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UNIVERSITY OF ALBERTA

ASSET PRICING MODELS WITH TIME VARYING MOMENTS

BY

Harry J. Turtle ©

A thesis submitted to the Faculty of Grad··· Studies and Research in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

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DEDICATION

To Marie,

Joel and Cameron;

whose patience and understanding

have made this possible.

ABSTRACT

I. This paper explores the use of a modified multivariate Generalized AutoRegressive Conditional Heteroscedastic in Mean (GARCH-M) process in an asset pricing context. In general, the premise behind the GARCH-M process is that changing univariate variances of individual assets are unconditionally stationary; however, conditional upon observable data they are heteroscedastic. Moreover, this heteroscedasticity is predictable. The framework postulated in this paper models changing variances of individual portfolios, after which an asset pricing model is invoked to determine how changing variances in turn affect expected returns.

The proposed model results in a more parsimonious approach towards testing asset-pricing models in a time-series context. The findings demonstrate a clear time varying pattern to both excess returns and conditional variances. From an asset pricing context, the model is useful in that it preserves the relation between (conditional) risk and return in a familiar CAPM paradigm. From a statistical standpoint the model is unique in the manner in which covariances are modelled as products of (stationary) correlations and time-varying conditional variances, in a pairwise setting. Diagnostics support the model and provide motivation for serial dependence in excess returns.

II. Recent work in asset pricing has stressed the time varying nature of security return moments which characterize conditional stock return dynamics. This paper employs time varying moments using Merton's (1973) intertemporal asset pricing paradigm. Two versions of the model are presented. In the first, changes in the short term Treasury Bill rate are used as a single state variable to drive all changes in the investment opportunity set. This results in a closed model of asset dynamics with one exogenous state variable. In the second, changes in the investment opportunity set are modelled as functions of previous periods' conditional variances and squared residuals. In both cases, time varying moments are preserved and changes in the opportunity set are driven by elements of the optimizing agent's information set. The empirical results strongly support the second version of the model which implicitly allows for a greater number of factors at the cost of a diluted economic structure. To the asset pricing literature, this paper suggests that previous rejections of the static version of the Capital Asset Pricing Model (CAPM) may be due to model misspecification. Specifically, inclusion of a hedge term for interest rate risk leads to a significant parameter estimate and does not lead to rejection of the null hypothesis that the Intertemporal CAPM (ICAPM) holds.

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CHAPTER ONE:
THE CAPITAL ASSET PRICING MODEL WITH TIME VARYING MOMENTS

Harry J. Turtle

A large portion of the literature concerning the testing of asset pricing models uses statistical paradigms which assume that conditional moments are stationary. For example, when testing CAPM, researchers have traditionally nested economic relations in the context of unconditional means and variances. However, given that the true asset pricing model holds under a rich information set, say ϕ_F, all means and variances must be stated relative to this information set. If the asset pricing relation is stated relative to a weaker information set, say ϕ_P the relation tested need not hold in general. Moreover, even if the model is properly constructed in an unconditiona' setting, more powerful test procedures exist in a conditional setting. The purpose of this paper is to incorporate time varying conditional moments into an asset pricing context by making use of a Generalized AutoRegressive Conditional Heteroscedastic in Mean (GARCH-M) process for conditional portfolio variances. Thus, this paper attempts to develop a better description of the CAPM paradigm by more carefully specifying the dynamics and inter-relations of excess return moments.

The intuition underlying the model proposed herein is analogous to many asset pricing paradigms. Shocks to second moments (variances and covariances) impact expected excess returns through relations specified in an asset pricing model. For example, the traditional version of CAPM states that the conditional expectation of an asset's excess return is linearly related to the conditional expectation of the excess return on the market by a slope coefficient called β. Intuitively, the GARCH process is then used to model the time series behaviour of β, giving β_t for each time period t. The specific form of the CAPM presented is slightly different from the typical β version in order to resolve econometric problems associated with using ex post excess returns on the market index, rather than its ex ante conditional expectation. Specifically, the model is stated in terms of a Constant Coefficient of Relative Risk Aversion (CRRA) which reduces the market index problem but does require the assumption of iso-elastic utility or a similar assumption of constant return for risk trade-off.

The use of iso-elastic utility functions has been widely studied in the financial economics literature with the result that, under relatively weak assumptions, an investor's optimal investment policy converges to the optimal investment policy for a representative investor with an iso-elastic utility function (see for example Mossin (1968), Cass and Stiglitz (1970), Leland (1972), and Hakansson (1974)). Thus, only the class of iso-elastic utility functions is considered.

This paper is unique in the manner in which cross-sectional relations are developed. The existing literature in the area of multivariate GARCH-M models is scant. Most models require estimation of all unique elements of a time varying covariance matrix which results in a great number of parameters (see for example Bollerslev, Engle, and Woolridge (1988), or Engel and Rodrigues (1989)). Due to the difficulties associated with simultaneously estimating a large number of parameters, these models are virtually always reduced to a simpler form which typically involves many fewer parameters and, unfortunately, oversimplifies a truer model structure. As a result of the above parameterization problems, researchers using multivariate GARCH-M models typically find highly significant second moment parameters when all parameters are analyzed as a whole, yet they have difficulty establishing significance for any particular second moment parameters.

A premise of this paper is that by assuming correlation matrix stationarity the estimation problem is reduced considerably, as only the diagonal elements (i.e., conditional variances) of the covariance matrix need be estimated. The benefit of this assumption is that it is consistent with the well documented notion of covariance nonstationarity; however, it also allows estimation to proceed without an excessive number of parameters. In an independent study, Giovannini and Jorion (GJ,1989) also consider the constant correlation model. This study is unique in that only correlations with the market are needed to close the model, rather than all possible pairwise correlations between assets, as is necessary in the GJ model. Given the difficulty associated with estimating the model when all pairwise correlations are included, this model provides a great reduction in computational burden with little loss in generality.

The remainder of the paper is organized as follows. Section I provides a description of the ARCH, GARCH, ARCH-M, and GARCH-M literature. Section II presents the model, Section III describes the data and methodology, and Section IV presents the results. Diagnostic test results are reported in Section V. Finally, Section VI concludes the paper.

I. MODELS FROM THE LITERATURE

Given the speed with which advances in the area of GARCH, GARCH-M, and multivariate GARCH-M are occurring, any review of the literature cannot hope to be complete. Thus, the particular references discussed below are intended to be indicative of the area, and not an exhaustive description of all related research. All models have been specified in a manner which remains notationally equivalent to the original work yet still allows for straightforward comparison across specifications.

Before discussing particular models, it will be helpful to first introduce some standard notation.

Let

N be the number of variates included in the market index,
$\mathbf{R}_t$ be an Nx1 vector of nominal security returns in period t,
r_{ft} be a scalar valued riskless rate,
$\mathbf{y}_t = \mathbf{R}_t - r_{ft}\,\mathbf{e}$ be an Nx1 vector of excess returns, where $\mathbf{e}$ is an Nx1 vector of ones,
μ_t be an Nx1 conditional mean vector of these excess returns given the information set ,ϕ_{t-1},
$\mathbf{H}_t$ be an NxN conditional covariance matrix of the Nx1 vector of excess returns given ϕ_{t-1}, with particular element σ_{ijt}, and
ω_{t-1} be an Nx1 vector of value weights invested in the N assets at the end of period $t-1$.

Given the preceding definitions we can write the excess return on the market in period t as,

$$y_{Mt} = \mathbf{y}_t^T \omega_{t-1} , \tag{1}$$

where ^{T} denotes the operation of transposition. Similarly, the vector of covariances with the market can be defined as,

$$\mathbf{H}_t \omega_{t-1} . \tag{2}$$

For the market index, we also have that the conditional expected excess return of the market is given by,

$$\mu_{Mt} = \omega_{t-1}^T \mu_t \tag{3}$$

with conditional variance given by,

$$\sigma_{Mt}^2 = \omega_{t-1}^T \mathbf{H}_t \omega_{t-1} . \tag{4}$$

The following discussion is presented to lead the reader through a sampling of the important papers in the area of autoregressive conditional heteroscedasticity. Several types of models are considered. To properly evaluate these models, readers should pay specific attention to whether a model is univariate or multivariate, as well as to the model's ability in allowing second moments to impact first moments in a meaningful manner. For example, univariate GARCH-M processes have difficulty establishing cross-sectional mean relations except in the special case where the model is fit to the market index.

A. Engle (1982)

In Engle's original paper, which focused on the modelling of conditional univariate variances, the ARCH regression model is specified as,

$$\begin{aligned} z_t \mid \phi_{t-1} &\sim N(\mathbf{x}_t^T \beta, \sigma_t^2), \\ \sigma_t^2 &= \sigma^2 (\phi_{t-1}), \text{ and} \\ u_t &= z_t - \mathbf{x}_t^T \beta, \end{aligned} \tag{5}$$

where z_t is the normally distributed random variable of interest, $\mathbf{x}_t$ is a vector of exogenous and lagged endogenous variables, ^{T} is the operation of transposition, and β is a parameter vector such that $\mathbf{x}_t^T \beta$, is the mean of $z_t \mid \phi_{t-1}$. The notation $\sigma^2 (\phi_{t-1})$, is used to show that the conditional variance of z_t is a function of information at t-1.

Some particular examples for σ_t^2 are,

$$\sigma_t^2 = e^{(\alpha_0 + \alpha_1 u_{t-1}^2)} \text{ and} \tag{6}$$

$$\sigma_t^2 = \alpha_0 + \alpha_1 \mid u_{t-1} \mid . \tag{7}$$

B. Bollerslev (1986)

Bollerslev introduces his specification of the GARCH(p,q) process as,

$$u_t \mid \phi_{t-1} \sim N(0, \sigma_t^2),$$

$$\sigma_t^2 = \gamma + \sum_{i=1}^{q} \alpha_i u_{t-i}^2 + \sum_{i=1}^{p} \beta_i \sigma_{t-i}^2 \tag{8}$$

$$= \gamma + A(L) u_t^2 + B(L) \sigma_t^2,$$

where $p \geq 0$[1] , $q>0$, $\gamma>0$, $\alpha_i \geq 0 \ \forall \ i=1,\ldots,q$, and $\beta_i \geq 0 \ \forall \ i=1,\ldots,p$.

A similar regression model can be specified for nonzero mean models in a manner analogous to Engel's approach.

C. Engle, Lilien, and Robbins (ELR, 1987)

ELR are among the first authors to present a model where conditional variances are allowed to impact upon conditional means. Although this paper provides fundamental breakthroughs, it leaves two important issues unresolved. First, multivariate relations are not considered since the model is developed solely within a univariate setting. Secondly, the class of ARCH processes considered by ELR is a strict subset of the GARCH class and thus the general methodology deserves further study in a richer GARCH-M environment. In sum, the results of ELR should be viewed as fundamental in shifting the emphasis in the literature towards the modelling of means as functions of time varying second moments.

The specific form of ELR's model is,

$$y_t = b + \delta \log(\sigma_t) + u_t ,$$

[1] The notation $p \geq 0$ implies that when $p = 0$, the final summand in (8) is to be ignored.

$$u_t \mid \phi_{t-1} \sim N(0,\sigma_t^2),$$

$$\sigma_t^2 = \gamma + \alpha \sum_{\tau=1}^{4} w_\tau u_{t-\tau}^2, \text{ and where } w_\tau = \frac{5-\tau}{10}. \qquad (9)$$

where y_t is the excess holding yield on a long term bond relative to a one period Treasury Bill, with conditional mean $b + \delta \log(\sigma_t)$ and conditional variance σ_t^2. Notice this parameterization forces a declining weight structure on lags of squared residuals. The GARCH process remedies this problem by allowing the weight structure to be determined implicitly in estimation.

D. Bollerslev, Engle, and Woolridge (BEW, 1988)

BEW extend ELR's model by addressing the multivariate nature of asset pricing relations and also by generalizing relations somewhat by employing the GARCH process. Although BEW provide important contributions towards modelling multivariate processes, at least two important issues remain. First, the model must be restricted in a meaningful manner to allow for parsimonious estimation. Secondly, advances in asset pricing suggest that a richer multivariate formulation should allow for more meaningful cross-sectional mean relations.

The general multivariate GARCH(p,q)-M CAPM model of BEW is,

$$\mathbf{y}_t = \mathbf{b} + \delta \mathbf{H}_t \omega_{t-1} + \mathbf{u}_t,$$

$$\text{vech}(\mathbf{H}_t) = \mathbf{C} + \sum_{i=1}^{q} \mathbf{A}_i \text{vech}(\mathbf{u}_{t-i} \mathbf{u}_{t-i}^T) + \sum_{j=1}^{p} \mathbf{B}_j \text{vech}(\mathbf{H}_{t-j}) \qquad (10)$$

$$\mathbf{u}_t \mid \phi_{t-1} \sim N(\mathbf{0}, \mathbf{H}_t)$$

where $\mathbf{y}_t$ is the vector of excess returns in period t with conditional mean $\mathbf{b} + \delta \mathbf{H}_t \omega_{t-1}$; and δ is the estimate of constant relative risk aversion. Consistent with previous notation, $\mathbf{H}_t$ is the contemporaneous conditional covariance matrix of excess returns with particular element σ_{ijt}, and ω_{t-1} is the investment weight vector. To further clarify notation, vech( ) is the column stacking operator which stacks the lower portion of a symmetric matrix. Thus, the dimensions of $\mathbf{y}_t$, $\mathbf{b}_t$, ω_t, and $\mathbf{u}_t$ are NX1, while vech($\mathbf{H}_t$) and vech ($\mathbf{u}_{t-i}$, $\mathbf{u}_{t-i}^T$) are 1/2(N)(N+1)X1, and finally $\mathbf{A}_i$ and $\mathbf{B}_j$ are both 1/2(N)(N+1)X1/2(N)(N+1)$\forall i$ = 1, 2, ... q and j = 1, 2, ... p.

Intuitively, the model fits the evolution of each unique element of the covariance matrix as a GARCH process and then allows that estimate to impact upon expected returns through the parameterization of $\mathbf{y}_t$. Section II shows how this parameterization is in fact the familiar CAPM.

The obvious drawback with this approach is the large dimensionality of the parameter vector $\mathbf{P}_1^T = (\mathbf{b}^T, \delta, \mathbf{C}^T, \text{vech}(\mathbf{A}_1)^T, ..., \text{vech}(\mathbf{A}_q)^T, \text{vech}(\mathbf{B}_1)^T, ..., \text{vech}(\mathbf{B}_p)^T)$. In full generality, $\mathbf{P}_1^T$ is of dimension m_1X1, where $m_1 = (N+1) + (1/2)(N)(N+1) + (1/4)(N^2)(N+1)^2 (p+q)$. Notice that this parameterization implicitly allows conditional variances and correlations to be time varying. If correlations are constant, then the general model presented by BEW can be greatly reduced to allow estimation of even complex second moment dynamics. (This notion is examined further in section II.)

BEW now restrict their model by requiring $\mathbf{A}_i$, $\mathbf{B}_j$, to be diagonal and by further imposing p=q=1. This reduces the number of parameters considerably and allows BEW to write their model as,

$$y_{it} = b_i + \delta \sum_{j=1}^{N} \omega_{jt-1}\, \sigma_{ijt} + u_{it}$$

$$\sigma_{ijt} = \gamma_{ij} + \alpha_{ij} u_{it-1} u_{jt-1} + \beta_{ij}\sigma_{ijt-1} \qquad (11)$$

$$\forall i,j = 1, 2, \ldots N$$

$$\mathbf{u}_t \mid \phi_{t-1} \sim N(\mathbf{0}, \mathbf{H}_t)$$

where σ_{ijt} is only specified for $i \le j$ due to symmetry of $\mathbf{H}_t$.

This specification still involves estimating a parameter vector $\mathbf{P}_2^T = (\mathbf{b}^T, \delta, \mathbf{C}^T, (\mathbf{A}_1\mathbf{e})^T, (\mathbf{B}_1\mathbf{e})^T)$ of dimension m_2X1, where

$$m_2 = (N+1) + 1/2(N)(N+1) + N(N+1)$$

$$= ((3/2)N+1)(N+1)$$

and $\mathbf{e}$ is a (1/2)(N)(N+1)X1 vector of ones.

Without restricting p=q=1, a large number of parameters remain to be estimated. Specifically,

$$m_2 = (N+1) + (1/2)(N)(N+1) + (1/2)(N)(N+1)(p+q)$$
$$= (N+1) + (1/2)(N)(N+1)(1+p+q)$$
$$= (1/2)N^2(1+p+q) + (1/2)N(3+p+q) + 1$$

E. Engel and Rodrigues (ER, 1989)

The international CAPM model of ER can be written as,

$$\mathbf{y}_t = \mathbf{b} + \delta \mathbf{H}_t\, \omega_{t-1} + \mathbf{u}_t$$
$$\text{VAR}(\mathbf{u}_t) = \mathbf{H}_t = \mathbf{P}^T\mathbf{P} + \mathbf{G}\mathbf{u}_{t-1}\mathbf{u}_{t-1}^T\,\mathbf{G} \tag{12}$$
$$\mathbf{u}_t \mid \phi_{t-1} \sim N(\mathbf{0}, \mathbf{H}_t)$$

where **P** is an upper triangular matrix and **G** is a symmetric matrix.

This formulation is very similar to that given above in BEW. $\mathbf{H}_t$ is the contemporaneous covariance matrix of the vector of excess returns, $\mathbf{y}_t$; ω_{t-1} is the weight vector at the end of period t-1; and δ is the coefficient of relative risk aversion for the typical market agent. The difference between BEW and ER lies in the parameterization of $\mathbf{H}_t$.

ER restrict **G** to reduce the parameterization of their model. Initially, **G** is treated as a symmetric matrix after which they further constrain **G** to be diagonal. Once again notice that the dimensionality of the parameter vector is very large in the general setting, especially considering that conditional variances are specified only as first order ARCH processes. Clearly, the models of BEW and ER could be more parsimoniously specified if there were not such a tremendous loss in degrees of freedom in the estimation of the conditional covariance matrix. Also notice the specification of ER could be somewhat generalized by moving to a more flexible GARCH framework.

ER also model asset means as being interrelated through CAPM. The interesting portion of the relation specified is the parameterization of the covariance matrix $\mathbf{H}_t$. The model chosen implies that next period's covariance matrix is given by a constant matrix, $\mathbf{P}^T\mathbf{P}$, plus a variable term, $\mathbf{G}\mathbf{u}_{t-1}\mathbf{u}_{t-1}^T\mathbf{G}$, which relates the matrix of contemporaneous squared residuals, $\mathbf{u}_{t-1}\mathbf{u}_{t-1}^T$, to next period's covariance matrix through a parameter matrix, $\mathbf{G}$.

F. Akgiray (1989)

Akgiray filters returns, R_t, of first order autocorrelation before fitting his model. His GARCH(p,q) model may then be written as,

$$
\begin{aligned}
& R_t \mid \phi_{t-1} \sim F(\mu_t, \sigma_t^2), \\
& \mu_t = \rho_0 + \rho_1 R_{t-1}, \\
& \sigma_t^2 = \gamma + \sum_{i=1}^{q} \alpha_1 u_{t-i}^2 + \sum_{j=1}^{p} \beta_j \sigma_{t-j}^2, \text{ and} \\
& u_t = R_t - \rho_0 - \rho_1 R_{t-1}
\end{aligned}
\qquad (13)
$$

where Akgiray subsequently assumes F() to be normal.

Akgiray's paper shows how GARCH processes can be used to forecast conditional variances more appropriately than other currently available techniques. The manner in which Akgiray models conditional variances and means is, however, somewhat curious. An important advantage of the GARCH-M framework in finance is it allows time varying conditional variances to affect means in an endogenous manner. Akgiray appears to stress the GARCH process, but not the interaction between means and variances. Clearly, in a univariate context this lack of interaction presents difficulties given the multivariate nature of financial asset pricing models. The intuitive nature of GARCH-M models, especially in a multivariate context, is that changing variances may be viewed as shocks to an otherwise closed system which in turn affect means in a manner which is predictable, or at least modellable, by an asset pricing paradigm.

G. Giovannini and Jorion (GJ, 1989)

The basic model of GJ can be written as,

$$\mathbf{y}_t = \mathbf{b} + \delta \mathbf{H}_t \, \omega_{t-1} + \mathbf{u}_t$$

$$\mathrm{VAR}(\mathbf{u}_t) = \mathbf{H}_t$$

$$= \Gamma + \mathbf{A} * \mathbf{u}_{t-1}\mathbf{u}_{t-1}^T + \mathbf{B} * \mathbf{H}_{t-1} + \mathbf{C} * \mathbf{i}_{t-1}\mathbf{i}_{t-1}^T \qquad (14)$$

$$\mathbf{u}_t \mid \phi_{t-1} \sim N(\mathbf{0}, \mathbf{H}_t)$$

where

$*$ represents element by element matrix multiplication (Hadamard product),

$\mathbf{i}_t = \mathbf{i}_t^* - r_{ft}\mathbf{e}$, such that $\mathbf{i}_t^*$ is a vector containing a zero for the stock market and the interest rate for each foreign-currency asset, r_{ft} is the scalar valued riskless rate, and $\mathbf{e}$ is a vector of ones,

Γ, $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are positive definite symmetric matrices,

$\mathbf{H}_t$ is again the covariance matrix of excess returns, $\mathbf{y}_t$, with market weight vector at the end of period $t-1$ of ω_{t-1}, and finally,

δ is the coefficient of relative risk aversion for the typical market agent.

The model of GJ is very similar to that of ER. The primary differences lie in the parameterization of the conditional covariance matrix. GJ generalize the work of ER by modelling conditional covariances as GARCH processes rather than simply ARCH processes.

The other unique feature of GJ is the addition of interest rate differentials to the covariance formulation. This is motivated as an ad hoc search for a better variance specification. One problem with this approach is that second moments are functions of interest rate differentials, while first moments are simultaneously immune to such shifts.

GJ also discuss and briefly report the results obtained when the model is estimated under the assumption of constant correlations. The constant correlation model presented herein is distinct from GJ in that only correlations with the market portfolio are required rather than all possible pairwise correlations between portfolios. This results in a parameterization which allows computationally feasible model selection over the class of Pairwise GARCH(p,q)-M models. This approach is discussed in section II.

Although various advances have been achieved through the employment of GARCH processes in finance, it is clear that the large number of parameters involved in multivariate models presents an important dilemma. The proposed model directly addresses this quandary.

II. THE MODEL

In this section the Capital Asset Pricing Model (CAPM) is discussed as background to the development and estimation of the model presented in this paper. The framework tested is then contrasted with the models outlined in Section I. Intuitively, the model presented herein is intended as a richer alternative to typical tests of the static CAPM paradigm. The model reduces the number of parameters considerably, with little loss in generality of either second moment processes or asset pricing mean relations. Specifically, the model assumes constant correlations to reduce the number of parameters required in estimation of the covariance matrix. [For empirical evidence regarding constancy of correlations see Gibbons (1986), Elton and Gruber (1973), Elton, Gruber, and Padberg (1976,1977a,1977b), or Elton, Gruber, and Urich (1978)]. The underlying statistical excess return process implied by this assumption displays conditional variance nonstationarity and correlation matrix stationarity.

The methodology employed also collapses the fully generalized model to a series of pairwise models in which each portfolio is compared with the market index. This allows a computationally feasible search procedure to be employed to determine the order of the GARCH(p,q)-M model for each of the variates which has not been previously examined. Moreover, pairwise models allow researchers to avoid strong data set requirements including a weight vector for each portfolio considered, and a composite market index composed of the value weighted portfolios.

The ability to estimate models for individual variates using a broad-based market index is significant in two respects. First, it allows for a better representation of the underlying market index. Secondly, an equivalent to the 'unconditional market model' is a by-product of the analysis. For example, the chosen pairwise GARCH(p,q)-M model could also be used in an event study as a control for market risk and return.

The paradigm presented can be intuitively described in a traditional economic context of second moment shocks affecting first moment equilibrium values. Consider an asset pricing model which gives rise to cross-sectional restrictions on portfolio mean returns for given covariances amongst assets (e.g., CAPM or APT). The GARCH process is then used to model the behaviour of conditional variances, and covariance terms are then specified by simply multiplying conditional variances by correlations, to close the model. In this particular formulation, shocks to second moments are modelled by a GARCH process, which in turn impact on expected excess returns through relations specified by CAPM.

