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Calculation of Augmentation Terminals

BY

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled **Calculation of Augmentation Terminals** submitted by **John Paul Kmet** in partial fulfillment of the requirements for the degree of **Master of Science.**

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TO MY FAMILY AND FRIENDS

ABSTRACT

Over the last decade there has been considerable effort spent by several authors on the calculation of the augmentation terminal of certain groups. Due to the complexity of the problem, this research has been centered on abelian groups culminating in Alfred Hales giving a description of the augmentation terminal of a finite abelian p-group. This result shows the augmentation terminal to be isomorphic to a particular finite abelian p-group from which the invariants of the group must be calculated in a very tedious and time consuming manner. In this thesis a result is derived which gives the invariants of the augmentation terminal directly from the invariants of the original group.

PREFACE

In Chapter *I* basic definitions and some reduction theorems show how calculating the augmentation terminal of abelian groups can be reduced to calculating the augmentation terminals of finite abelian p-groups.

Chapter *II* breifly presents the complicated result of Hâles refered to above. Also a detailed description of maximal cyclic subgroups is given which is used in Hales' result as well as the main theorem of this thesis.

In Chapter *III* the main result of calculating the invariants of the augmentation terminal in a direct manner, is presented. This chapter also contains several examples illustrating the calculation of the invariants with and without the main theorem.

The last chapter presents a brief history and summary of the work and results accomplished in the non-abelian case.

Regarding notation, results within a chapter are referred to by two digits. A three point reference like (*II*.1.3) refers to (1.3) of Chapter *I*. References from the bibliography are enclosed within square brackets.

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PREREQUISITES

The reader is assumed to be familiar with basic group and ring theory. Unless stated otherwise, N and Z will denote the set of natural numbers and the set of integers respectively. p will denote a prime number.

Throughout this thesis, we will be dealing with the group ring ZG of the group G over the ring of integers Z. ZG is the ring of all formal sums

$$\lambda = \sum_{g \in G} z_g g, \quad z_g \in Z .$$

such that the set $\{g \in G \mid z_g \neq 0\}$ is finite; with the following rules

1) $\sum_{g \in G} z_g g = \sum_{g \in G} y_g g \iff z_g = y_g$ for all $g \in G$,

2) $\sum_{g \in G} z_g g + \sum_{g \in G} y_g g = \sum_{g \in G} (z_g + y_g) g$,

3) $(\sum_{g \in G} z_g g)(\sum_{g \in G} y_g g) = \sum_{g \in G} w_g g$ where $w_g = \sum_{h \in G} z_h y_{h^{-1} g}$.

Note, in this thesis almost all groups G are finite and so these are the natural rules of addition and multiplication.

If e_G is the identity of G then $1 = 1 \cdot e_G$ is the identity of ZG and each $z \in Z$ can be identified with $z \cdot e_G$. Hence $zg = gz$ for all $z \in Z$ and $g \in G$.

Define the map $\epsilon : ZG \to Z$ by

$$\epsilon(\sum_{g \in G} z_g g) = \sum_{g \in G} z_g \in Z.$$

Then ϵ is a homomorphism and its kernel

$$\Delta_Z(G) = \{\lambda = \sum_{g \in G} z_g g \in ZG \mid \sum_{g \in G} z_g = 0\}$$

is called the augmentation ideal of ZG. We sometimes denote $\Delta_Z(G)$ by Δ if the group G is understood. (The underlying ring is always Z.)

CHAPTER 1

Some Reductions of the Problem

The study of augmentation quotients was initiated by Passi in [Passi '68]. He reduced the problem of calculating the augmentation quotients of arbitrary finitely-generated abelian groups to that of calculating the structures of polynomial groups of finite abelian p-groups. In this chapter we shall present some of the reductions to this problem. In Section 1, definitions and some easy results are given. Section 2 shows how to take care of cyclic groups. Finally in the last section some of the reductions are proved.

Section 1 Definitions and some easy results.

1.1 Definition. *Let G be a group and $\Delta = \Delta_Z(G)$ the augmentation ideal of ZG. Define Δ^n for $n = 0, 1, 2, \ldots$, by*

$$\Delta^0 = ZG, \quad \text{and} \quad \Delta^n = \Delta^{n-1}\Delta \quad \text{for all } n \in N.$$

Δ^n are the <u>powers</u> of the ideal Δ.

1.2 Remark. Recall that if I and J are two ideals of ZG then

$$IJ = \left\{\sum x_i y_i, \text{ finite sums } \mid x_i \in I \text{ and } y_i \in J\right\}$$

is also an ideal of ZG.

Therefore, the sequence $\{\Delta^n\}_{n=0}^{\infty}$ is a descending sequence of ideals in ZG. We can now make the following definition.

1.3 Definition. *Let G be a group and Δ the augmentation ideal of ZG. By an <u>augmentation quotient</u> of G we mean the (left) ZG-modules*

$$Q_{n,r}(G) = \Delta^n/\Delta^{n+r}$$

where n and r are any non-negative integers. For simplicity we denote

$$Q_{n,1}(G) = \Delta^n/\Delta^{n+1} \quad by \quad Q_n(G), \qquad \text{and} \qquad Q_{1,n}(G) = \Delta/\Delta^{n+1} \quad by \quad P_n(G).$$

1.4 Remark. The modules $Q_n(G)$ and $P_n(G)$ are called *<u>polynomial groups</u>*.

1.5 Remark. Let $i : Q_n(G) \to P_n(G)$ be the inclusion map and let $proj_n : P_n(G) \to P_{n-1}(G)$ be the projection map. i.e. if $\alpha \in \Delta^n$ and $\beta \in \Delta$ then

$$i(\alpha + \Delta^{n+1}) = \alpha + \Delta^{n+1} \qquad (\text{as } \Delta^n \subseteq \Delta), \text{ and}$$
$$proj_n(\beta + \Delta^{n+1}) = \beta + \Delta^n \qquad (\text{this is well defined as } \Delta^n \supseteq \Delta^{n+1}).$$

Then we have the following exact sequence:

$$0 \longrightarrow Q_n(G) \xrightarrow{\;i\;} P_n(G) \xrightarrow{\;proj_n\;} P_{n-1}(G) \longrightarrow 0$$

Thus, although we are concerned with the augmentation quotients $Q_n(G)$, because of the above exact sequence we should also study the polynomial groups $P_n(G)$. However, because the results that are needed do not require the use of $P_n(G)$, the corresponding results for $P_n(G)$ will be stated without proof. The reader should consult [Passi '79, Chapter viii, Section G] for these.

