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HOMOMORPHISMS OF FOURIER ALGEBRA AND COSET SPACES OF A LOCALLY COMPACT GROUP

by ©

Monica Ilie

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Abstract

Let G, H be locally compact groups and $A(G)$ and $B(H)$ be respectively the Fourier algebra of G and Fourier Stieltjes algebra of H.

The first part of the thesis is dedicated to the study of completely bounded algebra homomorphism of $A(G)$ into $B(H)$. It is shown that any such algebra homomorphism can be described in terms of a piecewise affine map of a subset of H into G, when G and H are discrete groups with G amenable. This generalizes a result of B. Host. As a consequence, for such G and H, a concrete description of the range of a completely bounded algebra homomorphism of $A(G)$ into $A(H)$ is obtained.

In the last part of the thesis we turn our attention to the Fourier algebras associated to the coset spaces of a locally compact group G. For a compact subgroup K of G, a description of the dual of $A(G{:}\,K)$ is given. The natural analogues of the spaces $UBC(\hat{G})$, $W(\hat{G})$, $AP(\hat{G})$ in this new setting are defined. We study the inclusion relationships that exist between these spaces. Their behaviour with respect to the Arens product on their duals is also explored.

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Contents

Chapter 1

Introduction

This is a thesis in abstract harmonic analysis devoted to the study of properties of the Fourier algebras, which are among the most important Banach algebras arising from a locally compact group.

A *topological group* is a basic ingredient in abstract harmonic analysis. It is born from the merging of an algebraic object, the group, with a topological notion, a topology of our choice. To obtain a topological group, the topology must be compatible with the group operation, which means that multiplication and inverse taking are continuous operations. The topology we most often work with is locally compact, and we call the group *a locally compact group.*

The algebras of functions associated to a locally compact group G are numerous and intensely studied in harmonic analysis. Classical algebras of this type are, for example, the group algebra, $L^1(G)$ and the measure algebra, $M(G)$. In 1964, P. Eymard introduced in [9] the non-commutative analogues of these two algebras, namely the Fourier algebra, $A(G)$, and the Fourier-Stieltjes algebra, $B(G)$, of a locally compact group G.

The mappings between various topological algebras associated with a locally compact group have been intensely studied from the early beginning of harmonic analysis. The problem of determining all homomorphisms of group algebras of locally compact abelian groups has been studied by several authors including A. Beurling and H. Helson [4], W. Rudin [38], [39] and P. J. Cohen [6]. Cohen completely solved the problem for abelian group algebras in [6], where he showed that the only homomorphisms of $L^1(G)$ into $M(H)$ are essentially those induced by piecewise affine maps of certain subsets of the dual group of H into the dual group of G.

In the third chapter of the thesis we look at the analogous problem of determining all homomorphisms of the Fourier algebra of a locally compact group. In 1986 B. Host proved that when G is abelian, any algebra homomorphism of $A(G)$ into $B(H)$ can be described in terms of a piecewise affine map of a subset of H into G ([24]). It is natural to ask whether or not this result holds true for other classes of groups as well. This is precisely the goal of chapter 3.

We generalize B. Host's result by employing the powerful machinery of operator space theory. The results obtained in the recent years ([18], [19], [36], [43]) illustrate the deep connections between the properties of Fourier algebras and their operator space structure. Operator space theory is sometimes referred to as "quantized" theory of operators. Perhaps the first and the most common meaning of "quantization" originates in Heisenberg's idea of approaching quantum phenomena by replacing the functions of classical physics with matrices. In a broader sense, it means the replacement of commuting objects with non-commuting ones. This is by now a well-established and deep theory with roots in, and remarkable applications to, C^*-algebras and von Neumann algebras, Banach spaces and non-commutative harmonic analysis.

The Fourier algebras, seen as preduals of von Neumann algebras, have a natural operator space structure. In this context we consider completely bounded algebra homomorphisms of $A(G)$ into $B(H)$. We show that the same description as that given by Host is possible for completely bounded algebra homomorphisms, when G and H are discrete groups with G amenable (Theorem 3.3.4). In the last section of this chapter we explore a possible extension of this result to other classes of groups.

In Chapter 4, we apply the result of the previous chapter, to obtain a concrete description of the range of a completely bounded algebra homomorphism of $A(G)$ into $A(H)$. In general, given two semisimple commutative Banach algebras, $\mathcal{A}$ and $\mathcal{B}$, and an algebra homomorphism $\phi : \mathcal{A} \longrightarrow \mathcal{B}$ the range of ϕ is included in a well determined set, specifically

$$\phi(\mathcal{A}) \subseteq \left\{ b \in \mathcal{B} : \begin{array}{ll} \phi^*(\varphi) = 0 \Rightarrow \hat{b}(\varphi) = 0 & \\ \phi^*(\varphi_1) = \phi^*(\varphi_2) \Rightarrow \hat{b}(\varphi_1) = \hat{b}(\varphi_2) & \varphi, \varphi_1, \varphi_2 \in \Sigma_{\mathcal{B}} \end{array} \right\},$$

where $\Sigma_{\mathcal{B}}$ is the spectrum of B and $\hat{b}$ is the Gelfand transform of b.

In the case when $\mathcal{A} = L^1(G)$ and $\mathcal{B} = L^1(H)$, with G, H two locally compact abelian groups, A. Kepert proved in [25] that equality takes place in the above inclusion. We prove that when $\mathcal{A}$ and $\mathcal{B}$ are the Fourier algebras of two discrete groups G and H respectively, with G amenable, then the range of a completely bounded algebra homomorphism ϕ is as large as it can possibly be, in other words we have equality above (Theorem 4.3.1). Moreover, the same is true for a bounded algebra homomorphism of $A(G)$ into $A(H)$ when G, H are locally compact groups with G abelian, (Theorem 4.4.7).

The idea of the proof is close in spirit to the approach of Kepert. The difficulties appear due to the fact that we are not dealing with abelian groups anymore. The proof relies on our description of the completely bounded algebra homomorphisms from $A(G)$ to $B(H)$ in terms of piecewise affine maps. In Section 2 we present preliminary results concerning piecewise affine maps which will lead to the results in the next section. In Section 3 we reformulate the above equality in terms of extensions of functions in $A(G)$, using as an essential tool our description of the completely bounded algebra homomorphisms from $A(G)$ to $B(H)$. This comes down to the fact that Theorem 4.3.1 is equivalent to Theorem 4.3.2. In Section 4 we proceed to prove Theorem 4.3.2 in three stages.

In the last chapter of the thesis we turn our attention to the Fourier algebras associated to the coset spaces of a locally compact group. In 1998, B. Forrest defined in [16] the Fourier and Fourier-Stieltjes algebras, $A(G/H)$ and $B(G/H)$ respectively, on the coset space G/H where H is a closed subgroup of a locally compact group G. He has proved that for H compact it is possible to extend many classical results to this new setting. Our goal is to continue this investigation.

The algebras $A(G/H)$ and $B(G/H)$ can be identified with subalgebras of $B(G)$, denoted by $A(G : H)$ and $B(G : H)$ respectively. In the third section of the chapter we study the dual space $VN(G{:}\,K)$ of $A(G{:}\,K)$, when K is a compact subgroup of G. We give a description of the dual that leads to the fact that $VN(G{:}\,K)$ is a w^*-closed left ideal in $VN(G)$, the dual space of $A(G)$. A natural question that arises is whether or not one can characterize all w^*-closed left ideals of $VN(G)$ that are of this form. We provide such a characterization in Section 4.

The last three sections of the chapter are dedicated to the natural analogues in $VN(G{:}\,K)$, of the space of uniformly continuous functionals on $A(G)$, $UBC(\hat{G})$, the space of weakly almost periodic functionals on $A(G)$, $W(\hat{G})$, and the space of almost periodic functionals on $A(G)$, $AP(\hat{G})$ ([8], [20]). We respectively denote them by $UBC(\widehat{G : K})$, $W(\widehat{G : K})$ and $AP(\widehat{G : K})$. We obtain results that are analogous to the ones in the classical case due to F. Dunkl and D. E. Ramirez ([8]), E. Granirer ([20]) , A. T. Lau ([26]).

In Section 5 we present conditions under which various inclusion relationships between these three spaces occur. Also, we prove that when G is amenable, $UBC(\widehat{G : K})$ is isometrically isomorphic to a closed subspace of $B(G{:}\,K)^*$ (Theorem 5.5.13). In Section 6, we explore the behaviour of these spaces with respect to the Arens product on their duals. Among other things we characterizes $W(\widehat{G : K})$ as the maximal subspace X of $VN(G{:}\,K)$ for which the Arens product makes sense on X^* and the product is separately continuous with respect to the weak* topology on bounded spheres (Proposition 5.6.7). In the last section we study operators which commute with the action of $A(G{:}\,K)$ on subspaces of $VN(G{:}\,K)$.

Chapter 2

Preliminaries and notations

2.1 C^*-algebras and von Neumann algebras

A *Banach algebra* $\mathcal{A}$ is a Banach space together with a norm compatible algebra structure, namely $\|ab\| \le \|a\|\|b\|$ for all $a, b \in \mathcal{A}$. An *involution* is an operation $x \mapsto x^*$ on $\mathcal{A}$, that satisfies

i) $(a^*)^* = a$

ii) $(a+b)^* = a^* + b^*$

iii) $(\lambda a)^* = \overline{\lambda} a^*$

iii) $(ab)^* = b^* a^*$

for any $a, b \in \mathcal{A}, \lambda \in \mathbf{C}$. An *involutive Banach algebra* is a Banach algebra with an isometric involuiton, that is $\|a^*\| = \|a\|$ for any $a \in \mathcal{A}$. Finally, we say that $\mathcal{A}$ is a *C^*-algebra* if we also have $\|aa^*\| = \|a\|^2$.

Given X a locally Hausdorff space, the Banach algebra $C_0(X)$ of all continuous complex functions vanishing at ∞, with the usual $*$-algebraic operations and the supremum norm is a C^*-algebra. In fact, any commutative C^*-algebra essentially arises in this fashion. If $\mathcal{H}$ is a Hilbert space, then $B(\mathcal{H})$ with the usual $*$-algebraic operations and the operator norm is another example of C^*-algebra.

A linear functional f on an involutive Banach algebra $\mathcal{A}$ is called *positive* if $f(aa^*) \geq 0$ for every $a \in \mathcal{A}$. In general, for a linear functional f, on an involutive Banach algebra, the *adjoint* functional f^* of f is defined by $f^*(a) = \overline{f(a^*)}$, $a \in \mathcal{A}$. If $f = f^*$ we say that f is *self-adjoint* or *hermitian.*

Given a C^*-algebra $\mathcal{A}$, the Jordan decomposition of functionals tells us that every hermitian functional f can be represented in the form

$$f = f^+ - f^- \quad \text{and} \quad \|f\| = \|f^+\| + \|f^-\|$$

by some f^+, f^- positive functionals on $\mathcal{A}$.

Given an involutive Banach algebra $\mathcal{A}$, a *representation* of $\mathcal{A}$ is a $*$-homomorphism, $\pi : \mathcal{A} \longrightarrow B(\mathcal{H})$, of $\mathcal{A}$ into the algebra of bounded operators on a Hilbert space $\mathcal{H}$. We may associate with each positive functional f on $\mathcal{A}$, a Hilbert space $\mathcal{H}_f$, a vector $\xi_f \in H_f$ and a representation $\pi_f : \mathcal{A} \longrightarrow B(\mathcal{H}_f)$ such that

$$f(x) =< \pi_f(x)\xi_f, \xi_f >$$

and $\pi_f(\mathcal{A})\xi_f$ is norm dense in $\mathcal{H}_f$.

A *bounded approximate identity* for a Banach algebra $\mathcal{A}$ is a bounded net $\{e_\alpha\}_{\alpha \in I}$ in $\mathcal{A}$ such that $\|e_\alpha a - a\| \longrightarrow 0$ and $\|a e_\alpha - a\| \longrightarrow 0$ for every $a \in \mathcal{A}$. If the net satisfies only the first (respectively, the second) condition it is called a *bounded left* (respectively *right*) *approximate identity* for $\mathcal{A}$.

A *von Neumann algebra* $\mathcal{M}$ on a Hilbert space $\mathcal{H}$ is a C^*-algebra on $\mathcal{H}$, which is closed under the weak operator topology, that is the topology determined by the functionals

$$\mathcal{M} \longrightarrow \mathbf{C} : \ T \mapsto < T\eta, \xi >$$

for $\eta, \xi \in \mathcal{H}$. The abstract characterization of von Neumann algebras tells us that von Neumann algebras are precisely the class of C^*-algebras of $B(\mathcal{H})$ which can be identified with the dual space of some Banach space $\mathcal{X}$. Given a von Neumann algebra $\mathcal{M}$, its predual is denoted by $\mathcal{M}_*$.

If $\mathcal{S}$ is a collection of operators on a Hilbert space $\mathcal{H}$, then its *commutant* is the collection

$$\mathcal{S}' = \{T \in B(\mathcal{H}) : TS = ST \text{ for all } S \in \mathcal{S}\}$$

The von Neumann's double commutant theorem tells us that given a self-adjoint algebra $\mathcal{R}$ of operators on a Hilbert space $\mathcal{H}$ which contains the identity operator I, the weak operator closure is equal to the double comutant algebra $\mathcal{R}''$. A $*$-subalgebra $\mathcal{M}$ of $B(\mathcal{H})$ is von Neumann algebra iff $\mathcal{M} = \mathcal{M}''$.

The second dual $\mathcal{A}^{**}$ of a C^*-algebra $\mathcal{A}$, can be identified with a von Neumann algebra and $\mathcal{A}$ is a C^*-subalgebra of $\mathcal{A}^{**}$.

2.2 Harmonic Analysis

Let G be a locally compact group. We assume that a left Haar measure λ on G has been fixed and we denote by Δ the modular function. The integral of a Borel measurable function f on G with respect to this measure is denoted by

$$\int_G f(x)dx.$$

Let $L^p(G)$ be the space of all complex valued measurable functions f on G such that

$$\int_G |f(x)|^p dx < \infty, \quad (1 \le p < \infty)$$

Identifying functions that are equal λ-almost everywhere, $L^p(G)$ is a Banach space with norm

$$\|f\|_p = (\int_G |f(x)|^p dx)^{\frac{1}{p}}, \quad (f \in L^p(G))$$

The space $L^1(G)$ is called the *group algebra* and it is a Banach algebra under convolution

$$f * g = \int_G f(t)g(t^{-1}s)dt \text{ for } f, g \in L^1(G)$$

$L^1(G)$ has an isometric involution given by $f^*(s) = \frac{1}{\Delta(s)}\overline{f(s^{-1})}$.

The C^*-algebra of all essentially bounded complex valued Borel measurable functions on G equipped with the essential supremum norm is denoted by $L^\infty(G)$. The C^*-subalgebras of continuous bounded functions and the functions vanishing at infinity on G are respectively $C(G)$ and $C_0(G)$. The space of continuous functions with compact support is denoted with $C_{00}(G)$.

If f is a function on G and $a \in G$, we define

$$_af(x) = f(ax)$$

$$f_a(x) = f(xa)$$

$$\check{f}(x) = f(x^{-1})$$

$$\tilde{f}(x) = \overline{f(x^{-1})}$$

for all $x \in G$.

A *left invariant mean* on G is a continuous linear functional $m \in L^\infty(G)$ such that

1. $\|m\| = m(1) = 1$

2. $m(_af) = m(f)$ for all $f \in L^\infty(G)$ and all $a \in G$.

A locally compact group that has left invariant mean is called *amenable.* Compact groups and abelian groups are amenable. Many properties of G as well as various properties of algebraic structures on G are shown to be equivalent to amenability of G. The most comprehensive references for amenability are A. Paterson's A.M.S. monograph [33] and Pier's book [34].

A *continuous unitary representation* of G is a continuous homomorphism $\pi : G \longrightarrow \mathcal{U}(\mathcal{H})$ where $\mathcal{U}(\mathcal{H})$ is the group of unitaries on a Hilbert space $\mathcal{H}$, given the relative weak operator topology from $B(\mathcal{H})$. The *left regular representation* of G is denoted by λ and is given by

$$\lambda : G \longrightarrow B(L^2(G))$$

$$\lambda(s)(f)(x) = f(s^{-1}x) \text{ for all } s, x \in G, f \in L^2(G)$$

Two representations $\{\pi_1, \mathcal{H}_1\}$ and $\{\pi_2, \mathcal{H}_2\}$ are said to be *unitarily equivalent* if there exists a unitary operator $U : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ such that for $U\pi_1(x) = \pi_2(x)U$. Let Σ_G denote the set of equivalence classes of continuous unitary representations of G. Each $\pi \in \Sigma_G$ induces a continuous non-degenerate *-representation of $L^1(G)$ by means of the formula

$$< \pi(f)\xi, \mu >= \int_G f(x) < \pi(x)\xi, \mu > dx$$

for each $f \in L^1(G)$, $\xi, \mu \in \mathcal{H}_\pi$. In fact, every non-degenerate *-representation of $L^1(G)$ arises in this manner. Therefore we will also denote the set of equivalence classes of non-degenerate *-representations of $L^1(G)$ with Σ_G.

Now any representation $\{\pi, \mathcal{H}\}$ of G satisfies

$$\|\pi(f)\| \leq \|f\|_1 \quad (f \in L^1(G))$$

so we may define a norm on $L^1(G)$ by

$$\|f\|_{C^*(G)} = \sup\{\|\pi(f)\| : \pi \in \Sigma_G\}$$

The completion of $(L^1(G), \|\cdot\|_{C^*(G)})$ is a C^*-algebra called the *group C^*-algebra of G* and it is denoted by $C^*(G)$.

The dual of $C^*(G)$ will be denoted by $B(G)$. It is a linear space of continuous functions on G which becomes an algebra with respect to pointwise multiplication. $B(G)$ is called the *Fourier-Stieltjes algebra of G*. $B(G)$ may be realized either as the space of coefficient functions of Σ_G, that is, functions of the form

$$u(x) =< \pi(x)\xi, \mu > \text{ for } \pi \in \Sigma_G, \ \xi, \mu \in \mathcal{H}_\pi$$

or as the span of the continuous positive definite functions on G. *A positive function* on G is a function f that for every $x_1, \cdots, x_n \in G$ and $\lambda_1, \cdots, \lambda_n \in \mathbf{C}$ satisfies

$$\sum_{i,j} \lambda_i \overline{\lambda_j} f(x_i x_j^{-1}) \geq 0$$

The set $P(G)$ denotes the set of all continuous positive definite functions on G and corresponds to the set of positive functionals on $C^*(G)$. Let $\|\cdot\|_{B(G)}$ be the norm on $B(G)$ induced by $C^*(G)$ with the duality given by the formula

$$< u, f >= \int_G u(x)f(x)dx \quad (u \in B(G),\ f \in L^1(G))$$

If $u \in P(G)$, then $\|u\|_{B(G)} = u(e)$. With $\|\cdot\|_{B(G)}$ and pointwise multiplication $B(G)$ becomes a commutative, regular, semisimple Banach algebra.

The *Fourier algebra of* G, denoted by $A(G)$, is the set of coefficient functions of the left regular representation of G, that is

$$A(G) = \{u(x) =< \lambda(x)f, g >= \int_G f(x^{-1}y)\overline{g(y)}dy = (f * \tilde{g})^{\vee} \quad : \ f, g \in L^2(G)\}$$

It is a $\|\cdot\|_{B(G)}$ closed ideal in $B(G)$. We will write $\|\cdot\|_{A(G)}$ for the restriction of $\|\cdot\|_{B(G)}$ to $A(G)$. With this norm $A(G)$ becomes a commutative, regular, semisimple Banach algebra. It can also be realized as the norm closure in $B(G)$ of $B(G) \cap \mathcal{C}_{00}(G)$. The spectrum of $A(G)$ is homeomorphic to G. Furthermore, $A(G)$=$B(G)$ if and only if G is compact.

Let $VN(G)$ denote the von Neumann subalgebra of $B(L^2(G))$ generated by either $\{\lambda(x) : x \in G\}$ or by $\{\lambda(f) : f \in L^1(G)\}$. It is called *the group von Neumann algebra of G*. It can be identified with the dual of $A(G)$, and the duality is given by

$$< T, u >=< Tf, g >_{L^2(G)}$$

if u is of the form $u = (f * \tilde{g})^{\vee} \in A(G),\ T \in VN(G)$ and by

$$< T, u >= u(x)$$

if $T = \lambda(x)$ for some $x \in G$.

The dual of $B(G)$ it is also a von Neumann algebra called the *big algebra of* G and it is denoted by $W^*(G)$.

When G is abelian $A(G)$ and $B(G)$ can be identified, via the Fourier transform, with the commutative group algebra $L^1(\Gamma)$ and the measure algebra $M(\Gamma)$

respectivelly, where Γ is the dual group of G. The basic reference on Fourier and Fourier-Stieltjes algebras is [9].

2.3 Operator spaces

An *operator space* is a vector space V together with a family $\{\| \ \|_n\}_n$ of Banach space norms on $M_n(V)$ such that

$$\textbf{(OS1)} \quad \left\| \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \right\|_{m+n} = \max\{\|A\|_n, \|B\|_m\} \qquad \text{for } A \in M_n(V), B \in M_m(V)$$

$$\textbf{(OS2)} \quad \|\alpha A \beta\| \leq \|\alpha\| \cdot \|A\| \cdot \|\beta\| \qquad \text{for } \alpha, \beta \in M_n, A \in M_n(V)$$

In (OS2), $M_n = M_n(\mathbf{C})$ and it is normed by identifying matrices with operators on the Hilbert space $\mathbf{C}^n$. The family of norms $\{\| \cdot \|_n : M_n(V) \longrightarrow \mathbf{R}^+\}$ is called an *operator space structure.*

The prototipical operator space is the space $B(\mathcal{H})$ of all bounded operators on a Hilbert space $\mathcal{H}$. We identify $M_n(B(\mathcal{H})) \cong B(\mathcal{H}^n)$, where $\mathcal{H}^n$ is the Hilbertian direct sum of n copies of $\mathcal{H}$, by

$$[a_{ij}] \mapsto \left(\begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix} \mapsto \begin{bmatrix} \sum_{k=1}^n a_{1k}\xi_k \\ \vdots \\ \sum_{k=1}^n a_{nk}\xi_k \end{bmatrix} \right)$$

Thus, any subspace of $B(\mathcal{H})$ is naturally an operator space given the induced operator space structure. Any C^*-algebra $\mathcal{A}$ is an operator space when the unique norm making each $M_n(\mathcal{A})$ a C^*-algebra is chosen.

