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THE EFFECT OF BANK SIZE ON THE EXCESS CASH RATIO UNDER CONDITIONS OF UNCERTAINTY

by

© BARTON J. SISK

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled THE EFFECT OF BANK SIZE ON THE EXCESS CASH RATIO UNDER CONDITIONS OF UNCERTAINTY submitted by BARTON J. SISK in partial fulfilment of the requirements for the degree of Master of Arts.

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ABSTRACT

Stochastic models of bank behavior have become popular over the last two decades, but the seminal work in the area was penned by F.Y. Edgeworth in the 19th century. In this work, Edgeworth found that the introduction of uncertainty into models of bank behavior generated the theoretically plausible conclusion that large banks will hold proportionately smaller levels of total cash reserves than smaller banks.

The techniques of stochastic models of bank behavior were refined in the 1960's, largely by authors interested in an area related to Edgeworth's work. Where Edgeworth concerned himself with the optimization of cash reserves, these authors sought to investigate the effect of uncertainty on the optimal expansion of deposits. With the improved techniques of the latter authors and the basic assumptions of Edgeworth, E. Baltensperger refined the treatment of the question of economies of scale in reserve management and arrived at the same conclusion as Edgeworth. In a later work A. Knobel, using a substantially different model and investigating the relationship of bank size to excess reserves, achieved results tending to question those of Edgeworth and Baltensperger, but not directly refuting their conclusions.

In this thesis, a stochastic model of excess reserve management is presented and the model is used to

analyze the effects of bank size on excess reserve management. The conclusions of Edgeworth and Baltensperger are shown to result solely from very specialized restrictions that they placed on the range of values permitted to the optimal level of cash in their models. As well, for the first time, fixed transactions costs are introduced into the discussion of economies of scale in reserve management. Their introduction supports the results of Edgeworth and Baltensperger, but not conclusively. A final contribution attempted by this work is a critical comparison of the stochastic models seeking to optimize deposit expansion and those seeking to optimize the level of reserves.

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LIST OF NOTATION USED IN CHAPTER III AND IV

c = the mean of U_i $(i = 1...n)$,

D = the dollar value of "initial deposits",

d^2 = the variance of U_i $(i = 1...n)$,

f = the probability density function of the variable Z,

G = the dollar value of fixed transactions cost,

g = the ratio of fixed transactions cost to "initial deposits",

m = the rate of interest on borrowing from the Central Bank,

n = the number of accounts in the bank,

p = the ratio of excess reserves to "initial deposits",

R = the ratio of required reserves to "initial deposits",

U_i = the proportion of the ith account, withdrawn or augmented,

u_i = values of the random variable U_i,

v = the size of all accounts in the bank,

y = the rate of interest on loans,

Z = the ratio of the cash drain to "initial deposits",

z = values of the random variable Z,

μ = the mean of Z,

σ^2 = the variance of Z,

$*$ = indicates an optimal value.

CHAPTER I

INTRODUCTION

The subject matter of this work is essentially microeconomic in that it seeks to examine on a theoretical level a particular aspect of the behavior of an economic agent, namely, the commercial bank. To be more specific, this thesis deals with the analysis of the effects of a bank's size on its demand for excess primary reserves. It employs a simple, one period, two-asset, stochastic, bank portfolio optimization model as the vehicle for that analysis.

Two-asset stochastic models of bank behavior have become popular in the past twenty years, with authors concentrating on two basic investigations: (1) the optimization of deposit expansion from a position of excess reserves; and (2) the optimization of the level of bank reserves. Among the authors concerned with the second issue only a very few have sought to employ their model to investigate the effects of bank size on the management of a bank's reserves.

Two of these authors arrived at the conclusion that large banks will hold a smaller proportion of deposits as reserves than smaller banks, while a third author, using a substantially different model, arrived at results that tend to question this conclusion. It is the contention of this inquiry that larger banks will not hold a lower cash reserve ratio than smaller banks under all conditions. Furthermore, only minor changes in the models of the authors who concluded that such an economy of scale exists in reserve management will belie their result.

The subject matter of Chapter II of this study is a selective review of the literature pertaining to the optimization of bank behavior using stochastic models. A good deal of space is devoted to works which seek to optimize deposit expansion from a position of excess reserves. This is because many of the broad techniques used in the analysis of bank reserve optimization in stochastic models were originally developed in studies dealing with the former problem. Also, it is hoped that this thesis will serve to display to the reader that, for many of the problems encountered in the deposit expansion optimization models, there exists a related problem in the reserve optimization models.

Some of the contributions to the field of bank behavior optimization under conditions of uncertainty are analyzed in detail in this thesis, while other contributions are given a more generalized treatment. The criterion used to determine the detail in which any particular contribution is presented essentially relates to how much the detail will contribute to the understanding of the workings of my own model which is presented in Chapter III of this thesis. In general, on the question of the detail of presentation, models concerning reserve optimization are not favoured above models concerning optimal deposit expansion; nor are reserve optimization models which concern themselves with bank size necessarily favoured over those that do not.

In Chapter III, my own one period, two-asset, stochastic, bank portfolio model will be presented. It will be presented first in a basic form and then in a slightly augmented form. In Chapter IV the basic and augmented forms of my model will be used to analyze the conclusions regarding the effects of bank size on reserve management that emerged from the literature review.

Chapter V will concern itself with the conclusions and limitations of this study and with some suggestions for further research.

CHAPTER II

THE EFFECT OF BANK SIZE ON A BANK'S EXCESS CASH RATIO: A REVIEW OF THE LITERATURE

Edgeworth

As long ago as 1888, F.Y. Edgeworth made the first attempt to analyze the nature of a bank's demand for cash reserves within the framework of probability theory.[1] His analysis included the first exposition of the theoretical plausibility of economies of scale in bank reserve management.

Edgeworth observed that the net cash drain which a bank faces as a result of customer additions to, or withdrawals from, deposit accounts is a quantity which, under normal circumstances, varies according to the activities of numerous independent agents. He reasoned, therefore, that this net cash drain should fluctuate according to the normal probability distribution, though he was careful to recognize that, for the description to apply, certain regular fluctuations in net cash drains, due to such factors as seasonality, had to be abstracted from. He also recognized that under certain circumstances, such as those existing during financial panics, the depositors of a bank act in a far from independent fashion.

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[1]F.Y. Edgeworth, "The Mathematical Theory of Banking," Journal of the Royal Statistical Society 51 (March 1888): 113-27.

Assuming normal times, Edgeworth describes in a parable how a banker could go about scientifically choosing a suitable level of reserves to hold against cash withdrawals. The banker would begin by estimating the parameters of the (presumably normal) probability distribution of net cash drains he expects to face him in the current period. This would involve the banker's professional judgement and careful observation of historical data. The banker would next select an acceptable probability of being caught with insufficient cash to satisfy his depositors' demands. Subsequently, from the cumulative probability distribution of net cash drains, the banker can determine the level of net cash drains whereby such drains will exceed that value by no more than the probability decided upon. The final step involves the banker equating the level of reserves to that level of net cash drains found by the preceding method.

As a very simple example, suppose the banker found that his net cash drains would likely average zero and exhibit a standard deviation of $1 million. If the banker was willing to accept a one in one thousand chance of defaulting on his depositors' cash withdrawals, he would hold $3.09 million as a reserve. This is because given the mean and variance of the banker's normally distributed net cash drains, there is only a one in one thousand chance of net cash drains exceeding $3.09 million.

Edgeworth does not detail the factors which the banker should consider in determining an acceptable risk of default on depositors' demands. He does, however, assume throughout his work that the banker will always make provision for the expected value of cash drains in the selection of a satisfactory level of reserves. Hence, Edgeworth is really concerned with the selection of a reasonable level of reserves to hold in excess of the average value of net cash drains. In the simple example, for instance, Edgeworth would not permit the choice of a probability of default above one chance in two.

Besides bringing the techniques of probability analysis to bear on the problem of reserve management, Edgeworth makes another very important discovery. Arguing in parable once more, Edgeworth implies that, under some circumstances, if the size of a bank (volume of deposits) is increased by a factor of n, an increase in reserves by a factor of $\sqrt{n}$ will provide the same protection against default on depositors' demands as before the expansion. This is to say that an increase of $\sqrt{n}$ in reserves will supply the identical probability of being caught with insufficient cash. The important assumptions behind his conclusion are: (1) the net cash drains are distributed normally (or at least asymptotically normally); (2) the average size of the accounts are equal before the expansion; (3) all customers' accounts are increased by a factor of n and, more subtly, (4) the banker's chosen level of reserves always exceeds the expected value of net cash drains during the period in question.[2]

Edgeworth's discovery that an increase of $\sqrt{n}$ in reserve will result in no loss of security should the size of a bank be increased by a factor of n, given the above assumptions, is important in two ways. First, it implies that the ratio of reserves to deposits, and hence, the opportunity cost of holding reserves, can be lowered without additional risk as a bank increases in size--that is, there exist economies of scale in bank reserve management. Second, this discovery marks the first statement of the square-root law of the demand for cash.[3] It is the implication of economies of scale in reserve management that is important to the investigation at hand.

Although Edgeworth was the first economist to investigate the importance of uncertainty and the possible economies of scale in reserve management, he did not present a true model of the demand for reserves. In Edgeworth's discourse, the banker is presumed to determine exogenously the

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[2]Edgeworth does not provide a rigorous proof of his conclusion in his article but one will be provided later in this thesis in the consideration of a subsequent article in the literature review.

[3]For more information on Edgeworth's contribution to the square root law, see J.H.G. Olivera, "The Square Root Law of Precautionary Reserves," Journal of Political Economy 79 (September 1971): 1095-1105.

level of default risk on his depositors' cash withdrawals that he is willing to accept and to hold the appropriate level of reserves called for by his estimate of the parameters of his bank's cash drain distribution.. There is no explicit discussion of profit maximization or of the effects of interest rates on the banker's choice of the appropriate level of risk to be entertained.

The explicit introduction of profit maximization into the theory of bank behavior under conditions of uncertainty did not materialize until 1961, when Orr and Mellon[4] investigated the effect of stochastic reserve losses upon credit expansion. The optimization of credit expansion, given that excess reserves exist, is largely the mirror image of the problem analyzed in this thesis, namely, the optimization of the level of reserves to be held. Nevertheless, since the earliest applications of true stochastic models to banking behavior concerned optimal credit expansion, and since certain roughly parallel phenomena occur in the analysis of the two subjects, it is worthwhile to present an overview of the stochastic theory of credit expansion.

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[4]Daniel Orr and W.C. Mellon, "Stochastic Reserve Losses and Expansion of Bank Credit, "American Economic Review 51 (September 1961): 614-23.

Before describing Orr and Mellon's model and conclusions, a brief explanation of the deterministic theory of credit expansion which Orr and Mellon sought to challenge is in order. The classic work in the development of the role of an individual bank in credit creation under conditions of certainty is that of C.A. Phillips.[5]

C.A. Phillips

Phillips begins his analysis by separating a bank's total deposits into two categories: (1) primary deposits; and (2) derivative deposits. Primary deposits are those which arise when a customer deposits currency into his bank account or a cheque drawn on another bank, where this placement does not involve the repayment of a loan from the bank. A derivative deposit is one which arises when the bank grants a loan to a customer, or one made with the intention of accumulating a deposit balance sufficient to repay an outstanding loan from the bank at some future date.

Primary deposits are considered to be quite stable in that they represent funds deposited at the bank for the purpose of safekeeping and for the purpose of providing transactions balances for the customers. In aggregate these deposits are envisioned as being drawn down and augmented in a regular pattern so that a relatively uniform volume is maintained. In contrast, derivative deposits

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[5]C.A. Phillips, Bank Credit (New York: Macmillan, 1920).

are postulated to be highly volatile as a result of their origin. Typically, the procedure followed in granting a loan to a customer involves the creation of a chequeable deposit in the customer's name. This deposit is a derivative deposit; and, according to Phillips, it can be expected that a large part of the deposit will be drawn down within a short period of time, because most loans are contracted with a definite use for the funds in mind. Moreover, when the borrower draws down his deposit, the cheque will likely be in favour of a customer of a different bank, so that the draw down represents a loss of reserves to the bank granting the loan. Hence, Phillips expects that for the average loan, the associated derivative deposit quickly diminishes to a level representing a low percentage of the value of the loan. In a still later period, as the repayment date on the loan approaches, Phillips expects the borrower to begin accumulating balances with the bank, so that ultimately this deposit will grow relative to the size of the loan until it is large enough to retire the debt at maturity. The American bank custom of requiring minimum balances in relation to loans inspired Phillips to believe that a derivative deposit associated with a given loan would never reach zero.

To sum up, Phillips feels that the derivative deposit associated with a particular loan will briefly represent a high proportion of that loan in the period just after the granting of the loan. The derivative deposit will then fall to a low percentage of the value of the loan and ultimately the value of the derivative deposit will rise in proportion to the size of the loan as the loan approaches maturity.

As a bank matures, Phillips expects that the aggregate of derivative deposits will approach a stable proportion of total loans outstanding. This is because at any given point of time the loan portfolio will be composed of loans which are smoothly distributed, in respect of term to maturity, between newly granted loans and loans which are nearly due. The aggregate of derivative deposits will be the sum of individual derivative deposits, each associated with a loan in the portfolio. Some of these individual derivative deposits will represent a high proportion of the associated loan (where the loan is newly granted or near maturity), and some a low proportion (where the loan is in an intermediate stage). However, so long as the value of new loans granted is roughly equal to the value of maturing loans, and so long as the terms to maturity at inception of all loans are relatively uniform, total derivative deposits will tend to stand in an approximately fixed proportional relationship to aggregate loans.

In his derivation of the formula for the optimal expansion of loans, given the acquisition of a primary deposit, the ratio of derivative deposits to loans is designated the letter k. The additional primary deposit which brings an equal increase in reserves is termed c, while the customary ratio of reserves to deposits (which is assumed to remain stable over time) is labelled r. The addition to loans is designated the letter x, and c_1 represents the amount of reserves lost through the booking of new loans. The proportion of loans drawn down by cheques in favour of other banks' customers is (1-k) since a proportion of k of freshly created derivative deposits are assumed to remain with the bank; hence:

$$(2.1) \qquad c_1 = (1-k)x.$$

The reserve ratio will be preserved after loan expansion only if

$$(2.2) \qquad c_1 = c-(rc+krx),$$

where c was the additional reserves provided by the primary deposit, rc is the reserve to be held against that deposit, and krx represents reserves to be held against the incremental derivative deposits, kx. Solving equations (2.1) and (2.2) for x yields the formula,

$$(2.3) \qquad x = \frac{c(1-r)}{k(r-1)+1},$$

where c(1-r) can be regarded as the level of excess reserves before the expansion of credit.

Since (r-1) must be negative, Phillips' findings imply that a higher ratio of derivative deposits to loans leads to a higher level of credit expansion. Additionally, Phillips qualifies his original assumption which stipulated that when a borrower wrote cheques against the derivative deposit created pursuant to his loan, the cheques would always favour customers of another bank. Phillips concedes that in countries having a high concentration in the banking industry a proportion of such cheques would be likely to favour customers of the loaning bank and not imply a loss of reserves. Phillips feels that this effect can be introduced into his formula by raising the value of k.[6]

The major criticism of Phillips' work is that it is entirely deterministic and that it lacks an explicit profit maximization framework. It also relies heavily on "steady state" assumptions; such as, the assumption that the value of new loans granted is equal to the value of maturing loans, and the assumption that the terms to maturity at inception of all loans are uniform.

Orr and Mellon

Orr and Mellon's analysis introduces two important improvements to Phillips' analysis. First, the authors attempt to optimize the level of credit expansion in a profit maximizing framework; and, second, they abandon the

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[6]For a roughly contemporary model which concludes that a bank may lend out somewhat more than the value of excess reserves, but bases its conclusion entirely on this effect, see W.F. Crick, "The Genesis of Bank Deposits, "Economica 7 (June 1927): 191-202.

assumption that incremental derivative deposits will stand in a constant ratio to new loans. Instead, they make allowances for random reserve losses so that this ratio becomes a random variable.

Orr and Mellon assume that the bank whose behavior is being analyzed begins the period in equilibrium; the ratio of its reserves to deposits are exactly ρ, the legal reserve requirement. Thus, its reserve holdings justify neither an expansion nor a contraction in credit. The legal reserve requirement is assumed to be of the contemporaneous variety in that it regulates the ratio of current reserves (balances held with the central bank plus vault cash) to current deposit liabilities. This differs from the lagged legal reserve requirement currently in effect in Canada and the U.S.A., in which average reserves in the current period must stand in a legally required ratio to deposit liabilities averaged over a prior time period.

With the lagged reserve requirements the bank knows the exact dollar value of the reserve requirement it will face throughout the period. With the contemporaneous reserve requirement, however, the bank cannot know the dollar value of the reserve requirements to be held in the current period as it does not know for certain what its level of deposits will be.

Orr and Mellon's model is a one-period static model where the period considered roughly corresponds to a reserve averaging period in Canada or the U.S.A. the model is static in that decisions made in each period are presumed to be independent of decisions made in any previous period.

R is defined as the volume of excess reserves which the bank--formerly in equilibrium--finds itself in possession of early in the period. D is the volume of new deposit liabilities which the bank creates during the period as a result of an equal amount of credit expansion. Each dollar of additional credit yields interest of i per cent per period, so the iD represents the positive return to the bank associated with the volume of deposit/credit expansion, assuming that all loans earn the interest i in the period no matter when in the period these loans are created. L represents the loss of reserves during the period, and reserves are presumed to be legally sufficient if, at the end of the evaluation period,

$$R-L \geq \rho(D-L). \tag{2.4}$$

Of course, in reality, a contemporaneous reserve requirement usually demands that average reserves over the period stand in the legal proportion to average deposits over the period, not just that the reserve/deposit ratio be maintained at the end of the period.

Expected losses arising from credit expansion hinge on the fact that an increase in the volume of loans will increase the probability of the bank failing to meet its legal reserve requirements. The costs associated with running short of reserves are M, a lump sum involving the administrative costs of running short, and r, a penalty rate of interest, per period, on each dollar of reserve deficit. The expected value of the cost of credit expansion is expressed by the authors as:

$$(2.5) \qquad M \int_{v}^{\infty} \phi(L)\,dL + r \int_{v}^{\infty} L\phi(L)\,dL,$$

with $\phi(L)$ being the probability density function of L; and

$$(2.6) \qquad v = (R-\rho D)/(1-\rho)$$

being the largest volume of cash outflow that can be sustained without falling below legal reserve requirements.

The authors intend that the first expression in (2.5) represent the product of the value of the lump sum cost associated with reserve deficits and the probability that a reserve deficit will occur: it does. They also intend that the second expression represent the product of the per dollar cost of reserve deficit and the expected value of the reserve deficit: it does not. The second expression should read: $r \int_{v}^{\infty} (L-v)\phi(L)\,dL$.[7] As the authors write it, the expression gives the product of the per dollar penalty rate and the expected value of the entire reserve loss (not just that in excess of v), given that the reserve loss exceeds the critical level v.

Orr and Mellon write their expected profit function:

$$(2.7)\quad P = iD - M\int_v^\infty \phi(L)dL - r\int_v^\infty L\phi(L)dL.$$

They assume that the reserve loss L is normally distributed, with a mean linearly dependent on D and with a variance which is assumed to be independent of D. The variables and parameters can be scaled so that the standard deviation of $\Phi(L)$ equals 1 and $\phi(L)$ can then be expressed as $\phi(L) = n(kD,1)$ where k is a constant $(o \leq k \leq 1)$. This k in Orr and Mellon's analysis can be interpreted as (1-k) in Phillips' analysis, i.e., the proportion of newly created deposits on which cheques are drawn by the borrower and those cheques cleared in favour of other banks.

