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Substitutions, Model sets and Pure Point Spectra

by

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Chapter 1

Introduction

This thesis is about the mathematics of long-range aperiodic structure. This is a relatively new area which arose from the discovery of aperiodic tilings in mathematics and from the experimental discovery of quasicrystals in the field of materials science.

Quasicrystals were discovered in the mid 1980's in experiments of Dan Shechtmann and his colleagues [42]. Quasicrystals are metallic solids which are not crystals yet have diffraction patterns consisting of pure bright peaks without diffuse background. Before the discovery of quasicrystals it had been believed that only crystal structures, that is an atomic structure based on periodic repetition of a fundamental cell, could have diffraction patterns with pure bright peaks. So when these new materials were first discovered there was no theory to explain or support the phenomenon and there was a great deal of debate about their reality.

Experiments over the past 20 years have confirmed the existence of the quasicrystals beyond doubt and at the same time generated a great deal of interest about ordered structures in various different contexts, particularly mathematics.

In mathematics, as a first approximation, we can interpret atoms as points (or more realistically points of various 'colours', representing the different types of atoms) and this leads to the study of structured point sets. In a broad sense we mean by 'aperiodic order' structure which is in some way beyond periodic. In the context of point sets we are especially interested in studying structured point sets whose diffraction consists entirely of pure bright diffraction peaks. This is what is called pure point diffraction.

Except in Chapter 8 we deal here with point sets that are Delone sets. The Delone property is the weakest hypothesis consistent with our intuitive notion of an (idealized) infinitely extended atomic material solid, namely that the point set is coextensive with space (that is there are not large regions of the space which are empty of points) and also there is minimum separation between the points.

The properties of Delone sets are well reflected in tilings. A tiling is a covering of space by a set of tiles in such a way that distinct tile interiors do not meet each other. Especially when we think about tilings with only finitely many tile types (prototiles) under translation, it is easy to construct corresponding Delone point sets. For example, the vertices of the tiles if they are polyhedral, form such a set. By choosing one point from each prototile and then selecting the corresponding point each time that tile appears in the tiling we can obtain a Delone set. On the other hand, given a Delone set Λ and then taking Voronoi cells, each of which consists of all points in the space which are closer to some point of Λ than to any other, we can easily move from Delone sets to tilings.

Tilings have their own long history in the form of arts, designs, and the practical problems of covering areas and spaces with building materials. Typically they are patterns with periodic structure. Since Robert Berger [6] proved the computational undecidability of the tiling problem - the problem of deciding whether a given finite set of polygonal tile shapes will or will not tile the plane - the existence of finite sets of prototiles that tile a plane only in aperiodic way has been confirmed. The Penrose tiles are the best-known example of this kind.

Just around the time of the discovery of quasicrystals strong evidence was given for the pure point diffractivity of the aperiodic Penrose tilings. This brought new attention to the theory of aperiodic tilings since they were apparently good models for trying to understand quasicrystals, and since that time tiling theory has had a great period of revival.

We usually consider both aperiodic point sets and tilings in this thesis. We have already suggested that there is a close connection between point sets and tilings. Points sets are more algebraic, tilings more geometric. Point sets are easy to create, describe, and manipulate algebraically, but it is hard to encode the space filling properties of their various patterns. On the other hand, tilings, especially in 2-dimensions have wonderfully visible geometrical attributes and also have spatial properties which make certain concepts very natural; for example, uniform patch frequencies (UPF)- that for any pattern consisting of a finite collection of the tiles there is a unique uniformly converging frequency for this pattern throughout the tiling. However, tilings are often very intricate and are almost beyond intuition or construction in higher dimensions. The study of point set structure is not far from the study of tiling structure and these two studies are often interrelated in the study of aperiodic order. In this thesis both concepts live side-by-side and we often depend on being able to pass between them.

We start by introducing definitions and notation for point sets and tilings in Chapter 2. We consider point sets which have finitely many different colours and call them *multisets.* We explain in this chapter the main basic concepts such as *finite local complexity* (FLC) that there are only finite local patterns up to translations; *repetitivity* that whatever (finite) patterns you see you will see them again without going too far in the space; and *uniform cluster frequencies* (UCF) which is a similar analogue to UPF except now in the setting of point sets .

There is a commonly used method for generating examples of aperiodic point sets which is called the cut and project method. It generates the so-called *model sets.* Roughly these are projections into a space of (parts of) a lattice that lies in some higher dimensional or enlarged 'super-space'. The 'cut' part refers to the selection process used in projection of the lattice. We will explain them in SubSec. 2.1.1. Much of the thesis is about these sets : how they connect with pure point diffractivity, and how they can be characterized.

In Sec. 2.2 we will define tiles and tilings precisely. Most of the basic concepts in point sets are convertible to tilings.

One of the most historically important methods of creating tilings is by substitutions. In fact the method has been used in the study of ordered alphabetic sequences for many years and it can also be used to create aperiodic point sets. We talk about substitutions in Chapter 3 from the point view of both point sets and tilings.

Just to get a brief idea about substitutions, let us first look at the so-called *period-doubling substitution.* We begin with two letters $\{a, b\}$ and define the substitution map Φ which acts so that $a \xrightarrow{\Phi} ab$ and $b \xrightarrow{\Phi} aa$; or repeating it twice, $a \xrightarrow{\Phi} ab \xrightarrow{\Phi} abaa$ and $b \xrightarrow{\Phi} aa \xrightarrow{\Phi} abab$. Starting from b and a and expanding b to the left and a to the right using Φ^2, we get a fixed bi-sequence

$$\cdots \; b \quad a \quad a \quad a \quad b \quad a \quad b \,|\, a \quad b \quad a \quad a \quad a \quad b \quad a \; \cdots$$

Attaching a unit length interval to each letter in the sequence consecutively starting with the interval $[0, 1]$ at the initial letter a and $[-1, 0]$ at the initial letter b we can tile a real line $\mathbb{R}$.

$\cdots$	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	$\cdots$
$\cdots$	b	a	a	a	b	a	b	a	b	a	a	a	b	a	$\cdots$

Let Λ_a (Λ_b) be the set of left end points from a (b) type intervals. This way we get two infinite aperiodic substitution point sets (although together they simply make up $\mathbb{Z}$).

In keeping with the idea of that Delone sets and tilings coexist in some sort of coherent relation, one can think about how this works at the level of substitution Delone sets and substitution tilings. In fact using the additional structure coming from iteration rules one can get closer relation between the two which allows us to switch from one to the other while keeping the same iteration rules. But this is not trivial.

When a substitution tiling is given, one can find an associated substitution point set using control points ([46], [18], [33], [43]). What about the converse? Given that we have a substitution point set, can we find an associated substitution tiling which represents the original point set structure and keeps the same substitution rules? Lagarias and Wang [20] characterize substitution point sets and study the subclass of the substitution Delone sets, analysing the structure in detail. Furthermore they give sufficient conditions for substitution Delone sets to be represented by the corresponding substitution tilings. This representation is important to our work. Using it we can associate to each point a corresponding tile and deal with either tiles or points, whichever is most convenient. In this way the theory which has been developed for substitution tilings can be used effectively for understanding substitution point set structure.

In Chapter 3 we define the substitution point sets and substitution tilings and extend the result of [20] on the relation between the two objects, finding necessary and sufficient conditions for substitution Delone sets to be represented by the corresponding substitution tilings and introducing the concept of legality. It is known that primitive substitution tilings (see Def. for primitive) have UPF. We show this property in this chapter for reader's convenience. Moreover in more general setting we connect substitution Delone sets with substitution multitilings. This connection was first raised in [20] as a question and we answer the question here. Using the connection we are able to prove that primitive substitution Delone sets have UCF.

Dynamical systems originally arose in the study of systems of differential equations used to model physical phenomena. The motions of the planets, or of mechanical systems, or of molecules in a gas can be modeled by such systems. Roughly speaking a dynamical system is a space with a group of transformations acting on it. Here we consider a collection of Delone multisets in $\mathbb{R}^d$ and define a metric on it. We define a dynamical hull as a closure of orbits of a Delone multiset $\mathbf{\Lambda}$ in the collection. Then $\mathbb{R}^d$ acts on the dynamical hull by translates and we get a dynamical system of the point set. A dynamical system of a tiling is defined in the same way. For any measure preserving system $(X_{\mathbf{\Lambda}}, \mu, \mathbb{R}^d)$ we can consider a group of unitary operators on $L^2(X_{\mathbf{\Lambda}}, \mu)$. Every function in $L^2(X_{\mathbf{\Lambda}}, \mu)$ defines a corresponding spectral measure and we study the spectral theory of the dynamical system. It turns out that the spectral theory is well connected with diffraction spectrum.

On this thesis we are interested in understanding point set structures with regard to their diffraction spectra. These are idealized mathematical interpretations of the diffraction patterns of physical experiments and are important since they encode information about the long-range order of the system. We explain the diffraction spectrum with a simple example in Sec. 4.2.

The spectral theory of dynamical systems has been studied for a long time and it turns out there is a way to connect this theory with the diffraction spectrum of point sets and tilings. One of the earliest uses of this idea appears in M. Queffélec [35] where she established an equivalence between pure point diffraction and dynamical spectra in the setting of symbolic (one dimensional) dynamics. The main step forward was provided by Stephen Dworkin [12] who showed how to connect the autocorrelation of a point set and particular spectral measures in the setting of its local hull. One result of this is that if a point set has pure point dynamical spectrum then it also has pure point diffraction spectrum. The converse (in the geometric setting of point sets) was first shown in [24].

We will talk about the equivalence in Chapter 5 in detail. Recently this equivalence has been improved in [15] and [1] in more general cases. Certainly due to the rich spectral theories on dynamical systems the relation between two spectra gives us better access to understanding the point set structure.

In Chapter 6 we will consider the dynamical spectrum in substitution tilings and point sets. Due to the hierarchical structure of substitutions we can find good characterizations for substitution tilings to have pure point dynamical spectra. This chapter is largely based on [43]. The main new feature is that here the dimension d is arbitrary, while [43] was focused on the case $d = 2$. We explain overlap coincidence which finds a geometric condition for substitution tilings to have pure point dynamical spectrum. Using the representable concept in substitution Delone sets we can derive the same spectral property for the dynamics of the substitution Delone sets.

In Chapter 7 we consider substitution point sets on lattices. Here we are dealing with coloured point sets which comprise in totally a lattice. The period-doubling example above is of this type since together all the points involved make up the integers. Applying our various equivalent characterizations of pure pointedness to lattice substitutions, we find a checkable condition for pure pointedness. This also enables us to connect pure pointedness with model sets.

Theorem 7.9 shows three different aspects of development in the subject of pure point diffractive point set structure.

One aspect is that it establishes the equivalence concept between pure pointedness and regular model sets (model sets whose windows have boundary measure 0). It has been known that any regular model sets are pure point diffractive [41] but the reverse is in general not yet known. In fact it is a challenging and natural problem to resolve the relation between pure pointedness and model sets in more general setting without assuming lattice situation.

Another aspect is that it extends Dekking's well-known 1D criterion for pure point spectrum in terms of coincidence [10]. The criterion says that an equal length symbolic substitution has pure point dynamical spectrum if and only if it admits a coincidence. Here coincidence means that there is an n-th column whose letters are same in all the iterated words. For example, the period-doubling substitution admits a coincidence, since a is the first letter in the iterated words ab and aa from a and b. It shows that the period-doubling sequence has pure point dynamical spectrum. We generalize the coincidence from its original one-dimensional setting in constant-length alphabetic substitutions to lattice substitutions in arbitrary dimensions, introducing modular coincidence. A number of interesting tilings such as Robinson, sphinx, and chair tilings ([38], [22]) fall into the setting of lattice substitutions.

The third aspect is that it gives condition in which model sets and substitution sets are connected. From the algebraic structure of cut and project schemes it is easy to get infinite point sets with certain properties which we would like to study. Especially, one can easily obtain the corresponding geometrical conditions for the important properties in the dynamics of infinite point sets such as FLC and repetitivity. Unlike substitution point sets it is not easy to configure local points completely in the sets coming from cut and project schemes. So the questions about when model sets can be viewed as substitution point sets and when substitution point sets become model sets are interesting problems. The studies for the former are on [16] and [28], and the studies for the latter are on [22] and [25] in lattice cases.

For the relation among coincidence and pure pointedness and model sets in substitution point sets, there is progressing work for more general cases in [21].

Due to their remarkable properties (which somehow capture all the essential features of aperiodic point sets), model sets have become a mainstay in the study of aperiodic order and in the building of theoretical models in the study of physical quasicrystals. On the other hand, it has been very hard to characterize model sets. Given a set of points, or a multiset, how can one know if this is a model set or model multiset?

When we have two infinite point sets, there are two different ways to determine the closeness of the two sets. One defines the closeness by how much the two sets exactly agree around origin, up to small translations. The other defines the closeness depending on how much they agree overall in a statistical sense, again up to small translations. In general these two different notions of closeness are not related each other. But when we look at model sets they are related concepts. Actually when we define two dynamical hulls with these two topologies, one defined with local topology and the other defined with autocorrelation topology, the existence of a continuous mapping between two hulls characterizes model sets. We will prove this characterization in Chapter 8.

This thesis is based on three papers [24], [25] and [23]. In [24] we prove the equivalence between pure point diffraction and dynamical spectra under UCF and FLC.

Theorem 5.1 Suppose that a Delone multiset $\mathbf{\Lambda}$ has FLC and UCF. Then the following are equivalent:

(i) $\mathbf{\Lambda}$ has pure point dynamical spectrum;

(iii) $\mathbf{\Lambda}$ has pure point diffraction spectrum.

In [25] we find the necessary and sufficient condition for a repetitive primitive substitution Delone multiset to be representable for a substitution tiling extending Lagarias and Wang's result in [20].

Theorem 3.10 Let $\mathbf{\Lambda}$ be a repetitive primitive substitution Delone multiset. Then every $\mathbf{\Lambda}$-cluster is legal if and only if $\mathbf{\Lambda}$ is representable.

In an extension of this work, which is not included in the three papers, I have been able to show that a repetitive primitive substitution Delone multiset is representable for a substitution multitiling, giving an answer to the question which was raised in the paper [20].

Theorem 3.21 If $\mathbf{\Lambda}$ is a repetitive primitive substitution Delone multiset, then $\mathbf{\Lambda}$ is representable for q-multitiling with some $q \in \mathbb{Z}_+$.

We also provide an answer in the case of lattice substitution systems to the question of "when is a pure point diffractive set a regular model set?", revising the modular coincidence which was first introduced in [22]. The result completes a circle of equivalences of modular coincidence, the existence of a model set interpretation, the notion of pure point spectrum, and a condition on almost-periods.

Theorem 7.9 Let $\mathbf{\Lambda}$ be a Delone multiset with expansive map Q such that $(\mathbf{\Lambda}, \Phi)$ is a primitive substitution system, $L = \bigcup_{i \le m} \Lambda_i$ for some lattice L in $\mathbb{R}^d$, and every $\mathbf{\Lambda}$-cluster is legal. Let $L' = L_1 + \cdots + L_m$, where $L_i = < \Lambda_i - \Lambda_i >$. The following are equivalent:

(i) $\mathbf{\Lambda}$ has pure point diffraction spectrum;

(ii) $\mathbf{\Lambda}$ has pure point dynamical spectrum;

(iii) $\mathrm{dens}(\mathbf{\Lambda} \triangle (Q^n \alpha + \mathbf{\Lambda})) \xrightarrow{n \to \infty} 0$ for all $\alpha \in L'$;

(iv) A modular coincidence relative to $Q^M L'$ occurs in Φ^M for some M;

(v) Each Λ_i is a regular model set for $i \le m$, relative to the CPS (7.25).

In [23] we characterize a model multiset by a map between two dynamical hulls, modifying arguments of [2] and [41], which apply only to single-coloured point sets. This characterization is especially useful when one is working with substitution point sets which usually deal with finitely many different colours of point sets.

Theorem 8.1 Let $\mathbf{\Lambda}$ be a repetitive Meyer multiset in $\mathbb{R}^d$. Then there is a continuous $\mathbb{R}^d$-map $\beta : X_{\mathbf{\Lambda}} \to \mathbb{A}(\mathbf{\Lambda})$ which is one-to-one a.e. with respect to $\mathbb{A}(\mathbf{\Lambda})$ if and only if $\mathbf{\Lambda}$ (or equivalently each element of $X_{\mathbf{\Lambda}}$) is a regular model multiset.

Chapter 2

Preliminaries

Here we present notation and basic definitions on point sets and tilings. Readers can come back to this chapter for the meaning of terminology.

2.1 Delone point sets

A *multiset* [1] or *m-multiset* in $\mathbb{R}^d$ is a subset $\mathbf{\Lambda} = \Lambda_1 \times \cdots \times \Lambda_m \subset \mathbb{R}^d \times \cdots \times \mathbb{R}^d$ (m copies) where $\Lambda_i \subset \mathbb{R}^d$. We also write $\mathbf{\Lambda} = (\Lambda_1, \ldots, \Lambda_m) = (\Lambda_i)_{i \le m}$. Although $\mathbf{\Lambda}$ is a product of sets, it is convenient to think of it as a set with types or colours, i being the colour of points in Λ_i.

Definition 2.1 We say that $\Lambda \subset \mathbb{R}^d$ is a *Delone set* if there exists $r > 0$ such that for any $x \in \Lambda$, $B_r(x) \cap \Lambda = \{x\}$ (uniformly discrete) and there exists $R > 0$ such that for any $y \in \mathbb{R}^d$, $B_R(y) \cap \Lambda \neq \emptyset$ (relatively dense). We say that $\mathbf{\Lambda} = (\Lambda_i)_{i \le m}$ is a *Delone multiset* in $\mathbb{R}^d$ if each Λ_i is Delone and $\mathrm{supp}(\mathbf{\Lambda}) := \bigcup_{i=1}^m \Lambda_i \subset \mathbb{R}^d$ is Delone.

Definition 2.2 A set $\Lambda \subset \mathbb{R}^d$ is *Meyer* if Λ is Delone and there is a finite set F so that $\Lambda - \Lambda \subset \Lambda + F$. We say that $\mathbf{\Lambda}$ is Meyer if each Λ_i is Meyer.

There are many known equivalent concepts to Meyer. Another common characterization of Meyer for Delone set Λ is that $\Lambda - \Lambda$ is uniformly discrete [19]. For more about the characterizations of Meyer sets, see [31].

Definition 2.3 A *cluster* of $\mathbf{\Lambda}$ is a family $\mathbf{P} = (P_i)_{i \le m}$ where $P_i \subset \Lambda_i$ is finite for all $i \le m$.

[1]Caution : In [20], which we occasionally cite below, the word multiset refers to a set with multiplicities.

For example, in the period doubling sequence (1), $(\{-2,0\},\{-1,1\})$ is a cluster of $\mathbf{\Lambda} = (\Lambda_a, \Lambda_b)$.

Many of the clusters that we consider have the form $A \cap \mathbf{\Lambda} := (A \cap \Lambda_i)_{i \le m}$, for a bounded set $A \subset \mathbb{R}^d$. There is a natural translation $\mathbb{R}^d$-action on the set of Delone multisets and their clusters in $\mathbb{R}^d$. The translate of a cluster $\mathbf{P}$ by $x \in \mathbb{R}^d$ is $x + \mathbf{P} = (x + P_i)_{i \le m}$. We say that two clusters $\mathbf{P}$ and $\mathbf{P}'$ are *translationally equivalent* if $\mathbf{P} = x + \mathbf{P}'$ for some $x \in \mathbb{R}^d$. For any two Delone m-multisets $\mathbf{\Lambda}, \mathbf{\Gamma}$ we define $\mathbf{\Lambda} \cap \mathbf{\Gamma} = (\Lambda_i \cap \Gamma_i)_{i \le m}$ and $\mathbf{\Lambda} \triangle \mathbf{\Gamma} = (\Lambda_i \triangle \Gamma_i)_{i \le m}$, where $\Lambda_i \triangle \Gamma_i = (\Lambda_i \backslash \Gamma_i) \cup (\Gamma_i \backslash \Lambda_i)$. We write $B_R(y)$ for the *closed* ball of radius R centered at y and use also B_R for $B_R(0)$.

Definition 2.4 A Delone multiset $\mathbf{\Lambda}$ has *finite local complexity* (FLC) if for every $R > 0$ there exists a finite set $Y \subset \mathrm{supp}(\mathbf{\Lambda}) = \bigcup_{i=1}^m \Lambda_i$ such that

$$\forall x \in \mathrm{supp}(\mathbf{\Lambda}),\ \exists y \in Y:\ B_R(x) \cap \mathbf{\Lambda} = (B_R(y) \cap \mathbf{\Lambda}) + (x - y).$$

In plain language, for each radius $R > 0$ there are only finitely many translational classes of clusters whose support lies in some ball of radius R.

Definition 2.5 A Delone multiset $\mathbf{\Lambda}$ is *repetitive* if for any compact set $K \subset \mathbb{R}^d$, $\{t \in \mathbb{R}^d : \mathbf{\Lambda} \cap K = (t + \mathbf{\Lambda}) \cap K\}$ is relatively dense; i.e. there exists $R = R(K) > 0$ such that every open ball $B_R(y)$ contains at least one element of $\{t \in \mathbb{R}^d : \mathbf{\Lambda} \cap K = (t + \mathbf{\Lambda}) \cap K\}$.

For a cluster $\mathbf{P}$ and a bounded set $A \subset \mathbb{R}^d$ denote

$$L_{\mathbf{P}}(A) = \sharp\{x \in \mathbb{R}^d :\ x + \mathbf{P} \subset A \cap \mathbf{\Lambda}\},$$

where $\sharp$ means the cardinality. In plain language, $L_{\mathbf{P}}(A)$ is the number of translates of $\mathbf{P}$ contained in A, which is clearly finite. For a bounded set $F \subset \mathbb{R}^d$ and $r > 0$, let

$$\begin{aligned} F^{+r} &:= \{x \in \mathbb{R}^d : \mathrm{dist}(x, F) \le r\}, \\ F^{-r} &:= \{x \in F : \mathrm{dist}(x, \partial F) \ge r\} \supset F \setminus (\partial F)^{+r}. \end{aligned}$$

A *van Hove sequence* for $\mathbb{R}^d$ is a sequence $\mathcal{F} = \{F_n\}_{n \ge 1}$ of bounded measurable subsets of $\mathbb{R}^d$ satisfying

$$\lim_{n \to \infty} \mathrm{Vol}((\partial F_n)^{+r}) / \mathrm{Vol}(F_n) = 0, \text{ for all } r > 0 \tag{2.1}$$

(this term is used in statistical mechanics, see [40] and [14]).

Throughout this thesis we deal with concepts that depend on some sort of averaging sequence for their very definition: densities, frequencies, autocorrelation and diffraction measures. Even when not explicitly mentioned, we will always have in mind that these concepts have been defined in terms of some predetermined van Hove sequence $\{F_n\}_{n\geq 1}$.

Definition 2.6 Let $\{F_n\}_{n\geq 1}$ be a van Hove sequence. A Delone multiset $\mathbf{\Lambda}$ has *uniform cluster frequencies* (UCF) (relative to $\{F_n\}_{n\geq 1}$) if for any cluster $\mathbf{P}$, there is the limit

$$\mathrm{freq}(\mathbf{P}, \mathbf{\Lambda}) = \lim_{n\to\infty} \frac{L_{\mathbf{P}}(x + F_n)}{\mathrm{Vol}(F_n)} \geq 0,$$

uniformly in $x \in \mathbb{R}^d$.

For any subset $\mathbf{\Lambda}' \subset \mathbf{\Lambda}$, we define

$$\mathrm{dens}(\mathbf{\Lambda}') := \lim_{n\to\infty} \frac{\sharp(\mathbf{\Lambda}' \cap F_n)}{\mathrm{Vol}(F_n)},$$

if the limit exists.

2.1.1 Model sets

We define a cut and project scheme with $\mathbb{R}^d$ for a physical space here, but in general one can define it with a locally compact Abelian group. The definition of a model set here is little different from conventional definition but more general. For more about model sets, see [30].

Definition 2.7 A *cut and project scheme* (CPS) consists of a collection of spaces and mappings as follows;

$$\begin{array}{ccccc} \mathbb{R}^d & \xleftarrow{\pi_1} & \mathbb{R}^d \times H & \xrightarrow{\pi_2} & H \\ & & \cup & & \\ & & \widetilde{L} & & \end{array} \tag{2.2}$$

where $\mathbb{R}^d$ is a real Euclidean space, H is some locally compact Abelian group, π_1 and π_2 are the canonical projections, $\widetilde{L} \subset \mathbb{R}^d \times H$ is a lattice, i.e. a discrete subgroup for which the quotient group $(\mathbb{R}^d \times H)/\widetilde{L}$ is compact, $\pi_1|_{\widetilde{L}}$ is injective, and $\pi_2(\widetilde{L})$ is dense in H.

For a subset $V \subset H$, we denote $\Lambda(V) := \{\pi_1(x) \in \mathbb{R}^d : x \in \widetilde{L}, \pi_2(x) \in V\}$.

Definition 2.8 A *model set* in $\mathbb{R}^d$ is a subset Γ of $\mathbb{R}^d$ for which, up to translation in $\mathbb{R}^d$, $\Lambda(W^\circ) \subset \Gamma \subset \Lambda(W)$ where W is compact in H and $W = \overline{W^\circ} \neq \emptyset$. The model set Γ is *regular* if the boundary $\partial W = \overline{W} \backslash W^\circ$ of W is of (Haar) measure 0. We say that $\mathbf{\Gamma}$ is a *model multiset* (resp. *regular model multiset*) if each Γ_i is a model set (resp. regular model set) with respect to the same CPS.

One should note here that since π_2 need not be $1-1$ on $\widetilde{L}$, the model set Γ need not actually be of the form $\Lambda(V)$ for any set $V \subset H$. Nonetheless it is hemmed in between two such sets differing only by points on the boundary of the window W.

When we need to be more precise we explicitly mention the cut and project scheme from which a model set arises. This is quite important in some of the theorems below.

To get an idea on what the cut and project schemes and model sets are about, we present an example.

Example 2.9 We consider the point sets Λ_a and Λ_b from the period doubling sequence (1). We define the 4-adic completion

$$\mathbb{Z}_4 := \lim_{\leftarrow k} \mathbb{Z}/4^k\mathbb{Z}$$

of $\mathbb{Z}$. When $\mathbb{Z}_4$ is supplied with the usual topology of a profinite group, the cosets $a + 4^k\mathbb{Z}_4$, $a \in \mathbb{Z}$, $k \in \mathbb{Z}_{\geq 0}$ form a basis of open sets of $\mathbb{Z}_4$ and each of these cosets is both open and closed. We note here that $\mathbb{Z}_4$ is a compact Abelian group. Since $\mathbb{Z}$ is embedded in $\mathbb{Z}_4$, we can identify $\mathbb{Z}$ with its image in $\mathbb{Z}_4$. We can construct a following cut and project scheme taking $\mathbb{Z}_4$ as an internal space :

$$\begin{array}{ccccc} \mathbb{R} & \xleftarrow{\pi_1} & \mathbb{R} \times \mathbb{Z}_4 & \xrightarrow{\pi_2} & \mathbb{Z}_4 \\ & & \cup & & \\ \mathbb{Z} & \longleftarrow & \widetilde{\mathbb{Z}} & \longrightarrow & \mathbb{Z} \\ x & \longleftarrow & (x,x) & \longrightarrow & x\,, \end{array} \tag{2.3}$$

where $\tilde{\mathbb{Z}} = \{(x,x) \in \mathbb{R} \times \mathbb{Z}_4 | x \in \mathbb{Z}\}$. It is well described in [4] that we get

$$\Lambda_a = \cup_{n \geq 0}(2 \cdot 4^n\mathbb{Z} + (4^n - 1))\,,$$

$$\Lambda_b = \cup_{n \geq 1}(4^n\mathbb{Z} + (2 \cdot 4^{n-1} - 1)) \cup \{-1\}\,.$$

We can describe $\Lambda_a = \Lambda(W_a)$ and $\Lambda_b = \Lambda(W_b)$ where $W_a = \overline{\Lambda_a} \backslash \{-1\}$ and $W_b = \overline{\Lambda_b}$ in $\mathbb{Z}_4$. We will be able to easily see after Theorem 7.9 that Λ_a and Λ_b are regular model sets.

2.2 Tilings

This section briefly reviews the basic definitions of tilings. We begin with a set of types (or colours) $\{1, \dots, m\}$, which we fix once and for all. A *tile* in $\mathbb{R}^d$ is defined as a pair $T = (A, i)$ where $A = \mathrm{supp}(T)$ (the support of T) is a compact set in $\mathbb{R}^d$ which is the closure of its interior, and $i = l(T) \in \{1, \dots, m\}$ is the type of T. We let $g + T = (g + A, i)$ for $g \in \mathbb{R}^d$. We say that a set P of tiles is a *patch* if the number of tiles in P is finite and the tiles of P have mutually disjoint interiors (strictly speaking, we have to say "supports of tiles," but this abuse of language should not lead to confusion). The *support of a patch* is the union of the supports of the tiles that are in it. Note that the support of a patch need not be connected. The *diameter of a patch* is the diameter of its support. The *translate of a patch* P by $g \in \mathbb{R}^d$ is $g + P := \{g + T : T \in P\}$. We say that two patches P_1 and P_2 are *translationally equivalent* if $P_2 = g + P_1$ for some $g \in \mathbb{R}^d$. A *tiling* of $\mathbb{R}^d$ is a set $\mathcal{T}$ of tiles such that $\mathbb{R}^d = \cup\{\mathrm{supp}(T) : T \in \mathcal{T}\}$ and distinct tiles have disjoint interiors. Given a tiling $\mathcal{T}$, finite sets of tiles of $\mathcal{T}$ are called $\mathcal{T}$-patches.

We define FLC, repetitivity, and uniform patch frequencies (UPF), which is the analog of UCF, on tilings in the same way as the corresponding properties on Delone multisets.

We always assume that any two $\mathcal{T}$-tiles with the same colour are translationally equivalent (hence there are finitely many $\mathcal{T}$-tiles up to translation).

Chapter 3

Substitutions

Substitutions have proven to be a powerful way of constructing aperiodic tilings and, in fact, all of the famous aperiodic tilings (Penrose, Ammann-Beenker, octagonal, chair and sphinx) can be constructed by this means. The idea is simple. A finite set $\mathcal{A}$ of prototiles are given and also an expansion map Q (most often just a scaling map). The tiles are such that when they are inflated by Q they can be decomposed into some non-overlapping translates of the prototiles (see Def. 3.4). Iteration of this process can be arranged to fill out larger and larger patches of space, and ultimately leads to a tiling of space. Most often we are interested in a "fixed point" of this process, that is to say, a tiling which is fixed under the inflation/decomposition rule.

Substitutions that create Delone multisets are, by contrast, more recent. The idea is similar but there is a significant difference. One is looking at a multiset $\mathbf{\Lambda}$ and an expansion map Q. But now Q spreads out the points of $\mathbf{\Lambda}$ and disjoint translates of the inflated colour sets $Q\Lambda_j$'s reconstruct $\mathbf{\Lambda}$ (see Def. 3.1).

In both cases there are substitution equations which consist of affine maps with the expansion Q and the translations, and it is natural to look for a connection between the two. In particular, given a substitution Delone multiset is there a corresponding substitution tiling with the same substitution rules whose tiles are "centred" on the points of the multiset, or as we say it, is the substitution Delone multiset *representable* for a substitution tiling? As it turns out, there is a natural candidate set of tiles for the job – namely the solution (consisting of compact subsets in $\mathbb{R}^d$) to an adjoint system of equations (associated iterated function system). Even so the answer is not always "yes". In Sec. 3.2 we provide a necessary and sufficient condition for the representability of substitution Delone multisets based on the work of Lagarias and Wang and legality which is a new concept we introduce.

An important property of primitive substitution tilings is that given any finite patch of tiles from the tiling, this patch has a well-defined frequency of occurrence. We give a proof of this fact in Sec. 3.3, partly for convenience of the reader and partly because it does not appear in the literature in quite the form we want. Of importance to us, though, is that we can use this result to prove a corresponding property in the setting of multisets : the existence of uniform cluster frequencies for repetitive primitive substitution Delone multisets with finite local complexity (see Cor. 3.24). We have mentioned that a substitution Delone multiset cannot always represent for a tiling with tiles coming from the associated iterated function system. What goes wrong is that when we use the tiles and place them on the points of the substitution Delone multiset, they may actually overlap. However, this overlapping happens in a nice way for the case of repetitive primitive substitution Delone multiset – it is almost everywhere q to 1 for some positive integer q. In Sec. 3.4 we prove this and show that we can use it to create a new tiling for a modified substitution, and it is to this new tiling substitution system that we apply the theorem of uniform patch frequencies to finally get the existence of the desired uniform cluster frequencies of the substitution Delone multiset.

3.1 Substitutions on point sets and tilings

We say that a linear map $Q : \mathbb{R}^d \to \mathbb{R}^d$ is *expansive* if there is a $c > 1$ with

$$d(Qx, Qy) \geq c \cdot d(x, y) \tag{3.1}$$

for all $x, y \in \mathbb{R}^d$ and some metric d on $\mathbb{R}^d$ compatible with the standard topology. This is equivalent to saying that all the eigenvalues of Q lie outside the closed unit disk in $\mathbb{C}$.

Definition 3.1 $\mathbf{\Lambda} = (\Lambda_i)_{i \leq m}$ is called a *substitution Delone multiset* if $\mathbf{\Lambda}$ is a Delone multiset and there exist an expansive map $Q : \mathbb{R}^d \to \mathbb{R}^d$ and finite sets $\mathcal{D}_{ij}$ for $i, j \leq m$ such that

$$\Lambda_i = \bigcup_{j=1}^{m} (Q\Lambda_j + \mathcal{D}_{ij}), \quad i \leq m, \tag{3.2}$$

where the unions on the right-hand side are disjoint.

Example 3.2 The Delone multiset $\mathbf{\Lambda} = (\Lambda_a, \Lambda_b)$ from the periodic-doubling sequence (1) is a substitution Delone multiset. In fact, $\mathbf{\Lambda}$ satisfies the following equations

$$\begin{aligned} \Lambda_a &= 4\Lambda_a \cup (4\Lambda_a + 2) \cup (4\Lambda_a + 3) \cup 4\Lambda_b \cup (4\Lambda_b + 2) \\ \Lambda_b &= (4\Lambda_a + 1) \cup (4\Lambda_b + 1) \cup (4\Lambda_b + 3). \end{aligned}$$

To the substitution Delone multiset $\mathbf{\Lambda}$ we associate its $m \times m$ substitution matrix S, with $S_{ij} := \sharp(\mathcal{D}_{ij})$. We say that $\mathbf{\Lambda}$ is *primitive* if the corresponding substitution matrix S is primitive, i.e. there is an $l > 0$ for which S^l has no zero entries.