Given two operator spaces V, W and a linear map $\phi : V \longrightarrow W$, for each n, we define the n^{th} *amplification* of ϕ

$$\phi^{(n)} : M_n(V) \longrightarrow M_n(W)$$

by the means of the formula

$$\phi^{(n)}([v_{ij}]) = [\phi(v_{ij})]$$

We say that ϕ is *completely bounded* if

$$\|\phi\|_{cb} = \sup\{\|\phi^{(n)}\| : n \in \mathbf{N}\} < \infty$$

Furthermore, ϕ will be called *complete isometry* if each $\phi^{(n)}$ is an isometry, and *complete contraction* if each $\phi^{(n)}$ is a contraction. The family of completely bounded maps, as above, will be denoted by $CB(V, W)$. It is a linear space with norm $\|\cdot\|_{cb}$.

The space $CB(V, W)$ has a natural operator space structure given by the identification $M_n(CB(V, W)) \cong CB(V, M_n(W))$ via

$$[\phi_{ij}] \mapsto (v \mapsto [\phi_{ij} v])$$

Note that $M_n(V)$ admits a natural operator space structure via the identification $M_m(M_n) \cong M_{nm}(V)$. A linear functional f on V is completely bounded if and only if it is bounded, and $\|f\| = \|f\|_{cb}$. Thus V^* is canonically an operator space. If $\mathcal{F} = [f_{ij}] \in M_n(V^*)$, the norm $\|\cdot\|_n$ on $M_n(V^*)$ is given by

$$\|\mathcal{F}\|_n = \sup\{\|[f_{ij}(v_{kl})]\|_{nm} : \ [v_{kl}] \in M_m(V), \ \|[v_{kl}]\|_m \leq 1 \ \ m \in \mathbf{N}\}.$$

Extending the result above for linear functionals we obtain that any linear map $T : V \longrightarrow \mathcal{A}$, where $\mathcal{A}$ is a commutative C^*-algebra, is completely bounded if and only if it is bounded, also with $\|T\| = \|T\|_{cb}$. Thus any operator space W that imbeds completely isometrically into a commutative C^*-algebra, has $CB(V, W) = B(V, W)$ isometrically, for any operator space V. We say that such W is a *min space* and write $W = \min W$. Using adjoints, any operator space V which imbeds completely isometrically into the dual of a commutative C^*-algebra, has the property that $CB(V, W) = B(V, W)$ isometrically, for any operator space W. We say that such a V is a *max space* and write $V = \max V$.

Given dual operator spaces V and V^* there can be defined two matrix pairings *the scalar pairing* $<,>: M_n(V) \times M_n(V^*) \longrightarrow \mathbf{C}$ given by

$$< \mathcal{V}, \mathcal{F} >= \sum f_{ij}(v_{ij})$$

the matrix pairing $\ll,\gg: M_n(V) \times M_n(V^*) \longrightarrow M_{n^2}$ given by

$$\ll \mathcal{V}, \mathcal{F} \gg= [f_{ij}(v_{kl})]$$

Let V be an operator space. A new norm, denoted by $\|\cdot\|_{1n}$ is defined on $\mathcal{M}_n(V)$ by

$$\|\cdot\|_{1n} : \mathcal{M}_n(V) \longrightarrow [0,\infty)$$

$$\|\mathcal{V}\|_{1n} = \inf\{\|\alpha\|_2\|\tilde{\mathcal{V}}\|\|\beta\|_2 \ : \ \mathcal{V} = \alpha\tilde{\mathcal{V}}\beta\}$$

where $\alpha \in HS_{n,r}, \beta \in HS_{r,n}, \tilde{\mathcal{V}} \in M_r(\mathcal{V})$ with r arbitrary and HS_p being the space of scalar matices with the Hilbert-Schmidt norm. We let $T_n(V)$ denote the space $\mathcal{M}_n(V)$ endowed with the norm $\|\cdot\|_{1n}$. The following isometric identifications take place

$$M_n(V^*) \simeq T_n(V)^*$$

$$T_n(V^*) \simeq M_n(V)^*$$

both given by the scalar pairing.

Given a net $\mathcal{F}_\lambda = [f_{ij}^\lambda] \in M_n(V^*)$, with λ in a directed index set Λ, and $\mathcal{F} = [f_{ij}] \in M_n(V^*)$ the following are equivalent

1) $\mathcal{F}_\lambda \longrightarrow \mathcal{F}$ in the w^*-topology

2) $< \mathcal{F}_\lambda, \mathcal{V} > \longrightarrow < \mathcal{F}, \mathcal{V} >$ in the norm topolgy, for any $\mathcal{V} \in T_n(V)$

3) $\ll \mathcal{F}_\lambda, \mathcal{V} \gg \longrightarrow \ll \mathcal{F}, \mathcal{V} \gg$ in the norm topolgy, for any $\mathcal{V} \in T_n(V)$

4) $f_{ij}^\lambda \longrightarrow f_{ij}$ in the w^*-topology, for any $i,j = 1,n$

Given a linear mapping of operator spaces $\phi : V \longrightarrow W$, we define

$$T_n(\phi) : T_n(V) \longrightarrow T_n(W)$$

given by

$$T_n(\phi)([v_{ij}]) = [\phi(v_{ij})]$$

Then $\|\phi^{(n)}\| = \|T_n(\phi)\|$, so the completely bounded norm of ϕ can also be calculated as

$$\|\phi\|_{cb} = \sup\{\|T_n(\phi)\| \ : \ n \in \mathbf{N}\}$$

If $\mathcal{M}$ is a von Neumann algebra, its predual $\mathcal{M}_*$ is naturally an operator space via the inclusion $\mathcal{M}_* \hookrightarrow \mathcal{M}^*$. Moreover, $(\mathcal{M}_*)^* \cong \mathcal{M}$ completely isometrically.

If X and Y are two Banach spaces, denote their algebraic tensor product by $X \otimes Y$. If V and W are operator spaces, $v = [v_{ij}] \in M_n(V)$ and $w = [w_{kl}] \in M_n(W)$, we let their *tensor product*, $v \otimes w$ in $M_{nm}(V \otimes W)$, be given by the doubly indexed matrix

$$v \otimes w = [v_{ij} \otimes w_{kl}]$$

Given an element u in $M_n(V \otimes W)$, we define

$$\|u\|_\wedge = \inf\{\|\alpha\|\|v\|\|w\|\|\beta\| : u = \alpha(v \otimes w)\beta\}$$

where the infimum is taken over arbitrary decompositions with $v \in M_p(V), w \in M_q(W), \alpha \in M_{n,pq}$ and $\beta \in M_{pq,n}(W)$ with $p, q \in \mathbf{N}$ arbitrary. We let

$$V \otimes_\wedge W = (V \otimes W, \| \cdot \|_\wedge)$$

and we define the *operator space projective tensor product* $V \widehat{\otimes} W$ to be the completion of this space.

Given operator spaces V, W and Z, and $p, q \in \mathbf{N}$, each bilinear mapping $\phi : V \times W \longrightarrow Z$ determines a bilinear mapping

$$\phi^{(p,q)} : M_p(V) \times M_q(W) \longrightarrow M_{pq}(Z)$$

where

$$\phi^{(p,q)}(v,w) = [\phi(v_{ij}, w_{kl}]$$

We say that ϕ is *completely bounded* (respectively *completely contractive*) if

$$||\phi||_{cb} = \sup\{||\phi^{(p,q)}|| : \ p, q \in \mathbf{N}\} = \sup\{\phi^{(p,p)} : \ p \in \mathbf{N}\} < \infty$$

(respectively $||\phi||_{cb} \leq 1$). The family of all such completely bounded bilinear mappings will be denoted by $CB(V \times W, Z)$. It is a linear space with norm $||\cdot||_{cb}$. It has a natural operator space structure, given by the identification $M_n(CB(V \times W, Z) \cong CB(V \times W, M_n(Z))$. The universal property of the operator projective tensor product is given by the completely isometric identification

$$CB(V \hat{\otimes} W, Z) \cong CB(V \times W, Z)$$

A Banach algebra $\mathcal{A}$ which is also an operator space and it is such that the multiplication $m : \mathcal{A} \hat{\otimes} \mathcal{A} \longrightarrow \mathcal{A}$, given by $m(a \otimes b) = ab$, is completely contractive, is called a *completely contractive Banach algebra.*

Chapter 3

Fourier algebra homomorphisms

3.1 Introduction

Let G and H be locally compact groups. Given an algebra homomorphism $\phi : A(G) \longrightarrow B(H)$, we can always find a set $Y \subset H$ and a map $\alpha : Y \longrightarrow G$ such that

$$\phi(f) = \begin{cases} f \circ \alpha & \text{on } Y \\ 0 & \text{off } Y \end{cases}$$

B. Host showed in ([24]) that if G and H are locally compact groups with G abelian, then Y is in the coset ring of H and the map α is piecewise affine (see Section 3.2). It is a natural question to consider whether or not the result holds true for other classes of groups as well. This is precisely the goal of this chapter.

We take into account the operator structure of the Fourier algebras and in this context we consider completely bounded algebra homomorphisms. We show that Host's description remains valid when G and H are discrete groups with G amenable.

In Section 2 we introduce the coset ring of a locally compact group and piecewise affine maps. In Section 3 we present the generalization of Host's characterization of homomorphisms between $A(G)$ and $B(H)$. The main result of this section is Theorem 3.3.4. In the last section we give a direction towards a generalization of Theorem 3.3.4.

3.2 Piecewise affine maps

We start with the definitions of affine and piecewise affine maps which will play an important role in the sequel.

A set $E \subseteq G$ is called a *left coset* in G if E is a left coset of some subgroup G_0 of G. The *coset ring of* G, denoted by $\Omega(G)$, is the smallest ring of sets of G which contains all open left cosets in G. As in the abelian case (see [37]) it can be shown that E is a left coset in G if and only if $EE^{-1}E \subseteq E$. Moreover, if E is a left coset and $\gamma_0 \in E$, then $H_0 = E^{-1}E = {\gamma_0}^{-1}E$ is a subgroup of G and $E = \gamma_0 H_0$.

Definition 3.2.1 *Let G, H be locally compact groups and $E \subseteq H$ a left coset. A continuous map $\alpha : E \longrightarrow G$ is called affine if*

$$\alpha(x_1 x_2^{-1} x_3) = \alpha(x_1)\alpha(x_2)^{-1}\alpha(x_3)$$

for any $x_1, x_2, x_3 \in E$.

Remark 3.2.2 There is a connection between affine maps and group homomorphisms as follows: α is an affine map if and only if for every $\gamma_0 \in E$, the map

$$\beta : \gamma_0^{-1}E \longrightarrow G$$

defined by

$$\beta(h) = \alpha(\gamma_0)^{-1}\alpha(\gamma_0 h) \quad \forall \quad h \in {\gamma_0}^{-1}E$$

is a group homomorphism.

Definition 3.2.3 *Let G, H be locally compact groups and $Y \subseteq H$ a set. A map $\alpha : Y \longrightarrow G$ is said to be piecewise affine if* *(i) there are $Y_i \in \Omega(H)$, $i = 1, ..., n$, pairwise disjoint such that*

$$Y = \bigcup_{i=1}^{n} Y_i$$

(ii) each $Y_i \subseteq K_i$, where K_i is an open left coset in H

(iii) for each i, there is $\alpha_i : K_i \longrightarrow G$ affine map such that $\alpha|_{Y_i} = \alpha_i$.

Taking into account Remark 3.2.2, it is not hard to see that the above definition is equivalent to the one given by B. Host in [24]:

Definition 3.2.4 *Let α be as above. Then α is piecewise affine if*

(i) there are $Y_i \in \Omega(H)$, $i = 1, ..., n$, pairwise disjoint such that

$$Y = \bigcup_{i=1}^{n} Y_i$$

(ii) each $Y_i \subseteq K_i = g_i H_i$, where K_i is an open left coset in H

(iii) for each i, there is a continuous homomorphism $\alpha_i : H_i \longrightarrow G$ and an element $g_i \in G$ such that $\alpha(y) = g_i \alpha_i(h_i^{-1} y)$, $y \in Y_i$.

Note that if G and H are abelian, the above definitions coincide with the ones in the abelian case, as they can be found in [37].

Remark 3.2.5 *(i)* If $E \subseteq G$ is a left coset, then it is a right coset as well, and vice versa. Indeed, if $\gamma_0 \in E$, then $E = \gamma_0 H_0$ with $H_0 = E^{-1}E$. Then $E = \gamma_0 H_0 \gamma_0^{-1} \gamma_0 = (\gamma_0 H_0 \gamma_0^{-1})\gamma_0 = H_1 \gamma_0$ is a right coset in G.

(ii) If $E \subseteq G$ is a left coset in G and $y \in G$, then Ey is left coset in G. Indeed, if $E = \gamma_0 H_0$, then $Ey = \gamma_0 H_0 y = \gamma_0 y(y^{-1} H_0 y) = \gamma_0 y H_1$, which is a left coset in G.

Let E be a closed coset of a locally compact group G and let $f : E \longrightarrow \mathbf{C}$ be a function on E. For any $\gamma \in E$, we let ${}_{\gamma^{-1}}f$ be the function $f : E^{-1}E \longrightarrow \mathbf{C}$

given by ${}_{\gamma^{-1}}f(x) = f(\gamma^{-1}x)$. We can define the Fourier Stieltjes algebra on a closed coset $E \subseteq G$ and denote it by $B(E)$, as follows :

$$B(E) = \{f : E \longrightarrow \mathbf{C} \mid {}_{\gamma^{-1}}f \text{ belongs to } B(E^{-1}E) \text{ for any } \gamma \in E\}.$$

where $B(E^{-1}E)$ is the Fourier Stieltjes algebra of the locally compact group $E^{-1}E$. The topological and algebraic structure of $B(E^{-1}E)$ can be carried over to $B(E)$ so that $B(E)$ is isomorphic to $B(E^{-1}E)$.

Recall that the coset ring of G can be viewed as a family of characteristic functions, that is

$$X \in \Omega(G) \text{ if and only if } 1_X \in B(G)$$

where 1_X is the characteristic function of X ([24]). We will use this to give the following definition for the coset ring of E.

Definition 3.2.6 *A set X is in the coset ring of E if and only if 1_X is in $B(E)$.*

Remark 3.2.7 It is not hard to see, given the definition of $B(E)$, that X is in the coset ring of E if and only if $\gamma^{-1}X \in \Omega(E^{-1}E)$ for each $\gamma \in E$.

3.3 Characterization of Fourier algebra homomorphisms

Given a von Neumann algebra $\mathcal{M}$, its predual $\mathcal{M}_*$ is naturally an operator space via the inclusion $\mathcal{M}_* \hookrightarrow \mathcal{M}^*$. Moreover, $(\mathcal{M}_*)^* \cong \mathcal{M}$ completely isometrically.

If G is a locally compact group, then the Fourier and Fourier-Stieltjes algebra, $A(G)$ and $B(G)$ respectively, have a natural operator space structure as preduals of von Neumann algebras. For $U = [u_{ij}] \in M_n(A(G))$ the norm is given by

$$\|U\|_n = \sup\{\|[T_{kl}(u_{ij})]\| \ : \ T = [T_{kl}] \in M_m(VN(G)), \|T\|_m \leq 1, \ m \in \mathbf{N}\},$$

and similarly for $V = [v_{ij}] \in M_n(B(G))$ the norm is given by

$$\|V\|_n = \sup\{\|[F_{kl}(u_{ij})]\| \ : \ F = [F_{kl}] \in M_m(W^*(G)), \|F\|_m \leq 1, \ m \in \mathbf{N}\}.$$

In each case the operator structure results in a completely contractive Banach algebra.

The following proposition will be used in the proof of the main theorem of this section, Theorem 3.3.4.

Proposition 3.3.1 *Let G, H be locally compact groups. The map*

$$S : B(G) \longrightarrow B(H \times G)$$

$$u \mapsto 1_H \cdot u$$

where $(1_H \cdot u)(h, g) = u(g)$, is completely contractive.

Before we present the proof, we give the following universality property (see e.g. [40]).

Lemma 3.3.2 *Let $\omega : G \longrightarrow W^*(G) = C^*(G)^{**}$ be the universal representation of a locally compact group G. If $\pi : G \longrightarrow B(\mathcal{H}_\pi)$ is a continuous representation of G, then there exists a unique w^*-continuous *-homomorphism*

$$\theta : W^*(G) \longrightarrow \pi(G)''$$

such that $\pi = \theta \circ \omega$.

Proof of Proposition 3.3.1 Let $\omega_{H\times G} : H \times G \longrightarrow W^*(H \times G)$ be the universal representation of $H \times G$. Let

$$i : H \times G \longrightarrow G$$

$$(h, g) \mapsto g$$

be the canonical projection on G. Let $\omega_G : G \longrightarrow W^*(G)$ be the universal representation of G. Then $\omega_G \circ i : H \times G \longrightarrow W^*(G)$ is a representation of

$H \times G$. We apply Lemma 3.3.2 for $\omega_{H\times G}$ and $\omega_G \circ i$ and we get a unique w^*-continuous *-homomorphism

$$\theta : W^*(H \times G) \longrightarrow W^*(G)$$

such that $\omega_G \circ i = \theta \circ \omega_{H\times G}$. This map yields a complete contraction

$$\theta_* : B(G) \longrightarrow B(H \times G).$$

Our next claim is that $\theta_* = S$. To show this we will prove that we have $S^* = \theta$. We have the formula

$$< S^*(F), u >=< F, S(u) > \text{ for each } F \in W^*(H \times G).$$

Let $F = \omega_{H\times G}(h, g) \in W^*(H \times G)$. Then

$$< S^*(\omega_{H\times G}(h, g)), u >=< \omega_{H\times G}(h, g), 1_H \cdot u >= u(g) =< \omega_G(g), u >$$

and we get

$$S^*(\omega_{H\times G}(h, g)) = \omega_G(g). \tag{3.1}$$

On the other hand, we know $\omega_G \circ i = \theta \circ \omega_{H\times G}$. If we apply this to F we obtain

$$\theta(\omega_{H\times G}(h, g)) = (\omega_G \circ i)(h, g) = \omega_G(g). \tag{3.2}$$

Combining (3.1) and (3.2) we get

$$S^*(\omega_{H\times G}(h, g)) = \theta(\omega_{H\times G}(h, g)) \ \forall (h, g) \in H \times G.$$

Since $\{\omega_{H\times G}(h, g) : h \in H, g \in G\}$ generates $W^*(H \times G)$ and S^*, θ are w^*-w^* continuous, it follows that $S^* = \theta$, as claimed. □

The next result is an immediate application of Proposition 3.3.1.

Corollary 3.3.3 *The map*

$$J : A(G) \times B(H) \longrightarrow B(H \times G)$$

$$(u, v) \rightarrow u \cdot v$$

where $u \cdot v(h, g) = u(g)v(h)$, *is completely contractive.*

Proof As $B(H \times G)$ is a completely contractive operator algebra, the map

$$B(H \times G) \times B(H \times G) \longrightarrow B(H \times G)$$

$$(w, v) \mapsto w \cdot v$$

is completely contractive. Now, taking into account Proposition 3.3.1, the conclusion follows. □

The next theorem is a generalization of Host's characterization of algebra homomorphisms between Fourier algebras.

Theorem 3.3.4 *Let G, H be discrete groups. Suppose that G is amenable and $\phi : A(G) \longrightarrow B(H)$ is a completely bounded algebra homomorphism. Then there exists $Y \in \Omega(H)$ and a piecewise affine map $\alpha : Y \longrightarrow G$ such that*

$$\phi(u) = \begin{cases} u \circ \alpha & \text{on } Y \\ 0 & \text{off } Y \end{cases}$$

Proof Let $h \in H$. Define the following map

$$\phi_h : A(G) \longrightarrow \mathbf{C}$$

$$u \mapsto \phi(u)(h)$$

Then ϕ_h is a multiplicative functional on $A(G)$. Since the spectrum of $A(G)$ can be identified with G by point evaluation, if $\phi_h \neq 0$ there is an element in G, that we denote by $\alpha(h)$, such that $\phi_h = \alpha(h)$ as elements of the spectrum of $A(G)$. Thus we have $\phi(u)(h) = u(\alpha(h))$ whenever $\phi_h \neq 0$ and zero otherwise. The map

$$\alpha : Y \longrightarrow G$$

$$h \mapsto \alpha(h)$$

where $Y = \{h \in H : \phi_h \neq 0\}$, satisfies the required equality. It remains to show that $Y \in \Omega(H)$ and α is piecewise affine.

To accomplish this we will use the following lemma from [37].

Lemma 3.3.5 *Let G, H be discrete groups and $\alpha : Y \subset H \longrightarrow G$ a map. Then α is piecewise affine if and only if $\mathcal{G}_\alpha = \{(y, \alpha(y)) : y \in Y\} \in \Omega(H \times G)$.*

Returning to the proof, define the map

$$J : A(G) \times A(G) \longrightarrow B(H \times G)$$

$$(u, v) \mapsto (\phi(u)) \cdot v$$

Our first claim is that J is jointly completely bounded.

Indeed, let $n_1, n_2 \in \mathbf{N}$. Let

$$J^{(n_1,n_2)} : M_{n_1}(A(G)) \times M_{n_2}(A(G)) \longrightarrow M_{n_1 n_2}(B(H \times G))$$

be the amplification. We will compute the norm of the amplification

$$\|J^{(n_1,n_2)}\| = \sup \left\{ \|J^{(n_1,n_2)}(U, V)\|_{n_1 n_2} : \begin{array}{c} U \in M_{n_1}(A(G)), \|U\|_{n_1} \leq 1 \\ V \in M_{n_2}(A(G)), \|V\|_{n_2} \leq 1 \end{array} \right\}.$$

Using the definition of the operator norm on the predual of a von Neumann algebra, we have

$$\begin{aligned} &\|J^{(n_1,n_2)}(U, V)\|_{n_1 n_2} = \\ &\quad = \sup\{\|[< F_{st}, J(u_{ij}, v_{kl}) >]\|_{n_1 n_2 n} : \mathcal{F} = [F_{st}]_{s,t} \in M_n(W^*(H \times G))\} \\ &\quad = \sup\{\|[< F_{st}, \phi(u_{ij}) \cdot v_{kl} >]\|_{n_1 n_2 n} : \mathcal{F} = [F_{st}]_{s,t} \in M_n(W^*(H \times G))\} \\ &\quad = \sup\{\|[< v_{kl} \cdot F_{st}, \phi(u_{ij}) >]\|_{n_1 n_2 n} : \mathcal{F} = [F_{st}]_{s,t} \in M_n(W^*(H \times G))\} \end{aligned}$$

where all the suprema are taken over $\|\mathcal{F}\|_n \leq 1$.

In the above, the product $v \cdot F$ with $F \in W^*(H \times G)$ and $v \in A(G)$, is given by

$$v \cdot F : B(H) \longrightarrow \mathbf{C}, \quad < v \cdot F, u > = < F, u \cdot v > \quad \text{for } u \in B(H)$$

and it is an element in $W^*(H)$. Then $V \cdot \mathcal{F} \in M_{n_2 n}(W^*(H))$, where

$$V \cdot \mathcal{F} = [v_{kl} \cdot F_{st}]_{s,t,k,l}$$

with $\mathcal{F} = [F_{st}] \in M_n(W^*(B(H \times G))), V = [v_{kl}] \in M_{n_2}(A(G))$. Moreover, if $\|\mathcal{F}\|_n \leq 1$ and $\|V\|_{n_2} \leq 1$, by Corollary 3.3.3, we obtain that $\|V \cdot \mathcal{F}\|_{n_2 n} \leq 1$.

