Optimal Portfolio-Consumption with Habit Formation under Partial Observations

by

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Abstract

The aim of my thesis consists of characterizing explicitly the optimal consumption and investment strategy for an investor, when her habit level process is incorporated in the utility formulation. For a continuous-time market model, I maximize the expected utility from terminal wealth and/or consumption. For this optimization problem, the thesis presents three novel contributions.

Using the Kalman-Bucy filter theorem, I transform the optimization problem under the partial information into an equivalent optimization problem within a full information context. Using the stochastic control techniques, this latter problem is reduced to solve an associated Hamilton-Jacobi-Bellman equation (HJB hereafter). For the exponential utility, the solution to the HJB is explicitly described, while the optimal policies/controls as well as the optimal wealth process are described by a stochastic differential equation. Furthermore, I discuss qualitative analysis on the optimal policies for the exponential utility. These achievements constitute my first contribution in this thesis. The second contribution lies in considering a stochastic volatility model and addressing the same optimization problem using again the techniques of stochastic control. The third contribution of my thesis resides in combining the filtering techniques with the martingale approach to solve the optimization problem when the investor is endowed with the logarithm, power or exponential utility.

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Contents

Chapter 1

Introduction

Mathematical finance is a multidisciplinary field, which draws on and extends classical applied mathematics, stochastic and probabilistic methods, and numerical techniques to enable models of financial systems to be constructed, analysed and interpreted. This methodology underpins applications to derivatives pricing, portfolio structuring, risk management, insurance analysis and many more.

Portfolio-Consumption Optimization and Asset pricing are two most popular topics in mathematical finance. Under optimization theory, the aim of the investors is to minimize the risk while seeking for the highest return. Or, they maximize their return for their acceptable level of risk. The pioneer of this field is Harry Markowitz, who stated in 1952 that it is possible for different portfolios to have varying levels of risk and return. Each investor must decide how much risk he or she can tolerate, and allocate their portfolio according to the efficient frontier which shows a set of optimal portfolios that offers the highest expected return for a defined level of risk.

The other popular topic is Asset Pricing Theory, which is built on mathematical models of bond and stock prices and has two important directions. The first direction is Black-Scholes arbitrage pricing of options and other derivative securities, while the second direction lies in the Capital Asset Pricing Model.

All this topics in mathematical finance have an enormous impact on the way modern financial markets operate. In this thesis, I mainly focus on the optimal consumption-portfolio choice with the effect of habit formation. I study the corresponding utility optimization problem for an investor within the finite time horizon, for both cases of full information and partial information.

1.1 Habit Formulation Utilities

Over the past decades, habit formation has become a popular topic and draws attention from many researchers. The time seperable von Neumann-Morgenstern preference on consumption has been observed to be conflict with some empirical experiments such as the Premium Puzzle (see [17]), the Joneses effect (see [1]) and the Exchange Economy with Habit Formation (see [9]).

Therefore, both the empirical and theoretical literatures have confirmed that the past consumption pattern plays a role in determing on individual's current consumption decisions. Based on this, a vast literature recommends this time non-separable preference as the new economic paradigm. In 1930, I. Fisher examined the measurability of the utility function, and emphasized the importance of nonseparable utility formation in [10]. Sundaresan, in [19], constructed a model in which consumer's utility depends on the consumption history. By applying the Hamilton-Jacobi-Bellman equation, he gave a feedback form consumption in a simple example. With simulation method, the consumption paths generated from this model is formed to be less fluctuating compared with the case of separable utility function. Detemple and Zapatero, in [8], proved the existence of optimal consumption-portfolio policies for utility function involving a general dependence on past consumption or the standard of living. Using the martingale approach, they calculated the optimal consumption rate and the feedback form of optimal portfolios under different utility functions. When the asset market is incomplete, the convex duality approach becomes an important method to deal with the utility maximization problems. (see [23] and [14]).

1.2 Information and Financial markets

Partial Information means investors can not observe the drift process and Brownian motion appearing in the stochastic differential equation for the security prices. Why it is important to study the partial information? Because partial information is more consistent with reality. We can easily attain the information about the stock price and interest rate but we can not know the pattern of drift process and the paths of Brownian motions.

Optimal investment problems under incomplete information was discussed already by Lakner in [16] where a formula was presented for the optimal level of terminal wealth, and the existence of a corresponding trading strategy has been shown. Lakner studies this problem using the martingale approach. In this way, the problem can be reduced to the calculation of a certain expected value. The main objective of the present paper is to work out explicit formula for the optimal trading strategy. Brennan and Xia, in [4], assume that the drift process is constant, but cannot be observed by the investor. They show that the Bellman equation can be reduced to a system of ordinary differential equations, which is solved numerically. Yu (see [22]) assume the drift process is unknown and satisfies the Ornstein Uhlenbeck stochastic differential equation. By using the dynamic programming arguments, he solved the optimal problems under partial information with power preference. Ibrahim and Abergel, in [12], studied the question of filtering and maximizing terminal wealth from expected utility in a stochastic volatility models by both martingale approach and partial differential equation method. This problem becomes more complicated, as it is a non linear filtering problem when transforming the volatility models under partial information into complete information.

1.3 Summary of the Thesis

In this thesis, we study the utility maximization problem of an investor with habit formation and incomplete information. The aim of the investor is to maximize the expected utility from her consumption and/or terminal wealth in a simple financial market with finite investment horizon T.

This thesis contains five chapters including the current chapter of the introduction. In the next chapter (Chapter 2), we introduce the mathematical tools as well as the fundamental financial market concepts that will be used throughout the thesis. In Chapter 3, we specify our market model with partial information. After transforming this model into an equivalent complete information model by filtering techniques, the optimal problem can be reduced to solve a partial differential equation (PDE). Then, we solve it explicitly for the exponential preference.

In Chapter 4, we focus on the stochastic volatility model. This case falls into the context of complete information even we can just observe the stock price and interest rate. So we can get the optimal consumption and portfolio by solving a PDE.

Chapter 5 develops the martingale approach for both cases of complete information and incomplete information. For the case of complete information, we extend the model of Detemple and Zapatero by relaxing some boundedness assumption and focusing on the case of exponential utility. In contrast to the case of complete information up to our knowledge, the case of incomplete information was not addressed using this martingale approach. Using this approach, we analyse the three cases of utilities (namely the logarithmic, power and exponential) and discuss many particular situations.

Chapter 2

Mathematical Preliminaries and Financial Market

In this chapter, we introduce some financial concepts, stochastic basis and other mathematical techniques used throughout the rest of the thesis.

2.1 Stochastic Basis and Calculus

The Financial modelling of system starts with a given filtered probability space

$$(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P),$$

which is called in the probabilistic literature as stochastic basis. Here, P is a probability measure and $\mathcal{F}$ is a σ-algebra that contains all negligible sets. The family $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$ is called filtration, where $\mathcal{F}_t$ is a σ-fields and

$$\mathcal{F}_0 \subseteq \mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F} \quad \text{for all } t \geq s \geq 0.$$

Financially speaking, $\mathcal{F}_t$ represents the aggregate information about the market/ agent/ financial products up to time t.

2.1.1 Brownian Motion and Martingales

Definition 2.1: 1) A stochastic process $X = (X_t)_{t\geq 0}$ is a family of random variable indexed by time.

2) A process X is said to be $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ adapted if for all $t \geq 0$, X_t is $\mathcal{F}_t$-measurable.

As an important example of adapted stochastic process is the Brownian motion.

Definition 2.2: A process $W = (W_t)_{t\geq 0}$ is said to be a Brownian motion if

1) $W_0 = 0 \quad P - a.s.$,

2) $t \to W_t(w)$ is continuous for almost all $w \in \Omega$,

3) $W_t - W_s \sim \mathcal{N}(0, t-s)$ for all $0 \leq s < t$,

4) $W_t - W_s$ independent of $W_u - W_r$ for all $0 \leq r \leq u \leq s < t$.

By an n-dimensional Brownian motion we mean a process with values in $\mathbb{R}^n$

$$W(t) = (W_1(t), W_2(t), ..., W_n(t)),$$

where the components W_i are independent one-dimensional Brownian motions. In the literature, the requirement 3) of Definition 2.2 with respect to a given filtration $\{\mathcal{F}_t\}_{t\geq 0}$ is typically stated as

3)* $W_t - W_s$ independent of $\mathcal{F}_s$, for $0 \le s \le t$.

When we consider a Brownian motion $\{W_t, \mathcal{F}_t\}_{t\ge 0}$ with arbitrary filtration $\{\mathcal{F}_t\}_{t\ge 0}$ we implicitly assume requirement 3)* to be fulfilled.

For technical reasons in the theory of stochastic integration a filtration $\sigma(\cup_{s>t}\mathcal{F}_s)$ is usually required to be right-continuous. Thus the Brownian filtration satisfies the usual condition in the sense of following.

Definition 2.3: A filtration $\{\mathcal{F}_t\}_{t\ge 0}$ satisfies the usual conditions if it is right-continuous (i.e. $\mathcal{F}_t = \vee_{s>t}\mathcal{F}_s$ for all $t \ge 0$) and $\mathcal{F}_0$ contains all P-null sets of $\mathcal{F}$.

Throughout the thesis, $\{\mathcal{F}_t\}_{t\ge 0}$ will be assumed to satisfy the usual conditions. Now we introduce a class of stochastic processes which will be fundamental for our analysis in this thesis.

Definition 2.4: A real-valued process $X = (X_t)_{t\ge 0}$, which is $\mathbb{F}$-adapted and satisfies $E\,|X_t| < \infty$ for all $t \ge 0$, is called

1) a super-martingale, if we have

$$E(X_t \mid \mathcal{F}_s) \le X_s \quad P - a.s., \quad 0 \le s \le t.$$

1) a sub-martingale, if we have

$$E(X_t \mid \mathcal{F}_s) \ge X_s \quad P - a.s., \quad 0 \le s \le t.$$

1) a martingale, if we have

$$E(X_t \mid \mathcal{F}_s) = X_s \quad P - a.s., \quad 0 \leq s \leq t.$$

The following theorem is Doob's inequality.

Theorem 2.1: *Let (X_t) be an non-negative continuous sub-martingale and $X^* = \sup_{t \geq 0} X_t$. Then*

$$E[X^*] \leq \frac{e}{e-1}\Big(1 + \sup_{t \leq 0} E[X_t \log^+ X_t]\Big),$$

$$||X^*||_p \leq q \sup_{t \geq 0} ||X_t||_q,$$

where $p > 1$ and $q > 1$ are a couple of conjugate indices.

Corollary 2.1.1: A process $X = (X_t)_{t \geq 0}$ is a martingale if and only if is a super-martingale and a sub-martingale.

Theorem 2.2: *A one-dimensional Brownian motion $W = (W_t)_{t \geq 0}$ is a martingale.*

Remark 2.1: The Brownian motion with drift μ and volatility σ

$$X_t := \mu t + \sigma W_t$$

is a martingale if $\mu = 0$, a super-martingale if $\mu \leq 0$ and a sub-martingale if $\mu \geq 0$.

Now we introduce the famous theorem demonstrating the way a Q-Brownian motion $W^Q(t)$ can be constructed from a P-Brownian motion $W(t)$ via a change of measure from P to Q.

Theorem 2.3: *(Girsanov's theorem) Define the process*

$$Z(t,X) := \exp\Big(-\sum_{i=1}^{m}\int_0^t X_i(s)dW_i(s) - \frac{1}{2}\int_0^t \|X(s)\|^2\, ds\Big), \quad t \in [0,T], \tag{2.1}$$

and $Z(t,X)$ is a martingale and define the process $\{(W^Q(t), \mathcal{F}_t)\}_{t\geq 0}$ by

$$W_t^Q(t) := W_i(t) + \int_0^t X_i(s)ds, \quad 1 \leq i \leq m, \quad t \geq 0.$$

Then, for each fixed $T \in [0,\infty]$ the process $\{(W^Q(t), \mathcal{F}_t)\}t \in [0,T]$ is an m-dimensional Brownian motion on $(\Omega, \mathcal{F}_T, Q)$ where the probability measure Q is defined as

$$Q(A) := E(1_A \cdot Z(T,X)) \quad for A \in \mathcal{F}_T.$$

2.1.2 Stochastic Integral and Itô's Formula

Before introducing the Itô's Formula, we need to define the stochastic integral. We shall start by constructing it for so-called simple process X_t.

Definition 2.5: A stochastic process $\{X_t\}_{t\in[0,T]}$ is called a simple process if there exist real number $0 = t_0 < t_1 < ... < t_p = T,\ p \in \mathbb{N}$, and bounded random variables $\Phi_i : \Omega \to \mathbb{R}$ with

$$\Phi_0 \quad \mathcal{F}_0 \text{ -measurable}, \quad \Phi_i \quad \mathcal{F}_{t_{i-1}}\text{-measurable, for all } i = 1, ..., p.$$

such that $X_t(w)$ has the following representation

$$X_t(w) = X(t,w) = \Phi_0(w)1_{\{0\}}(t) + \sum_{i=1}^{p} \Phi_i(w)1_{(t_{i-1},t_i]}(t)$$

for each $w \in \Omega$.

Definition 2.6: For a simple process $\{X_t\}_{t\in[0,T]}$ the stochastic integral $I_t(X)$ for $t \in (t_k, t_{k+1}]$ is defined according to

$$I_t(X) := \int_0^t X_s dW_s := \sum_{1\leq i\leq k} \Phi_i(W_{t_i} - W_{t_i-1}) + \Phi_{k+1}(W_t - W_{t_k}),$$

or more generally for $t \in [0,T]$:

$$I_t(X) := \int_0^t X_s dW_s := \sum_{1\leq i\leq p} \Phi_i(W_{t_i\wedge t} - W_{t_{i-1}\wedge t}).$$

Hence, on each interval where X is constant, the increments of the Brownian motion on that interval are multiplied with the corresponding value of X_t. namely Φ_i..

In most of cases, simple process is a strict condition. So we need to define the stochastic process in a more general level. We have to take a closer look at measurability assumptions for the stochastic process X to be able to define the stochastic integral for more general integrands in a reasonable way.

Definition 2.7: Let $\{(X_t, \mathcal{G}_t)\}_{t\in[0,\infty)}$ be a stochastic process. This stochastic process will be called measurable if the mapping

$$[0,\infty) \times \Omega \to \mathbb{R}^n : \quad (s,w) \mapsto X_s(w)$$

is $\mathcal{B}([0,\infty)) \otimes \mathcal{F} - \mathcal{B}(\mathbb{R}^n)$ measurable.

Remark 2.2: Measurability of the process X in particular implies that for a fixed $w \in \Omega$, $X(\cdot, w)$ is $\mathcal{B}([0,\infty)) - \mathcal{B}(\mathbb{R}^n)$-measurable. Thus, for all $t \in [0,\infty), i = 1, ..., n$ the integral $\int_0^t X_t^2(s)ds$ is defined.

Definition 2.8: Let $\{(X_t, \mathcal{G}_t)\}_{t\in[0,\infty]}$ be a stochastic process. This stochastic process will be called progressively measurable if for all $t \geq 0$ the mapping

$$[0,t) \times \Omega \to \mathbb{R}^n : \quad (s,w) \mapsto X_s(w)$$

is $\mathcal{B}([0,t)) \otimes \mathcal{F}_t - \mathcal{B}(\mathbb{R}^n)$ measurable.

According to the above discussion we require integrands to be progressively measurable when we want to extend the stochastic integral for a larger class of integrands than simple processes. Further to be able to define a norm for stochastic integrals, we consider the following vector space:

$$\begin{aligned} L^2[0,T] :&= L^2\Big([0,T], \Omega, \mathbb{F}, P\Big) \\ :&= \Big\{\{(X_t, \mathcal{F}_t)\}_{t\in[0,T]} \text{ real-valued stochastic process } | \\ &\quad \{X_t\}_{t\in[0,T]} \text{ progressively measurable}, E\Big(\int_0^T X_t^2 dt\Big) < \infty\Big\} \end{aligned}$$

Theorem 2.4: *(Construction of the Iô integral for process in $L^2[0,T]$) There exist a unique linear mapping J from $L^2[0,T]$ into the space of continuous martingales on $[0,T]$ with respect to $\{\mathcal{F}_t\}_{t\in[0,T]}$ satisfying*

1) For any simple process, $X = \{X_t\}_{t \in [0,T]}$*,*

$$J_t(X) = I_t(X), \quad \textit{for all } t \in [0,T], \quad P - a.s.$$

2) For any $X \in L^2[0,T]$*, we have*

$$E\Big(J_t(X)^2\Big) = E\Big(\int_0^t X_s^2 ds\Big)$$

Definition 2.9: For $X \in L^2[0,T]$ and J as Theorem 2.4 we denote

$$\int_0^t X_s dW_s := J_t(X), \quad t \geq 0$$

and $J(x)$ is called the stochastic integral or the Iô integral of X with respect to W.

Now, we introduce Itô's Formula for n-dimensional Itô process having the form of

$$X_i(t) = X_i(0) + \int_0^t K_i(s)ds + \sum_{j=1}^{m} \int_0^t H_{ij}(s)dW_j(s), \quad i = 1, ..., n.$$

Theorem 2.5: *(Itô's Formula) Let* $f : [0,\infty) \times \mathbb{R}^n \mapsto \mathbb{R}$ *be a* $C^{1,2}$*-function. That is,* f *is continuous, continuously differentiable with respect to the first variable (time), and twice continuously differentiable with respect to the other* n *variables (space).*

Then, for every $t \geq 0$,

$$
\begin{aligned}
&f(t, X_1(t), ..., X_n(t)) \\
=&f(0, X_1(0), ..., X_n(0)) \\
&+ \int_0^t f_t(s, X_1(s), ..., X_n(s))ds + \sum_{i=1}^n \int_0^t f_{x_i}(s, X_1(s), ..., X_n(s))dX_i(s) \\
&+ \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \int_0^t f_{x_i x_j}(s, X_1(s), ..., X_n(s))d\langle X_i, X_j \rangle_s.
\end{aligned}
$$

2.2 Market Structure

With all knowledge of the stochastic calculus, we can describe and introduce the financial market. A financial market is a market in which people can trade financial derivatives, commodities, and other financial products. In the market, everything for trade has a corresponding price. Price is the expense for different goods or services. The law of the markets determines that a suitable price is the one which can keep a balance between supply and demand. Usually, there exists low transaction costs when trading activities happen in real life.

However, to make the research concise, we need to simplify the market structure in this thesis. Therefore, I will focus on the single-investor economy. In other words, I will consider the case where transaction fees or costs have no significant influence on the market equilibrium. Hence, throughout this thesis, transaction fees can be neglected.

2.2.1 Modelling the Security Prices

We also consider this single-investor economy with frictionless markets and no taxes in time interval $[0, T]$. It is only stocks and bonds that are tradeable in our model. Bond is a riskless asset, and its rate of return is a positive number $r(t)$. $r(t)$ is also called risk-free interest rate at time t. The price process of the bond is denoted by $S_0(t)$, and follows

$$dS_0(t) = r(t)S_0(t)dt, \quad t \in [0, T]. \tag{2.2}$$

Equivalently, given the initial bond price $S_0(0) = s_0^*$,

$$S_0(t) = s_0^* \exp\Big(\int_0^t r(s)ds\Big), \ t \in [0, T]. \tag{2.3}$$

Different from bonds, usually many kinds of stocks exist in the financial market. We assume that the market consists of m stocks. We denote by $S_i(t)$ the price of the i^{th} stock at time t $(i = 1, 2..., m)$. The dynamic of the stock price process is given by:

$$dS_i(t) = \mu_i(t)S_i(t)dt + S_i(t)\sum_{j=1}^{d} \sigma_{ij}(s)dW_t^j. \tag{2.4}$$

Equivalently, given the initial stock price $S_i(0) = s_i$, we have

$$S_i(t) = s_i \exp\left\{\int_0^t \left[\mu_i(s) - \frac{1}{2}\sum_{j=1}^{d} \sigma_{ij}^2(s)\right] ds + \sum_{j=1}^{d}\int_0^t \sigma_{ij}(s)dW_s^j\right\}, \quad t \in [0, T]. \tag{2.5}$$

In the equations above, the interest rate $r(t)$, the stock return rate $\mu(t) \triangleq$

$(\mu_1(t), ..., \mu_m(t))$, the volatility matrix $\sigma(t) = \{\sigma_{ij}(t)\}_{1\leq i\leq m, 1\leq j\leq d}$ are the volatilities of stocks. Precisely, $r(t) : [0,T] \times \Omega \to \mathbb{R}$ and $\mu_i(t) : [0,T] \times \Omega \to \mathbb{R}$ are positive scalars, while the volatility of i^{th} stock $\sigma_i(t) : [0,T] \times \Omega \to \mathbb{R}^m$ describes the price dispersion rate. All these processes are assumed to be $(\mathcal{F}_t)_{t\geq 0}$-adapted.

The process $W_t \triangleq \left(W_t^1, W_t^2, ..., W_t^d\right)$ is a d-dimensional standard Brownian motion. It is assumed that $m \leq d$. If $m = d$ and the volatility matrix is non-singular, those stocks create what is called a complete market. A financial market is said to be complete where every payoff can be replicated. Otherwise, the market is incomplete such as the case $m < d$, where an infinite number of risk neutral probability measures exist.

2.2.2 Trading Strategy and Wealth Process

We further assume that the investors can buy stocks and bonds with their capital. The investment activity is characterized by portfolio $\pi(t) \triangleq (\pi_0(t), \pi_1(t), ..., \pi_m(t))$, where $\pi_i(t) \triangleq N_i(t)S_i(t)$. $N_0(t)$ represents the amount of bond, and $N_i(t)$ represents the amount of i^{th} stock at time t, $i = 1, ..., m$. In our model, short-selling is allowed, which means that N_i can be any real number for $i = 0, ..., m$. Moreover, it is assumed that the investor have to make a continuous living expense at the note $c(t)$ to cover his or her expense, $c(t)$ represents the consumption rate of the investor at time t.

In our economy, the investor starts with an initial capital x_0, and no endowment will be added at any time $t \in [0,T]$. We use $X(t)$ to represent the wealth of the agent at time t. At any time t, the consumer must decide his consumption rate $c(t)$ and investment strategy $\pi(t)$. Then, the wealth process is given by the following stochastic differential equation

$$\begin{cases} dX(t) = \left[r(t)X(t) + \sum_{i=1}^{m} (\mu_i(t) - r(t))\, \pi_i(t) - c(t) \right] dt + \sum_{j=1}^{d} \sum_{i=1}^{m} \pi_i(t)\sigma_{ij}(t) dW_t^j, \\ X(0) = x_0, \quad t \in [0, T]. \end{cases} \tag{2.6}$$

Equivalently,

$$X(t) = \exp\left[\int_0^t r(s)ds \right] \left\{ x_0 + \int_0^t \exp\left[- \int_0^s r(u)du \right] \left[\pi(s)(\mu(s) - r(s) \cdot 1)^\top \right.\right.$$
$$\left.\left. - \; c_s \right] ds + \sum_{j=1}^{d} \int_0^t \exp\left[- \int_0^s r(u)du \right] \pi(s)\sigma_{\cdot j}(s) dW_s^j \right\}, \quad t \in [0, T]. \tag{2.7}$$

Definition 2.10: A pair (π, c) consisting of a portfolio process π and a consumption rate c will be called admissible for the initial wealth $x_0 > 0$, if

1) (π_t, c_t) are $(\mathcal{F}_t)_{t\in[0,T]}$- progressively measurable and satisfies the integrability conditions

$$\int_0^T (\pi_t^2 + c_t^2) dt < +\infty, P - a.s..$$

2) the corresponding wealth process satisfies

$$X(t) \geq 0 \quad P - a.s., \text{ for all } t \in [0, T]$$

The set of admissible pairs (π, c) will denote by $\mathcal{A}(x_0)$.

This condition is to make the wealth process well defined and to avoid bankruptcy.

