THE BULLETIN OF SYMBOLIC LOGIC
Volume 6, Number 3, Sept. 2000

REVIEWS

The Association for Symbolic Logic reviews selected books and articles in the field of symbolic logic. The reviews were published in ***The journal of symbolic logic*** from the founding of the JOURNAL in 1936 until the end of 1999. The Association moved the reviews to this BULLETIN, beginning in 2000.

The Reviews Section is edited by Herbert Enderton (Coordinating Editor), Matthew Foreman, Gerhard Jäger, Penelope Maddy, and Philip Scowcroft. Authors and publishers are requested to send, for review, copies of books to *Herbert Enderton, Association for Symbolic Logic, Mathematical Sciences Bldg. 7332, UCLA, Los Angeles, California 90095-1566, USA*.

In a review, a reference "JSL XLIII 148," for example, refers either to the publication reviewed on page 148 of volume 43 of the JOURNAL, or to the review itself (which contains full bibliographical information for the reviewed publication). Analogously, a reference "BSL VI 94" refers to the review beginning on page 94 in volume 6 of this BULLETIN, or to the publication there reviewed. "JSL LV 347" refers to one of the reviews or one of the publications reviewed or listed on page 347 of volume 55 of the JOURNAL, with reliance on the context to show which one is meant. The reference "JSL LIII 318(3)" is to the third item on page 318 of volume 53 of the JOURNAL, that is, to van Heijenoort's *Frege and vagueness*, and "JSL LX 684(8)" refers to the eighth item on page 684 of volume 60 of the JOURNAL, that is, to Tarski's *Truth and proof*.

References such as 186***4*** or 280***1*** are to entries so numbered in *A bibliography of symbolic logic* (the JOURNAL, vol. 1, pp. 121–218).

PETR HÁJEK. ***Metamathematics of fuzzy logic.*** Trends in logic, vol. 4. Kluwer Academic Publishers, Dordrecht, Boston, and London, 1998, viii + 297 pp.

'Fuzzy logic' means different things to different people. For some it is a philosophy of life—"a way to break the stranglehold that the black-and-white thinking of the Western tradition has upon us." For others it is a more accurate way of describing our ordinary language (and thought) "in which we do not think that everything is either true or false but where we recognize the shades of grey that populate our thought and linguistic communication." Some see fuzzy logic as an ontological device that "more accurately describes reality than claiming that, for every existent item a and for every property F, either a manifests F or a manifests *not*-F." Some theorists view the fuzzy–classical debate as being a modern analogue–digital debate; and in various engineering fields, fuzzy logic is viewed as the background theoretical framework in which varying a dial to degree n results in a greater change somewhere else in the factory (or machine) than had the dial not been varied quite to degree n.

Although it is undeniable that the engineering feats accomplished by clever use of this last technique have led to wonderful advances in camera and refrigerator technology, many people have questioned whether it really is "applied fuzzy logic" or whether it even presupposes anything more than classical two-valued logic. (Does it really require fuzzy logic to model a dial that maps minutes-on-barbecue into degree-of-doneness?) And needless to say, the claims made by fuzzy-theorists in the other areas cited above have not been met with universal acceptance. In fact, it is not much of an exaggeration to say that the phrase 'fuzzy logic' divides the world into two: those who advocate it with missionary zeal and those who think

1079-8986/00/0603-0005/$2.90

that it is logical pornography. (It is intriguing and exciting at first sight, but quickly is seen as shallow and just a little disgusting.)

Hájek's book is a welcome item in this stewpot full of differing frameworks. His first maneuver is to distinguish "fuzzy logic in the wide sense" from "fuzzy logic in the narrow sense"; and he claims that the aim of the present book is to investigate the narrow notion, which he sees as one with the study of formal calculi of many-valued logic. Nonetheless, despite this concern in the main part of the book with the formal properties of fuzzy logic, Hájek spends some opening pages discussing such topics as whether fuzzy logic actually has any applications, or whether it can be used as a model of vagueness and uncertainty. (Hájek is adamant that it does not mirror probability theory.) And there is a short but interesting discussion concerning "where the truth values come from" and "how the values can be interpreted."

Hájek claims that his readers are "assumed to be able to follow mathematical arguments and to have at least a basic mathematical knowledge." They further need "at least a partial experience with the classical (Boolean) propositional logic [and] some knowledge of predicate calculus." He claims that he "do[es] not assume any deep knowledge of mathematical logic or algebra." This reviewer's experience is that Hájek overestimates how much can be easily comprehended by a reader with only this much background.

