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ON THE RISE OF VERTICAL PLUMES

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ON THE RISE OF VERTICAL PLUMES

ABSTRACT

The equations expressing the conservation of mass, vertical momentum and energy are solved analytically for the rise of buoyant plumes in calm atmospheric conditions. The separate and joint effects of entrainment and drag on plume behavior are discussed.

INTRODUCTION

Hot stack plumes rise vertically in a calm atmosphere mainly owing to their exit velocity and buoyancy. The three approximate equations expressing the conservation of mass (continuity), vertical momentum and energy are considered to adequately describe mean flow conditions in these plumes. These equations, valid at each point within the plume should be capable of solution given appropriate boundary and initial conditions if the turbulent stress terms can be expressed in terms of mean flow variables. However, so far this procedure has met with little success. Instead, the detailed equations are integrated with respect to cross-sectional area (or radius if cylindrical coordinates are used) to yield the flux equations for the plume. This integrated set of equations is no longer dependent on the turbulent stress terms, since these yield zero contribution upon area integration. Because of this it follows that the integrated equations contain less information than the detailed equations. In addition there appears, in the continuity equation, the unknown quantity <u>entrainment</u> about which some assumption must be made to enable closure of the system of equations.

Turbulent entrainment consists of drawing ambient fluid across the plume boundary, i.e., across the interface separating rotational and irrotational motions. As a consequence, the radial inflow Rv_e is greater than zero and the plume mass flux increases monotonically with height. Most models have introduced a relationship between the entrainment and the local mean axial flow of the plume; $Rv_e = R\alpha w$. The entrainment constant α must be determined experimentally but an order-of-magnitude analysis will show it to be relatively close to 0.1. Turbulent entrainment normally has a small axial component as well as an inward component (Taylor, 1958) but the axial component invariably has been ignored as insignificant by plume rise modelers.

GOVERNING EQUATIONS

In the following, the three conservation equations governing the behavior of buoyant plumes will be written down. In the present application these equations imply steady state conditions, "top hat" profiles for the plume variables and circular sources of heat. The Boussinesq approximation is assumed valid and the physical properties of the plume are assumed to be those of the ambient air. The mean flow is axisymmetric and irrotational. Relative to a set of cylindrical polar coordinates with the vertical axis (z) directed upward the integrated continuity equation then becomes:

$$\frac{d}{dz}(R^2 w) = 2Rv_e \qquad (1)$$

Eq. (1) states that the rate of increase of mass flux in the plume is equal to the rate of entrainment of ambient fluid.

The vertical component of the momentum equation requires that the rate of change of momentum flux in the plume equals the sum of the vertical forces. Relevant forces are: body force, static pressure force and drag force. We then have:

$$\frac{d}{dz}\left(R^2 w^2\right) = R^2 g\left[\frac{T_p}{T_a} - 1\right] - C_D R w^2 \tag{2}$$

where C_D is an unknown drag coefficient and where the body and static pressure forces were combined into a buoyancy force.

The energy equation, being a statement of the first law of thermodynamics requires that the rate of change of energy flux in the plume equals the energy entrained plus the heat added within the plume less the net work done by the plume. In this treatment only dry plumes are considered (no heat added by condensation processes in the plume) and the transfer of heat is ignored because of difficulties in determining a realistic transfer coefficient. The work done by the plume is against gravity, the pressure force and against the drag force. Work against the pressure force is normally included in the enthalpy term since; $C_v T_p + P/\rho_p = (C_v + R_i)T_p = C_p T_p$ = enthalpy. The energy equation becomes:

$$\begin{aligned} &\frac{d}{dz}\left[R^2 w\left(\frac{w^2}{2} + gz + C_p T_p\right)\right] = 2Rv_e E_a - C_D R w^3 \\ &E_a = v_e^2/2 + gz + C_p T_a \; . \end{aligned} \tag{3}$$

In the following, the effects of different terms on plume behavior will be assessed.