2.3 Habit Utility Formation

In this section, we define some mathematical variables that describe the state of agent. When investors consume some money they will gain happiness from their consumption, and we call this effect as utility. In economics, utility is a description of preferences over some set of goods and services. In mathematics, utility is a function $U : [0, \infty) \to \mathbb{R}$ that is increasing and concave. Usually, it is a single variable function with respect to consumption rate or wealth. In macroeconomics, the utility function must satisfy the Inada's condition.

Assumption 1: *Let $U : [0, \infty) \to \mathbb{R}$ be a strictly concave and continuously differentiable function satisfying*

$$U'(0) := \lim_{x \downarrow 0} U'(x) = +\infty, \quad U'(\infty) := \lim_{x \uparrow \infty} U'(x) = 0. \tag{2.8}$$

Then U is called a utility function.

From a financial view, the marginal utility is strictly decreasing, and it goes to zero as consumption rate or wealth approaches positive infinity.

In our model, we study the problem of optimal consumption and investment rules for an agent with habit formation. Therefore, we expand the utility U to a dual-variable function $U(c, z)$ with respect to consumption rate $c(t)$ and consumption habit level $z(t)$.

Similarly, the habit-related utility function has to satisfy the following condition.

Assumption 2: *$U(c, z)$ is continuously differentiable and satisfies:*

1. $\dfrac{\partial U(c,z)}{\partial c} > 0$. *For fixed historical consumption rate, an increase in current consumption will increase utility.*

2. $\dfrac{\partial U(c,z)}{\partial z} < 0$. *For fixed current consumption, an increase in historical consumption rate will decrease utility.*

3. $\dfrac{\partial^2 U(c,z)}{\partial c^2} < 0$. *Marginal utility will decreases as current consumption increases. It indicates that utility function $U(c, z)$ is concave down for c.*

4. $\lim\limits_{c \to +\infty} \dfrac{\partial U(c,z)}{\partial c} = 0$. *Marginal utility approaches 0 as consumption rate goes to infinity.*

One simple example of the habit index process $z(t)$ is given by

$$dz(t) = \beta(c(t) - z(t))dt, \quad z(0) = z_0, \quad t \in [0, T]. \tag{2.9}$$

Equivalently, given the initial consumption habit z_0,

$$z(t) = z_0 e^{-\beta t} + \int_0^t \beta e^{\beta(s-t)} c(s)ds, \quad t \in [0, T]. \tag{2.10}$$

In this formulation, z_0 is the initial consumption preference level. β is the habit formulation factor, and it represents the weight of nearby consumption in the formulation of habit. As time passes, the preference places less weight on historical consumption at a given past date. From the differential form, we can see that the consumption habit will increase if the momentary consumption rate exceeds the consumption habit. The higher β is, the fast $z(t)$ is adjusted to current consumption rate. If $\beta = 0$, the preference index is a constant and stays at z_0.

A more general example of z is given by

$$dz(t) = (\delta(t)c(t) - \alpha(t)z(t))dt, \quad z(0) = z_0, \quad t \in [0, T]. \tag{2.11}$$

That is,

$$z(t) = z_0 e^{-\int_0^t \alpha(u)du} + \int_0^t \delta(u) e^{-\int_u^t \alpha(v)dv} c(u)du, \quad t \in [0, T]. \tag{2.12}$$

Here, $\alpha(t)$ is the persistence of the past level, and $\delta(t)$ is the intensity (weight) of consumption history.

The most common utility functions can be specialized as follows: (just examples, not limited to those cases)

1. Exponential utility function: $u(c, z) = -\dfrac{1}{\Phi_1} e^{-\Phi_1 c + \Phi_2 z}$, where $\Phi_1 > 0, \Phi_2 \geqslant 0$. The parameter Φ_2 describes the strength of intertemporal dependence.

2. Power utility function: $u(c, z) = \dfrac{\{c - z\}^A}{A}$, $\quad A < 1$. This utility formation has the property that as c approaches z, the marginal utility goes to infinity. Therefore, the agent would never allow his consumption level to be lower than his consumption habit.

3. Logarithmic utility function: $u(c, z) = \log\{c - z\}$. Same as power utility function, as $c \to z$, the marginal utility goes to infinity. Therefore, the consumption habit determines the lower limit of consumption rate.

2.4 Filtering Techniques

In section 2.2, we establish a model under full information. Actually, it is more realistic to assume that investors have only partial information since stock prices and interest rates are published and available to the public, but drifts and paths of Brownian motions are just mathematical tools for model description and not observable. Moreover, investors have only partial information, so the consumption rate and portfolio of investors are adapted to the filtration generated by the stock prices, which is smaller than the one we talked at section 2.2.

Filtering problems concern estimating something about an unobserved stochastic process Y given observations of a related process Λ. It is an important tool to transfer a partial information problem into a complete information problem.

The setting is a probability space $(\Omega, \mathbb{F}, P)$ equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$. All processes are assumed to be $\mathbb{F}$-adapted. Note that $\mathbb{F}$ is not the observation filtration. Let us call $\mathbb{F}$ the background filtration. We consider two processes, both taken to be one-dimensional:

- a signal process $Y := (Y_t)_{t\in[0,T]}$ which is not directly observable;
- an observation process $\Lambda = (\Lambda_t)_{t\in[0,T]}$ which is observable and somehow correlated with Y, so that by observing Λ we can say something about the distribution of Y.

Let $\mathbb{F}^\Lambda := (\mathcal{F}_t^\Lambda)_{t\in[0,T]}$ denote the observation filtration generated by Λ and

$$\mathcal{F}_t^\Lambda := \sigma(\Lambda_s; 0 \leq s \leq T).$$

The filtering problem is to compute the conditional distribution of the signal $Y_t, t \in [0,T]$, given observations up to that time.

To proceed further, we need to specify some particular model for the observation process.

2.4.1 Observation process

Let $W = (W_t)_{t\in[0,T]}$ be an $\mathbb{F}$-Brownian motion, let $G = (G_t)_{t\in[0,T]}$ be an $\mathbb{F}$-adapted process satisfying

$$E\int_0^T G_t^2 dt < \infty,$$

and we shall assume the observation process Λ is of the linear form

$$d\Lambda_t = G(t)Y_t dt + dW_t, \quad t \in [0,T]. \quad \text{(linear observation model)}$$

2.4.2 Innovation process

We introduce the filter estimate process $\widehat{Y}$, for any $\mathbb{F}$-adapted process Y, as the optional projection of Y onto the $\mathbb{F}^\Lambda$ filtration, i.e.

$$\widehat{Y}_t = E[Y_t \mid \mathcal{F}_t^\Lambda], \quad t \in [0,T]. \tag{2.13}$$

Define the $\mathbb{F}^\Lambda$-adapted innovation process

$$N_t := \Lambda_t - \int_0^t (\widehat{G(t)Y_t})ds, \quad t \in [0,T]. \tag{2.14}$$

Proposition 2.1: *The innovation process N is an $\mathbb{F}^{\Lambda}$-Brownian motion.*

The proof of above proposition can be found in [18].

2.4.3 Kalman-Bucy filter

Theorem 2.6: *(One-dimensional Kalman-Bucy filter) On a filtered probability space $(\Omega, \mathbb{F}, P)$, with background filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$, let $Y = (Y_t)_{t\in[0,T]}$ be an $\mathbb{F}$-adapted signal process satisfying*

$$dY_t = A(t)Y_t dt + C(t)dB_t,$$

and let $\Lambda = (\Lambda_t)_{t\in[0,T]}$ be an $\mathbb{F}$-adapted observation process satisfying

$$d\Lambda_t = G(t)Y_t dt + dW_t.$$

Here W and B are $\mathbb{F}$-Brownian motions with correlation ρ, and the coefficients $A(\cdot), C(\cdot)$ and $G(\cdot)$ are deterministic functions satisfying

$$\int_0^T (|A(t)| + C^2(t) + G^2(t))dt < \infty.$$

Define the observation filtration $\mathbb{F}^{\Lambda} = (\mathcal{F}_t^{\Lambda})_{t\in[0,T]}$ by

$$\mathcal{F}_t^{\Lambda} = \sigma(\Lambda_s : 0 \le s \le t).$$

Suppose Y_0 is an $\mathcal{F}_0$-measurable random variable, and that the distribution of Y_0 is Gaussian with mean μ_0 and variance η_0, independent of W and B.

Then, the conditional expectation $\widehat{Y}_t := E[Y_t \mid \mathcal{F}_t^\Lambda]$*, for* $t \in [0,T]$ *satisfies*

$$d\widehat{Y}_t = A(t)\widehat{Y}_t dt + [G(t)V_t + \rho C(t)]dN_t, \quad \widehat{Y}_t = \eta_0,$$

where $N = (N_t)_{t\in[0,T]}$ *is the innovations process, which is an* $\mathbb{F}^\Lambda$*-Brownian motion satisfying*

$$dN_t = d\Lambda_t - G(t)\widehat{Y}_t dt.$$

Furthermore, $V_t = \text{VAR}[Y_t|\mathcal{F}_t^\Lambda] = E[(Y_t - \widehat{Y}_t)^2 \mid \mathcal{F}_t^\Lambda]$*, for* $t \in [0,T]$*, is the conditional variance, which is independent of* $\mathcal{F}_t^\Lambda$ *and satisfies the deterministic Riccati equation*

$$\frac{dV_t}{dt} = (1-\rho^2)C^2(t) + 2[A(t) - \rho C(t)G(t)]V_t - G^2(t)V_t^2, \quad V_0 = \theta_0.$$

Chapter 3

Exponential Utility with Incomplete Information

In this chapter, we consider a model of optimal investment and consumption with both habit formation and partial information. The investor chooses her consumption rate and portfolio using the information from the stock price only. Herein, we assume that the investor has exponential utility towards consumption and terminal wealth.

At first, we mathematically describe the financial market structure. We consider a continuous-time economy on a finite time horizon $[0, T]$. Given a filtered probability space $(\Omega, \mathbb{F}, P)$ where the filtration $\mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]}$ satisfies the usual conditions. The investors in this market are assumed to be "small investors", and as a consequence their actions have no influence on the market prices. In addition, the transaction is smooth, which indicates all transaction costs are ignored. This market consists of a riskless bond and one stock. The stock price is driven by a one-dimensional Brownian motion W (m = d = 1), which indicates the completeness of the market. Without loss of generality, we assume that the bond process $S_t^0 \equiv 1$ for all $t \in [0, T]$, or equivalently the interest rate $r(t) \equiv 0, t \in [0, T]$. The stock price process S_t follows the following dynamic

$$dS_t = \mu_t S_t d_t + \sigma_S S_t dW_t,\ \ t \in [0, T], \quad S_0 = s_0 > 0. \tag{3.1}$$

Similar to [22], we assume that the drift process μ_t satisfies the Ornstein-Uhlenbeck stochastic differential equation

$$d\mu_t = -\lambda(\mu_t - \bar{\mu})dt + \sigma_\mu dB_t,\ \ t \in [0, T]. \tag{3.2}$$

Here, W and B are two $\mathbb{F}$-adapted Brownian motions with correlation coefficient $\rho \in [-1, 1]$. The initial value of the drift process μ_0 is assume to be an $\mathcal{F}_0$-measurable Gaussian random variable and $\mu_0 \sim N(\eta_0, \theta_0)$, which is independent of Brownian motions W and B. We also assume that other coefficients $\sigma_S, \sigma_\mu, \lambda, \bar{\mu}$ are non-negative constants.

If we denote by x_0 the investor's initial wealth, then at time t, the investor's wealth X_t equals to this initial wealth plus the gain or loss from investment activities and less the accumulated consumption. We denote by c_t the consumption rate at time t and by π_t the amount of wealth invested in the stock. The investor's total wealth at time t is given by

$$X(t) = x_0 + \int_0^t \frac{\pi_s}{S_s} dS_s - \int_0^t c_s ds, \quad t \in [0, T].$$

Or equivalently

$$dX_t = (\pi_t \mu_t - c_t)dt + \sigma_S \pi_t dW_t, \quad t \in [0, T], \quad X_0 = x_0. \tag{3.3}$$

3.1 Model for Habit formation

Now, we study the problem of optimal consumption and investment rules for an investor with Habit formation. We expand the original utility $U(x)$ to a bivariate function $U(c, z)$ with respect to consumption rate c_t and consumption habit level z_t. The process z_t satisfies

$$dz_t = (\delta(t)c_t - \alpha(t)z_t)dt, \quad t \in [0, T], \tag{3.4}$$

or equivalently

$$z_t = z_0 e^{-\int_0^t \alpha(u)du} + \int_0^t \delta(u) e^{-\int_u^t \alpha(v)dv} c_u du, \quad t \in [0, T].$$

According to Section 2.3, $\alpha(t)$ represents the persistence of the past level, while $\delta(t)$ models the intensity (weight) of consumption history.

3.2 Model for Partial observations

We assume that investors can only observe the stock price process S_t, while μ_t, W_t and B_t are unknown for investors. Thus, our goal is to find the optimal investment strategy π_t and consumption policy c_t under partial observation filtration $\mathbb{F}^S = (\mathcal{F}_t^S)_{t\in[0,T]}$. This pair (π_t, c_t) is adapted to $\mathcal{F}_t^S = \sigma(S_u, u \in [0,t])$ only. The key ideal here is to transform the partial information problem to an equivalent problem with complete observations. To this end, we need to project the unknown process μ and Brownian motions W and B onto the observable filtration $\mathbb{F}^S = (\mathcal{F}_t^S)_{t\in[0,T]}$. In other words, we need to estimate μ, B and W using $\mathbb{F}^S$.

We use the filtering techniques in Section 3.2. According to (2.14), we define the innovation process $\widehat{W}$ associate to W as follows

$$d\widehat{W}_t := \frac{1}{\sigma_S}[(\mu_t - \widehat{\mu}_t)dt + \sigma_S dW_t] = \frac{1}{\sigma_S}(\frac{dS_t}{S_t} - \widehat{\mu}_t dt), \quad t \in [0,T]. \tag{3.5}$$

Here $\widehat{\mu}_t := E[\mu_t \mid \mathcal{F}_t^S]$. Thanks to Proposition 2.1, $\widehat{W} := (\widehat{W}_t)_{t\in[0,T]}$ is a Brownian Motion under $\mathbb{F}^S$, and due to Theorem 2.6 (One-dimensional Kalman-Bucy filter), the process $\widehat{\mu}_t$ satisfies

$$d\widehat{\mu}_t = -\lambda(\widehat{\mu}_t - \bar{\mu})dt + (\frac{\widehat{\Omega}_t + \sigma_S\sigma_\mu\rho}{\sigma_S})d\widehat{W}_t, \tag{3.6}$$

$$\widehat{\mu}_0 = E[\mu_0 \mid \mathcal{F}_0^S] = \eta_0.$$

In addition, the conditional variance $\widehat{\Omega}_t = E[(\mu_t - \widehat{\mu}_t)^2 \mid \mathcal{F}_t^S]$ satisfies the following Riccati ordinary differential equation (ODE):

$$d\widehat{\Omega}_t = \Big[-\frac{1}{\sigma_S^2}\widehat{\Omega}_t^2 + (-\frac{2\sigma_\mu\rho}{\sigma_S} - 2\lambda)\widehat{\Omega}_t + (1-\rho^2)\sigma_\mu^2\Big]dt, \tag{3.7}$$

$$\widehat{\Omega}_0 = E[(\mu_0 - \widehat{\mu}_0)^2 \mid \mathcal{F}_0^S] = \theta_0.$$

The solution to the above Riccati ODE is already derived in [22] and is

$$\widehat{\Omega}_t = \widehat{\Omega}(t;\theta_0) = \sqrt{k}\sigma_s \frac{k_1 e^{\frac{2\sqrt{k}t}{\sigma_S}} + k_2}{k_1 e^{\frac{2\sqrt{k}t}{\sigma_S}} - k_2} - (\lambda + \frac{\sigma_\mu \rho}{\sigma_S})\sigma_S^2, \quad t \in [0,T], \tag{3.8}$$

where

$$k = \lambda^2 \sigma_S^2 S + 2\sigma_S \sigma_\mu \lambda \rho + \sigma_\mu^2,$$

$$k_1 = \sqrt{k}\sigma_S + (\lambda \sigma_S^2 + \sigma_S \sigma_\mu \rho) + \theta_0,$$

$$k_2 = -\sqrt{k}\sigma_S + (\lambda \sigma_S^2 + \sigma_S \sigma_\mu \rho) + \theta_0.$$

It is easy to see that $\widehat{\Omega}(t)$ converges monotonically to the value

$$\theta^* = \sigma_S \sqrt{\lambda^2 \sigma_S^2 + 2\sigma_S \sigma_\mu \lambda \rho + \sigma_\mu^2} - (\lambda \sigma_S^2 + \sigma_S \sigma_\mu \rho) > 0, \tag{3.9}$$

as time $t \to \infty$. The convergence property of $\widehat{\Omega}(t)$ tells us the precision of the drift estimate goes from an initial condition to a steady state in the long time run. By the evolution of Riccati ODE, we obtain that the monotone solution $\widehat{\Omega}(t)$ on $(0,\infty)$ has the bounds

$$\min(\theta_0, \theta^*) \leq \widehat{\Omega}(t) \leq \max(\theta_0, \theta^*), \quad \forall t \in [0,T], \tag{3.10}$$

notice that θ_0, θ^* are independent of t.

Under the observation filtration $\mathbb{F}^S$, the stock price process can be derived using the innovation process of (3.5) as follows

$$dS_t = \widehat{\mu}_t S_t dt + \sigma_S S_t d\widehat{W}_t, \quad t \in [0,T]. \tag{3.11}$$

The habit formation process z_t still satisfies (3.4). However, the pair (π, c) is now $\mathbb{F}^S$-progressively measurable.

By the same procedure, as in the full information case, the dynamic of the wealth process under $\mathbb{F}_t^S$ are given by

$$dX_t = (\pi_t \widehat{\mu}_t - c_t)dt + \sigma_S \pi_t d\widehat{W}_t, \quad X_0 = x_0, \quad t \in [0, T]. \tag{3.12}$$

And no bankruptcy is allowed, that means the investor's wealth remains non-negative: $X_t \geq 0, t \in [0, T]$.

3.3 Utility Maximization and HJB Equation

Our goal is to maximize the consumption with habit formation and the terminal wealth, for investors endowed with exponential utility preference, under the partial observation filtration $\mathbb{F}^S$. Mathematically, this objective can be stated as follows

$$v(x_0, z_0, \eta_0, \theta_0) = \sup_{(\pi,c)\in\mathcal{A}(x_0)} E[\int_0^T -e^{-(c_s - z_s)} ds - e^{-X_T}], \tag{3.13}$$

where $\mathcal{A}(x_0)$ is the set of admissible pairs defined in Definition 2.10. This optimization problem is a stochastic control problem which can be reduced to solve a Hamilton-Jacobi-Bellman (HJB hereafter) equation.

3.3.1 The form of HJB Equation

We look for a smooth function defined as

$$\tilde{v}(t,x,z,\eta,\theta) = \sup_{(\pi,c)\in\mathcal{A}(x_0)} E\Big(\int_t^T -e^{-(c_s-z_s)}ds - e^{-X_T}\Big| X_t = x, z_t = z, \widehat{\mu}_t = \eta, \widehat{\Omega}_t = \theta\Big), \tag{3.14}$$

and notice that

$$\tilde{v}(0,x,z,\eta,\theta) = v(x_0,z_0,\eta_0,\theta_0).$$

By Definition 2.4, on an appropriate domain the following process

$$\widetilde{Y}^{(\pi,c)}(t) := \int_0^t -e^{-(c_s-z_s)}ds + \tilde{v}(t,X_t,z_t,\widehat{\mu}_t,\widehat{\Omega}_t), \quad t\in[0,T],$$

is a local super-martingale for each admissible control $(\pi_t,c_t)\in\mathcal{A}(x_0)$. Because for each $(\pi_t,c_t)\in\mathcal{A}(x_0)$ and $\forall t\in[0,T]$ we have

$$E[\widetilde{Y}^{(\pi,c)}(t)\mid\mathcal{F}_0] \le \widetilde{Y}^{(\pi,c)}(0) = v(x_0,z_0,\eta_0,\theta_0),$$

For the optimal control pair $(\pi_t^*,c_t^*)\in\mathcal{A}(x_0)$:

$$\widetilde{Y}^{(\pi^*,c^*)}(t) := \int_0^t -e^{-(c_s^*-z_s^*)}ds + \tilde{v}(t,X_t^*,z_t^*,\widehat{\mu}_t,\widehat{\Omega}_t), \quad t\in[0,T],$$

is a local martingale,because for $(\pi^*,c^*)\in\mathcal{A}(x_0)$ and $\forall t\in[0,T]$ we have

$$E[\widetilde{Y}^{(\pi^*,c^*)}(t)\mid\mathcal{F}_0] = \widetilde{Y}^{(\pi^*,c^*)}(0) = v(x_0,z_0,\eta_0,\theta_0).$$

From Section 3.2 , we know that the conditional variance process $\widehat{\Omega}_t = \widehat{\Omega}(t, \theta_0)$ is a deterministic function of time. Therefore, the variable θ in the definition of $\tilde{v}$ can be set as a deterministic function of t, and $\theta = \theta(t, \theta_0)$ depending on the parameter θ_0. So the dimension of the function $\tilde{v}$ can be reduced. Hence, we can define a new function $V(t, x, z, \eta; \theta_0)$ as

$$V(t, x, z, \eta; \theta_0) := \tilde{v}(t, x, z, \eta, \theta(t, \theta_0)), \tag{3.15}$$

and our goal can be simplified into finding a smooth enough function $V(t, x, z, \eta; \theta_0)$ on some appropriate domain, denoted by $V(t, x, z, \eta)$, such that for each fixed initial value $\widehat{\Omega}(0) = \theta_0$

$$Y^{(\pi,c)}(t) = \int_0^t -e^{-(c_s - z_s)ds} + V(t, X_t, z_t, \widehat{\mu}_t), \quad \forall t \in [0, T], \tag{3.16}$$

is a super-martingale for each $(\pi_t, c_t) \in \mathcal{A}(x_0)$, and is a martingale for the optimal control $(\pi_t^*, c_t^*) \in \mathcal{A}(x_0)$.

Therefore, the control problem (3.13) can be reduced to an HJB equation.

Theorem 3.1: *The optimal value function $V(t, x, z, \eta)$ defined in (3.15), is the solution for the following HJB equation*

$$\begin{aligned} V_t - \alpha(t) z V_z - \lambda(\eta - \bar{\mu}) V_\eta + \frac{(\widehat{\Omega}(t) + \sigma_S \sigma_\mu \rho)^2}{2\sigma_S^2} V_{\eta\eta} + \max_c \Big[-c V_x + c\delta(t) V_z \\ -e^{-(c-z)} \Big] + \max_\pi \Big[\pi \eta V_x + \frac{1}{2} \sigma_S^2 \pi^2 V_{xx} + V_{x\eta} \Big(\widehat{\Omega}(t) + \sigma_S \sigma_\mu \rho \Big) \pi \Big] = 0, \end{aligned} \tag{3.17}$$

with the terminal condition $V(T, x, z, \eta) = -e^{-x}$.