Then the norm can be written as

$$\|J^{(n_1,n_2)}(U,V)\|_{n_1 n_2} = \sup \left\{ \|[< E_{rp}, \phi(u_{ij}) >]\|_{n_1 n_2 n} : \begin{array}{l} \mathcal{E} = [E_{rp}] \in M_{n_2 n}(W^*(H)) \\ \mathcal{E} = V \cdot \mathcal{F}, \ \|\mathcal{E}\|_{n_2 n} \leq 1 \end{array} \right\}.$$

But now,

$$\|[< E_{rp}, \phi(u_{ij}) >]\|_{n_1 n_2 n} \leq \|\phi(u_{ij})\|_{n_1} = \|\phi^{(n_1)}(U)\|_{n_1} \leq \|\phi^{(n_1)}\| \leq \|\phi\|_{cb}$$

where the last inequality holds because ϕ is completely bounded. Therefore we have

$$\|J^{(n_1,n_2)}\| \leq \|\phi\|_{cb}$$

which shows that J is jointly completely bounded.

By the universal property of the operator projective tensor product, there exists a completely bounded map

$$\tilde{J} : A(G)\hat{\otimes}A(G) \longrightarrow B(H \times G)$$

such that $\tilde{J}(u \otimes v) = \phi(u) \cdot v$. From the identification $A(G \times G) \simeq A(G)\hat{\otimes}A(G)$ (see [12]), we obtain a map

$$\psi : A(G \times G) \longrightarrow B(H \times G)$$

such that $\psi(u \cdot v) = \phi(u) \cdot v, \ \forall u, v \in A(G)$.

Our next claim is that the following formula holds:

$$\psi(\omega)(h,g) = \omega(\alpha(h), g) \quad \text{for} \quad h \in Y, g \in G. \tag{3.3}$$

Clearly, if $\omega = u \cdot v$ with $u, v \in A(G)$, then

$$\psi(\omega) = \psi(u \cdot v) = \phi(u) \cdot v \Rightarrow \psi(\omega)(h,g) = u(\alpha(h)) \cdot v(g) = \omega(\alpha(h), g).$$

If $\omega = \sum_{i=1}^{n} \omega_i$ where $\omega_i = u_i \cdot v_i$, then

$$\psi(\omega)(h,g) = \sum_{i=1}^{n} \psi(\omega_i)(h,g) = \sum_{i=1}^{n} \omega_i(\alpha(h), g) = \omega(\alpha(h), g).$$

Now suppose that $\omega = \lim_i \psi_i$, where $\psi_i = \sum_{j=1}^{k_i} u_j \cdot v_j$. Then $\psi(\omega) = \lim_i \psi(\omega_i)$. In particular we have pointwise convergence, therefore

$$\psi(\omega)(h,g) = \lim_i \psi(\omega_i)(h,g) = \lim_i \omega_i(\alpha(h),g).$$

But since $\omega = \lim_i \psi_i$, it follows in particular that ψ_i converges pointwise to ω, and therefore $\lim_i \omega_i(\alpha(h),g) = \omega(\alpha(h),g)$. So $\psi(\omega)(h,g) = \omega(\alpha(h),g)$ and our last claim is proved.

The set $\Delta = \{(x,x) : x \in G\}$ is an open subgroup in $G \times G$, so the characteristic function $\chi_\Delta \in B(G \times G)$. Because G is amenable, we can find $\{u_j\}_j \in A(G \times G)$ such that $\|u_j\| \leq 1$ for all j and $u_j \longrightarrow \chi_\Delta$ pointwise .

Then $\psi(u_j)(h,g) \overset{(3.3)}{=} u_j(\alpha(h),g) \longrightarrow \chi_\Delta(\alpha(h),g) = \chi_{\mathcal{G}_\alpha}$ Therefore $\chi_{\mathcal{G}_\alpha}$ is the pointwise limit of a bounded net in $B(H \times G)$. By Corollary 2.25 from [9] it follows that $\chi_{\mathcal{G}_\alpha} \in B(H \times G)$, which is equivalent to $\mathcal{G}_\alpha \in \Omega(H \times G)$. $\square$

Remark 3.3.6 We note that B. Host's result is for continuous algebra homomorphisms as opposed to completely bounded algebra homomorphisms. However when G is abelian then $A(G)$ has MAX operator space structure. In this case every continuous homomorphism is automatically completely bounded, so in particular, we see that Theorem 3.3.4 is a true generalization of Host's result. For the moment we are restricted to discrete groups, but there are hopes that we can drop the assumption of discreteness (see §3.4).

In the case that $\phi : A(G) \longrightarrow B(H)$ has range inside $A(H)$, the corresponding map α is also proper, which means that it returns compacts to compacts.

Proposition 3.3.7 *The homomorphism $\phi : A(G) \longrightarrow B(H)$ maps $A(G)$ into $A(H)$ if and only if $\alpha^{-1}(C)$ is compact for every compact set C of G.*

Proof Suppose there is a compact set $C \subset G$ such that $\alpha^{-1}(C)$ is not compact. Choose $f \in A(G)$ such that $f = 1$ on C. Then, the set $S = \{\gamma \in Y : f(\alpha(\gamma)) =$

1} contains $\alpha^{-1}(C)$ which is closed, non-compact, therefore S is not compact. On the other hand, we can write $S = \phi f^{-1}\{1\}$. If S is not compact, it follows that $\phi f \notin A(H)$, contradiction.

Conversely, suppose $\alpha^{-1}(C)$ is compact for every compact $C \subset G$. If $f \in A(G)$ we can find $f_n \in A(G) \cap C_{00}(G)$ such that

$$f_n \longrightarrow f \quad \text{in} \quad \|\cdot\|.$$

Then each $f_n \circ \alpha$ has compact support, and so $f_n \circ \alpha = \phi f_n \in A(H)$. Since $A(H)$ is closed in $B(H)$ and $\phi f_n \longrightarrow \phi f$, we obtain $\phi f \in A(H)$. □

3.4 Towards a generalization of Theorem 3.3.4

Let G, H be locally compact groups and $\phi : A(G) \longrightarrow B(H)$ a completely bounded algebra homomorphism. In the previous section we have seen, in the first part of the proof of Theorem 3.3.4, that there always exists a continuous map $\alpha : Y \subset H \longrightarrow G$ such that

$$\phi(u) = \begin{cases} u \circ \alpha & \text{on } Y \\ 0 & \text{off } Y \end{cases}$$

Moreover, we have shown that when the groups G and H are discrete with G amenable, the map α is piecewise affine.

A natural question that arises is whether there are other classes of groups for which Theorem 3.3.4 holds true, that is, the map α is piecewise affine. A natural approach to answering this question is to build on the discrete case. This is what we will do here. We show that the condition that α is piecewise affine is equivalent to the complete boundedness of a certain algebra homomorphism $\tilde{\phi}$. We also present a sufficient condition for $\tilde{\phi}$ to be completely bounded.

We begin with the construction of the new algebra homomorphism $\tilde{\phi}$, from the given homomorphism $\phi : A(G) \longrightarrow B(H)$. Let G_d and H_d be the groups G and H respectively, endowed with the discrete topology.

Proposition 3.4.1 *The map* $\tilde{\phi} : A(G_d) \longrightarrow B(H_d)$ *given by*

$$\tilde{\phi}(v) = \begin{cases} v \circ \alpha & \text{on } Y \\ 0 & \text{off } Y \end{cases}$$

is a well defined algebra homomorphism.

To prove this, we first need a lemma, that is probably known, but we present a proof here for the sake of completeness.

Lemma 3.4.2 *Let* G *be a locally compact group. Then for any* $v \in A(G_d)$ *there is a net* $\{u_j\}_j \in A(G)$ *such that* $\|u_j\| \leq 2\|v\|$ *and* $u_j \xrightarrow{w^*} v$*, where* $w^* = \sigma(B(G_d), C^*(G_d))$.

Proof The left regular representation of G, λ_G viewed as a unitary representation of G_d, weakly contains λ_{G_d} ([2]). Then, it follows that any positive functional v associated with λ_{G_d} is w^*-limit of a net $\{u_\alpha\}_\alpha$, where each u_α is a finite sum of positive functionals associated with λ_G, with $\|u_\alpha\| \leq \|v\|$ ([15]). This means that any positive definite function $v \in A(G_d)$ is the w^*-limit of such a net $\{u_\alpha\}_\alpha \in A(G)$.

Let $v \in A(G_d)$. Then v can be written as the sum of two hermitian elements

$$v = v_1 + iv_2, \text{ where } v_1 = \frac{v + \tilde{v}}{2}, v_2 = \frac{v - \tilde{v}}{2i}.$$

By the Jordan decomposition, each v_i, $i = 1, 2$ can be written as

$$v_1 = v_1^+ - v_1^- \text{ such that } \|v_1\| = \|v_1^+\| + \|v_1^-\|$$

$$v_2 = v_2^+ - v_2^- \text{ such that } \|v_2\| = \|v_2^+\| + \|v_2^-\|$$

where each of v_i^+, v_i^- are positive definite functions in $A(G_d)$.

Let $\{u_{i\alpha}^+\}_\alpha, \{u_{i\alpha}^-\}_\alpha \in A(G), i = 1, 2$ be the nets corresponding respectively to v_i^+, v_i^- as above:

$$u_{i\alpha}^+ \xrightarrow{w^*} v_i^+ \text{ with } \|u_{i\alpha}^+\| \leq \|v_i^+\|,$$

$$u_{i\alpha}^{-} \xrightarrow{w^*} v_i^{-} \text{ with } \|u_{i\alpha}^{-}\| \leq \|v_i^{-}\|.$$

Let $u_\alpha = (u_{1\alpha}^{+} - u_{1\alpha}^{-}) + i(u_{2\alpha}^{+} - u_{2\alpha}^{-}) \in A(G)$. We claim that this net satisfies the requirements of the lemma.

Indeed, clearly $u_\alpha \xrightarrow{w^*} v$. Also, we have

$$\begin{aligned}
\|u_\alpha\| &\leq \|u_{1\alpha}^{+}\| + \|u_{1\alpha}^{-}\| + \|u_{2\alpha}^{+}\| + \|u_{2\alpha}^{-}\| \\
&\leq \|v_1^{+}\| + \|v_1^{-}\| + \|v_2^{+}\| + \|v_2^{-}\| \\
&\leq \|v_1\| + \|v_2\| \\
&\leq \frac{\|v\| + \|\tilde{v}\|}{2} + \frac{\|v\| + \|\tilde{v}\|}{2} \\
&\leq 2\|v\|
\end{aligned}$$

so the condition of the norm is satisfied as well. □

Proof of Proposition 3.4.1. We will show that $\tilde{\phi}$ is well defined, that is, $\tilde{\phi}(v) \in B(H_d)$ for any $v \in A(G_d)$.

By Lemma 3.4.2, if $v \in A(G_d)$ there is a net $\{u_j\}_j \in A(G)$, $\|u_j\| \leq 2\|v\|$ such that $u_j \xrightarrow{w^*} v$, where $w^* = \sigma(B(G_d), C^*(G_d))$. In particular, the net $\{u_j\}_j$ converges pointwise to v and therefore we have

$$\phi(u_j)(h) = u_j(\alpha(h)) \longrightarrow v(\alpha(h)) = \tilde{\phi}(v)(h) \text{ for any } v \in A(G_d), h \in Y$$

If $h \notin Y$, then $\phi(u_j)(h) = 0 = \tilde{\phi}(v)(h)$, hence $\tilde{\phi}(v)$ is the pointwise limit of $\phi(u_j) \in B(H)$. In conclusion, $\tilde{\phi}(v)$ is the pointwise limit of a bounded net in $B(H)$, so by Corollary 2.25 of [9] it follows that it belongs to $B(H_d)$. It can easily be checked that $\tilde{\phi}$ is an algebra homomorphism. □

Remark 3.4.3 We should note that $\tilde{\phi}(v) = \text{pointwise} - \lim_j \phi(u_j)(v)$ is equivalent to $\tilde{\phi}(v) = w^* - \lim_j \phi(u_j)(v)$ since the groups are discrete and the net is bounded.

In the next theorem we shall present the equivalence mentioned at the beginning of the section.

Theorem 3.4.4 *Let G, H be locally compact groups such that G_d is amenable. Let $\phi : A(G) \longrightarrow B(H)$ be a completely bounded algebra homomorphism, $\alpha : Y \subset H \longrightarrow G$ the continuous map corresponding to ϕ as above, and $\tilde{\phi} : A(G_d) \longrightarrow B(H_d)$ the algebra homomorphism constructed using α. Then the following are equivalent:*

1) α is piecewise affine

2) $\tilde{\phi}$ is a completely bounded algebra homomorphism

Proof "1) $\Rightarrow$ 2)" We will prove this in three steps:

i) α is a group homomorphism

ii) α is an affine map

iii) α is a piecewise affine map.

The first step is given by the next lemma

Lemma 3.4.5 *Let G, H be locally compact groups and $\psi : A(G) \longrightarrow B(H)$ an algebra homomorphism defined by*

$$\psi(u) = u \circ \alpha$$

where $\alpha : H \longrightarrow G$ a group homomorphism. Then ψ is completely bounded.

Proof Let $\omega_H : H \longrightarrow W^*(H)$ be the universal representation of H and $\lambda_G : G \longrightarrow VN(G)$ the left regular representation of G. Then

$$\lambda_G \circ \alpha : H \longrightarrow VN(G)$$

is another representation of H. By Lemma 3.3.2, there is a w^*-continuous *-homomorphism

$$\theta : W^*(H) \longrightarrow (\lambda_G \circ \alpha)(H)'' = VN(G)$$

such that

$$\lambda_G \circ \alpha = \theta \circ \omega_H. \tag{3.4}$$

Let $\psi^* : W^*(H) \longrightarrow VN(G)$ be the dual map of ψ. Our claim is that $\theta = \psi^*$. Indeed, let $F = \omega_H(h) \in W^*(H)$. Then, for $u \in A(G)$, we have on the one hand

$$\begin{aligned} < \theta(F), u > &= < \theta \circ \omega_H(h), u > \overset{(3.4)}{=} < \lambda_G \circ \alpha(h), u > \\ &= < \lambda_G(\alpha(h)), u > = u(\alpha(h)), \end{aligned} \tag{3.5}$$

and on the other hand

$$\begin{aligned} < \psi^*(F), u > &= < F, \psi(u) > = < \omega_H(h), \psi(u) > \\ &= \psi(u)(h) = u(\alpha(h)) \end{aligned} \tag{3.6}$$

Therefore we have equality

$$\theta(F) = \psi^*(F) \quad \text{for all} \quad F \in \{\omega_H(h) : h \in H\}.$$

Now, since θ and ψ^* are w^*-continuous and equal on a w^*-dense subset of $W^*(H)$, it follows that they are equal everywhere and the claim is proved.

The map θ is a *-homomorphism between C^*-algebras, so it is a completely bounded map. It follows that ψ^* is completely bounded, which is equivalent to saying that ψ completely bounded, and the lemma is proved. □

Using the previous lemma, we will prove the next step.

Lemma 3.4.6 *Let G, H be locally compact groups and $\psi : A(G) \longrightarrow B(H)$ an algebra homomorphism given by*

$$\psi(u) = \begin{cases} u \circ \alpha & \text{on } Y \\ 0 & \text{off } Y \end{cases}$$

where $\alpha : Y \subset H \longrightarrow G$ is an affine map defined on an open coset Y in H. Then ψ is a completely bounded map.

Proof Let $Y = \gamma H_1$, where $\gamma \in H$ and H_1 is an open subgroup of H. Since α is affine, by Remark 3.2.2, there is a group homomorphism

$$\beta : H_1 = \gamma^{-1}Y \longrightarrow G$$

given by the formula

$$\beta(h_1) = \alpha(\gamma)^{-1}\alpha(\gamma h_1). \tag{3.7}$$

Let $\psi_1 : A(G) \longrightarrow B(H_1)$ be the algebra homomorphism defined by β

$$\psi_1(u) = u \circ \beta.$$

By Lemma 3.4.5, this is a completely bounded map. Given $v \in B(H_1)$, denote with $\tilde{v}$ the function on H that is equal to v on H_1 and is zero elsewhere. The map $\sigma : B(H_1) \longrightarrow B(H)$ that sends v into $\tilde{v}$ is a complete isometry ([46]). Then, the map

$$\psi_2 = \sigma \circ \psi_1 : A(G) \longrightarrow B(H)$$

is a completely bounded map, given by

$$\psi_2(u) = \begin{cases} u \circ \beta & \text{on } H_1 \\ 0 & \text{off } H_1 \end{cases}$$

Our claim is that $\psi = L_{\gamma^{-1}} \circ \psi_2 \circ L_{\alpha(\gamma)}$, where $L_{\alpha(\gamma)} : A(G) \longrightarrow A(G)$ is the translation on the left by $\alpha(\gamma)$ on $A(G)$, and $L_{\gamma^{-1}} : B(H) \longrightarrow B(H)$ is the left translation by γ^{-1} on $B(H)$. Indeed, let $y \in Y = \gamma H_1$ and $u \in A(G)$. Then

$$\begin{aligned}(L_{\gamma^{-1}} \circ \psi_2 \circ L_{\alpha(\gamma)})(u)(y) &= L_{\gamma^{-1}}(\psi_2(L_{\alpha(\gamma)}(u)))(y) = \psi_2(L_{\alpha(\gamma)}(u))(\gamma^{-1}y) \\ &= L_{\alpha(\gamma)}(u))(\beta(\gamma^{-1}y)) \quad (\text{since } \gamma^{-1}y \in H_1) \\ &= u(\alpha(\gamma)\beta(\gamma^{-1}y)) \overset{(3.7)}{=} u(\alpha(\gamma)\alpha(\gamma)^{-1}\alpha(y)) \\ &= u(\alpha(y)) = u \circ \alpha(y) = \psi(y). \end{aligned} \tag{3.8}$$

If $y \notin H_1$ the same calculation as above shows that $(L_{\gamma^{-1}} \circ \psi_2 \circ L_{\alpha(\gamma)})(u)(y) = 0 = \psi(y)$. Therefore the claim is proved.

As shown by P.J. Wood in ([46]), the translation maps $L_{\gamma^{-1}}$ and $L_{\alpha(\gamma)}$ are completely bounded. Hence, taking relation (3.8) into account, we obtain that ψ is completely bounded. □

The next lemma gives us the last step of the proof.

Lemma 3.4.7 *Let G, H be locally compact groups and $\psi : A(G) \longrightarrow B(H)$ an algebra homomorphism given by*

$$\psi(u) = \begin{cases} u \circ \alpha & \text{on } Y \\ 0 & \text{off } Y \end{cases}$$

where $\alpha : Y \subset H \longrightarrow G$ is a piecewise affine map. Then ψ is a completely bounded map.

Proof By the definition of a piecewise affine map, there are pairwise disjoint sets $Y_i \in \Omega(H)$, $i = 1, ..., n$ such that $Y = \cup_i^n Y_i$ and for each i there is an open coset K_i, that contains Y_i, and an affine map $\alpha_i : K_i \longrightarrow G$ such that $\alpha_i | Y_i = \alpha$.

For each i, let $\psi_i : A(G) \longrightarrow B(H)$ be the algebra homomorphism given by

$$\psi_i(u) = \begin{cases} u \circ \alpha_i & \text{on } K_i \\ 0 & \text{off } K_i \end{cases}$$

By the previous lemma, this is a completely bounded map.

For any $u \in A(G)$, $\psi(u)$ can be written as follows

$$\psi(u) = \sum_1^n 1_{Y_i} \psi_i(u) \tag{3.9}$$

where 1_{Y_i} is the characteristic function of Y_i, which by ([24]) belongs to $B(H)$.

Let $\omega_i : B(H) \longrightarrow B(H)$ be the map given by $\omega_i(v) = 1_{Y_i} v$. This map is a completely bounded map, since $B(H)$ is a completely contractive Banach algebra. Now relation (3.9) can be rewritten as

$$\psi = \sum_1^n \omega_i \circ \psi_i$$

therefore ψ, as a sum of completely bounded maps, is completely bounded.

□

The implication "1) $\Rightarrow$ 2)" now follows from Lemma 3.4.7.

"2) $\Rightarrow$ 1)" Suppose that $\tilde{\phi}$ is completely bounded. We are then in the hypothesis of Theorem 3.3.4 since G_d is amenable. Therefore, we obtain that $\alpha : Y \subset H_d \longrightarrow G_d$ is piecewise affine, seen as a map on the discrete groups. The only thing left to show is that this remains true if we delete the "d" subscript. This follows from an argument in [6] or [37], that can be adapted to the non-abelian case without many modifications, since it is mostly topological and does not depend on the abelian structure of the group. This completes the proof of Theorem 3.4.4. □

The question that arises now is under what conditions is $\tilde{\phi}$ completely bounded. In the remainder of this section we give a sufficient condition for the completely bounded norm of $\tilde{\phi}$ to be finite.

Definition 3.4.8 *We say that a locally compact group G has property (S) if*

(S) *for any $\mathcal{V} \in T_n(A(G_d))$ there is a net $\{\mathcal{U}\}_t \in T_n(A(G))$ such that*

$$\mathcal{U}_t \xrightarrow{\sigma_n} \mathcal{V} \quad \text{and} \quad \|\mathcal{U}_t\| \leq 2\|\mathcal{V}\| \tag{3.10}$$

where $\sigma_n = \sigma(T_n(B(G_d)), M_n(C^*(G_d)))$, $n \in \mathbf{N}$.

We note that for $n = 1$, relation (3.10) is satisfied for any locally compact group, by Lemma 3.4.2. We will need the following remarks to calculate the completely bounded norm of $\tilde{\phi}$.

Remark 3.4.9 Given an operator space V, we have the following isometric identification

$$T_n(V^*) \simeq (M_n(V))^*$$

given by

$$\mathcal{F} \mapsto <\mathcal{F}, \mathcal{V}> = \sum_{i,j=1}^{n} f_{ij}(v_{ij})$$

where $\mathcal{F} = [f_{ij}] \in T_n(V^*), \mathcal{V} = [v_{ij}] \in M_n(V)$. For $V = C^*(G)$, we thus have

$$T_n(B(G)) \simeq (M_n(C^*(G)))^*.$$

Remark 3.4.10 By the previous remark, the norm of $\mathcal{W} \in T_n(B(G))$ is given by

$$\|\mathcal{W}\|_{T_n(B(G))} = \sup\{| < \mathcal{W},\mathcal{U} > | \ : \ \mathcal{U} = [u_{ij}] \in M_n(C^*(G)), \|\mathcal{U}\|_n \leq 1\}. \tag{3.11}$$

The next result shows that condition (S) is a sufficient condition for $\tilde{\phi}$ to be completely bounded.