Formal development of the model requires an asset pricing paradigm to specify equilibrium cross-sectional relations. Throughout, this paper uses the static CAPM which in its simplest form requires that expected excess returns are linear (with zero intercept in the Sharpe-Lintner case) in expected market excess returns (see for example Sharpe (1964,1970) or Lintner (1965)). Merton (1973) extends the CAPM to an intertemporal continuous time setting and finds a similar relation given a fixed investment opportunity set. More recently, Longstaff (1989) discusses the problems associated with the temporal aggregation of a continuous-time model using discrete return data. Longstaff shows that the simple linear CAPM relation may fail in a discrete time setting although the continuous-time CAPM is in fact the underlying paradigm. Longstaff's specific discrete time 'approximate' CAPM is developed under very strong assumptions and thus little can be said regarding the true discrete CAPM version generally. In fact, even if all of Longstaff's assumptions hold, his results are specific to the continuous-time version of the CAPM. As discussed by Longstaff, the appropriate CAPM paradigm uses the representative investor's portfolio revision period, which is not necessarily instantaneous as is assumed under the continuous-time version of the model.

Monthly return data is used to balance the concerns regarding normality of excess return data with those regarding the appropriate portfolio revision period. The CAPM restriction on means then requires,

$$\mu_t = \mathbf{b} + \beta_t \, \mu_{Mt} \qquad (15)$$

$$\text{where } \beta_t = \frac{H_t \omega_{t-1}}{\sigma^2_{Mt}}, \qquad (16)$$

with typical element of β_t given by,

$$\beta_{it} = \left(\frac{\sigma_{iMt}}{\sigma^2_{Mt}} \right) \qquad (17)$$

and if correlations are constant, this can be simplified to.

$$\beta_{it} = \left(\frac{\sigma_{it} \rho_{iM}}{\sigma_{Mt}} \right) \qquad (18)$$

The constant term, **b** in (15), is included to control for effects such as differential taxes and preferred habitats of investors not modelled by the static version of CAPM.

Thus, CAPM requires that excess returns for any portfolio i satisfy,

$$y_{it} = b_i + \beta_{it} \mu_{Mt} + u_{it}, \qquad \forall \; i=1,2,\ldots,N \qquad (19)$$

where μ_{Mt} is the conditional expected excess return on the market index in period t.

A. The Traditional CAPM

Notice the relation specified in (19) above is not testable in its current form because the conditional market mean in (19), μ_{Mt}, is not observable. Typically, rational expectations are assumed to hold to the extent that actual market excess returns equal expected market excess return plus a mean zero white noise error term. That is, if y_{Mt} is unbiased for μ_{Mt}, agents do not systematically deviate from CAPM.

Without loss of generality we can write,

$$y_{Mt} = b_M + \mu_{Mt} + u_{Mt}. \qquad (20)$$

Reversing the above relation, and substituting μ_{Mt} in the CAPM equilibrium relation, (20), we obtain,

$$\begin{aligned} y_{it} &= b_i + \beta_{it}(y_{Mt} - b_M - u_{Mt}) + u_{it} \\ &= b_i + \beta_{it} y_{Mt} + \varepsilon_{it} \end{aligned} \qquad (21)$$

where $\varepsilon_{it} = u_{it} - \beta_{it}(b_M + u_{Mt})$.

This substitution will induce time variability into ε_{it} because β_{it} is time varying. If a researcher is willing to assume that individual security excess return residuals swamp market excess return residuals and that b_M is zero (i.e.,$\varepsilon_{it} \approx u_{it}$), the model can then be stated as,

$$y_{it} = b_i + \left(\frac{\sigma_{it}\rho_{iM}}{\sigma_{Mt}}\right) y_{Mt} + u_{it},$$

$$\sigma^2_{it} = \gamma_i + \sum_{k=1}^{q} \alpha_{ik} u^2_{it-k} + \sum_{l=1}^{p} \beta^*_{il} \sigma^2_{it-l}, \qquad (22)$$

$$\forall\ i=1,2,\ldots,N$$

$$u_t \mid \phi_{t-1} \sim N(\mathbf{0}, \mathbf{H}_t) \text{ where } \mathbf{H}_t = [\sigma_{ijt}].$$

In this specification, the conditional mean of any excess return series y_{it} is given by CAPM as $b_i + \left(\frac{\sigma_{it}\rho_{iM}}{\sigma_{Mt}}\right) y_{Mt}$, with conditional standard deviation obtained by taking the square root of the appropriate variance. Each conditional variance is modelled as evolving according to the specification above where γ_i represents the constant portion of conditional variance, α_{ik} represents linearity of this period's variance in previous squared residuals, and β^*_{il} is the coefficient reflecting linearity of current conditional variance in previous conditional variances[2].

This specification is undesirable due to autocorrelation in the residuals and also because it requires estimation of parameters which enter the model through the denominator (i.e., through σ_{Mt}). To resolve these issues an alternative but comparable specification is employed.

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[2] The "*" is included only to differentiate this coefficient from β_{it}, asset i's covariance with the market.

B. CAPM under Constant Relative Risk Aversion

The alternative specification of CAPM is developed under the assumption of a constant coefficient of relative risk aversion, δ. Under this specification, δ is equal to the aggregate measure of Relative Risk Aversion (RRA) given by the harmonic mean of an agent's RRA weighted by an agent's share of aggregate wealth in equilibrium [see BEW (1988), p. 118].

Using δ as above, the CAPM relation can be written as,

$$\mu_t = \mathbf{b} + \delta H_t\, \omega_{t-1}. \tag{23}$$

Now, by comparison of (23) with (15) and (16), we get,

$$\delta = \frac{\mu_{Mt}}{\sigma^2_{Mt}}. \tag{24}$$

Notice the assumption of a constant coefficient of relative risk aversion is simply a restriction that the equilibrium trade-off of conditional excess market return to risk is constant over time. The value of an assumption of this sort is it allows model builders to avoid concern regarding changes in tastes over the period of estimation. Given both theoretical[3] and empirical[4] results supporting convergence to iso-elastic utility, this restriction seems reasonable for the lengthy data series analyzed herein.

Assuming δ is constant, the CAPM relation can be written for any particular portfolio as,

$$\mu_{it} = b_i + \delta\sigma_{iMt}. \tag{25}$$

Then, by specifying a GARCH process for σ_{iMt}, the model is complete. To this point the model is similar to others found in the literature [see for example Bollerslev, Engle and Woolridge (1988), or Engel and Rodrigues (1989)]. Giovannini and Jorion (GJ,1989), in a work independent from that herein, also model a similar generalized constant correlation model. The discussion which

[3] See for example Mossin (1968), Cass and Stiglitz (1970), Leland (1972), and Hakansson (1974).

[4] For example, see Fama and MacBeth [(1973), pp. 628-629] who report empirical evidence that $\frac{\mu_{Mt}}{\sigma_{Mt}}$ is relatively stable over time.

follows distinguishes this paper from existing approaches through its simplicity and parsimonious specification of σ_{iMt}.

The general version of the model to be tested assumes the conditional covariance matrix, $\mathbf{H}_t$, is determined by the multiplication of constant correlations and changing variances (not changing correlations and variances as is implicit in BEW, or ER). This approach has the benefit of pragmatically allowing for more specialized GARCH(p,q) processes to specify variances without exploding parameterizations. The most general version of the model postulated herein is,

$$y_{it} = b_i + \delta \sum_{j=1}^{N} \omega_{jt-1}\, \sigma_{ijt} + u_{it}$$

$$\sigma_{ijt} = \sigma_{it}\sigma_{jt}\,\rho_{ij} \qquad \forall\ i,j,t \tag{26}$$

where

$$\sigma^2_{it} = \gamma_i + \sum_{k=1}^{q} \alpha_{ik}\, u^2_{it-k} + \sum_{l=1}^{p} \beta_{il}\, \sigma^2_{it-l}$$

$$\forall\ i=1,2,\ldots,N$$

$$\mathbf{u}_t \mid \phi_{t-1} \sim N(\mathbf{0}, \mathbf{H}_t)$$

The notation and development to this point is similar to BEW to allow comparison between the models. However, notice that this specification involves estimation of a considerably reduced parameter vector $\mathbf{P}_3^T = (\mathbf{b}^T, \delta, \text{vech}(\rho), \gamma^T, \alpha_1^T, \alpha_2^T, \ldots \alpha_q^T, \beta_1^T, \beta_2^T, \ldots \beta_p^T)$ of dimension m_3X1, where

$$m_3 = (N+1) + (1/2)(N)(N-1) + N + qN + pN$$

$$= (1/2)N^2 + ((3/2)+p+q)N + 1$$

Recalling BEW's reduced model parameter vector without restricting $p=q=1$, $\mathbf{P}_2^T$, we can see that the proposed formulation reduces the number of parameters by,

$$m_2 - m_3 = [(1/2)N^2(1+p+q) + (1/2)(N)(3+p+q) + 1] - [(1/2)N^2 + ((3/2)+p+q)N + 1]$$

$$= (1/2)N^2(p+q) - (1/2)N(p+q)$$

As an example, consider five portfolios with $p=q=1$, then

$m_2 = 51$, and $m_3 = 31$ so, $m_2 - m_3 = 20$.

That is, for five portfolios the above methodology results in a reduction of 20 model parameters. For 10 portfolios,

$m_2 - m_3 = 90$.

Clearly, this reparameterization results in a considerable savings in degrees of freedom.

Giovannini and Jorion (1989) report results for an independent but similar constant correlation model. The analogous general model results from this study are reported in section IV-C for completeness.

The reduced version of the model compares each portfolio to the market individually using the market index and the identity $\sum_{j=1}^{N} \omega_{jt-1}\sigma_{ijt} = \sigma_{iMt}$. This approach allows for a computationally feasible search over GARCH(p,q)-M models for each of the variates. Alternative studies in this area have simply imposed an ad hoc ARCH or GARCH process upon portfolio variances. Moreover, given that σ_{iMt} does not involve ω_{jt-1}, this procedure allows any broad-based market index to be employed rather than simply a composite index formed from the portfolios considered; however, to maintain comparability with earlier work, a similar composite index is used. The pairwise model can now be written as,

$$y_{it} = b_i + \delta\sigma_{iMt} + u_{it}.$$

$$\sigma_{iMt} = \sigma_{it}\sigma_{Mt}\,\rho_{iM} \qquad \forall\ i,t \tag{27}$$

where

$$\sigma^2_{it} = \gamma_i + \sum_{k=1}^{q} \alpha_{ik}\, u^2_{it-k} + \sum_{l=1}^{p} \beta_{il}\, \sigma^2_{it-l}$$

$$\forall\ i=1,2,\ldots,N, \text{ and } M$$

$$u_t \mid \phi_{t-1} \sim N(\mathbf{0}, \mathbf{H}_t), \text{ for } \mathbf{H}_t = \begin{pmatrix} \sigma^2_{it} & \sigma_{iMt} \\ \sigma_{iMt} & \sigma^2_{Mt} \end{pmatrix}$$

This section has developed the theoretical background for the paradigm to be tested. The specific benefits of the constant correlation model have been discussed and examination of the empirical results of the proposed approach can now be undertaken.

III. DATA AND METHODOLOGY

The data set used in this paper is chosen to achieve as lengthy a series of returns as possible in order to capture the dynamic behaviour of second moments. Monthly percentage returns (with adjustments to include dividends, capital gains, splits and stock dividends) are employed to produce excess return distributions close to normality. Notice, however, monthly series will likely exhibit less of a time series pattern than a similar model for more dynamic weekly returns. For example, if trading volume displays more detectable time series behaviour over shorter periodicities, then weekly returns will likely display stronger time series behaviour in both means and variances. Thus, an alternative methodology might suggest a similar approach using weekly excess returns; however, the results might then be subject to criticisms of nonsynchronous trading over shorter intervals as well as criticisms regarding normality of the multivariate distribution. Thus, to counter these concerns this paper explores monthly returns while a subsequent paper employs weekly returns.

Stocks selected for the sample contain no missing observations for either monthly adjusted returns with dividends, or the number of shares outstanding over the period January 1926, through December 1986. Price data is based upon the closing trade price if it is available or the average of bid and ask prices if the closing price is not available. Total returns on one month U.S. Treasury Bills are subtracted from stock returns to convert nominal returns to excess returns and the result is then multiplied by 100.

To maintain comparability with the multivariate models found in the literature an index is required based upon the stocks found in the selection process described above. The Standard

Industry Code market index (IC) is created as a weighted average of all stocks for any given period. Weights are computed as the product of the number of shares outstanding multiplied by the price per share, divided by the sum of the same quantity over all stocks. The IC market proxy is thus a limited proxy of the true market, however it does carefully monitor each portfolio's weight in the market over time. The alternative approach of using a broader based market index leads to similar results, as will be shown, but does not allow the model to be naturally extended to a more general multivariate setting. Standard and Poor's 500 index (SP) from Ibbotson and Associates is used for comparison purposes relative to the IC index.

Next, stocks are grouped into portfolios based upon the first two digits of the Standard Industry Code to reduce the dimension of the covariance matrix while maintaining economically meaningful heterogeneity amongst the portfolios. The final portfolios are described fully in Table 1-1.

The estimation procedure used in this paper is in accordance with the estimation methodology found in the literature [see for example Engle (1982), Bollerslev (1986), Engle, Lilien and Robbins (1987), Bollerslev, Engle, and Woolridge (1988), Akgiray (1989), Hsieh (1989), or Giovannini and Jorion (1989)]. The methodology assumes excess return residuals follow a multivariate normal distribution. Then, for any particular specification of the model with N variates, the log of the likelihood function is,

$$L(\theta) = \sum_{t=1}^{T} l_t(\theta) ,$$

where

$$l_t(\theta) = -(N/2)\log(2\pi) - (1/2)\log|\mathbf{H}_t(\theta)| - (1/2)\,\mathbf{u}_t^T(\theta)\mathbf{H}_t^{-1}(\theta)\,\mathbf{u}_t(\theta) \qquad (28)$$

is the log of the likelihood for any particular observation t. It is to be understood that all parameters have been combined into a parameter vector θ. For example, in the most general version of the model we have, $\theta^T = [\,\mathbf{b}^T, \delta, [\mathrm{vech}(\rho_M)]^T, \gamma^T, \alpha_1^T,...,\alpha_N^T, \beta_1^T,...,\beta_N^T\,]$ where, α_i^T (resp. β_i^T is the transposed vector of coefficients reflecting linearity of conditional variance in previous period's squared residuals (resp. conditional variances) for variate i. The dimension of α_i^T (resp. β_i^T) is q (resp. p), although in general p and q need not be the same across variates.

Clearly, from the specification of the model in section II [see equation (27)], the likelihood function depends upon θ in a highly nonlinear manner. This paper maximizes (28) using numerical derivatives in conjunction with the method of Berndt et al. (1974). As Hsieh (1989) points out, maximum likelihood procedures provide numerous benefits over least squares estimation. These include the joint estimation of parameters affecting both means and variances, and the ability to impose restrictions in a straightforward manner. One common restriction imposed is nonnegativity of conditional variances, which can be assured through a penalty on the likelihood function. It should be noted that all estimates provided in the next section converge quickly and uniformly to their final values.

To aid in model selection both Akaike's and Schwartz's Information Criterion are reported (AIC and SIC) where

$$AIC = -2*\max L(\theta) + 2*k, \text{ and}$$

$$SIC = -2*\max L(\theta) + k*\ln(T),$$

where k is the number of parameters in the estimated model, and T is the total number of observations. These criterion are used to choose between nested versions of the model in sections IV.*A* and IV.*B*. In both cases the model with the minimum value of the information criterion is the preferred model. Given the tendency for AIC to overparameterize models, SIC will be employed when conflicts occur. A survey of the issues and concerns associated with model selection can be found in de Gooijer, Abraham, Gould, and Robinson (1985) and Mizon (1977).

Section IV presents the specific versions of the model tested and the resultant estimates and standard errors. The findings appear to be both economically meaningful and statistically significant. One important facet of the results is the highly significant second moment coefficients which suggests very dynamic conditional variance processes.

IV. EMPIRICAL RESULTS

In subsection *A*, the estimation methodology determines the optimal p, q lag lengths for the IC market index. For comparison purposes, the analogous results for the SP proxy are also reported. The optimal lag lengths for the univariate IC market index are then imposed upon the pairwise models to determine the optimal p, q lags for each of the portfolios considered in subsection *B*. This procedure consistently determines the optimal lag lengths for each of the industry portfolios at the cost of a loss in efficiency. The benefit of this approach is that the dimensionality of the search procedure is reduced from four to two dimensions producing a tremendous computational savings. Furthermore, consistency of the final test statistics is maintained in contrast to models which impose ad hoc lag lengths. The fully generalized multivariate model of (co)variance evolution is then reported in subsection *C* for completeness.

A. Univariate GARCH Models

The importance of the results from this subsection are twofold. First, they demonstrate that the time series behaviour of the composite IC index and the broader SP index are very similar. This is not to suggest that similar time series behaviour assures appropriate asset pricing relations; however, it is a promising result. Second, and of primary interest, is the development of an appropriate GARCH(p,q)-M for the IC market index. Because the univariate GARCH(p,q)-M for the IC index is nested within each of the pairwise models, the univariate model for market variance provides a straightforward methodology to reduce the search dimension in each pairwise model. That is, the univariate GARCH(p,q)-M for the market index provides the appropriate (p,q) orders for the market index within each of the pairwise GARCH-M models. Thus, when the pairwise GARCH-M models are estimated, the search over optimal lag lengths may be reduced by two dimensions since the market lags have already been determined. The resultant loss in efficiency is countered by the consistency of the resultant test statistics.

A.1. GARCH(1,1)

The first results considered are those based upon the simplest process consistent with the methodology[5] herein, namely, the GARCH(1,1) process for both the SP and IC market proxy excess return series. That is, for each series the GARCH(1,1) model,

$$\sigma^2_{Mt} = \gamma_M + \beta_M \sigma^2_{Mt-1} + \alpha_M\, (\, y_{Mt-1} - b_M)^2$$

is estimated[6]. Notice in this simple specification b_M is the unconditional mean of the series (assumed constant by construction), γ_M is the constant portion of the conditional variance, β_M is the coefficient which multiplies last period's conditional variance, and α_M is the coefficient reflecting linearity of conditional variance in last period's squared deviation from expected return. Empirical results for the Univariate GARCH(1,1) process are reported in Table 1-2.

One important aspect of the results is that both market indices are strikingly similar with respect to parameter estimates and significance. Another positive aspect of the results is that the coefficients reflecting the time variability of conditional variance are extremely significant and are in accordance with very dynamic second moments for both series.

Notice the importance of β_M, which reflects linearity of conditional variance in previous period's conditional variance. This term is ignored in the simple ARCH specification although it is straightforward to show that the simple GARCH(1,1) can be used to model geometrically declining weights on previous squared residuals by writing the relationship solely in terms of previous period's squared residuals. For example, making use of the lag operator, B

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[5] Notice that this process is not strictly consistent with the methodology in this paper because it does not allow conditional variances to affect means. However, the process does demonstrate how univariate models, like Akgiray (1989), relate to more specialized processes in mean.

[6] The models are stated in this reduced form for succinctness. This formulation is identical to that presented earlier, where residuals are folded into the conditional variance equation. For example, the term y_{t-1} - b is the deviation from the expected excess return for portfolio y's excess return. Thus, in this simple case, the expected excess return is simply b_M.

(e.g., $B\sigma^2_{Mt} = \sigma^2_{Mt-1}$), the IC market results can be written solely in terms of previous squared residuals,

$$\sigma^2_{Mt} = 1.35 + .802\sigma^2_{Mt-1} + .16u^2_{Mt-1}$$

$$\sigma^2_{Mt} = (1.35)/(1-.802) + (.16u^2_{Mt-1})/(1-.802B)$$

$$= 6.82 + (.16u^2_{Mt-1})\,[1+.802B+.643B^2+.516B^3+.414B^4+.332B^5+...]$$

$$\approx 6.82 + .16u^2_{Mt-1} + .13u^2_{Mt-2} + .10u^2_{Mt-3} + .08u^2_{Mt-4} + .07u^2_{Mt-5} + ...$$

Notice that the use of one parameter for lags of conditional variance allows researchers greater flexibility in parsimoniously parameterizing a decaying lag structure on residuals squared. Thus, any attempt to force an ad hoc decay pattern to squared residuals must justify not employing a more flexible GARCH process.

A.2. GARCH(1,1)-M

In this section the optimal lag lengths for p and q are established for the IC market proxy. Given the two dimensional nature of the search procedure, AIC and SIC seem best suited to determine the optimal lag orders (although an exhaustive search procedure using likelihood ratio test statistics with adjusted significance levels could also be considered). Previous work has shown under various circumstances that AIC tends to overparameterize models [see for example Geweke and Meese (1981), or de Gooijer, Abraham, Gould, and Robinson (1985)], thus SIC will be employed when the methods conflict. The search procedure for the IC index GARCH(p,q)-M processes is concentrated over $p,q \in \{1,2\}$ for parsimony. The SIC and AIC values are reported in Table 1-3. As shown in italics in Table 1-3, the optimal univariate model for the market is seen to be the GARCH(1,1)-M,

$$\sigma^2_{Mt} = \gamma_M + \beta_M\sigma^2_{Mt-1} + \alpha_M\,[y_{Mt-1} - b_M - \delta\,\sigma^2_{Mt-1}]^2.$$

This process is distinct from the simple GARCH(1,1) because it allows second moments to impact upon means through CAPM mean relations and thus expected excess returns are also time varying. Notice for this process, $b_M + \delta\,\sigma^2_{Mt}$ is the conditional mean of the series, γ_M is the constant portion of the conditional variance, β_M is the coefficient which multiplies last period's conditional variance, and α_M reflects linearity of conditional variance in the previous period's squared deviation from expected return.

Although the estimated coefficient of relative risk aversion, δ, is only marginally significant for the IC index based upon a one-tailed test, all coefficients appear to have the appropriate sign and magnitude relative to literature standards [see for example Akgiray (1989), Hsieh (1989), or Engel and Rodrigues (1989)]. Given the admitted inefficiency in the estimate of δ in this univariate setting, judgments regarding significance should be deferred until the pairwise models are presented. The primary importance of the univariate model is to establish the market lag lengths to employ in the pairwise estimation in subsection *B*. It is interesting to observe that the GARCH(1,1) and GARCH(1,1)-M frameworks lead to very comparable results in that both suggest highly dynamic conditional variances.

B. Pairwise GARCH-M Models

In this section, each of the five industry portfolios is compared to the IC market index in a pairwise GARCH(p,q)-M model. The notation p and q represent the order of the industry portfolio lag lengths, not market lag lengths (recall subsection *A* determined the optimal market lag lengths to be $p=1$ and $q=1$). Thus, for each portfolio i the pairwise GARCH(p,q)-M model specifies the variance process for the industry portfolio as,

$$\sigma_{it}^2 = \gamma_i + \sum_{j=1}^{p} \beta_{ij}\,\sigma_{it-j}^2 + \sum_{k=1}^{q} \alpha_{ik}\,[y_{it-k} - b_i - \delta\,\rho_{iM}\sigma_{it-k}\sigma_{Mt-k}]^2$$

and the conditional variance of the market as,

$$\sigma_{Mt}^2 = \gamma_M + \beta_M\sigma_{Mt-1}^2 + \alpha_M\,[y_{Mt-1} - b_M - \delta\,\sigma_{Mt-1}^2]^2$$

That is, the conditional mean of any portfolio i is given by the relation, $b_i + \delta\,\rho_{iM}\sigma_{it}\sigma_{Mt}$. In addition γ_i is the constant portion of the conditional variance, β_{ij} is the coefficient which multiplies the j^{th} lag of conditional variance, and α_{ik} is the coefficient reflecting linearity of conditional variance in the k^{th} lag's squared deviation from expected return.

Using the above specification, without requiring that $\delta_i=\delta_M$, each IC portfolio excess return is jointly estimated with the market index which is constructed as the weighted average of all the portfolios for orders of $p,q \in \{1,2\}$. The AIC and SIC criterion are then used to select the favoured model and the results are reported in italics in Table 1-4. As shown in the table, the Pairwise GARCH(1,1)-M is the selected model for portfolios 1, 3, and 5. However, portfolio 2 selects the Pairwise GARCH(1,2)-M, while portfolio 4 selects the Pairwise GARCH(2,1)-M process[7]. In all cases the AIC criterion chooses at least as large values for both p and q.