We shall now prove the following easy but useful results.

1.6 Proposition. *Let G be a group. Then $\Delta = \Delta_Z(G)$ is a free Z-module with the Z-basis $\{g - 1 \mid 1 \neq g \in G\}$.*

Proof. Let $\alpha = \sum_{i=1}^{n} z_i g_i \in \Delta$. Then

$$\begin{aligned}
\alpha &= \sum_{i=1}^{n} z_i[(g_i - 1) + 1] \\
&= \sum_{i=1}^{n} z_i(g_i - 1) + \sum_{i=1}^{n} z_i \\
&= \sum_{i=1}^{n} z_i(g_i - 1) \qquad \text{as } 0 = \epsilon(\alpha) = \sum_{i=i}^{n} z_i.
\end{aligned}$$

Hence, $\{g-1 \mid 1 \neq g \in G\}$ spans Δ. Now suppose $\sum_{i=1}^{n} a_i(g_i - 1) = 0$ for some $g_i \in G\backslash\{1\}$, $a_i \in Z$, the g_i all distinct. Then comparing coffecients of the g's in ZG and using the fact that $\{g \in G\}$ are a basis for ZG we get that all $a_i = 0$. Therefore $\{g - 1 \mid 1 \neq g \in G\}$ is linearly independent and hence is a basis. ∎

1.7 Proposition. *Let G be a cyclic group, $G = \langle x \rangle$. Then $\Delta = ZG(x-1) = \{(x-1)\}$, the ideal generated by $(x - 1)$.*

Proof. Using the augmentation map it is easy to see $ZG(x - 1) \subseteq \Delta$. It suffices to show that $x^i - 1 \in ZG(x - 1)$ $\forall i \in Z\backslash\{0\}$. Now let $i \in N$. Then

$$x^i - 1 = (x^{i-1} + x^{i-2} + \ldots + 1)(x - 1) \in ZG(x - 1), \text{ and this gives}$$
$$x^{-i} - 1 = -x^{-i}(x^i - 1) \in ZG(x - 1).$$

∎

1.8 Corollary. *Let $G = < x >$. Then $\Delta^n = ZG(x-1)^n$ $\forall n \in N$.*

Proof. Follows from above. ∎

1.9 Proposition. *Let G be a finite group. Then $Q_n(G)$ (and $P_n(G)$) are finite $\forall n \geq 1$.*

Proof. Let $|G| = m < \infty$. Thus $x^m = 1$ $\forall x \in G$. Then

$$0 = x^m - 1 = [\sum_{i=0}^{m} \binom{m}{i}(x-1)^i] - 1 = \sum_{i=1}^{m} \binom{m}{i}(x-1)^i,$$

consequently,

$$m(x-1) = -\sum_{i=2}^{m} \binom{m}{i}(x-1)^i \in \Delta^2.$$

Since $x - 1$, $x \in G\backslash\{1\}$ generate Δ (by proposition 1.6) we have $m\Delta \subseteq \Delta^2$. Let $n \in N$. Then

$$m\Delta^n = (m\Delta)\Delta^{n-1} \subseteq \Delta^2\Delta^{n-1} = \Delta^{n+1}.$$

i.e. $m\Delta^n \subseteq \Delta^{n+1}$ $\forall n \in N$. Thus every element of $Q_n(G)$ has finite order. But $Q_n(G)$ is generated by the finite number of elements

$$(x_1 - 1)(x_2 - 1)\ldots(x_n - 1) + \Delta^{n+1}, \quad x_i \in G.$$

Since $Q_n(G)$ is an abelian group generated by a finite number of elements of finite order, we have that $Q_n(G)$ is a finite abelian group. ∎

Section 2 The cyclic case.

We now wish to start examining the augmentation quotients of some groups. Therefore the easiest case, the cyclic group, should be dealt with first. In this section we present a modified version of the proof given in [Passi '68 p127]. First, we shall need the following well-known result.

2.1 Theorem. *If G is an abelian group and Δ its augmentaion ideal, then $G \cong \Delta/\Delta^2$.*

Proof. By proposition 1.6, Δ is a free abelian group with basis $\{g - 1 \mid 1 \neq g \in G\}$. Also, since G is abelian, if we define $\Theta : \Delta \to G$ by $\Theta(g-1) = g$, $\forall g \in G\backslash\{1\}$ then Θ extends linearly to a homomorphism, namely,

$$\Theta(\sum z_g(g-1)) = \prod g^{z_g}.$$

Let $K = kernel(\Theta)$. We need to show $K = \Delta^2$. Let $v, w \in \Delta$, say $v = \sum a_g(g-1)$, $w = \sum b_h(h-1)$. Then

$$vw = \sum a_g b_h (g-1)(h-1) = \sum a_g b_h [(gh-1) - (g-1) - (h-1)]$$

and so

$$\Theta(vw) = \prod (gh)^{a_g b_h} (g)^{-a_g b_h} (h)^{-a_g b_h} = 1 \qquad \text{as } G \text{ is abelian.}$$

Hence $vw \in K$ and by addition we have $\Delta^2 \subseteq K$. Conversely, if $\sum a_g(g-1) \in K$ then

$$1 = \Theta(\sum a_g(g-1)) = \prod g^{a_g}.$$

Hence,

$$\prod (1+x_g)^{a_g} = 1 \qquad \text{where} \quad x_g = g-1.$$

Evaluating the product we get

$$1 + \sum a_g x_g \equiv 1 \mod \Delta^2 \qquad \text{or} \qquad \sum a_g x_g \equiv 0 \mod \Delta^2.$$

This implies that

$$\sum a_g(g-1) \equiv 0 \mod \Delta^2.$$

Hence $K \subseteq \Delta^2$ and so $K = \Delta^2$. ∎

2.2 Remark. The above result is a specialization of the more general result that $\Delta(G)/\Delta^2(G) \cong G/G'$ where $G' = [G,G]$ is the derived subgroup.

2.3 Proposition. *Let G be a group and let $x \in G$.*

$$\textit{Define} \quad \hat{x} = \begin{cases} 1 + x + \ldots + x^{o(x)-1}, & \textit{if } o(x) < \infty; \\ 0, & \textit{otherwise.} \end{cases}$$

Then $\{\alpha \in ZG \mid \alpha(x-1) = 0\} = ZG\hat{x}$.