Theorem 3.4.11 *Let G, H be locally compact groups. Let $\phi : A(G) \longrightarrow B(H)$ be a completely bounded algebra homomorphism, $\alpha : Y \subset H \longrightarrow G$ the continuous map corresponding to ϕ as above, and $\tilde{\phi} : A(G_d) \longrightarrow B(H_d)$ the algebra homomorphism constructed using α. If G satisfies condition (S) then $\tilde{\phi}$ is a completely bounded algebra homomorphism.*

Proof For any $n \in \mathbf{N}$ we define the map

$$T_n(\tilde{\phi}) : T_n(A(G_d)) \longrightarrow T_n(B(H_d)), \text{ given by } T_n(\tilde{\phi})([v_{ij}]) = [\tilde{\phi}(v_{ij})].$$

The completely bounded norm of $\tilde{\phi}$ can be calculated by means of the formula

$$\|\tilde{\phi}\|_{cb} = \sup\{\|T_n(\tilde{\phi})\| : n \in \mathbf{N}\}$$

Next, we will compute the norm of the map $T_n(\tilde{\phi})$. We have

$$\begin{aligned} \|T_n(\tilde{\phi})\| &= \sup_{\|\mathcal{V}\|\leq 1} \{\|T_n(\tilde{\phi})(\mathcal{V})\|_{T_n(B(H_d))} : \mathcal{V} \in T_n(A(G_d))\} \\ &\overset{(3.11)}{=} \sup_{\|\mathcal{V}\|\leq 1} \sup_{\|\mathcal{F}\|\leq 1} \{| < T_n(\tilde{\phi})(\mathcal{V}), \mathcal{F} > | : \mathcal{F} \in M_n(C^*(H_d))\} \quad (3.12) \end{aligned}$$

By property (S), for $\mathcal{V} = [v_{ij}] \in T_n(A(G_d))$ there is a net $\{\mathcal{U}_t = [u^t_{ij}]\}_t \in T_n(A(G))$ such that

$$\mathcal{U}_t \xrightarrow{\sigma_n} \mathcal{V} \quad \text{and} \quad \|\mathcal{U}_t\| \leq 2\|\mathcal{V}\|$$

where $\sigma_n = \sigma(T_n(B(G_d)), M_n(C^*(G_d)))$, $n \in \mathbf{N}$. Then we have

$$u_{ij}^t \xrightarrow{w^*} v_{ij} \text{ for any } i, j = 1, n.$$

By Remark 3.4.3, it follows that $\phi(u_{ij}^t) \xrightarrow{w^*} \tilde{\phi}(v_{ij})$, which in turn implies that

$$T_n(\phi)(\mathcal{U}_t) \xrightarrow{w_n^*} T_n(\tilde{\phi})(\mathcal{V})$$

where $w_n^* = \sigma(T_n(B(H_d)), M_n(C^*(H_d)))$. The last relation means nothing other than

$$< T_n(\tilde{\phi})(\mathcal{V}), \mathcal{F} >= \lim_t < T_n(\phi)(\mathcal{U}_t), \mathcal{F} > \tag{3.13}$$

for any $\mathcal{F} \in M_n(C^*(H_d))$.

We return now to the computation of the norm of $T_n(\tilde{\phi})$. Using (3.13), relation (3.12) becomes

$$||T_n(\tilde{\phi})|| = \sup_{||\mathcal{V}|| \leq 1} \sup_{||\mathcal{F}|| \leq 1} \lim_t \{ | < T_n(\phi)(\mathcal{U}_t), \mathcal{F} > | : \mathcal{F} \in M_n(C^*(H_d)) \}$$

Because ϕ is completely bounded, we have

$$\begin{aligned} | < T_n(\phi)(\mathcal{U}_t), \mathcal{F} > | &\leq ||T_n(\phi)(\mathcal{U}_t)||_{T_n(B(H_d))} = ||T_n(\phi)(\mathcal{U}_t)||_{T_n(B(H))} \\ &\leq ||T_n(\phi)|| ||\mathcal{U}_t|| \leq ||\phi||_{cb} ||\mathcal{U}_t|| \\ &\leq 2||\phi||_{cb} ||\mathcal{V}|| \leq 2||\phi||_{cb} \end{aligned}$$

Therefore

$$||T_n(\tilde{\phi})|| \leq 2||\phi||_{cb} \quad \Rightarrow ||\tilde{\phi}||_{cb} \leq 2||\phi||_{cb} < \infty$$

which shows that $\tilde{\phi}$ is completely bounded. □

In conclusion we have proved the following theorem

Theorem 3.4.12 *Let G, H be locally compact groups and $\phi : A(G) \longrightarrow B(H)$ a completely bounded algebra homomorphism. If G_d is amenable and G satisfies condition (S), then there is $Y \in \Omega(H)$ and a piecewise affine map $\alpha : Y \subset H \longrightarrow G$ such that*

$$\phi(u) = \begin{cases} u \circ \alpha & \text{on } Y \\ 0 & \text{off } Y \end{cases}$$

It remains to be seen what classes of groups satisfy relation (S).

Chapter 4

The range of Fourier algebra homomorphisms

4.1 Introduction

The result we obtained in Chapter 3 is instrumental for the study of the range of a completely bounded algebra homomorphism between the Fourier algebras of two locally compact groups. The range of such a homomorphism sits inside a well determined set. This particular situation is not exclusive to the Fourier algebras, but takes place in a more general setting. More precisely, if $\mathcal{A}$ and $\mathcal{B}$ are two commutative semisimple Banach algebras and $\phi : \mathcal{A} \longrightarrow \mathcal{B}$ is an algebra homomorphism then

$$\phi(\mathcal{A}) \subseteq \left\{ b \in \mathcal{B} : \begin{array}{l} \phi^*(\varphi) = 0 \Rightarrow \hat{b}(\varphi) = 0 \\ \phi^*(\varphi_1) = \phi^*(\varphi_2) \Rightarrow \hat{b}(\varphi_1) = \hat{b}(\varphi_2) \quad \varphi, \varphi_1, \varphi_2 \in \Sigma_{\mathcal{B}} \end{array} \right\}$$

where $\hat{b}$ is the Gelfand transform of b, $\Sigma_{\mathcal{B}}$ is the spectrum of $\mathcal{B}$.

We prove that when $\mathcal{A}$ and $\mathcal{B}$ are the Fourier algebras of two discrete groups G and H respectively, with G amenable, then the range is as large as it can possibly be, in other words we have equality above. Also, we show that the same description of the range can be achieved, if ϕ is a bounded algebra homomorphism between two Fourier algebras, $A(G)$ and $B(H)$, when G, H are locally compact groups with G abelian. The statement is similar to the one obtained by A. Kepert ([25]) in the case when $\mathcal{A} = L^1(G)$, $\mathcal{B} = L^1(H)$ with G and H two locally compact abelian groups.

The idea of the proof is close in spirit to the approach of Kepert. The difficulties appear due to the fact that we are not dealing with abelian groups anymore. The proof relies on the description of the completely bounded algebra homomorphisms from $A(G)$ to $B(H)$ in terms of piecewise affine maps.

Section 2 contains preliminary results concerning piecewise affine maps which will lead to the results in the next section. In Section 3 we present a reduction of the problem, which comes down to the fact that Theorem 4.3.1 is equivalent to Theorem 4.3.2. In Section 4 we proceed to prove Theorem 4.3.2 in three stages.

4.2 On piecewise affine maps

We present here some technical results which will allow us to get information about the affine pieces of a piecewise affine map. The results are the non-abelian analog of those found in [25].

Definition 4.2.1 *If E is a left coset of a subgroup G_1 in G, we define the index of E to be the index of G_1 in G. If E_1, E_2 are cosets in G of the subgroups G_1 and G_2 respectively, then the index of E_1 in E_2 is defined to be the index of $G_1 \cap G_2$ in G_2.*

When G is abelian any set in $\Omega(G)$ is a finite disjoint union of sets in

$$\Omega_0(G) = \left\{ E_0 \setminus (\bigcup_1^m E_k) : \begin{array}{l} E_0 \subset G \text{ open coset} \\ E_1, ... E_m \text{ open subcosets of infinite index in } E_0 \end{array} \right\}$$

(see [6]). It is not a hard task to verify that this is valid for all locally compact groups. The following algebraic result assures us that $\emptyset \notin \Omega_0(G)$.

Lemma 4.2.2 ([31]) *A group is not a finite union of cosets of infinite index.*

Remark 4.2.3 If $U = E_0 \setminus (\bigcup_1^m E_k) \in \Omega_0(G)$ then $Aff(U)$, the coset generated by U, is exactly E_0.

Indeed, for every $y \in E_0^{-1}E_0$, we have

$$\begin{aligned} U \cap Uy &= (E_0 \setminus \bigcup_1^m E_k) \bigcap (E_0 y \setminus \bigcup_1^m E_k y) \\ &= (E_0 \setminus \bigcup_1^m E_k) \bigcap (E_0 \setminus \bigcup_1^m E_k y) \\ &= E_0 \setminus (\bigcup_1^m E_k \bigcup \bigcup_1^m E_k y). \end{aligned}$$

By Remark 3.2.5 *(i)* we have

$$E_k y = (xG_k)y = xyG_0 \text{ where } G_0 = y^{-1}G_k y \subseteq E_0^{-1}E_0$$

so it is a subcoset of E_0. Since G_k has infinite index in $E_0^{-1}E_0$, so does G_0. Therefore $E_k y$ is an open subcoset of infinite index in E_0. So, $U \cap Uy \in \Omega_0(G)$. Since $\emptyset \notin \Omega_0(G)$, it follows that $U \cap Uy \neq \emptyset$. Hence, if E is any coset containing U, then

$$\emptyset \neq U \cap Uy \subseteq E \cap Ey$$

and therefore $y \in E^{-1}E$. That is, $E_0^{-1}E_0 \subseteq E^{-1}E$ which is equivalent to $E_0 \subseteq E$. Thus $E_0 = Aff(U)$.

The next proposition is obtained by combining the above with the definition of piecewise affine maps. From now on this is the way that we will look at piecewise affine maps.

Proposition 4.2.4 *Let $Y \in \Omega(H)$. Then $\psi : Y \longrightarrow G$ is piecewise affine if and only if $Y = \bigcup_{i=1}^n Y_i$, with $Y_1, ..., Y_n \in \Omega_0(H)$ and $\psi|_{Y_i}$ has a continuous affine extension $\psi_i : Aff(Y_i) \longrightarrow G$, for each i.*

We now present a few technical results.

Lemma 4.2.5 *Let G be a locally compact group. Then*

(i) for $Y \in \Omega_0(G)$ and $E_0 = Aff(Y)$, there is a finite subset $F \subseteq E_0^{-1}E_0$ such that $E_0 = YF$.

(ii) for $Y \in \Omega(G)$ and Λ a compact subgroup of G, there is a finite set $F \subseteq \Lambda$ such that $Y\Lambda = YF$.

Proof *(i)* We will use induction by m to prove the statement for $Y = E_0 \setminus (\bigcup_1^m E_k)$. If $m = 0$, then $Y = E_0$ and we take $F = \{e\}$. Suppose the statement is true for $m-1$. Let $Y_1 = E_0 \setminus (\bigcup_1^{m-1} E_k)$. Then there is a finite set $F_1 \subseteq E_0^{-1}E_0$ such that $E_0 = Y_1F_1$. Now we look at the set $F_1^{-1}E_m^{-1}E_mF_1$. This can be written as follows :

$$\begin{aligned} F_1^{-1}E_m^{-1}E_mF_1 &= \bigcup_{z\in F_1^{-1}, y\in F_1} z(E_m^{-1}E_m)y = \bigcup_{z\in F_1^{-1}, y\in F_1} (zE_m^{-1}E_mz^{-1}z)y \\ &= \bigcup_{z\in F_1^{-1}, t\in F_1^{-1}F_1} G_zt \end{aligned}$$

where $G_z = zE_m^{-1}E_mz^{-1} \subseteq E_0^{-1}E_0$, which is of infinite index in $E_0^{-1}E_0$, since $E_m^{-1}E_m$ is. This means that $F_1^{-1}E_m^{-1}E_mF_1$ is finite union of cosets of infinite index in $E_0^{-1}E_0$. Since a group cannot be written as a finite union of cosets of infinite index, it follows that there is $\gamma \in E_0^{-1}E_0 \setminus (F_1^{-1}E_m^{-1}E_mF_1)$. Therefore, there is $\gamma \in E_0^{-1}E_0$ such that

$$E_mF_1 \cap E_mF_1\gamma = \emptyset. \tag{4.1}$$

Now put $F = F_1 \cup F_1\gamma$. Since $Y_1 \subseteq Y \cup E_m$, we have

$$E_0 = Y_1F_1 \subseteq (Y \cup E_m)F_1 = YF_1 \cup E_mF_1 \subseteq E_0.$$

Therefore equality holds :

$$E_0 = YF_1 \cup E_mF_1. \tag{4.2}$$

Next, note that $E_0 \subseteq E_0\gamma$, since $\gamma \in E_0^{-1}E_0$ and so we have

$$E_0 \subseteq E_0\gamma = YF_1\gamma \cup E_mF_1\gamma \subseteq E_0.$$

Therefore equality holds :

$$E_0 = YF_1\gamma \cup E_mF_1\gamma. \tag{4.3}$$

Combining (4.1),(4.2) and (4.3) we get

$$E_0 = YF_1 \cup YF_1\gamma = Y(F_1 \cup F_1\gamma) = YF.$$

and we are done.

(ii) This follows from *(i)*, as in [25].

□

The next proposition is the non-abelian analogue of Proposition 2.4 in [25]. The proof in [25] can be adapted without any modifications (using now Proposition 4.2.5*(i)*). We include the proof for completeness.

Proposition 4.2.6 *Let G, H be locally compact groups, $Y \in \Omega_0(H)$ and ψ : $Aff(Y) \longrightarrow G$ an affine map such that $\psi|_Y$ is proper. Then ψ is proper.*

Proof It suffices to prove the proposition in the case that $Aff(Y)$ is a subgroup and ψ is a homomorphism. The general case follows by translation. Under these assumptions, by Proposition 4.2.5 (i), there is a finite set $F \subseteq Aff(Y)$ such that $YF = Aff(Y)$. We have the following:

$$\begin{aligned}
\psi^{-1}(C) &= \psi^{-1}(C) \cap Aff(Y) = \psi^{-1}(C) \cap (\bigcup_{y \in F} Y\gamma) \\
&= \bigcup_{\gamma \in F} (\psi^{-1}(C) \cap Y\gamma) = \bigcup_{\gamma \in F} [\psi^{-1}(C)\gamma^{-1}\gamma \cap Y\gamma)] \\
&= \bigcup_{\gamma \in F} [\psi^{-1}(C\psi(\gamma^{-1})) \cap Y]\gamma = \bigcup_{\gamma \in F} [(\psi|_Y)^{-1}(C\psi(\gamma^{-1}))]\gamma
\end{aligned}$$

where we used the fact that $(\psi|_Y)^{-1}(\cdot) = \psi^{-1}(\cdot) \cap Y$. Now, since $\psi|_Y$ is proper, it follows that $\psi^{-1}(C)$ is compact. □

We end this section with the following corollary.

Corollary 4.2.7 *Suppose G, H are locally compact groups. For $Y \in \Omega_0(H)$ let $\psi : Y \longrightarrow G$ be a proper map with affine extension $\psi' : Aff(Y) \longrightarrow G$. Then $E_1 := \psi'(Aff(Y))$ is a closed coset in G and $\psi(Y) \in \Omega(E_1)$.*

Proof Let $E_0 = Aff(Y)$, $E_1 = \psi'(E_0)$. By Proposition 4.2.6 ψ' is proper. Thus $\psi'(E_0) = E_1$ is a closed coset in G. By Remark 3.2.2, since ψ' is affine, for any $\gamma_0 \in E_0$, the map

$$\beta : \gamma_0^{-1}E_0 \longrightarrow G, \quad \beta(h) = \psi'(\gamma_0)^{-1}\psi'(\gamma_0 h)$$

is a group homomorphism.

Let $\Lambda = \text{Ker}\ \beta = \gamma_0^{-1}\psi'^{-1}\{\psi'(\gamma_0)\}$. Then Λ is a compact normal subgroup of $H_0 := \gamma_0^{-1}E_0$. Let $Q_\Lambda : H_0 \longrightarrow H_0/\Lambda$ be the quotient map.

We have to show that $\psi(Y) \in \Omega(E_1)$. From the definition of $\Omega(E_1)$, this is equivalent to

$$\gamma^{-1}\psi(Y) \in \Omega(E_1^{-1}E_1) \ \ \forall \gamma \in E_1 \Leftrightarrow (\psi'(\gamma_0))^{-1}\psi'(Y) \in \Omega(E_1^{-1}E_1) \quad \forall \gamma_0 \in E_0.$$

We can rewrite the above using β :

$$\psi'(\gamma_0)^{-1}\psi'(Y) = \beta(\gamma_0^{-1}Y)$$

$$E_1^{-1}E_1 = \beta(H_0)^{-1}\beta(H_0) = \beta(H_0)$$

where the latter is obtained from the equality

$$\beta(H_0) = \psi'(\gamma_0)^{-1}\psi'(E_0) = \psi'(\gamma_0)^{-1}E_1.$$

We can now write

$$\psi(Y) \in \Omega(E_1) \Leftrightarrow \beta(\gamma_0^{-1}Y) \in \Omega(\beta(H_0)).$$

Once we show that $\beta(\gamma_0^{-1}Y) \in \Omega(\beta(H_0))$, we are done. By Lemma 4.2.5 *(ii)*, there is $F \subset H_0$ such that $Y\Lambda = YF$ which is equivalent to $\gamma_0^{-1}Y\Lambda = \gamma_0^{-1}YF$.

First notice that $\gamma_0^{-1}YF \in \Omega(H_0)$. Indeed, since $Y \in \Omega(E_0^{-1}E_0)$ so does $\gamma_0^{-1}Y$. By Remark 3.2.5, by multiplying on the right with elements of H_0 we stay in $\Omega(H_0)$. Since F is finite we get that $\gamma_0^{-1}YF \in \Omega(H_0)$.

It is not hard to see that $\beta \circ Q_\Lambda^{-1} : H_0/\Lambda \longrightarrow \beta(H_0)$ is a continuous bijective homeomorphism (for details see Lemma 4.4.1). Denote by M the following set

$$M = Q_\Lambda(\gamma_0^{-1}Y) = Q_\Lambda(\gamma_0^{-1}Y\Lambda).$$

Since by above $\gamma_0^{-1}Y\Lambda = \gamma_0^{-1}YF$ and the latter is in $\Omega(H_0)$, we obtain that $\gamma_0^{-1}Y\Lambda \in \Omega(H_0)$. Hence $M \in \Omega(H_0/\Lambda)$. But then

$$\beta(\gamma_0^{-1}Y) = \beta \circ Q_\Lambda^{-1} \circ Q_\Lambda(\gamma_0^{-1}Y) = \beta \circ Q_\Lambda^{-1}(M) \in \Omega(\beta(H_0)).$$

□

Define $\Omega_d(H) = \Omega(H_d)$, where H_d is H with the discrete topology, and $\Omega_c(H) = \{X \subset H : X \text{ closed and } X \in \Omega_d(H)\}$. The next corollary now follows easily.

Corollary 4.2.8 *Let G, H be locally compact groups.*

If $Y \in \Omega(H)$ and $\psi : Y \longrightarrow G$ is a proper, piecewise affine map, then $\psi(Y) \in \Omega_c(G)$.

4.3 An equivalent theorem

In this section we study the range of completely bounded algebra homomorphisms between the Fourier algebra of two groups. We prove the following result

Theorem 4.3.1 *Let G, H be two discrete groups with G amenable. If $\phi : A(G) \longrightarrow A(H)$ is a completely bounded algebra homomorphism, then*

$$\phi(A(G)) = \left\{ f \in A(H) : \begin{array}{l} \phi^*(h_1) = 0 \Rightarrow f(h_1) = 0 \\ \phi^*(h_1) = \phi^*(h_2) \Rightarrow f(h_1) = f(h_2) \quad h_1, h_2 \in H \end{array} \right\}.$$

Let us denote the right hand set by U_ϕ. Theorem 3.3.4 allows us to reformulate the above equality in terms of extensions of functions in $A(G)$.

Indeed, given a map $\alpha : Y \subset H \longrightarrow G$, let $k(\alpha)$ be the set

$$k(\alpha) = \left\{ g \in A(H) : \begin{array}{cc} g = 0 & \text{off} \quad Y \\ g(y_1) = g(y_2) & \text{whenever} \quad \alpha(y_1) = \alpha(y_2) \end{array} \right\}.$$

Note that for $g \in k(\alpha)$, the second condition implies that $g \circ \alpha^{-1}$ is well defined. If α is the map from Theorem 3.3.4, it is not hard to see that

$$\phi(A(G)) \subset U_\phi \subset k(\alpha).$$

Therefore, in order to prove Theorem 4.3.1, that is, to show the equality $\phi(A(G)) = U_\phi$, it suffices to prove

$$k(\alpha) \subset \phi(A(G)).$$

Notice that the latter is now equivalent with the following : for any $g \in k(\alpha)$, $g \circ \alpha^{-1}$ has an extension in $A(G)$.

Indeed, if $g \in k(\alpha) \subset \phi(A(G))$, then there is $f \in A(G)$ such that

$$g = \phi f = \begin{cases} f \circ \alpha & \text{on } Y \\ 0 & \text{off } Y \end{cases}$$

Therefore, $g \circ \alpha^{-1} = f|_{\alpha(Y)}$ so $g \circ \alpha^{-1}$ has an extension in $A(G)$. The other direction follows similarly.

We should also remember that the map α corresponding to $\phi : A(G) \longrightarrow A(H)$ is proper (see Proposition 3.3.7).

In conclusion, we see that Theorem 4.3.1 is in fact equivalent to the following theorem

Theorem 4.3.2 *Let G, H be discrete groups with G amenable and suppose that $\alpha : Y \subset H \longrightarrow G$ is a proper piecewise affine map. Then for any $g \in k(\alpha)$, $g \circ \alpha^{-1}$ has an extension in $A(G)$.*

For the sake of generality we will prove Theorem 4.3.2′ below, which is stated with a more general hypothesis than Theorem 4.3.2.

Theorem 4.3.2′ *Let G, H be locally compact groups with G discrete and amenable and suppose that $\alpha : Y \subset H \longrightarrow G$ is a proper piecewise affine map. Then for any $g \in k(\alpha)$, $g \circ \alpha^{-1}$ has an extension in $A(G)$.*

4.4 The range of the Fourier algebra homomorphisms

This section is dedicated to the proof of Theorem 4.3.2′. The proof consists of the following three steps:

(i) Y an open coset and α an affine map

(ii) $Y \in \Omega_0(H)$

(iii) $Y \in \Omega(H)$ and α a piecewise affine map

We start with the first case, Y an open coset and α an affine map, which is a key step of the proof.