Proof. By applying Itô's formula to $V(t, X_t, z_t, \widehat{\mu}_t)$, we get

$$
\begin{aligned}
dV(t, X_t, z_t, \widehat{\mu}_t) &= V_t dt + V_x dX_t + V_z dz_t + V_\eta d\widehat{\mu}_t \\
&\quad + \frac{1}{2} V_{xx} d < X_t, X_t > + \frac{1}{2} V_{zz} d < z_t, z_t > + \frac{1}{2} V_{\eta\eta} d < \widehat{\mu}_t, \widehat{\mu}_t > \\
&\quad + V_{xz} d < X_t, z_t > + V_{x\eta} d < X_t, \widehat{\mu}_t > + V_{z\eta} d < z_t, \widehat{\mu}_t > \\
&= V_t dt + V_x \Big((\pi_t \widehat{\mu}_t - c_t) dt + \sigma_S \pi_t d\widehat{W}_t \Big) + V_z \Big((\delta(t) c_t - \alpha(t) z_t) dt \Big) \\
&\quad + V_\eta \Big(-\lambda(\widehat{\mu}_t - \bar{\mu}) dt + (\frac{\widehat{\Omega}_t + \sigma_S \sigma_\mu \rho}{\sigma_S}) d\widehat{W}_t \Big) + \frac{1}{2} \sigma_S^2 \pi^2 V_{xx} dt \\
&\quad + \frac{(\widehat{\Omega}(t) + \sigma_S \sigma_\mu \rho)^2}{2\sigma_S^2} V_{\eta\eta} dt + V_{x\eta} \pi_t \Big(\widehat{\Omega}_t + \sigma_S \sigma_\mu \rho \Big) dt \\
&= \Big[V_t + V_x \Big(\pi_t \widehat{\mu}_t - c_t \Big) + V_z \Big(\delta(t) c_t - \alpha(t) z_t \Big) - V_\eta (\lambda(\widehat{\mu}_t - \bar{\mu})) \\
&\quad + \frac{1}{2} \sigma_S^2 \pi_t^2 V_{xx} + \frac{(\widehat{\Omega}(t) + \sigma_S \sigma_\mu \rho)^2}{2\sigma_S^2} V_{\eta\eta} + V_{x\eta} \pi_t \Big(\widehat{\Omega}_t + \sigma_S \sigma_\mu \rho \Big) \Big] dt \\
&\quad + \Big(V_x (\sigma_S \pi_t) + V_\eta (\frac{\widehat{\Omega}_t + \sigma_S \sigma_\mu \rho}{\sigma_S}) \Big) d\widehat{W}_t.
\end{aligned}
$$

As a result, for each $(\pi_t, c_t) \in \mathcal{A}(x_0)$,

$$
Y^{(\pi,c)}(t) = \int_0^t -e^{-(c_s - z_s) ds} + V(t, X_t, z_t, \widehat{\mu}_t), \quad \forall t \in [0, T],
$$

is a local super-martingale if and only if V satisfies

$$
\begin{aligned}
V_t - \alpha(t) z V_z - \lambda(\eta - \bar{\mu}) V_\eta + \frac{(\widehat{\Omega}(t) + \sigma_S \sigma_\mu \rho)^2}{2\sigma_S^2} V_{\eta\eta} + \Big[-c V_x + c\delta(t) V_z \\
-e^{-(c-z)} \Big] + \Big[\pi \eta V_x + \frac{1}{2} \sigma_S^2 \pi^2 V_{xx} + V_{x\eta} \Big(\widehat{\Omega}(t) + \sigma_S \sigma_\mu \rho \Big) \pi \Big] \leq 0,
\end{aligned}
\tag{3.18}
$$

for all $(\pi_t, c_t) \in \mathcal{A}(x_0)$. And for optimal control pair $(\pi_t^*, c_t^*) \in \mathcal{A}(x_0)$,

$$
Y^{(\pi^*,c^*)}(t) = \int_0^t -e^{-(c_s^* - z_s^*) ds} + V(t, X_t^*, z_t^*, \widehat{\mu}_t), \quad \forall t \in [0, T],
$$

is a local martingale if and only if V satisfies

$$\begin{aligned} V_t - \alpha(t) z V_z - \lambda(\eta - \bar{\mu}) V_\eta + \frac{(\widehat{\Omega}(t) + \sigma_S \sigma_\mu \rho)^2}{2\sigma_S^2} V_{\eta\eta} + \Big[-c^* V_x + c^* \delta(t) V_z \\ - e^{-(c^* - z)} \Big] + \Big[\pi^* \eta V_x + \frac{1}{2} \sigma_S^2 (\pi^*)^2 V_{xx} + V_{x\eta} \Big(\widehat{\Omega}(t) + \sigma_S \sigma_\mu \rho \Big) \pi^* \Big] = 0. \end{aligned} \tag{3.19}$$

Then, by combining (3.18) and (3.19), we conclude that (3.17) holds.

From (3.14) and (3.15), we get

$$\begin{aligned} V(T, x, z, \eta; \theta) = \tilde{v}(T, x, z, \eta, \theta(T, \theta_0)) &= \sup_{\pi, c \in \mathcal{A}} E\Big[-e^{-X_T} \mid X_T = x, z_T = z, \widehat{\mu}_T = \eta, \widehat{\Omega}_T = \theta \Big] \\ &= -e^{-x}. \end{aligned}$$

This ends the proof of this theorem. □

3.3.2 Explicit Solution of the HJB Equation

We use the first order condition to find feedback form for the optimal control pair $(\pi^*, c^*) \in \mathcal{A}(x_0)$. If $V(t, x, z, \eta)$ is smooth enough, we get

$$-V_x + \delta(t) V_z + e^{-(c - z)} = 0,$$

$$\eta V_x + \pi \sigma_S^2 V_{xx} + V_{x\eta}(\widehat{\Omega}(t) + \sigma_S \sigma_\mu \rho) = 0.$$

Therefore, the optimal control pair is given by

$$\begin{aligned} c^*(t, x, z, \eta) &= z - \ln(V_x - \delta(t) V_z), \\ \pi^*(t, x, z, \eta) &= \frac{-\eta V_x - (\widehat{\Omega}(t) + \sigma_S \sigma_\mu \rho) V_{x\eta}}{\sigma_S^2 V_{xx}}. \end{aligned} \tag{3.20}$$

By inserting (3.20) into (3.17), we get the following PDE

$$
\begin{aligned}
0 \equiv\ & V_t - \alpha(t) z V_z - \lambda(\eta - \bar{\mu}) V_\eta + \Big(\delta V_z - V_x\Big)\Big(z + 1 - \ln(V_x - \delta V_z)\Big) \\
& + \frac{(\widehat{\Omega}(t) + \sigma_S \sigma_\mu \rho)^2}{2\sigma_S^2} V_{\eta\eta} - \frac{1}{2} \frac{(\eta V_x + (\widehat{\Omega}(t) + \sigma_S \sigma_\mu \rho) V_{x\eta})^2}{\sigma_S^2 V_{xx}}.
\end{aligned}
\tag{3.21}
$$

In order to solve this PDE explicitly, we propose the following candidate function of $V(t, x, z, \eta)$

$$
V(t, x, z, \eta) = -M(t, \eta) \exp\Big(-\phi(t, \eta) x + m(t, \eta) z\Big). \tag{3.22}
$$

Here, $\phi(t, \eta), m(t, \eta)$ and $M(t, \eta)$ are functions to be determined.

Thanks to the terminal condition $V(T, x, z, \eta) = -e^{-x}$, we get the following

$$
\phi(T, \eta) = 1, \quad m(T, \eta) = 0 \text{ and } M(T, \eta) = 1.
$$

We calculate the derivatives of (3.22)

$$
\begin{aligned}
V_t &= \frac{M_t}{M} V + (m_t z - \phi_t x) V, \\
V_z &= m V, \\
V_\eta &= \frac{M_\eta}{M} V + (m_\eta z - \phi_\eta x) V, \\
V_{\eta\eta} &= \frac{M_{\eta\eta}}{M} V + 2(m_\eta z - \phi_\eta x) \frac{M_\eta}{M} V + (m_{\eta\eta} z - \phi_{\eta\eta} x) V + (m_\eta z - \phi_\eta x)^2 V, \\
V_x &= -\phi V, \\
V_{xx} &= \phi^2 V, \\
V_{x\eta} &= -\phi_\eta V - \phi \frac{M_\eta}{M} V - \phi(m_\eta z - \phi_\eta x) V,
\end{aligned}
$$

and substitute them in (3.21). Then, after dividing the resulting equation by V, we get

$$\frac{M_t}{M} + (m_t z - \phi_t x) - \alpha z m + \beta_1 \frac{M_\eta}{M} + \beta_1(m_\eta z - \phi_\eta x)$$
$$+\frac{\beta_2^2}{2\sigma_S^2}\frac{M_{\eta\eta}}{M} + \frac{\beta_2^2}{\sigma_S^2}(m_\eta z - \phi_\eta x)\frac{M_\eta}{M} + \frac{\beta_2^2}{2\sigma_S^2}(m_{\eta\eta} z - \phi_{\eta\eta} x) + \frac{\beta_2^2}{2\sigma_S^2}(m_\eta z - \phi_\eta x)^2$$
$$+z(1-m)(\phi+\delta m) + \phi x(\phi+\delta m) - (\ln(\phi+\delta m) + \ln(M) - 1)(\phi+\delta m)$$
$$-\frac{1}{2\sigma_S^2}(\eta + \beta_2\frac{\phi_\eta}{\phi} + \beta_2\frac{M_\eta}{M} - \beta_2\phi_\eta x + \beta_2 m_\eta z)^2 = 0,$$

where

$$\begin{cases} \beta_1(\eta) = -\lambda(\eta - \bar{\mu}), \\ \beta_2(t) = \widehat{\Omega}_t + \sigma_S\sigma_\mu\rho. \end{cases}$$

After arranging terms in the previous PDE, we get

$$z\Big[m_t - \alpha m + \beta_1 m_\eta + \frac{\beta_2^2}{\sigma_s^2}m_\eta\frac{M_\eta}{M} + \frac{\beta_2^2}{2\sigma_S^2}m_{\eta\eta} - \frac{\beta_2 m_\eta}{\sigma_S^2}(\eta + \beta_2\frac{\phi_\eta}{\phi} + \beta_2\frac{M_\eta}{M})+$$
$$(1-m)(\phi+\delta m)\Big] + x\Big[-\phi_t - \beta_1\phi_\eta - \frac{\beta_2^2}{\sigma_s^2}\phi_\eta\frac{M_\eta}{M} - \frac{\beta_2^2}{2\sigma_S^2}\phi_{\eta\eta} + \frac{\beta_2\phi_\eta}{\sigma_S^2}(\eta+$$
$$\beta_2\frac{\phi_\eta}{\phi} + \beta_2\frac{M_\eta}{M}) + \phi(\phi+\delta m)\Big] + \frac{M_t}{M} + \beta_1\frac{M_\eta}{M} + \frac{\beta_2^2}{2\sigma_S^2}\frac{M_{\eta\eta}}{M} -$$
$$(\ln(\phi+\delta m) + \ln(M) - 1)(\phi+\delta m) - \frac{\eta^2}{2\sigma_s^2} - \frac{\beta_2^2}{2\sigma_S^2}\frac{\phi_\eta^2}{\phi^2} - \frac{\beta_2^2}{2\sigma_S^2}\frac{M_\eta^2}{M^2} - \frac{\beta_2^2}{\sigma_S^2}\frac{\phi_\eta M_\eta}{\phi M}$$
$$-\frac{\beta_2\eta}{\sigma_S^2}\frac{\phi_\eta}{\phi} - \frac{\beta_2\eta}{\sigma_S^2}\frac{M_\eta}{M} = 0.$$

Since this equation holds for all $x \geq 0$ and $z \geq 0$, then the following hold

$$m_t - \alpha m + \beta_1 m_\eta + \frac{\beta_2^2}{\sigma_S^2}m_\eta\frac{M_\eta}{M} + \frac{\beta_2^2}{2\sigma_S^2}m_{\eta\eta} - \frac{\beta_2 m_\eta}{\sigma_S^2}(\eta + \beta_2\frac{\phi_\eta}{\phi} + \beta_2\frac{M_\eta}{M})$$
$$+(1-m)(\phi+\delta m) = 0, \tag{3.23}$$

$$-\phi_t - \beta_1\phi_\eta - \frac{\beta_2^2}{\sigma_S^2}\phi_\eta\frac{M_\eta}{M} - \frac{\beta_2^2}{2\sigma_S^2}\phi_{\eta\eta} + \frac{\beta_2\phi_\eta}{\sigma_S^2}(\eta + \beta_2\frac{\phi_\eta}{\phi} + \beta_2\frac{M_\eta}{M}) \\ +\phi(\phi+\delta m) = 0, \tag{3.24}$$

and

$$\frac{M_t}{M} + \beta_1\frac{M_\eta}{M} + \frac{\beta_2^2}{2\sigma_S^2}\frac{M_{\eta\eta}}{M} - (\ln(\phi+\delta m) + \ln(M) - 1)(\phi+\delta m) \\ -\frac{\eta^2}{2\sigma_S^2} - \frac{\beta_2^2}{2\sigma_S^2}\frac{\phi_\eta^2}{\phi^2} - \frac{\beta_2^2}{2\sigma_S^2}\frac{M_\eta^2}{M^2} - \frac{\beta_2^2}{\sigma_s^2}\frac{\phi_\eta M_\eta}{\phi M} - \frac{\beta_2\eta}{\sigma_S^2}\frac{\phi_\eta}{\phi} - \frac{\beta_2\eta}{\sigma_S^2}\frac{M_\eta}{M} = 0. \tag{3.25}$$

In order to simplify the problem, we propose to set the unknown functions $\phi(t,\eta) = \phi(t)$ and $m(t,\eta) = m(t)$ as functions of time t only with the terminal condition $\phi(T) = 1$ and $m(T) = 0$. Throughout the rest of this section, we have the following assumption.

Assumption 3: *We assume that $\alpha(t) = \alpha$ and $\delta(t) = \delta$ are constant numbers.*

Then, (3.23) and (3.24) take the following forms

$$m_t - \alpha m + (1-m)(\phi+\delta m) = 0, \quad \phi_t - \phi(\phi+\delta m) = 0, \quad m(T) = 0, \text{ and } \phi(T) = 1. \tag{3.26}$$

Lemma 3.1: *The pair (ϕ, m) solution to (3.26) is given by*

$$\phi(t) = \begin{cases} \dfrac{1}{1+(T-t)+\frac{1}{2}\alpha(T-t)^2}, & \textit{if} \quad \alpha = \delta, \\[2ex] \dfrac{(\delta-\alpha)^2}{(\delta-\alpha)^2 - \delta - \alpha(\delta-\alpha)(T-t) + \delta e^{(\delta-\alpha)(T-t)}}, & \textit{if} \quad \alpha \neq \delta, \end{cases} \tag{3.27}$$

and

$$m(t) = \begin{cases} \dfrac{T-t}{1+(T-t)+\frac{1}{2}\alpha(T-t)^2}, & \textit{if} \quad \alpha = \delta, \\[2ex] \dfrac{(\delta-\alpha)(e^{\alpha(T-t)} - e^{-\alpha(T-t)} + e^{(\delta-\alpha)(T-t)} - 1)}{(\delta-\alpha)^2 - \delta - \alpha(\delta-\alpha)(T-t) + \delta e^{(\delta-\alpha)(T-t)}}, & \textit{if} \quad \alpha \neq \delta. \end{cases} \tag{3.28}$$

Proof. We rewrite equation (3.26) as

$$\frac{m_t - \alpha m}{m-1} = \frac{\phi_t}{\phi} = \phi + \delta m = \Gamma(t), \tag{3.29}$$

and derive

$$\begin{aligned} \phi(t) &= e^{-\int_t^T \Gamma(u)du} \\ m(t) &= \int_t^T \Gamma(s) e^{\int_s^t \Gamma(u)du + \alpha(t-s)} ds. \end{aligned} \tag{3.30}$$

Now, we insert these in

$$\phi + \delta m = \Gamma(t)$$

and multiply both side by $e^{-\int_0^t \Gamma(u)du - \alpha t}$ afterwards, and get

$$e^{-\int_0^T \Gamma(u)du - \alpha t} + \delta \int_0^T \Gamma(s) e^{-\int_0^s \Gamma(u)du - \alpha s} ds - \delta \int_0^t \Gamma(s) e^{-\int_0^s \Gamma(u)du - \alpha s} ds = \Gamma(t) e^{-\int_0^t \Gamma(u)du - \alpha t}. \tag{3.31}$$

Put

$$g(t) := \Gamma(t) e^{-\int_0^t \Gamma(u)du - \alpha t}. \tag{3.32}$$

Then (3.31) becomes

$$g(t) = A_1 e^{-\alpha t} + \delta A_2 - \delta \int_0^t g(s) ds,$$

where $A_1 = e^{-\int_0^T \Gamma(u)du}$ and $A_2 = \displaystyle\int_0^T \Gamma(s)e^{-\int_0^s \Gamma(u)du - \alpha s}ds$ are constant numbers.

By differentiating the above equality, we get

$$g'(t) = -\delta g(t) - \alpha A_1 e^{-\alpha t}. \tag{3.33}$$

1) For $\alpha \neq \delta$, the solution to above ODE is given by

$$g(t) = A_1 e^{(\delta-\alpha)T} e^{-\delta t} + \alpha A_1 (e^{-\delta t}) \int_t^T e^{(\delta-\alpha)s} ds. \tag{3.34}$$

(3.34) and (3.32) leads to

$$\Gamma(t) e^{\int_t^T \Gamma(u)du} = e^{(\delta-\alpha)(T-t)} + \frac{\alpha}{\delta-\alpha}(e^{(\delta-\alpha)(T-t)} - 1). \tag{3.35}$$

By integrating the above equation on $[t, T]$, we get

$$e^{\int_t^T \Gamma(u)du} = 1 - \frac{\delta}{(\delta-\alpha)^2} - \frac{\alpha}{\delta-\alpha}(T-t) + \frac{\delta}{(\delta-\alpha)^2} e^{(\delta-\alpha)(T-t)}. \tag{3.36}$$

Then, by combining (3.35) and (3.36), $\Gamma(t)$ is given by

$$\Gamma(t) = \frac{\delta(\delta-\alpha)e^{(\delta-\alpha)(T-t)} - \alpha(\delta-\alpha)}{(\delta-\alpha)^2 - \delta - \alpha(\delta-\alpha)(T-t) + \delta e^{(\delta-\alpha)(T-t)}}.$$

.

(2) For $\alpha = \delta$, the solution to (3.33) is given by

$$g(t) = A_1 e^{-\alpha t}(1 + \alpha(T - t)). \tag{3.37}$$

(3.37) and (3.32) leads to

$$\Gamma(t) e^{\int_t^T \Gamma(u)du} = 1 + \alpha(T - t). \tag{3.38}$$

By integrating the above equation on $[t, T]$, we get

$$e^{\int_t^T \Gamma(u)du} = 1 + (T - t) + \frac{1}{2}\alpha(T - t)^2. \tag{3.39}$$

Then, by combining (3.38) and (3.39), $\Gamma(t)$ is given by

$$\Gamma(t) = \frac{\alpha(T - t) + 1}{1 + (T - t) + \frac{1}{2}\alpha(T - t)^2}.$$

.

Combining the above two scenarios, $\Gamma(t)$ is given by

$$\Gamma(t) = \begin{cases} \dfrac{\alpha(T - t) + 1}{1 + (T - t) + \frac{1}{2}\alpha(T - t)^2}, & \alpha = \delta \\ \\ \dfrac{\delta(\delta - \alpha)e^{(\delta-\alpha)(T-t)} - \alpha(\delta - \alpha)}{(\delta - \alpha)^2 - \delta - \alpha(\delta - \alpha)(T - t) + \delta e^{(\delta-\alpha)(T-t)}}, & \alpha \neq \delta. \end{cases} \tag{3.40}$$

Denote $f(t)$ as the denominator of $\Gamma(t)$

$$f(t) = \begin{cases} 1 + (T-t) + \frac{1}{2}\alpha(T-t)^2, & \alpha = \delta \\ \\ (\delta-\alpha)^2 - \delta - \alpha(\delta-\alpha)(T-t) + \delta e^{(\delta-\alpha)(T-t)}, & \alpha \neq \delta, . \end{cases} \tag{3.41}$$

Then, from (3.40), we deduce that

$$\Gamma(t) = -\frac{f'(t)}{f(t)}.$$

By inserting the above equation into (3.30) and solving afterwards, we get

$$\begin{cases} \phi(t) = \dfrac{f(T)}{f(t)}, \\ m(t) = -\dfrac{e^{\alpha t}}{f(t)} \displaystyle\int_t^T f'(s) e^{-\alpha s} ds. \end{cases} \tag{3.42}$$

Thus, by integration by parts, the proof of the lemma follows immediately.

□

Now, we focus on solving the remaining PDE (3.25). By substituting $m_\eta = 0$ and $\phi_\eta = 0$ into (3.25), we get

$$\frac{M_t}{M} - (\lambda(\eta - \bar{\mu}) + \frac{\beta_2(t)\eta}{\sigma_s^2})\frac{M_\eta}{M} + \frac{\beta_2^2}{2\sigma_s^2}\frac{M_{\eta\eta}}{M} - (\ln(\Gamma(t)) - 1)\Gamma(t) - \ln(M)\Gamma(t) - \frac{\eta^2}{2\sigma_s^2} = 0 \tag{3.43}$$

The solution of this PDE is given by the following.

Lemma 3.2: *The function*

$$M(t,\eta) = \exp\Big(A(t)\eta^2 + B(t)\eta + C(t)\Big), \tag{3.44}$$

is the solution of equation (3.43).

Here, $A(t)$ satisfies the following ODE equation,

$$A_t - (\frac{2\beta_2}{\sigma_S^2} + 2\lambda + \Gamma)A + \frac{2\beta_2^2}{\sigma_S^2}A^2 - \frac{1}{2\sigma_S^2} = 0, \tag{3.45}$$

and B and C are given by

$$\begin{aligned} B(t) &= 2\bar{\mu}\lambda \int_t^T A(s) \exp\Big(\int_s^t [\lambda + \frac{\beta_2(u)}{\sigma_S^2} + \Gamma(u) - \frac{2\beta(u)_2^2}{\sigma_S^2}A(u)]du\Big)ds, \\ C(t) &= \int_t^T \Gamma(s) \exp\Big(\int_s^t [\lambda\bar{\mu}B(u) + \frac{\beta(u)_2^2}{2\sigma_S^2}B(u)^2 + \frac{\beta(u)_2^2}{\sigma_S^2}A(u) \\ &\quad -\Gamma(u)\ln(\Gamma(u)) + \Gamma(u)]du\Big)ds. \end{aligned} \tag{3.46}$$

Proof. We assume $M(t, \eta)$ takes the following form

$$M(t, \eta) = \exp\Big(A(t)\eta^2 + B(t)\eta + C(t)\Big).$$

By calculating the derivatives of M and inserting them in (3.43), we get

$$\begin{aligned} &\eta^2\Big[A_t - (\frac{2\beta_2}{\sigma_S^2} + 2\lambda + \Gamma)A + \frac{2\beta_2^2}{\sigma_S^2}A^2 - \frac{1}{2\sigma_S^2}\Big] + \eta\Big[B_t - (\lambda + \frac{\beta_2}{\sigma_S^2} + \Gamma)B \\ &+\frac{2\beta_2^2}{\sigma_S^2}AB + 2\bar{\mu}\lambda A\Big] + C_t - \Gamma C + \lambda\bar{\mu}B + \frac{\beta_2^2}{2\sigma_S^2}B^2 + \frac{\beta_2^2}{\sigma_S^2}A - (\ln(\Gamma) - 1)\Gamma = 0. \end{aligned} \tag{3.47}$$

Since η is arbitrary, (3.43) holds if and only if the following equations hold

$$\begin{aligned} &A_t - (\frac{2\beta_2}{\sigma_S^2} + 2\lambda + \Gamma)A + \frac{2\beta_2^2}{\sigma_S^2}A^2 - \frac{1}{2\sigma_S^2} = 0, \\ &B_t - (\lambda + \frac{\beta_2}{\sigma_S^2} + \Gamma)B + \frac{2\beta_2^2}{\sigma_S^2}AB + 2\bar{\mu}\lambda A = 0, \\ &C_t - \Gamma C + \lambda\bar{\mu}B + \frac{\beta_2^2}{2\sigma_S^2}B^2 + \frac{\beta_2^2}{\sigma_S^2}A - (\ln(\Gamma) - 1)\Gamma = 0. \end{aligned} \tag{3.48}$$

As a result, we finish the proof of this lemma. □

Theorem 3.2: *The function*

$$V(t,x,z,\eta) = -\exp\Big\{-\phi(t)x + m(t)z + A(t)\eta^2 + B(t)\eta + C(t)\Big\} \quad (3.49)$$

is the solution of the HJB equation (3.17), where $\phi(t)$ and $m(t)$ are given by (3.27) and (3.28) respectively, while $A(t), B(t)$ and $C(t)$ are given by (3.45) and (3.46).