Proof. Suppose $\alpha(x-1) = 0$ where $\alpha \in ZG$, $\alpha = \sum_{h \in G} a_h h$. Then

$$\begin{aligned} \alpha(x-1) = 0 &\iff (\sum a_h h)(x-1) = 0 \iff \sum a_h(hx) = \sum a_h h \\ &\iff \sum a_{hx^{-1}} h = \sum a_h h \iff a_{hx^{-1}} = a_h \quad \forall h \in G \\ &\iff a_h = a_{hx^i} \quad \forall i \in Z, \quad h \in G. \end{aligned}$$

Suppose $o(x) = \infty$. Then hx^i, $i \in Z$ are all distinct elements of G. So if some $a_h \neq 0$ then all $a_{hx^i} = a_h \neq 0$ and α is not a finite sum; contradiction to $\alpha \in ZG$. Thus $\alpha = 0$. Now suppose $o(x) = m < \infty$. Hence

$$\begin{aligned} \alpha &= a_h h(1 + x + \ldots + x^{m-1}) + a_{h_2} h_2(1 + x + \ldots + x^{m-1}) + \ldots \qquad \text{(finite sum)} \\ &= \beta \hat{x} \qquad \text{for some} \quad \beta \in ZG. \end{aligned}$$

In both cases, if $\alpha(x-1) = 0$ then $\alpha \in ZG\hat{x}$. Conversely, if $\alpha \in ZG\hat{x}$ then $\alpha = \beta\hat{x}$ for some $\beta \in ZG$ and $\alpha(x-1) = \beta\hat{x}(x-1) = 0$ as $\hat{x}(x-1) = 0 \; \forall x \in G$. ∎

We are now ready to prove the main result of this section.

2.4 Theorem. *If $G = < x >$ is a cyclic group, then $Q_n(G) \cong G \; \forall n \in N$.*

Proof. By theorem 2.1 we have $Q_1(G) \cong G$. Thus we need only establish an isomorphism $Q_n(G) \cong Q_1(G) \; \forall n \geq 2$. Since G is cyclic, by corollary 1.8 we have that $\Delta^n = ZG(x-1)^n$ $\forall n \in N$. For $n \geq 2$ consider the additive group homomorphism $f : \Delta \to \Delta^n$ given by

$$f(\beta(x-1)) = \beta(x-1)^n. \qquad (\beta \in Z).$$

(This is well defined and a homomorphism.) Now note that

$$(1) \qquad ZG\hat{x} \cap ZG(x-1) = \{0\}.$$

If $o(x) = \infty$ then $ZG\hat{x} = \{0\}$ and the claim holds. So let $o(x) = m < \infty$ and notice that $(\hat{x})^2 = m\hat{x}$. Let $\alpha \in ZG\hat{x} \cap ZG(x-1)$. Then $\alpha = \alpha_1 \hat{x}$ and $\alpha = \alpha_2(x-1)$ for some $\alpha_1, \alpha_2 \in ZG$. But

$$\alpha_1 \hat{x} m = \alpha\hat{x} = \alpha_2(x-1)\hat{x} = 0.$$

So $\alpha_1 \hat{x} = 0$. Then for $\beta \in ZG$,

$$\begin{aligned}
f(\beta(x-1)) \in \Delta^{n+1} &\iff \beta(x-1)^n = \gamma(x-1)^{n+1} && \text{for some } \gamma \in ZG \\
&\iff [\beta - \gamma(x-1)](x-1)^n = 0 && \text{for some } \gamma \in ZG \\
&\iff [\beta - \gamma(x-1)](x-1)^{n-1} \in ZG\hat{x} && \text{for some } \gamma \in ZG \\
&\iff [\beta - \gamma(x-1)](x-1)^{n-1} = 0 && \text{for some } \gamma \in ZG, \text{ by (1)} \\
&\iff [\beta - \gamma(x-1)](x-1) = 0 && \text{for some } \gamma \in ZG \\
& \qquad \text{(By repeating the argument.)} \\
&\iff \beta(x-1) = \gamma(x-1)^2 && \text{for some } \gamma \in ZG
\end{aligned}$$

Thus f induces a monomorphism $f^* : \Delta/\Delta^2 \longrightarrow \Delta^n/\Delta^{n+1}$. But f is clearly onto and so f^* is an isomorphism. Thus $Q_1(G) \cong Q_n(G)$ and we are done. ∎

Therefore we see that the structure of $Q_n(G)$ for G cyclic is relatively simple. However the calculations and the corresponding result for $P_n(G)$ are extremely complicated. (See [Passi '68].) For future reference it should be noted that $P_n(Z) \cong (Z)^n$.

Section 3 Reduction Theorems.

We shall give some reductions for the computation of $Q_n(G)$.

The first case we examine is when the group is the direct sum of two groups of coprime exponent.

3.1 Definition. *The exponent of a group G is the smallest integer $r > 0$ such that $x^r = 1\ \forall x \in G$, if it exists and ∞ otherwise.*

3.2 Theorem. *Let G and H be groups of finite exponent r and s respectively. Suppose r and s are relatively prime. Then:*

i) $(x-1)(y-1) \in \Delta^i(G \oplus H) \quad \forall\, i \geq 1, x \in G, y \in H,$

ii) $Q_n(G \oplus H) \cong Q_n(G) \oplus Q_n(H),$ and

iii) $P_n(G \oplus H) \cong P_n(G) \oplus P_n(H).$

Proof i) First note that if $x \in G$ then

$$0 = x^r - 1 = ((x-1)+1)^r - 1 = \sum_{j=0}^{r} \binom{r}{j}(x-1)^j - 1$$
$$= \sum_{j=1}^{r} \binom{r}{j}(x-1)^j.$$

Hence:

$$r(x-1) = -\sum_{j=2}^{r} \binom{r}{j}(x-1)^j \in \Delta^2(G) \quad \forall x \in G.$$

Thus $r\Delta(G) \subseteq \Delta^2(G)$. By induction we have $r^{i-1}\Delta(G) \subseteq \Delta^i(G)\ \forall i \geq 2$. Likewise $s^{i-1}\Delta(H) \subseteq \Delta^i(H)\ \forall i \geq 2$. Since r and s are relatively prime we have $(r^{i-1}, s^{i-1}) = 1$ $\forall i \geq 2$. Hence there exists integers u_i and v_i such that

$$r^{i-1}u_i + s^{i-1}v_i = 1 \qquad \forall i \geq 2.$$

Let $i \geq 2$ be fixed and let $u = u_i, v = v_i$. Now if $x \in G$ and $y \in H$, and regarding G and H as actually being contained in $G \oplus H$, we have

$$\begin{aligned}(x-1)(y-1) &= \left(r^{i-1}u + s^{i-1}v\right)\left[(x-1)(y-1)\right]\\ &= u\{r^{i-1}(x-1)\}(y-1) + v(x-1)\{s^{i-1}(y-1)\}\\ &\in \Delta^{i+1}(G \oplus H).\end{aligned}$$