Differentiating (2.7) with respect to D yields the first order condition for maximizing profit with respect to credit expansion:

$$(2.8)\quad 0 = i + (M+rv)\phi(v)\frac{dv}{dD} - (Mk+rv)\phi(v) - r[1-\Phi(v)],$$

where $\Phi(v)$ is the cumulative distribution of L at v. Substituting for v and $\frac{dv}{dD}$ from (2.6) Orr and Mellon arrive at the expression

$$(2.9)\quad 0 = i - \{\frac{r(R-\rho D)}{(1-\rho)^2} + M[k + \frac{\rho}{(1-\rho)}]\}\phi[\frac{(R-\rho D)}{(1-\rho)}] - r\{1 - \Phi[\frac{(R-\rho D)}{(1-\rho)}]\}.$$

Equation (2.8), however, is incorrect and should read:

$$(2.10)\quad 0 = i + (M+rv)(\frac{dv}{dD} - k)\phi(v) - rk[1-\Phi(v)].$$ [8]

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[8] Sho-Chiek Tsiang, "Errata," American Economic Review 53 (September 1963): 745.

This alteration corrects a slip in the authors' mathematics, but does not correct the conceptual error in equation (2.5) which was mentioned earlier.

From equation (2.9) Orr and Mellon solve for D, the optimal level of credit expansion, by assuming values for M, r, i, ρ, k, and R. They compare values of D arrived at through this process with the values generated by the certainty model in which the optimal value of D in terms of R is obtained by solving the equations:

$$(2.11) \qquad R-L = \rho(D-L),$$

$$(2.12) \qquad L = kD,$$

to yield,

$$(2.13) \qquad D = \frac{R}{k(1-\rho) + \rho}.$$

Equation (2.13) is equivalent to equation (2.3) in Phillips' model, though Orr and Mellon's k is equal to Phillips' (1-k). Additionally, R in Orr and Mellon's work must be compared to c(1-r) in Phillips' model; since Orr and Mellon define R as the volume of excess reserves, and the volume of excess reserves in Phillips' analysis is c(1-r), where c represents a new primary deposit.

Under most combinations of the chosen values of M, r, i, ρ, k, and R, Mellon and Orr arrive at results which suggest that credit expansion would be less under conditions of uncertainty than under conditions of certainty, with the notable exception of the case where i, the rate of interest on loans, is allowed to exceed r, the penalty rate per dollar of reserve deficit. In this case the bank is found to expand credit indefinitely, regardless of the value of M, k, or R.

Orr and Mellon's work came under serious criticism. The obvious faults concerning the specification of the profit function and the error in mathematics leading to the incorrect equation (2.9) detracted from the presentation, but it was the infinite expansion of credit case which drew the greatest attention. Infinite credit expansion had never been observed, even though the penalty rate (in the U.S.A. this is normally 2 per cent above the discount rate) was usually below the rate of return on bank loans. The non-existence of this phenomenon was carefully explained away by consideration of the rate of return on loans as net of administrative costs and by reference to rationing of central bank credit, and the importance to a bank of its reputation for soundness.[9]

Brown and Lloyd

Besides correcting the misspecifications in Orr and Mellon's work, Brown and Lloyd, the authors of the next piece of work to be discussed, introduce lagged reserve

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[9]For a general discussion of criticisms of Mellon and Orr's work see C.F. Brown and R.M. Lloyd, "Static Models of Bank Credit Expansion," p. 1001.

requirements into their analysis.[10] This step is important because both Canada and the United States presently use this type of requirement and because the model to be introduced in Chapter III of this thesis employs lagged reserve requirements.

Brown and Lloyd's major objectives are to determine whether credit expansion under uncertainty can exceed credit expansion under conditions of certainty and to define the circumstances which will engender that result. At this point their certainty and uncertainty results under contemporaneous reserve requirements will be examined.

Brown and Lloyd's model, like Orr and Mellon's, is a one-period static model. R_o is defined as reserves held by the bank at the beginning of the period, while D_o is the level of deposits at the beginning of the period. The required reserve ratio is ρ. The fraction of new deposits created in the period which will be withdrawn from the bank in the certainty case is k. The rate of interest per dollar of loans is i, and r is the penalty rate assessed per dollar of reserve shortage. A fixed penalty of size M is incurred (in addition to the proportional penalty r) whenever the reserve requirement is violated. M consists of a fixed sum representing administrative costs and loss of "goodwill". L represents reserve losses resulting from

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[10] Ibid., 996-1014.

credit expansion, and λ is the level of new loans (deposits) created.

In the contemporaneous reserve system of Brown and Lloyd, the reserve requirement will be met if, at the end of the evaluation period,

$$(2.14) \quad R_o - L \geq \rho(D_o + \lambda - L).$$

Additionally, in the certainty case it is known that

$$(2.15) \quad L = k\lambda.$$

Therefore, λ^*, the credit expansion under conditions of certainty for which the reserve requirement is met with equality, can be found by solving for λ in equations (2.14) and (2.15) to yield:

$$(2.16) \quad \lambda^* = \frac{R_o - \rho D_o}{k + \rho(1-k)}.$$

Under conditions of uncertainty, reserve losses, L, are considered to be distributed normally around a mean linearly dependent on the level of new credit. This mean is written as $k\lambda$. The variance of this distribution is a constant σ^2. $\phi(L)$ designates the probability of incurring reserve losses of level L, and $\Phi(L)$ designates the cumulative distribution at L of the probability density function $\phi(L)$.

The profit function in the uncertainty case can be written as:

$$(2.17) \quad \Pi = i\lambda - M\int_V^\infty \phi(L)\,dL$$
$$- r\int_V^\infty [\rho(D_0+\lambda-L) - (R_o-L)]\phi(L)\,dL,$$

where v is equal to $\frac{R_o - \rho(D_o+\lambda)}{1-\rho}$, the largest reserve loss sustainable without violating reserve requirements. The first term in (2.17) represents total interest revenue on new loans. The second term represents the fixed costs involved in violating reserve requirements weighted by the probability of this situation occurring, and the third term represents the penalty rate weighted by the expected value of the reserve deficit.

For the first and second order derivatives of (2.17) with respect to credit expansion, Brown and Lloyd arrive at:

$$(2.18)\quad \frac{\partial \Pi}{\partial \lambda} = i - M[\frac{\rho}{1-\rho} + k]\phi(v) - r[k+\rho(1-k)][1-\Phi(v)],$$

$$(2.19)\quad \frac{\partial^2 \Pi}{\Pi \lambda^2} = -M\frac{1}{\sigma^2}(\frac{\rho}{1-\rho})v\phi(v) - r[k+\rho(1-k)][\frac{\rho}{1-\rho}+k]\phi(v).$$

While their first order derivative is correctly stated, the second order derivative appears to be in error and should read:

$$(2.20)\quad \frac{\partial^2 \Pi}{\partial \lambda^2} = -(k + \frac{\rho}{1-\rho})^2\phi(v)[\frac{M}{\sigma^2}(v-\lambda k)+r(1-\rho)].$$

The corrected second order derivative will be less than zero, as required for a maximum only if $\frac{M}{\sigma^2}(v-\lambda k)+r(1-\rho)$ is positive. After replacing $\frac{R_o-\rho(D_o+\lambda)}{1-\rho}$ for v and simplifying, this condition can be expressed as

$$(2.21)\quad \lambda-\lambda^* < \frac{r\sigma^2(1-\rho)^2}{M[k+\rho(1-k)]}.$$

This implies that the second order condition for a maximization of (2.17) with respect to λ is only satisfied if λ does not exceed λ^* by more than the magnitude on the right hand side of equation (2.21). Setting equation (2.18) equal to zero allows the profit maximizing level of credit expansion λ' to be found, and λ' will satisfy the equation:

$$(2.22) \qquad \Phi(v) = 1 - \frac{i}{r[k+\rho(1-k)]} + \frac{M}{r(1-\rho)}\phi(v).$$

Since the expression on the left hand side of equation (2.22) is a cumulative probability distribution and cannot take on values less than zero, optimal credit expansion will be finite only if $\frac{i}{r[k+\rho(1-k)]} - \frac{M}{r(1-\rho)}\phi(v) < 1$.

By evaluating equation (2.22) at λ^* Brown and Lloyd seek to compare the optimal credit expansion under conditions of uncertainty with that of the certainty case. They find that

$$\Phi\left(\frac{R_o-\rho(D_o+\lambda^*)}{1-\rho}\right) = \Phi\left(\frac{k(R_o-\rho D_o)}{k+\rho(1-k)}\right) = \Phi(k\lambda^*) = 1/2,$$

and that

$$\phi\left[\left(\frac{R_o-\rho(D_o+\lambda^*)}{1-\rho}\right)\Big|k\lambda,\sigma^2\right] = \phi(k\lambda^*|k\lambda^*,\sigma^2) = \frac{1}{\sqrt{2\pi}\,\sigma},$$

so that:

$$(2.23) \quad 1 - \frac{i}{r[k+\rho(1-k)]} + \frac{M}{r(1-\rho)\sqrt{2\pi}\,\sigma} > 1/2$$

implies $\lambda'<\lambda^*$,

$$(2.24) \quad 1 - \frac{i}{r[k+\rho(1-k)]} + \frac{M}{r(1-\rho)\sqrt{2\pi}\,\sigma} = 1/2$$

implies $\lambda'=\lambda^*$,

$$(2.25)\quad 1 - \frac{i}{r[k+\rho(1-k)]} + \frac{M}{r(1-\rho)\sqrt{2\pi}\,\sigma} < 1/2$$

implies $\lambda' > \lambda^*$.

Hence, Brown and Lloyd's work agrees with the efforts of Orr and Mellon in that infinite credit expansion is a definite possibility in both of their respective models. However, Brown and Lloyd's model clearly indicates that uncertainty does not necessarily tend to result in increased expansion. Equations (2.23), (2.24), and (2.25) show that credit expansion under conditions of uncertainty may exceed, equal, or fall short of expansion under certainty conditions, depending on the values of the parameters i, r, k, ρ, σ, and M.

Besides correcting the work of Orr and Mellon, Brown and Lloyd examine the implications of lagged reserve requirements on credit expansion. In the case of Brown and Lloyd's lagged reserve requirement model, the dollar value of reserve which the bank is required to hold at the end of the period is known with certainty at the onset of the period. This quantity is specified as ρD_O where D_O is the level of deposits at the beginning of the period. In reality, a lagged reserve requirement is based on the average level of deposits on the books of the bank over some portion of the immediately preceding reserve averaging period (or two preceding reserve periods). Furthermore, in reality, it is the average of reserves in the current period that must stand in the legal relationship to reservable deposits, not just reserves held at the end of the period. Brown and Lloyd's utilization of the initial level of deposits in the current period as the base for the period's reserve requirement is acceptable in the interests of simplification, as is their assumption that the reserve requirement need only be met at the end of the period. The important consideration is that deposits created during the current period will not increase the dollar value of reserve requirements during the current period; they will, however, affect the dollar value of reserve requirements in subsequent periods. A one period, static model employing lagged reserve requirements will tend to overestimate the profits associated with credit expansion by neglecting the costs associated with the higher levels of required reserves in subsequent periods which credit expansion will engender.

In the certainty case, Brown and Lloyd write their legal reserve restriction as

$$(2.26) \quad R_o - L \geq \rho D_o .$$

Reserve losses due to credit expansion are assumed to be

$$(2.27) \quad L = k\lambda ,$$

and using (2.27) the authors find λ^{**}, the level of loans at which (2.26) is satisfied with equality, to be

$$(2.28) \quad \lambda^{**} = \frac{w}{k} ,$$

where w is equal to $R_o - \rho D_o$.

At this juncture Brown and Lloyd illustrate a very profit maximization. Where the fixed penalty cost M is assumed to be zero for simplicity's sake, the certainty case profit function is described as

$$(2.29) \qquad \Pi = \begin{cases} i\lambda \text{ for } \lambda \leq \lambda^{**} \\ i\lambda - r[\rho D_o - R_o + k\lambda] \text{ for } \lambda > \lambda^{**}, \end{cases}$$

and in order to compare the magnitudes of the marginal revenue and the marginal cost we take the derivative of profit with respect to credit expansion:

$$(2.30) \qquad \frac{\partial \Pi}{\partial \lambda} \begin{cases} i \text{ for } \lambda \leq \lambda^{**} \\ i - rk \text{ for } \lambda > \lambda^{**}. \end{cases}$$

From (2.30) it can be seen that λ^{**} is optimal as long as $i<rk$, but that indefinite expansion of credit is warranted if $rk<i$. This can be seen by noting that as long as $i>0$ the optimal level of λ must be at least λ^{**}, because each dollar of deposit expansion yields a positive increment to profits. If $i<rk$, λ^{**} must be the optimal level of deposit expansion because the marginal revenue i, associated with each further dollar of credit expansion falls short of rk, the marginal cost. If $i>rk$ the marginal revenue everywhere exceeds the marginal cost and there is no finite optimal value for λ. In the certainty case with profit maximization it is always profitable to expand credit if excess reserves are being held, but it also may be profitable to continue credit expansion indefinitely if the penalties associated with running short of reserves are less than the interest rate accrued on loans. Here we see that it is not uncertainty which admits of the possibility of infinite credit expansion--it is profit maximization!

Now let us consider Brown and Lloyd's uncertainty results under lagged reserve requirements. First, the uncertainty results in the absence of M (the fixed penalty cost), will be considered and second, the results with M included will be examined. This is done not only because Brown and Lloyd themselves affect this maneouver, but also because a similar procedure is carried out in Chapter III of this thesis.

[illegible]en fixed penalty costs are ignored, the profit [illegible] the uncertainty case with lagged reserve re-q[illegible]s can be written as:

$$\Pi = i\lambda - r\int_{w}^{\infty}\{\rho D_o - (R_o - L)\}\phi(L)\,dL, \quad (2.31)$$

where w is equal to $R_o - \rho D_o$, the largest reserve loss sustainable without violating reserve requirements. The first term in (2.31) is the total interest revenue on new loans, while the second term is the penalty rate weighted by the expected value of the reserve deficit.

Taking the first and second order derivatives of (2.31) with respect to loan expansion, we get:

$$\frac{\partial \Pi}{\partial \lambda} = i - rk[1 - \Phi(w)], \quad (2.32)$$

$$\frac{\partial^2 \Pi}{\partial^2 \lambda} = -rk^2 \phi(w). \quad (2.33)$$

Equation (2.33), the second derivative, is necessarily negative, since r is assumed to be positive, and the probability that reserve losses will exactly equal w is assumed to be some positive fraction less than one. Setting (2.32) equal to zero allows the profit maximizing level of credit expansion λ'' to be deduced. λ'' will satisfy the equation:

$$(2.34) \qquad \Phi(w) = 1 - \frac{i}{rk} .$$

Clearly, λ'' will be finite only if $i<rk$; otherwise infinite credit expansion is warranted. To compare the certainty (without profit maximization) and uncertainty results, Brown and Lloyd substitute $\lambda^{**} = \frac{w}{k}$ from equation (2.28) into equation (2.34) to reveal that

$$(2.35) \qquad \Phi(w|k\lambda^{**},\sigma^2) = 1/2,$$

since w would then be the mean of the distribution, and the cumulative distribution at the mean is equal to 1/2. Thus, the uncertainty case will call for the same credit expansion as the certainty case when $\frac{i}{rk}$ is equal to 1/2. If $\frac{i}{rk}$ is greater than 1/2, λ'' must be greater than $\frac{w}{k}$, since the mean of L must be greater than w if $\Phi(w|k\lambda,\sigma^2)$ is to be less than 1/2, (the normal distribution being symmetric about the mean). By equation (2.28), this means that λ'' is greater than λ^{**} if $\frac{i}{rk}$ is greater than 1/2. Similarly, λ'' is less than λ^{**} if $\frac{i}{rk}$ is less than 1/2.

When fixed penalty costs are included, the profit function in the uncertainty case with lagged reserve requirements is expressed as:

$$(2.36) \quad \Pi = i\lambda - M\int_w^\infty \phi(L)\,dL - r\int_w^\infty (\rho D_o - (R_o - L))\,\phi(L)\,dL.$$

Here the first and third terms are interpreted as in equation (2.31), and the second term in (2.36) is the fixed lump sum penalty cost for violating the reserve requirement, weighted by the probability that random reserve flows force such a violation given the level of credit expansion chosen.

The first and second order derivatives of (2.36) with respect to λ are:

(2.37) $$\frac{\partial \Pi}{\partial \lambda} = i - Mk\phi(w) - rk[1-\Phi(w)]\text{, and}$$

(2.38) $$\frac{\partial^2 \Pi}{\partial \lambda^2} = -\frac{Mk^2}{\sigma 2}(w-k\lambda)\phi(w) - rk^2\phi(w).$$

As in the contemporaneous reserve model with fixed penalty cost included, the second order derivative is not necessarily negative as is required for a maximum. The second order derivative will be negative only if

(2.39) $$\lambda < \lambda^{**} + \frac{r\sigma^2}{Mk}.$$

Equating (2.37) to zero allows the profit maximizing level of credit expansion λ''' to be found. λ''' will satisfy the equation,

(2.40) $$\Phi(w) = 1 - \frac{i}{rk} + \frac{M}{r}\phi(w).$$

In this case optimal credit expansion will be finite only if $\frac{i}{rk} - \frac{M}{r}\phi(w)$ is less than one.

Evaluating (2.37) at λ^{**} yields:

$$\Phi(w) = \Phi(k\lambda^{**}) = 1/2\ ,$$

and

$$\Phi((w)|k\lambda^{**},\sigma^2) = \phi(k\lambda^{**}|k\lambda^{**},\sigma^2) = \frac{1}{\sqrt{2\pi}\ \sigma} .$$

This allows the optimal credit expansion under conditions of certainty (without profit maximization) to be compared with optimal credit expansion under conditions of uncertainty. Their relative magnitudes can be summed up by the following relations:

$$(2.41) \quad 1 - \frac{i}{rk} + \frac{M}{r\sqrt{2\pi}\ \sigma} > 1/2 \text{ implies } \lambda''' < \lambda^{**},$$

$$(2.42) \quad 1 - \frac{i}{rk} + \frac{M}{r\sqrt{2\pi}\ \sigma} = 1/2 \text{ implies } \lambda''' = \lambda^{**},$$

$$(2.43) \quad 1 - \frac{i}{rk} + \frac{M}{r\sqrt{2\pi}\ \sigma} < 1/2 \text{ implies } \lambda''' > \lambda^{**}.$$

Brown and Lloyd, therefore, contributed to the state of the art by correcting the many errors associated with Orr and Mellon's work and by introducing lagged reserve requirements. They also introduced profit maximization into their credit expansion model under conditions of certainty. This step allowed them to reveal that the profit maximization assumption, rather than the uncertainty assumption, generated the possibility of infinite credit expansion. The other major contribution of Brown and Lloyd was the discovery that optimal credit expansion under conditions of uncertainty could equal, be greater than, or be less than the optimal credit expansion under certainty conditions (without profit maximization), depending on the values of the parameters of the system.

The mathematical error in Brown and Lloyd's presentation (corrected in this paper) detracted from the work, especially since it prevented them from realizing that the inclusion of the lump sum penalty cost, M, always prevents the conditions for global profit maximization from being satisfied. Additionally, in their lagged reserve requirement models, they were guilty of underestimating the costs of credit expansion by not making allowances for the higher levels of required reserves in future periods induced by credit expansion. In a later paper, Brown utilized a dynamic linear programming model to examine bank credit expansion behavior under uncertainty.[11] In this dynamic, multi-period model, which takes into account the inter-period effects of credit expansion, Brown concluded that the results obtained by Brown and Lloyd extended well to the dynamic model. Brown also concluded that bank credit expansion, and, hence, bank reserve behavior, in a contemporaneous reserve requirement system differed from bank behavior in a lagged reserve requirement system only in the short term.

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[11] G.F. Brown, "Optimal Management of Banks Reserves," Journal of Financial and Quantitative Analysis 7 (December 1972): 2031-54.

Ratti

Mellon and Orr's work was also criticized in an article by R.A. Ratti.[12] Ratti's model is developed using Orr and Mellon's notation, and he concerns himself solely with the analysis of credit expansion under contemporaneous requirements. The only changes from Orr and Mellon's notation in Ratti's work involves his definition of the lump sum penalty cost as m, rather than the upper case M used by Orr and Mellon.