SOLUTIONS

Case 1 $v_e = C_D = 0$

Since there is no entrainment the mass flux remains constant with height, i.e., $R^2 w$ = constant. This relationship can also be written in the form:

$$\left(\frac{R}{R_o}\right)^2 = \frac{w_o}{w} \tag{4}$$

where index o indicates values measured at the source. In view of Eq. (4) the momentum equation (2) may now be written as:

$$\frac{1}{2}\frac{dw^2}{dz} = g\left[\frac{T_p}{T_a} - 1\right] . \tag{5}$$

Thus, in the absence of entrainment and drag the motion of the plume is solely determined by the plume buoyancy. Similarly the energy equation (3) becomes:

$$\frac{1}{2}\frac{dw^2}{dz} + g + C_p \frac{dT_p}{dz} = 0 \quad . \tag{6}$$

By eliminating the velocity terms between Eqs. (5) and (6) the plume temperature distribution may be obtained. We find that:

$$-\frac{dT_p}{dz} = \frac{g}{C_p}\frac{T_p}{T_a} = \Gamma \frac{T_p}{T_a}$$

the solution of which is:

$$T_p = T_{po}\, e^{-\Gamma z/\overline{T}_a} . \tag{7}$$

An average ambient temperature, $T_a = \overline{T}_a$, was considered in the plume rise layer. As would be expected the temperature decreases essentially dry adiabatically with height except at small distances away from the source where the large excess plume temperature causes a decrease somewhat greater than that given by the dry adiabatic lapse rate.

The plume's motion is obtained from either of Eqs.(5) or (6). We have:

$$\frac{w^2}{2} = \frac{w_o^2}{2} + C_p\, T_{po}[1 - e^{-\Gamma z/\overline{T}_a}] - gz \,. \tag{8}$$

The development of the above case is illustrated in Figure 1. The plume velocity can be seen to increase initially, take a maximum, then decrease monotonically in the region of reversed buoyancy. Because of the absence of entrainment there is little plume growth except at the top where $w \rightarrow 0$. At this point the plume exhibits a mushrooming effect. Although hot tall stack plumes can occasionally be seen to behave in this fashion it is probably more indicative of the behavior of volcanic eruptions and nuclear explosions. Furthermore, it is highly unlikely that tall stack plumes will reach the ultimate heights predicted by this case. Note: Once $w \rightarrow 0$, it is clear that the plume will begin to accelerate downward because $T_p < \overline{T}_a$. It is expected that a solution to the set of equations can be found to describe the subsequent development of the plume but this has not been explored quantitatively.

<u>Case 2</u> $v_e = 0 \quad C_D \neq 0$

The continuity equation remains as in the previous case. Therefore, the plume can be expected to grow as before and to retain the shape displayed in Figure 1. However, in this case the plume will reach only moderate heights since the drag force now affects its motion. The momentum and energy equations become:

$$\frac{1}{2}\frac{dw^2}{dz} + \frac{C_D}{R}w^2 = g[\frac{T_p}{T_a} - 1] \tag{9}$$

$$\frac{1}{2}\frac{dw^2}{dz} + g + C_p\frac{dT_p}{dz} = -\frac{C_D}{R}w^2 . \tag{10}$$

The plume's temperature distribution (with height) may be obtained by eliminating the velocity terms from Eqs. (9) and (10). Thus:

$$-\frac{dT_p}{dz} = \Gamma\frac{T_p}{T_a}$$

which is identical to the distribution given in the previous case. We draw the conclusion that drag influences the motion of the plume but has no effect on its temperature distribution, certainly in agreement with expectations. To resolve the plume motion the drag constant C_D must be determined and an assumption about the radius R must also be made. From Eq. (4) it is clear that R depends on w and therefore on z. Little is known about the constant C_D except that it depends on the Reynolds number and therefore on both w and R. However, for large Reynolds numbers C_D is essentially constant. Furthermore from Figure 1 we conclude that $R \approx$ constant except in the final stages of plume rise. Therefore, to a good approximation it is possible to write $R/2C_D \equiv \ell_m$ = constant. The constant ℓ_m is interpreted as a characteristic momentum length which is to be determined empirically. The momentum equation (9) can now be solved and the result is:

$$\frac{w^2}{2} = \frac{w_o^2}{2}e^{-z/\ell_m} + g[\frac{T_{po}}{\overline{T}_a} - 1](\frac{1}{\ell_m} - \frac{\Gamma}{\overline{T}_a})^{-1}[e^{-\Gamma z/\overline{T}_a} - e^{-z/\ell_m}] . \tag{11}$$

It was again assumed that $T_a = \overline{T}_a$ and the momentum equation was adjusted so as to reflect the boundary condition $w = 0$ at $z = \infty$. In general $\Gamma/\overline{T}_a \ll 1/\ell_m$ so Eq. (11) may finally take the form:

$$\frac{w^2}{2} = \frac{w_o^2}{2} e^{-z/\ell_m} + g\ell_m[\frac{T_{po}}{\overline{T}_a} - 1][e^{-\Gamma z/\overline{T}_a} - e^{-z/\ell_m}] \quad . \qquad (12)$$