Lemma 4.4.1 *Let G, H be locally compact groups with G discrete. Let Y be an open coset in H and $\alpha : Y \longrightarrow G$ a proper affine map. Then, for any $g \in k(\alpha)$, the map $g \circ \alpha^{-1} : \alpha(Y) \longrightarrow \mathbf{C}$ has an extension in $A(G)$.*

Proof Let $\gamma_0 \in Y$. Then $\gamma_0^{-1}Y$ is an open subgroup of H, and will be denoted by H_0. Consider the map $\beta : \gamma_0^{-1}Y \longrightarrow G$, given by

$$\beta(h) = \alpha(\gamma_0)^{-1}\alpha(\gamma_0 h).$$

This is a proper group homomorphism. Also $\beta(H_0)$ is a closed and open subgroup of G.

Let $g \in k(\alpha)$. Consider ${}_{\gamma_0}g(x) = g(\gamma_0 x)$ the left translation of g by γ_0. Then g has the following properties:

1) ${}_{\gamma_0}g \in A(H)$

2) ${}_{\gamma_0}g = 0$ off $\gamma_0^{-1}Y = H_0$

3) ${}_{\gamma_0}g \circ \beta^{-1}$ is well defined.

The first two are clear. To see the last one, suppose $\beta(h_1) = \beta(h_2)$. Then $\alpha(\gamma_0 h_1) = \alpha(\gamma_0 h_2)$ which implies that $g(\gamma_0 h_1) = g(\gamma_0 h_2)$, since $g \in k(\alpha)$, and we are done.

To end the proof it suffices to show that ${}_{\gamma_0}g \circ \beta^{-1}$ has an extension in $A(G)$. Indeed, if this happens, then there is $f \in A(G)$ such that

$${}_{\gamma_0}g \circ \beta^{-1} = f|_{\beta(H_0)}.$$

Then the map ${}_{\alpha(\gamma_0)^{-1}}f \in A(G)$ will be the extension needed for $g \circ \alpha^{-1}$. To see this, let $x = \alpha(y) \in \alpha(Y)$. Then we have

$$\begin{aligned}
{}_{\alpha(\gamma_0)^{-1}}f(x) &= f(\alpha(\gamma_0)^{-1}x) = f(\beta(\gamma_0^{-1}y)) \\
&= {}_{\gamma_0}g \circ \beta^{-1}(\beta(\gamma_0^{-1}y)) = {}_{\gamma_0} g(\gamma_0^{-1}y) \\
&= g(y) = g \circ \alpha^{-1}(x).
\end{aligned}$$

We obtain $g \circ \alpha^{-1} = {}_{\alpha(\gamma_0)^{-1}} f|_{\alpha(Y)}$.

Next we will show that ${}_{\gamma_0}g \circ \beta^{-1}$ has an extension in $A(G)$.

Let $\Lambda = \mathrm{Ker}\beta$. It is not hard to check that $\Lambda = \gamma_0^{-1}\alpha^{-1}\{\alpha(\gamma_0)\}$. It follows that Λ is a compact normal subgroup of H_0. Because of the definition of Λ and the fact that $g \in k(\alpha)$, it follows that g is constant on the cosets of Λ. Then ${}_{\gamma_0}g$ is constant on the cosets of Λ as well. Note that ${}_{\gamma_0}g \in A(H_0)$ because of *2)*. Then, by a result due to Eymard ([9], Theorem 3.25) there is $\tilde{g} \in A(H_0/\Lambda)$ such that

$${}_{\gamma_0}g = \tilde{g} \circ Q_\Lambda$$

where $Q_\Lambda : H_0 \longrightarrow H_0/\Lambda$ is the canonical map.

Our next claim is that $\beta \circ Q_\Lambda^{-1} : H_0/\Lambda \longrightarrow \beta(H_0)$ is a well defined bijection. Indeed, suppose that $Q_\Lambda(h_1) = Q_\Lambda(h_2)$ where $h_i = \gamma_0^{-1} y_i$, $y_i \in Y$. Then

$$h_1^{-1}h_2 \in \Lambda \Leftrightarrow y_1^{-1}y_2 \in \Lambda.$$

Therefore, there is $x \in \alpha^{-1}\{\alpha(\gamma_0)\}$ such that $y_1^{-1}y_2 = \gamma_0^{-1}x$. Now we have

$$\beta(h_i) = \alpha(\gamma_0)^{-1}\alpha(\gamma_0 h_i) = \alpha(\gamma_0)^{-1}\alpha(y_i).$$

On the other hand, $\alpha(y_2) = \alpha(y_1\gamma_0^{-1}x) = \alpha(y_1)\alpha(\gamma_0)^{-1}\alpha(x) = \alpha(y_1)$. Therefore $\beta(h_1) = \beta(h_2)$. So $\beta \circ Q_\Lambda^{-1}$ is well defined. Similarly it can be shown that $\beta \circ Q_\Lambda^{-1}$ is bijective. Moreover, since Q_Λ is an open map and β is proper it follows that $\beta \circ Q_\Lambda^{-1}$ is a homeomorphism.

It follows that the map

$$\Phi : A(\beta(H_0)) \longrightarrow A(H_0/\Lambda)$$

$$f \to f \circ (\beta \circ Q_\Lambda^{-1})$$

is an isomorphism.

Therefore we have

$${}_{\gamma_0}g \circ \beta^{-1} = \tilde{g} \circ (\beta \circ Q_\Lambda^{-1})^{-1} \in A(\beta(H_0))$$

since $\tilde{g} \in A(H_0/\Lambda)$. Since $\beta(H_0)$ is an open subgroup of G, by Theorem 3.21 of [9], there is $F \in A(G)$ such that ${}_{\gamma_0}g \circ \beta^{-1} = F|_{\beta(H_0)}$, which concludes the proof.

□

Remark 4.4.2 We need the fact that G is discrete in order to apply Theorem 3.21 from [9]. If G is not discrete, the subgroup $\beta(H_0)$ is only a closed subgroup and the theorem cannot be applied. Nevertheless, if G is abelian we can apply Theorem 2.7.4 of [37], and then Lemma 4.4.1 holds true for this case as well. We will need this remark later in the proof of Theorem 4.4.7.

We will now make use of Lemma 4.4.1 together with the results proved in Section 4.2 on piecewise affine maps, to complete the second step of the proof of Theorem 3.2′.

Lemma 4.4.3 *Let G, H be locally compact groups with G discrete. Suppose $Y \in \Omega_0(H)$ and $\alpha : Y \longrightarrow G$ is a proper map that has an affine extension $\alpha_1 : Aff(Y) \longrightarrow G$. Then for any $g \in k(\alpha)$, $g \circ \alpha^{-1}$ has an extension in $A(G)$.*

Proof Recall

$$k(\alpha) = \left\{ g \in A(H) : \begin{array}{ccc} g = 0 & \text{off} & Y \\ g(y_1) = g(y_2) & \text{whenever} & \alpha(y_1) = \alpha(y_2) \end{array} \right\}.$$

Let $E = Aff(Y)$. By Proposition 4.2.6, α_1 is a proper map. As in Lemma 4.4.1 there is a compact normal subgroup $\Lambda \subset E^{-1}E$. Let $g \in k(\alpha)$. Define $\tilde{g} : H \longrightarrow \mathbf{C}$ by

$$\tilde{g}(h) = \begin{cases} g(h\lambda^{-1}) & \text{if} \quad h \in Y\Lambda \\ 0 & \text{if} \quad h \notin Y\Lambda \end{cases}$$

We claim that $\tilde{g}$ is well defined on $Y\Lambda$.

Indeed, let $h = y_1\lambda_1 = y_2\lambda_2, \quad y_i \in Y, \ \lambda_i \in \Lambda, \ i = 1, 2$. We will show that $\alpha(h\lambda_1^{-1}) = \alpha(h\lambda_2^{-1})$, from which it will follow that $g(h\lambda_1^{-1}) = g(h\lambda_2^{-1})$, proving the claim.

We have

$$y_1\lambda_1 = y_2\lambda_2 \Rightarrow y_2^{-1}y_1 = \lambda_2\lambda_1^{-1} \in \Lambda.$$

Since $\Lambda = \gamma_0^{-1}\alpha^{-1}\{\alpha(\gamma_0)\}$ for some $\gamma_0 \in E$, it follows that there is $x \in \alpha^{-1}\{\alpha(\gamma_0)\}$ such that

$$y_2^{-1}y_1 = \gamma_0^{-1}x \Rightarrow y_1 = y_2\gamma_0^{-1}x.$$

Then, since α is affine we get

$$\alpha(y_1) = \alpha(y_2\gamma_0^{-1}x) = \alpha(y_2)\alpha(\gamma_0)^{-1}\alpha(x) = \alpha(y_2).$$

and the claim is proved.

The following two properties of $\tilde{g}$ are straightforward and the third one can be obtained in a similar way as that shown above.

i) $g = \tilde{g}$ on Y

ii) $\tilde{g} = 0$ outside $Y\Lambda$

iii) $\tilde{g}$ is constant on left cosets of Λ

Moreover, since $Y \subset E$ and $\Lambda \subset E^{-1}E$ we have $Y\Lambda \subset E$, so we can say that

$$\tilde{g} = 0 \quad \text{off E.} \tag{4.4}$$

From definition of Λ as the $\mathrm{Ker}(\beta_1)$ and *(iii)* it follows easily that

$$\alpha_1(h_1) = \alpha_1(h_2) \Rightarrow \tilde{g}(h_1) = \tilde{g}(h_2). \tag{4.5}$$

Our next claim is that

$$\tilde{g} \in A(H). \tag{4.6}$$

By Lemma 4.2.5 *(ii)* we can find a finite set $F \subset \Lambda$, such that $Y\Lambda = YF = \bigcup_{\lambda \in F} Y\lambda$. Note that for each $\lambda \in F$ we have

$$\tilde{g}1_{Y\lambda}(h) = g(h\lambda^{-1}) = g_{\lambda^{-1}}(h) \in A(H).$$

Then

$$\tilde{g} = \tilde{g}1_{Y\lambda_1 \cup \ldots \cup Y\lambda_n} = \sum_{k=1}^{n} \tilde{g}1_{Y\lambda_k} \prod_{j=1}^{k-1} 1_{H \setminus Y\lambda_k} \in A(H)$$

by the above and the fact that $Y\lambda \in \Omega(H)$ for any λ.

From (4.4),(4.5) and (4.6) we obtain that $\tilde{g} \in k(\alpha_1)$. Therefore we are in the hypothesis of Lemma 4.4.1, so $\tilde{g} \circ \alpha_1^{-1}$ has an extension in $A(G)$. Since $\tilde{g} \circ \alpha_1^{-1}$ is an extension of $f \circ \alpha^{-1}$, this concludes the proof. □

Looking back at the decomposition in Proposition 4.2.4, if we have only one piece we have proved the theorem. For the general case, when we have more pieces, we have to put them together. The amenability of G is essential for this step of the proof. The following lemma will allow us to glue everything together.

Lemma 4.4.4 *Let G be an amenable locally compact group. If $X, Y \in \Omega_c(G)$ then*

$$I(X) + I(Y) = I(X \cap Y).$$

where $I(X) = \{f \in A(G) : f|_X = 0\}$, closed ideal in $A(G)$.

Proof By [18], Theorem 2.3, $I(X)$ has a bounded approximate identity, therefore $I(X) + I(Y)$ is a closed ideal in $A(G)$ (see [38], Theorem 4.2). Given an ideal E in $A(G)$, we let $Z(E) = \{x \in G : u(x) = 0 \text{ for any } u \in E\}$. Then

$$Z(I(X) + I(Y)) = X \cap Y \in \Omega_c(G).$$

But $X \cap Y$ is a set of spectral synthesis ([18], Lemma 2.2), that is $I(X \cap Y)$ is the only ideal whose hull is $X \cap Y$, giving $I(X) + I(Y) = I(X \cap Y)$. □

The next result follows immediately.

Lemma 4.4.5 *Let G be a locally compact amenable group. If $X, Y \in \Omega_c(G)$ and $g_1, g_2 \in A(G)$ are such that*

$$g_1|_{X \cap Y} = g_2|_{X \cap Y}$$

then there exists $g \in A(G)$ such that $g|_X = g_1|_X$ and $g|_Y = g_2|_Y$.

Proof By hypothesis it follows that $g_1 - g_2 \in I(X \cap Y)$, so by Lemma 4.4.4, there exists $f_1 \in I(X), f_2 \in I(Y)$ such that $f_1 - f_2 = g_1 - g_2$. Then $g = g_1 - f_1 = g_2 - f_2$ satisfies all of the requirements and we are done.

□

We are now ready to conclude the proof of Theorem 4.3.2′.

Proof of Theorem 4.3.2′ By Proposition 4.2.4 there are disjoint sets $S_1, \ldots, S_m \in \Omega_0(H)$ such that $Y = \bigcup_1^n S_i$ and each $\alpha|_{S_i}$ is proper with an affine extension $\alpha_i : \mathit{Aff}(S_i) \longrightarrow G$.

For each i, $g\chi_{S_i} \in k(\alpha|_{S_i})$ so by Lemma 4.4.3, there is $g_i \in A(G)$ such that

$$g_i|_{\alpha(S_i)} = g \circ \alpha^{-1}|_{\alpha(S_i)}$$

We have seen in Corollary 4.2.8 that $\alpha(S_i) \in \Omega_c(G)$. Repeatedly applying Lemma 4.4.5 we obtain $f \in A(G)$ such that

$$f|_{\alpha(Y)} = g_k|_{\alpha_{S_i}} \; \forall 1 \leq i \leq n$$

Then $f|_{\alpha(Y)} = g \circ \alpha^{-1}$ as required.

□

Theorem 4.3.2 is now proved as well, since it is a particular case of Theorem 4.3.2′. Since the former result is equivalent to Theorem 4.3.1, we have obtained the result announced at the beginning of this section.

Remark 4.4.6 Theorem 3.3.4 has played a central rôle in the description of the range of a completely bounded algebra homomorphism $\phi : A(G) \longrightarrow A(H)$. We can obtain the same description of the range of a bounded algebra homomorphisms using now B. Host's result and the hypothesis that G, H are locally compact groups with G abelian.

Theorem 4.4.7 *Let G, H be two locally compact groups. Suppose that G is abelian and $\phi : A(G) \longrightarrow A(H)$ is an algebra homomorphism. Then*

$$\phi(A(G)) = \left\{ f \in A(H) : \begin{array}{l} \phi^*(h_1) = 0 \Rightarrow f(h_1) = 0 \\ \phi^*(h_1) = \phi^*(h_2) \Rightarrow f(h_1) = f(h_2) \quad h_1, h_2 \in H \end{array} \right\}.$$

Proof B. Host's result ([24]) allows us to follow the same procedure as that given in the proof of Theorem 4.3.1. The proof can be carried over without many modifications. As we have noticed in Remark 4.4.2, Lemma 4.4.1 holds for G abelian as well, therefore Lemma 4.4.3 follows immediately. Since G is abelian, it is also amenable, so we can apply Lemma 4.4.5 to end the proof.

□

Chapter 5

Dual Banach algebras associated to the coset space of a locally compact group

5.1 Introduction

Let G be a locally compact group and let H be a closed subgroup of G. B. Forrest has defined in [16] the Fourier and Fourier-Stieltjes algebra associated to the coset space of a locally compact group G, $A(G/H)$ and $B(G/H)$ respectively. He has proved that when H compact it is possible to extend many classical results to this new setting. Our goal is to continue this investigation.

Given a continuous function $\tilde{u}$ on G/H we can identify $\tilde{u}$ with the continuous function u on G defined by $u = \tilde{u} \circ q$, where $q : G \longrightarrow G/H$ is the canonical map. This provides us with an isomorphism between $C(G/H)$ and $C(G : H)$, the subalgebra of $C(G)$ consisting of functions which are constant on the left cosets of H in G. Under this isomorphism $A(G/H)$ and $B(G/H)$ correspond to two subspaces of $C(G : H)$, denoted by $A(G : H)$ and $B(G : H)$.

In the third section we study the dual space $VN(G{:}\,K)$ of $A(G{:}\,K)$, when

K is a compact subgroup of G. We give a description of the dual that leads to the fact that $VN(G{:}\,K)$ is a w^*-closed left ideal in $VN(G)$. A natural question that arises is whether we can characterize all w^*-closed left ideals of $VN(G)$ that are of this form. We answer this question in Section 4.

P. Eymard [9] proved that $VN(G)$ can be identified with the dual of $A(G)$. There is a natural module action of $A(G)$ on $VN(G)$ given by $< \phi \cdot T, \gamma > = < T, \phi\gamma >$ for each $\phi, \gamma \in A(G)$ and $T \in VN(G)$. E. Granirer [20] defined the subspace $UBC(\hat{G})$ as the norm closure of $A(G) \cdot VN(G)$. F. Dunkl and D. Ramirez [8] have defined $W(\hat{G})$ (respectively $AP(\hat{G})$), the space of weakly almost periodic (respectively almost periodic) functionals on $A(G)$, to be the set of all T in $VN(G)$ for which the operator from $A(G)$ to $VN(G)$ given by $\phi \mapsto \phi \cdot T$ is weakly compact (respectively compact).

The last three sections of this chapter are dedicated to the study of the natural analogues of the spaces $UBC(\hat{G})$, $W(\hat{G})$, $AP(\hat{G})$ in $VN(G{:}\,K)$, which we denote by $UBC(\widehat{G{:}\,K})$, $W(\widehat{G{:}\,K})$ and $AP(\widehat{G{:}\,K})$ respectively. We obtain results that are analogous to the ones in the classical case due to F. Dunkl and D. E. Ramirez ([8]), E. Granirer ([20]) , A. T. Lau ([26]). The proofs are motivated by the ones in [26], [20], [8]. We adapt them to our new setting.

In Section 5 we study the various inclusion relationships that exist between these spaces. Also, we prove that when G is amenable, $UBC(\widehat{G{:}\,K})$ is isometrically isomorphic to a closed subspace of $B(G{:}\,K)^*$ (Theorem 5.5.13).

In Section 6, we explore the behaviour of these spaces with respect to the Arens product on their duals. Among other things, we characterize $W(\widehat{G{:}\,K})$ as the maximal subspace X of $VN(G{:}\,K)$ for which the Arens product makes sense on X^* and the product is separately continuous with respect to the weak* topology on bounded spheres (Proposition 5.6.7). In the last section we study operators commuting with the action of $A(G{:}\,K)$ on subspaces of $VN(G{:}\,K)$.

5.2 Preliminaries

Let G be a locally compact group. Let H be a closed subgroup of G. B. Forrest defined in [16] the following sets:

$$B(G:H) = \{ \quad u \in B(G) : \text{u constant on left cosets of } H\}$$

$$A(G:H) = \overline{\{ \quad u \in B(G:H) : q(\text{supp } u) \text{ compact in } G/H\}}^{\|\cdot\|_{B(G)}}$$

If H is normal, then $A(G:H) \simeq A(G/H)$, $B(G:H) \simeq B(G/H)$.

From now on K will denote a compact subgroup of G. We assume that a measure $\mu_{G/K}$ has been chosen on the coset space G/K, such that for every $f \in L^1(G)$,

$$\int_G f(x)dx = \int_{G/K}\int_K f(kx)dkd\mu_{G/K}(\tilde{x})$$

where dk is a fixed left Haar measure on K. There exists a projection

$$P_K : B(G) \longrightarrow B(G{:}K), \quad P_K(u)(x) = \int_K u(xk)dk$$

such that P_K restricted to $A(G)$ is a projection onto $A(G{:}K)$ (see [16]). We can define a similar projection on $L^1(G)$,

$$P_K : L^1(G) \longrightarrow L^1(G), \quad P_K(f)(x) = \int_K f(xk)dk$$

whose image is the set denoted by

$$L^1(G{:}K) = \{f \in L^1(G) \ : \ f \ \text{ is constant on left cosets of } K\}.$$

$L^1(G:K)$ is a closed subalgebra of $L^1(G)$ which is not self adjoint.

Similarly we can define

$$P_K^r : L^1(G) \longrightarrow L^1(G), \quad P_K^r(f)(x) = \int_K f(kx)dk.$$

In this case the image is the set of all functions $f \in L^1(G)$ that are constant on the right cosets of K. The following results are straightforward calculations.

Proposition 5.2.1 *Let $f, g \in C_{00}(G)$. Then:*

i) $P_K(f * g) = f * P_K g$ *i)'* $P^r_K(f * g) = P^r_K f * g$

ii) $P_K(f^\vee) = (P^r_K f)^\vee$ *ii)'* $P^r_K(f^\vee) = (P_K f)^\vee$

iii) $P_K(\tilde{f}) = (P^r_K f)^\sim$ *iii)'* $P^r_K(\tilde{f}) = (P_K f)^\sim$

where $f^\vee(x) = f(x^{-1})$, $\tilde{f}(x) = \overline{f(x^{-1})}$.

Proof Let $f, g \in C_{00}(G)$. Given $x \in G$, we have on the one hand,

$$P_K(f * g)(x) = \int_K (f * g)(xk)dk = \int_K \int_G f(y)g(y^{-1}xk)dydk, \tag{5.1}$$

and on the other hand,

$$\begin{aligned}(f * P_K g)(x) &= \int_G f(y)P_K g(y^{-1}x)dy = \int_G \int_K f(y)g(y^{-1}xk)dkdy \\ &= \int_K \int_G f(y)g(y^{-1}xk)dydk \end{aligned} \tag{5.2}$$

Thus, from (5.1) and (5.2), we get the equality $P_K(f * g) = f * P_K g$, which proves *i)*.

To prove *i)'* we proceed similarly and calculate

$$P^r_K(f * g)(x) = \int_K (f * g)(kx)dk = \int_K \int_G f(y)g(y^{-1}kx)dydk \tag{5.3}$$

and also

$$\begin{aligned}(P^r_K f * g)(x) &= \int_G P^r_K f(y)g(y^{-1}x)dy = \int_G \int_K f(ky)g(y^{-1}x)dkdy \\ &= \int_K \int_G f(ky)g(y^{-1}x)dydk \qquad (5.4) \\ &= \int_K \int_G f(y)g(y^{-1}kx)dydk \text{ (change of variable } y = k^{-1}y) \end{aligned}$$

Therefore we have equality.

We will prove only one of the remaining equalities and the other ones can be proved by similar calculations. To prove *ii)*, let $x \in G$. Then

$$\begin{aligned}P_K(f^\vee)(x) &= \int_K f^\vee(xk)dk = \int f(k^{-1}x^{-1})dk \\ &= \int_K f(kx^{-1})dk = (P^r_K f)^\vee(x)\end{aligned}$$

and which concludes the proof. □

5.3 The space $VN(G{:}K)$

Let Γ_K be the Banach space adjoint of the projection $P_K : A(G) \longrightarrow A(G{:}K)$. We then have

$$\begin{aligned} \Gamma_K : VN(G) &\longrightarrow VN(G) \\ < \Gamma_K(T), \gamma > &= < T, P_K\gamma > \end{aligned} \tag{5.5}$$

for every $\gamma \in A(G)$ and $T \in VN(G)$.