Ratti begins his analysis by pointing out that, if L is less than or equal to v, profit is given by

$$P_1 = iD.$$

If there is a reserve deficit, i.e., if L is greater than v, profit is given by

$$P_2 = iD - m - r(\rho D - R + (1-\rho)L).$$

Since Ratti is troubled by the fact that P_2 is not defined where L is equal to v he defines a profit equation for L greater than or equal to v,

$$P_2^* = iD - M(L) - r(\rho D - R + (1-\rho)L),$$

where

$$M(L) = 0, \text{ for } L \leq v,$$

$$= m, \text{ for } L > v.$$

This implies that

$$P_2^* = P_1, \text{ for } L = v,$$

$$= P_2, \text{ for } L < v.$$

[12]R.A. Ratti, "Stochastic Reserve Losses and Bank Credit Expansion," Journal of Monetary Economics 5 (April 1979): 283-94.

Expected profits can now be defined as

$$(2.44) \quad E(P) = \int_{-\infty}^{v} P_1\phi(L)dL + \int_{v}^{\infty} P_2{*}\phi(L)dL,$$

or

$$(2.45) \quad E(P) = iD - \int_{v}^{\infty}[M(L) + r(\rho D-R+(1-\rho)L)]\phi(L)dL.$$

Differentiating equation (2.45) with respect to D and equating the result to zero yields the first order maximization condition

$$(2.46) \quad \frac{\partial E(P)}{\partial D} = i-r\rho\int_{v}^{\infty}\phi(L)dL - \int_{v}^{\infty}[M(L)+r(\rho D-R+(1-\rho)L)]\frac{\partial\phi(L)}{\partial D}dL = 0.$$

It is instructive to point out at this stage that defining the M(L) function to be equal to zero where L is equal to v has caused a term to fall out of the first order maximization condition that would have been present under the techniques of Orr and Mellon or Brown and Lloyd. This term is $\frac{-m\rho}{1-\rho}\phi(v)$. The lower boundary of the integral in equation (2.45) is v which is clearly a function of D so that the proper differentiation of the integral with respect to D should include a term which is the negative of the function within the integral, evaluated at v, multiplied by the derivative of v with respect to D. The second term within the square brackets of equation (2.45) is equal to zero at v; however, the first term is equal to zero as well using Ratti's technique, thus the term drops out. In Orr and Mellon's or Brown and Lloyd's work, the two terms within the square brackets of equation (2.45) are isolated in separate integrals each with the same upper and lower bounds, and each multiplied by $\phi(L)$ within their separate integrals. Since these authors do not define an M(L) function to cause the lump sum penalty cost to vanish at v the notational equivalent of $\frac{-m\rho}{1-\rho}\phi(v)$ is present in their work.

In order to obtain useful results, Ratti must simplify the third term on the right hand side of equation (2.46), but the task is complicated by the fact that the function within the integral is not continuous where L is equal to v. Consequently, he defines an arbitrarily small positive number, θ, and integrates,

$$\int_{v+\theta}^{\infty}[M(L) + r(\rho D - R + (1-\rho)L)]\frac{\partial\phi(L)}{\partial D}\,dL,$$

by parts to yield an equivalent expression

$$k[m + r(1-\rho)\theta]\phi(v+\theta) + r(1-\rho)k\int_{v+\theta}^{\infty}\phi(L)\,dL.$$

Since,

$$\int_{v}^{\infty}[M(L) + r(\rho D - R + (1-\rho)L)]\frac{\partial\phi(L)}{\partial D}\,dL$$

is equal to

$$\lim_{\theta\to 0}\int_{v+\theta}^{\infty}[M(L) + r(\rho D - R + (1-\rho)L)]\frac{\partial\phi(L)}{\partial D}dL,$$

it is also equal to

$$mk\phi(v) + r(1-\rho)k\int_{v}^{\infty}\phi(L)\,dL,$$

so that equation (2.46) can be expressed as

$$(2.47)\quad \frac{\partial E(P)}{\partial D} = i - r(\rho+k(1-\rho))\int_{v}^{\infty}\phi(L)\,dL - mk\phi(v) = 0.$$

This compares with

$$(2.48)\quad \frac{\partial E(P)}{\partial D} = i-r(\rho+k(1-\rho))\int_v^{\infty}\phi(L)dL-m(k+\frac{\rho}{1-\rho})\phi(v),$$

which is the equation (2.18) attributable to Brown and Lloyd, translated into Ratti's notation and format.

Second order conditions are given by

$$(2.49)\quad \frac{\partial^2 E(P)}{\partial D^2} = -r(1-\rho)(\frac{\rho}{1-\rho}+k)^2\phi(v) - mk(\frac{\rho}{1-\rho}+k)\frac{(v-kD)}{\sigma^2}\phi(v) < 0,$$

and Ratti indicates that this condition will be met only if

$$(2.50)\quad D < D^* + \frac{\sigma^2 r(1-\rho)}{km},$$

where

$$(2.51)\quad D^* = \frac{R}{\rho + k(1-\rho)},$$

the familiar optimal level of credit expansion under conditions of certainty.

The profit maximizing level of credit expansion $\bar{D}$ can be determined from equation (2.47) and will satisfy the equation

$$(2.52)\quad \Phi(v) = 1 + \frac{mk\phi(v) - i}{r[k + \rho(1-k)]},$$

where $\Phi(v)$ is the cumulative distribution of $\phi(L)$ at v. From equation (2.52) it can be seen that optimal credit expansion will be finite only if $\frac{i-mk\phi(v)}{r[k+\rho(1-k)]}$ is less than one. The conditions under which $\bar{D}$ will equal, be less than or exceed D* can be stated as:

$$(2.53)\quad 1 - \frac{i}{r[k+\rho(1-k)]} + \frac{mk}{r[k+\rho(1-k)]\sqrt{2\pi}\,\sigma}$$

$$> 1/2 \text{ implies } \bar{D} < D^*,$$

$$(2.54)\quad 1 - \frac{i}{r[k+\rho(1-k)]} + \frac{mk}{r[k+\rho(1-k)]\sqrt{2\pi}\,\sigma}$$

$$= 1/2 \text{ implies } \bar{D} = D^*,$$

$$(2.55)\quad 1 - \frac{i}{r[k+\rho(1-k)]} + \frac{mk}{r[k+\rho(1-k)]\sqrt{2\pi}\,\sigma}$$

$$< 1/2 \text{ implies } \bar{D} > D^*.$$

A comparison of equations (2.50), (2.52), (2.53), (2.54), and (2.55) with equations (2.21), (2.22), (2.23), (2.24), and (2.25) reveals that the model of Ratti does not differ fundamentally from that of Brown and Lloyd. The models will yield different numerical results depending upon the values of the parameters but both agree on the basic points. Both models allow for infinite credit expansion as an optimal result. They also conclude that optimal credit expansion under conditions of uncertainty may exceed, fall short of, or equal optimal credit expansion under conditions of certainty. Both models also display difficulties in the second order maximization conditions, caused by the introduction of the lump sum penalty cost.

Although Ratti does not examine the case of lagged reserve requirements in his paper, it is interesting to point out that his results would be identical to those of Brown and Lloyd. This can be seen by noting that in equation (2.36) the lower bound of the integrals comprising the second and third terms on the right hand side of the equation is $w=R_o-\rho D_o$ which is clearly not a function of λ, the level of credit expansion. This being the case, Brown and Lloyd's work would not contain the "extra" term arising from the failure to define an M(L) function that emerged in the contemporaneous reserve case. The problem would be bypassed because the derivative or w with respect to λ is zero, i.e., w and λ are independent.

Since Ratti's work differs from that of Brown and Lloyd only in the definition of the M(L) function, the relative merits of the two models can be quickly established by evaluating the efficacy of this technique used by Ratti. Ratti defines the M(L) function because of concern over the fact that P_1, (the profit function where L is less than, or equal to, v) is not equal to P_2 (the profit function where L is greater than v) when L is equal to v. The existence of this discontinuity in the profit function, however, is not a matter of real concern. The discontinuity exists in the mathematical model because the discontinuity exists conceptually, given the institutional framework of bank reserve management.

In taking the derivative of equation (2.45) to yield (2.46), Ratti applies Liebnitz's rule for differentiation under the integral sign in a situation which violates one of the conditions necessary for the application of that rule. The function in the variable L and the

the parameter D, within the integral of equation (2.45), is not continuous in L at the lower bound v, as it must be for Liebnitz's rule to apply. It is not continuous at this point because one component of the expression under the integral, namely M(L), is equal to zero where L is equal to v, but is equal to m for any value of L in excess of v. So it seems that to make his profit function continuous in L, which it should not be, Ratti has made a component of his expected profit function discontinuous in L, which it must not be if Liebnitz's rule is to be applicable.

It is curious that Ratti uses integration by parts "at the limit" to simplify the third term on the right hand side of equation (2.46) because he recognizes that the integral in that expression is not continuous where L is equal to v, but he does not seem to recognize the same problem in his differentiation of equation (2.45) with respect to D. Differentiating equation (2.45) by D with the lower bound of the integral set at $v+\theta$ and then solving for the limiting case where θ approaches zero will yield results which are identical to those of Brown and Lloyd. In other words, Ratti's unique results derive from the use of this limiting technique on only a part of the differentiation of equation (2.45) by D, where the results would be the same as Brown and Lloyd's if the technique were applied throughout the differentiation, (of course the most efficient action would be to abandon the M(L) function rather than to take measures to circumvent its effects). In conclusion, it seems that Ratti's efforts do not represent an improvement over Brown and Lloyd's, (a work that Ratti did not acknowledge having encountered).

Morrison

The first attempt to examine optimal bank portfolio behavior under conditions of uncertainty which emphasized bank reserve management was undertaken by George Morrison.[13] Morrison's objective was to optimize the level of cash reserves held by a bank under conditions of uncertainty using loss minimization. That is to say, he attempted to minimize the opportunity cost of holding reserves. Whereas Orr and Mellon, Brown and Lloyd, and Ratti examined marginal changes in a bank's portfolio given that excess reserves were held by a bank, Morrison was concerned with discovering the optimal level of reserves under uncertainty while making no assumptions about the initial level of reserves held. Morrison develops a one-period static bank behavior model in which a bank may hold two types of assets: non-interest bearing cash reserves and interest bearing loans. These loans are also subject to capital gains or losses during the period, which gives them some of the features of a bond

[13] G.R. Morrison, Liquidity Preferences of Commercial Banks (Chicago: University of Chicago Press, 1966).

or a fixed interest rate installment loan. The management of the bank is assumed to construct a probability distribution of possible capital gains or losses. It is also assumed to construct a probability distribution of stochastic cash drains or accruals which may occur during the period. The rate of interest per dollar of loans, per period, is labelled y and is assumed to be known prior to the planning period and fixed throughout it. Loans also have a capital gain or loss component, g, which is defined as a proportion of loans. This component is stochastic, with a probability density of $\phi(g)$, and g is distributed over the range $-1 \leq g < \infty$. The stochastic cash inflow is represented by v, which is expressed as a proportion of initial deposits (negative values of v designate cash drains).

To explain the definition of initial deposits, it is necessary to digress somewhat to present Morrison's assumptions concerning deposit creation through the extension of credit. Morrison assumes that all deposits created in the process of making loans are drawn down immediately at the beginning of the period, and that no borrower deals with merchants who use the lending bank. This is equivalent to the assumption that the lending bank has a negligible market share of banking business, and corresponds to the small bank case of the usual total banking system deposit expansion multiplier exposition which graces modern textbooks on money and banking. This assumption contrasts with those in the models of Orr and Mellon, Brown a[illegible] Llo[illegible]d Ratti which emphasized the retention of a p[illegible]tion [illegible] deposits created through credit expansion. Si[illegible]e a[illegible] deposits created by loan extension are transitor[illegible] Morrison's assumption, initial deposits refer to both [illegible]e level of deposits inherited from the previous peri[illegible] and the level of deposits existing after loan extension. The level of deposits may alter during the planning period but only as a result of the stochastic cash inflow (or drain).

Morrison designates the probability density of v, $f(v)$, [illegible]d v is assumed to be distributed over the interval [illegible], where $c \geq -1$. The lower limit on v must be -1 because that represents the unlikely event that every dollar of initial deposits are drawn down, leaving the bank with zero deposits. This cash drain or inflow occurs at the end of the period, after loan creation but before any outstanding loans are repaid. All cash deficiencies, which occur because of the inadequacy of reserve to cover cash withdrawals, are assumed to be made up by short-term borrowing at a rate n, which is known with certainty. Morrison defines ρ to be cash reserves as a proportion of initial deposits, so that the expected loss function $E[L(\rho)]$ is:

$$(2.56) \quad E[L(\rho)] = y\rho + \rho\int_{-1}^{\infty} g\phi(g)\,dg + \int_{c}^{-\rho} n(-v-\rho)f(v)\,dv.$$

The first two terms of equation (2.56) represent the opportunity cost of holding reserves, while the third term is the expected cost of cash drains exceeding cash reserves held.

Cash reserves may not be negative in Morrison's model. This is because Morrison assumes away a legal reserve requirement, so that n represents a cost of borrowing short-term to cover an actual reserve deficit, rather than a penalty for failing to meet reserve requirements--such as the rate referred to in the work of Orr and Mellon, Brown and Lloyd, and Ratti.

f(v) is assumed to be the probability density function of the uniform distribution and therefore, $f(v) = \frac{1}{b-c}$. Substituting this into equation (2.56), and evaluating the integrals, yields:

$$(2.57) \quad E[L(\rho)] = y\rho + \bar{g}\rho - \frac{n\rho^2}{2(b-c)} + \frac{nc^2}{2(b-c)} + \frac{n\rho^2}{b-c} + \frac{n\rho c}{b-c},$$

where $\bar{g}$ is the mean of g. Differentiating equation (2.57) with respect to ρ yields:

$$(2.58) \quad \frac{\partial E}{\partial \rho} = y + \bar{g} + \frac{n\rho + nc}{b - c}$$

Setting this expression equal to zero and solving for ρ yields:

$$(2.59) \quad \rho = \frac{(c - b)(y + \bar{g})}{n} - c$$

The second order derivative of the expected loss function with respect to ρ is:

$$(2.60)\quad \frac{\partial^2 E}{\partial \rho^2} = \frac{n}{b - c},$$

which is greater than zero, since $n>0$ by assumption and $b-c>0$ by definition. Therefore, setting the first derivative to zero necessarily determines a minimum point.

Morrison then sets $b=c+k$ where $k>0$, to represent a crude measure of the dispersion of v. Equation (2.59) can then be written as:

$$(2.61)\quad \rho = \frac{-k(y + \bar{g})}{n} - c .$$

The mean of v is $\bar{v} = \frac{b + c}{2}$, since $b = c + k$, $\bar{v} = c + \frac{k}{2}$, and

$$(2.62)\quad c = \bar{v} - \frac{k}{2} .$$

Substituting equation (2.62) into equation (2.61) yields Morrison's final demand for cash reserves equation:

$$(2.63)\quad \rho = k[1/2 - \frac{y + \bar{g}}{n}] - \bar{v}.$$

Differentiating with respect to parameters y, $\bar{g}$, n, $\bar{v}$ and k yields:

$$\frac{\partial \rho}{\partial y} = \frac{-k}{n} < 0,$$

$$\frac{\partial \rho}{\partial \bar{g}} = -1 < 0,$$

$$\frac{\partial \rho}{\partial \bar{v}} = \frac{-k}{n} < 0,$$

$$\frac{\partial \rho}{\partial k} = \frac{1}{2} - \frac{y + \bar{g}}{n} \lesseqgtr 0,$$

$$\frac{\partial \rho}{\partial n} = \frac{k(y + \bar{g})}{n^2} \geq 0.$$

Morrison finds, therefore, that the demand for reserves varies inversely with the interest rate on loans, the expected capital gain on loans, and the expected stochastic cash inflow, while it varies directly with the borrowing rate n. Of interest is the result that the relationship between k and ρ may be of either sign. If twice the rate of return on loans, $2(y+\bar{g})$, is greater than n, the demand for reserves varies inversely with k; while, if $2(y+\bar{g})$ is less than n, it varies directly with k. Morrison explains this phenomenon by noting that as the dispersion of the stochastic cash inflow or drain increases, whether the bank holds more or less reserves depends on the relative costs of running short of reserves and foregoing income from loans. If k = o, the optimal cash ratio wil be $-\bar{v}$, since the cash inflow ratio will be $\bar{v}$ with certainty. Now, if k is increased, the bank will face a risk of holding too high, or too low, a cash ratio. The higher the rate of return on loans relative to the cost of running short of reserves, the more likely it is that an increase in k will encourage the bank to reduce the ratio below $-\bar{v}$, thus taking a high risk of holding too small a cash ratio. Obversely, the higher the cost of running short of reserves relative to the rate of return on loans, the more likely it is that an increase in k will encourage the bank to increase the ratio above $-\bar{v}$, thereby taking a high risk of holding too large a cash ratio.

In the handling of the stochastic reserve drain, there is an interesting point of contrast between the credit expansion models of Mellon and Orr, Brown and Lloyd, and Ratti and the typical reserve optimization model, of which Morrison's is representative. The credit expansion models are concerned with marginal changes in loan and reserve holdings. They assume that excess reserves mysteriously emerge to disturb a portfolio in perfect equilibrium, and they describe the optimal behavior to re-establish equilibrium. The most important quality of their random reserve loss element is how it will alter the amount of reserve retention after credit expansion. The mean of their stochastic reserve drain is specified as kD (or $k\lambda$ in Brown and Lloyd's model), where k is a constant and can be interpreted as one minus the bank's share of the total banking market. The deviation of this stochastic drain from the mean value is considered to be a result of random withdrawals or deposits on the part of the bank's customers for any reason whatever. However, the fact that the mean is specified as kD implies that the expected value of the customer's purely random withdrawals or deposits is zero. The mean of their stochastic cash drain is strictly linearly dependent on the volume of credit expansion of the bank and cannot encompass such situations as expected exogenous inflow or outflow of reserves emanating, say, from a definite policy on the part of the monetary authorities to encourage deposit expansion or contraction.

This contrasts with the nature of the stochastic reserve drain in the reserve optimization models. In those models, an absolute portfolio allocation between loans and reserves is sought, not just a marginal one. Additionally, as pointed out earlier in the discussion of Morrison's model, a typical assumption specifies that all deposits created in the process of credit extension are immediately withdrawn. In the works of Orr and Mellon, Brown and Lloyd, and Ratti, this would be equivalent to the assumption that the k in their models was equal to one with certainty. Therefore, the typical reserve optimization model totally excludes that aspect of the random reserve loss which was most emphasized by the credit expansion models. The mean value of the random reserve loss in the typical reserve optimization model, ($\bar{v}$ in Morrison's model), represents, therefore, only the expected value of exogenous inflows or outflows of reserves resulting from such factors as conscious intervention on the part of the monetary authorities.

Morrison's model, in particular, is subject to criticism on several accounts. First of all, the model makes no attempt to accommodate legal reserve requirements, which are, after all, quite fundamental to bank reserve management behavior. Another difficulty with the model arises from the fact that total reserve holdings can never be negative; therefore, the values of the parameters y, g, and n are restricted in a fashion similar to the models of bank credit expansion that were reviewed previously. For example, in equation (2.63), if the mean of v is equal to zero and k is other than zero, $(y+\bar{g})$ cannot take on values greater than $\frac{n}{2}$, because this would call for negative reserve holdings. This peculiarity is the counterpart of the situation in the credit expansion models which generates infinite credit expansion.

It is difficult to understand how capital gains and losses are possible on bank loans as they are assumed to be in Morrison's model. The rate of interest on bank loans y is fixed throughout the period so that changes in y cannot be the cause of these capital fluctuations. Nor can it be changes in the rate of interest on competing financial assets which initiate these capital value movements, since banks may hold only one type of asset and, therefore, it is difficult to imagine a competing asset which could seriously affect the capital value of bank loans.

