University of Alberta

Chow Motives of Del Pezzo Surfaces of Degree 5 and 6

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Abstract

We determine the decomposition of the Chow motive of a Del Pezzo surface S of degree 5 or 6 with a K-rational point $\mathrm{pt} : K \to S$ into a direct sum of Chow motives. In each case, we give a $\mathrm{Gal}\left(\overline{K}/K\right)$-permutation resolution of the Picard group $\mathrm{Pic}\left(\overline{K} \times_K S\right)$ and deduce that there is some étale algebra E such that the corresponding twisted motive $(\mathrm{Spec}\, E, \mathrm{id}_{\mathrm{Spec}\, E})\,(1)$ is isomorphic to the direct sum of $(S, \mathrm{id}_S - (\mathrm{pt} \times S + S \times \mathrm{pt}))$ and $(\mathrm{Spec}\, K, \mathrm{id}_{\mathrm{Spec}\, K})\,(1)$.

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Chapter 1

Introduction

The notion of motives was introduced by Grothendieck in a letter to Serre in 1964 and a corresponding homology theory, called motivic cohomology, was conjectured around that time. His idea was to take the category of smooth, projective schemes over a field K and to replace the usual morphisms of schemes by correspondences, i. e. algebraic cycles modulo an adequate equivalence relation. In general, there are many possible equivalence relations that one might consider, for example rational, algebraic, homological, or numerical equivalence, which yield different categories of motives.

Also in the 1960s, Grothendieck conjectured about the existence of a category known as the category of mixed motives $MM(K)$ over a fixed field K. To this day, it is an unresolved question as to whether or not this category exists. In an attempt to find a suitable definition for $MM(K)$, a category known as the category of geometric motives, denoted $DM(K)$ was created. The category $DM(K)$ has been and continues to be actively studied by many authors and has proven to be a very significant category in the area of algebraic geometry. One of the most important applications is in Voevodsky's proof of the Milnor conjecture.

In this thesis, we are concerned with the category of Chow motives $\mathrm{Chow}(K)$ which uses rational equivalence. The construction of the category $\mathrm{Chow}(K)$ is far simpler than that of $DM(K)$ which means that results are more accessible in this setting than in $DM(K)$. Since there is a fully faithful embedding from $\mathrm{Chow}(K)$ into a subcategory of $DM(K)$, results concerning $\mathrm{Chow}(K)$ contribute to the understanding of $DM(K)$.

The goal of this thesis is to describe how the Chow motive of a Del Pezzo surface of degree 5 or 6 with a K-rational point splits into a direct sum of Chow motives.

In the first chapter, we review important notions from various branches of mathematics. The geometric prerequisites include Picard groups and Chow groups, which can be found in Hartshorne's book on algebraic geometry [9] and Fulton's book on intersection theory [3]. The algebraic prerequisites include material from [2] on permutation modules and basic results from representation theory on root systems which can be found in Humphreys' book [10]. We also formally define Del Pezzo surfaces, the schemes whose motives we are interested in.

In Chapter 2, we present the definition of the category of Chow motives as given in Manin's paper [13]. Furthermore, we state the general decomposition of the motive of an equidimensional variety X with a K-rational point, i. e. a morphism $\mathrm{pt} : \operatorname{Spec} K \to X$, and show that

$$\begin{aligned}(X, \mathrm{id}_X) \cong{} & (\operatorname{Spec} K, \mathrm{id}_{\operatorname{Spec} K}) \\ & \oplus (X, \mathrm{id}_X - (\mathrm{pt} \times X + X \times \mathrm{pt})) \oplus (\operatorname{Spec} K, \mathrm{id}_{\operatorname{Spec} K})\,(\dim X).\end{aligned}$$

Since the motives of $\operatorname{Spec} K$ are well understood, this thesis focuses on a detailed description of the middle term $(S, \mathrm{id}_S - (\mathrm{pt} \times S + S \times \mathrm{pt}))$ for a Del Pezzo surface of degree 5 or 6 with a K-rational point.

In the third chapter, we briefly recall the structure of the Picard group of Del Pezzo surfaces, which is based on Manin's work [14]. We present two permutation resolutions that we found for the Picard groups of Del Pezzo surfaces of degree 5 and 6.

Etale algebras are introduced in Chapter 4 where we state relevant results from Bourbaki's elements [1] and the standard book on involutions [12]. For the Galois extension $\overline{K}/K$, we prove that the category of $\mathrm{Gal}(\overline{K}/K)$-permutation modules is equivalent to the category of motives of étale algebras over K.

In the fifth chapter, we use the explicit permutation resolutions from Chapter 3 to show how to construct an étale algebra E over K in such a way that the twisted motive of $\operatorname{Spec} E$ decomposes into the direct sum of the middle term (S, ρ) and a twisted motive of $\operatorname{Spec} K$, i. e. we prove that

$$(\operatorname{Spec} E, \mathrm{id}_{\operatorname{Spec} E})(1) \cong (S, \mathrm{id}_S - (\mathrm{pt} \times S + S \times \mathrm{pt})) \oplus (\operatorname{Spec} K, \mathrm{id}_{\operatorname{Spec} K})\,(1).$$

Chapter 2

Preliminaries

Let K be a perfect field with algebraic closure $\overline{K}$, X a projective scheme over K with structure sheaf $\mathcal{O}_X$. We write $\overline{X} = X \times_K \overline{K}$ and in general, if L/K is a field extension, we write $X_L = L \times_K X$. We use *(projective) surface* to denote a smooth, projective, geometrically integral scheme over K of dimension 2.

2.1 Algebraic Geometry: The Picard Group, the Intersection Form and Chow Groups

The Picard Group

Definition. An *invertible sheaf over* X is a locally free $\mathcal{O}_X$-module of rank 1.

The set of isomorphism classes of invertible sheaves forms a group with multiplication induced by the sheaf tensor product, i. e. for two sheaves $\mathcal{L}_1$ and $\mathcal{L}_2$ over X, we have

$$[\mathcal{L}_1] \cdot [\mathcal{L}_2] := [\mathcal{L}_1 \otimes \mathcal{L}_2],$$

where $[\mathcal{L}_i]$ denotes the isomorphism class of the invertible sheaf $\mathcal{L}_i$. The neutral element is the sheaf $\mathcal{O}_X$ regarded as an $\mathcal{O}_X$-module and the inverse of an invertible sheaf $\mathcal{F}$ is the dual

$$\mathcal{F}^\vee = \mathcal{H}\mathrm{om}_{\mathcal{O}_X}\left(\mathcal{F}, \mathcal{O}_X\right),$$

where the right hand side is the Hom-sheaf from $\mathcal{F}$ into $\mathcal{O}_X$.

Definition. The group of isomorphism classes of invertible sheaves with the tensor product is called the *Picard group*, $\operatorname{Pic} X$, of X.

We can also describe the Picard group in terms of homology groups in the Zariski homology and the étale homology (c. f. Chapter III, Example 2.22 in [15]): Let $\mathcal{O}_X^\times$ denote the sheaf of invertible elements in $\mathcal{O}_X$, and $\mathbb{G}_m$ the multiplicative group of K; then

$$\mathrm{H}^1_{\text{ét}}\left(X, \mathbb{G}_m\right) \cong \operatorname{Pic} X \cong \mathrm{H}^1_{\text{Zar}}\left(X, \mathcal{O}_X^\times\right).$$

The Intersection Form

We consider a symmetric bilinear form on the Picard group for projective surfaces S called the intersection form following Mumford [16, Lecture 12]. A more general approach for intersections of pseudo-divisors with an algebraic cycle in the Chow ring can be found in Fulton's book [3].

Definition. A sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules is called *of finite type* over $\mathcal{O}_X$, if for every $x \in X$, there is an open neighborhood $U \subset X$ of x such that the restriction $\mathcal{F}(U)$ is generated by a finite number of sections.

The sheaf $\mathcal{F}$ is called *coherent* if it is of finite type and for every open $U \subset X$, every integer $n \geq 0$, and every morphism $\phi : \mathcal{O}_X^n(U) \to \mathcal{F}(U)$, the kernel of ϕ is of finite type.

Let S be a surface over K. The global section functor Γ_S from the category of coherent sheaves on S into the category of abelian groups is given by

$$\Gamma_S : \mathcal{L} \mapsto \mathcal{L}(S).$$

Since Γ_S is left exact, this functor admits a sequence of right derived functors denoted by $H^i(S, \mathcal{L})$. If S is proper over K, then each of the spaces $H^i(S, \mathcal{L})$ are finite dimensional K-vector spaces with $H^i(S, \mathcal{L}) = 0$ for all $i > \dim S$, for details see Grothendieck [6]. Therefore, we can define the *Euler-Poincaré-Characteristic* of $\mathcal{L}$ to be

$$\chi(\mathcal{L}) := \sum_{i \in \mathbb{Z}_{\geq 0}} (-1)^i \dim_K \left(H^i(S, \mathcal{L})\right),$$

where $\mathbb{Z}_{\geq 0} := \{i \in Z \mid i \geq 0\}$. As a direct consequence, if $\mathcal{L}' \cong \mathcal{L}$, then $H^i(S, \mathcal{L}') \cong H^i(S, \mathcal{L})$. Thus these vector spaces have the same dimensions and hence

$$\chi(\mathcal{L}') = \chi(\mathcal{L}),$$

which proves that the Euler-Poincaré-Characteristic is independent of the choice of the representative of an isomorphism class of sheaves.

Since invertible sheaves are coherent, we can define the intersection form on S as follows:

Definition. The *intersection form* on S is given by

$$
\begin{aligned}
(-,-) &: \operatorname{Pic} S \times \operatorname{Pic} S \to \mathbb{Z} \\
([\mathcal{L}_1], [\mathcal{L}_2]) &:= (\mathcal{L}_1 \cdot \mathcal{L}_2) := \chi(\mathcal{O}_S) - \chi\left(\mathcal{L}_1^{-1}\right) - \chi\left(\mathcal{L}_2^{-1}\right) + \chi\left(\mathcal{L}_1^{-1} \otimes \mathcal{L}_2^{-1}\right).
\end{aligned}
$$

Proposition 2 in [16] states that the intersection form on S is a symmetric bilinear form on $\operatorname{Pic} S$, i. e. we have

$$
\begin{aligned}
([\mathcal{L}_2], [\mathcal{L}_1]) &= ([\mathcal{L}_1], [\mathcal{L}_2]) \\
\left(\left[\mathcal{L}_1^{-1}\right], [\mathcal{L}_2]\right) &= -([\mathcal{L}_1], [\mathcal{L}_2]) \\
([\mathcal{L}_1 \otimes \mathcal{L}_1'], [\mathcal{L}_2]) &= ([\mathcal{L}_1], [\mathcal{L}_2]) + ([\mathcal{L}_1'], [\mathcal{L}_2])
\end{aligned}
$$

for the equivalence classes of any three invertible sheaves $\mathcal{L}_1$, $\mathcal{L}_1'$ and $\mathcal{L}_2$.

Chow Groups

Definition. For any integer $m \geq 0$, the free abelian group $Z_m(X)$ generated by the m-dimensional integral subvarieties of X is called the *group of cycles of dimension* m. Its elements, known as m-*cycles*, are finite formal sums $\sum_i n_i[V_i]$ where $n_i \in \mathbb{Z}$ and V_i is an m-dimensional integral subvariety of X.

The Chow group of dimension m will be defined shortly as a factor group of the group of cycles of dimension m. To that end, let $W \subset X$ be an integral subvariety, $V \subset W$ an irreducible subvariety of codimension 1 in W, $K(W)$ the corresponding field of rational functions and $R := \mathcal{O}_{W,V} \subseteq K(W)$ the local ring of X along V. Then R is a local ring without zero divisors with fraction field isomorphic to $K(W)$. Further, since R has dimension 1, for any element $0 \neq r \in R$, we have $\dim R/rR = 0$. As any noetherian ring of dimension 0 is Artinian, we can define the length of R/rR as the length of a composition series of the R-module R/rR. This induces a well defined map

$$
\begin{aligned}
l_{W,V} : R \backslash \{0\} &\to \mathbb{Z} \\
r &\mapsto \operatorname{length}_R(R/rR)
\end{aligned}
$$

In the fraction field $K(W)$, any element can be written as a quotient of two elements of R. Since the length function is additive on short exact sequences, this induces a well defined map

$$\begin{aligned}\operatorname{ord}_{W,V} : K(W)^\times &\to \mathbb{Z}\\ \frac{r}{s} &\mapsto l_{W,V}(r) - l_{W,V}(s),\end{aligned}$$

see [3, Appendix A] for details.

Definition. For an irreducible subvariety W of X of dimension $m+1$ and $f \in K(W)^\times$, define the *divisor of f* to be

$$\operatorname{div}(f) := \sum_{\substack{V\subset W \text{ irreducible,}\\ \dim(V)=m}} \operatorname{ord}_{W,V}(f)[V].$$

This sum is finite since for any $f \in K(W)^\times$, there are only finitely many irreducible subvarieties $V \subset W$ of codimension 1 such that $\operatorname{ord}_{W,V}(f) \neq 0$ (c. f. Fulton [3, Appendix B.4.3]).

Definition. We denote by $\operatorname{Rat}_m(X)$ the group generated by the divisors $\operatorname{div}(f)$ for all $f \in K(W)^\times$ for all $(m+1)$-dimensional subvarieties $W \subset X$. The *Chow group of cycles of dimension m* is the factor group

$$\mathrm{CH}_m(X) := Z_m(X)/\operatorname{Rat}_m(X).$$

Elements in the same equivalence class are said to be *rationally equivalent*. If $X = \operatorname{Spec}(R)$ is affine, we simply write $\mathrm{CH}_m(R)$ instead of $\mathrm{CH}_m(\operatorname{Spec} R)$. For a smooth scheme X, define the *Chow ring*

$$\mathrm{CH}_*(X) := \bigoplus_{m=0}^{\dim X} \mathrm{CH}_m(X).$$

This is a ring with multiplication induced by the composition of the exterior product

$$\begin{aligned}\mathrm{CH}_*(X) \otimes \mathrm{CH}_*(X) &\to \mathrm{CH}_*(X \times_K X)\\ [V] \otimes [T] &\mapsto [V \times T]\end{aligned}$$

with the refined Gysin homomorphism $\Delta^! : \mathrm{CH}_*(X \times_K X) \to \mathrm{CH}_*(X)$ for the diagonal map $\Delta_X : X \hookrightarrow X \times_K X$, which is a regular immersion since X is smooth over K. This turns $\mathrm{CH}_*(X)$ into a unital ring with multiplicative identity denoted by $[X]$ or 1_X in $\mathrm{CH}_{\dim X}(X)$.

If X is smooth with irreducible components $X_1, \dots, X_e$, we define

$$\mathrm{CH}^m(X) := \bigoplus_i \mathrm{CH}^m(X_i) = \bigoplus_i \mathrm{CH}_{\dim X_i - m}(X_i)$$

and if X is equidimensional, we let $\mathrm{CH}^*(X) := \bigoplus_{m=0}^{\dim X} \mathrm{CH}^m(X)$. The ring $\mathrm{CH}^*(X)$ is a graded ring. The multiplication therefore satisfies

$$\begin{aligned} \mathrm{CH}_i(X) \times \mathrm{CH}_j(X) &\to \mathrm{CH}_{i+j-\dim X}(X) \\ ([V],[T]) &\mapsto [V]\cdot[T]. \end{aligned} \tag{2.1}$$

In the remaining part of this section, we will explain the connection between the Picard group $\mathrm{Pic}(X)$ and the Chow group $\mathrm{CH}_{n-1}(X)$ for an equidimensional variety X. A *Cartier divisor* is defined as a pair (U_α, f_α), where the U_α form an open covering of X, and f_α is a non-zero function in the field of rational functions $K(U_\alpha)$ such that f_α/f_β is a unit on $U_\alpha \cap U_\beta$. Two pairs (U_α, f_α) and (V_β, g_β) *represent the same Cartier divisor* if f_α/g_β is a unit in $K[U_\alpha \cap V_\beta]$. With the addition of Cartier divisors $D = (U_\alpha, f_\alpha)$ and $D' = (V_\beta, g_\beta)$ defined as follows:

$$D + D' = (U_\alpha \cap V_\beta, f_\alpha g_\beta),$$

the Cartier divisors form the group $\mathrm{CaDiv}(X)$ of Cartier divisors on X. Furthermore, a Cartier divisor D is called *principal* if there is $f \in K[X]^\times$ such that D and (X, f) represent the same Cartier divisor. Since the principal Cartier divisors form a subgroup of $\mathrm{CaDiv}(X)$, we can define the *group of classes of Cartier Divisors*, denoted $\mathrm{CaCl}(X)$, as the quotient

$$\mathrm{CaCl} := \mathrm{CaDiv}\,/\text{ group of principal Cartier divisors.}$$

Our first step is to define an isomorphism from the Picard group of X into the group $\mathrm{CaCl}(X)$. To that end, recall that there is an equivalence of categories of the category of locally free sheaves on X and the category of vector bundles over X, hence we can identify any invertible sheaf $\mathcal{L}$ with the corresponding line bundle L.

Lemma 2.1. $\mathrm{Pic}(X)$ *is isomorphic to* $\mathrm{CaCl}(X)$ *via*

$$\begin{aligned} \mathrm{Pic}(X) &\to \mathrm{CaCl}(X) \\ [\mathcal{L}] &\mapsto [(U_\beta, g_{\alpha\beta})] \end{aligned}$$

for some fixed α, where U_β is the open covering and $g_{\alpha\beta}$ are the transition functions of the line bundle L associated to the invertible sheaf $\mathcal{L}$

Let $D = (U_\alpha, f_\alpha)$ denote a Cartier divisor. If $V \subset X$ is an irreducible subvariety of codimension 1, let

$$\mathrm{ord}_V(D) := \mathrm{ord}_{U_\alpha, V\cap U_\alpha}(f_\alpha)$$

for any α such that $U_\alpha \cap V \neq \emptyset$. This definition is independent of α since for two such indices α, β, the functions f_α and f_β only differ by units.

Lemma 2.2 (Corollaire 21.6.10 in [7])**.** *If X is locally factorial, then* $\mathrm{CaCl}(X)$ *is isomorphic to* $\mathrm{CH}_{n-1}(X)$ *via*

$$\begin{aligned} \mathrm{CaCl}(X) &\to \mathrm{CH}_{n-1}(X) \\ D &\mapsto [D] := \sum_V \mathrm{ord}_V(D)[V]. \end{aligned}$$

Corollary 2.3. *Let S be a smooth projective surface. The two previous lemmas yield that* $\mathrm{Pic}(S) \overset{\cong}{\to} \mathrm{CaCl}(X) \overset{\cong}{\to} \mathrm{CH}_1(S)$ *via the given isomorphisms.*

Furthermore, also the intersection form on $\mathrm{Pic}(S)$ for smooth, projective surfaces S described in this section corresponds to an operation on the Chow group $\mathrm{CH}_1(S)$. To that end, let X be a complete scheme and $\alpha = \sum_P n_P[P]$ a zero-cycle on X. Define the *degree* of α to be

$$\deg(\alpha) := \int_X \alpha := \sum_P n_P[K(P) : K],$$

where $[K(P) : K]$ denotes the degree of the field extension of $K(P)$ over K. One can then prove (c. f. [16]) for the isomorphism classes of two line bundles $\mathcal{L}_1, \mathcal{L}_2$ in $\mathrm{Pic}(S)$ and their corresponding cycles λ_1, λ_2 in $\mathrm{CH}_1(S)$, we have an identity

$$(\mathcal{L}_1, \mathcal{L}_2) = \deg(\lambda_1 \cdot \lambda_2).$$

2.2 Del Pezzo Surfaces

In this thesis, we will mainly work with a special type of surface known as a Del Pezzo surface. We will first recall the definition of the anticanonical sheaf and an anticanonical divisor.

Definition. Let Ω_X be the cotangent bundle on an equidimensional scheme X. The *canonical bundle* on X is $\bigwedge^{\dim X} \Omega_X$. The corresponding (invertible) sheaf of sections ω_X is called the *canonical sheaf* and its inverse ω_X^{-1} is referred to as *anticanonical sheaf*. Any divisor K_X such that $\mathcal{O}(K_X) \cong \omega_X$ is called a *canonical divisor*. An *anticanonical divisor* is a divisor $-K_X$ such that K_X is a canonical divisor.

Definition. An invertible sheaf $\mathcal{L}$ is said to be *ample* if for every coherent sheaf $\mathcal{F}$ on X, there is an integer $n_0 > 0$ such that for every $n \geq n_0$, the sheaf $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is generated by its global sections.

Definition. A *Del Pezzo surface* is a smooth surface S such that $\overline{S} = S \times_K \overline{K}$ is birationally equivalent to $\mathbb{P}^2_{\overline{K}}$ and the anticanonical sheaf ω_S^{-1} is ample. The *degree* of a Del Pezzo surface is $d = (\omega_S, \omega_S)$, where $(-,-)$ is the intersection form on S.

Example. (a) The projective plane $\mathbb{P}^2_K$ is a Del Pezzo surface of degree 9.

(b) Let $K = \overline{K}$ be algebraically closed. We say that points $P_1, \ldots, P_r \in \mathbb{P}^2_{\overline{K}}$ for $2 \leq r \leq 8$ are *in general position* if no three of them are collinear, no six of them lie on a conic and no eight of them on a cubic having a node at one of them. For $1 \leq d \leq 7$, the blow up of $\mathbb{P}^2_{\overline{K}}$ in $r := 9 - d$ points $P_1, \ldots, P_r$ in general position is a Del Pezzo surface of degree d.

(c) The image of the rational map $\varphi : \mathbb{P}^2_K \to \mathbb{P}^6_K$ given by

$$[x : y : z] \mapsto [x^2y : x^2z : xy^2 : xyz : xz^2 : y^2z : yz^2]$$

is a Del Pezzo surface of degree 6 (c. f. [18]).

We are particularly interested in Del Pezzo surfaces of degrees 5 and 6 that have a K-rational point, i. e. there is a K-morphism $\mathrm{pt} : \operatorname{Spec} K \to S$. As was proven in [17], any Del Pezzo surface of degree 5 has a K-point but in general this does not hold for Del Pezzo surfaces of degree 6. In the remainder of this chapter we will provide important lemmas which can be applied to Del Pezzo surfaces of those degrees.

Lemma 2.4 (Künneth formula, Lemma 27.1.10 in [14])**.** *Let $\overline{S}_1$, $\overline{S}_2$ be rational surfaces over an algebraically closed field $\overline{K}$. Then*

$$\bigoplus_{i=0}^{n} \mathrm{CH}_i(\overline{S}_1) \otimes \mathrm{CH}_{n-i}(\overline{S}_2) \xrightarrow{\cong} \mathrm{CH}_n\left(\overline{S}_1 \times_{\overline{K}} \overline{S}_2\right)$$
$$\sum_i \alpha_i \otimes \beta_{n-i} \mapsto \sum_i \alpha_i \times \beta_{n-i}$$

for $0 \leq n \leq 4$.