We denote by $VN(G{:}K)$ the w^*-closure of $\lambda(L^1(G{:}K))$, where λ is the left regular representation of G. This is a w^*-closed subalgebra of $VN(G)$ which is not closed under involution in general.

Lemma 5.3.1 *(i) Γ_K is a projection of $VN(G)$ onto $VN(G{:}K)$.*

(ii) Furthermore, we have $VN(G{:}K) \simeq A(G{:}K)^*$ isomorphically.

Proof *(i)* Indeed, let us assume that $T = \lambda(\mu)$, where μ is a measure in $M(G)$ and let us view the Haar measure on K, dk, as an element μ_K of $M(G)$. We then have for every $u \in A(G)$

$$\begin{aligned} < \Gamma_K(T), u > &= < \lambda(\mu), P_K u >= \int_G P_K u(x) d\mu(x) = \int_G \int_K u(xk) d\mu_K(k) d\mu(x) \\ &= \int_G \int_G u(xk) d\mu_K(k) d\mu(x) =< \lambda(\mu * \mu_K), u > . \end{aligned}$$

Therefore $\Gamma_K(T) = \lambda(\mu * \mu_K)$. In particular, for $\mu = \mu_f$ with $f \in L^1(G)$ we obtain that

$$\Gamma_K(\lambda(\mu_f)) = \lambda(\mu_f * \mu_K) = \lambda(f * \mu_K) = \lambda(P_K(f)) \in \lambda(L^1(G{:}K)).$$

Therefore $\Gamma_K(\lambda(L^1(G))) = \lambda(L^1(G{:}K))$. Since $VN(G) = \lambda(L^1(G))^{-w^*}$ and the map Γ_K is w^*-w^* continuous, it follows that $\Gamma_K(VN(G)) = \lambda(L^1(G{:}K))^{-w^*} = VN(G{:}K)$ and we have proved the claim.

(ii) Clearly we have $\mathrm{Ker}\Gamma_K = A(G{:}K)^{\perp}$. Then

$$VN(G) = \Gamma_K(VN(G{:}K)) \oplus \mathrm{Ker}\Gamma_K = VN(G{:}K) \oplus A(G{:}K)^{\perp}$$

and hence $VN(G{:}K) \simeq VN(G)/A(G{:}K)^{\perp} \simeq A(G{:}K)^*$. □

Remark 5.3.2 We explain now, in more detail, the isomorphism from Lemma 5.3.1 *(ii)*. Let $T_0 \in VN(G{:}K)$. As an element in $VN(G)$, T_0 can be seen as a functional on $A(G)$. Then the functional on $A(G{:}K)$ given by the isomorphism is nothing else than the restriction of T_0 to $A(G{:}K)$.

Conversely, if we start with a functional $F_0 \in A(G{:}K)^*$, it can be extended to a functional F on $A(G)$. If $T_F \in VN(G)$ is the operator given by the duality $A(G)^* \simeq VN(G)$, then the corresponding operator T_{F_0} in $VN(G{:}K)$ is given by $\Gamma_K(T_F)$.

It is known that $VN(G)$ is an $A(G)$-module with the multiplication given by the duality $VN(G) \simeq A(G)^*$ as follows

$$A(G) \times VN(G) \longrightarrow VN(G)$$

$$(\varphi, T) \to \varphi \cdot T$$

where $< \varphi \cdot T, \gamma > = < T, \varphi\gamma >$ for every $\gamma \in A(G)$. The next result shows that $VN(G{:}K)$, as a subspace of $VN(G)$, is invariant under this multiplication with respect to $A(G{:}K)$.

Proposition 5.3.3 *Let K be a compact subgroup of a locally compact group G. Then*

$$A(G{:}K)VN(G{:}K) \subset VN(G{:}K).$$

Proof Let $x = \lambda(f)$, $f \in L^1(G{:}K)$ and $\varphi \in A(G{:}K)$. Let $T_0 = \varphi \cdot x$. We will show that $T_0 \in VN(G{:}K)$.

For every $\gamma \in A(G)$, we have

$$< T_0, \gamma > = < x, \varphi\gamma > = < \lambda(f), \varphi\gamma > = \int_G f(t)\varphi(t)\gamma(t)dt.$$

Let $\gamma = (h * \tilde{g})^{\vee}$ with $h, g \in C_{00}(G)$. On the one hand we have

$$\begin{aligned}
< T_0, \gamma > &= \int_G f(t)\varphi(t)\left(\int_G \overline{g}(y)h^{\vee}(y^{-1}t)dy\right)dt \\
&= \int_G\int_G f(t)\varphi(t)\overline{g}(y)h(t^{-1}y)dydt \\
&= \int_G\left(\int_G f(t)\varphi(t)h(t^{-1}y)dt\right)\overline{g}(y)dy \\
&= \int_G (f\varphi * h)(y)\overline{g}(y)dy \\
&= < f\varphi * h, g > .
\end{aligned}$$

On the other hand $< T_0, (h * \tilde{g})^{\vee} >=< T_0 h, g >$. Therefore $< T_0 h, g >=< f\varphi * h, g >$, for every $h, g \in C_{00}(G)$. It follows that

$$T_0 h = f\varphi * h = \lambda(f\varphi)h \qquad \text{which is} \qquad T_0 = \lambda(f\varphi).$$

Since $f\varphi \in L^1(G{:}\,K)$, it follows that $T_0 \in VN(G{:}\,K)$.

Now, let $x \in VN(G{:}\,K)$. Then there is a net $\{x_\alpha\}_\alpha \in L^1(G{:}\,K)$ such that

$$x_\alpha \xrightarrow{w^*} x$$

Let $\varphi \in A(G{:}\,K)$. Then $\varphi \cdot x_\alpha \xrightarrow{w^*} \varphi \cdot x$. Indeed,

$$< \varphi \cdot x_\alpha, \gamma >=< x_\alpha, \varphi\gamma > \longrightarrow < x, \varphi\gamma >=< \varphi \cdot x, \gamma > .$$

By the above $\varphi \cdot x_\alpha \in VN(G{:}\,K)$ and since $VN(G{:}\,K)$ is w^*-closed, it follows that $\varphi \cdot x \in VN(G{:}\,K)$. □

Remark 5.3.4 Let $\psi \in A(G{:}\,K)$, $T \in VN(G{:}\,K)$. As a consequence of the above result and of the duality $A(G{:}\,K)^* \simeq VN(G{:}\,K)$, we can define two module actions of $A(G{:}\,K)$ on $VN(G{:}\,K)$ by

1) $\psi \cdot T$ as above, viewing $\psi \in A(G), T \in VN(G)$

$$< \psi \cdot T, \gamma >=< T, \psi\gamma > \quad \text{for all } \gamma \in A(G)$$

2) $\psi \circ T$ given by the duality $A(G{:}K)^* \simeq VN(G{:}K)$

$$<\psi_K \circ T, \gamma_K>=<T, \psi\gamma_K> \quad \text{for all } \gamma_K \in A(G{:}K).$$

Clearly $\psi \circ T = \psi \cdot T|_{A(G:K)}$, so by Remark 5.3.2 and Proposition 5.3.3, the corresponding operators in $VN(G)$ and $VN(G{:}K)$ are the same:

$$T_{\psi\circ T} = \Gamma_K(T_{\psi\cdot T}) = T_{\psi\cdot T}$$

Therefore the above multiplications are the same. In the sequel when we talk about the module action of $A(G{:}K)$ on $VN(G{:}K)$ we will use any of these two, as necessary.

We now give a description of $VN(G{:}K)$ and we show that it is a left ideal of $VN(G)$.

Theorem 5.3.5 *Let K be a compact subgroup of a locally compact group G. Then:*

i) $VN(G{:}K) = \{T \in VN(G) \ : <T, \gamma>=<T, P_K\gamma> \ \text{for all } \gamma \in A(G)\}$

ii) $VN(G{:}K) = \{T \in VN(G) \ : \ T = TT_K\}$

where $T_K : L^2(G) \longrightarrow L^2(G), \quad T_K f(x) = \int_K f(kx)dk$. *In particular, it follows that* $VN(G{:}K) = VN(G)T_K$, *which implies that* $VN(G{:}K)$ *is a left ideal in* $VN(G)$.

Proof *i)* Let $T \in VN(G{:}K)$. Then

$$\Gamma_K(T) = T \quad \text{i.e.} \ <\Gamma_K(T), \gamma>=<T, \gamma> \quad \text{for each } \gamma \in A(G).$$

This means that $<T, P_K\gamma>=<T, \gamma>$ for each $\gamma \in A(G)$. Therefore

$$VN(G{:}K) \subset \{T \in VN(G) \ : <T, \gamma>=<T, P_K\gamma>\}.$$

Conversely, let $T \in VN(G)$ such that $< T, \gamma > = < T, P_K\gamma >$. Then

$$< \Gamma_K(T), \gamma > = < T, P_K\gamma > = < T, \gamma > \quad \text{for each } \gamma \in A(G)$$

and therefore $\Gamma_K(T) = T$, which means that $T \in VN(G{:}K)$.

ii) Let $T \in VN(G{:}K)$. Let $u \in A(G)$ such that $u = (h * \tilde{k})^\vee$ with $h, k \in C_{00}(G)$. By *i)*, T satisfies

$$< T, u > = T, P_K u > . \tag{5.6}$$

Using Proposition 5.2.1 we have

$$P_K u = P_K[(h * \tilde{k})^\vee] = [P_K^r(h * \tilde{k})]^\vee = [P_K^r h * \tilde{k}]^\vee = (g * \tilde{k})^\vee$$

where $g := P_K^r h$. Then

$$< T, P_K u > = < T, (g * \tilde{k})^\vee > = < Tg, k > \tag{5.7}$$

Combining (5.6) and (5.7) we get

$$< Th, k > = < Tg, k >$$

which is equivalent to

$$< Th, k > = < TP_K^r h, k > \quad \Leftrightarrow \quad Th = TP_K^r h \quad \Leftrightarrow \quad T = TT_K$$

Conversely, let $T \in VN(G)$ such that $T = TT_K$. For any $h, k \in C_{00}(G)$ we have

$$< Th, k > = < TP_K^r h, k > \quad \Leftrightarrow \quad < T, (h * \tilde{k})^\vee > = < T, (P_K^r h * \tilde{k})^\vee > .$$

Hence we get that $< T, u > = < T, P_K u >$ where $u = (h * \tilde{k})^\vee$.
Since elements of the form $u = (h * \tilde{k})^\vee$ are dense in $A(G)$, we get that

$$< T, \gamma > = < T, P_K\gamma > \quad \text{for each } \gamma \in A(G)$$

which shows that $T \in A(G)$. The last part of the statement follows easily using *ii)*. □

Remark 5.3.6 From the proof we obtain the following formula for the projection Γ_K

$$\Gamma_K(T) = TT_K \quad \text{for each } T \in VN(G)$$

5.4 Left ideals of the form $VN(G{:}\,K)$ in $VN(G)$

In this section we will characterize all w^*-closed left ideals in $VN(G)$ that are of the form $VN(G{:}\,K)$, for some compact subgroup K of G.

Note that as the predual of a von Neumann algebra, $A(G)$ becomes a right $VN(G)$-module, with the multiplication

$$A(G) \times VN(G) \longrightarrow A(G)$$

$$(\phi, T) \mapsto \phi \diamond T \quad \text{where} \quad <\phi \diamond T, T'> = <\phi, TT'>$$

and a left $VN(G)$-module with

$$VN(G) \times A(G) \longrightarrow A(G)$$

$$(T, \phi) \mapsto T \diamond \phi \quad \text{where} \quad <T \diamond \phi, T'> = <\phi, T'T>.$$

A set $\mathcal{A} \subset A(G)$ is said to be *right (*respectively *left) invariant* if

$$\mathcal{A} \diamond VN(G) \subset \mathcal{A} \quad (\text{respectively } VN(G) \diamond \mathcal{A} \subset \mathcal{A}).$$

Remark 5.4.1 Let E be a projection in $VN(G)$. Then $\mathcal{A} = E \diamond A(G)$ is a right invariant subspace of $A(G)$. Indeed, it is easy to see that $(E \diamond \phi) \diamond T = E \diamond (\phi \diamond T)$. Furthermore, we have the following characterization of $E \diamond A(G)$:

$$E \diamond A(G) = \{\phi \in A(G) : <\phi, T> = <\phi, TE> \quad \text{for each } T \in VN(G)\}. \quad (5.8)$$

Define P_E to be the projection on $A(G)$ given by

$$P_E : A(G) \longrightarrow A(G)$$

$$\phi \mapsto E \diamond \phi$$

Let Γ_E be the Banach space adjoint of P_E

$$\Gamma_E : VN(G) \longrightarrow VN(G)$$

$$< \Gamma_E(T), \gamma >=< T, P_E\gamma > \quad \text{for each } T \in VN(G),\ \gamma \in A(G).$$

Remark 5.4.2 *i)* Note that Γ_E is a projection onto $VN(G)E$. Indeed, we have

$$< \Gamma_E(T), \gamma >=< T, P_E\gamma >=< T, E \diamond \gamma >=< TE, \gamma > \quad \text{for each } \gamma \in A(G)$$

which implies that $\Gamma_E(T) = TE$ and therefore the image of Γ_E is $VN(G)E$.
ii) Furthermore, $\mathrm{Ker}\Gamma_E = E \diamond A(G)^\perp = VN(G)(I - E)$.

Proposition 5.4.3 *Let G be a locally compact group and let E be a projection in $VN(G)$. The following are equivalent*

(i) $E \diamond A(G)$ is a subalgebra of $A(G)$

ii) $VN(G)(I-E)$ is invariant under the module action of $E \diamond A(G)$ on $VN(G)$.

Proof *i)* $\Rightarrow$ *ii)* Suppose $E \diamond A(G)$ is an algebra. Let $\varphi_i = E \diamond \tilde{\varphi}_i$, $\varphi_i \in A(G)$, $i = 1, 2$. Then $\varphi_1\varphi_2$ must be in $E \diamond A(G)$, so by (5.8), we must have

$$< \varphi_1\varphi_2, T >=< \varphi_1\varphi_2, TE > \quad \forall\, T \in VN(G).$$

On the one hand ,

$$< \varphi_1\varphi_2, T >=< \varphi_2, \varphi_1 T >=< E \diamond \tilde{\varphi}_2, \varphi_1 T >=< \tilde{\varphi}_2, (\varphi_1 T)E > . \tag{5.9}$$

On the other hand,

$$< \varphi_1\varphi_2, TE >=< \varphi_2, \varphi_1(TE) >=< \tilde{\varphi}_2, [\varphi_1(TE)]E > . \tag{5.10}$$

Since $\tilde{\varphi}_2$ is arbitrary, by combining (5.9) and (5.10), we obtain that

$$(\varphi_1 T)E = [\varphi_1(TE)]E \quad \text{or equivalently} \quad \Gamma_E(\varphi_1 T) = \Gamma_E(\varphi_1 \Gamma_E(T))$$

for all $T \in VN(G)$, $\varphi_1 \in E \diamond A(G)$. This means that

$$\varphi_1 T - \varphi_1 \Gamma_E(T) \in \mathrm{Ker}\Gamma_E$$

and therefore $\varphi_1(T - \Gamma_E(T)) \in VN(G)(I-E)$ for all $T \in VN(G)$, $\varphi_1 \in A(G)$. Hence, we have shown that

$$(E \diamond A(G)) \cdot VN(G)(I-E) \subset VN(G)(I-E).$$

ii) $\Rightarrow$ *i)* This follows similarly. □

Remark 5.4.4 If $E = T_K$, then $E \diamond A(G) = T_K \diamond A(G) = A(G{:}\,K)$ since

$$< T_K \diamond \varphi, T >=< \varphi, TT_K >=< \varphi, \Gamma_K(T) >=< P_K\varphi, T > \text{ for each } T \in VN(G).$$

Since $A(G{:}\,K)$ is clearly an algebra, it follows from the previous proposition that $VN(G)(I - T_K)$ is invariant with respect to $A(G{:}\,K) = T_K \diamond A(G)$.

We denote by C the following operator :

$$C : L^2(G) \longrightarrow L^2(G)$$

$$h \mapsto \overline{h}$$

It can be shown that $CVN(G)C = VN(G)$ (see [44]). We can define an anti-automorphism ω as follows:

$$\omega : VN(G) \longrightarrow VN(G), \quad \omega(T) = CT^*C.$$

This defines an involution $\sharp$ on $A(G)$ as follows

$$< \varphi^\sharp, T >= \overline{< \omega(T^*), \varphi >}.$$

Remark 5.4.5 If $T = \lambda(s)$, $s \in G$, then

$$< \varphi^\sharp, \lambda(s) >= \overline{< \lambda(s), \varphi >} \Leftrightarrow \varphi^\sharp = \overline{\varphi(s)}$$

so this involution is nothing else but the complex conjugation of functions.

Proposition 5.4.6 *Let G be a locally compact group and let E be a projection in $VN(G)$. The following are equivalent:*

i) $E \diamond A(G)$ *is closed under the involution* $\sharp$

ii) E *has the property that* $CE = ECE$.

Proof *i)* $\Rightarrow$ *ii)* Let $\varphi = E \diamond \varphi_1 \in E \diamond A(G)$. Then, by hypothesis, $\varphi^\sharp \in E \diamond A(G)$ which means

$$< \varphi^\sharp, T >=< \varphi^\sharp, TE > \quad \text{for any} \quad T \in VN(G).$$

Equivalently,

$$\overline{< \varphi, CTC >} = \overline{< \varphi, CTEC >}$$

The conjugation can be removed, and so we get $< \phi, CTC >=< \phi, CTEC >$ for all $T \in VN(G)$. Taking into account that $\varphi = E \diamond \varphi_1$ we get that

$$< \varphi_1, CTCE >=< \varphi_1, CTECE >$$

which can be rewritten, using that $C^2 = 1$, as

$$< \varphi_1, CTCE >=< \varphi_1, CTC(CECE) > .$$

The last equality shows that E satisfies

$$< \varphi_1 \diamond CTC, E >=< \varphi_1 \diamond CTC, CECE >$$

for each $\varphi_1 \in A(G)$ and $T \in VN(G)$.

Now, since $CVN(G)C = VN(G)$, it follows that CTC spans all of $VN(G)$ if T does, and therefore we have

$$< \phi_1 \diamond T, E >=< \phi_1 \diamond T, CECE > \quad \text{for each } T \in VN(G).$$

Because $\phi_1 \diamond T$ spans all of $A(G)$ we obtain that E satisfies the identity

$$< \psi, E >=< \psi, CECE > \quad \text{for all } \psi \in A(G).$$

In conclusion, we obtain that $E = CECE$ which is equivalent to $CE = ECE$.

ii) $\Rightarrow$ *i)* This follows similarly. □

Remark 5.4.7 Note that if $E = T_K$, then it satisfies the condition *ii)* of Proposition 5.4.6.

We now characterize all w^*-closed left ideals in $VN(G)$ which are of the form $VN(G\colon K)$, for some compact subgroup K of G.

Theorem 5.4.8 *Let G be a locally compact group. Let J be a w^*-closed left ideal in $VN(G)$, given by the projection E, that is, $J = VN(G)E$. Also, suppose that*

i) $VN(G)(I - E)$ is invariant with respect to $E \diamond A(G)$, and

ii) $CE = ECE$, that is $\overline{E(f)} = E(\overline{E(f)})$.

Then there exists a compact subgroup K of G such that $J = VN(G\colon K)$ (or equivalently, $E = T_K$).

Proof Given a von Neumann algebra $\mathcal{M}$, there is a one to one correspondence between w^*- closed left ideals and right invariant subspaces of the predual $\mathcal{M}_*$ (see Theorem 2.9, [45]). If we let $\mathcal{M} = VN(G)$, $\mathcal{M}_* = A(G)$, then the map

$$\mathcal{F} : \left\{ J : \begin{array}{c} w^* - \text{closed left} \\ \text{ideal in } \ VN(G) \end{array} \right\} \longrightarrow \left\{ \mathcal{A} : \begin{array}{c} \text{right invariant} \\ \text{subspace of } A(G) \end{array} \right\}$$

$$J = VN(G)E \mapsto J^{\perp} = (I - E) \diamond A(G)$$

is a bijection. Denote by $\mathcal{G}$ the following one-to-one map:

$$\mathcal{G} : \{(I - E) \diamond A(G) \ : \ E \in \mathcal{P}(VN(G))\} \longrightarrow \{E \diamond A(G) \ : \ E \in \mathcal{P}(VN(G))\}$$

$$(I - E) \diamond A(G) \mapsto E \diamond A(G)$$

where $\mathcal{P}(VN(G))$ is the set of all projections in $VN(G)$. And finally, let $\mathcal{H}$ denote the following map:

$$\mathcal{H} : \left\{ \begin{array}{c} \mathcal{A} : \text{self adjoint right invariant} \\ \text{subalgebra of } A(G) \end{array} \right\} \longrightarrow \{K : \text{compact subgroup of } G\}$$

$$\mathcal{A} \mapsto K_{\mathcal{A}} = \{s \in G : \lambda(s) \diamond \varphi = \varphi \, \forall \varphi \in \mathcal{A}\}.$$

By Theorem 9 in [44], $\mathcal{H}$ is a one-to-one map.

Notice that under our assumptions, by Proposition 5.4.3 and Proposition 5.4.6, $E \diamond A(G)$ is a self adjoint right invariant subalgebra of $A(G)$. So, if we denote by $\mathcal{J}$ the following set

$$\mathcal{J} = \left\{ J \in VN(G) : \begin{array}{l} J = VN(G)E \text{ such that } \overline{E(h)} = E(\overline{E(h)}) \\ \text{and } (E \diamond A(G))(VN(G)(I-E)) \subset VN(G)(I-E) \end{array} \right\},$$

then the map

$$\Lambda : \mathcal{J} \longrightarrow \{K : K \text{ compact subgroup of } G\}$$

$$\Lambda = \mathcal{H} \circ \mathcal{G} \circ (\mathcal{F}|\mathcal{J})$$

is one-to-one. Moreover, if we denote by $\mathcal{J}_K = \{VN(G{:}\,K) : K \subset G \text{ compact subgroup}\}$ we have $\mathcal{J}_K \subset \mathcal{J}$ and

$$\Lambda(\mathcal{J}_K) = \{K : K \text{ compact subgroup in } G\}$$

Indeed, the fact that $\mathcal{J}_K \subset \mathcal{J}$ follows from Remark 5.4.4 and Remark 5.4.7. The equality follows from fact that:

$$\begin{aligned} \Lambda(VN(G{:}\,K_0)) &= \mathcal{H}(\mathcal{G}((I - T_{K_0}) \diamond A(G))) \\ &= \mathcal{H}(T_{K_0} \diamond A(G)) \\ &= \mathcal{H}(A(G{:}\,K_0)) \\ &= K_0 \end{aligned}$$

for every $K_0 \subset G$ compact subgroup.