Thus, $(x-1)(y-1) \in \Delta^{i+1}(G \oplus H)\ \forall i \geq 1, x \in G, y \in H$.

ii) Let $f_1 : G \oplus H \to G$ and $f_2 : G \oplus H \to H$ be given by the projection maps

$$f_1(xy) = x \qquad \text{and} \qquad f_2(xy) = y.$$

Consider the obvious homomorphisms:

$$Q_n(f_1) : Q_n(G \oplus H) \to Q_n(G) \qquad \text{and} \qquad Q_n(f_2) : Q_n(G \oplus H) \to Q_n(H)$$

given by

$$Q_n(f_1)(\alpha+\Delta^{n+1}(G\oplus H)) = f_1(\alpha)+\Delta^{n+1}(G) \quad \text{and}$$
$$Q_n(f_2)(\alpha+\Delta^{n+1}(G\oplus H)) = f_2(\alpha)+\Delta^{n+1}(H) \quad \text{for } \alpha\in\Delta^n(G\oplus H).$$

Now consider the homomorphism

$$\Theta : Q_n(G\oplus H)\to Q_n(G)\oplus Q_n(H)$$

given by

$$\Theta(z) = Q_n(f_1)(z)\oplus Q_n(f_2)(z), \qquad \text{for } z\in Q_n(G\oplus H).$$

We need to show that Θ is an isomorphism. Let z be a general element of $Q_n(G\oplus H)$. Then

$$z=\sum_{\substack{x_i\in G, y_i\in H\\ t\in Z}} t(x_1y_1-1)(x_2y_2-1)\cdots(x_ny_n-1)+\Delta^{n+1}(G\oplus H).$$

Using the identity $xy-1=(x-1)+(y-1)+(x-1)(y-1)$ and using i), we have

$$(*)\qquad z=\sum_{\substack{x_i\in G, y_i\in H\\ t\in Z}} t\{(x_1-1)(x_2-1)\cdots(x_n-1)+(y_1-1)(y_2-1)\cdots(y_n-1)\}$$
$$\in\Delta^{n+1}(G\oplus H).$$

Now suppose $z\in kernel(\Theta)$. Then $\Theta(z)=0$ which says $Q_n(f_1)(z)=0$ and $Q_n(f_2)(z)=0$. Applying this to the original z we have

$$\sum t(x_i-1)(x_2-1)\cdots(x_n-1)\in\Delta^{n+1}(G), \text{ and } \sum t(y_i-1)(y_2-1)\cdots(y_n-1)\in\Delta^{n+1}(H).$$

But from (*) this says that $z\in\Delta^{n+1}(G\oplus H)$. Thus Θ is a monomorphism. But Θ is obviously onto and so Θ is an isomorphism. Thus $Q_n(G\oplus H)\cong Q_n(G)\oplus Q_n(H)$.

iii) This is similiar (and easier) than the proof of ii). ∎

Now that we can handle a direct sum of two finite groups of relatively prime orders, it would be nice if we could split off direct factors of infinite cyclic groups. To this end we have the following reduction [Passi '79 p117].

3.3 Theorem. *Let G be an arbitrary group. Then for all $n\ge 1$ we have:*

i) $Q_n(Z\oplus G)\cong Q_n(Z)\oplus Q_n(G)\oplus Q_{n-1}(G)\oplus\cdots\oplus Q_2(G)\oplus Q_1(G)$ *and*
ii) $P_n(Z\oplus G)\cong P_n(Z)\oplus P_n(G)\oplus P_{n-1}(G)\oplus\cdots\oplus P_2(G)\oplus P_1(G)$.

Proof. Define $M=X\oplus G$ where X is an infinite cyclic group with generator x. (Hence $X\cong Z$.) Consider ZM as a group ring of X over ZG. i.e. $ZM=(ZG)X$. Consider the ZG-homomorphism $\eta: ZG\to(ZG)X$ given by $\eta(1)=1x^0$. Let ε be the augmentation map of $(ZG)X$ (so ε maps $x \mapsto 1$). Then it is easy to see that $\varepsilon \circ \eta : ZG \to ZG$ is the identity homomorphism. Hence

$$ZM = (ZG)X = image(\eta) \oplus kernel(\varepsilon) = ZGx^0 \oplus \Delta_{ZG}(X).$$

Now (as in the proof of proposition 1.7) we have

$$\Delta_{ZG}(X) = [(ZG)X](x-1) = ZM(x-1).$$

Hence, we have

$$(*) \qquad ZM = ZG \oplus ZM(x-1)$$

Taking the augmentation of this (in ZM) we have $\Delta(M) = \Delta(G) \oplus ZM(x-1)$. Using $(*)$ and by induction on $n \geq 1$ we have

$$ZM = ZG \oplus ZG(x-1) \oplus ZG(x-1)^2 \oplus \cdots \oplus ZG(x-1)^{n-1} \oplus ZM(x-1)^n \text{ and}$$
$$\Delta(M) = \Delta(G) \oplus ZG(x-1) \oplus ZG(x-1)^2 \oplus \cdots \oplus ZG(x-1)^{n-1} \oplus ZM(x-1)^n.$$

Therefore

$$\Delta^n(M) = \Delta^n(G) + \Delta^{n-1}(G)(x-1) + \Delta^{n-2}(G)(x-1)^2 + \cdots \Delta(G)(x-1)^{n-1} + ZM(x-1)^n.$$

But $\Delta^i(G)(x-1)^{n-i} \subseteq ZG(x-1)^{n-i}$ $\forall i = 1,2,\ldots,n$ and since ZM is a direct product of these factors, so is $\Delta^n(M)$. That is

$$\Delta^n(M) = \Delta^n(G) \oplus \Delta^{n-1}(G)(x-1) \oplus \cdots \oplus \Delta(G)(x-1)^{n-1} \oplus ZM(x-1)^n \quad \forall n \geq 1.$$

