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THE UNIVERSITY OF ALBERTA

BIRKHOFF INTERPOLATION FOR CHEBYSHEVIAN SPLINES

by

Zuowei Shen

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled BIRKHOFF INTERPOLATION FOR CHEBYSHEVIAN SPLINES submitted by Zuowei Shen in partial fulfilment of the requirements for the degree of Master of Science.

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ABSTRACT

In this thesis, we mainly deal with Chebyshevian splines, and interpolation by Chebyshevian splines. Polynomial splines and interpolation by polynomial splines were given a unified and extensive treatment in the book Birkhoff Interpolation, written by G.G. Lorentz, K. Jetter and S.D. Riemenschneider [1]. The main purpose of our investigation is to carry the theory of Birkhoff splines and Birkhoff spline interpolation as presented in [1] over to Chebyshevian splines.

In the first part of the thesis, from Chapter One to Chapter Seven, we discuss the Birkhoff kernel for Birkhoff interpolation by extended complete Chebyshevian systems. We also investigate Chebyshevian splines based on an interpolation matrix, and the zeros of Chebyshevian splines. We get similar results to ones obtained in Chapter seven of the book Birkhoff Interpolation for algebraic polynomial interpolation.

The Second part of the thesis consists of Chapter Eight through Eleven. We discuss the generalized spline interpolation matrix. In these Chapters, we depart from the book Birkhoff Interpolation in two ways. Not only do we carry out the analysis for Chebyshevian splines, but we also permit one-sided interpolation at the knot points when the splines are permitted to have a jump discontinuity at these points. And we get the generalized Goodman Theorem.

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Introduction

Over the past more than twenty years, the concept of a polynomial spline has been generalized in many ways, resulting in many kinds of splines. Chebyshevian splines are of particular interest, because they have almost all of the nice properties of the classical polynomial splines. Interpolation of functions is a much older topic, but it too has seen rapid development in the last twenty years. Polynomial interpolation and interpolation by polynomial splines were given a unified and extensive treatment in the book Birkhoff Interpolation, written by G. G. Lorentz, K. Jetter, and S. D. Riemenschneider [1]. The main purpose of our investigation is to carry the theory of Birkhoff splines and Birkhoff spline interpolation as presented in [1] over to Chebyshevian splines. Interpolation by Chebyshevian splines and other generalized splines has been treated by many authors, for example L. L. Schumaker [2], but not in terms of the general theory presented in [1].

In Chapters 1 and 2, we introduce extended complete Chebyshev systems, a concept which generalizes the polynomials while retaining many of their nice properties. We give some of their basic properties that will be necessary for our investigation. Most of these properties are proven in [2, Chapter 9] where a much more extensive treatment can be found. Also in Chapter 1, we define an interpolation matrix and some other basic concepts from Birkhoff interpolation. Birkhoff interpolation is the interpolation function and derivative values at a set of points in a pattern that is described by the matrix.

Chapter 3 is devoted to the notion of a Birkhoff kernel for Birkhoff interpolation by extended complete Chebyshev systems. We get similar results to the ones obtained in [1, Chapter 7] for algebraic polynomial interpolation.

Chebyshevian splines based on an interpolation matrix are introduced in Chapter 4. The Chebyshevian splines are piecewise Chebyshevian polynomials, i.e. they are piecewise linear combinations of functions from an extended complete Chebyshev system connected at points called knots. The continuity conditions at the knots are determined by the matrix. Most previous considerations of Chebyshevian splines correspond to special types of matrices called Hermitian matrices. We compare our definition in this last case to the old one as given in [2].

In Chapters 5 to 7, we discuss the zeros of Chebyshevian splines. Actually, we get the same theorems as in [1, Chapter 7], although we use the more general notion of splines. Most proofs of these theorems can be done by checking the proofs of the corresponding theorems in [1] essentially word by word once some properties about extended complete Chebyshev systems are known.

The second part of the thesis consists of Chapters 8 through 11. We discuss generalized Chebyshevian spline interpolation by introducing a generalized spline interpolation matrix. In these chapters we depart from the book [1] in two ways. Not only do we carry out the analysis for Chebyshevian splines, but we also permit one-sided interpolation at the knot points when the splines are permitted to have jump discontinuities at these points. We succeed in carrying this through far enough to obtain the analogy of the most general theorem known the well-poisedness of spline interpolation, the theorem of Goodman. In the final chapter, we present the notion of duality for our generalized spline interpolation, and prove a duality theorem.

CHAPTER 1

BASIC CONCEPTS AND NOTATIONS

In this chapter, we introduce some basic concepts and notations. Given positive functions $w_i \in C^{N-i+1}[a,b]$, $i = 1, 2, \ldots, N$, define

$$u_1 = w_1(x)$$

$$u_2 = w_1(x) \int_a^x w_2(s_2)\, ds_2$$

$$\vdots$$

$$u_N = w_1(x) \int_a^x w_2(s_2) \int_a^{s_2} \cdots \int_a^{s_{N-1}} w_N(s_N)\, ds_N \ldots ds_2.$$

From the reference book [2], we know these functions are the canonical representation for an extended complete Chebyshevian (*ECT*) system on $[a,b]$. We shall write $\mathcal{U}_N$ for both the system $\{u_i\}_{i=1}^N$ and its span. For emphasis, a linear combination $u = \sum_{j=1}^N a_j u_j$ with real number a_j will be called a $\mathcal{U}_N$-polynomial in the ECT-system $\mathcal{U}_N$.

A matrix

$$E = [e_{i,k}]_{i=1}^{m}{}_{k=0}^{n}, \qquad m \geq 1, \qquad n > 0, \qquad (n = N - 1),$$

is an interpolation matrix, if its elements $e_{i,k}$ are 0 or 1 and if the number of 1's in E is equal to N, that is $|E| = \sum e_{i,k} = N$. In general, we do not allow empty rows, that is, an i for which $e_{i,k} = 0$, $k = 0, 1, \ldots, n$.

A set of knots $X = \{x_1, \ldots, x_m\}$ consists of m distinct points from an interval $[a,b]$. The elements $E, X, \mathcal{U}_N$ and the data $(c_{i,k})$ (defined for $e_{i,k} = 1$) determine a Birkhoff interpolation problem which is to find an unique $\mathcal{U}_N$ polynomial S that satisfies

$$L_k S(x_i) = c_{i,k}, \qquad e_{i,k} = 1, \tag{1.1}$$

where L_k are differential operators defined inductively by

$$D_0 f = f, \qquad D_i f = D\left(\frac{f}{w_i}\right), \tag{1.2}$$

and

$$L_i f = D_i \cdots D_0 f. \tag{1.3}$$

The system (1.1) consists of N linear equations with N unknowns a_j. The pair E, X, is called regular if the equations (1.1) have an unique solution for each given set of $c_{i,k}$, otherwise the pair E, X is singular. A pair E, X is regular if and only if the determinant of the system

$$D(E, X) := \det\left[L_k u_1(x_i) \cdots L_k u_N(x_i); \qquad e_{i,k} = 1\right] \tag{1.4}$$

is different from zero.

A matrix is a regular interpolation matrix, if the pair E, X is regular for each set of knots $X \subseteq [a, b]$. We call an interpolation matrix normal if it has as many 1's as columns.

Formula (1.4) displays only one row of the determinant, namely, the row corresponding to a pair (i, k) with $e_{i,k} = 1$. We order the pairs in (1.4) lexicographically; the pair (i, k) precedes (i', k') if and only if $i < i'$ or $i = i'$ and $k < k'$. By $A(E, X)$ we denote the $N \times N$ matrix that appears in (1.4).

Let E be an $m \times (n+1)$ interpolation matrix. Then $m_k = \sum_{i=1}^{m} e_{i,k}$ is the number of 1's in columns k and

$$M_r = \sum_{k=0}^{r} m_k = \sum_{k=0}^{r} \sum_{i=1}^{m} e_{i,k}$$

is the number of 1's in columns of E numbered $0, 1, \ldots, r$. For normal matrices, the condition

$$M_r \geq r+1, \qquad r = 0, 1, 2, \ldots, n \tag{1.5}$$

is called the Polya condition. Then automatically, $M_n = n+1$. Subtracting this from the inequality (1.5), we see that for normal matrices (1.5) is equivalent to

$$\sum_{k=r+1}^{n} m_k \leq n - r, \qquad r = 0, 1, 2, \ldots, n \tag{1.6}$$

CHAPTER 2

PROPERTIES OF ECT SYSTEMS

In this chapter, we will give some basic properties of ECT systems. Some of them are from [2], so we omit the proof of those theorems.

Theorem 2.1. *Suppose that $f_0, f_1, \ldots, f_{N-1}$ are given real numbers. Then for any $c \in [a,b]$ there exists an unique member u of U_N such that*

$$L_i u(c) = f_i, \qquad i = 0, 1, \ldots, N-1$$

Proof: Let

$$u = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_N u_N$$

Since

$$L_k u_i(c) = \begin{cases} 0, & \text{if } k \geq i, \\ w_i(c), & \text{if } k = i-1, \\ w_{k+1}(c) \int_a^c \cdots \int_a^{s_{i-1}} w_{k+2}(s_{k+2}) \cdots w_i(s_i)\, ds_i \cdots ds_{k+2}, & \text{if } k < i-1, \end{cases}$$

the $N \times N$ coefficient matrix for the system of equations

$$L_k u(c) = f_k, \qquad k = 0, \cdots, N-1$$

in the unknowns $\alpha_1, \ldots, \alpha_N$ is upper triangular with diagonal elements $a_{i,i} = w_i(c)$. Hence, the determinant of this system equals $\prod_{i=1}^{N} w_i(c) \neq 0$. This completes the proof of uniqueness. ∎

Define

$$g_j(x,y) = \begin{cases} h_j(x,y), & x \geq y \\ 0, & \text{otherwise} \end{cases} \tag{2.1}$$

where

$$h_j(x,y) = w_1(x)\int_y^x w_2(s_2)\int_y^{s_2}\cdots\int_y^{s_{j-1}} w_j(s_j)\,ds_j\cdots ds_2$$

Theorem 2.2. *Let $h \in L_1[a,b]$ and suppose $f_0, \ldots, f_{N-1}$ are given real numbers. Let u be the unique member of $\mathcal{U}_N$ such that*

$$L_i u(a) = f_i \qquad i = 0, 1, \cdots, N-1$$

Then

$$f(x) = u(x) + \int_a^b g_N(x,y)h(y)\,dy$$

is the unique solution of the initial-value problem

$$L_N f(x) = h(x) \qquad x \in [a,b]$$

$$L_i f(a) = f_i \qquad i = 0, 1, \ldots, N-1$$

Proof. [2] Page 376 Theorem 9.15. ∎

Theorem 2.3. *Suppose that $L_N f \in L_1[a,b]$, then for all $a \le x \le b$*

$$f(x) = u_f(x) + \int_a^b g_N(x,y)L_N f(y)\,dy$$

where u_f is the function in $\mathcal{U}_N$ such that

$$L_i u_f(a) = L_i f(a) \qquad i = 0, 1, \ldots, N-1 \;.$$

Proof: [2] page 375 Theorem 9.16. ∎

Now we introduce a dual set of functions associated with a canonical ECT-system $\{u_i\}_1^N$. Given $u_1, \cdots, u_N$ defined as in the beginning of the Chapter 1, with the weight functions $w_1, \ldots, w_N$, we define the dual canonical ECT system $\mathcal{U}_N^* = \{u_i^*\}_{i=1}^N$ by

$$
\begin{aligned}
u_1^* &= 1 \\
u_2^* &= \int_a^x w_N(s_N)\, ds_N \\
&\ \vdots \\
u_N^* &= \int_a^x w_N(s_N) \int_a^{s_N} \cdots \int_a^{s_3} w_2(s_2)\, ds_2 \cdots ds_N .
\end{aligned}
\tag{2.2}
$$

Associated with this dual ECT system, we have the operators

$$
L_i^* = D_i^* \cdots D_0^*, \qquad i = 0, 1, \ldots, N, \tag{2.3}
$$

where $D_0^* f = f$ and

$$
D_i^* = \frac{1}{w_{N-i+1}} Df, \qquad i = 1, 2, \ldots, N. \tag{2.4}
$$

Clearly, span $\{u_i^*\}_{i=1}^N$ is the null space of L_N^* and

$$
L_j^* u_i^*(a) = 0, \qquad j = 0, 1, \ldots, i-2, \quad i = 1, 2, \ldots, N.
$$

Given an ECT-system $\mathcal{U}_N$ and its dual canonical ECT system $\mathcal{U}_N^*$ as in (2.2), we define their jth reduced systems, $j = 0, 1, \ldots, N-1$, by

$$
\begin{aligned}
u_{j,1} &= w_{j+1}(x) \\
u_{j,2} &= w_{j+1}(x) \int_a^x w_{j+2}(s_{j+2})\, ds_{j+2} \\
&\ \vdots \\
u_{j,N-j}(x) &= w_{j+1}(x) \int_a^x w_{j+2}(s_{j+2}) \int_a^{s_{j+2}} \cdots \int_a^{s_{N-1}} w_N(s_N)\, ds_N \cdots ds_{j+2},
\end{aligned}
\tag{2.5}
$$

for U_N, and

$$
\begin{aligned}
u^{*}_{j,1} &= 1 \\
u^{*}_{j,2} &= \int_a^x w_{N-j}(s_{N-j})\, ds_{N-j} \\
&\vdots \\
u^{*}_{j,N-j}(x) &= \int_a^x w_{N-j}(s_{N-j}) \int_a^{s_{N-j}} \cdots \int_a^{s_3} w_2(s_2)\, ds_2 \cdots ds_{N-j},
\end{aligned}
\tag{2.6}
$$

for U^{*}_N. Note that when $j = 0$ the reduced systems are just the original systems.

A very useful formula that involves the reduced systems is

$$
\begin{aligned}
w_{r+1}(x) \int_y^x \int_y^{s_{r+2}} \cdots \int_y^{s_{j-1}} w_j(s_j) \cdots w_{r+2}(s_{r+2})\, ds_j \cdots ds_{r+2} \\
= \sum_{i=r+1}^{j} (-1)^{j-i} u_{r,i-r}(x) u^{*}_{N-j,j-i+1}(y).
\end{aligned}
\tag{2.7}
$$

A proof of this appears in [2, p374]. It is proved by using the following equation, not proven in [2], which will also be used later:

Theorem 2.4. *For $r+2 < j-1$,*

$$
\begin{aligned}
\int_a^y \int_y^{s_{r+2}} \cdots \int_y^{s_{j-1}} w_j(s_j) \cdots w_{r+2}(s_{r+2})\, ds_j \cdots ds_{r+2} \\
= (-1)^{j-r-2} \int_a^y \int_a^{s_j} \cdots \int_a^{s_{r+3}} w_{r+2}(s_{r+2}) \cdots w_j(s_j)\, ds_{r+2} \cdots ds_j.
\end{aligned}
$$

Proof:(cf [2] page 375) We prove it by induction. For $j = r+2+1$ we have

$$
\begin{aligned}
&\int_a^y \int_y^{s_{r+2}} w_{r+3}(s_{r+3}) w_{r+2}(s_{r+2})\, ds_{r+3}\, ds_{r+2} \\
&\quad = \int_y^a \int_y^{s_{r+2}} w_{r+3}(s_{r+3}) w_{r+2}(s_{r+2})\, ds_{r+3}\, ds_{r+2} \\
&\quad = (-1) \int_a^y \int_a^{s_{r+3}} w_{r+3}(s_{r+3}) w_{r+2}(s_{r+2})\, ds_{r+3}\, ds_{r+2}.
\end{aligned}
$$

We suppose that for $j = r + 2 + k$, the result holds and prove it for $j = r + 2 + k + 1$.

Let

$$A = \int_a^y \int_y^{s_{r+2}} \cdots \int_y^{s_{r+2+k}} w_{r+k+3}\,(s_{r+k+3}) \cdots w_{r+2}\,(s_{r+2})\, ds_{r+k+3} \cdots ds_{r+2}$$

Further, suppose that

$$w_{r+k+2}\,(s_{r+k+2}) \int_y^{s_{r+k+2}} w_{r+k+3}\,(s_{r+k+3})\, ds_{r+k+3} = w^{*}_{r+k+2}\,(s_{r+k+2}).$$

Then

$$
\begin{aligned}
A &= (-1)^{k+2} \int_a^y \int_a^{s_{r+k+2}} \cdots \int_a^{s_{r+3}} w_{r+2}\,(s_{r+2}) \cdots w^{*}_{r+k+2}\,(s_{r+k+2})\, ds_{r+2} \cdots ds_{r+k+2} \\
&= (-1)^{k} \int_a^y w^{*}_{r+k+2}\,(s_{r+k+2}) \left(\int_a^{s_{r+k+2}} \cdots \int_a^{s_{r+3}} w_{r+2}\,(s_{r+2}) \cdots \right. \\
&\qquad \left. \cdots w_{r+k+1}\,(s_{r+k+1})\, ds_{r+2} \cdots ds_{r+k+1} \right) ds_{r+k+2} \\
&= (-1)^{k} \int_a^y w_{r+k+2}\,(s_{r+k+2}) \int_y^{s_{r+k+2}} w_{r+k+3}\,(s_{r+k+3})\, ds_{r+k+3} \left(\int_a^{s_{r+k+2}} \right. \\
&\qquad \left. \cdots \int_a^{s_{r+3}} w_{r+2}\,(s_{r+2}) \cdots w_{r+k+1}\,(s_{r+k+1})\, ds_{r+2} \cdots ds_{r+k+1} \right) ds_{r+k+2}.
\end{aligned}
$$

We consider

$$w_{r+k+2}\,(s_{r+k+2}) \int_a^{s_{r+k+2}} w_{r+2}\,(s_{r+2}) \cdots w_{r+k+1}\,(s_{r+k+1})\, ds_{r+2} \cdots ds_{r+k+1}$$

as a function of s_{r+k+2} and use the result for $j = r + 2 + 1$ to obtain

$$A =$$

$$(-1)^{k+1} \int_a^y \int_a^{s_{r+k+3}} \cdots \int_a^{s_{r+3}} w_{r+2}\,(s_{r+2}) \cdots w_{r+k+3}\,(s_{r+k+3})\, ds_{r+2} \cdots ds_{r+k+3}\,.$$

This completes the equality. ∎

For the dual ECT system $\mathcal{U}_N^*$, we also need functions $g_j^*(x,y)$ and $h_j^*(x,y)$ corresponding to $g_j(x,y)$ and $h_j(x,y)$ for the system $\mathcal{U}_N$:

$$g_j^*(x,y) = \begin{cases} h_j^*(x,y), & x \geq y \\ 0, & \text{otherwise,} \end{cases} \tag{2.8}$$

where

$$h_j^*(x,y) = w_{j+1}(x) \int_y^x \int_y^{s_{j+2}} \cdots \int_y^{s_{N-1}} w_N(s_N) \cdots w_{j+2}(s_{j+2})\, ds_N \cdots ds_{j+2}.$$

Theorem 2.5. *For all x and y, $a \leq y \leq b$, we have*

$$h_j(x,y) = \sum_{i=1}^{j} u_i(x) u^*_{N-j,j-i+1}(y)(-1)^{j-i}, \qquad j = 1, \ldots, N$$

and

$$h_j^*(x,y) = \sum_{i=j+1}^{N} u^*_{j,i-j}(x) u^*_{N-i+1}(y)(-1)^{N-i}, \qquad j = 1, \ldots, N.$$

Moreover, for each fixed y,

$$L_i^{(x)} g_j(x,y)\big|_{x=y} = \delta_{i,j-1} w_j(y) \qquad i = 0, 1, \ldots, j-1,$$

and for each fixed x,

$$L^{*(y)}_{N-k}\, g_j^*(x,y)\big|_{x=y} = \delta_{k,j+1}(-1)^{N-j-1}, \qquad k = 0, \ldots, j+1.$$

Proof: [2] page 374 Theorem 9.13. ∎.

