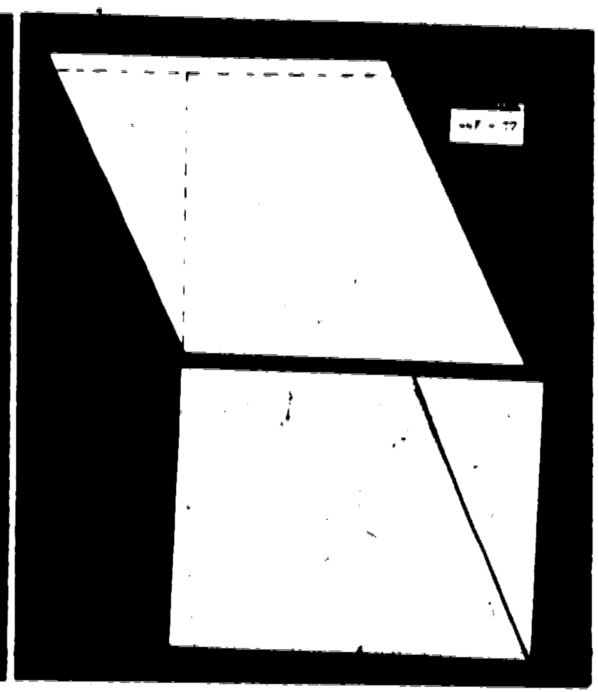
30F



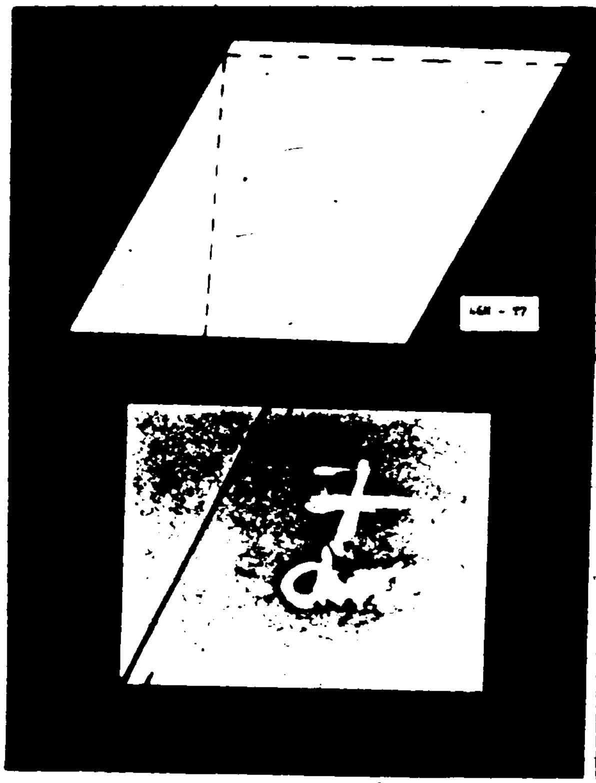
34F



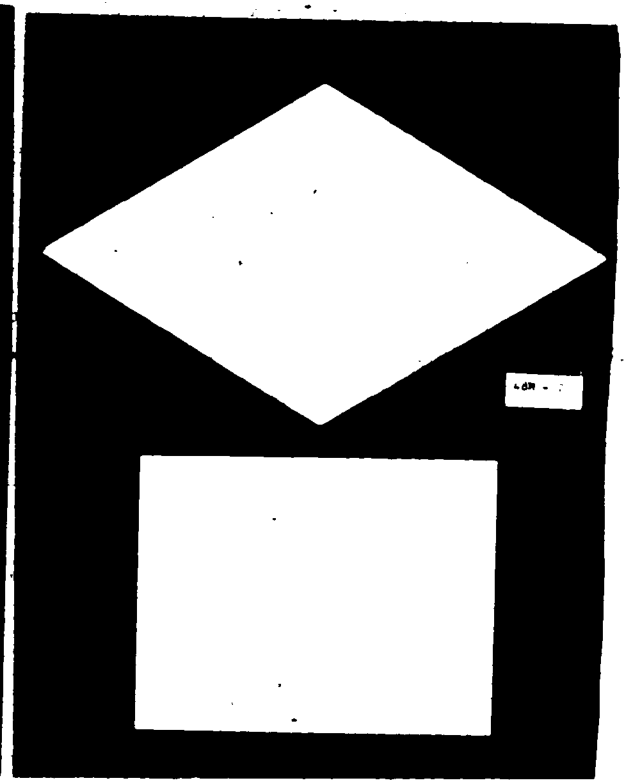
41F



44F

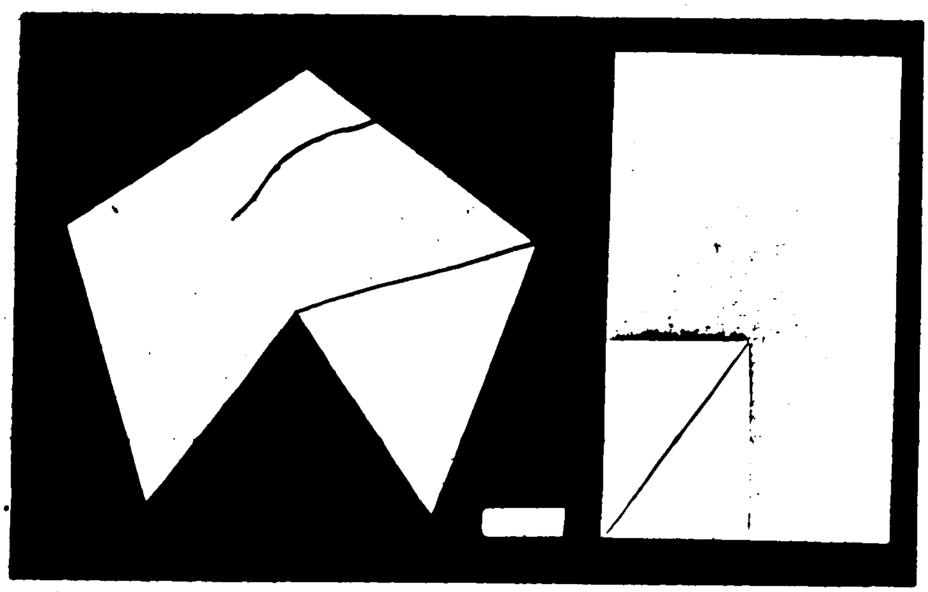


46M

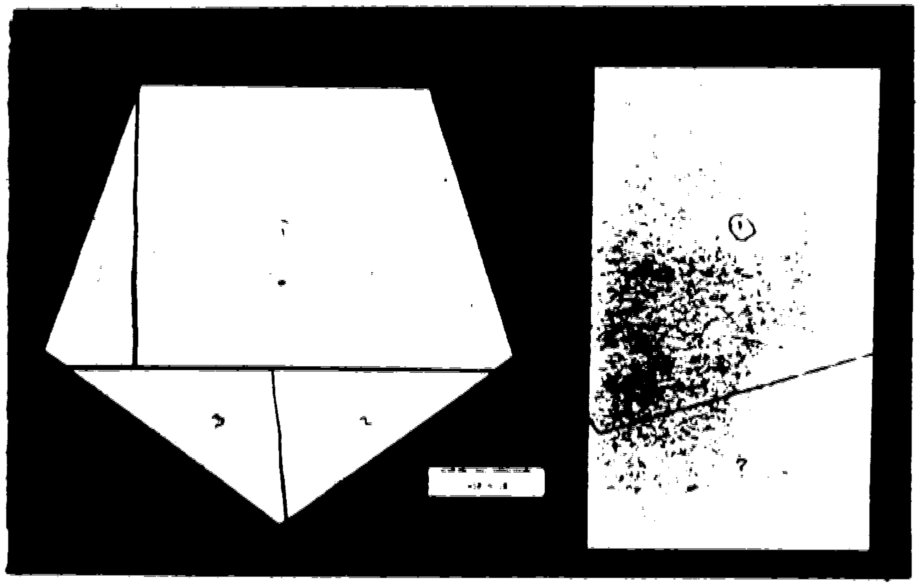


48M

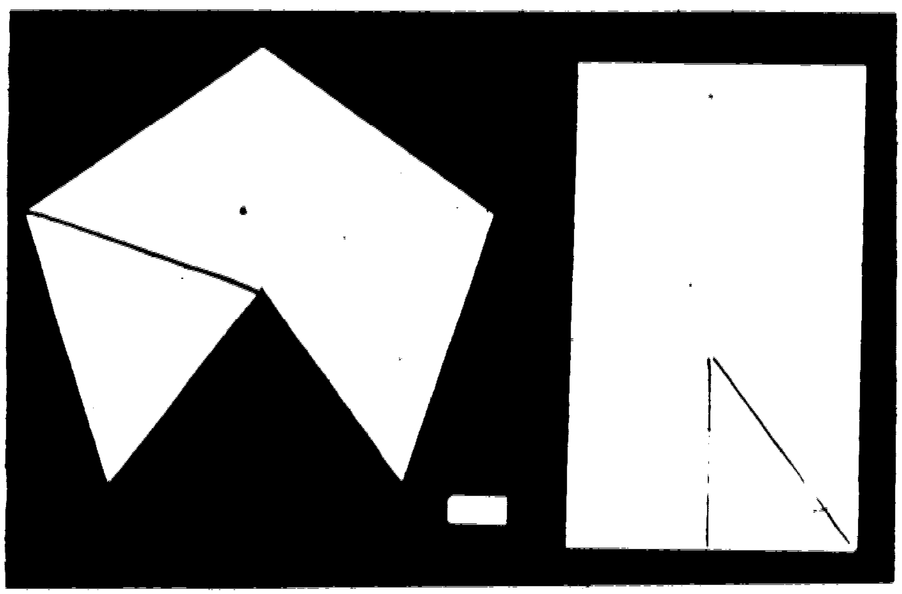
TASK 8



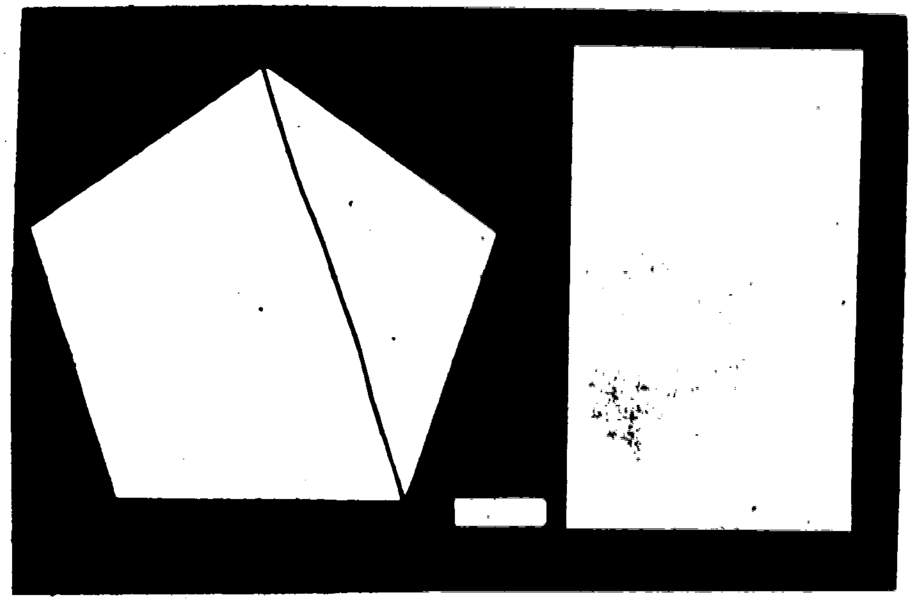
03M



05F



12M



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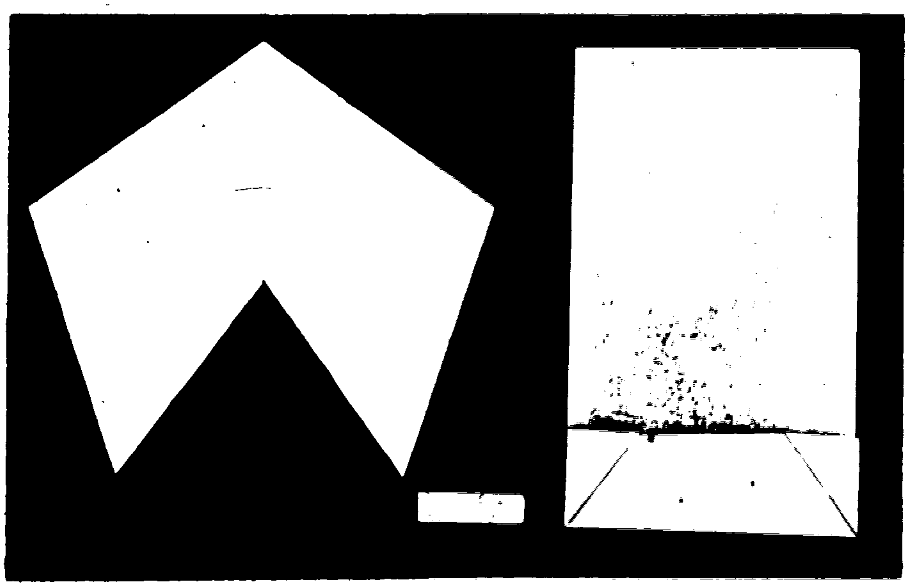