Note that the effect of the exit velocity (w_o) on the plume motion is limited to distances close to the source. We may conclude that a balance is established relatively quickly between buoyancy and drag and that the motion is adequately given by:

$$\frac{w^2}{2} = g\ell_m[\frac{T_{po}}{\overline{T}_a} - 1] e^{-\Gamma z/\overline{T}_a} \qquad \text{when} \quad z \gg \ell_m \; . \qquad (13)$$

<u>Case 3</u> $v_e \neq 0$ $C_D = 0$

Since in this case $v_e = \alpha w$ the continuity equation may be expanded into the form:

$$\frac{dR}{dz} + \frac{R}{2w}\frac{dw}{dz} = \alpha \qquad (14)$$

which cannot be integrated without a previous knowledge of the velocity distribution. This is given by the momentum equation which in this case becomes:

$$\frac{1}{2}\frac{dw^2}{dz} + \frac{2\alpha}{R} w^2 = g[\frac{T_p}{T_a} - 1] \quad . \qquad (15)$$

We note immediately that equation (15) is of the same form as the corresponding equation (9). In fact they become identical when $2\alpha = C_D$. Therefore, the effect of entrainment on the plume motion is the same as that of drag, i.e., to retard the motion. The energy equation is:

$$\frac{1}{2}\frac{dw^2}{dz} + g + C_p \frac{dT_p}{dz} = -\frac{2\alpha}{R}\left[C_p(T_p - T_a) + \frac{w^2}{2}\right] \quad (16)$$

where the assumption was made that $\alpha^2 \ll 1$. To proceed, an assumption must be made about α/R . We shall assume, perhaps without justification, that $2\alpha = C_D$ and that the concept of a constant momentum length is applicable also here. Then for the purpose of evaluating T_p it will further be assumed that a balance between buoyancy and "drag" has been achieved. The temperature distribution is now given by:

$$\frac{dT_p}{dz} + \frac{T_p}{2\ell_m} = \frac{T_a}{2\ell_m}$$

the solution of which is:

$$T_p = T_{po}\, e^{-z/2\ell_m} + \overline{T}_a(1 - e^{-z/2\ell_m}) \quad \text{for} \quad T_a = \overline{T}_a \ . \quad (17)$$

With the temperature distribution given by Eq. (17) the plume motion becomes:

$$\frac{w^2}{2} = \frac{w_o^2}{2} e^{-z/\ell_m} + 2g\ell_m\left[\frac{T_{po}}{\overline{T}_a} - 1\right]\left[e^{-z/2\ell_m} - e^{-z/\ell_m}\right] . \quad (18)$$

A balance between buoyancy and "drag" has been established when:

$$\frac{w^2}{2} = g\ell_m\left[\frac{T_{po}}{\overline{T}_a} - 1\right] e^{-z/2\ell_m} . \quad (19)$$

We are now in the position of solving Eq. (14) for the plume growth. For simplicity the approximation (19) is used to represent the plume motion in favor of the more exact Eq. (18). The change of R with height now becomes:

$$\frac{dR}{dz} - \frac{R}{8\ell_m} = \alpha$$

the solution of which is:

$$R = R_o e^{z/8\ell_m} + 4C_D \ell_m (e^{z/8\ell_m} - 1) . \tag{20}$$

Figure 2 illustrates the plume behavior in the above case. Both T_p and w show rapid decrease with height in the early stages of plume rise. This is caused by cold ambient air being entrained into the plume at a rate proportional to the plume velocity itself. Since the plume velocity is greatest in the vicinity of the source the observed behavior of T_p and w follows. The plume is shown to expand realistically with height although the actual growth rate may have to be determined experimentally.