The action of the Galois group $G = \mathrm{Gal}(\overline{K}/K)$ on $\overline{K}$ induces an action on $\operatorname{Spec} \overline{K}$, on $S \times_K \overline{K}$ and therefore also on $\operatorname{Pic}\left(S \times_K \overline{K}\right)$. Let $H \subset G$ be an open subgroup. We then let $\operatorname{Pic}\left(S \times_K \overline{K}\right)^H$ denote the invariant elements of $\operatorname{Pic}\left(S \times_K \overline{K}\right)$ under the action of H.

Lemma 2.5. *Let S be a surface with a K-rational point and $H \subseteq G$ a closed subgroup. Then*

$$\operatorname{Pic}\left(S \times_K \overline{K}^H\right) \overset{\cong}{\to} \left(\operatorname{Pic}\left(S \times_K \overline{K}\right)\right)^H$$

induced by the inclusion $\overline{K}^H \hookrightarrow \overline{K}$.

Sketch of the proof. It suffices to prove that $\operatorname{Pic} S \cong \left(\operatorname{Pic} \overline{S}\right)^G$. The Hochschild-Serre spectral sequence

$$\mathrm{H}^p\left(G, \mathrm{H}^q_{\text{ét}}\left(\overline{S}, \mathbb{G}_m\right)\right) \Rightarrow \mathrm{H}^{p+q}_{\text{ét}}\left(S, \mathbb{G}_m\right),$$

(c. f. Theorem III.2.20 in [15]) gives an exact sequence, the low term exact sequence is

$$\begin{aligned} 0 &\to \mathrm{H}^1\left(G, \mathrm{H}^0_{\text{ét}}\left(\overline{S}, \mathbb{G}_m\right)\right) \to \mathrm{H}^1_{\text{ét}}\left(S, \mathbb{G}_m\right) \\ &\to \mathrm{H}^0\left(G, \mathrm{H}^1_{\text{ét}}\left(\overline{S}, \mathbb{G}_m\right)\right) \to \mathrm{H}^2\left(G, \mathrm{H}^0_{\text{ét}}\left(\overline{S}, \mathbb{G}_m\right)\right) \\ &\to \ker\left(\mathrm{H}^2_{\text{ét}}\left(S, \mathbb{G}_m\right) \to \mathrm{H}^0\left(G, \mathrm{H}^2_{\text{ét}}\left(\overline{S}, \mathbb{G}_m\right)\right)\right) \end{aligned} \tag{2.2}$$

Since $\mathrm{H}^0_{\text{ét}}\left(\overline{S}, \mathbb{G}_m\right) = \overline{K}^\times$, we have by Hilbert 90

$$\mathrm{H}^1\left(G, \mathrm{H}^0_{\text{ét}}\left(\overline{S}, \mathbb{G}_m\right)\right) = \mathrm{H}^1\left(G, \overline{K}^\times\right) = 0$$

and

$$\mathrm{H}^2\left(G, \mathrm{H}^0_{\text{ét}}\left(\overline{S}, \mathbb{G}_m\right)\right) = \mathrm{H}^2\left(G, \overline{K}^\times\right) = \operatorname{Br}(K),$$

where $\operatorname{Br}(K)$ denotes the Brauer group of K. From $\mathrm{H}^1_{\text{ét}}\left(\overline{S}, \mathbb{G}_m\right) = \operatorname{Pic} \overline{S}$, we obtain

$$\mathrm{H}^0\left(G, \mathrm{H}^1_{\text{ét}}\left(\overline{S}, \mathbb{G}_m\right)\right) = \mathrm{H}^0\left(G, \operatorname{Pic} \overline{S}\right) = \left(\operatorname{Pic} \overline{S}\right)^G.$$

Finally, let $\operatorname{Br}(X)$ denote the Brauer group of a scheme X. Then $\mathrm{H}^2_{\text{ét}}\left(S, \mathbb{G}_m\right) = \operatorname{Br}(S)$ and $\mathrm{H}^2_{\text{ét}}\left(\overline{S}, \mathbb{G}_m\right) = \operatorname{Br}(\overline{S})$ imply that

$$\begin{aligned} &\ker\left(\mathrm{H}^2_{\text{ét}}\left(S, \mathbb{G}_m\right) \to \mathrm{H}^0\left(G, \mathrm{H}^2_{\text{ét}}\left(\overline{S}, \mathbb{G}_m\right)\right)\right) \\ &\qquad = \ker\left(\operatorname{Br}(S) \to \mathrm{H}^0\left(G, \operatorname{Br}(\overline{S})\right)\right). \end{aligned}$$

The exact sequence (2.2) then takes the form

$$0 \to \operatorname{Pic} S \to \left(\operatorname{Pic} \overline{S}\right)^G \to \operatorname{Br}(K) \to \ker\left(\operatorname{Br}(S) \to \mathrm{H}^0\left(G, \operatorname{Br}(\overline{S})\right)\right).$$

Since S has a K-point, the map $\mathrm{Br}(K) \to \ker\left(\mathrm{Br}(S) \to \mathrm{H}^0\left(G, \mathrm{Br}(\overline{S})\right)\right)$ is split injective and hence we have an isomorphism

$$\operatorname{Pic} S \xrightarrow{\cong} \left(\operatorname{Pic} \overline{S}\right)^G. \qquad \square$$

2.3 Permutation Modules and Permutation Resolutions

In this section, let G be a profinite group and $H \subset G$ a closed subgroup of finite index. Since every profinite group is compact, this is equivalent to $H \subset G$ being open.

Definition. Let M be a $\mathbb{Z}[G]$-module which is finitely generated as a $\mathbb{Z}$-module. Consider the discrete topology on M and the product topology on $G \times M$. The module M is called a *continuous module* if the group action $G \times M \to M, (g, m) \mapsto g.m$ is continuous. We simply write *G-module* for a continuous $\mathbb{Z}[G]$-module.

Lemma 2.6. *Let M be a $\mathbb{Z}[G]$-module which is finitely generated as a $\mathbb{Z}$-module. The following are equivalent:*

(a) M is a G-module.

(b) For all $m \in M$, the stabilizer

$$G_m := \{g \in G \mid g.m = m\}$$

is open in G.

Proof. (a) $\Rightarrow$ (b): Fix $m \in M$. Since M is a continuous G-module with the discrete topology, the composition of maps

$$\begin{aligned} G &\to G \times M \to M \\ g &\mapsto (g, m) \mapsto g.m \end{aligned}$$

is continuous. Hence, the preimage $G_m \subset G$ of $\{m\} \subset M$ is open in G.

(b) $\Rightarrow$ (a): We have to show that the map $G \times M \to M, (g, m) \mapsto g.m$ is continuous. Since M is equipped with the discrete topology, it suffices to show that for any $m \in M$, the preimage of $\{m\}$ is open in $G \times M$.

The preimage of $\{m\}$ is given by

$$\{(g,n) \in G \times M \mid g.n = m\} = \bigcup_{g \in G} gG_{g^{-1}.m} \times \{g^{-1}.m\},$$

hence a union of open sets in $G \times M$. $\square$

Let M and N be two G-modules, then the action of G on M and N induces an action on

(a) $M \otimes_{\mathbb{Z}} N$ via
$$g.(m \otimes n) := (g.m) \otimes (g.n)$$
for all $g \in G$, $m \in M$, $n \in N$.

(b) $\mathrm{Hom}_{\mathbb{Z}}(M,N)$ via
$$(g.\alpha)(m) := g.\alpha(g^{-1}.m)$$
for all $g \in G$, $\alpha \in \mathrm{Hom}_{\mathbb{Z}}(M,N)$, $m \in M$. In particular, since $\mathbb{Z}$ is a trivial G-module, G acts on $M^\vee := \mathrm{Hom}_{\mathbb{Z}}(M,\mathbb{Z})$ via
$$(g.\alpha)(m) = \alpha(g^{-1}.m)$$
for all $g \in G$, $\alpha \in M^\vee$, $m \in M$.

Since M and N are finitely generated $\mathbb{Z}$-modules, let H_M and H_N denote the intersection of the stabilizers of the $\mathbb{Z}$-generators of M and N, resp., which are open subgroups of G. Hence, $H := H_M \cap H_N$ is an open subgroup of G that fixes $M \otimes_{\mathbb{Z}} N$ and $\mathrm{Hom}_{\mathbb{Z}}(M,N)$ and the given G-action turns $M \otimes_{\mathbb{Z}} N$ and $\mathrm{Hom}_{\mathbb{Z}}(M,N)$ into G-modules.

Definition. Let M be a G-module. If M is a free $\mathbb{Z}$-module which admits a $\mathbb{Z}$-basis that is permuted by the G-action, then M is called a *G-permutation module*. We will simply write *permutation module* if the group G is clear from the context.

Example. Let $H \subset G$ be a closed subgroup of finite index and let G act on $\mathbb{Z}[G/H]$ via left multiplication. Then $\mathbb{Z}[G/H]$ is a permutation module.

Proposition 2.7. *The following are equivalent:*

(a) M is a permutation module.

(b) There are finitely many closed subgroups $H_i \subset G$ of finite index such that M is isomorphic to the finite direct sum $\bigoplus_i \mathbb{Z}[G/H_i]$.

Remark. The subgroups H_i of the previous proposition need not be normal in G, in particular, the sets G/H_i are sets of left cosets, but not necessarily groups.

Proof. (a)$\Rightarrow$(b): Since M is a permutation module, we may choose a finite $\mathbb{Z}$-basis X of M that is permuted by the G-action. Let Y be a finite G-set, then let $\mathbb{Z}[Y] := \bigoplus_{y\in Y} \mathbb{Z}\cdot y$ with the G-action induced by the action of G on Y. If we split X into its finitely many disjoint orbits $G.x_1, \dots, G.x_n$ under the G-action, we obtain that

$$M \cong \bigoplus_{i=1}^{n} \mathbb{Z}[G.x_i]$$

as G-modules. It therefore suffices to show that every G-module of the form $\mathbb{Z}[Y]$ for a finite G-set Y on which G acts transitively (i. e. $Y = G.y$ for some $y \in Y$), is isomorphic to $\mathbb{Z}[G/K]$ for some subgroup $K \subset G$ of finite index. To prove this, let $K \subset G$ be the stabilizer of y in G, i. e.

$$K = \{k \in G \mid k.y = y\}.$$

Since M is continuous, $K \subset G$ is a closed subgroup of finite index. We claim that $\mathbb{Z}[Y] \cong \mathbb{Z}[G/K]$ as G-modules: Define a bijection $\tilde{\alpha} : G/K \to Y$ as G-sets via

$$gK \mapsto g.y.$$

The map $\tilde{\alpha}$ is then

- well defined: If $g_1K = g_2K$, then there is a $k \in K$ such that $g_1 = g_2k$ and therefore
$$g_1.y = (g_2k).y = g_2.(k.y) = g_2.y$$
since k stabilizes y.
- injective: Assume that $\tilde{\alpha}(g_1K) = \tilde{\alpha}(g_2K)$ for some $g_1K, g_2K \in G/K$, then
$$\begin{aligned} g_1.y &= g_2.y \\ y &= g_1^{-1}g_2.y, \end{aligned}$$
i.e. $g_1^{-1}g_2 \in K$ or equivalently $g_1K = g_2K$.
- surjective: Let $z \in Y$, then there is a $g \in G$ such that $z = g.y$, hence $z = \tilde{\alpha}\,(gK)$.
- a G-map: For $g \in G$, $\tilde{g}K \in G/K$, we have
$$\tilde{\alpha}(g.\tilde{g}K) = \tilde{\alpha}(g\tilde{g}K) = (g\tilde{g}).y = g.(\tilde{g}.y) = g.\tilde{\alpha}\,(\tilde{g}K)\,.$$

We define a G-module homomorphism $\alpha : \mathbb{Z}[G/K] \to \mathbb{Z}[Y]$ by extending $\tilde{\alpha}$ $\mathbb{Z}$-linearly. Since $\tilde{\alpha}$ is a bijection of G-sets, α is a G-module isomorphism.

(b)⇒(a): Each of the summands $Z\,[G/H]$ is a G-permutation module and therefore, their finite direct sum is a G-permutation module as well. □

If M, N are G-modules, then we use the notation

$$\begin{aligned}\mathrm{Hom}_G(M,N) &:= \mathrm{Hom}_{\mathbb{Z}[G]}(M,N)\\ &= \{\alpha \in \mathrm{Hom}_{\mathbb{Z}}(M,N) \mid g.\alpha(m) = \alpha(g.m) \text{ for all } g \in G, m \in M\}\,.\end{aligned}$$

Definition. Let M be a G-module. A *G-permutation resolution* of M is a long exact sequence

$$\dots \xrightarrow{\alpha_2} P_2 \xrightarrow{\alpha_1} P_1 \xrightarrow{\alpha_0} P_0 \xrightarrow{\beta} M \longrightarrow 0,$$

in which the P_i are G-permutation modules, $\alpha_i \in \mathrm{Hom}_G(P_{i+1}, P_i)$ for all i and $\beta \in \mathrm{Hom}_G(P_0, M)$. We will simply write *permutation resolution* if the group G is clear from the context.

We denote the $\mathbb{Z}$-module consisting of all H-invariant elements of M by M^H, i. e.

$$M^H = \{m \in M \mid h.m = m \text{ for all } h \in H\}\,.$$

In the remaining part of this section, we will identify some G-modules that are isomorphic to M^H. To do this, we will use the following Proposition:

Proposition 2.8. *Let M and N be G-modules. The elements of* $\mathrm{Hom}_G(M,N)$ *are the G-invariant elements of* $\mathrm{Hom}_{\mathbb{Z}}(M,N)$ *under the above action, i. e.*

$$\mathrm{Hom}_G(M,N) = \big[\,\mathrm{Hom}_{\mathbb{Z}}(M,N)\big]^G.$$

Proof. $\alpha \in \mathrm{Hom}_{\mathbb{Z}}(M,N)$ is G-invariant if and only if

$$\alpha(m) = (g.\alpha)(m) = g.\alpha(g^{-1}.m)$$

for all $g \in G$, $m \in M$, which is equivalent to

$$g.\alpha(m) = \alpha(g.m)$$

for all $g \in G, m \in M$, that is $\alpha \in \mathrm{Hom}_G(M,N)$. □

The following theorem yields the first of these modules isomorphic to M^H.

Theorem 2.9. *Let M be a finitely generated G-module. Then M^H is isomorphic to* $\mathrm{Hom}_G(\mathbb{Z}[G/H], M)$ *as $\mathbb{Z}$-modules.*

Proof. Define a map

$$\begin{aligned} f : \mathrm{Hom}_G(\mathbb{Z}[G/H], M) &\to M^H \\ \alpha &\mapsto \alpha(H). \end{aligned}$$

For f to be well defined, we have to show that $\alpha(H) \in M^H$. By definition, $\alpha(H) \in M$ and for all $h \in H$, we have

$$h.\alpha(H) = \alpha(h.H) = \alpha(hH) = \alpha(H).$$

Further, f is a $\mathbb{Z}$-homomorphism since for all numbers a, $b \in \mathbb{Z}$ and homomorphisms α, $\beta \in \mathrm{Hom}_G(\mathbb{Z}[G/H], M)$ we have

$$f(a\alpha - b\beta) = (a\alpha - b\beta)(H) = a\alpha(H) - \beta(H) = af(\alpha) - bf(\beta).$$

To find an inverse map $k : M^H \to \mathrm{Hom}_G(\mathbb{Z}[G/H], M)$, we first note that for any $m \in M^H$, we can define a $\mathbb{Z}$-linear map $\alpha_m \in \mathrm{Hom}_G(\mathbb{Z}[G/H], M)$ via

$$gH \mapsto g.m.$$

The map α_m is well defined since if $g_1, g_2 \in G$ such that $g_1H = g_2H$, then there is $h \in H$ such that $g_1 = g_2h$ and therefore

$$\alpha(g_1H) = g_1.m = (g_2h).m = g_2.(h.m) = g_2.m = \alpha(g_2H).$$

Additionally, α is G-invariant since for all generators $\tilde{g}H$ of G/H, we have

$$g.\alpha(\tilde{g}H) = g.(\tilde{g}.m) = (g\tilde{g}).m = \alpha((g\tilde{g}).H) = \alpha(g.(\tilde{g}H)).$$

Hence $\alpha \in \mathrm{Hom}_G(\mathbb{Z}[G/H], M)$ and consequently, the map

$$\begin{aligned} k : M^H &\to \mathrm{Hom}_G(\mathbb{Z}[G/H], M) \\ m &\mapsto (\alpha_m : gH \mapsto g.m) \end{aligned}$$

is well defined. It is also is a $\mathbb{Z}$-homomorphism since for any a, $b \in \mathbb{Z}$ and

$m,\, n \in M^H$, we have

$$\begin{aligned} k(am-bn) &= (\alpha_{am-bn} : gH \mapsto g.(am-bn) \\ &= (gH \mapsto a(g.m) - b(g.n)) \\ &= a(gH \mapsto g.m) - b(gH \mapsto g.n) \\ &= a\alpha_m - b\alpha_n \\ &= ak\,(m) - bk\,(n)\,. \end{aligned}$$

Finally, it remains to prove that k is actually an inverse of f. To see this, we note that for all $m \in M^H$

$$f(k(m)) = f(\alpha_m) = \alpha_m(H) = \alpha_m(eH) = e.m = m$$

and for all $\beta \in \mathrm{Hom}_G\,(\mathbb{Z}[G/H], M)$

$$k(f(\beta)) = k(\beta(H)) = \alpha_{\beta(H)} = \beta.$$

The last equality holds since

$$\alpha_{\beta(H)}(gH) = g.\beta(H) = \beta\,(gH)$$

for all $gH \in G/H$. $\square$

Definition. Let M be a G-module. A symmetric bilinear form $b : M \times M \to \mathbb{Z}$ is called

(a) *non-degenerate* if the map

$$\begin{aligned} &M \to M^\vee \\ &m \mapsto (m^* : n \mapsto b(m,n)) \end{aligned}$$

is an isomorphism.

(b) *G-invariant*, if

$$b(g.m, g.n) = b(m,n)$$

for all $g \in G$, $m, n \in M$.

Lemma 2.10. *Let M be a finitely generated G-module which is free as a $\mathbb{Z}$-module. If the bilinear form $b : M \times M \to \mathbb{Z}$ is non-degenerate and G-invariant, then M and $M^\vee$ are isomorphic as G-modules.*

Proof. The definition of a non-degenerate bilinear form implies that the map

$$\begin{aligned} M &\to M^\vee \\ m &\mapsto (m^* : n \mapsto b(m,n)) \end{aligned}$$

is an isomorphism. If b is also G-invariant, we have

$$(g.m^*)(n) = b(m, g^{-1}.n) = b(g.m, n) = (g.m)^*(n)$$

for all $g \in G$ and $m, n \in M$, so that the given isomorphism is compatible with the G-action. □

Lemma 2.11. *$\mathbb{Z}[G/H]$ is isomorphic to $\left[\mathbb{Z}[G/H]\right]^\vee$ as G-modules.*

Proof. By the previous lemma, it suffices to show the existence of a non-degenerate, G-invariant bilinear form $c : \mathbb{Z}[G/H] \times \mathbb{Z}[G/H] \to \mathbb{Z}$. We define c on generators as follows

$$c(gH, \tilde{g}H) = \begin{cases} 1, & \text{if } gH = \tilde{g}H \\ 0, & \text{otherwise} \end{cases}$$

and then extend this bilinearly to $\mathbb{Z}[G/H]$. To show that this bilinear form is non-degenerate, we fix a generating set $g_1H, \dots g_sH$ of G/H which is a free basis for $\mathbb{Z}[G/H]$. The representing matrix for this bilinear form is then the $(s \times s)$-identity matrix, hence non-degenerate. □

Theorem 2.12. *Let M be a finitely generated G-module which is free es a $\mathbb{Z}$-module. If M admits a non-degenerate, G-invariant bilinear form $b : M \times M \to \mathbb{Z}$, then M^H is also isomorphic to $\mathrm{Hom}_G(M, \mathbb{Z}[G/H])$ as $\mathbb{Z}$-modules.*

Proof. We have

$$\begin{aligned} M^H &\xrightarrow{\cong} \mathrm{Hom}_G(\mathbb{Z}[G/H], M) \\ m &\mapsto (\alpha_m : gH \mapsto g.m) \end{aligned}$$

by Theorem 2.9, further

$$\begin{aligned} \mathrm{Hom}_G(\mathbb{Z}[G/H], M) &\xrightarrow{\cong} \mathrm{Hom}_G\left(M^\vee, \left[\mathbb{Z}[G/H]\right]^\vee\right) \\ \alpha &\mapsto (\alpha^* : \beta \mapsto \beta \circ \alpha) \end{aligned}$$

and from $M \cong M^\vee$ by Lemma 2.10 and $\mathbb{Z}[G/H] \cong \mathbb{Z}[G/H]^\vee$ by Lemma 2.11, it follows that

$$\operatorname{Hom}_G\left(M^\vee, \left[\mathbb{Z}\left[G/H\right]\right]^\vee\right) \cong \operatorname{Hom}_G\left(M, \mathbb{Z}\left[G/H\right]\right),$$

hence

$$M^H \cong \operatorname{Hom}_G\left(M, \mathbb{Z}[G/H]\right). \qquad \square$$

Now assume that M has a permutation basis $x_1, \ldots, x_n$ and a G-invariant bilinear form $b : M \times M \to \mathbb{Z}$ with $b(x_i, x_j) = \pm\delta_{ij}$. As we will see later, the Picard group of a Del Pezzo surface of degree 5 or 6 satisfies those conditions. Let $f \in \operatorname{Hom}_G(M, \mathbb{Z}[G/H])$. We will denote the corresponding element in $\operatorname{Hom}_G(\mathbb{Z}[G/H], M)$ by f^*, hence the isomorphisms in the proof of theorem 2.12 are given by

$$\begin{aligned} \operatorname{Hom}_G\left(M, \mathbb{Z}[G/H]\right) &\xrightarrow{\cong} M^H \\ f &\mapsto f^*(H). \end{aligned}$$

2.4 Root Systems and the Weyl Group

This section is a short overview of the most important properties of a root system and its Weyl group. A detailed presentation can be found in [10], chapter 10.

Let E be a Euclidean space, that is a real vector space with a positive definite, symmetric bilinearform $(-,-) : E \times E \to \mathbb{R}$, called *scalar product*.