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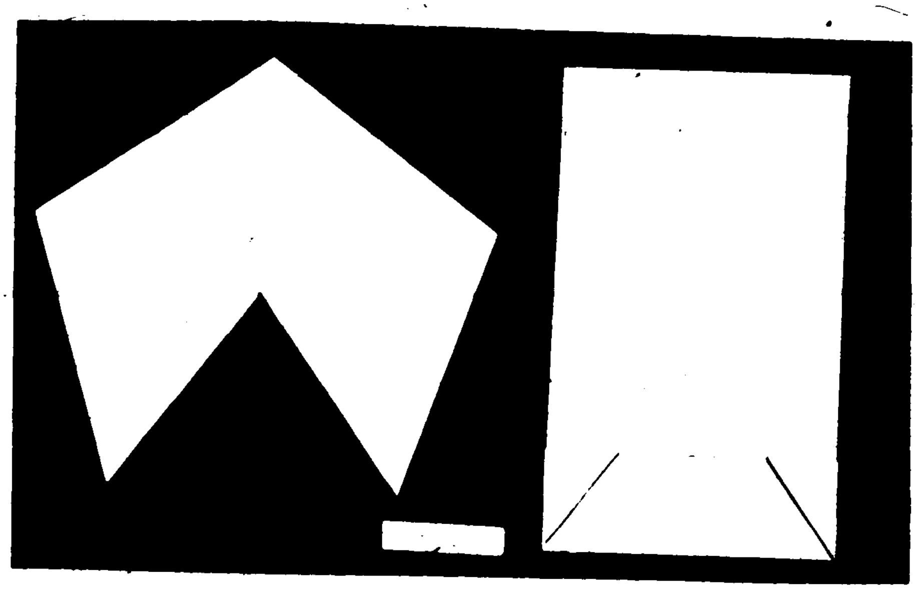
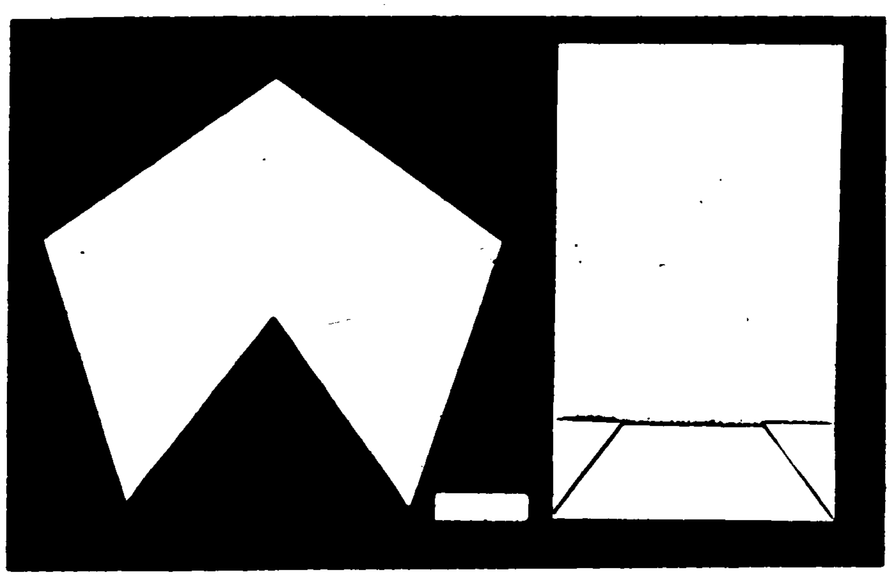
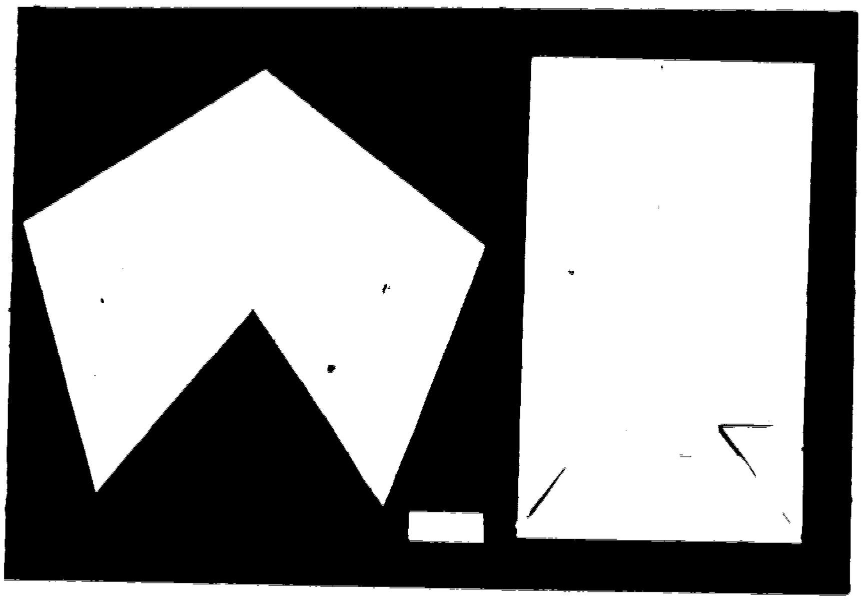


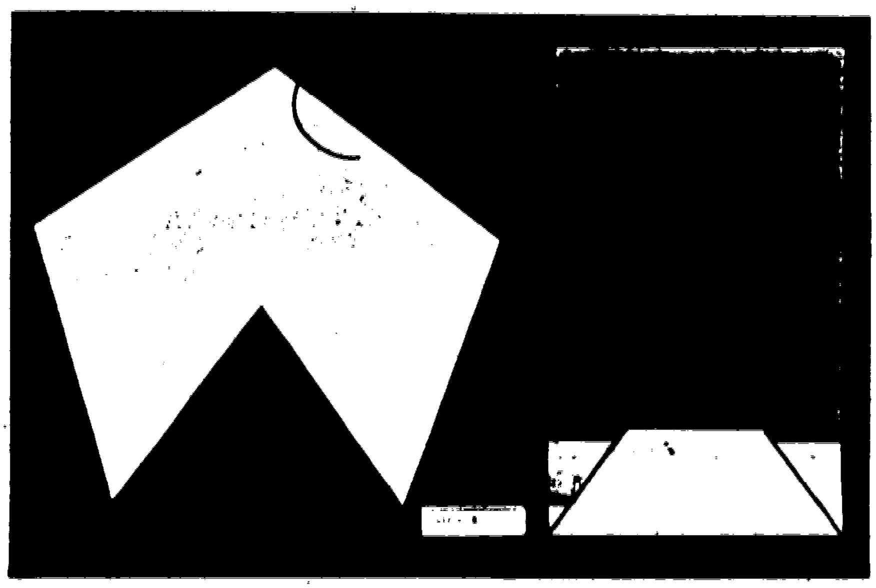
27M



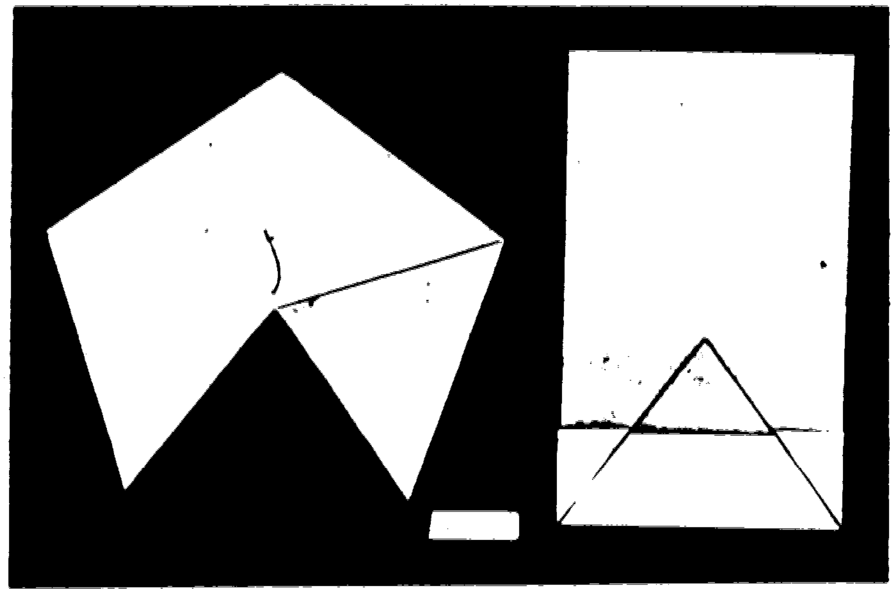
30F



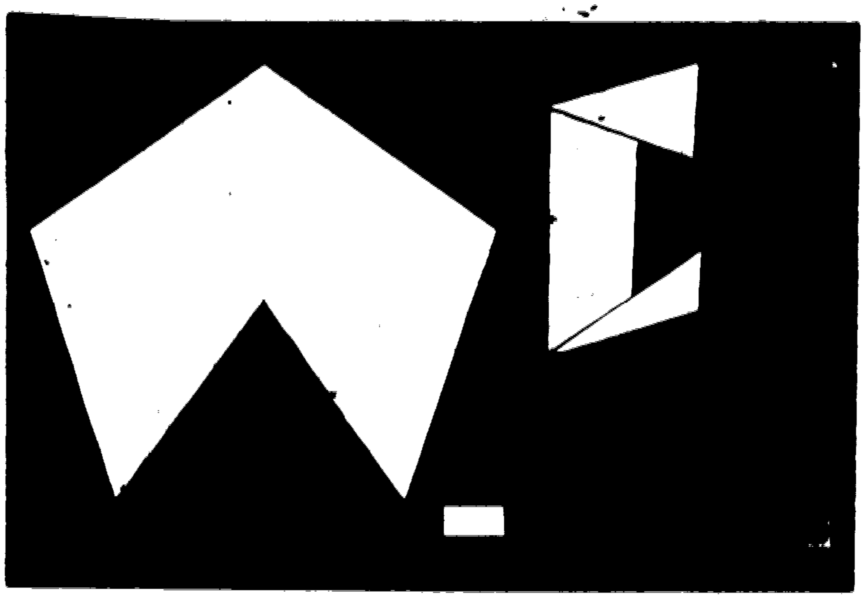
34F



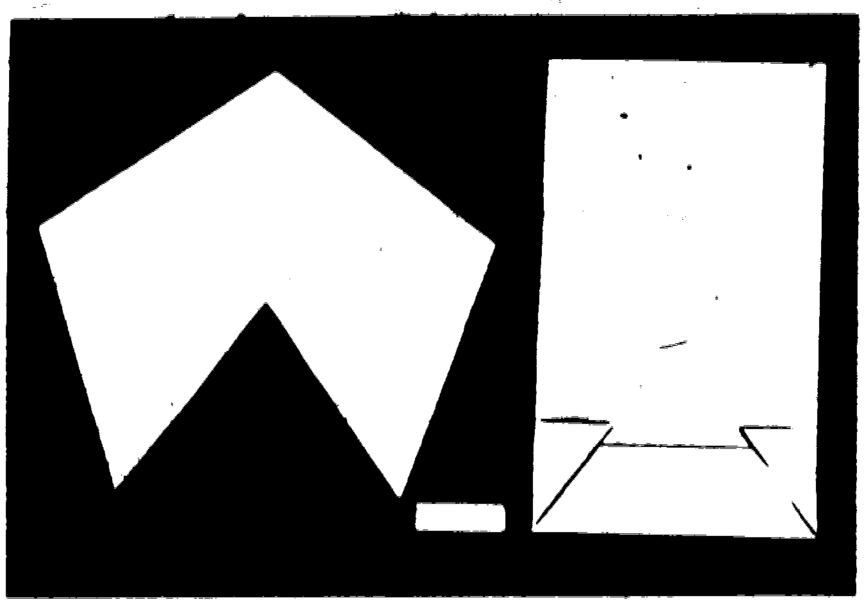
41F



44F



46M



48M

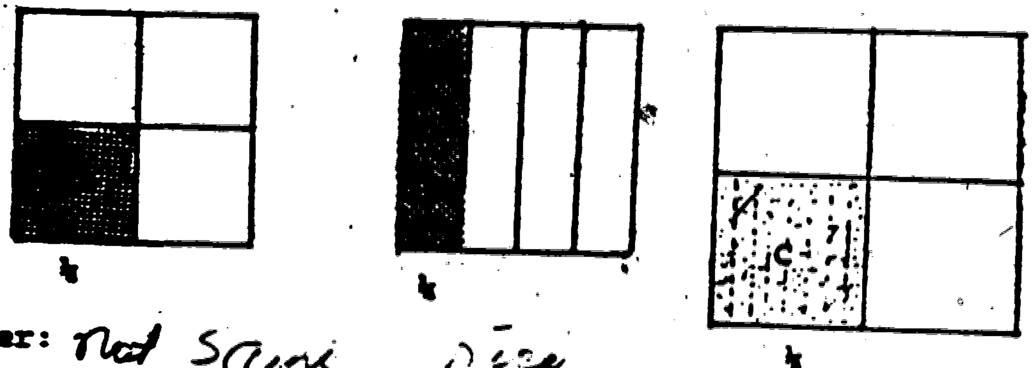
APPENDIX I

SOME STUDENTS' ORIGINAL ACTIONS ON GEOMETRY TEST VI:
PIECE-WISE CONGRUENCY-RATIONAL NUMBERS TEST

Below are six groups of figures each of which has three figures. The three regions in each group have shaded subregions A, B, and C. Now, which of these subregions A, B, and C do you think are equal? Why?