<u>Case 4</u> $v_e \neq 0$ $C_D \neq 0$

The continuity equation is the same as in the previous case. Therefore, drag has no influence whatsoever on plume growth; this is due solely to entrainment. However, the momentum and energy equations are affected. They become:

$$\frac{1}{2}\frac{dw^2}{dz} + [\frac{2\alpha}{R} + \frac{C_D}{R}]w^2 = g[\frac{T_p}{T_a} - 1] \tag{21}$$

$$\frac{1}{2}\frac{dw^2}{dz} + g + C_p \frac{dT_p}{dz} = - \frac{2\alpha}{R} [C_p(T_p - T_a) + \frac{w^2}{2}] - \frac{C_D}{R} w^2 . \tag{22}$$

Following the treatment of the previous case (with respect to assumptions made) the plume temperature can be found to be distributed in accordance with Eq. (17). Thus, again drag has been shown to have no effect on the plume's temperature distribution. The motion of the plume is given by:

$$\frac{w^2}{2} = \frac{w_o^2}{2} e^{-2z/\ell_m} + \frac{2}{3} g\ell_m[\frac{T_{po}}{T_a} - 1][e^{-z/2\ell_m} - e^{-2z/\ell_m}] \quad . \tag{23}$$

The effect of drag is then to further retard the motion resulting in a final plume rise somewhat below that of the previous case.

REFERENCE

Taylor, G.I., Flow Induced by Jets, J. Aero Space Sci., 25, 1958.