Another difficulty with Morrison's model is its assumption that all deposits created in the process of credit extension are immediately withdrawn. While it is an unfortunate assumption to have to make, it is a necessary assumption if the model is not to become so unwieldy as to be useless. If Morrison had not assumed away the power of a single bank to create money, as it were, the mean of v would depend on the marginal change in ρ from the level inherited at the onset of the current period. Of course, this would require a multi-period dynamic model, since the adjustment to the optimal level of ρ in the current period would affect the level of initial deposits in the next period and, hence, the level of ρ in the next period. Additionally, it becomes difficult to define the opportunity cost of a single dollar of reserves as simply y, the rate of return on a single dollar of loans, because the existence of reserve retention after credit expansion permits the situation in which the keeping of one dollar less of reserves implies the creation of more than a single dollar of loans.

Morrison also assumes that the stochastic reserve loss is distributed according to the uniform distribution. Although this is quite an unreasonable assumption, it does allow the technical essence of the process of reserve optimization to be observed at its simplest level.

Poole

The next model to be examined is that of William Poole.[14] Poole's model is deeply entrenched in the rules

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[14]William Poole, "Commercial Bank Reserve Management in a Stochastic Model: Implications for Monetary Policy," Journal of Finance 23 (December 1968): 769-91.

and regulations of the American banking system that existed before the inception of lagged reserve requirements. It is assumed that banks must meet a required reserve ratio each day and that the reserve requirement applies to the ratio of end of day reserves to end of day deposits. At the end of the day, the bank may borrow from the central bank at the discount rate. It also may borrow or lend on the Federal funds market at mid-day at a competitively determined rate. The problem posed is that the bank faces a stochastic reserve loss as a result of cheque clearing which occurs after the Federal funds market has closed for the day. Poole assumes that the Federal funds market closes at noon, so that at noon the bank's reserve manager must make the decision whether to borrow or lend funds on the Federal funds market. This decision will be based on the anticipated reserve gains or losses which will occur from cheque clearing at the end of the day, on the discount rate, and on the prevailing Federal funds rate. It is assumed that the discount rate is always greater than the Federal funds rate, though in reality this is seldom the case. The bank is assumed to borrow at the discount rate if it finds itself short of reserves at the end of the day.

D represents the bank's deposits at noon; R, the bank's unborrowed reserves at noon; q, the required reserve ratio on deposits; i_d, the discount rate; and i_f, the Federal funds rate. Z is the net deposit and, hence, reserve accretion between noon and the close of business. Z_U represents the upper limit on Z, and Z_L represents the lower limit. The probability density function of Z is f(Z). F is the quantity of Federal funds borrowed at noon; F* is the optimal size of F; and C is the expected cost arising from uncertainty about reserve flows. The legal reserve requirement will be met at day's end if

$$(2.64)\quad R + F + Z \geq q(D + Z).$$

If (2.64) is satisfied, the bank will have excess reserves, E, described by

$$(2.65)\quad E = R + F + Z - q(D + Z)$$
$$= R + F - qD + (1 - q)Z.$$

Otherwise, the bank will have to borrow at the discount rate. B, the amount of borrowing, will be described by

$$(2.66)\quad B = q(d + Z) - (R + F + Z)$$
$$= qD - R - F - (1 - q)Z.$$

The expected cost arising from uncertainty is

$$(2.67)\quad C = i_f \int_{\hat{Z}}^{Z_U} [R+F-qD+(1-q)Z]f(Z)dZ$$
$$+ (i_d - i_f) \int_{Z_L}^{\hat{Z}} [qD-R-F-(1-q)Z]f(Z)dZ,$$

where $\hat{Z} = (qD-R-F)/(1-q)$, the level of reserve loss which would leave excess reserves at exactly zero. The first expression in (2.67) represents the opportunity cost of holding excess reserves which could have been lent out at the Federal funds rate at noon. The second expression is the cost of running short of reserves and being forced to borrow at the discount rate, when extra funds could have been borrowed on the Federal funds market at noon.

Taking the first derivative of (2.67) with respect to F and setting it equal to zero yields:

$$(2.68) \quad 0 = i_f + i_d \int_{Z_L}^{\hat{Z}} f(Z)dZ, \text{ or}$$

$$(2.69) \quad \text{Prob } [Z < (qD - R - F^*)/(1 - q)] = \frac{i_f}{i_d}$$

F^* can be considered the demand for Federal funds and is chosen so that the probability given by the left hand side of equation (2.69) is equal to the ratio of i_f to i_d, which, by assumption, is less than one. After presenting a discussion of reserve averaging, Poole calls attention to the position of a U. S. bank on Wednesday morning under the old contemporaneous reserve requirements. Reserve averaging periods in the U.S. ended on a Wednesday--they still do--but the bank's actual reserves were tallied at the end of each day, while required reserves were computed on deposits existing at the beginning of each day. If the daily sum of the actual reserves exceeded the daily sum of reserve requirements over the entire period, the reserve requirement was satisfied. However, on the last Wednesday of the averaging period, the bank would know the requirements for that day and every previous day in the averaging period, and it would know the cumulative amount of actual reserves held for all but Wednesday. Therefore, the dollar value of reserves necessary to meet the averaging period's requirements would be known on Wednesday. Poole designated this quantity R_o. The bank will now meet its requirements if

(2.70) $R + F + Z \geq R_o$,

and the optimal solution for F* is modified to

(2.71) $\text{Prob}[Z < R_o - R - F^*] = \frac{i_f}{i_d}$.

This solution is equivalent to one based on a lagged reserve requirement.

Poole concerns himself mainly with the supply and demand for Federal funds, rather than the demand for excess reserves per se, but the behavior of both of these aggregates is easily obtainable from equation (2.71). First, the demand for both Federal funds and excess reserves increases as i_f decreases or i_d increases. The demand for Federal funds and excess reserves, (R_o-R-F^*), as a function of i_f/i_d, will shift to the right or left as E(Z) decreases or increases, respectively. The demand for Federal funds, as a function of i_f/i_d, will take on the shape of the mirror image of the cumulative distribution function of Z, and will shift to the right or left as R_o-R increases or decreases. The demand for excess reserves will be exactly the shape of the mirror image of the cumulative probability distribution of Z. The variance of Z, σ_Z^2, makes the demand for either Federal funds or excess reserves, as function of i_f/i_d, flatter or steeper as σ_Z^2 increases or decreases. Poole's diagram of the effect of the variance of Z on the demand for Federal funds is reproduced below as Figure I.[15]

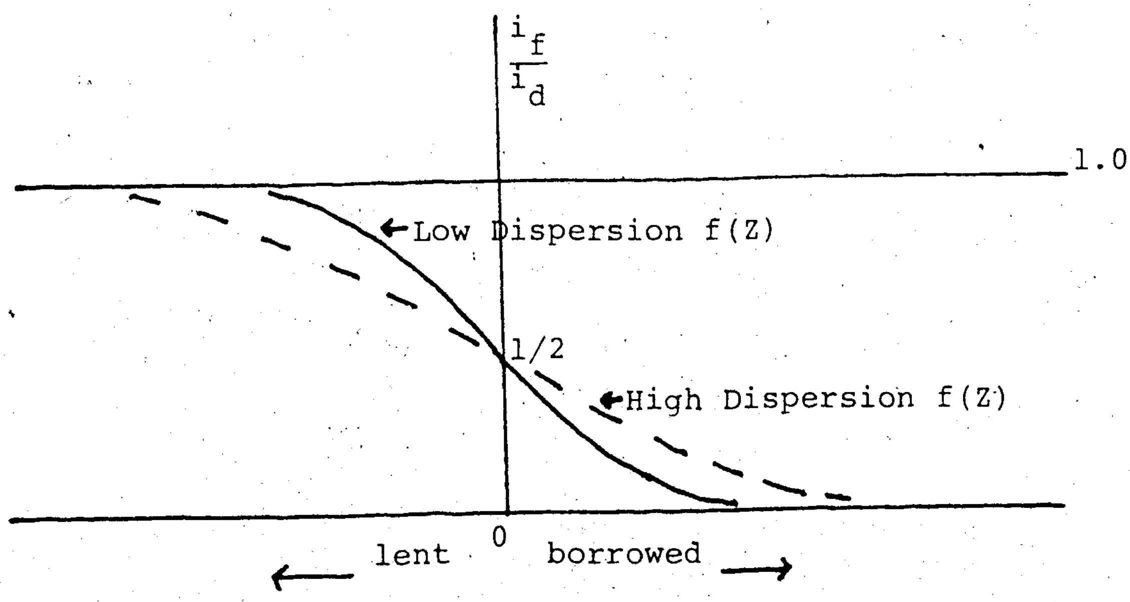

Figure I:

Demand for Federal Funds by Bank

This Figure represents the case in which $R_o - R = 0$, $E(Z) = o$, and f(Z) is symmetric. In this case, since $R_o - R = 0$, the demand for excess reserves and the demand

[15] Ibid., p. 784.

for Federal funds are identical. It will be noticed that an increase in uncertainty (i.e., an increase in σ_Z^2) will result in a differential effect on the demand for Federal funds, depending on the value of $\frac{i_f}{i_d}$. Where $\frac{i_f}{i_d} > \frac{1}{2}$, the higher the variance of f(Z), the smaller is the demand for Federal funds. As was noted in Morrison's explanation of the effects of dispersion, this result is obtained because the increased uncertainty of cash fluctuations causes less reserves to be held when the opportunity cost of holding reserves is high relative to the cost of borrowing from the central bank. If $\frac{i_f}{i_d} < \frac{1}{2}$, the increased uncertainty will cause more borrowing on the Federal funds market or, equivalently more excess reserve to be held. In other words, it becomes relatively more attractive to risk holding too high a level of contingency reserves than to risk borrowing from the central bank when the opportunity cost of holding reserves is low in relation to the cost of borrowing from the central bank.

The main drawback of Poole's work is the assumption that a bank's reserve management is virtually equivalent to the management of its operations on the market for Federal funds. The Federal funds rate is always assumed to be less than the discount rate, a situation which is very unlikely by historical experience. No attempt is made to examine the effects of ordinary domestic loan policies on the demand for reserves. Poole does not have to refer to the problem of deposit expansion induced by a combination of loan expansion and low reserve losses because the only loans considered in the portfolio are loans of Federal funds. Since such loans are inter-bank and are largely for reserve management purposes, it is a reasonable assumption that reserves will be lost, the amount being exactly equal to the size of the loan of Federal funds.

Poole's major contributions include the introduction of a reserve requirement (omitted by Morrison), much less stringent restrictions on the distribution of the random reserve loss component, and a slightly more rigorous examination of the effects of changes in the level of uncertainty on reserve management.

Frost

The first work in the literature which attempts to optimize the level of excess reserves was penned by Peter Frost.[16] His model is explicitly designed to provide a theoretical rationale for a kinked demand curve for excess reserves as a function of interest rates. The empirical testing of the existence of such a kinked demand curve comprises the bulk of his article.

[illegible]'s model is a one-period inventory model in

[illegible] Frost, "Bank's Demand for Excess Reserves," [illegible]ical Economy 79 (July/August 1971): [illegible]05-25.

which a bank's objective is to maximize profits. A bank's earning assets at the beginning of the period are designated securities, labelled S_o, where the subscript o indicates a level held at the beginning of the period. E_o is used to label the level of excess reserves at the beginning of ther period, and F_o, the level of free assets (assets other than required reserves), is the sum of S_o and E_o.

Securities earn a return of r per cent per period, and the cost to the bank of buying or selling these securities is composed of a lump sum cost of G dollars and a variable adjustment cost of v per cent per period. N is the change in the bank's excess reserves during the period as a result of deposits or withdrawals by its customers (negative values of N designate cash drains). It is assumed to be a random variable with a probability distribution $\phi(N)$. By assumption, $\phi(N)$ is symmetric and definable by the mean of N, μ, and the standard deviation of N, σ. Q is the change in the bank's excess reserve brought about by the buying or selling of securities. The level of excess reserves at the end of the period is

$$(2.72) \quad E_1 = E_o + N + Q,$$

where the subscript 1 indicates an end of period level.[17]

17Frost begins by assuming that the bank will, at the beginning of the period, make the transactions necessary to adjust its level of excess reserves from that inherited from the end of the previous period to the level optimal for the current period. In effect, this requires that inter-period adjustment costs are zero. This simpli-

An important assumption made by Frost, the one ultimately crucial in his development of a kinked demand curve for excess reserves, is that excess reserves may not be negative, either at the beginning or the end of the period. Thus a bank is assumed never to have the desire to hold a level of reserves which is less than required reserves regardless of the relative levels of v and r. Note that Poole's model placed no restriction on negative excess reserves at the beginning of the period, though the model was constructed so as to forestall the possibility of negative excess reserves at the end of the period. In Poole's model, if cost minimization indicated that the optimal level of excess reserves was negative in value, (perhaps because a large inflow of funds was expected during the period), the bank was not restricted from adjusting its excess reserve level at the beginning of the period to a level below zero. If, at the end of the period, however, the random reserve flows resulted in the level of excess reserve being negative, Poole's assumption was that

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fying assumption is characteristic of simple inventory optimization models and was tacitly used by all of the models which we have encountered thus far and will be employed in the model particular to this thesis. This point has been brought up in the present context only because Frost relaxes this assumption at a later point in the development of his model so as to completely justify his kinked demand curve for excess reserves on a theoretical basis. This later development in his model will not be discussed in this literature review and all observations concerning inter-period adjustments costs will be relegated to the footnotes.

the bank would borrow at the discount rate to eliminate this reserve deficit.

Equation (2.72) must be written as

(2.73) $E_1 = E_o + N + Q \geq 0,$

to incorporate Frost's assumption of the impossibility of negative excess reserves.

If, in equation (2.73), $E_o+N \geq 0$, there will be no need for adjusting the bank's portfolio because the bank has been able to weather the random reserve loss without jeopardizing its meeting of its reserve requirements. In this case, Q will be equal to zero, in that any attempt to increase reserves would be both costly and unnecessary. If $E_o+N < 0$, however, the bank must sell securities in an amount equal to $-(E_o+N)$ in order that reserve requirements be met. Frost writes the expected profit function of the bank, E(P), as

(2.74) $$E(P) = rS_o - G\int_{-F_o}^{-E_o} \phi(N)dN - v\int_{-F_o}^{-E_o} (-E_o - N)\phi(N)dN.$$

The first term in equation (2.74) represents the return on securities; the second term, the lump sum adjustment cost weighted by the probability of the bank being required to sell securities to meet end of period reserve requirements; and the third term, the total variable adjustment cost of security sales necessary to meet end of period reserve requirements.

This expected profit function is misspecified. Note that the lower limit on the integrals in equation (2.74) is $-F_o$. Referring to the third expression in equation (2.74), we see that this lower limit is associated with a value of the function under the integral of $(-E_o+F_o) = S_o$. This appears to be necessary in that a bank cannot sell off more securities to meet its cash requirements than it possesses. However, the lower limit on this expression should really represent the maximum volume of deposits that are subject to drawdown by the bank's customers. By setting up his model in terms of free assets so as to avoid as much as possible the handling of reserve requirements, Frost has implicitly assumed that bank liabilities free of random fluctuations (namely, capital and debentures) are always exactly equal to the level of required reserves. This is the misspecification referred to above.

Frost maximizes the bank's expected profit function with respect to E_o, under the restraint that E_o must be non-negative, arriving at the first order condition,

$$(2.75) \quad \frac{\partial E(P)}{\partial E_o} = -r + G\phi(-E_o) + v\int_{-F_o}^{-E_o} \phi(N)\,dN \leq 0,$$

and

$$(2.76) \quad E_o > 0.$$

The optimal level of excess reserves $E_o{}^*$, is the level of E_o that satisfies both equations (2.75) and (2.76). E_o^* is thus a function of r, G, and v. It is also a functional of $\phi(N)$ which, by assumption, can be completely defined by its mean and standard deviation. Thus, the desired level of excess reserves can be expressed as

$$(2.77) \quad E_o^* = \max[f(r, G, v, \mu, \sigma,), \text{zero}].$$

It must be emphasized at this point that E_o^* represents the ex ante demand for excess reserves at the beginning of the period. As the stochastic cash drain occurs, the actual level of excess reserves will alter, but E_o^* is by far the more relevant variable in that it is directly related to the effects of uncertainty on the bank's planning behavior. Note that in the models we have treated so far, it was usual to assume that actual reserves would never be allowed to fall short of requirements (with the bank borrowing short-term instead). In Frost's model, however, a more stringent assumption is made. Frost assumes that ex ante reserves may never fall short of requirements. In models with no legal reserve requirement, neither of these assumptions need be made explicit, since neither ex ante nor ex post total cash can be negative.

From the manner in which Frost has taken the derivative of equation (2.74) to yield (2.75), two implicit assumptions can be deduced. The first is that a lagged reserve requirement has been assumed. Otherwise, the random deposit fluctuations would affect the level of required reserves and, hence, E_o. The derivative of expected profits with respect to excess reserves has been taken, but in a manner which treats E_o as independent of $\phi(N)$. The second implicit assumption is that the bank is very small, so that the addition of one dollar of securities to the portfolio costs exactly one dollar of reserves--i.e., there is no possibility of retaining any reserves jeopardized by security purchases. The easiest way to see that this assumption has been implicitly made in Frost's model is to notice that one dollar of security sales is presumed to supply enough reserves to offset one dollar of reserve deficit--this is evident from the third expression in equation (2.74). In reality, if a bank sold securities, it would run the risk of selling them to one of its customers who would pay for them by drawing down deposits at that bank. This means that the bank could expect to increase its reserves only by an amount equal to one minus that bank's share of the market times the volume of securities sold.[18]

The second derivative of the expected profit function is given by

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[18]If Frost's profit function had been set up with Ratti's treatment of the lump sum transaction cost it would

$$(2.78)\quad \frac{\partial^2 E(P)}{\partial E_o^2} = G[\partial\phi(-E_o)]/\partial E_o - [v\phi(-E_o)].$$

This equation need not be negative as is required for a maximum. If the first term on the right hand side is greater than zero and exceeds the second term in absolute value, the second derivative of expected profits with respect to excess reserves will take on a positive value. Frost comments that this second derivative will certainly be negative if $-E_o<\mu$ and $\phi(N)$ is unimodal and symmetric. This is true because the derivative in the first term on the right hand side of (2.78) will be negative under those conditions. The inclusion of the lump sum transactions cost clearly results in a failure of the second order conditions of maximization in both the reserve optimization

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be stated as,

$$E(P) = rS_o - \int_{-F_o}^{-E_o}[G(N) + v(-E_o - N)]\phi(N)dN,$$

where $G(N) = 0,$ for $N \geq -E_o$

$= G,$ for $N < -E_o$

the first order conditions would then read,

$$\frac{\partial E(P)}{\partial E_o} = -r + v\int_{-F_0}^{-E_o} \phi(N)dN \leq 0,$$

$$E_o > 0,$$

remembering that Frost has implicitly assumed lagged reserve requirements so that E_o and $\phi(N)$ are independent. This result indicates that the lump sum transactions cost would play no part in determining the optimal level of excess reserves, no matter how large that lump sum cost might be. This conclusion alone casts doubt on the efficacy of Ratti's technique. Now, it is true that in the credit expansion models, the mean of the random reserve and the credit expansion models.

When the optimal level of excess reserves is positive, Frost finds that E_o* increases when G or v increases and decreases when r or μ increases. Additionally, he states at one point that the derivative of E_o* with respect to σ is negative when E_o* is large and that this derivative is indeterminate when E_o* is close to zero.[19] Throughout most of his paper, however, Frost manipulates his model as if E_o* is a positive function of σ.[20] A careful examination of equation (2.75) and the results of Morrison and Poole should convince the reader that the derivative of E_o* with respect to σ can be either positive or negative, depending on whether E_o* is less than μ or greater than μ,

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drain is a function of the level of new deposits. Thus, the lump sum cost will appear in the derivative of profits with respect to deposit expansion, even if the lower bound of the integrals in the expected profit function is not a function of deposit expansion. In the reserve optimization models, it is typically assumed that each dollar of reserves nets only one dollar of loans so that the mean of the random reserve loss is independent of the level of reserves held. In this case, by assumption, the lump sum cost may only appear in the derivative of profits with respect to reserves if the lower bound of the integrals of the profit function are functions of reserves. Even so, it is hard to understand how the assumption that a bank is so small that no reserves return to it after lending can engender, by itself, the result that the optimum level of reserves is independent of the lump sum transaction cost.