Let Y be a nonempty set in $\mathbb{R}^d$. For $m \in \mathbb{Z}_+$, an $m \times m$ *matrix function system* (MFS) on Y is an $m \times m$ matrix $\Phi = (\Phi_{ij})$, where each Φ_{ij} is a finite set (possibly empty) of mappings Y to Y.

Any MFS Φ on Y induces a mapping on Delone multiset $\mathbf{\Lambda} = (\Lambda_i)_{i \le m}$, where $\Lambda_i \subset Y$, $i \le m$, by

$$\Phi \begin{bmatrix} \Lambda_1 \\ \vdots \\ \Lambda_m \end{bmatrix} = \begin{bmatrix} \bigcup_{j \le m} \bigcup_{f \in \Phi_{1j}} f(\Lambda_j) \\ \vdots \\ \bigcup_{j \le m} \bigcup_{f \in \Phi_{mj}} f(\Lambda_j) \end{bmatrix}, \tag{3.3}$$

which we call the *substitution determined by* Φ. We often write $\Phi_{ij}(\Gamma_j)$ for $\bigcup_{f \in \Phi_{ij}} f(\Gamma_j)$, $\Phi(\Gamma_j)$ for $(\Phi_{ij}(\Gamma_j))_{i \le m}$, and $\Phi(\mathbf{\Gamma})$ for $(\bigcup_{j \le m} \Phi_{ij}(\Gamma_j))_{i \le m}$ for any subset $\mathbf{\Gamma} = (\Gamma_j)_{j \le m} \subset \mathbf{\Lambda}$. In particular, we often write $\Phi_{ij}(x)$ for $\Phi_{ij}(\{x\})$, where $x \in \Lambda_j$. We associate to Φ its substitution matrix $S(\Phi)$, with $(S(\Phi))_{ij} = \sharp(\Phi_{ij})$.

Let Φ, Ψ be $m \times m$ MFS's on Y. Then we can compose them :

$$\Psi \circ \Phi = ((\Psi \circ \Phi)_{ij}), \tag{3.4}$$

where $(\Psi \circ \Phi)_{ij} = \bigcup_{k=1}^{m} \Psi_{ik} \circ \Phi_{kj}$ and $\Psi_{ik} \circ \Phi_{kj} := \begin{cases} \{ g \circ f \ : \ g \in \Psi_{ik}, f \in \Phi_{kj} \} \\ \emptyset \quad \text{if } \Psi_{ik} = \emptyset \text{ or } \Phi_{kj} = \emptyset . \end{cases}$
Evidently, $S(\Psi \circ \Phi) \le S(\Psi)\, S(\Phi)$.

For any given substitution Delone multiset $\mathbf{\Lambda} = (\Lambda_i)_{i \le m}$, we can always find the corresponding MFS Φ such that $\Phi(\mathbf{\Lambda}) = \mathbf{\Lambda}$. Indeed, by Def. 3.1 we can take $\Phi_{ij} = \{f : f : x \mapsto Qx + a,\ a \in \mathcal{D}_{ij}\}$. So $\Phi(\Lambda_j) = (Q\Lambda_j + \mathcal{D}_{ij})_{i \le m}$, $j \le m$.

Theorem 3.3 [20, Theorem 2.3] *If $\mathbf{\Lambda}$ is a primitive substitution Delone multiset with expansive map Q, then the Perron-Frobenius (PF) eigenvalue of its substitution matrix S equals $|\det(Q)|$.*

Definition 3.4 Let $\mathcal{A} = \{T_1, \dots, T_m\}$ be a finite set of tiles in $\mathbb{R}^d$ such that $T_i = (A_i, i)$; we will call them *prototiles.* Denote by $\mathcal{P}_{\mathcal{A}}$ the set of patches made of tiles each of which is a translate of one of T_i's. We say that $\omega : \mathcal{A} \to \mathcal{P}_{\mathcal{A}}$ is a *tile-substitution* (or simply *substitution*) with expansive map Q if there exist finite sets $\mathcal{D}_{ij} \subset \mathbb{R}^d$ for $i, j \le m$, such that

$$\omega(T_j) = \{u + T_i : \ u \in \mathcal{D}_{ij},\ i = 1, \dots, m\} \quad \text{for } j \le m, \tag{3.5}$$

with

$$Q(A_j) = \bigcup_{i=1}^{m} (\mathcal{D}_{ij} + A_i).$$

Here all sets in the right-hand side must have disjoint interiors; it is possible for some of the $\mathcal{D}_{ij}$ to be empty.

The substitution (3.5) is extended to all translates of prototiles by $\omega(x + T_j) = Qx + \omega(T_j)$, and to patches and tilings by $\omega(P) = \cup\{\omega(T) : T \in P\}$. The substitution ω can be iterated, producing larger and larger patches $\omega^k(T_j)$. The substitution ω is called *primitive* if the substitution matrix S, with $S_{ij} := \sharp(\mathcal{D}_{ij})$, is primitive.

Definition 3.5 A patch will be called *legal* if it is a translate of a subpatch of $\omega^k(T_i)$ for some $i \leq m$ and $k \geq 1$.

A tiling $\mathcal{T}$ with FLC is said to be *self-affine* with the prototile set $\mathcal{A}$, expansive map Q and primitive substitution ω, if every $\mathcal{T}$-patch is legal. The set of self-affine tilings associated with $(\mathcal{A}, \omega)$ will be denoted by $X_{\mathcal{A},\omega}$. A tiling $\mathcal{T}$ is called a *fixed point* of the substitution ω if $\omega(\mathcal{T}) = \mathcal{T}$. It turns out that a tiling which is a fixed point need not be repetitive, even though the substitution is primitive, see [44]. It is well-known (and easy to see) that one can always find a *periodic point* for ω in the space $X_{\mathcal{A},\omega}$, i.e. there is $\mathcal{T} \in X_{\mathcal{A},\omega}$ such that $\omega^N(\mathcal{T}) = \mathcal{T}$ for some $N \geq 1$. In this case we can always use ω^N in place of ω to obtain a tiling which is a fixed point.

Proposition 3.6 *Let $\mathcal{T}$ be a fixed point of a primitive substitution ω with expansive map Q and prototiles $\mathcal{A}$. Then $\mathcal{T}$ is repetitive if and only if every $\mathcal{T}$-patch is legal, i.e. $\mathcal{T} \in X_{\mathcal{A},\omega}$.*

Proof. Suppose $\mathcal{T}$ is repetitive. Then for any patch P of $\mathcal{T}$ there exists $R > 0$ such that every open ball $B_R(y)$ contains a patch translationally equivalent to P. Since each tile of $\mathcal{T}$ has non-empty interior and ω is with expansive map Q, there is $M \geq 1$ such that for any $i \leq m$, $Q^M(A_i)$ contains $B_R(y)$ for some $y \in \mathbb{R}^d$. This means that $\omega^M(T_i)$ contains a patch translationally equivalent to P for any $i \leq m$. Thus every $\mathcal{T}$-patch is legal.

Conversely, suppose every $\mathcal{T}$-patch is legal. Then for every $\mathcal{T}$-patch P there is $K \geq 1$ such that for any $i \leq m$, P is a translate of a subpatch of $\omega^K(T_i)$ by the primitivity of ω. Choose $r > \max\{\mathrm{diam}(T_i) : i \leq m\}$. Every open ball $B_r(y)$ contains at least one tile. So every open ball $B_{\|Q\|^K r}(y)$, where $\|Q\|$ is the operator norm, contains at least one supertile $\omega^K(T)$, which contains a translate of P. Therefore $\mathcal{T}$ is repetitive. $\square$

3.2 Substitution Delone multisets to tilings

For each primitive substitution Delone multiset (3.1) one can set up an *adjoint system* of equations

$$QA_j = \bigcup_{i=1}^{m} (\mathcal{D}_{ij} + A_i), \quad j \leq m. \tag{3.6}$$

From Hutchinson's Theory (or rather, its generalization to the "graph-directed" setting), it follows that (3.6) always has a unique solution for which $(A_i)_{i \leq m}$ is a family of non-empty compact sets of $\mathbb{R}^d$ (see for example [3], Prop. 1.3). It is proved in [20, Theorem 2.4 and Theorem 5.5] that if $\mathbf{\Lambda}$ is a primitive substitution Delone multiset, then all the sets A_i from (3.6) have non-empty interiors and, moreover, each A_i is the closure of its interior.

Definition 3.7 A Delone multiset $\mathbf{\Lambda} = (\Lambda_i)_{i \leq m}$ is called *representable* (by tiles) for a tiling if there exists a set of prototiles $\mathcal{A} = \{T_i : i \leq m\}$ so that

$$\mathbf{\Lambda} + \mathcal{A} := \{x + T_i : \ x \in \Lambda_i, \ i \leq m\} \quad \text{is a tiling of } \ \mathbb{R}^d, \tag{3.7}$$

that is, $\mathbb{R}^d = \bigcup_{i \leq m} \bigcup_{x \in \Lambda_i} (x + A_i)$ where $T_i = (A_i, i)$ for $i \leq m$, and the sets in this union have disjoint interiors. In the case that $\mathbf{\Lambda}$ is a primitive substitution Delone multiset we will understand the term representable to mean relative to the tiles $T_i = (A_i, i)$, for $i \leq m$, arising from the solution to the adjoint system (3.6).

Definition 3.8 A cluster $\mathbf{P}$ will be called *legal* if it is a translate of a subcluster of $\Phi^k(x_j)$ for some $x_j \in \Lambda_j$, $j \leq m$ and $k \in \mathbb{Z}_+$.

Not every Delone multiset is representable (see Ex.3.22 below). In [20] Lagarias and Wang give a condition, namely existence of a fundamental cycle of period 1, which ensures representability. Legality generalizes this and in fact our Theorem 3.9 is based on [20, Theorem 7.1].

In the same paper [20, Lemma 3.2] it is shown that if $\mathbf{\Lambda}$ is a substitution Delone multiset, then there is a finite multiset (cluster) $\mathbf{P} \subset \mathbf{\Lambda}$ for which $\Phi^{n-1}(\mathbf{P}) \subset \Phi^n(\mathbf{P})$ for $n \geq 1$ and $\mathbf{\Lambda} = \lim_{n \to \infty} \Phi^n(\mathbf{P})$. We call such a multiset $\mathbf{P}$ a *generating multiset.*

Theorem 3.9 *Let $\mathbf{\Lambda}$ be a primitive substitution Delone multiset such that every $\mathbf{\Lambda}$-cluster is legal. Then $\mathbf{\Lambda}$ is representable.*

Proof. Let Φ be the MFS satisfying $\Phi(\Lambda) = \Lambda$ and suppose $\mathcal{A} = \{T_1, \ldots, T_m\}$ arises from the solution to the adjoint system (3.6). So for any $M \in \mathbb{Z}_+$,

$$Q^M(A_j) = \bigcup_{i=1}^{m}((\mathcal{D}^M)_{ij} + A_i), \quad j \leq m, \tag{3.8}$$

where $A_i = \mathrm{supp}(T_i)$ and

$$(\mathcal{D}^M)_{ij} = \bigcup_{k_1,k_2,\ldots,k_{(M-1)} \leq m} (\mathcal{D}_{ik_1} + Q\mathcal{D}_{k_1k_2} + \cdots + Q^{M-1}\mathcal{D}_{k_{(M-1)}j}).$$

On the other hand, for any $M \in \mathbb{Z}_+$ and $i \leq m$,

$$(\Phi^M)_{ij}(x_j) = Q^M x_j + (\mathcal{D}^M)_{ij}, \quad \text{for any } x_j \in \Lambda_j, j \leq m. \tag{3.9}$$

So putting (3.8) and (3.9) together, we get

$$\begin{aligned} Q^M(x_j \;\; + \;\; A_j) &= \bigcup_{i=1}^{m}(Q^M x_j + (\mathcal{D}^M)_{ij} + A_i) \\ &= \bigcup_{i=1}^{m}((\Phi^M)_{ij}(x_j) + A_i) \\ &= \bigcup_{i=1}^{m} \bigcup_{y \in (\Phi^M)_{ij}(x_j)} (y + A_i), \quad \text{for any } x_j \in \Lambda_j, \; j \leq m. \end{aligned} \tag{3.10}$$

From [20, Theorem 2.4 and Theorem 5.5], $\mu(A_j) > 0$ and A_j is the closure of its interior for any $j \leq m$. Let $\tilde{\mu} := (\mu(A_1), \ldots, \mu(A_m)) > 0$. Taking measures in (3.10),

$$|\det Q|^M \tilde{\mu} \leq \tilde{\mu} S(\Phi^M) \leq \tilde{\mu} S(\Phi)^M, \tag{3.11}$$

where $S(\Phi)$ is the substitution matrix of Φ. From 3.3, we know that $|\det Q| =$ PF eigenvalue of $S(\Phi)$. So we can derive

$$|\det Q|^M \tilde{\mu} = \tilde{\mu} S(\Phi)^M$$

from (3.11), see [22, Lemma 1]. Thus for any $x_j \in \Lambda_j, j \leq m$,

$$\begin{aligned} \mu(Q^M(x_j + A_j)) &= \mu\left(\bigcup_{i=1}^{m}((\Phi^M)_{ij}(x_j) + A_i)\right) \\ &= \sum_{i=1}^{m}(S(\Phi^M))_{ij}\mu(A_i), \end{aligned} \tag{3.12}$$

and this shows that the unions on the right-hand side of (3.10) are measure-wise disjoint. So we get a (tile)-substitution $\omega : \mathcal{A} \to \mathcal{P}_{\mathcal{A}}$ associated with $\mathcal{A}$ and $\mathbf{\Lambda}$.

Next, let $\mathbf{P}$ be a generating multiset. From the assumption that every cluster in $\mathbf{\Lambda}$ is legal, there is $a_l \in \Lambda_l$ for some $l \leq m$ such that $z + \mathbf{P} = (z + P_j)_{j \leq m} \subset ((\Phi^K)_{jl}(a_l))_{j \leq m}$ for some $K \in \mathbb{Z}_+$ and some $z \in \mathbb{R}^d$. So

$$\{x + T_j : x \in (\Phi^K)_{jl}(a_l), j \leq m\} \supset \{x + T_j : x \in (z + P_j), j \leq m\}. \tag{3.13}$$

By (3.12), all the tiles in the left-hand side of (3.13) are measure-wise disjoint. So then are the tiles in the right-hand side of (3.13). This implies that all the tiles in the set $\{p_j + T_j : p_j \in P_j, j \leq m\}$ are measure-wise disjoint. Thus for any $n \in \mathbb{Z}_+$, all the super tiles in the set $\{\omega^n(p_j + T_j) : p_j \in P_j, j \leq m\}$ are measure-wise disjoint. Noting that

$$\omega^n(p_j + T_j) = \{x + T_i : x \in (\Phi^n)_{ij}(p_j), i \leq m\} \ \text{ for each } p_j \in P_j, \ j \leq m,$$

we get that

$$\Phi^n(\mathbf{P}) + \mathcal{A} := \{x + T_i : x \in (\Phi^n)_{ij}(p_j), p_j \in P_j, i, j \leq m\}$$

also consists of tiles which are measure-wise disjoint. Since $\Phi^{n-1}(\mathbf{P}) \subset \Phi^n(\mathbf{P})$ for $n \geq 1$ and $\mathbf{\Lambda} = \lim_{n\to\infty} \Phi^n(\mathbf{P})$, $\mathbf{\Lambda} + \mathcal{A} = \{x_j + T_j : x_j \in \Lambda_j, j \leq m\}$ consists of tiles which are measure-wise disjoint. Thus distinct tiles in $\mathbf{\Lambda} + \mathcal{A}$ have disjoint interiors.

Now we need prove that $\mathbf{\Lambda} + \mathcal{A}$ is a tiling. Let us define $\text{supp}(\mathbf{\Lambda} + \mathcal{A}) := \cup\{x_j + A_j : x_j \in \Lambda_j, j \leq m\}$. Then

$$\begin{aligned} Q(\text{supp}(\mathbf{\Lambda} + \mathcal{A})) &= \cup\{Qx_j + QA_j : x_j \in \Lambda_j, j \leq m\} \\ &= \cup\{Qx_j + \mathcal{D}_{ij} + A_i : x_j \in \Lambda_j, i, j \leq m\} \\ &= \cup\{\Phi_{ij}(x_j) + A_i : x_j \in \Lambda_j, i, j \leq m\} \\ &= \cup\{x_i + A_i : x_i \in \Lambda_i, i \leq m\} \\ &= \text{supp}(\mathbf{\Lambda} + \mathcal{A}). \end{aligned} \tag{3.14}$$

Suppose $\mathbb{R}^d \backslash \text{supp}(\mathbf{\Lambda} + \mathcal{A}) \neq \emptyset$. Then there is $z \in \mathbb{R}^d \backslash \text{supp}(\mathbf{\Lambda} + \mathcal{A})$ and a ball $B_r(z)$ with radius r centered at z such that $B_r(z) \subset \mathbb{R}^d \backslash \text{supp}(\mathbf{\Lambda} + \mathcal{A})$, since $\text{supp}(\mathbf{\Lambda} + \mathcal{A})$ is a closed set. So for any $N \in \mathbb{Z}_+$,

$$\begin{aligned} Q^N(B_r(z)) \cap \text{supp}(\mathbf{\Lambda} + \mathcal{A}) &= Q^N(B_r(z)) \cap Q^N(\text{supp}(\mathbf{\Lambda} + \mathcal{A})) \quad \text{by (3.14)} \\ &= Q^N(B_r(z) \cap \text{supp}(\mathbf{\Lambda} + \mathcal{A})) = \emptyset. \end{aligned}$$

But this is a contradiction, since $\mathbf{\Lambda}$ is Delone. Therefore $\mathbf{\Lambda} + \mathcal{A}$ is a tiling and so $\mathbf{\Lambda}$ is representable. □

Theorem 3.10 *Let $\mathbf{\Lambda}$ be a repetitive primitive substitution Delone multiset. Then every $\mathbf{\Lambda}$-cluster is legal if and only if $\mathbf{\Lambda}$ is representable.*

Proof. We only need to prove the sufficiency direction. Suppose $\mathbf{\Lambda}$ is representable. Then we have a tiling $\mathbf{\Lambda} + \mathcal{A} = \{x_i + T_i : x_i \in \Lambda_i, i \le m\}$ with a unique solution $\mathcal{A} = \{T_1, \dots, T_m\}$ of the adjoint system of equations such that $\omega(\mathbf{\Lambda} + \mathcal{A}) = \mathbf{\Lambda} + \mathcal{A}$, where $\omega : \mathcal{A} \to \mathcal{P}_{\mathcal{A}}$ is a (tile)-substitution. So $\mathbf{\Lambda} + \mathcal{A}$ is a fixed point of a primitive substitution ω with expansive map Q. By Prop. 3.6, every $(\mathbf{\Lambda} + \mathcal{A})$-patch is legal, since $\mathbf{\Lambda} + \mathcal{A}$ is repetitive. Recall that for any $M \in \mathbb{Z}_+$,

$$\omega^M(x_j + T_j) = \{x + T_i : x \in (\Phi^M)_{ij}(x_j), i \le m\} \quad \text{for any } x_j \in \Lambda_j,\ j \le m,$$

where Φ is the MFS satisfying $\Phi(\mathbf{\Lambda}) = \mathbf{\Lambda}$. So the legality in the tiling $\mathbf{\Lambda} + \mathcal{A}$ shows the legality in $\mathbf{\Lambda}$. Therefore every $\mathbf{\Lambda}$-cluster is legal. $\square$

Remark 3.11 Note that, in order to check that every $\mathbf{\Lambda}$-cluster is legal, we only need to see if some cluster that contains a finite generating multiset for $\mathbf{\Lambda}$ is legal.

Example 3.12 (A substitution Delone multiset in $\mathbb{R}^2$ with gasket tiles.)
Consider the substitution on $\mathbb{R}^2$ with the following MFS Φ;

$$\Phi = \begin{pmatrix} \{f_1, f_4\} & \{f_1\} & \{f_1\} & \emptyset \\ \{f_2\} & \{f_2, f_3\} & \emptyset & \{f_2\} \\ \{f_3\} & \emptyset & \{f_2, f_3\} & \{f_3\} \\ \emptyset & \{f_4\} & \{f_4\} & \{f_1, f_4\} \end{pmatrix},$$

where $f_1(x) = 2x$, $f_2(x) = 2x + (1, 0)$, $f_3(x) = 2x + (0, 1)$, and $f_4(x) = 2x + (-1, -1)$.

The Delone multiset $\mathbf{\Lambda} = (\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4)$ generated from $(\{(0,0),(1,1)\}, \{(0,-1)\}, \{(-1,0)\}, \emptyset)$ is fixed under Φ. We observe that $\bigcup_{i \le 4} \Lambda_i = \mathbb{Z}^2$. The generating multiset $(\{(0,0),(1,1)\}, \{(0,-1)\}, \{(-1,0)\}, \emptyset)$ is legal, indicating representability. The solution from the adjoint system consists of four copies of the gasket tile(T) [47]. Thus $\{x + T : x \in \Lambda_i, i \le 4\}$ is a tiling of $\mathbb{R}^2$ by gaskets pinned down on the standard lattice $\mathbb{Z}^2$, see Fig. 3.1. Note that the tiles are triangular gaskets in 4 colours, indicated by 3 shades of grey and black. The largest solid triangle of one colour (e.g. black) are caused by the meshing of two gaskets of the same colour.

3.3 UPF on substitutions

In this section we show that if $\mathcal{T}$ is a fixed point of a primitive substitution, then $\mathcal{T}$ has *uniform patch frequencies* (UPF), the analog of UCF, see Def. 2.6. This is a bit

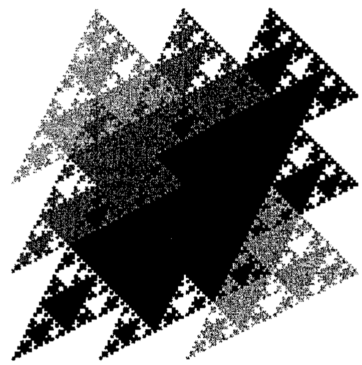

Figure 3.1: Gasket Tiling

more general than [43, Theorem 3.1] where it was also assumed that $\mathcal{T}$ is repetitive. We present complete details here, in part because the proof of UPF was omitted in [43], but we should note that in a similar but slightly different setting the existence of UPF was established in [14, Prop. 1] (see also [14, p.182] for references to earlier results of this kind).

Throughout this section, we fix $\mathcal{T}$ — a tiling satisfying $\omega(\mathcal{T}) = \mathcal{T}$ for a primitive tile-substitution ω.

Lemma 3.13 (see, e.g., [34, Prop. 1.1]) *Every prototile in a primitive tile-substitution has the boundary of zero Lebesgue measure.*

Lemma 3.14 *Let $\mathcal{T}$ be a fixed point of a substitution with expansive map Q. Then for any tile $T \in \mathcal{T}$ with $T = (A, i)$ for some $i \le m$, $\{Q^n A\}_{n \ge 1}$ is a van Hove sequence.*

Proof. This is pretty straightforward from Lemma 3.13. □

The following is proved in [43]. Note that it does not require repetitivity. Below PF is an abbreviation for "Perron-Frobenius."

Corollary 3.15 *Let ω be a primitive tile-substitution with prototiles $T_i = (A_i, i)$, for $i \le m$, and expansive map Q. Then the PF eigenvalue of the substitution matrix S is $|\det(Q)|$ and the vector $(\mathrm{Vol}(A_i))_{i \le m}$ is a left PF eigenvector. Thus,*

$$\lim_{n \to \infty} |\det(Q)|^{-n} (S^n)_{ij} = r_i \mathrm{Vol}(A_j),$$

where $(r_i)_{i \le m}$ is the right PF eigenvector of S such that $\sum_{i=1}^{m} r_i \mathrm{Vol}(A_i) = 1$.

Notation. For a patch P and a bounded set $F \subset \mathbb{R}^d$ denote

$$L_P(F) = \sharp\{g \in \mathbb{R}^d : \ g + P \subset \mathcal{T}, \ (g + \mathrm{supp}(P)) \subset F\}$$

and

$$N_P(F) = \sharp\{g \in \mathbb{R}^d : \ g + P \subset \mathcal{T}, \ (g + \mathrm{supp}(P)) \cap F \neq \emptyset\}.$$

Let $V_{\min}$ and $V_{\max}$ be the minimal and maximal volumes of $\mathcal{T}$-tiles respectively, let $\|Q\|$ be the operator norm, and let $t_{\max}$ be the maximal diameter of $\mathcal{T}$-tiles.

Inflated Tilings. Given a tiling $\mathcal{T}$ and an expansive map Q' on $\mathbb{R}^d$ we let

$$Q'\mathcal{T} = \{(Q'(\mathrm{supp}(T)), l(T)) : \ T \in \mathcal{T}\}.$$

In other words, we blow up the tiles and retain their labels. Usually we will take $Q' = Q^k$. If $\mathcal{T}$ is a fixed point of ω, then the tilings $Q^k\mathcal{T}$, $k = 1, 2, \ldots$, form an hierarchical family of order k *super-tilings* in the sense that every tile of a higher-order tiling can be decomposed into tiles of a lower order tiling.

Lemma 3.16 *Let $\mathcal{T}$ be a fixed point of a primitive substitution. Let $F \subset \mathbb{R}^d$ be an arbitrary bounded set and let $\{F_n\}_{n\geq 1}$ be a van Hove sequence in $\mathbb{R}^d$. Then for any $\mathcal{T}$-patch P and any $h \in \mathbb{R}^d$ we have*

(i) *$L_P(F) \leq c_1 \mathrm{Vol}(F)$, where c_1 depends only on $\mathcal{T}$;*

(ii) *If P is a legal patch, then there are $c_2, n_0 > 0$ (depending on P and $\{F_n\}_{n\geq 1}$, but not on h) so that $L_P(h + F_n) \geq c_2 \mathrm{Vol}(F_n)$, for all $n \geq n_0$;*

(iii) *If P is a legal patch, then $\lim_{n\to\infty} N_P(\partial(h+F_n))/L_P(h+F_n) = 0$ uniformly in h.*

Proof. (i) Select any tile from the patch P. Then distinct $\mathcal{T}$-patches equivalent to P will have distinct selected tiles. Therefore,

$$\mathrm{Vol}(F) \geq L_P(F) V_{\min},$$

so we can take $c_1 = V_{\min}^{-1}$.

(ii) If P is a legal patch, then its translate occurs in some patch $\omega^k(T_i)$. Let $\ell \in \mathbb{N}$ be such that $\omega^\ell(T_j)$ contains tiles of all types, for all $j \leq m$. (This exists by the primitivity of the substitution.) Then every patch $\omega^{k+\ell}(T)$, $T \in \mathcal{T}$, contains a translate of P. We consider the super-tiling $Q^{k+\ell}\mathcal{T}$. It follows that for any set F, $L_P(F)$ is at least the number of $Q^{k+\ell}\mathcal{T}$-tiles whose supports are contained in F. Therefore, for $r = \|Q\|^{k+\ell} \cdot t_{\max}$,

$$\begin{aligned} & L_P(h + F_n) \cdot |\det(Q)|^{k+\ell} V_{\max} \\ \geq\ & \mathrm{Vol}(h + F_n^{-r}) = \mathrm{Vol}(F_n^{-r}) \geq \mathrm{Vol}(F_n) - \mathrm{Vol}((\partial F_n)^{+r}). \end{aligned}$$

This implies the desired statement in view of (2.1).

(iii) Let $t = \text{diam}(P)$. Then

$$\frac{N_P(h + \partial F_n)}{L_P(h + F_n)} \leq \frac{L_P(h + \partial F_n{}^{+t})}{L_P(h + F_n)} \leq \frac{c_1 \text{Vol}(h + \partial F_n{}^{+t})}{c_2 \text{Vol}(h + F_n)} \to 0, \text{ from (i) and (ii).}$$

□

Proposition 3.17 *Let $\mathcal{T}$ be a fixed point of a primitive substitution with expansive map Q. Let P be a $\mathcal{T}$-patch. Then*

$$c_P := \lim_{n\to\infty} \frac{L_P(Q^n A)}{\text{Vol}(Q^n A)}$$

exists uniformly in A, a support of a $\mathcal{T}$-tile.

Proof. If P is non-legal, then $L_P(Q^n A) = 0$ for every n and every tile support A, so $c_P = 0$.

Assume now P is a legal patch. Let $\{T_1, \dots, T_m\}$ be representatives of all $\mathcal{T}$-tile types, having supports A_i, $i \leq m$. Fix $\varepsilon > 0$. By Lemma 3.16(iii) and Lemma 3.14, we can find $k_0 \in \mathbb{N}$ so that for any $k \geq k_0$ and any tile support A on $\mathcal{T}$,

$$N_P(\partial Q^k A) \leq \varepsilon L_P(Q^k A)\,. \tag{3.15}$$

Choose a tile $T \in \mathcal{T}$. Then $T = (A, j)$ for some $j \leq m$, where $A = \text{supp}(T)$. Consider the subdivision of $Q^n A = \text{supp}(Q^n T)$, $n > k > k_0$, into the tiles of $Q^k \mathcal{T}$. By the definition of the substitution matrix S, there are $(S^{n-k})_{ij}$ tiles equivalent to $Q^k T_i$ in the subdivision of $Q^n T$. Therefore, in view of (3.15),

$$\sum_{i=1}^{m} L_P(Q^k A_i)(S^{n-k})_{ij} \leq L_P(Q^n A) \leq (1+\varepsilon) \sum_{i=1}^{m} L_P(Q^k A_i)(S^{n-k})_{ij}. \tag{3.16}$$

By Cor. 3.15,

$$\lim_{n\to\infty} \frac{(S^{n-k})_{il}}{\text{Vol}(Q^n A_l)} = r_i |\det(Q)|^{-k}, \quad \text{for any} \quad l \leq m.$$

Thus, dividing (3.16) by $\text{Vol}(Q^n A)$ and letting $n \to \infty$ we obtain

$$\limsup_{n\to\infty} \frac{L_P(Q^n A)}{\text{Vol}(Q^n A)} - \liminf_{n\to\infty} \frac{L_P(Q^n A)}{\text{Vol}(Q^n A)} \leq \varepsilon |\det(Q)|^{-k} \sum_{i=1}^{m} r_i L_P(Q^k A_i).$$

By Lemma 3.16(i), the right-hand side does not exceed $c_1\varepsilon\sum_{i=1}^m r_i\mathrm{Vol}(A_i) = c_1\varepsilon$. Since $\varepsilon > 0$ is arbitrary, this proves the existence of the limit. Moreover,

$$\sum_{i=1}^{m} L_P(Q^k A_i) r_i |\det(Q)|^{-k} \le c_P \le (1+\varepsilon)\sum_{i=1}^{m} L_P(Q^k A_i) r_i |\det(Q)|^{-k}$$

and it does not depend on the choice of the tile T on $\mathcal{T}$. Since $L_P(Q^k A_i) > 0$ for k sufficiently large, we have $c_P > 0$. □

Theorem 3.18 *Let $\mathcal{T}$ be a fixed point of a primitive substitution. For any $\mathcal{T}$-patch P and for any van Hove sequence $\{F_n\}_{n\ge1}$, there exists*

$$\mathrm{freq}(P,\mathcal{T}) := \lim_{n\to\infty} \frac{L_P(h+F_n)}{\mathrm{Vol}(F_n)} = c_P, \tag{3.17}$$

uniformly in $h \in \mathbb{R}^d$.

Proof. Consider the decomposition of the space $\mathbb{R}^d$ into the tiles of $Q^k\mathcal{T}$ for k large. Then $L_P(h+F_n)$ is roughly the sum of $L_P(Q^kA)$ where Q^kA ranges over the supports of those $Q^k\mathcal{T}$-tiles which intersect $h+F_n$. For large n the "boundary effects" from ∂F_n become small by the definition of van Hove sequence. Note also that the "boundary effects" from ∂Q^kA become small. This is the idea; now let us give the details.

Let $G_{k,n} = \{A : Q^kA \cap (h+F_n) \ne \emptyset,\ A = \mathrm{supp}(T) \text{ for } T \in \mathcal{T}\}$ and $H_{k,n} = \{A : Q^kA \subset (h+F_n),\ A = \mathrm{supp}(T) \text{ for } T \in \mathcal{T}\}$ for $k, n \ge 1$. We have

$$\sum_{A\in H_{k,n}} L_P(Q^kA) \le L_P(h+F_n) \le \sum_{A\in G_{k,n}} [L_P(Q^kA) + N_P(\partial(Q^kA))]. \tag{3.18}$$

Fix $\epsilon > 0$. Using Prop. 3.17 and Lemma 3.16(iii) choose k so that for any tile support A,

$$|L_P(Q^kA)/\mathrm{Vol}(Q^kA) - c_P| < \epsilon \quad\text{and}\quad N_P(\partial(Q^kA)) < \epsilon L_P(Q^kA). \tag{3.19}$$

Combining (3.18) and (3.19) we obtain

$$(c_P - \epsilon)\sum_{A\in H_{k,n}} \mathrm{Vol}(Q^kA) \le L_P(h+F_n) \le (1+\epsilon)(c_P+\epsilon)\sum_{A\in G_{k,n}} \mathrm{Vol}(Q^kA).$$

Let $t_k := \max\{\mathrm{diam}(Q^kA) : A = \mathrm{supp}(T), T \in \mathcal{T}\}$. Observe that

$$\sum_{A\in H_{k,n}} \mathrm{Vol}(Q^kA) \ge \mathrm{Vol}(F_n^{-t_k}) \quad\text{and}\quad \sum_{A\in G_{k,n}} \mathrm{Vol}(Q^kA) \le \mathrm{Vol}(F_n^{+t_k}).$$

Since k is fixed, by the van Hove property we have for n sufficiently large:

$$\mathrm{Vol}(F_n^{-t_k}) \geq (1-\epsilon)\mathrm{Vol}(F_n) \quad \text{and} \quad \mathrm{Vol}(F_n^{+t_k}) \leq (1+\epsilon)\mathrm{Vol}(F_n).$$

Combining everything, we obtain for n sufficiently large:

$$(c_P - \epsilon)(1-\epsilon) \leq \frac{L_P(h+F_n)}{\mathrm{Vol}(F_n)} \leq (c_P+\epsilon)(1+\epsilon)^2.$$

Since ϵ was arbitrary, this implies (3.17) as desired. □

3.4 Multitilings

Let $\mathbf{\Lambda} = (\Lambda_i)_{i \leq m}$ be a primitive substitution Delone multiset. We have seen in the previous section that there exists a unique solution $\mathcal{A} = \{T_1, \ldots, T_m\}$ of the adjoint system of equations to $\mathbf{\Lambda}$ consisting of a set of compact subsets $A_i = \mathrm{supp}(T_i)$ of $\mathbb{R}^d$. In this section we define a multitiling and show that a repetitive primitive substitution Delone multiset can be representable for a multitiling. From the representability for a multitiling we are able to prove that a repetitive primitive substitution Delone multiset has UCF.