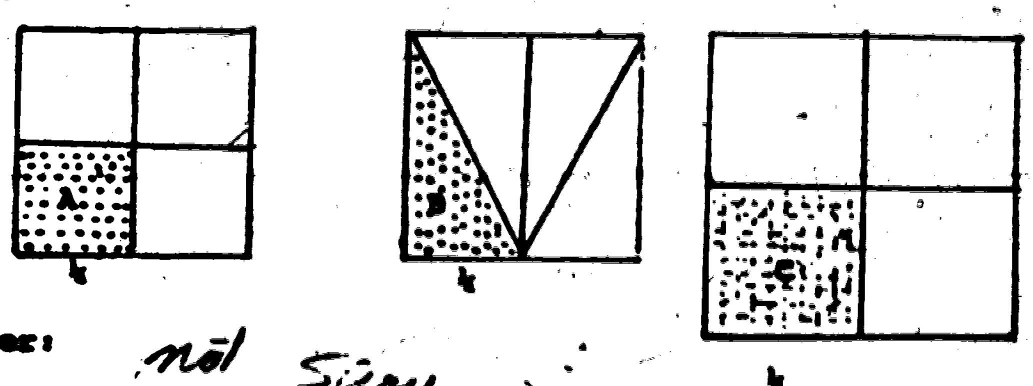
Student 07F
Pretest- Geometry Test (vi)

1.



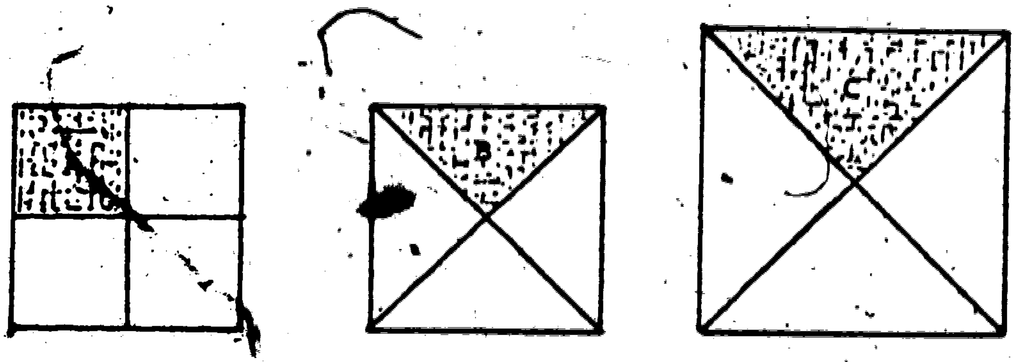
Answer: not same size

2.



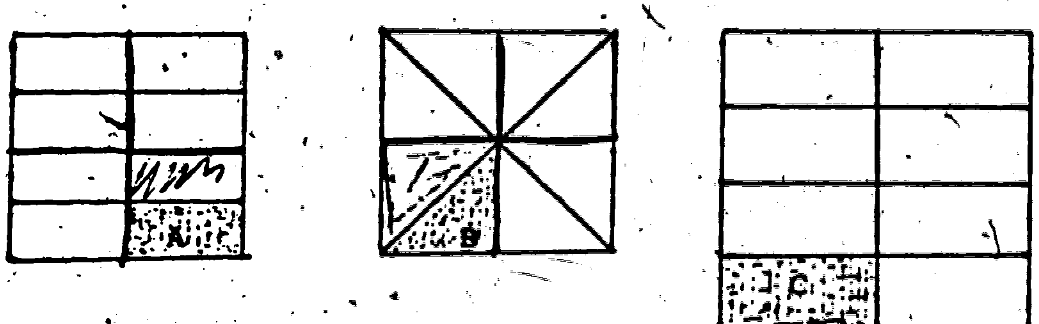
Answer: not same size

3.



Answer: none

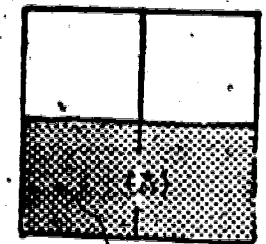
4.

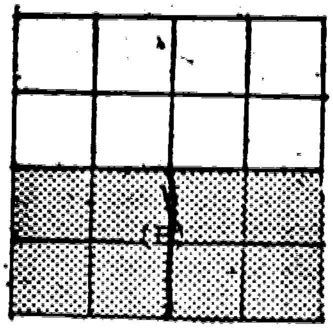


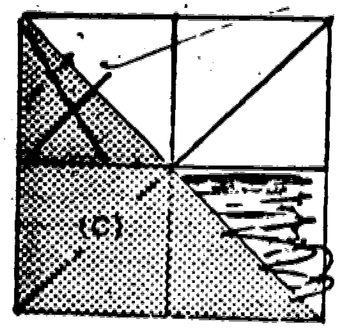
Answer: A + B } when you [illegible]

Student 07F - Pretest continues-

5.

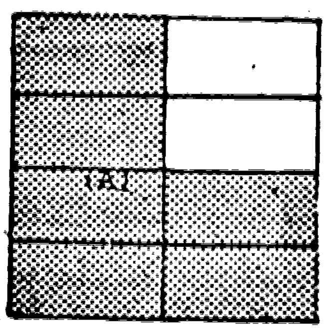


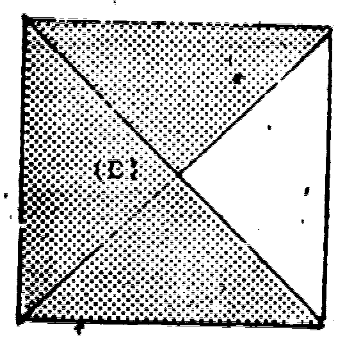


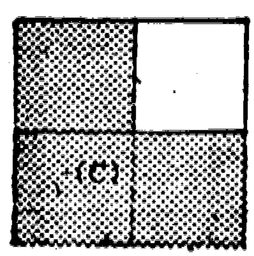


Answer: B&C see diagram

6.





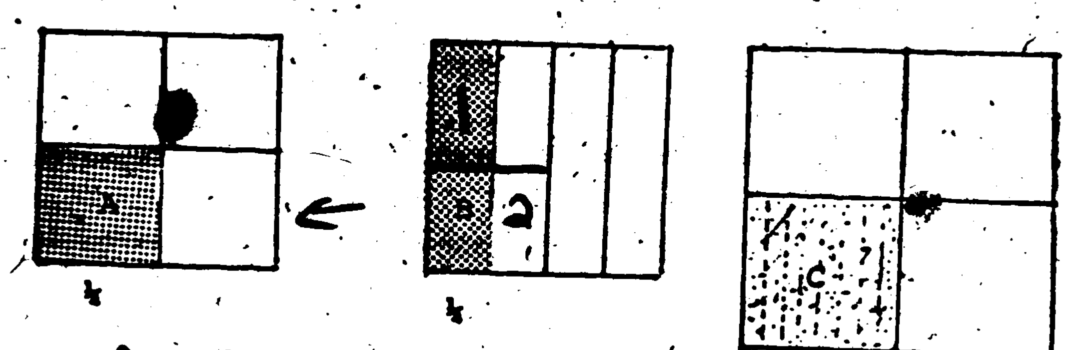


Answer:

Below are six groups of figures each of which has <u>three</u> figures. The three regions in each group have shaded subregions A, B, and C. Now, which of these subregions A, B, and C do you think are <u>equal</u>? Why?

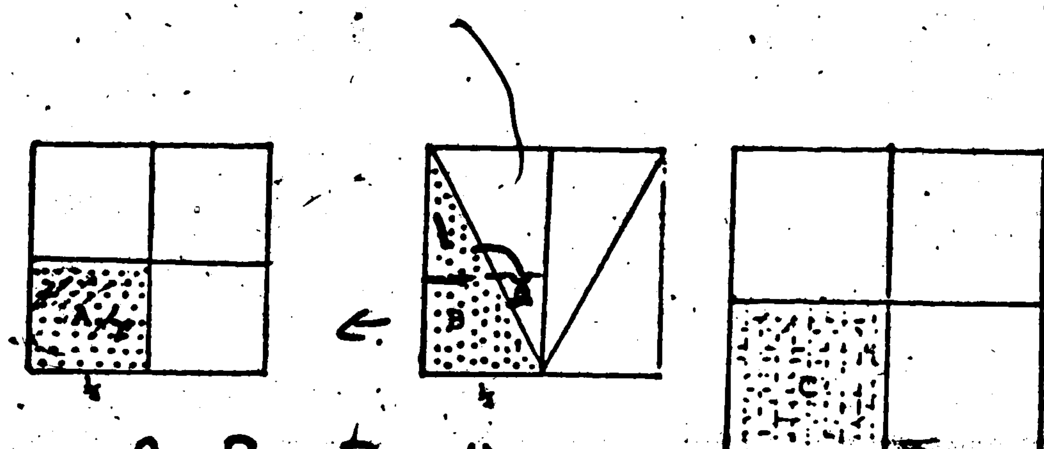
Student 07F
Posttest - Geometry Test (vi)

1.



Answer: A + B see diagram

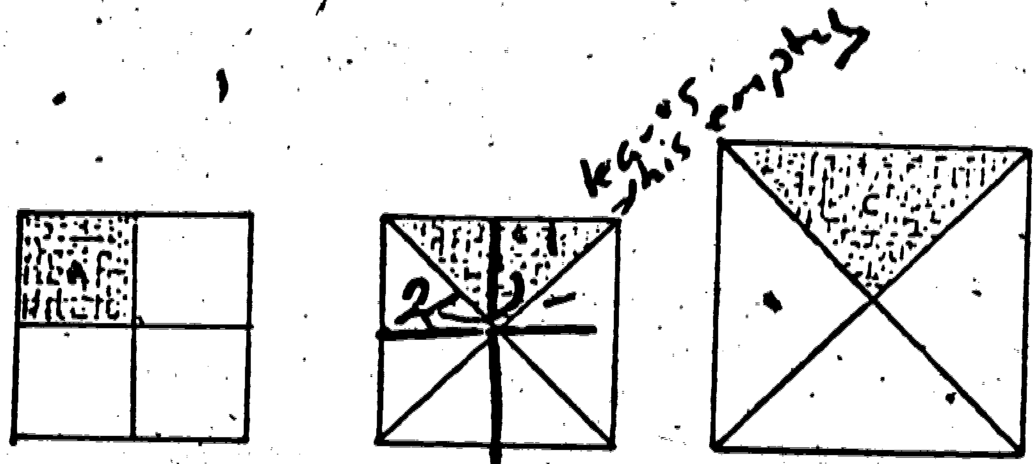
2.