Definition. Every element $\alpha \in E$ defines a *reflection*

$$\begin{aligned} \sigma_\alpha : E &\to E \\ \beta &\mapsto \beta - 2\frac{(\beta, \alpha)}{(\alpha, \alpha)}\alpha. \end{aligned}$$

We can visualize this map geometrically as follows: if we decompose our space E into the direct sum of the linear subspace $L(\alpha)$ spanned by α and its orthogonal complement $L(\alpha)^\perp$, i. e.

$$E = L(\alpha) \oplus L(\alpha)^\perp,$$

then σ_α maps

- any element $\beta \in L(\alpha)$ to its negative $-\beta$. To see this, write $\beta = k\alpha$ for some $k \in \mathbb{R}$, then

$$\sigma_\alpha(\beta) = \sigma_\alpha(k\alpha) = k\alpha - 2\frac{(k\alpha, \alpha)}{(\alpha, \alpha)}\alpha = k\alpha - 2k\alpha = -k\alpha = -\beta.$$

- any element $\beta \in L(\alpha)^\perp$ to itself since $(\beta, \alpha) = 0$.

From this, we immediately obtain two basic properties of reflections:

Lemma 2.13. *(a) Reflections are self-inverse.*

(b) Reflections preserve the scalar product. □

Definition. A subset $\Phi \subset E$ is called a *root system* in E if

(R1) Φ is finite, Φ spans E and $0 \notin \Phi$.

(R2) If $\alpha \in \Phi$, then $-\alpha \in \Phi$ and Φ does not contain any other scalar multiples of α, i. e. $L(\alpha) \cap \Phi = \{\pm\alpha\}$.

(R3) If $\alpha \in \Phi$, then the reflection σ_α leaves Φ invariant.

(R4) If α, $\beta \in \Phi$, then $2\frac{(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$.

Definition. A subset $\Delta \subset \Phi$ of a root system is called a *base* if

(B1) Δ is a basis of E.

(B2) For every $\beta \in \Phi$, we can write $\beta = \sum_{\alpha\in\Delta} k_\alpha \alpha$ with integral coefficients k_α such that either $k_\alpha \geq 0$ or $k_\alpha \leq 0$ for all $\alpha \in \Delta$.

An element $\alpha \in \Delta$ is called a *simple root*.

Definition. For a given root system $\Phi \subset E$, the *Weyl group* $\mathcal{W}(\Phi) \subset GL(E)$, or simply $\mathcal{W}$, is the group generated by the reflections $\sigma_\alpha, \alpha \in \Phi$.

The following theorem lists important properties of bases and the Weyl group, which we use in subsequent chapters. A proof of this theorem can be found in [10, Theorem 10.3].

Theorem 2.14. *Let Φ be a root system in E with Weyl group $\mathcal{W}(\Phi)$ and a base Δ.*

(a) $\mathcal{W}$ is generated by the reflections σ_α, $\alpha \in \Delta$.

(b) $\mathcal{W}$ acts transitively on bases.

(c) If $\alpha \in \Phi$ is a root, then there is $\sigma \in \mathcal{W}$ such that $\sigma(\alpha) \in \Delta$.

Chapter 3

Effective Chow Motives

3.1 Intersection Theory

Intersection theories were formally introduced by Grothendieck in [5]. This section is a short summary of the definitions.

Definition. An *intersection theory* with coefficients in a commutative ring R is a contravariant functor C from the category $\mathrm{PSm}(K)$ of smooth projective K-schemes into the category of commutative R-algebras

$$\begin{array}{rcl} X & \mapsto & C(X), \\ \phi : (X \to Y) & \mapsto & \phi^* : C(Y) \to C(X), \end{array}$$

subject to the following axioms:

(a) Every proper morphism $\phi : X \to Y$ induces an R-algebra homomorphism $\phi_* : C(X) \to C(Y)$ such that $(\mathrm{id}_X)_* = \mathrm{id}_{C(X)}$ and $(\phi \circ \psi)_* = \phi_* \circ \psi_*$.

(b) For any two schemes $X, Y \in \mathrm{PSm}(K)$, we have an R-module homomorphism

$$C(X) \otimes_R C(Y) \to C(X \times_K Y),$$
$$x \otimes y \mapsto x \times y.$$

(c) For an irreducible scheme $X \in \mathrm{PSm}(K)$, we have an augmentation homomorphism $\epsilon : C(X) \to R$, which is an isomorphism for $X = \operatorname{Spec} K$.

In addition, these morphisms have to satisfy certain conditions as listed in [5, I.1-I.9]. In particular, we have

1. The *Multiplication Axiom*: Let $X \in \mathrm{PSm}(K)$ and let $\Delta_X : X \hookrightarrow X \times X$ be the diagonal morphism. Then the composition

$$C(X) \otimes_R C(X) \to C(X \times X) \xrightarrow{\Delta_X^*} C(X)$$

coincides with the multiplication homomorphism $\Delta_X^*(x \times y) = xy$.

2. The *Projection Formula*: Let $\phi : X \to Y$ be a morphism and $x \in C(X)$, $y \in C(Y)$. Then

$$\phi_* \left(x \phi^*(y) \right) = \phi_*(x) y.$$

For example, the Chow ring $\mathrm{CH}_*(X)$, as defined in Section 2.1, yields an intersection theory, known as *Chow Theory*, with $R = \mathbb{Z}$ and

$$C(X) := \mathrm{CH}_*(X) = \bigoplus_{i=0}^{\dim X} \mathrm{CH}_i(X).$$

If X and Y are equidimensional and $\phi : X \to Y$ is a morphism, then:

(e) The homomorphisms $\phi^* : \mathrm{CH}_*(Y) \to \mathrm{CH}_*(X)$ respect the grading by codimension, i. e.

$$\phi^* : \mathrm{CH}_i(Y) \to \mathrm{CH}_{\dim X - \dim Y + i}(X). \tag{3.1}$$

(f) The homomorphisms $\phi_* : \mathrm{CH}_*(X) \to \mathrm{CH}_*(Y)$ respect the grading by dimension, i. e.

$$\phi_* : \mathrm{CH}_i(X) \to \mathrm{CH}_i(Y). \tag{3.2}$$

3.2 The Category of Effective Chow Motives

We will describe a covariant version of Grothendieck's construction of the category of effective Chow(K)-motives following Karpenko [11] and Manin [13].

Definition. Let X and Y be smooth projective schemes over K. A *correspondence* between X and Y is an element of $\mathrm{CH}_*(X \times_K Y)$. The *composition of two correspondences* $f \in \mathrm{CH}_*(X \times Y)$, $g \in \mathrm{CH}_*(Y \times_K Z)$ is

$$g \circ f = p_{XZ*} \left(p_{YZ}^*(g) \cdot p_{XY}^*(f) \right) \in \mathrm{CH}_*(X \times_K Z),$$

where p_{XY}, p_{XZ}, p_{YZ} denote the projections

$$\begin{tikzcd} & X \times Y \times Z \arrow[dl, "p_{XY}"'] \arrow[d, "p_{XZ}"] \arrow[dr, "p_{YZ}"] & \\ X \times Y & X \times Z & Y \times Z. \end{tikzcd}$$

This composition preserves dimensions: Let X and Y be equidimensional, $f \in \mathrm{CH}_{\dim X}(X \times_K Y)$ and $g \in \mathrm{CH}_{\dim Y}(Y \times_K Z)$, then from the maps (2.1), (3.1) and (3.2), we obtain that $g \circ f \in \mathrm{CH}_{\dim X}(X \times_K Z)$. Further, let $\phi : X \to Y$ be a morphism in $\mathrm{PSm}(K)$ and let $\mathrm{id}_X : X \to X$ be the identity morphism. We define the *graph class* of ϕ to be

$$\Gamma_\phi := (\mathrm{id}_X, \phi)_* (1_X) \in \mathrm{CH}_{\dim X}(X \times_K Y).$$

The diagonal class of a scheme X is the graph class Γ_{id_X} of the identity morphism, which we will often again denote by id_X.

We define the *category $C^0(K)$ of correspondences of degree 0* to be the category with

- objects: objects in $\mathrm{PSm}(K)$
- morphisms: $\mathrm{Hom}_{C^0(K)}(X, Y) = \bigoplus_{i=1}^{e} \mathrm{CH}_{\dim X_i}(X_i \times_K Y)$,

where $X_1, \ldots, X_e$ are the irreducible components of X. If X is an object in $\mathrm{PSm}(K)$, then let $\underline{X}$ denote the corresponding object in $C^0(K)$. The category $C^0(K)$ is an additive category with direct sums $\underline{X} \oplus \underline{Y} = \underline{X \sqcup Y}$ and tensor products $\underline{X} \otimes \underline{Y} = \underline{X \times_K Y}$, but is not an abelian category.

Definition. Let $\mathcal{D}$ be an additive category, X an object in $\mathcal{D}$, and $p \in \mathrm{Hom}_{\mathcal{D}}(X, X)$. The category $\mathcal{D}$ is said to be *pseudo-abelian* if $p^2 = p$ implies the existence of an object Y in $\mathcal{D}$ and morphisms $f \in \mathrm{Hom}_{\mathcal{D}}(X, Y)$ and $g \in \mathrm{Hom}_{\mathcal{D}}(Y, X)$ such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = p$. We also use the notation $\ker p$ for Y.

Let $\mathcal{D}$ be a pseudo-abelian category and let $p \in \mathrm{Hom}_{\mathcal{D}}(X, X)$ be a projector, i. e. idempotent, and we have the maps f and g as in the above definition. Since p is a projector implies that $\mathrm{id}_X - p$ is a projector, we also have an object Z and morphisms $h \in \mathrm{Hom}_{\mathcal{D}}(X, Z)$ and $k \in \mathrm{Hom}_{\mathcal{D}}(Z, X)$ such that $h \circ k = \mathrm{id}_Z$ and $k \circ h = \mathrm{id}_X - p$. These maps induce isomorphisms

$$X \cong Y \oplus Z = \ker p \oplus \ker(\mathrm{id}_X - p)$$

Definition. The *pseudo-abelian completion* of an additive category $\mathcal{D}$ is the category $\widetilde{\mathcal{D}}$ with

- objects: pairs (X, p) where X is an object in $\mathcal{D}$ and p is an idempotent element in $\mathrm{Hom}_{\mathcal{D}}(X, X)$
- morphisms: $\mathrm{Hom}_{\widetilde{\mathcal{D}}}((X, p), (Y, q)) = q \circ \mathrm{Hom}_{\mathcal{D}}(X, Y) \circ p$

We have a completely faithful functor $G : \mathcal{D} \to \widetilde{\mathcal{D}}$ that extends the mapping $X \mapsto (X, \mathrm{id})$ and $f \mapsto f$ uniquely. For any pseudo-abelian category $\mathcal{E}$ and additive functor $F : \mathcal{D} \to \mathcal{E}$, there is an additive functor $\widetilde{F} : \widetilde{\mathcal{D}} \to \mathcal{E}$ such that F and $\widetilde{F}G$ are equivalent.

We define the *category of effective Chow motives over* K, denoted $\mathrm{Chow}(K) = \widetilde{C^0(K)}$, to be the pseudo-abelian completion of the category $C^0(K)$ of correspondences of degree 0. Hence we have

- objects: pairs (X, p) where X is an object in $\mathrm{PSm}(K)$ and p is a projector in $\mathrm{Hom}_{C^0(K)}(X, X)$
- morphisms: $\mathrm{Hom}_{\mathrm{Chow}(K)}((X, p), (Y, q)) = q \circ \mathrm{Hom}_{C^0(K)}(X, Y) \circ p$

The notion of the graph class allows us to define a covariant functor from $\mathrm{PSm}(K)$ to $\mathrm{Chow}(K)$ that maps

- objects: $X \mapsto \tilde{X} = (X, \mathrm{id}_X)$, often again denoted by X, and
- morphisms: $(\phi : X \to Y) \mapsto \Gamma_\phi$.

Let $\mathbb{P}^1 := \mathbb{P}^1_K$ denote the projective line over K. Let further $\mathrm{pt} \in \mathrm{CH}_0(\mathbb{P}^1)$ denote the graph class of a point $\mathrm{Spec}\, K \to \mathbb{P}^1$ and $\mathbb{P}^1 \in \mathrm{CH}_1(\mathbb{P}^1)$. Define two projectors p_0, $p_1 \in \mathrm{Hom}_{\mathrm{Chow}(K)}(\mathbb{P}^1, \mathbb{P}^1)$ as

$$p_0 = \mathrm{pt} \times \mathbb{P}^1 \qquad \text{and} \qquad p_1 = \mathbb{P}^1 \times \mathrm{pt}.$$

Then

$$\begin{aligned} p_0^2 &= (p_{\mathbb{P}^1\mathbb{P}^1})_* \left(\mathbb{P}^1 \times \mathrm{pt} \times \mathbb{P}^1 \ \cdot\ \mathrm{pt} \times \mathbb{P}^1 \times \mathbb{P}^1\right) \\ &= \mathrm{pt} \times \mathbb{P}^1 \\ &= p_0 \end{aligned}$$

and

$$\begin{aligned} p_0 \circ p_1 &= (p_{\mathbb{P}^1\mathbb{P}^1})_* \left(\mathbb{P}^1 \times \mathrm{pt} \times \mathbb{P}^1 \ \cdot \ \mathbb{P}^1 \times \mathrm{pt} \times \mathbb{P}^1\right) \\ &= (p_{\mathbb{P}^1\mathbb{P}^1})_* \left(\mathbb{P}^1 \times 0 \times \mathbb{P}^1\right) \\ &= 0. \end{aligned}$$

Similarly, one shows that

$$p_1^2 = p_1 \qquad \text{and} \qquad p_1 \circ p_0 = 0.$$

The Künneth formula yields the identity

$$\mathrm{CH}_1\left(\mathbb{P}^1 \times \mathbb{P}^1\right) \cong \mathrm{CH}_0\left(\mathbb{P}^1\right) \otimes \mathrm{CH}_1\left(\mathbb{P}^1\right) \oplus \mathrm{CH}_1\left(\mathbb{P}^1\right) \otimes \mathrm{CH}_0\left(\mathbb{P}^1\right).$$

Since $\mathrm{CH}_0(\mathbb{P}^1) \otimes \mathrm{CH}_1(\mathbb{P}^1)$ is generated by p_0 and $\mathrm{CH}_1(\mathbb{P}^1) \otimes \mathrm{CH}_0(\mathbb{P}^1)$ is generated by p_1, we can write

$$\mathrm{id}_{\mathbb{P}^1} = ap_0 + bp_1$$

for some $a, b \in \mathbb{Z}$. Then

$$\begin{aligned} p_0 &= \mathrm{id}_{\mathbb{P}^1} \circ p_0 \\ &= (ap_0 + bp_1) \circ p_0 \\ &= ap_0, \end{aligned}$$

so $a = 1$ and similarly

$$\begin{aligned} p_1 &= \mathrm{id}_{\mathbb{P}^1} \circ p_1 \\ &= (ap_0 + bp_1)\, p_1 \\ &= bp_1, \end{aligned}$$

so $b = 1$, hence

$$\mathrm{id}_{\mathbb{P}^1} = p_0 + p_1,$$

where p_0 and p_1 are orthogonal projectors. Since the category of effective Chow Motives is pseudo-abelian, this yields the decomposition

$$\begin{aligned} (\mathbb{P}^1, \mathrm{id}_{\mathbb{P}^1}) &\cong \left(\mathbb{P}^1, p_0\right) \oplus \left(\mathbb{P}^1, \mathrm{id}_{\mathbb{P}^1} - p_0\right) \\ &\cong (K, \mathrm{id}_K) \oplus \left(\mathbb{P}^1, p_1\right) \end{aligned}$$

where the isomorphisms $(\mathbb{P}^1, p_0) \to K$ and $K \to (\mathbb{P}^1, p_0)$ in $\mathrm{Chow}(K)$ are given by $(\mathbb{P}^1 \times K) \circ p_0 = \mathrm{pt} \times K$ and $p_0 \circ (K \times \mathrm{pt}) = K \times \mathbb{P}^1$, respectively.

We define the *Tate motive* $\mathbf{L} := (\mathbb{P}^1, p_1)$ and can now introduce *twisting* with

$\mathbf{L}$: For any integer $i \geq 0$, we have a functor $-(i)$ of Chow(K) into itself via

- objects: $U \mapsto U(i) = U \otimes \mathbf{L}^i = U \otimes \underbrace{\mathbf{L} \otimes \ldots \otimes \mathbf{L}}_{i \text{ times}}$
- morphisms: $f \mapsto f(i) = f \otimes id_{\mathbf{L}^i}$.

Proposition 3.1 (Chapter 8 in [13])**.** *Let U and V be effective motives. The functor $-(i)$ is completely faithful, i. e.*

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Chow(K)}}(U,V) &\to \mathrm{Hom}_{\mathrm{Chow(K)}}(U(i),V(i)) \\ f &\mapsto f(i) \end{aligned}$$

is an isomorphism for all $i \geq 0$. If X and Y are equidimensional schemes, we further have an isomorphism

$$\begin{aligned} \mathrm{CH}_{\dim X+i-j}(X \times_K Y) &\to \mathrm{Hom}_{\mathrm{Chow(K)}}((X,\mathrm{id}_X)(i),(Y,\mathrm{id}_Y)(j)) \\ \alpha &\mapsto \alpha \times \delta_{ij}, \end{aligned} \tag{3.3}$$

where $\delta_{ij} \in \mathrm{CH}_j(\mathbf{L}^i)$ denotes the morphism corresponding to the identity in $\mathrm{Hom}_{\mathrm{Chow(K)}}(L^n, L^n)$ for large n.

We can extend the category of effective Chow motives over K to the category of Chow motives over K by adding the formal inverse $\mathbf{T} = \mathbf{L}^{-1}$ of the Tate motive and, as a consequence, can also extend the definition of twisting to negative integers by tensoring with this inverse. For this thesis, it is sufficient to work in the category of effective Chow motives.

3.3 The Chow Motive of a Variety with a K-rational Point

Let X be an irreducible variety of dimension n with a K-point $\mathrm{pt} : \operatorname{Spec} K \to X$. We will also denote the corresponding graph class of $\mathrm{pt} : \operatorname{Spec} K \to X$ by

$$\mathrm{pt} \in \mathrm{CH}_0(K \times_K X) \cong \mathrm{CH}_0(X).$$

Lemma 3.2. *The two algebraic cycles $\mathrm{pt} \times X$ and $X \times \mathrm{pt}$ in $\mathrm{CH}_n(X \times_K X)$ are orthogonal.*

Proof. Let p_{12}, p_{13}, p_{23} denote the projections onto the respective factors of

$X \times X \times X$,

$$\begin{tikzcd}
& X \times X \times X \arrow[dl, "p_{12}"'] \arrow[d, "p_{13}"] \arrow[dr, "p_{23}"] & \\
X \times X & X \times X & X \times X.
\end{tikzcd}$$

Then

$$\begin{aligned}
(\mathrm{pt} \times X) \circ (X \times \mathrm{pt}) &= (p_{13})_* \left(X \times \mathrm{pt} \times X \ \cdot \ X \times \mathrm{pt} \times X\right) \\
&= (p_{13})_* \left(X \cdot X \times \underbrace{\mathrm{pt} \cdot \mathrm{pt}}_{\in \mathrm{CH}_{-n}(X)=0} \times X \cdot X \right) \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
(X \times \mathrm{pt}) \circ (\mathrm{pt} \times X) &= (p_{13})_* \left(X \times X \times \mathrm{pt} \ \cdot \ \mathrm{pt} \times X \times X\right) \\
&= (p_{13})_* \left(\mathrm{pt} \times X \times \mathrm{pt}\right) \\
&= 0
\end{aligned}$$

for dimensional reasons. □

Lemma 3.3. *The cycle $X \times \mathrm{pt} + \mathrm{pt} \times X$ is a projector.*

Proof. We will first show that $X \times \mathrm{pt}$ and $\mathrm{pt} \times X$ are projectors:

$$\begin{aligned}
(X \times \mathrm{pt}) \circ (X \times \mathrm{pt}) &= (p_{13})_* \left(X \times X \times \mathrm{pt} \ \cdot \ X \times \mathrm{pt} \times X\right) \\
&= (p_{13})_* \left(X \times \mathrm{pt} \times \mathrm{pt}\right) \\
&= X \times \mathrm{pt}, \\
(\mathrm{pt} \times X) \circ (\mathrm{pt} \times X) &= (p_{13})_* \left(X \times \mathrm{pt} \times X \ \cdot \ \mathrm{pt} \times X \times X\right) \\
&= (p_{13})_* \left(\mathrm{pt} \times \mathrm{pt} \times X\right) \\
&= \mathrm{pt} \times X.
\end{aligned}$$

We use these two computations to show that $X \times \mathrm{pt} + \mathrm{pt} \times X$ is a projector:

$$\begin{aligned}
(X \times \mathrm{pt} + \mathrm{pt} \times X) &\circ (X \times \mathrm{pt} + \mathrm{pt} \times X) \\
&= (X \times \mathrm{pt}) \circ (X \times \mathrm{pt}) \ + \ (X \times \mathrm{pt}) \circ (\mathrm{pt} \times X) \ + \\
&\quad + \ (\mathrm{pt} \times X) \circ (X \times \mathrm{pt}) \ + \ (\mathrm{pt} \times X) \circ (\mathrm{pt} \times X) \\
&= (X \times \mathrm{pt}) \circ (X \times \mathrm{pt}) \ + \ (\mathrm{pt} \times X) \circ (\mathrm{pt} \times X) \\
&= X \times \mathrm{pt} \ + \ \mathrm{pt} \times X.
\end{aligned}$$
□

Since the category of Chow(K)-motives is pseudo-abelian, this yields

$$\begin{aligned}(X, \mathrm{id}_X) &\cong (X, \mathrm{id}_X - (X \times \mathrm{pt} + \mathrm{pt} \times X)) \oplus (X, X \times \mathrm{pt} + \mathrm{pt} \times X) \\ &\cong (X, \mathrm{id}_X - (X \times \mathrm{pt} + \mathrm{pt} \times X)) \oplus (X, X \times \mathrm{pt}) \oplus (X, \mathrm{pt} \times X)\,.\end{aligned} \tag{3.4}$$

The latter two summands can be described in terms of motives of the field K, as computed in the following two Lemmas.