After the optimal p and q have been established, CAPM statistics test the restriction, $\delta_i = \delta_M = \delta$ for $i = 1, 2, 3, 4$, and 5 versus the alternative unrestricted process where $\delta_i \neq \delta_M$. CAPM then tests the restriction that these δ_i coefficients are equal. For completeness the Lagrange Multiplier, Likelihood Ratio, and Wald statistics are reported in Table 1-4[8]; all test statistics are asymptotically $\chi^2(1)$. All portfolios except portfolio 2 do not reject the null hypothesis, providing support for the proposed model. Portfolio 2 rejects the CAPM restriction at the 10% level of significance for the Lagrange Multiplier and Likelihood Ratio test statistics but not for the Wald statistic.

The results provide support for the static Sharpe-Lintner version of the model in a pairwise setting. Given the strong rejection of CAPM in studies such as Giovannini and Jorion (1989) or Engel and Rodrigues (1989), it remains to be seen whether these rejections are a result of more powerful multivariate methods, or misspecification errors over lag length choices for p

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[7] It is worth noting that methods which arbitrarily impose GARCH or ARCH lag lengths (such as ER, 1989 or GJ, 1989) and then subsequently test economic relations, may reject solely due to model misspecification, rather than failure of the tested paradigm. In fact, Engel and Rodrigues (1989) cite evidence that if the conditional covariance matrix is misspecified estimates of δ are quite likely inconsistent [see Pagan (1984), and Pagan and Ullah (1988)].

[8] Earlier studies have solely employed the Lagrange Multiplier test statistic [see for example ER (1989) and GJ (1989)]. It is interesting to note that for all five portfolios $\xi_{LM} > \xi_{LR} > \xi_W$. This empirical result suggests the familiar numerical inequality $\xi_W \geq \xi_{LR} \geq \xi_{LM}$ established when testing linear restrictions on parameters in the classical regression model with normal errors (see for example p. 43 Godfrey, 1988) does not extend to this nonlinear heteroscedastic setting.

and q. However, it is interesting to note that similar studies have resulted in negative point estimates of δ, while this study finds δ to be strictly positive in all cases (although not highly significant). For example, Engel and Rodrigues (1989) estimate δ to be -18.8 in their most general model while Giovannini and Jorion (GJ,1989) report $\delta = -142.91$ in their constant covariance model (although GJ's general model leads to an insignificant positive δ estimate).

The estimates and standard errors reported in Table 1-4 are also supportive of the model. Several items deserve mention. First, the estimates of relative risk aversion across pairwise models are fairly constant relative to their standard errors. Also notice that b_M, γ_M, β_M, and α_M, the market parameters, are comparatively stable relative to their standard errors. Moreover, all of these parameters are highly significant individually. Finally, notice that for each of the variates, the second moment coefficients γ_i, β_{i1}, β_{i2}, α_{i1}, and α_{i2} are all individually highly significant with the exception of β_{i1} for portfolio 4.

The second moment parameters lend considerable explanatory power to the evolution of conditional variances. In addition, the parameters determining the evolution of conditional variance are remarkably similar for both the market and the individual IC portfolios, as expected. However, there are important differences across portfolios in terms of the appropriate orders of p and q.

Figure 1-1 plots the conditional variance and excess return of the IC market index against time to demonstrate the highly dynamic nature of the market proxy over time. Notice that the conditional variance based upon the model responds very well to volatility in the actual excess return series, as desired. The depression years lend tremendous support to the notion of dynamic conditional moments although it is interesting to note that in preliminary work, removal of this early period did not drastically alter the resultant estimates. To exclude a period of extreme volatility seems irresponsible, especially in light of October 1987.

Figure 1-2 shows the evolution of conditional expected returns based upon the model presented. Clearly, any model based upon the assumption of constant expected returns will be difficult to justify in anything but an unconditional setting of very weak information. However, in this context, tests lack power even if they can be correctly specified. Notice that, even when the pre-1940 data is ignored, expected returns can vary by as much as 30 to 50 basis points over relatively short periods of time. This model suggests that, if powerful asset pricing tests are to be constructed, time varying moments must be considered[9]. It may be argued that expected returns may be safely treated as stationary over the post 1940 years. However, this stability might just as likely be due to a temporary lull in an inherently volatile series.

Finally, Figure 1-3 is included to heuristically show that beta estimates need not be constant, as is commonly assumed. That is, this figure simply plots portfolio 1's beta over time, constructed as $\beta_{1t} = \rho_{1m} \left[\frac{\sigma_{1t}}{\sigma_{Mt}}\right]$, where all quantities in β_{1t} are replaced by their sample estimates. The invariance principle of maximum likelihood estimation ensures that the estimate of β_{1t} will also be the maximum likelihood estimate for β_{1t}. This figure shows how the assumption of conditional beta constancy is seriously violated by the data.

C. Generalized Constant Correlation Multivariate GARCH-M Model

The fully generalized constant correlation model is presented next for completeness and to allow comparison with similar studies in the area. Given the computational burden in estimation of this fully generalized model, only the GARCH(1,1)-M version is presented[10]. As discussed in section II, the most general version of the model under the CAPM restriction is,

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[9] Plots of actual excess returns and conditionally expected excess returns against time display the desired pattern of expected returns being larger in periods of greater excess return volatility. However, due to noise in the excess return series relative to expected excess returns, plots including actual excess returns are not as informative in demonstrating the time variability of expected returns as plots of conditionally expected excess returns alone.

[10] Similar studies in the area such as Engel and Rodrigues (1989), and Giovannini and Jorion (1989) also estimate only the Multivariate GARCH(1,1)-M process. To further consider the methodology of this paper in a fully generalized multivariate setting, the pairwise models could be naturally extended by imposing the optimal lag lengths determined upon the generalized model.

$$y_{it} = b_i + \delta \sum_{j=1}^{N} \omega_{jt-1}\, \sigma_{ijt} + u_{it}$$

$$\sigma_{ijt} = \sigma_{it}\sigma_{jt}\, \rho_{ij} \qquad \forall\ i,j,t \tag{29}$$

where

$$\sigma^2_{it} = \gamma_i + \sum_{k=1}^{q} \alpha_{ik}\, u^2_{it-k} + \sum_{l=1}^{p} \beta_{il}\, \sigma^2_{it-l}$$

$$\forall\ i=1,2,\ldots,N$$

$$u_t \mid \phi_{t-1} \sim N(\mathbf{0}, \mathbf{H}_t)$$

The general multivariate GARCH(1,1)-M specification results in the estimates reported in Table 1-5. Due to difficulties in obtaining convergence of maximum likelihood estimates, unconditional correlation estimates are employed[11]. The estimate of δ from the generalized model is very similar to that obtained under the pairwise models. Moreover, the coefficients reflecting linearity of period t's conditional variance in past conditional variances, β_i's, and past squared residuals, α_i's, are also very similar.

V. DIAGNOSTICS

A number of diagnostic tests are employed in this section to examine the merit of the pairwise GARCH(p,q)-M process. Diagnostic summary statistics will be presented first, followed by tests for parameter stability over the estimation period. The diagnostics suggest the proposed pairwise GARCH(p,q)-M process adequately models conditional means and variances; however, parameter stability is strongly rejected over this lengthy series of observations. A pragmatic solution to parameter instability may be to employ weekly data sets and/or shorter calendar data

[11] Engel and Rodrigues (1989), and Giovannini and Jorion (1989) report similar issues in the estimation of multivariate ARCH and GARCH specifications. The solution employed by both of these alternative papers is to ensure positive definiteness of all contemporaneous covariance matrices. The similarity between restricting correlations and ensuring positive definiteness by alternative means should be apparent. It should also be emphasized that the simple version of the model which makes pairwise comparisons with the market encounters no such difficulties.

series in future studies. However, under these circumstances the long term predictive ability of the model may be suspect as well as the normality of the initial data set.

Table 1-6 presents numerous summary statistics of interest in evaluating the model's success. Skewness and kurtosis values are centered about zero under the assumption of normality. A positive value for skewness indicates a lengthy right tail, similarly, a positive value of kurtosis indicates a leptokurtic distribution with long tails. Panel A presents summary statistics for the raw excess return data and replicates Table 1-1 to some extent. Panel B presents statistics for squared residuals from unconditional sample means. That is, for each series the unconditional mean is subtracted from each observation and the result is then squared. Panel C is based upon standardized residuals computed as model residuals divided by conditional standard deviation estimates for each portfolio (i.e., $\frac{\hat{u}_{it}}{\hat{\sigma}_{it}}$). Finally, Panel D statistics are computed from squaring the standardized residuals of Panel C.

The sample autocorrelations of the series levels and residuals squared are $\rho_x(r)$ and $\rho_{xx}(r)$ respectively; both are asymptotically standard normal random variables. Similarly, the Box-Pierce Portmanteau statistics, $Q_x(r)$ and $Q_{xx}(r)$ are asymptotically χ^2_r random variables under the null hypothesis of independent identically distributed random variables [see McLeod and Li (1983)].

Panel A shows the raw excess return series to be right skewed and leptokurtic as expected. However, comparison with Panel C shows that the model removes virtually all systematic skewness and substantially reduces leptokurtism. Moreover, the standardized residuals appear to be very close to mean zero and unit variance, although there is a slight tendency towards negative residuals or overprediction of the conditional mean.

Comparison of Panel A with Panel C also shows that virtually all autocorrelation in the series levels has been removed. The Box-Pierce Portmanteau statistic further demonstrates that, for all but portfolio 3, autocorrelation in the series levels has been virtually eliminated. It is worth emphasizing that the point of Akgiray's inclusion of autocorrelations is explainable in this pairwise CAPM GARCH-M model without explicitly modelling correlations in the series levels. That is, conditional means are significantly affected by time varying second moments, resulting in autocorrelation in excess returns themselves if considered in isolation. Thus, the correlation estimates found by Akgiray may be due to varying second moments filtering down on first moments.

Panels B and D are included to monitor the performance of the model in fitting second moment dynamics. The unconditional squared residuals of Panel B show large individual correlations causing both $Q_{xx}(12)$ and $Q_{xx}(24)$ to strongly reject at very high significance levels. Panel D demonstrates the ability of the model to remove the time variability in the second moments of the standardized residuals. Comparison of the autocorrelations in Panel D with those in Panel B suggests the model is quite successful. In addition, the $Q_{xx}(12)$ and $Q_{xx}(24)$ statistics show that, for all portfolios except portfolio 2, second moment dynamics have been adequately captured by the model.

The results from Table 1-7 present strong evidence of changing parameters over the period of estimation. To test parameter stability, the entire sample is split into two equal halves. The CAPM version of the model is estimated for each data subset as well as for the entire sample period. The log likelihood for the entire sample (say L_R) is then subtracted from the sum of the log likelihoods for each subset (say $L_{u1} + L_{u2}$) and the sum is multiplied by two. The resultant quantity ξ_{LR} [= $2(L_{u1} + L_{u2} - L_R)$] is distributed as a χ^2 random variable with degrees of freedom given by the number of parameters for each pairwise model. As is seen in Table 1-7, all χ^2 statistics are highly significant and soundly reject the null hypothesis of parameter stability. Given the extreme volatility of the late 1920's and the extremely lengthy series analyzed, this result is not surprising[12].

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[12] Further research will examine a similar model over a shorter calendar time period using weekly data.

VI. CONCLUSIONS

The literature concerning autoregressive conditional heteroscedasticity in asset pricing paradigms appears to have arrived at a conundrum. Univariate models seem to fail in financial contexts because they are difficult to nest within multivariate asset pricing models. Similarly, multivariate models suffer due to the difficulty in parameterizing a time varying covariance matrix. This paper presents an alternative in which a multivariate model is restricted in a manner consistent with current financial thought, yet allows for a rich class of processes for conditional portfolio variances. The approach is computationally feasible and allows information criterion to be used to determine the appropriate order of time series processes.

The results of the research presented herein must be viewed as supportive of time varying moments in asset pricing. The pairwise model demonstrates the importance of variation in excess return moments. In addition, the methodology employed allows for interesting second moment dynamics and yet greatly reduces the number of parameters required in estimation through the assumption of constant correlations.

Diagnostics provide support for the ability of the model to capture the essential elements of excess return moments. However, likelihood ratio tests for changes in model parameters over the estimation period result in rejection. Given the nature of time series models to 'fit' rather than 'explain', this conclusion is not surprising. Pragmatically, this rejection may be handled by using a weekly data set over a shorter period of calendar time[13]. It remains to be seen, however, if weekly data will result in distributions as close to normality as those displayed by the standardized residuals of this data set over monthly data.

The analysis also suggests an apparent resolution of an issue put forth by Akgiray (1989). Akgiray stresses that autocorrelated excess returns violate the assumptions necessary for maximum likelihood procedures which require independent observations. However,

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[13] The lengthy time series of monthly data including the depression years may also exacerbate the problem. Similarly, using a monthly data set over a shorter period of calendar time might resolve the stability issue; however, some of the benefits associated with a large number of observations might also be lost.

diagnostics show that second moment dynamics and the relation between first and second moments in CAPM induces autocorrelation in excess returns although there is no such autocorrelation in the standardized residuals and therefore maximum likelihood procedures are acceptable. An asset pricing paradigm may thus present an alternative method to correct for autocorrelation in raw excess returns instead of explicit estimation of autocorrelations.

The pairwise model employed herein has numerous benefits over the fully generalized constant correlation model [see for example Giovannini and Jorion (1989)]. The use of correlations with the market rather than all possible pairwise correlations results in correlation estimates which are in accordance with unconditional estimates, as desired (more general models lead to larger than expected correlation estimates). In addition, the use of the fully generalized model requires a greater number of parameters, restricted data sets (due to the necessity of defining a weight vector), and tremendous computational expense. The proposed pairwise model produces a nested approach to determine GARCH(p,q)-M lag lengths in an asset pricing context while still maintaining a feasible search procedure.

Empirically, the main difference between the pairwise approach and the fully generalized model is that the generalized multivariate model strongly rejects CAPM while the pairwise approach does not. This lack of rejection may be due to a less powerful procedure or a more careful search methodology over GARCH(p,q)-M lag lengths before testing the asset pricing relations.

Two areas for future research appear worth examination, first, in respecifying the model for different processes such as the Exponential GARCH, and secondly, in theoretically respecifying the asset-pricing paradigm to explicitly consider temporal aggregation issues, and hedging state variables.

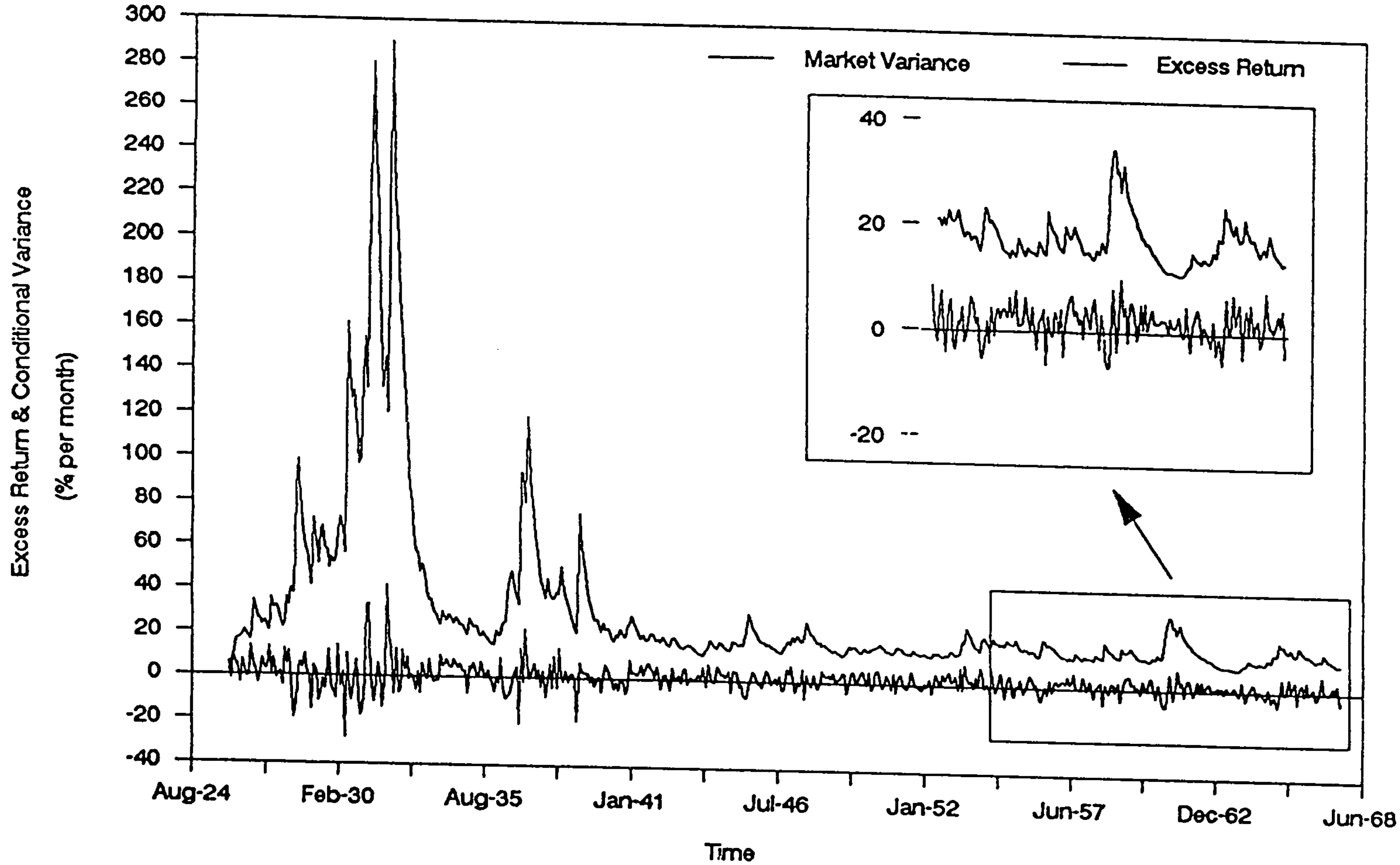

FIGURE 1-1 Conditional Market Variance and Actual Excess Returns for 1926-1966.

Conditional Market variance is based upon the Multivariate GARCH(1,1)-M proces.. Actual excess returns are computed as the difference between nominal monthly returns and the 1 month Treasury bill rate (% per month).

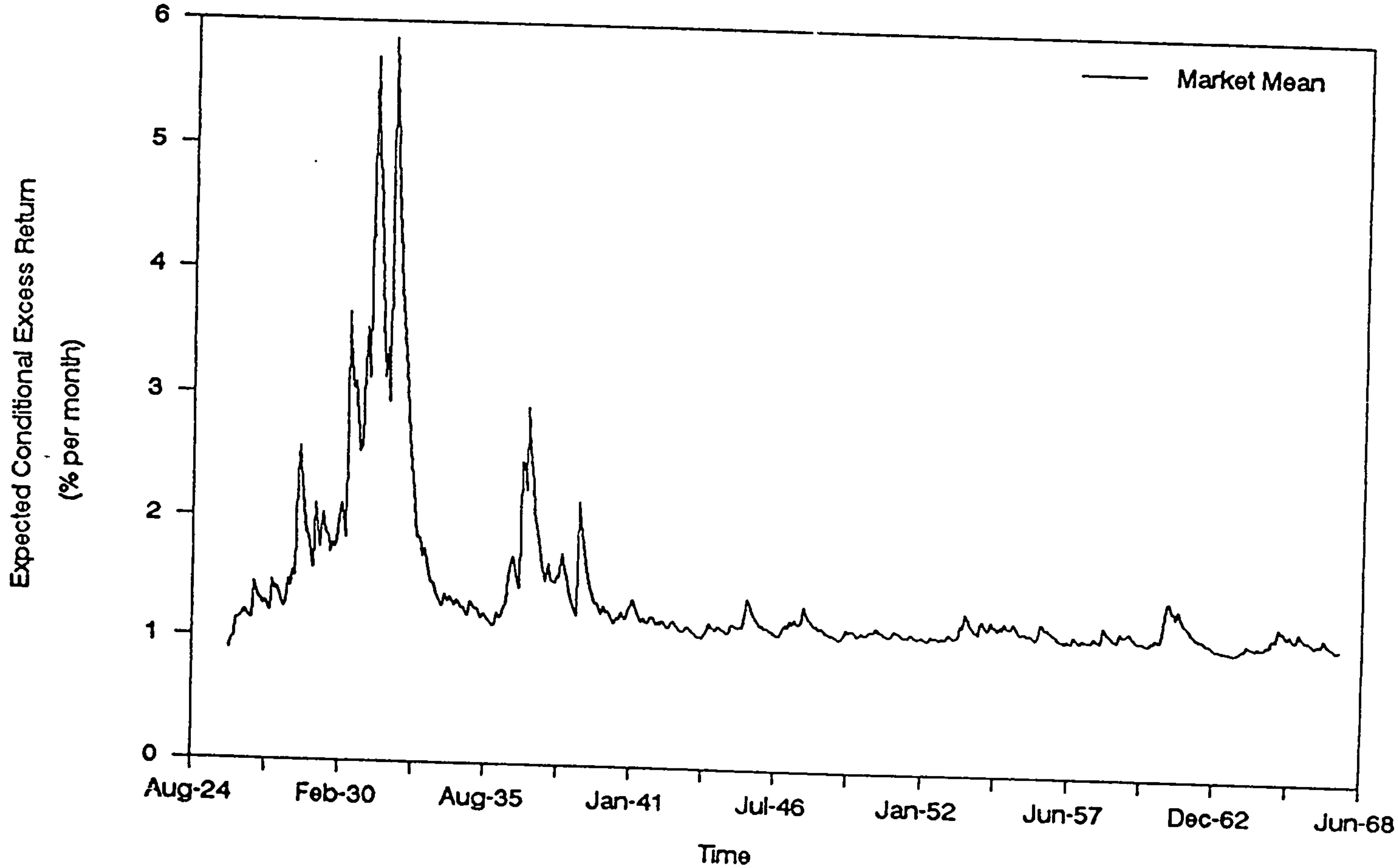

FIGURE 1-2 Conditional Market Excess Returns for the period 1926-1966.

The conditional market excess return is a one-step 'forecast' of the market excess return series based upon the Multivariate GARCH(1,1)-M model parameters fit over the entire sample period (% per month).