FIGURE 1

PLOT OF EQUATIONS IN CASE 1

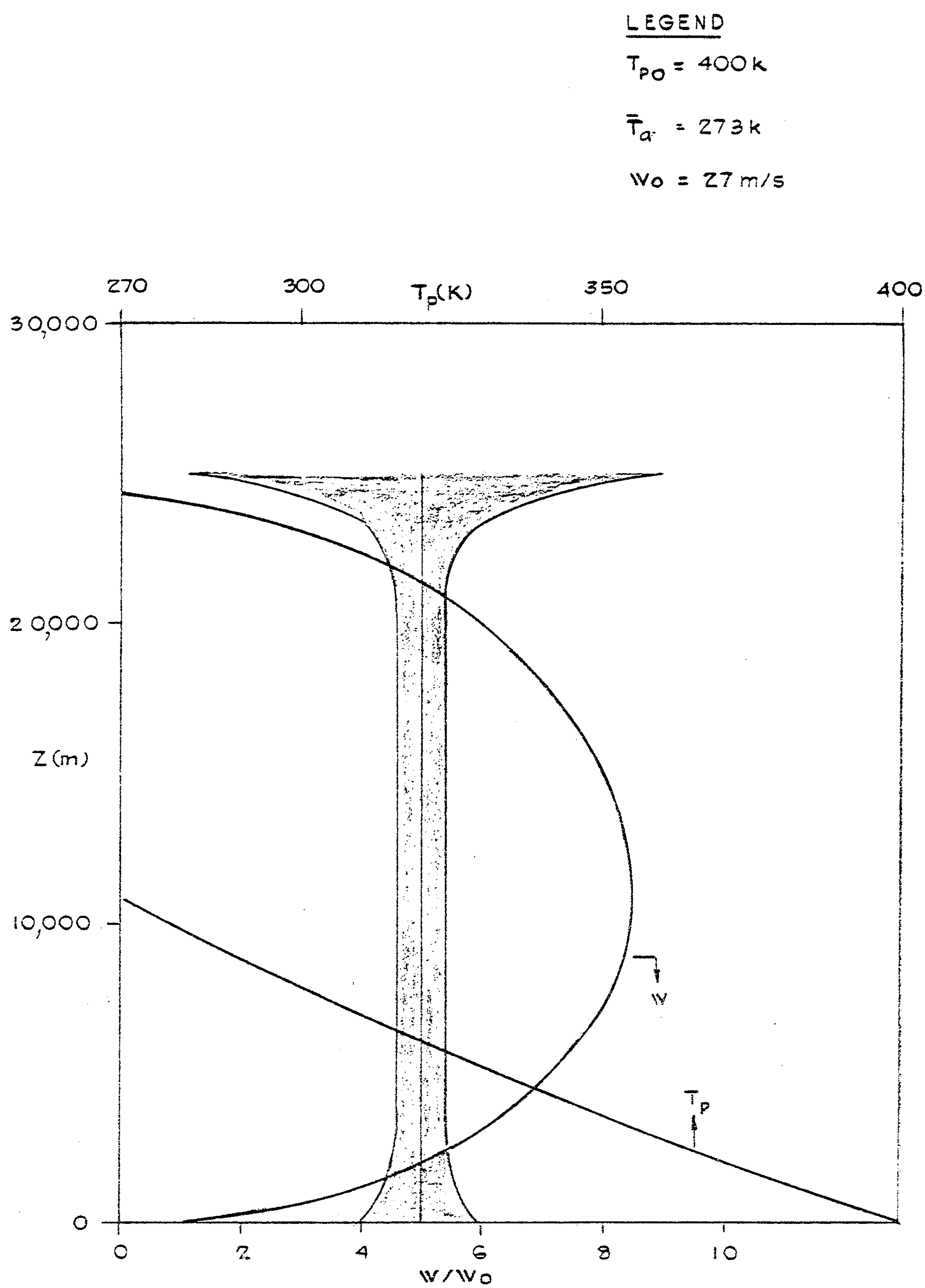

FIGURE 2

PLOT OF EQUATIONS IN CASE 3

LEGEND

T_{PO} = 400 K

$\bar{T}_a$ = 273 K

W_0 = 27 m/s

R_a = 4 m

C_D = 0.1

lm = 50m

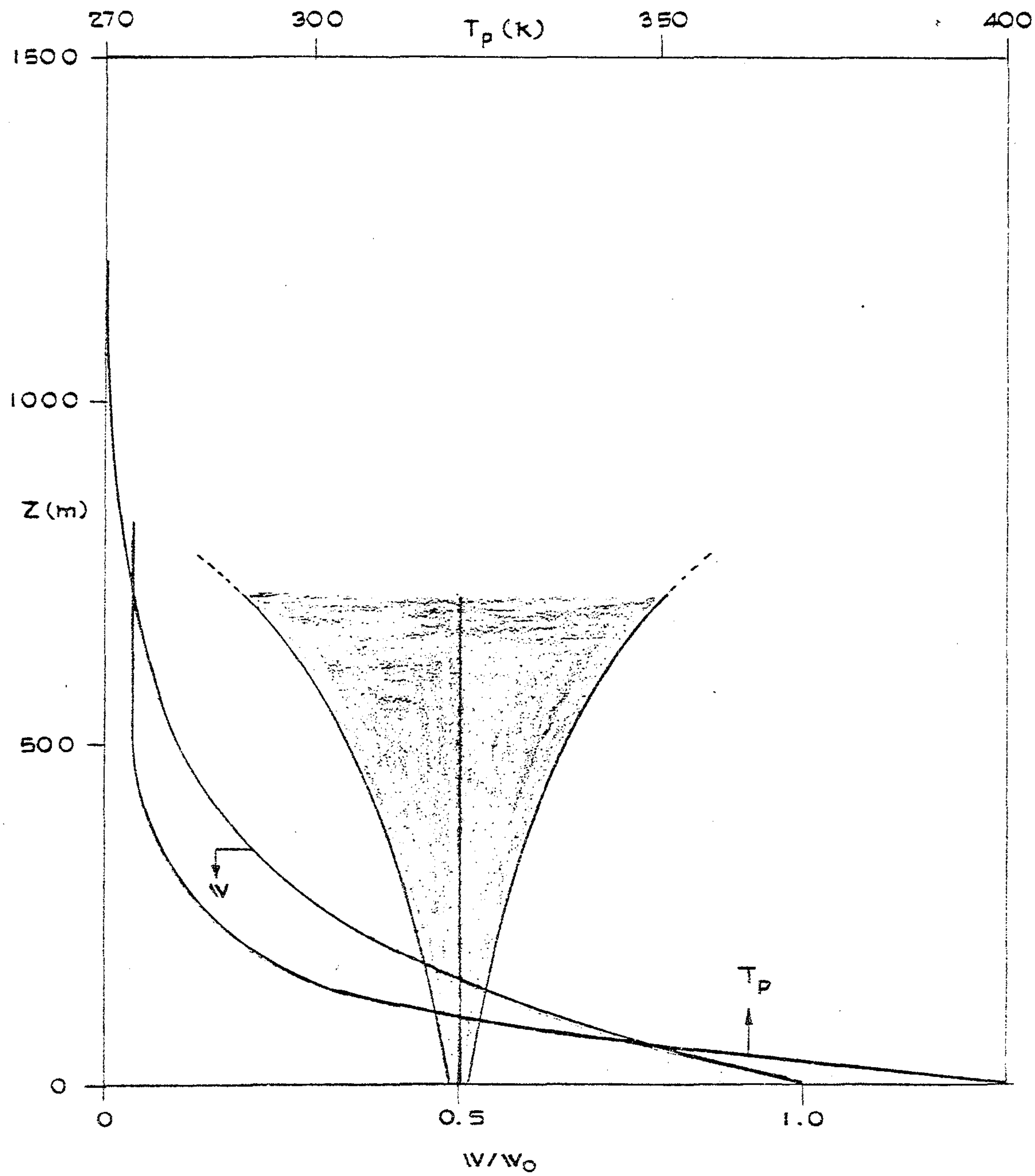

FIGURE 1