Lemma 3.4.

$$(X, X \times \mathrm{pt}) \cong (\mathrm{Spec}\, K, \mathrm{id}_{\mathrm{Spec}\, K})\,.$$

Proof. We need

$$\begin{aligned} f &\in \mathrm{Hom}\left((X, X \times \mathrm{pt}), (\mathrm{Spec}\, K, \mathrm{id}_{\mathrm{Spec}\, K})\right) \\ &\quad = \mathrm{id}_{\mathrm{Spec}\, K} \circ \mathrm{CH}_n (X \times_K K) \circ (X \times \mathrm{pt})\,, \\ g &\in \mathrm{Hom}\left((\mathrm{Spec}\, K, \mathrm{id}_{\mathrm{Spec}\, K}), (X, X \times \mathrm{pt})\right) \\ &\quad = (X \times \mathrm{pt}) \circ \mathrm{CH}_0 (K \times_K X) \circ \mathrm{id}_{\mathrm{Spec}\, K} \end{aligned}$$

such that

$$f \circ g = \mathrm{id}_{\mathrm{Spec}\, K} \qquad \text{and} \qquad g \circ f = X \times \mathrm{pt}.$$

To that end, let

$$\begin{aligned} f &= \mathrm{id}_{\mathrm{Spec}\, K} \circ (X \times K) \circ (X \times \mathrm{pt}) \\ &= (p_{XK})_* (X \times X \times K \;\cdot X \times \mathrm{pt} \times K) \\ &= X \times K, \\ g &= (X \times \mathrm{pt}) \circ (K \times \mathrm{pt}) \circ \mathrm{id}_{\mathrm{Spec}\, K} \\ &= p_{KX*} (K \times X \times \mathrm{pt} \;\cdot\; K \times \mathrm{pt} \times X) \\ &= K \times \mathrm{pt}, \end{aligned}$$

then

$$\begin{aligned} f \circ g &= (p_{KK})_* (K \times X \times K \;\cdot\; K \times \mathrm{pt} \times K) \\ &= K \times K \\ &= \mathrm{id}_{\mathrm{Spec}\, K} \end{aligned}$$

and

$$\begin{aligned} g \circ f &= (p_{XX})_* \left(X \times K \times \mathrm{pt} \;\cdot\; X \times K \times X \right) \\ &= X \times \mathrm{pt}. \end{aligned}$$

□

Lemma 3.5.

$$(X, \mathrm{pt} \times X) \cong (\operatorname{Spec} K, \mathrm{id}_{\operatorname{Spec} K})\,(n)$$

Proof. We need

$$\begin{aligned} f &\in \operatorname{Hom}\left((X, \mathrm{pt} \times X), (\operatorname{Spec} K, \mathrm{id}_{\operatorname{Spec} K})\,(n)\right) \\ &= \mathrm{id}_{\operatorname{Spec} K} \circ \mathrm{CH}_0 \left(X \times_K K\right) \circ (\mathrm{pt} \times X), \\ g &\in \operatorname{Hom}\left((\operatorname{Spec} K, \mathrm{id}_{\operatorname{Spec} K})\,(n), (X, \mathrm{pt} \times X)\right) \\ &= (\mathrm{pt} \times X) \circ \mathrm{CH}_n \left(K \times_K X\right) \circ \mathrm{id}_{\operatorname{Spec} K}, \end{aligned}$$

such that

$$f \circ g = \mathrm{id}_{\operatorname{Spec} K} \qquad \text{and} \qquad g \circ f = \mathrm{pt} \times X.$$

Therefore, take

$$\begin{aligned} f &= \mathrm{id}_{\operatorname{Spec} K} \circ (\mathrm{pt} \times K) \circ (\mathrm{pt} \times X) \\ &= (p_{XK})_* \left(X \times \mathrm{pt} \times K \;\cdot\; \mathrm{pt} \times X \times K \right) \\ &= \mathrm{pt} \times K, \\ g &= (\mathrm{pt} \times X) \circ (K \times X) \circ \mathrm{id}_{\operatorname{Spec} K} \\ &= p_{KX*} \left(K \times \mathrm{pt} \times X \;\cdot\; K \times X \times X \right) \\ &= K \times X, \end{aligned}$$

then

$$\begin{aligned} f \circ g &= (p_{KK})_* \left(K \times \mathrm{pt} \times K \;\cdot\; K \times X \times K \right) \\ &= K \times K \\ &= \mathrm{id}_{\operatorname{Spec} K} \end{aligned}$$

and

$$\begin{aligned} g \circ f &= (p_{XX})_* \left(X \times K \times X \;\cdot\; \mathrm{pt} \times K \times X \right) \\ &= \mathrm{pt} \times X. \end{aligned}$$

□

Let $\rho := \mathrm{id}_X - (X \times \mathrm{pt} + \mathrm{pt} \times X)$. Using the two previous lemmas, equation

(3.4) becomes

$$(X, \mathrm{id}_X) \cong (\operatorname{Spec} K, \mathrm{id}_{\operatorname{Spec} K}) \oplus (X, \rho) \oplus (\operatorname{Spec} K, \mathrm{id}_{\operatorname{Spec} K})\,(n).$$

If S is a Del Pezzo surface of degree 5 or 6 with a K-rational point, this implies

$$(S, \mathrm{id}_S) \cong (\operatorname{Spec} K, \mathrm{id}_{\operatorname{Spec} K}) \oplus (S, \rho) \oplus (\operatorname{Spec} K, \mathrm{id}_{\operatorname{Spec} K})\,(2). \tag{3.5}$$

In Chapter 6, we will describe the middle term (S, ρ).

Chapter 4

Permutation Resolutions of the Picard Group of Del Pezzo Surfaces of Degree 5 and 6

4.1 The Picard Group of Del Pezzo Surfaces of Degree d

In this section, we will present the basic results for Picard groups of Del Pezzo surfaces $\overline{S}$ over algebraically closed field $\overline{K}$ of degree d. The proofs can be found in [14, Chapter IV].

Proposition 4.1 (Proposition 25.1 in [14]). *Let $r := 9 - d$. There exists a $\mathbb{Z}$-basis $l_0, \ldots, l_r$ of $\operatorname{Pic} \overline{S}$ such that the intersection form is given by the bilinear form $(-,-) : \operatorname{Pic} \overline{S} \times \operatorname{Pic} \overline{S} \to \mathbb{Z}$ with*

$$(l_0, l_0) = 1, \qquad (l_i, l_i) = -1 \text{ for } i \geq 1, \qquad (l_i, l_j) = 0 \text{ for } i \neq j.$$

To be able to use the results of Section 2.4 on root systems, we will identify a root system in $\operatorname{Pic} \overline{S}$ and a system of simple roots in $\operatorname{Pic} \overline{S}$.

Proposition 4.2 (Propositions 25.1 and 25.2 in [14]). *Let $r \geq 3$, i. e. $d \leq 6$. The canonical sheaf is given by $\omega = -3l_0 + \sum_{i=1}^{r} l_i$. The set*

$$R_r := \left\{ l \in \operatorname{Pic} \overline{S} \mid (l, \omega) = 0,\ (l, l) = -2 \right\}$$

is a root system in $\omega^{\perp}$, the orthogonal complement of ω in $\operatorname{Pic} \overline{S} \otimes_{\mathbb{Z}} \mathbb{R}$.

Due to the importance of this fact, we will prove that R_r is a root system in

$\operatorname{Pic} \overline{S} \otimes_{\mathbb{Z}} \mathbb{R}$.

Proof as given in [14]. We start by proving that the bilinear form in Proposition 4.1 is negative definite on $\omega^\perp$, yielding that $\omega^\perp$ is a Euclidean space with inner product $-(-,-)$. We first note that since $\operatorname{Pic} \overline{S} \otimes_{\mathbb{Z}} \mathbb{R}$ is generated by $\omega, l_1, \ldots, l_r$, we can write any element as $l = a\omega + \sum_{i=1}^r b_i l_i$. The condition of being in the orthogonal complement of ω in $\operatorname{Pic} \overline{S} \otimes_{\mathbb{Z}} \mathbb{R}$ is then equivalent to

$$\begin{aligned} 0 &= (\omega, l) \\ &= \left(\omega, a\omega + \sum_{i=1}^r b_i l_i\right) \\ &= a(\omega,\omega) + \sum_{i=1}^r b_i(\omega, l_i) \\ &= (9-r)a - \sum_{i=1}^r b_i, \end{aligned}$$

hence

$$(9-r)a = \sum_{i=1}^r b_i. \tag{4.1}$$

Applying our bilinear form (l,l) to this, we obtain

$$\begin{aligned} (l,l) &= \left(a\omega + \sum_{i=1}^r b_i l_i, a\omega + \sum_{i=1}^r b_i l_i\right) \\ &= a^2(\omega,\omega) + 2a\sum_{i=1}^r b_i(\omega, l_i) + \sum_{i=1}^r \sum_{j=1}^r b_i b_j (l_i, l_j) \\ &= (9-r)a^2 - 2a\sum_{i=1}^r b_i - \sum_{i=1}^r b_i^2 \\ &\overset{(4.1)}{=} (9-r)a^2 - 2a\cdot(9-r)a - \sum_{i=1}^r b_i^2 \\ &= -\underbrace{(9-r)}_{\geq 0} a^2 - \sum_{i=1}^r b_i^2 \\ &\leq 0. \end{aligned}$$

Consequently, $-(-,-)$ is a symmetric, positive bilinear form on $\omega^\perp$. To prove that R_r is a root system in $\omega^\perp$, we verify the conditions of the definition on page 19:

(R1) $\omega^\perp \cap \mathbb{Z}^{r+1}$ is a lattice and therefore contains only a finite number of elements of length 2, so R_r is finite. Further, since R_r contains the r linearly independent vectors $l_2 - l_1, \dots, l_r - l_{r-1}$ and $-l_0 + l_1 + l_2 + l_3$, it spans the r-dimensional space $\omega^\perp$. Finally, R_r does not contain 0 since $(0,0) = 0 \neq -2$.

(R2) Let $l \in R_r$ and assume that we have $al \in R_r$, for some $a \in \mathbb{R}$. Then

$$(al, \omega) = a(l, \omega) = 0$$

and

$$(al, al) = a^2(l,l) = -2a^2 = -2 \quad \text{if and only if } a = \pm 1.$$

Hence $al \in R_r \quad \Leftrightarrow \quad a = \pm 1$.

(R3) Let $l \in R_r$, then $(l,l) = -2$ and hence the reflection σ_l is of the form

$$\sigma_l(x) = x - 2\frac{(x,l)}{(l,l)}l = x + (x,l)l. \tag{4.2}$$

We will now prove that for any $k \in R_r$, we also have $\sigma_l(k) \in R_r$. For that we check

$$\begin{aligned}(\sigma_l(k), \omega) &= \big(k + (k,l)l, \omega\big) \\ &= (k,\omega) + (k,l)(l,\omega) \\ &= 0 + 0 \qquad \text{since } k, l \in R_r \\ &= 0\end{aligned}$$

and

$$(\sigma_l(k), \sigma_l(k)) \overset{\text{Lemma 2.13}(b)}{=} (k,k) = -2.$$

(R4) Let $k, l \in R_r$, then

$$2\frac{(k,l)}{(l,l)} = -(k,l) \in \mathbb{Z}.$$ □

Proposition 4.3 (Proposition 25.4 in [14])**.** *The root system R_r is of type $A_1 \times A_2$, A_4 for $d = 6$, 5, respectively.*

Theorem 4.4 (Proposition 26.5 in [14])**.** *The group of automorphisms of the Picard group which preserves both ω and the intersection form is equal to the Weyl group $W_r := W(R_r)$. In particular, $W_3 \cong S_2 \times S_3$ and $W_4 \cong S_5$.*

We conclude this chapter with explicit equations defining the roots, i. e. the elements of

$$R_r = \left\{ l \in \operatorname{Pic} \overline{S} \mid (l, \omega) = 0,\ (l, l) = -2 \right\}.$$

Let $l = al_0 - \sum_{i=1}^r b_i l_i \in R_r$. Then

$$\begin{aligned} 0 &= (l, \omega) \\ &= \left(al_0 - \sum_{i=1}^r b_i l_i\ ,\ \ -3l_0 + \sum_{i=1}^r l_i \right) \\ &= -3a(l_0, l_0) - \sum_{i=1}^r b_i (l_i, l_i) \\ &= -3a + \sum_{i=1}^r b_i, \end{aligned}$$

$$\sum_{i=1}^r b_i = 3a \tag{4.3}$$

and

$$\begin{aligned} -2 &= (l, l) \\ &= \left(al_0 - \sum_{i=1}^r b_i l_i\ ,\ al_0 - \sum_{i=1}^r b_i l_i \right) \\ &= a^2 - \sum_{i=1}^r b_i^2, \end{aligned}$$

$$\sum_{i=1}^r b_i^2 = a^2 + 2 \tag{4.4}$$

We can restrict the possibilities for a using the Cauchy-Schwartz inequality: For any real numbers $b_1, \ldots, b_r$, we know that

$$\begin{gathered} \left(\sum_{i=1}^r 1 \cdot b_i \right)^2 \le \left(\sum_{i=1}^r 1^2 \right) \cdot \left(\sum_{i=1}^r b_i^2 \right) \\ \left(\sum_{i=1}^r b_i \right)^2 \le r \sum_{i=1}^r b_i^2. \end{gathered}$$

From (4.3) and (4.4), it follows that

$$(3a)^2 = \left(\sum_{i=1}^{r} b_i\right)^2 \leq r \sum_{i=1}^{r} b_i^2 = r(a^2 + 2),$$

which is equivalent to

$$9a^2 \leq r(a^2 + 2). \tag{4.5}$$

For $r \leq 5$, i. e. $d = 9 - r \geq 4$, this inequality yields

$$\begin{aligned} 9a^2 &\leq r(a^2 + 2) \leq 5(a^2 + 2) \\ a^2 &\leq \frac{5}{2}. \end{aligned}$$

Since $a \in \mathbb{Z}$, we must have

$$a \in \{-1, 0, 1\}. \tag{4.6}$$

The inequality (4.5) also gives an estimation for the remaining degrees, yielding $|a| \leq 4$ for $r \leq 8$. In fact, one can prove that $|a| \leq 3$ (c. f. [14, Proposition 25.5.3]).

4.2 A Permutation Resolution for Degree 6

If the degree of the Del Pezzo surface is $d = 6$, we get $r = 9 - 6 = 3$. In this section, we identify one example of a system of simple roots for the Picard group of a Del Pezzo surface of degree 6 over an algebraically closed field $\overline{K}$ and use this to compute the action of the Weyl group on the Picard group. It is then straightforward to verify that the short exact sequence given in Theorem 4.6 is a permutation resolution of $\operatorname{Pic} \overline{S}$ with respect to the Weyl group $W_3 \cong S_2 \times S_3$.

The equations (4.3), (4.4) and (4.6) of the preceding section lead to the following possibilities for the coefficients of $l = al_0 - \sum_{i=1}^{3} b_i l_i \in R_3$:

a	arbitrary permutation of the b_i
-1	-1, -1, -1
0	1, -1, 0
1	1, 1, 1

Hence, we get a root system

$$R_3 = \big\{l_2 - l_1,\ l_1 - l_2,\ l_3 - l_2,\ l_2 - l_3,\ l_3 - l_1,\ l_1 - l_3,\\ -l_0 + l_1 + l_2 + l_3,\ l_0 - l_1 - l_2 - l_3\big\}.$$

Theorem 4.5. *A system of simple roots (i. e. a base) is given by*

$$s_1 := l_2 - l_1, \quad s_2 := l_3 - l_2, \quad s_3 := -l_0 + l_1 + l_2 + l_3.$$

Proof. s_1, s_2 and s_3 are linearly independent and $\omega^\perp$ is $r = 3$-dimensional, hence s_1, s_2 and s_3 are a basis and (B1) is satisfied. For (B2), we check

$$\begin{aligned} l_2 - l_1 &= s_1, & l_1 - l_2 &= -s_1,\\ l_3 - l_2 &= s_2, & l_2 - l_3 &= -s_2,\\ l_3 - l_1 &= s_1 + s_2, & l_1 - l_3 &= -s_1 - s_2,\\ -l_0 + l_1 + l_2 + l_3 &= s_3, & l_0 - l_1 - l_2 - l_3 &= -s_3. \end{aligned}$$ □

Because of Theorem 2.14(a), we know that the Weyl group is generated by the simple roots. To completely describe the action of the Weyl group on the Picard group, it is therefore sufficient to compute the action of the simple roots on our given basis of the Picard group. Let w_i be the element of the Weyl group that reflects at s_i, i. e. with (4.2)

$$w_i(x) = x - 2\frac{(x, s_i)}{(s_i, s_i)} s_i = x + (x, s_i) s_i,$$

then

$$\begin{aligned} w_1(l_0) &= l_0 + (l_0, s_1)s_1 = l_0,\\ w_1(l_1) &= l_1 + (l_1, s_1)s_1 = l_1 + (l_2 - l_1) = l_2,\\ w_1(l_2) &= l_2 + (l_2, s_1)s_1 = l_2 - (l_2 - l_1) = l_1,\\ w_1(l_3) &= l_3 + (l_3, s_1)s_1 = l_3,\\ \\ w_2(l_0) &= l_0 + (l_0, s_2)s_2 = l_0,\\ w_2(l_1) &= l_1 + (l_1, s_2)s_2 = l_1,\\ w_2(l_2) &= l_2 + (l_2, s_2)s_2 = l_2 + (l_3 - l_2) = l_3,\\ w_2(l_3) &= l_3 + (l_3, s_2)s_2 = l_3 - (l_3 - l_2) = l_2, \end{aligned}$$

and

$$\begin{aligned}
w_3(l_0) &= l_0 + (l_0, s_3)s_3 = l_0 - (-l_0 + l_1 + l_2 + l_3) = 2l_0 - l_1 - l_2 - l_3,\\
w_3(l_1) &= l_1 + (l_1, s_3)s_3 = l_1 - (-l_0 + l_1 + l_2 + l_3) = l_0 - l_2 - l_3,\\
w_3(l_2) &= l_2 + (l_2, s_3)s_3 = l_2 - (-l_0 + l_1 + l_2 + l_3) = l_0 - l_1 - l_3\\
w_3(l_3) &= l_3 + (l_3, s_3)s_3 = l_3 - (-l_0 + l_1 + l_2 + l_3) = l_0 - l_1 - l_2.
\end{aligned}$$

Let the Weyl group act

- on $\mathbb{Z}$: trivially.
- on $\mathbb{Z}^5$: Choose a basis $e_1, \dots, e_5$, then

  w_1 transposes e_1 and e_2 and fixes e_3, e_4 and e_5,
  w_2 transposes e_2 and e_3 and fixes e_1, e_4 and e_5
  w_3 transposes e_4 and e_5 and fixes e_1, e_2 and e_3.
- on the Picard group $\operatorname{Pic}\overline{S}$ as computed above,

Theorem 4.6. *A W_3-permutation resolution of the Picard group $\operatorname{Pic}\overline{S}$ of a Del Pezzo surface $\overline{S}$ of degree 6 is given by*

$$0 \longrightarrow \mathbb{Z} \xrightarrow{h} \mathbb{Z}^5 \xrightarrow{g} \operatorname{Pic}\overline{S} \longrightarrow 0, \tag{4.7}$$

where $h,\ g$ are $\mathbb{Z}$-linear maps defined as $h : \mathbb{Z} \to \mathbb{Z}^5$,

$$1 \mapsto e_1 + e_2 + e_3 - e_4 - e_5$$

and $g : \mathbb{Z}^5 \to \operatorname{Pic}\overline{S}$,

$$\begin{aligned}
&e_1 \mapsto l_0 - l_1, \qquad e_2 \mapsto l_0 - l_2, \qquad e_3 \mapsto l_0 - l_3,\\
&e_4 \mapsto 2l_0 - l_1 - l_2 - l_3, \qquad e_5 \mapsto l_0.
\end{aligned} \tag{4.8}$$

For the proof of this theorem, will identify a more symmetric generating system of $\operatorname{Pic}\overline{S}$. Let

$$m_1 := l_0 - l_2 - l_3, \qquad m_2 := l_0 - l_1 - l_3, \qquad m_3 := l_0 - l_1 - l_2,$$

then

$$l_1,\ l_2,\ l_3,\ m_1,\ m_2,\ m_3$$

also generate the Picard group. Geometrically, these generators correspond to the classes of the exceptional divisors: If $\overline{S}$ is the blow up of the three points

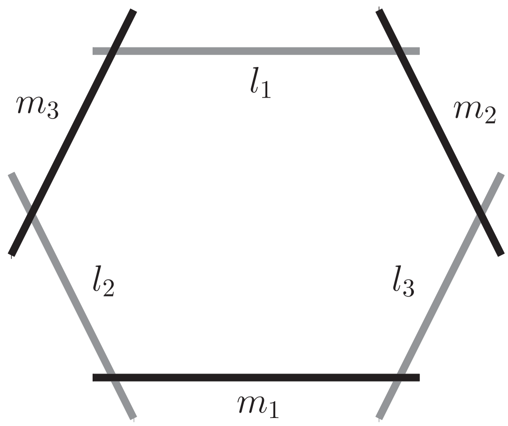

Figure 4.1: A generating system for the Picard group of a Del Pezzo surface of Degree 6

P_1, P_2 and P_3 in $\mathbb{P}^2$, then l_1, l_2 and l_3 are the pullbacks of those points and for any permutation i, j, k of 1, 2, 3, m_i is the strict transform of the line joining P_j and P_k.

We have the identities

$$m_i + l_j = (l_0 - l_1 - l_2 - l_3 + l_i) + l_j = (l_0 - l_1 - l_2 - l_3 + l_j) + l_i = m_j + l_i. \tag{4.9}$$

The Weyl group acts on our new generating system by

$$\begin{aligned}
w_1(m_1) &= w_1(l_0 - l_2 - l_3) = l_0 - l_1 - l_3 = m_2,\\
w_1(m_2) &= w_1(l_0 - l_1 - l_3) = l_0 - l_2 - l_3 = m_1,\\
w_1(m_3) &= w_1(l_0 - l_1 - l_2) = l_0 - l_2 - l_1 = m_3,\\
\\
w_2(m_1) &= w_2(l_0 - l_2 - l_3) = l_0 - l_3 - l_2 = m_1,\\
w_2(m_2) &= w_2(l_0 - l_1 - l_3) = l_0 - l_1 - l_2 = m_3,\\
w_2(m_3) &= w_2(l_0 - l_1 - l_2) = l_0 - l_1 - l_3 = m_2,
\end{aligned}$$

and

$$\begin{aligned}
w_3(m_1) &= w_3(l_0 - l_2 - l_3)\\
&= (2l_0 - l_1 - l_2 - l_3) - (l_0 - l_1 - l_3) - (l_0 - l_1 - l_2)\\
&= l_1,\\
w_3(m_2) &= w_3(l_0 - l_1 - l_3)
\end{aligned}$$

$$
\begin{aligned}
&= (2l_0 - l_1 - l_2 - l_3) - (l_0 - l_2 - l_3) - (l_0 - l_1 - l_2) \\
&= l_2, \\
w_3(m_3) &= w_3(l_0 - l_1 - l_2) \\
&= (2l_0 - l_1 - l_2 - l_3) - (l_0 - l_2 - l_3) - (l_0 - l_1 - l_3) \\
&= l_3.
\end{aligned}
$$

We can rewrite the map $g : \mathbb{Z}^5 \to \operatorname{Pic} \overline{S}$ in terms of our new generating system to get

$$
\begin{aligned}
&e_1 \mapsto m_2 + l_3, \qquad e_2 \mapsto m_1 + l_3, \qquad e_3 \mapsto m_1 + l_2, \\
&e_4 \mapsto m_1 + m_2 + l_3, \qquad e_5 \mapsto m_1 + l_2 + l_3.
\end{aligned} \tag{4.8$'$}
$$

For the proof of Theorem 4.6, we will be using both descriptions (4.8) and (4.8′) of the map $g : \mathbb{Z}^5 \to \operatorname{Pic} \overline{S}$.