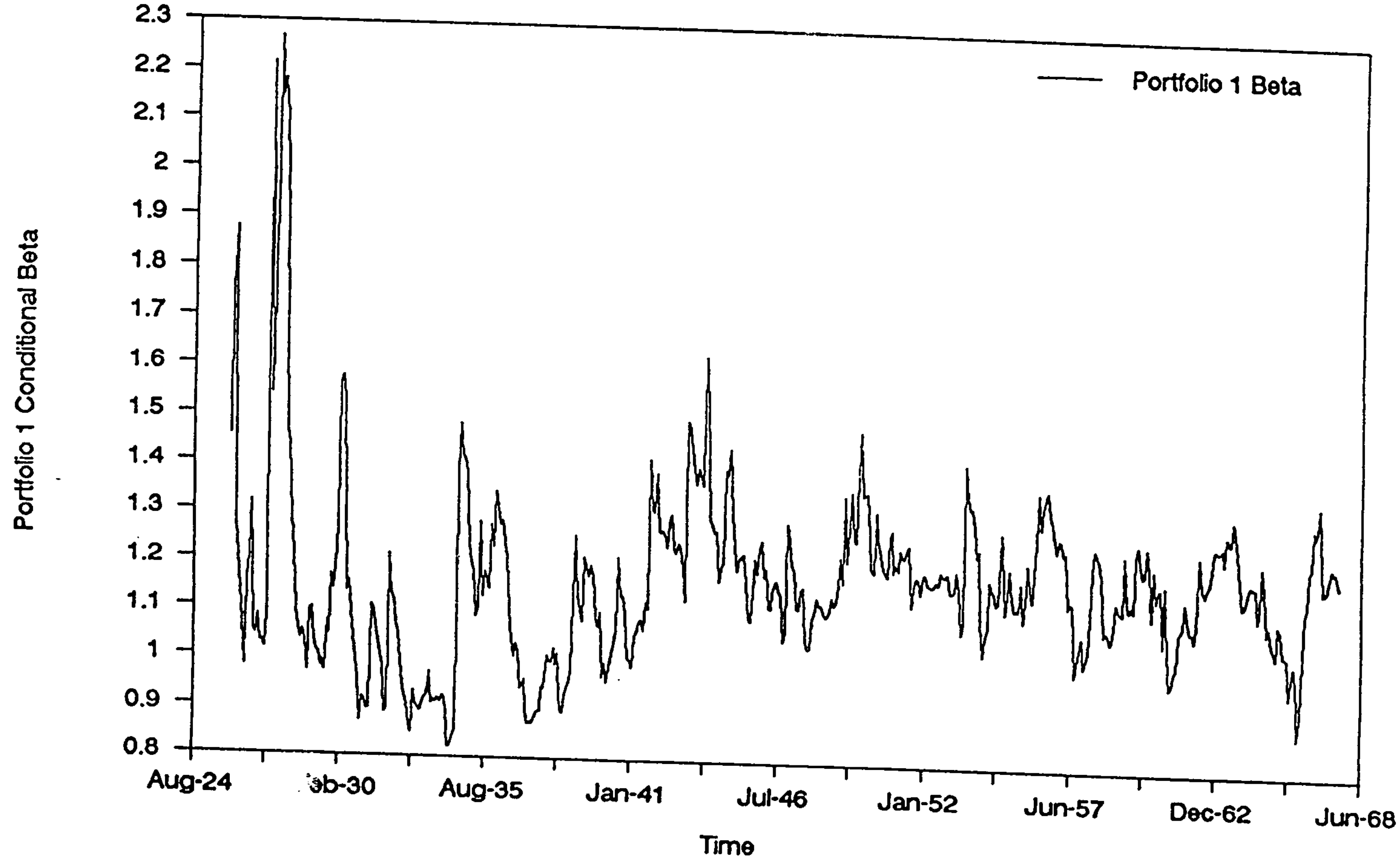

FIGURE 1-3 Conditional Beta Evolution for Portfolio 1 over the period 1926-1966.

Portfolio 1's conditional beta is generated as the product of portfolios 1's correlation with the market times the standard deviation of portfolio 1's excess returns divided by the market excess return's standard deviation.

Table 1-1

Descriptive Statistics for Excess Returns on Standard Industry Code (IC) Portfolios and Market Proxies *

The five IC portfolios are constructed from the first two digits of the underlying stocks' Standard Industry Codes. The IC market index is then constructed as the weighted average of the individual portfolios' excess returns. Weights are time varying and are computed by dividing the total market value of the industry by the total market value across all industries. All excess returns are stated in percent per month form. The Standard and Poor's (SP) 500 index is taken from the Ibbotson and Associates Data base.

STANDARD INDUSTRY CODE PORTFOLIO	MEAN	(% per month) MINIMUM	MAXIMUM	STANDARD DEVIATION
1. Mining and Construction	1.3391	-31.853	66.784	8.4369
2. Manufacturing	1.2679	-30.463	53.259	6.6614
3. Transportation and Public Utilities	.72145	-25.272	32.526	5.0517
4. Wholesale and Retail Trade	1.0300	-31.103	39.750	6.7251
5. Finance, Insurance and Real Estate	.98435	-22.695	35.925	6.0916
IC Market Index	1.1384	-28.322	41.831	5.8963
SP Market Index	.87205	-29.752	42.466	6.4648

* All results are based upon the 500 observations from January 1926 through August 1966. The remaining observations through December 1986 are reserved for forecasting in a paper to follow. Means and standard deviations in the above table are unconditional estimates.

Table 1-2

Estimates of the Univariate GARCH(1,1) Process for the period January 1926-August 1966.

$$\sigma^2_{Mt} = \gamma_M + \beta_M \sigma^2_{Mt-1} + \alpha_M (y_{Mt-1} - b_M)^2$$

The univariate GARCH(1,1) process is estimated for the Standard and Poor's (SP) 500 market index and the Standard Industry Code (IC) market index. All excess returns are stated in % per month form. Asymptotic standard errors are given in parentheses.

	SP 500 Index	IC Index
b_M	1.20593 (.197648)	1.33051 (.192618)
γ_M	1.44873 (.212469)	1.34691 (.189157)
β_M	.795868 (.021097)	.802242 (.019713)
α_M	.167595 (.028518)	.159978 (.026390)
Log-Likelihood =	-1523.609	-1494.700

Table 1-3

Estimates of the Univariate GARCH(1,1)-M Process for the period January 1926 - August 1966

$$\sigma^2_{Mt} = \gamma_M + \beta_M \sigma^2_{Mt-1} + \alpha_M (y_{Mt-1} - b_M - \delta\sigma^2_{Mt-1})^2$$

The univariate GARCH(1,1)-M process is the selected model for the Standard Industry Code (IC) market index based upon Schwartz's Information Criterion (SIC). Akaike's Information Criterion (AIC) is also stated for completeness. The results for the univariate GARCH(1,1)-M are reported for both the IC index and the Standard and Poor's 500 market index. All excess returns are stated in % per month form. Asymptotic standard errors are given in parentheses.

	SP 500 Index	IC Index
b_M	1.063982 (.269359)	1.080205 (.278395)
γ_M	1.465258 (.232272)	1.454126 (.237764)
β_M	.793174 (.021415)	.790872 (.020489)
α_M	.170332 (.029577)	.167062 (.028240)
δ	.005999 (.008499)	.013687 (.009210)
Log-Likelihood =	-1523.4365	-1494.1295

GARCH(p,q)	AIC	SIC
(1,1)	2998.26	3019.33
(1,2)	2994.11	3019.40
(2,1)	2996.55	3021.84
(2,2)	2995.00	3026.50

Table 1-4

Estimates of the Pairwise GARCH(p,q)-M Process for the period January 1926-August 1966.

$$\sigma^2_{it} = \gamma_i + \sum_{j=1}^{p} \beta_{ij}\sigma^2_{it-j} + \sum_{l=1}^{q} \alpha_{il}(y_{it-l} - b_i - \delta\rho_{iM}\sigma_{it-1}\sigma_{Mt-1})^2$$

$$\sigma^2_{Mt} = \gamma_M + \beta_M\sigma^2_{Mt-1} + \alpha_M(y_{Mt-1} - b_M - \delta\sigma^2_{Mt-1})^2$$

Each Pairwise GARCH(p,q)-M model reported is the selected process based upon Schwartz's Information Criterion (SIC) for each of the Standard Industry Code (IC) portfolios against the IC market index. Akaike's Information Criterion (AIC) is also reported for completeness. All excess returns are stated in % per month form. Asymptotic standard errors are given in parentheses.

	PORTFOLIO				
	ONE	TWO	THREE	FOUR	FIVE
b_i	1.1376 (.4127)	.61262 (.3060)	.57358 (.2025)	.67339 (.3314)	.99438 (.3462)
γ_i	5.5653 (.8719)	2.1849 (.2786)	1.0677 (.1555)	2.8700 (.8447)	7.5702 (1.015)
β_{i1}	.76028 (.0197)	.80049 (.0164)	.82556 (.0205)	.03755 (.0395)	.64456 (.0499)
β_{i2}	——— ———	——— ———	——— ———	.73506 (.0380)	——— ———
α_{i1}	.14672 (.0231)	.08095 (.0173)	.11981 (.0196)	.15108 (.0233)	.15688 (.0377)
α_{i2}	——— ———	.05336 (.0130)	——— ———	——— ———	——— ———
b_M	1.0382 (.3099)	.56910 (.2869)	.91817 (.2627)	.79857 (.3163)	1.0204 (.3034)
γ_M	1.5586 (.2862)	1.7438 (.2112)	1.7005 (.2758)	2.2971 (.3242)	1.7704 (.3940)
β_M	.81886 (.0188)	.81608 (.0138)	.81529 (.0219)	.77589 (.0259)	.79900 (.0243)
α_M	.12560 (.0200)	.11815 (.0146)	.12202 (.0212)	.13767 (.0284)	.13782 (.0249)
δ	.00807 (.0121)	.02521 (.0110)	.01779 (.0103)	.02042 (.0107)	.01004 (.0110)
ρ_{iM}	.78294 (.0127)	.98653 (.0012)	.82886 (.0106)	.76644 (.0158)	.67934 (.0202)
Log-Likelihood =	-2948.3816	-2144.2731	-2612.5840	-2859.0925	-2936.7065

Model Selection Statistics *:

GARCH (p,q)	AIC	SIC	AIC	SIC	AIC	SIC	AIC	SIC	AIC	SIC
(1,1)	5917.4	5963.8	4323.2	4369.6	5247.2	5293.5	5750.8	5797.2	5894.9	5941.3
(1,2)	5919.3	5969.8	4309.3	4359.8	5249.2	5299.7	5752.8	5803.4	5895.1	5945.7
(2,1)	5914.6	5965.1	4325.2	4375.8	5248.5	5299.1	5741.7	5792.2	5893.8	5944.3
(2,2)	5911.8	5966.6	4308.9	4363.7	5248.9	5303.7	5741.9	5796.7	5892.5	5947.3

CAPM Test Statistics **:

ξ_{LM} =	2.46083	3.82297	.00536	.77296	.73592
ξ_{LR} =	1.37744	3.28279	.00567	.52646	.50484
ξ_W =	.99284	2.08284	.00015	.21609	.29096

---

* Model selection statistics are based upon the unrestricted models where δ_i and δ_M are not restricted to be equal.

** CAPM statistics test the null hypothesis $H_0 : \delta_i = \delta_M$ versus $H_A : \delta_i \neq \delta_M$. To develop these statistics let $\hat{\theta}$ be the unrestricted parameter vector under the alternative and let θ be the restricted parameter vector under the null. Also let $s(\theta)$ be the score vector, $L(\theta)$ be the likelihood function, let $V(\theta)$ be the covariance matrix of parameters, and let $\mathbf{R}$ be the restriction vector. The Lagrange Multiplier test statistic is then $\xi_{LM} = s(\theta)^T V(\theta) s(\theta)$. The Likelihood Ratio test statistic is $\xi_{LR} = 2[L(\hat{\theta}) - L(\theta)]$. The Wald test statistic is then $\xi_W = (R\hat{\theta})^T [RV(\hat{\theta})R^T]^{-1} (R\hat{\theta})$. All test statistics have a limiting χ^2 distribution with 1 degree of freedom.

Table 1-5

Estimates of the General Multivariate GARCH(1,1)-M Process for the period January 1926-August 1966

$$\sigma_{it}^2 = \gamma_i + \beta_i \sigma_{it-1}^2 + \alpha_i \left[y_{it-1} - b_i - \delta \sum_{j=1}^{N} \omega_{jt-1} \rho_{ij} \sigma_{it-1} \sigma_{jt-1} \right]^2$$

$$\sigma_{ijt} = \rho_{ij} \sigma_{it} \sigma_{jt}$$

$$H_t = (\sigma_{ijt})$$

The General Multivariate GARCH(1,1)-M process estimates all model parameters simultaneously with the exception of the pairwise correlation estimates. Unconditional correlation estimates are reported below. All excess returns are stated in % per month form. Asymptotic standard errors are given in parenthesis for all parameters except correlations.

	PORTFOLIO				
	ONE	TWO	THREE	FOUR	FIVE
b_i	.6732 (.3398)	.8223 (.2247)	.4138 (.1740)	.5708 (.2864)	.6090 (.3057)
γ_i	1.777 (.5774)	.5733 (.1305)	.3524 (.0706)	1.212 (.2345)	3.192 (.6509)
β_i	.8074 (.0358)	.8414 (.0247)	.8479 (.0243)	.8481 (.0227)	.7467 (.0463)
α_i	.0871 (.0178)	.0444 (.0089)	.0473 (.0090)	.0452 (.0110)	.0878 (.0238)
δ	.042915 (.013983)				
ρ_{ij}		.82515	.71554	.67295	.63795
			.80975	.79912	.68377
				.73774	.61818
					.63067

Log of the Likelihood = -7002.23

Note: A simple t-test of $H_0: \rho_{ij} = 0$ results in rejection for all correlations at the 1% level.

Table 1-6

Diagnostic Summary Statistics for Pairwise Garch(p,q) Models for the period January 1926-August 1966

Portfolio excess returns are as described in Table 1-1 and are reproduced below along with autocorrelations and Box-Pierce Portmanteau χ^2 statistics. Unconditional squared residuals are simply computed as actual excess returns less their respective unconditional means. Standardized residuals are computed as $\frac{\hat{u}_{it}}{\hat{\sigma}_{it}}$ where $\hat{u}_{it}$ is the residual from the selected pairwise GARCH(p,q)-M model under the null restriction and $\hat{\sigma}_{it}$ is the respective conditional variance estimate. The Box-Pierce Portmanteau statistic is distributed as a χ^2 random variable with degrees of freedom given by the number of autocorrelations evaluated.

	PORTFOLIO					STANDARD INDUSTRY CODE MARKET INDEX
	ONE	TWO	THREE	FOUR	FIVE	
PANEL A. EXCESS RETURNS						
Mean	1.339	1.268	0.721	1.030	0.984	1.138
Variance	71.182	44.374	25.519	45.227	37.108	34.766
Skewness	1.529	1.048	0.385	0.275	0.265	0.678
Kurtosis	11.556	11.614	7.135	5.751	2.865	9.393
First five Autocorrelations						
1	0.07	0.12	0.16	0.06	0.00	0.11
2	0.02	-0.02	-0.01	-0.05	0.00	-0.02
3	-0.12	-0.13	-0.15	-0.07	-0.04	-0.15
4	0.07	0.06	-0.01	-0.01	0.04	0.05
5	0.04	0.04	0.05	0.06	0.04	0.05
Box-Pierce Portmanteau Test ξ						
$Q_x(12)$	28.51**	35.66**	47.51**	39.14**	14.11**	38.86**
$Q_{xx}(24)$	58.21**	63.39**	75.77**	63.13**	32.69**	65.82**

PANEL B. UNCONDITIONAL SQUARED RESIDUALS						
Mean	71.039	44.286	25.468	45.136	37.034	34.697
First five Autocorrelations						
1	0.15	0.20	0.35	0.28	0.14	0.24
2	0.11	0.11	0.20	0.29	0.09	0.15
3	0.12	0.16	0.26	0.22	0.14	0.20
4	0.09	0.13	0.18	0.07	0.09	0.13
5	0.01	0.03	0.14	0.05	0.05	0.05
Box-Pierce Portmanteau Test						
$Q_x(12)$	144.21**	186.95**	359.12**	357.62**	85.84**	251.32**
$Q_{xx}(24)$	193.10**	247.31**	432.04**	423.27**	111.09**	319.58**

PANEL C. STANDARDIZED RESIDUALS						
Mean	-0.004	-0.003	-0.034	-0.007	-0.027	-0.058
Variance	1.001	1.006	1.001	1.007	1.002	1.000
Skewness	0.685	-0.071	-0.032	0.103	0.224	-0.246
Kurtosis	3.636	1.729	3.251	3.956	2.276	2.010
First five Autocorrelations						
1	0.03	0.05	0.09	0.01	-0.03	0.03
2	0.02	0.04	0.10	0.02	0.02	0.05
3	-0.03	-0.03	0.01	0.01	-0.01	-0.01
4	0.06	0.08	0.07	0.08	0.04	0.08
5	0.05	0.06	0.07	0.07	0.04	0.07
Box-Pierce Portmanteau Test						
$Q_x(12)$	11.78	15.83	26.57**	16.24	15.47	15.88
$Q_{xx}(24)$	28.36	26.94	37.08*	25.60	31.40	27.42

PANEL D. SQUARED STANDARDIZED RESIDUALS						
Mean	0.999	1.004	1.000	1.076	1.001	1.002
First five Autocorrelations						
1	0.03	-0.02	0.02	0.00	-0.01	-0.04
2	0.00	0.04	0.00	0.07	-0.04	0.02
3	0.03	0.03	0.02	0.01	0.02	0.00
4	-0.04	0.00	0.01	0.03	0.01	-0.01
5	-0.04	-0.01	0.00	-0.02	-0.01	-0.02
Box-Pierce Portmanteau Test						
$Q_x(12)$	12.05	27.67**	17.69	8.50	6.97	13.82
$Q_{xx}(24)$	35.74	53.15**	34.19	18.85	19.22	34.47

ξ ** Significant at the 1% level.
* Significant at the 5% level.

Table 1-7

Likelihood Ratio Tests for Parameter Stability for each of the Pairwise Garch(p,q) Models for the period January 1926-August 1966

Parameter stability for each of the selected Pairwise GARCH(p,q) models from Table 1-4 is considered using a Likelihood Ratio test statistic. Each Likelihood Ratio statistic ξ_{LR} is asymptotically distributed as a χ^2_r random variable, where r is the number of parameters allowed to vary over the subsample periods.*

	PORTFOLIO				
	ONE	TWO	THREE	FOUR	FIVE
Selected GARCH(p,q) Model	(1,1)	(1,2)	(1,1)	(2,1)	(1,1)
r	10	11	10	11	10
ξ_{LR} ζ =	85.075	59.476	55.180	40.101	60.056

* Each Likelihood Ratio statistic tests the null hypothesis H_0: $\theta_1=\theta_2$ versus H_A :$\theta_1 \neq \theta_2$ where θ_1 is the parameter vector for the first half of the sample, and θ_2 is the parameter vector for the second half of the sample period. To develop these statistics let $\hat{\theta} = \left(\hat{\theta}_1^T \ \hat{\theta}_2^T\right)^T$ be the unrestricted parameter under the alternative and let $\tilde{\theta}$ be the restricted parameter vector under the null. Letting $L(\theta)$ be the likelihood function gives $\xi_{LR} = 2[L(\hat{\theta}) - L(\tilde{\theta})] \overset{a}{\sim} \chi^2_r$ where r is the number of restrictions.

ζ ** Significant at the 1% level.

CHAPTER TWO:
AN INTERTEMPORAL ASSET PRICING MODEL WITH TIME VARYING MOMENTS

Harry J. Turtle

A vast amount of literature has dealt with appropriately specifying conditional asset pricing relations in terms of both conditional and unconditional asset moments. Previous research has handled the issue of time varying moments in one of two manners. First, some researchers have been content to use estimation techniques which are consistent in spite of time varying moments although they may result in less powerful test statistics[14]. Others have postulated a statistical framework which explicitly models the time varying nature of asset return moments[15]. This paper stresses the use of conditional moments in the hopes of maintaining powerful test statistics.[16] Although many paradigms in the literature have used time varying conditional moments, little work has simultaneously extended the static two-period asset pricing paradigm into a more general intertemporal framework. This paper examines an Intertemporal Capital Asset Pricing Model (ICAPM) in the context of time varying conditional moments.

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[14] A good example of this is Shanken's (1990) use of heteroscedasticity-consistent standard errors in multiple regressions to develop asset pricing test statistics. Shanken uses a methodology adapted from Jobson (1982) and Jobson and Korkie (1982). Shanken's methodology is similar to this paper; however, there are important differences. First, as discussed above, Shanken does not explicitly model changing variances to construct his asset pricing test statistics and thus his resultant statistics are likely understated. In fact, in conclusion Shanken suggests using a multivariate GARCH process for residuals in future research. Secondly, Shanken develops his test statistics from intercept restrictions on a series of multiple regressions. Although this approach is simple, intuitive, and does not require the assumption of a constant price of market or bond risk, Shanken implicitly assumes agents can perfectly forecast excess returns (see footnote 4 for further discussion and clarification of this point). Finally, all of Shanken's test results are based upon intercept restrictions and thus may result in rejection solely due to misspecification of the slope relations.

[15] See for example Engle, Lillien and Robbins (1987), Bollerslev, Engle and Woolridge (1988), Giovannini and Jorion (1989), or Engel and Rodrigues (1989).

[16] It is important to note that, although the approach of modelling time varying moments can increase the power of tests, it can also lead to inappropriate tests if the statistical process for moment evolution is not suitable. This paper attempts to mitigate these concerns by first selecting an appropriate process using Schwartz's Information Criterion [see de Gooijer, Abraham, Gould, and Robinson (1985)]. Previous papers which simply impose an ad hoc process on moment evolution may result in misspecified models and test statistics.

One body of literature exploring the use of conditional moments in asset pricing involves the use of GARCH-M (Generalized AutoRegressive Conditional Heteroscedastic in Mean) processes. These papers nest static versions of CAPM within mean relations and test the restrictions implied [see for example Giovannini and Jorion (1989), or Engel and Rodrigues (1989)]. However, nesting static models in a setting of dynamic moments is only appropriate under restrictive assumptions [see Fama (1970), Merton (1971 and 1973), or Rubinstein (1976), for a complete discussion]. Meaningful mean relations must include terms which provide for hedging against unfavourable shifts in the investment opportunity set. Ignoring the intertemporal aspect of asset pricing while simultaneously modelling time-varying covariances may misspecify the framework for analysis.

Two specific formulations are posited for conditional asset variances. The first is intended in the spirit of the single state variable world of Merton (1973). In this case, changes in the one week riskless Treasury Bill rate are used as a state variable to drive all changes in the investment opportunity set. The second formulation is also consistent with Merton's analysis; but conditional variances are now functions of previous conditional variances and squared residuals. The motivation for the latter approach is to fit, rather than model in a strict economic sense, changes in the investment opportunity set.

The remainder of the paper is organized as follows. In section I the basic model is motivated and discussed in detail. The data, estimation methodology, and primary empirical results of the paper are presented in section II. In section III, diagnostic analysis is performed to verify the integrity of the statistical assumptions as well as to ascertain the success of the model in capturing moment dynamics. Concluding comments are offered in section IV.

I. THE MODEL

Merton's (1973) Intertemporal Capital Asset Pricing Model (ICAPM) is employed to develop mean relations. Merton's analysis allows researchers to relax the typical normality assumptions with the somewhat weaker notion of continuous sample price paths. Locally, then, the natural logarithm of gross returns will be normally distributed. In addition to the typical market risk term of the static CAPM, ICAPM posits risk terms which result from investors hedging against unfavourable shifts in the investment opportunity set.

The general framework considers an agent maximizing expected lifetime utility over consumption and portfolio choices. Merton (1973) solves the problem generally under numerous state variables and subsequently derives a relatively simple solution for the case in which changes in the investment opportunity set are driven by a single state variable. Appendix One replicates the relevant work of Merton and specializes the analysis to the case of constant covariance risk premiums for both the market (M) and long term bond portfolio (B)[17].

The choice of Merton's ICAPM in conjunction with constant risk premiums is motivated by a desire to model moment dynamics in a manner consistent with economic theory. To meaningfully model time varying moments requires restrictions on beliefs or tastes[18]. This paper combines both types of restrictions. Tastes are restricted to satisfy constant relative risk aversion as discussed in Appendix One. Moreover, correlations between all assets are assumed to be constant over time. This implies all time variation in covariances is due to changing

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[17] Shanken criticizes this constancy assumption; however, he fails to point out the effects of assuming perfect market expectations in his framework. Specifically, Shanken posits stock returns can be written as a constant, plus a time varying 'market beta' multiplied by the actual market excess return, plus a 'bond beta' times the actual bond portfolio excess return. Using the notation developed below and in Appendix One,

$$y_{jt} = b_{jot} + b_{j1t}*y_{Mt} + b_{j2t}*y_{Bt} + u_{jt}.$$

Although this approach is intuitive it is not a strict representation of the asset pricing relation based upon conditional expectations. That is, the use of actual excess returns, if strictly interpreted, implies that agents can perfectly forecast market and bond excess returns (y_{Mt} and y_{Bt} respectively). If this were not the case, the model should be posited with conditional expectations of the market excess return and bond excess return. If the market and bond excess returns in Shanken's model are partitioned into expected and unexpected components, notice the residual term is equal to u_{it} as above plus the time varying parameters (b_{j1t} and b_{j2t}) multiplied by the unexpected portion of market and bond excess returns. As an aside, it should be noted that this criticism arises largely because the parameters in Shanken's model are time varying. Traditional studies using constant parameters simply add greater noise to residuals by ignoring the expectation problem.