Answer: A+B See diagram

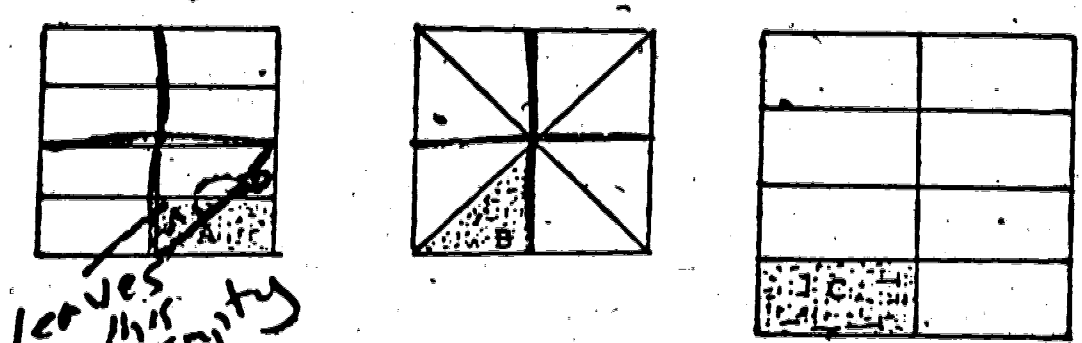
Student 07F - Posttest continues -

3.



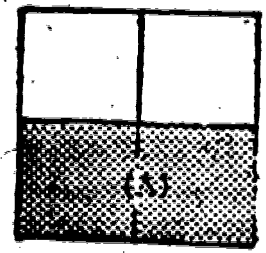
Answer: A+B
see diagram

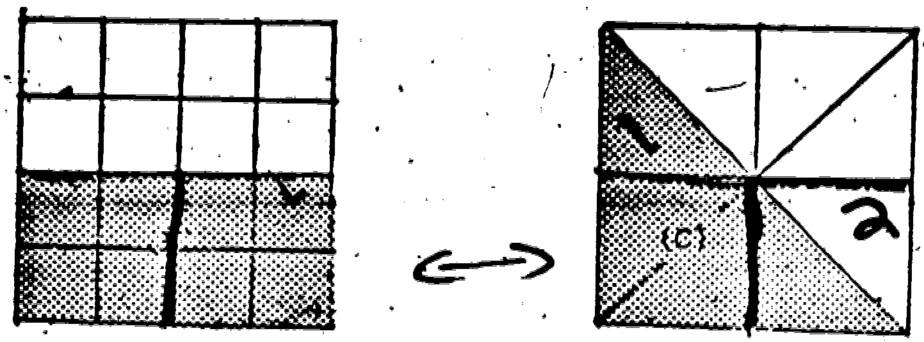
4.



Answer: A+B

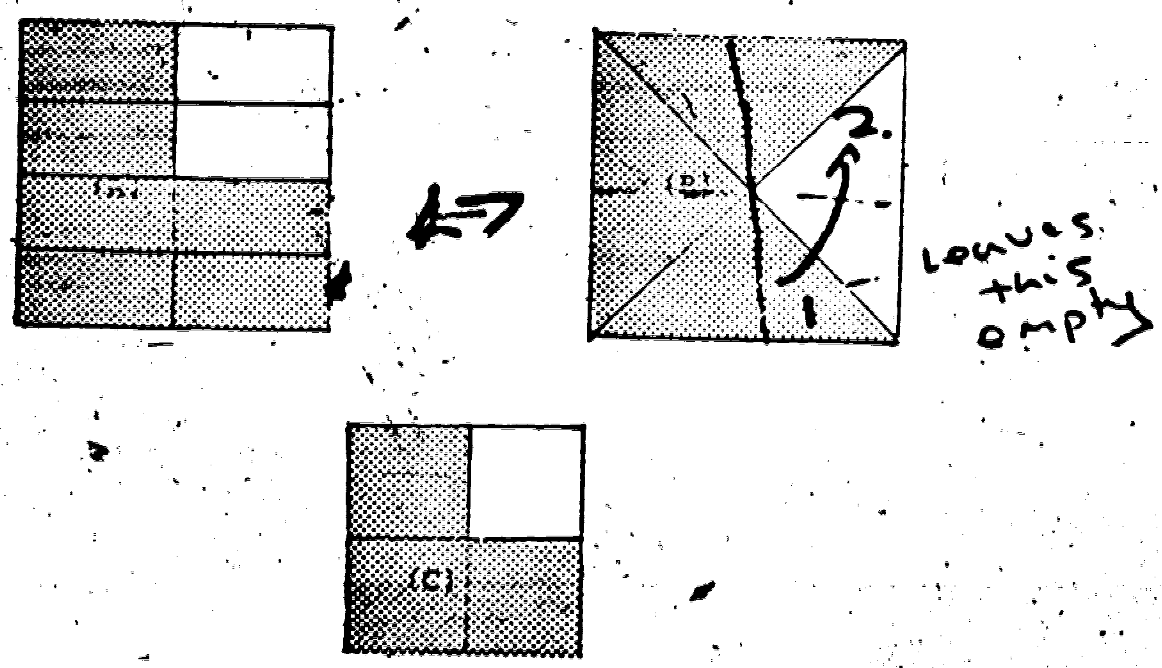
5.





Answer: B+C.

6.

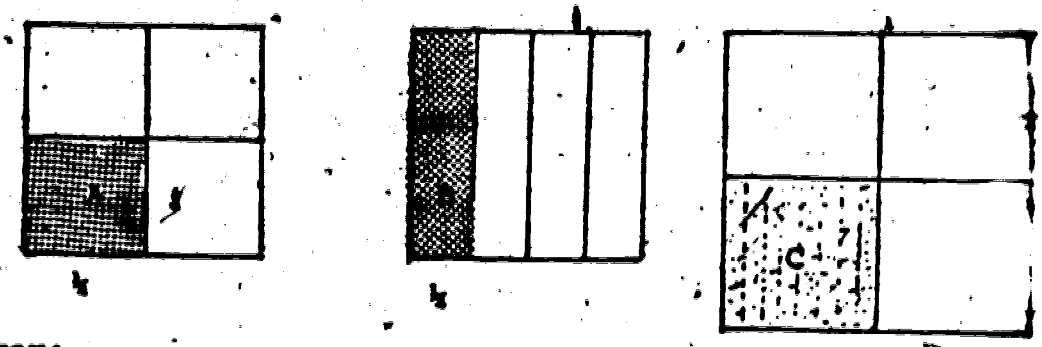


Answer: A+B see diagram

Below are six groups of figures each of which has <u>three</u> figures. The three regions in each group have shaded subregions A, B, and C. Now, which of these subregions A, B, and C do you think are <u>equal</u>? Why?

Student 51F
Pretest - Geometry Test (vi)

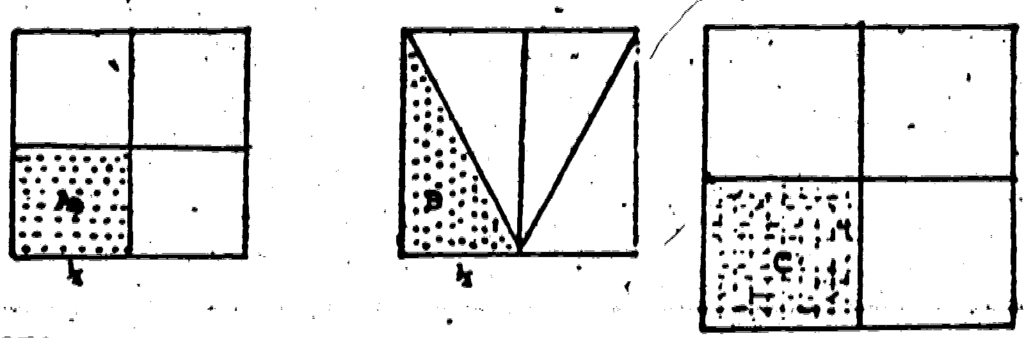
1.



Answer:

A & B they are the same size

2.

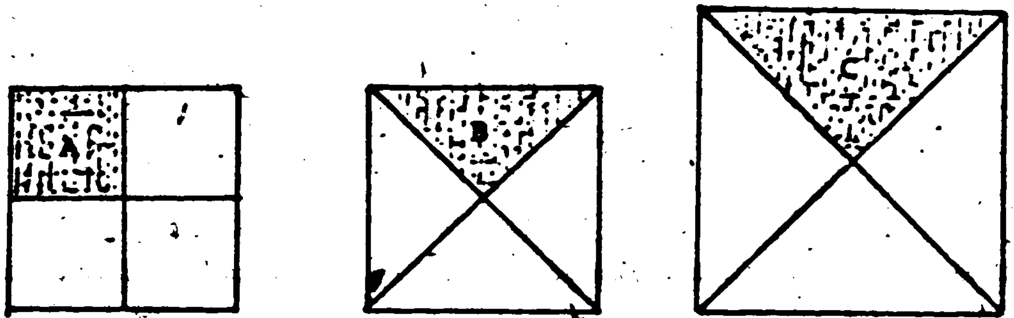


Answer:

A & B same size

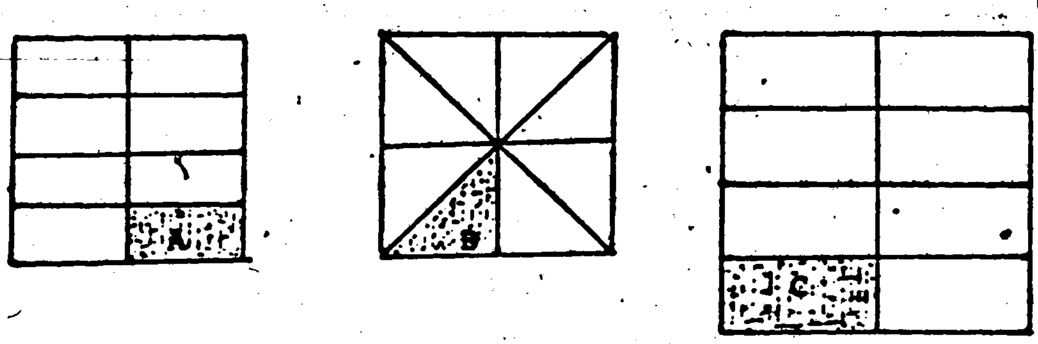
Student 51F - Pretest continues -

3.



Answer: A e B e C

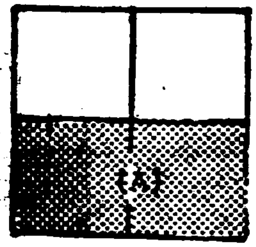
4.

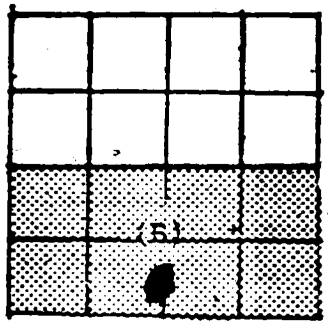


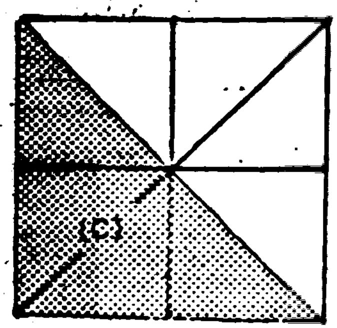
Answer: A e B e C

Student 51F - Pretest continues -

5.

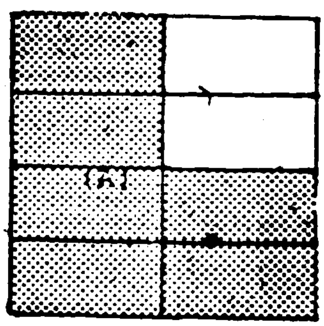


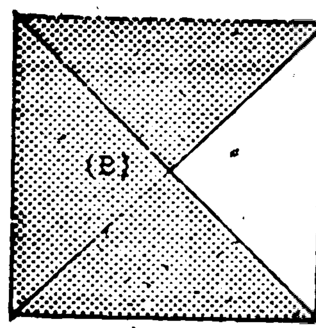


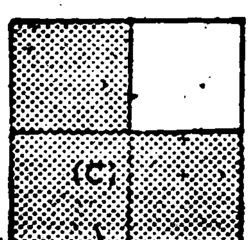


Answer: A, B, C

6.