Proof of Theorem 4.6. $W_3 \cong S_2 \times S_3$ permutes the chosen bases of $\mathbb{Z}$ and $\mathbb{Z}^5$. Further, the sequence (4.7) is exact because h is injective, g is surjective and $\operatorname{im}(h) = \ker(g)$: Consider the transformation matrix A of g for the ordered basis $(e_1, \ldots, e_5)$ and $(l_0, \ldots, l_3)$ using (4.8). It is

$$
A = \begin{pmatrix} 1 & 1 & 1 & 2 & 1 \\ -1 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & -1 & 0 \end{pmatrix},
$$

hence for an element $a_1 e_1 + \ldots + a_5 e_5$ with $a_1, \ldots, a_5 \in \mathbb{Z}$ to be in the kernel of g, we get the equations

$$
\begin{aligned}
a_3 &= -a_4, \\
a_2 &= -a_4, \\
a_1 &= -a_4, \\
a_5 &= -a_1 - a_2 - a_3 - 2a_4 = a_4 + a_4 + a_4 - 2a_4 = a_4.
\end{aligned}
$$

With $a_4 := -1 \in \mathbb{Z}^\times$, this yields

$$
\ker(g) = \mathbb{Z}(e_1 + e_2 + e_3 - e_4 - e_5) = \operatorname{im}(h).
$$

Finally, one can see directly that h is W_3-linear since $e_1 + e_2 + e_3 - e_4 - e_5$ is invariant. The map g is W_3-invariant since:

$$\begin{aligned}
w_1.g(e_1) &= m_1 + l_3 = g(w_1.e_1),\\
w_1.g(e_2) &= m_2 + l_3 = g(w_1.e_2),\\
w_1.g(e_3) &= m_2 + l_1 = m_1 + l_2 = g(w_1.e_3),\\
w_1.g(e_4) &= m_2 + m_1 + l_3 = g(w_1.e_4),\\
w_1.g(e_5) &= m_2 + l_1 + l_3 = m_1 + l_2 + l_3 = g(w_1.e_5),
\end{aligned}$$

$$\begin{aligned}
w_2.g(e_1) &= m_3 + l_2 = m_2 + l_3 = g(w_2.e_1),\\
w_2.g(e_2) &= m_2 + l_3 = g(w_2.e_2),\\
w_2.g(e_3) &= m_1 + l_3 = g(w_2.e_3),\\
w_2.g(e_4) &= m_1 + m_3 + l_2 = m_1 + m_2 + l_3 = g(w_2.e_4),\\
w_2.g(e_5) &= m_1 + l_3 + l_2 = g(w_2.e_5),
\end{aligned}$$

and

$$\begin{aligned}
w_3.g(e_1) &= l_2 + m_3 = m_2 + l_3 = g(w_3.e_1),\\
w_3.g(e_2) &= l_1 + m_3 = m_1 + l_3 = g(w_3.e_2),\\
w_3.g(e_3) &= l_1 + m_2 = m_1 + l_2 = g(w_3.e_3),\\
w_3.g(e_4) &= l_1 + l_2 + m_3 = g(w_3.e_4),\\
w_3.g(e_5) &= l_1 + m_2 + m_3 = g(w_3.e_5).
\end{aligned}$$

□

Theorem 4.7. *The permutation resolution is a split exact sequence.*

Proof. Define a $\mathbb{Z}$-linear homomorphism $k : \operatorname{Pic}\overline{S} \to \mathbb{Z}^5$,

$$\begin{aligned}
&l_0 \mapsto -e_1 - e_2 - e_3 + e_4 + 2e_5,\\
&l_1 \mapsto -e_1 + e_5, \qquad l_2 \mapsto -e_2 + e_5, \qquad l_3 \mapsto -e_3 + e_5,
\end{aligned} \tag{4.10}$$

hence also

$$\begin{aligned}
k(m_1) &= k(l_0 - l_2 - l_3) \\
&= -e_1 - e_2 - e_3 + e_4 + 2e_5 - (e_5 - e_2) - (e_5 - e_3) \\
&= -e_1 + e_4, \\
k(m_2) &= k(l_0 - l_1 - l_3) \\
&= -e_1 - e_2 - e_3 + e_4 + 2e_5 - (e_5 - e_1) - (e_5 - e_3) \\
&= -e_2 + e_4, \\
k(m_3) &= k(l_0 - l_1 - l_2) \\
&= -e_1 - e_2 - e_3 + e_4 + 2e_5 - (e_5 - e_1) - (e_5 - e_2) \\
&= -e_3 + e_4.
\end{aligned}$$

For l_1, l_2, l_3, m_1, m_2 and m_3, it is straightforward to check that k is W_3-linear. Further, k is a right-inverse of g since

$$\begin{aligned}
g(k(l_i)) &= g(-e_i + e_5) \\
&= -(l_0 - l_i) + l_0 \\
&= l_i, \\
g(k(m_i)) &= g(-e_i + e_4) \\
&= -(l_0 - l_i) + (2l_0 - l_1 - l_2 - l_3) \\
&= l_0 - l_1 - l_2 - l_3 + l_i \\
&= m_i.
\end{aligned}$$

for all $1 \leq i \leq 3$. □

Similarly, the map $h : \mathbb{Z} \to \mathbb{Z}^5$ has a left-inverse $f : \mathbb{Z}^5 \to \mathbb{Z}$ defined via

$$e_i \mapsto 1 \quad \text{for all } i.$$

It is clear that f is W_3-invariant. It is also a left-inverse of h since

$$\begin{aligned}
f(h(1)) &= f(e_1 + e_2 + e_3 - e_4 - e_5) \\
&= 1 + 1 + 1 - 1 - 1 \\
&= 1.
\end{aligned}$$

4.3 A Permutation Resolution for Degree 5

If the degree of the Del Pezzo surface is $d = 5$, we get $r = 9 - 5 = 4$. In this section, similarly to the previous one, we identify one example of a system of simple roots for the Picard group of a Del Pezzo surface of degree 5 over an algebraically closed field $\overline{K}$ and use this to compute the action of the Weyl group W_4 on the Picard group $\operatorname{Pic}\overline{S}$. It is then straightforward to verify that the short exact sequence given in Theorem 4.9 is a permutation resolution of $\operatorname{Pic}\overline{S}$ with respect to the Weyl group $W_4 \cong S_5$.

The equations (4.3), (4.4) and (4.6) of Section 4.1 lead to the following possibilities of the coefficients of $l = al_0 - \sum_{i=1}^{4} b_i l_i \in R_4$:

a	arbitrary permutation of the b_i
-1	-1, -1, -1, 0
0	1, -1, 0, 0
1	1, 1, 1, 0

Hence, we get a root system

$$\begin{aligned} R_4 = \big\{ & l_2 - l_1,\ l_1 - l_2,\ l_3 - l_2,\ l_2 - l_3,\ l_4 - l_3,\ l_3 - l_4, \\ & l_3 - l_1,\ l_1 - l_3,\ l_4 - l_1,\ l_1 - l_4,\ l_4 - l_2,\ l_2 - l_4, \\ & - l_0 + l_1 + l_2 + l_3,\ l_0 - l_1 - l_2 - l_3, \\ & - l_0 + l_1 + l_2 + l_4,\ l_0 - l_1 - l_2 - l_4 \\ & - l_0 + l_1 + l_3 + l_4,\ l_0 - l_1 - l_3 - l_4, \\ & - l_0 + l_2 + l_3 + l_4,\ l_0 - l_2 - l_3 - l_4 \big\}. \end{aligned}$$

Theorem 4.8. *A system of simple roots is given by*

$$s_1 := l_2 - l_1, \quad s_2 := l_3 - l_2, \quad s_3 := l_4 - l_3, \quad s_4 := -l_0 + l_1 + l_2 + l_3.$$

Proof. s_1, s_2, s_3 and s_4 are linearly independent and $\omega^\perp$ is $r = 4$-dimensional, hence it is a basis and (B1) is satisfied. For (B2), we check

$$\begin{aligned} l_2 - l_1 &= s_1, & l_1 - l_2 &= -s_1, \\ l_3 - l_2 &= s_2, & l_2 - l_3 &= -s_2, \\ l_4 - l_3 &= s_3, & l_3 - l_4 &= -s_3, \\ l_3 - l_1 &= s_1 + s_2, & l_1 - l_3 &= -s_1 - s_2, \\ l_4 - l_1 &= s_1 + s_2 + s_3, & l_1 - l_4 &= -s_1 - s_2 - s_3, \\ l_4 - l_2 &= s_2 + s_3, & l_2 - l_4 &= -s_2 - s_3, \end{aligned}$$

$$\begin{aligned}
-l_0 + l_1 + l_2 + l_3 &= s_4, & l_0 - l_1 - l_2 - l_3 &= -s_4,\\
-l_0 + l_1 + l_2 + l_4 &= s_3 + s_4, & l_0 - l_1 - l_2 - l_4 &= -s_3 - s_4,\\
-l_0 + l_1 + l_3 + l_4 &= s_2 + s_3 + s_4, & l_0 - l_1 - l_3 - l_4 &= -s_2 - s_3 - s_4,\\
-l_0 + l_2 + l_3 + l_4 &= s_1 + s_2 + s_3 + s_4, & l_0 - l_2 - l_3 - l_4 &= -s_1 - s_2 - s_3 - s_4.
\end{aligned}$$

□

Because of Theorem 2.14(a), we know that the Weyl group is generated by the simple roots. To completely describe the action of the Weyl group on the Picard group, it is therefore sufficient to compute the action of the simple roots on our given basis of the Picard group. Let w_i be the element of the Weyl group that reflects at s_i, i. e. with (4.2)

$$w_i(x) = x - 2\frac{(x, s_i)}{(s_i, s_i)} s_i = x + (x, s_i)\, s_i,$$

then

$$\begin{aligned}
w_1(l_0) &= l_0 + (l_0, s_1)\, s_1 = l_0,\\
w_1(l_1) &= l_1 + (l_1, s_1)\, s_1 = l_1 + (l_2 - l_1) = l_2,\\
w_1(l_2) &= l_2 + (l_2, s_1)\, s_1 = l_2 - (l_2 - l_1) = l_1,\\
w_1(l_3) &= l_3 + (l_3, s_1)\, s_1 = l_3,\\
w_1(l_4) &= l_4 + (l_4, s_1)\, s_1 = l_4,
\end{aligned}$$

$$\begin{aligned}
w_2(l_0) &= l_0 + (l_0, s_2)\, s_2 = l_0,\\
w_2(l_1) &= l_1 + (l_1, s_2)\, s_2 = l_1,\\
w_2(l_2) &= l_2 + (l_2, s_2)\, s_2 = l_2 + (l_3 - l_2) = l_3,\\
w_2(l_3) &= l_3 + (l_3, s_2)\, s_2 = l_3 - (l_3 - l_2) = l_2,\\
w_2(l_4) &= l_4 + (l_4, s_2)\, s_2 = l_4,
\end{aligned}$$

$$\begin{aligned}
w_3(l_0) &= l_0 + (l_0, s_3)\, s_3 = l_0,\\
w_3(l_1) &= l_1 + (l_1, s_3)\, s_3 = l_1,\\
w_3(l_2) &= l_2 + (l_2, s_3)\, s_3 = l_2,\\
w_3(l_3) &= l_3 + (l_3, s_3)\, s_3 = l_3 + (l_4 - l_3) = l_4,\\
w_3(l_4) &= l_4 + (l_4, s_3)\, s_3 = l_4 - (l_4 - l_3) = l_3,
\end{aligned}$$

and

$$w_4(l_0) = l_0 + (l_0, s_4)\, s_4 = l_0 - (-l_0 + l_1 + l_2 + l_3) = 2l_0 - l_1 - l_2 - l_3,$$

$$\begin{aligned}
w_4(l_1) &= l_1 + (l_1, s_4)\, s_4 = l_1 - (-l_0 + l_1 + l_2 + l_3) = l_0 - l_2 - l_3,\\
w_4(l_2) &= l_2 + (l_2, s_4)\, s_4 = l_2 - (-l_0 + l_1 + l_2 + l_3) = l_0 - l_1 - l_3,\\
w_4(l_3) &= l_3 + (l_3, s_4)\, s_4 = l_3 - (-l_0 + l_1 + l_2 + l_3) = l_0 - l_1 - l_2,\\
w_4(l_4) &= l_4 + (l_4, s_4)\, s_4 = l_4.
\end{aligned}$$

Let the Weyl group act

- on $\mathbb{Z}$: trivially.
- on $\mathbb{Z}^6$: Choose a basis $e_1, \dots, e_6$ and let w_i transpose e_i and e_{i+1} for $1 \leq i \leq 4$ and leave the rest fixed.
- on the Picard group $\operatorname{Pic} \overline{S}$ as computed above.

Theorem 4.9. *A W_4-permutation resolution of the Picard group $\operatorname{Pic} \overline{S}$ of a Del Pezzo surface $\overline{S}$ of degree 5 is given by*

$$0 \longrightarrow \mathbb{Z} \xrightarrow{h} \mathbb{Z}^6 \xrightarrow{g} \operatorname{Pic} \overline{S} \longrightarrow 0, \tag{4.11}$$

where $h,\ g$ are $\mathbb{Z}$-linear maps defined as $h : \mathbb{Z} \to \mathbb{Z}^6$,

$$1 \mapsto e_1 + e_2 + e_3 + e_4 + e_5 - 2e_6$$

and $g : \mathbb{Z}^6 \to \operatorname{Pic} \overline{S}$,

$$\begin{gathered}
e_1 \mapsto l_0 - l_1, \qquad e_2 \mapsto l_0 - l_2, \qquad e_3 \mapsto l_0 - l_3, \qquad e_4 \mapsto l_0 - l_4,\\
e_5 \mapsto 2l_0 - l_1 - l_2 - l_3 - l_4, \qquad e_6 \mapsto 3l_0 - l_1 - l_2 - l_3 - l_4.
\end{gathered} \tag{4.12}$$

Similarly to the case when the degree is equal to 6, we now let

$$m_{ij} := l_0 - l_i - l_j$$

for $1 \leq i, j \leq 4$. Then

$$l_1,\ l_2,\ l_3,\ l_4,\ m_{ij} \quad \text{for } 1 \leq i < j \leq 4$$

also generate the Picard group. The geometric interpretation is the same as for degree 6: If $\overline{S}$ is the blow up of the four points P_1, P_2, P_3 and P_4 in $\mathbb{P}^2$, then l_1, l_2, l_3 and l_4 are the pullbacks of those points and for any $1 \leq i, j \leq 4$, m_{ij} is the strict transform of the line joining P_i and P_j.

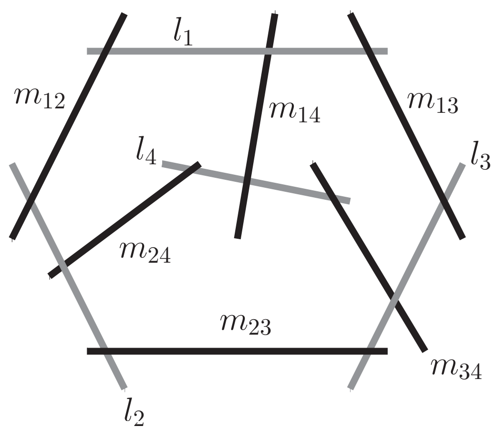

Figure 4.2: A generating system for the Picard group of a Del Pezzo surface of Degree 5

The Weyl group acts on our new generating system via

w_i for $1 \leq i \leq 3$:	The element w_i transposes indices i and $i+1$.
w_4 :	For any permutation i, j, k of $1, 2, 3$, we have $w_4(m_{ij}) = l_k$ and $w_4(m_{i4}) = m_{i4}$ for all $1 \leq i \leq 3$.

As before, we also rewrite the map $g : \mathbb{Z}^6 \to \operatorname{Pic} \overline{S}$ in terms of our new generating set:

$$\begin{aligned} &e_1 \mapsto m_{12} + l_2, \quad e_2 \mapsto m_{23} + l_3, \quad e_3 \mapsto m_{13} + l_1, \quad e_4 \mapsto m_{14} + l_1, \\ &e_5 \mapsto m_{12} + m_{34}, \quad e_6 \mapsto m_{14} + m_{24} + m_{34} + 2l_4 \end{aligned} \tag{4.12'}$$

Proof of Theorem 4.9. W_4 permutes the chosen bases of $\mathbb{Z}$ and $\mathbb{Z}^6$. Further, the sequence is exact because h is injective, g is surjective and $\operatorname{im}(h) = \ker(g)$: Consider the transformation matrix A of g for the ordered basis $(e_1, \ldots, e_6)$ and $(l_0, \ldots, l_4)$ using (4.12). It is

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 2 & 3 \\ -1 & 0 & 0 & 0 & -1 & -1 \\ 0 & -1 & 0 & 0 & -1 & -1 \\ 0 & 0 & -1 & 0 & -1 & -1 \\ 0 & 0 & 0 & -1 & -1 & -1 \end{pmatrix},$$

hence for an element $a_1e_1 + \dots a_6e_6$ with $a_1, \dots, a_6 \in \mathbb{Z}$ to be in the kernel of g, we get the equations

$$\begin{aligned}
a_4 &= -a_5 - a_6,\\
a_3 &= -a_5 - a_6,\\
a_2 &= -a_5 - a_6,\\
a_1 &= -a_5 - a_6,\\
a_6 &= -a_1 - a_2 - a_3 - a_4 - 2(a_5 + a_6)\\
&= (a_5 + a_6) + (a_5 + a_6) + (a_5 + a_6) + (a_5 + a_6) - 2(a_5 + a_6)\\
&= 2(a_5 + a_6),
\end{aligned}$$

hence

$$\begin{aligned}
a_6 &= -2a_5,\\
a_4 &= -a_5 + 2a_5 = a_5,\\
a_3 &= a_5,\\
a_2 &= a_5,\\
a_1 &= a_5.
\end{aligned}$$

With $a_5 := 1 \in \mathbb{Z}^\times$ this yields

$$\ker(g) = \mathbb{Z}(e_1 + e_2 + e_3 + e_4 + e_5 - 2e_6) = \mathrm{im}(h).$$

Finally, one can see that h is W_4-linear since $e_1 + e_2 + e_3 + e_4 + e_5 - 2e_6$ is invariant. The map g is invariant since:

$$\begin{aligned}
w_1.g(e_1) &= m_{12} + l_1 = m_{23} + l_3 = g(w_1.e_1),\\
w_1.g(e_2) &= m_{13} + l_3 = m_{12} + l_2 = g(w_1.e_2),\\
w_1.g(e_3) &= m_{23} + l_2 = m_{13} + l_1 = g(w_1.e_3),\\
w_1.g(e_4) &= m_{24} + l_2 = g(w_1.e_4),\\
w_1.g(e_5) &= m_{12} + m_{34} = g(w_1.e_5),\\
w_1.g(e_6) &= m_{24} + m_{13} + m_{34} + 2l_4 = g(w_1.e_6),
\end{aligned}$$

$$\begin{aligned}
w_2.g(e_1) &= m_{13} + l_3 = m_{12} + l_2 = g(w_2.e_1),\\
w_2.g(e_2) &= m_{23} + l_2 = m_{13} + l_1 = g(w_2.e_2),\\
w_2.g(e_3) &= m_{12} + l_1 = m_{23} + l_3 = g(w_2.e_3),\\
w_2.g(e_4) &= m_{14} + l_1 = g(w_2.e_4),\\
w_2.g(e_5) &= m_{13} + m_{24} = m_{12} + m_{34} = g(w_2.e_5),
\end{aligned}$$

$$w_2.g(e_6) = m_{14} + m_{34} + m_{24} + 2l_4 = g(w_2.e_6),$$

$$\begin{aligned}
w_3.g(e_1) &= m_{12} + l_2 = g(w_3.e_1),\\
w_3.g(e_2) &= m_{24} + l_4 = m_{23} + l_3 = g(w_3.e_2),\\
w_3.g(e_3) &= m_{14} + l_1 = g(w_3.e_3),\\
w_3.g(e_4) &= m_{13} + l_1 = g(w_3.e_4),\\
w_3.g(e_5) &= m_{12} + m_{34} = g(w_3.e_5),\\
w_3.g(e_6) &= m_{13} + m_{23} + m_{34} + 2l_3 = m_{14} + m_{24} + m_{34} + 2l_4 = (w_3.e_6),
\end{aligned}$$

and

$$\begin{aligned}
w_4.g(e_1) &= l_3 + m_{13} = m_{12} + l_2 = g(w_4.e_1),\\
w_4.g(e_2) &= l_1 + m_{12} = m_{23} + l_3 = g(w_4.e_2),\\
w_4.g(e_3) &= l_2 + m_{23} = m_{13} + l_1 = g(w_4.e_3),\\
w_4.g(e_4) &= m_{14} + m_{23} = m_{12} + m_{34} = g(w_4.e_4),\\
w_4.g(e_5) &= l_3 + m_{34} = m_{14} + l_1 = g(w_4.e_5),\\
w_4.g(e_6) &= m_{14} + m_{24} + m_{34} + 2l_4 = g(w_4.e_6),
\end{aligned}$$ □

Theorem 4.10. *The permutation resolution of Theorem 4.9 is a split exact sequence.*

Proof. Define a $\mathbb{Z}$-linear homomorphism $k : \operatorname{Pic} \overline{S} \to \mathbb{Z}^6$ via

$$\begin{aligned}
&l_0 \mapsto 2e_1 + 2e_2 + 2e_3 + 2e_4 + e_5 - 3e_6, \qquad l_1 \mapsto e_2 + e_3 + e_4 - e_6,\\
&l_2 \mapsto e_1 + e_3 + e_4 - e_6, \qquad l_3 \mapsto e_1 + e_2 + e_4 - e_6,\\
&l_4 \mapsto e_1 + e_2 + e_3 - e_6,
\end{aligned} \tag{4.13}$$

hence also

$$\begin{aligned}
k(m_{ij}) &= k(l_0 - l_i - l_j)\\
&= (2e_1 + 2e_2 + 2e_3 + 2e_4 + e_5 - 3e_6) - (e_1 + e_2 + e_3 + e_4 - e_i - e_6)\\
&\qquad - (e_1 + e_2 + e_3 + e_4 - e_j - e_6)\\
&= e_i + e_j + e_5 - e_6.
\end{aligned}$$

For l_1, l_2, l_3, l_4, m_{ij}, it is straightforward to check that k is S_5-linear. Further, k is a right-inverse of g: For $1 \le i, j \le 4$, we have

$$g(k(l_i)) = g(e_1 + e_2 + e_3 + e_4 - e_i - e_6)$$

$$
\begin{aligned}
&= (l_0 - l_1) + (l_0 - l_2) + (l_0 - l_3) + (l_0 - l_4) \\
&\quad - (l_0 - l_i) - (3l_0 - l_1 - l_2 - l_3 - l_4) \\
&= l_i
\end{aligned}
$$

and

$$
\begin{aligned}
g(k(m_{ij})) &= g(e_i + e_j + e_5 - e_6) \\
&= (l_0 - l_i) + (l_0 - l_j) + (2l_0 - l_1 - l_2 - l_3 - l_4) \\
&\quad - (3l_0 - l_1 - l_2 - l_3 - l_4) \\
&= l_0 - l_i - l_j \\
&= m_{ij}
\end{aligned}
$$

□

Similarly, the map $h : \mathbb{Z} \to \mathbb{Z}^6$ has a left-inverse $f : \mathbb{Z}^6 \to \mathbb{Z}$ defined via

$$
\begin{aligned}
e_i &\mapsto 1 \quad \text{for all } 1 \leq i \leq 5, \\
e_6 &\mapsto 2
\end{aligned}
$$

It is again clear that f is W_4-invariant. It is also a left-inverse of h since

$$
\begin{aligned}
f(h(1)) &= f(e_1 + e_2 + e_3 + e_4 + e_5 - 2e_6) \\
&= 1 + 1 + 1 + 1 + 1 - 2 \cdot 2 \\
&= 1.
\end{aligned}
$$

Chapter 5

The Chow Motives of Etale Algebras

We begin by defining étale algebras and identifying their properties following Bourbaki [1].