Definition 3.19 A *q-multitiling* of $\mathbb{R}^d$ is a set $\mathcal{T}$ of tiles for which every point of $\mathbb{R}^d$ is covered by at least q tiles of $\mathcal{T}$ and the interior of each tile meets with the interiors of q tiles of $\mathcal{T}$ including itself.

We say that a set Π of tiles is a *packing* of $\mathbb{R}^d$ if the tiles of Π have mutually disjoint interiors.

Definition 3.20 A Delone multiset $\mathbf{\Lambda}$ is *representable for a q-multitiling* if $\mathbf{\Lambda} + \mathcal{A} = \{x + T_i : \ x \in \Lambda_i, \ i \leq m\}$ is a q-multitiling for some prototile set $\mathcal{A} = \{T_1, \ldots, T_m\}$.

We say that $\mathbf{\Lambda}$ is *irreducible* under Φ if $\Phi(\mathbf{\Lambda}) = \mathbf{\Lambda}$ and it cannot be partitioned as the disjoint union of two nonempty multiset $\mathbf{\Lambda}^1$ and $\mathbf{\Lambda}^2$ in $\mathbb{R}^d$ for which $\Phi(\mathbf{\Lambda}^i) = \mathbf{\Lambda}^i, i = 1, 2$.

Theorem 3.21 *If $\mathbf{\Lambda}$ is a repetitive primitive substitution Delone multiset, then $\mathbf{\Lambda}$ is representable for q-multitiling with some $q \in \mathbb{Z}_+$.*

Proof. Let Φ be the MFS such that $\Phi(\mathbf{\Lambda}) = \mathbf{\Lambda}$. By [20, Th 2.1], we can uniquely partition $\mathbf{\Lambda}$ into a finite number of irreducible discrete multisets each of which is fixed by Φ. Let $\mathbf{\Lambda} = \mathbf{\Lambda}^1 \cup \cdots \cup \mathbf{\Lambda}^k$, where $\mathbf{\Lambda}^s$ is irreducible under Φ for $1 \le s \le k$. By [20, Th 3.3], each $\mathbf{\Lambda}^s$ contains exactly one directed generating cycle $Y = (x_{s,1}, \ldots, x_{s,p_s})$ with multiplicity one, that is to say, $x_{s,t+1} \in \Phi(x_{s,t})$ for $1 \le t < p_s$, $x_{s,1} \in \Phi(x_{s,p_s})$, and $\lim_{n\to\infty} \Phi^n(\mathbf{Y}) = \mathbf{\Lambda}^s$, where $\mathbf{Y} = \{x_{s,1}, \ldots, x_{s,p_s}\}$. Note that $x_{s,t} \in \Phi^{p_s}(x_{s,t})$, for all $1 \le s \le k$ and $1 \le t \le p_s$. We replace Φ by $\Phi^{p_1 \cdots p_k}$ (Q by $Q^{p_1 \cdots p_k}$, $\mathcal{D}$ by $\mathcal{D}^{p_1 \cdots p_k}$, and ω by $\omega^{p_1 \cdots p_k}$). This replacement of Φ remains throughout the proof. Now we define

$$\mathbf{\Omega}_{s,t} := \lim_{n\to\infty} \Phi^n(x_{s,t}), \quad \text{for } 1 \le s \le k \text{ and } 1 \le t \le p_s.$$

We can write $\mathbf{\Omega}_{s,t} = ((\Omega_{s,t})_i)_{i \le m}$, with $(\Omega_{s,t})_i \subset \Lambda_i$. By the definition of substitution Delone multiset, no element in $\mathbf{\Lambda}$ can have two different preimages. So $\mathbf{\Omega}_{s,t}$, $1 \le s \le k$, $1 \le t \le p_s$, are all disjoint as multisets of which colours are also a factor for distinguishing points in $\mathbb{R}^d$ and

$$\mathbf{\Lambda} = \cup_{s=1}^{k} \cup_{t=1}^{p_s} \mathbf{\Omega}_{s,t}.$$

Let

$$\Pi_{s,t} := \{x_i + T_i : x_i \in (\Omega_{s,t})_i, i \le m\}.$$

Then $\mathbf{\Lambda} + \mathcal{A} = \cup_{s=1}^{k} \cup_{t=1}^{p_s} \Pi_{s,t}$, where all the sets in the union on the right-hand side are distinct from each other.

Firstly, we claim that each $\Pi_{s,t}$ is a packing of $\mathbb{R}^d$, for $1 \le s \le k$ and $1 \le t \le p_s$. In fact, the unique solution $\{A_1, \ldots, A_m\}$ of the adjoint system of equations of $\mathbf{\Lambda}$ satisfies

$$Q(A_j) = \bigcup_{i=1}^{m} (\mathcal{D}_{ij} + A_i), \quad j \le m,$$

where $A_i = \mathrm{supp}(T_i)$, $i \le m$. So for $x_j \in \Lambda_j$, $M \in \mathbb{Z}_+$, $j \le m$,

$$\begin{aligned} Q^M(x_j + A_j) &= \bigcup_{i=1}^{m} (Q^M x_j + (\mathcal{D}^M)_{ij} + A_i) \\ &= \bigcup_{i=1}^{m} (\Phi_{ij}^M(x_j) + A_i). \end{aligned} \tag{3.20}$$

By the same argument as in Theorem 3.9, the unions on the right side of (3.20) are measure-wise disjoint. Taking the limit as $M \to \infty$ in (3.20),

$$\lim_{M\to\infty} Q^M(x_j + A_j) = \lim_{M\to\infty} (\bigcup_{i=1}^{m} (\Phi_{ij}^M(x_j) + A_i)) \tag{3.21}$$

and the unions on the right side of (3.21) are measure-wise disjoint. Since

$$\boldsymbol{\Omega}_{s,t} = \lim_{n\to\infty} \Phi^n(x_{s,t}) = \lim_{n\to\infty} ((\Phi^n)_{ij}(x_{s,t}))_{i\le m}$$

with some j for which $x_{s,t} \in \Lambda_j$, $\Pi_{s,t}$ is a packing of $\mathbb{R}^d$.

Secondly, each $\Pi_{s,t}$ is invariant under ω, for $1 \le s \le k$ and $1 \le t \le p_s$. Recall that

$$\omega(T_j) = \{u + T_i : u \in \mathcal{D}_{ij}, i \le m\}$$

with

$$Q(A_j) = \bigcup_{i=1}^{m} (\mathcal{D}_{ij} + A_i) \quad \text{for } j \le m.$$

So

$$\begin{aligned}
\omega(\Pi_{s,t}) &= \bigcup \{\omega(x_j + T_j) : x_j \in (\Omega_{s,t})_j, j \le m\} \\
&= \{Qx_j + \mathcal{D}_{ij} + T_i : x_j \in (\Omega_{s,t})_j, j \le m, i \le m\} \\
&= \{\Phi_{ij}(x_j) + T_i : x_j \in (\Omega_{s,t})_j, j \le m, i \le m\} \\
&= \{x_i + T_i : x_i \in (\Omega_{s,t})_i, i \le m\} \quad \text{since } \Phi(\boldsymbol{\Omega}_{s,t}) = \boldsymbol{\Omega}_{s,t} \\
&= \Pi_{s,t}. \qquad (3.22)
\end{aligned}$$

It implies additionally that

$$\omega(\boldsymbol{\Lambda} + \mathcal{A}) = \boldsymbol{\Lambda} + \mathcal{A}. \qquad (3.23)$$

Let

$$\mathrm{supp}(\Pi_{s,t}) = \bigcup \{x_i + A_i : x_i \in (\Omega_{s,t})_i, A_i = \mathrm{supp}(T_i), i \le m\}.$$

Observe also here that

$$Q^N(\mathrm{supp}(\Pi_{s,t})) = \mathrm{supp}(\omega^N(\Pi_{s,t})) = \mathrm{supp}(\Pi_{s,t}) \quad \text{for any } N \in \mathbb{Z}_+. \qquad (3.24)$$

Finally, we now prove that $\boldsymbol{\Lambda} + \mathcal{A}$ is a q-multitiling for some $q \ge 1$. Note first that since $\boldsymbol{\Lambda}$ is a Delone, $\boldsymbol{\Lambda} + \mathcal{A}$ covers $\mathbb{R}^d$ by the same argument as in Theorem 3.9. Next we prove that every point of $\mathbb{R}^d$ is covered by at least q tiles of $\overline{\mathcal{T}}$ and the interior of each tile in $\overline{\mathcal{T}}$ meets with the interiors of q tiles of $\overline{\mathcal{T}}$ including itself for some $q \in \mathbb{Z}_+$. For

$$x, y \in \mathbb{R}^d \backslash (\bigcup_{s=1}^{k} \bigcup_{t=1}^{p_s} (\bigcup_{i=1}^{m} ((\Omega_{s,t})_i + \partial A_i)))$$

we will show that x and y intersect with the same number of the interiors of tiles in $\Lambda + \mathcal{A}$. Suppose that x is covered by q tiles and y is by l tiles. Then we can find the covering tiles $R^1, \ldots, R^q$ of x and $S^1, \ldots, S^l$ of y in $\Lambda + \mathcal{A}$. For each tile T in $\Lambda + \mathcal{A}$ there exists a unique $\Pi \in \{\Pi_{s,t} : 1 \le s \le k, 1 \le t \le p_s\}$ which contains the tile T, since all $\Pi_{s,t}$'s are disjoint. So we can find $\{\Pi(x,1), \ldots, \Pi(x,q)\}$ for which $R^i \in \Pi(x,i) \in \{\Pi_{s,t} : 1 \le s \le k, 1 \le t \le p_s\}$, $1 \le i \le q$. Since $x \in \bigcap_{i=1}^q (\mathrm{supp}(R^i))^\circ$, there is an open ball $B_\varepsilon(x)$ around x with radius $\varepsilon > 0$ such that

$$B_\varepsilon(x) \subset \bigcap_{i=1}^{q} (\mathrm{supp}(R^i))^\circ \subset \bigcap_{i=1}^{q} \mathrm{supp}(\Pi(x,i)).$$

So

$$Q^n(B_\varepsilon(x)) \subset \bigcap_{i=1}^{q} \mathrm{supp}(\Pi(x,i)) \quad \text{for any } n \in \mathbb{Z}_+.$$

Since Λ is repetitive, $E := \{T \in \Lambda + \mathcal{A} : \mathrm{supp}(T) \subset Q^M(B_\varepsilon(x))\}$ with some $M \in \mathbb{Z}_+$ contains $\{t+S^1, \ldots, t+S^l\}$ of which interiors contain $t+y$ for some $t \in \mathbb{R}^d$. However $t + y \in Q^M(B_\varepsilon(x))$ should be covered by the interiors of q tiles, since $B_\varepsilon(x)$ was covered by q tiles, by (3.24). Thus $q = l$. Then for

$$x \in \bigcup_{s=1}^{k} \bigcup_{t=1}^{p_s} (\bigcup_{i=1}^{m} ((\Omega_{s,t})_i + \partial A_i))),$$

there are at least q tiles which contain x. Therefore $\Lambda + \mathcal{A}$ is a q-multitiling for some $q \in \mathbb{Z}_+$.

Example 3.22 (A substitution Delone multiset representable for a 2-multitiling.) Consider the MFS Φ

$$\Phi = \begin{pmatrix} \{5x, 5x+2, 5x+4\} & \{5x+1, 5x+3\} \\ \{5x+1, 5x+3\} & \{5x, 5x+2, 5x+4\} \end{pmatrix}$$

which generates the bi-infinite sequence shown below with the a and b point sets starting from the generating set $(\{0\}, \{-\frac{1}{2}, -1\})$. This leads to the Delone multiset $\Lambda = (\Lambda_a, \Lambda_b)$ which is fixed under Φ.

$\cdots$	-3	$-\frac{5}{2}$	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3	$\frac{7}{2}$	4	$\cdots$
$\cdots$	b	b	a	a	b	b	a	a	b	b	a	a	b	b	a	$\cdots$

Note that the generating set $(\{0\}, \{-\frac{1}{2}, -1\})$ is not legal, even though Λ is periodic and so repetitive. Thus Λ is not representable for a tiling by tiles arising from the solution to the adjoint system. In fact the solution to the adjoint system is $\mathcal{A} = \{[0,1],[0,1]\}$ so that $\mathbf{\Lambda} + \mathcal{A}$ double tiles the line.

Proposition 3.23 *If $\mathbf{\Lambda} + \mathcal{A}$ is a q-multitiling which is repetitive and a fixed point of a primitive substitution ω, then $\mathbf{\Lambda} + \mathcal{A}$ can be associated with a tiling $\mathcal{T}'$ which is repetitive and a fixed point of a primitive substitution ω'.*

Proof. Now we define new tiles from the overlaps in $\mathbf{\Lambda} + \mathcal{A}$ as follows;
Let $T^1, \ldots, T^q$ be overlapping tiles in $\mathbf{\Lambda} + \mathcal{A}$ for which $\bigcap_{l=1}^{q} (A^l)^\circ \neq \emptyset$, where $A^l = \mathrm{supp}(T^l)$ for $1 \leq l \leq q$. Define $B := \overline{\bigcap_{l=1}^{q} (A^l)^\circ}$. Notice

$$B^\circ \; \subset \; (\bigcap_{l=1}^{q} \overline{(A^l)^\circ})^\circ = (\bigcap_{l=1}^{q} A^l)^\circ = \bigcap_{l=1}^{q} (A^l)^\circ \subset \; B^\circ.$$

Thus $B = \overline{B^\circ}$. Note here that

$$\mu(\partial(\bigcap_{l=1}^{q} A^l)) = 0. \tag{3.25}$$

We define the colour of B depending on the cluster of all the points of which corresponding tile interiors contribute to form B. Then we can define new tile $S := (B, \text{the colour of } B)$ in $\mathbb{R}^d$.

We claim that these new tiles form a tiling of $\mathbb{R}^d$, which means that all the new tiles have mutually disjoint interiors and cover $\mathbb{R}^d$. In fact, for any tile T with $A = \mathrm{supp}(T)$ in $\mathbf{\Lambda} + \mathcal{A}$, let $\{B^1, \ldots, B^e\}$ be the set of supports of all new tiles which are contained in A. This set is at most finite by FLC. Then $A^\circ \subset B^1 \cup \cdots \cup B^e$. Otherwise there is support of other new tile which is contained in A, since $A^\circ \backslash (B^1 \cup \cdots \cup B^e)$ is an open set, but this contradicts the choice of $\{B^1, \ldots, B^e\}$. Thus

$$A = \overline{A^\circ} = B^1 \cup \cdots \cup B^e. \tag{3.26}$$

Since $\mathbf{\Lambda} + \mathcal{A}$ covers $\mathbb{R}^d$, all the new tiles cover $\mathbb{R}^d$ also. The disjointness of the new tile interiors comes naturally from the definition of the new tiles. So we have a new tiling and denote it by $\mathcal{T}'$.

Since $\mathbf{\Lambda}$ has FLC, there are only finite new prototiles for $\mathcal{T}'$. Let $\{S_1, \ldots, S_n\}$ be the set of all the new prototiles. From $B^\circ = \bigcap_{l=1}^{q} (A^l)^\circ$,

$$(Q(B))^\circ = Q(B^\circ) = \bigcap_{l=1}^{q} Q((A^l)^\circ) = (\bigcap_{l=1}^{q} Q(A^l))^\circ,$$

since Q is a homeomorphism. Each $Q(A^l)$ is subdivided by the supports of tiles in $\Lambda + \mathcal{A}$. So $\bigcap_{l=1}^{q} Q(A^l)$ can be subdivided by new tile supports in $\mathcal{T}'$, in other words,

$$\bigcap_{l=1}^{q} Q(A^l) = E^1 \cup \cdots \cup E^J,$$

where $E^i = \bigcap_{l=1}^{q} (A^{i,l}) \neq \emptyset$ and $(E^i)^\circ$'s are disjoint for $1 \leq i \leq J$. Claim that

$$\overline{(E^1 \cup \cdots \cup E^J)^\circ} = \overline{(E^1)^\circ} \cup \cdots \cup \overline{(E^J)^\circ}.$$

It is easy to see "$\supset$" containment. Suppose $x \in \overline{(E^1 \cup \cdots \cup E^J)^\circ}$, then for any open neighbourhood V around x

$$V \cap (E^1 \cup \cdots \cup E^J)^\circ \neq \emptyset.$$

Since $\mu(\partial E^1 \cup \cdots \cup \partial E^J) = 0$ from (3.25), $V \cap ((E^1)^\circ \cup \cdots \cup (E^J)^\circ) \neq \emptyset$. So there is a converging sequence $\{x_n\}_{n=1}^{\infty}$ to x such that $\{x_n\}_{n=1}^{\infty} \subset (E^i)^\circ$ for some $1 \leq i \leq J$. This shows that $x \in \overline{(E^1)^\circ} \cup \cdots \cup \overline{(E^J)^\circ}$.

Note that each $\overline{(E^i)^\circ}$ is the support of some new tile, $1 \leq i \leq J$. Thus we can get

$$Q(B) = \overline{(Q(B))^\circ} = B^1 \cup \cdots \cup B^J,$$

where $B^i = \overline{(E^i)^\circ} = \overline{(\cap_{l=1}^{q} (A^{i,l}))^\circ}$, $1 \leq i \leq J$. Therefore $Q(B)$ can be subdivided by the supports of new tiles in $\mathcal{T}'$.

We can define a new substitution ω' with the new prototiles and the subdividing shown above. Since $\omega(\Lambda + \mathcal{A}) = \Lambda + \mathcal{A}$,

$$\omega'(\mathcal{T}') = \mathcal{T}'.$$

Since $\Lambda + \mathcal{A}$ is repetitive, there is $M \in \mathbb{Z}_+$ such that ${\omega'}^M(S)$ contains all types of new tiles for any $S \in \mathcal{T}'$. Thus the substitution ω' is primitive. Therefore the tiling $\mathcal{T}'$ which $\Lambda + \mathcal{A}$ is associated with is repetitive and a fixed point of a primitive substitution ω'.

Corollary 3.24 *If Λ is a repetitive primitive substitution Delone multiset with FLC, then Λ has UCF.*

PROOF. From Prop. 3.21 and 3.23, we know that there is a fixed point $\mathcal{T}'$ of a primitive substitution which $\Lambda + \mathcal{A}$ is associated with. Note that by Theorem 3.18, $\mathcal{T}'$ has uniform patch frequency. We need to prove that Λ has UCF. Let $\mathbf{P} = (P_i)_{i \leq m}$

be a cluster of $\mathbf{\Lambda}$ and let $x \in \mathbb{R}^d$ be any point so that $x + \mathbf{P} \subset \mathbf{\Lambda}$. Then there exists a unique patch R in $\mathcal{T}'$ for which $\text{supp}(R) = \text{supp}((x + \mathbf{P}) + \mathcal{A})$ from (3.26), where $(x + \mathbf{P}) + \mathcal{A} = \{(x + P_i) + T_i : i \leq m\}$. Note that $\mathcal{T}'$ has FLC, since $\mathbf{\Lambda}$ has FLC. So there are only finite types of patches in $\mathcal{T}'$, say $\{R_1, \ldots, R_r\}$, such that $\text{supp}(R_i) = \text{supp}((x + \mathbf{P}) + \mathcal{A})$ for some translate patch $x + \mathbf{P} \subset \mathbf{\Lambda}$ of the patch $\mathbf{P}$. Thus

$$\text{freq}(\mathbf{P}, \mathbf{\Lambda}) = \sum_{i=1}^{r} \text{freq}(R_i, \mathcal{T}').$$

Therefore $\mathbf{\Lambda}$ has UCF.

Chapter 4

Dynamical Systems and Diffraction

4.1 Dynamical systems

Basically a dynamical system is a space X together with a group of transformations $\{T_x\}_{x\in G} : X \to X$ acting on it. The most familiar example is that of the phase space of some physical system with the action of a single parameter, time, acting on it.

In our study, the system consists of a collection of related point sets (so each point of X represents a point set in $\mathbb{R}^d$) and G is $\mathbb{R}^d$ which acts on this space by translation, so that if $\Lambda \in X$ then $T_x(\Lambda) = x + \Lambda$. It is implicit that in speaking of a dynamical system one is interested in orbits of the motion, so in our case this means subsets of X of the form $\{x + \Lambda : x \in \mathbb{R}^d\}$.

If X is a compact metric space and T_x is a continuous map for all $x \in \mathbb{R}^d$, we call $(X, \mathbb{R}^d)$ a topological dynamical system. If X is endowed with a σ-algebra $\mathcal{B}$ of subsets of X and a probability measure μ on $\mathcal{B}$, and each T_x is a μ-invariant map, we call $(X, \mathbb{R}^d)$ a measure theoretical dynamical system. We will find that our systems have both these types of structures.

Here we start with a collection of all Delone multisets in $\mathbb{R}^d$ and define a metric on it so that we get a topological space. Then the collection is a complete space with respect to this metric (see [41]). For each Delone multiset $\mathbf{\Lambda}$ of $\mathbb{R}^d$ we define its dynamical hull $X_{\mathbf{\Lambda}}$ as a closure of $\mathbb{R}^d$-orbit of $\mathbf{\Lambda}$. It is known that if $\mathbf{\Lambda}$ has finite local complexity (FLC) then the metric space $X_{\mathbf{\Lambda}}$ is compact. So from the action of $\mathbb{R}^d$ on $X_{\mathbf{\Lambda}}$ by translates we get a topological dynamical system $(X_{\mathbf{\Lambda}}, \mathbb{R}^d)$.

In SubSec. 4.1.1, we define cylinder sets, which are sets of elements in $X_{\mathbf{\Lambda}}$ determined by clusters in $\mathbf{\Lambda}$ along with translations from Borel sets in $\mathbb{R}^d$. We show that $X_{\mathbf{\Lambda}}$ can be decomposed into a finite number of cylinder sets with any given size clusters and Borel sets. We will make use of this property especially in proving Theorem 5.1 when we approximate continuous functions in $X_{\mathbf{\Lambda}}$ by step functions.

When a topological dynamical system has a unique invariant probability measure, we say that the system is uniquely ergodic. Ergodic measures play an important role in the theory of dynamical systems such as the Birkhoff ergodic theorem. Especially the existence of a unique ergodic measure of a system provides a good deal of information about the system as we will see in Chapter 5. We will show in SubSec. 4.1.2 that in our situation the corresponding equivalent geometrical property of unique ergodicity is uniform cluster frequencies. As a consequence we get an equation between frequencies of clusters and measure of cylinders.

With any measure theoretical dynamical system one has a corresponding unitary representation of the group on $L^2(X_{\mathbf{\Lambda}}, \mu)$. The resulting spectral theory turns out to be fundamentally connected with the problem of diffraction. We introduce these concepts here in preparation for the next chapter which deals with the connection between dynamical and diffraction spectra.

4.1.1 Dynamical systems with local topology

Let $\mathbf{\Lambda}$ be a Delone multiset and let X be the collection of all Delone multisets. We introduce a metric on Delone multisets in a simple variation of the standard way: for Delone multisets $\mathbf{\Lambda}_1$, $\mathbf{\Lambda}_2 \in X$,

$$\varrho(\mathbf{\Lambda}_1, \mathbf{\Lambda}_2) := \min\{\tilde{\varrho}(\mathbf{\Lambda}_1, \mathbf{\Lambda}_2), 2^{-1/2}\}, \tag{4.1}$$

where

$$\begin{aligned}\tilde{\varrho}(\mathbf{\Lambda}_1, \mathbf{\Lambda}_2) \;=\; & \inf\{\varepsilon > 0 : \exists\, x, y \in B_\varepsilon(0),\\ & \qquad B_{1/\varepsilon}(0) \cap (-x + \mathbf{\Lambda}_1) = B_{1/\varepsilon}(0) \cap (-y + \mathbf{\Lambda}_2)\}\,.\end{aligned}$$

Let us indicate why this is a metric. Clearly, the only issue is the triangle inequality. Suppose that $\varrho(\mathbf{\Lambda}_1, \mathbf{\Lambda}_2) \le \varepsilon_1$, $\varrho(\mathbf{\Lambda}_2, \mathbf{\Lambda}_3) \le \varepsilon_2$; we want to show that $\varrho(\mathbf{\Lambda}_1, \mathbf{\Lambda}_3) \le \varepsilon_1 + \varepsilon_2$. We can assume that $\varepsilon_1, \varepsilon_2 < 2^{-1/2}$, otherwise the claim is obvious. Then

$$(-x_1 + \mathbf{\Lambda}_1) \cap B_{1/\varepsilon_1}(0) = (-x_2 + \mathbf{\Lambda}_2) \cap B_{1/\varepsilon_1}(0) \;\text{ for some } x_1, x_2 \in B_{\varepsilon_1}(0),$$

$$(-x_2' + \mathbf{\Lambda}_2) \cap B_{1/\varepsilon_2}(0) = (-x_3' + \mathbf{\Lambda}_3) \cap B_{1/\varepsilon_2}(0) \;\text{ for some } x_2', x_3' \in B_{\varepsilon_2}(0).$$

It follows that

$$(-x_1 - x_2' + \Lambda_1) \cap B_{1/\varepsilon_1}(-x_2') = (-x_2 - x_2' + \Lambda_2) \cap B_{1/\varepsilon_1}(-x_2').$$

Since $B_{1/\varepsilon_1}(-x_2') \supset B_{(1/\varepsilon_1)-\varepsilon_2}(0)$, this implies

$$(-x_1 - x_2' + \Lambda_1) \cap B_{(1/\varepsilon_1)-\varepsilon_2}(0) = (-x_2 - x_2' + \Lambda_2) \cap B_{(1/\varepsilon_1)-\varepsilon_2}(0). \tag{4.2}$$

Similarly,

$$(-x_2 - x_2' + \Lambda_2) \cap B_{(1/\varepsilon_2)-\varepsilon_1}(0) = (-x_2 - x_3' + \Lambda_3) \cap B_{(1/\varepsilon_2)-\varepsilon_1}(0). \tag{4.3}$$

A simple computation shows that $\frac{1}{\varepsilon_1} - \varepsilon_2 \geq \frac{1}{\varepsilon_1+\varepsilon_2}$ and $\frac{1}{\varepsilon_2} - \varepsilon_1 \geq \frac{1}{\varepsilon_1+\varepsilon_2}$ when $\varepsilon_1, \varepsilon_2 < 2^{-1/2}$, so by (4.2) and (4.3),

$$(-x_1 - x_2' + \Lambda_1) \cap B_{1/(\varepsilon_1+\varepsilon_2)}(0) = (-x_2 - x_3' + \Lambda_3) \cap B_{1/(\varepsilon_1+\varepsilon_2)}(0),$$

hence $\varrho(\Lambda_1, \Lambda_3) \leq \varepsilon_1 + \varepsilon_2$.

We define $X_\Lambda := \overline{\{-h + \Lambda : h \in \mathbb{R}^d\}}$ with the metric ϱ. In spite of the special role played by 0 in the definition of ϱ, any other point of $\mathbb{R}^d$ may be used as a reference point, leading to an equivalent metric and more importantly the same topology on X_Λ. The following lemma is standard.

Theorem 4.1 ([36], [41]) *If a Delone multiset Λ has FLC, then the metric space X_Λ is compact.*

The group $\mathbb{R}^d$ acts on X_Λ by translations which are obviously homeomorphisms, and we get a topological dynamical system $(X_\Lambda, \mathbb{R}^d)$. The dynamical system is *minimal* if the orbit of every element of X_Λ is dense in X_Λ.

Theorem 4.2 ([13]) *Let Λ is a Delone multiset with FLC. The minimality of dynamical system $(X_\Lambda, \mathbb{R}^d)$ is equivalent to the repetitivity of Λ.*

Definition 4.3 *Let $\mathbf{P}$ be a non-empty cluster of a Delone multiset Λ or some translate of Λ, and let $V \subset \mathbb{R}^d$ be a Borel set. Define the* cylinder set *$X_{\mathbf{P},V} \subset X_\Lambda$ by*

$$X_{\mathbf{P},V} := \{\Lambda' \in X_\Lambda : -g + \mathbf{P} \subset \Lambda' \text{ for some } g \in V\}.$$

Let $\eta(\Lambda) > 0$ be chosen so that every ball of radius $\frac{\eta(\Lambda)}{2}$ contains at most one point of $\mathrm{supp}(\Lambda)$, and let $b(\Lambda) > 0$ be such that every ball of radius $\frac{b(\Lambda)}{2}$ contains at least a point in $\mathrm{supp}(\Lambda)$. These exist by the Delone set property.

The following technical result will be quite useful.

Lemma 4.4 *Let $\mathbf{\Lambda}$ be a Delone multiset with FLC. For any $R \geq \frac{b(\mathbf{\Lambda})}{2}$ and $0 < \delta < \eta(\mathbf{\Lambda})$, there exist Delone multisets $\mathbf{\Gamma}_j \in X_{\mathbf{\Lambda}}$ and Borel sets V_j with* $\mathrm{diam}(V_j) < \delta$, $\mathrm{Vol}(\partial V_j) = 0$, $1 \leq j \leq N$, *such that*

$$X_{\mathbf{\Lambda}} = \bigcup_{j=1}^{N} X_{\mathbf{P}_j, V_j}$$

is a disjoint union, where $\mathbf{P}_j = B_R(0) \cap \mathbf{\Gamma}_j$.

Proof. For any $R \geq \frac{b(\mathbf{\Lambda})}{2}$ consider the clusters $\{B_R(0) \cap \mathbf{\Gamma} : \mathbf{\Gamma} \in X_{\mathbf{\Lambda}}\}$. They are non-empty, by the definition of $b(\mathbf{\Lambda})$. By FLC, there are finitely many such clusters up to translations. This means that there exist $\mathbf{\Gamma}_1, \ldots, \mathbf{\Gamma}_K \in X_{\mathbf{\Lambda}}$ such that for any $\mathbf{\Gamma} \in X_{\mathbf{\Lambda}}$ there are *unique* $n = n(\mathbf{\Gamma}) \leq K$ and $u = u(\mathbf{\Gamma}) \in \mathbb{R}^d$ satisfying

$$B_R(0) \cap \mathbf{\Gamma} = -u + (B_R(0) \cap \mathbf{\Gamma}_n).$$

For $j = 1, \ldots, K$ let

$$W_j = \{u(\mathbf{\Gamma}) : \mathbf{\Gamma} \in X_{\mathbf{\Lambda}} \text{ such that } n(\mathbf{\Gamma}) = j\}.$$

By construction, $X_{\mathbf{\Lambda}} = \bigcup_{j=1}^{K} X_{\mathbf{P}_j, \mathrm{W}_j}$, and this is a disjoint union.

Next we show that the sets W_j are sufficiently "nice," so that they can be obtained from a finite number of closed balls using operations of complementation, intersection, and union.

Let $b = b(\mathbf{\Lambda})$ and fix j. Since every ball of radius $b/2$ contains a point in $\mathrm{supp}(\mathbf{\Lambda})$, we have that $W_j \subset B_b(0)$. Indeed, shifting a cluster of points in $B_R(0)$ by more than b would move at least one point out of $B_R(0)$. Let $\mathbf{P}_j := B_R(0) \cap \mathbf{\Gamma}_j$. The set W_j consists of vectors u such that $-u + \mathbf{P}_j$ is a $B_R(0)$-cluster for some Delone multiset in $X_{\mathbf{\Lambda}}$. Thus $u \in W_j$ if and only if the following two conditions are met. The first condition is that for each $x \in \mathrm{supp}(\mathbf{P}_j)$, we have $-u + x \in B_R(0)$. The second condition is that no points of $\mathbf{\Gamma}_j$ outside of $B_R(0)$ move inside after the translation by $-u$. Since $W_j \subset B_b(0)$, only the points in $B_{R+b}(0)$ have a chance of moving into $B_R(0)$. Thus we need to consider the $B_{R+b}(0)$ extensions of $\mathbf{P}_j$. By FLC, in the space $X_{\mathbf{\Lambda}}$ there are finitely many $B_{R+b}(0)$-clusters that extend the cluster $\mathbf{P}_j$. Denote these clusters by $\mathbf{Q}_1, \ldots, \mathbf{Q}_L$. Summarizing this discussion we obtain

$$W_j = \bigcap_{x \in \mathrm{supp}(\mathbf{P}_j)} (-B_R(0) + x) \ \cap \ \bigcup_{i \leq L} \left[\bigcap_{x \in \mathrm{supp}(\mathbf{Q}_i) \setminus B_R(0)} (-(\mathbb{R}^d \setminus B_R(0)) + x) \right].$$

This implies that W_j is a Borel set, with $\mathrm{Vol}(\partial W_j) = 0$.

It remains to partition each W_j such that $W_j = \bigcup_{k=1}^{n_j} V_{jk}$, where $\operatorname{diam}(V_{jk}) \le \delta$, $0 < \delta < \eta(\mathbf{\Lambda})$. To this end, consider, for example, a decomposition of the cube $[-b, b]^d$ into a disjoint union of (half-open and closed) grid boxes of diameter less than $\delta < \eta(\mathbf{\Lambda})$. Let $\mathcal{Q}$ denote the (finite) collection of all these grid boxes. Then

$$W_j = \bigcup_{D \in \mathcal{Q}} (W_j \cap D) = \bigcup_{k=1}^{n_j} V_{jk},$$

where V_{jk}'s are disjoint and $\operatorname{Vol}(\partial V_{jk}) = 0$. Note that the union $X_{\mathbf{P}_j, \mathrm{W}_j} = \bigcup_{k=1}^{n_j} X_{\mathbf{P}_j, \mathrm{V}_{jk}}$ is disjoint, from the definition of W_j and $\operatorname{diam}(V_{jk}) < \eta(\mathbf{\Lambda})$ for all $k \le n_j$. So the lemma is proved. □

4.1.2 Unique ergodicity and UCF

A topological dynamical system is *uniquely ergodic* if there is a unique invariant probability measure.