For the determinant

$$\widetilde{D}(E,X) = \det\left[L_k u_1(x_i) \cdots L_k u_N(x_i); \quad e_{i,k-1}\right]$$

and

$$u_i(x) = w_1(x) \int_a^x w_2(s_2) \int_a^{s_2} \cdots \int_a^{s_{i-1}} w_i(s_i)\, ds_i \cdots ds_2$$

we can consider $D(E,X)$ as a function of a sometimes. To emphasize that $D(E,X)$ is a function of a, we denote $D(E,X)$ by $D(E,X)_a$.

Theorem 2.6. *The determinant*

$$D(E,X) = \det\left[L_k u_1(x_i) \cdots L_k u_N(x_i); \qquad e_{i,k} \quad 1\right]$$

where $L_k u_i(x) = L_k u_i(x)_a$ *and*

$$L_k u_i(x)_a := \begin{cases} 0, & \text{if } k > i, \\ w_i(x), & \text{if } k = i - 1, \\ w_{k+1}(x) \int_a^x \cdots \int_a^{s_{i-1}} w_{k+2}(s_{k+2}) \cdots w_i(s_i)\, ds_i \cdots ds_{k+2}, & \text{if } k < i - 1, \end{cases}$$

is independent of a.

Proof: First we notice that for

$$G(u) = \int_{a(u)}^{b(u)} f(x,u)\, dx$$

we have

$$G'(u) = \int_{a(u)}^{b(u)} f'_u(x,u)\, dx + f(b(u),u)b'(u) - f(a(u),u)a'(u).$$

Therefore, if $k < i - 1$

$$\frac{dL_k u_i(x)_a}{da} = w_{k+1}(x) \int_a^x w_{k+2}(s_{k+2}) \frac{d}{da}\left[\int_a^{s_{k+2}} w_{k+3}(s_{k+3}) \cdots \int_a^{s_{i-1}} w_i(s_i)\, ds_i \cdots ds_{k+3}\right] ds_{k+2}$$
$$- w_{k+1}(x) w_{k+2}(a) \int_a^a w_{k+3}(s_{k+3}) \cdots \int_a^{s_{i-1}} w_i(s_i)\, ds_i \cdots ds_{k+3},$$

and is 0 otherwise. The second term of the above formula is 0, so that

$$\frac{dL_k u_i(x)_a}{da} =$$
$$w_{k+1}(x) \int_a^x w_{k+2}(s_{k+2}) \frac{d}{da}\left[\int_a^{s_{k+2}} w_{k+3}(s_{k+3}) \cdots \int_a^{s_{i-1}} w_i(s_i)\, ds_1 \cdots ds_{k+3}\right] ds_{k+2}$$

when $k = i - 1$. Continuing this process, we get

$$\frac{dL_k u_i(x)_a}{da} = \begin{cases} 0, & \text{if } k \neq i, \\ -w_i(a) L_k u_{i-1}(x), & \text{if } k = i - 1, \end{cases}$$

$$= -w_i(a) L_k u_{i-1}(x).$$

For $D(E, X)$, we compute the derivative with respect to a for the ith column $i > 0$. The derivative of each column is the proceding column multiplied by $-w_i(a)$, and the derivative of the first column is 0. Hence, $\frac{d(D(E,X))}{da} = 0$. Therefore, $D(E, X)$ is independent of a. ∎

As a consequence of this Theorem and its proof, we note the following:

Corollary 2.7. *If u is an U_N-polynomial on an interval $[c, d] \subset [a, b]$ and u has more than $N - 1$ distinct zeros in $[c, d]$, then $u \equiv 0$.*

Proof: Suppose that $x_1 < \cdots < x_N$ are distinct zeros of $u = \alpha_1 u_1 + \cdots + \alpha_N u_N$ where $U_N = u_1, \ldots, u_N$ is an ECT-system with weight functions $w_1(x), \ldots, w_N(x)$. The determinant for the system of equations $u(x_i) = 0$, $i = 1, \ldots, N$, is

$$\det\left[u_1(x_i) \cdots u_N(x_i); \quad i = 1, \ldots, N\right].$$

By Theorem 2.6, we may replace a by x_1 in the definition of u_i in this determinant. The first row of the resulting determinant is $w_1(x_1), 0, \ldots, 0$. Hence, the determinant equals

$$w_1(x_1) \det \begin{bmatrix} u_2(x_2)_{x_1} & \cdots & u_N(x_2)_{x_1} \\ \vdots & \vdots & \vdots \\ u_2(x_N)_{x_1} & \cdots & u_N(x_N)_{x_1} \end{bmatrix}.$$

The same argument as in Theorem 2.6 shows that the remaining determinant is independent of x_1. Therefore, we may continue this process to find that the determinant of our original system is $\prod_{i=1}^{N} w_i(x_i) > 0$. ∎

CHAPTER 3

BIRHKOFF'S KERNEL FOR ECT-SYSTEMS

In this chapter we discuss the Birkhoff kernel for ECT systems. The presentation follows closely that of [1, Chapter 7], but the details are quite different.

Let

$$E = [e_{i,k}]_{i=1}^{m}{}_{k=0}^{n}$$

be a normal interpolation matrix with 0's in the last column, and let $X = \{a < x_1 < \cdots < x_m < b\}$ be an arbitrary set of knots. We assume that $x_1 < x_2 < \cdots < x_m$. In the determinant $D(E, X)$ defined in (1.4), we replace the entries in the last column by the element $L_k g_{N-1}(x_i, t)$. The resulting real function

$$K_E(X,t) = K(t)$$

$$= \det \left[L_k u_1(x_i) \cdots L_k u_{N-1}(x_i) \quad L_k g_{N-1}(x_i,t); \qquad e_{i,k} = 1 \right]$$

is the Birkhoff kernel associated with E, X and $\mathcal{U}_N$. The Birkhoff kernel will be a Chebyshevian spline (see Chapter 4) for the system $\mathcal{U}_N^*$.

Theorem 3.1. *(Properties of Birkhoff kernel) Let E be an $m \times n$, $(n = N-1)$, normal interpolation matrix with 0's in the last column and let $X = \{a < x_1 < \cdots < x_m < b\}$. Then Birkhoff's kernel*

$$K_E(X,t) = \det \left[L_k u_1(x_i) \cdots L_K u_{N-1}(x_i) \quad L_k g_{N-1}(x_i,t); \quad e_{i,k} = 1 \right]$$

has the following properties:

a. *$K_E(X,t)$ is a $\mathcal{U}_{N-1}^*$-polynomial on each interval $(x_i, \ldots, x_{i+1})$, $i = 1, \ldots, m-1$,* *and* $L_q^{*(t)} K_E(X, x_{i'}+) \neq L_q^{*(t)} K_E(X, x_{i'}-)$ *is possible only if* $e_{i',n-1-q} = 1$ *and* $D_{i,n-1-q}(X) \neq 0$, *where*

$$D_{i,n-1-q}(X) = \det\left[L_k u_1(x_i) \cdots L_k u_{N-1}(x_i); \quad e_{i,k} = 1, \quad i \neq i', \quad k \neq n-1-q\right].$$

b. $\operatorname{supp}[K_E(X,t)] \subset [x_1, x_m]$.

c. $\int_{-\infty}^{\infty} w_N(t) K_E(X,t)\, dt = D(E,X)$.

d. If E *is not a Pólya matrix, then* $K_E(X,t) = 0$.

Proof: a. By the definition of $K_E(X,t)$ and $g_{N-1}(x,t)$, and evaluating the determinant by the last column, we know that $K_E(X,t)$ is the sum of the

$$\pm L_{n-1-k'}\, g_{N-1}(x_i', t) \det\left[L_k u_1(x_i) \cdots L_k u_{N-1}(x_i);\right.$$
$$\left. e_{i,k} = 1, \quad i \neq i', \quad k \neq n-1-k'\right]$$

for $e_{i',n-1-k'} = 1$.

Since each $L_{n-1-k'}\, g_{N-1}(x_{i'}, t)$ is a piecewise continuous $\mathcal{U}_{N-1}^*$-polynomial with knots $x_1, \ldots, x_m$ and $L_{n-1-k'}\, g_{N-1}(x_{i'}, t) \in \mathcal{U}_{N-1}^*$ on each interval (x_i, x_{i+1}), we have that $K_E(X,t)$ is a $\mathcal{U}_{N-1}^*$-polynomial on each interval (x_i, x_{i+1}), $1 \leq i < m$.

It is clear that $L_q^{*(t)} K_E(X, x_i+) \neq L_q^{*(t)} K_E(X, x_i-)$ only if there is not equality as $t \to x_i+$ and $t \to x_i-$ in one of the terms of the form

$$L_q^* L_{n-1-k}\, g_{N-1}(x_i, t) D_{i,n-1-k}(X)$$

for $e_{i,n-1-k} = 1$. If $k \leq q$, we have

$$L_q^{*(t)} L_{n-1-k}\, g_{N-1}(x_i, t) = \begin{cases} \sum_{N-1-k}^{N-1-q} u_{k,i-k}(x_i) u_{q,N-q-i}^*(t)(-1)^{N-i}, & \text{if } t < x_i; \\ 0, & \text{otherwise,} \end{cases}$$

and $L_q^{*(t)} L_{n-1-k}\, g_{N-1}(x_i,t) = 0$ if $k > q$. Therefore, we have

$$L_q^{*(t)} L_{n-1-k}\, g_{N-1}(x_i, x_i+) = L_q^{*(t)} L_{n-1-k}\, g_{N-1}(x_i, x_i-)$$

for $k \neq q$. If $k = q$,

$$L_q^{*(t)} L_{n-1-k}\, g_{N-1}(x_i,t) = \begin{cases} 1, & \text{if } t > x_i; \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, $L_q^{*(t)} K_E(X, x_i+) \neq L_q^{*(t)} K_E(X, x_i-)$ is possible only if $e_{i,n-1-q} = 1$ and $D_{i,n-1-q}(X) \neq 0$.

b. If $t < x_1$, by using the same argument as in Theorem 2.6, for arbitary real values y the determinant

$$K_y(t) := \det\Big[L_k u_1(x_i)_y \cdots L_k u_{N-1}(x_i)_y \quad L_k g_{N-1}(x_i,t); \qquad e_{i,k} = 1\Big]$$

is independent of y where

$$u_i(x)_y := w_1(x)\int_y^x w_2(s_2)\cdots\int_y^{s_{i-1}} w_i(s_i)\,ds_i\cdots ds_2.$$

So that $K_y(t) = K_E(X,t)$. If we choose $y = t$, if $t < x_i$ for all i, we get that the last column and the second to last column of the determinant are identical. Hence $K_y(t)\Big|_{y=t} = 0 = K_E(X,t)$. If $t > x_m$, the last column of $K_E(X,t)$ is all zeros. That completes the proof of b.

c. By Theorem 2.4 we get,

$$\int_{x_1}^{x_m} w_N(t) L_k g_{N-1}(x_i,t)\,dt = \int_{x_1}^{x_i} w_N(t) w_{k+1}(x_i)\int_t^{x_i}\int_t^{s_{k+2}}\cdots$$

$$\cdots\int_t^{s_{N-1}} w_{N-1}(s_{N-1})\cdots w_{k+2}(s_{k+2})\,ds_{N-1}\cdots ds_{k+2}\,dt$$

$$= w_{k+1}(x_i)(-1)^{N-k}\int_{x_1}^{x_i}\int_{x_i}^{t}\int_{x_i}^{s_{N-1}}\cdots\int_{x_i}^{s_{k+3}} w_N(t) w_{N-1}(s_{N-1})\cdots$$

$$\cdots w_{k+2}(s_{k+2})\,ds_{k+2}\cdots ds_{N-1}\,dt.$$

Using Theorem 2.4 again with $s_N = t$, we get

$$\int_{x_1}^{x_m} w_N(t) L_k g_{N-1}(x_i, t)\, dt$$
$$= w_{k+1}(x_i) \int_{x_1}^{x_i} \cdots \int_{x_1}^{s_{N-1}} w_N(s_N) \cdots w_{k+2}(s_{k+2})\, ds_N \cdots ds_{k+2}$$
$$= L_k u_N(x_i).$$

Therefore,

$$\int_{-\infty}^{\infty} w_N(t) K_E(X,t)\, dt = \int_{x_1}^{x_m} w_N(t) K_E(X,t)\, dt$$
$$= \det \left[L_k u_1(x_i) \cdots L_k u_{N-1}(x_i) \quad \int_{x_1}^{x_m} w_N(t) L_k g_{N-1}(x_i, t)\, dt; \qquad e_{i,k} = 1 \right]$$
$$= \det \left[L_k u_1(x_i) \cdots L_k u_{N-1}(x_i) \quad L_k u_N(x_i); \quad e_{i,k} = 1 \right]$$
$$= D(E, X).$$

d. If E is not a Pólya matrix, then let r, $0 \le r \le N-3$, be the first integer for which $M_r < r+1$. By rearranging the rows, we get

$$K_E(X,t) = \det \begin{pmatrix} \overbrace{*}^{r+1} & \overbrace{*}^{N-r-1} \\ 0 & * \end{pmatrix} \begin{matrix} \}M_r \\ \}N - M_r \end{matrix}.$$

This shows that K vanishes identically. ∎

Theorem 3.2. *Let E be an $m \times N$ normal interpolation matrix with 0's in the last column, and let $X = \{a = x_1 < x_2 < \cdots < x_m = b\}$. Then for all f such that $L_{N-1} f \in L_1[a,b]$ we have*

$$\sum_{e_{i,k}=1} D_{i,k}(X) L_k f(x_i) = \int_{x_1}^{x_m} L_{N-1} f(t) K_E(X,t)\, dt,$$

where $D_{i,k}(X)$ denotes the algebraic complements of the entries of the last column of the determinant defining $K_E(X,t)$.

Proof: Since $L_{N-1} f \in L_1[a,b]$, by Theorems 2.1 and 2.3, there is $u_f \in U_{N-1}$ such that

$$f(x) = u_f(x) + \int_{x_1}^{x_m} g_{N-1}(x,y) L_{N-1} f(y)\, dy.$$

Therefore,

$$u_f(x) = c_1 u_1(x) + \cdots + c_{N-1} u_{N-1}(x)$$

and

$$L_k f(x) = L_k u_f(x) + \int_{x_1}^{x_m} L_k^{(x)} g_{N-1}(x,y) L_{N-1}^{(y)} f(y)\, dy.$$

Hence,

$$\begin{aligned}
\sum_{e_{i,k}=1} & D_{i,k}(X) L_k f(x_i) \\
&= \det\left[L_k u_1(x_i) \cdots L_k u_{N-1}(x_i) \quad L_k f(x_i); \quad e_{i,k} = 1\right] \\
&= \det\left[L_k u_1(x_i) \cdots L_k u_{N-1}(x_i) \quad \sum_{j=1}^{N-1} c_j L_k u_j(x_i); \quad e_{i,k} = 1\right] \\
&\quad + \det\left[L_k u_1(x_i) \cdots L_k u_{N-1}(x_i) \quad \int_{x_1}^{x_m} L_k^{(x)} g_{N-1}(x_i,t) L_{N-1}^{(y)} f(y)\, dy; \quad e_{i,k} = 1\right].
\end{aligned}$$

The first term vanishes since the last column is a linear combination of the preceding columns. The second term gives

$$\begin{aligned}
\int_{x_1}^{x_m} & L_{N-1} f(y) \det\left[L_k u_1(x_i) \cdots L_k u_{N-1}(x_i) \quad L_k g_{N-1}(x_i,y); \quad e_{i,k} = 1\right] dy \\
&= \int_{x_1}^{x_m} L_{N-1} f(y) K_E(X,y)\, dy. \blacksquare
\end{aligned}$$

Remark: The algebraic complements $D_{i,k}$ will differ only by a sign from $D(E_{i,k}, X)$, where the matrix $E_{i,k}$ is derived from E by replacing $e_{i,k} = 1$ by 0 and by omitting the last column of E.