[19]Peter Frost, "Bank's Demand for Excess Reserves," p 809.

[20]See especially p. 811.

respectively.

Frost's demand for excess reserves as a function of r is depicted as the line A B C in the Figure II below.[21]

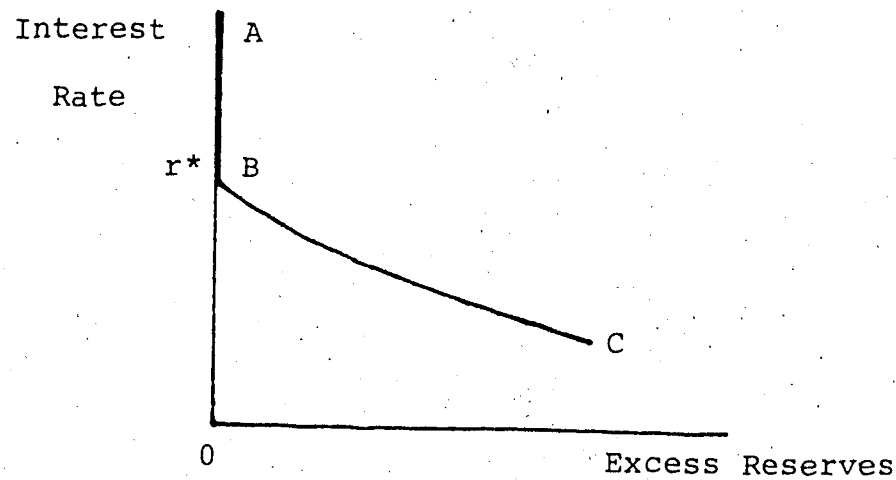

Figure II

Demand for Excess Reserves

Notice that the demand for excess reserves as a function of r is kinked, being zero for values of r greater than a critical value of

$$(2.79) \quad r^* = G\phi(0) + v\int_{-F_o}^{o} \phi(N)dN$$

and positive for interest rates below that critical value. When r is less than r*, the derivative of excess reserves with respect to r is negative throughout. The line B C, however, need not be smoothly downward sloping, as is the one depicted by Frost and reproduced in Figure II. In fact, the relationship between r and E_o* shown in Figure II is peculiar to the case where μ is equal to zero.

[21] Ibid., p. 810.

In a discussion of possible effects of bank size on reserve management, Frost presumes that r* (which varies directly with lump sum and variable adjustment costs) should be lower for larger banks because their average adjustment costs are lower owing to their larger volume of money market operations and their nearness to the main financial markets. Since σ is likely to be larger for large banks than for small banks, he also feels that the partial derivative of excess reserves with respect to the interest rate on securities should be larger for larger banks. Note that this last observation on bank size does not truly represent a diseconomy of scale because the variance per unit of deposits would have to be higher for the larger bank in order that this be the case. However, if average adjustment costs are lower for larger banks, this represents a legitimate claim for economies of scale in reserve management.

Throughout his work, Frost seems to have difficulty in interpreting the effects of σ in his model. Nowhere is this difficulty more pronounced than in his expectation that the higher σ of larger banks should result in a larger partial derivative of excess reserves with respect to the interest rate on securities than would be evident for a smaller bank. In Figure III below, the demand curve for excess reserve with respect to the interest rate on securities is shown for the case where $\mu<\sigma$.

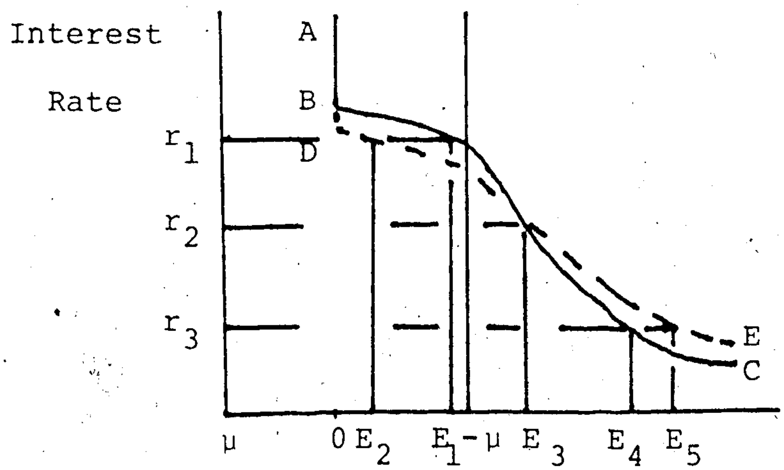

Figure III

Demand for Excess Reserves: $\mu < 0$

Let us say that, in Figure III, the Line A B C represents the demand for excess reserves of a bank with a stochastic reserve change variance of σ_1^2. Increasing the variance of this bank to a higher σ_2^2 will result in its demand curve shifting to A D E. The increase in σ results in a combined rotation and downward shift of the function, in a manner which will be explained in Chapter III of this thesis. If the rate of interest on securities is r_1, both before and after the change in variance involved in the shift of the demand function, it is clear that the increased variance has led to a decline in excess reserves from E_1 to E_2. If the relevant rate of interest was r_2, however, excess reserves would remain without change at E_3. With a rate of interest r_3, excess reserves rise from E_4 to E_5 as σ^2 rises from σ_1^2 to σ_2^2. For any rate of interest below r_2, Frost's presumption concerning the effect of a higher variance on the derivative of excess reserves with respect to the rate of interest on securities is true. For rates of interest above r_2, his contention is clearly false.[22]

For our purposes, the important features of Frost's model are the possibility of economies of scale in reserve management through lower average adjustment costs on the part of larger banks, and the assumption that banks never consider negative levels of excess reserves as optimal at any point in the planning period. It is this last point that is the crucial one for the development of his kinked demand curve for excess reserves.

The assumption that banks never consider holding negative excess reserves at any time in the averaging period is fairly unrealistic. During periods when the monetary base is rising rapidly so that the bank can expect reserve accretions during the period, and especially when the discount rate is low relative to short term interest rates, holding negative excess reserve balances in the early stages of a reserve averaging period must be too

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[22]The demand curves in Figure III are drawn with negative slopes throughout for $E_o > o$. This need not be the case. The second derivative of $E(P)$ with respect to E_o is not everywhere negative. For values of r leading to values of $E_o > -\mu$, where $\mu < o$ Frost's model can produce a demand curve that is vertical at zero and then positively sloping for some range in the region $-\mu > E_o > 0$. The relative sizes of G and v are instrumental in determining whether this occurs, as can be seen from an investigation of equation (2.78).

great a temptation for many banks to resist. In an American context, recent history shows that, when the Federal funds rate exceeds the discount rate by more than one half of a percentage point, member bank borrowing from the Federal Reserve System rises very rapidly, sometimes by such a large amount that free reserves become negative. This behavior is not consistent with an assumption that banks necessarily shy away from negative excess reserve holdings within the averaging period, though the infrequency of penalty rate charges indicates that banks shy away from actually violating the reserve requirement for the period as a whole.

Baltensperger

The most thorough analysis of the effects of bank size on reserve management in the context of stochastic models of bank behavior was undertaken by Ernst Baltensperger.[23] Basically, the work represents a refinement of the work of Edgeworth.

A bank is assumed to accept one type of non-interest bearing deposit, the volume of which is represented by D. This is the only liability of the bank. The bank has a choice of two assets, reserves R and loans L. The

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[23]Ernst Baltensperger, "Economies of Scale, Firm Size and Concentration in Banking," Journal of Money, Credit and Banking 4 (August 1972): 467-87.

interest rate on loans is designated by i_a, and the cost per dollar of reserve deficit is labelled i_p. Let R and D represent beginning of period reserves and deposits. Let V represent the reserve loss during the period, where V is a random variable with density function v(V). The expected costs of holding reserves, C_u can be described by

$$(2.80) \quad C_u = i_a R + i_p \int_R^\infty (V - R)v(V)dV.$$

Optimizing the level of reserves, R, can be accomplished by differentiating C_u with respect to R and setting the derivative equal to zero:

$$(2.81) \quad \frac{\partial C_u}{\partial R} = i_a - i_p \int_R^\infty v(V)dV = 0.$$

The optimal level of reserves R* can then be expressed as

$$(2.82) \quad \int_{R*}^\infty v(V)dV = \frac{i_a}{i_p}.$$

Two immediate limitations of this model, which were recognized by Baltensperger, are that i_a must be less than i_p for a solution to exist, and that, to get a positive value for optimal reserves, $\int_o^\infty v(V)dV$ must be greater than i_a/i_p. Baltensperger must have a positive value for optimal reserves because, in reality, total reserves cannot be negative. If Baltensperger were dealing with excess reserves, he would have the option of permitting negative excess reserves. It should also be noted that Baltensperger's model tacitly assumes that every dollar of deposits created during loan expansion is drawn down immediately and that reserves lost by loan expansion do not return to the bank.

The most crucial assumptions in Baltensperger's analysis of economies of scale in bank reserve management concern the effects of bank size upon the mean and variance of v(V). He supposes that a bank has n deposit accounts of identical size $\bar{v}$. x_i is the proportion of account i that is withdrawn; it is considered to be a random variable, with mean k and variance a^2. All accounts are assumed to be homogeneous so that the probability of a given withdrawal is the same for all accounts. An assumption of the independence of accounts allows Baltensperger to define V, the reserve loss variable, as $V=\Sigma_i x_i \bar{v}$, where V is approximately normally distributed by the Central Limit Theorem, with $E(V)=nkv=kD$, $Var(V)=\bar{v}^2 na^2=a^2D^2/n$, and $\sigma_V = aD/\sqrt{n}$. If all deposits are assumed to be the same size D_o, then $D = nD_o$ and σ_V can be written as $\hat{a}\sqrt{D_o}$. This establishes a proportionality between the number of accounts and the size of total deposits.

Baltensperger explains that the reserve loss in a period can be affected by two factors: deposit fluctuations, and the net repayment or granting of loans. He assumes that the second cause is foreseeable with certainty and, therefore, has no effect on the variance of V, but will be able to affect the mean. Baltensperger then assumes that either the bank expects on average no outflows or inflows of deposits and grants new loans only as old ones are repaid, or the bank adjusts its lending policy so that average reserve changes are zero. This implies, of course, that $k = 0$.

Given equation (2.82) and the assumption that $k = 0$, R^* can be expressed as a multiple of σ_V, namely:

$$(2.83) \quad R^* = b\sigma_V = b\hat{a}\sqrt{D}, \text{ or } R^*/D = b\hat{a}/\sqrt{D}.$$

Here, b is a parameter, depending on i_a and i_p, that must be greater than or equal to zero. Substituting the results of equation (2.83) into equation (2.80), Baltensperger gets

$$(2.84) \quad C_u = i_a b\sigma_V + i_p b\int_V^\infty (V - b\sigma_V)v(V)dV.$$[24]

This equation, however, is clearly misprinted and should read:

$$(2.85) \quad C_u = i_a b\sigma_V + i_p \int_{b\sigma_V}^\infty (V - b\sigma_V)v(V)dV.$$

Baltensperger then expresses the cost of reserves equation (2.84) in terms of the standard normal distribution, arriving at

$$(2.86) \quad C_u = i_a b\sigma_V + i_p \sigma_V \int_b^\infty (t-b)f(t)dt = \delta\sigma_V = \delta\hat{a}\sqrt{D},$$

where δ is a constant, and $f(t)$ represents the standard normal probability density function. Equation (2.86) implies that

$$(2.87) \quad C_u/D = \hat{a}/\sqrt{D}.$$

From equations (2.83) and (2.87), Baltensperger draws two conclusions: first, that the level of optimal

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[24] Ibid., p. 471

reserves, as a proportion of deposits, varies inversely with bank size; and, second, that the costs of optimal reserve management vary inversely with bank size.[25]

Baltensperger's conclusion that large banks will, under all circumstances, hold smaller proportional reserves is based on two crucial features of his model. The first feature is that total reserves, which cannot be less than zero, are being optimized; and the second feature is that k is assumed to be equal to zero.

The assumption that k is equal to zero and the requirement that R* must be greater than, or equal to, zero enables Baltensperger to duplicate Edgeworth's assumption that the banker always makes provision for the expected

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[25]Baltensperger also considers the effects of abandoning the assumption of zero inter-period transactions costs. The effects of the relaxation of that assumption creates a special zone above and below the level of optimal reserves; the level of reserves will be adjusted to the lower level of the zone if inherited reserves are much lower than the optimal level of reserves, and adjusted to the upper level of the zone if inherited reserves are much higher than the optimal level or reserves. If the inherited level of reserves are within the zone itself, no adjustments will be profitable because of the introduction of inter-period transactions costs. Baltensperger finds that the zone will be proportionately smaller for large banks, and the large bank will therefore stay closer to its optimal level of reserves from period to period. Baltensperger feels that this effect enhances his case for economies of scale in banking because it means large banks will, on average, be operating with reserve levels relatively nearer to the optimal level. Baltensperger's findings, however, rest on the same assumptions as his proof that large banks have a proportionately smaller optimal level of reserves than small banks and are not necessarily true in all cases. If the financial environment

value of cash drains in his reserves. If Baltensperger allowed k to be a positive number, meaning that a cash drain on average was to be expected, smaller banks would hold relatively smaller levels of reserves than large banks if the value of i_a/i_p exceeded 1/2. Likewise, if Baltensperger had concerned himself with excess reserves and did not specifically assume that banks would not consider negative levels of excess reserves as optimal, his conclusions could have been rejected, in spite of the assumption of a zero expected value of cash drains. In this case, where i_a/i_p exceeded 1/2, small banks would hold relatively smaller levels of excess reserves than large banks, but the optimal levels of excess reserves in both cases would be less than zero. These points will be made clearer in Chapter IV of this thesis.

One final point of criticism of Baltensperger's work is his failure to deal with non-proportional transactions costs in his model. This type of transactions cost clearly has implications for economies of scale in reserve management and can be handled in a relatively straightforward manner, as will be seen in Chapter IV.

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was such that the smaller bank's optimal reserves were proportionately less than that of the larger bank, the larger bank's keeping reserve levels nearer to this optimal level because of inter-period adjustment costs, would not necessarily militate for the large bank's holding ~~smaller~~ relative reserve levels in any case.

Knobel

In a Master's Thesis and a resulting published article, Abraham Knobel investigated a bank's demand for excess reserves.[26,27] Because of the lengthy space necessary to present Knobel's model and its extreme dissimilarity to the other models appearing in this thesis, Knobel's model will not be presented in detail. However, a brief description of his model and conclusions will be given.

In Knobel's model a bank has two assets: reserves and loans. Fluctuations in the bank's reserves are assumed to take the form of a Bernuolli random walk, rising or falling every 1/t of a day in increments or decrements of m dollars. The economy is assumed to have a constant monetary base, so that reserve changes reflect purely random phenomena. The bank manages its reserves by setting an upper boundary H and a lower boundary A for its reserves. These boundaries are subject to optimization. When reserves rise to the upper boundary H, the bank grants loans valued at H-A dollars, and, when reserves fall to the lower boundary, the bank retires loans to the value of H-A dollars.

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[26] Abraham Knobel, "The Demand for Reserves by Commercial Banks," (Master's Thesis, Tel-Aviv University, 1974).

[27] Abraham Knobel, "The Demand for Reserves by Commercial Banks," Journal of Money, Credit and Banking 9 (February 1977): 32-47.

A deposition function and redeposition function are specified. The deposition function is a probability density function applying to the amount of reserves gained by the retirement of H-A dollars of loans. It reflects the fact that some of the funds used by the customers of the bank to retire their loans may ultimately be drawn from deposits at the bank. Similarly, the redeposition function is a probability density function applying to the amount of reserves lost through the granting of H-A dollars of loans, given that some of the deposits being created in the loan extension may find their way back to the lending bank. The bank's share of the market is assumed to be the sole factor determining the distribution of the deposition and redeposition functions. The bank faces a reserve requirement that is defined as a constant fraction of the average amount of deposits during the previous time period. The requirement is to be met at all times, not just on average over a specified period. A fine of n dollars is incurred for each dollar of reserve deficit. The rate of interest on loans, which represents the opportunity cost of reserves, is v. The lump sum transactions cost, incurred every time the bank is forced to retire or grant H-A dollars of loans, is y. Since the monetary base is assumed to be constant, Knobel feels that L, the amount of required reserves, is a constant. If the averaging period is sufficiently long he feels that the average deposit holdings in the previous period (upon which the reserve requirement is based) will approach the expected value of deposits in the current period. Likewise, the expected value of deposits in the forthcoming period should approximate the average of deposits held over the current period.

Gathering together all of the cost factors, specifying deposition and redeposition functions, and solving for the expected values of reserve balances, transfer costs, and penalty costs generated by the Bernoulli process, Knobel arrives at the expected daily cost of holding reserves. This cost can be expressed in the two equations:

$$(2.88) \quad E_1(C) = \frac{H+A}{2}v + \frac{6y\sigma^2}{(H-A)^2} + n\left(L - \frac{H+A}{2}\right), \text{ if } H \leq L,$$

$$(2.89) \quad E_2(C) = \frac{H+A}{2}v + \frac{6y\sigma^2}{(H-A)^2} + n\left(\frac{(L-A)^3}{(H-A)^3}\right)\left(H - \frac{L+A}{2}\right),$$

$$\text{if } H \geq L,$$

where σ^2 is the variance of daily reserve flows. The expected value of reserves held is $\frac{H+A}{2}$, given that the mean of any steady-state distribution generated by a symmetric random walk, and the particular deposition and redisposition functions specified by Knobel is $\frac{H+A}{2}$. The deposition and redeposition functions are applicable to banks with a 25 to 50 per cent share of the banking industry. Knobel analyzes the effects of changes in L, v, n, y, and σ^2 on the optimal values of A, H, and a new variable, $G \equiv H-A$.

The results differ depending on whether $n<v$ or $n>v$. Where $n<v$, Knobel proves that A equals zero, so that only H need be considered. A change in reserve requirements, expressed as a change in L, will not affect H if $H<L$, but, if $H>L$, H increases as L increases. Excess reserves may decrease, increase, or remain the same, however. The changes in H with respect to v, n, y, and σ^2 can be summarized by:

$$\frac{dH}{dv} < 0, \quad \frac{dH}{dn} > 0, \quad \frac{dH}{dy} > 0, \quad \frac{dH}{d\sigma^2} > 0.$$

Where $n>v$, an increase in reserve requirements will raise H and A by an amount equal to the change in L, leaving G unchanged. Excess reserves will also remain unchanged with changes in L. The changes in H, A, and G, with respect to v, n, y, and σ^2 can be summarized by:

$$\frac{dA}{dV} < 0, \quad \frac{dH}{dv} < 0, \quad \frac{dG}{dv} \gtrless 0, \text{ if } L \gtrless \frac{(H+A)}{2}.$$

$$\frac{dA}{dn} > 0, \quad \frac{dH}{dn} > 0, \quad \frac{dG}{dn} < 0.$$

$$\frac{dA}{dy} < 0, \quad \frac{dH}{dy} > 0, \quad \frac{dG}{dy} > 0, \quad \frac{d(H+A)}{dy} \gtrless 0, \text{ if } L \lessgtr \frac{H+A}{2}.$$

$$\frac{dA}{d\sigma^2} < 0, \quad \frac{dH}{d\sigma^2} > 0, \quad \frac{dG}{d\sigma^2} > 0, \quad \frac{d(H+A)}{d\sigma^2} \gtrless 0, \text{ if } L \lessgtr \frac{H+A}{2}$$

$\frac{H+A}{2}$ -L is the optimal level of excess reserv[illegible]re-fore, the effects of the parameters on excess reserves will be of the same sign as the effects on H, where $n<v$, except where changes in L are concerned. Also, where $n>v$, the effects of the parameters on excess reserves will be of the same sign as the effects on H+A, except for changes in L.