3.3.3 The Optimal Control Policies

The solution (3.49) to the HJB equation (3.17) coincides with the optimal value function defined in (3.13):

$$V(0,x_0,z_0,\eta_0;\theta_0) = v(x_0,z_0,\eta_0,\theta_0). \quad (3.50)$$

Theorem 3.3: *The optimal investment policy π_t^*, optimal consumption rate c_t^* and the optimal habit level are given by*

$$\pi_t^* = \frac{\widehat{\mu}_t + (\widehat{\Omega}_t + \sigma_S\sigma_\mu\rho)(2\widehat{\mu}_t A(t) + B(t))}{\sigma_S^2\phi(t)}, \quad (3.51)$$

$$c_t^* = \Big(1 - m(t)\Big)z_t^* + \phi(t)X_t^* + Y(t,\widehat{\mu}_t), \quad (3.52)$$

and

$$z_t^* = z_0 e^{\int_0^t \Big(\delta(1-m(s))-\alpha\Big)ds} + \int_0^t \delta\Big(\phi_s X_s^* + Y(s,\widehat{\mu}_s)\Big)e^{\int_s^t \Big(\delta(1-m(u))-\alpha\Big)du} ds. \quad (3.53)$$

Here, X^* is the optimal wealth process and is the solution of following SDE

$$dX_t^* = dK_t - \phi(t)X_t^* dt + \Big((m(t)-1)\int_0^t \delta\phi_s X_s^* ds\Big)dt, \quad X_0^* = x_0, \quad (3.54)$$

where

$$\begin{aligned} dK_t = &\frac{\pi_t^*}{S_t}dS_t - Y(t,\widehat{\mu}_t)dt + \Big((m(t)-1)z_0 e^{\int_0^t \Big(\delta(1-m(s))-\alpha\Big)ds}\Big)dt \\ &+\Big((m(t)-1)\int_0^t \delta Y(s,\widehat{\mu}_s)e^{\int_s^t \Big(\delta(1-m(u))-\alpha\Big)du}ds\Big)dt, \end{aligned}$$

and for any $\eta \in \mathbb{R}$

$$Y(t,\eta) = -\Big(A(t)\eta^2 + B(t)\eta(t) + C(t)\Big) - \ln(\phi(t)+\delta m).$$

Proof. Apply the first order condition to the HJB equation, we can easily get the feedback forms of c^* and π^*. Now we need to find out the optimal wealth process and the optimal habit formation process. First, we define

$$Y(t,\eta) = -\Big(A(t)\eta^2 + B(t)\eta + C(t)\Big) - \ln(\phi(t)+\delta m).$$

Since

$$c^*(t,x,z,\eta) = (1-m)z + \phi(t)x + Y(t,\eta).$$

Then, we have

$$c_t^* = c^*(t, X_t^*, z_t^*, \widehat{\mu}_t) = \Big(1-m(t)\Big)z_t^* + \phi(t)X_t^* + Y(t,\widehat{\mu}_t). \quad (3.55)$$

The optimal habit formation process satisfies

$$dz_t^* = (\delta c_t^* - \alpha z_t^*)dt.$$

By substituting the (3.55) into the above equation , we get

$$\begin{aligned} dz_t^* &= \Big(\delta\Big((1-m(t))z_t^* + \phi(t)X_t^* + Y(t,\widehat{\mu}_t)\Big) - \alpha z_t^*\Big)dt \\ &= \Big(\delta(1-m(t)) - \alpha\Big)z_t^* dt + \delta\Big(\phi(t)X_t^* + Y(t,\widehat{\mu}_t)\Big)dt, \end{aligned} \tag{3.56}$$

which is equivalent to (3.53). Moreover, we also know that the optimal wealth process satisfies

$$dX_t^* = \frac{\pi_t^*}{S_t} dS_t - c_t^* dt,$$

By inserting (3.55) into above equation, we get the SDE for the optimal wealth process

$$\begin{aligned} dX_t^* &= \frac{\pi_t^*}{S_t} dS_t - \Big((1-m(t))z_t^* + \phi(t)X_t^* + Y(t,\widehat{\mu}_t)\Big)dt \\ &= \frac{\pi_t^*}{S_t} dS_t - \Big(\phi(t)X_t^* + Y(t,\widehat{\mu}_t)\Big)dt + (m(t)-1)\times \\ &\Big[\ z_0 e^{\int_0^t \left(\delta(1-m(s))-\alpha\right)ds} + \int_0^t \delta\Big(\phi_s X_s^* + Y(s,\widehat{\mu}_s)\Big)e^{\int_s^t \left(\delta(1-m(u))-\alpha\right)du} ds\Big]dt. \end{aligned}$$

This proves (3.54), and the proof of this theorem is completed. □

Based on the explicit structures, we can easily provide some qualitative analysis on the optimal policies and optimal value function.

Corollary 3.3.1: The following properties hold:

1) The optimal value function $V(t, x, z, \eta)$ is strictly increasing and concave

in x, while is decreasing and concave in z, just like the utility function for consumption and terminal wealth $U(c, z)$.

2) The feedback form $\pi^*(t, x, z, \eta)$ dose not depend on x nor on z. As a result, the optimal portfolio dose not change when wealth process/standard living process is increasing or decreasing.

3) The feedback form $c^*(t, x, z, \eta)$ is increasing in x and increasing in z. As a result, the optimal consumption rate becomes higher when investor's wealth/ standard living is increasing.

Proof. 1) The function $f(t)$ defined in (3.41) is a decreasing function on $[0, T]$ and notice that

$$f(T) = \begin{cases} 1, & \alpha = \delta \\ (\delta - \alpha)^2, & \alpha \neq \delta \end{cases},$$

is always positive. Thus, $f(t)$ is positive for all $t \in [0, T]$. Moreover, from (3.27), (3.28) and (3.44), we conclude that $\phi(t)$ and $M(t, \eta)$ are positive, and $m(t)$ is non-negative on $[0, T]$.

By calculating the derivatives of V, we get

$$V_x = \frac{\partial V}{\partial x} = \phi M e^{-\phi x + mz} > 0,$$

$$V_{xx} = \frac{\partial V^2}{\partial^2 x} = -\phi^2 M e^{-\phi x + mz} < 0,$$

$$V_z = \frac{\partial V}{\partial z} = -mMe^{-\phi x + mz} \leq 0,$$

and

$$V_{zz} = \frac{\partial V^2}{\partial^2 z} = -m^2 M e^{-\phi x + mz} \leq 0.$$

Therefore, $V(t, x, z, \eta)$ is strictly increasing and concave in x, while is decreasing and concave in z.

2) From (3.51), we conclude that $\pi^*(t, x, z, \eta) = \pi^*(t, \eta)$ does not depend on x nor on z.

3) On the one hand, according to (3.52), we know that:

$$c^* \text{ is increasing in } z, \text{ if } 1 - m > 0;$$

$$c^* \text{ does not depend on } z, \text{ if } m = 1;$$

$$c^* \text{ is decreasing in } z, \text{ if } 1 - m < 0.$$

On the other hand, from (3.42), we have

$$m(t) - 1 = \frac{-e^{\alpha t} \int_t^T f'(s) e^{-\alpha s} ds - f(t)}{f(t)}.$$

Put

$$y(t) := -e^{\alpha t} \int_t^T f'(s) e^{-\alpha s} ds - f(t),$$

which is the numerator of $m(t) - 1$. Then, the derivative of y is given by

$$y'(t) = -\alpha e^{\alpha t} \int_t^T f'(s) e^{-\alpha s} ds.$$

Since $f'(t) < 0$, we know that $y'(t) > 0$. The maximum value of y on $[0, T]$ is $y(T) = -f(T) < 0$. Thus, y is negative for all $t \in [0, T]$.

Moreover, $f(t)$ is positive for all $t \in [0, T]$. Then, $1 - m > 0$ for all $t \in [0, T]$. That means c^* is always increasing in z.

$\square$

Chapter 4

Stochastic Volatility Model

In this chapter, we address the problem of optimal investment and consumption under the volatility model. To this end, we start by introducing the stochastic volatility model.

4.1 The Volatility Model

In this chapter, we introduce the stochastic volatility model with constant drift μ in time horizon $[0, T]$. We still consider $(\Omega, \mathbb{F}, P)$ as a complete probability space equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ satisfying the usual conditions. And the filtration $\mathbb{F}^S = (\mathcal{F}_t^S)_{t\in[0,T]}$ is the aggregate information about the stock price. Since the volatility of the stock price can be estimated from the quadratic variation of $\ln(S_t)$, we actually know everything just from the information of stock price. Thus, in this model, filtration $\mathbb{F} = \mathbb{F}^S$.

The price of the stock satisfies the following SDE

$$\frac{dS_t}{S_t} = \mu dt + \sigma_t dW_t, \quad S(0) = s_0 \geq 0, \quad t \in [0, T],$$

and the volatility is given by

$$d\sigma_t = -\lambda(\sigma_t - \bar{\sigma})dt + \theta \sigma_t dB_t, \quad \sigma(0) = \sigma_0, \quad t \in [0, T].$$

Here, W and B are two $\mathbb{F}$-adapted Brownian motion with correlation coefficient $\rho \in [-1, 1]$. And σ_0 is a positive constant. Other parameters μ, λ, θ and $\bar{\sigma}$ are non-negative constants.

Assume c_t is the consumption rate of investor at time t and π_t is the portfolio or trading strategy at time t. Then, the wealth process is given by

$$dX_t = (\pi_t \mu - c_t)dt + \pi_t \sigma_t dW_t, \quad X(0) = x_0 \geq 0, \quad t \in [0, T].$$

And the habit level still satisfies the following dynamic

$$dz_t = (\delta(t)c_t - \alpha(t)z_t)dt, \quad t \in [0, T].$$

Throughout the rest of this chapter, we consider investors with either power preference or exponential preference.

4.2 The Case of Power Preference

In this section, we assume that investors are endowed with power utility preference. Mathematically, our objective can be formulated as follows

$$v(x_0, z_0, \sigma_0) = \sup_{(\pi,c)\in\mathcal{A}(x_0)} E\Big(\int_0^T \frac{(c_s - z_s)^p}{p} ds + \frac{(X_T)^p}{p}\Big), \tag{4.1}$$

here, $p \neq 0$ and $p < 1$ represents the risk aversion coefficient. To solve this optimal problem, we define a smooth function

$$V(t, x, z, \sigma) = \sup_{(\pi,c)\in\mathcal{A}(x_0)} E\Big(\int_t^T \frac{(c_s - z_s)^p}{p} ds + \frac{(X_T)^p}{p}\Big| X_t = x, z_t = z, \sigma_t = \sigma\Big), \tag{4.2}$$

with the terminal condition $V(T, x, z, \sigma) = \dfrac{x^p}{p}$.

Theorem 4.1: *The optimal value function $V(t, x, z, \sigma)$ defined in (4.2) is the solution for the following HJB equation*

$$\begin{aligned} V_t - \alpha(t)zV_z - \lambda(\sigma - \bar{\sigma})V_\sigma + \frac{\theta^2\sigma^2}{2}V_{\sigma\sigma} + \max_c[-cV_x + c\delta(t)V_z + \frac{(c-z)^p}{p}] \\ + \max_\pi[\pi\mu V_x + \frac{1}{2}\pi^2\sigma^2 V_{xx} + \pi\rho\theta\sigma^2 V_{x\sigma}] = 0. \end{aligned} \tag{4.3}$$

Moreover, the optimal control pair has the following feedback forms

$$\begin{aligned} c^*(t, x, z, \eta) &= z + (V_x - \delta(t)V_z)^{\frac{1}{p-1}}, \\ \pi^*(t, x, z, \eta) &= \frac{-\mu V_x - \sigma^2\theta\rho V_{x\sigma}}{\sigma^2 V_{xx}}. \end{aligned} \tag{4.4}$$

Proof. Following the same procedure as in Theorem 3.1, we can easily get the above HJB equation by using the properties of martingale and Itô's formula. Applying first order condition to this HJB equation, we derive the feedback forms of the optimal consumption and optimal portfolio. □

By inserting the optimal control pair (4.4) in (4.3), the HJB equation becomes the following PDE

$$V_t - \alpha(t)zV_z - \lambda(\sigma-\bar{\sigma})V_\sigma + \frac{\theta^2\sigma^2}{2}V_{\sigma\sigma} + (V_x - \delta V_z)(\frac{1-p}{p}(V_x-\delta V_z)^{\frac{1}{p-1}} - z) - \frac{(\mu V_x + \sigma^2\theta\rho V_{x\sigma})^2}{2\sigma^2 V_{xx}} = 0. \tag{4.5}$$

Assumption 4: *For the rest of this section, we assume the above PDE, (4.5), has a unique classical (twice continuously differentiable) solution.*

Lemma 4.1: $V(t,x,z,\sigma)$ *defined in (4.2) is homogeneous in* (x,z).

Proof. Due to the homogeneity property of the power utility function and the linearity of dynamics for X_t and z_t, for any $x \geq 0, z \geq 0$ and the positive constant k, we have

$$V(t,kx,kz,\sigma) = \sup_{\pi,c}[\int_t^T \frac{(kc_s - kz_s)^p}{p}ds + \frac{(kX_T)^p}{p}] = k^p V(t,x,z,\sigma).$$

□

Since $V(t,x,z,\sigma)$ is homogeneous in (x,z) with degree p, it makes sense for us to guess that V takes the following form

$$V(t,x,z,\sigma) = \frac{[x-m(t,\sigma)z]^p}{p}M(t,\sigma), \tag{4.6}$$

here, $m(t,\sigma)$ and $M(t,\sigma)$ are functions to be determined. Thanks to the terminal condition $V(T,x,z,\sigma) = \dfrac{x^p}{p}$, we have

$$m(T,\sigma) = 0 \quad \text{and} \quad M(T,\sigma) = 1.$$

We calculate the derivatives of (4.6)

$$\begin{aligned}
V_t &= -m_t z[x-mz]^{p-1}M + M_t\frac{V}{M},\\
V_z &= -m[x-mz]^{p-1}M,\\
V_\sigma &= -m_\sigma z[x-mz]^{p-1}M + M_\sigma\frac{V}{M},\\
V_{\sigma\sigma} &= -m_{\sigma\sigma} z[x-mz]^{p-1}M + m_\sigma^2 z^2[x-mz]^{p-2}(p-1)M - 2m_\sigma z[x-mz]^{p-1}M_\sigma + M_{\sigma\sigma}\frac{V}{M},\\
V_x &= [x-mz]^{p-1}M,\\
V_{xx} &= (p-1)[x-mz]^{p-2}M,\\
V_{x\sigma} &= -m_\sigma z[x-mz]^{p-2}(p-1)M + [x-mz]^{p-1}M_\sigma,
\end{aligned}$$

and substitute them in (4.5). After dividing the resulting equation by $[x-mz]^p$, we get

$$\begin{aligned}
\Big[-m_t + \alpha(t)m + \lambda(\sigma_t-\bar{\sigma})m_\sigma - \frac{\theta^2\sigma_t^2}{2}m_{\sigma\sigma} + \frac{\theta^2\sigma_t^2 m_\sigma^2 z}{2[x-mz]}(p-1)(1-\rho^2)&\\
-(1+\delta(t)m) + \mu\theta\rho m_\sigma + \sigma_t^2\theta^2(\rho^2-1)\frac{M_\sigma}{M}m_\sigma\Big]\frac{zM}{x-mz}&\\
+\frac{M_t}{p} - \lambda(\sigma_t-\bar{\sigma})\frac{M_\sigma}{p} + \frac{\theta^2\sigma_t^2}{2p}M_{\sigma\sigma} + M^{\frac{p}{p-1}}(1+\delta(t)m)^{\frac{p}{p-1}}\frac{1-p}{p}&\\
+\frac{\sigma_t^2\theta^2\rho^2}{2(1-p)}\frac{M_\sigma^2}{M} + \frac{\mu^2}{2\sigma_t^2(1-p)}M + \frac{\mu\theta\rho}{1-p}M_\sigma = 0.&
\end{aligned}$$

Since for every $x \geq 0$ and $z \geq 0$, above equation holds. Thus, we have

$$\begin{aligned}
-m_t + \alpha(t)m + \lambda(\sigma_t-\bar{\sigma})m_\sigma - \frac{\theta^2\sigma_t^2}{2}m_{\sigma\sigma}&\\
-(1+\delta(t)m) + \mu\theta\rho m_\sigma + \sigma_t^2\theta^2(\rho^2-1)\frac{M_\sigma}{M}m_\sigma = 0,&
\end{aligned} \tag{4.7}$$

$$\frac{M_t}{p} - \lambda(\sigma - \bar{\sigma})\frac{M_\sigma}{p} + \frac{\theta^2\sigma^2}{2p}M_{\sigma\sigma} + M^{\frac{p}{p-1}}(1+\delta(t)m)^{\frac{p}{p-1}}\frac{1-p}{p}$$
$$+\frac{\sigma^2\theta^2\rho^2}{2(1-p)}\frac{M_\sigma^2}{M} + \frac{\mu^2}{2\sigma^2(1-p)}M + \frac{\mu\theta\rho}{1-p}M_\sigma = 0. \tag{4.8}$$

and

$$\frac{\theta^2\sigma^2 m_\sigma^2}{2}(p-1)(1-\rho^2) = 0. \tag{4.9}$$

From (4.9), we know $m_\sigma = 0$ for $\rho \neq \pm 1$. Therefore, we conclude m is just a function of t. Then (4.7) is reduced to the following ODE

$$-m_t + \alpha(t)m - (1+\delta(t)m) = 0, \quad m(T) = 0.$$

The solution to above equation is

$$m(t) = \int_t^T \exp\Big(\int_t^s (\delta(u) - \alpha(u))du\Big)ds, \quad t \in [0,T]. \tag{4.10}$$

We set the following power transform

$$M(t,\sigma) = N(t,\sigma)^{1-p}.$$

Then, the non-linear PDE for $M(t,\sigma)$ is transferred to a PDE for $N(t,\sigma)$

$$N_t + \Big(\frac{\mu\theta\rho p}{1-p} - \lambda(\sigma-\bar{\sigma})\Big)N_\sigma + \frac{\theta^2\sigma^2}{2}N_{\sigma\sigma} + \frac{\theta^2\sigma^2 p}{2}(\rho^2-1)\frac{N_\sigma^2}{N}$$
$$+\frac{\mu^2 p}{2\sigma^2(1-p)^2}N + (1+\delta(t)m)^{\frac{p}{p-1}} = 0. \tag{4.11}$$

with $N(T,\sigma) = 1$.

Theorem 4.2: *If the Assumption 4 holds, then the function*

$$V(t,x,z,\sigma) = \frac{[x - m(t,\sigma)z]^p}{p} N(t,\sigma)^{1-p}, \tag{4.12}$$

is the solution to the HJB equation (4.3), where $m(t)$ is given by (4.10) and $N(t,\sigma)$ is the solution to (4.11).

The solution (4.12) to the HJB equation coincides with the optimal value function defined in (4.1):

$$V(0, x_0, z_0, \sigma_0) = v(x_0, z_0, \sigma_0).$$

Theorem 4.3: *The optimal consumption rate c_t^*, optimal investment policy π_t^*, the optimal habit level z_t^* and the optimal wealth process X_t^* are given by*

$$c_t^* = z_t^* + \frac{Y_t^*}{N(t,\sigma_t)}(1 + \delta(t)m(t))^{\frac{1}{p-1}}, \tag{4.13}$$

$$\pi_t^* = \Big(\theta\rho \frac{N_\sigma(t,\sigma_t)}{N(t,\sigma_t)} + \frac{\mu}{\sigma_t^2(1-p)}\Big) Y_t^*, \tag{4.14}$$

$$\begin{aligned} z_t^* &= z_0 e^{\int_0^t \left(\delta(u)-\alpha(u)\right)du} + \int_0^t \Big(\delta(s)\frac{Y_s^*}{N(s,\sigma_s)}(1+\delta(s)m(s))^{\frac{1}{p-1}}\Big) e^{\int_s^t \left(\delta(u)-\alpha(u)\right)du} ds \\ &:= F(t, Y_t^*), \end{aligned} \tag{4.15}$$

and

$$X_t^* = Y_t^* + m(t)z_t^*. \tag{4.16}$$

Here, Y_t^* is the solution of following stochastic differential equation

$$dY_t^* = \Big(\theta\mu\rho\frac{N_\sigma(t,\sigma_t)}{N(t,\sigma_t)} + \frac{\mu^2}{\sigma_t^2(1-p)} - \frac{1}{N(t,\sigma_t)}(1+\delta(t)m(t))^{\frac{p}{p-1}}\Big)Y_t^* dt$$
$$-\Big(1+m(t)\Big(\delta(t)-\alpha(t)\Big)\Big)F(t,Y_t^*)dt + \Big(\theta\rho\sigma_t\frac{N_\sigma(t,\sigma_t)}{N(t,\sigma_t)} + \frac{\mu}{\sigma_t(1-p)}\Big)Y_t^* dW_t. \tag{4.17}$$

Proof. Apply the first order condition to HJB equation, we can easily get the feedback form of c^* and π^*. Define $Y_t^* := X_t^* - m(t)z_t^*$. Then, we have

$$dY_t^* = d(X_t^* - m(t)z_t^*) = \Big((\pi_t^*\mu - c_t^*)dt + \sigma_t\pi_t^* dW_t\Big) - m(t)\Big(\delta(t)c_t^* - \alpha(t)z_t^*\Big)dt. \tag{4.18}$$

Put (4.13) and (4.14) into above equation, we derive (4.17). Moreover, the optimal habit formation process satisfies

$$dz_t^* = (\delta(t)c_t^* - \alpha(t)z_t^*)dt.$$

By substituting the (4.14) into above equation, we get

$$\begin{aligned} dz_t^* &= \Big(\delta(t)\Big(z_t^* + Y_t^*\frac{1}{N(t,\sigma_t)}(1+\delta(t)m(t))^{\frac{1}{p-1}}\Big) - \alpha(t)z_t^*\Big)dt \\ &= \Big(\delta(t)-\alpha(t)\Big)z_t^* dt + \delta(t)Y_t^*\frac{1}{N(t,\sigma_t)}(1+\delta(t)m(t))^{\frac{1}{p-1}}dt, \end{aligned}$$

which is equivalent to (4.15). This ends the proof of this theorem □

4.3 The Case of Exponential Preference

Now, we take the utility function under exponential preference. In this section, our objective can be formulated as follows

$$v(x_0, z_0, \sigma_0) = \sup_{\pi, c \in \mathcal{A}} E[\int_0^T -e^{-(c_s - z_s)} ds - e^{-X_T}]. \tag{4.19}$$

To solve this optimization problem, we define a smooth function

$$V(t, x, z, \sigma) = \sup_{(\pi, c) \in \mathcal{A}(x_0)} E\Big(\int_t^T -e^{-(c_s - z_s)} ds - e^{-X_T} \Big| X_t = x, z_t = z, \sigma_t = \sigma\Big), \tag{4.20}$$

with terminal condition $V(T, x, z, \sigma) = -e^{-x}$.