Theorem 3.3. *Let E be an $m \times N$ normal interpolation matrix with 0's in the last column, and let $X = \{a \le x_1 < x_2 < \cdots < x_m \le b\}$. If at least one of the algebraic complements $D_{i,k}$ is not 0 and if the constants $c_{i,k}$ and the integrable function g on $[x_1, x_m]$ satisfy*

$$\sum_{e_{i,k}=1} c_{i,k} D_{i,k}(X) = \int_{x_1}^{x_m} g(t) K_E(X,t)\, dt, \tag{3.1}$$

then there is a unique function f such that $L_{N-1} f(x) \in L_1[x_1, x_m]$ with the properties

$$L_k f(x_i) = c_{i,k} \tag{3.2a}$$

for $e_{i,k} = 1$ and

$$L_{N-1} f(x) = g(x) \quad a.e. \tag{3.2b}$$

Proof: Since $L_{N-1} f(x) \in L_1[x_1, x_m]$, by Theorems 2.3 and 2.1 there exists $u_f \in \mathcal{U}_{N-1}$ such that for all $x \in [x_1, x_m]$ we have

$$f(x) = u_f(x) + \int_{x_1}^{x_m} g_{N-1}(x,y) L_{N-1} f(y)\, dy$$

where $u_f(x) = \sum_{i=1}^{N-1} a_i u_i(x)$ and

$$L_i u_f(x_1) = L_i f(x_1) = a_{i+1} w_i(x_1), \qquad 0 \le i \le N-2.$$

The conditions (3.2) are equivalent to the system

$$\begin{aligned} L_k f(x_i) &= c_{i,k} \\ &= \sum_{j=1}^{N-1} a_j L_k u_j(x_i) + \int_{x_1}^{x_m} L_k^{(x)} g_{N-1}(x_i, y) L_{N-1}^{(y)} f(y)\, dy \\ &= \sum_{j=1}^{N-1} a_j L_k u_j(x_i) + \int_{x_1}^{x_m} L_k^{(x)} g_{N-1}(x_i, y) g(y)\, dy. \end{aligned} \tag{3.3}$$

So that

$$\sum_{j=1}^{N-1} a_j L_k u_j(x_i) = c_{i,k} - \int_{x_1}^{x_m} L_k^{(x)} g_{N-1}(x_i, y) g(y)\, dy \qquad (3.4)$$

We have to show that the at most N equations are uniquely solvable for the $N-1$ unknowns

$$a_1 = \frac{L_0 f(x_1)}{w_1(x_1)}, \ldots, a_{N-1} = \frac{L_{N-2} f(x_1)}{w_{N-1}(x_1)}.$$

We notice that (3.4) means the vanishing of the determinant

A

$$\det \left[L_k u_1(x_i) \ldots L_k u_{N-1}(x_i) \quad c_{i,k} - \int_{x_1}^{x_m} L_{N-1}^{(x)} g_{N-1}(x_i, y) g(y)\, dy, \quad c_{i,k} \quad 1 \right].$$

Since one of the $D_{i,k}(X)$ does not vanish,

$$\text{Rank}(A) = N-1$$

Therefore, the equations (3.4) have an unique solution. By Theorem 2.2, a function f with $L_{N-1} f \in L^1[x_1, x_m]$ is uniquely defined by $L_0 f(x_1), \ldots, L_{N-2} f(x_1)$ and $L_{N-1} f = g$ up to $a.e.$. ∎

Let E, X be a regular pair, $E = [e_{i,k}]_{i=1}^{m}{}_{k=0}^{n}$, $n+1 = N$, and $X = \{a = x_1 < x_2 < \cdots < x_m = b\}$. Let $x \in [x_1, x_m]$ and $q = 0, 1, \ldots, N-1$ be fixed for the moment. We define the extended set of knots

$$\tilde{X} = \{x_1 < x_2 \cdots < x_j < x < x_{j+1} < \cdots < x_m\}$$

and the extended interpolation matrix $\tilde{E}$ as follows: if $x_j < x < x_{j+1}$ we put

$$\tilde{E} = [\tilde{e}_{i,k}]_{i=1}^{m+1}{}_{k=0}^{N} = \begin{pmatrix} e_{1,0} & e_{1,1} & \cdots & e_{1,q} & \cdots & e_{1,N-1} & 0 \\ \vdots & \vdots & & \vdots & & \vdots & \vdots \\ e_{j,0} & e_{j,1} & \cdots & e_{j,q} & \cdots & e_{j,N-1} & 0 \\ 0 & 0 & \cdots & 1 & \cdots & 0 & 0 \\ e_{j+1,0} & e_{j+1,1} & \cdots & e_{j+1,q} & \cdots & e_{j+1,N-1} & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ e_{m,0} & e_{m,1} & \cdots & e_{m,q} & \cdots & e_{m,N-1} & 0 \end{pmatrix}$$

where the 1 in the row $j+1$ is $\tilde{e}_{j+1,q} = 1$.

If $x_j = x$ and $e_{j,q} = 0$, then $\tilde{E} = [\tilde{e}_{i,k}]_{i=1}^{m+1}{}_{k=0}^{N}$ is obtained from E by replacing the entry $e_{j,q} = 0$ by $\tilde{e}_{j,q} = 1$ and adding a column of zeros. If $x = x_j$ and $e_{j,q} = 1$, we do not define $\tilde{E}$ but put $K_{\tilde{E}}(\tilde{X}, t) = 0$. Then we get

Theorem 3.4. *Suppose that $u_f(x) \in U_N$ is the U_N-polynomial which interpolates a function f with $L_N f \in L_1[a,b]$ as specified by the regular pair E, X. If $\tilde{E}, \tilde{X}$ and q are as described before, then*

$$L_q f(x) - L_q u_f(x) = \frac{(-1)^{n+1+\epsilon}}{D(E,X)} \int_{x_1}^{x_m} L_N f(t) K_{\tilde{E}}(\tilde{X}, t)\, dt,$$

where ϵ is the number of 1's in $\tilde{E}$ that precede the new 1 (in the lexicographic order).

Proof: For the situation $x = x_j$ for some j, and $e_{j,q} = 1$, since we put $K_{\tilde{E}}(\tilde{X}, t) = 0$, and since u_f interpolates f at x_j, the result is obviously true.

For the other cases, by using Theorem 3.2 for the function $f(x) - u_f(x)$, we get

$$(L_q f(x) - L_q u_f(x)) \tilde{D}_{j,q} = \int_{x_1}^{x_m} (L_N f(t) - L_N u_f(x)) K_{\tilde{E}}(\tilde{X}, t)\, dt.$$

Since $u_f(x) \in \mathcal{U}_N$, $L_N u_f(x) = 0$. So that

$$(L_q f(x) - L_q u_f(x)) D_{j,q} = \int_{x_1}^{x_m} L_N f(t) K_E(X,t)\, dt.$$

By the definition of E we have

$$D_{i,q} = (-1)^{n+1+\epsilon} D(E,X),$$

where ϵ is the number of 1's in E that precede the new 1 (in the lexicographic order).

Therefore,

$$L_q f(x) - L_q u_f(x) = \frac{(-1)^{n+1+\epsilon}}{D(E,X)} \int_{x_1}^{x_m} L_N f(t) K_E(X,t)\, dt. \quad \blacksquare$$

Theorem 3.5. *For matrices E as in Theorem 3.2 and for all functions f with $L_{N-1} f(x) \in C[x_1, x_m]$, we have*

$$\sum_{e_{i,k}=1} D_{i,k} L_k f(x_i) = \frac{L_{N-1} f(\xi)}{w_N(\xi)} D(E,X)$$

for some $\xi \in (x_1, x_m)$, if and only if Birkhoff's kernel $K_E(X,t)$ does not change sign.

Proof: Sufficiency: By Theorem 3.2

$$\sum_{e_{i,k}=1} D_{i,k} L_k f(x_i) = \int_{x_1}^{x_m} L_{N-1} f(t) K_E(X,t)\, dt = \int_{x_1}^{x_m} \frac{L_{N-1} f(t)}{w_N(t)} w_N(t) K_E(X,t)\, dt.$$

Since $w_N > 0$, $L_{N+1}f(t)/w_N(t) \in C[x_1, x_m]$ and $w_N(t)K_E(X,t)$ does not change sign. Therefore, by the mean value theorem and Theorem 3.1c, there is $\xi \in (x_1, x_m)$ such that

$$\sum_{i,k=1} D_{i,k} L_k f(x_i) = \frac{L_{N+1}f(\xi)}{w_N(\xi)} \int_{x_1}^{x_m} w_N(t) K_E(X,t)\,dt$$
$$= \frac{L_{N+1}f(\xi)}{w_N(\xi)} D(E,X).$$

On the other hand, we shall show that if the piecewise continuous function K changes sign, then there is a continuous function g, for which the formula

$$\int_{x_1}^{x_m} g(t) K_E(X,t)\,dt = \frac{g(\xi)}{w_N(\xi)} \int_{x_1}^{x_m} w_N(t) K_E(X,t)\,dt$$

does not hold, for any $\xi \in [x_1, x_m]$.

Assume K changes sign, that is, $K > 0$ and $K < 0$ both hold on sets of positive measure. Let

$$\delta = \operatorname{sgn} \int_a^b w_n(t) K_E(X,t)\,dt.$$

We assume $\delta \neq 0$. Since $K_E(X,t)$ is a piecewise continuous function, we may find a continuous function $g > 0$ such that

$$\emptyset \neq \operatorname{supp} g \subset \{t : \operatorname{sgn} K_E(X,t) = -\delta\}.$$

Then

$$\operatorname{sgn} \int_{x_1}^{x_m} K_E(X,t) g(t)\,dt = -\delta, \tag{3.5}$$

but

$$\operatorname{sgn}\left(\frac{g(\xi)}{w_N(\xi)} \int_{x_1}^{x_m} w_N(t) K_E(X,t)\,dt\right) = \delta \text{ or } 0$$

for any $\xi \in [x_1, x_m]$. If $\delta = 0$, then one simply chooses g so that the integral in (3.5) is not zero. ∎

CHAPTER 4

CHEBYSHEVIAN SPLINES

Let $E = [e_{i,k}]_{i=1}^{m}{}_{k=0}^{n}$ be an interpolation matrix, and let $X : a < x_1 < x_2 < \cdots < x_m < b$. Instead of E, the inverted matrix $\hat{E} = [\hat{e}_{i,k}]$ with elements $\hat{e}_{i,k} = e_{i,n-k}$ is often more convenient. The Chebyshevian spline space $S = S(E, X, U_N)$ consists of the functions

$$S(x) = u(x) + \sum_{\substack{\hat{e}_{i,k}=1 \\ 1\le i\le m}} \alpha_{i,k}\, g_{k+1}(x, x_i)$$

where $u(x) \in U_N$, and $n = N - 1$. The real numbers $\alpha_{i,k}$ are defined whenever $\hat{e}_{i,k} = 1$, while $g_i(x, y)$ is defined in (2.1) of Chapter 2.

Proposition 4.1. *The Chebyshevian splines $S \in S(E, X, U_N)$ are precisely the piecewise U_N-polynomials with knots x_i, $1 \le i \le m$, for which $L_k S(x_i+) = L_k S(x_i-)$, $1 \le i \le m$, except possibly when $\hat{e}_{i,k} = 1$ and $\alpha_{i,k} \neq 0$.*

Proof: Let

$$S(x) = u(x) + \sum_{\substack{e_{i,k}=1 \\ 1\le i\le m}} \alpha_{i,k}\, g_{k+1}(x, x_i)$$

be a Chebyshevian spline in $S(E, X, U)$. Since each $g_j(x, x_i)$ is a piecewise U_N-polynomial with knots x_i, $1 \le i \le m$, and since $u(x)$ is a U_N-polynomial, $S(x)$ is a piecewise U_N-polynomial with knots x_i, $1 \le i \le m$.

Since

$$L_{k'}\, g_{k+1}(x, x_i)\,|_{x=x_i} = \begin{cases} \delta_{k',k}\, w_{k+1}(x_i) & \text{if } k' = 1, \cdots, k \\ 0 & \text{if } k' > k+1 \end{cases},$$

$L_{k'} g_{k+1}(x, x_i)$ may have a (jump) discontinuity only if $x = x_i$ and only if $k = k'$. Therefore, $L_k S(x+) \neq L_k S(x-)$ only at knots x_i, $1 \le i \le m$, and only if $\bar{e}_{i,k} = 1$ with $\alpha_{i,k} \neq 0$. ∎

We will now compare the Chebyshevian splines $S(E, X, U_N)$ with the Chebyshevian splines with multiplicities as discussed in [2].

Let $X = \{a = x_1 < x_2 < \cdots < x_m = b\}$ be a partition of the interval $[a, b]$, and let $R = \{r_2, \ldots, r_{m-1}\}$ be a vector of integers with $1 \le r_i \le N$, $i = 2, \ldots, m-1$. Suppose $U_N = \{u_i\}_{i=1}^N$ is defined as in Chapter 1, we say that an element S belongs to $S(R, X, U_N)$, if there exist $S_1, \cdots, S_{m-1}$ in U_N such that

$$S|_{(x_i, x_{i+1})} = S_i, \qquad i = 1, \ldots, m-1,$$

and

$$D^{j-1} S_{i-1}(x_i) = D^{j-1} S_i(x_i), \qquad j = 1, \ldots, N - r_i, \qquad i = 2, \ldots, m-1.$$

The space $S(R, X, U_N)$ forms the space of Chebyshevian spline functions with knots $x_2, \cdots, x_{m-1}$ of multiplicities $r_2, \cdots, r_{m-1}$.

From the reference book [2] page 365, we have for any function $f(x) \in C^{N-1}$

$$L_i f(x) = \frac{D^i f(x)}{w_1(x) \cdots w_i(x)} + \sum_{j=0}^{i-1} a_{i,j}(x) D^j f(x).$$

Thus for an arbitary function $f(x)$ the conditions

$$f(t) = L_1 f(t) = \cdots = L_{z-1} f(t) = 0 \neq L_z f(t)$$

are actually equivalent to

$$f(t) = Df(t) = \cdots = D^{z-1} f(t) = 0 \neq D^z f(t).$$

Therefore, in the definition of $S(R, X, \mathcal{U}_N)$, in place of the required continuity of the ordinary derivatives across the knots we could just as well have required continuity of the corresponding L_j's for an equivalent definition.

Let $E = [e_{i,k}]_{i=1\ k=0}^{m\ \ n}$, $n+1 = N$, be an interpolation matrix, we call a row i of the matrix E Hermitian, if for some r_i, $e_{i,k} = 1$ for $k < r_i$ and $e_{i,k} = 0$ for $k \geq r_i$. A matrix E is quasi-Hermitian if its rows, $i = 2, \ldots, m-1$, are Hermitian.

Theorem 4.2. *Let E be a $m \times N$ quasi-Hermitian interpolation matrix and let r_i denote the number of 1's in each row of E for $i = 2, \cdots, m-1$. Then*

$$S(R, X, \mathcal{U}_N) = S(E, X, \mathcal{U}_N).$$

Proof: That $S(E, X, \mathcal{U}_N) \subseteq S(R, X, \mathcal{U}_N)$ follows from Proposition 4.1.

Suppose now that $S(x) \in S(R, X, \mathcal{U}_N)$. We want to show that S has the representation

$$S = u(x) + \sum_{\substack{\bar{e}_{i,k}=1 \\ 1<i<m}} \alpha_{i,k}\, g_{k+1}(x, x_i).$$

This is certainly true for $x \in (x_1, x_2)$. Assume that $\alpha_{i,k}$, $1 < i < j$, have been chosen so that

$$S(x) = u(x) + \sum_{\substack{\bar{e}_{i,k}=1 \\ 1<i<j}} \alpha_{i,k}\, g_{k+1}(x, x_i)$$

holds on $[x_1, x_j] \backslash \boldsymbol{X}$.

Suppose that $L_k S(x)$ has a jump at x_j when $e_{j,k} = 1$. Set $\alpha_{i,k} = L_k(x_i+) - L_k(x_i-)$ and put

$$f(x) = u(x) + \sum_{\substack{e_{i,k}=1 \\ 1\le i\le j+1}} \alpha_{i,k}\, g_{k+1}(x, x_i).$$

Then $S(x) - f(x) \in \mathcal{U}_N$ for $x \in (x_j, x_{j+1})$, and $L_k S(x_j+) - L_k f(x_j+) = 0$ for $k = 0, \cdots, N-1$. Therefore, $S(x) - f(x) = 0$ for $x \in (x_j, x_{j+1})$, by Theorem 2.1. Hence,

$$S(x) = u(x) + \sum_{\substack{e_{i,k}=1 \\ 1\le i\le j+1}} \alpha_{i,k}\, g_{k+1}(x, x_i)$$

holds on $[x_1, x_{j+1}]\setminus X$.

By the induction principle, we get

$$S(x) = u(x) + \sum_{\substack{e_{i,k}=1 \\ 1\le i\le m}} \alpha_{i,k}\, g_{k+1}(x, x_i)$$

holds for $[x_1, x_m]\setminus X$. ∎

Chebyshevian Birkhoff splines S, the class $S^0 = S^0(E, X, \mathcal{U}_N) \subset S(E, X, \mathcal{U}_N)$, are defined as the splines (4.1) that are $\mathcal{U}_N$-polynomials on each of the intervals $(-\infty, x_1), (x_1, x_2), \cdots, (x_m, \infty)$, are identically 0 outside of the interval (x_1, x_m), and $L_k S(x_i+) \neq L_k S(x_i-)$ only if $e_{i,n-k} = 1$, $i = 1, \ldots, m$. Examples of Chebyshevian Birkhoff splines are given by

Proposition 4.3. *If E is as in Theorem 3.1, then $K_E(X,t) \in S^0(E, X, \mathcal{U}_N^*)$.*

Proof: This follows from Theorem 3.1. ∎

CHAPTER 5

DIAGRAMS AND ZEROS OF CHEBYSHEVIAN SPLINES

In this chapter we give forms of Rolle's Theorem for Chebyshevian splines and use them to count the number of zeros that a Chebyshevian spline can have in its support. Our discussion parallels [1, Sections 7.3 and 7.4]. We repeat many of the definitions from [1] that are required for this discussion.

A Chebyshevian spline could have interval zeros. To distinquish this possibility, we say that a point c, $a < c < b$, is a point zero of the spline S, only if S satisfies $S(c+h)S(c-h) \neq 0$ for all sufficiently small h. We say that a point of discontinuity c of $S(x)$ is a discontinuous zero (of multiplicity 1) if and only if $S(x)$ changes sign at c, that is, if and only if $S(c+h)S(c-h) < 0$ for all sufficiently small h. A continuous zero of $S(x)$ is a point c at which $S(x)$ is continuous ($S(x+) = S(x-)$ for knots) and $S(c) = 0$.

The multiplicity of the zero at a point c is defined as follows: Let

$$S(x), S^{(1)}(x), \cdots, S^{(l)}(x)$$

be a maximal sequence with the property that c is a continuous zero of $S^{(k)}(x)$, $k = 0, 1, \cdots, l-1$, and that c is continuous or discontinuous zero of $S^{(l)}(x)$. Then the multiplicity of c as a zero of S is $l+1$.

From the formula

$$L_i f(x) = \frac{D^i f(x)}{w_1(x)\cdots w_i(x)} + \sum_{j=0}^{i-1} a_{i,j}(x) D^j f(x)$$

and the fact that $w_i(x) > 0$ on $[a,b]$ for all i, we see that the above definition of multiplicity is equivalent to replacing $S(x), \cdots, S^{(l)}(x)$ by $S(x), L_1 S(x), \cdots, L_l S(x)$ in that definition.

Let Γ be the lattice points (i,k), $i = 1,\ldots,m$, $k = 0,\ldots,n$. For each k, let a set of disjoint subintervals $[j,j']$ of $[1,m]$ be defined; we call them intervals of level k. This set of intervals is a lower set Φ (or upper set Ψ) in Γ, if each interval $[j,j']$ of level k, $k \geq 1$, is contained in an interval of level $k - 1$ or correspondingly, if each interval of level k, $k < n$, is contained in an interval of level $k+1$. Simple examples of lower or upper sets are provided by rectangles $i_1 \leq i \leq i_2$; $0 \leq k \leq n$ or $i_1 \leq i \leq i_2$; $r_2 \leq k \leq n$.

A lower set Φ defines a set all of points (i,k) that belong to the intervals of Φ. We also denote it by Φ. However, to a set of points there may correspond several lower sets by virtue of different definitions of the disjoint subintervals $[j,j']$ for each k.