Definition. Let K be a field and A a K-algebra. A is called

- *diagonalizable* over K, if there is a non-negative integer n such that A is isomorphic to the product algebra K^n.
- *diagonalized* by an extension L of K if $A_L = L \otimes_K A$ is diagonalizable over L. We say that L *diagonalizes* A.
- *étale* if there is an extension of K that diagonalizes A.

Let K be a perfect field, $\overline{K}$ the algebraic closure of K and A a commutative K-algebra of finite degree.

Proposition 5.1 (Chapter V, Sections 6-8 in [1])**.** *The following are equivalent:*

(1) A is an étale algebra.

(2) There is a finite extension L of K that diagonalizes A, i. e.

$$L \otimes_K A \cong L^m$$

for some non-negative integer m.

(3) $\overline{K}$ diagonalizes A.

(4) For every extension L of K, the ring $L \otimes_K A$ is reduced.

(5) There are finite separable extensions $L_1, \ldots, L_n$ of K such that

$$A \cong L_1 \times \ldots \times L_n.$$

(6) $A \cong K[X]/(f)$ for some separable polynomial $f \in K[X]$.

(7) For every $0 \neq x \in A$, there is $y \in A$ such that $\mathrm{Tr}_{A/K}(xy) \neq 0$.

Etale algebras are of special interest to us since they are closely related to G-permutation modules. Let L/K be a Galois extension and $G = \mathrm{Gal}(L/K)$. We first define a G-action on Chow groups: For $g \in G$ and a smooth projective scheme X, let σ_g denote the induced morphism $\overline{K} \times_K X \to \overline{K} \times_K X$. Define the G-action on $\mathrm{CH}_i\left(\overline{X}\right)$ via the morphism σ_g^*. Now let A be an étale algebra. Since we can write $A \cong L_1 \times \ldots \times L_m$ for some intermediate fields $K \subseteq L_i \subseteq \overline{K}$ such that L_i/K is finite and separable, i. e. $A \cong \overline{K}^{H_1} \times \ldots \overline{K}^{H_n}$ for some open subgroups $H_i \subset G$, then

$$\mathrm{CH}_0\left(\overline{K} \otimes_K A\right) \cong \bigoplus_{i=1}^{n} \mathrm{CH}_0\left(\overline{K} \otimes_K \overline{K}^{H_i}\right)$$

as $\mathbb{Z}[G]$-modules. Furthermore, since for any open $H \subset G$, there is $f \in K[X]$ such that $\overline{K}^H \cong K[X]/fK[X]$, hence

$$\overline{K} \otimes_K \overline{K}^H \cong \overline{K} \otimes_K K[X]/(f) \cong \overline{K}[X]/(f) \cong \overline{K}^{\deg f}$$

since $\overline{K}$ is algebraically closed. The G-action on $\overline{K} \otimes_K \overline{K}^H$ induces a G-action on $\overline{K}^{\deg f}$ that permutes the different factors of $\overline{K}$. The prime ideals in $\overline{K}^{\deg f}$ are given by

$$\tilde{A}_j := \overline{K} \times \ldots \times \overline{K} \times \underbrace{\{0\}}_{j-\text{th position}} \times \overline{K} \times \ldots \times \overline{K}$$

for $1 \leq j \leq \deg f$, which get permuted by the G-action. Let $A_j \subset \overline{K} \otimes_K \overline{K}^H$ denote the corresponding ideals in $\overline{K} \otimes_K \overline{K}^H$. Then

$$\mathrm{CH}_0\left(\overline{K} \times_K \overline{K}^H\right) \cong \bigoplus_{j=1}^{\deg f} \mathbb{Z} \cdot \left[\mathrm{Spec}\left(\left(\overline{K} \otimes_K \overline{K}^H\right)/A_j\right)\right]$$

is a G-permutation module. Since G acts transitively on the set

$$\{A_j \mid 1 \leq j \leq \deg f\}$$

and the stabilizer of any A_j in G is given by H, we get

$$\mathrm{CH}_0\left(\overline{K} \otimes_K \overline{K}^H\right) \cong \mathbb{Z}[G/H],$$

c. f. the proof of Proposition 2.7. This yields

$$\mathrm{CH}_0\left(\overline{K} \otimes_K A\right) \cong \bigoplus_{i=1}^{n} \mathbb{Z}[G/H_i]. \tag{5.1}$$

Lemma 5.2. *Let A be an étale algebra over K, $G = \mathrm{Gal}\left(\overline{K}/K\right)$ and X a smooth projective scheme over K. Then the map*

$$\begin{gathered}\mathrm{CH}_0\left(\overline{K} \otimes_K A\right) \otimes_{\mathbb{Z}} \mathrm{CH}_m\left(\overline{X}\right) \to \mathrm{CH}_m\left(\mathrm{Spec}\left(\overline{K} \otimes_K A\right) \times_{\overline{K}} \overline{X}\right) \\ \alpha \otimes \beta \mapsto \alpha \times \beta\end{gathered}$$

is an isomorphism of G-modules.

Proof. Let n be an integer such that $\overline{K} \otimes_K A \cong \overline{K}^n$. Then

$$\begin{aligned}\mathrm{Spec}\left(\overline{K} \otimes_K A\right) &\cong \mathrm{Spec}\,\overline{K}^n \\ &\cong \coprod_{i=1}^{n} \mathrm{Spec}\,\overline{K},\end{aligned}$$

and similarly

$$\begin{aligned}\mathrm{Spec}\left(\overline{K} \otimes A\right) \times_{\overline{K}} \overline{X} &\cong \left(\coprod_{i=1}^{n} \mathrm{Spec}\,\overline{K}\right) \times_{\overline{K}} \overline{X} \\ &\cong \coprod_{i=1}^{n} \left(\mathrm{Spec}\,\overline{K} \times_{\overline{K}} \overline{X}\right) \\ &\cong \coprod_{i=1}^{n} \overline{X},\end{aligned}$$

hence

$$\begin{aligned}\mathrm{CH}_0\left(\overline{K} \otimes_K A\right) &\cong \bigoplus_{i=1}^{n} \mathrm{CH}_0(\overline{K}) \\ &\cong \bigoplus_{i=1}^{n} \mathbb{Z}\end{aligned}$$

and

$$\mathrm{CH}_m\left(\mathrm{Spec}\left(\overline{K} \otimes_K A\right) \times_{\overline{K}} \overline{X}\right) \cong \bigoplus_{i=1}^{n} \mathrm{CH}_m\left(\overline{X}\right),$$

where each copy of $\mathrm{CH}_0\left(\overline{K}\right) \otimes_{\mathbb{Z}} \mathrm{CH}_m\left(\overline{X}\right) \cong \mathrm{CH}_m\left(\overline{X}\right)$ in

$$\mathrm{CH}_0\left(\overline{K} \otimes_K A\right) \otimes_Z \mathrm{CH}_m\left(\overline{X}\right) \cong \bigoplus_{i=1}^{n} \mathrm{CH}_0\left(\overline{K}\right) \otimes_{\mathbb{Z}} \mathrm{CH}_m\left(\overline{X}\right)$$

gets mapped into the corresponding copy of $\mathrm{CH}_m\left(\overline{X}\right)$ in

$$\mathrm{CH}_m\left(\mathrm{Spec}\left(\overline{K} \otimes_K A\right) \times_{\overline{K}} \overline{X}\right) \cong \bigoplus_{i=1}^{n} \mathrm{CH}_m\left(\overline{X}\right),$$

hence the map is an isomorphism of $\mathbb{Z}$-modules. It further is G-invariant since $g \in G$ acts on $\alpha \otimes \beta \in \mathrm{CH}_0\left(\overline{K} \otimes_K A\right) \otimes_{\mathbb{Z}} \mathrm{CH}_m\left(\overline{X}\right)$ via

$$g.\left(\alpha \otimes \beta\right) = g.\alpha \otimes g.\beta$$

and on $\alpha \times \beta \in \mathrm{CH}_m\left(\mathrm{Spec}\left(\overline{K} \otimes_K A\right) \times_{\overline{K}} \overline{X}\right)$ via

$$g.\left(\alpha \times \beta\right) = \left(g.\alpha\right) \times \left(g.\beta\right). \qquad \square$$

Lemma 5.3. *Let X and Y be smooth projective schemes over K and $G = \mathrm{Gal}\left(\overline{K}/K\right)$. For any $\alpha \in \mathrm{CH}_m\left(X \times_K Y\right)$, we define $\alpha_{\overline{K}} := \mathrm{res}_{\overline{K}/K}\left(\alpha\right) \in \mathrm{CH}_m\left(\overline{X} \times_{\overline{K}} \overline{Y}\right)$. Then the map*

$$\begin{aligned} \mathrm{CH}_m\left(X \times_K Y\right) &\to \mathrm{Hom}_G\left(\mathrm{CH}_0\left(\overline{K} \times_{\overline{K}} \overline{X}\right), \mathrm{CH}_m\left(\overline{K} \times_{\overline{K}} \overline{Y}\right)\right) \\ \alpha &\mapsto \left(\left(\alpha_{\overline{K}}\right)_* : \beta \mapsto \alpha_{\overline{K}} \circ \beta\right) \end{aligned}$$

is well defined.

Proof. We have to show that $\left(\alpha_{\overline{K}}\right)_*$ is G-invariant for any $\alpha \in \mathrm{CH}_m\left(X \times_K Y\right)$: Let $\beta \in \mathrm{CH}_0\left(\overline{X}\right) \cong \mathrm{CH}_0\left(\overline{K} \times_{\overline{K}} \overline{X}\right)$. Since the diagram

$$\begin{tikzcd} \overline{X} \times_{\overline{K}} \overline{Y} \arrow[r, "\sigma_g"] \arrow[d, "p_{\overline{Y}}"'] & \overline{X} \times_{\overline{K}} \overline{Y} \arrow[d, "p_{\overline{Y}}"] \\ \overline{Y} \arrow[r, "\sigma_g"'] & \overline{Y} \end{tikzcd}$$

commutes, we can compute

$$\begin{aligned}
(\alpha_{\overline{K}})_* (g.(\beta)) &= (\alpha_{\overline{K}})_* \left(\sigma_g^*(\beta)\right) \\
&= \alpha_{\overline{K}} \circ \left(\sigma_g^*(\beta)\right) \\
&= (p_{\overline{KY}})_* \left(\alpha_{\overline{K}} \cdot \sigma_g^* (\beta) \times \overline{Y}\right) \\
&= (p_{\overline{KY}})_* \left(\alpha_{\overline{K}} \cdot \sigma_g^* \left(\beta \times \overline{Y}\right)\right) \\
&= (p_{\overline{KY}})_* \left(\sigma_g^* (\alpha_{\overline{K}}) \cdot \sigma_g^* \left(\beta \times \overline{Y}\right)\right) \\
&= (p_{\overline{KY}})_* \sigma_g^* \left(\alpha_{\overline{K}} \cdot \beta \times \overline{Y}\right) \\
&= \sigma_g^* (p_{\overline{KY}})_* \left(\alpha_{\overline{K}} \cdot \beta \times \overline{Y}\right) \\
&= \sigma_g^* \left((\alpha_{\overline{K}})_* (\beta)\right) \\
&= g. (\alpha_{\overline{K}})_* (\beta) .
\end{aligned}$$

□

Lemma 5.4. *Let X and Y be smooth projective schemes over K and $G = \operatorname{Gal}\left(\overline{K}/K\right)$. The map*

$$\begin{aligned}
\mathrm{CH}_m \left(\overline{X} \times_{\overline{K}} \overline{Y}\right) &\to \operatorname{Hom}_{\mathbb{Z}} \left(\mathrm{CH}_0 \left(\overline{K} \times_{\overline{K}} \overline{X}\right), \mathrm{CH}_m \left(\overline{K} \times_{\overline{K}} \overline{Y}\right)\right) \\
\alpha &\mapsto (\alpha_* : \beta \mapsto \alpha \circ \beta)
\end{aligned}$$

is G-invariant.

Proof. Let $g \in G$ and σ_g denote the induced morphism on schemes. We can then compute

$$\begin{aligned}
(g.(\alpha)_*) (\beta) &= g.\alpha_* \left(g^{-1}.\beta\right) \\
&= \sigma_g^* (p_{\overline{KY}})_* \left(\overline{K} \times \alpha \cdot \sigma_{g^{-1}}^* \left(\beta \times \overline{Y}\right)\right) \\
&= \sigma_g^* (p_{\overline{KY}})_* \sigma_{g^{-1}}^* \left(\sigma_g^* \left(\overline{K} \times \alpha\right) \cdot \beta \times \overline{Y}\right) \\
&= \sigma_g^* \sigma_{g^{-1}}^* (p_{\overline{KY}})_* \left(\overline{K} \times \sigma_g^* (\alpha) \cdot \beta \times \overline{Y}\right) \\
&= (g.\alpha)_*(\beta).
\end{aligned}$$

□

Lemma 5.5. *Let K be algebraically closed, A an étale algebra over K and X a smooth projective scheme over K. Then the following maps are isomorphisms of G-modules:*

(a) $$\begin{aligned}
\Phi : \mathrm{CH}_m (A \times_K X) &\to \operatorname{Hom}_{\mathbb{Z}} (\mathrm{CH}_0 (K \times_K A), \mathrm{CH}_m (K \times_K X)) \\
\alpha &\mapsto (\alpha_* : \beta \mapsto \alpha \circ \beta)
\end{aligned}$$

(b) $$\begin{aligned}
\Psi : \mathrm{CH}_m (X \times_K A) &\to \operatorname{Hom}_{\mathbb{Z}} (\mathrm{CH}_m (K \times_K X), \mathrm{CH}_0 (K \times_K A)) \\
\alpha &\mapsto (\alpha_* : \beta \mapsto \alpha \circ \beta) .
\end{aligned}$$

Proof. We will only show part (a), part (b) can be proven similarly. Since A is an étale algebra over an algebraically closed field, we can assume that $A = K^n$ for some positive integer n. Hence,

$$A \times_K X = K^n \times_K X \cong \underbrace{X \dot{\cup} \dots \dot{\cup} X}_{n \text{ times}}$$

and therefore

$$\mathrm{CH}_i\left(A \times_K X\right) \cong \bigoplus_{i=1}^{n} \mathrm{CH}_i\left(X\right).$$

Similarly,

$$\begin{aligned} &\mathrm{Hom}_{\mathbb{Z}}\left(\mathrm{CH}_0\ \left(K \times_K A\right), \mathrm{CH}_m\left(K \times_K X\right)\right) \\ &\qquad = \mathrm{Hom}_{\mathbb{Z}}\left(\bigoplus_{i=1}^{n} \mathrm{CH}_0\left(K \times_K K\right), \mathrm{CH}_m\left(K \times_K X\right)\right) \\ &\qquad = \bigoplus_{i=1}^{n} \mathrm{Hom}_{\mathbb{Z}}\left(\mathrm{CH}_0\left(K \times_K K\right), \mathrm{CH}_m\left(K \times_K X\right)\right) \end{aligned}$$

Let pr_i denote the projection onto the i-th component. Since the following diagram commutes

$$\begin{tikzcd} \bigoplus_{i=1}^{n} \mathrm{CH}_m\left(K \times_K X\right) \arrow[r, "\Phi"] \arrow[d, "pr_i"'] & \bigoplus_{i=1}^{n} \mathrm{Hom}_{\mathbb{Z}}\left(\mathrm{CH}_0\left(K \times_K K\right), \mathrm{CH}_m\left(K \times_K X\right)\right) \arrow[d, "pr_i"] \\ \mathrm{CH}_m\left(K \times_K X\right) \arrow[r, "\Phi"'] & \mathrm{Hom}_{\mathbb{Z}}\left(\mathrm{CH}_0\left(K \times_K K\right), \mathrm{CH}_m\left(K \times_K X\right)\right) \end{tikzcd} \tag{5.2}$$

it suffices to notice that

$$\Phi : \mathrm{CH}_m\left(K \times_K X\right) \to \mathrm{Hom}_{\mathbb{Z}}\left(\mathrm{CH}_0\left(K \times_K K\right), \mathrm{CH}_m\left(K \times_K X\right)\right)$$

is the isomorphism that maps $\alpha \in \mathrm{CH}_m\left(K \times_K X\right)$ onto the $\mathbb{Z}$-homomorphism α_* mapping $\mathrm{id}_K \in \mathrm{CH}_0\left(K \times_K K\right)$ onto α. Hence, Φ is an isomorphism of $\mathbb{Z}$-modules. Since the previous lemma yields that Φ is G-invariant, Φ is an isomorphism of G-modules. $\square$

Theorem 5.6. *(a) Let*

$$\Phi : \mathrm{CH}_m\left(A \times_K X\right) \to \mathrm{Hom}_G\left(\mathrm{CH}_0\left(\overline{K} \times_{\overline{K}} \overline{A}\right), \mathrm{CH}_m\left(\overline{K} \times_{\overline{K}} \overline{X}\right)\right)$$
$$\alpha \mapsto \left(\left(\alpha_{\overline{K}}\right)_* : \beta \mapsto \alpha_{\overline{K}} \circ \beta \right).$$

Then the following diagram

$$\begin{tikzcd}
\mathrm{CH}_m\left(\overline{A} \times_{\overline{K}} \overline{X}\right) \arrow[r, "\Phi_{\overline{K}}"] & \mathrm{Hom}_{\mathbb{Z}}\left(\mathrm{CH}_0\left(\overline{K} \times_{\overline{K}} \overline{A}\right), \mathrm{CH}_m\left(\overline{K} \times_{\overline{K}} \overline{X}\right)\right) \\
\mathrm{CH}_m\left(A \times_K X\right) \arrow[u, "\mathrm{res}_{\overline{K}/K}"] \arrow[r, "\Phi"'] & \mathrm{Hom}_G\left(\mathrm{CH}_0\left(\overline{K} \times_{\overline{K}} \overline{A}\right), \mathrm{CH}_m\left(\overline{K} \times_{\overline{K}} \overline{X}\right)\right) \arrow[u, hook, "\textit{forget } G\textit{-structure}"']
\end{tikzcd} \tag{5.3}$$

commutes.

(b) Similarly, let

$$\Psi : \mathrm{CH}_m\left(X \times_K A\right) \to \mathrm{Hom}_G\left(\mathrm{CH}_m\left(\overline{K} \times_{\overline{K}} \overline{X}\right), \mathrm{CH}_0\left(\overline{K} \times_{\overline{K}} \overline{A}\right)\right)$$
$$\alpha \mapsto \left(\left(\alpha_{\overline{K}}\right)_* : \beta \mapsto \alpha_{\overline{K}} \circ \beta \right).$$

Then the following diagram

$$\begin{tikzcd}
\mathrm{CH}_m\left(\overline{X} \times_{\overline{K}} \overline{A}\right) \arrow[r, "\Psi_{\overline{K}}"] & \mathrm{Hom}_{\mathbb{Z}}\left(\mathrm{CH}_m\left(\overline{K} \times_{\overline{K}} \overline{X}\right), \mathrm{CH}_0\left(\overline{K} \times_{\overline{K}} \overline{A}\right)\right) \\
\mathrm{CH}_m\left(X \times_K A\right) \arrow[u, "\mathrm{res}_{\overline{K}/K}"] \arrow[r, "\Psi"'] & \mathrm{Hom}_G\left(\mathrm{CH}_m\left(\overline{K} \times_{\overline{K}} \overline{X}\right), \mathrm{CH}_0\left(\overline{K} \times_{\overline{K}} \overline{A}\right)\right) \arrow[u, hook, "\textit{forget } G\textit{-structure}"']
\end{tikzcd} \tag{5.4}$$

commutes.

The proof follows directly from the definitions of the maps Φ and Ψ.