In "the narrow sense," fuzzy logic is infinitely-many valued logic. Despite the belief of many "popular fuzzy logicians" that Lotfi Zadeh invented infinite-valued logic in 1965 (see JSL XXXVIII 656), it was in fact investigated in the 1920's and 1930's by Łukasiewicz and Tarski and there was a very strong resurgence of interest in the 1950's and early 1960's. Hájek cites some of this historically relevant material in a very short concluding chapter to the book (only two pages of it to cover the period before Zadeh's 1965 paper). This reviewer would have preferred to see the author develop fuzzy logic as a development of the historical quest for infinite-valued logics; but then to each his or her own. Hájek has a clear idea of how to investigate the topic of infinitely-many valued logics, and he does a nice job within his own terms, and indeed, a nice job even with no qualifications.

Every other presentation of fuzzy logic seen by this reviewer starts with a claim like this: "A propositional letter takes a value in the real interval [0, 1]. A conjunction of sentences is the minimum value of its conjuncts, while a disjunction of sentences is the maximum value of its disjuncts. A negation is interpreted as 1 minus the value of the unnegated component." Hájek instead starts his semantic account by describing a very general notion of a conjunctive truth function, $*$. It must agree with classical conjunction on the values 0 and 1; it must be commutative and associative; it must be non-decreasing with respect to both arguments; $(1*x) = x$ and $(0*x) = 0$; and it must be a continuous function on $[0, 1]^2$ into $[0, 1]$. Hájek's strategy is to develop a semantic theory for a basic logic (BL) that has just this "weak" type of conjunction, and then to add on various restrictions to the conjunctive operator which will generate semantics for various different fuzzy logics. Different ways to add restrictions to this operator give rise to distinct "t-norms," as Hájek calls them. Three different restrictions are considered:

Łukasiewicz:	$(x * y) = \max(0, x + y - 1)$
Gödel:	$(x * y) = \min(x, y)$
Goguen:	$(x * y) = xy$ (product of reals)

(It should be noted that Hájek sometimes uses 'Goguen' for this last, and sometimes uses 'product'.) On these definitions of conjunction in a fuzzy logic, only the Gödel logic resembles fuzzy logic as traditionally understood.

Given any one of these t-norms, one can define a (semantic) conditional, $\Rightarrow$, as its *residuum* in the following way:

$$(x \Rightarrow y) = \text{the maximal } z \text{ satisfying } (x * z) \leqq y$$

Corresponding to the three different definitions of $*$, we have three different conditionals. Each conditional yields the same value, namely 1, whenever the antecedent has the same or smaller truth-value than the consequent, but they differ in the cases where the antecedent is "truer" than the consequent:

If $x \leqq y$, then $(x \Rightarrow y) = 1$; otherwise:
Łukasiewicz: $(x \Rightarrow y) = 1 - x + y$
Gödel: $(x \Rightarrow y) = y$
Goguen: $(x \Rightarrow y) = y/x$ (division of reals)

There has long been (and still is) disagreement in the fuzzy logic community about the correct definition of the conditional, and we see here how three of the ones used turn out to be consequences of the choice of conjunction. (There are other common suggestions in the literature for implication that are not generated like this; some of these are even simply 2-valued.)

Hájek uses $\Rightarrow$ to define $(-)$, a semantic negation operator: $(-)x =_{\mathrm{df}} (x \Rightarrow 0)$. For our three different types of $\Rightarrow$ this yields two negations:

Łukasiewicz: $(-)x = 1 - x$
Gödel: $(-)x = 1$, if $x = 0$; $= 0$ otherwise
Goguen: $(-)x = 1$, if $x = 0$; $= 0$ otherwise

It will be noted that only the Łukasiewicz negation corresponds to the traditional fuzzy logic definition of negation, and that the other negation is essentially a 2-valued operator that says whether the unnegated sentence has the value 0 or not.

Nonetheless, given any one of these t-norms, using & to be the connective in the language that represents $*$ and using $\rightarrow$ as the connective to represent $\Rightarrow$, we can define the traditional fuzzy $\wedge$ and $\vee$ (min and max operators):

$$A \wedge B \quad \text{is} \quad A \mathbin{\&} (A \rightarrow B)$$
$$A \vee B \quad \text{is} \quad ((A \rightarrow B) \rightarrow B) \wedge ((B \rightarrow A) \rightarrow A)$$

(Notice that $\wedge$ rather than & is the conjunctive operator used in defining $\vee$.) Given the constant **0** in the language, then negation and equivalence are defined:

$$\neg A \quad \text{is} \quad A \rightarrow \mathbf{0}$$
$$A \equiv B \quad \text{is} \quad (A \rightarrow B) \mathbin{\&} (B \rightarrow A)$$

(Note that & rather than $\wedge$ is the conjunctive operator used in defining $\equiv$.)