PLOT OF EQUATIONS IN CASE 1

LEGEND

T_{PO} = 400k

$\bar{T}_a$ = 273k

W_o = 27 m/s

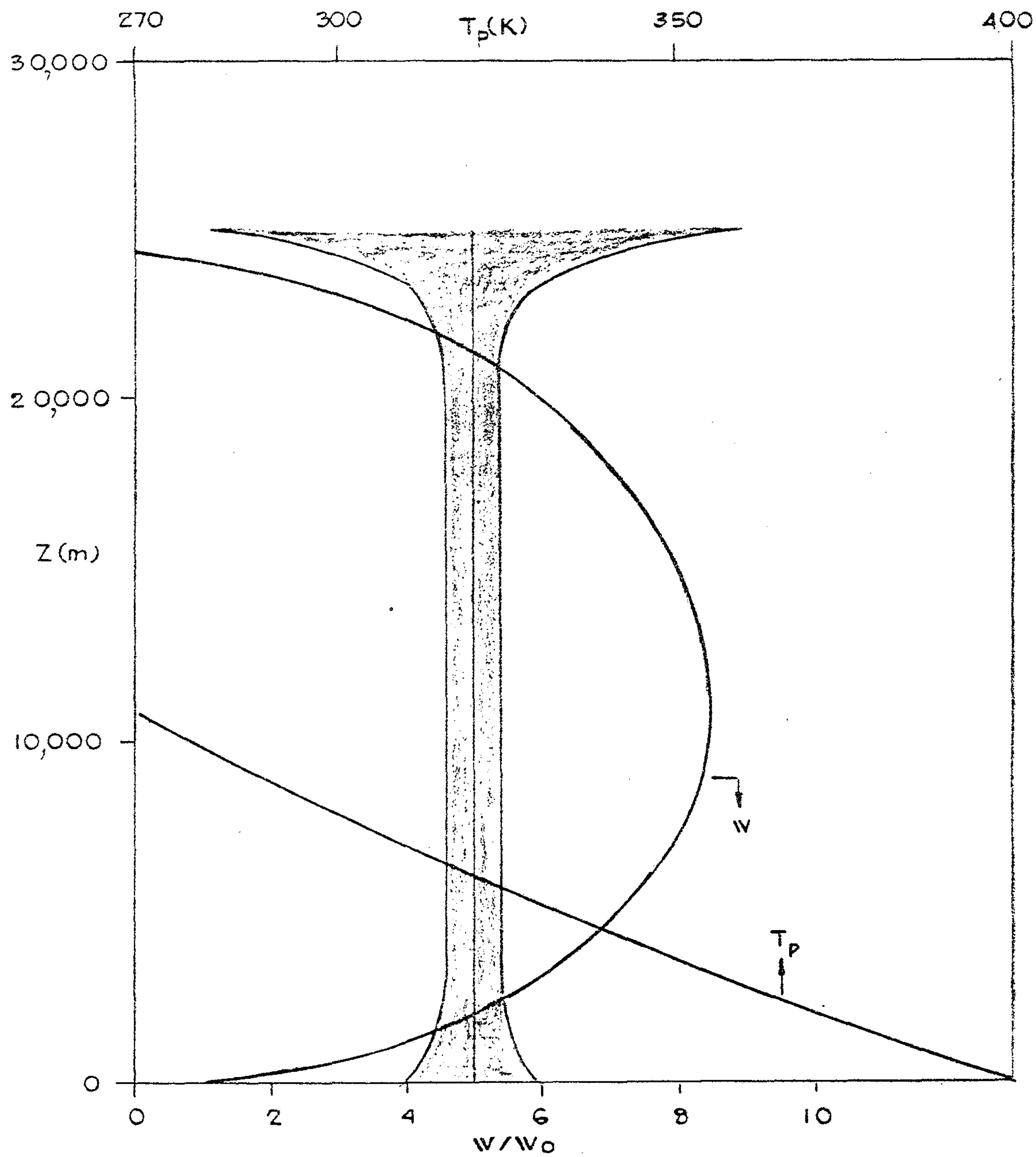

FIGURE 2

PLOT OF EQUATIONS IN CASE 3

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R_o = 4 m

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