Corollary 5.7. *Let S be a surface with a K-rational point. Then the following maps are isomorphisms of G-modules:*

(a) $\Phi : \mathrm{CH}_1\left(A \times_K S\right) \to \mathrm{Hom}_G\left(\mathrm{CH}_0\left(\overline{K} \times_{\overline{K}} \overline{A}\right), \mathrm{CH}_1\left(\overline{K} \times_{\overline{K}} \overline{S}\right)\right)$
$$\alpha \mapsto \left(\left(\alpha_{\overline{K}}\right)_* : \beta \mapsto \alpha_{\overline{K}} \circ \beta \right),$$

(b) $\Psi : \mathrm{CH}_1\left(S \times_K A\right) \to \mathrm{Hom}_G\left(\mathrm{CH}_1\left(\overline{K} \times_{\overline{K}} \overline{S}\right), \mathrm{CH}_0\left(\overline{K} \times_{\overline{K}} \overline{A}\right)\right)$
$$\alpha \mapsto \left(\left(\alpha_{\overline{K}}\right)_* : \beta \mapsto \alpha_{\overline{K}} \circ \beta \right).$$

Proof. We will again only prove part (a). It suffices to show this statement for $A = L$, where L/K is a finite separable field extension. Since the Diagram 5.3 commutes, $\Phi_{\overline{K}}$ is an isomorphism and

$$\mathrm{Hom}_G\left(\mathrm{CH}_0\left(\overline{K} \times_{\overline{K}} \overline{L}\right), \mathrm{CH}_1\left(\overline{K} \times_{\overline{K}} \overline{S}\right)\right)$$

consists of the G-invariant elements in

$$\mathrm{Hom}_{\mathbb{Z}}\left(\mathrm{CH}_0\left(\overline{K} \times_{\overline{K}} \overline{L}\right), \mathrm{CH}_1\left(\overline{K} \times_{\overline{K}} \overline{S}\right)\right),$$

it remains to prove that $\mathrm{CH}_1(L \times_K S)$ consists of the G-invariant elements in $\mathrm{CH}_1\left(\overline{L} \times_{\overline{K}} \overline{S}\right)$: This identity follows from Lemma 2.5, which shows that

$$\begin{aligned} \mathrm{Pic}(S \times_K L) &\cong \mathrm{Pic}(S_L \times_K K) \\ &\cong \left[\mathrm{Pic}\left(S_L \times_K \overline{K}\right)\right]^G \\ &\cong \left[\mathrm{Pic}\left(\left(S \times_K \overline{K}\right) \times_{\overline{K}} \left(L \times_K \overline{K}\right)\right)\right]^G \\ &\cong \left[\mathrm{Pic}\left(\overline{S} \times_{\overline{K}} \overline{L}\right)\right]^G \end{aligned}$$

and thus

$$\mathrm{CH}_1(L \times_K S) \cong \left[\mathrm{CH}_1\left(\overline{L} \times_{\overline{K}} \overline{S}\right)\right]^G. \qquad \square$$

Corollary 5.8. *Let A and B be étale algebras over K. Then*

$$\begin{aligned} \Phi : \mathrm{CH}_0(A \times_K B) &\to \mathrm{Hom}_G\left(\mathrm{CH}_0\left(\overline{K} \times_{\overline{K}} \overline{A}\right), \mathrm{CH}_0\left(\overline{K} \times_{\overline{K}} \overline{B}\right)\right) \\ \alpha &\mapsto \Big((\alpha_{\overline{K}})_* : \beta \mapsto \alpha_{\overline{K}} \circ \beta \Big), \end{aligned}$$

is an isomorphism of G-modules.

Proof. Since the Diagram 5.3 commutes, $\Phi_{\overline{K}}$ is an isomorphism and

$$\mathrm{Hom}_G\left(\mathrm{CH}_0\left(\overline{K} \times_{\overline{K}} \overline{A}\right), \mathrm{CH}_0\left(\overline{K} \times_{\overline{K}} \overline{B}\right)\right)$$

consists of the G-invariant elements in

$$\mathrm{Hom}_{\mathbb{Z}}\left(\mathrm{CH}_0\left(\overline{K} \times_{\overline{K}} \overline{A}\right), \mathrm{CH}_0\left(\overline{K} \times_{\overline{K}} \overline{B}\right)\right),$$

it remains to show that $\mathrm{CH}_0(A \times_K B)$ consists of the G-invariant elements in $\mathrm{CH}_0\left(\overline{A} \times_{\overline{K}} \overline{B}\right)$. It is clear that $\mathrm{CH}_0(A \times_K B) \subseteq \mathrm{CH}_0\left(\overline{A} \times_{\overline{K}} \overline{B}\right)^G$. For the other inclusion, recall that $\mathrm{CH}_0\left(\overline{A} \times_{\overline{K}} \overline{B}\right)$ is the free abelian group of zero cycles in $\overline{A} \times_{\overline{K}} \overline{B}$, which, in turn, are given by the prime ideals in $\overline{A} \otimes_{\overline{K}} \overline{B}$. Since the extension $K \subset \overline{K}$ is algebraic, every prime ideal of $A \otimes_K B$ lies under some prime ideal of $\overline{A} \otimes_{\overline{K}} \overline{B}$. Further, since the Galois group acts transitively on the prime ideals in $\overline{A} \otimes_{\overline{K}} \overline{B}$ above a given prime ideal in $A \otimes_K B$, this yields $\mathrm{CH}_0\left(\overline{A} \times_{\overline{K}} \overline{B}\right)^G \subseteq \mathrm{CH}_0\left(A \times_K B\right)$ and therefore

$$\mathrm{CH}_0\left(\overline{A} \times_{\overline{K}} \overline{B}\right)^G = \mathrm{CH}_0\left(A \times_K B\right). \qquad \square$$

Now define the full additive subcategory $\mathrm{Chow}(K)^{\overline{K}}$ of $\mathrm{Chow}(K)$ with objects generated by $\operatorname{Spec} L$ for a finite Galois extension L/K, hence we have

- objects: $\operatorname{Spec} A$ for an étale K-algebra A,
- morphisms: $\mathrm{Hom}_{\mathrm{Chow}(K)^{\overline{K}}}(A, B) := \mathrm{Hom}_{\mathrm{Chow}(K)}(A, B) = \mathrm{CH}_0\left(A \times_K B\right)$.

Definition. We define a functor from $\mathrm{Chow}(K)^{\overline{K}}$ into the category of G-permutation modules via

- objects: $X \mapsto \mathrm{Hom}_{\mathrm{Chow}(K)}\left(\overline{K}, X\right) = \mathrm{CH}_0\left(\overline{K} \times_K X\right) = \mathrm{CH}_0\left(\overline{K} \times_{\overline{K}} \overline{X}\right)$,
- morphisms:

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Chow}(K)^{\overline{K}}}\left(X, Y\right) &\to \mathrm{Hom}_G\left(\mathrm{CH}_0\left(\overline{K} \times_{\overline{K}} \overline{X}\right), \mathrm{CH}_0\left(\overline{K} \times_{\overline{K}} \overline{Y}\right)\right) \\ \alpha &\mapsto \left(\mathrm{res}_{\overline{K}/K}(\alpha)\right)_* . \end{aligned}$$

Theorem 5.9. *This functor is an equivalence of categories.*

Proof. Corollary 5.7 shows that for two objects X, Y in $\mathrm{Chow}(K)^{\overline{K}}$, the induced maps

$$\mathrm{Hom}_{\mathrm{Chow}(K)}\left(X, Y\right) \to \mathrm{Hom}_G\left(\mathrm{CH}_0\left(\overline{K} \times_{\overline{K}} \overline{X}\right), \mathrm{CH}_0\left(\overline{K} \times_{\overline{K}} \overline{Y}\right)\right)$$

are isomorphisms of G-modules. It remains to show that each G-permutation module M is isomorphic to the image of $\operatorname{Spec} A$ for some étale algebra A: If M is a permutation module, then by Proposition 2.7, M is isomorphic to the direct sum $\bigoplus_{i=1}^n \mathbb{Z}[G/H_i]$ for some open subgroups $H_i \subseteq G$. Let $E_i := \overline{K}^{H_i}$ denote the fixed subfield of $\overline{K}$ under the action of the subgroup H_i on $\overline{K}$. Since $H_i \subset G$ is open, E_i/K is a finite Galois extension. If we let $A := E_1 \times \ldots \times E_n$, then $\bigoplus_{i=1}^n \mathbb{Z}[G/H_i]$ is isomorphic to the image of $\operatorname{Spec}(A)$, c. f. Equation (5.1). $\square$

Chapter 6

The Chow Motive of a Del Pezzo Surface of Degree 5 or 6 with a K-rational Point

In this chapter, we are going to prove our main theorem:

Theorem 6.1. *Let S be a Del Pezzo surface of degree 5 or 6 over a perfect field K. Let further S have a K-rational point* $\mathrm{pt} : \operatorname{Spec} K \to S$ *and define* $\rho := \mathrm{pt} \times S + S \times \mathrm{pt} \in \mathrm{CH}_2 (S \times_K S)$. *Then there exists some étale algebra E such that*

$$(\operatorname{Spec} E, \mathrm{id}_{\operatorname{Spec} E}) (1) \cong (\operatorname{Spec} K, \mathrm{id}_{\operatorname{Spec} K}) (1) \oplus (S, \rho) .$$

We will first explain how to get the étale algebra E and the injection and projection maps of the direct sum, then prove that (S, ρ) is a direct summand in $E(1)$ for algebraically closed fields K and, in the third part, prove our theorem for any perfect field K.

6.1 Definition of the étale algebra E

Recall the two W_r-permutation resolutions (4.7) and (4.11) of the Picard group of a Del Pezzo surface of degree 5 or 6 which are of the form

$$0 \to \mathbb{Z} \to \bigoplus_{\text{finite}} \mathbb{Z} \to \operatorname{Pic} \overline{S} \to 0.$$

Theorem 4.4 enables us to extend the group action of the Weyl group W_r to an action on the Picard group $\operatorname{Pic}\left(\overline{S}\right) \otimes_{\mathbb{Z}} \mathbb{R} \cong \omega^{\times} \oplus \mathbb{R}\omega$, where any element of the Weyl group acts trivially on ω. Let $G = \operatorname{Gal}\left(\overline{K}/K\right)$. Then G acts on $\operatorname{Pic}\left(S \times_K \overline{K}\right)$ via the induced action on $\overline{K}$ and the trivial action on S. Theorem 23.9 in [14] then shows that the Galois group acts trivially on ω and preserves the intersection form. Hence, we have a representation $\operatorname{Gal}\left(\overline{K}/K\right) \to \operatorname{Aut}\left(\operatorname{Pic}\overline{S}\right)$ that factors through the Weyl group W_r. Thus the diagram

$$\begin{tikzcd} \operatorname{Gal}\left(\overline{K}/K\right) \arrow[rr] \arrow[dr] & & \operatorname{Aut}\left(\operatorname{Pic}\overline{S}\right) \\ & W_r \arrow[ur] & \end{tikzcd} \tag{6.1}$$

commutes. The map $\operatorname{Gal}\left(\overline{K}/K\right) \to W_r$ allows us to view the W_r-permutation resolutions as G-permutation resolutions. By the results in Section 2.3, the sequences can therefore be written as

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r, shift left, "h"] & P \arrow[l, shift left, "f"] \arrow[r, shift left, "g"] & \operatorname{Pic}\bar{S} \arrow[l, shift left, "k"] \arrow[r] & 0. \end{tikzcd} \tag{6.2}$$

for some G-permutation module P, i. e. $P \cong \bigoplus_j \mathbb{Z}[G/H_j]$ for some open subgroups $H_j \subseteq G$. Then $\overline{K}^{H_j}$ is a finite field extension of K and $E := \prod_j \overline{K}^{H_j}$ an étale algebra. The results in the previous chapter, in particular Theorem 5.9, then show that this sequence is of the form

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{CH}_0\left(\overline{K} \times_{\overline{K}} \overline{K}\right) \arrow[r, shift left, "h"] & \mathrm{CH}_0\left(\overline{K} \times_{\overline{K}} \overline{E}\right) \arrow[l, shift left, "f"] \arrow[r, shift left, "g"] & \mathrm{CH}_1\left(\overline{K} \times_{\overline{K}} \overline{S}\right) \arrow[l, shift left, "k"] \arrow[r] & 0. \end{tikzcd} \tag{6.3}$$

Corollary 5.7 then implies that there are algebraic cycles $\alpha \in \mathrm{CH}_1\left(E \times_K S\right)$ and $\beta \in \mathrm{CH}_1\left(S \times_K E\right)$ such that

$$\begin{aligned} \Phi : \mathrm{CH}_1\left(E \times_K S\right) &\to \operatorname{Hom}_G\left(\mathrm{CH}_0\left(\overline{K} \times_{\overline{K}} \overline{E}\right), \mathrm{CH}_1\left(\overline{K} \times_{\overline{K}} \overline{S}\right)\right) \\ \alpha &\mapsto \left(\alpha_{\overline{K}}\right)_* = g \end{aligned}$$

and

$$\begin{aligned} \Psi : \mathrm{CH}_1\left(S \times_K E\right) &\to \operatorname{Hom}_G\left(\mathrm{CH}_1\left(\overline{K} \times_{\overline{K}} \overline{S}\right), \mathrm{CH}_0\left(\overline{K} \times_{\overline{K}} \overline{E}\right)\right) \\ \beta &\mapsto \left(\beta_{\overline{K}}\right)_* = k \end{aligned}$$

We further know that

$$\begin{aligned}
\alpha \in \mathrm{CH}_1\left(E \times_K S\right) &= \mathrm{Hom}_{\mathrm{Chow}(K)}\left(E(1), S\right) \\
&\cong \underbrace{\mathrm{Hom}_{\mathrm{Chow}(K)}\left(E(1), K\right)}_{\mathrm{CH}_1(E\times_K K)=0} \oplus \mathrm{Hom}_{\mathrm{Chow}(K)}\left(E(1), (S,\rho)\right) \\
&\qquad\qquad \oplus \underbrace{\mathrm{Hom}_{\mathrm{Chow}(K)}\left(E(1), K(2)\right)}_{\mathrm{CH}_{-1}(E\times_K K)=0} \\
&= \mathrm{Hom}_{\mathrm{Chow}(K)}\left(E(1), (S,\rho)\right)
\end{aligned}$$

and

$$\begin{aligned}
\beta \in \mathrm{CH}_1\left(S \times_K E\right) &= \mathrm{Hom}_{\mathrm{Chow}(K)}\left(S, E(1)\right) \\
&\cong \underbrace{\mathrm{Hom}_{\mathrm{Chow}(K)}\left(K, E(1)\right)}_{\mathrm{CH}_1(K\times_K E)=0} \oplus \mathrm{Hom}_{\mathrm{Chow}(K)}\left((S,\rho), E(1)\right) \\
&\qquad\qquad \oplus \underbrace{\mathrm{Hom}_{\mathrm{Chow}(K)}\left(K(2), E(1)\right)}_{\mathrm{CH}_{-1}(K\times_K E)=0} \\
&= \mathrm{Hom}_{\mathrm{Chow}(K)}\left((S,\rho), E(1)\right).
\end{aligned}$$

Similarly, we can find algebraic cycles $\gamma \in \mathrm{CH}_0\left(E \times_K K\right)$ and $\delta \in \mathrm{CH}_0\left(K \times_K E\right)$ such that

$$\begin{aligned}
\mathrm{CH}_0\left(E \times_K K\right) &\to \mathrm{Hom}_G\left(\mathrm{CH}_0\left(\overline{K} \times_{\overline{K}} \overline{E}\right), \mathrm{CH}_0\left(\overline{K} \times_{\overline{K}} \overline{K}\right)\right) \\
\gamma &\mapsto \left(\gamma_{\overline{K}}\right)_* = f
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{CH}_0\left(K \times_K E\right) &\to \mathrm{Hom}_G\left(\mathrm{CH}_0\left(\overline{K} \times_{\overline{K}} \overline{K}\right), \mathrm{CH}_0\left(\overline{K} \times_{\overline{K}} \overline{E}\right)\right) \\
\delta &\mapsto \left(\delta_{\overline{K}}\right)_* = h
\end{aligned}$$

and again, those can be viewed as morphisms in the category of Chow Motives over K as described in Theorem 5.9

$$\gamma \in \mathrm{CH}_0\left(E \times_K K\right) = \mathrm{Hom}_{\mathrm{Chow}(K)}\left(E, K\right) = \mathrm{Hom}_{\mathrm{Chow}(K)}\left(E(1), K(1)\right)$$

and

$$\delta \in \mathrm{CH}_0\left(K \times_K E\right) = \mathrm{Hom}_{\mathrm{Chow}(K)}\left(K, E\right) = \mathrm{Hom}_{\mathrm{Chow}(K)}\left(K(1), E(1)\right)$$

To prove Theorem 6.1, we will show that there is an algebraic cycle $\tilde{\alpha} \in \mathrm{Hom}_{\mathrm{Chow(K)}}\left(E(1),(S,\rho)\right)$ associated to α such that

$$\mathrm{id}_{(S,\rho)} = \tilde{\alpha} \circ \beta \tag{6.4}$$

$$\mathrm{id}_{K(1)} = \gamma \circ \delta \tag{6.5}$$

$$\mathrm{id}_{E(1)} = \beta \circ \tilde{\alpha} + \delta \circ \gamma. \tag{6.6}$$

6.2 Proof of the main theorem for K algebraically closed

Let now K be algebraically closed, hence we can assume that $E = K^n$. We will firstly focus on equation (6.4) and show that

$$\mathrm{id}_{(S,\rho)} = \alpha \circ \beta.$$

Since Diagram 5.2 in the proof of Lemma 5.5 commutes, we know that we can write

$$\begin{aligned} g = (g_j)_j &\in \mathrm{Hom}_{\mathbb{Z}}\left(\mathrm{CH}_0\left(K \times_K E\right), \mathrm{CH}_1\left(K \times_K S\right)\right) \\ &= \bigoplus_{j=1}^{n} \mathrm{Hom}_{\mathbb{Z}}\left(\mathrm{CH}_0\left(K \times_K K\right), \mathrm{CH}_1\left(K \times_K S\right)\right) \\ &\cong \bigoplus_{j=1}^{n} \mathrm{Hom}_{\mathbb{Z}}\left(\mathbb{Z}, \mathrm{Pic}\, S\right) \end{aligned}$$

via

$$\begin{aligned} \mathrm{CH}_0\left(K \times_K K\right) &\to \mathbb{Z} \\ K \times_K K &\mapsto 1 \end{aligned}$$

and

$$\begin{aligned} \mathrm{Pic}\, S = \mathrm{CH}_1\, S &\to \mathrm{CH}_1\left(K \times_K S\right) \\ \mu &\mapsto K \times \mu. \end{aligned}$$

Further,

$$\begin{aligned}\alpha = (\alpha_j)_j \in & \operatorname{CH}_1(E \times_K S) \\ &= \bigoplus_{j=1}^{n} \operatorname{CH}_1(K \times_K S) \\ &\cong \bigoplus_{j=1}^{n} \operatorname{Pic} S\end{aligned}$$

such that

$$\Phi(\alpha_j) = (\alpha_j)_* = g_j.$$

Similarly, we have

$$\begin{aligned}k = (k_j)_j \in & \operatorname{Hom}_{\mathbb{Z}}(\operatorname{CH}_1(K \times_K S), \operatorname{CH}_0(K \times_K E)) \\ &= \bigoplus_{j=1}^{n} \operatorname{Hom}_{\mathbb{Z}}(\operatorname{CH}_1(K \times_K S), \operatorname{CH}_0(K \times_K K)) \\ &\cong \bigoplus_{j=1}^{n} \operatorname{Hom}_{\mathbb{Z}}(\operatorname{Pic} S, \mathbb{Z})\end{aligned}$$

via

$$\begin{aligned}\operatorname{CH}_1(K \times_K S) &\to \operatorname{CH}_1 S = \operatorname{Pic} S \\ K \times \mu &\mapsto \mu\end{aligned}$$

and

$$\begin{aligned}\mathbb{Z} &\to \operatorname{CH}_0(K \times_K K) \\ 1 &\mapsto K \times K.\end{aligned}$$

Further,

$$\begin{aligned}\beta = (\beta_j)_j \in \mathrm{CH}_1\left(S \times_K E\right) &= \bigoplus_{j=1}^{n} \mathrm{CH}_1\left(S \times_K K\right) \\ &\cong \bigoplus_{j=1}^{n} \operatorname{Pic} S\end{aligned}$$

such that

$$\Psi(\beta_j) = (\beta_j)_* = k_j.$$

Given the maps $g_j : \mathbb{Z} \to \operatorname{Pic} S$ and $k_j : \operatorname{Pic} S \to \mathbb{Z}$, we can compute the components α_j and β_j via the isomorphisms

$$\begin{array}{ccccccc}
\operatorname{Pic} S & \longrightarrow & \mathrm{CH}_1\left(K \times_K S\right) & \longrightarrow & \operatorname{Hom}_{\mathbb{Z}}\left(\mathrm{CH}_0\left(K \times_K K\right), \mathrm{CH}_1\left(K \times_K S\right)\right) & \longrightarrow & \operatorname{Hom}_{\mathbb{Z}}\left(\mathbb{Z}, \operatorname{Pic} S\right) \\
\alpha_j & \longmapsto & K \times \alpha_j & \longmapsto & \left(K \times \alpha_j\right)_* & \longmapsto & g_j,
\end{array}$$

and

$$\begin{array}{ccccccc}
\operatorname{Pic} S & \longrightarrow & \mathrm{CH}_1\left(S \times_K K\right) & \longrightarrow & \operatorname{Hom}_{\mathbb{Z}}\left(\mathrm{CH}_1\left(K \times_K S\right), \mathrm{CH}_0\left(K \times_K K\right)\right) & \longrightarrow & \operatorname{Hom}_{\mathbb{Z}}\left(\operatorname{Pic} S, \mathbb{Z}\right) \\
\beta_j & \longmapsto & \beta_j \times K & \longmapsto & \left(\beta_j \times K\right)_* & \longmapsto & k_j.
\end{array}$$

For α_j, this yields

$$\begin{aligned}
K \times g_j(1) &= \left(K \times \alpha_j\right)_* \left(\mathrm{id}_K\right) \\
&= K \times \alpha_j
\end{aligned}$$

and hence

$$\alpha_j = g_j(1). \tag{6.7}$$

Recall that we have a $\mathbb{Z}$-basis $l_0, \ldots, l_r$ of $\operatorname{Pic} \overline{S}$. If we write $\beta_j = \sum_{a=0}^{r} b_{aj} l_a$, then

$$\begin{aligned} k_j(l_i)\ \mathrm{id}_K &= (\beta_j \times K)_* (K \times l_i) \\ &= (\beta_j \times K) \circ (K \times l_i) \\ &= \sum_{a=0}^{r} b_{aj} (l_a \times K) \circ (K \times l_i) \\ &= \sum_{a=0}^{r} b_{aj} (p_{KK})_* (K \times (l_a, l_i)\, l_i \times K) \\ &= \sum_{a=0}^{r} b_{aj} (l_a, l_i)\ \cdot \mathrm{id}_K \\ &= \begin{cases} b_{ij}\ \mathrm{id}_K & \text{if } i = 0 \\ -b_{ij}\ \mathrm{id}_K & \text{if } i > 0 \end{cases} \end{aligned}$$

and hence

$$\beta_j = k_j(l_0) l_0 - \sum_{i=1}^{r} k_j(l_i) l_i \tag{6.8}$$

Next, we also want to express $\mathrm{id}_{(S,\rho)}$ (which is equal to ρ) in terms of the $\mathbb{Z}$-basis $l_0, \ldots, l_r$ of $\operatorname{Pic} S = \mathrm{CH}_1(S)$.