Theorem 4.4: *The optimal value function $V(t, x, z, \sigma)$ defined in (4.20) is the solution for the following HJB equation*

$$\begin{aligned} &V_t - \alpha(t) z V_z - \lambda(\sigma - \bar{\sigma}) V_\sigma + \frac{\theta^2 \sigma^2}{2} V_{\sigma\sigma} + \max_c [-c V_x + c\delta(t) V_z - e^{-(c-z)}] \\ &+ \max_\pi [\pi \mu V_x + \frac{1}{2} \pi^2 \sigma^2 V_{xx} + \pi \rho \theta \sigma^2 V_{x\sigma}] = 0. \end{aligned} \tag{4.21}$$

And the optimal control pair takes the following form

$$\begin{aligned} c^*(t, x, z, \eta) &= z - \ln(V_x - \delta(t) V_z), \\ \pi^*(t, x, z, \eta) &= \frac{-\mu V_x - \sigma^2 \theta \rho V_{x\sigma}}{\sigma^2 V_{xx}}. \end{aligned} \tag{4.22}$$

By inserting (4.22) in (4.21), the HJB equation becomes the following PDE

$$V_t - \alpha(t) z V_z - \lambda(\sigma - \bar{\sigma}) V_\sigma + \frac{\theta^2\sigma^2}{2} V_{\sigma\sigma} + \Big(\delta V_z - V_x\Big)\Big(z + 1 - \ln(V_x - \delta V_z)\Big) - \frac{(\mu V_x + \sigma^2\theta\rho V_{x\sigma})^2}{2\sigma^2 V_{xx}} = 0. \tag{4.23}$$

Assumption 5: *For the rest of this section, we assume the above PDE, (4.23), has a unique classical (twice continuously differentiable) solution.*

We propose the following candidate function of $V(t, x, z, \sigma)$

$$V(t, x, z, \sigma) = -M(t, \sigma) \exp\Big(-\phi(t, \sigma) x + m(t, \sigma) z\Big), \tag{4.24}$$

here, $\phi(t, \sigma), m(t, \sigma)$ and $M(t, \sigma)$ are functions to be determined. We calculate the derivatives of (4.24)

$$\begin{aligned}
V_t &= \frac{M_t}{M} V + (m_t z - \phi_t x) V, \\
V_z &= m V, \\
V_\sigma &= \frac{M_\sigma}{M} V + (m_\sigma z - \phi_\sigma x) V, \\
V_{\sigma\sigma} &= \frac{M_{\sigma\sigma}}{M} V + 2(m_\sigma z - \phi_\sigma x) \frac{M_\sigma}{M} V + (m_{\sigma\sigma} z - \phi_{\sigma\sigma} x) V + (m_\sigma z - \phi_\sigma x)^2 V, \\
V_x &= -\phi V, \\
V_{xx} &= \phi^2 V, \\
V_{x\sigma} &= -\phi_\sigma V - \phi \frac{M_\sigma}{M} V - \phi(m_\sigma z - \phi_\sigma x) V.
\end{aligned} \tag{4.25}$$

By substituting them in (4.23) and dividing V on the resulting equation, we get

$$\frac{M_t}{M} + (m_t z - \phi_t x) - \alpha z m - \lambda(\sigma - \bar{\sigma})\frac{M_\sigma}{M} - \lambda(\sigma - \bar{\sigma})(m_\sigma z - \phi_\sigma x)$$
$$+\frac{\theta^2\sigma^2}{2}(\frac{M_{\sigma\sigma}}{M} + 2(m_\sigma z - \phi_\sigma x)\frac{M_\sigma}{M} + (m_{\sigma\sigma} z - \phi_{\sigma\sigma} x) + (m_\sigma z - \phi_\sigma x)^2)$$
$$+(z + 1 + \ln(\phi V + \delta m V))(\delta m + \phi) - \frac{\mu^2}{2\sigma^2} - \frac{\sigma^2\theta^2\rho^2}{2\phi^2}(\phi_\sigma + \phi\frac{M_\sigma}{M} + \phi(m_\sigma z$$
$$-\phi_\sigma x))^2 - \mu\theta\rho(\frac{\phi_\sigma}{\phi} + \frac{M_\sigma}{M} + (m_\sigma z - \phi_\sigma x)) = 0.$$

To simplify our calculation, we assume that $m = m(t)$ and $\phi = \phi(t)$ are just functions of t. Then, above equation can be reduced to the following

$$\frac{M_t}{M} + (m_t z - \phi_t x) - \alpha z m - \lambda(\sigma - \bar{\sigma})\frac{M_\sigma}{M} + \frac{\theta^2\sigma^2}{2}\frac{M_{\sigma\sigma}}{M}$$
$$+(z + 1 + \ln(\phi + \delta m) + \phi x - m z + \ln M)(\delta m + \phi) - \frac{\mu^2}{2\sigma^2} - \frac{\sigma^2\theta^2\rho^2}{2}\frac{M_\sigma^2}{M^2} - \mu\theta\rho\frac{M_\sigma}{M} = 0. \tag{4.26}$$

Since for any $x \geq 0$ and $z \geq 0$, this equation is satisfied. Thus, we get the following equations

$$m_t - \alpha m + (1 - m)(\delta m + \phi) = 0, \tag{4.27}$$

$$\phi_t - \phi(\delta m + \phi) = 0, \tag{4.28}$$

and

$$\frac{M_t}{M} - \Big(\lambda(\sigma - \bar{\sigma}) + \mu\theta\rho\Big)\frac{M_\sigma}{M} - \frac{\sigma^2\theta^2\rho^2}{2}\frac{M_\sigma^2}{M^2} + \frac{\theta^2\sigma^2}{2}\frac{M_{\sigma\sigma}}{M}$$
$$+(1 + \ln(\phi + \delta m) + \ln M)(\phi + \delta m) - \frac{\mu^2}{2\sigma^2} = 0. \tag{4.29}$$

For (4.29), we set the following exponential transform

$$M(t,\sigma) = e^{N(t,\sigma)}.$$

Then the above PDE for $M(t,\sigma)$ is transferred to a PDE for $N(t,\sigma)$

$$\begin{aligned} &N_t - \Big(\lambda(\sigma-\bar{\sigma}) + \mu\theta\rho\Big)N_\sigma + \frac{\sigma^2\theta^2}{2}(1-\rho^2)N_\sigma^2 + \frac{\theta^2\sigma^2}{2}N_{\sigma\sigma} \\ &+(\phi+\delta m)N + \Big(1+\ln(\phi+\delta m)\Big)(\phi+\delta m) - \frac{\mu^2}{2\sigma^2} = 0. \end{aligned} \tag{4.30}$$

Theorem 4.5: *If the Assumption 5 holds, then the function*

$$V(t,x,z,\sigma) = -\exp\Big(-\phi(t)x + m(t)z + N(t,\sigma)\Big), \tag{4.31}$$

is the solution of the HJB equation (4.21), here, (4.27) and (4.28) are already solved in Chapter 3 Lemma 3.1. And $N(t,\sigma)$ is the solution of (4.30).

The solution (4.31) of the HJB equation coincides with the optimal value function defined in (4.20):

$$V(0,x_0,z_0,\sigma_0) = v(x_0,z_0,\sigma_0). \tag{4.32}$$

Theorem 4.6: *The optimal investment policy π_t^*, optimal consumption rate c_t^* and the optimal habit level are given by*

$$\pi_t^* = \frac{\mu}{\sigma_t^2\phi(t)} + \frac{\theta\rho}{\phi(t)}N_\sigma(t,\sigma_t), \tag{4.33}$$

$$c_t^* = (1-m(t))z_t^* + \phi(t)X_t^* - N(t,\sigma_t) - \ln(\phi(t)+\delta(t)m(t)), \tag{4.34}$$

$$
\begin{aligned}
z_t^* = z_0 \exp\Big(\int_0^t \Big(\delta(u) - \delta(u)m(u) - \alpha(u)\Big)du\Big) \\
+ \int_0^t \Big[\Big(\delta(s)\Big(\phi(s)X_s^* - N(s,\sigma_s) - \ln(\phi(s) + \delta(s)m(s))\Big)\Big) \\
\times \exp\Big(\int_s^t \Big(\delta(u) - \delta(u)m(u) - \alpha(u)\Big)du\Big)\Big]du := F(t, X_t^*).
\end{aligned} \tag{4.35}
$$

Here, X_t^* is the optimal wealth process and is the solution of following SDE

$$
\begin{aligned}
dX_t^* = \Big(\frac{\mu^2}{\sigma_t^2\phi(t)} + \frac{\theta\rho\mu}{\phi(t)}N_\sigma(t,\sigma_t)\Big)dt + (m(t)-1)F(t,X_t^*)dt - \phi X_t^* dt \\
+\Big(N(t,\sigma_t) + \ln(\phi(t) + \delta(t)m(t))\Big)dt + (\frac{\mu}{\sigma_t\phi(t)} + \frac{\theta\rho\sigma_t}{\phi(t)}N_\sigma(t,\sigma_t))dW_t, \\
X_0^* = x_0.
\end{aligned} \tag{4.36}
$$

Proof. Apply the first order condition to the HJB equation (4.21), we can easily get the feedback forms of c^* and π^*. The optimal habit level z_t^* satisfies

$$dz_t^* = (\delta(t)c_t^* - \alpha(t)z_t^*)dt.$$

By substituting (4.34) into above equation , we get

$$
\begin{aligned}
dz_t^* &= \Big(\delta(t)\Big((1-m(t))z_t^* + \phi(t)X_t^* - N(t,\sigma) - \ln(\phi(t) + \delta(t)m(t)\Big) - \alpha(t)z_t^*\Big)dt \\
&= \Big(\delta(t) - m(t)\delta(t) - \alpha(t)\Big)z_t^* dt + \delta(t)\Big(\phi(t)X_t^* - N(t,\sigma_t) - \ln(\phi(t) + \delta(t)m(t))\Big)dt,
\end{aligned} \tag{4.37}
$$

which is equivalent to (4.35). Moreover, by inserting (4.34) and (4.33) in

$$dX_t^* = (\pi_t^*\mu - c_t^*)dt + \sigma_t\pi_t^* dW_t,$$

we derive (4.36). This ends the proof of this theorem. □

Chapter 5

The Martingale Approach

In the previous two chapters, we use the stochastic control techniques to reduce the optimization problem into solving an associated HJB equation. In this chapter, we introduce the martingale approach. This method is based on the completeness of the market and the optimization problem can be reduced to the calculation of expected values.

5.1 The Case of Complete Information

5.1.1 The Economy and the Optimal Problem

We still assume risk-free interest rate $r \equiv 0$, and the process of risky asset price S_t satisfies the following stochastic differential equation

$$\frac{dS_t}{S_t} = \mu_t dt + \sigma_S dW_t, \quad S_0 = s_0, \quad t \in [0, T], \tag{5.1}$$

and the drift process μ_t is given by the following dynamic

$$d\mu_t = -\lambda(\mu_t - \bar{\mu})dt + \sigma_\mu dB_t, \quad t \in [0, T]. \tag{5.2}$$

The two processes W and B are standard Brownian Motion with correlation coefficient $\rho \in [-1, 1]$. The standard of living is given by

$$z_t = z_0 \exp\Big(-\int_0^t \alpha_u du\Big) + \int_0^t \delta_s \exp\Big(-\int_s^t \alpha_u du\Big) c_s ds, \tag{5.3}$$

which equivalent to

$$dz_t = (\delta_t c_t - \alpha_t z_t)dt, \quad z_0 \geq 0; \quad t \in [0, T].$$

The wealth process X satisfies the no-bankruptcy condition $X_t \geq 0$ for all $t \in [0, T]$, and X_t solves the SDE below

$$dX_t = (\pi_t \mu_t - c_t)dt + \pi_t \sigma_S dW_t, \quad X_0 = x_0 > 0; \quad t \in [0, T]. \tag{5.4}$$

In this chapter, preferences are defined over consumption plans and can be represented by the non-separable von Neumann-Morgenstern index $U(c) = E[\int_0^T u(c_t, z_t)]$. Our goal now is to find the optimal consumption c^* to maximize $U(c)$:

$$V(x_0; \pi, c) = \sup_{(\pi,c)\in\mathcal{A}(x_0)} U(c) = \sup_{(\pi,c)\in\mathcal{A}(x_0)} E[\int_0^T u(c_t, z_t)dt], \tag{5.5}$$

here, $\mathcal{A}(x_0)$ is the set of admissible pairs defined in Definition 2.10.

5.1.2 Useful Intermediate Results

In this subsection, we derive new results that will play important roles in the rest of the thesis.

Remark 5.1: The market price of risk process $\theta = \{\theta_t\}_{t\in[0,T]}$ under complete information model is given by

$$\theta_t = \frac{\mu_t}{\sigma_S}, \quad t \in [0, T]. \tag{5.6}$$

Since $\int_0^T \theta_t^2 dt \leq \sup_{0\leq t\leq T} |\mu_t|^2 \frac{T}{\sigma_S^2} < \infty$. Hence, θ is integrable with respect to W.

Lemma 5.1: *The market price of risk θ can be written as*

$$\theta_t = h(t) + \frac{\sigma_\mu}{\sigma_S} G(t), \quad t \in [0, T], \tag{5.7}$$

Here,

$$\begin{cases} h(t) = \dfrac{\mu_0}{\sigma_S} e^{-\lambda t} + \dfrac{\bar{\mu}}{\sigma_S}(e^{\lambda t} - 1), \\ G(t) = \displaystyle\int_0^t e^{\lambda s} dB_s. \end{cases} \tag{5.8}$$

To proof Lemma 5.1, we need to introduce the following lemma.

Lemma 5.2: *The unique solution of the following stochastic differential equation:*

$$dP_t = -\lambda P_t dt + dZ_t, \quad t \in [0, T],$$

is

$$P_t = e^{-\lambda t}(P_0 + \int_0^t e^{\lambda s} dZ_s), \quad t \in [0, T],$$

where, Z_t is a Itô process.

Proof of Lemma 5.1. Applying the above Lemma 5.2 to (5.2), the drift process can be calculated as follows

$$\mu_t = \mu_0 e^{-\lambda t} + \bar{\mu}(e^{\lambda t} - 1) + \sigma_\mu \int_0^t e^{\lambda s} dB_s, \quad t \in [0, T].$$

According to (5.6), the proof of (5.7) follows immediately. □

Throughout the rest of the thesis, we consider a process $\eta = (\eta_t)_{t \in [0,T]}$ defined as follows

$$\eta_t := \exp\Big[- \int_0^t \theta_s dW_s - \frac{1}{2} \int_0^t \theta_s^2 ds \Big], \quad t \in [0, T]. \tag{5.9}$$

From Remark 5.1, we know θ is integrable with respect to W, so η is well-defined on $[0, T]$.

Proposition 5.1: *The following assertions hold:*

1) *The process η_t defined in (5.9) is a martingale. And the probability measure Q defined as*

$$Q(A) = E[\eta_T 1_A], \quad A \in \mathcal{F}_T, \tag{5.10}$$

is a risk-neutral probability. This measure is equivalent to P and unique due to the completeness of the market.

2) *The following processes:*

$$\widetilde{W}_t := W_t + \int_0^t \theta_s ds, \quad t \in [0, T], \tag{5.11}$$

and

$$\widetilde{B}_t := B_t + \int_0^t \rho\theta_s ds, \quad t \in [0,T], \tag{5.12}$$

are both standard Q-Brownian motions.

The proof of Proposition 5.1 segments a technical but interesting lemma which is a version of Gronwall's lemma (see [7]).

Lemma 5.3: *Suppose β is a positive number, f and g are two differentiable functions. If f satisfies*

$$\int_0^T m(s)df(s) \leq \int_0^T m(s)dg(s) + \beta \int_0^T m(s)f(s)ds, \tag{5.13}$$

for any non-negative and bounded Borel function m, then we have

$$f(t) \leq e^{\beta t} \int_0^t e^{-\beta s} g'(s)ds.$$

Proof. For $\varepsilon > 0$, put $m(s) := \frac{1}{\varepsilon} 1_{]t,t+\varepsilon]}$. Then, (5.13) becomes

$$\frac{f(t+\varepsilon) - f(t)}{\varepsilon} \leq \frac{g(t+\varepsilon) - g(t)}{\varepsilon} + \beta \int_t^{t+\varepsilon} f(s)ds.$$

By letting ε go to zero, we get

$$f'(t) \leq g'(t) + \beta f(t), \quad t \geq 0.$$

Obviously there exists an non-negative function k such that

$$f'(t) = \beta f(t) + g'(t) - k(t), \quad t \geq 0.$$

The solution to above differential equation is given by

$$f(t) = e^{\beta t} \int_0^t e^{-\beta s}(g'(s) - k(s))ds \le e^{\beta t} \int_0^t e^{-\beta s} g'(s)ds, \quad t \ge 0.$$

Since $k(t) \ge 0$, the proof of the lemma follows immediately. □

Proof of Proposition 5.1.

1) By applying Itô formula to (5.9), we get

$$d\eta_t = -\theta_t \eta_t dW_t, \quad \eta_0 = 1, \quad t \in [0, T].$$

Since η_t is continuous and $\int_0^T \theta_t^2 dt < \infty$. We derive

$$\int_0^T \theta_t^2 \eta_t^2 dt \le \sup_{0 \le t \le T} (\eta_t)^2 \int_0^T \theta_t^2 dt < \infty.$$

This leads to conclude that is η a positive local martingale and is locally bounded as it is continuous. Let $(T_n)_{n \ge 1}$ be a sequence of stopping times that increasing to infinity and

$$\sup_{t \le T_n} \eta_t \le c,$$

where c is a constant. Put

$$f_n(t) = E[\eta_{t \wedge T_n} \ln(\eta_{t \wedge T_n}) - \eta_{t \wedge T_n} + 1]. \tag{5.14}$$

Since $\eta_{t \wedge n}$ is bounded and $x \ln(x) - x + 1 \ge 0$, the function $f_n(t)$ is non-negative and is well defined. Furthermore, by applying Itô formula to $\eta \ln(\eta) - \eta + 1$, we get

$$\begin{aligned} f_n(t) = E[\eta_{t\wedge T_n} \ln(\eta_{t\wedge T_n}) - \eta_{t\wedge T_n} + 1] &= \frac{1}{2} E \int_0^{t\wedge T_n} (\eta_{s\wedge T_n} \theta_s^2) ds \\ = \frac{1}{2} E \int_0^t 1_{\{s\leq T_n\}} \eta_s \theta_s^2 ds &= \frac{1}{2} \int_0^t E(1_{\{s\leq T_n\}} \eta_s \theta_s^2) ds. \end{aligned} \tag{5.15}$$

Consider a non-negative and bounded Borel function m. Then, using above inequality, we get

$$\begin{aligned} \int_0^T m(s) df_n(s) &= \frac{1}{2} \int_0^T m(s) E(1_{\{s\leq T_n\}} \eta_s \theta_s^2) ds = \frac{1}{2} \int_0^T E(m(s) 1_{\{s\leq T_n\}} \eta_s \theta_s^2) ds \\ &= \frac{1}{2} E \int_0^{T\wedge T_n} m(s) \eta_{s\wedge T_n} \theta_s^2 ds. \end{aligned}$$

Since $\eta_t > 0$ for all $t \in [0, T]$, by applying the Young's inequality ($ab \leq e^a + b \ln b - b$, for $b > 0$), for any $\alpha > 0$ we have

$$\begin{aligned} \int_0^T m(s) df_n(s) &= \frac{1}{2} E \int_0^{T\wedge T_n} m(s) \eta_{s\wedge T_n} \theta_s^2 ds \\ &\leq E \int_0^{T\wedge T_n} m(s) \Big[e^{\frac{\alpha}{2}\theta_s^2} + \Big(\frac{\eta_{s\wedge T_n}}{\alpha} \ln(\frac{\eta_{s\wedge T_n}}{\alpha}) - \frac{\eta_{s\wedge T_n}}{\alpha} \Big) \Big] ds \\ &\leq E \int_0^T m(s) e^{\frac{\alpha}{2}\theta_s^2} ds + \int_0^T m(s) E \Big(\frac{\eta_{s\wedge T_n}}{\alpha} \ln(\frac{\eta_{s\wedge T_n}}{\alpha}) - \frac{\eta_{s\wedge T_n}}{\alpha} \Big) ds. \end{aligned} \tag{5.16}$$

Thanks to

$$\begin{aligned} E\Big(\frac{\eta_{s\wedge T_n}}{\alpha} \ln(\frac{\eta_{s\wedge T_n}}{\alpha}) - \frac{\eta_{s\wedge T_n}}{\alpha} \Big) &= \frac{1}{\alpha} E\Big(\eta_{s\wedge T_n} \ln(\eta_{s\wedge T_n}) - \eta_{s\wedge T_n} + 1 \Big) - \frac{1}{\alpha} - \frac{1}{\alpha} E\Big(\eta_{s\wedge T_n} \ln(\alpha) \Big) \\ &= \frac{1}{\alpha} f_n(s) - \frac{1}{\alpha} - \frac{\ln \alpha}{\alpha}, \end{aligned}$$

the inequality (5.16) becomes

$$\int_0^T m(s) df_n(s) \leq \int_0^T m(s) E(e^{\frac{\alpha}{2}\theta_s^2} - \frac{1 + \ln \alpha}{\alpha}) ds + \frac{1}{\alpha} \int_0^T m(s) f_n(s) ds.$$

A direct application of Lemma 5.3, we conclude that

$$f_n(T) \leq e^{\frac{T}{\alpha}} \int_0^T e^{-\frac{s}{\alpha}} E(e^{\frac{\alpha}{2}\theta_s^2} - \frac{1+\ln\alpha}{\alpha}) ds \leq e^{\frac{T}{\alpha}} \int_0^T e^{-\frac{s}{\alpha}} E(e^{\frac{\alpha}{2}\theta_s^2} - \frac{\ln\alpha}{\alpha}) ds. \tag{5.17}$$

Now, we just need to prove $E(e^{\frac{\alpha}{2}\theta_t^2})$ is finite. To this end, we remark that

$$\theta_t^2 \leq 2h^2(t) + 2\frac{\sigma_\mu^2}{\sigma_S^2} G^2(t),$$

where $h(t)$ and $G(t)$ are given by (5.8). Then we obtain

$$E\exp\Big(\frac{\alpha}{2}\theta_t^2\Big) \leq \exp\Big(\alpha h^2(t)\Big) E\exp\Big(\alpha\frac{\sigma_\mu^2}{\sigma_S^2} G^2(t)\Big). \tag{5.18}$$

Since $\frac{\sigma_\mu}{\sigma_S} G(t) = \frac{\sigma_\mu}{\sigma_S}\int_0^t e^{\lambda s} dB_s \sim N(0, r_t^2)$, $r_t^2 = \frac{\sigma_\mu^2}{\sigma_S^2}\int_0^t e^{2\lambda u} du$ and using the distribution density function, we derive

$$\begin{aligned} E[\exp\Big(\alpha\frac{\sigma_\mu^2}{\sigma_S^2} G^2(t)\Big)] &= \int_{-\infty}^{+\infty} \exp\Big(\alpha y^2\Big) \exp\Big(-\frac{y^2}{2r_t^2}\Big) \frac{dy}{\sqrt{2\pi} r_t} \\ &= \int_{-\infty}^{+\infty} \exp\Big(-\frac{1-2r_t^2\alpha}{2r_t^2} y^2\Big) \frac{dy}{\sqrt{2\pi} r_t} \\ &= \frac{1}{\sqrt{1-2r_t^2\alpha}}. \end{aligned}$$

Since α is arbitrary positive number, we set

$$\alpha = \frac{1}{4r_T^2} < \frac{1}{2r_t^2},$$

Therefore, for any $t \in [0, T]$, we conclude that

$$E\exp\Big(\frac{\alpha}{2}\theta_t^2\Big) \leq \frac{e^{\alpha h^2(t)}}{\sqrt{1-2r_t^2\alpha}} < \frac{e^{\alpha h^2(t)}}{2} \leq C_1, \tag{5.19}$$

where $C_1 = \frac{1}{2}\exp\Big((\frac{\mu_0+\bar{\mu}}{\sigma_S})^2 e^{2\lambda T}\Big)$ is a positive constant. By inserting (5.19) into (5.17), we get

$$f_n(t) \leq e^{\frac{T}{\alpha}}(C_1 + \frac{\ln\alpha}{\alpha})T, \quad \text{for all } \ t \geq 0.$$

Hence, due to Fatou's lemma and $x\ln x - x + 1 \geq 0$, we get

$$E(\eta_T \ln(\eta_T) - \eta_T + 1) \leq \lim_{n\to\infty} f_n(T) < +\infty$$

According to Doob's inequality (see Theorem 2.1), we have $E(\sup_{t\leq T}\eta_t) < +\infty$. This proves that η_t is an uniformly integrable martingale.