Let $n \geq 1$ be given. Then

$$\Delta^n(M) = [\bigoplus_{i=0}^{n-1} \Delta^{n-i}(G)(x-1)^i] \oplus ZM(x-1)^n$$
$$= [\bigoplus_{i=0}^{n-1} \Delta^{n-i}(G)(x-1)^i] \oplus ZG(x-1)^n \oplus ZM(x-1)^{n+1}$$

and

$$\Delta^{n+1}(M) = [\bigoplus_{i=0}^{n} \Delta^{n+1-i}(G)(x-1)^i] \oplus ZM(x-1)^{n+1}.$$

Hence,

$$Q_n(M) = \Delta^n(M)/\Delta^{n+1}(M) = [\bigoplus_{i=0}^{n-1} \Delta^{n-i}(G)/\Delta^{n-i+1}(G)] \oplus ZG/\Delta(G) \oplus ZM/ZM$$
$$= [\bigoplus_{i=0}^{n-1} Q(G)] \oplus ZG/\Delta(G).$$

Since $ZG/\Delta(G) \cong Z$ we have

$$Q_n(M) \cong Q_n(G) \oplus Q_{n-1}(G) \oplus \cdots \oplus Q_1(G) \oplus Z.$$

But, by theorem 2.4 we have $Q_n(Z) \cong Z$. Hence we have

$$Q_n(M) \cong Q_n(Z) \oplus Q_n(G) \oplus Q_{n-1}(G) \oplus \cdots \oplus Q_1(G).$$

In a very similiar manner we get ii). ∎

3.4 Remark. Passi also calculated the augmentation quotients for the following two types of groups:

i) If $n \geq (m-1)(p-1)+1$ then $Q_n(Z_p^{(m)}) \cong Z_p^{\left(\frac{1-p^m}{1-p}\right)}$ for $m \geq 1$.

ii) If F is the free group on a set X then $Q_n(F)$ is the free abelian group on the set of all cosets $a_n + \Delta^{n+1}(F)$ where a_n runs through all elements of the type

$$(x_1-1)(x_2-1)\cdots(x_n-1) \text{ where } x_1, x_2, \ldots, x_n \in F.$$

We are now ready to handle the case of abelian groups.

CHAPTER 2

The Abelian Case

Let G be a finite abelian group. Then by the Fundamental Theorem on Finite Abelian Groups [Herstein '75], $G = G_1 \oplus H$ where G_1 is a finite direct sum of cyclic p-groups and H is a finite direct sum of groups whose exponent q is relatively prime to p. By theorem I.3.2 we have $Q_n(G) \cong Q_n(G_1) \oplus Q_n(H)$. Hence it is sufficient to study finite abelian p-groups.

In this chapter, for a finite abelian p-group, we will define a group Q_G which will be isomorphic to $Q_n(G)$ for all n large enough. From this group Q_G we can then find the invariants of $Q_n(G)$ for large n in a straight forward but time-consuming manner. In Chapter 3 we will present a theorem for determining these invariants. Right now, we shall use the results of [Hales '83] and [Hales '85] to show the isomorphism $Q_G \cong Q_n(G)$ for large n.

In section 1, Hales' conjecture will be presented. Section 2 will define and determine some results about maximal cyclic subgroups. These results will be needed for both Chapters 2 and 3. Section 3 will present a condensed proof of Hales' conjecture which has been included for completeness. This section is very detailed and can be skimmed over without affecting the understanding of Chapter 3.

Section 1 *The conjecture.*

The following result of Bachmann and Grunenfelder [Bachmann and Grunenfelder '74] (stated for nilpotent groups) is an extension of Passi [Passi '68].

1.1 Theorem. *Let G be a finite nilpotent group of class c.*
Let $\bar{c}$ = least common multiple of $\{1, 2, \ldots, c\}$. Then there exist positive integers N and π such that π divides $\bar{c}$ and $Q_{n+\pi}(G) \cong Q_n(G)$ for all $n \geq N$.

Note that an abelian group is nilpotent of class 1. Hence for a finite abelian p-group G there exists a $n_0(G)$ such that

$$Q_{n_0}(G) \cong Q_{n_0+1}(G) \cong \cdots$$

as abelian groups. This leads to the following definition.

1.2 Definition. *Let G be a finite abelian p-group. Let $n_0(G)$ be as above. Let $Q_\infty(G) \cong Q_{n_0}(G)$. Then $Q_\infty = Q_\infty(G)$ is the augmentation terminal of G.*

We can now define Q_G.

1.3 Definition. *Let G be a finite abelian p-group. Define an abelian group Q_G by the following generators and relations. Let P_G denote the poset of cyclic subgroups H of G. (So P_G is a tree with root $H_0 = \{1\}$.) Take one generator x_H for each $H \in P_G$, and relations $px_H = x_K$ whenever H covers K in P_G (i.e. $K \subsetneq H$ and $[H : K] = p$). Also define $x_{H_0} = 0$.*

1.4 Example. Let $G = < x >$ where $o(x) = p^m$. Then define $H_0 = < 1 >$ and $H_i = < x_{H_i} >$ where $x_{H_i} = x^{p^{m-i}}$ for $i = 1, 2, \ldots, m$. Thus

$$P_G = \{H_i \mid i = 0, 1, \ldots, m\} \text{ and}$$
$$Q_G = \{x_{H_i} \mid i = 0, 1, \ldots, m, \; px_{H_i} = x_{H_{i-1}}\}.$$

This gives $Q_G = \{x_{H_m} \mid p^m x_{H_m} = 0\} \cong Z_{p^m} \cong G$. Note by theorem *I.2.4* we had $Q_\infty(< x >) \cong < x >$. So in this case $Q_\infty(< x >) = Q_{<x>}$.

Partially motivated by the above example, A. Hales stated the following conjecture [Hales '83]:

1.5 Conjecture. *Let G be a finite abelian p-group. Then $Q_G \cong Q_\infty(G)$.*

The rest of the chapter will be concerned with proving the above conjecture.

Section 2 *Maximal Cyclic Subgroups.*

As the above hypothesis depends extensively on knowing what the structure of Q_G is, we should have a good understanding of the maximal cyclic subgroups in a finite abelian group. In this section we present an inductive way of creating the generators of such groups.

2.1 Definition. *Let G be a group and H a cyclic subgroup of G. Then H is a maximal cyclic subgroup (M.C.S) of G if there does not exist any cyclic subgroup N of G such that $H \subset N$ (proper containment). The set of all maximal cyclic subgroups is denoted by $MCS(G)$.*

It should be noted that the only M.C.S. of a cyclic group G is G itself. This slight variation from the standard meaning of maximal should not cause the reader many problems.