[18] Rubinstein (1976) first discussed the tradeoff between restrictive beliefs (about subjective probability distributions) and tastes (utility functions) in micro-economic models of uncertainty. This paper restricts utility functions in an effort to maintain the critical links between the asset pricing relation and economic thought.

variances and not changing correlations. Constancy of correlations has been extensively studied by Elton, and Gruber (1973), Elton, Gruber, and Padberg (1976, 1977a and 1977b), Elton, Gruber and Urich (1978), and Gibbons (1986) with the finding that correlations are relatively constant, especially over relatively short intervals such as 5 years. Gibbons (1986) also presents evidence that the nonstationarity in asset return covariances is due to changing variances, not correlations. The assumption of a constant coefficient of relative risk aversion is supported by empirical evidence in Fama and MacBeth (1973) as well as theoretically by Mossin (1968), Cass and Stiglitz (1970), Leland (1972), and Hakansson (1974). The assumption of constant relative risk aversion has intuitive appeal in this model because it allows second moments to vary over time while maintaining constancy of tastes over the estimation period.

Using the assumptions discussed above, along with the development in Appendix One, results in the basic mean relation for all assets,

$$y_{it} = \delta_1 \left[\frac{\rho_{iM} - \rho_{iB}\rho_{BM}}{1-\rho_{BM}^2} \right] \sigma_{it}\sigma_{Mt} + \delta_2 \left[\frac{\rho_{iB} - \rho_{iM}\rho_{BM}}{1-\rho_{BM}^2} \right] \sigma_{it}\sigma_{Bt} + u_{it} \tag{1}$$

where y_{it} is the excess return on security i, ρ_{ij} is the correlation between asset i and j assumed constant by construction, σ_{ijt} is the time varying covariance between asset i and j, constructed as, $\sigma_{ijt} = \rho_{ij}\,\sigma_{it}\sigma_{jt}$, and u_{it} is a mean zero normal variate with time varying conditional variance, σ_{it}^2. The vector of excess return residuals, $u = \{ u_{it}\}$, is distributed as a multivariate normal random variable with a zero mean vector and time varying covariance matrix given by $\{ \sigma_{ijt}\}$.

The parameters δ_1 and δ_2 are developed from the relations,

$$\delta_{1t} \equiv \frac{(\alpha_{Mt} - r_{ft})}{\sigma_{Mt}^2} \text{ and } \delta_{2t} \equiv \frac{(\alpha_{Bt} - r_{ft})}{\sigma_{Bt}^2}.$$

Now, as developed in Appendix One, using the assumptions of constant relative risk aversion and constant correlations between assets allows us to write $\delta_{1t} \equiv \delta_1$ and $\delta_{2t} \equiv \delta_2$. The intuition behind these parameterizations is simply that the reward per unit of risk is constant over time. This particular parameterization is beneficial as it allows for straightforward interpretation of estimated coefficients. Specifically, the magnitude of the parameters gives a direct interpretation of the importance of the different risk prices. The price of market and bond risk are also affected by the correlation between market and bond portfolio returns, but this parameterization allows a categorization of the differences between naive reward to risk tradeoffs and the effects of correlations.

To close the system as specified in (1) requires a method of parameterizing the conditionally nonstationary variances. Two different specifications for variance evolution are suggested to close the system. Both processes assume period t's conditional variance is a deterministic function of the information set available to investors at t. The first process is developed under the assumption that all changes in the investment opportunity set are driven by changes in the riskless Treasury Bill rate (the single state variable). The premise of this parameterization is that unanticipated changes in the short term Treasury Bill rate are the only state variable of importance to investors. In the second case, excess return variances are fit as GARCH processes. The GARCH methodology attempts to model variances as deterministic functions of previous periods' squared residuals and conditional variances[19]. This approach presumes a myriad of factors may affect conditional variances and thus, the best way to model variances may be to use information implicit in previous own variances and previous own squared disturbances.

[19] Mandelbrot (1963, and 1966) first considered the notion that periods of instability tend to persist. Thus large (small) shocks tend to be followed by further large (small) shocks of either sign.

Regardless of which variance evolution method is chosen, it is clear that the mean relation to be subsequently tested will be affected by the choice in variance processes. That is, the ICAPM mean relation given in equation (1) is clearly dependent upon the selected GARCH variance processes, as it posits asset returns are functions of second moments. Therefore, the choice of the variance processes must be general enough to capture the essential aspects of variance evolution. If the variance structure is not appropriately fit , misspecified mean relations may result[20]. Thus, it is critical to choose processes which capture variance evolution without overparameterizing the model. To determine the appropriate lag lengths in the variance models discussed below, Schwartz's Information Criterion (SIC) is employed.

The entire research methodology is complicated by the fact that all parameters should be jointly estimated. This means that the variance process lag lengths are determined simultaneously with the parameters in (1). Unfortunately, the parameters in (1) are the essential elements of the subsequent asset pricing tests. Therefore the methodology must take care not to find a variance process which is only optimal conditional upon the ICAPM mean relation (1). If the search procedure was blindly performed under the mean relation (1), then subsequent tests would tend to find (1) to be less restrictive than if the search occurred under another mean relation. However, ignoring the mean relation when searching for variances results in a misspecified search procedure. To mitigate problems due to the mean relation confounding subsequent tests, (1) is relaxed to allow parameters to deviate from ICAPM as follows,

$$y_{it} = b_i + d_{1i}\sigma_{it}\sigma_{Mt} + d_{2i}\sigma_{it}\sigma_{Bt} + u_{it} \qquad (1^*)$$

This approach allows a variance process search over GARCH parameterizations without imposing the ICAPM mean relation (1). This generalized mean relation is restricted in the sections which follow to incorporate additional economic information about the ICAPM mean

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20 Engel and Rodrigues (1989), Pagan (1984), and Pagan and Ullah (1988) present evidence that if the covariance matrix is misspecified, estimates of the mean relation [such as (1)] may be inconsistent.

relation for the bond and market portfolio of excess returns and to avoid problems due to nonconvergence in (sub)systems of numerous variates.[21] The essential problem with nonconvergence is apparent from (1*). By not imposing any restrictions on the parameters across equations, the search procedure chooses b_i, d_{1i}, and d_{2i} values for each variate which maximizes the likelihood in each (sub)system. As an example, consider the bivariate system of the market and bond portfolio using (1) and (1*). Under the unrestricted (1*), 6 mean parameters must be estimated; however, using the restricted ICAPM relation (1), only 2 mean parameters need be estimated. The restrictions on (1*) in the sections which follow will be carefully discussed as they are implemented.

A. The Case of a Single State Variable

In the case of a single state variable, the variance specification is a function of unexpected changes in the state. Expected changes in the state variable are removed using a time series approach explained below. The sole use of unexpected changes to drive variances implies, for example, that large expected deviations in changes of the riskless Treasury Bill rate will not make the economy more uncertain, or forecasts more difficult[22]. Thus, the variance specification for the state variable only models unexpected shocks to the economy.

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[21] The mean relation in (1*) results in nonconvergence difficulties in the largest subsystems. To overcome this difficulty the alternative mean relation (1*) is restricted to exclude the intercept term for all variates. Notice , there are still (N-1)*2 restrictions imposed upon the slope coefficients, where N is the number of variates in the system.

[22] Statistically, to show that any variate X depends solely on unexpected changes in Y requires more care. A sufficient set of assumptions to show,

$$E(X|Y) = E[X|E(Y)+D(Y)] = E[X|D(Y)]$$

where Y is partitioned into an expected component, E(Y), and a deviation from expectation, D(Y).

are that X and E(Y) are independent, that E(Y) and D(Y) are independent, and that X, E(Y), and D(Y) are jointly normally distributed.

The first assumption is treated as a behavioural assumption, and motivated by economic arguments. Specifically, new information may only arrive through unanticipated shocks. Thus, changes in all moments of X may only be related to unanticipated shocks to Y. The latter two assumptions follow from the methodology employed in the paper.

The analogous result for variances follows without loss of generality by defining $Z = [X-E(X|D(Y))]^2$ and considering $E[Z|Y] = E[Z|E(Y)+D(Y)]$.

To further examine the motivation for only considering unexpected changes in the state variable, consider equation (1). Using the arguments set forth in Appendix One, and a desire to maintain a relatively parsimonious modelling construct, the parameters representing market and bond risk prices, $\delta_1 \left(\equiv \frac{(\alpha_{Mt} - r_{ft})}{\sigma^2_{Mt}} \right)$ and $\delta_2 \left(\equiv \frac{(\alpha_{Bt} - r_{ft})}{\sigma^2_{Bt}} \right)$ respectively, are assumed constant.

Essentially this requires that the reward to risk tradeoff for both of these types of risk (cet. par.) remains constant over the period of estimation. That is, changes in the state variable may impact expected market return and conditional market risk, but it is assumed the impact on the ratio of expected excess return to risk is constant.

To operationalize this notion, both the conditional mean and variance of the state variable are jointly estimated with the variates of concern in any particular (sub)system[23]. Now define the state variable, τ_t, as the first difference of Treasury Bill returns, $r_{ft} - r_{ft-1}$, and write the mean relation for the state variable using a Box-Jenkins (1976) type relation,

$$\tau_t \equiv r_{ft} - r_{ft-1} = b_f + \sum_{j=1}^{J} \phi_j (r_{ft-j} - r_{ft-1-j}) + \sum_{k=1}^{K} \theta_k (u_{ft-k}) + u_{ft} , \qquad (2)$$

where b_f is the constant portion of the state variable, ϕ_j is the impact of the state variable of j periods' past upon the current state variable, and θ_k is the impact of unanticipated changes in the state variable k periods' past. Because the state variable is the first difference of short term Treasury Bill returns, a positive b_f represents an upward trend in short rates. The other coefficients used to model state dynamics are intended to capture the tendency for changes in short rates to be functions of previous short rate changes ($r_{ft-j} - r_{ft-1-j}$) and previous deviations

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[23] In a related paper which tests a static version of CAPM, Engel and Rodrigues (1989) model state variables as ARIMA processes and then square the resultant residuals to use as regressors in a Multivariate ARCH specification. This methodology has two serious drawbacks. First, the use of generated regressors will likely cause all test statistics to be overstated (see Murphy and Topel (1985) for a discussion of the impact of using generated regressors from an auxiliary econometric model as though they were true random variables of interest). Secondly, this formulation has the unpalatable implication that although the excess returns are conditionally heteroscedastic, the state variable is homoscedastic (since it is fit using an ARIMA methodology).

from expectations of short rate changes (u_{ft-k}). The term u_{ft} is a zero mean conditionally normally distributed random disturbance term with variance given by the GARCH process,

$$\sigma_{ft}^2 = \gamma_f + \sum_{p=1}^{Pf} \beta_{fp} \sigma_{ft-p}^2 + \sum_{q=1}^{Qf} \alpha_{fq} u_{ft-q}^2 \tag{3}$$

where γ_f represents the constant portion of conditional variance, β_{fp} represents the portion of this period's variance attributed to the conditional variance p periods ago, and α_{fq} represents the impact of squared disturbances q periods ago from the mean relation specified in (2)[24]. Equations (2) and (3) are intended to model the first and second moment dynamics for the single state variable in the model. It is important to recognize that this specification is used to model the evolution of the state variable to separate expected changes in the state variable, $b_f + \sum_{j=1}^{J} \phi_j (r_{ft-j} - r_{ft-1-j}) + \sum_{k=1}^{K} \theta_k u_{ft-k}$, from unexpected changes, u_{ft}. Unexpected changes are then used to model unexpected state variable shocks through the variance specification (3).

For all versions of the single state variable model, equations (2) and (3) are used to describe the evolution of the state variable with the other variates in the model. The novelty of the single state variable model is that second moments for all other variates in the system are functions of unexpected changes in the state variable, u_{ft}, through σ_{ft-p}^2 and u_{ft-q}^2. Specifically, the variance of the excess return, y_{it} is written as,

$$\sigma_{it}^2 = \gamma_i + \sum_{p=1}^{Pi} \beta_{ip} \sigma_{ft-p}^2 + \sum_{q=1}^{Qi} \alpha_{iq} u_{ft-q}^2 . \tag{4}$$

It is important to note that the variance specification of asset i is solely a function of previous values of the single state variable's conditional variances, σ_{ft-p}^2, and residuals squared,

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24 The model development to this point uses unobservable disturbances (u_{it}). Implementation of this specification requires the use of model residuals; that is, observable ex post realizations of differences between model expectations and sample data. The empirics of the paper use residuals in place of unobservable disturbances.

u^2_{ft-q}, not its own previous variances, σ^2_{it-p}, and residuals squared, u^2_{it-q}. The traditional GARCH process discussed in section B, models changes in variances, and hence the opportunity set, through inclusion of previous own variances, σ^2_{it-p}, and own residuals squared, u^2_{it-q}.

To integrate the traditional GARCH methodology with the state variable approach, a researcher might simply add terms in own variances, σ^2_{it-k}, and own squared residuals, u^2_{it-l}, to equation (4) for each excess return variance specification. Unfortunately, this approach would have at least two serious implications. First, from a pragmatic point of view, the dimensionality of the search over lag lengths would be in two additional directions for each variance function. [illegible]y, the parameter estimates would not easily lend themselves to interpretation due to [illegible] between regressors and the combination of two inherently distinct methodologies. To [illegible], suppose own previous variances and squared residuals are included and result in significant parameter estimates, but insignificant parameters result for unexpected state shocks. Own regressors might then be claimed to efficiently capture variance dynamics; unfortunat little could be said about state shocks. The idea behind the GARCH time series approach is to statistically fit variances using previous own variances and squared residuals, assuming the relationship will remain stable over time. Unfortunately, previous own variances and own squared residuals are likely to be highly related to previous state variable shocks. This collinearity between regressors may result in serious difficulties in obtaining significance of state variable shock parameters.

B. The Case of Traditional GARCH Variances

The GARCH variance process writes all variances, σ_{it}^2, as functions of past variances, σ_{it-p}^2, and squared residuals, u_{it-q}^2. That is, all versions of the GARCH formulation for variances use the mean relation specified in (1) or (1*), in conjunction with the variance process (5),

$$\sigma_{it}^2 = \gamma_i + \sum_{p=1}^{Pi} \beta_{ip} \sigma_{it-p}^2 + \sum_{q=1}^{Qi} \alpha_{iq} u_{it-q}^2 . \quad (5)$$

The intuition underlying the GARCH formulation of the model is that a single state variable may have difficulty explicitly modelling changing variances. Thus, the inclusion of past conditional variances and squared residuals in the variance equation are intended to 'fit' variances, rather than strictly modelling variances in an economic sense. This approach is consistent with the time series philosophy.

Both of the above specifications model moment evolution in a manner consistent with Merton's (1973) ICAPM, and simultaneously allow variances to time vary and impact upon the mean. The next section discusses how the appropriate lags are determined for all of the statistical processes given in equations (2), (3), (4) and (5). The ICAPM tests statistics are then carefully detailed.

II. DATA, METHODOLOGY, AND EMPIRICAL RESULTS

Weekly returns for the period July 14, 1983 to December 15, 1989 are studied. Stock return data is obtained from the Center for Research in Security Prices (CRSP) at the University of Chicago. Weekly stock returns are computed by compounding the total daily returns reported by CRSP from Thursday close until Thursday close. Bond returns and Treasury Bill rates are obtained from Reuters Canada. All data sources are fully described in Appendix Two. The decision to examine weekly returns is motivated by the desire to avoid weekend effects, while maintaining a reasonably short holding period for testing a continuous time model. Because effective weekly returns display more skewness than similar effective monthly returns, and because the continuous time asset pricing theory is developed using continuously compounded returns, 100 times the natural logarithm of 1 plus the effective return is examined to preserve near normality of returns.

Table 2-1 presents descriptive statistics for the variables examined. The first interesting fact to note from the table is that the continuously compounded Treasury Bill return is highly nonstationary. The slow decay of the sample autocorrelations shown in the table, as well as the omitted autocorrelations which continue to decay slowly, demonstrate clear nonstationarity in mean. First differencing of this series appears to result in a stationary series for analysis.

One method of considering whether the first K autocorrelations as a group display significant differences from zero is to use the Box-Pierce Portmanteau test statistic,

$$Q(K) = T \sum_{l=1}^{K} \rho_l^2 ,$$

where T is the number of observations in the series being considered, ρ_l^2 is the squared sample autocorrelation at lag l, and K is the number of autocorrelations being considered in the statistic. This statistic is distributed as a χ^2 random variable with degrees of freedom given by the number of sample autocorrelations considered less the number of parameters estimated[25].

The Q(36) statistic shows the initial 36 autocorrelations for bond returns, market returns, and the first difference of Treasury Bill returns are significantly different from zero when considered as a whole. The market portfolio presents the weakest evidence of time variability in means as it only rejects at the 5% level. Moreover, none of the autocorrelations for the first four lags suggests serious time variability in means for the market portfolio. Both the bond portfolio

[25] The notation $Q \equiv Q_X$ is introduced later in the paper to denote the fact that the statistic relates to the first moment of the series considered. Q_{XX} is introduced to consider time variability in second moments in the diagnostics section.

and the first difference of Treasury Bill returns indicate strong evidence of changing conditional means according to both the individual sample autocorrelations and the Q statistic.

Another interesting result from the table is the pattern of long term bond portfolio returns over the sample period. Notice the long term bond portfolio performed extremely well over the sample period ex post, relative to the value weighted market portfolio. Moreover, the standard deviation for the long term bond portfolio was much less than for the market portfolio. The likely cause of this was the heightened interest rates of the early eighties. As the general rate of interest subsequently fell, long term bondholders realized generous capital gains. The pricing of interest rate risk after these events is of particular concern as investors were made cognizant of its effects.

Because the mean and variance specifications in section I are very general, they are difficult to estimate for even moderate numbers of variates. To overcome this problem, subsets of the total set of assets are initially considered to determine the number of lags to be included in each of the statistical processes. Specifically, in the case of a single state variable, the lag lengths for both the mean and variance of the state variable are first determined by searching over alternative lengths for mean and variance processes. Having determined these lengths, the system is enlarged to include the market portfolio and the appropriate lag lengths for the market portfolio variance process are established, conditional on the previously determined lag lengths for the state variable process. The lag lengths for the bond portfolio variance process are then established by jointly estimating the bond and state variable processes. Next, the system of the state variable, bond portfolio, and market portfolio are jointly estimated using the lag lengths found earlier. This procedure is employed to generate a computationally feasible search procedure. The determined lag lengths will be consistent although not efficient; however, the final parameter estimates are efficient given the lag lengths chosen. That is, this subset procedure entails numerous searches over relatively few dimensions, rather than a single search over a large number of dimensions. The specifics of the procedure are discussed below.

The case of traditional GARCH variance processes is similar to the single state process except the state variable process is excluded from the analysis. Specifically, the market is first analyzed as a univariate process to determine variance lag lengths. Next, the bond portfolio variance process is established in isolation. The joint bond and market subsystem is then estimated using the lag lengths established in the univariate systems. Finally, each of the size portfolios is added to the bond and market subsytem to determine and efficiently estimate parameter estimates in a system of three variates. For convenience the following terminology is employed: univariate analysis refers to the consideration of one variate in isolation, pairwise or bivariate analysis refers to the joint investigation of two variates, and trivariate analysis refers to the joint investigation of three variates.

In each of the (sub)systems discussed previously all variance processes, except the state variable variance process, are determined in conjunction with the unrestricted mean relation given by (1*). This is important, as discussed in section I, to mitigate the impact that the variance search procedure has on subsequent asset pricing tests based upon the ICAPM mean relation (1). For each (sub)system, the model displaying the minimum value for Schwartz's Information Criterion (SIC),

where SIC = -2•max(Likelihood) + (number of parameters)•ln(number of sample obs.)[26]

is chosen. This method consistently determines the appropriate lag structure for each subset of processes examined. The system of assets is then expanded and the search over lag lengths continues for undetermined systems until all of the appropriate lag orders have been determined under (1*)[27].

The remainder of section II is organized in two subsections. First, the case of a single state variable is developed. The estimation methodology is detailed and the resultant parameter

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[26] For more discussion see de Gooijer, Abraham, Gould, and Robinson (p. 319-320, 1985). See also Geweke and Meese (1981) for Monte Carlo evidence that SIC consistently estimates correct lag lengths.
[27] It is recognized that this method is not the most efficient way of determining process lag lengths but it does ease the tremendous computational burden of jointly estimating every process for all possible lag lengths while maintaining a consistent estimation methodology. The efficiency of the estimators for the chosen lag lengths improves as the subsytem is enlarged.

estimates and test statistics are discussed. Next, GARCH variance processes are used to develop the model and test the intertemporal asset pricing model under the alternative variance specification.

A. The Case of a Single State Variable

The first process to be considered is that of the riskless Treasury Bill rate. Preliminary identification runs using Box-Jenkins methodology clearly indicates nonstationarity for the Treasury Bill process in levels. Supplementary analysis of the riskless variable allowing both the mean and variance to change also supports the notion that differencing of r_{ft} is necessary[28]. Thus, the state variable used for analysis is the first difference of 100 times the natural logarithm of one plus the effective weekly Treasury Bill rate. To determine the appropriate time series process for the state variable, the system of equations (2) and (3) are considered in isolation to determine lag lengths (i.e. J, K, Pf, and Qf).

To implement the above methodology, the search for the Treasury Bill process given by equations (2) and (3) is concentrated over the following lag lengths,

$$(J,K,Pf,Qf) \in \{(1,1,2,2),(0,1,2,2),(1,0,2,2),(1,1,2,1),(0,1,2,1),(1,0,2,1),(1,1,1,2),(0,1,1,2),(1,0,1,2),(1,1,1,1),(0,1,1,1),(1,0,1,1)\}.$$

Models with J=K=0 are excluded from the search due to the strong time variability in the mean of the state variable as reported in Table 2-1.

Table 2-2 reports the estimated model for the chosen state variable process as well as summary measures for the remaining models including their likelihood values at the optimum and Schwartz's Information Criterion (SIC). As discussed above, first differences of the Treasury Bill rate are used as the single state variable which describes changes in the investment opportunity set. The lower portion of Table 2-2 presents the results which determine the chosen state

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28 The estimated coefficient of first order autocorrelation, ρ_1, from the state variable system (2) and (3) for the level of r_{ft} (not $r_{ft} - r_{ft-1}$), with J=1, K=1, Pf=1, and Qf=1 leads to an estimate of ρ_1 = .998715 with a standard error of .004299.

variable process. Each of the lag structures for past first differences, residuals, conditional variances, and squared residuals are employed to find the best process to describe the state variable based upon the SIC value. The state variable model with lag lengths resulting in the minimum value for Schwartz's Information Criterion (SIC) is given by the mean process,

$$\tau_t = r_{ft} - r_{ft-1} = b_f + \theta_1 u_{ft-1} + u_{ft},$$

and variance process,

$$\sigma^2_{ft} = \gamma_f + \beta_{f1} \sigma^2_{ft-1} + \alpha_{f1} u^2_{ft-1}.$$

The maximum likelihood parameter estimates demonstrate first differences in the Treasury Bill rate are best modelled as a constant less 88% of last periods' residual plus a mean zero normal residual with time varying second moment given by a constant plus 58% of last periods' conditional variance and 20% of last periods residual squared.

The state variable process is only important in that it is used to purge expectations from the state variable series. The residuals from the model can then be used as unexpected state shocks in larger subsystems. Table 2-2 does not determine the shocks for the larger subsystems; rather, it determines the process for the state variable in these larger subsystems. The larger subsystems use the process found in Table 2-2 for the state variable in conjunction with the other variates in the system.