2) Since η is a martingale, it is easy to prove that these two processes defined in (5.11) and (5.12) are Brownian motions under Q by using Girsanov's theorem (see Theorem 2.3).

□

5.1.3 Optimal Policies for General Case

We consider a generalization of the "linear" parametric utility. For the class of "linear" utilities $u(c, z) = v(c - z)$, we have the following assumption.

Assumption 6: *The function $v(\cdot) : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}$ is increasing and strictly concave and has the following properties:*

1) $v'(\infty) = \lim_{x\to\infty} v'(x) = 0,$

2) $v'(0) = \lim_{x\to 0} v'(x) = \infty.$

Remark 5.2: The functions that we studied before like power utility $u(c,z) = \dfrac{(c-z)^p}{p}$ and exponential utility $u(c,z) = -e^{-(c-z)}$ satisfy Assumption 6.

For the utilities satisfied the above assumption, we provide the following explicit solutions for optimal policies.

Theorem 5.1: *Consider the economy introduced in Subsection 5.1.1 and suppose that Assumption 6 hold. Define the non-negative, adapted process $\gamma = \{\gamma_t\}_{t\in[0,T]}$ as*

$$\gamma_t(y) := y\eta_t\Big(1 + \delta_t E^Q\Big[\int_t^T \exp\Big(-\int_t^s (\alpha_u - \delta_u)du\Big)ds \mid \mathcal{F}_t\Big]\Big) \tag{5.20}$$

and denote

$$\phi_t := 1 + \delta_t E^Q\Big[\int_t^T \exp\Big(-\int_t^s (\alpha_u - \delta_u)du\Big)ds \mid \mathcal{F}_t\Big], \tag{5.21}$$

then we have $\gamma_t(y) = y\eta_t\phi_t$. Let I represents the inverse of the function $v'(\cdot)$ and let y^* denote the unique solution of the following equation:

$$\begin{aligned}\chi(y) \equiv E^Q\Big[\int_0^T \Big(&z_0 \exp\Big(-\int_0^t (\alpha_u - \delta_u)du\Big) + I(y\eta_t\phi_t)\\ &+ \int_0^t \delta_s \exp\Big(-\int_s^t (\alpha_u - \delta_u)du\Big) I(y\eta_s\phi_s)ds\Big)dt\Big]\\ = x_0&\end{aligned} \tag{5.22}$$

The optimal control pair (π^*, c^*) *is given by*

$$
\begin{aligned}
c_t^* = z_0 \exp\Big(-\int_0^t (\alpha_u - \delta_u) du\Big) + I(y^* \eta_t \phi_t) \\
+ \int_0^t \delta_s \exp\Big(-\int_s^t (\alpha_u - \delta_u) du\Big) I(y^* \eta_s \phi_s)) ds,
\end{aligned} \tag{5.23}
$$

$$
\pi_t^* = \frac{\psi_t^*}{\sigma_S}.
$$

where $\psi^* = \{\psi_t^*\}_{t \in [0,T]}$ *is square-integrable, adapted process that uniquely represents the martingale* $E^Q[\int_0^T c_s^* ds \mid \mathcal{F}_t] - E^Q[\int_0^T c_s^* ds]$, *i.e.*

$$
\int_0^t \psi_s^* d\widetilde{W}_s = E^Q[\int_0^T c_s^* ds \mid \mathcal{F}_t] - E^Q[\int_0^T c_s^* ds].
$$

The associated optimal standard of living process and optimal wealth process are given by

$$
\begin{aligned}
z_t^* &= z_0 \exp\Big(-\int_0^t (\alpha_u - \delta_u) du\Big) \\
&\quad + \int_0^t \delta_s \exp\Big(-\int_s^t (\alpha_u - \delta_u) du\Big) I(y^* \eta_s \phi_s)) ds, \qquad (5.24) \\
X_t^* &= x_0 - \int_0^t c_s^* ds + \int_0^t \psi_s^* d\widetilde{W}_s.
\end{aligned}
$$

Remark 5.3: The process $\gamma_t(y)$ is the state price density η_t adjusted by $\delta_t E^Q\Big[\int_t^T \exp\Big(-\int_t^s (\alpha_u - \delta_u) du\Big) ds \mid \mathcal{F}_t\Big]$. The factor $\delta_t E^Q\Big[\int_t^T \exp\Big(-\int_t^s (\alpha_u - \delta_u) du\Big) ds \mid \mathcal{F}_t\Big]$ represents the cost at time t of the subsistence consumption policy $c_s = z_s, s > t$ per unit of time t standard of living.

The above Theorem 5.1 was first introduced in [8]. We cite the proof of this theorem and develop as follows.

Proof of Theorem 3.1. The following equation has the recursive linear structure:

$$\gamma_t = y\eta_t + \delta_t E\Big[\int_t^T \exp\Big(-\int_t^s \alpha_u du\Big)\gamma_s ds \mid \mathcal{F}_t\Big]. \tag{5.25}$$

This equation can be solved by repeated iteration, letting the number of iterations tend to ∞. Alternatively, it can be verified by substitution that (5.20) is a solution. Indeed, the process $\gamma_t(y) = y\eta_t\phi_t$ of (5.20) solves (5.25) if and only if

$$\eta_t\Big(1 + \delta_t E^Q\Big[\int_t^T \exp\Big(-\int_t^s (\alpha_u - \delta_u)du\Big)ds \mid \mathcal{F}_t\Big]\Big) = \eta_t$$
$$+ \delta_t E\Big[\int_t^T \exp\Big(-\int_t^s \alpha_u du\Big)\eta_s\Big(1 + \delta_s E^Q\Big[\int_s^T \exp\Big(-\int_s^l (\alpha_u - \delta_u)du\Big)dl \mid \mathcal{F}_s\Big]\Big)ds \mid \mathcal{F}_t\Big].$$

Simplifying and using the properties of η leads to

$$\begin{aligned} & E^Q\Big[\int_t^T \exp\Big(-\int_t^s (\alpha_u - \delta_u)du\Big)ds \mid \mathcal{F}_t\Big] \\ &= E^Q\Big[\int_t^T \exp\Big(-\int_t^s \alpha_u du\Big)\frac{\eta_s}{\eta_t}\Big(1 + \delta_s E^Q\Big[\int_s^T \exp\Big(-\int_s^l (\alpha_u - \delta_u)du\Big)dl \mid \mathcal{F}_s\Big]\Big)ds \mid \mathcal{F}_t\Big] \\ &= E^Q\Big[\int_t^T \exp\Big(-\int_t^s \alpha_u du\Big)\Big(1 + \delta_s E^Q\Big[\int_s^T \exp\Big(-\int_s^l (\alpha_u - \delta_u)du\Big)dl \mid \mathcal{F}_s\Big]\Big)ds \mid \mathcal{F}_t\Big]. \end{aligned} \tag{5.26}$$

Define the following processes

$$Y_t \equiv \int_t^T \exp\Big(-\int_t^s \alpha_u du\Big)\Big(1 + \delta_s \int_s^T \exp\Big(-\int_s^l (\alpha_u - \delta_u)du\Big)dl\Big)ds,$$

and

$$H_t \equiv \int_t^T \exp\Big(-\int_t^s (\alpha_u - \delta_u)du\Big)ds.$$

(5.26) can be expressed as $E^Q[H_t \mid \mathcal{F}_t] = E^Q[Y_t \mid \mathcal{F}_t]$. To prove the result, we show that Y_t coincides with H_t (P - a.s). Standard computations yield

$$\begin{aligned}
Y_t &= \int_t^T \exp\Big(-\int_t^s \alpha_u du\Big) ds + \int_t^T \Big\{\exp\Big(-\int_t^s \alpha_u du\Big)\delta_s \int_s^T \exp\Big(-\int_s^l (\alpha_u - \delta_u)du\Big) dl\Big\} ds \\
&= \int_t^T \exp\Big(-\int_t^s \alpha_u du\Big) ds + \int_t^T \delta_s \int_t^T \exp\Big(-\int_t^l \alpha_u du + \int_0^l \delta_u du\Big) \exp\Big(-\int_0^s \delta_u du\Big) dl\Big) ds \\
&- \int_t^T \int_t^s \delta_s \exp\Big(-\int_t^l \alpha_u du + \int_0^l \delta_u du\Big) \exp\Big(-\int_0^s \delta_u du\Big) dl\Big) ds.
\end{aligned} \tag{5.27}$$

The second integral of (5.27) equals:

$$\begin{aligned}
&\int_t^T \delta_s \int_t^T \exp\Big(-\int_t^l \alpha_u du + \int_0^l \delta_u du\Big) \exp\Big(-\int_0^s \delta_u du\Big) dl\Big) ds \\
&= \Big(-\int_t^T \exp(-\int_t^l \alpha_u du + \int_0^l \delta_u du) \exp(-\int_0^s \delta_u du) dl\Big)\Big|_t^T \\
&= \int_t^T \exp\Big(-\int_t^l (\alpha_u - \delta_u) du\Big)\Big(1 - \exp(-\int_t^T \delta_u) du\Big) dl.
\end{aligned}$$

By using the integration by parts, the third integral of (5.27) can be written as

$$\begin{aligned}
&-\int_t^T \int_t^s \delta_s \exp\Big(-\int_t^l \alpha_u du + \int_0^l \delta_u du\Big) \exp\Big(-\int_0^s \delta_u du\Big) dl\Big) ds \\
&= \int_t^T \exp(-\int_t^s \alpha_u du)\Big(\exp(-\int_s^T \delta_u) du - 1\Big) ds
\end{aligned}$$

Substituting back into (5.27) then we have:

$$Y_t = \int_t^T \exp\Big(-\int_t^s (\alpha_u - \delta_u) du\Big) ds = H_t.$$

For the standard of living process, $z_t(y)$ is the unique solution to $dz_t =$

$\Big(\delta_t(y\eta_t\phi_t)) - \alpha_t z_t\Big)dt$. Then, the candidate optimal habit level is given by

$$z_t(y) = z_0 \exp\Big(-\int_0^t (\alpha_u - \delta_u)du\Big) + \int_0^t \delta_s \exp\Big(-\int_s^t (\alpha_u - \delta_u)du\Big) I(y\eta_s\phi_s)ds.$$

The candidate optimal policy is $c_t(y) = z_t + v'^{-1}(\gamma_t(y))$, so our candidate consumption process becomes

$$c_t(y) = z_0 \exp\Big(-\int_0^t (\alpha_u - \delta_u)du\Big) + \int_0^t \delta_s \exp\Big(-\int_s^t (\alpha_u - \delta_u)du\Big) I(y\eta_s\phi_s)ds + I(y\eta_t\phi_t).$$

To complete the proof, we show existence of a unique multiplier that satisfies the static budget constraint. It is straightforward to verify that the function

$$\begin{aligned}\chi(y) \equiv E^Q\Big[\int_0^T \Big(z_0 \exp\Big(-\int_0^t (\alpha_u - \delta_u)du\Big) I(y\eta_t\phi_t) \\ + \int_0^t \delta_s \exp\Big(-\int_s^t (\alpha_u - \delta_u)du\Big) I(y\eta_s\phi_s)ds\Big)dt\Big] : \\ [0,\infty] \to \Big[z_0 E^Q\Big(\int_0^T \exp\Big(-\int_0^t (\alpha_u - \delta_u)du\Big)dt\Big), \infty\Big]\end{aligned}$$

is continuous and strictly decreasing with limiting values $\chi(0) = \infty$ and $\chi(\infty) = z_0 E^Q\Big(\int_0^T \exp\Big(-\int_0^t (\alpha_u - \delta_u)du\Big)dt\Big)$. Since $\chi(y)$ is strictly decreasing, y^* is unique .

□

5.1.4 The Case of Exponential Preference

Under complete information with habit formation case, Detemple and Zapatero have already calculated the optimal policies for logarithmic and power utilities in [8]. Therefore, in this subsection, we consider that investors are endowed with exponential utility preference. To get the the explicit solution under this case, we need to calculate the following expectations.

Lemma 5.4: *We have the following expectations under Q:*

$$E^Q[\theta_t] = \int_0^t (\frac{\lambda\bar{\mu}}{\sigma_S}e^{\lambda s} - \frac{\lambda\mu_0}{\sigma_S}e^{-\lambda s})\exp\Big(\frac{\sigma_\mu\rho}{\sigma_S\lambda}e^{\lambda(s-t)}\Big)ds + \frac{\mu_0}{\sigma_S}\exp\Big(\frac{\sigma_\mu\rho}{\sigma_S\lambda}e^{1-\lambda t}\Big), \tag{5.28}$$

$$\begin{aligned} E^Q[\int_0^t e^{\lambda s}dB_s] = E^Q[G(t)] = & \int_0^t (\frac{\lambda\bar{\mu}}{\sigma_\mu}e^{\lambda s} - \frac{\lambda\mu_0}{\sigma_\mu}e^{-\lambda s})\exp\Big(\frac{\sigma_\mu\rho}{\sigma_S\lambda}e^{\lambda(s-t)}\Big)ds \\ & + \frac{\mu_0}{\sigma_\mu}\sigma_\mu\exp\Big(\frac{\sigma_\mu\rho}{\sigma_S\lambda}e^{1-\lambda t}\Big) - \frac{\mu_0}{\sigma_\mu}e^{-\lambda t} - \frac{\bar{\mu}}{\sigma_\mu}(e^{\lambda t}-1), \end{aligned} \tag{5.29}$$

and

$$E^Q[G^2(t)] = \frac{1}{2\lambda}(e^{2\lambda t}-1) - 2\rho\int_0^t h(s)E^Q[G(s)]\exp\Big(\frac{2\rho}{\lambda}\frac{\sigma_\mu}{\sigma_S}(e^{\lambda s}-e^{\lambda t})\Big)ds. \tag{5.30}$$

Proof. From (5.7), the expectation of θ_t becomes

$$E^Q[\theta_t] = h(t) + \frac{\sigma_\mu}{\sigma_S}E^Q[G(t)]. \tag{5.31}$$

Here, $h(t)$ and $G(t)$ are defined in (5.8). To compute the above expectation, we need to calculate $E^Q[G(t)]$ first. From Proposition 5.1, we know

$$E^Q[G(t)] = E^Q[\int_0^t e^{\lambda s}d\widetilde{B}_t] - E^Q[\int_0^t e^{\lambda s}\rho\theta_s ds] = -\rho\int_0^t e^{\lambda s}E^Q[\theta_s]ds. \tag{5.32}$$

Combining above equation with (5.31), we get

$$E^Q[\theta_t] = h(t) - \rho\frac{\sigma_\mu}{\sigma_S}\int_0^t e^{\lambda s}E^Q[\theta_s]ds. \tag{5.33}$$

If we denote $f(t) := E^Q[\theta_t]$, and differentiate above equation, we get the following differential equation

$$f'(t) = h'(t) - \rho\frac{\sigma_\mu}{\sigma_S}e^{\lambda t}f(t).$$

The solution to above ODE is

$$f(t) = E^Q[\theta_t] = \int_0^t (\frac{\lambda\bar{\mu}}{\sigma_S}e^{\lambda s} - \frac{\lambda\mu_0}{\sigma_S}e^{-\lambda s})\exp\Big(\frac{\sigma_\mu\rho}{\sigma_S\lambda}e^{\lambda(s-t)}\Big)ds + \frac{\mu_0}{\sigma_S}\exp\Big(\frac{\sigma_\mu\rho}{\sigma_S\lambda}e^{1-\lambda t}\Big).$$

Put the above result into (5.32), we can get the (5.29). Now, we focus on calculating the expectation of $G^2(t)$. Applying Itô formula to $G^2(t)$, we get

$$G^2(t) = 2\int_0^t G(s)(e^{\lambda s}d\widetilde{B}_s - \rho\theta_s e^{\lambda s}dt) + \int_0^t e^{2\lambda s}ds.$$

Therefore, the expectation of $G^2(t)$ becomes

$$E^Q[G^2(t)] = -2\rho\int_0^t E^Q[G(s)\theta_s]e^{\lambda s}dt + \int_0^t e^{2\lambda s}ds. \tag{5.34}$$

Since

$$E^Q[G(t)\theta_t] = h(t)E^Q[G(t)] + \frac{\sigma_\mu}{\sigma_S}E^Q[G^2(t)].$$

Inserting above expectation into (5.34), we get

$$E^Q[G^2(t)] = -2\rho \int_0^t \Big(h(s)E^Q[G(s)] + \frac{\sigma_\mu}{\sigma_S}E^Q[G^2(s)]\Big)e^{\lambda s}dt + \int_0^t e^{2\lambda s}ds. \quad (5.35)$$

Again, by denoting $g(t) := E^Q[G^2(t)]$ and differentiating above equation, we get

$$g'(t) = e^{2\lambda t} - 2\rho h(t)E^Q[G(t)] - 2\rho\frac{\sigma_\mu}{\sigma_S}e^{\lambda t}f(t).$$

The solution to above ODE is given by

$$\begin{aligned}E^Q[G^2(t)] &= \int_0^t \Big(e^{2\lambda s} - 2\rho h(s)E^Q[G(s)]\exp\Big(\frac{2\rho}{\lambda}\frac{\sigma_\mu}{\sigma_S}(e^{\lambda s} - e^{\lambda t}\Big)\Big)ds \\ &= \frac{1}{2\lambda}(e^{2\lambda t} - 1) - 2\rho\int_0^t h(s)E^Q[G(s)]\exp\Big(\frac{2\rho}{\lambda}\frac{\sigma_\mu}{\sigma_S}(e^{\lambda s} - e^{\lambda t}\Big)\Big)ds.\end{aligned}$$

□

Suppose the utility function takes the exponential form, we have the following theorem.

Theorem 5.2: *If the utility function takes the form of $u(c-z) = -e^{-(c-z)}$ or $v(x) = -e^{-x}$. Then, the following assertions hold:*

- *The optimal consumption rate c_t^* is*

$$\begin{aligned}c_t^* = z_0 e^{-\int_0^t(\alpha_u-\delta_u)du} - \ln(y^*)(1 + \int_0^t \delta_s e^{-\int_s^t(\alpha_u-\delta_u)du}ds) \\ - \ln(\eta_t\phi_t) - \int_0^t \delta_s \ln(\eta_s\phi_s)e^{-\int_s^t(\alpha_u-\delta_u)du}ds\end{aligned} \quad (5.36)$$

- *The optimal standard of living z_t^* is*

$$z_t^* = z_0 e^{-\int_0^t (\alpha_u - \delta_u)du} - \ln(y^*) \int_0^t \delta_s e^{-\int_s^t (\alpha_u - \delta_u)du} ds - \int_0^t \delta_s \ln(\eta_s \phi_s) e^{-\int_s^t (\alpha_u - \delta_u)du} ds. \tag{5.37}$$

- *The optimal wealth process X_t^* is*

$$X_t^* = x_0 + E^Q[\int_t^T c_s^* ds \mid \mathcal{F}_t] - E^Q[\int_0^T c_s^* ds]. \tag{5.38}$$

Here,

$$y^* = \exp\Big\{ \Big[z_0 E^Q \int_0^T e^{-\int_0^t (\alpha_u - \delta_u)du} dt - x_0 - E^Q \int_0^T \ln(\eta_t \phi_t) dt - E^Q \int_0^T (\int_0^t \delta_s e^{-\int_s^t (\alpha_u - \delta_u)du} \ln(\eta_s \phi_s) ds) dt \Big] \times \Big[E^Q \int_0^T (1 + \int_0^T \delta_s e^{-\int_s^t (\alpha_u - \delta_u)du} ds) dt \Big]^{-1} \Big\}. \tag{5.39}$$

Proof. For exponential utility $v(x) = -e^{-x}$, we obtain $I(y\eta_t\phi_t) = -\ln(y\eta_t\phi_t)$. Following the construction outlined in the proof of Theorem 5.1 produces the following candidate of consumption process

$$\begin{aligned} c_t(y) &= z_0 e^{-\int_0^t (\alpha_u - \delta_u)du} - \ln(\gamma_t(y)) - \int_0^t \delta_s e^{-\int_s^t (\alpha_u - \delta_u)du} \ln(\gamma_s(y)) ds \\ &= z_0 e^{-\int_0^t (\alpha_u - \delta_u)du} - \ln(y)(1 + \int_0^t \delta_s e^{-\int_s^t (\alpha_u - \delta_u)du} ds) \\ &\quad - \ln(\eta_t \phi_t) - \int_0^t \delta_s \ln(\eta_s \phi_s) e^{-\int_s^t (\alpha_u - \delta_u)du} ds. \end{aligned}$$

Substituting $c_t(y)$ in (5.22) and solving the equation $\chi(y) = x_0$ leads to

$$\begin{aligned} -\ln(y^*)\Big[E^Q \int_0^T (1 + \int_0^T \delta_s e^{-\int_s^t (\alpha_u - \delta_u)du} ds)dt\Big] \\ = \Big[x_0 - z_0 E^Q \int_0^T e^{-\int_0^t (\alpha_u - \delta_u)du} dt + E^Q \int_0^T \ln(\eta_t \phi_t) dt \\ + E^Q \int_0^T \Big(\int_0^t \delta_s e^{-\int_s^t (\alpha_u - \delta_u)du} \ln(\eta_s \phi_s) ds\Big) dt\Big]. \end{aligned}$$

From above equation, we get

$$\begin{aligned} y^* = \exp\Big\{\Big[z_0 E^Q \int_0^T e^{-\int_0^t (\alpha_u - \delta_u)du} dt - x_0 \\ - E^Q \int_0^T \ln(\eta_t \phi_t) dt - E^Q \int_0^T (\int_0^t \delta_s e^{-\int_s^t (\alpha_u - \delta_u)du} \ln(\eta_s \phi_s) ds) dt\Big] \\ \times \Big[E^Q \int_0^T (1 + \int_0^t \delta_s e^{-\int_s^t (\alpha_u - \delta_u)du} ds) dt\Big]^{-1}\Big\}. \end{aligned}$$

Therefore, the optimal consumption rate is $c_t^* = c_t(y^*)$. And put the optimal c_t^* into (5.24) derive (5.37) immediately. □

For the case when the parameters α and δ are deterministic, we get more simplified results.