Lemma 6.2.

$$\rho = l_0 \times l_0\ - \sum_{i=1}^{r} l_i \times l_i \tag{6.9}$$

Proof. The Künneth Formula (c. f. Lemma 2.4) yields that

$$\mathrm{CH}_2 (S \times_K S) \cong \bigoplus_{j=0}^{2} \mathrm{CH}_j (S) \otimes_{\mathbb{Z}} \mathrm{CH}_{2-j} (S)\,,$$

hence we can write

$$\mathrm{id}_S = a \cdot S \times \mathrm{pt}\ +\ b \cdot \mathrm{pt} \times S\ +\ \sum_{i,j=0}^{r} c_{ij} \cdot l_i \times l_j$$

with integer coefficients a, b and c_{ij}. Since $\mathrm{id}_S \circ (S \times \mathrm{pt}) = S \times \mathrm{pt}$, that is

$$\left(a \cdot\ S \times \mathrm{pt}\ +\ b \cdot\ \mathrm{pt} \times S\ +\ \sum_{i,j=0}^{r} c_{ij} \cdot\ l_i \times l_j \right) \circ (S \times \mathrm{pt}) = (S \times \mathrm{pt})\,,$$

we can compute

$$\begin{cases} a\,(S \times \mathrm{pt}) \circ (S \times \mathrm{pt}) = a \cdot p_*\,(S \times S \times \mathrm{pt}\ \cdot\ S \times \mathrm{pt} \times S) = a \cdot S \times \mathrm{pt} \\ b\,(\mathrm{pt} \times S) \circ (S \times \mathrm{pt}) = b \cdot p_*\,(S \times \mathrm{pt} \times S\ \cdot\ S \times \mathrm{pt} \times S) = 0 \\ c_{ij}\,(l_i \times l_j) \circ (S \times \mathrm{pt}) = c_{ij} \cdot p_*\,(S \times l_i \times l_j\ \cdot\ S \times \mathrm{pt} \times S) = 0, \end{cases}$$

and hence $a = 1$. Similarly, since $\mathrm{id}_S \circ (\mathrm{pt} \times S) = \mathrm{pt} \times S$, that is

$$\left(a \cdot\ S \times \mathrm{pt}\ +\ b \cdot\ \mathrm{pt} \times S\ +\ \sum_{i,j=0}^{r} c_{ij} \cdot\ l_i \times l_j \right) \circ (\mathrm{pt} \times S) = (\mathrm{pt} \times S)\,,$$

we can compute

$$\begin{cases} a\,(S \times \mathrm{pt}) \circ (\mathrm{pt} \times S) = a \cdot p_*\,(S \times S \times \mathrm{pt}\ \cdot\ \mathrm{pt} \times S \times S) = 0 \\ b\,(\mathrm{pt} \times S) \circ (\mathrm{pt} \times S) = b \cdot p_*\,(S \times \mathrm{pt} \times S\ \cdot\ \mathrm{pt} \times S \times S) = b \cdot \mathrm{pt} \times S \\ c_{ij}\,(l_i \times l_j) \circ (\mathrm{pt} \times S) = c_{ij} \cdot p_*\,(S \times l_i \times l_j\ \cdot\ \mathrm{pt} \times S \times S) = 0, \end{cases}$$

and hence $b = 1$. Finally, since $\mathrm{id}_S \circ (l_m \times l_n) = l_m \times l_n$, that is

$$\left(a \cdot\ S \times \mathrm{pt}\ +\ b \cdot\ \mathrm{pt} \times S\ +\ \sum_{i,j=0}^{r} c_{ij} \cdot\ l_i \times l_j \right) \circ (l_m \times l_n) = (l_m \times l_n)\,,$$

we can compute

$$\begin{cases} a\,(S \times \mathrm{pt}) \circ (l_m \times l_n) = a \cdot p_*\,(S \times S \times \mathrm{pt}\ \cdot\ l_m \times l_n \times S) = 0 \\ b\,(\mathrm{pt} \times S) \circ (l_m \times l_n) = b \cdot p_*\,(S \times \mathrm{pt} \times S\ \cdot\ l_m \times l_n \times S) = 0 \\ c_{ij}\,(l_i \times l_j) \circ (l_m \times l_n) = c_{ij} \cdot p_*\,(S \times l_i \times l_j\ \cdot\ l_m \times l_n \times S) \\ \qquad\qquad\qquad\qquad\qquad\ = c_{ij}(l_i, l_n) \cdot l_m \times l_j. \end{cases}$$

The last set of equations yields that

$$\begin{aligned} l_m \times l_n &= \mathrm{id}_S \circ (l_m \times l_n) \\ &= \sum_{i,j} c_{ij}\,(l_i, l_n) \cdot l_m \times l_j \\ &= \sum_{j} c_{nj}\,(l_n, l_n) \cdot l_m \times l_j. \end{aligned}$$

As $(l_n, l_n) = \pm 1$ is its own inverse in $\mathbb{Z}$, we obtain $c_{nj} = (l_n, l_n)\delta_{nj}$. Therefore,

$$\begin{aligned} \mathrm{id}_S &= S \times \mathrm{pt} \,+\, \mathrm{pt} \times S \,+\, \sum_{i=0}^{r} (l_i, l_i) \cdot l_i \times l_i \\ &= S \times \mathrm{pt} \,+\, \mathrm{pt} \times S \,+\, l_0 \times l_0 \,-\, \sum_{i=1}^{r} l_i \times l_i, \end{aligned}$$

and

$$\begin{aligned} \rho &= \mathrm{id}_S \,-\, (S \times \mathrm{pt} \,+\, \mathrm{pt} \times S) \\ &= l_0 \times l_0 \,-\, \sum_{i=1}^{r} l_i \times l_i. \end{aligned}$$

□

Proposition 6.3. *(S, ρ) is a direct summand in $E(1)$.*

Proof. We will show that $g \circ k = \mathrm{id}_{\mathrm{Pic}\,S}$ implies that $\alpha \circ \beta = \rho$. For this, we compute

$$\begin{aligned} \alpha \circ \beta &= \sum_{j=1}^{n} (K \times \alpha_j) \circ (\beta_j \times K) \\ &= \sum_{j} (p_{SS})_* \,(S \times K \times \alpha_j \,\cdot\, \beta_j \times K \times S) \\ &= \sum_{j} (p_{SS})_* \,(\beta_j \times K \times \alpha_j) \\ &= \sum_{j} \beta_j \times \alpha_j \\ &= \sum_{j} \left(k_j(l_0) l_0 - \sum_{i=1}^{r} k_j(l_i) l_i \right) \times g_j(1) && \text{by (6.7) and (6.8)} \\ &= \sum_{j} k_j(l_0) l_0 \times g_j(1) - \sum_{i} k_j(l_i) l_i \times g_j(1) \end{aligned}$$

$$\begin{aligned}
&= \sum_j l_0 \times (k_j(l_0)g_j(1)) - \sum_i l_i \times (k_j(l_i)g_j(1)) \\
&= \sum_j l_0 \times g_j(k_j(l_0)) - \sum_i l_i \times g_j(k_j(l_i)) \\
&= l_0 \times g(k(l_0)) - \sum_i l_i \times g(k(l_i)) \\
&= l_0 \times l_0 - \sum_i l_i \times l_i \\
&= \rho \\
&= \mathrm{id}_{(S,\rho)}\,. \qquad \square
\end{aligned}$$

Proposition 6.4 (Theorem 6.1 for K algebraically closed)**.** *Let K be algebraically closed, then*

$$E(1) \cong K(1) \oplus (S, \rho).$$

Proof. Since we have already proven Proposition 6.3, it suffices to prove that $K(1)$ is a direct summand in $E(1)$ and that $E(1)$ doesn't have any other summands, i. e. to verify equations (6.4) and (6.6). Since the functor described in Theorem 5.9 induces an equivalence of categories, equation $f \circ h = \mathrm{id}_{\mathbb{Z}}$ immediately implies $\gamma \circ \delta = \mathrm{id}_{K(1)}$. The split exact sequence (6.3) further yields that $k \circ g + h \circ f = \mathrm{id}_P$. The following two commuting diagrams

$$\begin{tikzcd}
\mathrm{Hom}_G\left(\mathrm{CH}_1\left(K \times_K S\right), \mathrm{CH}_0\left(K \times_K E\right)\right) \times \mathrm{Hom}_G\left(\mathrm{CH}_0\left(K \times_K E\right), \mathrm{CH}_1\left(K \times_K S\right)\right) \arrow[r] & \mathrm{End}_G\left(\mathrm{CH}_0\left(K \times_K E\right)\right) \\
\mathrm{CH}_1\left(S \times_K E\right) \times \mathrm{CH}_1\left(E \times_K S\right) \arrow[r] \arrow[u] & \mathrm{CH}_0\left(E \times_K E\right) \arrow[u]
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathrm{Hom}_G\left(\mathrm{CH}_0\left(K \times_K K\right), \mathrm{CH}_0\left(K \times_K E\right)\right) \times \mathrm{Hom}_G\left(\mathrm{CH}_0\left(K \times_K E\right), \mathrm{CH}_0\left(K \times_K K\right)\right) \arrow[r] & \mathrm{End}_G\left(\mathrm{CH}_0\left(K \times_K E\right)\right) \\
\mathrm{CH}_0\left(K \times_K E\right) \times \mathrm{CH}_0\left(E \times_K K\right) \arrow[r] \arrow[u] & \mathrm{CH}_0\left(E \times_K E\right) \arrow[u]
\end{tikzcd}$$

with the horizontal maps being composition and vertical maps being Φ and Ψ, enable us to translate this equation into

$$\beta \circ \alpha + \delta \circ \gamma = \mathrm{id}_{E(1)} . \qquad \square$$

6.3 Proof of the main theorem for K perfect

We will extend the proofs of the previous section. Let now K be a perfect field with algebraic closure $\overline{K}$.

Proposition 6.5. *(S, ρ) is a direct summand in $E(1)$.*

Proof. In Theorem 5.6, we show that the two diagrams

$$\begin{tikzcd} \mathrm{CH}_1\left(\overline{E} \times_{\overline{K}} \overline{S}\right) \arrow[r, "\Phi_{\overline{K}}"] & \mathrm{Hom}_{\mathbb{Z}}\left(\mathrm{CH}_0\left(\overline{K} \times_{\overline{K}} \overline{E}\right), \mathrm{CH}_1\left(\overline{K} \times_{\overline{K}} \overline{S}\right)\right) \\ \mathrm{CH}_1\left(E \times_K S\right) \arrow[u, "\mathrm{res}_{\overline{K}/K}"] \arrow[r, "\Phi"'] & \mathrm{Hom}_G\left(\mathrm{CH}_0\left(\overline{K} \times_{\overline{K}} \overline{E}\right), \mathrm{CH}_1\left(\overline{K} \times_{\overline{K}} \overline{S}\right)\right) \arrow[u, hook] \end{tikzcd} \tag{6.10}$$

and

$$\begin{tikzcd} \mathrm{CH}_1\left(\overline{S} \times_{\overline{K}} \overline{E}\right) \arrow[r, "\Psi_{\overline{K}}"] & \mathrm{Hom}_{\mathbb{Z}}\left(\mathrm{CH}_1\left(\overline{K} \times_{\overline{K}} \overline{S}\right), \mathrm{CH}_0\left(\overline{K} \times_{\overline{K}} \overline{E}\right)\right) \\ \mathrm{CH}_1\left(S \times_K E\right) \arrow[u, "\mathrm{res}_{\overline{K}/K}"] \arrow[r, "\Psi"'] & \mathrm{Hom}_G\left(\mathrm{CH}_1\left(\overline{K} \times_{\overline{K}} \overline{S}\right), \mathrm{CH}_0\left(\overline{K} \times_{\overline{K}} \overline{E}\right)\right) \arrow[u, hook] \end{tikzcd} \tag{6.11}$$

commute. Hence, also the following diagram commutes:

$$
\begin{tikzcd}[column sep=small]
 & \mathrm{CH}_1\left(\overline{E}\times_{\overline{K}}\overline{S}\right)\times\mathrm{CH}_1\left(\overline{S}\times_{\overline{K}}\overline{E}\right) \arrow[rr, "\text{composition}"'] \arrow[dl, "\Phi"'] \arrow[dl, "\Psi"] & & \mathrm{CH}_2\left(\overline{S}\times_{\overline{K}}\overline{S}\right) \\
\mathrm{Hom}_{\mathbb{Z}}\left(\mathrm{CH}_0\left(\overline{K}\times_{\overline{K}}\overline{E}\right),\mathrm{CH}_1\left(\overline{K}\times_{\overline{K}}\overline{S}\right)\right)\times\mathrm{Hom}_{\mathbb{Z}}\left(\mathrm{CH}_1\left(\overline{K}\times_{\overline{K}}\overline{S}\right),\mathrm{CH}_0\left(\overline{K}\times_{\overline{K}}\overline{E}\right)\right) & & & \\
 & \mathrm{CH}_1\left(E\times_K S\right)\times\mathrm{CH}_1\left(S\times_K E\right) \arrow[uu, "\mathrm{res}_{\overline{K}/K}"] \arrow[rr, "\text{composition}"'] \arrow[dl, "\Phi"] \arrow[dl, "\Psi"'] & & \mathrm{CH}_2\left(S\times_K S\right) \arrow[uu, "\mathrm{res}_{\overline{K}/K}"'] \\
\mathrm{Hom}_G\left(\mathrm{CH}_0\left(\overline{K}\times_{\overline{K}}\overline{E}\right),\mathrm{CH}_1\left(\overline{K}\times_{\overline{K}}\overline{S}\right)\right)\times\mathrm{Hom}_G\left(\mathrm{CH}_1\left(\overline{K}\times_{\overline{K}}\overline{S}\right),\mathrm{CH}_0\left(\overline{K}\times_{\overline{K}}\overline{E}\right)\right) \arrow[uu] & & &
\end{tikzcd}
$$

Since Proposition 6.3 proves that $\alpha_{\overline{K}} \circ \beta_{\overline{K}} = \mathrm{id}_{\left(\overline{S},\rho_{\overline{K}}\right)}$, we can use Rost nilpotence (c. f. Theorem 2.8 in [4]) to deduce that there is a nilpotent $\mu \in \mathrm{End}_{\mathrm{Chow}(K)}(S,\rho)$ such that

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Chow(K)}}(E(1),(S,\rho)) \times \mathrm{Hom}_{\mathrm{Chow(K)}}((S,\rho),E(1)) &\to \mathrm{End}_{\mathrm{Chow}(K)}(S,\rho) \\ (\alpha,\beta) &\mapsto \alpha\circ\beta = \mathrm{id}_{(S,\rho)} + \mu. \end{aligned}$$

Since then

$$\left(\mathrm{id}_{(S,\rho)} + \mu\right)^{-1} = \sum_j (-\mu)^j$$

is a finite sum, we can define $\tilde{\alpha} := \left(\left(\sum_j (-\mu)^j\right) \circ \alpha\right)$ and compute

$$\begin{aligned} \tilde{\alpha}\circ\beta &= \left(\sum_j (-\mu)^j\right) \circ (\alpha\circ\beta) \\ &= \left(\sum_j (-\mu)^j\right) \circ \left(\mathrm{id}_{(S,\rho)} + \mu\right) \\ &= \mathrm{id}_{(S,\rho)}. \end{aligned}$$

This yields that (S,ρ) is a direct summand in $E(1)$. □

Now we will be able to prove the main theorem given on page 57.

Theorem 6.1. *Let S be a Del Pezzo surface of degree 5 or 6 over a perfect field K. Let further S have a K-rational point* $\mathrm{pt} : \mathrm{Spec}\, K \to S$ *and define* $\rho := \mathrm{pt} \times S + S \times \mathrm{pt} \in \mathrm{CH}_2(S \times_K S)$. *Then there exists some étale algebra E such that*

$$\left(\mathrm{Spec}\, E, \mathrm{id}_{\mathrm{Spec}\, E}\right)(1) \cong \left(\mathrm{Spec}\, K, \mathrm{id}_{\mathrm{Spec}\, K}\right)(1) \oplus (S,\rho).$$

Proof. We will mimic the proof of Proposition 6.4. Again, the Proposition 6.5 shows equation (6.4), i. e. that (S,ρ) is a direct summand in $E(1)$. Equation 6.5 is again a consequence of the equivalence of categories given in Theorem 5.9 and it remains to proof equation 6.6. Similarly to the proof of Proposition 6.5, the two commuting Diagrams (6.10) and (6.11) imply that

$$
\begin{tikzcd}
\mathrm{CH}_1\left(\overline{S}\times_{\overline{K}}\overline{E}\right)\times\mathrm{CH}_1\left(\overline{E}\times_{\overline{K}}\overline{S}\right) \arrow[r, "\text{composition}"'] \arrow[d, "\Psi,\ \Phi"', shift right] & \mathrm{CH}_0\left(\overline{E}\times_{\overline{K}}\overline{E}\right) \\
\mathrm{Hom}_{\mathbb{Z}}\left(\mathrm{CH}_1\left(\overline{K}\times_{\overline{K}}\overline{S}\right),\mathrm{CH}_0\left(\overline{K}\times_{\overline{K}}\overline{E}\right)\right)\times\mathrm{Hom}_{\mathbb{Z}}\left(\mathrm{CH}_0\left(\overline{K}\times_{\overline{K}}\overline{E}\right),\mathrm{CH}_1\left(\overline{K}\times_{\overline{K}}\overline{S}\right)\right) \arrow[u, shift right] & \\
\mathrm{CH}_1\left(S\times_K E\right)\times\mathrm{CH}_1\left(E\times_K S\right) \arrow[r, "\text{composition}"'] \arrow[uu, bend right, "\mathrm{res}_{\overline{K}/K}"'] \arrow[d, "\Psi,\ \Phi"', shift right] & \mathrm{CH}_0\left(E\times_K E\right) \arrow[uu, "\mathrm{res}_{\overline{K}/K}"'] \\
\mathrm{Hom}_{G}\left(\mathrm{CH}_1\left(\overline{K}\times_{\overline{K}}\overline{S}\right),\mathrm{CH}_0\left(\overline{K}\times_{\overline{K}}\overline{E}\right)\right)\times\mathrm{Hom}_{G}\left(\mathrm{CH}_0\left(\overline{K}\times_{\overline{K}}\overline{E}\right),\mathrm{CH}_1\left(\overline{K}\times_{\overline{K}}\overline{S}\right)\right) \arrow[uu, bend left] &
\end{tikzcd}
$$

also commutes. This allows us to translate the equation

$$k \circ g + h \circ f = \mathrm{id}_P$$

into

$$\beta_{\overline{K}} \circ \alpha_{\overline{K}} + \delta_{\overline{K}} \circ \gamma_{\overline{K}} = \mathrm{id}_{\overline{E(1)}}\,. \tag{6.12}$$

Furthermore, we know that $-\mu$ is in the kernel of $\mathrm{res}_{\overline{K}/K}$, i.e. $\mu_{\overline{K}} = 0$ and hence,

$$\begin{aligned}
(\tilde{\alpha})_{\overline{K}} &= \left(\left(\sum_j (-\mu)^j\right) \circ \alpha\right)_{\overline{K}} \\
&= \left((-\mu_{\overline{K}})^0 + \sum_{j>0} (-\mu_{\overline{K}})^j\right) \circ \alpha_{\overline{K}} \\
&= \left(\mathrm{id}_{(\overline{S},\rho_{\overline{K}})} + \sum_{j>0} 0\right) \circ \alpha_{\overline{K}} \\
&= \alpha_{\overline{K}}
\end{aligned}$$

Therefore, Equation (6.12) can also be written as

$$\beta_{\overline{K}} \circ \tilde{\alpha}_{\overline{K}} + \delta_{\overline{K}} \circ \gamma_{\overline{K}} = \mathrm{id}_{\overline{E(1)}}\,.$$

Since $\mathrm{res}_{\overline{K}/K}$ is injective on étale algebras, this implies

$$\beta \circ \tilde{\alpha} + \delta \circ \gamma = \mathrm{id}_{E(1)},$$

which proves Equation (6.6) and hence our theorem. □

6.4 More information on the étale algebra E

As described in Section 6.1, the permutation module $P \cong \bigoplus_j \mathbb{Z}[G/H_j]$ is completely determined by the G-action on P. Furthermore, the G-action is induced by the action of W_r via the map $\mathrm{Gal}(\overline{K}/K) \to W_r$ in Diagram 6.1. The complete decomposition of the étale algebra E into its factors of field extensions of K depends on the image of $\mathrm{Gal}(\overline{K}/K)$ in W_r and is a case-by-case consideration. We will only describe the decomposition without any further assumptions regarding the image of this map.

The main idea of this section is to decompose our permutation module P into a direct sum of permutation modules such that W_r acts transitively on the given basis of each summand. If the map $\mathrm{Gal}(\overline{K}/K) \to W_r$ is surjective, this is all there is to say. Otherwise, the permutation module might decompose even further.

For degree 6, recall the W_3-action on $\mathbb{Z}^5$ on page 36. The $\mathbb{Z}$-basis $\{e_1, \dots, e_5\}$ can be split into the two disjoint W_3-sets $\{e_1, e_2, e_3\}$ and $\{e_4, e_5\}$, hence

$$\mathbb{Z}^5 \cong \mathbb{Z}[e_1, e_2, e_3] \oplus \mathbb{Z}[e_4, e_5]$$

as W_3-permutation modules. The corresponding étale algebra E is therefore given by a product of an étale algebra of degree 3 and an étale algebra of degree 2 over K.

Similarly, for degree 5, recall the W_4-action on $\mathbb{Z}^6$ on page 43. The $\mathbb{Z}$-basis $\{e_1, \dots, e_6\}$ can be split into the two disjoint W_4-sets $\{e_1, \dots, e_5\}$ and $\{e_6\}$, hence

$$\mathbb{Z}^6 \cong \mathbb{Z}[e_1, \dots, e_5] \oplus \mathbb{Z}[e_6]$$

as W_4-permutation modules. The étale algebra E is therefore given by a product of an étale algebra of degree 5 and an étale algebra of degree 1 over K.

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