The endpoints of the maximal intervals $[j,j']$ of Φ at level k make up the horizontal boundary of Φ at level k. The number of boundary points is an even number $2L(\Phi)$. We also need the notion of the vertical boundary of Φ. An interval $[j,j']$ is part of the vertical boundary of Φ at level k, $k = 0,1,\ldots,n$, if it is a maximal connected union of intervals $[i,i+1]$ with the following property: Each $[i,i+1]$ is contained in an interval of k of Φ, but not contained in an interval of higher level.

Outside and inside corners of Φ are easily defined. An outside corner is a point (i,k) in the horizontal boundary of Φ for which $(i,k+1)$ is not in the horizontal boundary. For an inside corner (i,k), the point $(i,k+1)$ belongs to the horizontal boundary but (i,k) does not.

A spline $S(x) \in S(E,X,\mathcal{U}_N)$ defines a lower set in a natural way. For each k, we take the maximal interval $[x_j, x_{j'}]$ on which $L_k S(x)$ does not vanish except at isolated points. We denote the collection of these intervals by X_S. The corresponding intervals $[j,j']$ form a lower set called the diagram, $\Phi(S)$, of S. For the interval $[j,j']$ of the vertical boundary of $\Phi(S)$ at kth level, $L_k S(x)$ consists of piecewise nonzero constant multiples of $w_{k+1}(x)$ on each interval $[x_i, x_{i+1}] \subset [x_j, x_{j'}]$. Observe that $L_k S(x)$ is discontinuous at x_j if (i,k) is an outside corner.

Lemma 5.1. *Let $S(x) \in S(E,X,\mathcal{U}_N)$, then each interior position $(i,k) \in \Phi(S)$ (i.e. a position not on the horizontal or vertical boundary) is supported on the right in $\widetilde{E}_\Phi$.*

Here we say that a position $(i,k) \in \Phi(S)$ is supported from the right (or left) in $\widetilde{E}_\Phi$, if there are $\widetilde{e}_{i_1,k_1} = \widetilde{e}_{i_2,k_2} = 1$ in $\widetilde{E}_\Phi$ with $i_1 < i \leq i_2$, $k_1, k_2 > k$ (or respectively $k_1, k_2 < k$).

Proof: Since $L_k S$ is discontinuous at x_j if (i,k) is an outside corner, we must have $\widetilde{e}_{i,k} = 1$ by Proposition 4.1. Therefore, the proof of Lemma 7.9 in [1] is applicable in this case as well. ∎

To investigate the zeros of a spline, we need following definition. If $L_j f(x)$ is an absolutely continuous function on interval (c,d), then we say that c is a left

Rolle's point of $L_j f(x)$ and $L_{j+1} f(x)$ provided that either $L_j f(c+) = 0$ or for every ϵ there exists some t, $c < t < c + \epsilon$, with $L_j f(t) L_{j+1} f(t) > 0$. Similarly, we say that d is a right Rolle's point of $L_j f(x)$ and $L_{j+1} f(x)$ provided either $L_j f(d-) = 0$ or for every ϵ there exists some t, $d - \epsilon < t < d$, with $L_j f(t) L_{j+1} f(t) < 0$.

Theorem 5.2. *(Extended Rolle's Theorem) Suppose $L_{j+1} f(x)$ exists on (c, d) (except for jump discontinuities at isolated points) and that c and d are left and right Rolle's points respectively of $L_j f(x)$ and $L_{j+1} f(x)$, then $L_{j+1} f(x)$ has at least one sign change on (c, d). If $L_{j+1} f(x)$ is continuous on (c, d), then it has at least one zero there.*

Proof: In [2] page 371. ∎

Let $E = [e_{i,k}]_{i=1}^{m}{}_{k=0}^{n}$ be an $m \times n+1$, $N = n+1$, interpolation matrix, and let $X = \{a = x_1 < x_2 < \cdots < x_m = b\}$ be a corresponding set of knots. For the Birkhoff spline $S(x) \in S^0(E, X, \mathcal{U}_N)$, we would like to estimate the number of its zeros. We conduct our investigation within the diagram $\Phi(S)$ of S. This will require another version of Rolle's Theorem, Lemma 5.3, and will lead to an estimate for the number of zeros based on $\widetilde{E}_{\Phi(S)}$, Theorem 5.4.

If the Chebyshevian spline $S(x)$ defined on $[a, b]$ does not vanish on intervals, then for each $\xi \in (a, b)$, the product $\pi = S(\xi + h_1)S(\xi - h_2)$ is either always > 0 or always < 0 for small $h_1, h_2 > 0$. In the first case, $S(x)$ does not change sign at ξ. In the second case, it does change sign there. This definition applies also to arbitrary Chebyshevian splines $S(x)$, but there is a third possibility $\pi = 0$ for all small $h_1 > 0$ or $h_2 > 0$.

Lemma 5.3. *Let $S(x)$ be a Chebyshevian spline on (ξ_1, ξ_2) with $L_jS(\xi_1+) = L_jS(\xi_2-) = 0$. Suppose that both $L_jS(x)$ and $L_{j+1}S(x)$ do not vanish on intervals, then there is a point ξ, $\xi_1 < \xi < \xi_2$, such that either A) ξ is zero of $L_{j+1}S(x)$ with change of sign, but not zero of $L_jS(x)$, or B) ξ is discontinuous zero of $L_jS(x)$ with change of sign, on which $L_{j+1}S(x)$ preserves sign. (Continuity here means $g(x+) = g(x-)$ at knot points.)*

Proof: First we can assume that (ξ_1, ξ_2) contains no other continuous zeros of $L_jS(x)$, for otherwise, we could replace the interval by a smaller one.

Secondly we notice that, by our definition of Chebyshevian spline, for arbitary $S(x) \in S(E, X, \mathcal{U}_N)$, the splines $L_kS(x)$, $0 < k < n$, have their only possible discontinuity points at the knots x_i, $1 < i < m$.

Therefore, for ξ_1 we have ϵ_1 such that when $\xi_1 < x < \xi_1 + \epsilon_1$, $L_jS(x)$ and $L_{j+1}S(x)$ are continuous and do not change sign (see Corollary 2.7). So that if $\xi_1 < x < \xi_1 + \epsilon_1$, and $L_jS(x) > 0$, then $L_{j+1}S(x)$ should be greater than zero, since $w_{j+1}(x) > 0$. Therefore, $L_jS(x)L_{j+1}S(x) > 0$ as $\xi_1 < x < \xi_1 + \epsilon_1$. Likewise, if $L_jS(x) < 0$, then $L_{j+1}S(x) < 0$ for $\xi_1 < x < \xi_1 + \epsilon_1$. Similarly, there is an $\epsilon_2 > 0$ such that $L_jS(x)$ and $L_{j+1}S(x)$ do not change sign on $\xi_2 - \epsilon_2 < x < \xi_2$, and one can deduce that $L_jS(x)L_{j+1}S(x) < 0$ on this interval.

Let $\epsilon = \min\{\epsilon_1, \epsilon_2\}$, we have for all $0 < h < \epsilon$, $L_jS(\xi_1 + h)L_{j+1}S(\xi_1 + h) > 0$ and $L_jS(\xi_2 - h)L_{j+1}(\xi_2 - h) < 0$. Therefore, $L_jS(x)$ and $L_{j+1}S(x)$ have numbers of changes of sign of different parity in (ξ_1, ξ_2) . Hence, there exists a point ξ at which $L_{j+1}S(x)$ but not $L_jS(x)$, or conversely $L_jS(x)$ but not $L_{j+1}S(x)$ changes sign. In the first case ξ cannot be a zero of $L_j S(x)$ (continuous or discontinuous) and we have A). In the second case, ξ is a discontinuous zero of $L_{j+1}S(x)$ as in B). ∎

Remark: We could check every knot in interval (ξ_1, ξ_2) if necessary. If there is some knot x_i, at which $L_j S(x)$ changes sign, but $L_{j+1}S(x)$ does not, then take $x_i = \xi$ for case B). If there is some knot x_j, such that at x_j, $L_{j+1}S(x)$ changes sign but $L_j S(x)$ does not, then take $\xi = x_j$ for a special case of case A). If there are no such knots x_i or x_j, then for any knots in interval (ξ_1, ξ_2), $L_j S(x)$ and $L_{j+1}S(x)$ change or do not change sign at the same time. Then the point ξ at which $L_{j+1}S(x)$ but not $L_j S(x)$ or conversely $L_j S(x)$ but not $L_{j+1}S(x)$ changes sign is not a knot. But we have assumed that $L_j S(x)$ has no continuous zeros in (ξ_1, ξ_2), hence, the second of these cases does not hold, and ξ is a continuous zero of $L_{j+1}S$. The latter is a special case of A).

For a Chebyshevian spline $S(x)$ that does not vanish on the interval (a, b) we call $\xi \in (a, b)$ an even point of $S(x)$, if $S(x)$ does not change sign at ξ and odd point if $S(x)$ changes sign at ξ. We say that $L_j S(x)$ and $L_{j+1}S(x)$ alternate at ξ, if ξ is a point of different parity for $L_j S(x)$ and $L_{j+1}S(x)$. Lemma 5.3 asserts the existence of a zero ξ of $L_j S(x)$ or $L_{j+1}S(x)$, at which $L_j S(x)$, $L_{j+1}S(x)$ alternate.

For a Birkhoff spline $S(x) \in S^0(E, X, \mathcal{U}_N)$, we define its diagram $\Phi(S)$ as above. A sequence in $\Phi(S)$

$$B : \tilde{e}_{i,s} = \cdots = \tilde{e}_{i,s+t} = 1 \qquad (5.1)$$

is a maximal sequence of 1's in row i of the matrix E_Φ, it is interior if each (i,k), $s < k < s+t$ is interior to $\Phi(S)$. By $Z(S(c,d))$, we denote the number of point zeros (counting multiplicities) of the Chebyshevian spline $S(x)$ defined on $[c,d]$.

Theorem 5.4. *Let $S(x) \in S^0(E, X, U_N)$ have support $[c,d] = [x_{i_1}, x_{i_2}]$, if Φ is the diagram of $S(x)$, $\Phi = \Phi(S)$, then*

$$Z(S(c,d)) = |E_\Phi| - L(\Phi) - 1 + \gamma(E_\Phi) \tag{5.2}$$

where $\gamma(E_\Phi)$ is the number of odd interior sequences in E_Φ.

Proof: (Even though the proof requires only small changes to the proof of the Theorem 7.11 in [1], we give all the details.) Let the matrix E_Φ and the Chebyshevian spline $S(x)$ with all its zeros in (c,d) be given. We introduce the following notations:

l_k: the number of intervals of X_S (or of Φ) of level k,

u_k : the number of continuous zeros of $L_k S(x)$ inside X_S which are also zeros of $S(x)$ of multiplicity $\geq k+1$;

v_k : the number of $\tilde{e}_{i,k} = 1$ in the interior of level k of Φ;

ϵ_k := the number of sequences of 1's ending at point (i,k) of Φ, but with the property that $(i, k+1)$ is inside Φ and $L_{k+1} S(x) \neq 0$;

η_k := the number of $\tilde{e}_{i,k} = 1$ for which x_i is a zero of $S(x)$ of multiplicity $k+1$;

h_k := the number of 1's on the horizontal boundary of $\tilde{E}_\Phi$ of level k.

Suppose that $L_{q+1}S(x) = 0$. We note that $u_q = 0$, $e_q = 0$ and $l_0 = 1$. We would like to find a lower bound, μ_k, for the number of continuous Rolle zeros of $L_kS(x)$ (they are disjoint from the zeros counted by u_k). We begin with $\mu_0 = 0$. We apply Lemma 5.3 to derive a lower estimate μ_{k+1} from μ_k. Then $L_kS(x)$ has $\mu_k + u_k$ continuous zeros inside the intervals of X_S. These continuous zeros may or may not be at the knots. We add $2l_k - h_k$ continuous zeros at the end points of intervals. This will give:

$$\mu_k + u_k + 2l_k - h_k$$

continuous zeros of $L_kS(x)$ with

$$\mu_k + u_k + 2l_k - h_k - 1 \tag{5.3}$$

intervals between them.

The endpoints of these intervals cannot belong to the vertical boundary of Φ. For $L_kS(x)$ consists of nonzero piecewise constant multiples of $w_{k+1}(x)$ in contradiction to $L_kS(x)$ having a continuous zero at the endpoints of the intervals. Neither can endpoints of the intervals in (5.3) belong to the $l_{k+1} - 1$ complements $\tilde{I}$ to intervals of X_S of level $k+1$, but intervals in (5.3) may contain some $\tilde{I}$. If we omit all of the latter intervals, we end up with at least

$$\mu_k + u_k + 2l_k - h_k - 1 - (l_{k+1} - 1) = \mu_k + u_k + 2l_k - l_{k+1} - h_k \tag{5.4}$$

intervals I of level k to which we can apply Lemma 5.3, since then $L_kS(x)$ and $L_{k+1}S(x)$ vanish only in isolated points on each of I. Applying Lemma 5.3 gives a point ξ inside each I. This ξ is even inside the intervals of X_S of level $k+1$.

In particular, ξ may be a continuous zero of $L_{k+1}S(x)$ (special case of A) in Lemma 5.3). By the remark after Lemma 5.3, any other possibility would require $\xi = x_i$ for some x_i inside of the intervals of X_S of level $k+1$. To single out continuous zeros, we first discard all I that contain x_i where $L_kS(x)$ could be discontinuous. There are at most $(v_k - \eta_{k+1})$ such x_i. Then discard x_i's at which $L_kS(x)$ is discontinuous, $L_{k+1}S$ is continuous, but $L_{k+1}s(x_i) \neq 0$. There are at most ϵ_k such x_i. Thus the difference between the number given in (5.4) and $(v_{k+1} - \eta_{k+1}) + \epsilon_k$ will be a lower estimate for μ_{k+1}.

It is important that this estimate can be improved. We call a pair (i,k) nonconfirming (to Lemma 5.3) if x_i is counted by $(v_{k+1} - \eta_{k+1})$ or ϵ_k, but cannot be obtained as a continuous zero ξ of level k by Lemma 5.3 and its remark. This will be the case if x_i does not belong to any I, or in case $L_kS(x), L_{k+1}S(x)$ do not alternate at x_i. For each k, $k = 0, 1, \cdots, q-1$, we examine all pairs $(i,k) \in \Phi$ and denote by N_k the number of noncomfirming (i,k) for this k. Since ϵ_k and $v_{k+1} - \eta_{k+1}$ count disjoint pairs, $N = \sum_{k=1}^{q-1} N_k$ will be the total number of nonconfirming pairs in Φ.

At step k, we need to discard only

$$(v_{k+1} - \eta_{k+1}) + \epsilon_k - N_k$$

intervals I. The remaining intervals will contain continuous Rolle zeros of $L_{k+1}S(x)$. Thus

$$\mu_{k+1} \geq \mu_k + u_k + 2l_k - l_{k+1} - h_k - (v_{k+1} - \eta_{k+1}) + N_k - \epsilon_k.$$

Since $l_0 = 1$, $2l_q = h_q$, we obtain by summation

$$0 \le \mu_q = \sum_{k=0}^{q}(u_k + \eta_k) - \sum_{k=1}^{q} v_k - \sum_{k=0}^{q-1} \epsilon_k +$$
$$+ N - \eta_0 - \sum_{k=0}^{q} h_k + \sum_{k=0}^{q} l_k + 1.$$

The first sum is $Z(S(c,d))$, while $\sum_{k=0}^{q}(v_k + h_k) = |\tilde{E}_\Phi|$, $\sum_{k=0}^{q} l_k = L(\Phi)$. Hence

$$Z(S(c,d)) \ge |\tilde{E}_\Phi| - L(\Phi) - 1 + \Delta - N$$

where $\Delta = \sum_{k=0}^{q-1} \epsilon_k + \eta_0 - v_0$.

We want to find an upper bound for $\Delta - N$ for each sequence B of $\tilde{E}_\Phi$ given by (5.1). We study the contributions of B, denoted by Δ_B and N_B, to the numbers Δ and N respectively. Here N_B is the number of nonconfirming pairs in the sequence

$$(i, s-1), (i, s), \cdots, (i, s+t+1) \qquad (s \ge 0). \tag{5.5}$$

Clearly, $N_B \ge 0$. The contribution of the sequence B to $\eta_0 - v_0$ is ≤ 0, whereas its contribution to $\sum_{k=0}^{q-1} \epsilon_k$ it is the same as its contibution to ϵ_{s+t} which may be 0 or 1. Hence,

$$\Delta_B - N_B < 1 \qquad \forall B. \tag{5.6}$$

We shall discard many sequences B for which the difference (5.6) is ≤ 0. In first place, we can assume that none of the pairs (i,k) in (5.5) is nonconfirming. Since $(i, s-1)$, (i,s) is not nonconfirming, that is x_i can be obtained by Lemma 5.3 for $L_{s-1}\, S(x)$ and $L_s S(x)$, x_i cannot be a continuous zero of $L_{s-1}\, S(x)$. Moreover, since none of the pairs $(i, s-1), \cdots, (i, s+t+1)$ is nonconfirming, the sequence $L_{s-1}\, S(x_i), \cdots, L_{s+t}\, S(x_i), L_{s+t+1}\; S(x_i)$ will alternate.

We can further assume that the contribution of the sequence B to e_{s+t} is 1, then $(i, s+t)$ will be an interior position in Φ. Hence the whole sequence B will be interior to $\tilde{E}_{\Phi}$, because $\Phi(S)$ is a lower set. Moreover, x_i will be an odd point of $L_{s+t}S(x)$, for if it were an even point , then alternation of $L_{s+t}S(x)$ and $L_{s+t+1}S(x)$ at x_i and $\tilde{e}_{i,s+t+1} = 0$ would imply $L_{s+t+1}S(x_i) = 0$ in contradiction to $e_{s+t} = 1$.

Let B be one of the remaining even sequences (for which t is odd). First, let $s > 0$, by the alternation of the sequence $L_{s-1}S(x_i), \cdots, L_{s+t+1}S(x_i)$, x_i is an odd point of $L_{s-1}S(x)$ and since $\tilde{e}_{i,s-1} = 0$, we have x_i is continuous zero of $L_{s-1}S(x)$, a contradiction. If $s = 0$, then x_i is an even point of $S(x)$, hence x_i is not a discontinuous zero of $S(x)$, then the contribution of B to η_0 is 0, and its contribution to v_0 is 1. So we have $\Delta_B < 0$

Therefore, we need only count the odd interior sequences B of $\tilde{E}_{\Phi}$, and consequently, we obtain

$$\Delta - N < \gamma(\tilde{E}_{\Phi}). \quad \blacksquare$$

CHAPTER 6

ZERO COUNTS FOR CHEBYSHEVIAN SPLINES

We now formulate some consequences of Theorem 5.4 to results that count zeros of Chebyshevian splines. The presentation in this and the next chapter parallels the discussion in the book [1, Section 7.5] for polynomial splines. In many cases the proofs are virtually unchanged.