Corollary 5.2.1: If parameters α_t and δ_t are deterministic. Then, the following assertions hold:

- The optimal consumption rate c_t^* is

$$\begin{aligned} c_t^* = z_0 e^{-\int_0^t (\alpha_u - \delta_u)du} - \ln(y^*)\Big(1 + \int_0^t \delta_t e^{-\int_s^t (\alpha_u - \delta_u)du} ds\Big) \\ - \Big(\ln(\eta_t) + \ln(1 + \delta_t \int_t^T e^{-\int_t^s (\alpha_u - \delta_u)du} ds)\Big) \\ - \int_0^t \delta_s e^{-\int_s^t (\alpha_u - \delta_u)du}\Big(\ln(\eta_s) + \ln(1 + \delta_s \int_s^T e^{-\int_s^l (\alpha_u - \delta_u)du} dl)\Big) ds. \end{aligned} \quad (5.40)$$

- The optimal standard of living z_t^* is

$$z_t^* = z_0 e^{-\int_0^t(\alpha_u-\delta_u)du} - \ln(y^*)\Big(\int_0^t \delta_t e^{-\int_s^t(\alpha_u-\delta_u)du} ds\Big) - \int_0^t \delta_s e^{-\int_s^t(\alpha_u-\delta_u)du}\Big(\ln(\eta_s) + \ln(1+\delta_s \int_s^T e^{-\int_s^l(\alpha_u-\delta_u)du} dl)\Big) ds. \tag{5.41}$$

- The optimal wealth process X_t^* is

$$X_t^* = x_0 + E^Q[\int_t^T c_s^* ds \mid \mathcal{F}_t] - E^Q[\int_0^T c_s^* ds]. \tag{5.42}$$

Here,

$$\begin{aligned} y^* = \exp\Big\{\Big[z_0 \int_0^T e^{-\int_0^t(\alpha_u-\delta_u)du} dt - x_0 \\ - \int_0^T E^Q \ln(\eta_t) dt - \int_0^T \Big(\ln(1+\delta_t \int_t^T e^{-\int_t^s(\alpha_u-\delta_u)du} ds)\Big) dt \\ - \int_0^T (\int_0^t \delta_s e^{-\int_s^t(\alpha_u-\delta_u)du}\Big(E^Q \ln(\eta_s) + \ln(1+\delta_s \int_s^T e^{-\int_s^l(\alpha_u-\delta_u)du} dl)\Big) ds) dt\Big] \\ \times\Big[\int_0^T (1+\int_0^t \delta_s e^{-\int_s^t(\alpha_u-\delta_u)du} ds) dt\Big]^{-1}\Big\}. \end{aligned} \tag{5.43}$$

and we have

$$E^Q \ln(\eta_t) = \frac{1}{2}\int_0^t h(s)^2 ds + \frac{\sigma_\mu^2}{2\sigma_S^2}\int_0^t E^Q[G(s)^2] ds + \frac{\sigma_\mu}{\sigma_S}\int_0^t h(s) E^Q[G(s)] ds,$$

here $h(t)$ and G_t are defined in (5.8). Moreover, $E^Q[G^2(t)]$ and $E^Q[G(t)]$ are calculated in (5.30) and (5.29).

Proof. If α and δ are deterministic, we have

$$E^Q\Big[\int_t^T \exp\Big(-\int_s^t (\alpha_u - \delta_u)du\Big)ds \mid \mathcal{F}_t] = \int_t^T \exp\Big(-\int_s^t (\alpha_u - \delta_u)du\Big)ds.$$

Thus,

$$\phi(t) = 1 + \delta_t \int_t^T \exp\Big(-\int_t^s (\alpha_u - \delta_u)du\Big)ds\big). \tag{5.44}$$

Substituting (5.44) into (5.40), we get

$$\begin{aligned} c_t^* = {} & z_0 e^{-\int_0^t (\alpha_u - \delta_u)du} - \ln(y^*)\Big(1 + \int_0^t \delta_t e^{-\int_s^t (\alpha_u - \delta_u)du} ds\Big) \\ & - \Big(\ln(\eta_t) + \ln(1 + \delta_t \int_t^T e^{-\int_t^s (\alpha_u - \delta_u)du} ds)\Big) \\ & - \int_0^t \delta_s e^{-\int_s^t (\alpha_u - \delta_u)du}\Big(\ln(\eta_s) + \ln(1 + \delta_s \int_s^T e^{-\int_s^l (\alpha_u - \delta_u)du} dl)\Big)dt, \end{aligned}$$

then substituting (5.44) into equation (5.43)

$$\begin{aligned} y^* = {} & \exp\Big\{\Big[z_0 \int_0^T e^{-\int_0^t (\alpha_u - \delta_u)du} dt - x_0 \\ & - \int_0^T E^Q\Big(\ln(\eta_t \phi_t)\Big)dt - \int_0^T (\int_0^t \delta_s e^{-\int_s^t (\alpha_u - \delta_u)du} E^Q\Big(\ln(\eta_s \phi_s)\Big)ds)dt\Big] \\ & \times \Big[\int_0^T (1 + \int_0^t \delta_s e^{-\int_s^t (\alpha_u - \delta_u)du} ds)dt\Big]^{-1}\Big\} \\ = {} & \exp\Big\{\Big[z_0 \int_0^T e^{-\int_0^t (\alpha_u - \delta_u)du} dt - x_0 \\ & - \int_0^T E^Q \ln(\eta_t)dt - \int_0^T \Big(\ln(1 + \delta_t \int_t^T e^{-\int_t^s (\alpha_u - \delta_u)du} ds)\Big)dt \\ & - \int_0^T (\int_0^t \delta_s e^{-\int_s^t (\alpha_u - \delta_u)du}\Big(E^Q \ln(\eta_s) + \ln(1 + \delta_s \int_s^T e^{-\int_s^l (\alpha_u - \delta_u)du} dl)\Big)ds)dt\Big] \\ & \times \Big[\int_0^T (1 + \int_0^t \delta_s e^{-\int_s^t (\alpha_u - \delta_u)du} ds)dt\Big]^{-1}\Big\}. \end{aligned} \tag{5.45}$$

Moreover, from (5.9) and (5.7), we get

$$\begin{aligned}
E^Q \ln(\eta_t) &= E^Q\Big[-\int_0^t \theta_s dW_s - \frac{1}{2}\int_0^t \theta_s^2 ds\Big] \\
&= E^Q\Big[-\int_0^t \theta_s d\widetilde{W}_s + \frac{1}{2}E^Q\int_0^t \theta_s^2 ds\Big] = -E^Q\Big[\int_0^t \theta_s d\widetilde{W}_s\Big] + \frac{1}{2}E^Q\Big[\int_0^t \theta_s^2 ds\Big] \\
&= \frac{1}{2}E^Q\Big[\int_0^t h(s)^2 ds + \frac{\sigma_\mu^2}{\sigma_S^2}\int_0^t G(s)^2 ds + 2\frac{\sigma_\mu}{\sigma_S}\int_0^t h(s)G(s)ds\Big] \\
&= \frac{1}{2}\Big[\int_0^t h(s)^2 ds + \frac{\sigma_\mu^2}{\sigma_S^2}\int_0^t E^Q[G(s)^2]ds + 2\frac{\sigma_\mu}{\sigma_S}\int_0^t h(s)E^Q[G(s)]ds\Big].
\end{aligned}$$

Here, $h(t)$ and $G(t)$ are defined in (5.8). And $E^Q[G(t)], E^Q[G(t)^2]$ have already been calculated in Lemma 5.4.

□

For the case when parameters α and δ are constant, we have the following corollary.

Corollary 5.2.2: If parameters $\alpha_t = \alpha$ and $\delta_t = \delta$ are constants, the following assertions hold:

- The optimal consumption rate c_t^* is

$$\begin{aligned}
c_t^* = z_0 e^{-(\alpha-\delta)t} &- \ln(y^*)\Big(\frac{\alpha}{\alpha-\delta} - \frac{\delta}{\alpha-\delta}e^{-(\alpha-\delta)t}\Big) \\
&-\Big(\ln(\eta_t) + \ln\big(\frac{\alpha}{\alpha-\delta} - \frac{\delta}{\alpha-\delta}e^{-(\alpha-\delta)(T-t)}\big)\Big) \\
&-\int_0^t \delta e^{-(\alpha-\delta)(t-s)}\Big(\ln(\eta_s) + \ln\big(\frac{\alpha}{\alpha-\delta} - \frac{\delta}{\alpha-\delta}e^{-(\alpha-\delta)(T-S)}\big)\Big)ds.
\end{aligned}$$

- The optimal standard of living z_t^* is

$$\begin{aligned}
z_t^* = z_0 e^{-(\alpha-\delta)t} &- \ln(y^*)\Big(\frac{\delta}{\alpha-\delta}(1 - e^{-(\alpha-\delta)t})\Big) \\
&-\int_0^t \delta e^{-(\alpha-\delta)(t-s)}\Big(\ln(\eta_s) + \ln\big(\frac{\alpha}{\alpha-\delta} - \frac{\delta}{\alpha-\delta}e^{-(\alpha-\delta)(T-s)}\big)\Big)ds.
\end{aligned}$$

- The optimal wealth process X_t^* is

$$X_t^* = x_0 + E^Q[\int_0^T c_s^* ds \mid \mathcal{F}_t] - E^Q[\int_0^T c_s^* ds].$$

Here,

$$\begin{aligned}
y^* = \exp\Big\{\Big[&\frac{z_0}{\alpha-\delta}(1-e^{-(\alpha-\delta)T}) - x_0 \\
&- \int_0^T E^Q \ln(\eta_t)dt - \int_0^T \Big(\frac{\alpha}{\alpha-\delta} - \frac{\delta}{\alpha-\delta}e^{-(\alpha-\delta)t}\Big) \\
&- \int_0^T (\int_0^t \delta e^{-(\alpha-\delta)(t-s)}\Big(E^Q \ln(\eta_s) + \ln(\frac{\alpha}{\alpha-\delta} - \frac{\delta}{\alpha-\delta}e^{-(\alpha-\delta)(T-s)})\Big)ds)dt\Big] \\
&\times \Big[\frac{\alpha}{\alpha-\delta}T + \frac{\delta}{(\alpha-\delta)^2}(e^{-(\alpha-\delta)T} - 1)\Big]^{-1}\Big\}.
\end{aligned}$$

Proof. If parameters $\alpha_t = \alpha$ and $\delta_t = \delta$ are constants, we have

$$E^Q\Big[\int_t^T \exp\Big(-\int_s^t (\alpha_u - \delta_u)du\Big)ds \mid \mathcal{F}_t] = \frac{1}{\alpha-\delta}(1-e^{-(\alpha-\delta)(T-t)}).$$

Thus,

$$\phi(t) = 1 + \frac{\delta}{\alpha-\delta}(1-e^{-(\alpha-\delta)(T-t)}).. \tag{5.46}$$

Inserting above equation into (5.40), (5.43) and (5.42) ends the proof of this corollary immediately. □

5.2 The Case of Partial Information

In this chapter, we study the case under partial observations setting. In Section 3.2, we use the filtering technique to transform our optimization problem from partial information context to full information context. For $t \in [0,T]$, the processes after transforming are given as follows

$$
\begin{aligned}
\frac{dS_t}{S_t} &= \widehat{\mu}_t dt + \sigma_S d\widehat{W}_t, \quad S_0 = s_0, \\
d\widehat{\mu}_t &= -\lambda(\widehat{\mu}_t - \bar{\mu})dt + (\frac{\widehat{\Omega}_t + \sigma_S\sigma_\mu\rho}{\sigma_S})d\widehat{W}_t, \quad \widehat{\mu}_0 = E[\mu_0 \mid \mathcal{F}_0^S], \\
d\widehat{\Omega}_t &= \Big[-\frac{1}{\sigma_S^2}\widehat{\Omega}_t^2 + \Big(-\frac{2\sigma_\mu\rho}{\sigma_S} - 2\lambda\Big)\widehat{\Omega}_t + (1-\rho^2)\sigma_\mu^2\Big]dt, \quad \widehat{\Omega}_0 = E[(\mu_0 - \widehat{\mu}_0)^2 \mid \mathcal{F}_0^S] = \theta_0, \\
dz(t) &= (\delta(t)\widehat{c}_t - \alpha(t)z_t)dt, \quad z_0 = z_0, \\
dX_t &= (\widehat{\pi}_t\widehat{\mu}_t - \widehat{c}_t)dt + \sigma_S\widehat{\pi}_t d\widehat{W}_t, \quad X_0 = x_0,
\end{aligned}
$$

where $\widehat{c}_t$ and $\widehat{\pi}_t$ are adapted to $\mathcal{F}_t^S$ only. From (3.7), we know $\widehat{\Omega}_t$ is a function of time.

Remark 5.4: The market price of risk process $\widehat{\theta} = \{\widehat{\theta}_t\}_{t\in[0,T]}$ under partial observations model is given by

$$
\widehat{\theta}_t = \frac{\widehat{\mu}_t}{\sigma_S}, \quad t \in [0,T]. \tag{5.47}
$$

Since $\widehat{\theta}$ takes the similar form with θ. And from (3.10) in Section 3.2, we know

$$
\frac{(\widehat{\Omega}_t + \sigma_S\sigma_\mu\rho)}{\sigma_S^2} \leq \frac{(\max(\theta_0, \theta^*) + \sigma_S\sigma_\mu\rho)}{\sigma_S^2},
$$

where, constant $\max(\theta_0, \theta^*)$ is the upper bound of $\widehat{\Omega}_t$. Thus, similar to Remark 5.1, we conclude that $\widehat{\theta}$ is integrable with respect to $\widehat{W}$.

Moreover, we have the following lemma towards $\widehat{\theta}$.

Lemma 5.5: *The market price of risk $\widehat{\theta}$ can be calculated as follows*

$$\widehat{\theta}_t = \widehat{h}(t) + \frac{(\widehat{\Omega}_t + \sigma_S\sigma_\mu\rho)}{\sigma_S^2}\widehat{G}(t), \quad t \in [0, T], \tag{5.48}$$

Here,

$$\begin{cases} \widehat{h}(t) = \dfrac{\widehat{\mu}_0}{\sigma_S}e^{-\lambda t} + \dfrac{\bar{\mu}}{\sigma_S}(e^{\lambda t} - 1), \\ \qquad G_t = \displaystyle\int_0^t e^{\lambda s} d\widehat{W}_t. \end{cases} \tag{5.49}$$

Proof. Applying Lemma 5.2 to (5.47), then the proof of this lemma is completed. □

Since $\widehat{\theta}_t$ is integrable with respect to $\widehat{W}_t$, the process

$$\widehat{\eta}_t := \exp\Big(-\int_0^t \widehat{\theta}_u d\widehat{W}_u - \frac{1}{2}\int_0^t \widehat{\theta}_u^2 du\Big),$$

is well-defined on $[0, T]$. And we have the following important proposition.

Proposition 5.2: *The following assertions hold:*

1) The process $\widehat{\eta}$ defined above is a martingale. The probability measure $\widehat{Q}$ defined by

$$\widehat{Q}(A) = E[\widehat{\eta}_T 1_A], \quad A \in \mathcal{F}_T^S, \tag{5.50}$$

is a risk-neutral probability. This measure is equivalent to P and unique due to the completeness of the market.

2) The following process

$$W_t^* := \widehat{W}_t + \int_0^t \widehat{\theta}_s ds, \quad t \in [0, T]. \tag{5.51}$$

is a standard $\widehat{Q}$-Brownian motion.

Proof. Follow the same procedure, as in Proposition 5.1, we can prove this proposition easily by using Lemma 5.3 and Theorem 2.3. □

To prepare the next section, we denote

$$\widehat{\phi}_t := 1 + \delta_t E^{\widehat{Q}}\Big[\int_t^T \exp\Big(-\int_t^s (\alpha_u - \delta_u)du\Big)ds \mid \mathcal{F}_t^S\Big]. \tag{5.52}$$

And we introduce the following lemma.

Lemma 5.6: *The following expectations under $\widehat{Q}$ can be calculated:*

$$E^{\widehat{Q}}[\widehat{\theta}_t] = \widehat{h}(t) = \frac{\widehat{\mu}_0}{\sigma_S}e^{-\lambda t} + \frac{\bar{\mu}}{\sigma_S}(e^{\lambda t} - 1), \tag{5.53}$$

and

$$E^{\widehat{Q}}[\int_0^t e^{\lambda s} d\widehat{W}_s] = 0. \tag{5.54}$$

Proof. From the (5.48), we get

$$E^{\widehat{Q}}[\widehat{\theta}_t] = \widehat{h}(t) + \frac{(\widehat{\Omega}_t + \sigma_S\sigma_\mu\rho)}{\sigma_S^2}E^{\widehat{Q}}[\widehat{G}_t]. \tag{5.55}$$

And

$$E^{\widehat{Q}}[\widehat{G}_t] = E^{\widehat{Q}}[\int_0^t e^{\lambda s} d\tilde{W}_s] = -\int_0^t e^{\lambda s}E^{\widehat{Q}}[\widehat{\theta}_s]ds. \tag{5.56}$$

By inserting above equation in (5.55), we get the following differential equation by denoting $f(t) := E^{\widehat{Q}}[\widehat{\theta}_t]$

$$f(t) = \widehat{h}(t) - \frac{(\widehat{\Omega}_t + \sigma_S\sigma_\mu\rho)}{\sigma_S^2}\int_0^t e^{\lambda s} f(s)ds.$$

To solve above equation, we define $F(t) := \int_0^t e^{\lambda s} f(s)ds$. Then above equation becomes

$$F'(t) = e^{\lambda t}\widehat{h}(t) - \frac{(\widehat{\Omega}_t + \sigma_S\sigma_\mu\rho)}{\sigma_S^2} e^{\lambda t} F(t).$$

The solution to above ODE is given by

$$F(t) = \int_0^t e^{\lambda s}\widehat{h}(s) \exp\Big(\int_t^s \frac{(\widehat{\Omega}_u + \sigma_S\sigma_\mu\rho)}{\sigma_S^2} e^{\lambda u} du\Big) ds.$$

Therefore,

$$f(t) = \widehat{h}(t) \exp\Big(\int_t^t \frac{(\widehat{\Omega}_u + \sigma_S\sigma_\mu\rho)}{\sigma_S^2} e^{\lambda u} du\Big) = \widehat{h}(t).$$

This ends the proof of this lemma. □

5.2.1 The Case of Logarithmic Preference

In this subsection, we consider that investors are endowed with the exponential utility of $u(c-z) = \ln(c-z)$ or $v = \ln(x)$. Then we have the following theorem.

Theorem 5.3: *The following assertions hold:*

- *The optimal consumption rate* $\widehat{c}_*(t)$ *is*

$$\widehat{c}_*(t) = z_0 e^{-\int_0^t (\alpha_u - \delta_u) du} + (\widehat{y}_*)^{-1}\Big[(\widehat{\eta}_t\widehat{\phi}_t)^{-1} + \int_0^t \delta_s e^{-\int_s^t (\alpha_u - \delta_u) du} (\widehat{\eta}_s\widehat{\phi}_s)^{-1} ds\Big]. \tag{5.57}$$

- *The optimal standard of living* $\widehat{z}_*(t)$ *is*

$$\widehat{z}_*(t) = z_0 e^{-\int_0^t (\alpha_u - \delta_u) du} + (\widehat{y}_*)^{-1} \int_0^t \delta_s e^{-\int_s^t (\alpha_u - \delta_u) du} (\widehat{\eta}_s\widehat{\phi}_s)^{-1} ds. \tag{5.58}$$

- *The optimal wealth process* $\widehat{X}_t^*$ *is*

$$\widehat{X}_t^* = x_0 + E^{\widehat{Q}}[\int_t^T \widehat{c}_*(s)ds \mid \mathcal{F}_t^S] - E^{\widehat{Q}}[\int_0^T \widehat{c}_*(s)ds]. \tag{5.59}$$

Here,

$$\begin{aligned}\widehat{y}_* = &\Big[x_0 - z_0 E^{\widehat{Q}} \int_0^T e^{-\int_0^t (\alpha_u - \delta_u)du} dt\Big]^{-1} \\ &\times \Big[E^{\widehat{Q}} \int_0^T \Big((\widehat{\eta}_t \widehat{\phi}_t)^{-1} + \int_0^t \delta_s e^{-\int_s^t (\alpha_u - \delta_u)du} (\widehat{\eta}_s \widehat{\phi}_s)^{-1} ds\Big) dt\Big].\end{aligned} \tag{5.60}$$

Proof. For power utility $v(x) = \ln(x)$, we obtain $I(\widehat{y}\widehat{\eta}_t\widehat{\phi}_t)) = (\widehat{y}\widehat{\eta}_t\widehat{\phi}_t)^{-1}$. Then it is easy to prove above theorem by applying Theorem 5.1. □

For the case when the parameters α and δ are deterministic, the above theorem can be simplified as follows.

Corollary 5.3.1: The following hold for α and δ are deterministic :

- The optimal consumption rate $\widehat{c}_*(t)$ is

$$\begin{aligned}\widehat{c}_*(t) = z_0 e^{-\int_0^t (\alpha_u - \delta_u)du} + (\widehat{y}_*)^{-1} \times \Big[\widehat{\eta}_t^{-1} \Big(1 + \delta_t \int_t^T e^{-\int_t^s (\alpha_u - \delta_u)du} ds\Big)^{-1} \\ + \int_0^t \delta_s e^{-\int_s^t (\alpha_u - \delta_u)du} \widehat{\eta}_s^{-1} \Big(1 + \delta_s \int_s^T e^{-\int_s^l (\alpha_u - \delta_u)du} dl\Big)^{-1} ds\Big].\end{aligned} \tag{5.61}$$

- The optimal standard of living $\widehat{z}_*(t)$ is

$$\widehat{z}_*(t) = z_0 e^{-\int_0^t (\alpha_u - \delta_u)du} + (\widehat{y}_*)^{-1} \int_0^t \delta_s e^{-\int_s^t (\alpha_u - \delta_u)du} \widehat{\eta}_s^{-1} \Big(1 + \delta_s \int_s^T e^{-\int_s^l (\alpha_u - \delta_u)du} dl\Big)^{-1} ds. \tag{5.62}$$

- The optimal wealth process $\widehat{X}_t^*$ is

$$\widehat{X}_t^* = x_0 + E^{\widehat{Q}}[\int_t^T \widehat{c}_*(s)ds \mid \mathcal{F}_t^S] - E^{\widehat{Q}}[\int_0^T \widehat{c}_*(s)ds].$$

Here,

$$\widehat{y}_* = \Big(x_0 - z_0 \int_0^T e^{-\int_0^t(\alpha_u-\delta_u)du}dt\Big)^{-1} \times \Big(\int_0^T \Big[\Big(1+\delta_t \int_t^T e^{-\int_t^s(\alpha_u-\delta_u)du}ds\Big)^{-1} + \int_0^t \delta_s e^{-\int_s^t(\alpha_u-\delta_u)du}\Big(1+\delta_s\int_s^T e^{-\int_s^l(\alpha_u-\delta_u)du}dl\Big)^{-1}ds\Big]dt\Big). \tag{5.63}$$

Proof. For deterministic parameters α_t and δ_t, we have $\widehat{\gamma}_t(y) = \widehat{y}\widehat{\eta}_t\widehat{\phi}_t$. Here

$$\widehat{\phi}_t = 1 + \delta_t \int_t^T \exp\Big(-\int_t^s(\alpha_u-\delta_u)du\Big)ds.$$

Substituting it into equation (5.57) and (5.60), we get

$$\begin{aligned}
\widehat{y}_* &= \Big(x_0 - z_0 \int_0^T e^{-\int_0^t(\alpha_u-\delta_u)du}dt\Big)^{-1} \\
&\quad \times E^{\widehat{Q}}\Big[\int_0^T \Big((\widehat{\eta}_t\widehat{\phi}_t)^{-1} + \int_0^t \delta_s e^{-\int_s^t(\alpha_u-\delta_u)du}\widehat{\phi}_s^{-1}ds\Big)dt\Big] \\
&= \Big(x_0 - z_0 \int_0^T e^{-\int_0^t(\alpha_u-\delta_u)du}dt\Big)^{-1} \times \Big(\int_0^T \Big[E^{\widehat{Q}}[\widehat{\eta_t}^{-1}]\Big(1+\delta_t\int_t^T e^{-\int_t^s(\alpha_u-\delta_u)du}ds\Big)^{-1} \\
&\quad + \int_0^t \delta_s e^{-\int_s^t(\alpha_u-\delta_u)du}E^{\widehat{Q}}[\widehat{\eta_s}^{-1}]\Big(1+\delta_s\int_s^T e^{-\int_s^l(\alpha_u-\delta_u)du}dl\Big)^{-1}ds\Big]dt\Big) \\
&= \Big(x_0 - z_0 \int_0^T e^{-\int_0^t(\alpha_u-\delta_u)du}ds\Big)^{-1} \times \Big(\int_0^T \Big[\Big(1+\delta_t\int_t^T e^{-\int_t^s(\alpha_u-\delta_u)du}ds\Big)^{-1} \\
&\quad + \int_0^t \delta_s e^{-\int_s^t(\alpha_u-\delta_u)du}\Big(1+\delta_s\int_s^T e^{-\int_u^s(\alpha_v-\delta_v)dv}du\Big)^{-1}ds\Big]dt\Big).
\end{aligned} \tag{5.64}$$

and the optimal consumption rate is $\widehat{c}_*(t) = \widehat{c}_t(\widehat{y}^*)$. The proof of this corollary is completed. □

For the case when the parameters α and δ are constants, we have the following corollary.