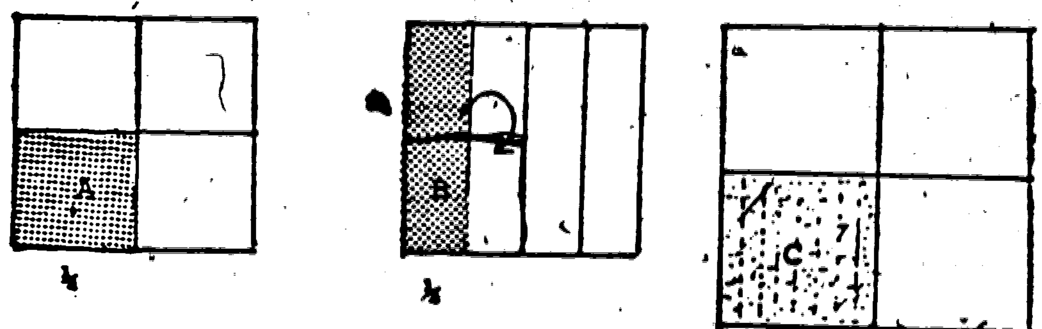
Answer:

A, B, C if you inlarge them all to size C they be all the same

Below are six groups of figures each of which has <u>three</u> figures. The three regions in each group have shaded subregions A, B, and C. Now, which of these subregions A, B, and C do you think are <u>equal</u>? Why?

Student 51F
Posttest - Geometry Test (vi)

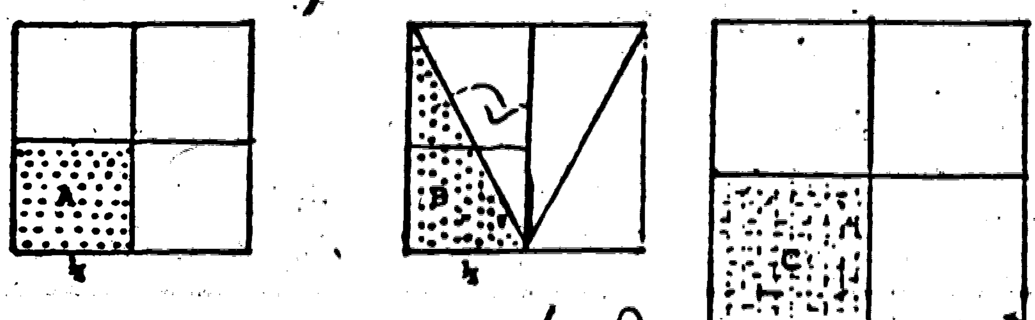
1.



Answer:

A, B if you cut B in half it will equal A

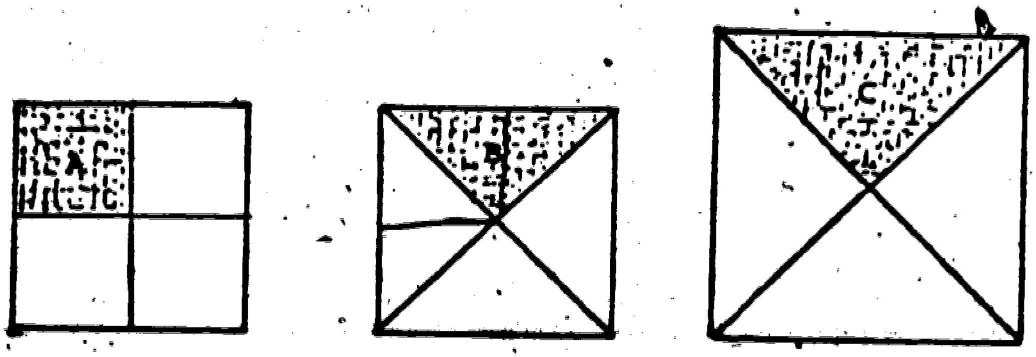
2.



Answer: A, B if you cut B turn half it will equal A

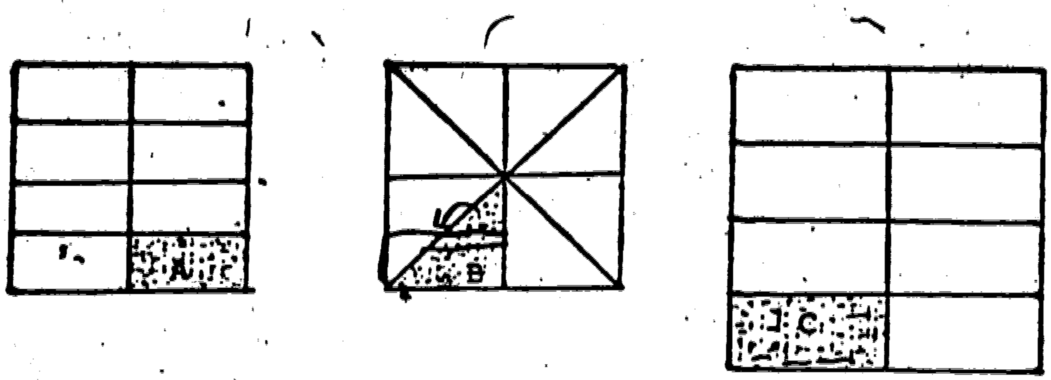
Student 51F - Posttest continues -

3.



Answer: A, B cut B in half half turn equals A

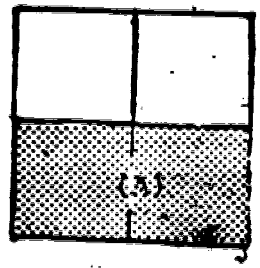
4.

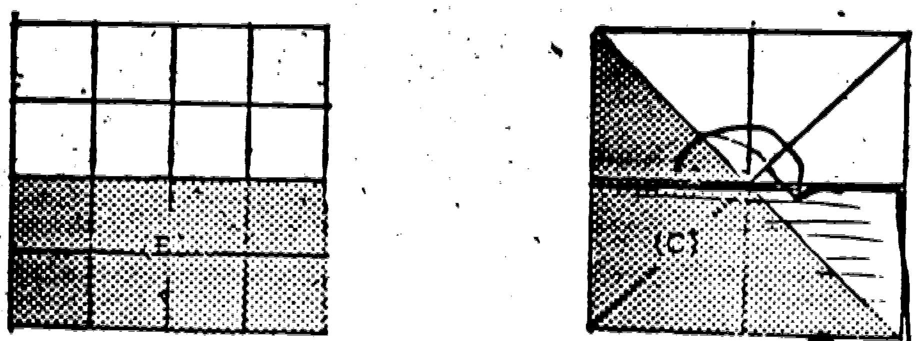


Answer: A, B cut B in half turn equals

Student 51 - Posttest continues -

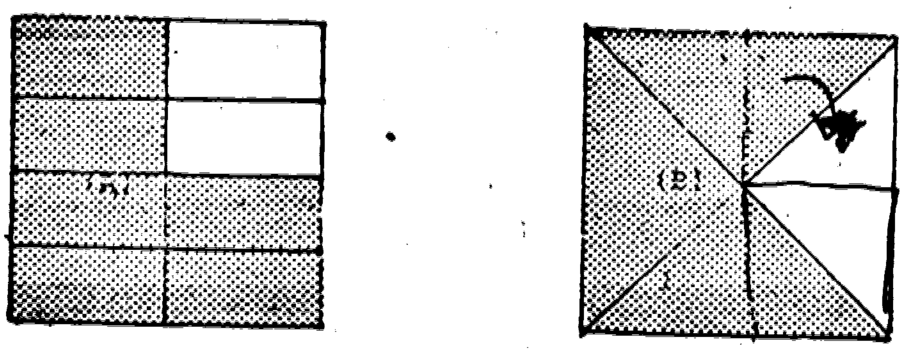
5.

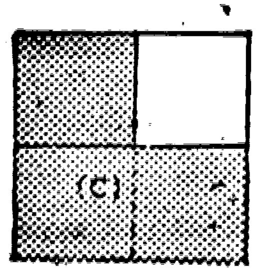




Answer: B, C if you cut C in half turn it half it equal B

6.





Answer: A, B cut B in half turn equal A

APPENDIX J

SOME STUDENTS' PERFORMANCES ON PROGRESS CHECKING TEST #2

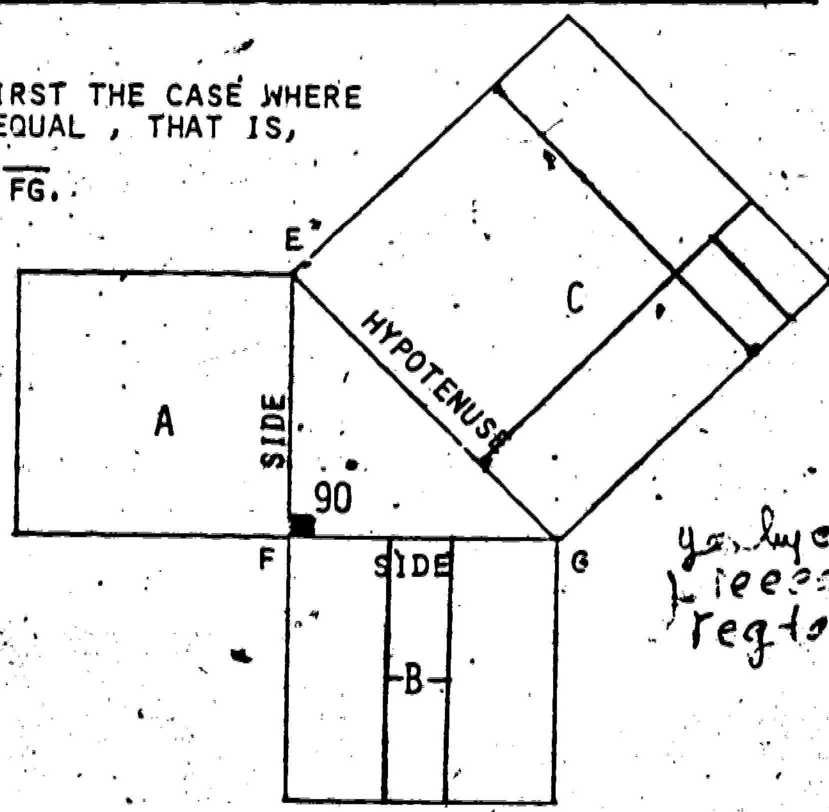
PYTHAGOREAN THEOREM - CASE 1 - (Student 04M)

PYTHAGOREAN THEOREM IS ONE OF THE MOST IMPORTANT THEOREMS, NOT ONLY IN GEOMETRY AND ELEMENTARY MATHEMATICS, BUT IN ADVANCED MATHEMATICS AS WELL. THE STATEMENT OF THIS THEOREM IS THE FOLLOWING:

> IN A RIGHT ANGLE TRIANGLE, THE SQUARE OF THE HYPOTENUSE IS EQUAL TO THE SUM OF THE SQUARES OF THE OTHER TWO SIDES.

LET US TAKE FIRST THE CASE WHERE THE SIDES ARE EQUAL, THAT IS,

$\overline{EF} = \overline{FG}$.



NOW,

TRACE AND CUT OUT A COPY OF EACH OF THE SQUARES A & B. TRY TO COVER THE SQUARE C USING THE COPIES OF THE SQUARES A AND B.

(*) SUCCEEDED IN THIS EXCERCISE, YOU HAVE, THEN, ALREADY PROVED THIS IMPORTANT THEOREM!

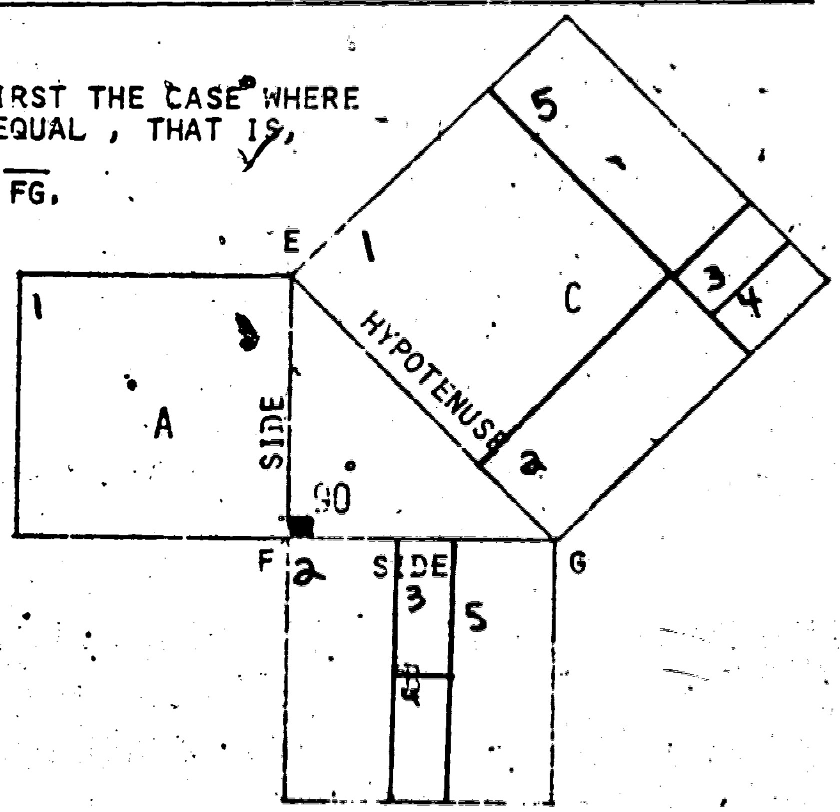
PYTHAGOREAN THEOREM: - CASE 1 - (Student 13F)

PYTHAGOREAN THEOREM IS ONE OF THE MOST IMPORTANT THEOREMS, NOT ONLY IN GEOMETRY AND ELEMENTARY MATHEMATICS, BUT IN ADVANCED MATHEMATICS AS WELL. THE STATEMENT OF THIS THEOREM IS THE FOLLOWING:

> IN A RIGHT ANGLE TRIANGLE, THE SQUARE OF THE HYPOTENUSE IS EQUAL TO THE SUM OF THE SQUARES OF THE OTHER TWO SIDES.