We say that a Chebyshevian spline, $S(x)$, on $[a,b]$ has degree q, if $L_q S(x)$ is nonzero somewhere in the interval, but $L_{q+1} S(x)$ vanishes identically in the interval.

Theorem 6.1. *For an arbitrary Chebyshevian Birkhoff spline $S(x) \in S^0(E, X, \mathcal{U}_N)$ on $[a,b]$, we have*

$$Z(S(a,b)) \le |\tilde{E}_\Phi| + \gamma(\tilde{E}_\Phi) - L(\Phi) - r \tag{6.1}$$

where r is the number of intervals in the support of $S(x)$.

Proof: We can write $S(x) = S_1(x) + \cdots + S_r(x)$ where the $S_j(x)$ are supported on disjoint intervals I_j, $j = 1, \ldots, r$. We obtain (6.1) by adding the relation (5.2) of Theorem 5.4 for $j = 1, \cdots, r$, and by noticing that for each $S_j(x)$ and its corresponding matrix $\tilde{E}_{\Phi(S_j)}$, we have $|\tilde{E}_{\Phi(S)}| = \sum_{j=1}^{r} |\tilde{E}_{\Phi(S_j)}|$ and $\gamma(\tilde{E}_\Phi) = \sum_{j=1}^{r} \gamma(\tilde{E}_{\Phi(S_j)})$. ∎

An important consequence of Theorem 5.4 is the following result stated in terms of the degree of the spline $S(x)$ rather than its diagram $\Phi(S)$. It is formulated in terms of the matrix E rather than $\tilde{E}$.

Theorem 6.2. *If $S(x) \in S^0(E, X, \mathcal{U}_N)$ is a Chebyshevian spline of degree q, $q < n$, whose support is an interval, then*

$$Z(S(a,b)) \leq |E_q| + \gamma(E) - q - 2. \tag{6.2}$$

Here $\gamma(E)$ is the number of odd sequences in E supported on the left and E_q is the truncated matrix consisting of the last $q+1$ columns of E.

Proof: By Lemma 5.1, we know any interior sequence of $\tilde{E}_{\Phi(S)}$ is supported on the right. So it is supported on the left in E. And for the lower set $\Phi = \Phi(S)$ in the truncated matrix, $L(\Phi) \geq q+1$ since $S(x)$ has degree of q. So that by the fact that $S(x) \in S^0(E, X, \mathcal{U}_N)$ and Theorem 5.4, we get the result. ∎

Going further, we can replace $|E_q|$ by $|E|$ in (6.2). This gives a formulation that requires no *a priori* knowledge of the degree of S. Observe that if E satisfies the Pólya condition, then $|E| - |E_q| \geq n - q$. Hence,

Theorem 6.3. *If S is as in Theorem 6.2 and if the matrix E satisfies the Pólya condition, then*

$$Z(S(a,b)) \leq |E| + \gamma(E) - n - 2.$$

To go further, we need the notion of multiplicity μ for an interval zero $\xi = [c,d]$ of a spline $S(x)$. A spline $S(x)$ on $[a,b]$ can have interval zero $[c,d], a < c < d < b$ only if $S(x) = 0$, $c < x < d$ and only if this is a maximal interval of this type. Thus, $S(c-h)S(d+h) \neq 0$ for all small $h > 0$. In this case, $[c,d]$ is a continuous zero of $S(x)$ if $S(x)$ is continuous at c and d, and a discontinuous zero of multiplicity 1 if $S(x)$ is discontinuous at one of the points and changes sign on $[c,d]$. The multiplicity μ of the zero $[c, d]$ is defined by means of the maximal sequence $S(x), L_1 S(x), \cdots, L_l S(x)$, with the property that $[c, d]$ is a zero for $L_j S$, $0 \le j \le l$, and is a continuous zero if $j < l$. Then $\mu = l + 1$. Clearly,

$$\mu \le \min(q, q') + 1$$

where q and q' are the degrees of $S(x)$ on the intervals adjoining $[c, d]$.

Theorem 6.4. *If $S(x) \in \mathcal{S}^0(E, X, U_N)$, and if q is the the degree of $S(x)$, then*

$$Z^*(S(a,b)) \le |E| + \gamma(E) - r - q - 1 \tag{6.4}$$

where r is the number of intervals in the support of $S(x)$.

Proof: Let $S(x) = \sum_{j=1}^{r} S_j(x)$, where $S_j(x)$ are Chebyshevian splines of degree q_j supported on disjoint intervals I_j, $j = 1, 2, \cdots, r$. If J_j is the interval between I_j and I_{j+1} and μ_j is its multiplicity as a zero of $S(x)$, then

$$L(\Phi(S_j)) \ge q_j + 1$$

and

$$\sum_{j=1}^{r-1} \mu_j \le \sum_{j=1}^{r-1} \min(q_j, q_{j+1}) + r - 1$$
$$\le \sum_{j=1}^{r} q_j - q + r - 1 \le L(\Phi(S)) - q - 1.$$

Adding this to (6.1), we obtain (6.4). ∎

CHAPTER 7

A BUDAN-FOURIER THEOREM

Let $v = (v_1, \cdots, v_n)$ be a vector of real numbers. We define the number of strong sign changes of v by $G^-(v) :=$ the number of sign changes in the sequence $v_1, v_2, \cdots, v_n$, where zeros are ignored. Similarly, we define the number of weak sign changes of v by $G^+(v) :=$ the maximum number of sign changes of v in the sequence $v_1, \cdots, v_n$, where each zero can be regarded as either $+1$ or -1 whichever makes the count largest.

It is clear that $G^-(v) \le G^+(v)$ for all v. From [2, page 25], we have the result that for all vectors v

$$G^+(v_1, -v_2, \cdots, (-1)^{r-1} v_r) + G^-(v_1, \cdots, v_r) = r - 1.$$

Let $f(x)$ be a function on (c, d). We say that c is a zero of $L_j f(x)$ on $[c, d]$ if either $L_j f(c+) = 0$ or else $L_j f(c+) L_{j+k} f(c+) > 0$, where k is the smallest integer $k = 1, 2, \cdots, n$ with $L_{j+k} f(c+) \neq 0$

Similarly, d is a zero of $L_j f(x)$ on $[c, d]$, if $L_j f(d-) = 0$ or if

$$L_j f(d-)(-1)^k L_{j+k} f(d-) > 0$$

with k defined as the smallest integer for which $L_{j+k} f(d-) \neq 0$.

Theorem 7.1. *Let $S(x)$ be a Chebyshevian spline on $(c, d) \subseteq [a, b]$. If the point c (or d) is a zero of $L_j S(x)$ on $[c, d]$ in the sense defined above, then c (or d) is a left (or right) Rolle's point for $L_j S(x)$.*

Proof: First notice that for a spline $S(x) \in S(E, X, U_N)$ and for each k, $0 \le k \le N-1$, $L_k S(x)$ is piecewise continuous. So that there exists ϵ_1, such that for $c < x < c + \epsilon_1$, $L_k S(x)$ is continuous and non-zero for $0 \le k \le N-1$ unless $L_k S(x) = 0$ in an interval $[x_i, x_{i+1})$ containing c.

Now we prove the theorem. If $L_j S(c+) = 0$, then c is automatically a left Rolle's point. If $L_j S(c+) \neq 0$, say $L_j S(c+) > 0$, then there exists r, such that

$$L_{j+1} S(c+) = \cdots = L_{j+r} S(c+) = 0, \qquad L_{j+r+1} S(c+) \neq 0$$

and $L_j S(c+) L_{j+r+1} S(c+) > 0$. So $L_{j+r+1} S(c+) > 0$. Therefore,

$$L_{j+1} S(x) = w_{j+2}(x) \int_c^x \cdots \int_c^{s_{j+r}} w_{j+3}(s_{j+2}) \cdots$$
$$\cdots w_{j+r+1}(s_{j+r}) L_{j+r+1} S(s_{j+r+1})\, ds_{j+r+1} \cdots ds_{j+2}$$
$$> 0$$

for $c < x < c + \epsilon$, if ϵ is sufficiently small. This proves that c is a left Rolle's point for $L_j S$ in this case. If $L_j S(c+) < 0$, the proof is similar. The same kind of argument may be given to show that if d is a zero of $L_j S(x)$ in the sense above, then d is a right Rolle's point for $L_j S(x)$. ∎

Theorem 7.2. *Let $S(x)$ be a Chebyshevian spline on $(\xi_1, \xi_2) \subseteq [a, b]$ with zeros, $L_j S(\xi_1+)$ and $L_j S(\xi_2-)$, of $L_j S(x)$ on $[\xi_1, \xi_2]$ as above. Suppose that both $L_j S(x)$ and $L_{j+1} S(x)$ do not vanish on subintervals of $[\xi_1, \xi_2]$, then there is a point ξ, $\xi_1 < \xi < \xi_2$, such that either A) ξ is a point zero of $L_{j+1} S(x)$ with change of sign,* *but not a point zero of $L_j S(x)$, or B) ξ is a discontinuous zero of $L_j S(x)$ (with change of sign) over which $L_{j+1} S(x)$ preserves sign.*

Proof: By checking the proof of the Lemma 5.3, we find that we only use the fact that ξ_1, ξ_2 are Rolle's points and Theorem 5.2 can be applied. Thus, the result follows by Theorem 7.1. ∎

Remark: The remark to Lemma 5.3 also holds for Theorem 7.2.

Now we come to the main theorem of this chapter.

Theorem 7.3. *Let $S(x)$ be a Birkhoff Chebyshevian spline of degree q that corresponds to an $m \times (q+1)$ matrix E and a set of knots X, and suppose that $S(x)$ does not vanish identically in neighborhoods of x_1 and x_m. Let $S_i(x)$ denote the restriction of $S(x)$ on (x_i, x_{i+1}), $i = 1, \cdots, m-1$. Suppose that at least one of these $S_i(x)$ is of exact degree q, and that $S_1(x)$ and $S_{m-1}(x)$ are of exact degree d_1 and d_{m-1} respectively. Then*

$$Z(S(x_1, x_m)) \leq G^-\{S(x_1+), \cdots, L_q S(x_1+)\} + G^+\{S(x_m-) \cdots L_q S(x_m-)\}$$

$$+ |E^0| + \gamma(E)$$

where G^+, G^- are defined as in the beginning of this chapter, E^0 is the matrix $[e_{i,k}]_{i=2}^{m-1}{}_{k=0}^{N-1}$ and $\gamma(E)$ is the number of odd sequences in E supported on the left.

Proof: From the given E and the spline $S(x)$ we get the corresponding $\widetilde{E}$ and the diagram $\Phi(S)$. We introduce the following notation:

l_k := the number of intervals of X_Φ (or of Φ) of level k;

u_k := the number of continuous zeros of $L_k S(x)$ inside X_S which are ~~also~~ zeros of $S(x)$ of multiplicity $\geq k+1$;

ν_k : the number of $\tilde{e}_{i,k} = 1$ in the interior of intervals of level k of Φ;

ϵ_k : the number of sequences of 1's ending at some point (i,k) of Φ with the property that $(i,k+1)$ is inside Φ and $L_{k+1}\,S(x_i) \neq 0$ and $i \neq 1, i \neq m$;

η_k : the number of $\tilde{e}_{i,k} = 1$ for which x_i is a zero of multiplicity $k+1$ and $i \neq 1, i \neq m$;

h_k : the number of 1's on the horizontal boundary of E_Φ of level k, but not counting the 1's on the first row and the last row.

Let

$$A_j = G^+[(-1)^j L_j S(x_1+), \cdots, (-1)^{d_1} L_{d_1} S(x_1+)]$$

$$B_j = G^+[L_j S(x_m-), \cdots, L_{d_m-1} S(x_m-)]$$

and let $\alpha_{k_1} = A_{k_1-1} - A_{k_1}$ and $\beta_{k_2} = B_{k_2-1} - B_{k_2}$, $1 \le k_1 \le d_1$, $1 \le k_2 \le d_m$. Clearly, α_{k_1} and β_{k_2} can take only the values 0 or 1. We claim that $\alpha_{k_1} = 1$ is possible only if x_1 is a left Rolle's point of $L_{k_1-1}\,S(x)$. If $L_{k_1-1}\,S(x_1+) = 0$, then x_1 is automatically a Rolle's point of $L_{k_1-1}\,S(x)$. Now suppose that $L_{k_1-1}\,S(x_1) > 0$, then, since $k_1 < d_1$, A_{k_1-1} must have the pattern :

$$\{(-1)^{k_1-1}, \overbrace{0, \cdots, 0}^{r}, (-1)^{k_1-1+r+1}, \cdots\}.$$

If $(-1)^{k_1-1} = 1$ and r is even, then, since $A_{k_1-1} - A_{k_1} = 1$ and $(-1)^{k_1+r} = -1$, we have $L_{k_1-1+r+1}\ S(x_1) > 0$. For the case when r is odd, the proof is the same. And for $(-1)^{k_1-1} = -1$, we also can show that $L_{k_1-1+r+1}\ S(x_1) > 0$. Since $L_{k_1-1}\,S(x)$ is a Chebyshevian spline, we know by Theorem 7.1 that $L_{k_1-1}\,S(x)$ has x_1 as a left Rolle's point.

If $L_{k_1-1}S(x_1) < 0$, the proof is the same. Moreover, the same kind of argument may be given to show that $\beta_{k_2} = 1$ is possible only if b is a right Rolle's point of $L_{k_2-1}S(x)$.

Suppose $S(x)$ has degree q, so that $S(x) \in S^0_{q+1}(E, X, U_N)$. We note $\mu_q = 0, \epsilon_q = 0$ and $l_0 = 1$. We would like to find a lower bound, μ_k, for the number of the continuous Rolle zeros of $L_k S(x)$. (They are disjoint from the zeros counted by u_k.) We begin with $\mu_0 = 0$. We apply Lemma 7.2 to derive a lower estimate μ_{k+1} from μ_k. The function $L_k S(x)$ has $\mu_k + u_k$ continuous zeros inside the intervals of X_S of level k. Endpoints of X_S for which $\tilde{e}_{i,k} = 0$ and $i \neq 0, i \neq m$, have $L_k S(x_i+) = 0$, and are continuous zeros of $L_k S(x)$. Thus, we get for $0 \leq k \leq \min(d_1, d_{m-1})$, $L_k S(x)$ has

$$\mu_k + u_k + 2l_k - h_k - 2$$

continuous zeros. For $\min(d_1, d_{m-1}) < k \leq \max(d_1, d_{m-1})$, $L_k S(x)$ has

$$\mu_k + u_k + 2l_k - h_k - 1$$

continuous zeros. And for $k > \max(d_1, d_{m-1})$, $L_k S(x)$ has

$$\mu_k + u_k + 2l_k - h_k$$

continuous zeros. For simplication of notation, we will assume that $d_1 \leq d_{m-1}$. The proof in the opposite case would proceed in an analogous manner.

We add the endpoints x_1, x_m to this collection of continuous zeros if they are Rolle's points. The intervals determined by this collection of points will be candidates for the application of Rolle's Theorem 7.2. The number of such intervals are

$$\mu_k + u_k + 2l_k - h_k - 3 + \alpha_{k+1} + \beta_{k+1} \qquad \text{if} \quad k < d_1,$$

$$\mu_k + u_k + 2l_k - h_k - 2 + \beta_{k+1} \qquad \text{if} \quad d_1 < k < d_{m-1}$$

$$\mu_k + u_k + 2l_k - h_k - 1 \qquad \text{if} \quad d_{m-1} < k$$

We omit those intervals for which Theorem 7.2 will not apply. For example, the endpoints of the intervals cannot belong to the vertical boundary of Φ, for there, $L_k S(x)$ consists of nonzero piecewise constant multiples of $w_{k+1}(x)$ which contradicts the fact that $L_k S(x)$ equals zero at a corresponding endpoint of the interval. For the same reason, the endpoint of the intervals cannot belong to the $l_{k+1} - 1$ complements $\tilde{I}$ of the intervals of X_S of level $k+1$. But the intervals may contain some $\tilde{I}$, if we omit all these, we end up with at least

$$\mu_k + u_k + 2l_k - h_k - l_{k+1} - 2 + \alpha_{k+1} + \beta_{k+1} \qquad \text{if} \quad k < d_1,$$

$$\mu_k + u_k + 2l_k - h_k - l_{k+1} - 1 + \beta_{k+1} \qquad \text{if} \quad d_1 < k < d_{m-1}, \qquad (7.1)$$

$$\mu_k + u_k + 2l_k - h_k - l_{k+1} + \beta_{k+1} \qquad \text{if} \quad d_{m-1} < k.$$

intervals I of level k, to which we can apply Theorem 7.2. Application of Theorem 7.2 gives a point ξ inside each I. This ξ is even inside of X_S at level $k+1$.

In particular, ξ may be a continuous zero of $L_{k+1} S$ (special case of A in Theorem 7.2). By the remark of Theorem 7.2, any other possibility would require $\xi = x_i$ for some x_i inside of the intervals of X_S at level $k+1$. To single out continuous zeros, we first discard all I that contain x_i where $L_{k+1} S(x)$ could be discontinuous. There are at most $(v_{k+1} - \eta_{k+1})$ such x_i. We next discard x_i's at which $L_k S(x)$ is discontinuous but $L_{k+1} S(x)$ is continuous and $L_{k+1} S(x_i) \neq 0$. There are at most

ϵ_k such x_i. Thus the difference between (7.1) and $(v_{k+1} - \eta_{k+1}) + \epsilon_k$ will be a lower estimate for μ_{k+1}.