Theorem 4.5 *Let $\mathbf{\Lambda}$ be a Delone multiset with FLC and $\{F_n\}_{n \ge 1}$ be a van Hove sequence. The system $(X_{\mathbf{\Lambda}}, \mathbb{R}^d)$ is uniquely ergodic if and only if for all continuous functions $f : X_{\mathbf{\Lambda}} \to \mathbb{C}$ ($f \in C(X_{\mathbf{\Lambda}})$),*

$$(I_n)(\mathbf{\Gamma}, f) := \frac{1}{\operatorname{Vol}(F_n)} \int_{F_n} f(-g + \mathbf{\Gamma})\, dg \to \ \textit{const}, \quad n \to \infty, \tag{4.4}$$

uniformly in $\mathbf{\Gamma} \in X_{\mathbf{\Lambda}}$, with the constant depending on f.

We include a proof of the needed direction for the reader's convenience (both directions, see e.g. [48, Theorem 6.19],[11, (5.15)], or [35, Theorem IV.13] for the case of $\mathbb{Z}$-actions).

Proof of sufficiency in Theorem 4.5. For any invariant measure μ, exchanging the order of integration yields

$$\int_{X_{\mathbf{\Lambda}}} I_n(\mathbf{\Gamma}, f)\, d\mu(\mathbf{\Gamma}) = \int_{X_{\mathbf{\Lambda}}} f\, d\mu,$$

so by the Dominated Convergence Theorem, the constant in (4.4) is $\int_{X_{\mathbf{\Lambda}}} f\, d\mu$. If there is another invariant measure ν, then $\int_{X_{\mathbf{\Lambda}}} f\, d\mu = \int_{X_{\mathbf{\Lambda}}} f\, d\nu$ for all $f \in C(X_{\mathbf{\Lambda}})$, hence $\mu = \nu$. □

Now we prove that FLC and UCF imply unique ergodicity of the system $(X_{\mathbf{\Lambda}}, \mathbb{R}^d)$ (see e.g. [35, Cor. IV.14(a)] for the case of $\mathbb{Z}$-actions).

Theorem 4.6 *Let $\mathbf{\Lambda}$ be a Delone multiset with FLC. Then the dynamical system $(X_{\mathbf{\Lambda}}, \mathbb{R}^d)$ is uniquely ergodic if and only if $\mathbf{\Lambda}$ has UCF.*

Proof. Let $X_{\mathbf{P},\mathrm{V}}$ be a cylinder set with $\mathrm{diam}(V) \leq \eta(\mathbf{\Lambda})$ and f be the characteristic function of $X_{\mathbf{P},\mathrm{V}}$. Then we have by the definition of the cylinder set:

$$\begin{aligned} J_n(h,f) &:= \int_{F_n} f(-x-h+\mathbf{\Lambda})\,dx \\ &= \mathrm{Vol}\{x \in F_n :\ -x-h+\mathbf{\Lambda} \in X_{\mathbf{P},\mathrm{V}}\} \\ &= \mathrm{Vol}\{x \in h+F_n : -y+\mathbf{P} \subset -x+\mathbf{\Lambda} \text{ for some } y \in V\} \\ &= \mathrm{Vol}\left[\bigcup_{\nu}((h+F_n)\cap(x_\nu+V))\right] \end{aligned}$$

where x_ν are all the vectors such that $x_\nu + \mathbf{P} \subset \mathbf{\Lambda}$. Note that the distance between any two vectors x_ν is at least $\eta(\mathbf{\Lambda})$, so the sets $x_\nu + V$ are disjoint. Let

$$r = \max\{|y| :\ y \in V\} + \max\{|x| :\ x \in \mathrm{supp}(\mathbf{P})\}.$$

Then

$$\mathrm{Vol}(V) L_{\mathbf{P}}(h+F_n^{-r}) \leq J_n(h,f) \leq \mathrm{Vol}(V) L_{\mathbf{P}}(h+F_n^{+r}). \tag{4.5}$$

Note that

$$L_{\mathbf{P}}(h+F_n^{+r}) - L_{\mathbf{P}}(h+F_n^{-r}) \leq L_{\mathbf{P}}(h+\partial F_n{}^{+2r}) \leq \frac{\mathrm{Vol}(\partial F_n{}^{+2r})}{\mathrm{Vol}(B_{\frac{\eta(\mathbf{\Lambda})}{2}})}.$$

So

$$\lim_{n\to\infty}\left(\frac{J_n(h,f)}{\mathrm{Vol}(F_n)} - \frac{\mathrm{Vol}(V)\cdot L_{\mathbf{P}}(h+F_n)}{\mathrm{Vol}(F_n)}\right) = 0 \text{ uniformly in } h \in \mathbb{R}^d. \tag{4.6}$$

If $(X_{\mathbf{\Lambda}}, \mathbb{R}^d)$ is uniquely ergodic,

$$\lim_{n\to\infty} \frac{J_n(h,f')}{\mathrm{Vol}(F_n)} \text{ exists uniformly in } h \in \mathbb{R}^d$$

for continuous functions f' approximating the characteristic function f of the cylinder set. Thus for any cluster $\mathbf{P}$,

$$\lim_{n\to\infty} \frac{L_{\mathbf{P}}(h+F_n)}{\mathrm{Vol}(F_n)} \text{ exists uniformly in } h \in \mathbb{R}^d,$$

i.e. $\mathbf{\Lambda}$ has UCF.

On the other hand, we assume that $\mathbf{\Lambda}$ has UCF. By Lemma 4.4, $f \in C(X_{\mathbf{\Lambda}})$ can be approximated in the supremum norm by linear combinations of characteristic functions of cylinder sets $X_{\mathbf{P},\mathbf{V}}$. Thus, it is enough to check (4.4) for f the characteristic function of $X_{\mathbf{P},\mathbf{V}}$ with $\mathrm{diam}(V) < \eta(\mathbf{\Lambda})$. We can see in the above (4.6) that (4.4) holds for all $-h + \mathbf{\Lambda}$ uniformly in $h \in \mathbb{R}^d$ under the assumption that $\mathbf{\Lambda}$ has UCF. Then we can approximate the orbit of $\mathbf{\Gamma} \in X_{\mathbf{\Lambda}}$ on F_n by $-h_n + \mathbf{\Lambda}$ as closely as we want, since the orbit $\{-h + \mathbf{\Lambda} : h \in \mathbb{R}^d\}$ is dense in $X_{\mathbf{\Lambda}}$ by the definition of $X_{\mathbf{\Lambda}}$. So we compute all those integrals (4.4) of $-h_n + \mathbf{\Lambda}$ over F_n and use the fact that independent of h_n they are going to a constant. Since each of these is uniformly close to $(I_n)(\mathbf{\Gamma}, f)$ in (4.4), we get that $(I_n)(\mathbf{\Gamma}, f)$ too goes to a constant. Therefore $(X_{\mathbf{\Lambda}}, \mathbb{R}^d)$ is uniquely ergodic. $\square$

Denote by μ the unique invariant probability measure on $X_{\mathbf{\Lambda}}$. As already mentioned, the constant in (4.4) must be $\int_{X_{\mathbf{\Lambda}}} f \, d\mu$. Thus, the proof of unique ergodicity yields the following result.

Corollary 4.7 *Let $\mathbf{\Lambda}$ be a Delone multiset with FLC and UCF. Then for any $\mathbf{\Lambda}$-cluster $\mathbf{P}$ and any Borel set V with* $\mathrm{diam}(V) < \eta(\mathbf{\Lambda})$, *we have*

$$\mu(X_{\mathbf{P},V}) = \mathrm{Vol}(V) \cdot \mathrm{freq}(\mathbf{P}, \mathbf{\Lambda}).$$

Let $\mathbf{\Lambda}$ be a Delone multiset with FLC. From the theory of group actions on a compact space we know that there is an invariant measure, that is, a Borel probability measure on $X_{\mathbf{\Lambda}}$. So we have the measure-preserving system $(X_{\mathbf{\Lambda}}, \mu, \mathbb{R}^d)$ associated with $\mathbf{\Lambda}$. Consider the associated group of unitary operators $\{U_x\}_{x \in \mathbb{R}^d}$ on $L^2(X_{\mathbf{\Lambda}}, \mu)$:

$$U_x f(\mathbf{\Lambda}') = f(-x + \mathbf{\Lambda}').$$

Every $f \in L^2(X_{\mathbf{\Lambda}}, \mu)$ defines a function on $\mathbb{R}^d$ by $x \mapsto (U_x f, f)$. This function is positive definite on $\mathbb{R}^d$, so its Fourier transform is a positive measure σ_f on $\mathbb{R}^d$ called the *spectral measure* corresponding to f

$$\sigma_f = \widehat{(U_{(\cdot)} f, f)}\,. \tag{4.7}$$

We say that the Delone multiset $\mathbf{\Lambda}$ has *pure point dynamical spectrum* if σ_f is pure point for every $f \in L^2(X_{\mathbf{\Lambda}}, \mu)$. We say that $f \in L^2(X_{\mathbf{\Lambda}}, \mu)$ is an *eigenfunction* for the $\mathbb{R}^d$-action if for some $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d$,

$$U_x f = e^{2\pi i x \cdot \alpha} f, \quad \text{for all } \; x \in \mathbb{R}^d,$$

where $\cdot$ is the standard inner product on $\mathbb{R}^d$.

Theorem 4.8 *Let Λ be a Delone multiset with FLC. Then σ_f is pure point for every $f \in L^2(X_\Lambda, \mu)$ if and only if the eigenfunctions for the $\mathbb{R}^d$-action span a dense subspace of $L^2(X_\Lambda, \mu)$.*

This is a consequence of the spectral theorem, see e.g. Theorem 7.27 and §7.6 in [49] for the case $d = 1$. The spectral theorem for unitary representations of arbitrary locally compact Abelian groups, including $\mathbb{R}^d$, is discussed in [27, §6].

4.2 Diffraction

When we say diffraction, we are actually talking about diffraction measure and it is the Fourier transform of volume averaged autocorrelation of point set measure.

For an idea on what diffraction is about let us look at a simple point set $\mathbb{Z}$ in real line. Define $\nu = \sum_{x \in \mathbb{Z}} \delta_x$, where δ_x is the delta function such that $\delta_x(f) = f(x)$ for any continuous $\mathbb{C}$-valued function f with compact support. Let $F_n = \{x \in \mathbb{R} | |x| \leq n\}$ and $\nu|_{F_n} = \sum_{x \in F_n \cap \mathbb{Z}} \delta_x$. The autocorrelation of $\nu|_{F_n}$ is $\nu|_{F_n} * \widetilde{\nu|_{F_n}}$, where

$$(\nu|_{F_n} * \widetilde{\nu|_{F_n}})(f) = \int_{\mathbb{R}} \int_{\mathbb{R}} f(x+y) d\nu|_{F_n}(x) d\widetilde{\nu|_{F_n}}(y).$$

From several steps of calculation we get

$$(\nu|_{F_n} * \widetilde{\nu|_{F_n}})(f) = \sum_{x,y \in F_n \cap \mathbb{Z}} \delta_{x-y}(f).$$

Then the volume averaged autocorrelation of ν is

$$\gamma = \lim_{n \to \infty} \frac{1}{\mathrm{Vol}(F_n)} (\nu|_{F_n} * \widetilde{\nu|_{F_n}})(f) = \lim_{n \to \infty} \frac{1}{\mathrm{Vol}(F_n)} \sum_{x,y \in F_n \cap \mathbb{Z}} \delta_{x-y}(f).$$

Since f has a compact support, there exists $n_0 \in \mathbb{R}$ such that

$$\sum_{x,y \in F_n \cap \mathbb{Z}} \delta_{x-y}(f) = \sum_{\substack{x, x-z \in F_n \cap \mathbb{Z} \\ z \in F_{n_0}}} \delta_z(f).$$

So

$$\sum_{x,y \in F_n \cap \mathbb{Z}} \delta_{x-y}(f) = \sum_{z \in F_{n_0}} \Big(\sum_{x, x-z \in F_n \cap \mathbb{Z}} 1 \Big) f(z).$$

Since z is fixed and $n \to \infty$, n can be assumed to be very large with respect to z. Thus

$$\lim_{n\to\infty} \frac{1}{\mathrm{Vol}(F_n)} \Big(\sum_{x, x-z \in F_n \cap \mathbb{Z}} 1 \Big) = 1.$$

Therefore

$$\gamma = \lim_{n\to\infty} \frac{1}{\mathrm{Vol}(F_n)} (\nu|_{F_n} * \widetilde{\nu|_{F_n}})(f) = \sum_{z \in F_{n_0}} f(z) = \sum_{z \in \mathbb{Z}} \delta_x(f),$$

since f has a compact support contained in F_{n_0}. The diffraction measure

$$\hat{\gamma} = \widehat{(\sum_{z\in\mathbb{Z}} \delta_z)} = \sum_{z\in\mathbb{Z}} \delta_z$$

by Poisson summation formula [5]. Thus the point set $\mathbb{Z}$ on the real line has pure point diffraction measure.

Now we define diffraction on general point sets. Suppose that $\mathbf{\Lambda} = (\Lambda_i)_{i \le m}$ is a Delone multiset. Given a translation-bounded measure ν on $\mathbb{R}^d$, let $\gamma(\nu)$ denote its autocorrelation (assuming it is unique), that is, the vague limit

$$\gamma(\nu) = \lim_{n\to\infty} \frac{1}{\mathrm{Vol}(F_n)} \left(\nu|_{F_n} * \widetilde{\nu|_{F_n}} \right), \tag{4.8}$$

where $\{F_n\}_{n\ge 1}$ is a van Hove sequence. [1] In particular, for the Delone multiset $\mathbf{\Lambda}$ with UCF we see that the autocorrelation is unique for any measure of the form

$$\nu = \sum_{i \le m} a_i \delta_{\Lambda_i}, \quad \text{where } \delta_{\Lambda_i} = \sum_{x \in \Lambda_i} \delta_x \text{ and } a_i \in \mathbb{C}\,. \tag{4.9}$$

Indeed, a simple computation shows

$$\gamma(\nu) = \sum_{i,j=1}^{m} a_i \overline{a}_j \sum_{y \in \Lambda_i, z \in \Lambda_j} \mathrm{freq}((y,z), \mathbf{\Lambda}) \delta_{y-z}. \tag{4.10}$$

Here (y, z) stands for a cluster consisting of two points $y \in \Lambda_i$, $z \in \Lambda_j$. The measure $\gamma(\nu)$ is positive definite, so by Bochner's Theorem [39] the Fourier transform $\widehat{\gamma(\nu)}$ is a positive measure on $\mathbb{R}^d$, called the *diffraction measure* for ν. We say that the measure ν has *pure point diffraction spectrum* if $\widehat{\gamma(\nu)}$ is a pure point or discrete measure. [2]

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[1]Recall that if f is a function in $\mathbb{R}^d$, then $\tilde{f}$ is defined by $\tilde{f}(x) = \overline{f(-x)}$. If μ is a measure, $\tilde{\mu}$ is defined by $\tilde{\mu}(f) = \overline{\mu(\tilde{f})}$ for all $f \in C_0(\mathbb{R}^d)$. In particular for ν in (4.9), $\tilde{\nu} = \sum_{i\le m} \overline{a_i} \delta_{-\Lambda_i}$.

[2]We also say that Λ_i (resp. $\mathbf{\Lambda}$) has pure point diffraction spectrum if $\widehat{\gamma(\delta_{\Lambda_i})}$ (resp. each $\widehat{\gamma(\delta_{\Lambda_i})}, i = 1, \ldots, m$) is a pure point measure.

Chapter 5

Pure Point Dynamical and Diffraction Spectra

We have defined two notions of pure pointedness, one in dynamical spectrum and the other in diffraction spectrum. Although they are defined very differently, they are in fact equivalent. We will show in Sec. 5.1 after a sequence of auxiliary lemmas that under the assumption of FLC and UCF, Λ has pure point dynamical spectrum if and only if Λ has pure point diffraction spectrum. In Sec. 5.2 we drop the condition of UCF and get the equivalent relation for almost every point set in X_{Λ}.

5.1 Equivalence of two notions of pure pointedness

In this section we prove the following theorem.

Theorem 5.1 *Suppose that a Delone multiset Λ has FLC and UCF. Then the following are equivalent:*

(i) *Λ has pure point dynamical spectrum;*

(ii) *The measure $\nu = \sum_{i \leq m} a_i \delta_{\Lambda_i}$ has pure point diffraction spectrum, for any choice of complex numbers $(a_i)_{i \leq m}$;*

(iii) *The measures δ_{Λ_i} have pure point diffraction spectrum, for $i \leq m$.*

Remark : This result is strictly about the situation of pure pointedness. There is no simple relationship known between the spectra in the case that they are not pure point.

The theorem is proved after a sequence of auxiliary lemmas. Fix complex numbers $(a_i)_{i\le m}$ and let $\nu = \sum_{i\le m} a_i \delta_{\Lambda_i}$. For $\mathbf{\Lambda}' = (\Lambda_i')_{i\le m} \in X_{\mathbf{\Lambda}}$ let

$$\nu_{_{\mathbf{\Lambda}'}} = \sum_{i\le m} a_i \delta_{\Lambda_i'},$$

so that $\nu = \nu_{_{\mathbf{\Lambda}}}$. To relate the autocorrelation of ν to spectral measures we need to do some "smoothing." Let $\omega \in C_0(\mathbb{R}^d)$ (that is, ω is continuous and has compact support). Denote

$$\rho_{\omega,\mathbf{\Lambda}'} := \omega * \nu_{_{\mathbf{\Lambda}'}}$$

and let

$$f_\omega(\mathbf{\Lambda}') := \rho_{\omega,\mathbf{\Lambda}'}(0) \quad \text{for} \quad \mathbf{\Lambda}' \in X_{\mathbf{\Lambda}}.$$

Lemma 5.2 $f_\omega \in \mathcal{C}(X_{\mathbf{\Lambda}})$.

Proof. We have

$$f_\omega(\mathbf{\Lambda}') = \int \omega(-x)\, d\nu_{_{\mathbf{\Lambda}'}}(x) = \sum_{i\le m} a_i \sum_{x\in -\mathrm{supp}(\omega)\cap\Lambda_i'} \omega(-x).$$

The continuity of f_ω follows from the continuity of ω and the definition of topology on $X_{\mathbf{\Lambda}}$. □

Denote by $\gamma_{\omega,\mathbf{\Lambda}}$ the autocorrelation of $\rho_{\omega,\mathbf{\Lambda}}$. Since under our assumptions there is a unique autocorrelation measure $\gamma = \gamma(\nu)$, see (4.8) and (4.9), we have

$$\gamma_{\omega,\mathbf{\Lambda}} = (\omega * \widetilde{\omega}) * \gamma.$$

Lemma 5.3 ([12], see also [17])

$$\sigma_{f_\omega} = \widehat{\gamma_{\omega,\mathbf{\Lambda}}}.$$

Proof. We provide a proof for completeness, following [17]. By definition,

$$f_\omega(-x + \mathbf{\Lambda}) = \rho_{\omega,\mathbf{\Lambda}}(x).$$

Therefore,

$$\begin{aligned}
\gamma_{\omega,\mathbf{\Lambda}}(x) &= \lim_{n\to\infty} \frac{1}{\mathrm{Vol}(F_n)} \int_{F_n} \rho_{\omega,\mathbf{\Lambda}}(x+y)\overline{\rho_{\omega,\mathbf{\Lambda}}(y)}\, dy \\
&= \lim_{n\to\infty} \frac{1}{\mathrm{Vol}(F_n)} \int_{F_n} f_\omega(-x-y+\mathbf{\Lambda})\overline{f_\omega(-y+\mathbf{\Lambda})}\, dy \\
&= \int_{X_{\mathbf{\Lambda}}} f_\omega(-x+\mathbf{\Lambda}')\overline{f_\omega(\mathbf{\Lambda}')}\, d\mu(\mathbf{\Lambda}') \\
&= (U_x f_\omega, f_\omega), \qquad (5.1)
\end{aligned}$$

where $\{F_n\}_{n\geq 1}$ is a van Hove sequence. Here the third equality is the main step; it follows from unique ergodicity and the continuity of f_ω, see Theorem 4.5. Thus,

$$\widehat{\gamma_{\omega,\mathbf{\Lambda}}} = \widehat{(U_{(\cdot)}f_\omega, f_\omega)} = \sigma_{f_\omega},$$

and the proof is finished. □

The introduction of the function f_ω and the series of equations (5.1) is often called Dworkin's argument.

Recall that $b(\mathbf{\Lambda}) > 0$ satisfies that every ball of radius $\frac{b(\mathbf{\Lambda})}{2}$ contains at least a point in $\mathrm{supp}(\mathbf{\Lambda})$. Fix ε with $0 < \varepsilon < \frac{1}{b(\mathbf{\Lambda})}$. Consider all the non-empty clusters of diameter $\leq 1/\varepsilon$ in $\mathbf{\Gamma} \in X_{\mathbf{\Lambda}}$. There are finitely many such clusters up to translation, by FLC. Thus, there exists $0 < \theta_1 = \theta_1(\varepsilon) < 1$ such that if $\mathbf{P}, \mathbf{P}'$ are two such clusters, then

$$\rho_H(\mathbf{P}, \mathbf{P}') \leq \theta_1 \ \Rightarrow\ \mathbf{P} = -x + \mathbf{P}' \quad \text{for some } x \in \mathbb{R}^d. \tag{5.2}$$

Here a variation of the Hausdorff metric

$$\rho_H(\mathbf{P}, \mathbf{P}') = \max\{\rho_H(P_i, P_i') : \ i \leq m\},$$

where

$$\rho_H(P_i, P_i') = \begin{cases} \max\{\mathrm{dist}(x, P_i'), \mathrm{dist}(y, P_i) : x \in P_i, y \in P_i'\}, & \text{if } P_i, P_i' \neq \emptyset; \\ 1, \quad \text{if } P_i = \emptyset \text{ and } P_i' \neq \emptyset \text{ (or vice versa)}; \\ 0, \quad \text{if } P_i = \emptyset = P_i', \end{cases}$$

with $\mathbf{P} = (P_i)_{i\leq m}$ and $\mathbf{P}' = (P_i')_{i\leq m}$.

Let

$$\theta = \theta(\varepsilon) := \min\{\varepsilon, \theta_1, \eta(\mathbf{\Lambda})\} \tag{5.3}$$

and

$$f_{i,\omega}(\mathbf{\Lambda}') = (\omega * \delta_{\Lambda_i'})(0) \quad \text{for } \mathbf{\Lambda}' = (\Lambda_i')_{i\leq m} \in X_{\mathbf{\Lambda}}.$$

Denote by $\mathbf{E}_i$ the cluster consisting of a single point of type i at the origin; formally,

$$\mathbf{E}_i = (\emptyset, \ldots, \emptyset, \underbrace{\{0\}}_{i}, \emptyset, \ldots, \emptyset).$$

Let $\chi_{\mathbf{E}_i, V}$ be the characteristic function for the cylinder set $X_{\mathbf{E}_i, \mathrm{V}}$.

Lemma 5.4 *Let $V \subset \mathbb{R}^d$ be a bounded set with* $\operatorname{diam}(V) < \theta$*, where θ is defined by (5.3), and $0 < \zeta < \theta/2$. Let $\omega \in C_0(\mathbb{R}^d)$ be such that*

$$\begin{cases} \omega(x) = 1, & x \in V^{-\zeta}; \\ \omega(x) = 0, & x \in \mathbb{R}^d \setminus V; \\ 0 \le \omega(x) \le 1, & x \in V \setminus V^{-\zeta}. \end{cases}$$

Then

$$\|f_{i,\omega} - \chi_{\mathbf{E}_i,V}\|_2^2 \le \operatorname{freq}(\mathbf{E}_i, \mathbf{\Lambda}) \cdot \operatorname{Vol}((\partial V)^{+\zeta}).$$

Proof. We have by the definition of $\mathbf{E}_i$ and Def. 4.3:

$$\chi_{\mathbf{E}_i,V}(\mathbf{\Lambda}') = \begin{cases} 1, & \text{if } \Lambda_i' \cap (-V) \neq \emptyset; \\ 0, & \text{otherwise}, \end{cases} \qquad \text{where } \mathbf{\Lambda}' \in X_{\mathbf{\Lambda}}.$$

On the other hand, since ω is supported in V and there is at most one point of Λ_i' in V,

$$f_{i,\omega}(\mathbf{\Lambda}') = \int \omega(-x)\, d\delta_{\Lambda_i'}(x) = \begin{cases} \omega(-x), & \text{if } \exists\, x \in \Lambda_i' \cap (-V); \\ 0, & \text{otherwise}. \end{cases}$$

It follows that

$$f_{i,\omega}(\mathbf{\Lambda}') - \chi_{\mathbf{E}_i,V}(\mathbf{\Lambda}') = 0 \quad \text{if} \quad \Lambda_i' \cap (-V^{-\zeta}) \neq \emptyset.$$

Thus,

$$\begin{aligned} \|f_{i,\omega} - \chi_{\mathbf{E}_i,V}\|_2^2 &\le \int_{X_{\mathbf{E}_i,V\setminus V^{-\zeta}}} |f_{i,\omega}(\mathbf{\Lambda}') - 1|^2 \, d\mu(\mathbf{\Lambda}') \\ &\le \mu(X_{\mathbf{E}_i,V\setminus V^{-\zeta}}) \\ &= \operatorname{freq}(\mathbf{E}_i, \mathbf{\Lambda}) \cdot \operatorname{Vol}(V \setminus V^{-\zeta}) \\ &\le \operatorname{freq}(\mathbf{E}_i, \mathbf{\Lambda}) \cdot \operatorname{Vol}((\partial V)^{+\zeta}), \end{aligned}$$

as desired. □

Lemma 5.5 *Let $\mathbf{P} = (P_i)_{i \le m} = B_{1/\varepsilon}(0) \cap \mathbf{\Gamma}$ with $\mathbf{\Gamma} \in X_{\mathbf{\Lambda}}$, and* $\operatorname{diam}(V) < \theta$*, where θ is defined by (5.3). Then the characteristic function $\chi_{\mathbf{P},V}$ of $X_{\mathbf{P},V}$ can be expressed as*

$$\chi_{\mathbf{P},V} = \prod_{i \le m} \prod_{x \in P_i} \chi_{x+\mathbf{E}_i,V}\ .$$

Proof. We just have to prove that

$$X_{\mathbf{P},\mathrm{V}} = \bigcap_{i \le m} \bigcap_{x \in P_i} X_{x+\mathbf{E}_i, V}.$$

A Delone multiset $\mathbf{\Gamma}$ is in the left-hand side whenever $-v + \mathbf{P} \subset \mathbf{\Gamma}$ for some $v \in V$. A Delone multiset $\mathbf{\Gamma}$ is in the right-hand side whenever for each $i \le m$ and each $x \in P_i$ there is a vector $v(\mathbf{x}) \in V$ such that $-v(\mathbf{x}) + \mathbf{x} \subset \mathbf{\Gamma}$, where $\mathbf{x} = (\emptyset, \ldots, \emptyset, \underbrace{\{x\}}_{i}, \emptyset, \ldots, \emptyset)$ stands for a single element cluster. Thus, "$\subset$" is trivial. The inclusion "$\supset$" follows from the fact that $\mathrm{diam}(V) < \theta$, see (5.3) and (5.2). □

Denote by $\mathcal{H}_{pp}$ the closed linear span in $L^2(X_{\mathbf{\Lambda}}, \mu)$ of the eigenfunctions for the dynamical system $(X_{\mathbf{\Lambda}}, \mu, \mathbb{R}^d)$. The following lemma is certainly standard, but since we do not know a ready reference, a short proof is provided.

Lemma 5.6 *If ϕ and ψ are both in $L^\infty(X_{\mathbf{\Lambda}}, \mu) \cap \mathcal{H}_{pp}$, then their product $\phi\psi$ is in $L^\infty(X_{\mathbf{\Lambda}}, \mu) \cap \mathcal{H}_{pp}$ as well.*

Proof. Fix arbitrary $\epsilon > 0$. Since $\phi \in \mathcal{H}_{pp}$, we can find a finite linear combination of eigenfunctions $\widetilde{\phi} = \sum a_i f_i$ such that

$$\|\phi - \widetilde{\phi}\|_2 < \frac{\epsilon}{\|\psi\|_\infty}.$$

Since the dynamical system is ergodic, the eigenfunctions have constant modulus, hence $\widetilde{\phi} \in L^\infty$. Thus, we can find another finite linear combination of eigenfunctions $\widetilde{\psi} = \sum b_j f_j$ such that

$$\|\psi - \widetilde{\psi}\|_2 < \frac{\epsilon}{\|\widetilde{\phi}\|_\infty}.$$

Then

$$\begin{aligned} \|\phi\psi - \widetilde{\phi}\widetilde{\psi}\|_2 &\le \|\widetilde{\phi}(\psi - \widetilde{\psi})\|_2 + \|(\phi - \widetilde{\phi})\psi\|_2 \\ &\le \|\widetilde{\phi}\|_\infty \|\psi - \widetilde{\psi}\|_2 + \|\psi\|_\infty \|\phi - \widetilde{\phi}\|_2 \\ &\le 2\epsilon. \end{aligned}$$

It remains to note that $\widetilde{\phi}\widetilde{\psi} \in \mathcal{H}_{pp}$ since the product of eigenfunctions for a dynamical system is an eigenfunction. Since ϵ is arbitrarily small, $\phi\psi \in \mathcal{H}_{pp}$, and the lemma is proved. □

Proof of Theorem 5.1. (i) $\Rightarrow$ (ii) This is essentially proved by Dworkin in [12], see also [17] and [4]. By Lemma 5.3, pure point dynamical spectrum implies that $\widehat{\gamma_{\omega,\mathbf{\Lambda}}}$ is pure point for any $\omega \in C_0(\mathbb{R}^d)$. Note that

$$\widehat{\gamma_{\omega,\mathbf{\Lambda}}} = |\widehat{\omega}|^2 \widehat{\gamma}. \tag{5.4}$$

Choosing a sequence $\omega_n \in C_0(\mathbb{R}^d)$ converging to the delta measure δ_0 in the vague topology, we can conclude that $\widehat{\gamma}$ is pure point as well, as desired. (This approximation step requires some care; it is explained in detail in [4].)

(ii) $\Rightarrow$ (iii) obvious.

(iii) $\Rightarrow$ (i) This is relatively new, although it is largely a generalization of Queffelec [35, Prop. IV.21].

We are given that δ_{Λ_i} has pure point diffraction spectrum, that is, $\widehat{\gamma}_i := \widehat{\gamma(\delta_{\Lambda_i})}$ is pure point, for all $i \leq m$. In view of (5.4) and Lemma 5.3, we obtain that $\sigma_{f_{i,\omega}}$ is pure point for all $i \leq m$ and all $\omega \in C_0(\mathbb{R}^d)$. So $f_{i,\omega} \in \mathcal{H}_{pp}$ for all $i \leq m$ and all $\omega \in C_0(\mathbb{R}^d)$. Fix $\varepsilon > 0$ and let V be a bounded set with $\mathrm{diam}(V) < \theta = \theta(\varepsilon)$, where θ is defined by (5.3), and $\mathrm{Vol}(\partial V) = 0$. Find $\omega \in C_0(\mathbb{R}^d)$ as in Lemma 5.4. Since $\mathrm{Vol}((\partial V)^{+\zeta}) \to \mathrm{Vol}(\partial V) = 0$ in Lemma 5.4, as $\zeta \to 0$, we obtain that $\chi_{\mathbf{E}_i,V} \in \mathcal{H}_{pp}$. Therefore, also $U_x \chi_{\mathbf{E}_i,V} = \chi_{x+\mathbf{E}_i,V} \in \mathcal{H}_{pp}$. Then it follows from Lemma 5.5 and Lemma 5.6 that $\chi_{\mathbf{P},V} \in \mathcal{H}_{pp}$ where $\mathbf{P} = B_{1/\varepsilon}(0) \cap \mathbf{\Gamma}$ for any $\mathbf{\Gamma} \in X_{\mathbf{\Lambda}}$, $\mathrm{diam}(V) < \theta$, and $\mathrm{Vol}(\partial V) = 0$.

Our goal is to show that $\mathcal{H}_{pp} = L^2(X_{\mathbf{\Lambda}}, \mu)$. Since $(X_{\mathbf{\Lambda}}, \mu)$ is a regular measure space, $C(X_{\mathbf{\Lambda}})$ is dense in $L^2(X_{\mathbf{\Lambda}}, \mu)$. Thus, it is enough to show that all continuous functions on $X_{\mathbf{\Lambda}}$ belong to $\mathcal{H}_{pp}$. Fix $f \in C(X_{\mathbf{\Lambda}})$. Using the decomposition $X_{\mathbf{\Lambda}} = \bigcup_{j=1}^{N} X_{\mathbf{P}_j,V_j}$ from Lemma 4.4 we can approximate f by linear combinations of characteristic functions of cylinder sets $X_{\mathbf{P}_j,V_j}$. So it suffices to show that these characteristic functions are in $\mathcal{H}_{pp}$, which was proved above. This concludes the proof of Theorem 5.1. $\square$

5.2 Equivalence of two notions of pure pointedness when the UCF fails

Here we present a version of the main theorem for Delone multisets which do not necessarily have uniform cluster frequencies. For this we must assume that in addition to the van Hove property (2.1) our averaging sequence $\{F_n\}$ is a sequence of compact neighbourhoods of 0 satisfying the Tempel'man condition:

$$\begin{aligned} &\text{(i) } \cup F_n = \mathbb{R}^d \\ &\text{(ii) } \exists\, K \geq 1 \text{ so that } \mathrm{Vol}(F_n - F_n) \leq K \cdot \mathrm{Vol}(F_n) \text{ for all } n. \end{aligned} \tag{5.5}$$

Let $\mathbf{\Lambda}$ be a Delone multiset with FLC in $\mathbb{R}^d$. Consider the topological dynamical system $(X_{\mathbf{\Lambda}}, \mathbb{R}^d)$ and an ergodic invariant Borel probability measure μ (such measures always exist). The ergodic measure μ will be fixed throughout the section.