Also every cyclic subgroup of G is contained in a M.C.S. Hence for the rest of this section we will only be concerned with the maximal cyclic subgroups of G.

2.2 Remark. Throughout this section (unless otherwise specified) we let G be a finite abelian p-group with the form

$$G = \prod_{i=1}^{m} < g_i > = < g_1 > \times < g_2 > \times \cdots \times < g_m >$$

where $m \geq 1$, $o(g_i) = p^{e_i}$ for $i = 1, 2, \ldots, m$, and $e_1 \geq e_2 \geq \cdots \geq e_m \geq 1$.

2.3 Proposition. *Let G be as in remark 2.2. Let $x = (g_1^{\beta_1}, g_2^{\beta_2}, \dots, g_m^{\beta_m})$ where $0 \leq \beta_i < p^{e_i}$ for $i = 1, 2, \dots, m$.*

a) $< x > \in MCS(G)$ if and only if p does not divide β_i for some $i \in \{1, 2, \dots, m\}$.

b) If $H \in MCS(G)$ then $p^{e_m} \leq |\, H \,| \leq p^{e_1}$.

<u>*Proof.*</u> a) ($\Longrightarrow$). For contradiction purposes suppose $p \mid \beta_i$ for all i.
Let $y = (g_1^{\beta_1/p}, g_2^{\beta_2/p}, \dots, g_m^{\beta_m/p})$. If $\beta_i = 0$ for all i then $x = (1, 1, \dots, 1)$ and so $< x > \notin MCS(G)$ as $|\, G \,| > 1$ and so there exists a $y \in G \backslash \{x\}$ and $< x > \subset < y >$. If some $\beta_i \neq 0$ then $x \neq y$ (defined as above) and $x = y^p$. Hence $< x > \subset < y >$ as $o(x) = o(y^p) = \frac{o(y)}{p} < o(y)$. In either case we get a contradiction.
($\Longleftarrow$). Suppose $< x >$ is not a M.C.S. Then there exists $y = (g_1^{\alpha_1}, \cdots, g_m^{\alpha_m})$ such that $< x > \subset < y >$. Hence $x = y^j$ for some $0 \leq j < o(y)$. Suppose $(j, p) = 1$. Then $(j, o(y)) = 1$ and so there exist integers s and t such that $sj + to(y) = 1$. i.e. $x^s = y^{sj} = y^{(1 - to(y))} = y$ implies that $< y > \subseteq < x >$, which is a contradiction. Thus $x = y^{pk}$ for some $0 \leq k < \frac{o(y)}{p}$. Therefore $\beta_i = pk\alpha_i \pmod{p^{e_i}}$ for $i = 1, 2, \dots, m$ and $p \mid \beta_i \quad \forall i$. This is a contradiction. Hence $< x >$ is a M.C.S.

b) Let $H = < x >$ where $x = (g_1^{\beta_1}, \dots, g_m^{\beta_m})$ for some $0 \leq \beta_i < p^{e_i}$. Then

$$x^{p^{e_1}} = ((g_1^{p^{e_1}})^{\beta_1}, (g_2^{p^{e_2}})^{p^{e_1 - e_2}\beta_2}, \dots, (g_m^{p^m})^{p^{e_1 - e_2}\beta_m})) = (1, 1, \dots, 1).$$

i.e. $|\, H \,| \leq p^{e_1}$. Now suppose $|\, H \,| < p^{e_m}$. Then

$$(1, 1, \dots, 1) = x^{p^{e_m - 1}} = (g_1^{\beta_1 p^{e_m - 1}}, \dots, g_m^{\beta_m p^{e_m - 1}}).$$

Therefore,

$$\beta_i p^{e_m - 1} \equiv 0 \pmod{p^{e_i}} \text{ for } 1 \leq i \leq m,$$
$$\beta_i \equiv 0 \pmod{p^{e_i - e_m + 1}} \text{ for } 1 \leq i \leq m,$$
$$\beta_i \equiv 0 \pmod{p} \text{ for } 1 \leq i \leq m \text{ as } e_i - e_m \geq 0.$$

Therefore $p \mid \beta_i$ for $1 \leq i \leq m$ and part a) gives us that H is not a M.C.S. Contradiction. Hence $p^{e_m} \leq |\, H \,| \leq p^{e_1}$. ∎

2.4 Definition. *Let G be as in remark 2.2 and let $x = (g_1^{\beta_1}, \dots, g_m^{\beta_m})$ be a generator of a maximal cyclic subgroup of G. Let $o(x) = p^l$. Choose $1 \leq j \leq m$ minimally such that $o(g_j^{\beta_j}) = p^l$. Let $0 < k < p^{e_j}$ be such that $g_j^{\beta_j k} = g_j^{p^{e_j - l}}$. Define $y = x^k$. Then $< y > = < x >$ and y is called the <u>standardization of x</u> or y is <u>standardized</u>. j is called the <u>index of standardization</u>.*

2.5 Remark. The above definition makes sense. By proposition 2.3 we have $1 \leq e_m \leq l$ and since $o(g_j) = p^{e_j}$ we have $l \leq e_j$. Hence

$$o(g_j^{\beta_j}) = p^l \quad \Longrightarrow \quad \beta_j p^l \equiv 0 \pmod{p^{e_j}}$$
$$\Longrightarrow \quad \beta_j \equiv 0 \pmod{p^{e_j - l}}$$
$$\Longrightarrow \quad \beta_j = \beta p^{e_j - l} \text{ for some } 0 < \beta < p^{e_j - (e_j - l)} = p^l.$$

Also if $p \mid \beta$ then

$$\beta_j p^{l-1} = (\frac{\beta}{p})p^{e_j-l+1}p^{l-1} = (\frac{\beta}{p})p^{e_j} \equiv 0 \pmod{p^{e_j}}$$

$$\Longrightarrow \quad o(g_j^{\beta_j}) \le p^{l-1}. \quad \text{Contradiction.}$$

Hence $\beta_j = \beta p^{e_j-l}$ where $(\beta, p) = 1$ or $(\beta, p^l) = 1$. Now there exist integers s and t such that $s\beta + tp^l = 1$. Then

$$(g_j^{\beta_j})^s = g_j^{(\beta s)p^{e_j-l}} = g_j^{(1-tp^l)p^{e_j-l}} = g_j^{p^{e_j-l}} g_j^{-tp^{e_j}} = g_j^{p^{e_j-l}}.$$

Let $k \equiv s \pmod{p^{e_j}}$. Then $g_j^{\beta_j k} = g_j^{p^{e_j-l}}$ and $< y > = < x^k > \subseteq < x >$.
But $o(y) \ge o(g_j^{p^{e_j-l}}) = p^l = o(x)$. i.e. $< y > = < x >$.