As in the proof of Theorem 5.4, it is important that this estimate can be improved. We call a pair (i, k) nonconfirming (to Theorem 7.2), if x_i is counted by $(v_{k+1} - \eta_{k+1})$ or ϵ_k, but cannot be obtained as a ξ of level k by the Theorem 7.2 and its remark. This will be the case if x_i does not belong to any I, or if $L_k S(x), L_{k+1} S(x)$ do not alternate at x_i. For each k, $k = 0, 1, \ldots, q-1$, we examine all pairs $(i, k) \in \Phi$, and denote by N_k the number of nonconfirming (i, k) for the level k. Since ϵ_k and $(v_{k+1} - \eta_{k+1})$ count disjoint pairs, $N = \sum_{k=1}^{q-1} N_k$ will be the total number of nonconfirming pairs in Φ. At step k, we need to discard only

$$(v_{k+1} - \eta_{k+1}) + \epsilon_k - N_k$$

intervals I. The remaining intervals will contain continuous Rolle zeros of $L_{k+1} S(x)$. Thus, we have the estimates

$$\mu_{k+1} \geq \mu_k + u_k + 2l_k - l_{k+1} - h_k - 2 + \alpha_{k+1} + \beta_{k+1} - (v_{k+1} - \eta_{k+1}) - \epsilon_k + N_k$$

for $k < d_1$;

$$\mu_{k+1} \geq \mu_k + u_k + 2l_k - l_{k+1} - h_k - 1 + \beta_{k+1} - (v_{k+1} - \eta_{k+1}) - \epsilon_k + N_k$$

for $d_1 < k < d_{m-1}$; and

$$\mu_{k+1} \geq \mu_k + u_k + 2l_k - l_{k+1} - h_k - (v_{k+1} - \eta_{k+1}) - \epsilon_k + N_k$$

for $k \leq d_{m-1}$. Since $l_0 = 1$, $2l_q = h_q$, we obtain by summation

$$
\begin{aligned}
0 \quad \mu_q \quad & \sum_{k=1}^{q}(u_k + \eta_k) \quad \sum_{k=1}^{q} v_k \quad \sum_{k=0}^{q-1} \epsilon_k + N + \sum_{k=1}^{d_1} \alpha_k \\
& + \sum_{k=1}^{d_m} \beta_k \quad d_1 \quad d_{m-1} \quad 1 \quad \eta_0 \quad \sum_{k=0}^{q} h_k + \sum_{k=0}^{q} l_k
\end{aligned}
$$

The first sum is $Z(S(x_1, x_m))$ while

$$
\begin{aligned}
\sum_{k=0}^{q}(v_k + h_k) &= |E^0| \\
\sum_{k=0}^{q} l_k &= L(\Phi) - q + 1 \\
\sum_{k=1}^{d_1} \alpha_k &= G^+(S(x_1+), \cdots, (-1)^{d_1} L_{d_1} S(x_1+)) \\
\sum_{k=1}^{d_{m-1}} \beta_k &= G^+(S(x_m-), \cdots, L_{d_{m-1}} S(x_m-))
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
Z(S(x_1, x_m)) - |E^0_\Phi| = L(\Phi) + d_1 + d_{m-1} + (\Delta - N) + 1 \\
- G^+(S(x_1+), \cdots, (-1)^{d_1} L_{d_1} S(x_1+)) \\
- G^+(S(x_m-), \cdots, L_{d_{m-1}} S(x_m-)),
\end{aligned}
$$

where

$$
\Delta = \sum_{k=0}^{q-1} \epsilon_k + \eta_0 - v_0.
$$

Since

$$
d_1 < G^+(S(x_1+), \cdots, (-1)^{d_1} L_{d_1} S(x_1)) + G^-(S(x_1+), \cdots, L_{d_1} S(x_1+)),
$$

we obtain

$$Z(S(x_1, x_m)) \cdot |E^{0}| \quad q + d_{m-1} + (\Delta \quad N) + G \quad (S(x_1 +), \quad , L_{d_1} S(x_1 +))$$

$$G^{+}(S(x_m \quad), \cdots, L_{d_{m-1}} S(x_m \quad))$$

$$|E^{0}| + G \quad (S(x_1 +), \cdots, L_q S(x_1 +))$$

$$G^{+}(S(x_m \quad), \quad , L_q S(x_m +)) + (\Delta \quad N)$$

By the same argument as in the proof of the Theorem 5.4, we get $\Delta \quad N \cdot \quad \gamma(E_{\Phi})$, where $\gamma(\tilde{E}_{\Phi})$ is the number of odd interior sequences of $\tilde{E}_{\Phi}$. Since Φ is a lower set, an interior sequence supported on the right in $\tilde{E}_{\Phi}$ is a sequence supported on left in E. Therefore, $\gamma(\tilde{E}_{\Phi}) \cdot \quad \gamma(E)$, so

$$(\Delta \quad N) \cdot \quad \gamma(E). \quad \blacksquare$$

CHAPTER 8

GENERALIZED SPLINE INTERPOLATION MATRICES

A generalized spline interpolation matrix is any matrix

$$F = [f_{i,k}]_{i=1,k=0}^{m,\ N-1} \qquad f_{i,k} = 0, 1, -1 \text{ or } F_{i,k},$$

where $F_{i,k} = \binom{1}{-1}$ or $\binom{-1}{1}$. We consider $F_{i,k}$ as an entry of the matrix F, and call it a box. The spline interpolation matrices considered in [1] only contained entries $0, \pm 1$. The usefulness of boxes will become clear later. The definitions and concepts useful for ordinary spline interpolation matrices are given in this chapter in modified form for generalized spline interpolation matrices.

For technical reasons, we shall always assume that the exterior rows of F contain no 0's and boxes. That is $f_{i,k} = 1$ or -1 if $i = 1$ or $i = m$.

We denote $[f_{i,k}]$ as the value of $f_{i,k}$, and set $[f_{i,k}] = f_{i,k}$, if $f_{i,k} = 0, 1$, or -1, and $[f_{i,k}] = 0$, if $f_{i,k}$ is a box.

The sum of the value of all entries of F is denoted by $|F|$. That is,

$$|F| = \sum_{i,k} [f_{i,k}].$$

This definition is the same as the definition given in [1] when the entries of F are only $0, \pm 1$.

For an arbitary spline interpolation matrix F, the matrix F^+ is obtained by replacing in rows $i = 1$ and $i = m$ of F all -1's by 0's. A matrix F is normal, if $|F^+| = N$.

We consider a space of Chebyshevian splines, $S(F, X, \mathcal{U}_N)$, associated with F. It is the linear span of the functions

$$\mathcal{B} = \{u_i\}_{i=1}^{N} \bigcup \left\{ g_{q+1}(x, x_p) : f_{p,q} \in \{-1, \binom{1}{1}, \binom{1}{1}\}; 1 \leq p \leq m, q = 0, \cdots, N-1 \right\}.$$

The functions $\mathcal{B}$ form a basis for the space $S(F, X, \mathcal{U}_N)$ i.e. a Chebyshevian spline $S \in S(F, X, \mathcal{U}_N)$ has the unique representation

$$S(x) = \sum_{q=1}^{N} \alpha_q u_q(x) + \sum_{\substack{f_{p,q} \in \{-1, \binom{1}{1}, \binom{1}{1}\} \\ 1 \leq p \leq m}} \alpha_{p,q} \, g_{q+1}(x, x_p). \qquad (8.1)$$

As a comparison with earlier chapters, we observe that $S(F, X, \mathcal{U}_N)$ is the space $S(E, X, \mathcal{U}_N)$ where

$$e_{i,k} = 1 <=> f_{i,N-1-k} \in \{-1, \binom{1}{1}, \binom{1}{1}\}.$$

For an arbitary set of knots $X : a = x_1 < x_2 \cdots < x_m = b$, we consider the interpolation functionals for a given function $g(x)$ defined by:

$$\Lambda_{i,k} g(x) = L_k g(x_i) \qquad \forall f_{i,k} = 1;$$

$$\Lambda_{i,k} g(x) = L_k g(x_i -) \qquad \forall f_{i,k} = \binom{1}{1}; \text{ and} \qquad (8.2)$$

$$\Lambda_{i,k} g(x) = L_k g(x_i +) \qquad \forall f_{i,k} = \binom{1}{1}.$$

This definition allows one-sided interpolation by splines at their knots. The one-sided interpolation by $L_k S$ at x_i and the fact that $L_k S$ may have a jump at x_i are both specified by the box in position (i, k). The orientation of the one-sided interpolation is determined by the form of the box.

A pair F, X is regular for spline interpolation, if for each choice of values $\{c_{i,k}; f_{i,k} = -1$ or $f_{i,k}$ is a box $\}$ there is a unique spline $S(x) \in S(F, X, \mathcal{U}_N)$, for which

$$L_k S(x_i) = c_{i,k}, \qquad f_{i,k} = -1;$$

$$L_k S(x_i -) = c_{i,k}, \qquad f_{i,k} = \binom{1}{-1}; \tag{8.3}$$

$$L_k S(x_i +) = c_{i,k}, \qquad f_{i,k} = \binom{-1}{1}.$$

Equivalently, F, X is regular if $S(x) \in S(F, X, \mathcal{U}_N)$ and $L_k S(x_i) = 0$ for all $f_{i,k} = -1$, $L_k S(x_i -) = 0$ for all $f_{i,k} = \binom{1}{-1}$, and $L_k S(x_i +) = 0$ for all $f_{i,k} = \binom{-1}{1}$ in F, implies $S(x) \equiv 0$. This can happen only if the functionals $\Lambda_{i,k}$, $f_{i,k} = -1$, or a box are linearly independent on the space $S(F, X, \mathcal{U}_N)$, and if their total number is equal to the dimension of the space. That is, only if F is normal. In this case, the equations

$$L_k S(x_i) = \sum_{q=1}^{N} \alpha_q L_k u_q(x_i) + \sum_{\substack{f_{p,q} \in \{-1, \binom{1}{-1}, \binom{-1}{1}\} \\ 1<p<m}} \alpha_{p,q} L_k g_{q+1}(x_i, x_p), \qquad f_{i,k} = -1, \tag{8.4}$$

$$L_k S(x_i -) = \sum_{q=1}^{N} \alpha_q L_k u_q(x_i -) + \sum_{\substack{f_{p,q} \in \{-1, \binom{1}{-1}, \binom{-1}{1}\} \\ 1<p<m}} \alpha_{p,q} L_k g_{q+1}(x_i -, x_p), \qquad f_{i,k} = \binom{1}{-1}, \tag{8.5}$$

and

$$L_k S(x_i +) = \sum_{q=1}^{N} \alpha_q L_k u_q(x_i +) + \sum_{\substack{f_{p,q} \in \{-1, \binom{1}{-1}, \binom{-1}{1}\} \\ 1<p<m}} \alpha_{p,q} L_k g_{q+1}(x_i +, x_p), \qquad f_{i,k} = \binom{-1}{1} \tag{8.6}$$

will have a unique solution.

Let $\mathcal{B}$ be any basis for $S(F, X, \mathcal{U}_N)$ written in some order. The system (8.3) of the equations (8.4), (8.5) and (8.6) has the coefficient matrix

$$A = A(F, X, \mathcal{B}) = \left[\Lambda_{i,k}(g);\, f_{i,k} \in \{1, \binom{1}{-1}, \binom{-1}{1}\},\quad g(r) \in \mathcal{B}\right]$$

with the rows corresponding to (i, k), $f_{i,k} \in \{1, \binom{1}{-1}, \binom{-1}{1}\}$, ordered lexicographically and columns corresponding to the elements of $\mathcal{B}$. The regularity of F, X is equivalent to the condition

$$D(F, X, \mathcal{B}) = \det A(F, X, \mathcal{B}) \neq 0.$$

The matrix is regular if the pair F, X is regular for each X.

Let Φ be a lower set, F be a spline interpolation matrix, $F_\Phi = [f_{i,k}]_{i,k \in \Phi}$ be the restriction of F to Φ (that is F with lower set Φ). The notions of horizontal boundary, vertical boundary, and inside and outside corners are carried over to F_Φ in a natural way. However, some additional discussion about boxes on the horizontal boundary is necessary. We can imagine that there are horizontal lines passing through the entries on each segment of the horizontal boundary of F_Φ. The lines pass between 1's and -1's, when they meet boxes. The lines have two sides. One side of the line is outside of the intervals of the lower set Φ, we call it the outside of the line. The other side is inside of the intervals of the lower set Φ, we call it the inside of the line. We can divide the boxes on the horizontal boundary into two types. Type I boxes are the boxes for which the number on the inside of the line is 1, and on the outside of the line is -1. Type O boxes are the boxes with 1 and -1 interchanged in above definition.

We define F^{1}_{Φ} by first replacing the 0's and the boxes of type I on the horizontal boundary of Φ by 1's, and then replacing the 1's and the boxes of type O on the horizontal boundary of Φ by 0's. Finally, F°_{Φ} is the submatrix of F_{Φ} restricted to pairs (i, k) that are not on the horizontal boundary of Φ.

We mention that the definition of the degree of the spline $S(x) \in S(F, X, \mathcal{U}_N)$ is the same as in the preceding chapter. That is , if $L_J S(x)$ does not vanish in some subinterval of $[a, b]$, but $L_{J+1} S(x)$ is identically zero in $[a, b]$, then S has degree J.

First, we prove a theorem about the space of splines that "live" on a lower set Φ. Let X be a set of knots. The set of intervals X_{Φ} is given by: $[x_j, x_{j'}]$ is an interval of kth level in X_{Φ} iff $[j, j']$ is an interval of kth level in Φ. The space S $S_{\Phi, X, \mathcal{U}_N}$, associated with Φ, X, consists of all $S(x)$ on $[a, b]$, that satisfy the following conditions: $S(x)$ is a Chebyshevian spline from $\mathcal{U}_N$ on each $[x_j, x_{j'}]$; $L_k S(x)$ vanishes outside of the intervals of level k of X_{Φ}; and, $L_k S(x_i\ ) \neq L_k S(x_i +)$ at an interior point x_i of an interval of level k only if $f_{i,k} \in \{ 1, (\,^{1}_{1}), (\,^{1}_{1})\}$. No restriction is made on the continuity of $L_k S(x)$ at the endpoints of intervals of level k.

Theorem 8.1. *For each lower set Φ, and each set of knots X, the dimension of $S_{\Phi, X, \mathcal{U}_N}$ is*

$$\dim S_{\Phi, X, \mathcal{U}_N} = L(\Phi) + (\text{number of } 1\text{'s}, \binom{1}{1}\text{'s, and } \binom{-1}{1}\text{'s in } F^{\circ}_{\Phi}). \qquad (8.7)$$

Proof: We prove this by induction in $h :=$ the order of the ECT-system and the number of columns in F.

If $h = 1$, let $I = [x_{i_1}, x_{i_2}]$ be one of the p intervals of X_{Φ} of level 0, since $h = 1$, the Chebyshevian splines of degree is 0 on I are spanned by $w_1(x)$ and functions

$g_1(x, x_j)$ corresponding to all $f_{j,0} = 1$, $\binom{1}{1}$, and $\binom{1}{1}$, $i_1 < j < i_2$. So that for the $h = 1$ the result holds.

Suppose we have proved the theorem for $h = N - 1$, now we try to prove it for $h = N$.

Let Φ' be Φ restricted to columns 1 to $N - 1$ and let F' be the restriction of the matrix F to columns $1, \ldots, N - 1$. The collection $\mathcal{U}'_N = \{L_1 u_2, \ldots, L_1 u_N\}$ is an ECT system of order $N - 1$ for the weight functions $w_2(x), \ldots, w_N(x)$. By our induction hypothesis and (8.7)

$$d' = \dim S_{\Phi', X, \mathcal{U}'_N} = L(\Phi') + (\text{number of } 1'\text{s}, \binom{1}{1}'\text{s, and } \binom{1}{1}'\text{s in } F'^{\,c}_{\Phi'}).$$

Now let $S \in S_{\Phi, X, \mathcal{U}_N}$. Since $L_1 u_1 = 0$, we have that $L_1 S \in S_{\Phi', X, \mathcal{U}'_N}$. On the other hand, suppose that $S^* \in S_{\Phi', X, \mathcal{U}'_N}$ and define $\tilde{S}(x)$ by

$$\tilde{S}(x) := \begin{cases} w_1(x) \int_a^x S^*(t)\, dt, & \text{if } x \in I_j, \quad j = 1, \ldots, p, \\ 0, & \text{otherwise,} \end{cases}$$

where I_j, $j = 1, \ldots, p$ are the intervals of level 0 in X_Φ. We claim that $\tilde{S} \in S_{\Phi, X, \mathcal{U}_N}$. Since $\tilde{S}$ is continuous inside each interval I_j, vanishes outside $\cup_{j=1}^p I_j$, and $L_1 \tilde{S} = S^*$, it remains to prove only that $\tilde{S}$ is a piecewise $\mathcal{U}_N$-polynomial. But this is obvious since $S^*|_{(x_i, x_{i+1})} = \sum_{j=2}^N \alpha_j L_1 u_j$ implies

$$\begin{aligned} \tilde{S}(x)|_{(x_i, x_{i+1})} &= w_1(x) \int_a^{x_i} S^*(t)\, dt + w_1(x) \int_{x_i}^x \sum_{j=2}^N \alpha_j L_1 u_j(t)\, dt \\ &= w_1(x) \left(\int_a^{x_i} S^*(t)\, dt + \sum_{j=2}^N \frac{u_j(x_i)}{w_1(x_i)} \right) + \sum_{j=2}^N \alpha_j u_j(x). \end{aligned}$$

Let $S_1^*(x), \ldots, S_{d'}^*(x)$ be a basis for $S_{\Phi', X, \mathcal{U}'_N}$. Then the correspondingly defined $\tilde{S}_1(x), \ldots, \tilde{S}_{d'}(x)$ form a basis for a basis for the subspace of all $\tilde{S} \in S_{\Phi, X, \mathcal{U}_N}$.

On the other hand, Chebyshevian splines of degree 0 on each $I = [x_{i_1}, x_{i_2}]$ are spanned by $w_1(x)$ and by the functions $g_1(x, x_j)$ corresponding to all $f_{j,0} = 1, \binom{1}{-1}$, and $\binom{-1}{1}$, $i_1 < j < i_2$. Since each $S(x)$ is uniquely representable on each I by the linear combination of a spline of degree 0 and the splines $\tilde{S}_1(x), \tilde{S}_2(x), \cdots, \tilde{S}_{d'}(x)$ restricted to I, the dimension of $S_{\Phi,X,\mathcal{U}_N}$

$$d' + p + (\text{number of } \quad f_{i,0} = 1, \binom{1}{-1}, \text{and } \binom{-1}{1} \quad \text{in } F_{\Phi}^{0}). \tag{8.8}$$

Here p is the portion of the number $L = L(\Phi)$ for column 0. Substituting the value of d' into (8.8), we obtain our statement. ∎

The generalized lower Goodman-Pólya conditions for a spline interpolation matrix F are

$$|F_{\Phi}^{1}| \geq L(\Phi) \text{ for each lower set } \Phi. \tag{8.9}$$

If the entries of F are only 0, ± 1, then this definition of F_{Φ}^{1} and the lower Goodman-Pólya conditions agree with the ones given in [1] for spline interpolation matrices.