Theorem 5.7 *Suppose that a Delone multiset $\mathbf{\Lambda}$ has FLC. Then the following are equivalent:*

(i) *The measure-preserving dynamical system $(X_{\mathbf{\Lambda}}, \mu, \mathbb{R}^d)$ has pure point spectrum;*

(ii) *For μ-a.e. $\mathbf{\Gamma} \in X_{\mathbf{\Lambda}}$, the measure $\nu = \sum_{i\le m} a_i \delta_{\Gamma_i}$ has pure point diffraction spectrum, for any choice of complex numbers $(a_i)_{i\le m}$;*

(iii) *For μ-a.e. $\mathbf{\Gamma} \in X_{\mathbf{\Lambda}}$, the measures δ_{Γ_i} have pure point diffraction spectrum, for $i \le m$.*

In fact, this formulation is closer to the work of Dworkin [12] who did not assume unique ergodicity. The proof is similar to that of Theorem 5.1, except that we have to use the Pointwise Ergodic Theorem instead of the uniform convergence of averages in the uniquely ergodic case (4.4).

Theorem 5.8 *(Pointwise Ergodic Theorem for $\mathbb{R}^d$-actions (see, e.g. [45], [8]) [1]) Suppose that a Delone multiset $\mathbf{\Lambda}$ has FLC and $\{F_n\}$ is a van Hove sequence satisfying (5.5). Then for any $f \in L^1(X_{\mathbf{\Lambda}}, \mu)$,*

$$\frac{1}{\mathrm{Vol}(F_n)} \int_{F_n} f(-x + \mathbf{\Gamma})\, dx \to \int f(\mathbf{\Lambda}')\, d\mu(\mathbf{\Lambda}'), \quad \text{as } n \to \infty, \tag{5.6}$$

for μ-a.e. $\mathbf{\Gamma} \in X_{\mathbf{\Lambda}}$.

For a cluster $\mathbf{P} \subset \mathbf{\Lambda}$, a bounded set $A \subset \mathbb{R}^d$, and a Delone multiset $\mathbf{\Gamma} \in X_{\mathbf{\Lambda}}$, denote

$$L_{\mathbf{P}}(A, \mathbf{\Gamma}) = \#\{x \in \mathbb{R}^d : \ x + \mathbf{P} \subset A \cap \mathbf{\Gamma}\}.$$

Lemma 5.9 *For μ-a.e. $\mathbf{\Gamma} \in X_{\mathbf{\Lambda}}$ and for any cluster $\mathbf{P} \subset \mathbf{\Lambda}$,*

$$\mathrm{freq}'(\mathbf{P}, \mathbf{\Gamma}) := \lim_{n\to\infty} \frac{L_{\mathbf{P}}(F_n, \mathbf{\Gamma})}{\mathrm{Vol}(F_n)} \tag{5.7}$$

exists for μ-a.e. $\mathbf{\Gamma} \in X_{\mathbf{\Lambda}}$. Moreover, if $\mathrm{diam}(V) < \eta(\mathbf{\Lambda})$, then the cylinder set $X_{\mathbf{P},V}$ satisfies, for μ-a.e. $\mathbf{\Gamma} \in X_{\mathbf{\Lambda}}$:

$$\mu(X_{\mathbf{P},V}) = \mathrm{Vol}(V) \cdot \mathrm{freq}'(\mathbf{P}, \mathbf{\Gamma}). \tag{5.8}$$

[1]For recent developments of this theorem in the direction of general locally compact amenable groups, see [26].

Note that we no longer can claim uniformity of the convergence with respect to translation of $\mathbf{\Gamma}$.

Sketch of the proof. Fix a cluster $\mathbf{P} \subset \mathbf{\Lambda}$ and let $X_{\mathbf{P},\mathrm{V}}$ be a cylinder set, with $\mathrm{diam}(V) < \eta(\mathbf{\Lambda})$. Applying (5.6) to the characteristic function of $X_{\mathbf{P},\mathrm{V}}$ and arguing as in the proof of Theorem 4.6 (with $-h + \mathbf{\Lambda}$ replaced by $\mathbf{\Gamma}$), we obtain (5.7) and (5.8) for μ-a.e. $\mathbf{\Gamma}$. Since there are countably many clusters $\mathbf{P} \subset \mathbf{\Lambda}$, we can find a set of full μ-measure on which (5.7) and (5.8) hold for all $\mathbf{P}$. $\square$

For a Delone multiset $\mathbf{\Gamma} = (\Gamma_i)_{i \le m}$, let $\nu = \sum_{i=1}^m a_i \delta_{\Gamma_i}$. Then, for μ-a.e. $\mathbf{\Gamma}$, the autocorrelation $\gamma(\nu)$ exists as the vague limit of measures $\frac{1}{\mathrm{Vol}(F_n)}(\nu|_{F_n} * \widetilde{\nu|_{F_n}})$, and

$$\gamma(\nu) = \sum_{i,j=1}^{m} a_i \overline{a}_j \sum_{y \in \Gamma_i, z \in \Gamma_j} \mathrm{freq}'((y,z), \mathbf{\Gamma}) \delta_{y-z},$$

for μ-a.e. $\mathbf{\Gamma}$, which is the analogue of (4.10). Again, $\widehat{\gamma(\nu)}$ is a positive measure, called the diffraction measure, giving the meaning to the words "pure point diffraction spectrum" in Theorem 5.7.

Sketch of the proof of Theorem 5.7. For μ-a.e. $\mathbf{\Gamma} \in X_{\mathbf{\Lambda}}$, the Pointwise Ergodic Theorem 5.8 holds for all functions $f \in C(X_{\mathbf{\Lambda}})$ (since the space of continuous functions on $X_{\mathbf{\Lambda}}$ is separable).

The $\nu_{\mathbf{\Lambda}'}$, $\rho_{\omega,\mathbf{\Lambda}'}$, and f_ω are defined the same way as in Sec. 5.1. Lemma 5.2 applies to our situation. Next we can show that

$$\sigma_{f_\omega} = \widehat{\gamma_{\omega,\mathbf{\Gamma}}} \tag{5.9}$$

for μ-a.e. $\mathbf{\Gamma}$. This is proved by the same chain of equalities as in (5.1), except that we average over F_n defined in (5.5) and use Theorem 5.8 instead of Theorem 4.6. Lemma 5.4 goes through, after we replace $\mathrm{freq}(\mathbf{E}_i, \mathbf{\Lambda})$ by $\mathrm{freq}'(\mathbf{E}_i, \mathbf{\Gamma})$, for μ-a.e. $\mathbf{\Gamma}$. There are no changes in Lemmas 5.5 and 5.6, since we did not use UCF or unique ergodicity in them. The proof of Theorem 5.7 now follows the scheme of the proof of Theorem 5.1. We only need to replace $\mathbf{\Lambda}$ by μ-a.e. $\mathbf{\Gamma}$, for which hold all the "typical" properties discussed above. $\square$

Chapter 6

Pure Pointedness for Substitutions

In this chapter we study pure point spectrum on dynamical systems generated by substitution Delone multisets and tilings. In Sec. 6.1 we briefly talk about the basic concepts and properties in tiling dynamical systems which are similar as in point set dynamical systems. We deduce a necessary condition on the density of overlaps between the tiling and its translate from pure point spectrum of the tiling dynamical system. The argument of Theorem 6.2 in [43] is quite based on substitution tilings. So when we start from substitution Delone multisets, we make use of the connection to associated substitution tilings in order to induce the same property. In Sec. 6.2 we present the concept of overlap coincidence. We make an assumption that the set of translation vectors forms a Meyer set. Under the assumption we get the finite number of equivalence classes of overlaps of tiles for the tiling. So we can show that the density condition in the previous section is not only necessary for pure point dynamical spectrum but also sufficient.

There are many different concepts of coincidence in the literature of substitutions and the coincidence turns out to be a crucial factor with regard to pure point spectrum. In this thesis we will see another coincidence called "modular coincidence" in the next chapter. Overlap coincidence and modular coincidence are defined quite differently but they are relevant and actually equivalent concepts (see [21]). In Sec. 6.3 we consider substitution Delone multisets. Using the concept of representability (see Def. 3.7) which links between substitution tilings and substitution Delone multisets, we derive the corresponding properties on substitution Delone multisets from substitution tilings.

6.1 Pure pointedness for substitution tilings

Let $\mathcal{T}$ be a tiling. We define the tiling space $X_{\mathcal{T}}$ as the set of all tilings $\mathcal{S}$ of $\mathbb{R}^d$ with the property that every $\mathcal{S}$-patch is equivalent to some $\mathcal{T}$-patch. We equip the space with a metric analogous to (4.1) for $X_{\boldsymbol{\Lambda}}$. Note that $X_{\mathcal{T}} = \overline{\{-g+\mathcal{T} : g \in \mathbb{R}^d\}}$ and $X_{\mathcal{T}}$ is compact if $\mathcal{T}$ has FLC. So we have a natural action of $\mathbb{R}^d$ on $X_{\mathcal{T}}$ which makes it a topological dynamical system. The set $\{-g+\mathcal{T} : g \in \mathbb{R}^d\}$ is the orbit of $\mathcal{T}$. As for Delone multiset dynamical systems, the minimality of dynamical system $(X_{\mathcal{T}}, \mathbb{R}^d)$ is equivalent to the repetitivity of $\mathcal{T}$. We define the cylinder set $X_{P,W}$ by

$$X_{P,W} := \{\mathcal{S} \in X_{\mathcal{T}} : -g+P \text{ is an } \mathcal{S}\text{-patch for some } g \in W\}.$$

where P is a patch of $\mathcal{T}$ or some translate of $\mathcal{T}$ and $W \subset \mathbb{R}^d$ is a Borel set.

In Theorem 3.18 we established the existence of uniform patch frequencies in primitive substitution tilings and in Theorem 4.6 we proved that UPF implies unique ergodicity in the setting of arbitrary Delone multisets with FLC. So we get the following corollary.

Corollary 6.1 *Let $\mathcal{T}$ be a fixed point of a primitive substitution with FLC.*[1] *The dynamical system $(X_{\mathcal{T}}, \mathbb{R}^d)$ is uniquely ergodic, i.e. ergodic with respect to the unique invariant probability measure on $X_{\mathcal{T}}$.*

We will denote the unique invariant probability measure on $X_{\mathcal{T}}$ by μ.

Let $\eta(\mathcal{T}) > 0$ be chosen such that every tile support contains a ball of diameter $\eta(\mathcal{T})$. Note also from Cor. 4.7 that for any $\mathcal{T}$-patch P and any Borel set V with $\mathrm{diam}(V) < \eta(\mathcal{T})$, we have

$$\mu(X_{P,V}) = \mathrm{Vol}(V) \cdot \mathrm{freq}(P, \mathcal{T}). \tag{6.1}$$

We consider the associated group of unitary operators $\{U_g\}_{g\in\mathbb{R}^d}$ on $L^2(X_{\mathcal{T}}, \mu)$:

$$U_g f(\mathcal{S}) = f(-g+\mathcal{S}).$$

We recall that $f \in L^2(X_{\mathcal{T}}, \mu)$ is an eigenfunction for the $\mathbb{R}^d$-action if for some $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d$,

$$U_x f = e^{2\pi i x\cdot\alpha} f, \quad \text{for all } x \in \mathbb{R}^d,$$

[1] Ludwig Danzer [9] has given an example of a primitive substitution tiling which does not satisfy FLC.

where $\cdot$ is the standard inner product on $\mathbb{R}^d$. The dynamical system $(X_{\mathcal{T}}, \mu, \mathbb{R}^d)$ is said to have pure point spectrum if the linear span of the eigenfunctions is dense in $L^2(X_{\mathcal{T}}, \mu)$.

Let $\Xi(\mathcal{T})$ be the set of translation vectors between $\mathcal{T}$-tiles of the same type:

$$\Xi(\mathcal{T}) = \{x \in \mathbb{R}^d : \exists T, T' \in \mathcal{T},\ T' = x + T\}. \tag{6.2}$$

Since $\mathcal{T}$ has the inflation symmetry with the expansive map Q, we have that $Q\Xi(\mathcal{T}) \subset \Xi(\mathcal{T})$. Note also that $\Xi(\mathcal{T}) = -\Xi(\mathcal{T})$. If $\mathcal{T} = \mathcal{T}(\mathbf{\Lambda})$ is a tiling for a representable Delone multiset $\mathbf{\Lambda}$, then $\Xi(\mathcal{T}) = \bigcup_{i=1}^m (\Lambda_i - \Lambda_i)$. The following is proved in [43, §4].

Theorem 6.2 *Suppose that $\mathcal{T}$ is a repetitive fixed point of a primitive substitution with expansive map Q and FLC. If $\alpha \in \mathbb{R}^d$ is an eigenvalue for $(X_{\mathcal{T}}, \mu, \mathbb{R}^d)$, then for any $x \in \Xi(\mathcal{T})$ we have*

$$\lim_{n\to\infty} e^{2\pi i (Q^n x)\cdot\alpha} = 1.$$

This theorem yields necessary conditions on the expansive map Q for the dynamical system to have non-trivial eigenfunctions. The simplest is that if Q is a diagonal matrix with diagonal entries $\lambda > 1$, then λ has to be a Pisot number. This follows from the algebraicity of λ and the classical theorem of Pisot. Other conditions can be found in [43].

Let $\mathcal{T}$ be a repetitive fixed point of a primitive substitution. For $x \in \Xi(\mathcal{T})$ consider the infinite subset

$$D_x := \mathcal{T} \cap (x + \mathcal{T}).$$

It is non-empty by (6.2), and $\mathrm{supp}(D_x)$ is relatively dense by repetitivity. Observe that D_x has a well-defined density given by

$$\begin{aligned} \mathrm{dens}(D_x) &= \lim_{n\to\infty} \frac{\mathrm{Vol}(D_x \cap F_n)}{\mathrm{Vol}(F_n)}, \\ &= \sum_{i=1}^m \mathrm{freq}(\{T_i, (x+T_i)\}, \mathcal{T}) \cdot \mathrm{Vol}(A_i) > 0 \end{aligned} \tag{6.3}$$

where $D_x \cap F_n = \{T \in D_x : \mathrm{supp}(T) \subset F_n\}$, $\{F_n\}_{n\ge1}$ is a van Hove sequence, T_i's are representatives of all tile types in $\mathcal{T}$ and A_i's are their supports. For this reason we may call the elements of $\Xi(\mathcal{T})$ the *almost-periods* of $\mathcal{T}$. Of course an almost-period only really "looks like" a period if the corresponding density $\mathrm{dens}(D_x)$ is close to 1.

Below we consider the cylinder set $X_{\{T\},V}$ for a $\mathcal{T}$-tile T and a Borel set V, which we denote $X_{T,V}$ to simplify the notation.

Lemma 6.3 *Suppose that $\mathcal{T}$ is a repetitive fixed point of a primitive substitution with expansive map Q such that $\Xi(\mathcal{T})$ is uniformly discrete. Then there is $r > 0$ and $n_0 > 0$ such that for any Borel set V with* $\mathrm{diam}(V) < r$, *all $x \in \Xi(\mathcal{T})$, and every $\mathcal{T}$-tile T,*

$$X_{T,V} \cap X_{Q^n x+T,V} = X_{\{T,(Q^n x+T)\},V}\,, \quad \textit{for all } n \geq n_0. \tag{6.4}$$

Proof. Note that for any $x \in \Xi(\mathcal{T})$ there is $n_0 > 0$ such that $\{T, (Q^n x + T)\}$ is a $\mathcal{T}$-patch for all $n \geq n_0$, from the primitivity. We only need to check "$\subset$" since the other inclusion is obvious in all cases. A tiling $\mathcal{S} \in X_{\mathcal{T}}$ is in the left-hand side of (6.4) if and only if there are two, possibly distinct, vectors $v_1, v_2 \in V$ such that $-v_1 + T \in \mathcal{S}$ and $-v_2 + Q^n x + T \in \mathcal{S}$. Hence $v_1 - v_2 + Q^n x \in \Xi(\mathcal{S}) = \Xi(\mathcal{T})$. But $Q^n x \in \Xi(\mathcal{T})$ since $Q\Xi(\mathcal{T}) \subset \Xi(\mathcal{T})$. Since $\Xi(\mathcal{T})$ is uniformly discrete, $v_1 - v_2 = 0$ if $\mathrm{diam}(V)$ is sufficiently small. Then $\{T, (Q^n x + T)\} \subset v_1 + \mathcal{S}$, and hence $\mathcal{S}$ is in the right-hand side of (6.4). The lemma is proved. □

Proposition 6.4 *Suppose that $\mathcal{T}$ is a repetitive fixed point of a primitive substitution with expansive map Q such that $\Xi(\mathcal{T})$ is uniformly discrete and $\mathcal{T}$ has FLC. If $(X_{\mathcal{T}}, \mu, \mathbb{R}^d)$ has pure point dynamical spectrum, then*

$$\lim_{n\to\infty} \mathrm{dens}(D_{Q^n x}) = 1, \quad \textit{for all } \; x \in \Xi(\mathcal{T}). \tag{6.5}$$

Proof. Fix $x \in \Xi(\mathcal{T})$. By Theorem 6.2, for every eigenvalue $\alpha \in \mathbb{R}^d$ we have $e^{2\pi i (Q^n x)\cdot\alpha} \to 1$. This implies

$$(U_{Q^n x} - I) f_\alpha \to 0, \tag{6.6}$$

in the norm of $L^2(X_{\mathcal{T}}, \mu)$, for the corresponding eigenfunction f_α. Since $\|U_{Q^n x} - I\| \leq 2$, the sequence of operators $\{U_{Q^n x} - I\}_{n\geq 0}$ is uniformly bounded, so by (6.6) we have $U_{Q^n x} f \to f$ for any f in the closed linear span of the eigenfunctions. Let T be a $\mathcal{T}$-tile and let V be a Borel set satisfying (6.4). Denote by f the characteristic function of the cylinder set $X_{T,V}$, which is in the closed linear span of the eigenfunctions by assumption. We can write

$$\begin{aligned}
\|U_{Q^n x} f - f\|_2^2 &= \int_{X_{\mathcal{T}}} |f(-Q^n x + \mathcal{S}) - f(\mathcal{S})|^2 \, d\mu(\mathcal{S}) \\
&= \mu(X_{T,V} \,\triangle\, X_{Q^n x+T,V}) \\
&= 2[\mu(X_{T,V}) - \mu(X_{T,V} \cap X_{Q^n x+T,V})].
\end{aligned}$$

The last equality uses the fact that μ is translation-invariant. It follows that

$$\mu(X_{T,V} \cap X_{Q^n x+T,V}) \to \mu(X_{T,V}) \tag{6.7}$$

for each $\mathcal{T}$-tile T.

Now, combining the Lemma 6.3, (6.7), and (6.1), we obtain that

$$\text{freq}(\{T, (Q^n x + T)\}, \mathcal{T}) \to \text{freq}(T, \mathcal{T}), \quad \text{as } n \to \infty,$$

for any tile $T \in \mathcal{T}$. In view of (6.3) this implies that $\lim_{n\to\infty} \text{dens}(D_{Q^n x}) = 1$, and the proposition is proved. □

Remark 6.5 In this subsection we have been assuming that the fixed point $\mathcal{T}$ of our primitive substitution is *repetitive*. We note that this assumption is not restrictive if we are interested in the measure-theoretic dynamical system $(X_{\mathcal{T}}, \mu, \mathbb{R}^d)$. Indeed, if $\mathcal{T}$ is non-repetitive, then the inclusion mapping from the set of self-affine tilings $X_{\mathcal{A},\omega}$ associated with $(\mathcal{A}, \omega)$ to $X_{\mathcal{T}}$ induces an isomorphism of measure-preserving systems (this follows from the proofs of Prop. 3.17 and Theorem 3.18 which show that non-legal patches have frequency 0 and then from (6.1) the measure of the corresponding cylinder set in $X_{\mathcal{T}}$ is 0). All the tilings in $X_{\mathcal{A},\omega}$ are repetitive; further, we can find a periodic point in $X_{\mathcal{A},\omega}$ for some power of ω and work with it.

6.2 Overlap coincidence

Here we use the notion of overlaps and the subdivision graph of overlaps from [43, p.721], with some modifications.

Definition 6.6 Let $\mathcal{T}$ be a tiling. A triple (T, y, S), with $T, S \in \mathcal{T}$ and $y \in \Xi(\mathcal{T})$, is called an *overlap* if the intersection $\text{supp}(y+T) \cap \text{supp}(S)$ has non-empty interior. We say that two overlaps (T, y, S) and (T', y', S') are *equivalent* if for some $g \in \mathbb{R}^d$ we have $y + T = g + y' + T'$, $S = g + S'$. Denote by $[(T, y, S)]$ the equivalence class of an overlap. An overlap (T, y, S) is a *coincidence* if $y + T = S$. The *support* of an overlap (T, y, S) is $\text{supp}(T, y, S) = \text{supp}(y + T) \cap \text{supp}(S)$.

Lemma 6.7 *Let $\mathcal{T}$ be a tiling such that $\Xi(\mathcal{T})$ is a Meyer set. Then the number of equivalence classes of overlaps for $\mathcal{T}$ is finite.*

Proof. Let T_i, $i \leq m$, be the representatives of all tile types for $\mathcal{T}$. Let Λ_i be the Delone set such that $\Lambda_i + T_i$ is the collection of all tiles of type i. Thus we have

$\mathcal{T} = \bigcup_{i \le m}(\Lambda_i + T_i)$. Let (T, y, S) be an overlap. We have $T = u_i + T_i$ and $S = u_j + T_j$ for some $i, j \le m$ (possibly equal) and some $u_i \in \Lambda_i$, $u_j \in \Lambda_j$. The equivalence class of the overlap is completely determined by i, j, and the vector $u_i + y - u_j$. Since the interiors of the supports of $y + T$ and S must intersect, we have

$$|u_i + y - u_j| \le C, \tag{6.8}$$

where $C = 2\max\{\text{diam(T)} : \text{T} \in \mathcal{T}\}$. Note that $u_i, y, u_j \in \Xi$. By the definition of $\Xi := \Xi(\mathcal{T})$ we have $\Xi = -\Xi$. By the definition of Meyer set, $\Xi - \Xi \subset \Xi + F$ for some finite set F. This implies $\Xi + \Xi - \Xi \subset \Xi + (F + F)$, which is a discrete set, so there are finitely many possible vectors $u_i + y - u_j$ in (6.8). This proves the lemma. □

Next we define the *subdivision graph* $\mathcal{G}_{\mathcal{O}}(\mathcal{T})$ *for overlaps.* Its vertices are equivalence classes of overlaps. Let $\mathcal{O} = (T, y, S)$ be an overlap. We will specify directed edges leading from the equivalence class $[\mathcal{O}]$. Recall that we have the tile-substitution ω, see Def. 3.4. Then $\omega(y + T) = Qy + \omega(T)$ is a patch of $Qy + \mathcal{T}$, and $\omega(S)$ is a $\mathcal{T}$-patch, and moreover,

$$\text{supp}(Qy + \omega(T)) \cap \text{supp}(\omega(S)) = Q(\text{supp}(T, y, S)).$$

For each pair of tiles $T' \in \omega(T)$ and $S' \in \omega(S)$ such that $\mathcal{O}' := (T', Qy, S')$ is an overlap, we draw a directed edge from $[\mathcal{O}]$ to $[\mathcal{O}']$.

Proposition 6.8 *Let $\mathcal{T}$ be a repetitive fixed point of a primitive substitution with expansive map Q such that $\Xi(\mathcal{T})$ is a Meyer set. Let $x \in \Xi(\mathcal{T})$. The following are equivalent:*

(i) $\lim_{n\to\infty} \text{dens}(D_{Q^n x}) = 1$;

(ii) $1 - \text{dens}(D_{Q^n x}) \le Cr^n$, $n \ge 1$, *for some* $C > 0$ *and* $r \in (0, 1)$;

(iii) *From each vertex of the graph $\mathcal{G}_{\mathcal{O}}(\mathcal{T})$ there is a path leading to a coincidence.*

Proof. We have for each $n \ge 0$:

$$\mathbb{R}^d = \bigcup_{T\in\mathcal{T}} \bigcup_{S\in\mathcal{T}} \text{supp}(T, Q^n x, S),$$

where the support is considered empty if $(T, Q^n x, S)$ is not an overlap. Notice that $\text{supp}(D_{Q^n x})$ is exactly the union of supports of coincidences in this formula. It is clear that all edges from the overlaps-coincidences lead to other coincidences. Thus

$$Q(\text{supp}(D_{Q^n x})) \subset \text{supp}(D_{Q^{n+1} x})$$

and

$$\text{dens}(D_{Q^n x}) \le \text{dens}(D_{Q^{n+1} x}). \tag{6.9}$$

Now, if (iii) holds, then there exists $\ell \in \mathbb{N}$ such that for each overlap $\mathcal{O}$ the inflation $Q^\ell(\text{supp}(\mathcal{O}))$ contains a coincidence. The volume of the support of a coincidence is at least $V_{\min}$. Thus

$$1 - \text{dens}(D_{Q^{n+\ell} x}) \le \left(1 - \frac{V_{\min}}{V_{\max}|\det(Q)|^\ell}\right)(1 - \text{dens}(D_{Q^n x})).$$

For any $n \ge 0$, $n = k\ell + s$ for some $k \in \mathbb{N}$ and $0 \le s < \ell$. So

$$\begin{aligned} 1 - \text{dens}(D_{Q^n x}) &= 1 - \text{dens}(D_{Q^{k\ell+s} x}) \\ &\le b^k(1 - \text{dens}(D_{Q^s x})), \text{ where } b = 1 - \frac{V_{\min}}{V_{\max}|\det(Q)|^\ell} \\ &= (b^{1/\ell})^{k\ell+s}\frac{(1 - \text{dens}(D_{Q^s x}))}{b^{s/\ell}} \\ &\le r^n C, \text{ for some } r \in (0,1) \text{ and } C > 0. \end{aligned} \tag{6.10}$$

Then (ii) follows.

It is straightforward that (ii) implies (i).

It remains to prove that (i) implies (iii). Suppose to the contrary, that there is an overlap $\mathcal{O}$ from which there is no path to a coincidence. Then $Q^n(\text{supp}(\mathcal{O})) \subset \mathbb{R}^d \setminus \text{supp}(D_{Q^n x})$ for all n. By the repetitivity of $\mathcal{T}$, the overlaps equivalent to $\mathcal{O}$ occur relatively dense in $\mathbb{R}^d$. Therefore,

$$1 - \text{dens}(D_{Q^n x}) \ge \text{dens}(Q^n(\text{supp}(\mathcal{O})) = \text{dens}(\text{supp}(\mathcal{O})) > 0,$$

which contradicts (i). This completes the proof of the lemma. □

As the next theorem shows, under the additional assumption that $\Xi(\mathcal{T})$ is a Meyer set, the converse of Prop. 6.4 is also true. This theorem extends [43, Theorem 6.2] to the case of $d \ge 3$. Notice that we do not add FLC to the assumption, since $\Xi(\mathcal{T})$ being a Meyer set implies it.

Theorem 6.9 *Suppose that $\mathcal{T}$ is a repetitive fixed point of a primitive substitution with expansive map Q such that $\Xi(\mathcal{T})$ is a Meyer set. Then $(X_\mathcal{T}, \mu, \mathbb{R}^d)$ has pure point dynamical spectrum if and only if*

$$\lim_{n\to\infty} \text{dens}(D_{Q^n x}) = 1, \quad \textit{for all } x \in \Xi(\mathcal{T}).$$

Proof. The assumption gives us through Prop. 6.8 that there exists a basis $\mathcal{B}$ for $\mathbb{R}^d$ such that for all $x \in \mathcal{B}$,

$$\sum_{n=0}^{\infty}(1 - \mathrm{dens}(D_{Q^n x})) < \infty. \tag{6.11}$$

Then [43, Theorem 6.1] implies that the dynamical system $(X_{\mathcal{T}}, \mu, \mathbb{R}^d)$ has pure point spectrum. □

6.3 Pure pointedness on substitution Delone multisets

Any tiling $\mathcal{T}$ can be converted into a Delone multiset by simply choosing a point $x_{(A,i)}$ for each tile (A, i) so that the chosen points for tiles of the same type are in the same relative position in the tile: $x_{(g+A,i)} = g + x_{(A,i)}$. We define $\Lambda_i := \{x_{(A,i)} : (A, i) \in \mathcal{T}\}$ and $\mathbf{\Lambda} := (\Lambda_i)_{i \le m}$. Clearly $\mathcal{T}$ can be reconstructed from $\mathbf{\Lambda}$ given the information about how the points lie in their respective tiles. This bijection establishes a topological conjugacy of $(X_{\mathbf{\Lambda}}, \mathbb{R}^d)$ and $(X_{\mathcal{T}}, \mathbb{R}^d)$. Concepts and theorems can clearly be interpreted in either language (FLC, UCF, unique ergodicity, pure point dynamical spectrum, etc.).

In the case that $\mathbf{\Lambda}$ is a representable primitive substitution Delone multiset, we make use of the following bijection. We consider $T_i = (A_i, i)$, $i \le m$, as prototiles, where A_i's are defined by (3.6). Let $\mathcal{T} = \mathcal{T}(\mathbf{\Lambda})$ be the tiling in (3.7), with the colours added, that is, $\mathcal{T} = \{x_i + T_i : x_i \in \Lambda_i, i \le m\}$, and let $\mathcal{A} = \{T_1, \ldots, T_m\}$. By (3.6) and the definition of representable primitive substitution Delone multiset, we have a tile-substitution $\omega : \mathcal{A} \to \mathcal{P}_{\mathcal{A}}$. For any Delone multiset $\mathbf{\Gamma} = (\Gamma_i)_{i \le m} \in X_{\mathbf{\Lambda}}$ we let $\phi(\mathbf{\Gamma}) = \{x_i + T_i : x_i \in \Gamma_i, i \le m\}$. Then $\phi(\mathbf{\Gamma}) \in X_{\mathcal{T}}$ and ϕ is a homeomorphism commuting with the translation action. So the dynamical systems $(X_{\mathbf{\Lambda}}, \mathbb{R}^d)$ and $(X_{\mathcal{T}}, \mathbb{R}^d)$ are topologically conjugate.

Transferring the results of the previous subsections about tilings to Delone multisets, we obtain three quick corollaries.

Proposition 6.10 *If $\mathbf{\Lambda}$ is a primitive substitution Delone multiset with FLC such that every $\mathbf{\Lambda}$-cluster is legal, then the dynamical system $(X_{\mathbf{\Lambda}}, \mathbb{R}^d)$ is uniquely ergodic.*

Proof. From Cor. 3.24 we note that $\mathbf{\Lambda}$ has UCF. Thus the dynamical system $(X_{\mathbf{\Lambda}}, \mathbb{R}^d)$ is uniquely ergodic by Theorem 4.6. □

Proposition 6.11 *Suppose that* $\mathbf{\Lambda} = (\Lambda_i)_{i \le m}$ *is a primitive substitution Delone multiset with expansive map* Q *such that* $\Xi(\mathbf{\Lambda}) := \cup_{i \le m}(\Lambda_i - \Lambda_i)$ *is a Meyer set and every* $\mathbf{\Lambda}$*-cluster is legal. Then the dynamical system* $(X_{\mathbf{\Lambda}}, \mu, \mathbb{R}^d)$ *has pure point spectrum if and only if*

$$\lim_{n \to \infty} \mathrm{dens}(\mathbf{\Lambda} \cap (Q^n x + \mathbf{\Lambda})) = \mathrm{dens}(\mathbf{\Lambda}), \quad \textit{for all } \ x \in \Xi(\mathbf{\Lambda}).$$

Proof. By Theorem 3.9 $\mathbf{\Lambda}$ is representable by a tiling $\mathcal{T}$. The legality of $\mathbf{\Lambda}$ implies that of $\mathcal{T}$ and Prop. 3.6 shows that $\mathcal{T}$ is repetitive. The result is now a direct consequence of Theorem 6.9, in view of the fact that $\Xi(\mathcal{T}) = \Xi(\mathbf{\Lambda})$. □

Proposition 6.12 *Suppose that* $\mathbf{\Lambda} = (\Lambda_i)_{i \le m}$ *is a primitive substitution Delone multiset with expansive map* Q *such that* $\bigcup_{i=1}^m \Lambda_i$ *lies in a lattice* L *in* $\mathbb{R}^d$ *and every* $\mathbf{\Lambda}$*-cluster is legal. Then the dynamical system* $(X_{\mathbf{\Lambda}}, \mu, \mathbb{R}^d)$ *has pure point spectrum if and only if*

$$\lim_{n \to \infty} \mathrm{dens}(\mathbf{\Lambda} \triangle (Q^n x + \mathbf{\Lambda})) = 0, \quad \textit{for all } x \in L', \tag{6.12}$$

where $L' = L_1 + \cdots + L_m$*, and* L_i *is the Abelian group generated by* $\Lambda_i - \Lambda_i$ *for* $i \le m$.

Proof. The Meyer set condition is obvious, since all the sets Λ_i lie in the lattice L. We have $\Xi(\mathbf{\Lambda}) = \cup_{i \le m}(\Lambda_i - \Lambda_i) \subset L'$, so the necessity follows from Prop. 6.11. The condition (6.12) for $x \in \Lambda_i - \Lambda_i$ follows from Prop. 6.11. In order to prove it for all $x \in L'$ we note that $\mathbf{\Lambda} \triangle (y + z + \mathbf{\Lambda}) \subset (\mathbf{\Lambda} \triangle (y + \mathbf{\Lambda})) \cup ((y + \mathbf{\Lambda}) \triangle (y + z + \mathbf{\Lambda}))$ for any $y, z \in \mathbb{R}^d$, hence

$$\mathrm{dens}(\mathbf{\Lambda} \triangle (y + z + \mathbf{\Lambda})) \le \mathrm{dens}(\mathbf{\Lambda} \triangle (y + \mathbf{\Lambda})) + \mathrm{dens}(\mathbf{\Lambda} \triangle (z + \mathbf{\Lambda})),$$

and the statement follows. □

Remark 6.13 Note that the density in (6.12) does not depend on the choice of the van Hove sequence, since this is true for the density $\mathrm{dens}(D_{Q^n x})$ of the associated tiling by (6.1) and (6.3).