Given the appropriate orders of J=0, K=1, Pf=1, and Qf=1, the system is expanded to include either the market index or the bond process. To develop the market index subsystem, consider equation (1) for y_{Mt} and notice the second term is zero for i = M. That is,

$$y_{Mt} = \delta_1 \left[\frac{\rho_{MM} - \rho_{MB}\rho_{BM}}{1-\rho^2_{BM}} \right] \sigma_{Mt}\sigma_{Mt} + \delta_2 \left[\frac{\rho_{MB} - \rho_{MM}\rho_{BM}}{1-\rho^2_{BM}} \right] \sigma_{Mt}\sigma_{Bt} + u_{Mt}$$

$$= \delta_1 \sigma^2_{Mt} + u_{Mt}.$$

Similarly, for the bond subsystem consider equation (1) for i = B and notice the first risk term is zero. Thus, for given values of J, K, Pf, and Qf, the optimal orders of PM, QM, PB and QB are determined using (1*) and omitting the redundant risk term. For example, the specific form of (1*) for the market subsystem is,

$$y_{Mt} = b_M + d_1\sigma^2_{Mt} + u_{Mt}.$$

Similarly, for the bond subsystem, b_B is included in (1*) to relax the restrictions imposed by the ICAPM model[29]. The market portfolio is then jointly estimated with the state variable, to determine the appropriate variance lag structure using (1*) for y_{Mt} as discussed above, (2) with J=0 and K=1, (3) with Pf=1 and Qf=1, and (4) to determine PM and QM. To close the model, constant correlations are also estimated and used to construct covariances as the product of time varying standard deviations and constant correlations. The subsystem of the bond portfolio and the state variable are jointly estimated in a similar manner.

Table 2-3 presents the results for the bond and market subsytems. The left portion of the table shows results for the market portfolio, and the bond portfolio results are displayed on the right. In both cases, the market or bond portfolios are combined with the riskless state variable to find the most appropriate variance structure using (1*) as the relevant mean relation The maximum likelihood estimates shown are those obtained when the restrictions from equation (1) are imposed upon the model. That is, for each of the subsystems the variance process is determined using (3) and (4) in conjunction with (1*). The reported values are those obtained under the restrictions of (1)[30].

It is critical to note that the regressors for the market and bond portfolio variances are based upon the state variable series, not the market or bond portfolios own previous variances or squared residuals. The results from these subsystems are not particularly satisfying. Obtaining convergence of these models is very difficult and expanding lag structures has only moderate benefit for even low order models. The lack of significance for the important variance evolution parameters, β_{M1} and β_{B1} in Table 2-3 suggest that the single state variable does not adequately

29 Further relaxations of the system to allow for less restrictive mean relations when searching for variance processes such as adding a redundant risk term would likely result in nonconvergence, as even the addition of a constant leads to difficulty in estimation.

30 It is interesting to note that the reported parameter values do not change considerably when an intercept is added to the ICAPM mean relation (1).

capture either market or bond portfolio variance evolution. Because the variances for both the market and bond portfolio are based upon shocks to state variable evolution, the only stationarity requirement on parameters is that $\beta_{f1} + \alpha_{f1}$ be less than unity which ensures σ^2_{ft} is defined. The extreme parameter values for β_{M1} and β_{B1} along with their lack of significance suggests perhaps changes in the riskless rate are simply not large enough to explain the variability of market or bond variances. Table 2-1 shows that the unconditional standard deviations for both the levels and first differences of Treasury Bill returns are much less variable than those for the market or bond portfolio excess returns. This lack of variability in both levels and first differences of Treasury Bill rates supports both the magnitude and significance of the impact of unexpected shocks on bond and market variances (i.e. the large values for β_{M1} and β_{B1}). Unfortunately, the use of weekly data may have resulted in a series of Treasury Bill rates which are so close to maturity that price differences are minimal. This results in a lack of variability in the riskless rate and hence in first differences of the riskless rate. This lack of variability leads to explosive, insignificant coefficients as obtained above. In the future, it would be interesting to consider a similar type of model with multiple state variables based upon different points on the yield curve.

Given the optimal values for PM, QM, PB, and QB as determined in Table 2-3, and the earlier determined values for J, K, Pf, and Qf from Table 2-2, the trivariate model based upon equations (1*), (2), (3) and (4) is estimated using the state variable, the market, and the long term bond portfolio. The model based upon equations (1), (2), (3), and (4) results when the restrictions $b_M = 0$ and $b_B = 0$ are imposed. These restrictions are used to test whether the asset pricing model adequately captures mean relations. Specifically, the null hypothesis to be tested is Ho: $b_M = b_B = 0$. To test these restrictions, a likelihood ratio statistic is used. The likelihood value under the alternative is compared to the (restricted) likelihood value under the null to give the test statistic,

$\chi^2(2) = 2 \cdot$(Unrestricted Likelihood - Restricted Likelihood),

which is distributed as a χ^2(d.f.) random variable with degrees of freedom given by the number of restrictions imposed. The results are reported in Table 2-4.

This trivariate joint system is also of only limited success, although a number of conclusions can be drawn. First notice that the parameter estimates from the joint system under the null are very similar to the less efficient estimates given in Table 2-3. The mean and variance parameters for state variable evolution are virtually unchanged with moderate decreases in standard errors as expected. The only exception to this is α_{f1}, which shows a slight increase in magnitude in comparison to the bivariate systems. This may be a direct result of the relative importance of the state variable increasing as another variate enters the analysis. That is, because the state variance process determines second moments for the state variable and both the market and bond portfolios, it is not surprising to see the parameters of σ^2_{ft} change as another variate is added to the subsystem. Both variance process slope parameters for the bond and market portfolio (β_{B1} and β_{M1}) remain insignificant in the joint system of Table 2-4. Thus, although the state variable process is clearly time varying, the impact of unanticipated state changes is minimal. The intuition behind this result is discussed above. The unexpected portion of Treasury Bill returns represents a very limited attempt at capturing changes in the investment opportunity set. Clearly the entire term structure may be important in determining the risks due to changing interest rates at any point in time. Rather than attempt to parameterize this elusive relationship, the risks associated with holding a long term bond portfolio are used to capture the important elements of interest rate risk facing investors in the development which follows. Using the traditional GARCH methodology for the long term bond portfolio may allow the relevant interest rate risk facing investors to be parsimoniously fit.

Notice that the ICAPM likelihood ratio test statistic of 3.3 from Table 2-4 is also insignificant at even the 10% level, which lends support to the intertemporal asset pricing model. Other author's have rejected the static CAPM [see for example Giovannini and Jorion (1989), or Engel and Rodrigues (1989)] using similar asymptotic test statistics. The significance of the risk premiums in this model may begin to explain these differences. Notice the market risk term, δ_1, is clearly insignificant even after imposing the null. However, the interest rate risk term, δ_2, is significant at the 1% level for a one tailed t-test of H_A: $\delta_2 > 0$. Thus, it is possible that earlier rejections may have been due to model misspecification by ignoring interest rate risk in the mean relation (1). That is, testing a static CAPM using time varying variances implies risks are changing unless very restrictive assumptions are met[31].

The remainder of this section uses the traditional GAHCH methodology to further pursue the ICAPM model with time varying second moments. As discussed above, long term bond portfolio returns may better capture risks associated with interest rate changes. Similarly, own previous variances and squared residuals for the market portfolio may better capture market risk than unanticipated changes in the Treasury Bill rate. Variances remain functions of elements available in the investor's information set, and thus the basic modelling structure is unchanged[32].

B. The Case of Traditional GARCH Variances

For the traditional GARCH specification of the model, the same general approach is followed to determine the appropriate orders of PM, QM, PB, and QB. To determine PM and QM, equation (1) is considered for i = M. Noting the second term is zero, equations (1*) without the second risk term, and (5) result in a direct univariate GARCH(PM,QM)-M model. Specifically, the ICAPM mean relation is as shown in subsection *A* above,

$$y_{Mt} = \delta_1 \sigma^2_{Mt} + u_{Mt}.$$

Relaxing this relation to allow for a nonzero intercept, the variance process is estimated under (1*) given by,

$$y_{Mt} = b_M + d_1 \sigma^2_{Mt} + u_{Mt},$$

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[31] This point is discussed further in the introduction. See for example Fama (1970), Merton (1971 and 1973) or Rubinstein (1976) for further discussion.

[32] Although the GARCH model is similar, there is one critical difference. Above, variances were modelled as functions of a single state variable. The fact this single state variable model did not adequately capture variance dynamics suggests another parameterization is necessary, perhaps including additional state variables. The case of traditional GARCH processes specifies variances in a manner consistent with, but not restricted by the asset pricing model developed.

in conjunction with the variance evolution equation (5). Using i = B with equation (1*), leads to a similar univariate GARCH(PB,QB)-M for the bond process to determine the appropriate values for PB and QB. Given the optimal values of PM, QM, PB, and QB the bivariate GARCH(PM,QM,PB,QB) is estimated where all parameters for both the bond and market portfolio are jointly estimated.

Table 2-5 presents the results of the search procedure to determine the appropriate lag lengths for the market and bond variance processes. Each model is independently estimated using different lag structures, and Schwartz's Information Criterion (SIC) is again used to determine the optimal variance process. It should be noted that the estimation procedure has extreme difficulty converging for (sub)systems with large numbers of parameters. Models with many lags tend to result in nonconvergence due to excessive numbers of parameters. These models are important only insofar as they identify the most appropriate lower order system. To implement the lag length search procedure, large (sub)systems which do not converge are restricted to obtain convergence. The restrictions imposed are chosen to ensure the model does not result in a lower order system being estimated independently (e.g. estimating the GARCH(2,2)-M with $\beta_{M2} = 0$ results in a GARCH(1,2)-M). As an example, consider the GARCH(2,2)-M process as described in Table 2-5 with β_{M1} restricted to zero. This restriction leads to the model,

$$y_{Mt} = b_M + \delta_1 \sigma_{Mt}^2 + u_{Mt}$$

$$\sigma_{Mt}^2 = \gamma_M + \beta_{M1}\sigma_{Mt-1}^2 + \beta_{M2}\sigma_{Mt-2}^2 + \alpha_{M1}u_{Mt-1}^2 + \alpha_{M2}u_{Mt-2}^2 \quad \text{s.t. } \beta_{M1} = 0$$

$$= \gamma_M + \beta_{M2}\sigma_{Mt-2}^2 + \alpha_{M1}u_{Mt-1}^2 + \alpha_{M2}u_{Mt-2}^2$$

Alternative parameterizations such as the model which results when $\alpha_{M1} = 0$ are also estimated whenever nonconvergence occurs. The model with the minimum SIC is reported in all such cases.

As shown in the table, both the market and bond variance processes are best fit using a GARCH(1,1)-M process. In the case of the bond subsytem, after imposing the ICAPM intercept restriction $b_B = 0$, the chosen model states that this week's conditional variance is given by a constant, plus 59% of last periods' variance, plus 28% of last periods' residual squared. In addition, both subsytems result in significant parameter estimates for all parameters at the 1% level with the exception of δ_1, the price of market risk. The risk price in the market subsystem is significant at the 5% level. It is also interesting to note the variance processes for both excess return series are very similar. The bond portfolio variance process evolves more in relation to previous variances, than previous squared residuals in comparison with the market variance process although these differences are relatively minor.

Table 2-6 shows the results of the joint system for the market and bond portfolios using the selected bond and market variance processes from Table 2-5. The increase in efficiency obtained by jointly estimating the bivariate system of market and bond portfolio returns results in larger parameter values and both risk price terms are now significant at the 1% level. The actual parameter values are similar to their univariate counterparts but the reduction in standard errors is noticeable. For second moment parameters, the joint bivariate model affects bond and market portfolio variances differently. The market variance constant, γ_M, decreases whereas the bond variance constant, γ_B, increases. The important slope coefficients of the variance relations are less distinct. There is a uniform increase in the parameters reflecting the importance of previous own squared residuals as shown by α_{M1} and α_{B1}. However, the parameter reflecting previous own conditional variances increases for the market portfolio, β_{M1}, and decreases for the bond portfolio, β_{B1}. Clearly, these changes do not occur in isolation. For example a decrease in the constant portion of market variance and a simultaneous increase in both time variability components is consistent with a constant unconditional variance. Table 2-6 provides further support for the ICAPM model, in that the null hypothesis of Ho: $b_M = b_B = 0$ is not rejected at even the 50% level given the reported χ^2 value of 1.003.

Table 2-7 takes the lag structures for both the market and bond portfolios as given and adds three size portfolios to the analysis, one at a time, resulting in a trivariate system to determine size portfolio lag structures[33]. These results are particularly interesting. First, notice that the significance levels and point estimates for both risk premiums are remarkably similar across trivariate models which is comforting. As found in Tables 2-5 and 2-6, the price of market risk is the least significant of the risk price terms. In the models with the largest, mid and smallest third of all firms δ_1 is significant at the 10, 1, and 2% levels, respectively. However, also notice the variance behaviour for the largest size portfolio is markedly different from either of the similar two smaller size portfolios. The trivariate models for the smaller portfolios result in similar bond and market parameter estimates as found earlier in Table 2-6. The only meaningful differences between Table 2-6 and Table 2-7 for the smaller portfolios lies in the tendency for the constant portion of market variance, γ_M, to increase and for the portion due to last periods' squared residual, α_M, to decrease. This may be a direct result of the larger relative noise in the smaller size portfolios' conditional variances.

The trivariate model for the largest size portfolio leads to some very interesting results. First, the constant portion of market variance declines markedly and it becomes insignificant from zero at even the 10% level. The portion of variance due to last periods' market variance increases dramatically, but the portion due to last period's squared market residual falls to near zero. Also notice that the selected model does not include last periods' size portfolio residual squared. Inclusion of this squared residual led to nonconvergence. To see why this result occurred, notice that the large portfolio behaves very similarly to the market portfolio and the correlation between this portfolio and the market is considerably larger than the other size portfolios. It may be that the largest size portfolio and market portfolio residuals are so similar that the estimation procedure treats them as nearly identical. For discussion purposes, suppose both residuals are equal (i.e. $u_{Mt} = u_{it}$), then knowledge of either residual determines the other. Similarly, squared residuals are also equal for all previous lags, and hence, previous variances as

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33 Various trivariate models led to difficulty in estimation when the ICAPM mean relation is completely relaxed as in (1*). To resolve estimation problems the search and alternative version of the model are obtained when all intercepts are resticted to zero, and slope relations for the mean are unrestricted for each size portfolios' risk premiums. This implies two restrictions in each trivariate model for the likelihood ratio test statistic.

well. Thus, the search procedure may be parsimoniously generating a (1,2)-M process implicitly for both the market and largest size portfolio in the first trivariate model.

For discussion purposes, the conditional market variances developed according to the trivariate GARCH model for the largest portfolio are plotted in Figure 2-1. The figure clearly demonstrates the ability of the model to impound information in excess return residuals into conditional variances. That is, a large shock to market excess return leads to a quick response in the market's conditional variance. Moreover, shocks of differing magnitude impact conditional variances with comparable changes in the magnitude of variances.

Generated residuals and conditional variances for the bond portfolio under the trivariate GARCH model for the largest size portfolio are plotted in Figure 2-2. The results from Figure 2-2 are interesting because they further support the model's ability to capture periods of increased volatility using GARCH variances. Moreover, this figure concurs with Figure 2-1 with regards to the identification of periods with increased volatility. That is, shocks appear to affect the entire system of variates. This final point is important as it may allow future research to focus on a single source of economy wide uncertainty[34].

The ICAPM test statistics reported in Table 2-7 are once again not contrary to the intertemporal asset pricing model. In fact, the largest χ^2 value reported of 1.608 is well below its critical value of 4.605 at even the 10% level. The clear significance of risk price terms in all models, in conjunction with insignificant χ^2 values lends support to the ICAPM paradigm with time varying moments.

III. DIAGNOSTIC ANALYSIS

The purpose of this section is to examine the time series properties of the original data series and the standardized residuals from the model to determine the success of the paradigm

[34] This motivation led to the single state variable model presented in the paper. Unfortunately, first differences of the short term Treasury Bill rate do not fully capture economic shocks.

in capturing moment dynamics. The results of this analysis are presented in Table 2-8. The data presented in the table is used to evaluate the success of the model in capturing the time series nature of excess returns. Panel A considers the raw excess return series themselves, y_{it}. Panel B squares difference between each raw return and its unconditional sample mean for each observation and then reports summary statistics for the resultant series, $(y_{it} - \bar{y}_i)^2$. Standardized residuals, $\frac{u_{it}}{\sigma_{it}}$, are examined next and discussed in Panel C. Finally, panel D presents the results for the squared standardized model residuals, $\left(\frac{u_{it}}{\sigma_{it}}\right)^2$.

McLeod and Li (1983) show that sample autocorrelations for both residuals and squared residuals are asymptotically standard normal random variables and the Box-Pierce Portmanteau test statistics are asymptotically distributed as χ^2 random variables. The Box-Pierce Portmanteau test statistics are defined as,

$$Q(K) = T\sum_{l=1}^{K}\rho_l^2 \equiv Q_x(K) = T\sum_{l=1}^{K}\rho_{xl}^2\text{, and } Q_{xx}(K) = T\sum_{l=1}^{K}\rho_{xxl}^2\text{,}$$

where T is the number of observations in the series being considered, ρ_{xl}^2 is the sample autocorrelation at lag l squared. For $Q_x(K)$, ρ_{xl} is the sample autocorrelation between the current residual and the residual of l lags. Similarly for $Q_{xx}(K)$, ρ_{xxl} is the sample autocorrelation between the current squared residual and the squared residual of l periods past. K denotes the number of autocorrelations being considered in the statistic. Both $Q_x(K)$ and $Q_{xx}(K)$ are distributed as χ^2 random variables with degrees of freedom given by the number of sample autocorrelations considered less the number of parameters estimated. These statistics are used to determine if the first K autocorrelations are significantly different from zero as a group.

The first two panels of Table 2-8 present summary statistics for the raw data series. Panel A studies the time series properties of the raw excess returns prior to analysis. Th sample autocorrelations show that raw portfolio excess returns tend to display positive autocorrelation for initial lags. Notice all size portfolios, the market portfolio, and the bond portfolio display significant autocorrelation in raw excess returns before the model is employed. The results for Panel B show unconditional squared residuals tend to be positively related, as expected. However, the Q_{xx} statistics suggest the long term bond portfolio and smallest size portfolio are the dominant variates requiring GARCH modelling to remove variance persistence.

The latter two panels of Table 2-8 demonstrate the model's ability to capture first and second moment dynamics. Panel C shows the time series properties of the standardized model residuals. Comparison of panels A and C suggests the model effectively removes autocorrelations from the long term bond portfolio and is moderately successful in removing persistence from the market portfolio for all trivariate models. However, the only size portfolio which demonstrates any clear success in removal of autocorrelations is the largest portfolio. The final panel shows that the GARCH models have effectively removed any patterns of second moment persistence in the excess return data.

IV. CONCLUSIONS

The motivation for this study is twofold. First, the study of intertemporal asset pricing paradigms with time varying moments has been largely unexplored in the finance literature. Shanken (1990) provides an alternative framework for analysis which is markedly distinct from the GARCH methodology used herein, but has significant limitations. A second motivation for this paper is to examine the weekly variance processes associated with excess returns on stock portfolios (monthly analysis was performed in Turtle 1990). The primary result from this study is support for the intertemporal asset pricing model. A primary research question remaining is how to specify variance dynamics in a manner consistent with economic theory while still providing an adequate description of the underlying process.

The initial section of the paper employs changes in the short term Treasury Bill rate as a single state variable to explain the evolution of the investment opportunity set. Unfortunately, the preliminary research provided in this paper suggests this single state variable does not adequately capture second moment dynamics. Future research may be useful in any of three directions. First, it might be useful to specify variances which are functions of state variables related to the yield curve. Secondly, combining short rate changes with yield curve state variables and traditional GARCH terms may better describe evolving variances. This approach requires solving a serious methodological issue which is not addressed by previous researchers in this area. The underlying motivation behind the GARCH methodology is that variance dynamics can be captured by a deterministic statistical process. If pure economic theory motivates model variances, then the use of carefully specified state variables as variance function regressors is warranted. To lump both of these effects into one variance function may result in dubious conclusions. For example, if the traditional GARCH approach dominates and economic state variables prove insignificant, this does not necessarily imply economic states are not important. It may only imply that the GARCH process captures the importance of the economic state variables. A final avenue for future research is suggested by Figures 2-1 and 2-2 which reveal variance evolution patterns for market and bond portfolio conditional variances which have peaks at similar points in time[35]. This tendency suggests a single explanatory state variable or a composite state variable may be constructed to explain variance evolution. Earlier results using changes in the Treasury Bill rate as a state variable, may simply indicate this is not the best state variable for parameterizing variance evolution.

Some interesting conclusions can be drawn from this research. First, as mentioned above, variance dynamics are better explained by traditional GARCH processes than by unanticipated variances of changes in Treasury Bill returns. This may be largely due to the fact that for weekly data the variability of changes in weekly Treasury Bill returns is small relative to the variability in bond or stock portfolio returns. Second, none of the suggested models reject ICAPM for any reasonable level of significance. Finally, the strong significance of the interest

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35 Similar results were obtained for size portfolio conditional variances but were omitted for brevity.

rate risk premium throughout the paper may help to rationalize earlier rejections of static versions of CAPM [see Giovannini and Jorion (1989), or Engel and Rodrigues (1989)].

Unfortunately there are at least two other possible reasons for the lack of rejection of ICAPM. The test statistics in this paper are based upon systems of three variates, not the entire system of assets under consideration. This simplification is employed to allow a feasible search procedure over GARCH lag lengths; however, this procedure also leads to inefficient test statistics as the entire residual covariance structure is not considered. The search procedure over lag lengths is employed to ensure consistency of test statistics. Unfortunately, although the resultant test statistics are consistent, they may not be powerful in comparison with larger multivariate systems. Thus, rejection of asset pricing models in other papers may be due to misspecified asset pricing relations, misspecified variance processes, or failure to use the most powerful test procedures.

The methodology of this paper initially searches over GARCH specifications to find the best variance process for each variate. This procedure seeks a variance process consistent with, but not biased by, the asset pricing paradigm. The intent of this approach is to develop an appropriate variance structure for residuals. This will lead to consistent parameter estimates and will ultimately result in correctly specified test statistics. The model appears to be very successful at capturing second moment dynamics although the time series properties in the conditional means of all series are relatively unaffected by the modelling. In fact, with the exception of bond portfolio standardized residuals, there still appears to be evidence of autocorrelation in excess return standardized residuals. Market portfolio and size portfolio sample autocorrelations appear to be reduced, although not statistically eliminated. An interesting area for future research is to explore how moment dynamics change with the periodicity of the data being considered. For example, Turtle (1990) finds much stronger support for the GARCH methodology using a lengthier monthly return series in comparison to the weekly data set used herein. The nature of time aggregation in GARCH models warrants closer attention to help link the various publications employing GARCH methodology. The periodicity of data across various publications varies tremendously [see for example Bollerslev et al (1988) for quarterly, Morgan and Morgan (1987), Engel and Rodrigues (1989), Engle et al (1990), or Schwert and Seguin (1990) for monthly, Giovannini and Jorion (1989) for weekly, and Bollerslev (1987), Akgiray(1989), or Hsieh(1989), Lamoureux and Lastrapes (1990) for daily periodicity] but has not been carefully studied in the context of changing moments and an intertemporal asset pricing model.

A great deal of work remains to be examined in the general area of GARCH processes and asset pricing models. A more careful examination of essential multivariate restrictions on pairwise correlations to assist in the estimation and examination of multivariate structures would be useful for both theoretical and empirical research. Future research must concentrate upon reducing the number of parameters in large systems in a manner consistent with financial thought to allow meaningful empirical research to be undertaken.

Figure 2-1 -- Generated Residuals and Conditional Variances for the Market Portfolio from July 1983 to December 1989

Conditional Market Variances are generated from model residuals according to the Trivariate GARCH model for the largest size portfolio. The results are those obtained when the model is fit over the sample period from July 1983 to Dec. 1989.