Corollary 5.3.2: If $\alpha_t = \alpha$ and $\delta_t = \delta$ are constant, the following assertions hold:

- The optimal consumption rate $\widehat{c}_*(t)$ is

$$\widehat{c}_*(t) = z_0 e^{-(\alpha-\delta)t} + (\widehat{y}_*)^{-1} \times \Big[\widehat{\eta_t}^{-1}\Big(\frac{\alpha}{\alpha-\delta} - \frac{\delta}{\alpha-\delta}e^{-(\alpha-\delta)T}\Big)^{-1} + \int_0^t \delta e^{-(\alpha-\delta)(t-s)}\widehat{\eta_s}^{-1}\Big(\frac{\alpha}{\alpha-\delta} - \frac{\delta}{\alpha-\delta}e^{-(\alpha-\delta)(T-s)}\Big)^{-1} ds\Big].$$

- The optimal standard of living $\widehat{z}_*(t)$ is

$$\widehat{z}_*(t) = z_0 e^{-(\alpha-\delta)t} + (\widehat{y}_*)^{-1}\int_0^t \delta e^{-(\alpha-\delta)(t-s)}\widehat{\eta_s}^{-1}\Big(\frac{\alpha}{\alpha-\delta} - \frac{\delta}{\alpha-\delta}e^{-(\alpha-\delta)(T-s)}\Big)^{-1} ds.$$

- The optimal wealth process $\widehat{X}^*_t$ is

$$\widehat{X}^*_t = x_0 + E^{\widehat{Q}}[\int_t^T \widehat{c}_*(s)ds \mid \mathcal{F}^S_t] - E^{\widehat{Q}}[\int_0^T \widehat{c}_*(s)ds].$$

  Here,

$$\widehat{y}_* = \Big(x_0 - \frac{z_0}{\alpha-\delta}\Big(e^{-(\alpha-\delta)T} - 1\Big)\Big)^{-1} \times \Big(\int_0^T \Big[\Big(\frac{\alpha}{\alpha-\delta} - \frac{\delta}{\alpha-\delta}e^{-(\alpha-\delta)(T-t)}\Big)^{-1} + \int_0^t \delta e^{-(\alpha-\delta)(t-s)}\Big(\frac{\alpha}{\alpha-\delta} - \frac{\delta}{\alpha-\delta}e^{-(\alpha-\delta)(T-s)}\Big)^{-1} ds\Big] dt\Big).$$

5.2.2 The Case of Power Preference

For this subsection, we consider that investors are endow with the power utility $u(c-z) = \frac{(c-z)^p}{p}$ or $v(x) = \frac{x^p}{p}$, with the risk aversion coefficient $p < 1$ and

$p \neq 0$.

Theorem 5.4: *The following assertions hold:*

- *The optimal consumption rate* $\widehat{c}_*(t)$ *is*

$$\widehat{c}_*(t) = z_0 e^{-\int_0^t (\alpha_u - \delta_u) du} + (\widehat{y}_*)^{\frac{1}{p-1}} \Big[(\widehat{\eta}_t \widehat{\phi}_t)^{\frac{1}{p-1}} + \int_0^t \delta_s e^{-\int_s^t (\alpha_u - \delta_u) du} (\widehat{\eta}_s \widehat{\phi}_s)^{\frac{1}{p-1}} ds \Big]. \tag{5.65}$$

- *The optimal standard of living* $\widehat{z}_*(t)$ *is*

$$\widehat{z}_*(t) = z_0 e^{-\int_0^t (\alpha_u - \delta_u) du} + (\widehat{y}_*)^{\frac{1}{p-1}} \int_0^t \delta_s e^{-\int_s^t (\alpha_u - \delta_u) du} (\widehat{\eta}_s \widehat{\phi}_s)^{\frac{1}{p-1}} ds. \tag{5.66}$$

- *The optimal wealth process* $\widehat{X}_t^*$ *is*

$$\widehat{X}_t^* = x_0 + E^{\widehat{Q}}[\int_t^T \widehat{c}_*(s) ds \mid \mathcal{F}_t^S] - E^{\widehat{Q}}[\int_0^T \widehat{c}_*(s) ds]. \tag{5.67}$$

Here,

$$\begin{aligned} \widehat{y}_* = &\Big[x_0 - z_0 E^{\widehat{Q}} \int_0^T e^{-\int_0^t (\alpha_u - \delta_u) du} dt \Big]^{p-1} \\ &\times \Big[E^{\widehat{Q}} \int_0^T \Big((\widehat{\eta}_s \widehat{\phi}_s)^{\frac{1}{p-1}} + \int_0^t \delta_s e^{-\int_s^t (\alpha_u - \delta_u) du} (\widehat{\eta}_s \widehat{\phi}_s)^{\frac{1}{p-1}} ds \Big) dt \Big]^{1-p}. \end{aligned} \tag{5.68}$$

Proof. For power utility $v(x) = \dfrac{x^p}{p}$, we obtain $I(\widehat{y}\widehat{\eta}_t \widehat{\phi}_t) = (\widehat{y}\widehat{\eta}_t \widehat{\phi}_t)^{\frac{1}{p-1}}$.
By (5.22) we get the following equation for y^*

$$\begin{aligned} &E^{\widehat{Q}} \Big[\int_0^T \Big(z_0 \exp\Big(-\int_0^t (\alpha_u - \delta_u) du \Big) + (\widehat{y}_* \widehat{\eta}_t \widehat{\phi}_t)^{\frac{1}{p-1}} \\ &+ \int_0^t \delta_s \exp\Big(-\int_s^t (\alpha_u - \delta_u) du \Big) (\widehat{y}_* \widehat{\eta}_s \widehat{\phi}_s)^{\frac{1}{p-1}} ds \Big) dt \Big] \\ &= x_0. \end{aligned}$$

Solving above equation we get

$$\widehat{y}_* = \Big(x_0 - z_0 E^{\widehat{Q}}\Big[\int_0^T \exp\Big(-\int_0^t (\alpha_u - \delta_u)du\Big)dt\Big]\Big)^{p-1}$$
$$\times\Big(E^{\widehat{Q}}\Big[\int_0^T \Big((\widehat{\eta}_t\widehat{\phi}_t)^{\frac{1}{p-1}} + \int_0^t \delta_s \exp\Big(-\int_s^t (\alpha_u - \delta_u)du\Big)(\widehat{\eta}_s\widehat{\phi}_s)^{\frac{1}{p-1}}ds\Big)dt\Big]\Big)^{1-p}.$$

Put $\widehat{y}_*$ into equation (5.23), we get the optimal policies. □

For the case when the parameters α_t and δ_t are deterministic, we have the following corollary.

Corollary 5.4.1: If α_t and δ_t are deterministic, the following hold:

- The optimal consumption rate $\widehat{c}_*(t)$ is

$$\widehat{c}_*(t) = z_0 e^{-\int_0^t(\alpha_u-\delta_u)du} + (\widehat{y}_*)^{\frac{1}{p-1}} \times \Big[\widehat{\eta}_t^{\frac{1}{p-1}}\Big(1+\delta_t\int_t^T e^{-\int_t^s(\alpha_u-\delta_u)du}ds\Big)^{\frac{1}{p-1}}$$
$$+\int_0^t \delta_s e^{-\int_s^t(\alpha_u-\delta_u)du}\widehat{\eta}_s^{\frac{1}{p-1}}\Big(1+\delta_s\int_s^T e^{-\int_s^l(\alpha_u-\delta_u)du}dl\Big)^{\frac{1}{p-1}}ds\Big].$$

- The optimal standard of living $\widehat{z}_*(t)$ is

$$\widehat{z}_*(t) = z_0 e^{-\int_0^t(\alpha_u-\delta_u)du} + (\widehat{y}_*)^{\frac{1}{p-1}}\int_0^t \delta_s e^{-\int_s^t(\alpha_u-\delta_u)du}\widehat{\eta}_s^{\frac{1}{p-1}}\Big(1+\delta_s\int_s^T e^{-\int_s^l(\alpha_u-\delta_u)du}dl\Big)^{\frac{1}{p-1}}ds.$$

- The optimal wealth process $\widehat{X}_t^*$ is

$$\widehat{X}_t^* = x_0 + E^{\widehat{Q}}[\int_t^T \widehat{c}_*(s)ds \mid \mathcal{F}_t^S] - E^{\widehat{Q}}[\int_0^T \widehat{c}_*(s)ds].$$

Here,

$$\widehat{y}_* = \Big(x_0 - z_0 \int_0^T e^{-\int_0^t(\alpha_u-\delta_u)du}dt\Big)^{p-1} \times \Big(\int_0^T \Big[E[\widehat{\eta}_t^{\frac{p}{p-1}}]\Big(1+\delta_t \int_t^T e^{-\int_t^s(\alpha_u-\delta_u)du}ds\Big)^{\frac{1}{p-1}}$$
$$+ \int_0^t \delta_s e^{-\int_s^t(\alpha_u-\delta_u)du} E[\widehat{\eta}_s^{\frac{p}{p-1}}]\Big(1+\delta_s \int_s^T e^{-\int_s^l(\alpha_u-\delta_u)du}dl\Big)^{\frac{1}{p-1}} ds\Big] dt\Big)^{1-p}.$$

Proof. For deterministic coefficients α_t and δ_t, we have $\widehat{\gamma}_t(y) = \widehat{y}\widehat{\eta}_t\widehat{\phi}_t$, where

$$\widehat{\phi}_t = 1 + \delta_t \int_t^T \exp\Big(-\int_t^s(\alpha_u-\delta_u)du\Big)ds.$$

Substituting it into (5.68), then we get

$$\begin{aligned}\widehat{y}_* &= \Big(x_0 - z_0 \int_0^T e^{-\int_0^t(\alpha_u-\delta_u)du}dt\Big)^{p-1} \times \Big(\int_0^T \Big[E^{\widehat{Q}}[\widehat{\eta}_t^{\frac{1}{p-1}}]\Big(1+\delta_t \int_t^T e^{-\int_t^s(\alpha_u-\delta_u)du}ds\Big)^{\frac{1}{p-1}}\\
&\quad + \int_0^t \delta_s e^{-\int_s^t(\alpha_u-\delta_u)du} E^{\widehat{Q}}[\widehat{\eta}_s^{\frac{1}{p-1}}]\Big(1+\delta_s \int_s^T e^{-\int_s^l(\alpha_u-\delta_u)du}dl\Big)^{\frac{1}{p-1}} ds\Big] dt\Big)^{1-p}\\
&= \Big(x_0 - z_0 \int_0^T e^{-\int_0^t(\alpha_u-\delta_u)du}ds\Big)^{p-1} \times \Big(\int_0^T \Big[E[\widehat{\eta}_t^{\frac{p}{p-1}}]\Big(1+\delta_t \int_t^T e^{-\int_t^s(\alpha_u-\delta_u)du}ds\Big)^{\frac{1}{p-1}}\\
&\quad + \int_0^t \delta_s e^{-\int_s^t(\alpha_u-\delta_u)du} E[\widehat{\eta}_s^{\frac{p}{p-1}}]\Big(1+\delta_s \int_s^T e^{-\int_s^l(\alpha_u-\delta_u)du}dl\Big)^{\frac{1}{p-1}} ds\Big] dt\Big)^{1-p}.\end{aligned}$$

and inserting above equation into (5.65), (5.66) and (5.67) we prove the corollary. □

For the case when parameters α and δ are constant, the above corollary can be further simplified.

Corollary 5.4.2: If $\alpha_t = \alpha$ and $\delta_t = \delta$ are constant, the following assertions hold:

- The optimal consumption rate $\widehat{c}_*(t)$ is

$$\widehat{c}_*(t) = z_0 e^{-(\alpha-\delta)t} + (\widehat{y}_*)^{\frac{1}{p-1}} \times \Big[\widehat{\eta}_t^{\frac{1}{p-1}} \Big(\frac{\alpha}{\alpha-\delta} - \frac{\delta}{\alpha-\delta} e^{-(\alpha-\delta)T}\Big)^{\frac{1}{p-1}} + \int_0^t \delta e^{-(\alpha-\delta)(t-s)} \widehat{\eta}_s^{\frac{1}{p-1}} \Big(\frac{\alpha}{\alpha-\delta} - \frac{\delta}{\alpha-\delta} e^{-(\alpha-\delta)(T-s)}\Big)^{\frac{1}{p-1}} ds\Big].$$

- The optimal standard of living $\widehat{z}_*(t)$ is

$$\widehat{z}_*(t) = z_0 e^{-(\alpha-\delta)t} + (\widehat{y}_*)^{\frac{1}{p-1}} \int_0^t \delta e^{-(\alpha-\delta)(t-s)} \widehat{\eta}_s^{\frac{1}{p-1}} \Big(\frac{\alpha}{\alpha-\delta} - \frac{\delta}{\alpha-\delta} e^{-(\alpha-\delta)(T-s)}\Big)^{\frac{1}{p-1}} ds.$$

- The optimal wealth process $\widehat{X}_t^*$ is

$$\widehat{X}_t^* = x_0 + E^{\widehat{Q}}[\int_t^T \widehat{c}_*(s) ds \mid \mathcal{F}_t^S] - E^{\widehat{Q}}[\int_0^T \widehat{c}_*(s) ds].$$

Here,

$$\widehat{y}_* = \Big(x_0 - \frac{z_0}{\alpha-\delta}\Big(e^{-(\alpha-\delta)T} - 1\Big)\Big)^{p-1} \times \Big(\int_0^T \Big[E[\widehat{\eta}_t^{\frac{p}{p-1}}]\Big(\frac{\alpha}{\alpha-\delta} - \frac{\delta}{\alpha-\delta} e^{-(\alpha-\delta)(T-t)}\Big)^{\frac{1}{p-1}} + \int_0^t \delta e^{-(\alpha-\delta)(t-s)} E[\widehat{\eta}_s^{\frac{p}{p-1}}]\Big(\frac{\alpha}{\alpha-\delta} - \frac{\delta}{\alpha-\delta} e^{-(\alpha-\delta)(T-s)}\Big)^{\frac{1}{p-1}} ds\Big] dt\Big)^{1-p}.$$

5.2.3 The Case of Exponential Preference

In this subsection, we take the exponential form of $u(c-z) = -e^{-(c-z)}$ or $v(x) = -e^{-x}$.

Theorem 5.5: *The following assertions hold:*

- *The optimal consumption rate* $\widehat{c}_*(t)$ *is*

$$\widehat{c}_*(t) = z_0 e^{-\int_0^t(\alpha_u-\delta_u)du} - \ln(\widehat{y}_*)(1+\int_0^t \delta_s e^{-\int_s^t(\alpha_u-\delta_u)du}ds) \\ -\ln(\widehat{\eta}_t\widehat{\phi}_t) - \int_0^t \delta_s \ln(\widehat{\eta}_s\widehat{\phi}_s)e^{-\int_s^t(\alpha_u-\delta_u)du}ds.$$

- *The standard of living* $\widehat{z}_*(t)$ *is*

$$\widehat{z}_*(t) = z_0 e^{-\int_0^t(\alpha_u-\delta_u)du} - \ln(\widehat{y}_*)\int_0^t \delta_s e^{-\int_s^t(\alpha_u-\delta_u)du}ds \\ -\int_0^t \delta_s \ln(\widehat{\eta}_s\widehat{\phi}_s)e^{-\int_s^t(\alpha_u-\delta_u)du}ds.$$

- *The optimal wealth process* $\widehat{X}_t^*$ *is*

$$\widehat{X}_t^* = x_0 + E^{\widehat{Q}}[\int_t^T \widehat{c}_*(s)ds \mid \mathcal{F}_t^S] - E^{\widehat{Q}}[\int_0^T \widehat{c}_*(s)ds].$$

Here,

$$\widehat{y}_* = \exp\Big\{\Big[z_0 E^{\widehat{Q}}\int_0^T e^{-\int_0^t(\alpha_u-\delta_u)du}dt - x_0 \\ -E^{\widehat{Q}}\int_0^T \ln(\widehat{\eta}_t\widehat{\phi}_t)dt - E^{\widehat{Q}}\int_0^T(\int_0^t \delta_s e^{-\int_s^t(\alpha_u-\delta_u)du}\ln(\widehat{\eta}_s\widehat{\phi}_s)ds)dt\Big] \\ \times\Big[E^{\widehat{Q}}\int_0^T(1+\int_0^T \delta_s e^{-\int_s^t(\alpha_u-\delta_u)du}ds)dt\Big]^{-1}\Big\}.$$

For the case with deterministic parameters α_t and δ_t, we have the following corollary.

Corollary 5.5.1: If α_t and δ_t are deterministic, the following assertions hold:

- The optimal consumption rate $\widehat{c}_*(t)$ is

$$\widehat{c}_*(t) = z_0 e^{-\int_0^t(\alpha_u-\delta_u)du} - \ln(\widehat{y}_*)\Big(1+\int_0^t \delta_t e^{-\int_s^t(\alpha_u-\delta_u)du}ds\Big)$$
$$-\Big(\ln(\widehat{\eta}_t)+\ln(1+\delta_t\int_t^T e^{-\int_t^s(\alpha_u-\delta_u)du}ds)\Big)$$
$$-\int_0^t \delta_s e^{-\int_s^t(\alpha_u-\delta_u)du}\Big(\ln(\widehat{\eta}_s)+\ln(1+\delta_s\int_s^T e^{-\int_s^l(\alpha_u-\delta_u)du}dl)\Big)ds.$$

- The optimal standard of living $\widehat{z}_*(t)$ is

$$\widehat{z}_*(t) = z_0 e^{-\int_0^t(\alpha_u-\delta_u)du} - \ln(\widehat{y}_*)\Big(\int_0^t \delta_t e^{-\int_s^t(\alpha_u-\delta_u)du}ds\Big)$$
$$-\int_0^t \delta_s e^{-\int_s^t(\alpha_u-\delta_u)du}\Big(\ln(\widehat{\eta}_s)+\ln(1+\delta_s\int_s^T e^{-\int_s^l(\alpha_u-\delta_u)du}dl)\Big)ds.$$

- The optimal wealth process $\widehat{X}_t^*$ is

$$\widehat{X}_t^* = x_0 + E^{\widehat{Q}}[\int_t^T \widehat{c}_*(s)ds \mid \mathcal{F}_t^S] - E^{\widehat{Q}}[\int_0^T \widehat{c}_*(s)ds].$$

Here,

$$\widehat{y}_* = \exp\Big\{\Big[z_0\int_0^T e^{-\int_0^t(\alpha_u-\delta_u)du}dt - x_0$$
$$-\int_0^T E^{\widehat{Q}}\ln(\widehat{\eta}_t)dt - \int_0^T\Big(\ln(1+\delta_t\int_t^T e^{-\int_t^s(\alpha_u-\delta_u)du}ds)\Big)dt$$
$$-\int_0^T(\int_0^t \delta_s e^{-\int_s^t(\alpha_u-\delta_u)du}\Big(E^{\widehat{Q}}\ln(\widehat{\eta}_s)+\ln(1+\delta_s\int_s^T e^{-\int_s^l(\alpha_u-\delta_u)du}dl)\Big)ds)dt\Big]$$
$$\times\Big[\int_0^T(1+\int_0^t \delta_s e^{-\int_s^t(\alpha_u-\delta_u)du}ds)dt\Big]^{-1}\Big\}$$

and $E^{\widehat{Q}}\ln(\widehat{\eta}_t) = \dfrac{1}{2}\displaystyle\int_0^t\Big(\widehat{h}_s^2 + \frac{(\widehat{\Omega}_s+\sigma_S\sigma_\mu\rho)^2}{\sigma_S^4}\int_0^s e^{2\lambda u}du\Big)ds.$

Further, for the case with constant parameters $\alpha_t = \alpha$ and $\delta_t = \delta$, we have the following corollary.

Corollary 5.5.2: If $\alpha_t = \alpha$ and $\delta_t = \delta$ are constant, the following assertions hold:

- The optimal consumption rate $\widehat{c}_*(t)$ is

$$\begin{aligned}\widehat{c}_*(t) = z_0 e^{-(\alpha-\delta)t} - \ln(\widehat{y}_*)\Big(\frac{\alpha}{\alpha-\delta} - \frac{\delta}{\alpha-\delta}e^{-(\alpha-\delta)t}\Big)\\ -\Big(\ln(\widehat{\eta}_t) + \ln(\frac{\alpha}{\alpha-\delta} - \frac{\delta}{\alpha-\delta}e^{-(\alpha-\delta)(T-t)})\Big)\\ -\int_0^t \delta e^{-(\alpha-\delta)(t-s)}\Big(\ln(\widehat{\eta}_s) + \ln(\frac{\alpha}{\alpha-\delta} - \frac{\delta}{\alpha-\delta}e^{-(\alpha-\delta)(T-S)})\Big)ds.\end{aligned}$$

- The optimal standard of living $\widehat{z}_*(t)$ is

$$\begin{aligned}\widehat{z}_*(t) = z_0 e^{-(\alpha-\delta)t} - \ln(\widehat{y}_*)\Big(\frac{\delta}{\alpha-\delta}(1 - e^{-(\alpha-\delta)t})\Big)\\ -\int_0^t \delta e^{-(\alpha-\delta)(t-s)}\Big(\ln(\widehat{\eta}_s) + \ln(\frac{\alpha}{\alpha-\delta} - \frac{\delta}{\alpha-\delta}e^{-(\alpha-\delta)(T-s)})\Big)ds.\end{aligned}$$

- The optimal wealth process $\widehat{X}_t^*$ is

$$\widehat{X}_t^* = x_0 + E^{\widehat{Q}}[\int_t^T \widehat{c}_*(s)ds \mid \mathcal{F}_t^S] - E^{\widehat{Q}}[\int_0^T \widehat{c}_*(s)ds].$$

Here,

$$\begin{aligned}\widehat{y}_* = \exp\Big\{\Big[\frac{z_0}{\alpha-\delta}(1 - e^{-(\alpha-\delta)T}) - x_0\\ -\int_0^T E^{\widehat{Q}}\ln(\widehat{\eta}_t)dt - \int_0^T\Big(\frac{\alpha}{\alpha-\delta} - \frac{\delta}{\alpha-\delta}e^{-(\alpha-\delta)t}\Big)\\ -\int_0^T(\int_0^t \delta e^{-(\alpha-\delta)(t-s)}\Big(E^{\widehat{Q}}\ln(\widehat{\eta}_s) + \ln(\frac{\alpha}{\alpha-\delta} - \frac{\delta}{\alpha-\delta}e^{-(\alpha-\delta)(T-s)})\Big)ds)dt\Big]\\ \times\Big[\frac{\alpha}{\alpha-\delta}T + \frac{\delta}{(\alpha-\delta)^2}(e^{-(\alpha-\delta)T} - 1)\Big]^{-1}\Big\},\end{aligned}$$

and $E^{\widehat{Q}}\ln(\widehat{\eta}_t) = \frac{1}{2}\int_0^t \Big(\widehat{h}_s^2 + \frac{(\widehat{\Omega}_s + \sigma_S\sigma_\mu\rho)^2}{\sigma_S^4}\int_0^s e^{2\lambda u}du\Big)ds$, here $\widehat{h}_s = \frac{\widehat{\mu}_0}{\sigma_S}e^{-\lambda s} + \frac{\bar{\mu}}{\sigma_S}(e^{\lambda s} - 1)$.

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