Chapter 7

Substitutions on Lattices

We assume here that substitution Delone multisets are on a lattice. Due to the additional lattice structure we can find an equivalent relation between pure pointedness and regular model sets. It is still a conjecture to find the equivalence in general Delone multisets without assuming lattice substitutions. The main bridge which connects these two is modular coincidence (see Def. 7.2). We first introduce a Q-adic like profinite group and define the Q-adic completion in Sec. 7.1. It enables us to construct a cut and project scheme using the completion. In Sec. 7.2 we show that the density property which is derived from pure pointedness in the previous chapter ensures a substitution to admit modular coincidence. Then the modular coincidence separates all window interiors of original sets and it proves that each point set is a model set. Using Perron-Frobenius theorems we prove that the boundary of the window from each point set has measure 0. Thus actually each point set becomes a regular model set. Applying Schlottmann's theorem, we conclude that each point set is pure point diffractive. From all these results we show a cycle of equivalent conditions to pure point spectrum. Example 7.11 demonstrates how to check the modular coincidence.

7.1 Lattice substitutions

In this chapter, L will be a lattice in $\mathbb{R}^d$ and the mappings of Φ will always be affine linear mappings of the form $x \mapsto Qx + a$, where $Q \in \mathrm{End}_{\mathbb{Z}}(L)$ is the *same* for all the maps and $a \in L$. Such maps are restrictions of uniquely determined affine linear mappings on $\mathbb{R}^d$ and we will not distinguish them notationally. A mapping $Q \in \mathrm{End}_{\mathbb{Z}}(L)$ is called an *inflation* for L if $\det Q \neq 0$ and $\bigcap_{k=0}^{\infty} Q^k L = \{0\}$. So an expansive map Q for L is an inflation.

Definition 7.1 A *substitution system on* L with inflation Q is a pair $(\mathbf{\Lambda}, \Phi)$ consisting of

- a Delone multiset $\mathbf{\Lambda} = (\Lambda_i)_{i\leq m}$, for which each Λ_i is a subset of L and all Λ_i are mutually disjoint, and
- an $m \times m$ MFS Φ on L, for which

$$\Lambda_i = \bigcup_{j\leq m} \bigcup_{f\in\Phi_{ij}} f(\Lambda_j), \quad i \leq m, \tag{7.1}$$

  where the maps of Φ are affine linear mappings of the form $x \mapsto Qx+a$, $a \in L$, and the unions in (7.1) are disjoint.

The substitution system $(\mathbf{\Lambda}, \Phi)$ is *primitive* if $S(\Phi)$ is primitive. For any affine linear mapping $f : x \mapsto Qx+b$ on L we denote the translational part, b, of f by $t(f)$. Let $(\mathbf{\Lambda}, \Phi)$ be a primitive substitution system on L with inflation Q. Then $(\mathbf{\Lambda}, \Phi^\ell)$ is a primitive substitution system on L with inflation Q^ℓ, so without loss of generality we can assume that $S(\Phi)$ is a positive matrix. Let $L' := L_1 + L_2 + \cdots + L_m$, where $L_i := <\Lambda_i - \Lambda_i>$, i.e. the Abelian group generated by $\Lambda_i - \Lambda_i$.

Since Q is an inflation for L, Q is an inflation for L' also. In fact, $\bigcup_{j\leq m} \bigcup_{f\in\Phi_{ij}} (Q(\Lambda_j) + t(f)) = \Lambda_i$ implies $\bigcup_{j\leq m} Q(\Lambda_j - \Lambda_j) \subset (\Lambda_i - \Lambda_i)$ for any $i \leq m$, and thus

$$Q(L_1 + \cdots + L_m) \subset L_i, \quad \text{for any } i \leq m. \tag{7.2}$$

So $QL' \subset L'$, hence $Q \in \mathrm{End}_{\mathbb{Z}}(L')$. Note also $\bigcap_{k=0}^\infty Q^k L' \subset \bigcap_{k=0}^\infty Q^k L = \{0\}$.

Note that L/L' is finite, since $\mathbf{\Lambda}$ is a Delone multiset. Let $q := |\det Q| = [L' : QL'] > 1$. We define the Q-adic completion

$$\overline{L} = (\overline{L})_Q = \lim_{\leftarrow k} L/Q^k L' = \lim_{\leftarrow k}(\cdots \to L/Q^k L' \to \cdots \to L/QL' \to L/L')$$

of L and

$$\overline{L'} = (\overline{L'})_Q = \lim_{\leftarrow k} L'/Q^k L' = \lim_{\leftarrow k}(\cdots \to L'/Q^k L' \to \cdots \to L'/QL')$$

of L'. Each of $\overline{L}$ and $\overline{L'}$ will be supplied with the usual topology of a profinite group. We can identify $\overline{L}$ and $(L/L') \times \overline{L'}$ as topological spaces. In particular, the cosets $a + Q^k\overline{L'}$, $a \in L, k = 1, 2, \ldots$, form a basis of open sets of $\overline{L}$ and each of these cosets is open and closed. When we use the word coset in this paper, we mean either a coset of the form $a + Q^k\overline{L'}$ in $\overline{L}$ or $a + Q^kL'$ in L according to the context. An important observation is that any two cosets in $\overline{L}$ are either disjoint or one is contained in the other. The same applies to cosets of L.

We let μ denote the Haar measure on $\overline{L}$, normalized so that $\mu(\overline{L}) = 1$. Thus for cosets, $\mu(a + Q^k\overline{L'}) = \frac{1}{|\det Q|^k \cdot |L/L'|} = \frac{1}{q^k \cdot |L/L'|}$, $k = 1, 2, \ldots$. From $\bigcap_{k=0}^{\infty} Q^kL' = \{0\}$, we conclude that the mapping $x \to \{x \bmod Q^kL'\}_k$ embeds L in $\overline{L}$. We identify L with its image in $\overline{L}$ and note that $\overline{L}$ is then the closure of L. With this identification, L is a dense subgroup of $\overline{L}$, so we have a unique extension of Φ to a MFS on $\overline{L}$. Thus if $f \in \Phi_{ij}$ and $f : x \mapsto Qx + a$, this formula defines a mapping on $\overline{L}$, to which we give the same name. Furthermore defining the compact subsets in $\overline{L}$

$$W_i := \overline{\Lambda_i}, \quad i \leq m,$$

and using the relations (7.1) and the continuity of mappings, we have

$$W_i = \bigcup_{j \leq m} \bigcup_{f \in \Phi_{ij}} f(W_j), \quad i \leq m. \tag{7.3}$$

We call $(\mathbf{W}, \Phi)$ the associated Q-adic system.

Suppose $L = \bigcup_{i \leq m} \Lambda_i$. For any $i \leq m$, since $< \Lambda_i - \Lambda_i > \subset L'$, we have

$$\Lambda_i \subset x + L' \quad \text{for any } x \in \Lambda_i \subset L.$$

For $a \in L$, let

$$\Phi_{ij}[a] \quad = \quad \{f \in (\Phi)_{ij} : f(\Lambda_j) \subset a + QL'\}. \tag{7.4}$$

Then

$$\bigcup_{i,j \leq m} \bigcup_{f \in \Phi_{ij}[a]} f(\Lambda_j) = a + QL'.$$

Let $\Phi[a] := \cup_{i,j \leq m} \Phi_{ij}[a]$. This partitions Φ into congruence classes induced by L/QL'.

Definition 7.2 Let $(\mathbf{\Lambda}, \Phi)$ be a primitive substitution system on L with inflation Q and $L = \bigcup_{i \leq m} \Lambda_i$. We say that $(\mathbf{\Lambda}, \Phi)$ admits *a modular coincidence relative to* QL' if $\Phi[a]$ is contained entirely in one row of Φ for some $a \in L$.

It is easy to see that $(\mathbf{\Lambda}, \Phi)$ admits a modular coincidence relative to QL' if and only if $(a + QL') \subset \Lambda_i$ for some $a \in L$ and $i \leq m$.

7.2 Pure pointedness, modular coincidence, and model sets

This subsection contains a new result, namely, that in the setting of lattice substitution systems, pure point diffraction spectrum implies that there is a model set realization. Precise conditions are given in Theorem 7.9, which incorporates earlier results and completes the circle of equivalences started in [22]. The key new ingredient of the proof is Theorem 7.4. Some of the arguments in this subsection are similar to the corresponding parts of [22]. However, there is an important distinction: here we have to do everything modulo the sublattice L'. For instance, the notion of modular coincidence is not the same as in [22], $\overline{L}$ is different, etc.

Consider $(\mathbf{\Lambda}, \Phi)$ a primitive substitution system on L with an expansive map Q. Since Q is expansive (see (3.1)), for any bounded subset S of $\mathbb{R}^d$ containing 0 as an interior point and $\lambda > 1$, there is a large enough $k_0 \in \mathbb{Z}_+$ so that $Q^k S \supset \lambda S$ for $k > k_0$.

Recall that for any set $F \subset \mathbb{R}^d$ we have a cluster $F \cap \mathbf{\Lambda} = (F \cap \Lambda_i)_{i \le m} \subset \mathbf{\Lambda}$.

Suppose $0 \in \mathbf{\Lambda}$. Let $\mathbf{I} = (I_i)_{i \le m} := \Phi(0) = (\Phi_{ij}(0))_{i \le m}$ where $0 \in \Lambda_j$, and let $t(\Phi) := \{t(f) : f \in \Phi_{ij},\ f : x \mapsto Qx + t(f),\ i, j \le m\}$. For a cluster $\mathbf{P} = (P_i)_{i \le m}$ we write $\mathrm{supp}(\mathbf{P}) = \cup_{i \le m} P_i$. Let $\{\beta_i : i = 1, \dots, d\}$ be a basis of L'. Let D_0 be the parallelepiped in $\mathbb{R}^d$ of the form

$$D_0 = \{x_1\beta_1 + \cdots + x_i\beta_i + \cdots + x_d\beta_d : -1 \le x_i < 1, 1 \le i \le d\}. \tag{7.5}$$

We can always find $p \in \mathbb{Z}_+$ so that $S(\Phi^p)$ is positive and $Q^p D_0 \supset \lambda D_0$ for some $\lambda > 1$, and then $a > 0$ so that $D := aD_0 \supset (\mathrm{supp}(\Phi^p(0)) \cup t(\Phi^p))$. Replacing Φ by Φ^p, QD_0 by $Q^p D_0$ etc., we may, for the purposes of Theorem 7.4, assume at the outset that $S(\Phi)$ is positive, $Q(D) \supset \lambda D$, and $D \supset (\mathrm{supp}(\mathbf{I}) \cup t(\Phi))$ for some $\lambda > 1$ and convex D.

Lemma 7.3 *Let $(\mathbf{\Lambda}, \Phi)$ be a primitive substitution system on L with expansive map Q and $0 \in \mathbf{\Lambda}$. Let D be a convex set for which $Q(D) \supset \lambda D$ for some $\lambda > 1$ and $D \supset (\mathrm{supp}(\mathbf{I}) \cup t(\Phi))$. Then there is $r > 0$ such that $\mathrm{supp}(\Phi^n(\mathbf{I})) \subset Q^n(rD)$ for all $n \in \mathbb{Z}_+$.*

Proof. Choose $p \in \mathbb{Z}_+$ so that $\lambda^p \ge p + 1$. Since $\lambda D \subset Q(D)$, we have $(p+1)D \subset \lambda^p D \subset Q^p(D)$. So

$$(p+1)D \subset Q^p(D). \tag{7.6}$$

Note that $Q(D) \supset D$ and $kD = \underbrace{D + \cdots + D}_{k}$ for any $k \in \mathbb{Z}_+$, since D is convex.

Since we have $\text{supp}(\mathbf{I}) \subset D$, which means $\mathbf{I} \subset D \cap \mathbf{\Lambda}$, for any $n \in \mathbb{Z}_+$

$$\begin{aligned}
&\text{supp}(\Phi^n(\mathbf{I})) \\
&\subset \text{supp}(\Phi^n(D \cap \mathbf{\Lambda})) \\
&\subset t(\Phi) + Q(\text{supp}(\Phi^{n-1}(D \cap \mathbf{\Lambda}))) \\
&\subset t(\Phi) + \cdots + Q^{n-1}(t(\Phi)) + Q^n(D) \\
&\subset D + \cdots + Q^{n-1}(D) + Q^n(D).
\end{aligned} \tag{7.7}$$

Since $Q(D) \supset \lambda D \supset D$, we have $Q^i(D) \supset Q^j(D)$ for $i > j$. Thus, writing $n = lp+s$, where $1 \le s \le p$ and $l \in \mathbb{Z}_{\ge 0}$, we obtain

$$\begin{aligned}
\sum_{i=0}^{n} Q^i(D) &= \sum_{i=0}^{n-lp} Q^i(D) + \sum_{k=1}^{l} Q^{n-kp}\left(\sum_{i=1}^{p} Q^i(D)\right) \\
&\subset (p+1)Q^{n-lp}(D) + \sum_{k=1}^{l} Q^{n-kp}(pQ^p(D)) \\
&= Q^{n-lp}\left((p+1)D + p\sum_{k=1}^{l} Q^{(l-k+1)p}(D)\right) \\
&\subset Q^{n-lp}\left(Q^p(D) + p\sum_{j=1}^{l} Q^{jp}(D)\right) \quad \text{from (7.6).}
\end{aligned}$$

So

$$\begin{aligned}
\sum_{i=0}^{n} Q^i(D) &\subset Q^{n-lp}Q^p\left((p+1)D + p\sum_{j=1}^{l-1} Q^{jp}(D)\right) \\
&\subset Q^{n-lp}Q^p\left(Q^p(D) + p\sum_{j=1}^{l-1} Q^{jp}(D)\right) \\
&\vdots \\
&\subset Q^{n-lp}\overbrace{Q^p \cdots Q^p}^{l}((p+1)D) = Q^n((p+1)D).
\end{aligned}$$

In view of (7.7) this implies that $r := p+1$ satisfies the assertion of the lemma. □

Theorem 7.4 *Let $(\mathbf{\Lambda}, \Phi)$ be a repetitive primitive substitution system on L with expansive map Q and $L = \bigcup_{i \le m} \Lambda_i$. If $\text{dens}(\mathbf{\Lambda} \triangle (Q^n\alpha + \mathbf{\Lambda})) \overset{n\to\infty}{\longrightarrow} 0$ for all $\alpha \in L'$, then a modular coincidence relative to $Q^M L'$ occurs in Φ^M for some $M \in \mathbb{Z}_+$.*

Proof. Suppose that for all $n \in \mathbb{N}$, Φ^n does not admit any modular coincidence relative to $Q^n L'$. We assume that $S(\Phi)$ is a positive matrix without loss of the generality. Then the cluster $\mathbf{I} = (I_i)_{i \leq m} = (\Phi_{ij}(0))_{i \leq m}$, with $0 \in \Lambda_j$, has at least one element from each point set in $\mathbf{\Lambda} = (\Lambda_i)_{i \leq m}$. We claim that for all $n \in \mathbb{Z}_+$, $\text{supp}(\Phi^n(\mathbf{I}))$ intersects every coset $x + Q^n L'$ of $Q^n L'$ in L non-trivially. Indeed, $L = \bigcup_{i \leq m} \Lambda_i$ and each $\Lambda_i \subset I_i + L'$ for $i \leq m$. So

$$\begin{aligned} L &= \bigcup_{j \leq m} \bigcup_{i \leq m} (\Phi^n)_{ji}(I_i + L') \\ &= \bigcup_{j \leq m} \bigcup_{i \leq m} ((\Phi^n)_{ji}(I_i) + Q^n L') \\ &= \text{supp}(\Phi^n(\mathbf{I})) + Q^n L'. \end{aligned}$$

So the claim follows.

To test modular coincidence we need only know about the translation parts of Φ^n and to which coset of L' each Λ_i belongs (see Def. 7.2 and (7.4)). Since $\mathbf{I}$ has at least one element of each colour type $i = 1, \ldots, m$, modular coincidence can be tested on $\Phi^n(\mathbf{I})$: if $(a + Q^n L') \cap \text{supp}(\Phi^n(\mathbf{I})) \subset \Lambda_i$ for some i, then it means all the mappings in Φ^n which contribute to produce points in $a + Q^n L'$ lie on the i-row of Φ^n. So Φ^n has modular coincidence relative to $Q^n L'$.

Since Q is an expansive map, there is a parallelepiped $D = aD_0$ for which $Q(D) \supset \lambda D$ and $D \supset (\text{supp}(\mathbf{I}) \cup t(\Phi))$ for some $\lambda > 1$. Then by Lemma 7.3 there is $r > 0$ such that $\text{supp}(\Phi^n(\mathbf{I})) \subset Q^n(rD)$ for all $n \in \mathbb{Z}_+$.

We have assumed that there is no modular coincidence. Thus, for any $a \in L$ and any $n \geq 1$, there exists $i \leq m$ and

$$x, y \in \text{supp}(\Phi^n(\mathbf{I})) \cap (a + Q^n L') \text{ such that } x \in \Lambda_i,\ y \notin \Lambda_i. \tag{7.8}$$

Let $\beta_{-j} = -\beta_j$ for $j = 1, \ldots, d$. We can write $y - x$ as a non-negative integer linear combination of the vectors $Q^n \beta_j$, $j \in \{\pm 1, \ldots, \pm d\}$. Now $Q^n(rD)$ is a parallelepiped generated by $Q^n \beta_j$, $1 \leq j \leq d$, which contains $\text{supp}(\Phi^n(\mathbf{I}))$. Then there exists a path $x = x_1, x_2, \ldots, x_s = y$ entirely in $Q^n(rD)$ whose steps are each of the form $x_{l+1} - x_l = Q^n \beta_j, |j| \leq d$ and we see that there is a $x' \in Q^n(rD) \cap (a + Q^n L') \cap \Lambda_i$ such that $x' + Q^n \beta_j \in (Q^n(rD) \cap (a + Q^n L')) \backslash \Lambda_i$ for some j, $|j| \leq d$. It follows that

$$\sharp \left[\cup_{|j| \leq d} (\mathbf{\Lambda} \triangle (Q^n \beta_j + \mathbf{\Lambda})) \cap Q^n(rD) \right] \geq |\det Q^n| \cdot |L/L'|, \tag{7.9}$$

since there are $|\det Q^n| \cdot |L/L'|$ cosets of $Q^n L'$ in L and each of the cosets contributes at least one point to our count.

Furthermore for a parallelepiped $c + Q^n(rD)$ containing $\mathrm{supp}(c + \Phi^n(\mathbf{I}))$, for which $c + \Phi^n(\mathbf{I})$ is a translate of $\Phi^n(\mathbf{I})$, the argument goes in the same way. For x, y in (7.8) we have

$$(x+c) \in \mathrm{supp}(c + \Phi^n(\mathbf{I})) \cap \Lambda_i, \quad (y+c) \in \mathrm{supp}(c + \Phi^n(\mathbf{I})) \backslash \Lambda_i, \quad y - x \in Q^n L'.$$

As above, this implies that for some $x'' \in (c + Q^n(rD)) \cap (c + a + Q^n L') \cap \Lambda_i$ we have $x'' + Q^n \beta_j \in ((c + Q^n(rD)) \cap (c + a + Q^n L')) \backslash \Lambda_i$ for some j, $|j| \leq d$. Thus, similarly to (7.9),

$$\sharp\ [\cup_{|j|\leq d}(\mathbf{\Lambda} \triangle (Q^n\beta_j + \mathbf{\Lambda})) \cap (c + Q^n(rD))] \geq |\det Q^n| \cdot |L/L'|. \tag{7.10}$$

Let

$$H_k = \{x \in L : x + (rD \cap \mathbf{\Lambda}) \subset F_k \cap \mathbf{\Lambda}\}, \quad \text{where } \{F_k\} \text{ is a van Hove sequence.}$$

and let $\widetilde{H}_k$ be a maximal set of $x \in H_k$ such that $(x + rD)$ are mutually disjoint. Since Q is invertible, $Q^n(x + rD)$ and $Q^n(y + rD)$ are disjoint if and only if $x + rD$ and $y + rD$ are disjoint. So $\{Q^n(x + rD) \cap \mathbf{\Lambda} : \ x \in \widetilde{H}_k\}$ is a set of disjoint $\mathbf{\Lambda}$-clusters, which need not be translates of each other. We claim that each of the clusters $Q^n(x+rD) \cap \mathbf{\Lambda}$, for $x \in \widetilde{H}_k$, contains a translate of $\Phi^n(\mathbf{I})$. Indeed, if $x \in \widetilde{H}_k$, then $x + \mathbf{I} \subset F_k \cap \mathbf{\Lambda}$ since $\mathbf{I} \subset rD \cap \mathbf{\Lambda}$. It follows that $\Phi^n(x + \mathbf{I}) \subset \Phi^n(\mathbf{\Lambda}) = \mathbf{\Lambda}$, $\Phi^n(x + \mathbf{I}) = Q^n x + \Phi^n(\mathbf{I})$ since $x + \mathbf{I}$ is a translate of the cluster $\mathbf{I}$, and

$$\begin{aligned} \mathrm{supp}(\Phi^n(x + \mathbf{I})) &= Q^n x + \mathrm{supp}(\Phi^n(\mathbf{I})) \\ &\subset Q^n x + Q^n(rD) \\ &= Q^n(x + rD). \end{aligned}$$

This claim, together with (7.10), yields

$$\begin{aligned} &\sharp\ [\cup_{|j|\leq d}(\mathbf{\Lambda} \triangle (Q^n\beta_j + \mathbf{\Lambda})) \cap Q^n(F_k)] \\ &\geq\ \sharp\ \widetilde{H}_k \cdot |\det Q^n| \cdot |L/L'|. \end{aligned} \tag{7.11}$$

Recall that

$$\mathrm{freq}(rD \cap \mathbf{\Lambda}, \mathbf{\Lambda}) = \lim_{k\to\infty} \frac{\sharp\ H_k}{\mathrm{Vol}(F_k)},$$

see Def. 2.6. Let $e := \mathrm{diam}(rD)$. The set $x + rD$, for $x \in L$, can intersect at most $\sharp\ ((rD)^{+e} \cap L)$ translates $y + rD$, $y \in L$. Thus,

$$\sharp\ \widetilde{H}_k \geq \frac{\sharp\ H_k}{\sharp\ ((rD)^{+e} \cap L)}.$$

Combining this with (7.11) we obtain

$$\frac{\sharp\,[\bigcup_{|j|\le d}(\mathbf{\Lambda}\triangle(Q^n\beta_j+\mathbf{\Lambda}))\cap Q^n(F_k)]}{\mathrm{Vol}(Q^n(F_k))} \ge \frac{\sharp\, H_k\cdot|L/L'|}{\mathrm{Vol}(F_k)\cdot\sharp\,((rD)^e\cap L)}.$$

Letting $k\to\infty$, we conclude that

$$\sum_{|j|\le d}\mathrm{dens}\,(\mathbf{\Lambda}\,\triangle\,(Q^n\beta_j+\mathbf{\Lambda})) \ge \frac{\mathrm{freq}(rD\cap\mathbf{\Lambda},\mathbf{\Lambda})}{\sharp((rD)^e\cap L)}\cdot|L/L'|\,.$$

Note that when passing to the limit in the left-hand side we used a van Hove sequence $\{Q^n(F_k)\}_{k\ge1}$. This is legitimate in view of Remark 6.13. By assumption, $\mathrm{dens}\,(\mathbf{\Lambda}\,\triangle\,(Q^n\alpha+\mathbf{\Lambda}))\stackrel{n\to\infty}{\longrightarrow}0$ for all $\alpha\in L'$, and in particular

$$\sum_{|j|\le d}\mathrm{dens}(\mathbf{\Lambda}\triangle(Q^n\beta_j+\mathbf{\Lambda}))\stackrel{n\to\infty}{\longrightarrow}0\,.$$

But $\mathrm{freq}(rD\cap\mathbf{\Lambda},\mathbf{\Lambda})>0$ by the repetitive property. This is a contradiction. □

Theorem 7.5 *Let $(\mathbf{\Lambda},\Phi)$ be a primitive substitution system on L with inflation Q. Let $(\mathbf{W},\Phi)$ be the corresponding associated Q-adic system. Suppose $\overline{L}=\bigcup_{i\le m}W_i$. Then*

(i) $S(\Phi^r)=(S(\Phi))^r$, *for all $r\ge1$;*

(ii) $\mu(W_i)=\frac{1}{q^r}\sum_{j\le m}(S(\Phi^r))_{ij}\mu(W_j)$, *for all $i\le m$, $r\ge1$;*

(iii) $W_i^\circ\ne\emptyset$ *and* $\mu(\partial W_i)=0$, *for all $i\le m$.*

Proof. For every measurable set $E\subset\overline{L}$ and any $f\in\Phi_{ij}$ where $f:x\mapsto Qx+a$, $\mu(f(E))=\mu(a+Q(E))=\frac{1}{|\det Q|}\mu(E)$. In particular, $\mu(f(W_j))=\frac{1}{q}\,w_j$, where $w_j:=\mu(W_j)$ and $q=|\det Q|$. We obtain

$$w_i\le\sum_{j=1}^{m}\frac{1}{q^r}\,(S(\Phi^r)_{ij})\,w_j,\quad \text{for any } r\ge1$$

from (7.3).

Let $w=(w_i)_{i\le m}$. Since $\overline{L}=\bigcup_{i\le m}W_i$, the Baire category theorem assures that for at least one i,

$$W_i^\circ\ne\emptyset \tag{7.12}$$

and then the primitivity gives this for all i. So $w > 0$ and

$$w \leq \frac{1}{q^r} S(\Phi^r) w \leq \frac{1}{q^r} S(\Phi)^r w, \quad \text{for any } r \geq 1\,. \tag{7.13}$$

Since $S(\Phi)^r$ is primitive and the PF eigenvalue of $S(\Phi)^r$ is $q^r = |\det Q|^r$ by 3.3, we have from [22, Lemma 1] that

$$w = \frac{1}{q^r} S(\Phi^r) w = \frac{1}{q^r} S(\Phi)^r w, \quad \text{for any } r \geq 1\,. \tag{7.14}$$

The positivity of w together with $S(\Phi^r) \leq S(\Phi)^r$ shows that $S(\Phi^r) = S(\Phi)^r$. This proves (i) and (ii).

Fix any $i \leq m$, let $W_i{}^\circ$ contain a basis open set $a + Q^r\overline{L'}$ with some $r \in \mathbb{Z}_{\geq 0}$ by (7.12). Since (Λ, Φ^r) is a substitution system, $a + Q^r\overline{L'} \subset W_i{}^\circ \subset W_i = \bigcup_{j \leq m} (\Phi^r)_{ij} W_j$. In particular, $(a + Q^r\overline{L'}) \cap g(W_k) \neq \emptyset$ for some $k \leq m$ and some $g \in (\Phi^r)_{ik}$. However $g(z + \overline{L'}) = Q^r(z + \overline{L'}) + t(g)$, where $\Lambda_k \subset z + L'$. So $(a + Q^r\overline{L'}) \cap (t(g) + Q^r z + Q^r\overline{L'}) \neq \emptyset$. This means $a + Q^r\overline{L'} = t(g) + Q^r z + Q^r\overline{L'}$. Thus

$$g(W_k) \subset g(z + \overline{L'}) = a + Q^r\overline{L'} \subset W_i{}^\circ\,. \tag{7.15}$$

For all $f \in (\Phi^r)_{ij}$, $j \leq m$, f is clearly an open map, so $\bigcup_{j \leq m} (\Phi^r)_{ij}(W_j{}^\circ) \subset W_i{}^\circ$. Thus

$$\begin{aligned} \partial W_i = W_i \setminus W_i{}^\circ &= \left(\bigcup_{j \leq m} (\Phi^r)_{ij}(W_j) \right) \setminus W_i{}^\circ \\ &\subset \bigcup_{j \leq m} \left((\Phi^r)_{ij}(W_j) \setminus (\Phi^r)_{ij}(W_j{}^\circ) \right) \\ &\subset \bigcup_{j \leq m} (\Phi^r)_{ij}(\partial W_j)\,. \end{aligned} \tag{7.16}$$

Note that due to (7.15) at least one g in $(\Phi^r)_{ij}$ does not contribute to the relation (7.16).

Let $v_i := \mu(\partial W_i)$, $i \leq m$, and $v := (v_i)_{i \leq m}$. So $v \leq \frac{1}{q^r} S(\Phi^r) v$. Actually, by what we just said,

$$0 \leq v \leq \frac{1}{q^r} S' v \leq \frac{1}{q^r} S(\Phi^r) v = \frac{1}{q^r} S(\Phi)^r v, \tag{7.17}$$

where $S' \leq S(\Phi)^r, S' \neq S(\Phi)^r$. Now applying [22, Lemma 1] again we obtain equality throughout (7.17). But by [22, Lemma 2] the eigenvalues of $\frac{1}{q^r} S'$ are strictly less in absolute value than the PF eigenvalue of $\frac{1}{q^r} S(\Phi)^r$, which is 1. This forces $v = 0$, and hence $\mu(\partial W_i) = 0, i \leq m$. $\square$

Theorem 7.6 *Let (Λ, Φ) be a primitive substitution system on L with inflation Q and $L = \bigcup_{i \le m} \Lambda_i$. If there is a modular coincidence relative to $Q^M L'$ in Φ^M, then $W_s^\circ \cap W_t^\circ = \emptyset$, where $s, t \le m$, $s \ne t$.*

Proof. By assumption, there is Λ_i such that

$$a + Q^M L' = \bigcup_{j \le m} \bigcup_{f \in (\Phi^M)_{ij}[a]} f(\Lambda_j) \subset \Lambda_i \text{ for some } a \in L. \tag{7.18}$$

Assume $(\Phi^M)_{ik}[a] \ne \emptyset$ for some k. Pick $f \in (\Phi^M)_{ik}[a]$, where $f : x \mapsto Q^M x + t(f)$. Then $Q^M y + t(f) \equiv a \bmod Q^M L'$, where $\Lambda_k \subset y + L'$, and $f(\Lambda_k) \subset a + Q^M L'$.

Take any $s \le m$ and suppose $\Lambda_s \subset z + L'$. Pick $g \in \Phi_{ks} \ne \emptyset$, where $g : x \mapsto Qx + t(g)$. Then

$$\begin{aligned} f \circ g(\Lambda_s) &= Q^M(Q(\Lambda_s) + t(g)) + t(f) \\ &= Q^{M+1}(\Lambda_s) + Q^M(t(g)) + t(f) \\ &\subset Q^{M+1}(z) + Q^M(t(g)) + t(f) + Q^{M+1}L'. \end{aligned}$$

Let $c := Q^{M+1}(z) + Q^M(t(g)) + t(f)$. So

$$f \circ g(\Lambda_s) \subset c + Q^{M+1}L'.$$

Let $p := f \circ g \in (\Phi^{M+1})_{is}$. Since $f \circ g(\Lambda_s) \subset a + Q^M L' \subset \Lambda_i$,

$$(a + Q^M L') \cap (c + Q^{M+1}L') \ne \emptyset.$$

Thus

$$c + Q^{M+1}L' \subset a + Q^M L' \subset \Lambda_i.$$

Let

$$H_j := \{h \in (\Phi^{M+1})_{ij} : h(\Lambda_j) \subset c + Q^{M+1}L'\}, \quad \text{where } j \le m.$$

So

$$c + Q^{M+1}L' = \bigcup_{j \le m} \bigcup_{h \in H_j} h(\Lambda_j). \tag{7.19}$$

Note that for any $j \le m$ and $h \in H_j$, $Q^{M+1}x + t(h) \equiv c \bmod Q^{M+1}L'$, where $\Lambda_j \subset x + L'$. So we can write (7.19) more explicitly as follows;

$$\begin{aligned} c + Q^{M+1}L' &= \bigcup_{\substack{j \le m \\ h \in H_j}} \{c + Q^{M+1}\alpha_h + Q^{M+1}\Lambda_j : \\ &\qquad c + Q^{M+1}\alpha_h = t(h), \text{where } \alpha_h \in L\}. \end{aligned} \tag{7.20}$$

So

$$L' = \bigcup_{j \le m} \bigcup_{h \in H_j} (\alpha_h + \Lambda_j).$$

Note $\alpha_p + \Lambda_s \subset L'$. Separating off Λ_s, we get

$$-\alpha_p + L' = \Lambda_s \cup (\bigcup_{j \le m} \bigcup_{h \in H'_j} (-\alpha_p + \alpha_h + \Lambda_j)), \tag{7.21}$$

where $H'_j := H_j$ if $j \ne s$, and $H'_s := H_s \backslash \{p\}$. Note that the decompositions of (7.21) are disjoint. But we also know that Λ_s and $\bigcup\{\Lambda_j : \Lambda_j \subset -\alpha_p + L', j \ne s\}$ are disjoint. So it follows that

$$\bigcup\{\Lambda_j : \Lambda_j \subset -\alpha_p + L', j \ne s\} \subset \bigcup_{j \le m} \bigcup_{h \in H'_j} (-\alpha_p + \alpha_h + \Lambda_j). \tag{7.22}$$

Taking closures to both sides of (7.22),

$$\bigcup\{W_j : \Lambda_j \subset -\alpha_p + L', j \ne s\} \subset \bigcup_{j \le m} \bigcup_{h \in H'_j} (-\alpha_p + \alpha_h + W_j). \tag{7.23}$$

On the other hand, if we apply Theorem 7.5 to Φ^{M+1} and look at (7.20), we get

$$\mu(c + Q^{M+1}\overline{L'}) = \sum_{j \le m} \sum_{h \in H_j} \mu(Q^{M+1}(\alpha_h + W_j) + c).$$

Hence

$$\mu(\overline{L'}) = \sum_{j \le m} \sum_{h \in H_j} \mu(\alpha_h + W_j).$$

So

$$\mu(-\alpha_p + \overline{L'}) = \sum_{j \le m} \sum_{h \in H_j} \mu(-\alpha_p + \alpha_h + W_j).$$

Thus

$$\mu(-\alpha_p + \overline{L'}) = \mu(W_s) + \left(\sum_{j \le m} \sum_{h \in H'_j} -\alpha_p + \alpha_h + \mu(W_j) \right)$$

which, after taking closures in (7.21), gives us

$$\mu\left(W_s \cap \left(\bigcup_{j \le m} \bigcup_{h \in H'_j} (-\alpha_p + \alpha_h + W_j) \right) \right) = 0. \tag{7.24}$$

Finally from (7.23) and (7.24) we obtain

$$\mu(W_s \cap (\bigcup\{W_j : \Lambda_j \subset -\alpha_p + L', j \neq s\})) = 0.$$

It shows that $W_s{}^\circ \cap W_j{}^\circ = \emptyset$ for any j with $\Lambda_j \subset -\alpha_p + L'$, $j \neq s$. It is easy to see that $W_s{}^\circ \cap W_k{}^\circ = \emptyset$, where $\Lambda_k \not\subset -\alpha_p + L'$. Since s is arbitrary in $\{1, \ldots, m\}$, $W_s{}^\circ \cap W_t{}^\circ = \emptyset$ for all $s, t \leq m, s \neq t$. □

Taking the compact Abelian group $\overline{L}$ as an internal space, we consider the following cut and project scheme;

$$\begin{array}{ccccc} \mathbb{R}^d & \xleftarrow{\pi_1} & \mathbb{R}^d \times \overline{L} & \xrightarrow{\pi_2} & \overline{L} \\ & & \cup & & \\ L & \longleftarrow & \widetilde{L} & \longrightarrow & L \\ t & \longleftarrow & (t,t) & \longrightarrow & t \end{array} \tag{7.25}$$

where $\widetilde{L} := \{\ (t,t)\ :\ t \in L\} \subset \mathbb{R}^d \times \overline{L}$.