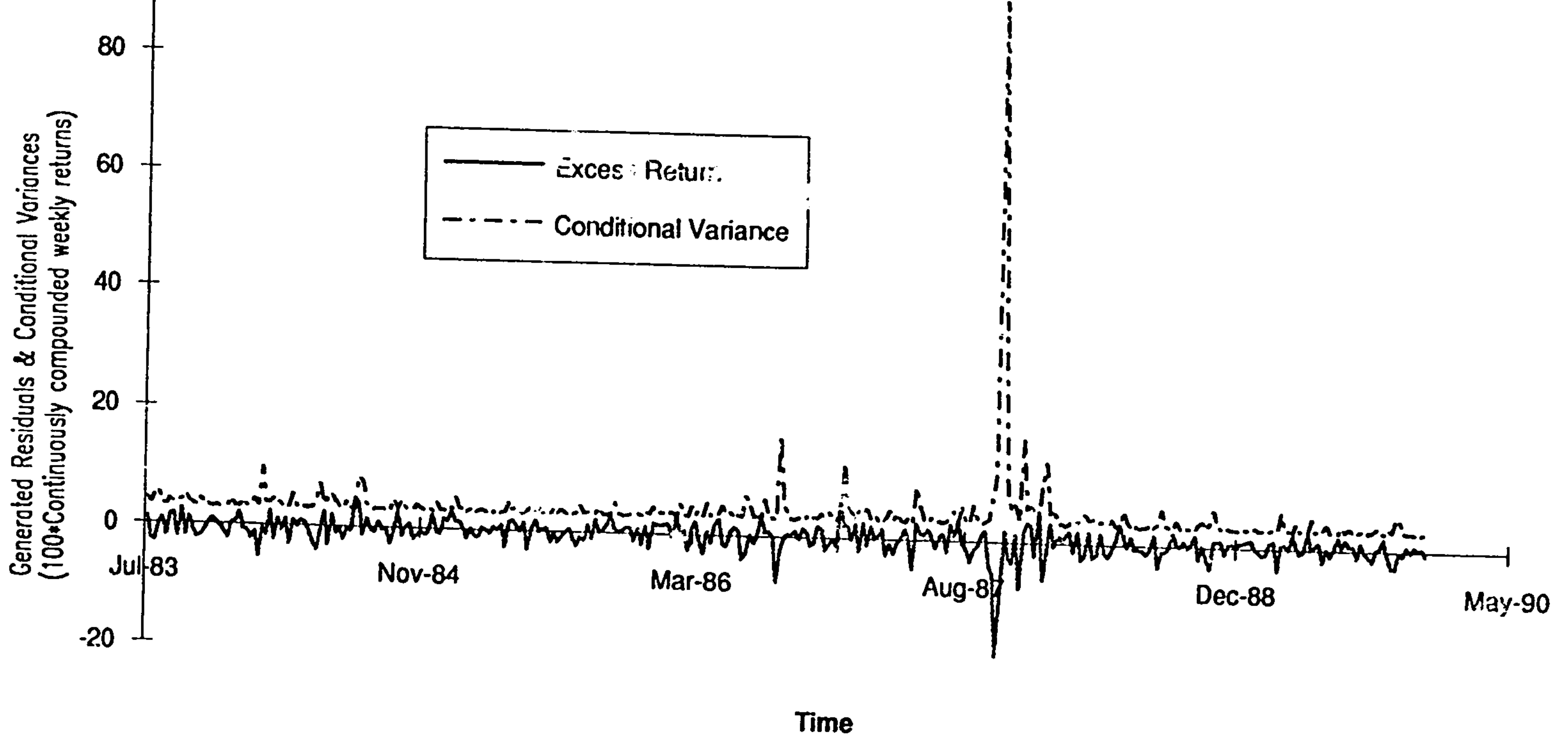

Figure 2-2 -- Generated Residuals and Conditional Variances for the Bond Portfolio from July 1983 to December 1989

Conditional Bond Variances are generated from the model residuals according to the Trivariate GARCH model for the largest size portfolio. The results are those obtained when the model is fit over the sample period from July 1983 to Dec. 1989.

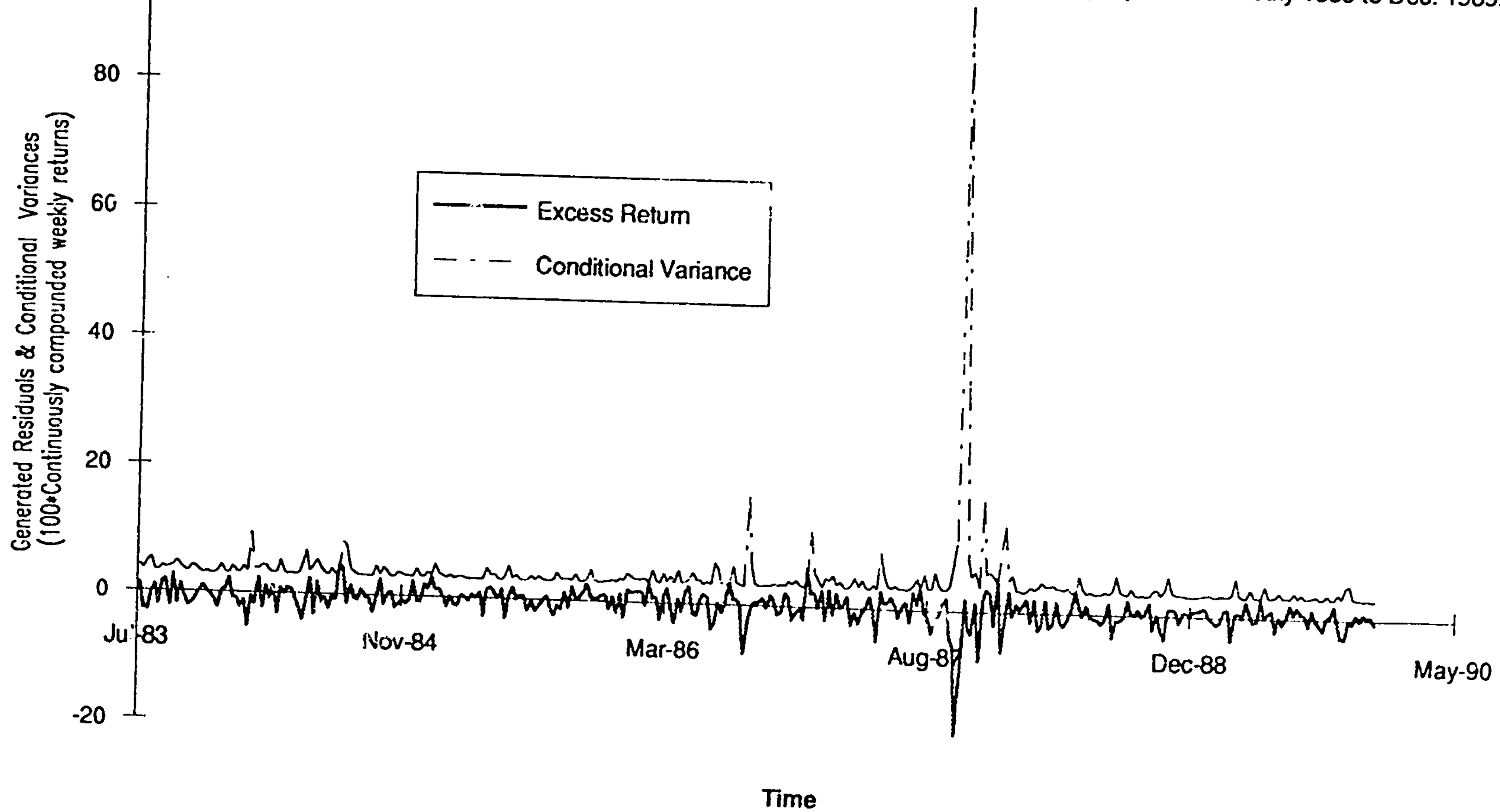

Table 2-1
Descriptive Statistics for Weekly Returns from July 1983 through December 1989

Unconditional summary statistics for the 337 continuously compounded weekly returns (multiplied by 100) from July 14, 1983 to December 15, 1989 are shown in the table. Weekly Treasury Bill (r_{ft}) and long term bond portfolio returns (r_{Bt}) are constructed from data provided by Reuters. The market portfolio (r_{Mt}) is derived from the daily value weighted market index provided by CRSP. All series are fully described in Appendix 1. Mean and S.D. are the unconditional maximum likelihood estimates of the sample mean and standard deviation. The Q(36) statistic reported is the Box-Pierce Portmanteau test statistic to determine if the first 36 autocorrelations are significantly different from zero as a group. Q(36) is distributed as a Chi-squared random variable with 36 degrees of freedom.

			Sample Autocorrelations				
Portfolio Return	Mean	S. D.	ρ_1	ρ_2	ρ_3	ρ_4	Q(36)
T-Bill return = r_{ft}	0.127	0.036	0.60**	0.55**	0.61**	0.69**	3252.42**
$100*(r_{ft} - r_{ft-1})$	0.048	3.354	-0.40**	-0.14**	-0.03	0.25**	230.63**
LT Bond Portfolio return = r_{Bt}	0.305	1.758	0.22**	0.25**	0.15**	0.14**	89.59**
Market Portfolio return = r_{Mt}	0.143	2.143	0.08	0.03	-0.02	-0.07	57.38*

* significant at the 5% level
** significant at the [illegible]% level

Table 2-2
The Selected Process for the Continuously Compounded Treasury Bill Rate from July 1983 through December 1989

First differences of the one week continuously compounded Treasury Bill rate ($r_{ft} - r_{ft-1}$) are used as the state variable to describe changes in the investment opportunity set in the initial portion of the paper. This table presents the results of the choice over J, K, Pf, and Qf which determine the appropriate lag structures for the state variable. Model selection is based upon the minimum value of Schwartz's Information Criterion (SIC). Standard errors are reported in parentheses below maximum likelihood parameter estimates.

$$\tau_t \equiv r_{ft} - r_{ft-1} = b_f + \sum_{j=1}^{J} \phi_j (r_{ft-j} - r_{ft-j-1}) + \sum_{k=1}^{K} \theta_k u_{ft-k} + u_{ft},$$

$$\sigma_{ft}^2 = \gamma_f + \sum_{p=1}^{Pf} \beta_{fp} \sigma_{ft-p}^2 + \sum_{q=1}^{Qf} \alpha_{fq} u_{ft-q}^2 .$$

	Maximum Likelihood Estimates			
b_f	θ_1	γ_f	β_{f1}	α_{f1}
-0.000234	-0.881934	0.000145	0.577209	0.203401
(.000168)	(.028640)	(.000046)	(.0912106)	(.0371528)

Model (Pf,Qf)-(J,K)	Likelihood	SIC
(2,2)-(1,1)	777.8519	-1509.1669
(2,2)-(0,1)	777.8534	-1514.9870
(2,2)-(1,0)	727.3958	-1414.0718
(2,1)-(1,1)	776.6069	-1512.4940
(2,1)-(0,1)	776.6069	-1518.3111
(2,1)-(1,0)	723.6177	-1412.3328
(1,2)-(1,1)	777.8542	-1514.9886
(1,2)-(0,1)	777.8536	-1520.8045
(1,2)-(1,0)	727.3961	-1419.8896
(1,1)-(1,1)	776.6069	-1518.3111
(1,1)-(0,1)	776.6069	-1524.1282
(1,1)-(1,0)	723.6179	-1418.1501

Table 2-3

Determining the Lag Orders for the Market and Bond Subsystems

To determine the lag orders for each of the market portfolio excess return (y_{Mt}) and long term bond portfolio excess return (y_{Bt}) subsystems, each is independently estimated with the state variable. Both subsystems treat the unanticipated portion of first differences of the Treasury Bill rate as a single state variable which determines all other variances in the model. First differences of the Treasury Bill rate ($r_{ft} - r_{ft-1}$) follow the time series behaviour given by a combination moving average process of order 1 and a GARCH(1,1) process for variances as discussed in 2-2. Given the state variable process, the market (PM and QM) and bond (PB and QB) portfolio variance lag structures are determined under the relaxed mean relation (1*). To close each subsystem, covariances are computed as the product of standard deviations and constant correlations. The maximum likelihood parameter estimates reported below obtain when the ICAPM restriction Ho: $b_M = 0.0$, or Ho: $b_B = 0.0$ respectively, is imposed. Standard errors are reported below parameter estimates in parentheses.

State Variable Dynamics

$$r_{ft} - r_{ft-1} = b_f + \theta_1 (u_{ft-k}) + u_{ft},$$

$$\sigma_{ft}^2 = \gamma_f + \beta_{f1} \sigma_{ft-1}^2 + \alpha_{f1} u_{ft-1}^2.$$

The Market and State Variable Subsytem

$$y_{Mt} = b_M + \delta_1 \sigma_{Mt}^2 + u_{Mt},$$

$$\sigma_{Mt}^2 = \gamma_M + \sum_{p=1}^{PM} \beta_{Mp} \sigma_{ft-p}^2 + \sum_{q=1}^{QM} \alpha_{Mq} u_{ft-q}^2, \text{ and}$$

The Bond and State Variable Subsytem

$$y_{Bt} = b_B + \delta_2 \sigma_{Bt}^2 + u_{Bt},$$

$$\sigma_{Bt}^2 = \gamma_B + \sum_{p=1}^{PB} \beta_{Bp} \sigma_{ft-p}^2 + \sum_{q=1}^{QB} \alpha_{Bq} u_{ft-q}^2, \text{ and}$$

Restricted Maximum Likelihood Estimates

Market and state variable subsystem:

b_f	θ_1	γ_f	β_{f1}	α_{f1}	δ_1	β_{M1}	ρ_{fM}
-.0003	-.903	.0002	.514	.255	.029	7848.	-.017
(.0001)	(.0207)	(.0000)	(.0655)	(.0327)	(.029)	(44550)	(.063)

Bond and state variable subsystem:

b_f	θ_1	γ_f	β_{f1}	α_{f1}	δ_2	γ_B	β_{B1}	ρ_{fB}
-.0002	-.890	.0002	.485	.255	.095	1.62	2355.	.084
(.0002)	(.0252)	(.0000)	(.0811)	(.0447)	(.036)	(.719)	(47690)	(.057)

Model Selection Criteria

Model (PM,QM)	Unrestricted Likelihood	Schwartz's Information Criterion (SIC)	Model (PB,QB)	Unrestricted Likelihood	Schwartz's Information Criterion (SIC)
(1,1)[a]			(1,1)	117.1932	-170.3981
(1,0)	46.7821	-41.2102	(1,0)	115.7254	-173.2798
(0,1)	-462.9425	1940.1239	(0,1)	114.0860	-170.2798

[a] The (1,1) model for the market subsytem could not be estimated due to nonconvergence problems. Numerous attempts were made to generate reasonable parameter starting values for the algorithm based upon both the (1,0) and (0,1) solutions. Also notice that the intercept in the market variance specification is restricted to zero for the selected model.

Table 2-4

The Market, Bond and State Variable Model

This model is developed under the assumption that all changes in the investment opportunity set are a direct result of unanticipated changes in the short term Treasury Bill rate ($r_{ft} - r_{ft-1}$). The state variable process and the market and bond excess return (y_{Mt} and y_{Bt}, respectively) processes are based upon the earlier developed lag structures for the three variates as shown in Tables 2-2 and 2-3. To close the system, covariances are determined as the product of standard deviations and constant correlations. The maximum likelihood estimates shown below are the estimates obtained when Ho: $b_M = b_B = 0.0$ is imposed. Standard errors are reported below parameter estimates in parentheses. The ICAPM test statistic is distributed as a χ^2 value with 2 degrees of freedom.

$$r_{ft} - r_{ft-1} = b_f + \theta_1 (u_{ft-k}) + u_{ft},$$

$$y_{Mt} = b_M + \delta_1 \sigma^2_{Mt} + u_{Mt}, \quad y_{Bt} = b_B + \delta_2 \sigma^2_{Bt} + u_{Bt},$$

$$\sigma^2_{ft} = \gamma_f + \beta_{f1} \sigma^2_{ft-1} + \alpha_{f1} u^2_{ft-1},$$

$$\sigma^2_{Mt} = \beta_{M1} \sigma^2_{ft-1}, \quad \sigma^2_{Bt} = \gamma_B + \beta_{B1} \sigma^2_{ft-1}.$$

Maximum Likelihood Estimates

b_f	θ_1	γ_f	β_{f1}	α_{f1}	δ_1	δ_2	γ_B	β_{B1}	β_{M1}	ρ_{fB}	ρ_{fM}	ρ_{BM}
-0.0004	-0.9089	0.00016	0.48641	0.29469	0.02788	0.08874	1.44627	2615.48	7776.87	0.08172	-0.0142	0.23605
(.00013)	(.01760)	(.000000)	(.057977)	(.035893)	(.030011)	(.038136)	(.593038)	(49669.0)	(59916.7)	(.057126)	(.064467)	(.044577)

ICAPM test statistic:

$\chi^2(2) = 2^*$(Unrestricted Likelihood - Restricted Likelihood) = 3.29575

• As expected, the jointly estimated parameter estimates in Table 2-4 differ somewhat from the pairwise model estimates reported in Table 2-3.

Table 2-5
Determining the Lag Orders for the Market and Bond Subsystems Using GARCH Variance Processes

To determine the lag orders for the market and bond portfolio excess returns (y_{Mt} and y_{Bt}, respectively), each is independently estimated under various GARCH parameterizations. The model with the minimum value for Schwartz's Information Criterion (SIC), is then selected. Maximum likelihood estimates for parameters are reported for the selected model under the null hypothesis in each subsystem (i.e. Ho: $b_M = 0$ or $b_B = 0$, respectively). Standard errors are reported below parameter estimates in parentheses.

The Market Subsytem

$$y_{Mt} = b_M + \delta_1 \sigma^2_{Mt} + u_{Mt},$$

$$\sigma^2_{Mt} = \gamma_M + \sum_{p=1}^{PM} \beta_{Mp} \sigma^2_{Mt-p} + \sum_{q=1}^{QM} \alpha_{Mq} u^2_{Mt-q}, \text{ and}$$

The Bond Subsytem

$$y_{Bt} = b_B + \delta_2 \sigma^2_{Bt} + u_{Bt},$$

$$\sigma^2_{Bt} = \gamma_B + \sum_{p=1}^{PB} \beta_{Bp} \sigma^2_{Bt-p} + \sum_{q=1}^{QB} \alpha_{Bq} u^2_{Bt-q}, \text{ and}$$

δ_1	γ_M	β_{M1}	α_{M1}	δ_2	γ_B	β_{B1}	α_{B1}
.048737	.901448	.547480	.259042	.092221	.465637	.585382	.280354
(.024122)	(.299985)	(.082484)	(.026657)	(.031335)	(.178141)	(.082948)	(.048639)

Model Selection Criterion

Model (PM,QM)	Unrestricted Likelihood	Schwartz's Information Criterion (SIC)	Model (PB,QB)	Unrestricted Likelihood	Schwartz's Information Criterion (SIC)
(2,2)[a]	-700.1942	1435.2911	(2,2)[c]	-650.7907	1330.6669
(1,2)[b]	-701.4725	1432.0306	(1,2)[d]	-648.4861	1326.0578
(2,1)	-699.3578	1433.6182	(2,1)	-634.2758	1303.4543
(1,1)	-700.1805	**1429.4466**	**(1,1)**	-635.6621	**1300.4098**

[a] To obtain convergence for this model required β_{M1} be restricted to zero.

[b] β_{M1} was also restricted to zero in this model.

[c] To obtain convergence for this model required both β_{B1} and α_{B1} be restricted to zero.

[d] α_{B1} was restricted to zero in this model.

Table 2-6

The Market and Bond Subsystem Using Weekly data over the period July 1983 through December 1989

Given the lag structures discussed in Table 2-5 for the market and bond excess returns (y_{Mt} and y_{Bt} respectively), both series are jointly estimated. To close the system covariances are computed as the product of standard deviations and constant correlations. The maximum likelihood estimates reported obtain when Ho: $b_M = b_B = 0$ is imposed. Standard errors are given in parentheses below parameter estimates. The Intertemporal Capital Asset Pricing Model (ICAPM) test statistic is the likelihood ratio statistic based upon the difference of the likelihood under the alternative and null hypothesis and is distributed as χ^2 with 2 degrees of freedom.

$$y_{Mt} = b_M + \delta_1\sigma^2_{Mt} + u_{Mt}, \quad y_{Bt} = b_B + \delta_2\sigma^2_{Bt} + u_{Bt},$$

$$\sigma^2_{Mt} = \gamma_M + \beta_{M1}\sigma^2_{Mt-1} + \alpha_{M1}u^2_{Mt-1}, \text{ and } \sigma^2_{Bt} = \gamma_B + \beta_{B1}\sigma^2_{Bt-1} + \alpha_{B1}u^2_{Bt-1}.$$

Restricted Maximum Likelihood Estimates

δ_1	δ_2	γ_M	β_{M1}	α_{M1}	γ_B	β_{B1}	α_{B1}	ρ_{MB}
0.05898	0.09982	0.41633	0.57404	0.32325	0.97481	0.46970	0.34392	0.37247
(.02125)	(.03038)	(.10072)	(.05542)	(.03727)	(.29321)	(.06000)	(.04756)	(.05217)

ICAPM test statistic:

$\chi^2(2)$ = 2*(Unrestricted Likelihood - Restricted Likelihood) = 1.00333

Table 2-7

Determining the Lag Orders for the Size Portfolios

To determine the lag orders for each of the three size portfolios, each portfolio of excess returns (y_{jt}, for j=1, 2, or 3) is jointly estimated with both the market and bond portfolio excess returns (y_{Mt} and y_{Bt} respectively). All three subsystems use the earlier determined variances processes for the market and bond excess return portfolios as discussed in Table 2-3. Given the processes for the market and bond portfolios, the size portfolio lag structures (PJ and QJ) are then determined by minimizing Schwartz's Information without imposing the restrictions of ICAPM. To close each subsystem, covariances are computed as the product of standard deviations and constant correlations. The maximum likelihood estimates reported are those obtained for the chosen model when the null hypothesis restrictions are imposed. The alternative unrestricted version of the model estimates an additional two parameters by replacing $\delta_1\left(\frac{\rho_{jM}-\rho_{jB}\rho_{BM}}{1-\rho_{MB}^2}\right)$ with d_1 and $\delta_2\left(\frac{\rho_{jB}-\rho_{jM}\rho_{BM}}{1-\rho_{MB}^2}\right)$ with d_2 below. Standard errors are reported below parameter estimates in parentheses.

$$y_{Mt} = \delta_1\sigma_{Mt}^2 + u_{Mt}, \quad y_{Bt} = \delta_2\sigma_{Bt}^2 + u_{Bt},$$

$$y_{jt} = \delta_1\left(\frac{\rho_{jM}-\rho_{jB}\rho_{BM}}{1-\rho_{MB}^2}\right)\sigma_{jt}\sigma_{Mt} + \delta_2\left(\frac{\rho_{jB}-\rho_{jM}\rho_{BM}}{1-\rho_{MB}^2}\right)\sigma_{jt}\sigma_{Bt} + u_{jt},$$

$$\sigma_{Mt}^2 = \gamma_M + \beta_{M1}\sigma_{Mt-1}^2 + \alpha_{M1}u_{Mt-1}^2, \quad \sigma_{Bt}^2 = \gamma_B + \beta_{B1}\sigma_{Bt-1}^2 + \alpha_{B1}u_{Bt-1}^2$$

$$\sigma_{jt}^2 = \gamma_j + \sum_{p=1}^{PJ}\beta_{jp}\sigma_{jt-p}^2 + \sum_{q=1}^{QJ}\alpha_{jq}u_{jt-q}^2 .$$

Restricted Maximum Likelihood Estimates

Panel A. The size portfolio of the largest third of all firms

δ_1	δ_2	γ_M	β_M	α_M	γ_B	β_B	α_B	γ_j	β_{j1}	α_{j2}	ρ_{MB}	ρ_{Mj}	ρ_{Bj}
.047928	.109908	.000402	.971942	.027876	.578982	.479985	.419267	.000156	.974389	.025565	.209956	.957376	.116507
(.02734)	(.03029)	(.00337)	(.00171)	(.00077)	(.19859)	(.09296)	(.08711)	(.01173)	(.00025)	(.00356)	(.03669)	(.00281)	(.03544)

Panel B. The size portfolio of the mid third of all firms

δ_1	δ_2	γ_M	β_M	α_M	γ_B	β_B	α_B	γ_j	β_j	α_j	ρ_{MB}	ρ_{Mj}	ρ_{Bj}
.065724	.098044	1.15944	.521171	.200533	.372395	.593596	.318770	1.21570	.390390	.310577	.354921	.934223	.279764
(.02341)	(.03093)	(.27546)	(.08524)	(.05547)	(.14129)	(.06567)	(.06538)	(.18090)	(.05669)	(.05247)	(.05123)	(.00677)	(.05384)

Panel C. The size portfolio of the smallest third of all firms

δ_1	δ_2	γ_M	β_M	α_M	γ_B	β_B	α_B	γ_j	β_j	α_j	ρ_{MB}	ρ_{Mj}	ρ_{Bj}
.058276	.093793	.933886	.571130	.211707	.347326	.600240	.324329	.885456	.363746	.470384	.355962	.817512	.152216
(.02293)	(.03111)	(.30139)	(.09548)	(.06583)	(.12738)	(.06036)	(.06395)	(.15692)	(.06839)	(.08385)	(.05198)	(.01855)	(.05737)

ICAPM test statistic:

$\chi^2(2) = 2^*$(Unrestricted Likelihood - Restricted Likelihood) =

1.60808	1.17576	1.12415

Model Selection Criterion

Panel A. The size portfolio of the largest third of all firms			Panel B. The size portfolio of the mid third of all firms			Panel C. The size portfolio of the smallest third of all firms		
Model (P1,Q1)	Unrestricted Likelihood	Schwartz's Information Criterion (SIC)	Model (P2,Q2)	Unrestricted Likelihood	Schwartz's Information Criterion (SIC)	Model (P3,Q3)	Unrestricted Likelihood	Schwartz's Information Criterion (SIC)
(2,2)[a]	-1612.12624	3323.14337	(2,2)	-1640.95932	3386.62664	(2,2)	-1768.93780	3642.58359
(2,1)[b]	-1658.16182	3409.39742	(2,1)	-1640.96185	3380.81458	(2,1)[e]	-1772.12753	3637.32884
(1,2)[c]	-1614.02529	3321.12435	(1,2)[d]	-1643.23414	3379.54205	(1,2)	-1769.18323	3637.25734
(1,1)	-1658.12773	3409.32923	(1,1)	-1641.91697	3376.90772	(1,1)	-1769.32070	3631.71577

[a] To obtain convergence for this model required α_{j1} be restricted to zero.