Theorem 8.2. *If the pair F, X is regular for some set of knots X, then F satisfies the lower Goodman-Pólya conditions.*

Proof: We subject the spline $S(x) \in S_{\Phi,X,\mathcal{U}_N}$ to the conditions $\Lambda_{i,k}(S) = 0$ for all $f_{i,k} = 1, \binom{1}{-1}$, and $\binom{-1}{1}$ in F_{Φ}^{1}, where $\Lambda_{i,k}$ are the functionals defined as in (8.2). If x_i is one of the endpoints of intervals of X_{Φ} of level k, this should be interpreted as the limit of $L_k S(x)$ from inside of the interval. Chebyshevian splines $S(x)$ satisfying these equations will be elements of $S(F, X, \mathcal{U}_N)$, annihilated by all $\Lambda_{i,k}$ corresponding to $f_{i,k} = 1, \binom{1}{-1}$, and $\binom{-1}{1}$ in F_{Φ}^{1}. If the lower Goodman-Pólya condition is violated for Φ, then by Theorem 8.1, the dimension of $S_{\Phi,X,\mathcal{U}_N}$ is greater than the number of 1's, $\binom{1}{1}$'s and $\binom{1}{1}$'s in F_{Φ}^{1}, that is, greater than the number of the equations. Then there is a nontrival $S(x) \in S(F, X, \mathcal{U}_N)$ annihilated by F, X and the pair is singular, which is a contradition. ∎

CHAPTER 9

ESTIMATION OF $|\hat{F}_{\Phi}^1|$

For a generalized spline interpolation matrix $F = [f_{i,k}]_{i=1,k=0}^{m,N-1}$ and a lower set Φ, we study the matrices F_{Φ}, F_{Φ}^1. This time, we consider blocks. A maximal sequence of $+1$'s, -1's and boxes in a row of F_{Φ}^1

$$B : f_{i,s}, \cdots, f_{i,s+t} \tag{9.1}$$

is a block in F_{Φ}^1. The block is interior to F_{Φ}^1 if it contains none of the horizontal or vertical boundary points of F_{Φ}^1. A block B in F_{Φ}^1 is supported on the left (or simply supported), if there are $f_{i_1,k_1} \in \{1, \binom{1}{1}\}$ and $f_{i_2,k_2} \in \{1, \binom{1}{1}\}$ in F_{Φ}^1 for which $i_1 < i < i_2$ and $\max(k_1, k_2) < s$. For an interior supported block, elements $f_{i,s-1}$ and $f_{i,s+t+1}$ belong to F_{Φ}^1 and are 0. A block is odd (or even) if the number of its entries is odd (or even). A block is supported on the right in F if there are $f_{i_3,k_3} \in \{-1, \binom{1}{1}\}$ and $f_{i_4,k_4} \in \{-1, \binom{-1}{1}\}$ in F with $i_3 < i < i_4$, $\min(k_3, k_4) > s + t$.

By using the same way of proof as for Lemma 5.1, we can get that an interior block of F_{Φ}^1 is supported on right in F, when $\Phi = \Phi(S)$ is the diagram of the Chebyshevian spline S.

Theorem 9.1. *Let F be an $m \times N$ generalized spline interpolation matrix and let $X : a = x_1 < \cdots < x_m = b$ be a set of knots. If a Chebyshevian spline $S(x) \in S(F, X, \mathcal{U}_N)$ with interval support is annhilated by F, X and if $\Phi = \Phi(S)$ is the diagram of S, then*

$$|F_{\Phi}^1| \leq L(\Phi) - 1 + \dot{\gamma}(F_{\Phi}^1)$$

where $\gamma(F^1_\Phi)$ is the number of interior odd blocks in F^1_Φ that are supported on the left.

Proof: We apply Lemma 5.3 to a Chebyshevian spline $S(x)$ of degree q, and derive numbers μ_k, $k = 0, 1, \cdots, q$, which are lower bounds for the number of continuous Rolle zeros ξ of $L_k S(x)$ not already stipulated by F^1_Φ. We start with $\mu_0 = 0$. (Since we only consider a lower bound for μ_k, we can assume this.)

Let $l_k :=$ the number of intervals in $\Phi(S)$ (or of X_S) of level k (in particular $l_0 = 1$);

$u_k :=$ the number of 1's in F^1_Φ in column k;

$v_k :=$ the number of 1's in the interior of F^1_Φ in column k;

$\epsilon_k :=$ the number of blocks (9.1) that are supported on the left in F^1_Φ and that begin in column k with a 1;

$b_k :=$ the number of blocks (9.1) that are supported on the left in F^1_Φ and that begin in column k with a box;

$\epsilon_k :=$ the number of sequences of 1's in F^1_Φ that end at a point (i, k) with $(i, k+1)$ in the interior of one of the intervals of Φ of level $k+1$ (in particular, points on the vertical boundary of Φ do not contribute to ϵ_k);

$\eta_k :=$ the number of -1's in column k that are not supported on the left or that are preceded by a 1 or a box;

$\omega_{k+1} :=$ the number of interior blocks in F^1_Φ that end with a 1 in column k for which $L_{k+1} S(x_i) = 0$.

At step k, there will be $\mu_k + u_k$ continuous zeros of $L_k S(x)$ with $\mu_k + u_k - 1$ intervals between them. If we omit at most $l_{k+1} - 1$ intervals from those of Λ's of level $k+1$, we obtain at least $\mu_k + u_k - l_{k+1}$ intervals I of level k to which we can apply Lemma 5.3. On each of them, $L_k S(x)$ and $L_{k+1} S(x)$ can vanish only at isolated points. For an interval $I = (\xi_1, \xi_2)$, Lemma 5.3 may produce a continuous zero ξ of $L_{k+1} S(x)$. This would be a special case of A) of Lemma 5.3. We notice that, if (ι, k) is a box between two continuous zeros ξ_1 and ξ_2 with $\xi_1 < x_\iota < \xi_2$ of $L_k S$ and $f_{\iota,k+1} = 1$, then we can replace Rolle interval (ξ_1, ξ_2) by (ξ_1, x_ι) or (x_ι, ξ_2), because one of intervals of (ξ_1, x_ι) and (x_ι, ξ_2) should be a Rolle interval. Furthermore, if there are several boxes between two continuous zeros ξ_1 and ξ_2 of $L_k S$, we can replace the Rolle interval (ξ_1, ξ_2) by a Rolle interval which does not contain any box. Hence, a continuous zero ξ obtained by Lemma 5.3 and equal to an x_ι can be specified already by F_Φ^1 only if α) $f_{\iota,k+1} = 1$ and x_ι is not a zero of $L_k S(x)$ and with (ι, k) counted either by ϵ_{k+1} or ϵ_k. Unspecified continuous zeros will be counted towards μ_{k+1}.

Otherwise, Lemma 5.3 will produce a ξ at which $L_k S(x)$ and $L_{k+1} S(x)$ alternate and that is either β) a discontinuous zero of $L_{k+1} S(x)$ corresponding to $f_{\iota,k+1} = 1$ or $f_{\iota,k+1}$ is a box, or γ) a discontinuous zero of $L_k S(x)$ with $f_{\iota,k} = 1$. In fact, if $f_{\iota,k+1} = 1$ or a box but $f_{\iota,k} = 1$, then $L_k S(x)$ has continuous zero at x_ι. If $f_{\iota,k}$ is a box, then we can change the Rolle interval as before if necessary, and get a Rolle interval which does not contain x_ι on the kth level. If $f_{\iota,k+1} = 1$ or a box but $(\iota, k+1)$ is not supported on the left, then either $f_{j_1,k_1} \notin \{1, \binom{1}{1}\}$

in E^1_Φ for $i_1 = i$ and $k_1 = k$, or $f_{i_2,k_2} \notin \{1, (^1_1)\}$ in E^1_Φ for $i_2 = i$ and $k_2 = k$. In either case, x_i could not be in a Rolle's interval of any lower level, and consequently could not be in a Rolle's interval at level k. So it cannot be in interval of kth level for which the Lemma 5.3 can be applied. Hence, if $f_{i,k+1} = 1$ or a box, then the disjoint cases β) and γ) are counted in $v_{k+1} - \eta_{k+1} + b_{k+1}$.

If $f_{i,k+1} = -1$, 0 or a box, then the disjoint cases α) and γ) are counted in $\epsilon_{k+1} + \epsilon_k$. Thus, we omit $\epsilon_{k+1} + \epsilon_k + b_{k+1} + (v_{k+1} - \eta_{k+1})$ intervals I of level k, the remaining intervals will contribute to μ_{k+1}.

Again we shall improve this estimate. We call a pair (i, k) in Φ confirming, if $\xi = x_i$ can be obtained from Lemma 5.3 applied to $L_k S(x)$ under α), β) or γ), otherwise it is nonconfirming. In particular, (i, k) is nonconfirming, if (i, k) is on the horizontal or vertical boundary of Φ, if $L_k S(x), L_{k+1} S(x)$ are not alternating at x_i, or if $(i, k+1)$ is counted by ϵ_{k+1} but either $(i, k+1)$ is not supported on the left or $L_k S(x_i) = 0$.

Let N_k be the number of nonconfirming pairs (i, k) among those counted by $v_{k+1} - \eta_{k+1}$ or $\epsilon_{k+1} + \epsilon_k + b_{k+1}$. For $k = 0, \ldots, q-1$ we will examine, without duplication, all positions (i, k).

Let $N = \sum_{k=1}^{q-1} N_k$ and $\Omega = \sum_{k=0}^{q} \omega_k$. Instead of removing $\epsilon_{k+1} + \epsilon_k + b_{k+1} + (v_{k+1} - \eta_{k+1})$ intervals I under α), β), and γ), we need only to remove at most $\epsilon_{k_1} + \bar{\epsilon}_k + (v_{k+1} - \eta_{k+1}) - N_k$ of them which contain confirming (i, k). This gives a lower estimate for μ_{k+1}. However, zeros of $L_{k+1} S(x)$ counted by ω_{k+1} cannot be obtained in this way. Therefore,

$$\mu_{k+1} - \omega_{k+1} = \mu_k + u_k - l_{k+1} - \epsilon_k - 1 - (v_{k+1} - \eta_{k+1}) - \epsilon_{k+1} - b_{k+1} + N_k. \quad (9.2)$$

Summing (9.2) for $k = 0, 1, \cdots, q-1$ and using $\epsilon_0 = 0, \eta_0 = v_0, \epsilon_q = 0, \mu_q = 0, \mu_0 = 0, b_k = 0$, we obtain

$$0 = \mu_q = \sum_{k=0}^{q} u_k - \sum_{k=0}^{q} v_k - \sum_{k=1}^{q} l_k - \sum_{k=0}^{q} (\epsilon_k - \eta_k + \epsilon_k + b_k) + N + \Omega. \quad (9.3)$$

Since $\sum_{k=0}^{q} u_k - \sum_{k=0}^{q} v_k = |F_\Phi^1|$ and $\sum_{k=1}^{q} l_k = L(\Phi) - 1$ (because $l_0 = 1$), we have

$$|F_\Phi^1| = L(\Phi) - 1 + \Delta - N - \Omega$$

where $\Delta = \sum_{k=0}^{q-1} \epsilon_k - \sum_{k=0}^{q} \eta_k + \sum_{k=0}^{q} \epsilon_k + \sum_{k=0}^{q} b_k$.

We estimate the contributions Δ_B, N_B, and Ω_B to Δ, N and Ω for each block (9.1). The block can end at the vertical boundary of F_Φ^1 only with $f_{i,s+t} = -1$. Since if $f_{i,s+t} = 1$ or a box, then $L_{s+t}S(x)$ has a continuous zero or one-sided zero interior to its support at x_i that contradicts the fact that $L_{s+t}S(x)$ consists of piecewise constant multiples of $w_{s+t}(t)$ and $w_{s+t}(t) > 0$.

Restricted to a block B, $\sum \epsilon_k$ counts the number of times -1 is followed by a 1, 0, or box in B; $\sum \eta_k$ counts the number of times -1 is preceded by a 1 or a box in B or B begins with -1 and is not supported from the left; and $\sum(\epsilon_k + b_k) = 0$, or 1 and is equal 1 precisely when B begins with a 1 or a box and is supported from the left. Therefore, $\sum(\bar{\epsilon}_k - \eta_k)$ restricted to B is either 0, 1 or -1. It can be 1 only if B is an interior supported (on the left) block that begins with a sequence of 1's, and it can be -1 only if B is not interior. Altogether we see that $\Delta_B \leq 1$ and that $\Delta_B = 1$ can happen only if B is an interior supported (on the left) block.

Now we prove that if the interior block B supported on the left is even, then $N_B = 0$ implies $\Omega_B = 1$. This will show that $\Delta_B - N_B - \Omega_B \leq 1$ with equality possible only when B is an interior supported odd block.

Let B be an interior block (8.2) with $s > 0$, $f_{i,s-1} = f_{i,s+t+1} = 0$, supported on the left. Since we assume $N_B = 0$ all pairs among $(i, s-1), \cdots, (i, s+t+1)$ counted by $e_k + e_{k+1} + b_{k+1} + (v_{k+1} - \eta_{k+1})$ are confirming. Then Lemma 5.3 and the properties of zero multiplicities show that the $L_j S(x)$ are alternating at x_i for $j = s-1, \cdots, s+t+1$. Since $f_{i,s-1} = 0$, $L_{s-1} S(x)$ is continuous at x_i, and since $(i, s-1), (i, s)$ is confirming, $L_{s-1} S(x_i) \neq 0$; for otherwise, x_i is a continuous zero of $L_{s-1} S(x)$. Since $L_{s-1} S(x)$ is continuous at x_i, we have x_i is an even point of $L_{s-1} S(x)$. Now assuming that the block is even, then x_i is an even point of $L_{s+t} S(x)$. This implies $f_{i,s+t} \neq -1$, because if $f_{i,s+t} = -1$, then $(i, s+t)$ could be confirming only in case γ), but in case γ) $L_{s+t} S(x)$ should be odd at x_i. We also have $f_{i,s+t} \neq \binom{1}{-1}$ or $\binom{-1}{1}$, because we have noted that we can change Rolle interval so that x_i no longer belongs to any Rolle intervals at level $s+t$, which contradicts the fact that the pair $(i, s+t)$ and $(i, s+t+1)$ are confirming. On the other hand, if $f_{i,s+t} = 1$, we have $\Omega_B = 1$.

Thus, we need only to consider odd supported blocks and obtain

$$\Delta - N - \Omega \leq \gamma(F_{\Phi}^{1}). \quad \blacksquare$$

CHAPTER 10

THE GENERALIZED GOODMAN THEOREM

We can now state and prove a generalized Goodman's theorem that holds for generalized spline interpolation matrices.

Theorem 10.1. *Let the normal generalized spline interpolation matrix F have no odd blocks that are supported from both the left and the right. Then F is regular for Chebyshevian spline interpolation if and only if it satisfies the lower Goodman Pólya conditions (8.9).*

Proof: By Theorem 8.2, we get the necessity. So we only need to prove the sufficiency of the condition. We assume that $S(x)$ is a nontrivial Chebyshevian spline in $S(F, X, \mathcal{U}_N)$ annihilated by F, X, that is, $\Lambda_{i,k} S = 0$ for all $f_{i,k} \in \{1, \binom{1}{1}, \binom{1}{1}\}$, obtain a contradiction.

Lemma 10.2. *If, in addition to the assumptions of Theorem 9.1, the matrix F satisfies the lower Goodman-Pólya conditions, then there is a restriction $S^*(x)$ of $S(x)$ to an interval $[x_{i_3}, x_{i_4}]$ and a matrix F^*, so that with $\Phi^* = \Phi(S^*)$: 1) $S^*(x)$ is not trivial in $S(F^*, X^*, \mathcal{U}_N)$ and is annihilated by F^*, X^*; 2) F^* satisfies the lower Goodman-Pólya conditions; 3) $\gamma(F^{*1}_{\Phi^*}) < \rho(F)$ the number of odd blocks in F supported from the left and from the right.*

Proof: Let $\Phi = \Phi(S)$. By the definition of a sequence supported on the left or on the right, we know that a block B of F^1_Φ may be supported from the left above by $f^1_{i_0,k_0} = 1$ in F^1_Φ without B being supported from the left above by a 1 or $\binom{-1}{1}$ in F. If this happens for some B, we say that $(i_0, k_0) \in U$. Similarly, we define $(i_0, k_0) \in V$ if $f^1_{i_0,k_0} = 1$ in F^1_Φ supports B from the left below, but B is not supported from the left below by a 1 or $\binom{1}{1}$ in F. Note that necessarily $(i,k) \in V \cup U$ belong to the horizontal boundary of Φ (for otherwise, $f^1_{i,k} = f_{i,k}$).

If $(i_0, k_0) \in U$, then none of the $f_{i,k}$ in rectangle $R : 1 \le i \le i_0; 0 \le k \le k_0$ is 1 or $\binom{1}{1}$ in F. For otherwise, the block B would be supported from the left above in F. Consequently, $f_{1,k} = 1$ for $k = 0, 1, \cdots, k_0$ by the definition of F. From the lower Goodman-Pólya conditions (8.9) for R, we have $|F^1_R| \le k_0 + 1$. It follows that all $f_{i,k} = 0$, or $\binom{1}{1}$, $1 < i \le i_0$, $0 \le k \le k_0$.

There is a similar conclusion, if $(i_0, k_0) \in V$. That is, $f_{m,k} = 1$ for $k = 0, \cdots, k_0$, and all $f_{i,k} = 0$, or $\binom{1}{1}$ for $1 \le i \le i_0$ and $k \le k_0$. We define i_3 to be the largest i_0 with $(i_0, k_0) \in U$. (If U is empty, this is to be interpreted as $i_3 = 1$.) We also define i_4 to be smallest i_0 with $(i_0, k_0) \in V$ (and $i_4 = m$, if V is empty). Then $i_3 < i_4$, for otherwise, let us suppose $(i_3, l_3) \in U$ and $(i_4, l_4) \in V$. If $i_3 < i_4$, let $r = \min(l_3, l_4)$, then the lower Goodman-Pólya condition (8.9) would be violated for the rectangle $1 < i < m$, $0 \le k \le r$. Now, let $k_3(k_4)$ be largest k_0 with $(i_3, k_0) \in U$ (or with $(i_4, k_0) \in V$). If q_3 is the degree of $S(x)$ on $[x_{i_3}, x_{i_3+1}]$, we put $k_3^* = \min(k_3, q_3)$ and define k_4^* similarly by using the degree, q_4 of $S(x)$ on $[x_{i_4-1}, x_{i_4}]$.

Let F^* be the matrix with rows i, $i_3 \le i < i_4$ obtained from the corresponding rows of F by changing the 0's and the boxes in positions (i, k), $i = i_3$, $0 \le k < k_3^*$, and $i = i_4$, $0 \le k \le k_4^*$, to -1's. Here, we notice that there are only $\binom{1}{1}$ form boxes on the i_3th row for $k \le k_3^*$, and only $\binom{-1}{1}$ form boxes on the i_4th row for $k \le k_4^*$.

Since $i_3 < i_4$ and $S(x)$ is not trivial in $\mathcal{S}(F, X, \mathcal{U}_N)$, we get $S^*(x)$, the restriction of $S(x)$ on $[x_{i_3}, x_{i_4}]$, is also not trivial (since (i_3, k_3) and (i_4, k_4) support a common block in F_Φ). We also have $S^*(x) \in \mathcal{S}(F^*, X^*, \mathcal{U}_N)$ where $X^* = \{x_{i_3} < \cdots < x_{i_4}\}$ and $S^*(x)$ is annihilated by F^*, X^*, by the definition of F^*. This proves 1).