2.6 Remark. Let H be a M.C.S. of G and suppose $< x_1 > = < x_2 > = H$. Let y_1 and y_2 be the standardizations of x_1 and x_2 respectively. Let $\mid H \mid = p^l$ and let $x_1 = (g_1^{\alpha_1}, \ldots, g_m^{\alpha_m})$ and $x_2 = (g_1^{\beta_1}, \ldots, g_m^{\beta_m})$. Since $< x_1 > = < x_2 >$ we must have

$$min\{i \in \{1, \ldots, m\} \mid o(g_i^{\alpha_i}) = p^l\} \quad = \quad min\{i \in \{1, \ldots, m\} \mid o(g_i^{\beta_i}) = p^l\} \quad = \quad j.$$

Hence

$$y_1 = (y'_1, \ldots, y'_{j-1}, g_j^{p^{e_j-l}}, y'_{j+1}, \ldots, y'_m) \text{ and } y_2 = (y''_1, \ldots, y''_{j-1}, g_j^{p^{e_j-l}}, y''_{j+1}, \ldots, y''_m).$$

Since $< y_1 > = < y_2 > = H$ and $o(g_j^{p^{e_j-l}}) = p^l = \mid H \mid$ we have $y_1 = y_2$.

Thus if H is a M.C.S. of G then the standardization of H is $< y >$ where y is the standardization of any $x \in H$ such that $< x > = H$.

2.7 Definitions. *Let G be as in remark 2.2 and assume $m \ge 2$. We define $e_1 - e_m + 2$ classes of maximal cyclic subgroups $T_0(G), T_1(G), \ldots, T_{e_1-e_m}(G)$, and $S(G)$ of G as follows:*

For each i, $0 \le i \le e_1 - e_m$, let $G_i = \prod_{i=1}^{m_i} < g_i >$ where $m_i < m$ is chosen maximal so that $e_{m_i} \ge e_m + i$.

For $0 \le i \le e_1 - e_m$ define $T_i(G)$ as the set of maximal cyclic subgroups of G, $< (g_1^{\beta_1}, \ldots, g_m^{\beta_m}) >$ where $< (g_1^{\beta_1}, \ldots, g_{m_i}^{\beta_{m_i}}) >$ is a M.C.S. of $G_i^{p^i}$; $0 \le \beta_m < p^{e_m}$; and for $m_i < j < m$ we have $o(g_j^{\beta_j}) < p^{e_m}$.

Also define $S(G)$ as the set of M.C.S. $< (g_1^{\beta_1}, \ldots, g_m^{\beta_m}) >$ where $o(g_j^{\beta_j}) < p^{e_m}$ for all $1 \le j \le m-1$. (If G is understood then let $S = S(G)$ and $T_i = T_i(G)$).

For future reference, let $x = (g_1^{\beta_1}, \ldots, g_{m_i}^{\beta_{m_i}})$ and note if $< x > \in MCS(G_i^{p^i})$ is standardized and if $< (g_1^{\beta_1}, \ldots, g_m^{\beta_m}) > \in MCS(G)$ (as derived above) then it is also standardized. This is because (using proposition 2.3)

$$o(x) \ge p^{e_m-i} \ (x \in G^{p^i}) \quad \Longrightarrow \quad o(x) \ge p^{e_m} \text{ (if } x \text{ is considered in } G).$$

Since $o((g_{m_i+1}^{\beta_{m_i+1}}, \ldots, g_m^{\beta_m})) \le p^{e_m}$ by definition, we have that both the old and new elements have the same index of standardization and so if x is already standardized so will be the new element.

2.8 Proposition. *$T_0, T_1, \ldots, T_{e_1-e_m}$, and S partition the set of maximal cyclic subgroups of G.*

Proof. Let $x = (g_1^{\beta_1}, \ldots, g_m^{\beta_m})$ generate a M.C.S of G. If $o(g_j^{\beta_j}) < p^{e_m}$ for all $1 \leq j \leq m-1$ then $< x > \in S$ by definition. Now suppose $< x > \notin S$. Then we can define

$$\text{and} \qquad \begin{aligned} i_0 &= min\{e_j - l_j \mid o(g_j^{\beta_j}) = p^{l_j} \text{ where } 0 \leq j \leq m-1 \text{ and } l_j \geq e_m\} \\ j_0 &= max\{i \in \{1, \ldots, m-1\} \mid e_i - l_i = i_0,\, l_i \geq e_m\}. \end{aligned}$$

Claim $< x > \in T_{i_0}$.
Now $m_{i_0} < m$ is chosen maximal so that $e_{m_{i_0}} \geq e_m + i_0$. Also j_0 is chosen maximal so that $i_0 = e_{j_0} - l_{j_0}$ and $l_{j_0} \geq e_m$. Hence

$$e_{j_0} = i_0 + l_{j_0} \geq i_0 + e_m \qquad \Longrightarrow \qquad j_0 \leq m_{i_0} \text{ as } m_{i_0} \text{ chosen maximally.}$$

Now suppose there exists $m_{i_0} < k < m$ such that $l_k \geq e_m$. Then since $k > m_{i_0} \geq j_0$ and j_0 was chosen maximally such that $i_0 = e_{j_0} - l_{j_0}$, is a minimum, we have

$$e_k - l_k > i_0 \qquad \Longrightarrow \qquad e_k > i_0 + l_k \geq i_0 + e_m \qquad \Longrightarrow \qquad k \leq m_{i_0}$$

a contradiction to $k > m_{i_0}$. Finally note that

$$o(g_{j_0}^{\beta_{j_0}}) = p^{l_{j_0}} = p^{e_{j_0} - i_0} = o(g_{j_0}^{p^{i_0}}) \quad \text{in } G_{i_0}^{p^{i_0}} = \prod_{i=1}^{m_{i_0}} < g_i^{p^{i_0}} >,$$

which implies that

$$(\frac{\beta_{j_0}}{p^{i_0}}, p) = 1 \quad . \quad \text{(Similiar to the proof of prop. 2.3a.)}$$

Hence by proposition 2.3, $< (g_1^{\beta_1}, \ldots, g_{i_0}^{\beta_{i_0}}) >$ is a M.C.S. of $G_{i_0}^{p^{i_0}}$. i.e. $< x > \in T_{i_0}$. Now since i_0 is uniquely defined we have each $T_0, T_1, \ldots, T_{e_1-e_m}$, and S are disjoint. ∎

2.9 Corollary. *With the same notation as above, we have:*

a) $\mid T_0 \mid = p^{e_m} \cdot$ *(number of M.C.S.'s of G_0),*

b) $\mid T_i \mid = (p^{e_m} - p^{e_m - 1}) \cdot (p^{e_m-1})^{m-m_i-1} \cdot$ *(number of M.C.S.'s in $G_i^{p^i}$)*
for $1 \leq i \leq e_1 - e_m$, and

c) $\mid S \mid = (p^{e_m-1})^{m-1}$.