In his thesis, but not his published work, Knobel sought to investigate the effects of bank size on the level of excess reserves. To analyze the effects of bank size Knobel examines the effects of L and σ^2 on the level of excess reserves. Since Knobel's cost functions (2.88) and (2.89) are compatible with banks encompassing anywhere from 25 to 50 per cent of the industry, Knobel can compare two banks that differ with respect to their share, but have the same cost function.

Knobel argues that the level of required reserves, L, is one factor that distinguishes a larger bank from a smaller bank. Where $n<v$, he finds that a rise in L causes excess reserves to decline in proportion to required reserves. The implication is that a larger bank will hold proportionately smaller levels of excess reserves than a small bank. This is an interesting result, since the larger bank is not holding proportionately more required reserves, only a larger dollar volume. In analyzing the effects of σ^2, Knobel refers to Baltensperger, claiming that the latter's analysis makes the variance of daily reserve flows an increasing function of bank size. Using this assumption in his analysis, Knobel concludes that, where $n<v$, the combined effect of L and σ^2 is to make indeterminate the relation between bank size and the absolute level of excess reserves held.

Where $n>v$, the demand for excess reserves is found to be independent of L. The assumption that larger banks display larger variances of reserve flows leads Knobel to the conclusion that larger banks will have higher levels of excess reserves than small banks if both are holding positive excess reserves, but lower levels of excess reserves if both are incurring deficits.

Knobel's work cannot be considered as a refutation of Baltensperger's work. For one thing, Knobel's study of bank size relates to an investigation of the differences between the absolute holdings of excess reserves of a large bank and those of a small bank. Baltensperger concerned himself with the differences in relative holdings of excess reserves.

Baltensperger did imply that large banks face larger variances of deposit flows than small banks, and this is the assumption employed by Knobel. However, what was most important to Baltensperger was his establishment of the plausibility that large banks face a relatively smaller reserve flow variance than small banks. Had Knobel sought to consider the relative differences between large and small banks, he would have had to accept Baltensperger's argument; that is, he would have had to assume that larger banks face a relatively smaller variance than small banks.

Interestingly, in the case where $n<v$, Knobel

found that the need for a higher absolute level of required reserves on the part of a larger bank led it to hold a proportionately lower level of excess reserves than a smaller bank. In fact, in Knobel's model, this result will occur where any single banks is faced with an increase in its absolute level of required reserves because of an increase in the required reserve ratio (where $n<v$). This result is intuitively quite reasonable. Where no non-monetary penalties derive from failing to meet reserve requirements and where the net rate of return on loans exceeds the penalty rate, an increase in required reserves can logically be expected to result in some economizing on excess reserves relative to required reserves.

In general, Knobel's model suffers from several shortcomings. The most serious relates to the extreme long-run, steady-state nature of the model. It hardly captures the importance of the extreme short-term reserve fluctuations that are the greatest challenge to bank reserve managers. Likewise, reserve drains and accretions occur in increments or decrements of a constant size in Knobel's model, an assumption that is very far from reality.

The largest potential advantage inherent in Knobel's model derives from the existence of the deposition and redeposition functions. It was possible for Knobel to include the effects of market size on the cost function in his analysis of the effects of bank size on excess reserve holdings; regrettably, he did not utilize this advantage.

CHAPTER III

THE PRESENTATION OF A STOCHASTIC MODEL OF BANK RESERVE MANAGEMENT

Introduction

The purpose of this chapter is to present a two-asset, stochastic model of bank behavior. The model will be presented first in a basic form and then in a slightly augmented form. The optimal level of excess reserves will be examined for each of these variants, as will the effects of changes in parameters on that optimal level. In Chapter IV, these results will be used to analyze the effects of bank size on reserve management.

The Basic Model

Let us begin by considering a representative bank in a hypothetical banking system and its reserve managing behavior during a typical planning period. The counterpart in reality to this planning period is the familiar reserve averaging period. It is assumed that the bank has two assets available to it. One asset is cash, which is perfectly liquid, but bears no interest. The other asset, called loans, is illiquid, but it bears a nominal rate of interest which is fixed during the planning period. This interest rate is known with certainty by the management of the bank at the beginning of the period.

It is denominated as a per cent period and is designated by y.[1] By assumption, all new loans are made at the beginning of the period, and either the bank or its customers may call in existing loans at the beginning of the period for immediate repayment. Thus, customers may not repay loans at any time other than the beginning of the period. Moreover, the bank may not call in loans at any other time. Under no circumstances may a bank sell loans for cash. At the beginning of the period, the lending rate on all existing loans is assumed to be adjusted to the current rate on new loans, so as to avoid the expense of needless roll-overs arising from the retractable feature on loans.

The bank's liabilities are of two types: the first consists of demand deposits bearing no interest, and the second consists of such interest bearing borrowings as the bank finds necessary to undertake during the planning period. These borrowings are analogous to advances from the central bank, and the nominal rate of interest, m per cent per period, that they bear is analogous to the central bank's re-discount rate.[2] Like the rate of return on bank

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[1]This rate of interest must be considered as net of costs associated with the administration of loans and loan defaults. Inflation is ignored in this model but could easily be handled by making all interest charges net of the expected rate of inflation and by deflating other variables such as deposits and loans by appropriate inflation indexes.

[2]Time deposits have been ignored here for simplicity's sake, even though the ratio of demand to time deposits

loans, m is assumed to be fixed over the planning period and known with certainty at the beginning of the period.

The bank is subject to a stochastic cash drain which is expressed as a proportion of "initial deposits", where initial deposits are defined as the volume of deposits existing at the beginning of the current period and are designated D. The cash drain ratio Z, as a ratio of two dollar values, is a pure number and may take on values in a range of $-\infty<z\leq 1$. The values of z must be greater than minus infinity because random cash accruals (negative values of z) in any period must be finite, while z must be no greater than one because that represents the limiting case where every single depositor withdraws his funds from the bank in the current period. It is assumed that the bank management constructs a probability distribution for the cash drain ratio at the beginning of the period, based on historical experience and current expectations. The probability density function, f(z) is associated with that distribution. The random variable Z has a variance of σ^2 and a mean of μ. The cash drain occurs at the end of the period; therefore, in the event that the cash drain is neg-

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is an important factor influencing the level of cash reserves held by banks. As well, time deposits bear a rate of interest which would augment the opportunity cost of holding cash. The bank's capital account has also been ignored for simplicity's sake. For a presentation of a model determining optimal capital account, see Ernst Baltensperger, "Economies of Scale," pp. 473-75.

ative (cash accruals), no part can be lent in the current period.

When loans are made at the beginning of the period, it is assumed that all deposits created by lending are withdrawn instantly and that no deposits lost in the process of making loans are recovered during the planning period. Similarly, it is assumed that customer loan repayments are made with cheques drawn on some other bank, so that loan repayments result in no loss of deposits to the lending bank and result in an increase in cash exactly equal to the size of the repayment.[3] These assumptions are associated with the standard model of deposit expansion in a perfectly competitive banking system. They ignore the effects of Phillips' "derivative deposits" which largely motivated the work of those authors investigating optimal deposit expansion in stochastic models. The assumptions must be made, however, to ensure that only one dollar of loans can

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[3]At first, this assumption appears entirely unrealistic. However, there is a reasonable grounding in reality for making this assumption in the framework of reserve management. In a simple two-asset model of bank behavior, all types of loans and interest bearing assets are lumped together. There is, however, a type of loan relevant to reserve management which exhibits the characteristic that virtually all repayments are paid with cheques drawn on other banks. The type of loan referred to is loans made to investment dealers and brokers to finance their inventories of stocks and securities. When a particular bank calls in its loans to dealers, the typical reaction on the part of the dealers is to seek refinancing with another bank. Only if this cannot be done will the dealer sell off securities to buyers who may pay with deposits at the bank calling in the loan.

can be created for each dollar of excess reserves, and that neither the making of new loans, nor the repayment of existing loans, may alter the level of deposits. This means that the level of deposits can be altered only by the stochastic cash drain. Therefore, the end of period deposits (post cash drain) are equal to the following period's initial deposits, and those initial deposits are, in turn, equal to the level of deposits following the loan transactions in that period.

One result of our assumptions is that the bank's distribution for the cash drain ratio, which is constructed at the beginning of the period, will in no way be influenced by changes in its loan portfolio because net changes in loans will leave initial deposits unaffected. The mean of Z, μ, is assumed to be affected by three factors: any anticipated alteration in the desired currency holdings of the bank's depositors, for seasonal or other reasons; any anticipated changes in the bank's market share of deposits, either through marketing strategy, or through the expectation of large changes in revenues earned by the bank's usual deposit customers; and, finally, any expected change in the monetary base resulting from monetary policy. In this simple model, the mechanism of monetary policy is presumed to operate through the shifting of government balances between accounts with the private banks and accounts with the central bank itself. Factors affecting

σ^2, the variance of Z, are the typical historical experience and the degree of uncertainty associated with the factors listed as determining μ above.

The bank is assumed to be subject to a mandatory cash reserve requirement which is determined on a lagged basis (as is typical of North American banking systems). A legal reserve ratio, R, is applied to the volume of deposits existing at the end of a planning period (after the random cash drain has occurred) to determine the dollar value of the legal reserve requirement in the following planning period. Hence, at the start of any planning period, the dollar value of the legal reserve requirement is fixed and known with certainty. Because this model is ultimately to be used to examine the effects of changes in parameters on the level of excess reserves relative to the level of deposits, it is convenient to express the current dollar value of the reserve requirement as a proportion of the current value of initial deposits, just as was done with the stochastic cash drain variable. Since, by assumption, the level of initial deposits in the current period is equal to the end of period deposits of the immediately preceding period, the task is simple; the required ratio of cash to initial deposits is R, the legal reserve ratio. The reserve requirement in this model is presumed to apply at a single point in time, just after the stochastic cash drain occurs. Of course, this represents a considerable simplification from reality, where reserve requirements typically apply to the average level of reserve over the entire period.

P is defined as the proportion of excess reserves to initial deposits; the excess reserves being those held in the bank's portfolio immediately after adjustments are made to the loan portfolio. P is an ex ante variable which will be the subject of optimization in the model. As the stochastic cash drain occurs, the actual excess reserve to initial deposit ratio will alter, but P is the more relevant variable here because it represents directly the effects of uncertainty on the bank's planning behavior. P + R represents the ratio of ex ante total cash reserves to initial deposits.

It is assumed in this model that cash drains exceeding P must be financed by short term borrowing from the central bank. Hence, there is no provision in this model for the actual excess cash reserve ratio to fall below zero during the period after the stochastic cash drain has occurred and the point in time when reserve requirements must be met. This assumes that m, the variable transactions cost associated with borrowing from the central bank, is lower than the penalty cost associated with failing to meet reserve requirements. No assumption, however, restricts P, the ex ante excess cash ratio, from being negative. Conceptually, P + R can be zero at the very least, so that P is restricted to a minimum value of -R. For convenience, it is postulated that all short-term borrowing from the central bank must be repaid at the beginning of the period following that in which the borrowing occurred.

The basic behavioral assumption is that the bank acts in a manner which minimizes the expected losses (per dollar of initial deposits) associated with holding excess reserves in the planned portfolio. The expected loss function associated with reserve management, E[L(z;P)], can be described by:

$$(3.1) \quad E[L(z;P)] = yP + m\int_{P}^{1} (z-P)f(z)dz,$$

where E[L(z;P)] is the expected value of the ratio of revenue lost, through the holding of P, to initial deposits. The first term of the expression on the right hand side of equation (3.1) represents the opportunity cost, per dollar of initial deposits, of holding excess reserves. The second term represents the cost, per dollar of initial deposits, that the bank expects to incur as a result of cash drains exceeding the level of excess reserves held in the portfolio.[4]

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[4]The limits on the integral are justified as follows: if z takes on values less than P, legal reserve requirements can be met without recourse to borrowing from the central bank, and no costs are incurred; while, if every depositor of the bank withdrew his deposits, the maximum amount of borrowing incurred would be 1-P and the

We will now proceed to optimize P, the excess reserve ratio. This is done by minimizing $E[L(z;P)]$. The first order condition for a minimum is found by differentiating $E[L(z;P)]$ with respect to P and setting the expression so obtained equal to zero, i.e.,

$$(3.2) \quad \frac{\partial E[L(z;P)]}{\partial P} = y - m\int_{P^*}^{1} f(z)dz = 0,$$

where P* refers to the optimal level of P. The second order condition for a minimum is that $\frac{\partial^2 E[L(z;P)]}{\partial P^2} > 0$. In our case,

$$(3.3) \quad \frac{\partial^2 E[L(z;P)]}{\partial P^2} = mf(P),$$

where f(P) is the probability that cash drains equal P. The expression mf(P) must be positive, since Z is distributed over the interval $-\infty<z\leq 1$, and m is the cost per dollar of borrowing from the central bank, which is assumed to be positive.

The formula for the optimal holdings of excess reserves, P*, can be written as:[5]

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stochastic cash drain would be 1. Hence, borrowing costs are incurred only for cash drains ranging in value from P to 1. Note that P is not treated as a function of z. This follows from the assumption of lagged reserve requirements. In a model with a contemporaneous reserve requirement, z would affect the current periods level of required reserves and, hence, P.

[5]The existence of a solution for P* requires y<m. This condition is necessary because, if $\int_{-R}^{1} f(z)dz \geq \int_{P^*}^{1} f(z)dz$ does not hold, the optimal volume of excess reserves will fall short of its minimum value, namely, -R. If y>m, the above condition cannot hold as P* approaches minus infinity.

(3.4) $\int_{P*}^{1} f(z)dz = \frac{Y}{m}$.

or, equivalently,

(3.5) Prob: $[Z > P^*] = \frac{Y}{m}$

As is customary in simple inventory models, the bank's adjustment of its excess cash ratio from the level at the end of the previous period to the optimal level in the current period is assumed to be costless. If the bank wishes to augment its excess reserve position, it will call in loans; if it wishes to reduce its excess reserves, it will lend out funds. Since adjustments in the loan portfolio do not, by assumption, affect the level of deposits, the mean of Z is independent of the adjustment to the optimal level of the excess cash ratio.

With the basic workings of the model set down, a calendar of the events in the planning period can now be made explicit. The first elements, which are known in the current period, are the dollar value of the reserve requirements and the level of initial deposits, both of which, by assumption, are known at the end of the previous period. The rate of return on loans and the cost of borrowing from the central bank are the next required items. Following this knowledge, the bank constructs its distribution for random cash drains and determines the optimal level of excess reserves. After this step, the bank must know the amount of loans being repaid by customers. This information, coupled with the knowledge of borrowings from the central bank which must be repaid, will allow the bank to alter loans in order to achieve the optimal level of excess reserves. Next comes the cash drain, and then any borrowings from the central bank that are necessary to meet the reserve requirement, which must be satisfied immediately after the cash drain. From the level of deposits existing after the cash drain, the next period's reserve requirement will be known, as will the level of initial deposits.

Carrying on in the spirit of Edgeworth and Baltensperger, a strong case can be made for considering the random cash drain ratio, Z, to be normally distributed. The crucial assumptions in the theoretical rationale behind this conclusion are that all deposits are approximately the same size, and all depositors act independently of one another. Furthermore, as the level of initial deposits increases, the number of independent depositors must rise proportionately.

The proof of the proposition in the preceding paragraph follows Baltensperger's presentation closely.[6] A bank is assumed to have n accounts of identical size v. U_i is defined as the proportion of the ith account which is

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[6] This presentation is adapted from Baltensperger, "Economies of Scale," p. 485.

either withdrawn ($u_i > 0$) or augmented ($u_i < 0$), where $-\infty < u_i \leq 1$. If it is assumed that all deposits are homogeneous, it follows that the probability of an account changing by a given proportion is identical for all accounts. If we specify the expected value of the U_i's as c and the variance of the U_i's as d^2, and if all the U_i's are independent of one another by the assumption of independent depositors, then the total reserve drain as a proportion of initial deposits, $Z = \frac{1}{nv}\Sigma_i U_i v$, is approximately normally distributed by the central limit theorem, with $E(Z)=c$ and $Var(Z) = \frac{d^2}{n}$, or $\sigma = \frac{d}{\sqrt{n}}$. This can be seen clearly by remembering that v is a constant, so that the distribution of Z is nothing but the distribution of the sample means of n identically distributed random variables $U_1, U_2, \ldots, U_n$.

The Effect of Parameters on the Optimal Level of Excess Reserves: The Simple Model

From equation (3.5), we see that the bank will hold an excess cash ratio which equalizes the probability of a reserve deficit and the ratio of the rate of return on loans to the cost of borrowing. It is clear that the smaller the value of y, the smaller the probability of a reserve deficit and, hence, the larger P* must be. The larger the value of m, the smaller must be the probability of a reserve deficit and, hence, the larger P* must be. Additionally, the larger the expected value of Z, the larger must P* be for any given value of the ratio on the right.

The effect of a change in σ^2 on P* will vary in direction according to the value of the ratio of y to m. The relationship between σ^2 and P* is depicted in Figure IV below.

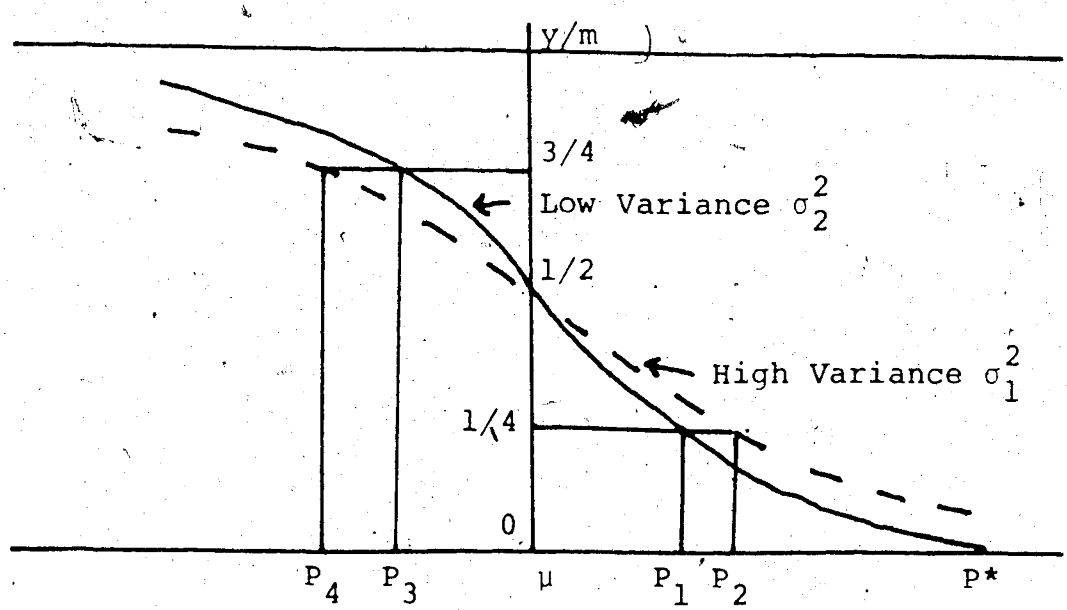

Figure IV

Excess Cash Ratio Demand Function: (Simple Model)

In Figure IV, the demand for the excess cash ratio as a function of y/m is depicted for the bank facing a high variance σ_1^2, (broken line), or a low variance σ_2^2 (solid line). The relationships depicted are nothing more than the mirror images of the cumulative distribution functions of the normal probability distribution under alternate variance assumptions. Where y/m is exactly equal to 1/2, the variance of Z will not affect the excess cash ratio, which will be μ under both variance assumptions. If, however, y/m is equal to 1/4, an increase in the variance of Z will result in an increase in the excess cash ratio to P_2 from P_1. Where y/m is equal to 3/4, an increase in the variance of Z will result in a decrease in the excess cash ratio to P_4 from P_3. This apparently paradoxical result (increased uncertainty causing higher reserve holdings if the ratio of the rate of return on loans to the cost of borrowing is less than 1/2, but lower reserve holdings if that ratio is greater than 1/2) was noted in the works of Morrison and Poole. The effect can easily be explained using the concepts of marginal cost and marginal return. From equation (3.2), we can isolate y as the marginal cost of P, and m as the marginal return on P, weighted by the probability of a reserve deficit. The expected value of revenue losses as a result of holding P is minimized by equating the marginal cost to the marginal return. If y is exactly one-half of m, then, in order to equate marginal cost and marginal return, the probability of a reserve deficit must be exactly one-half. Because Z has been assumed to be normally distributed, and the normal distribution is symmetric, P* will have to be equal to μ. Furthermore, since the mean of the normal distribution divides that distribution in half regardless of the variance, no change in P* can be caused by an increase in σ^2. If y is one-quarter of m, the probability of a reserve deficit must also be one-quarter in order to equate marginal cost and marginal revenue. This will require P* to be raised above μ to P_1. If σ^2 is increased in this case, the probability of Z exceeding P will be enhanced, since the increased uncertainty has made it more likely that relatively large cash drains will occur. To equate marginal cost and marginal return, the cash ratio must be raised to P_2 as a result of the greater risk of cash deficit engendered by the increased uncertainty. Where y is three-quarters of m, the probability of a reserve deficit must also be three-quarters, and, consequently, P* will fall below μ to P_3. If σ^2 is raised in this case, the probability of Z falling short of P_3 is enhanced because the increased uncertainty has made it more likely that relatively large cash accruals will occur. The excess cash ratio will, therefore, be lowered to P_4 to offset the increased likelihood of surplus cash.