[b] β_{j2} and α_{j1} were restricted to zero in this model.

[c] α_{j1} was restricted to zero in this model.

[d] β_{j1} was restricted to zero in this model.

[e] β_{j1} was restricted to zero in this model.

Panel D. Squared Standardized Residuals

First Five Autocorrelations									
1	.08	.07	-.03	.05	.05	.00	.02	.04	.00
2	.02	.03	-.06	-.01	.01	-.05	.02	.01	-.05
3	-.01	-.01	.03	-.02	-.04	.01	-.03	-.04	.01
4	.00	.00	.01	.00	.01	.00	-.01	.01	-.01
5	-.02	-.02	-.07	-.02	-.03	-.07	-.02	-.04	-.07
Box-Pierce Portmanteau Test									
$Q_{xx}(12)$	3.45	3.06	15.36	3.46	6.63	12.40	4.86	4.83	12.32
$Q_{xx}(24)$	4.94	5.40	29.57	10.08	15.39	23.70	7.55	14.40	23.58

a For this panel, the market and bond portfolio statistics are not affected by the trivariate model considered.
* significant at the 10% level
** significant at the 5% level
*** significant at the 1% level

Table 2-8

Diagnostic Summary Statistics for the Trivariate GARCH in Mean Models for the Weekly Observation Period July 1983 to December 1989.

This Table compares the unconditional excess return moments with the standardized residuals from the estimated models for each portfolio under the ICAPM mean restrictions as discussed in Table 2-7. Excess returns for portfolio j are the difference between continuously compounded portfolio returns and the continuously compounded Treasury Bill rate, y_j. Unconditional squared residuals are simply the difference of each excess return from its unconditional sample mean, $(y_{jt} - \bar{y}_j)^2$. Standardized residuals are constructed as the ratio of period t's residual over period t's conditional standard deviation, $\frac{u_{it}}{\sigma_{it}}$, where residuals and standard deviations are based on the trivariate model under the ICAPM restrictions. Finally, squared standardized residuals are constructed as $\left(\frac{u_{it}}{\sigma_{it}}\right)^2$. The Box-Pierce Portmanteau statistic is distributed as a χ^2 random variable.

	Trivariate Model for largest size Portfolio			Trivariate Model for mid-size Portfolio			Trivariate Model for smallest size Portfolio		
	Size Portfolio	Market Portfolio	Bond Portfolio	Size Portfolio	Market Portfolio	Bond Portfolio	Size Portfolio	Market Portfolio	Bond Portfolio
Panel A. Raw Excess Returns[a]									
First Five Autocorrelations									
1	.05	.08	.22	.23			.27		
2	.06	.03	.25	-.01			.01		
3	-.07	-.02	.15	.00			.06		
4	-.06	-.07	.14	-.07			-.02		
5	-.08	-.05	.15	-.04			.01		
Box-Pierce Portmanteau Test									
$Q_x(12)$	32.14***	31.81***	66.79***	45.90***			52.76***		
$Q_x(24)$	51.23***	49.11***	83.47***	66.26***			71.34***		

Panel B. Unconditional Squared Residuals[a]									
First Five Autocorrelations									
1	.08	.08	.18	.19			.29		
2	.05	.07	.17	.09			.09		
3	.00	.01	.14	.00			.00		
4	.02	.02	.12	.00			.00		
5	.00	.00	.07	.00			.00		
Box-Pierce Portmanteau Test									
$Q_{xx}(12)$	16.62	16.56	47.57***	19.32*			35.80***		
$Q_{xx}(24)$	18.23	18.54	54.54***	20.16			36.76**		
Panel C. Standardized Residuals									
Mean	-.013875	-.021981	-.055977	-.053881	-.047152	-.024629	-.051258	-.032861	-.018465
Variance	1.1320	1.1491	.93594	1.0049	1.0050	1.0045	1.0068	1.0064	1.0047
First Five Autocorrelations									
1	.09	.10	.09	.18	.06	.09	.20	.06	.09
2	.09	.06	.07	.04	.05	.07	.12	.05	.07
3	-.08	-.06	.07	-.03	-.05	.06	.03	-.05	.06
4	-.04	-.05	.05	-.05	-.08	.05	.00	-.08	.05
5	-.06	-.06	.11	-.06	-.05	.10	-.03	-.05	.10
Box-Pierce Portmanteau Test									
$Q_x(12)$	24.38**	24.08**	14.07	32.46***	23.98**	13.23	38.01***	23.06**	13.01
$Q_x(24)$	38.85**	35.87*	33.47*	48.65***	37.64**	32.02	51.73***	36.17*	31.68

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APPENDIX ONE
MODEL DEVELOPMENT

Consider the problem of maximizing expected lifetime utility over consumption and portfolio decisions [see Merton (1971 and 1973)],

$$\max_{(c^k, \omega_i^k)} E_0 \left[\int_0^{T^k} U^k[c^k(s),s]ds + B^k[W^k(T^k),T^k] \right] \tag{A1}$$

subject to the budget constraint

$$dW^k = \left[\sum_{i=1}^{n} \omega_i^k(\alpha_i - r_f) + r_f \right] W^k dt + \left[\sum_{i=1}^{n} \omega_i^k W^k \sigma_i dq_i \right] - c^k dt \tag{A2}$$

and the boundary condition

$$J^k(W^k,T,r_f) = B^k(W^k,T), \tag{A3}$$

where E_0 is the conditional expectation operator at time zero given initial wealth, $W^k(0)$, and the state variables of the investment opportunity set,

$c^k(t)$ is the consumption flow of agent k at time t,

$U^k[c^k(t),t]$ is the von Neumann-Morgenstern utility k at time t,

$B^k[W^k(T^k),T^k]$ is the strictly concave bequest function of agent k for the terminal date T^k which need not be known,

ω_i^k is the fraction of wealth agent k holds in asset i $\in \{1,2,...,n\}$ which may also vary over time, and

$J^k(\cdot)$ is agent k's derived utility of wealth function[36].

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[36] The derived utility of wealth function is the maximum value function,

$$J^k(W^k,t,r_f) = \max_{(c^k, \omega_i^k)} E_t \left[\int_t^{T^k} U^k[c^k(s),s]ds + B^k[W^k(T^k),T^k] \right].$$

That is, $J(\cdot)$ gives the maximum expected lifetime utility for agent k over the remainder of his/her life at any point in time t. Using $J(\cdot)$ allows Merton to employ the stochastic Bellman equation to solve the problem providing the solutions shown above. The interested reader is referred to Merton (1971, pp.380-381).

The remaining terms used in the above problem formulation (i.e. α_i, r_f, σ_i, and dq_i) relate to asset price dynamics and changes in the investment opportunity set. Asset price dynamics are assumed to satisfy,

$$\frac{dP_i}{P_i} = \alpha_i dt + \sigma_i dq_i, \tag{A4}$$

where time subscripts have been ignored for notational convenience, dq_i is a Wiener process and α_i is asset i's instantaneous expected rate of return with instantaneous standard deviation σ_i. The single stochastic state variable r_f (the instantaneous riskless rate) similarly satisfies the diffusion process,

$$dr_f = fdt + \sigma_f dq_f, \tag{A5}$$

where time subscripts are again ignored for simplicity, dq_f is a Wiener process and f is the expected change in the riskless rate in the next instant with instantaneous standard deviation σ_f. Further, let ρ_{ij} be the instantaneous correlation coefficient between dq_i and dq_j. It is assumed that (A4) and (A5) together form a Markov system.

Given the above problem formulation, the investor's solution satisfies the following two first order conditions [see Merton (1973)],

$$0 = U^k_c(c^k,t) - J^k_w(W^k,t,r_f), \tag{A6}$$

and

$$0 = J^k_w(\alpha_i - r_f) + J^k_{ww} \sum_{j=1}^{n} \omega^k_j W^k \sigma_{ij} + J^k_{wr_f}\sigma_f\sigma_i\rho_{if} \tag{A7}$$

for all i = 1,2,...,n.

The intuition underlying these n+1 first order conditions is straightforward and appealing.

Condition A6 simply requires the marginal utility of current consumption equal the marginal utility of wealth. If this condition fails and current consumption provides more utility at the margin than future consumption, then investors have an incentive to shift their consumption plan to include more current consumption and to use future wealth to finance this consumption. This will result in an increase in the marginal utility of future consumption and a reduction in the marginal utility of present consumption until the two equate.

The latter n first order conditions represented by A7 simply require that each security contribute equally to utility at the margin. This contribution to marginal utility can be partitioned into three components. First, the benefit each security provides to portfolio return in excess of the riskless rate. This term represents the payment made to investors for incurring the risks represented by the latter two terms. The first of the latter two terms illustrates the loss in utility associated with an increase in portfolio variance. The third term embodies the loss in utility associated with changes in the investor's opportunity set.

The general three-fund separation theorem developed by Merton assumes there exists an asset whose return is perfectly negatively correlated with the state variable. This relationship is not necessary for Merton's theorem but it does illuminate the underlying discussion and intuition. The asset presumed to have this intuitive relation with changes in short rates is the return on a default free long term Treasury bond. Clearly, this assumption is a simplification of reality, however, the intuition is appealing. Consider an investor holding long term Treasury bonds when an unanticipated decline in short rates occurs. If the decline in short rates is truly unanticipated and will continue for some period of time, then the price of the long term bond will rise until it is again priced in equilibrium. The expected return on the long term bond portfolio will clearly fall to reflect a general decline in rates; however, the existing long term bondholder will simultaneously experience a capital appreciation in bond price. Thus, there is an obvious role for an assumption of ex post long term bond returns being negatively correlated with changes in short term riskless rates. Unfortunately, as short rates change so does the entire yield curve in general. Thus, the assumption of an asset whose return is perfectly negatively correlated with changes in the riskless rate is nontrivial. Fortunately, neither Merton's three fund separation theorem, nor the specific application of Merton's theorem to include a long term Treasury bond, requires this perfect negative correlation assumption between changes in the riskless rate and bond returns. In fact, all that is required is that the correlation between changes in the short rate and the long term bond portfolio are scalar multiples. Clearly, this is a very restrictive assumption, but it does result in a closed and tractable system[37].

Using A6, A7, and the correlation assumption discussed above, Merton (1973, pp. 878-879) shows the following continuous time analogue to the static security market line can be posited,

$$\alpha_i - r_f = \left(\frac{W}{A}\right) \sigma_{iM} + \left(\frac{H\sigma_f}{A\sigma_B}\right) \sigma_{iB} \tag{A8}$$

$$= \frac{\sigma_i\,(\rho_{iM} - \rho_{iB}\rho_{BM})}{\sigma_M(1-\rho_{BM}^2)}(\alpha_M - r_f) + \frac{\sigma_i\,(\rho_{iB} - \rho_{iM}\rho_{BM})}{\sigma_B(1-\rho_{BM}^2)}(\alpha_B - r_f)$$

such that

$$W = \text{Total market wealth at time } t = \sum_{k=1}^{K} W^k ,$$

$$A = \sum_{k=1}^{K} A^k , \quad A^k = \frac{-J_w^k}{J_{ww}^k}, \tag{A9}$$

$$H = \sum_{k=1}^{K} H^k , \quad H^k = \frac{-J_{r_f w}^k}{J_{ww}^k}, \tag{A10}$$

$J^k(w,t,r_f) \equiv$ agent k's derived utility function.

Equation (A8) is the fundamental asset pricing relation of this paper. The second line of equation (A8) states that, in equilibrium, the expected excess return on any security i is given by a price of market risk, $\beta_{iM} = \dfrac{\sigma_i\,(\rho_{iM} - \rho_{iB}\rho_{BM})}{\sigma_M(1-\rho_{BM}^2)}$, multiplied by an expected market risk premium,

37 In future research, it would be interesting to consider extending the relationship between long term bonds and changes in the yield curve in general. However, this work would be better served by a data set of returns on bonds with differing terms to maturity at all points in time.

$(\alpha_M - r_f)$, plus a price of bond risk, $\beta_{iB} = \dfrac{\sigma_i\,(\rho_{iB} - \rho_{iM}\rho_{BM})}{\sigma_B(1-\rho_{BM}^2)}$, multiplied by an expected bond risk premium, $(\alpha_B - r_f)$.

To gain further insight into the equation, suppose for simplicity that the bond and market portfolio are independent, then;

$$
\begin{aligned}
(\alpha_B - r_f) &= \frac{\sigma_i\,\rho_{iM}}{\sigma_M}(\alpha_M - r_f) + \frac{\sigma_i\,\rho_{iB}}{\sigma_B}(\alpha_B - r_f) \\
&= \frac{\sigma_{iM}}{\sigma_M^2}(\alpha_M - r_f) + \frac{\sigma_{iB}}{\sigma_B^2}(\alpha_B - r_f) \\
&= \beta_{iM}(\alpha_M - r_f) + \beta_{iB}(\alpha_B - r_f)
\end{aligned}
$$

where β_{ij} is the familiar simple regression slope coefficient. Now notice that the simplification of independence above is unnecessary if β_{iM} and β_{iB} are treated as multiple regression coefficients.

Given the derived utility function of wealth for the k^{th} investor, $J^k(w,t,r_f)$, we can write agent k's coefficient of relative risk aversion with respect to wealth as,

$$\gamma_1^k = \frac{-J_{ww}^k}{J_w^k} W^k. \tag{A11}$$

Now the harmonic mean of all agents' coefficients of relative risk aversion weighted by their share of aggregate wealth, say γ_1, is defined according to,

$$\frac{1}{\gamma_1} = \sum_{k=1}^{K} \frac{W^k}{W}\frac{1}{\gamma_1^k} = \frac{1}{W}\sum_{k=1}^{K} \frac{W^k}{\cdot} = \frac{A}{W} \tag{A12}$$

Thus we have, $\gamma_1 = \dfrac{W}{A}$, is the harmonic mean of all agents' coefficients of relative risk aversion where each agent's coefficient is weighted by his/her share of aggregate wealth as desired. Notice this relation is based upon the derived utility function of wealth[38]. Given the

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[38] The equivalent relation with respect to utility of consumption can be derived using the implicit function theorem upon equation (A6).

above relation, γ_1 can be treated as a constant if power utility is assumed. It is further assumed that the second coefficient term which multiplies σ_{iB}, $\left(\frac{H\sigma_f}{A\sigma_B}\right)$, is constant[39].

$$\frac{\partial}{\partial c}\left(U^k_c(c,t)\right)\frac{\partial c}{\partial w} = \frac{\partial}{\partial w}\left(J^k_w(W,t,r_f)\right),$$

or using the implied notation for partials,

$$U^k_{cc}(c,t)\frac{\partial c}{\partial w} = J^k_{ww}(W,t,r_f).$$

Then using the above and equation (A6) we have,

$$\frac{-J^k_w}{J^k_{ww}} = \frac{-U^k_c}{U^k_{cc}\left(\frac{\partial c}{\partial w}\right)}.$$

39 To be more precise, the term $\left(\frac{H\sigma_f}{A\sigma_B}\right)$ can be considered similarly to $\left(\frac{W}{A}\right)$. Recall that the long term bond portfolio return is perfectly negatively correlated with changes in the instantaneously riskless rate in the Merton paper. Thus, the ratio $\left(\frac{\sigma_f}{\sigma_B}\right)$, is necessarily constant. However, in this case, the ratio $\left(\frac{H}{A}\right)$, cannot be simplified to a constant as in the case of γ_1. In fact, even in the simpler case of a representative investor, constancy of γ_2 is not reasonable. For the representative investor we have, by the implicit function theorem,

$$\left(\frac{H}{A}\right) = \frac{-\left(\frac{J_{wr_f}}{J_{ww}}\right)}{-\left(\frac{J_w}{J_{ww}}\right)} = \frac{J_{wr_f}}{J_w} = \cdot\frac{U'\left(\frac{\partial_c}{\partial_r}\right)}{U_c}$$

Therefore even in the relatively simple representative agent case there is no theoretical support for constancy of γ_2, as it is proportional to the coefficient of absolute risk aversion, not relative risk aversion.

To resolve this issue one could assume that $\left(\frac{H\sigma_f}{A\sigma_B}\right) \equiv \gamma_{2t}$, is given by the form,

$\gamma_{2t} = \gamma_2 + \lambda_{2t}$ where λ_{2t} is a mean zero iid normal random variable.

The model could then be written as,

$$y_{it} = \gamma_1\sigma_{iMt} + \gamma_2\sigma_{iBt} + (\lambda_{2t}\sigma_{iBt} + u_{it})$$

$$\sigma_{ijt} = \sigma_{it}\sigma_{jt}\rho_{ij} \text{ for all i,j, and t}$$

where

$$\sigma^2_{it} = \kappa_i + \sum_{k=1}^{qi}\alpha_{ik}\varepsilon^2_{it-k} + \sum_{l=1}^{pi}\beta_{il}\sigma^2_{it-l} \text{ for all i = 1,2, ..., N,M, and B,}$$

Now assuming rational expectations hold, equation (A8) can be relaxed to examine the actual excess returns, y_{it}, as

$$y_{it} = \delta_1 \left(\frac{\rho_{iM} - \rho_{iB}\rho_{BM}}{1-\rho_{BM}^2} \right) \sigma_{it}\sigma_{Mt} + \delta_2 \left(\frac{\rho_{iB} - \rho_{iM}\rho_{BM}}{1-_{|}} \right) \sigma_{it}\sigma_{Bt} + u_{it} \qquad \text{(A13)}$$

where ρ_{ij} is the correlation between asset i and j assumed constant by construction[40].

$$\varepsilon_t | \phi_{t-1} \sim N(0, COV_t), \quad \text{for } COV_t = (\sigma_{ijt}), \text{ and}$$

$$\varepsilon_{it} = \lambda_{2t}\sigma_{iBt} + u_{it}$$

Notice that under this formulation the new error term, ε_{it}, is heteroscedastic regardless of whether or not the original error u_{it}, is heteroscedastic. Thus without loss of generality we can assume γ_2 is constant within this GARCH paradigm. Moreover, the development also provides a motivation for the GARCH process if u_{Bt} is heteroscedastic, even if u_{it} is homoscedastic. One drawback with this approach is an efficiency loss if in fact the original errors, u_{it}, are GARCH processes. This latter approach is not explicitly followed due to the ad hoc nature of the random specification for γ_{2t} above. Instead, γ_2 is treated as a constant with the understanding that this specification is somewhat robust to misspecification due to the GARCH variance processes.

40 The constancy assumptions for γ_1 and γ_2 allow us to write the mean relation in (A13) with $\delta_1 = \frac{(\alpha_{Mt} - r_{ft})}{\sigma_{Mt}^2}$ and $\delta_2 = \frac{(\alpha_{Bt} - r_{ft})}{\sigma_{Bt}^2}$ constant using the assumption of constant correlations. This is done to simplify the interpretation of the estimation results.

APPENDIX TWO
DATA SOURCES

The data series employed are from two primary sources. Raw nominal stock returns are extracted from the CRSP daily data tapes, while the weekly nominal long term bond returns and the weekly riskless rates are from Reuters Canada Limited.

Nominal security weekly returns are computed from Thursday close to Thursday close for the period July 14, 1983 until December 15, 1989. If the exchange is closed on Thursday, then Wednesday's return is used. Similarly, if the exchange is closed on Wednesday, Friday's return is used A stock with a missing weekly return is a stock with no daily returns for any of Thursday, Wednesday, or Friday in at least one of the 337 weeks. Weekly returns for the CRSP value weighted index are computed similarly.

For each of the 1419 stocks selected at the outset, a weight is constructed as the number of shares outstanding times the price per share divided by the same quantity across all securities, for each week. Then for each week all stocks are size sorted into descending order according to the previous period's weights. Three size portfolios are subsequently formed. The first portfolio contains the largest third of all firms, the second contains the mid-third of all size sorted stocks, and the third portfolio contains the smallest third of all firms. The weekly returns for each of these portfolios is the value weighted return of all included securities.

All 1419 stocks are sorted for each week of the sample period. This procedure does not ensure that any given stock will remain in the same portfolio over time. For example, a rapidly expanding conglomerate would be expected to shift into larger size portfolios over time. Thus, although this procedure maintains the desired size rankings of firms over time, it also implicitly requires some portfolio revision. The alternative method of sorting every year, or at some other prespecified interval may be appealing because it requires less portfolio revision. However, the procedure of sorting all stocks every period is important in at least two respects. First, the intertemporal asset pricing model allows agents to continually revise portfolios, thus the data should reflect any possible portfolio changes each period. Second, this specific study considers

the behaviour of size portfolios over time, the selected sorting procedure does not mask any important size portfolio behaviour.

The short term riskless rate and long term U.S. Government bond return are computed using information obtained from Reuters Canada Limited. Specifically, Reuters provided the high, low, and closing prices on Treasury Bills with one week until maturity for each of the sample dates. To compute the 1 week nominal riskless rate, Treasury Bills with 1 week to maturity are selected for each sample date. Thursday closing prices are then used to compute the return on the bill as 100.00 less the 1 week prior closing price divided by the closing price. The data for each of the long term U.S. Treasury Bonds selected includes a cusip number, starting date, maturity date, coupon rate, and maturity date. For each bond, high, low, and closing prices are reported. To compute the return on the long term U.S. Government bond portfolio, a bond with a maturity as close to twenty years is selected for each sample date. The total return of each bond is then computed as the percentage change in flat price. The flat price is defined as the average of the bond's bid and ask prices plus accrued coupon. Accrued coupon is computed as,

$$\text{semiannual coupon rate} * \left(\frac{\text{number of days since the last coupon payment}}{\text{total number of days between consecutive coupon payments}}\right).$$ [41]

[41] This methodology is the same as that employed by Ibbotson Associates (1986, pp.18-19) in the construction of their monthly long term U.S. Government bond portfolio, but it is an approximation to the true weekly return.

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Current Research Interests:

I am currently interested in studying the role of time varying moments in asset pricing models. Understanding how asset return moments change in relation to the evolution of macroeconomic variables is a critical link in economic modelling which is not well understood. To carefully handle this problem requires a detailed statistical model in conjunction with a well structured causal economic model.

I believe that this path of research will assist in the integration of asset pricing models such as the Capital Asset Pricing Model (CAPM) or Intertemporal CAPM with return generating models such as the Arbitrage Pricing Theory (APT). The critical links between these areas lie in understanding how statistical and economic modelling assumptions relate.

Keywords to Describe Research Interests:

Asset Pricing Theory, Portfolio analysis, Multivariate statistics, International Finance, Corporate Control, Agency theory, Signalling problems, and Game theory.