If Φ^* is any lower set in $i_3 \le i \le i_4$, $0 \le k \le N-1$, and if (i_3, k), $k < k_3^*$ is on the boundary of Φ^*, we extend it to the left by adding the interval $1 \le i < i_3$ at level k. Similarly, if (i_4, k), $k < k_4^*$ is on the boundary of Φ^*, we extend it to the right by adding the interval $i_4 < m^*$ at level k. This gives a lower set Ψ of F with $L(\Psi) = L(\Phi^*)$. Since $f_{1,k} = 0$ in F_Ψ^1 and $f_{i_3,k} = 0$ in $F_{\Phi^*}^{*1}$ for $k < k_3^*$, and similarly, $f_{m,k} = 0$ in F_Φ^1 and $f_{i_4,k} = 0$ in $F_{\Phi^*}^{*1}$ for $k < k_4^*$, by the definition of F_Φ^1, F^*, $F_{\Phi^*}^{*1}$, we have $|F_\Phi^1| = |F_{\Phi^*}^{*1}|$. Hence, 2) follows by the lower Goodman-Pólya conditions (8.9) for F.

An interior odd block of $F_{\Phi^*}^{*1}$ is also a block of this type in F_Φ^1. If it is supported from the left above in $F_{\Phi^*}^{*1}$ by some 1 or $\binom{1}{1}$ in position (i_0, k_0), then also by a 1 or $\binom{1}{1}$ in F_Φ^1. By the definition of $F_{\Phi^*}^{*1}$ and the properties of F, we just proved, we know that if $k_0 \le k_3$, then $i_0 > i_3$, and if $k_0 > k_3$, then $i_0 \ge i_3$. In any case, by the definition of (i_3, k_3), we know that $(i_0, k_0) \notin U$. Therefore, the block is supported from the left above in F. Similarly, we can prove the situation for the block supported from the left below. This proves 3). ∎

Proof of the Theorem 10.1: Let $S(x) \in \mathcal{S}(F, X, \mathcal{U}_N)$ be a nontrivial Chebyshevian spline annihilated by F, X, and let $S^*(x), F^*$ be given as in the lemma.

We wish to apply Theorem 9.1. One small problem is that F^* may not be a spline interpolation matrix since there may be zeros or boxes in the first or last row. However, the proof of Theorem 9.1 requires only that S^* be annihilated by $F^{*1}_{\Phi^*}$, $X \quad \{x_{i_3}, \ldots, x_{i_4}\}$. For this we only have to check that $L_k S^*(x_{i_3}+) \quad 0$ if $f^*_{i_3,k} \quad 0$, $k \cdot q_3$, and $L_k S^*(x_{i_4} \quad) \quad 0$ if $f^*_{i_4,k} \quad 0$, $k \cdot q_4$. Since $f^*_{i_j,k} \quad 1$ for $k \cdot k^*_j$, $f^*_{i_j,k} \quad 0 \quad f_{i_j,k}$ implies that (i_j, k) belongs to the horizontal boundary of Φ and $L_k S(x_{i_j}+) \quad L_k S(x_{i_j} \quad) \quad 0$, $j \quad 3,4$. This shows that S^* is annihilated by $F^*_{\Phi^*}$.

Therefore, by Theorem 9.1 and Lemma 10.2,

$$|F^{*1}_{\Phi^*} \quad L(\Phi^*) \cdot \quad 1 + \gamma(F^{*1}_{\Phi^*})$$

$$< \quad 1 + \gamma(F) \cdot 0.$$

This contradicts the lower Goodman-Pólya conditions. ∎

CHAPTER II

DUALITY

The generalized $m \times N$ spline interpolation matrix F defines a spline space $S(F, X, \mathcal{U}_N)$ for any given ECT-system $\mathcal{U}_N = \{u_1, \ldots, u_N\}$ on $[a, b]$ and a set of knots $X = \{a < x_1 < \cdots < x_m < b\}$. A basis for this space is given by

$$\mathcal{B} = \{u_i\}_{i=1}^{N} \bigcup \left\{ g_{q+1}(x, x_p) : f_{p,q} \in \{1, (^{1}_{-1}), (^{-1}_{1})\}; 1 < p < m,\ q = 0, \cdots, N-1 \right\}.$$

The generalized matrix F also determines the interpolation problem as given by the functionals

$$\Lambda_{i,k} g(x) = L_k g(x_i) \qquad \forall f_{i,k} = 1;$$

$$\Lambda_{i,k} g(x) = L_k g(x_i -) \qquad \forall f_{i,k} = (^{1}_{-1}); \text{ and}$$

$$\Lambda_{i,k} g(x) = L_k g(x_i +) \qquad \forall f_{i,k} = (^{-1}_{1}).$$

For each generalized spline interpolation matrix we define a dual generalized matrix F^*, and a corresponding dual interpolation problem. The generalized matrix F^* is obtained by

$$F^* = [f^*_{i,k}], \qquad f^*_{i,k} = -f_{i,N-1-k} \tag{11.1}$$

where $-(^{-1}_{1}) = (^{1}_{-1})$ and $-(^{1}_{-1}) = (^{-1}_{1})$. Clearly,

$$(F^*)^* = [-f^*_{i,N-1-k}] = [-(-f_{i,N-1-(N-1-k)})] = [f_{i,k}] = F.$$

The pair F^*, X defines an interpolation problem connected with the dual ECT-system $\mathcal{U}^*_N$ and the corresponding space $S(F^*, X, \mathcal{U}^*_N)$ which is spanned by

$$\mathcal{B}^* = \{u^*_i\}_{i=1}^{N} \bigcup \left\{ g^*_{q+1}(x_p, y) : f^*_{p,q} \in \{-1, (^{1}_{-1}), (^{-1}_{1})\}; 1 < p < m,\ q = 0, \cdots, N-1 \right\}. \tag{11.2}$$

The interpolation functionals are determined by F^* and the dual operators L_k^*.

$$\Lambda^*_{i,k} g(x) = L^*_k g(x_i) \quad \forall f^*_{i,k} = 1 - f_{i,N-i-k};$$

$$\Lambda^*_{i,k} g(x) = L^*_k g(x_i -) \quad \forall f^*_{i,k} = \binom{1}{1} \text{ (i.e. } f_{i,N-i-k} = \binom{1}{1}\text{); and} \tag{11.3}$$

$$\Lambda^*_{i,k} g(x) = L^*_k g(x_i +) \quad \forall f^*_{i,k} = \binom{1}{1} \text{ (i.e. } f_{i,N-i-k} = \binom{1}{1}\text{)}.$$

It is an interesting and useful fact that regularity is invariant under duality:

Theorem 11.1. *A generalized pair F, X is regular if and only if the dual pair F^*, X is regular; that is, we have*

$$D(F, X, \mathcal{G}) = \pm D(F^*, X, \mathcal{G}^*)$$

for some bases $\mathcal{G}$ and $\mathcal{G}^*$ of $S(F, X, \mathcal{U}_N)$ and $S(F^*, X, \mathcal{U}^*_N)$ respectively. In particular, F is regular for a given Chebyshevian system if and only if F^* is regular for the corresponding dual Chebyshevian system.

The proof of this theorem requires the study of the quantity

$$K(x,y) = \sum_{i=1}^{N} u_i(x) u^*_{N-i+1}(y)(-1)^{N-i}.$$

Using (2.7) and Theorems 2.4 and 2.5 (see also [2, equations (9.9) and (9.30)]), we find that

$$\begin{aligned} L^{*(y)}_{N-j} K(x,y) &= \sum_{i=1}^{j} u_i(x) L^*_{N-j} u^*_{N-i+1}(y)(-1)^{N-i} \\ &= \sum_{i=1}^{j} u_i(x) u^*_{N-j,j-i+1}(y)(-1)^{N-i} \\ &= (-1)^{N-j} h_j(x,y) \end{aligned}$$

and

$$\begin{aligned} L^{(x)}_j K(x,y) &= \sum_{i=j+1}^{N} u_{j,i-j}(x) u^*_{N-i+1}(y)(-1)^{N-i} \\ &= h^*_j(x,y). \end{aligned}$$

Therefore,

$$h_j(x,y) = (-1)^{N-j} L_{N-j}^{*(y)} K(x,y) \quad \text{and} \quad h_j^*(x,y) = L_j^{(x)} K(x,y),$$

for $j = 1, 2, \ldots, N-1$.

Lemma 11.2. *For $K(x,y)$, we have*

$$L_k^{(x)} L_j^{*(y)} K(x,y) = L_j^{*(y)} L_k^{(x)} K(x,y).$$

Proof: Now by the above

$$L_j^{*(y)} K(x,y) = (-1)^j h_{N-j}(x,y),$$

so that

$$\begin{aligned} & L_k^{(x)} L_j^{*(y)} K(x,y) \\ & = L_k^{(x)} \sum_{i=1}^{N-j} u_i(x) u^*_{j,N-j-i+1}(y)(-1)^{N-i} \\ & = \begin{cases} \sum_{i=k+1}^{N-j} u_{k,i-k}(x) u^*_{j,N-j-i+1}(y)(-1)^{N-i}, & \text{if } k+1 \leq N-j, \\ 0 & \text{if } k+1 > N-j. \end{cases} \end{aligned}$$

Also,

$$L_k^{(x)} K(x,y) = h_j^*(x,y),$$

so that

$$L_j^{*(y)} L_k^{(x)} K(x,y) = \begin{cases} \sum_{i=k+1}^{N-j} u_{k,i-k}(x) u^*_{j,N-j-i+1}(y)(-1)^{N-i} & \text{if } k+1 \leq N-j \\ 0 & \text{if } k+1 > N-j. \end{cases}$$

Therefore,

$$L_j^{*(y)} L_k^{(x)} K(x,y) = L_k^{(x)} L_j^{*(y)} K(x,y). \quad \blacksquare$$

Proof of Theorem 11.1: Define

$$H(x,y) = \begin{cases} K(x,y), & \text{if } x > y; \\ 0, & \text{otherwise} \end{cases}$$

Then

$$L^{*}_{N-j}(t)H(x,t) = (-1)^{N-j} g_j(x,t) \qquad t \neq x,$$

and

$$L_k^{(x)} H(x,t) = g_k^{*}(x,t) \qquad t \neq x.$$

We set

$$\mathcal{G} = \left\{ \frac{u_{q+1}}{w_{q+1}(x_1)} \right\}_{f_{1,q}=1} \bigcup_{\substack{f_{p,q}=1 \\ 1<p\leq m}} L^{*(t)}_{N-(q+1)} H(x,t)\big|_{t=x_p} \bigcup_{f_{p,q}=\left(\begin{smallmatrix} -1 \\ 1 \end{smallmatrix}\right)} L^{*(t)}_{N-(q+1)} H(x,t)\big|_{t=x_p^-}$$
$$\bigcup_{f_{p,q}=\left(\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right)} L^{*(t)}_{N-(q+1)} H(x,t)\big|_{t=x_p^+}$$

and

$$\mathcal{G}^{*} = \left\{ \tilde{u}^{*}_{N-k} \right\}_{f_{m,k}=1} \bigcup_{\substack{f_{i,k}=1 \\ 1<p\leq m}} L_k^{(x)} H(x,t)\big|_{x=x_i} \bigcup_{f_{i,k}=\left(\begin{smallmatrix} -1 \\ 1 \end{smallmatrix}\right)} L_k^{(x)} H(x,t)\big|_{x=x_p^-}$$
$$\bigcup_{f_{i,k}=\left(\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right)} L_k^{(x)} H(x,t)\big|_{x=x_p^+}$$

where

$$\tilde{u}^{*}_{N-k}(t) = \int_{x_m}^{t} w_N(s_N) \int_{x_m}^{s_N} \cdots \int_{x_m}^{s_{N-k-1}} w_{N-k}(s_{N-k})\, ds_{N-k} \cdots ds_N.$$

The functions $\mathcal{G}$ and $\mathcal{G}^{*}$ are bases for $S(F, X, \mathcal{U}_N)$ and $S(F^{*}, X, \mathcal{U}^{*}_N)$ respectively.

We apply the functionals $\Lambda_{i,k}$ for $f_{i,k} \in \{1, \left(\begin{smallmatrix} -1 \\ 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right)\}$ to functions in $g \in \mathcal{G}$. For $i = 1$ the result will be 0 unless g is $u_{k+1}/w_{k+1}(x_1)$, in which case it will give ± 1. Hence, the resulting coefficient matrix for the system of equations can be brought to the form

$$A(F, X, \mathcal{G}) = \begin{bmatrix} I_\eta & 0 \\ * & A_2 \end{bmatrix}$$

where I_η is an $\eta \times \eta$ diagonal matrix with diagonal entries ± 1 and $\eta = \#\{f_{1,k} = 1\}$ and A_2 is a matrix described as follows: The rows of A_2 correspond to pairs (i, k) with $f_{i,k} \in \{1, \binom{-1}{1}, \binom{1}{-1}\}$, $1 \le k \le m$, and the columns correspond to pairs (p, q) with $f_{p,q} \in \{1, \binom{1}{-1}, \binom{-1}{1}\}$, $1 \le p \le m$. The entries of A_2 are given by the table

$$
\begin{array}{ll}
L_k^{(x)} L_{N-q-1}^{*(t)} H\big|_{x=x_i, t=x_p} & \text{if } f_{i,k} = 1 \text{ and } f_{p,q} = 1, \\
L_k^{(x)} L_{N-q-1}^{*(t)} H\big|_{x=x_i+, t=x_p} & \text{if } f_{i,k} = \binom{-1}{1} \text{ and } f_{p,q} = 1, \\
L_k^{(x)} L_{N-q-1}^{*(t)} H\big|_{x=x_i-, t=x_p} & \text{if } f_{i,k} = \binom{1}{-1} \text{ and } f_{p,q} = 1, \\
L_k^{(x)} L_{N-q-1}^{*(t)} H\big|_{x=x_i, t=x_p+} & \text{if } f_{i,k} = 1 \text{ and } f_{p,q} = \binom{1}{-1}, \\
L_k^{(x)} L_{N-q-1}^{*(t)} H\big|_{x=x_i+, t=x_p+} & \text{if } f_{i,k} = \binom{-1}{1} \text{ and } f_{p,q} = \binom{1}{-1}, \\
L_k^{(x)} L_{N-q-1}^{*(t)} H\big|_{x=x_i-, t=x_p+} & \text{if } f_{i,k} = \binom{1}{-1} \text{ and } f_{p,q} = \binom{1}{-1}, \\
L_k^{(x)} L_{N-q-1}^{*(t)} H\big|_{x=x_i, t=x_p} & \text{if } f_{i,k} = 1 \text{ and } f_{p,q} = \binom{-1}{1}, \\
L_k^{(x)} L_{N-q-1}^{*(t)} H\big|_{x=x_i+, t=x_p} & \text{if } f_{i,k} = \binom{-1}{1} \text{ and } f_{p,q} = \binom{-1}{1}, \\
L_k^{(x)} L_{N-q-1}^{*(t)} H\big|_{x=x_i-, t=x_p-} & \text{if } f_{i,k} = \binom{1}{-1} \text{ and } f_{p,q} = \binom{-1}{1}.
\end{array}
$$

Similarly, we apply the functionals $\Lambda^*_{p,N-q-1}$ for $f_{p,q} \in \{-1, \binom{-1}{1}, \binom{1}{-1}\}$ to functions in $g \in \mathcal{G}^*$. For $p = m$ the result will be 0 unless g is $\tilde{u}^*_{N-q}$, in which case it will give 1. Hence, the resulting coefficient matrix for the dual system of equations can be brought to the form

$$A(F^*, X, \mathcal{G}^*) = \begin{bmatrix} A_2^* & * \\ 0 & I_{\eta^*} \end{bmatrix}$$

where I_{η^*} is an $\eta^* \times \eta^*$ identity matrix with $\eta^* = \#\{f_{m,q} = 1\}$ and A_2^* is a matrix described as follows: The rows of A_2^* correspond to pairs (p,q) with $f_{p,q} \in \{-1, \binom{1}{-1}, \binom{-1}{1}\}$, $1 \le p < m$, and the columns correspond to pairs (i,k) with $f_{i,k} \in \{1, \binom{1}{-1}, \binom{-1}{1}\}$, $1 < i < m$. The entries of A_2^* are given by the table

$$
\begin{array}{lll}
L^{*(t)}_{N-q-1} L^{(x)}_k H\big|_{x=x_i,t=x_p} & \text{if} \quad f_{i,k} = 1 \text{ and } f_{p,q} = -1, \\
L^{*(t)}_{N-q-1} L^{(x)}_k H\big|_{x=x_i+,t=x_p} & \text{if} \quad f_{i,k} = \binom{-1}{1} \text{ and } f_{p,q} = -1, \\
L^{*(t)}_{N-q-1} L^{(x)}_k H\big|_{x=x_i-,t=x_p} & \text{if} \quad f_{i,k} = \binom{1}{-1} \text{ and } f_{p,q} = -1, \\
L^{*(t)}_{N-q-1} L^{(x)}_k H\big|_{x=x_i,t=x_p+} & \text{if} \quad f_{i,k} = 1 \text{ and } f_{p,q} = \binom{1}{-1}, \\
L^{*(t)}_{N-q-1} L^{(x)}_k H\big|_{x=x_i+,t=x_p+} & \text{if} \quad f_{i,k} = \binom{-1}{1} \text{ and } f_{p,q} = \binom{1}{-1}, \\
L^{*(t)}_{N-q-1} L^{(x)}_k H\big|_{x=x_i-,t=x_p+} & \text{if} \quad f_{i,k} = \binom{1}{-1} \text{ and } f_{p,q} = \binom{1}{-1}, \\
L^{*(t)}_{N-q-1} L^{(x)}_k H\big|_{x=x_i,t=x_p-} & \text{if} \quad f_{i,k} = 1 \text{ and } f_{p,q} = \binom{-1}{1}, \\
L^{*(t)}_{N-q-1} L^{(x)}_k H\big|_{x=x_i+,t=x_p-} & \text{if} \quad f_{i,k} = \binom{-1}{1} \text{ and } f_{p,q} = \binom{-1}{1}, \\
L^{*(t)}_{N-q-1} L^{(x)}_k H\big|_{x=x_i-,t=x_p-} & \text{if} \quad f_{i,k} = \binom{1}{-1} \text{ and } f_{p,q} = \binom{-1}{1}.
\end{array}
$$

It follows from Lemma 11.2 and the definition of $H(x,y)$ that

$$L^{*(t)}_{N-q-1} L^{(x)}_k H(x,t) = L^{(x)}_k L^{*(t)}_{N-q-1} H(x,t) \qquad x \neq t.$$

However, in the above tables for the entries of A_2 and A_2^*, $x_i \neq x_p$ unless $i = p$ and then only if $f_{i,k} = f_{p,q}$ is a box. In the latter case, the one-sided limits are defined and are equal. Consequently, we see that A_2 and A_2^* are transposes of one another and

$$D(F, X, \mathcal{G}) = \pm D(F^*, X, \mathcal{G}^*). \quad \blacksquare$$

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