LET US TAKE FIRST THE CASE WHERE THE SIDES ARE EQUAL, THAT IS,

$\overline{EF} = \overline{FG}$.



NOW,

TRACE AND CUT OUT A COPY OF EACH OF THE SQUARES A & B.

TRY TO COVER THE SQUARE C USING THE COPIES OF THE SQUARES A AND B.

(*) SUCCEEDED IN THIS EXCERCISE, YOU HAVE, THEN, ALREADY PROVED THIS IMPORTANT THEOREM!

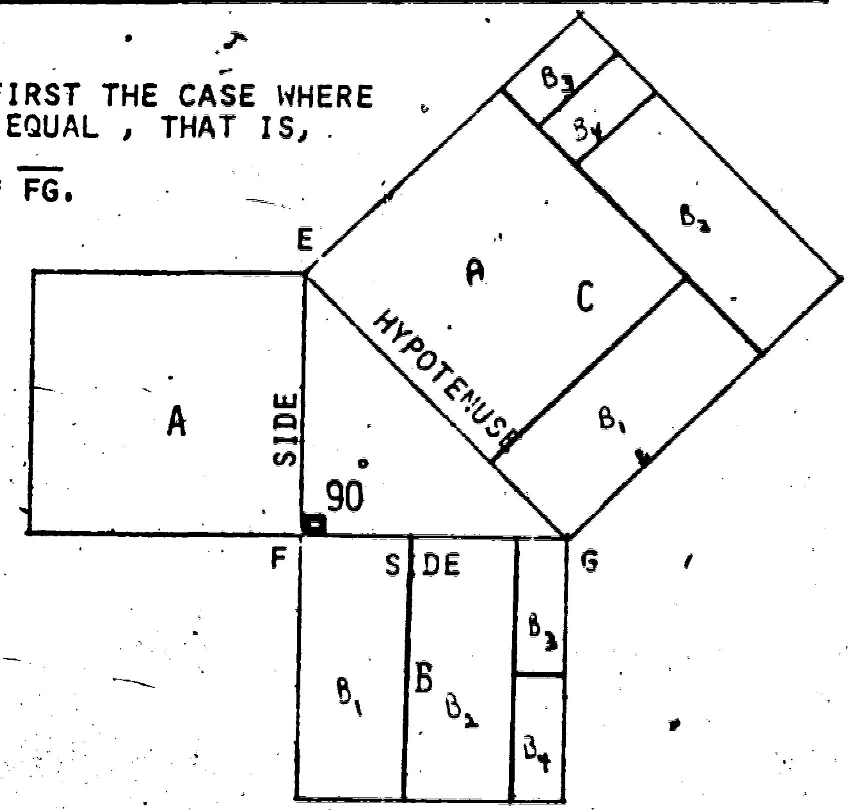
PYTHAGOREAN THEOREM: - CASE 1 - (Student 20F)

PYTHAGOREAN THEOREM IS ONE OF THE MOST IMPORTANT THEOREMS, NOT ONLY IN GEOMETRY AND ELEMENTARY MATHEMATICS, BUT IN ADVANCED MATHEMATICS AS WELL. THE STATEMENT OF THIS THEOREM IS THE FOLLOWING:

> IN A RIGHT ANGLE TRIANGLE, THE SQUARE OF THE HYPOTENUSE IS EQUAL TO THE SUM OF THE SQUARES OF THE OTHER TWO SIDES.

LET US TAKE FIRST THE CASE WHERE THE SIDES ARE EQUAL, THAT IS,

$\overline{EF} = \overline{FG}$.



NOW,

TRACE AND CUT OUT A COPY OF EACH OF THE SQUARES A & B.

TRY TO COVER THE SQUARE C USING THE COPIES OF THE SQUARES A AND B.

(*) SUCCEEDED IN THIS EXCERCISE, YOU HAVE, THEN, ALREADY PROVED THIS IMPORTANT THEOREM!

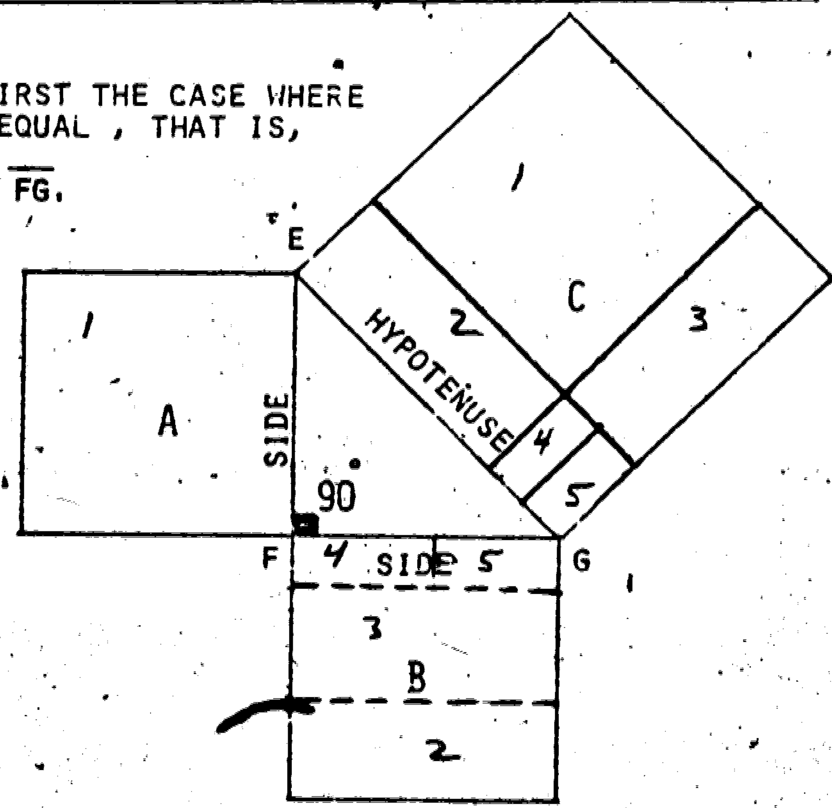
PYTHAGOREAN THEOREM: - CASE 1 - (Student 21M)

PYTHAGOREAN THEOREM IS ONE OF THE MOST IMPORTANT THEOREMS, NOT ONLY IN GEOMETRY AND ELEMENTARY MATHEMATICS, BUT IN ADVANCED MATHEMATICS AS WELL. THE STATEMENT OF THIS THEOREM IS THE FOLLOWING:

> IN A RIGHT ANGLE TRIANGLE, THE SQUARE OF THE HYPOTENUSE IS EQUAL TO THE SUM OF THE SQUARES OF THE OTHER TWO SIDES.

LET US TAKE FIRST THE CASE WHERE THE SIDES ARE EQUAL, THAT IS,

$\overline{EF} = \overline{FG}$.



NOW,

TRACE AND CUT OUT A COPY OF EACH OF THE SQUARES A & B. TRY TO COVER THE SQUARE C USING THE COPIES OF THE SQUARES A AND B.

(*) SUCCEEDED IN THIS EXCERCISE, YOU HAVE, THEN, ALREADY PROVED THIS IMPORTANT THEOREM!

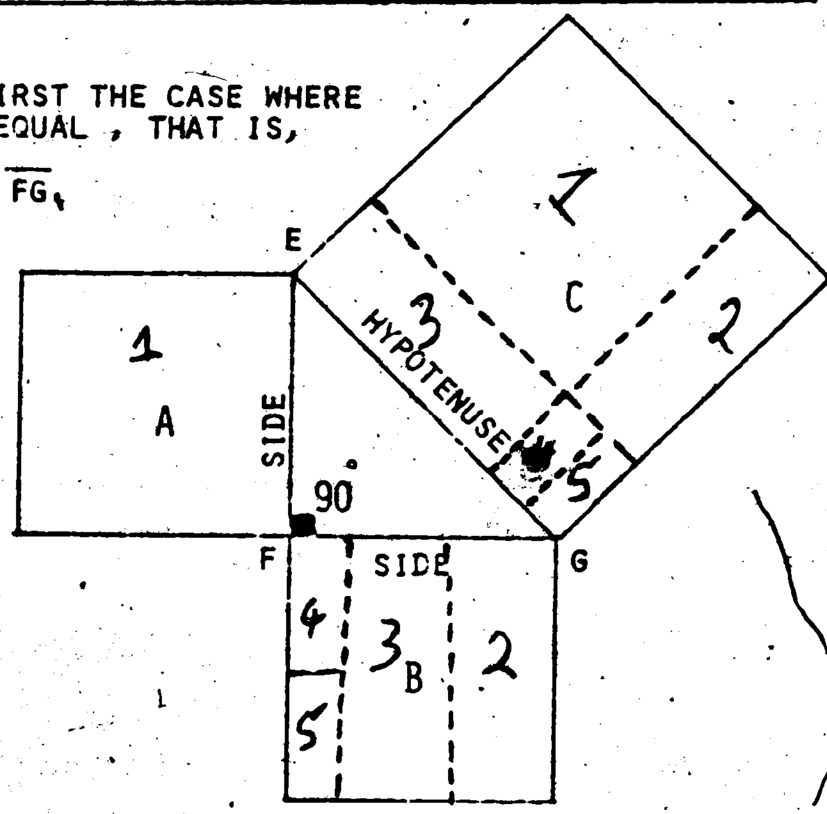
PYTHAGOREAN THEOREM: - CASE 1 - (Student 24F)

PYTHAGOREAN THEOREM IS ONE OF THE MOST IMPORTANT THEOREMS, NOT ONLY IN GEOMETRY AND ELEMENTARY MATHEMATICS, BUT IN ADVANCED MATHEMATICS AS WELL. THE STATEMENT OF THIS THEOREM IS THE FOLLOWING:

> IN A RIGHT ANGLE TRIANGLE, THE SQUARE OF THE HYPOTENUSE IS EQUAL TO THE SUM OF THE SQUARES OF THE OTHER TWO SIDES.

LET US TAKE FIRST THE CASE WHERE THE SIDES ARE EQUAL, THAT IS,

$\overline{EF} = \overline{FG}$,



NOW,

TRACE AND CUT OUT A COPY OF EACH OF THE SQUARES A & B.

TRY TO COVER THE SQUARE C USING THE COPIES OF THE SQUARES A AND B.

(*) SUCCEEDED IN THIS EXCERCISE, YOU HAVE, THEN, ALREADY PROVED THIS IMPORTANT THEOREM!