The reader might note that many of the standard properties of the different connectives are preserved in BL, such as the usual distributivity laws, commutativity laws, the De Morgan laws (for $\wedge$, $\vee$, and $\neg$), and the like. Note that $(p \rightarrow \neg\neg p)$ is a theorem but that $(\neg\neg p \rightarrow p)$ is not (and indeed it fails under the Gödelian-Goguenian interpretation). We note also that the deduction theorem does not hold in full generality in BL. What we have instead is

$$\Gamma \cup \{\varphi\} \vdash \psi \text{ iff there is an } n \text{ such that } \Gamma \vdash (\varphi^n \rightarrow \psi)$$

where φ^n is $(\varphi \mathbin{\&} \varphi \mathbin{\&} \cdots \mathbin{\&} \varphi)$, n times. (The unqualified deduction theorem only holds for the Gödel logic.) One might also wonder about semantic compactness: Is it true that

$$\Gamma \models \varphi \text{ iff } \Delta \models \varphi \text{ for some } \Delta \text{ which is a finite subset of } \Gamma?$$

Given the resources of fuzzy logic, it seems possible to construct an infinite contradictory set Γ that has no contradictory finite subsets. But then $\Gamma \models \varphi$ would be valid but $\Delta \models \varphi$ would not be, for any finite Δ. This would have consequences for completeness, for then there could be a valid argument $\Gamma \models \varphi$ for which there was no derivation (because derivations are finite).

As can be seen, many truth functions can be defined from these t-norms. But a simple cardinality argument shows that fuzzy logic is expressively incomplete, for there are uncountably many n-ary truth functions from $[0, 1]^n$ into $[0, 1]$. In a very interesting theorem, Hájek shows that every *continuous* t-norm (every reasonable definition of conjunction) is a combination of the three t-norms he has singled out.

Hájek spends two chapters on the three defined fuzzy logics, axiomatizing them and showing the cases in which they are inter-interpretable. At the end of Chapter 3 (on Łukasiewicz fuzzy logic) he discusses how this logic would deal with the sorites paradox. At the end of this discussion he claims that the example will be further dealt with in the following chapters on Gödel and Goguen logic as well as in the fuzzy predicate logic. The reviewer thinks it unfortunate that this very interesting discussion was not continued.

In Chapter 5 Hájek extends his analysis to predicate logic. Once again, we start with BL and consider the logic generated by adding predicates, constants, variables, and the quantifiers $\forall$ and $\exists$. (The logic is called BL$\forall$.) Of some interest are the theorems

$$\exists x\,\varphi \rightarrow \neg\forall x\neg\varphi \quad \text{and} \quad \neg\exists x\,\varphi \equiv \forall x\neg\varphi$$

and the fact that the converse of the first of these is not a theorem of BL$\forall$. Semantically the idea is that constants denote (precisely!) elements of a domain, while n-place predicates designate a "fuzzy n-ary relation" (which gives the truth value of that predicate for any n-tuple chosen from the domain). $\forall x\,\varphi$ has the truth value of the "least true instance of φ" while $\exists x\,\varphi$ has the truth value of the "most true instance of φ." If there are no such instances then the value of the formula is undefined. (This seems a departure from traditional fuzzy logic where such formulas are assigned the limit-value of the various φ's.) This restriction is carried out by having only "safe" models be relevant to considerations of completeness; completeness is attained with respect to the class of linearly ordered $\mathcal{C}$-algebras that are safe. Hájek thus avoids the result that fuzzy logic is not recursively axiomatizable by enlarging the class of truth-value algebras. A traditional fuzzy logician ought to say that this is not "real" completeness, but only completeness with respect to a limited set of models. Both the traditional fuzzy logician and the doubter of fuzzy logic ought to say that the importance or relevance of this new class of models needs to be made plausible before we can responsibly say that fuzzy logic has been given a complete axiomatization that is reasonable.

After developing BL$\forall$, Hájek moves on to the task of extending the Łukasiewicz, Gödel, and Goguen logics with predicates, names, and quantifiers. Hájek shows that tautologies of the Gödel predicate logic coincide with the tautologies over "the standard G-algebra $[0,1]_{\mathrm{G}}$ of truth functions [and] thus G$\forall$ is recursively axiomatized." The interested reader might note that this is done with respect to the *countable* subinterval of $[0,1]$. Hájek then shows that no similar result is possible for either the Łukasiewicz or Goguen logics: no recursive axiomatization for either of these logics is possible. (This follows from a result of Scarpellini's, JSL XXIX 145.) Many observers might think this should sound the death knell for fuzzy logic. This reviewer thinks Hájek might have spent much more time on this issue.