To sum up, where y/m is less than one-half and σ^2 rises, P* will increase because of the relevance of the increased likelihood of relatively large cash drains to the equation of marginal cost and marginal return. Where y/m is greater than one-half and σ^2 rises, P* will fall because of the relevance of the increased likelihood of relatively large cash accruals. Where y/m is exactly equal to one-half, however, and σ^2 rises, P* will be unaffected, because the increased likelihood of relatively large cash drains and relatively large cash accruals are equally relevant and totally offset one another.

The Augmented Model

In this somewhat more complicated model, the sole addition is the introduction of a fixed transactions cost imposed on the bank on occasions when it is required to borrow from the central bank. The counterpart in reality to this fixed transactions cost is the cost of paperwork and administration and telecommunications charges incurred in the act of borrowing from the central bank. In this model, the fixed transactions cost is expressed as a ratio of initial deposits and is designated g. It is assumed that the size of g does not change within the planning period and that it is known with certainty at the beginning of the period.

The expected revenue loss, per dollar of initial deposits, associated with reserve management, $E[L(z;P)]$, can now be described by:[7]

$$(3.6)\quad E[L(z;P)] = yP + g\int_P^1 f(z)dz + m\int_P^1 (z-P)f(z)dz,$$

where the second term on the right represents the fixed transactions cost, per dollar of initial deposits, weighted by the probability that a reserve deficit will occur. The

[7] Note that we continue to use 1 as the upper limit on the integrals in the loss function and the derivatives of the loss function. Since Z has been found to be approximately normal the upper limit on these and earlier integrals in this chapter could be changed to infinity. In a sense, the area under the normal curve beyond one, to positive infinity, where that normal curve is associated with μ, the mean of Z and σ^2, the variance of Z, reflects the error of the approximation.

first order condition for a minimum is found by differentiating E[L(z;P)] with respect to P and setting the expression so obtained equal to zero, i.e.,

$$(3.7)\quad \frac{\partial E[L(z;P)]}{\partial P} = y - gf(P^*) - m\int_{P^*}^{1} f(z)dz = 0$$

The second order condition for a minimum requires that $\frac{\partial^2 E[L(z;P)]}{\partial P^2} > 0$. In the case of the augmented model,

$$(3.8)\quad \frac{\partial^2 E[L(z;P)]}{\partial P^2} = mf(P) - g\frac{\partial f(P)}{\partial P},$$

which is clearly not everywhere greater than zero. Since Z has been assumed to be distributed normally, the conditions under which the second order condition will be satisfied can be stated more explicitly as,

$$(3.9)\quad P > \mu - \frac{m\sigma^2}{g}.$$

From the inequality (3.9), it can be seen that the second order conditions will always be met if $P > \mu$, and that the second order conditions are more likely to be met the larger are m and σ^2 and the smaller is g.

To simplify the analysis, equation (3.7) can be rearranged as:[8]

$$(3.10)\quad m\int_{P^*}^{1} f(z)dz + gf(P^*) = y,$$

which indicates that the bank will act in a manner that will equate the rate of return on loans to the probability that the cash drain exceeds P*, weighted by the proportion-

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[8]The existence of a solution for P* in the augmented model requires that $m\int_{-R}^{1} f(z)dz + gf(-R) \geq m\int_{P^*}^{1} f(z)dz + gf(P^*)$. This, of course, constrains the values of m, g, and y accordingly.

al cost of borrowing from the central bank; plus the probability that the cash drain equals P*, weighted by the fixed transactions cost of borrowing from the central bank.

The Effect of Parameters on the Optimal Level of Excess Reserves: The Augmented Model

In the more complex augmented model, the relationship between P* and the levels of the various parameters are more difficult to perceive than in the simple model, so a graphical presentation will be made from the outset. Figure V graphs the relationship between P* and y.

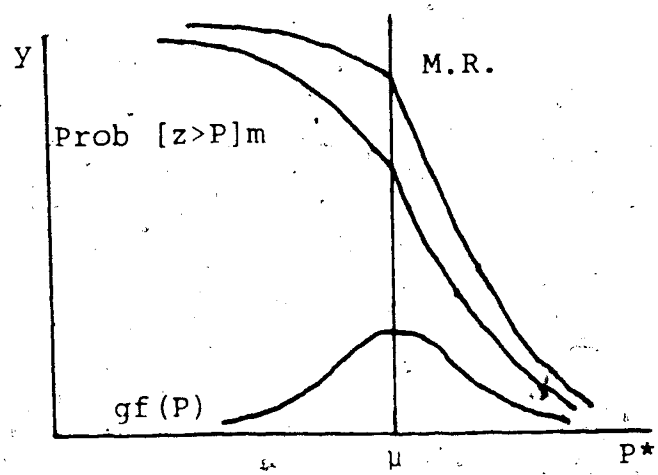

Figure V

Excess Cash Ratio Demand Function: (Augmented Model)

In Figure V, three functions are dislpayed. The line labelled Prob [Z>P]m is nothing but the mirror image of the cumulative distribution function of Z weighted by the proportional cost of borrowing, m. The line gf(P) is the probability density function $f(Z)=n(\mu, \sigma^2)$ weighted by the fixed cost of borrowing, g. The line labelled M.R.

is simply the vertical sum of the other two lines and constitutes the marginal return per unit of the excess cash ratio. Since y is the marginal cost per unit of the excess cash ratio, the optimal value of P will be determined by the equation of y with the M.R. function. Hence, the M.R. function can be considered to be the demand function for the excess cash ratio.

As in the simple model, the smaller y, the larger P* must be. This is immediately evident from Figure V. An increase in m, by shifting the Prob[Z>P]m function up proportionately, will shift the demand for the excess cash ratio upward, while at the same time imposing the shape of Prob[Z>P]m more forcefully on the demand function. This will flatten the angle of the demand function at $P^* = \mu$. Other things being equal, this shift will result in an increase in P*. An increase in g will also shift the demand curve upward by shifting gf(P) upward proportionately, but in a manner which will impose the shape of gf(P) more forcefully on the demand function. This will sharpen the angle of the demand function at $P^* = \mu$, and, if the effect is pronounced enough, will produce a positively sloped portion in the demand curve at low levels of P* within the visible confines of the graph. This occurrence would reflect the violation of the second order conditions, since a single large value of y could then call for two equally optimal values of P*. Of course, the second order conditions must be violated eventually as P* declines below

$\mu - \frac{m\sigma^2}{g}$. In sum, other things being equal, the rise in g will precipitate a higher level of P*. A larger value for μ will simply move all of the relations in Figure V to the right and, hence, result in an increase in P*.

As in the simple model, the relationship between σ^2 and P* is the most complex among the battery of parameters. The relationship between σ^2 and P* is graphed in Figure VI below.

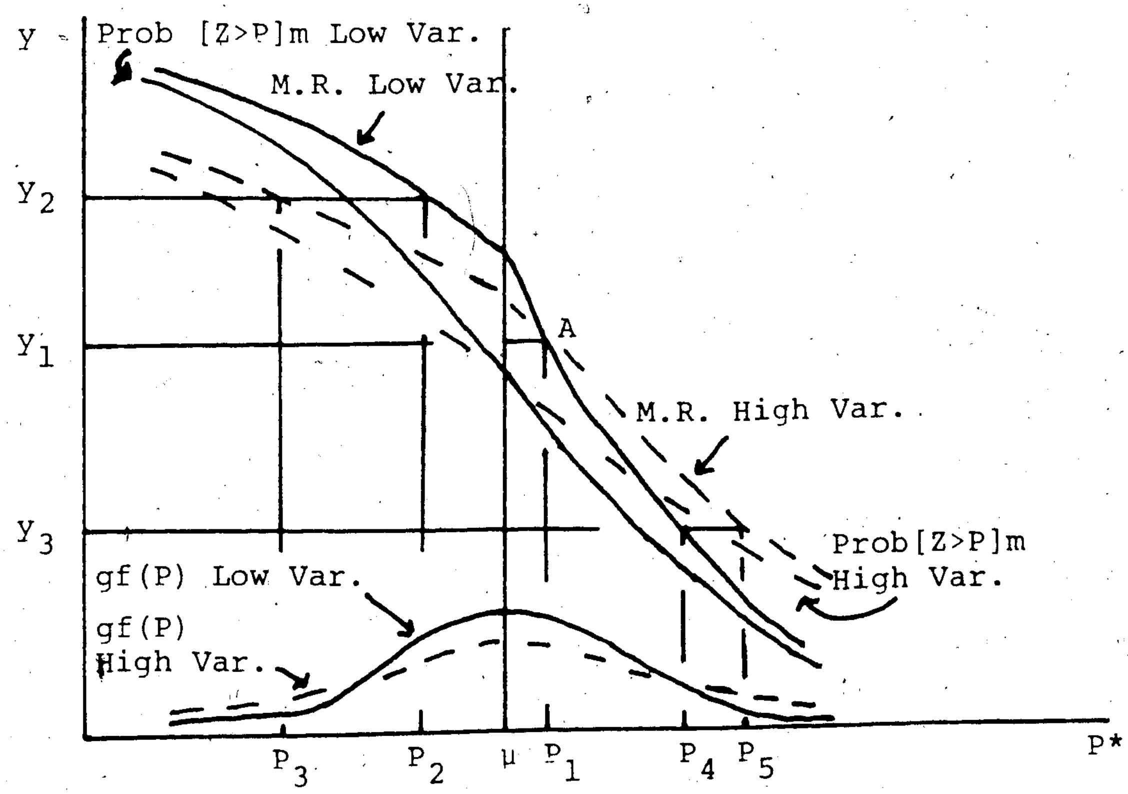

Figure VI

Excess Cash Ratio Demand Function (Augmented Model):

The Effect of Uncertainty

The effect of an increase in σ^2 is displayed in Figure VI by shifting the gf(P), Prob[Z>P]m, and M.R. functions from solid line positions to dashed line positions. The increase in uncertainty can be seen to stretch out the gf(P) function, causing it to shrink where $P^* = \mu$, yet rise eventually as $|P^*-\mu|$ increases. The Prob[Z>P]m function rotates around $P^* = \mu$ in the same fashion as was illustrated in the simple model. The demand function for P* becomes flatter and appears to rotate around the point A, though an even more drastic increase in variance would produce a rotation point even further down on the original demand function. As is suggested by equation (3.9), the increase in σ^2, by imparting a more pronounced negative slope to the demand function at low levels of P*, results in a decrease in that value of P*, beneath which the second order conditions are violated. If the yield on loans was y_1, before the increase in uncertainty, then P* would be P_1, both before, and after, the increase in uncertainty. If the yield was higher than y_1, say at y_2, then P* would decline from P_2 to P_3 with the increase in uncertainty. Obversely, if the yield was lower than y_1, say at y_3, then P* would rise from P_4 to P_5 with the increase in uncertainty. As in the case of the simple model, an increase in uncertainty may, or may not, result in the holding of a higher excess cash ratio.

CHAPTER IV

THE EFFECTS OF BANK SIZE ON THE OPTIMAL LEVEL OF EXCESS RESERVES

In this chapter, the model presented and analyzed in Chapter III will be used to evaluate the relationship between bank size and the level of excess reserves that emerged from the review of the literature. One effect of bank size on reserve holdings derives from the work of Edgeworth and Baltensperger. Their work suggests that the variance of cash drains does not increase proportionately with increases in the level of deposits, and that the demand for cash by banks likewise will not increase proportionately with increases in the level of deposits. The other effect to be discussed arises from the work of Frost, who postulated that the average adjustment cost of maintaining a bank portfolio decreases with bank size as a result of the larger bank's nearness to financial markets and its typically larger transactions size.[1]

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[1]Another possible effect of bank size that is not amenable to analysis using the model of Chapter III resulted from the work of Knobel. In Knobel's model, where the return of loans exceeds the cost of reserve shortfalls, an increase in required reserves will cause a decline in the ratio of excess to required reserves. This, of course, implies a decline in the excess reserves to deposit ratio. In Knobel's model, this result is equally true of the case where a single bank faces an increase in required reserves through a higher required reserve ratio, but not an increase in scale, and of the case where the increase in the level of required reserves results solely from an increase

The mechanics of the Edgeworth-Baltansperger effect are, no doubt, already familiar to the reader and no comments are necessary before the actual analysis of that effect, using the model of Chapter III. Some preliminary remarks are necessary, however, in regard to the handling of Frost's contribution to the possible effects of bank size on excess reserve holdings.

To say that the average adjustment cost of a larger bank is less than that of a smaller bank may mean that the variable adjustment cost is lower for larger banks, the fixed adjustment cost is lower, or both costs are lower. Since Frost was comparing large New York City banks with smaller New York State country banks, it is not

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in scale. The conditions under which Knobel arrived at this result are not duplicable in the model in Chapter III, since the return on loans in that model may never exceed the cost of borrowing from the central bank. The only effect changes in reserve requirements can have on our model arises through the conditions imposed on the model arising from the fact that P* may never be less than -R. If R, the required reserve ratio, rises, the sole effect on the model is that a higher level of y, (other things being equal), can be accommodated without defying that condition. This, in turn, means that a lower value of P* can be accommodated as well. However, the rise in R and the lower possible value of P* will always change proportionately to maintain a ratio of -1. The strict proportionality of this effect for changes in the level of required reserves arising from changes in the required reserve ratio implies that the effect will also be strictly proportional for changes in the level of required reserves arising solely from scale size. In the model of Chapter III, therefore, larger banks will not hold a lower excess reserve ratio because of their larger absolute size of reserve requirement.

unreasonable for him to have meant that both costs are lower. This is because there are serious qualitative differences between the two types of banks; namely, differences in the type of customer, differences in the type of typical bank business engaged in, and differences in the specialization of bank staff and bank functions. Any valid proof of the theoretical existence of economies of scale in reserve holdings arising from the introduction of uncertainty, however, must be capable of demonstrating that the effect applies to banks differing only quantitatively. For that reason, the analysis of the effects of adjustment costs on economies of scale in reserve holdings will be handled in a purely quantitative fashion, i.e., as if we were attempting to demonstrate that Citibank may, theoretically, be expected to hold a smaller excess cash ratio than the Chase Manhattan Bank.

Given our focus on the strictly quantitative differences among banks, it is unreasonable to assume differences among those banks in either the rate of return on loans or the variable transactions cost associated with borrowing to meet reserve requirements. Neither can a strong case be made for assuming that the larger bank will enjoy a fixed transactions cost which is lower in absolute size than that of a marginally smaller bank. It is, however, reasonable to assume that the fixed transactions cost is identical in dollar value for both the larger and

and marginally smaller banks, but that the level of fixed transactions cost per dollar of deposits is somewhat lower for the larger bank. In terms of the variables in our model, these assumptions can be summarized by stating that y and m are identical for the larger and smaller banks, but that g declines proportionately for increases in the scale of the bank.

To analyze the effects of bank size on the excess cash ratio in the simple model of Chatper III, we need only analyze the Edgeworth-Baltensperger effect, since the fixed transactions cost does not appear in the simple model. In Chapter III, we discovered that σ^2 was equal to $\frac{d^2}{n}$, where d^2 is a constant and n represents the scale size of the bank under the assumptions that depositors act independently and the size of deposits are uniform. This implies that the variance of Z declines proportionately with increases in the size of the bank. Both Edgeworth and Baltensperger used this phenomenon to prove that larger banks hold proportionately lower levels of reserves than smaller banks. We need not duplicate this analysis because the effect on an increase in σ^2 on P* has already been thoroughly analyzed in Chapter III, with Figure IV (p.94) clearly demonstrating that the effect of an increase in σ^2 on P* is variable depending on whether $y/m \lesseqgtr 1/2$. What is necessary to discuss, however, is the reason why our results are different from those of Edgeworth and

Baltensperger. Referring to Figure IV, we can see that the Edgeworth assumption--namely, the bank will always make accommodation for the mean of the cash drain in its reserve holdings--will restrict values of P* to those exceeding μ. In that region, his results are clearly quite correct, i.e., the larger bank with the smaller proportionate variance will always hold a proportionately smaller level of reserves.

Again, referring to Figure IV, we can see that Baltensperger's assumptions have a similar effect. Since Baltensperger dealt with total cash reserves rather than excess reserves, the minimum value attainable by his cash variable was zero. He combined this factor with an assumption that the mean of his cash drain was equal to zero. If we consider P* to be the optimal value of total cash and restrict it to be equal to $0 = \mu$, we can see from Figure IV that these assumptions will define the same range of values for P* as Edgeworth's assumptions, namely, the range in which the larger bank will always hold a proportionately smaller level of excess reserves.[2]

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[2]Baltensperger's model can be derived from the simple model of Chapter III very easily. In equation (3.2), we need only substitute (P+R)* for P*, perform a change of variable on the integral from z to dz and multiply both sides of the equation by D to yield:

$$yD - m\int_{D(P+R)^*}^{D} t(Dz)\,dDz = 0, \text{ where } t(Dz) \simeq n(D\mu/D^2\sigma^2).$$

Here again, we could consider the upper limit on the above integral to be infinity, as Baltensperger does.

There is yet another simple way of restricting the range of P* to the areas in which an increase in the variance will always result in a higher excess cash ratio. It can be done by restricting the excess cash ratio to zero ex ante (as was done in Frost's model) and by simultaneously restricting μ to be equal to zero. As Figure IV indicates, however, the assumption that banks will never consider a negative value of P* as optimal will not, in itself, be sufficient to ensure that larger banks hold proportionately smaller levels of excess reserves. This assumption leads only to the conclusion that cases in which small banks hold proportionately lower levels of excess reserves will be confined to situations in which a large cash drain is expected and the rate of return on loans is high relative to the discount rate. Such a situation is equivalent in reality to a period with a declining real monetary base and a low real discount rate--relative to the bank's real prime rate; it is usually associated with a cyclical downturn in economic activity.

Another situation which will result in large banks holding a lower excess cash ratio arises from the fact that P* may never be less than -R. If, in Figure IV, $\mu = -R$, the value of P* will be restricted to the range in which economies of scale in reserve management occur. Again, with reference to this phenomenon, it is clear that an expected cash drain ratio greater than zero and a high y/m ratio will still foil the conclusion concerning economies of scale. This further underscores the conclusion that conditions of cyclical economic decline are most condusive to small banks holding lower excess cash ratios than larger banks.

In the augmented mode, there are two effects to be considered in the analysis of the effect of bank size on the excess cash ratio. First, the variance of Z declines proportionately with increases in the size of the bank. Second, g will also decline proportionately with increases in the size of the bank. This follows from the fact that $g = \frac{G}{nv}$, where G is the dollar value of fixed transactions costs, n is the number of deposits and v is the size of all deposits. The effect of bank size on the level of the excess cash ratio is depicted graphically in Figure VII below.