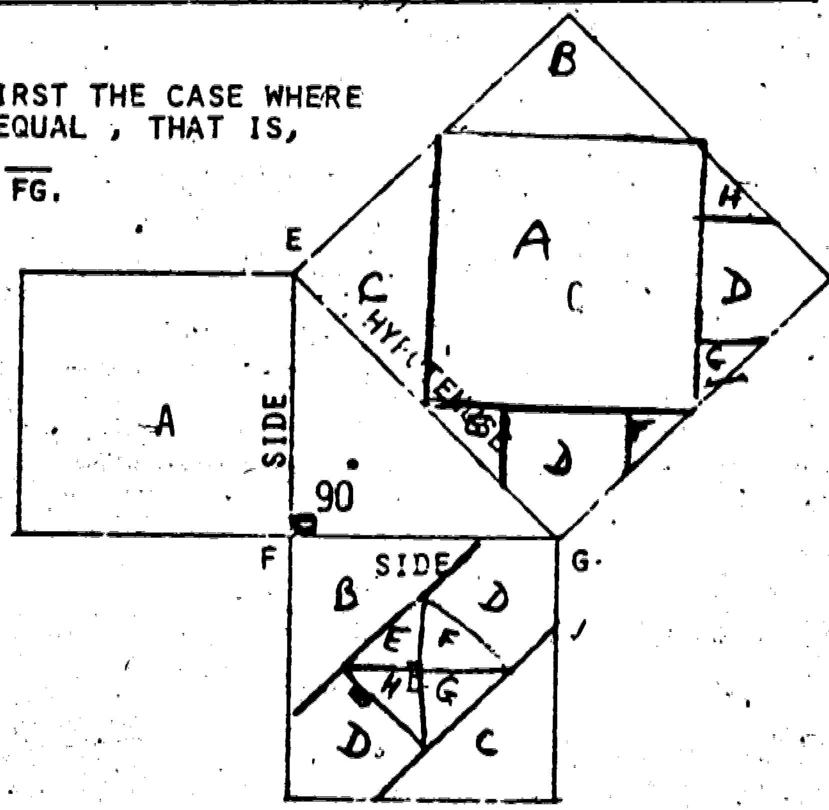
PYTHAGOREAN THEOREM: - CASE 1 - (Student 34F)

PYTHAGOREAN THEOREM IS ONE OF THE MOST IMPORTANT THEOREMS, NOT ONLY IN GEOMETRY AND ELEMENTARY MATHEMATICS, BUT IN ADVANCED MATHEMATICS AS WELL. THE STATEMENT OF THIS THEOREM IS THE FOLLOWING:

> IN A RIGHT ANGLE TRIANGLE, THE SQUARE OF THE HYPOTENUSE IS EQUAL TO THE SUM OF THE SQUARES OF THE OTHER TWO SIDES.

LET US TAKE FIRST THE CASE WHERE THE SIDES ARE EQUAL, THAT IS,

$\overline{EF} = \overline{FG}$.



NOW,

- TRACE AND CUT OUT A COPY OF EACH OF THE SQUARES A & B.
- TRY TO COVER THE SQUARE C USING THE COPIES OF THE SQUARES A AND B.

(*) SUCCEEDED IN THIS EXCERCISE, YOU HAVE, THEN, ALREADY PROVED THIS IMPORTANT THEOREM!

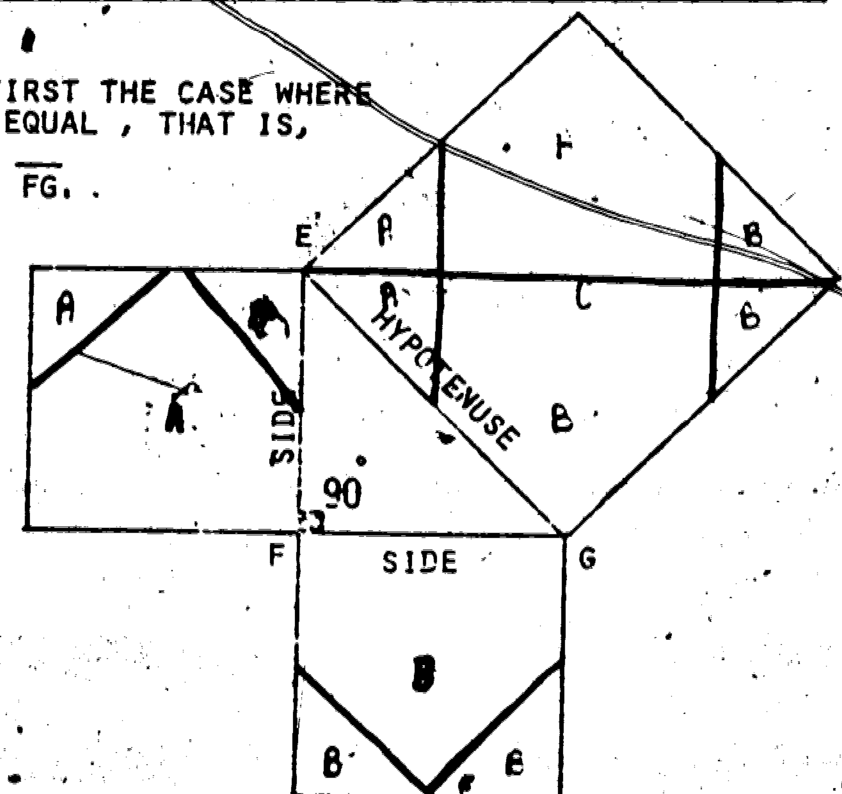
PYTHAGOREAN THEOREM: - CASE 1 - (Student 35F)

PYTHAGOREAN THEOREM IS ONE OF THE MOST IMPORTANT THEOREMS, NOT ONLY IN GEOMETRY AND ELEMENTARY MATHEMATICS, BUT IN ADVANCED MATHEMATICS AS WELL. THE STATEMENT OF THIS THEOREM IS THE FOLLOWING:

> IN A RIGHT ANGLE TRIANGLE, THE SQUARE OF THE HYPOTENUSE IS EQUAL TO THE SUM OF THE SQUARES OF THE OTHER TWO SIDES.

LET US TAKE FIRST THE CASE WHERE THE SIDES ARE EQUAL, THAT IS,

$\overline{EF} = \overline{FG}$.



NOW,

TRACE AND CUT OUT A COPY OF EACH OF THE SQUARES A & B.

TRY TO COVER THE SQUARE C USING THE COPIES OF THE SQUARES A AND B.

(*) SUCCEEDED IN THIS EXCERCISE, YOU HAVE, THEN, ALREADY PROVED THIS IMPORTANT THEOREM!

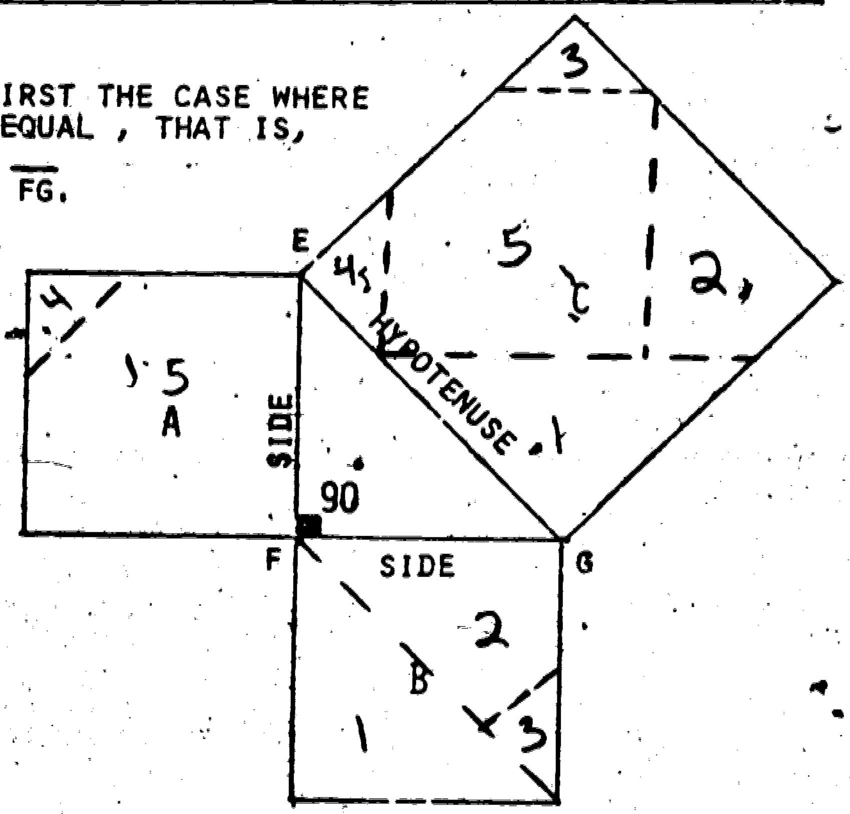
PYTHAGOREAN THEOREM: - CASE 1 - (Student 54F)

PYTHAGOREAN THEOREM IS ONE OF THE MOST IMPORTANT THEOREMS, NOT ONLY IN GEOMETRY AND ELEMENTARY MATHEMATICS, BUT IN ADVANCED MATHEMATICS AS WELL. THE STATEMENT OF THIS THEOREM IS THE FOLLOWING:

> IN A RIGHT ANGLE TRIANGLE, THE SQUARE OF THE HYPOTENUSE IS EQUAL TO THE SUM OF THE SQUARES OF THE OTHER TWO SIDES.

LET US TAKE FIRST THE CASE WHERE THE SIDES ARE EQUAL, THAT IS,

$\overline{EF} = \overline{FG}$.



NOW,

TRACE AND CUT OUT A COPY OF EACH OF THE SQUARES A & B.

TRY TO COVER THE SQUARE C USING THE COPIES OF THE SQUARES A AND B.

(*) SUCCEEDED IN THIS EXCERCISE, YOU HAVE, THEN, ALREADY PROVED THIS IMPORTANT THEOREM!

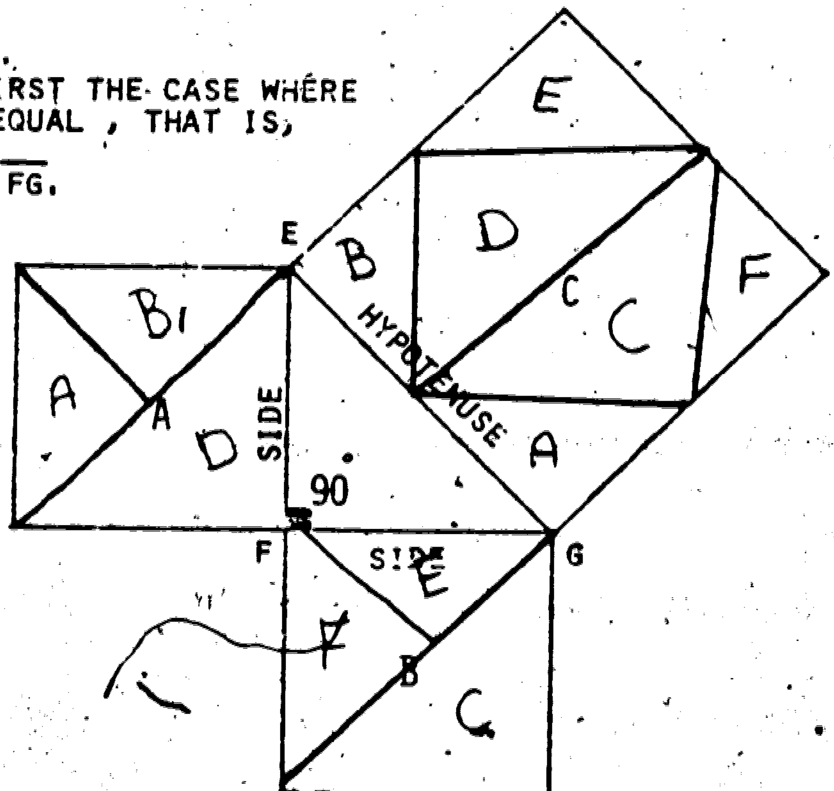
PYTHAGOREAN THEOREM: - CASE 1 - (Student 55M)

PYTHAGOREAN THEOREM IS ONE OF [redacted] MOST IMPORTANT THEOREMS, NOT ONLY IN GEOMETRY AND ELEMENTARY MATHEMATICS, BUT IN ADVANCED MATHEMATICS AS WELL. THE STATEMENT OF THIS THEOREM IS THE FOLLOWING:

> IN A RIGHT ANGLE TRIANGLE, THE SQUARE OF THE HYPOTENUSE IS EQUAL TO THE SUM OF THE SQUARES OF THE OTHER TWO SIDES.

LET US TAKE FIRST THE CASE WHERE THE SIDES ARE EQUAL, THAT IS,

$\overline{EF} = \overline{FG}$.



NOW,

TRACE AND CUT OUT A COPY OF EACH OF THE SQUARES A & B.
TRY TO COVER THE SQUARE C USING THE COPIES OF THE SQUARES A AND B.

(*) SUCCEEDED IN THIS EXCERCISE, YOU HAVE, THEN, ALREADY PROVED THIS IMPORTANT THEOREM!

APPENDIX K

Pythagorean Theorem and Transformation Geometry

Although there are many elegant proofs of the Pythagorean theorem (Loomis 1968), the one presented below is particularly appealing because it is based on only two transformations. Its very simplicity makes it easy to follow. Let's begin by assuming we have a plane figure (fig. 1) with $\triangle EFG$, $m\angle F = 90$, $EF = b$, $FG = a$, $GE = c$, and squares on sides $\overline{EF}$ and $\overline{FG}$.