There are many further very interesting topics covered in this book, such as the treatment of equality (similarity?) in fuzzy logic, issues of computational complexity and undecidability, a whole chapter on "approximate inference," generalized quantifiers, modalities, an exposition of other fuzzy logics, a discussion of the liar paradox in fuzzy logic ("This sentence is at least a little false"), and various other topics. There are many hidden gems in these discussions; the reviewer found Hájek's lengthy discussion in Chapter 7 of Zadeh's notion of a "compositional rule of inference" and the possibility of other "fuzzy rules of inference" to be especially interesting and of especial importance to those theorists who are not logicians but who are interested in developing accounts of reasoning or accounts of fuzzy controllers.

The reviewer found enough (inconsequential) typographical errors to believe that there are probably quite a few in the book. Some of the proofs seem like verbatim transcriptions of a proof-sketch off of Hájek's blackboard: the abbreviations and informal conventions employed are sometimes difficult to unravel.

Anyone interested in fuzzy logic—whether they are "true believers" or simply "logical pornographers"—should have this book and learn from it. It is not much of a compliment to say that it is the best book there is on the formal properties of fuzzy logic, since there are no others. But even if there were others, it is still fuzzily-valid to infer that this one would still be the best. (Thanks to Alasdair Urquhart for discussion.)

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Petr Hájek, Lluis Godo, and Francesc Esteva. *A complete many-valued logic with product-conjunction*. ***Archive for mathematical logic***, vol. 35 (1996), pp. 191–208.

A beautiful result essentially due to P. S. Mostert and A. I. Shields (*On the structure of semigroups on a compact manifold with boundary*, ***Annals of mathematics***, vol. 65 (1957), pp. 117–143; cf. also D. H. Ling, *Representation of associative functions*, ***Publicationes mathematicae*** (Debrecen), vol. 12 (1965), pp. 189–212), says that every continuous t-norm on $[0,1]$ is in some sense a combination of three fundamental t-norms: Łukasiewicz t-norm $x \cdot y = \max\{x + y - 1, 0\}$, Gödel's t-norm $x \wedge y = \min\{x, y\}$, and product t-norm $\odot$ (ordinary product).

In his book ***Metamathematics of fuzzy logic*** (see the preceding review), P. Hájek emphasizes the relationships between fuzzy logics and continuous t-norms with their residua. According to this (quite reasonable) point of view, four logics emerge: Łukasiewicz logic, Gödel logic, product logic, and basic logic. As shown recently by Cignoli, Esteva, Godo, and Torrens in *Basic fuzzy logic is the logic of continuous t-norms and their residua* (***Soft computing***, vol. 4 no. 2 (2000), pp. 106–112), basic logic is the logic of arbitrary continuous t-norms and their residua.

Whereas Łukasiewicz logic and Gödel logic have been investigated for many years (completeness proofs for these logics have been known since the fifties), product logic and basic logic are rather recent subjects.

The present paper contains an axiomatization and a completeness proof for product logic. For the axiomatization, the idea is that the structure $\langle (0,1), \odot, \rightarrow, 0, 1\rangle$ (with 0 excluded), where $\odot$ is product and $\rightarrow$ is its residuum, is a residuated cancellative commutative monoid, and that $0 \odot x = x \odot 0 = 0$, $0 \rightarrow x = 1$, and for $x \neq 0$, $x \rightarrow 0 = 0$. The crucial axiom expressing that the cancellation property holds with the exception of 0 is

$$\neg\neg\varphi \rightarrow ((\varphi \odot \psi) \rightarrow (\varphi \odot \gamma)) \rightarrow (\psi \rightarrow \gamma),$$

where $\neg\varphi$ is short for $\varphi \rightarrow 0$. Furthermore, the behavior of $\neg$ is described by the axiom $\neg(\varphi \wedge \neg\varphi)$ (where $\varphi \wedge \psi$ is short for $\varphi \odot (\varphi \rightarrow \psi)$).

The completeness proof is in three steps. First, the authors prove some syntactic results that establish some basic properties of product logic, and allow for the introduction of a complete algebraic semantics, based on product algebras.

As a second step, it is proved that every product algebra is isomorphic to a subdirect product of a family of linearly ordered product algebras. From this, it follows that product logic is complete with respect to evaluations on linearly ordered product algebras.

Finally, it is shown that any finite partial subalgebra of a linearly ordered product algebra can be embedded into the standard product algebra on the real interval $[0,1]$. The completeness of product logic with respect to evaluations on the standard product algebra is an easy consequence of the results mentioned above.

Although Łukasiewicz logic is probably deeper, and Gödel logic has a nicer proof theory, product logic is also very interesting. First of all, product is a quite natural t-norm. Moreover, the algebraic structure of product algebras is far from trivial: thinking of product algebras as negative cones of lattice-ordered Abelian groups with an added bottom element