In fact, $\overline{L}$ is a compact Abelian group and $\widetilde{L} \subset \mathbb{R}^d \times \overline{L}$ is a lattice, i.e. a discrete subgroup for which the quotient group $(\mathbb{R}^d \times \overline{L})/\widetilde{L}$ is compact. Furthermore, $\pi_1|_{\widetilde{L}}$ is injective and $\pi_2(\widetilde{L})$ is dense in $\overline{L}$.

Proposition 7.7 *Let $\Lambda_i, i \leq m$, be disjoint point sets of the lattice L in $\mathbb{R}^d$ for which $L = \bigcup_{i \leq m} \Lambda_i$. If $W_i{}^\circ \cap W_j{}^\circ = \emptyset$ for all $i \neq j$, then $\Lambda(W_i{}^\circ) \subset \Lambda_i \subset \Lambda(W_i)$ for all $i \leq m$. Furthermore if each Λ_i is repetitive, then $W_i = \overline{W_i{}^\circ}$. In other words, Λ_i is a model set for all $i \leq m$.*

Proof. For $i \leq m$, let $x \in \Lambda(W_i{}^\circ)$. From the assumption that $W_i{}^\circ \cap W_j{}^\circ = \emptyset$ for all $j \neq i$, we can find a neighbourhood U of x in $\overline{L}$ such that $U \cap W_j{}^\circ = \emptyset$ for all $j \neq i$. So $x \notin W_j$ for all $j \neq i$, which means $x \notin \bigcup_{j \neq i} \Lambda_j = L \backslash \Lambda_i$. Thus $x \in \Lambda_i$. Therefore $\Lambda(W_i{}^\circ) \subset \Lambda_i \subset \Lambda(W_i)$ for all $i \leq m$. If each Λ_i is repetitive, by [41, Cor. 4.4] we get $W_i = \overline{W_i{}^\circ}$. □

Theorem 7.8 (Schlottmann [41]) *If $\Gamma \subset \mathbb{R}^d$ is a regular model set, then Γ has pure point diffraction spectrum, i.e. the Fourier transform of its volume averaged autocorrelation measure is a pure point measure.*

This theorem was established for real internal spaces by [17] and in full generality, as stated here, in [41]. For a new simpler proof of this result see [4]. □

Gathering all the results, we can state the following theorem, which says, in particular, that $\mathbf{\Lambda}$ has pure point diffraction spectrum if and only if each Λ_i is a regular model set with respect to a certain cut-and-project scheme, and that generalizes Dekking's well-known criterion for pure point diffractivity to lattice substitutions in $\mathbb{R}^d$.

Theorem 7.9 *Let $\mathbf{\Lambda}$ be a Delone multiset with expansive map Q such that $(\mathbf{\Lambda}, \Phi)$ is a primitive substitution system, $L = \bigcup_{i \le m} \Lambda_i$ for some lattice L in $\mathbb{R}^d$, and every $\mathbf{\Lambda}$-cluster is legal. Let $L' = L_1 + \cdots + L_m$, where $L_i =< \Lambda_i - \Lambda_i >$. The following are equivalent:*

(i) *$\mathbf{\Lambda}$ has pure point diffraction spectrum;*
(ii) *$\mathbf{\Lambda}$ has pure point dynamical spectrum;*
(iii) $\mathrm{dens}(\mathbf{\Lambda} \triangle (Q^n \alpha + \mathbf{\Lambda})) \overset{n \to \infty}{\longrightarrow} 0$ *for all $\alpha \in L'$;*
(iv) *A modular coincidence relative to $Q^M L'$ occurs in Φ^M for some M;*
(v) *Each Λ_i is a regular model set for $i \le m$, relative to the CPS (7.25).*

Proof. It is easy to see that $\mathbf{\Lambda}$ has FLC since $\mathbf{\Lambda}$ lies in a lattice L in $\mathbb{R}^d$, that $\mathbf{\Lambda}$ is repetitive since every $\mathbf{\Lambda}$-cluster is legal, and that $\mathbf{\Lambda}$ has UCF. Thus the proof goes as follows:

(i) $\Leftrightarrow$ (ii) Theorem 5.1.
(ii) $\Leftrightarrow$ (iii) Prop. 6.12.
(iii) $\Rightarrow$ (iv) Theorem 7.4.
(iv) $\Rightarrow$ (v) Theorem 7.5 and 7.6, Prop. 7.7.
(v) $\Rightarrow$ (i) Theorem 7.8. □

Remark 7.10 If $\mathbf{\Lambda}$ is representable but not repetitive, then it still may be possible to use the criteria of Theorem 7.9 to check for pure point diffractivity, using the argument in Remark 6.5. One only has to find another $\mathbf{\Gamma} \in X_{\mathbf{\Lambda}}$ which does satisfy the conditions in Theorem 7.9, since $\mathbf{\Lambda}$ is pure point diffractive if and only if $\mathbf{\Gamma}$ is pure point diffractive (see Theorem 5.1).

Example 7.11 (Substitution Delone multiset with modular coincidence) Consider a substitution defined by $a \to abc$, $b \to dcb$, $c \to cda$, and $d \to dab$. We can consider a corresponding MFS Φ as follows;

$$\Phi = \begin{pmatrix} \{3x\} & \emptyset & \{3x+2\} & \{3x+1\} \\ \{3x+1\} & \{3x+2\} & \emptyset & \{3x+2\} \\ \{3x+2\} & \{3x+1\} & \{3x\} & \emptyset \\ \emptyset & \{3x\} & \{3x+1\} & \{3x\} \end{pmatrix}$$

A Delone multiset $\mathbf{\Lambda} = (\Lambda_a, \Lambda_b, \Lambda_c, \Lambda_d)$ generated from $(\{0\}, \{-1\}, \emptyset, \emptyset)$ is fixed under Φ. On the real line, $\mathbf{\Lambda}$ looks like

$\cdots$	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	$\cdots$
$\cdots$	c	d	a	d	c	b	a	b	c	d	c	b	c	d	a	$\cdots$

Note that $< \Lambda_\alpha - \Lambda_\alpha \ : \ \alpha \in \{a, b, c, d\} >= 2\mathbb{Z}$ and that $\Lambda_a \subset 2\mathbb{Z}$, $\Lambda_b \subset 1 + 2\mathbb{Z}$, $\Lambda_c \subset 2\mathbb{Z}$, and $\Lambda_d \subset 1 + 2\mathbb{Z}$.

$$\Phi \begin{pmatrix} \Lambda_a \\ \Lambda_b \\ \Lambda_c \\ \Lambda_d \end{pmatrix} \subset \begin{pmatrix} 6\mathbb{Z} & \cup & \emptyset & \cup & (6\mathbb{Z}+2) & \cup & (6\mathbb{Z}+4) \\ (6\mathbb{Z}+1) & \cup & (6\mathbb{Z}+5) & \cup & \emptyset & \cup & (6\mathbb{Z}+5) \\ (6\mathbb{Z}+2) & \cup & (6\mathbb{Z}+4) & \cup & 6\mathbb{Z} & \cup & \emptyset \\ \emptyset & \cup & (6\mathbb{Z}+3) & \cup & (6\mathbb{Z}+1) & \cup & (6\mathbb{Z}+3) \end{pmatrix}.$$

Since each of $\Phi[3]$ and $\Phi[5]$ lies in one row of Φ, the modular coincidence is confirmed. By Theorem 7.9 the sets $\Lambda_a, \Lambda_b, \Lambda_c, \Lambda_d$ are pure point diffractive and regular model sets.

Chapter 8

Model Multisets and Two Dynamical Hulls

We have defined model multisets in SubSec. 2.1.1 and introduced a dynamical hull $X_{\mathbf{\Lambda}}$ with a topology based on local structure in the SubSec. 4.1.1. We now introduce another dynamical hull $A(\mathbf{\Lambda})$ with a topology based on the average overall structure. Although the two dynamical hulls are quite different from each other in general, when the generating point set is a regular model multiset, there is a continuous map between the two dynamical hulls. Furthermore, under certain assumptions the existence of such a map ensures that the point set is a regular model multiset. Thus we can characterize regular model multisets using this mappings.

This characterization was first given in [2] but only in the context of single coloured point sets and with an analysis on dynamical systems. However a lot of physical samples and mathematical examples can be idealized and represented by multisets, and there is no direct way to obtain the similar properties on multisets from the properties on single coloured point sets. So we here extend the characterization in multiset cases.

A nice aspect of the characterization is that when we try to check if a multiset is a regular model multiset, instead of directly using the set, we can choose an easier set in the local hull $X_{\mathbf{\Lambda}}$ and work with it and then deduce information we need for the original set.

In Sec. 8.1 we define $A(\mathbf{\Lambda})$ the completion of $\mathbb{R}^d$ with the topology based on the average overall structure. When $A(\mathbf{\Lambda})$ is compact, we can construct a cut and project scheme. We prove several properties here which we are going to use in later subsections.

In Sec. 8.2 we show that the existence of a continuous surjective $\mathbb{R}^d$-map β :

$X_\Lambda \to \mathbb{A}(\Lambda)$ (so called '*torus parametrization*') which is one-to-one a.e. ensures that Λ is a regular model multiset.

In Sec. 8.3 we show the converse. That is, if Λ is a regular model multiset then the continuous surjective $\mathbb{R}^d$-map which is one-to-one a.e. exists.

Briefly we prove the following theorem in this chapter

Theorem 8.1 *Let Λ be a repetitive Meyer multiset in $\mathbb{R}^d$. Then there is a continuous $\mathbb{R}^d$-map $\beta : X_\Lambda \to \mathbb{A}(\Lambda)$ which is one-to-one a.e. with respect to $\mathbb{A}(\Lambda)$ if and only if Λ (or equivalently each element of X_Λ) is a regular model multiset.*

8.1 Dynamical system with autocorrelation topology

In this chapter we consider an averaging sequence $\mathcal{A} = \{A_n\}_{n\in\mathbb{N}}$ satisfying (i) each A_n is a compact set of $\mathbb{R}^d$, (ii) for all n, $A_n \subset A^\circ_{n+1}$, (iii) $\bigcup_{n\in\mathbb{N}} A_n = \mathbb{R}^d$, and (iv) (the *van Hove* property) for all $r > 0$, $\lim_{n\to\infty} \mathrm{Vol}(\partial(A_n))^{+r}/\mathrm{Vol}(A_n) = 0$, as in [32].

We say that Λ is locally finite if for any compact set K in $\mathbb{R}^d$, $K \cap \Lambda$ is finite (equivalently each Λ_i is discrete and closed). For the rest of this chapter we consider Λ a locally finite multiset in which each colour set Λ_i is relatively dense.

Now we construct the autocorrelation group $\mathbb{A}(\Lambda)$. Let Λ', Λ'' be locally finite multisets in $\mathbb{R}^d$. We define

$$d(\Lambda', \Lambda'') := \lim_{n\to\infty} \sup \frac{\sum_{i=1}^m \sharp((\Lambda'_i \triangle \Lambda''_i) \cap A_n)}{\mathrm{Vol}(A_n)}. \tag{8.1}$$

For each open neighbourhood V of 0 in $\mathbb{R}^d$ and each $\epsilon > 0$, define

$$U(V,\epsilon) := \{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d : d(-v + x + \Lambda, y + \Lambda) < \epsilon \text{ for some } v \in V\}.$$

Let $\mathcal{U} = \{U(V,\epsilon) \subset \mathbb{R}^d \times \mathbb{R}^d : V$ is an open neighbourhood of 0 in $\mathbb{R}^d$ and $\epsilon > 0\}$. Then $\mathcal{U}$ forms a fundamental set of entourages for a uniformity on $\mathbb{R}^d$. Since each $U(V,\epsilon)$ is $\mathbb{R}^d$-invariant, we obtain a topological group structure on $\mathbb{R}^d$. We will call it the *autocorrelation topology*. Let $\mathbb{A}(\Lambda)$ be the Hausdorff completion of $\mathbb{R}^d$ in this topology, which is a new topological group (see [7, Chapter III, §3.4]).

On the other hand, we define $\mathcal{D}$ as a collection of all locally finite multisets in $\mathbb{R}^d$. We obtain a metric by defining the equivalence relation

$$\Lambda' \equiv \Lambda'' \Leftrightarrow d(\Lambda', \Lambda'') = 0$$

and factoring d through it. Then $\mathcal{D}/\equiv$ is a complete space with the metric (this uses the property of the averaging sequence $\mathcal{A}$) [32]. So the closure of the orbit $\mathbb{R}^d + [\Lambda]$ in the autocorrelation topology, where $[\Lambda]$ is an equivalence class of Λ, is isomorphic to $\mathbb{A}(\Lambda)$ (see [7, Chapter II, §3.6, Cor.]).

Although $\mathbb{A}(\Lambda)$ is the completion of $\mathbb{R}^d$ under the autocorrelation topology, it may be more enlightening to think of $\mathbb{A}(\Lambda)$ as the hull (completion) of Λ under translation action by $\mathbb{R}^d$ when the topology is supplied by mixing that of pseudo-metric d and the given topology of $\mathbb{R}^d$. For $y \in \mathbb{R}^d$ and $U \in \mathcal{U}$, define $U[y] = \{x \in \mathbb{R}^d : (x,y) \in U\}$. Let $P_\epsilon = \{x \in \mathbb{R}^d : d(x + \Lambda, \Lambda) < \epsilon\}$ for each $\epsilon > 0$. Then $U(V,\epsilon)[0] = P_\epsilon + V$. Note that for any $\epsilon > 2d(\Lambda, \emptyset)$, $P_\epsilon = \mathbb{R}^d$.

Proposition 8.2 *Let Λ be a locally finite multiset in $\mathbb{R}^d$. Then $\mathbb{A}(\Lambda)$ is compact if and only if for all $\epsilon > 0$, P_ϵ is relatively dense in $\mathbb{R}^d$.*

PROOF. Suppose that $\mathbb{A}(\Lambda)$ is compact. Since $\mathbb{A}(\Lambda)$ is the completion of $\mathbb{R}^d$, $\mathbb{R}^d$ is precompact. So for any $\epsilon > 0$ and open neighbourhood V of 0 in $\mathbb{R}^d$ whose closure is compact, there are $t_j \in \mathbb{R}^d$ with $1 \le j \le M$ such that

$$\mathbb{R}^d \subset \bigcup_{j=1}^{M} (t_j + U(V,\epsilon)[0]) \subset \bigcup_{j=1}^{M} (t_j + P_\epsilon + V) \subset P_\epsilon + K,$$

where $K := \bigcup_{j=1}^{M} \overline{(t_j + V)}$ is compact. Therefore P_ϵ is relatively dense for all $\epsilon > 0$.

Conversely, we assume that P_ϵ is relatively dense for all $\epsilon > 0$. Let $\epsilon > 0$ and V' be an open neighbourhood of 0 in $\mathbb{R}^d$. From the assumption, $\mathbb{R}^d \subset P_\epsilon + K'$ for some compact set K'. We can cover K' with finite translations of V' i.e. there are $t_1, \ldots, t_L \in \mathbb{R}^d$ such that $K' \subset \bigcup_{j=1}^{L} (t_j + V')$. Thus

$$\mathbb{R}^d \subset P_\epsilon + K' \subset \bigcup_{j=1}^{L} (t_j + P_\epsilon + V') = \bigcup_{j=1}^{L} U(V',\epsilon)[t_j].$$

Hence $\mathbb{R}^d$ is precompact and $\mathbb{A}(\Lambda)$ is compact. □

Let us assume that P_ϵ is relatively dense for all $\epsilon > 0$. We define

$$L := \langle \Lambda_i - \Lambda_i, \Lambda_j : i, j \le m \rangle.$$

Then $P_\epsilon \subset L$ for any $\epsilon < 2d(\Lambda, \emptyset)$. We can define a uniformity on L using the pseudo metric d. Then the P_ϵ's are a basis for a fundamental system of neighbourhoods of 0 in the corresponding topology. We define H to be a (Hausdorff) completion of L

in this uniformity. Then the completion H is a locally compact Abelian group (see [4]). By definition of H there exists a uniformly continuous mapping $\phi : L \to H$ such that $\phi(L)$ is dense in H. Now we can construct a cut and project scheme:

$$\begin{array}{ccccc} \mathbb{R}^d & \xleftarrow{\pi_1} & \mathbb{R}^d \times H & \xrightarrow{\pi_2} & H \\ & & \cup & & \\ L & \longleftarrow & \widetilde{L} & \longrightarrow & \phi(L) \\ & & & & \\ x & \longleftarrow & (x, \phi(x)) & \longrightarrow & \phi(x), \end{array} \tag{8.2}$$

where $\widetilde{L} = \{(x, \phi(x)) : x \in L\}$. Here $\widetilde{L}$ is relatively dense and a discrete subgroup in $\mathbb{R}^d \times H$, and so the factor group $\mathbb{T}(\mathbf{\Lambda}) := (\mathbb{R}^d \times H)/\widetilde{L}$ is compact (see [4]). The pseudo-metric d on L determines a corresponding metric d^H on H. Let B^H_ϵ denote the corresponding open ball of radius ϵ in H. Then $\phi(P_\epsilon) = \phi(L) \cap B^H_\epsilon$. We define

$$\iota : \mathbb{R}^d \to (\mathbb{R}^d \times \phi(L))/\widetilde{L} \text{ by } \iota(x) = (x, 0) + \widetilde{L}.$$

Proposition 8.3 *Let $\mathbf{\Lambda}$ be a locally finite multiset in $\mathbb{R}^d$. Suppose that P_ϵ is relatively dense for all $\epsilon > 0$. Then $\mathbb{A}(\mathbf{\Lambda}) \cong \mathbb{T}(\mathbf{\Lambda})$.*

Proof. $\mathbb{T}(\mathbf{\Lambda})$ may be viewed as the completion of $\mathbb{R}^d$ under the uniform topology which is the coarsest topology on $\mathbb{R}^d$ for which the map $\iota : \mathbb{R}^d \longrightarrow (\mathbb{R}^d \times \phi(L))/\widetilde{L}$ is continuous. For an open neighbourhood $(V \times \phi(P_\epsilon) + \widetilde{L})$ of 0 in $(\mathbb{R}^d \times \phi(L))/\widetilde{L}$ with $\epsilon < 2d(\mathbf{\Lambda}, \emptyset)$,

$$\begin{aligned} V \times \phi(P_\epsilon) + \widetilde{L} &= (V - P_\epsilon) \times \{0\} + \widetilde{L} \\ &= (V + P_\epsilon) \times \{0\} + \widetilde{L} = \iota(V + P_\epsilon) = \iota(U(V, \epsilon)[0]). \end{aligned}$$

This is the same topology on $\mathbb{R}^d$ as the autocorrelation topology on $\mathbb{R}^d$. Thus $\mathbb{A}(\mathbf{\Lambda}) \cong \mathbb{T}(\mathbf{\Lambda})$. □

Corollary 8.4 *Let $\mathbf{\Lambda}$ be a locally finite multiset with FLC. Suppose that there exists a continuous $\mathbb{R}^d$-map $\beta : X_{\mathbf{\Lambda}} \to \mathbb{A}(\mathbf{\Lambda})$. Then P_ϵ is relatively dense in $\mathbb{R}^d$ for all $\epsilon > 0$ and $\mathbb{A}(\mathbf{\Lambda}) \cong \mathbb{T}(\mathbf{\Lambda})$.*

Proof. Since $\mathbf{\Lambda}$ has FLC, $X_{\mathbf{\Lambda}}$ is compact. Note that $\beta(X_{\mathbf{\Lambda}})$ is dense in $\mathbb{A}(\mathbf{\Lambda})$ because β is a $\mathbb{R}^d$-map. The continuity of β gives us that $\mathbb{A}(\mathbf{\Lambda})$ is compact. Thus P_ϵ is relatively dense in $\mathbb{R}^d$ for all $\epsilon > 0$. By Prop. 8.3 we can conclude that $\mathbb{A}(\mathbf{\Lambda}) \cong \mathbb{T}(\mathbf{\Lambda})$. □

Thus we may identify $\mathbb{A}(\mathbf{\Lambda})$ with $\mathbb{T}(\mathbf{\Lambda})$ when it is convenient.

Proposition 8.5 *Let $\mathbf{\Lambda}$ be a Meyer multiset in $\mathbb{R}^d$ and set $\Delta_i := \Lambda_i - \Lambda_i$ for each $i \leq m$. If P_ϵ is relatively dense for all $\epsilon > 0$ then, for each $i \leq m$, Δ_i is precompact in L with the pseudo-metric topology and $\overline{\phi(\Delta_i)}$ is compact in H .*

PROOF. Choose any $\epsilon > 0$. Since each Λ_i is a Meyer set, $\mathbf{\Lambda}$ is locally finite and there exist finite sets J_i with $\Lambda_i - \Lambda_i \subset \Lambda_i + J_i$. It is easy to see that $\Delta_i - \Delta_i \subset \Delta_i + F_i$ for some finite set F_i and $\Delta_i + F_i \subset \Lambda_i + J_i + F_i$ for each $i \leq m$. Note that $\Lambda_i + J_i + F_i$ is locally finite. From the assumption, for any given P_ϵ we can find a compact set K_ϵ such that $\mathbb{R}^d \subset P_\epsilon + K_\epsilon$. So $\Delta_i \subset P_\epsilon + K_\epsilon$ for each $i \leq m$. Since each Λ_i is relatively dense, for small enough $\epsilon > 0$, $P_\epsilon \subset \cap_{i=1}^m \Delta_i$. Then by the assumption

$$(\Delta_i - P_\epsilon) \cap K_\epsilon \subset (\Delta_i - \Delta_i) \cap K_\epsilon \subset (\Delta_i + F_i) \cap K_\epsilon = N_i,$$

where N_i is a finite subset of L. Thus $\Delta_i \subset P_\epsilon + N_i$ for each $i \leq m$. So Δ_i is precompact (totally bounded) in L with the pseudo-metric topology and $\overline{\phi(\Delta_i)}$ is compact in H for each $i \leq m$. □

8.2 Torus parameterizations for model multisets

We say that $\mathbf{\Gamma}$ is *non-singular* in $X_{\mathbf{\Lambda}}$ if $\beta^{-1}(\beta(\{\mathbf{\Gamma}\})) = \{\mathbf{\Gamma}\}$. The set of non-singular elements of $X_{\mathbf{\Lambda}}$ consists of full $\mathbb{R}^d$-orbits.

Proposition 8.6 *Let $\mathbf{\Lambda}$ be a multiset in $\mathbb{R}^d$ with FLC. Suppose that there exists a continuous $\mathbb{R}^d$-map $\beta : X_{\mathbf{\Lambda}} \longrightarrow \mathbb{A}(\mathbf{\Lambda})$. If $\mathbf{\Gamma}$ is non-singular in $X_{\mathbf{\Lambda}}$, then given any $M \in \mathbb{Z}_+$, there is $\epsilon = \epsilon(M) > 0$ such that for any $t \in P_\epsilon$, $(t + \mathbf{\Gamma}) \cap A_M = \mathbf{\Gamma} \cap A_M$.*

PROOF. Suppose that there is a positive integer M such that for any $n > 0$ there exists $t_n \in P_{(1/2^n)}$ for which

$$(t_n + \mathbf{\Gamma}) \cap A_M \neq \mathbf{\Gamma} \cap A_M.$$

Since $X_{\mathbf{\Gamma}} \subset X_{\mathbf{\Lambda}}$ is compact from FLC of $\mathbf{\Lambda}$, $\{t_n + \mathbf{\Gamma}\}_n$ has a convergent subsequence $\{t_{n_k} + \mathbf{\Gamma}\}_k$ such that $t_{n_k} + \mathbf{\Gamma} \to \mathbf{\Gamma}'$. On the other hand, identifying $\mathbb{A}(\mathbf{\Lambda})$ with $\mathbb{T}(\mathbf{\Lambda})$,

$$\beta(t_{n_k} + \mathbf{\Gamma}) = (t_{n_k}, 0) + \beta(\mathbf{\Gamma}) = (0, -\phi(t_{n_k})) + \beta(\mathbf{\Gamma}).$$

Since $\phi(t_{n_k}) \xrightarrow{k \to \infty} 0$, $\lim_{k \to \infty} \beta(t_{n_k} + \mathbf{\Gamma}) = \beta(\mathbf{\Gamma})$. And by the continuity of β,

$$\lim_{k \to \infty} \beta(t_{n_k} + \mathbf{\Gamma}) = \beta(\mathbf{\Gamma}').$$

Then $\beta(\mathbf{\Gamma}) = \beta(\mathbf{\Gamma}')$, and by the assumption that $\mathbf{\Gamma}$ is non-singular in $X_{\mathbf{\Lambda}}$ we get $\mathbf{\Gamma} = \mathbf{\Gamma}'$. It follows that $t_{n_k} + \mathbf{\Gamma} \xrightarrow{k \to \infty} \mathbf{\Gamma}$. This contradicts the choice of $\{t_{n_k}\}$. □

Proposition 8.7 *Let $\mathbf{\Lambda}$ be a multiset in $\mathbb{R}^d$ with FLC. Suppose that there exists a continuous $\mathbb{R}^d$-map $\beta : X_{\mathbf{\Lambda}} \to \mathbb{A}(\mathbf{\Lambda})$. If $\mathbf{\Gamma}$ is non-singular in $X_{\mathbf{\Lambda}}$ then there exist $s \in \mathbb{R}^d$ and non-empty open sets $U_i \subset H$ so that*

$$\Gamma_i = -s + \Lambda(U_i) \quad \textit{for each } i \le m$$

with respect to CPS (8.2). Furthermore each $\overline{U_i}$ is compact if and only if $\mathbf{\Gamma}$ is Meyer.

PROOF. Since $\mathbf{\Gamma} \in X_{\mathbf{\Lambda}}$, we can choose a compact set K for which $\mathbf{\Gamma} \cap K$ contains all colours of points in $\mathbf{\Gamma}$. Then we can find $s \in \mathbb{R}^d$ such that $\mathbf{\Gamma} \cap K = (-s + \mathbf{\Lambda}) \cap K$. For all $i \le m$, $s + \Gamma_i \subset \Lambda_i + L \subset L$. Furthermore, since β is a $\mathbb{R}^d$-map and $\mathbf{\Gamma}$ is non-singular, $s + \mathbf{\Gamma}$ is also non-singular. So we can translate and assume at the outset that $\mathrm{supp}(\mathbf{\Gamma}) \subset L$.

Let $i \le m$ and $x \in \Gamma_i$. For any $M > 0$ with $x \in \Gamma_i \cap A_M$, there is $\epsilon_x = \epsilon(M) > 0$ so that for any $y \in P_{\epsilon_x}$, $(y + \Gamma_i) \cap A_M = \Gamma_i \cap A_M$ by Prop. 8.6. This implies that $x - y \in \Gamma_i$ for any $y \in P_{\epsilon_x}$, and thus $x - P_{\epsilon_x} \subset \Gamma_i$. Since $P_{\epsilon_x} = -P_{\epsilon_x}$, $x + P_{\epsilon_x} \subset \Gamma_i$. Therefore

$$\Gamma_i = \bigcup_{x \in \Gamma_i} (x + P_{\epsilon_x}).$$

Recall that $\phi(P_{\epsilon_x}) = \phi(L) \cap B^H_{\epsilon_x}$ where $B^H_{\epsilon_x}$ is an open ball of radius ϵ_x in H. So

$$\Gamma_i = \Lambda(U_i) \quad \text{where } U_i = \bigcup_{x \in \Gamma_i} (\phi(x) + B^H_{\epsilon_x}) \text{ is open in } H.$$

From Prop. 8.2, P_ϵ is relatively dense for all $\epsilon > 0$. If $\mathbf{\Gamma}$ is Meyer, $\overline{U_i}$ is compact from Prop. 8.5. On the other hand, if each $\overline{U_i}$ is compact then $\Gamma_i - \Gamma_i \subset \Lambda(\overline{U_i} - \overline{U_i})$ which is uniformly discrete. So each Γ_i is a Meyer set. □

Proposition 8.8 *Let $\mathbf{\Lambda}$ be a multiset in $\mathbb{R}^d$ with FLC. Suppose that there exists a continuous $\mathbb{R}^d$-map $\beta : X_{\mathbf{\Lambda}} \to \mathbb{A}(\mathbf{\Lambda})$. If $\mathbf{\Gamma} \in X_{\mathbf{\Lambda}}$ is non-singular and for each $i \le m$ $\Gamma_i = \Lambda(U_i)$ with an open set $U_i \subset H$ with respect to CPS (8.2), then for each $i \le m$ $\phi(L) \cap \partial U_i = \emptyset$ with respect to CPS (8.2).*

PROOF. Suppose that $\phi(x) \in \phi(L) \cap \partial U_i$ for some $x \in L$. So $x \notin \Gamma_i = \Lambda(U_i)$. There is a sequence $\{y_n\}_n$ in Γ_i such that $\phi(y_n) \in \phi(L) \cap U_i$ and $\lim_{n\to\infty} \phi(y_n) = \phi(x)$. Since $X_{\mathbf{\Gamma}} \subset X_{\mathbf{\Lambda}}$ is compact, we can assume that $\{x - y_n + \mathbf{\Gamma}\}_n$ is a converging sequence in $X_{\mathbf{\Gamma}}$. Let $\mathbf{\Gamma}' := \lim_{n\to\infty}(x - y_n + \mathbf{\Gamma})$. Note that Γ'_i contains x and $\beta(\mathbf{\Gamma}') = \beta(\mathbf{\Gamma})$. Since $\mathbf{\Gamma}$ is non-singular in $X_{\mathbf{\Lambda}}$, $\mathbf{\Gamma}' = \mathbf{\Gamma}$. However $x \notin \Gamma_i$. This contradiction shows that the boundary of U_i cannot contribute to $\mathbf{\Gamma}$ and so $\phi(L) \cap \partial U_i = \emptyset$. □

Remark: In this last proposition it is possible to make the sets U_i a little nicer, replacing them by W_i° where $W_i := \overline{U_i}$, since these differ only from the U_i by boundary points, and these additional points do not affect the multiset $\mathbf{\Gamma}$. This set W_i has the nice property that W_i is the closure of its interior, a property often assumed in defining model sets.

Proposition 8.9 *Let $\mathbf{\Lambda}$ be a multiset in $\mathbb{R}^d$ for which $\Lambda(V_i^\circ) \subset \Lambda_i \subset \Lambda(\overline{V_i})$ where $\overline{V_i}$ is compact and $V_i^\circ \neq \emptyset$ for $i \leq m$, with respect to some CPS (see (2.7)). Then $\mathbf{\Lambda}$ has FLC, and for any $\mathbf{\Gamma} \in X_{\mathbf{\Lambda}}$ there exists $(-s, -h) \in \mathbb{R}^d \times H$ so that*

$$-s + \Lambda(h + V_i^\circ) \subset \Gamma_i \subset -s + \Lambda(h + \overline{V_i}) \ \ \textit{for each } i \leq m.$$

If $\bigcup_{i=1}^m \Gamma_i \subset L$, then we can take $s = 0$. Furthermore, if there exists a continuous $\mathbb{R}^d$-map $\beta : X_{\mathbf{\Lambda}} \to \mathbb{A}(\mathbf{\Lambda})$ such that $\beta(\mathbf{\Lambda}) = (0,0) + \widetilde{L}$, then $\beta(\mathbf{\Gamma}) = (-s, -h) + \widetilde{L}$.