To demonstrate the proof, we will describe a series of motions with figures. Each figure represents the preceding figure after a certain transformation (the physical term *motion* is used rather than the more formal term *transformation* because of its teaching appeal). Translations and rotations are the only transformations needed for this proof. For the proof it is assumed that these two

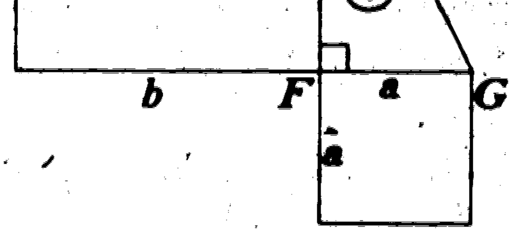
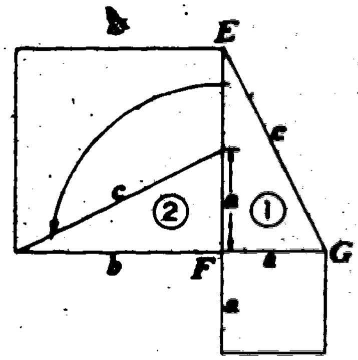
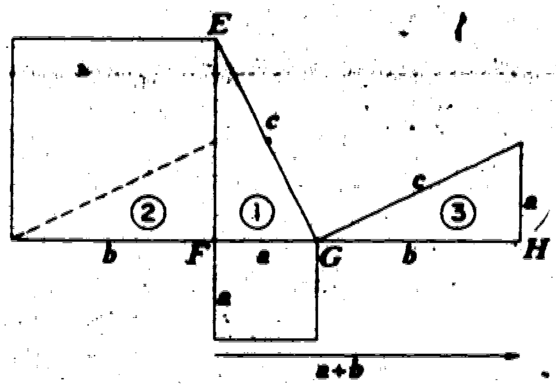


Fig. 1

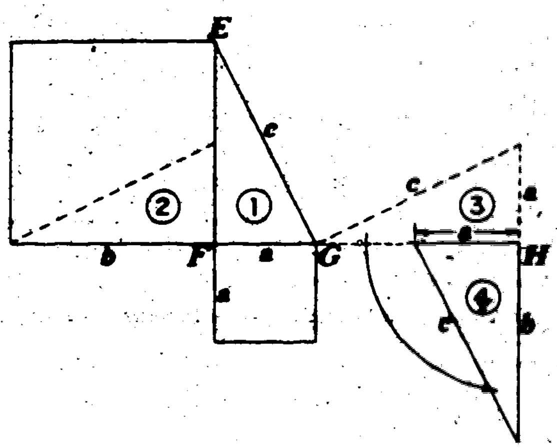
This article is dedicated for children around the world on the occasion of the International Year of the Child, 1979.



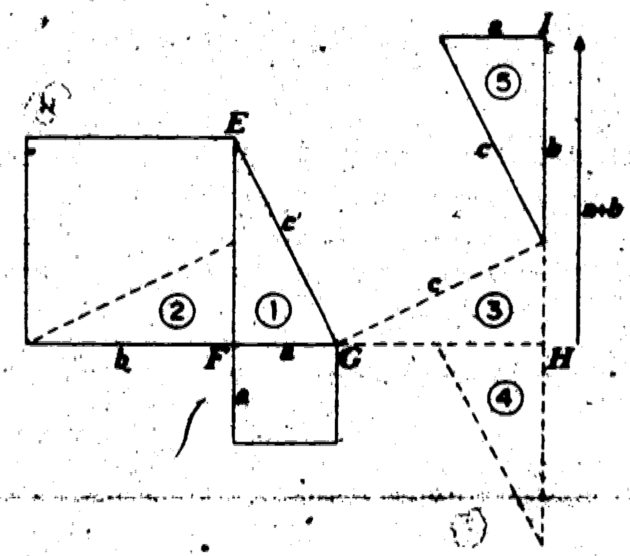
Motion 1: Rotate triangle ① 90° counterclockwise about *F* to obtain triangle ②.



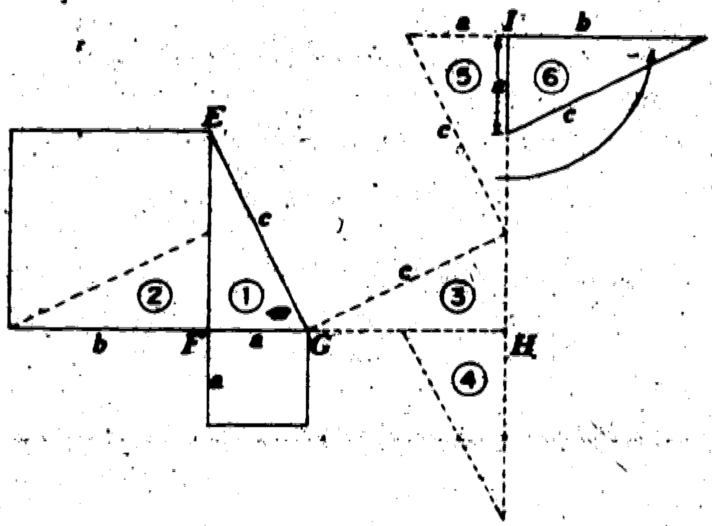
Motion 2: Translate triangle ② to the right *a* + *b* units to obtain triangle ③.



Motion 3: Rotate triangle ③ 90° counterclockwise about *H* to obtain triangle ④.

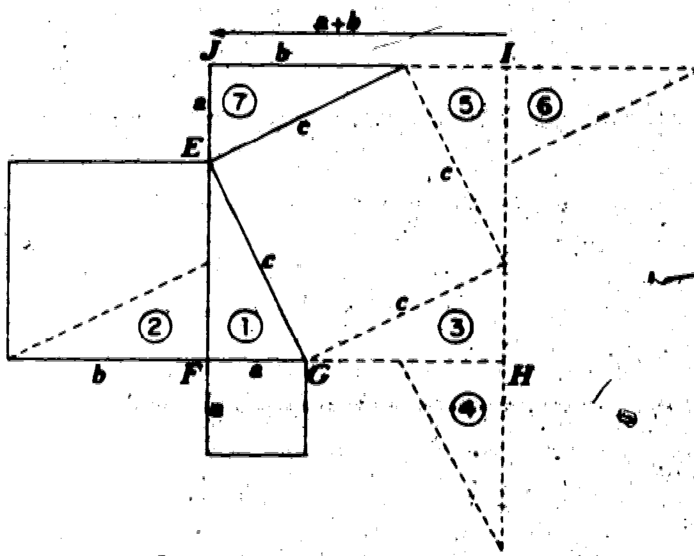


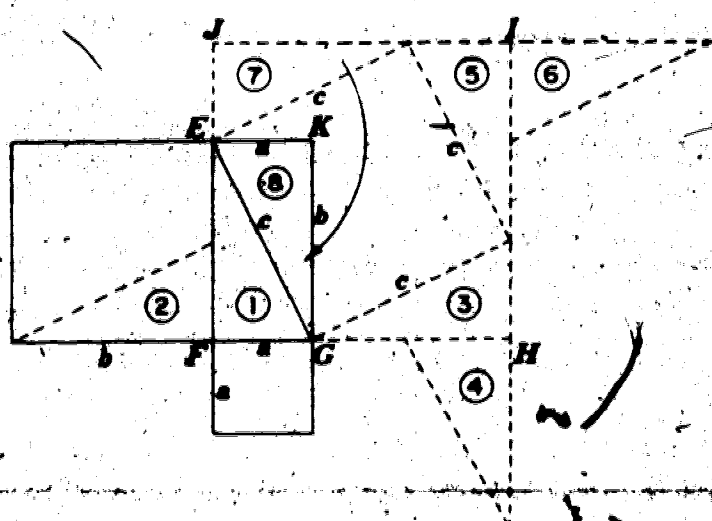
Motion 4: Translate triangle ④ upward *a* + *b* units to obtain triangle ⑤.



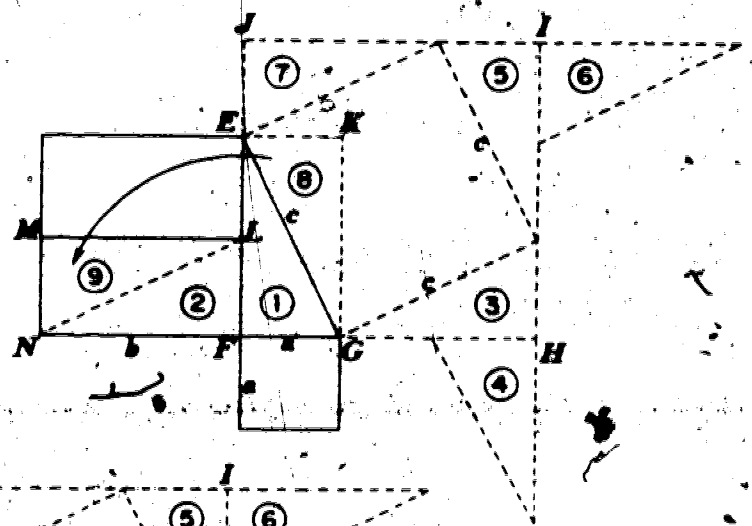
Motion 5: Rotate triangle ⑤ 90° counterclockwise about *I* to obtain triangle ⑥.

Motion 6: Translate triangle ⑥ to the left *a* + *b* units to obtain triangle ⑦. By the previous six transformations, we have obtained the square *FHIJ* whose sides are of measure *a* + *b* units and which contains the square whose side measures *c* units.

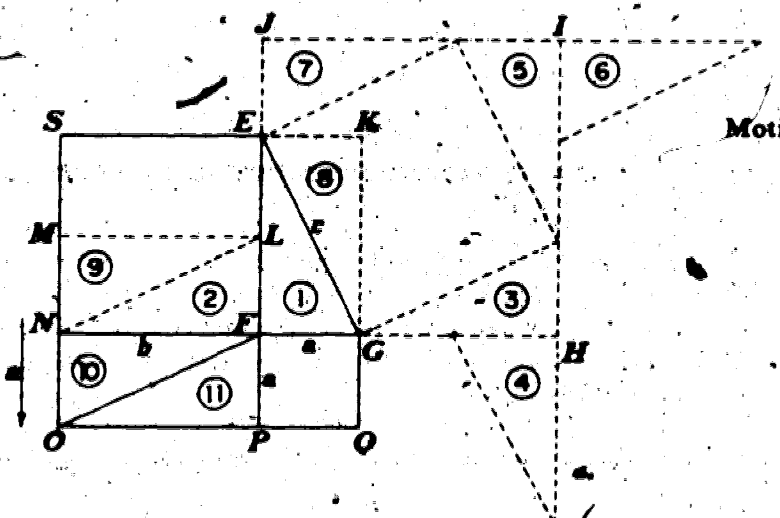




Motion 7: Rotate triangle ⑦ 90° clockwise about *E* to obtain triangle ⑧.



Motion 8: Rotate the rectangle *EFGK* 90° counterclockwise about *F* to put it in the position *MNFL* (triangles ⑧ and ⑨).



Motion 9: Translate the rectangle *MNFL* down *a* units to put it in the position *NOPF* (triangles ⑩ and ⑪). Thus using motions 7, 8, and 9, we obtain square *OQKS* whose sides are of measure *a* + *b* units.

operations preserve the measures of corresponding angles and segments (Gans 1969).

From figure 2, we see that the two squares *FHIJ* and *OQKS* are congruent as are triangles ①, ③, ④, ⑦, ⑧, ⑩, and ⑪. Thus subtracting the areas of triangles ①, ③, ④, and ⑦ from the area of square *FHIJ* and the areas of triangles ①, ⑧, ⑩, and ⑪ from the area of square *OQKS* yields another proof of the Pythagorean theorem.

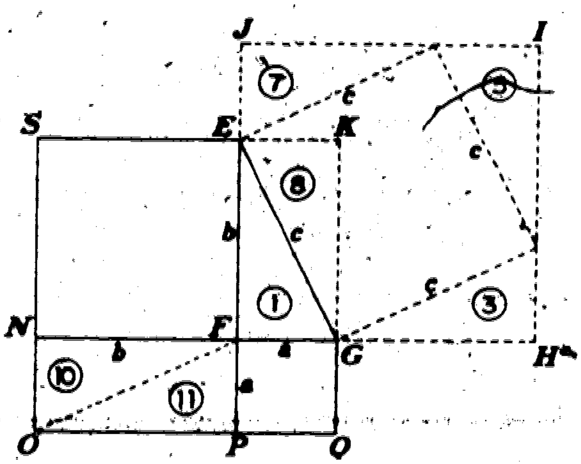


Fig. 2

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(3) "Pythagorean Theorem and Transformation Geometry." The Mathematics Teacher. Vol. 72, No. 7, October, 1979.