Given the fact that the map Λ is injective, we get that $\mathcal{J} = \mathcal{J}_K$. □

5.5 The subspaces $UBC(\widehat{G{:}\,K})$, $W(\widehat{G{:}\,K})$ and $AP(\widehat{G{:}\,K})$

Let G be a locally compact group and let K be a compact subgroup of G. In [20] E. Granirer defined the subspace $UBC(\hat{G})$ as the norm closure of $A(G) \cdot$

$VN(G)$. We define the analogue of $UBC(\hat{G})$, to be the norm closure of $A(G{:}\,K)\cdot VN(G{:}\,K)$ in $VN(G)$, and we denote it by $UBC(\widehat{G{:}\,K})$.

Remark 5.5.1 *(i)* When K is normal we have $UBC(\widehat{G{:}\,K}) \simeq UBC(\widehat{G/K})$, the space of bounded uniformly continuous functionals on $\widehat{G/K}$ as defined by E. Granirer in [20], based on the fact that $A(G{:}\,K) \simeq A(G/K)$ and $VN(G{:}\,K) \simeq VN(G/K)$ when K is normal.

(ii) Using Proposition 5.3.3 , we get that

$$UBC(\widehat{G{:}\,K}) \subset VN(G{:}\,K)$$

.

(iii) Since $A(G{:}\,K) \subset A(G)$ and $VN(G{:}\,K) \subset VN(G)$ it follows that

$$UBC(\widehat{G{:}\,K}) \subset UBC(\hat{G})$$

(iv) $UBC(\widehat{G{:}\,K})$ is a linear subspace. This follows by arguing as in [20]. Indeed, it is not hard to see that

$$A(G{:}\,K) = A_c(G{:}\,K)^{-\|\cdot\|_{B(G)}}$$

where $A_c(G{:}\,K) = \{u \in B(G{:}\,K) : \text{supp}\, u \text{ compact}\}$. Then $A_c(G{:}\,K)\cdot VN(G{:}\,K)$ is dense in $UBC(\widehat{G{:}\,K})$. Let $u_1T_1,\ u_2T_2 \in A_c(G{:}\,K)\cdot VN(G{:}\,K)$. Our claim is that $u_1T_1 + u_2T_2 \in A_c(G{:}\,K)\cdot VN(G{:}\,K)$.

Let $K_1 = \text{supp}\ u_1,\ K_2 = \text{supp}\ u_2$ compact sets and put $F = K_1 \cup K_2$. Let O be an open set such that $F \subset O$ and $\overline{O}$ is compact. There exists $u \in A(G{:}\,K)$ such that $u(F) = 1$ and $u = 0$ outside O, $0 \le u \le 1$. Then

$$u_1T_1 + u_2T_2 = u(u_1T_1 + u_2T_2) \in A_c(G{:}\,K)\cdot VN(G{:}\,K).$$

Therefore $A_c(G{:}\,K)\cdot VN(G{:}\,K)$ is linear space. Now $UBC(\widehat{G{:}\,K})$, its linear closure, is a linear space as well.

Theorem 5.5.2 *Let G be a locally compact group and K a compact subgroup such that G/K is an amenable coset space. Then $A(G{:}K)VN(G{:}K)$ is closed. In particular, $UBC(\widehat{G{:}K}) = A(G{:}K)VN(G{:}K)$.*

Proof If G/K is an amenable coset space then $A(G{:}K)$ has bounded approximate identity (see [16]). Since, by Proposition 5.3.3, $VN(G{:}K)$ is a left Banach $A(G{:}K)$-module, we can apply the Cohen's factorization theorem ([23], vol2, p.268) and we conclude the proof. □

Remark 5.5.3 When $K = \{e\}$ this was proved by E. Granirer ([20], Proposition 1).

Definition 5.5.4 *Let G be a locally compact group and K a compact subgroup of G. We define*

$$W(\widehat{G{:}K}) = \left\{ \begin{array}{c} T \in VN(G{:}K) : A(G{:}K) \longrightarrow VN(G{:}K) \text{ weakly compact} \\ \phi \mapsto \phi T \end{array} \right\}.$$

Remark 5.5.5 *(i)* It is easy to see that $W(\hat{G}) \cap VN(G{:}K) \subset W(\widehat{G{:}K})$.

(ii) If K is normal then $W(\widehat{G{:}K}) \simeq W(\widehat{G/K})$.

Definition 5.5.6 *Let G be a locally compact group and K a compact subgroup of G. We define*

$$AP(\widehat{G{:}K}) = \left\{ \begin{array}{c} T \in VN(G{:}K) : A(G{:}K) \longrightarrow VN(G{:}K) \text{ is compact} \\ \phi \mapsto \phi T \end{array} \right\}.$$

Remark 5.5.7 We have $AP(\widehat{G{:}K}) \subset W(\widehat{G{:}K})$. Also, when K is normal then $AP(\widehat{G{:}K}) \simeq AP(\widehat{G/K})$.

The next two propositions present conditions under which various inclusion relationships between these three spaces occur. The results are similar to the ones in the classical case, that are proved in the work of E. Granirer ([20]), and our proofs are similar to the ones given there.

Proposition 5.5.8 *Let G be a locally compact group and K a subgroup such that G/K is an amenable coset space. Then*

$$W(\widehat{G:K}) \subset UBC(\widehat{G:K})$$

Proof Since G/K is an amenable coset space it follows from [16] that $A(G{:}K)$ has a bounded approximate identity $\{e_\alpha\}_\alpha$ such that $\|e_\alpha\| \leq 1$ for every α. We claim that, for any $T \in W(\widehat{G:K})$, $e_\alpha T \xrightarrow{\alpha} T$ in the topology $\sigma = \sigma(VN(G{:}K), A(G{:}K))$.

Indeed, by the definition of the module action of $A(G{:}K)$ on $VN(G{:}K)$ we have

$$< e_\alpha T, \gamma_k >=< T, e_\alpha \gamma_k > .$$

On the other hand, $e_\alpha \gamma_k \longrightarrow \gamma_k$ since $\{e_\alpha\}_\alpha$ is a bounded approximate identity. Therefore we obtain

$$< e_\alpha T, \gamma > \xrightarrow{\alpha} < T, \gamma_k > \text{ for all } \gamma_k \in A(G{:}K)$$

and the claim is proved.

The set $\{e_\alpha T : \|e_\alpha\| \leq 1\}$ is weakly relatively compact, since $T \in W(\widehat{G:K})$, so there is $T_0 \in VN(G{:}K)$ such that $e_{\alpha_\beta} T \xrightarrow{\beta} T_0$ weakly. In particular, it converges in the σ topology. But then we must have $T = T_0$. Therefore $e_{\alpha_\beta} T \xrightarrow{\beta} T$ weakly. Therefore T is in the closure of $A(G{:}K)VN(G{:}K)$ in the weak topology, which is equal to the norm closure of $A(G{:}K)VN(G{:}K)$, that is $UBC(\widehat{G:K})$. □

Proposition 5.5.9 *Let G be a locally compact group and K a compact open subgroup. Then*

$$UBC(\widehat{G:K}) \subset AP(\widehat{G:K}).$$

In particular, it follows that $UBC(\widehat{G:K}) \subset W(\widehat{G:K})$.

Proof For $a \in G$, let 1_{aK} be the characteristic function of aK. Since K is open we have $1_{aK} \in A(G\colon K)$ (it has compact support and is constant on the cosets of K). Our first claim is that for $T \in VN(G\colon K)$, $1_{aK}T \in AP(\widehat{G\colon K})$.

Let $g \in G$. Define the map

$$l_g : A(G) \longrightarrow A(G)$$
$$u \mapsto l_g(u)$$

where $l_g(u)(x) = u(gx)$. Then the dual map is $l_g^* : VN(G) \longrightarrow VN(G)$. If I is the identity operator in $VN(G)$ it can be shown that $< I, u >= u(e)$ for any $u \in A(G)$. Let $\phi \in A(G\colon K)$. Then for any $\gamma \in A(G\colon K)$ one has

$$\begin{aligned} < \phi \cdot (1_{aK}T), \gamma > &= < 1_{aK}T, \gamma\phi >=< T, \gamma\phi|_{aK} > \\ &= < T, \gamma(a)\phi(a)1_{aK} > \ (\gamma, \phi \text{ are constants on cosets of } K) \\ &= \phi(a)T(1_{aK})\gamma(a) = \phi(a)T(1_{aK})l_a^*(I)\ (\gamma) \\ &\quad (\text{since } \gamma(a) = l_a^*(I)(\gamma)) \end{aligned}$$

So $\phi \cdot (1_{aK}T) = [\phi(a)T(1_{aK})]l_a^*(I)$. If $\|\phi\| \leq 1$, then

$$\phi \cdot (1_{aK}T) \in \{\alpha l_a^* I : |\alpha| \leq |T(1_{aK})|\} = \{\alpha T_a : |\alpha| \leq |T(1_{aK})|\}$$

and the last set is a compact set in $VN(G)$. Therefore $\{\phi \cdot (1_{ak}T) : \|\phi\| \leq 1,\ \phi \in A(G\colon K)\}$ is a compact set in $VN(G\colon K)$ which proves that $1_{ak}T \in AP(\widehat{G\colon K})$.

Our second claim is that if $T \in VN(G\colon K)$ and $h \in A_c(G\colon K)$ then $hT \in AP(\widehat{G\colon K})$. Since $h \in A_c(G\colon K)$, $q(\text{supp } u) \subset G/K$ is compact. By hypothesis G/K is discrete, therefore $q(\text{supp } u)$ is finite, so we can write it as $q(\text{supp } u) = \cup_{i=1}^n \hat{x}_i$ where $\hat{x}_i = x_iK \in G/K$. Since h is constant on the left cosets of K, there are $\alpha_i \in \mathbf{C}$ such that

$$h = \sum_{i=1}^{n} \alpha_i 1_{x_iK}$$

Then $hT = \sum_{i=1}^n \alpha_i 1_{x_iK}T$, so it is linear combination of compact operators, and therefore is itself a compact operator.

Our last claim is that for $T \in VN(G{:}K)$ and $\phi \in A(G{:}K)$ we have $\phi T \in AP(\widehat{G{:}K})$. There is $\{\phi_n\}_n \in A_c(G{:}K)$ such that $\phi_n \longrightarrow \phi$ in norm. So for $v \in A(G{:}K)$, $\|v\| \leq 1$ we have

$$|<(\phi_n - \phi) \cdot T, v>| \leq |<T, (\phi_n - \phi)v>| \leq \|T\|\|\phi_n - \phi\|\|v\|$$

which implies that $\|(\phi_n - \phi) \cdot T\| \leq \|\phi_n - \phi\|\|T\|$. Thus $\phi_n T \longrightarrow \phi T$ in the operator (uniform) norm (as operators from $A(G)$ to $VN(G)$). Since norm limits of compact operators are compact, it follows that $\phi T \in AP(\widehat{G{:}K})$.

We have shown that $A(G{:}K)VN(G{:}K) \subset AP(\widehat{G{:}K})$. Consequently, we obtain $UBC(\widehat{G{:}K}) \subset AP(\widehat{G{:}K})$. □

Corollary 5.5.10 *If G/K is an amenable coset space and K is compact and open, then*

$$UBC(\widehat{G{:}K}) = W(\widehat{G{:}K}) = AP(\widehat{G{:}K}).$$

Proof This follows from Proposition 5.5.8 and Proposition 5.5.9 □

B. Forrest has defined in [16] the space $C^*(G{:}K)$ to be the closure of $L^1(G{:}K)$ in $C^*(G)$ in the $\|\cdot\|_{C^*(G)}$ norm. This is a non-selfadjoint subalgebra of $C^*(G)$ and one can show that $[C^*(G{:}K)]^* = B(G{:}K)$. We define now the following subspace of $VN(G{:}K)$

$$C^*_\lambda(G{:}K) = \{\lambda(f) : f \in L^1(G{:}K)\}^{-\|\cdot\|_{VN(G:K)}}$$

Remark 5.5.11 When G/K is amenable we have $C^*(G{:}K) \simeq C^*_\lambda(G{:}K)$ since in this case the two norms agree.

Proposition 5.5.12 *Let G be a locally compact group and K a compact subgroup. Then*

$$C^*_\lambda(G{:}K) \subset UBC(\widehat{G{:}K}).$$

Proof Let $f \in C_{00}(G{:}K)$ such that supp $f = F \subset G$. Choose $\phi \in A(G{:}K)$ such that $\phi(t) = 1$ on F. Then

$$< \phi\cdot\lambda(f), \gamma > = < \lambda(f), \phi\gamma > = \int_G f(t)\phi(t)\gamma(t)dt = \int_G f(t)\gamma(t)dt = < \lambda(f), \gamma >$$

for any $\gamma \in A(G{:}K)$. Therefore $\phi\cdot\lambda(f) = \lambda(f)$ so $\lambda(f) \in A(G{:}K)VN(G{:}K) \subset UBC(\widehat{G{:}K})$. It follows that $C^*_\lambda(G{:}K) \subset UBC(\widehat{G{:}K})$. □

Suppose that G is an amenable locally compact group. Then $x \in UBC(\hat{G})$ is of the form $x = \phi \cdot y$ with $\phi \in A(G), y \in VN(G)$, and the map

$$\hat{x} : B(G) \longrightarrow \mathbf{C}$$

$$< \hat{x}, \psi > = < y, \phi \cdot \psi > \text{ for any } \psi \in B(G)$$

is well defined (see [26]). Now let $x_k \in UBC(\widehat{G{:}K}) \subset UBC(\hat{G})$ be of the form $x_k = \phi_k \cdot y_k$ with $\phi_k \in A(G{:}K), y_k \in VN(G{:}K)$. Define

$$\tilde{x}_k : B(G{:}K) \longrightarrow \mathbf{C}$$

$$\tilde{x}_k = \hat{x}_k|_{B(G:K)} \text{ that is , } < \tilde{x}, \psi > = < y_k, \phi_k \cdot \psi > \text{ for all } \psi \in B(G{:}K).$$

Clearly $\tilde{x}_k \in B(G{:}K)^*$ and it is well defined since $\hat{x}_k$ is well defined.

Now, we can define the map

$$\Pi_K : UBC(\widehat{G{:}K}) \longrightarrow B(G{:}K)^*$$

$$x \mapsto \tilde{x}$$

The next theorem proves that when G is amenable, $UBC(\widehat{G{:}K})$ is isometrically isomorphic to a closed subspace of $B(G{:}K)^*$. This theorem is proved for $K = \{e\}$ by A. Lau in [26] (Theorem 4.6). Our proof is motivated by the one given there.

Theorem 5.5.13 *Let G be an amenable locally compact group and let K be a compact subgroup of G. Then Π_K is an isometry onto a closed subspace of $B(G{:}K)^*$, such that Π_K extends the natural embedding of $C^*(G{:}K)$ into $B(G{:}K)^*$. Furthermore, if G is compact then Π_K is onto.*

Proof Let $x \in UBC(\widehat{G:K})$, $x = \phi \cdot y$ with $\phi \in A(G:K)$ and $y \in VN(G:K)$. We show first that $\|x\| = \|\tilde{x}\|$. We have

$$\|\tilde{x}\| = \sup\{|< y, \psi\phi >| : \ \psi \in B(G:K), \|\psi\| \leq 1\}$$

and

$$\|x\| = \sup\{|< y, \gamma\phi >| : \ \gamma \in A(G:K), \|\psi\| \leq 1\}.$$

Clearly, since $A(G:K) \subset B(G:K)$, we have $\|x\| \leq \|\tilde{x}\|$.

To show the converse inequality, we use the fact that $A(G:K)$ has a bounded approximate identity bounded by 1, since G is amenable (see [16]). Then, for any $\delta > 0$ there is $\phi_0 \in A(G:K)$ and $y_0 \in VN(G:K)$ with $\|\phi_0\| \leq 1$ such that $\|x - y_0\| \leq \delta$ and $x = \phi_0 y_0$ (Prop 32.50, [23], vol II). Hence, for $\psi \in B(G:K), \|\psi\| \leq 1$ we have

$$|< \tilde{x}, \psi >| = |< y_0, \phi_0\psi >| \leq \|y_0\| \leq \|x\| + \delta$$

which implies

$$\|\tilde{x}\| \leq \|x\| + \delta \ \Rightarrow \|x\| \geq \|\tilde{x}\|$$

Therefore, Π_K is a linear isometry.

Now notice that $C^*(G:K) \simeq C^*_\lambda(G:K)$ since G is amenable, and, since $C^*(G:K) \hookrightarrow B(G:K)^*$, it makes sense to say that Π_K extends the natural embedding $C^*(G:K) \hookrightarrow B(G:K)^*$.

Let $x = \lambda(f)$ with $f \in C_{00}(G:K)$, and denote the support of f by F. Let $\phi \in A(G:K)$ such that $\phi(t) = 1$ on F. Then $x = \phi \cdot x$. Hence

$$< \tilde{x}, \psi > = < x, \phi\psi > = \int f(t)\phi(t)\psi(t)dt = \int f(t)\psi(t)dt = < x, \psi >$$

for $\psi \in B(G:K)$. This shows that Π_K agrees with the natural embedding on a dense set of $C^*(G:K)$, and hence it must agree on $C^*(G:K)$.

Now we prove the last statement of the theorem. Suppose that G is compact. Then $1 \in A(G)$, therefore it is also in $A(G:K)$. Then any $x \in VN(G:K)$ can be written as $1 \cdot x$ so we have $VN(G\colon K) \subset UBC(\widehat{G\colon K})$. This gives $VN(G\colon K) = UBC(\widehat{G\colon K})$. Moreover we have $A(G\colon K) = B(G\colon K)$. Then

$$\Pi_K(UBC(\widehat{G\colon K})) = \Pi_K(VN(G\colon K)) = A(G\colon K)^* = B(G\colon K)^*$$

and therefore Π_K is onto. □

5.6 The Banach algebras $UBC(\widehat{G\colon K})^*$, $W(\widehat{G\colon K})^*$, and $AP(\widehat{G\colon K})^*$

In this section we shall define on each of the dual spaces $UBC(\widehat{G\colon K})^*$, $W(\widehat{G\colon K})^*$, and $AP(\widehat{G\colon K})^*$ the Arens product which turns them into a Banach algebra. The results obtained in this and the next section are generalizations of the those proved by A. Lau in [26], for $K = \{e\}$. The proofs are close in spirit to the ones given there.

We start with a few definitions.

Definition 5.6.1 *A subset $X \subset VN(G\colon K)$ is K-topologically invariant if*

$$A(G\colon K)X \subset X$$

If X is a K-topologically invariant subspace of $VN(G\colon K)$ we say that X is K-topologically introverted if for each $m \in X^*$, $x \in X$ the map

$$m \odot x : A(G\colon K) \longrightarrow \mathbf{C}$$

$$< m \odot x, \gamma >=< m, \gamma \cdot x >$$

defines an element of X.

Remark 5.6.2 Let $\sigma_1 = \sigma(VN(G\colon K), A(G\colon K))$ and $\sigma_2 = \sigma(VN(G), A(G))$ restricted to $VN(G\colon K)$. Then $\sigma_1 = \sigma_2$.

Indeed, let $\{T_i\}_i \in VN(G\colon K)$ be such that it converges in the σ_2-topology to T, which means that $< T_i, \gamma > \to < T, \gamma >$ for any $\gamma \in A(G)$. In particular, the above is true for any $\gamma \in A(G\colon K) \subset A(G)$. This means that $\{T_i\}_i$ converges in the σ_1 topology to T. Therefore $\sigma_2 \geq \sigma_1$. To show the other direction, suppose that $\{T_i\}_i$ converges in the σ_1-topology to T. This means that

$$< T_i, \gamma_k > \to < T, \gamma_k > \text{ for any } \gamma_k \in A(G\colon K).$$

Now, let $\gamma \in A(G)$. By Theorem 5.3.5 we have that $< T_i, \gamma > = < T_i, P_k\gamma >$. Therefore

$$< T_i, \gamma > = < T_i, P_k\gamma > \to < T, P_k\gamma > = < T, \gamma > \text{ for any } \gamma \in A(G),$$

so we obtain that $\sigma_2 \leq \sigma_1$ and the assertion is proved. We will denote both topologies by σ.

Next we give a characterization of the K-topologically introverted subspaces of $VN(G\colon K)$.

Lemma 5.6.3 *Let X be a K-topologically invariant subspace of $VN(G\colon K)$. Then X is K-topologically introverted if and only if $\overline{K(x)}^{\sigma} \subset X$ for any $x \in X$ where $K(x) = \{\phi \cdot x : \phi \in A(G\colon K),\ \|\phi\| \leq 1\}$.*

Proof "$\Rightarrow$" Suppose X is K-topologically introverted. Let $y \in \overline{K(x)}^{\sigma}$. Then there is a net $\{\phi_\alpha\} \subset A(G\colon K)$, $\|\phi_\alpha\| \leq 1$ such that $\phi_\alpha \cdot x \to y$ in the σ-topology which means that

$$< \phi_\alpha \cdot x, \gamma > \to < y, \gamma > \text{ for all } \gamma \in A(G\colon K).$$

Let m be a w^*-cluster point of $\{\phi_\alpha\}$ in $VN(G)^*$. Then

$$< \phi_\alpha, z > \to < m, z > \text{ for all } z \in VN(G\colon K)$$

Then for any $\gamma \in A(G{:}K)$

$$\begin{aligned} < y, \gamma > &= \lim_{\alpha} < \phi_\alpha \cdot x, \gamma > = \lim_{\alpha} < x, \phi_\alpha \gamma > \\ &= \lim_{\alpha} < \gamma \cdot x, \phi_\alpha > = < m, \gamma \cdot x > = < m \odot x, \gamma >, \end{aligned}$$

so $y = m \odot x \in X$. Hence $\overline{K(x)}^{\sigma} \subset X$.

"$\Leftarrow$" Now suppose that $\overline{K(x)}^{\sigma} \subset X$. Let $m \in VN(G{:}K)^*$, $\|m\| = 1$. By Goldstein's theorem, the unit ball of $A(G{:}K)$ is w^*-dense in the unit ball of $VN(G{:}K)^*$, so there is a net $\{\phi_\alpha\}_\alpha$ in the unit ball of $A(G{:}K)$ such that it converges to m in the w^*-topology.

Then $< m \odot x, \gamma > = < m, \gamma \cdot x > = \lim_\alpha < \phi_\alpha, \gamma \cdot x > = \lim_\alpha < \phi_\alpha \cdot x, \gamma >$ so $m \odot x = \sigma - \lim \phi_\alpha \cdot x$. Since $\phi_\alpha \cdot x \in K(x)$ it follows that $m \odot x \in \overline{K(x)}^{\sigma} \subset X$.
$\square$

The next theorem provides us with examples of K-topologically introverted subspaces of $VN(G{:}K)$.