Proof. For each $\mathbf{\Gamma} \in X_{\mathbf{\Lambda}}$ we can choose $s \in \mathbb{R}^d$ such that $s + \bigcup_{i=1}^m \Gamma_i \subset L$. We first claim that $\mathbf{\Lambda}$ has FLC. Since $\Lambda(V_i^\circ) \subset \Lambda_i \subset \Lambda(\overline{V_i})$ for each $i \leq m$,

$$\Lambda(\bigcup_{i \leq m} V_i^\circ) \subset \bigcup_{i \leq m} \Lambda_i \subset \Lambda(\bigcup_{i \leq m} \overline{V_i}).$$

Note that $W := \bigcup_{i \leq m} \overline{V_i}$ is compact. So for any compact set $K \subset \mathbb{R}^d$, $(\Lambda(W) - \Lambda(W)) \cap K$ has finite elements. It implies that for any $x \in \mathrm{supp}(\mathbf{\Lambda})$ there are, up to translation, only finitely many different subsets of the form $(K + x) \cap \mathbf{\Lambda}$. Therefore $\mathbf{\Lambda}$ has FLC. Then $X_{\mathbf{\Lambda}}$ is compact and we can find $\{t_n\}_n \subset L$ such that $\{t_n + \mathbf{\Lambda}\}$ converges to $s + \mathbf{\Gamma}$ in $X_{\mathbf{\Lambda}}$. There exists $n_0 \in \mathbb{Z}_+$ such that for any $k, l \geq n_0$, $(t_k + \mathbf{\Lambda}) \cap (t_l + \mathbf{\Lambda}) \neq \emptyset$ and so $t_k - t_l \in \Lambda_i - \Lambda_i$ for some $i \leq m$. Thus $\phi(t_k) - \phi(t_l) \in \overline{V_i} - \overline{V_i}$. Since $\overline{V_i} - \overline{V_i}$ is compact, we can find a convergent subsequence of $\{\phi(t_n)\}_n$. Without loss of generality, we can assume that

$$\lim_{n \to \infty} \phi(t_n) =: h \in H.$$

For each $i \leq m$, if $z \in \Lambda(h + V_i^\circ)$, then there is $n_{i,1} \in \mathbb{Z}_+$ such that $\phi(z) - \phi(t_n) \in V_i^\circ$ for any $n \geq n_{i,1}$. So $z - t_n \in \Lambda(V_i^\circ) \subset \Lambda_i$ i.e. $z \in t_n + \Lambda_i$. Thus $z \in s + \Gamma_i$. This implies

$$\Lambda(h + V_i^\circ) \subset s + \Gamma_i \ \text{ for each } i \leq m.$$

On the other hand, if $z \in s + \Gamma_i$, then $z \in t_n + \Lambda_i$ for large n. So $\phi(z) \in \phi(t_n) + \overline{V_i}$ for large n and $\phi(z) \in h + \overline{V_i}$. Thus $z \in \Lambda(h + \overline{V_i})$. Therefore

$$\Lambda(h + V_i^\circ) \subset s + \Gamma_i \subset \Lambda(h + \overline{V_i}) \ \text{ for each } i \leq m.$$

Note that $\beta(t_n + \mathbf{\Lambda}) = \iota(t_n) + \beta(\mathbf{\Lambda})$ and $\iota(t_n) = (t_n, 0) + \widetilde{L} = (0, -\phi(t_n)) + \widetilde{L}$. Thus

$$\begin{aligned}\beta(\mathbf{\Gamma}) &= \iota(-s) + \beta(s + \mathbf{\Gamma}) = \iota(-s) + \lim_{n\to\infty} \beta(t_n + \mathbf{\Lambda}) \\ &= \iota(-s) + \lim_{n\to\infty} \iota(t_n) + \beta(\mathbf{\Lambda}) = (-s, 0) + (0, -h) + \widetilde{L} \\ &= (-s, -h) + \widetilde{L}.\end{aligned}$$

□

Corollary 8.10 *Let $\mathbf{\Lambda}$ be a multiset with FLC. Suppose that there exists a continuous $\mathbb{R}^d$-map $\beta : X_{\mathbf{\Lambda}} \to \mathbb{A}(\mathbf{\Lambda})$, and $\Lambda_i = \Lambda(W_i)$ with respect to CPS (8.2) where $W_i \subset H$ is compact and $W_i = \overline{W_i^\circ} \neq \emptyset$ for each $i \leq m$. If $\mathbf{\Gamma}$ is non-singular in $X_{\mathbf{\Lambda}}$, then there is $(-s, -h) \in \mathbb{R}^d \times H$ so that*

$$\Gamma_i = -s + \Lambda(h + W_i^\circ) \quad \textit{for each } i \leq m.$$

PROOF. If $\mathbf{\Gamma}$ is non-singular in $X_{\mathbf{\Lambda}}$, from Prop. 8.7 and 8.8 there exists $s \in \mathbb{R}^d$ such that $\Gamma_i = -s + \Lambda(U_i)$ where $U_i \subset H$ is open and $\phi(L) \cap \partial U_i = \emptyset$ for each $i \leq m$. Then from Prop. 8.9 there exists $(-s, -h) \in \mathbb{R}^d \times H$ such that $-s + \Lambda(h + W_i^\circ) \subset \Gamma_i \subset -s + \Lambda(h + W_i)$ for each $i \leq m$. Thus for each $i \leq m$,

$$\Lambda(h + W_i^\circ) \subset \Lambda(U_i) \subset \Lambda(h + W_i).$$

Since $U_i \backslash (h + W_i)$ is open and $\phi(L)$ is dense in H, $U_i \subset (h + W_i)$. Replacing U_i by $(h + W_i^\circ) \cup U_i$, we can assume that $h + W_i^\circ \subset U_i \subset h + W_i$. Since U_i is an open set, $U_i \subset h + W_i^\circ$. Therefore $U_i = h + W_i^\circ$ and $\Gamma_i = -s + \Lambda(h + W_i^\circ)$. □

Proposition 8.11 *Let $\mathbf{\Lambda}$ be a multiset in $\mathbb{R}^d$ with FLC and repetitivity. Suppose that there exists a continuous $\mathbb{R}^d$-map $\beta : X_{\mathbf{\Lambda}} \to \mathbb{A}(\mathbf{\Lambda})$ and that $\Lambda(V_i^\circ) \subset \Lambda_i \subset \Lambda(\overline{V_i})$ where $\overline{V_i}$ is compact, $V_i^\circ \neq \emptyset$, and ∂V_i has empty interior for each $i \leq m$, with respect to CPS (8.2). Then there exists a non-singular element $\mathbf{\Lambda}'$ in $X_{\mathbf{\Lambda}}$ such that for each $i \leq m$, $\Lambda'_i = \Lambda(W_i)$ where W_i is compact and $W_i = \overline{W_i^\circ}$ with respect to the same CPS (8.2), and so for each $\mathbf{\Gamma} \in X_{\mathbf{\Lambda}}$ there exists $(-s, -h) \in \mathbb{R}^d \times H$ so that*

$$-s + \Lambda(h + W_i^\circ) \subset \Gamma_i \subset -s + \Lambda(h + W_i) \quad \textit{for each } i \leq m.$$

PROOF. From Prop. 8.9, for any $\mathbf{\Gamma} \in X_{\mathbf{\Lambda}}$ there exists $(-s, -h) \in \mathbb{R}^d \times H$ so that

$$-s + \Lambda(h + V_i^\circ) \subset \Gamma_i \subset -s + \Lambda(h + \overline{V_i}) \quad \text{for each } i \leq m.$$

By Baire category theorem, there exists $h' \in H$ such that

$$\phi(L) \cap (h' + \bigcup_{i \le m} \partial V_i) = \emptyset.$$

So we can find $\mathbf{\Lambda}' \in X_{\mathbf{\Lambda}}$ such that $\beta(\mathbf{\Lambda}') = (0, -h') + \widetilde{L}$ and $\Lambda_i' = \Lambda(h' + V_i^{\circ})$ for $i \le m$. For each $i \le m$, let $W_i = h' + \overline{V_i^{\circ}}$. Then $W_i = \overline{W_i^{\circ}}$, W_i is compact, and $\Lambda_i' = \Lambda(W_i)$. Note that $X_{\mathbf{\Lambda}} = X_{\mathbf{\Lambda}'}$ by the repetitivity of $\mathbf{\Lambda}$. Therefore applying Prop. 8.9 again, we can conclude the theorem. □

Proposition 8.12 *Let $\mathbf{\Lambda}$ be a multiset with FLC. Suppose that there exists a continuous $\mathbb{R}^d$-map $\beta : X_{\mathbf{\Lambda}} \to \mathbb{A}(\mathbf{\Lambda})$ which is one-to-one a.e. $\mathbb{A}(\mathbf{\Lambda})$, and $\Lambda_i = \Lambda(W_i)$ with respect to CPS (8.2) where W_i is compact and $W_i = \overline{W_i^{\circ}} \neq \emptyset$ for each $i \le m$. Then*

$$\theta_H(\partial W_i) = 0 \quad \textit{for each } i \le m,$$

where θ_H is a Haar measure in H.

PROOF. From Cor. 8.10, for any non-singular element $\mathbf{\Gamma} \in X_{\mathbf{\Lambda}}$, $\Gamma_i = -s + \Lambda(h + W_i^{\circ})$. By Prop. 8.8, we get $\phi(L) \cap (h + \partial W_i) = \emptyset$. So by the assumption on β, $-s + \Lambda(h + \partial W_i) = \emptyset$ a.e. $(-s, -h) + \widetilde{L} \in \mathbb{A}(\mathbf{\Lambda})$. By [29], $\text{dens}(-s + \Lambda(h + \partial W_i)) = \theta_H(\partial W_i)$ a.e. $\mathbb{A}(\mathbf{\Lambda})$. Thus $\theta_H(\partial W_i) = 0$ for each $i \le m$. □

We note that if $\mathbf{\Lambda}$ is a Meyer multiset then $\mathbf{\Lambda}$ had FLC.

Theorem 8.13 *Let $\mathbf{\Lambda}$ be a Meyer multiset with repetitivity. Suppose that there exists a continuous $\mathbb{R}^d$-map $\beta : X_{\mathbf{\Lambda}} \to \mathbb{A}(\mathbf{\Lambda})$ which is one-to-one a.e. $\mathbb{A}(\mathbf{\Lambda})$. Then there exists $\mathbf{\Lambda}' \in X_{\mathbf{\Lambda}}$ such that $\Lambda_i' = \Lambda(W_i)$ with respect to CPS (8.2) where $W_i = \overline{W_i^{\circ}} \neq \emptyset$ and W_i is compact for each $i \le m$, and so for each $\mathbf{\Gamma} \in X_{\mathbf{\Lambda}}$ there exists $(-s, -h) \in \mathbb{R}^d \times H$ so that*

$$-s + \Lambda(h + W_i^{\circ}) \subset \Gamma_i \subset -s + \Lambda(h + W_i) \quad \textit{for each } i \le m.$$

Furthermore

$$\theta_H(\partial W_i) = \emptyset \quad \textit{for each } i \le m.$$

In other words, for each $\mathbf{\Gamma} \in X_{\mathbf{\Lambda}}$, $\mathbf{\Gamma}$ is a regular model multiset.

PROOF. Since β is one-to-one a.e. $\mathbb{A}(\mathbf{\Lambda})$, there exists a non-singular element $\mathbf{\Lambda}' \in X_{\mathbf{\Lambda}}$. So from Prop. 8.7 and 8.8 $\Lambda_i' = \Lambda(W_i)$, $W_i = \overline{W_i^{\circ}} \neq \emptyset$, and W_i is compact for each $i \le m$. Note that $X_{\mathbf{\Lambda}'} = X_{\mathbf{\Lambda}}$ by the repetitivity of $\mathbf{\Lambda}$ (see [13, 41]). Thus applying Prop. 8.9 and Prop. 8.12 we can conclude that for each $\mathbf{\Gamma} \in X_{\mathbf{\Lambda}}$, $\mathbf{\Gamma}$ is a regular model multiset. □

8.3 Torus parameterizations from model multisets

We consider a cut and project scheme :

$$\begin{array}{ccccc} \mathbb{R}^d & \xleftarrow{\pi_1} & \mathbb{R}^d \times H & \xrightarrow{\pi_2} & H \\ & & \cup & & \\ & & \widetilde{L} & & \end{array} \tag{8.3}$$

where H is a locally compact Abelian group, $\widetilde{L}$ is a lattice in $\mathbb{R}^d \times H$, $\pi_1|_{\widetilde{L}}$ is one-to-one, and $\pi_2(\widetilde{L})$ is dense in H. Let $L = \pi_1(\widetilde{L})$. We define $\phi : L \to H$ by $\phi(x) = \pi_2(\pi_1^{-1}(x))$.

Suppose that $\mathbf{\Lambda}$ is a multiset in $\mathbb{R}^d$ and that for each $i \leq m$, $\Lambda(W_i^\circ) \subset \Lambda_i \subset \Lambda(W_i)$, W_i is compact in H, and $W_i = \overline{W_i^\circ} \neq \emptyset$ with respect to CPS (8.3).

Let $I = \{t \in H : t + W_i = W_i \text{ for all } i \leq m\}$. Translations t in H of this form indicate a certain redundancy in H which we can remove by factoring out the subgroup I. We define $H' := H/I$, $\psi : L \to H'$ by $\psi(x) = \phi(x) + I$, and $\widetilde{L'} := \{(x, \psi(x)) \in \mathbb{R}^d \times H' : x \in L\}$. Then $\widetilde{L'}$ is a lattice in $\mathbb{R}^d \times H'$, i.e. $\widetilde{L'}$ is a discrete subgroup for which $(\mathbb{R}^d \times H')/\widetilde{L'}$ is compact. Note that $W_i + I = W_i$ and $W_i^\circ + I = W_i^\circ$ for all $i \leq m$. Thus for all $i \leq m$

$$\Lambda(c + W_i + I) = \Lambda(c + W_i) \text{ and } \Lambda(c + W_i^\circ + I) = \Lambda(c + W_i^\circ) \text{ for any } c \in H.$$

We denote W_i' for $W_i + I$ in H'. Then we can construct a new cut and project scheme :

$$\begin{array}{ccccc} \mathbb{R}^d & \xleftarrow{\pi_1'} & \mathbb{R}^d \times H' & \xrightarrow{\pi_2'} & H' \\ & & \cup & & \\ L & \longleftarrow & \widetilde{L'} & \longrightarrow & \psi(L) \\ & & & & \\ x & \longleftarrow & (x, \psi(x)) & \longrightarrow & \psi(x) \end{array} \tag{8.4}$$

and we get $\Lambda(W_i'^\circ) \subset \Lambda_i \subset \Lambda(W_i')$, W_i' is compact in H', $W_i' = \overline{W_i'^\circ} \neq \emptyset$ for $i \leq m$, $\{t \in H' : t + W_i' = W_i' \text{ for all } i \leq m\} = \{0\}$, with respect to CPS (8.4). Furthermore since I is a closed subgroup of H, if $\theta_H(\partial W_i) = 0$ where θ_H is a Haar measure in H, then $\theta_{H'}(\partial W_i') = 0$ where $\theta_{H'}$ is a Haar measure in H' (see [37, Theorem 3.3.28]).

Thus without loss of generality we will assume

$$\{t \in H : t + W_i = W_i \text{ for all } i \leq m\} = \{0\},$$

a condition that we will refer to as *irredundancy.* All subsequent CPS will be assumed reduced into this form.

In this section we establish the existence of the torus parameterization from regular model multisets. The following theorem is proved after a sequence of auxiliary propositions.

Theorem 8.14 *Let $\mathbf{\Lambda}$ be a multiset in $\mathbb{R}^d$. Suppose that for each $i \le m$, $\Lambda(W_i^\circ) \subset \Lambda_i \subset \Lambda(W_i)$ where W_i is compact, $W_i = \overline{W_i^\circ} \neq \emptyset$, and $\theta_H(\partial W_i) = 0$ with respect to some irredundant CPS (see (8.3)). Then there is a continuous $\mathbb{R}^d$-map $\beta : X_{\mathbf{\Lambda}} \to \mathbb{A}(\mathbf{\Lambda})$ which is one-to-one a.e. $\mathbb{A}(\mathbf{\Lambda})$.*

Proposition 8.15 *Let $\mathbf{\Lambda}$ be a multiset in $\mathbb{R}^d$. Suppose that for each $i \le m$, $\Lambda(W_i^\circ) \subset \Lambda_i \subset \Lambda(W_i)$ where W_i is compact, and $W_i = \overline{W_i^\circ} \neq \emptyset$ with respect to some irredundant CPS. Then for any $\mathbf{\Gamma} \in X_{\mathbf{\Lambda}}$ with $\bigcup_{i=1}^m \Gamma_i \subset L$,*

$$\bigcap\{\phi(t) - W_i : t \in \Gamma_i, i \le m\}$$

contains exactly one element $c_{\mathbf{\Gamma}}$ in H and $\overline{\phi(\Gamma_i)} = c_{\mathbf{\Gamma}} + W_i$ for each $i \le m$. Furthermore, $c_{\mathbf{\Lambda}} = 0$.

PROOF. Let $\mathbf{\Gamma} \in X_{\mathbf{\Lambda}}$ with $\cup_{i=1}^m \Gamma_i \subset L$. We claim that

$$\bigcap\{\phi(t) - W_i : t \in \Gamma_i, i \le m\} \neq \emptyset.$$

Suppose that $\bigcap\{\phi(t) - W_i : t \in \Gamma_i, i \le m\} = \emptyset$. Since each $\phi(t) - W_i$ is compact, there exists a compact $K \subset \mathbb{R}^d$ such that $\bigcap\{\phi(t) - W_i : t \in \Gamma_i \cap K, i \le m\} = \emptyset$ and $\mathbf{\Gamma} \cap K \neq \emptyset$. Since $\mathbf{\Gamma} \in X_{\mathbf{\Lambda}}$ and $\bigcup_{i=1}^m \Gamma_i \subset L$, we can find $t_0 \in L$ such that $\mathbf{\Gamma} \cap K = (t_0 + \mathbf{\Lambda}) \cap K$. Then for any $i \le m$ and any $t \in \Gamma_i \cap K$, $t \in t_0 + \Lambda_i$ and $\phi(t_0) \in \phi(t) - W_i$. This is a contradiction.

For any $c \in \bigcap\{\phi(t) - W_i : t \in \Gamma_i, i \le m\}$, $\phi(\Gamma_i) \subset c + W_i$ for all $i \le m$. Since $W_i = \overline{(W_i)^\circ}$, there is $h \in H$ such that $\overline{\phi(\Gamma_i)} = h + W_i$ for all $i \le m$ by Prop. 8.9. So $h + W_i \subset c + W_i$. We claim that $h + W_i = c + W_i$. In fact, $W_i \subset (c - h) + W_i$. Let $x = c - h$. Since $W_i - W_i$ is compact, $\overline{\{nx : n \in \mathbb{Z}_+\}}$ is compact. For any neighbourhood V of 0 in H, $\{V + nx : n \in \mathbb{Z}_+\}$ is an open cover of $\overline{\{nx : n \in \mathbb{Z}_+\}}$. So there are $n_1, \ldots, n_k \in \mathbb{Z}_+$ such that $\{V + n_j x : 1 \le j \le k\}$ covers $\{nx : n \in \mathbb{Z}_+\}$. If $\{n_j x : 1 \le j \le k\} = \{nx : n \in \mathbb{Z}_+\}$, $m_1 x = 0$ for some $m_1 \in \mathbb{Z}_+$. So $W_i \subset x + W_i \subset m_1 x + W_i = W_i$. Thus $W_i = x + W_i = c - h + W_i$. If $\{n_j x : 1 \le j \le k\} \neq \{nx : n \in \mathbb{Z}_+\}$, V contains $m_2 x$ for some $m_2 \in \mathbb{Z}_+$. So $W_i \subset x + W_i \subset W_i + V$. Since V is arbitrary, $W_i = x + W_i = c - h + W_i$ for $i \le m$.

Thus $c = h$, since $0 = \{t \in H : t + W_i = W_i, i \leq m\}$. So there is a unique $c_{\Gamma} \in H$ such that

$$\{c_{\Gamma}\} = \bigcap\{\phi(t) - W_i : t \in \Gamma_i, i \leq m\}. \tag{8.5}$$

Since $0 \in \bigcap\{\phi(t) - W_i : t \in \Lambda_i, i \leq m\}$, $c_{\Lambda} = 0$. $\square$

We define $\mathbb{T}(\Lambda) := (\mathbb{R}^d \times H)/\widetilde{L}$ in an irredundant CPS (see (8.3)).

Corollary 8.16 *Let $\mathbf{\Lambda}$ be a multiset in $\mathbb{R}^d$. Suppose that for each $i \leq m$, $\Lambda(W_i^{\circ}) \subset \Lambda_i \subset \Lambda(W_i)$ where W_i is compact and $W_i = \overline{W_i^{\circ}} \neq \emptyset$ with respect to some irredundant CPS. Then the map $\gamma : X_{\mathbf{\Lambda}} \to \mathbb{T}(\mathbf{\Lambda})$ given by $\mathbf{\Gamma} \mapsto \gamma(\mathbf{\Gamma}) = (-s, -c_{s+\mathbf{\Gamma}}) + \widetilde{L}$, where s is any element of $\mathbb{R}^d$ for which $s + \bigcup_{i=1}^m \Gamma_i \subset L$ and $c_{s+\mathbf{\Gamma}}$ is given by (8.5), is a well-defined $\mathbb{R}^d$-map. Furthermore, for any $\mathbf{\Gamma} \in X_{\mathbf{\Lambda}}$,*

$$\overline{\phi(\Gamma_i)} = -\phi(s) + c_{s+\mathbf{\Gamma}} + W_i \quad \textit{for all } i \leq m$$

and $\gamma(\mathbf{\Lambda}) = (0, 0) + \widetilde{L}$.

Proof. Suppose that $s \neq s_1$ where $s + \bigcup_{i=1}^m \Gamma_i \subset L$ and $s_1 + \bigcup_{i=1}^m \Gamma_i \subset L$. Note that $s - s_1 \in L$ and

$$\begin{aligned} (-s_1, -c_{s_1+\mathbf{\Gamma}}) + \widetilde{L} &= (-(s+l), -c_{s+l+\mathbf{\Gamma}}) + \widetilde{L} \text{ for some } l \in L \\ &= (-s - l, -\phi(l) - c_{s+\mathbf{\Gamma}}) + \widetilde{L} \\ &= (-s, -c_{s+\mathbf{\Gamma}}) + \widetilde{L}. \end{aligned} \tag{8.6}$$

For any $g \in \mathbb{R}^d$,

$$\gamma(g + \mathbf{\Gamma}) = (-s + g, -c_{s+\mathbf{\Gamma}}) + \widetilde{L} = \iota(g) + \gamma(\mathbf{\Gamma}).$$

The rest follows directly from the Prop. 8.15. $\square$

Proposition 8.17 *Let $\mathbf{\Lambda}$ be a multiset in $\mathbb{R}^d$. Suppose that for each $i \leq m$, $\Lambda(W_i^{\circ}) \subset \Lambda_i \subset \Lambda(W_i)$ where W_i is compact, and $W_i = \overline{W_i^{\circ}} \neq \emptyset$ with respect to some irredundant CPS. The mapping $\gamma : X_{\mathbf{\Lambda}} \to \mathbb{T}(\mathbf{\Lambda})$ defined in Cor. 8.16 is continuous and surjective.*

Proof. Let $\mathbf{\Gamma} \in X_{\mathbf{\Lambda}}$ and $\gamma(\mathbf{\Gamma}) = (-s, -c_{s+\mathbf{\Gamma}}) + \widetilde{L}$ for some $s \in \mathbb{R}^d, c_{s+\mathbf{\Gamma}} \in H$. Let U be an open neighbourhood of 0 and U' be an open neighbourhood of $-c_{s+\mathbf{\Gamma}}$. Let $U_0 = U \cap (-U)$. Since $c_{s+\mathbf{\Gamma}} = \bigcap\{\phi(t) - W_i : t \in s + \Gamma_i, i \leq m\}$,

$$\bigcap\{(-\phi(t) + W_i) \backslash U' : t \in s + \Gamma_i, i \leq m\} = \emptyset.$$

Each of $(-\phi(t) + W_i)\backslash U'$ is closed and so there exists a compact K in $\mathbb{R}^d$ such that

$$\bigcap\{(-\phi(t) + W_i)\backslash U' : t \in (s + \Gamma_i) \cap K, i \leq m\} = \emptyset$$

i.e. $\bigcap\{-\phi(t) + W_i : t \in (s + \Gamma_i) \cap K, i \leq m\} \subset U'$. Let

$U_{-s+K,U_0}[\mathbf{\Gamma}] = \{\mathbf{\Gamma}' \in X_{\mathbf{\Lambda}} : (\mathbf{\Gamma}' + r) \cap (-s + K) = \mathbf{\Gamma} \cap (-s + K)$ for some $r \in U_0\}$.

For any $\mathbf{\Gamma}' \in U_{-s+K,U_0}[\mathbf{\Gamma}]$,

$$(\mathbf{\Gamma}' + r) \cap (-s + K) = \mathbf{\Gamma} \cap (-s + K) \text{ for some } r \in U_0.$$

So $(\mathbf{\Gamma}' + r + s) \cap K = (\mathbf{\Gamma} + s) \cap K$. Then

$$\bigcap\{-\phi(t) + W_i : t \in (\Gamma_i' + r + s) \cap K, i \leq m\} \subset U'.$$

It shows

$$\gamma(\mathbf{\Gamma}') \in (-r - s, U') + \widetilde{L} \subset (-s + U_0, U') + \widetilde{L} \subset (-s + U, U') + \widetilde{L}.$$

Note that there exists $\epsilon > 0$ such that an open set $\{\mathbf{\Gamma}' \in X_{\mathbf{\Lambda}} : \varrho(\mathbf{\Gamma}', \mathbf{\Gamma}) < \epsilon\}$ around $\mathbf{\Gamma}$ is a subset of $U_{-s+K,U_0}[\mathbf{\Gamma}]$. Therefore γ is continuous. Furthermore,

$$\gamma(\mathbb{R}^d + \mathbf{\Lambda}) = (\mathbb{R}^d, 0) + \widetilde{L} = (\mathbb{R}^d, \phi(L)) + \widetilde{L}$$

is dense in $\mathbb{T}(\mathbf{\Lambda})$. So γ is surjective. □

The proofs of the existence of the torus parameterization shown here are essentially due to Schlottmann [41]. However here we assume the condition on W_i's that each window W_i is the closure of its interior instead of the repetitivity on Λ_i in [41] and we place his results into multiset setting.

Let $X_g := \{\mathbf{\Gamma} \in X_{\mathbf{\Lambda}} : \phi(L) \cap (c_{\mathbf{\Gamma}} + \bigcup_{i=1}^m \partial W_i) = \emptyset\}$. Note that $X_g \neq \emptyset$ by Baire category theorem.

Proposition 8.18 *Let $\mathbf{\Lambda}$ be a multiset in $\mathbb{R}^d$. Suppose that for each $i \leq m$, $\Lambda(W_i^\circ) \subset \Lambda_i \subset \Lambda(W_i)$ where W_i is compact and $W_i = \overline{W_i^\circ} \neq \emptyset$ with respect to some irredundant CPS. Then $\gamma|_{X_g}$ is one-to-one.*

PROOF. From Prop. 8.9 we know that for any $\mathbf{\Gamma} \in X_g$ there exists $(-s, -c_{\mathbf{\Gamma}}) \in G \times H$ such that

$$-s + \Lambda(c_{\mathbf{\Gamma}} + W_i^\circ) \subset \Gamma_i \subset -s + \Lambda(c_{\mathbf{\Gamma}} + W_i) \text{ for all } i \leq m.$$

Here $\phi(L) \cap (c_{\mathbf{\Gamma}} + \bigcup_{i=1}^m \partial W_i) = \emptyset$. So $\Gamma_i = -s + \Lambda(c_{\mathbf{\Gamma}} + W_i^\circ)$ for all $i \leq m$. Thus $\gamma|_{X_g}$ is one-to-one. □

We define $\mathbb{T}_g := \gamma(X_g)$.

Proposition 8.19 *Let Λ be a multiset in $\mathbb{R}^d$. Suppose that for each $i \leq m$, $\Lambda(W_i^\circ) \subset \Lambda_i \subset \Lambda(W_i)$ where W_i is compact and $W_i = \overline{W_i^\circ} \neq \emptyset$ with respect to some irredundant CPS. Then $\mathbb{T}_g = \mathbb{T}(\boldsymbol{\Lambda})\backslash((\mathbb{R}^d \times \bigcup_{i=1}^m \partial W_i + \widetilde{L})/\widetilde{L})$.*

PROOF. Note that

$$\begin{aligned}
(-s, -c_\Gamma) + \widetilde{L} \in \mathbb{T}_g \quad &\Leftrightarrow \quad (c_\Gamma + \bigcup_{i=1}^m \partial W_i) \cap \phi(L) = \emptyset \\
&\Leftrightarrow \quad c_\Gamma \notin \phi(L) - \bigcup_{i=1}^m \partial W_i \\
&\Leftrightarrow \quad (-s, -c_\Gamma) \notin \mathbb{R}^d \times (-\phi(L) + \bigcup_{i=1}^m \partial W_i) \\
&\Leftrightarrow \quad (-s, -c_\Gamma) + \widetilde{L} \notin (\mathbb{R}^d \times \bigcup_{i=1}^m \partial W_i + \widetilde{L})/\widetilde{L}. \qquad (8.7)
\end{aligned}$$

Thus $\mathbb{T}_g = \mathbb{T}(\boldsymbol{\Lambda})\backslash((\mathbb{R}^d \times \bigcup_{i=1}^m \partial W_i + \widetilde{L})/\widetilde{L})$. □

Proposition 8.20 *Let Λ be a multiset in $\mathbb{R}^d$. Suppose that for each $i \leq m$, $\Lambda(W_i^\circ) \subset \Lambda_i \subset \Lambda(W_i)$ where W_i is compact and $W_i = \overline{W_i^\circ} \neq \emptyset$ with respect to some irredundant CPS. Suppose that $\theta_H(\partial W_i) = 0$ for all $i \leq m$, where θ_H is a Haar measure in H. Then $\lambda(\mathbb{T}_g) = 1$, where λ is a Haar measure in $\mathbb{T}(\boldsymbol{\Lambda})$.*

PROOF. We will show that $\lambda((\mathbb{R}^d \times \bigcup_{i=1}^m \partial W_i + \widetilde{L})/\widetilde{L}) = 0$. By [37, Theorem 3.3.28], we only need to show that $\nu(\mathbb{R}^d \times \bigcup_{i=1}^m \partial W_i + \widetilde{L}) = 0$, where ν is a Haar measure in $\mathbb{R}^d \times H$. Since $\mathbb{R}^d$ is compactly generated, there exists a sequence of compact sets $\{K_n\}$ such that $\mathbb{R}^d = \bigcup_{i=1}^\infty K_n$. Each $\nu(K_n \times \bigcup_{i=1}^m \partial W_i) = 0$ from the assumption that $\theta_H(\partial W_i) = 0$ for all $i \leq m$. Since $\widetilde{L}$ is countable, $\nu(\mathbb{R}^d \times \bigcup_{i=1}^m \partial W_i + \widetilde{L}) = 0$. Thus the assertion follows. □

Proposition 8.21 *Let Λ be a multiset in $\mathbb{R}^d$. Suppose that for each $i \leq m$, $\Lambda(W_i^\circ) \subset \Lambda_i \subset \Lambda(W_i)$ where W_i is compact and $W_i = \overline{W_i^\circ} \neq \emptyset$ with respect to some irredundant CPS. Suppose that $\theta_H(\partial W_i) = 0$ for all $i \leq m$, where θ_H is a Haar measure in H. Then $\mathbb{A}(\boldsymbol{\Lambda}) \cong \mathbb{T}(\boldsymbol{\Lambda})$.*

PROOF. $\mathbb{A}(\boldsymbol{\Lambda})$ is the completion of $\mathbb{R}^d$ under the autocorrelation topology. $\mathbb{T}(\boldsymbol{\Lambda})$ may be considered as the completion of $\mathbb{R}^d$ when it is given the coarsest topology for which the mapping $x \mapsto (x, 0) + \widetilde{L}$ of $\mathbb{R}^d$ into $\mathbb{T}(\boldsymbol{\Lambda})$ is continuous. It will suffice to show that these two topologies on $\mathbb{R}^d$ are the same. If $x \in \mathbb{R}^d$ is close to 0 in

$\mathbb{T}$-topology, then for small open neighbourhoods V of 0 in $\mathbb{R}^d$ and V_1 of 0 in H there exists $(t, \phi(t)) \in \widetilde{L}$ such that $x - t \in V$ and $\phi(t) \in V_1$. On the other hand if $x \in \mathbb{R}^d$ is close to 0 in $\mathbb{A}$-topology, then for a small open neighbourhood U of 0 in $\mathbb{R}^d$ and some $\epsilon > 0$ there exists $t \in L$ such that $x - t \in U$ and $d(t + \mathbf{\Lambda}, \mathbf{\Lambda}) < \epsilon$. So we need to show that for $t \in L$, $\phi(t)$ is close to 0 in H if and only if $d(t + \mathbf{\Lambda}, \mathbf{\Lambda})$ is close to 0.

For $t \in L$,

$$\begin{aligned} d(t+\mathbf{\Lambda}, \mathbf{\Lambda}) &= \limsup_{n\to\infty} \frac{\sum_{i=1}^m \sharp(((t+\Lambda_i) \triangle \Lambda_i) \cap A_n)}{\mathrm{Vol}(A_n)} \\ &= \sum_{i=1}^m \lim_{n\to\infty} \frac{\sharp(((t+\Lambda_i) \triangle \Lambda_i) \cap A_n)}{\mathrm{Vol}(A_n)} \\ &= \sum_{i=1}^m (\theta_H(W_i \backslash (\phi(t) + W_i)) + \theta_H(W_i \backslash (-\phi(t) + W_i))), \qquad (8.8) \end{aligned}$$

since each point set is a regular model set by the assumption (see [29]).

Note that

$$\theta_H(W_i \backslash (s + W_i)) = \theta_H(W_i) - \mathbf{1}_{W_i} * \widetilde{\mathbf{1}_{W_i}}(s)$$

is uniformly continuous in s. So if $\phi(t)$ converges to 0 in H, then $d(t + \mathbf{\Lambda}, \mathbf{\Lambda})$ converges to 0 in $\mathbb{R}$.

On the other hand, suppose that $\{t_n\}$ is a sequence such that $d(t_n + \mathbf{\Lambda}, \mathbf{\Lambda}) \to 0$ as $n \to \infty$. Then for each $i \le m$

$$\{\theta_H(W_i \backslash (\phi(t_n) + W_i))\}_n \to 0 \text{ as } n \to \infty.$$

Note that for large enough n, $W_i \cap (\phi(t_n) + W_i) \neq \emptyset$ and so $\phi(t_n) \in W_i - W_i$ for all $i \le m$. Since $W_i - W_i$ is compact, $\{\phi(t_n)\}_n$ has a converging subsequence $\{\phi(t_{n_k})\}_k$. For any such sequence define $t_0{}^* := \lim_{k\to\infty} \phi(t_{n_k})$. Then

$$\theta_H(W_i \backslash (t_0{}^* + W_i)) = 0$$

and so $\theta_H(W_i{}^\circ \backslash (t_0{}^* + W_i)) = 0$ for each $i \le m$. Thus $W_i{}^\circ \subset t_0{}^* + W_i$ and it implies $W_i \subset t_0{}^* + W_i$. On the other hand, $\lim_{k\to\infty} -\phi(t_{n_k}) = -t_0{}^*$ and $\theta_H(W_i{}^\circ \backslash (-t_0{}^* + W_i)) = 0$. So $W_i \subset -t_0{}^* + W_i$. Hence $W_i \subset t_0{}^* + W_i \subset t_0{}^* - t_0{}^* + W_i$ and

$$W_i = t_0{}^* + W_i.$$

This equality is for each $i \le m$. Thus $t_0{}^* = 0$. So all converging subsequences $\{\phi(t_{n_k})\}_k$ converges to 0 and

$$\{\phi(t_n)\}_n \to 0 \text{ as } n \to \infty.$$

This establishes the equivalence of the two topologies. By [7, Prop 5, III §3.3], there exists an isomorphism of $\mathbb{A}(\mathbf{\Lambda})$ onto $\mathbb{T}(\mathbf{\Lambda})$. $\square$

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