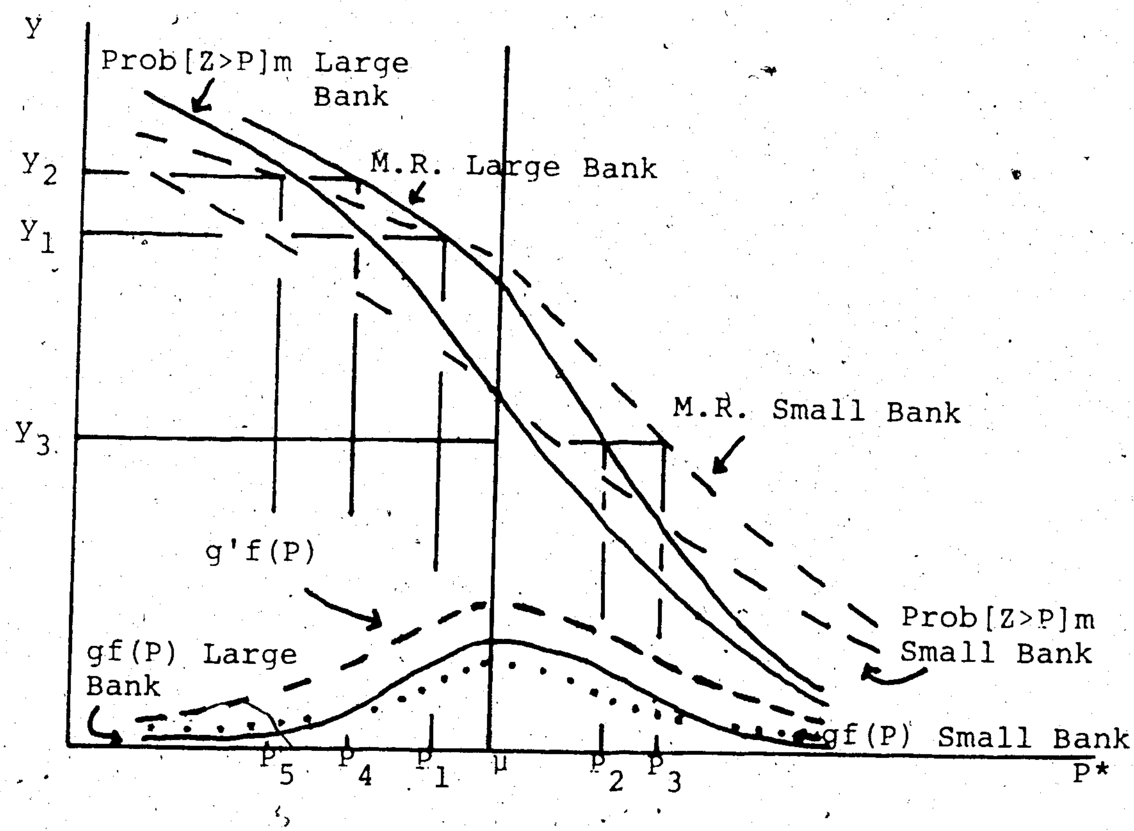

Figure VII

Bank Size and the Excess Cash Ratio Demand Function:

The Augmented Model

Figure VII graphs the excess cash ratio demand function of a large bank and a small bank. The gf(P) curve, Prob [Z>P]m curve, and the M.R. (demand) curve for the large bank are shown in solid line in Figure VII, where the demand curve is, again, simply the vertical sum of the other two functions. For the small bank, the Prob [Z>P]m curve is shown in dashed line, and it represents a shallower curve than that of the large bank's, with a rotation point at $P^* = \mu$. The $gf(P)$ curve resulting solely from the increased variance of the smaller bank is shown in dotted line, and it represents the familiar stretching of the function characteristic of an increased variance. The $g'f(P)$ curve, which represents the effect of the proportional increase in g accompanying decline in bank size, is shown in dashed line and represents a proportional upward shift in the dotted $gf(P)$ curve. The demand curve for the small bank is shown in dashed line and represents the vertical sum of the dashed line Prob[Z>P]m curve and the dashed line $g'f(P)$ curve.[3]

In Figure VII, we see that, if the rate of return on loans is y_1, both the large and small banks will hold an excess cash ratio of P_1. If the rate of return on loans is y_3, the larger bank will hold a cash ratio of P_2, which is lower than the ratio P_3 held by the smaller bank. At y_2, however, it is the smaller bank which holds the lower

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[3]Notice that in Figure VII the dashed line $g'f(P)$ represents an upward shift in the $gf(P)$ function of the small bank to a level which exceeds the $gf(P)$ function of the large bank at the point $P^* = \mu$. This must be so. The value at μ of the $gf(P)$ function can be described by $\frac{G}{nv} \frac{1}{\sqrt{2\pi\sigma}}$ because the distribution is normal. However, since $\sigma = d/\sqrt{n}$, a further simplification can be made to $G/\sqrt{2\pi n}\, vd$. This indicates that the function for the larger bank must be below the $g'f(P)$ function of the smaller bank. Similarly, since the Prob[Z>P]m functions will be equal, when valued at $P^* = \mu$, for both small and large banks, the demand function of the small bank must exceed that of the large bank at $P^* = \mu$. As is clearly shown in equation (3.9), reducing the size of a bank cannot enhance the range of values of P^* for which the second order conditions are satisfied; the

excess cash ratio P_5, while the larger bank holds P_4. Clearly then, even the additional assumption of a fixed transactions cost ratio that declines in proportion to an increase in bank size does not guarantee that larger banks will hold a smaller excess cash ratio.

The assumption that g rises proportionately with a decrease in bank size does not necessarily preclude the possibility that the smaller bank will hold a higher excess cash ratio than the larger bank. If, in Figure VII, the absolute size of the fixed transactions cost g were to be increased, other things equal, the gf(P) function would impact upon the demand functions more seriously, and the result could easily be that the demand function of the smaller bank was everywhere above that of the larger bank. Too much can be made of this effect, however, since g represents, at most, a few hundred dollars of transactions costs divided by the bank's entire deposit volume. The effect will be pronounced only if m, the discount rate which scales the Prob[Z>P]m function, is extremely low. Another factor to be considered is the constraint on P* to be greater than -R. If -R happens to be equal to P_1 (Figure VII), the small bank will hold a larger cash ratio than the larger bank under the entire admissable range of P*. As in the case of the simple model, however, the

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reduction merely increases both g and σ^2 proportionately, leaving P* unchanged.

higher the expected value of the cash drain, the more likely it is that $P^* = -R$ will occur far to the left of μ and P_1. From Figure VII, it can be seen that any of the situations which restrict the values of P^* to levels above μ will result in the restriction in the range of P^* to an area where economies of scale in reserve management occur. The same is true for Edgeworth's assumption, Baltensperger's assumptions, and the combined assumptions that the bank is reluctant to hold negative values of P^* (ex ante) and $\mu = 0$. In fact, these assumptions will now be more powerful than they need be; the introduction of the assumption that the fixed borrowing cost ratio declines proportionately with increases in scale always raises the demand curve for the smaller bank above that of the larger bank at μ and for some range of $P^* < \mu$. The assumption that banks are reticent about holding negative levels of P^* (ex ante) does not result in the conclusion that there are economies of scale in reserve management so long as μ exceeds zero by $\mu - P_1$, unless $P_1 < -R$, or the small bank's demand curve is everywhere above the large bank's. The augmented model also shows that, theoretically, periods of cyclical decline are most favorable to the occurrence of the situation in which small banks hold a smaller excess cash ratio than large banks.

To sum up, in the augmented model, the assumption that both the fixed transactions cost ratio and the var-does not guarantee that larger banks will hold a smaller excess ratio than smaller banks without the introduction of special restrictive assumptions. The assumption that the variance declines proportionately with increases in scale has a variable effect on the relative excess holdings of large and small banks. The assumption that the fixed transactions cost ratio declines proportionately with increased scale always militates for larger banks holding a smaller excess cash ratio. Only where the effect of the latter assumption outweighs the effect of the former assumption will the result be that the large bank holds a smaller excess cash ratio than the smaller bank at all levels of y. This situation will be the more probable, the higher is G, and the lower is m.

CHAPTER V

CONCLUSIONS

The contents of this chapter are divided into three parts. The first part contains a summary of the conclusions of Chapter IV regarding the effects of bank size on the optimal level of excess reserves. The second part embodies a discussion of the many similarities between the results obtained by other authors on the effects of uncertainty on deposit expansion and the results obtained in this thesis concerning the general effects of uncertainty on the management of excess reserves. The third section consists of a discussion of the weakness of the excess reserve optimization model presented in Chapter III of this thesis.

Bank Size and Excess Reserves

In the simple model presented in Chapter III, the only effect of bank size on the optimal holdings of excess reserves results from the assumption that the variance of the cash drain ratio decreases proportionately with an increase in bank size. Other authors, namely Edgeworth and Baltensperger, who confined themselves to discussing total reserves, concluded that this effect was sufficient under conditions of uncertainty to ensure that, in theory, a larger bank would hold a smaller cash ratio than a smaller bank. The analysis of the simple model, conducted in Chapter IV, questioned the results of Edgeworth and Baltensperger. In that chapter, the effects of bank size on excess reserve holdings varied according to whether the ratio of the rate of return on loans to the cost of borrowing reserves was greater than, equal to, or less than one-half. Where that ratio was less than one-half, the excess cash ratio would be larger than the mean of the cash drain ratio, and larger banks would hold a smaller excess cash ratio than smaller banks. When the return on loans to the cost of borrowing ratio was exactly equal to one-half, the excess cash ratio would be equal to the mean of the cash drain ratio, and banks of all sizes would hold an identical cash ratio. If the rate of return on loans to the cost of borrowing ratio was greater than one-half, the excess cash ratio would be less than the mean of the cash drain ratio, and the smaller bank would hold a smaller excess cash ratio than the larger bank.

It was pointed out in Chapter IV that the results of Edgeworth and Baltensperger derived from special restrictions on the admissable range of the cash ratio. In Edgeworth's model, it was the assumption that the bank would always hold reserves at least equal to the mean of the cash drain which resulted in his conclusion that larger banks would hold a smaller cash ratio than smaller banks. In Baltensperger's model, only total reserves (which cannot take on negative values) were considered.

Thus, the assumption that the expected value of the cash drain was equal to zero amounted to a restriction on the admissable range of the cash ratio identical to that in Edgeworth's model.

Another set of restrictions can also be constructed in the simple model which will result in the conclusion that larger banks will hold a smaller excess cash ratio than smaller banks. This set entails two assumptions: first, banks are reluctant to hold a negative excess reserve ratio (ex ante and ex post), and, second, the mean of the cash drain ratio is equal to zero. Should it be thought reasonable that banks will hesitate to hold a negative excess reserve ratio (ex ante and ex post), the relationship between the relative size of the excess cash ratio and scale size will still be inconclusive, so long as the mean of the cash drain ratio is greater than zero. Additionally, since the cash ratio can never be negative, the excess cash ratio can never be less than the negative of the required reserve ratio. This being the case, an inverse relationship between the excess cash ratio and bank scale will emerge when the mean of the cash drain ratio is less than minus the required reserve ratio (indicating an expected cash inflow).

Under any of these conditions, other than Edgeworth's assumption that the bank will always set the level of reserves at least equal to the mean of the cash drain, the relationship between the excess cash ratio and bank size will be conditional, as long as the expected value of the cash drain is greater than zero. Furthermore, small banks will tend to hold the smaller excess cash ratio when the rate of return on loans is high relative to the cost of borrowing reserves, and the mean of the cash drain is greater than zero--exactly the conditions that can be expected in a period of cyclical economic decline.

In the augmented model of Chapter III, which introduces a fixed transactions cost incurred by borrowing reserves, two assumptions are brought to bear in analyzing the effects of bank size on the level of the excess cash ratio. One is the assumption that the variance of the cash drain ratio declines proportionately with an increase in scale, and the other is that the fixed transactions cost ratio declines proportionately with an increase in scale. The analysis of Chapter IV leads to the conclusion that an inverse relationship between bank size and the excess cash ratio cannot be guaranteed in the absence of special assumptions restricting the admissable range of the excess cash ratio. It was found that the higher the dollar value of the fixed transaction costs and the lower the variable transactions cost associated with the borrowing of reserves, the more likely it is that larger banks will hold a smaller excess cash ratio than smaller banks.

In the augmented model, it was found that the special assumptions of Edgeworth and Baltensperger restricting the range of the excess cash ratio were more than enough to yield the conclusion that large banks hold a smaller excess cash ratio than smaller banks. This is also true for the combination of assumptions which restricts the value of the excess cash ratio (ex ante and ex post) to positive values and makes the cash drain ratio equal to zero.

As in the case of the simple model, neither the restriction that the excess cash ratio cannot be in deficit by more than the value of required reserves nor the assumption that banks will desire to hold positive excess ratios (ex ante and ex post) will be sufficient to ensure that large banks hold smaller excess cash ratios than small banks. This follows because a combination of high loan rate for any given discount rate, coupled with a relatively large expected cash drain ratio, will offset those two conditions, unless, of course, the dollar value of the fixed transaction cost is so high that the small bank will everywhere hold a higher excess cash ratio than the large bank. Once again, therefore, we see that the situation in which a small bank holds a lower excess cash ratio than a larger bank is most likely to occur in a period of cyclical economic decline.

Similarities Between the Deposit Expansion and Reserve Optimization Models

In this part of the concluding section, some of the very interesting similarities between the deposit expansion models, described in Chapter II, and the reserve optimization model of Chapter III will be discussed. It should come as no surprise that such similarities exist; the two types of model address problems which are the mirror image of each other. This can be seen in the perfect competition model of banking, in which uncertainty is ignored; here, the statement that deposits will expand by the reciprocal of the legal reserve requirements is precisely the same as the statement that excess reserves will be exactly equal to zero.

The first similarity to be discussed concerns the effect of the variance in the two models. In the simple model of Chapter III, the effect of the variance on the excess cash ratio depends upon whether the loan rate to discount rate ratio is less than, equal to, or greater than one-half. In the model of Brown and Lloyd, without the lump sum transactions cost, the conclusion is reached that the optimal level of deposit expansion in the uncertainty case can be less than, equal to, or greater than that of the certainty case, depending on whether the ratio of the loan rate to the penalty rate is less than, equal to, or greater than one-half. This similarity of the two models is far from chance. In the Brown and Lloyd model, the certainty case represents the special case in which the cash drain is equal to the expected value of the cash drain in the uncertainty case; in addition, the variance is equal to zero in the certainty case and to some positive number in the uncertainty case. In the Brown and Lloyd model, therefore, the sole difference between the certainty case and the uncertainty case is that the latter has a higher variance. In the simple versions of both models, changes in variance have an identical effect on the variables to be optimized. The same similarity exists in the two models when they are expanded to include the fixed transactions cost of borrowing reserves, but, in this case, the comparison is less graphic.

Another interesting comparison of the deposit expansion and reserve optimization models concerns their behavior when the rate of return on loans is very high relative to the discount rate. In Brown and Lloyd's model, for example, there is some high value of the loan rate to penalty rate ratio at which the bank will find it optimal to expand its level of deposits indefinitely. The exact counterpart of this situation exists in the model of Chapter III. There, high values of the loan rate to discount rate ratio will result in it being optimal for the bank to hold negative total reserves, a situation which is, of course, impossible. Conceptually, however, it is similar to the attempt to expand the level of deposits indefinitely.

The final point of comparison between the two types of model involves the fact that, in both models, the introduction of a fixed transactions cost, associated with borrowing reserves, will result in the failure to meet second order conditions for maximization.

A Discussion of Shortcomings of the Model of Chapter III

The model of Chapter III does not represent a complete description of bank portfolio behavior. Rather, it attempts only to arrive at a division of assets which allows a consideration of the marginal cost and marginal return of liquidity. This very narrow frame of reference has led to the neglect of several important elements of a bank's balance sheet. The capital account and assets of intermediate liquidity are completely ignored. The latter shortcoming is particularly critical in a Canadian context where there exists a secondary reserve requirement. Also ignored in the model are term and notice deposits. This is an important shortcoming, not only because the existence of these interest bearing deposits would enhance the opportunity cost of holding excess reserves (the existence of interest bearing demand deposits in the U.S. could easily supply that), but also because the payment of interest on these time deposits represents a cost paid in order to reduce the level of uncertainty associated with deposit roll-over.

No interperiod transactions costs were considered in the model of Chapter III, but their inclusion would only serve to reinforce the conclusions of Chapter IV. Similarly, the assumption that the reserve requirement be met at a single point in time, rather than on average over the period, is admittedly unrealistic, but the alteration of it to accommodate reality would not likely affect the conclusions of this thesis. The effect of price level changes is entirely ignored in the model, which is unfortunate, but not critical to the conclusions. The inclusion of price level changes in the model would open the possibility of either the real rate of return on loans, or the real discount rate, or both, being negative; such developments would quickly run afoul of the optimization conditions of the model.

Somewhat more serious shortcomings include the absence of non-monetary restrictions on borrowing from the central bank. Qualitative restrictions on borrowing could set further restrictions on the admissible level of the excess cash ratio, or at least present the problem of a higher discount rate for borrowings beyond a certain proportion of deposits (as is typical in Canada, for example). Another difficulty associated with borrowing from the central bank is that it could involve the loss of "goodwill". This would be a form of fixed transactions cost, but one which would be proportionate to the size of the bank, since the cost of losing the privilege of participating in the business of banking would rise in proportion to the resources invested in the bank.

Two of the most critical assumptions in this work, and in the work of Edgeworth and Baltensperger, are the assumptions of depositor independence and uniform deposit size. These are not realistic assumptions, but, in a sense, are not critical to the results of Chapter IV. Certainly the elimination of these assumptions will not permit the conclusion that the variance of the cash drain ratio will decline in exact proportion to bank size. Neverthel[illegible]variance of the cash drain ratio will probably [illegible]modestly with an increase in scale, partly because [illegible]gree of interdependence among depositors is not lik[illegible]to differ substantially at any given time among banks o[illegible]arginally different size, and partly because some of the l[illegible]ger size of the larger bank is likely to be the result o[illegible] more depositors.

Another shortcoming of the model of Chapter III concerns the total neglect of the demand for loans. It is always assumed that the bank can increase its loans at will, and that none of the excess reserves in the portfolio arises from the inability to place loans at the current rate of interest.

Like all of the reserve management models mentioned in the lite[illegible]ture review, except Knobel's, the model of Chapter II[illegible]sumes that only one dollar of loans can be made per dollar of excess reserves. This assumes that when a loan is extended none of the deposits extended in the granting of the loan will return to the bank after the loan customer has spent the proceeds of the loan. It also implies that if a bank calls in a loan none of the repayment will be made with funds on deposit with the bank. This is an unfortunate assumption, but is quite necessary. Without it, the opportunity cost of a dollar of reserves will be increased, and, typically, the effect will be more pronounced the larger is the bank. If we had defined our illiquid asset to be securities and the variable cost of the reserve deficit to the the spread between bid and asked prices in the selling of securities, the all-in cost associated with the reserve deficit would be similarly boosted by the possibility of the securities being sold to customers of the bank and being paid for with deposits at the bank. This effect, too, would be more pronounced the larger the bank. This would not cause exceptional difficulties. However, another problem associated with the relaxation of this assumption would be that the level of deposits would alter with the adjustment to the optimal level of reserves. This would result in the mean of the cash drain being a function of the management of the loan portfolio and, hence, also a function of the level of excess reserves. Additionally, the adjustment to the optimal level of excess reserves, by affecting the level of deposits in the current period, would affect the required reserves of subsequent periods and, therefore, require the model to be dynamic.

As a final comment, it is unfortunate that the results of the model have not been tested empirically. This step, however, awaits the development of an econometric technique capable of adequately encompassing a bank's ex ante demand for excess reserves by taking into consideration the expected mean and variance of its cash drain.

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