Theorem 5.6.4 *Let G be a locally compact group and K a compact subgroup. The subspaces $UBC(\widehat{G{:}K}), W(\widehat{G{:}K}), AP(\widehat{G{:}K})$ are K-topologically introverted.*

Proof It is easy to see that these spaces are K-topologically invariant. We will show that they are K-topologically introverted. We start with $UBC(\widehat{G{:}K})$.

Let $m \in UBC(\widehat{G{:}K})^*$ and $x \in UBC(\widehat{G{:}K}), x = \gamma_0 \cdot z$ with $\gamma_0 \in A(G{:}K), z \in VN(G{:}K)$. Then

$$\begin{aligned} < m \odot x, \gamma > &= < m, \gamma \cdot x > = < m, \gamma(\gamma_0 \cdot z) > = < m, (\gamma\gamma_0) \cdot z > \\ &= < m \odot z, \gamma\gamma_0 > = < \gamma_0(m \odot z), \gamma > . \end{aligned}$$

Therefore $m \odot x = \gamma_0(m \odot z) \in A(G{:}K)VN(G{:}K) \subset UBC(\widehat{G{:}K})$.

Now suppose that $x \in UBC(\widehat{G{:}K})$. Then there is a net $\{x_\alpha\}_\alpha$ that belongs to $A(G{:}K)VN(G{:}K)$, such that

$$\|x_\alpha - x\| \to 0.$$

Then $\|m \odot x - m \odot x_\alpha\| \leq \|m\| \cdot \|x - x_\alpha\| \to 0$. By the above each $m \odot x_\alpha \in UBC(\widehat{G{:}K})$ so the first claim is proved.

We will show next that $W(\widehat{G{:}K})$ is K-topologically introverted. In fact we will show that any K-topologically invariant closed subspace $X \subset W(\widehat{G{:}K})$ is K-topologically introverted.

Let $x \in X$. Then

$$K(x) = \{\phi \cdot x : \phi \in A(G{:}K),\ \|\phi\| \leq 1\}$$

is weakly relatively compact since $x \in W(\widehat{G{:}K})$. Therefore $\overline{K(x)}^{w}$ is compact. Since the weak topology is stronger than ultraweak topology, it follows that they coincide on $\overline{K(x)}^{w}$. Now since $K(x)$ is a convex set we have $\overline{K(x)}^{\|\cdot\|} = \overline{K(x)}^{w}$. So we get that $\overline{K(x)}^{\sigma} = \overline{K(x)}^{\|\cdot\|}$. $(\star)$

Now since $K(x) \subset X$ and X is norm closed, we have that $\overline{K(x)}^{\|\cdot\|} \subset X$. Taking $(\star)$ into account we get $\overline{K(x)}^{\sigma} \subset X$. By Lemma 5.6.3, it follows that X is K-topologically introverted. $\square$

Remark 5.6.5 We know that $AP(\widehat{G{:}K}) \subset W(\widehat{G{:}K})$. By the proof above, it follows that $AP(\widehat{G{:}K})$ is K-topologically introverted as well.

Arens has showed that given a Banach algebra B, it is possible to define a multiplication on B^{**} that extends the multiplication on B. In the case that $B = A(G{:}K)$ and $m, n \in VN(G{:}K)^*$, the Arens product $m \odot n$ is defined by the formula

$$< m \odot n, x >=< m, n \odot x > \text{ for each } x \in VN(G{:}K).$$

The same formula makes sense when $VN(G{:}K)$ is replaced by a K-topologically invariant and K-topologically introverted subspace X of $VN(G{:}K)$. This turns $UBC(\widehat{G{:}K})^*$, $W(\widehat{G{:}K})^*$, and $AP(\widehat{G{:}K})^*$ into Banach algebras.

If G is amenable and $\phi \in B(G{:}K)$, let $\tilde{\phi}$ denote the functional on $UBC(\widehat{G{:}K})$ defined by

$$< \tilde{\phi}, x >=< \Pi_K x, \phi >=< \tilde{x}, \phi > \tag{5.11}$$

for any $x \in UBC(\widehat{G:K})$.

The next proposition lists some properties of the Arens product on $UBC(\widehat{G:K})^*$.

Proposition 5.6.6 *Let G be a locally compact group and let K a compact subgroup.*

i) For each $m \in UBC(\widehat{G:K})^*$ the map $n \mapsto n \odot m$ is weak-weak* continuous.*

ii) If G is amenable, then for each $\phi \in B(G:K)$ the map

$$UBC(\widehat{G:K})^* \longrightarrow UBC(\widehat{G:K})^*$$

$$m \mapsto \tilde{\phi} \odot m$$

is weak-weak* continuous.*

iii) If G is amenable and $\phi, \gamma \in B(G:K)$ then $\widetilde{\phi \cdot \gamma} = \tilde{\phi} \odot \tilde{\gamma}$.

Proof *i)* Trivial

ii) Our first claim is that

$$< \tilde{\phi} \odot m, x > = < m, \phi \cdot x > \text{ for all } x \in UBC(\widehat{G:K})$$

when $m \in UBC(\widehat{G:K})^*, \phi \in B(G:K)$.

Indeed, if $x = \gamma_0 \cdot z$ with $\gamma_0 \in A(G:K)$ and $z \in VN(G:K)$, then

$$m \odot x = \gamma_0(m \odot z).$$

Then,

$$\begin{aligned} < \tilde{\phi} \odot m, x > &= < \tilde{\phi}, m \odot x > = < \tilde{\phi}, \gamma_0(m \odot z) > \\ &= < (\gamma_0(m \odot z))\tilde{}, \phi > = < m \odot z, \gamma_0\phi > \\ &= < m, (\gamma_0\phi), z > = < m, \phi(\gamma_0 z) > = < m, \phi \cdot x > \end{aligned}$$

and the claim is proved.

Next we will see that the map

$$UBC(\widehat{G:K})^* \longrightarrow UBC(\widehat{G:K})^*$$

$$m \mapsto \tilde{\phi} \odot m$$

is weak*-weak* continuous. Let $\{m_\alpha\} \in UBC(\widehat{G:K})^*$ such that

$$m_\alpha \overset{weak^*}{\longrightarrow} m \text{ i.e } < m_\alpha, x > \longrightarrow < m, x > \text{ for all } x \in UBC(\widehat{G:K}).$$

Now let $x \in UBC(\widehat{G:K})$. Then

$$< \tilde{\phi} \odot m_\alpha, x > = < m_\alpha, \phi \cdot x > \to < m, \phi \cdot x > = < \tilde{\phi} \odot m, x >$$

iii) Let $\phi, \gamma \in B(G:K)$. Let $x \in UBC(\widehat{G:K})$, $x = \gamma_0 \cdot z$, with $\gamma_0 \in A(G:K)$ and $z \in VN(G:K)$. Then

$$\begin{aligned} < \tilde{\phi} \odot \tilde{\gamma}, x > &= < \tilde{\phi}, \tilde{\gamma} \odot x > = < \tilde{\phi}, \gamma_0(\tilde{\gamma} \odot z) > \\ &\overset{(5.11)}{=} < (\gamma_0(\tilde{\gamma} \odot z))\tilde{}, \phi > = < \tilde{\gamma} \odot z, \gamma_0 \phi > \\ &= < \tilde{\gamma}, (\phi\gamma_0) z > = < (\phi\gamma_0)\tilde{} \cdot z, \gamma > = < z, \phi\gamma_0\gamma > \\ &= < \widetilde{\gamma_0 z}, \phi\gamma > = < \widetilde{\phi\gamma}, x > \end{aligned}$$

Therefore $\widetilde{\phi \cdot \gamma} = \tilde{\phi} \odot \tilde{\gamma}$ and we are done. □

The next theorem characterizes $W(\widehat{G:K})$ as the maximal subspace X of $VN(G:K)$ for which the Arens product makes sense on X^* and the product is separately continuous with respect to the weak*-topology on bounded spheres.

Proposition 5.6.7 *Let X be a closed K-topologically invariant and K-topologically introverted subspace of $VN(G:K)$. The following are equivalent:*

i) $X \subset W(\widehat{G:K})$.

ii) The product in X^* is separately continuous with respect to the weak topology on bounded spheres.*

iii) X^* is a commutative Banach algebra.

Proof *i)* $\Rightarrow$ *ii)* It is clear that for any $m \in X^*$, the map $n \mapsto n \odot m$ is weak*-weak* continuous from X^* to X^*.

Next we will prove that the map $n \mapsto m \odot n$ is weak*-weak* continuous from X^* to X^*, for any $m \in X^*$. Let $\{n_\alpha\}$ be a net in X^* converging to some $n \in X^*$ in the weak*-topology

$$n_\alpha \xrightarrow{w*} n \text{ and } \|n_\alpha\| \leq M, \ \|n\| \leq M.$$

We may assume $M = 1$. For each $x \in X$, we have

$$n_\alpha \odot x \longrightarrow n \odot x \text{ in the topology } \sigma(VN(G{:}K), A(G{:}K)).$$

Let $K(x) = \{\phi{\cdot}x : \phi \in A(G) \text{ and } \|\phi\| \leq 1\}$. Then $K(x)$ is relatively compact in the weak topology of $VN(G)$. If $\overline{K(x)}^{\sigma}$ is the closure in the σ-topology, it follows that the weak topology coincides with the σ-topology on $\overline{K(x)}^{w}$. Then $\overline{K(x)}^{\|\cdot\|} = \overline{K(x)}^{\sigma}$.

Consequently, the net $\{n_\alpha\}$ which is in $\overline{K(x)}^{\sigma}$, also converges to $n \odot x$ in the weak topology. So, if $m \in X^*$, we have

$$< m \odot n_\alpha, x >=< m, n_\alpha \odot x >\rightarrow< m, n \odot x >=< m \odot n, x >$$

for any $x \in X$. Hence the map $n \mapsto m \odot n$ is weak*-weak* continuous.

ii) $\Rightarrow$ *iii)* Let $m \in X^*$ and γ be the restriction of an element in $A(G)$ to X. Let $\{\phi_\alpha\} \in A(G{:}K)$ be a net such that

$$< \phi_\alpha, x >\longrightarrow< m, x > \forall x \in X \text{ and } \|\phi_\alpha\| \leq \|m\|.$$

Then we have

$$< m \odot \gamma, x >= \lim_\alpha < \phi \odot \gamma, x >= \lim_\alpha < \gamma \odot \phi_\alpha, x >=< \gamma \odot m, x >.$$

Hence $m \odot \gamma = \gamma \odot m$.

A second application of $ii)$ and the weak* density of $A(G\colon K)$(restricted to X) in X^* shows that $m \odot \gamma = \gamma \odot m$ for each $m, n \in X^*$.

$iii) \Rightarrow i)$ Suppose that X^* is a commutative Banach algebra. Let $x \in X$ and consider the map

$$p : X^* \longrightarrow X$$

$$m \mapsto m \odot x$$

We will show that p is weak*-weak continuous. Let $m_\alpha \in X^*$ such that $m_\alpha \xrightarrow{w*} m$. For each $n \in X^*$ we have $< m_\alpha \odot x, n > = < n \odot m_\alpha, x > \overset{(iii)}{=} < m_\alpha \odot n, x >$. But

$$< m_\alpha \odot n, x > \longrightarrow < m \odot n, x > \overset{iii)}{=} < n \odot m, x > = < m \odot x, n >$$

and we can conclude that p is weak*-weak continuous.

Then the set $\{m \odot x : m \in X^* \text{ and } \|m\| \leq 1\}$ is relatively compact in the weak topology of X (hence of $VN(G\colon K)$). Since

$$\{\phi \odot x : \phi \in A(G) \text{ and } \|\phi\| \leq 1\} \subset \{m \odot x : m \in X^* \text{ and } \|m\| \leq 1\},$$

it follows that $x \in W(\widehat{G\colon K})$. □

Theorem 5.6.8 *Assume that G is amenable. Then the map*

$$Q : B(G\colon K) \longrightarrow UBC(\widehat{G\colon K})^*$$

$$\phi \mapsto \tilde{\phi}$$

where $< \tilde{\phi}, x > = < \prod_k(x), \phi >$ for $x \in UBC(\widehat{G\colon K})$, is a linear isometry and an algebra homomorphism. The image of $B(G\colon K)$ under Q is contained in the centre of the algebra $UBC(\widehat{G\colon K})^*$. Furthermore, if G/K is discrete, then $UBC(\widehat{G\colon K})^*$ is commutative.

Proof That Q is an algebra homomorphism follows from Proposition 5.6.6 *iii*). Now if $\phi \in B(G\colon K)$ and $x \in UBC(\widehat{G\colon K}), \|x\| \leq 1$ we have

$$|<\tilde{\phi}, x>| = |<\Pi_K(x), \phi> \leq \|\Pi_K(x)\|\|\phi\| = \|x\|\|\phi\| \leq \|\phi\|$$

which implies $\|\tilde{\phi}\| \leq \|\phi\|$.

On the other hand, since $C^*(G\colon K) \subset UBC(\widehat{G\colon K})$ and Π_K extends the natural embedding, we have $<\tilde{\phi}, x> = <\phi, x>$ for $x \in C^*(G\colon K)$. Hence $\|\tilde{\phi}\| = \|\phi\|$.

If $m \in UBC(\widehat{G\colon K})^*$, let $\{\phi_\alpha\} \in A(G\colon K)$ be such that

$$\tilde{\phi}_\alpha \xrightarrow{w*} m.$$

Then, if $\gamma \in B(G\colon K), x \in UBC(\widehat{G\colon K})$ we have

$$\begin{aligned} < m \odot \tilde{\gamma} > &= < m, \tilde{\gamma} \odot x > = \lim_\alpha < \tilde{\phi}_\alpha, \tilde{\gamma} \odot x > \\ &= \lim_\alpha < \tilde{\phi}_\alpha \odot \tilde{\gamma}, x > = \lim_\alpha < \widetilde{\phi_\alpha \gamma}, x > \\ &= \lim_\alpha < \widetilde{\gamma \phi_\alpha}, x > = \lim_\alpha < \tilde{\gamma} \odot \tilde{\phi}_\alpha, x > = < \tilde{\gamma} \odot m, x > . \end{aligned}$$

Therefore $\tilde{\gamma} \in Z(UBC(\widehat{G\colon K})^*)$.

For the last statement, notice that if G/K is discrete, then by Proposition 5.5.9, it follows that $UBC(\widehat{G\colon K}) \subset W(\widehat{G\colon K})$. By Proposition 5.6.7 now it follows that $UBC(\widehat{G\colon K})^*$ is commutative. □

5.7 Operators commuting with the action by $A(G\colon K)$ on subspaces of $VN(G\colon K)$

Let X be a K-topologically invariant subspace of $VN(G\colon K)$. We say that an operator $T : X \longrightarrow X$ *commutes with the action by* $A(G\colon K)$ if

$$T(\phi \cdot x) = \phi \cdot T(x) \text{ for all } \phi \in A(G\colon K),\ x \in X$$

We will give a characterization of the space of all such operators in terms of the dual of certain subspaces of $UBC(\widehat{G{:}K})$.

Now assume that G is amenable. Then, Cohen's factorization theorem and the existence of bounded approximate identities in $A(G{:}K)$, show that $A(G{:}K)\cdot X$ is a closed linear subspace of $VN(G)$. Moreover, $A(G{:}K)\cdot X$ is K-topologically introverted if X is.

For each $m \in (A(G{:}K)\cdot X)^*$, define

$$m_L : X \longrightarrow X, \ < m_L(x), \gamma > = < m, \gamma \cdot x >$$

for any $\gamma \in A(G{:}K)$ and $x \in X$.

Lemma 5.7.1 *Assume that G is amenable. Then*

(i) m_L *commutes with the action of* $A(G{:}K)$ *on* X

(ii) $\|m_L\| = \|m\|$.

Proof (i) Let $\phi \in A(G{:}K)$. Then

$$< m_L(\phi \cdot x), \gamma > = < m, \gamma \cdot (\phi \cdot x) > = < m, (\gamma\phi) \cdot x >$$

and

$$< \phi \cdot m_L(x), \gamma > = < m_L(x), \gamma\phi > = < m, (\gamma\phi) \cdot x > .$$

Therefore we obtain $m_L(\phi \cdot x) = \phi \cdot m_L(x)$.

(ii) Clearly $\|m_L\| \leq \|m\|$. To prove the converse inequality, let $\{\phi_\alpha\}_\alpha$ be an approximate identity in $A(G{:}K)$ with $\|\phi_\alpha\| \leq 1$. For each $z \in A(G{:}K)\cdot X$ we have $\|\phi_\alpha \cdot z - z\| \to 0$. Hence

$$\|m_L(z)\| \geq | < m_L(z), \phi_\alpha > | = | < m, \phi_\alpha \cdot z > | \to | < m, z > |$$

Therefore $\|m_L\| \geq \|m\|$. □

Theorem 5.7.2 *Assume that G is amenable. Let X be a K-topologically invariant and K-topologically introverted subspace of $VN(G{:}\,K)$. Then the map*

$$\tau : (A(G{:}\,K)X)^* \to \{T : X \to X \mid T(\phi \cdot x) = \phi \cdot T(x) \text{ for all } \phi \in A(G{:}\,K)\}$$

$$m \mapsto m_L$$

is a linear isometry and algebra homomorphism onto the space of all bounded linear operators commuting with the action of $A(G{:}\,K)$ on X.

Proof By Lemma 5.7.1 it is sufficient to show that τ is onto. Let $T : X \to X$ be such that it commutes with the action of $A(G{:}\,K)$ on X, and let $\{\phi_\alpha\}_\alpha$ be an approximate identity of $A(G{:}\,K)$. Then T maps $A(G{:}\,K)X$ into $A(G{:}\,K)X$.

Let m be a weak* cluster point of the net $\{T^*(\phi_\alpha)\}_\alpha$ in $A(G{:}\,K)X^*$. Then, for $x \in X$ and $\gamma \in A(G{:}\,K)$, we have

$$\begin{aligned} < T(x), \gamma > &= \lim_\alpha < T(x), \phi_\alpha \gamma >= \lim_\alpha < \gamma \cdot T(x), \phi_\alpha > \\ &= \lim_\alpha < T(\gamma \cdot x), \phi_\alpha >= \lim_\alpha < \gamma \cdot x, T^*(\phi_\alpha) > \\ &= < \gamma \cdot x, m >=< m_L(x), \gamma > . \end{aligned}$$

Therefore $T = m_L$.

To see that τ is an isomorphism, let $m, n \in (A(G{:}\,K)X)^*$. Then for each $\gamma \in A(G{:}\,K)$ and $x \in X$,

$$\begin{aligned} < (n \odot m_L)(x), \gamma > &= < n \odot m, \gamma \cdot x >=< n, m_L(\gamma \odot x) > \\ &= < n, \gamma \cdot m_L(x) >=< n_L(m_L(x)), \gamma > . \end{aligned}$$

Therefore we obtain $(n \odot m)_L = n_L(m_L)$. □

Corollary 5.7.3 *If G is amenable and X is a K-topologically invariant and K-topologically introverted closed subspace of $UBC(\widehat{G{:}\,K})$, then*

$$X^* \simeq \{T : X \to X \mid T(\gamma \cdot x) = \gamma \cdot T(x) \text{ for all } \gamma \in A(G{:}\,K)\}$$

isometrically and algebra isomorphically.

Proof It suffices to show $A(G{:}K) \cdot X = X$. Now, let $x \in X$ and $\{\phi_\alpha\}_\alpha$ be a bounded approximate identity for $A(G{:}K)$. Then $\|\phi_\alpha \cdot x - x\| \to 0$. Since $A(G{:}K) \cdot X$ is closed, it follows that $x \in A(G{:}K) \cdot X$ and we are done. $\square$

Corollary 5.7.4 *If G is amenable, then*

$$UBC(\widehat{G{:}K})^* \simeq \{T : VN(G{:}K) \to VN(G{:}K) \mid T(\gamma{\cdot}x) = \gamma{\cdot}T(x) \;\; \forall\, \gamma \in A(G{:}K)\}$$

isometrically and algebra isomorphically.

Corollary 5.7.5 *If G is amenable, then $B(G{:}K)$ is isometric and algebra isomorphic to the space of all bounded operators commuting with action of $A(G{:}K)$ on $C^*(G{:}K)$.*

Chapter 6

Open Questions

Question 1. What classes of groups satisfy property (S)? (see Definition 3.4.8)

All we know is that for $n = 1$, relation (3.10) is satisfied for any locally compact group by Lemma 3.4.2. For these groups a generalization of Theorem 3.3.4 holds true, as it was seen in the discussion in Section 3.4.

Question 2. Let G, H be two locally compact groups, let $\phi : A(G) \longrightarrow B(H)$ a completely bounded algebra homomorphism and $\tilde{\phi} : A(G_d) \longrightarrow B(H_d)$ be the map constructed from ϕ as in Section 3.4. Under what conditions is $\tilde{\phi}$ completely bounded?

The answer to this question would provide us with an hypothesis under which Theorem 3.3.4 also holds true.

Question 3. Let G, H be locally compact groups and $\phi : A(G) \longrightarrow A(H)$ a completely bounded algebra homomorphism. When does the following equality hold true:

$$\phi(A(G)) = \left\{ f \in A(H) : \begin{array}{l} \phi^*(h_1) = 0 \Rightarrow f(h_1) = 0 \\ \phi^*(h_1) = \phi^*(h_2) \Rightarrow f(h_1) = f(h_2) \quad h_1, h_2 \in H \end{array} \right\}.$$

We have proved in Theorem 4.3.1 that if G and H are discrete groups with G amenable, this is true. To apply the methods of Chapter 4, we need G to be amenable in order to ensure the existence of bounded approximate identities in ideals of $A(G)$ (see [18]), and to ensure that given a closed subgroup G_0 of G, the map $u \mapsto u|_{G_0}$ from $A(G)$ into $A(G_0)$ is onto. Moreover, a generalization of Theorem 3.3.4 is instrumental, so the answers to Question 1 and Question 2 will be very useful.

Question 4. In [30] V. Losert showed that a locally compact group is amenable if and only if each multiplier on the Fourier algebra $A(G)$ is given by a function from the Fourier-Stieltjes algebra $B(G)$. Can we say the same in the context of the Fourier algebra associated to the coset space G/K, for a compact subgoup K of a locally compact group G?

In [16], B. Forrest showed one direction, namely if G/K is an amenable coset space, then each multiplier on the Fourier algebra $A(G/K)$ is given by a function from the Fourier-Stieltjes algebra $B(G/K)$. The inverse direction is still open.

Question 5. In Theorem 5.5.2 it is shown that if G/K is an amenable coset space then $A(G{:}K)VN(G{:}K)$ is closed. In particular, it follows that $UBC(\widehat{G{:}K}) = A(G{:}K)VN(G{:}K)$. Is the converse true?

For $K = \{e\}$ the converse was proved by A. Lau and V. Losert in [28] (Proposition 7.1). V. Losert's theorem [30] which was mentioned above plays a key role in the proof. A positive answer to Question 4, would give us hope for a positive answer here as well.

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