Proof. Let $< (g_i^{\beta_1}, \ldots, g_{m_i}^{\beta_{m_i}}) >$ be a M.C.S. of $G_i^{p^i}$ in standardized form. Let h_j, h'_j be chosen for $m_i + 1 \leq j \leq m$ such that $o(h_j) \leq p^{e_m}$ and $o(h'_j) \leq p^{e_m}$. Let

$$H_1 = < (g_1^{\beta_1}, \ldots, g_{m_i}^{\beta_{m_i}}, h_{j+1}, \ldots, h_m) > \quad \text{and} \quad H_2 = < (g_1^{\beta_1}, \ldots, g_{m_i}^{\beta_{m_i}}, h'_{j+1}, \ldots, h'_m) > .$$

Note H_1 and H_2 are standardized as $< (g_1^{\beta_1}, \ldots, g_{m_i}^{\beta_{m_i}}) >$ was standardized in $G_i^{p^i}$. Hence $H_1 = H_2$ iff $h_j = h_j'$ for all $m_i + 1 \leq j \leq m$, by remark 2.6. Thus we can choose each h_j for $m_i < j < m$ arbitrary such that $o(h_j) < p^{e_m}$. Also h_m can be chosen arbitrarily except that if $i \geq 1$ then $o(h_m) = p^{e_m}$ as otherwise by proposition 2.3a) we do not have a maximal cyclic subgroup. Note there are $(p^{e_m} - p^{e_m - 1})$ h_m's such that $o(h_m) = p^{e_m}$.

a) Thus $| T_0 | = p^{e_m} \cdot$ (number of M.C.S.'s of G_0), as $m_0 = m - 1$ and h_m can be chosen arbitrarily.

b) For $1 \leq i \leq e_1 - e_m$ we have

$| T_i | = (p^{e_m} - p^{e_m - 1}) \cdot (p^{e_m - 1})^{m - m_i - 1}$ · (number of M.C.S.'s in $G_i^{p^i}$) from above, and

c) $| S | = (p^{e_m - 1})^{m-1}$ as if $< x > \in S$ then $x = (g_1^{\beta_1}, \ldots, g_m^{\beta_m})$ where $o(g_i^{\beta_i}) < p^{e_m}$ for $i = 1, 2, \ldots, (m-1)$ and $o(g_m^{\beta_m}) = p^{e_m}$ since $< x >$ is a M.C.S. But x is standardized implies $\beta_m = 1$ as $o(g_m) = o(x)$. ∎

We now have a recursive method for calculating the generators of all maximal cyclic subgroups. For illustrative purposes, we now will calculate the standardized generators for a small class of abelian groups.

2.10 Example. Let $G = < g > \times < h >$ where $o(g) = p^{e_1}$, $o(h) = p^{e_2}$ and $e_1 \geq e_2$. Hence, in our previous notation, $m = 2$. thus $G_0 = G_1 = \cdots = G_{e_1 - e_2} = < g >$ as $1 \leq m_i < 2$ for all $0 \leq i \leq e_1 - e_2$. Note $< g^{p^i} >$ is the only M.C.S. of $< g^{p^i} >$ and is in standardized form. Hence

$$T_0 = \{ < (g, h^j) > \mid 0 \leq j < p^{e_2} \}$$

$$T_i = \{ < (g^{p^i}, h^j) > \mid 0 \leq j < p^{e_2}, p \text{ does not divide } j \}$$

$$\text{for } 1 \leq i \leq e_1 - e_2, \text{ and}$$

$$S = \{ < (g^{(p^{e_1 - e_2 + 1})j}, h) > \mid 0 \leq j < p^{e_2 - 1} \}.$$

Section 3 *Hales' Result.*

This section is concerned with showing the proof of Hales' conjecture (1.5). For brevity and because of the complexity involved in the proof of the result, we will not show the entire proof. The complete result can be found in [Hales '83] and [Hales '85].

The following two results provide the punch for the proof of the result. They are by no means obvious but quite complex and detailed. They are respectively lemma 1 and lemma 2 as found in [Hales '85].

3.1 Lemma. *Let $G = < g > \times < h >$ where g and h have orders p^m and p^n respectively with $m \geq n$. In the group ring ZG let $x = g - 1$ and $y = h - 1$. Then for each k, $0 \leq k < n$, we have that*

$$p^k y^{(m-n)(p^{n-k} - p^{n-k-1})} \left(x^{p^{n-k}} y^{p^{n-k-1}} - x^{p^{n-k-1}} y^{p^{n-k}} \right)$$

lies in

$$\Delta^{(m-n)(p^{n-k}-p^{n-k-1})+p^{n-k}+p^{n-k-1}+1}.$$

3.2 Lemma. *Let $G = \langle g \rangle \times \langle h \rangle \times \langle a_1 \rangle \times \cdots \times \langle a_r \rangle$ where $r \geq 0$, $o(g) = p^m$, $o(h) = p^n$, and $o(a_i) = p^{e_i}$ for $i = 1, 2, \ldots, r$. Also assume $m \geq n = e_0 > e_1 > \cdots > e_r \geq 1$. In ZG define $x = g - 1$, $y = h - 1$, and $z_i = a_i - 1$ for each i. Suppose the positive integers $K = \{k, k_1, \ldots, k_r\}$ exists such that $0 \leq k < e_r$, and for each i with $1 \leq i \leq r$ that k_i satisfies $e_i + k_i \leq e_{i-1}$. Define for each i*

$$d_i = d_i(G, K) = e_i - \sum_{j=i+1}^{r} k_j - k, \text{ and}$$

$$s_i = s_i(G, K) = p^{d_i - 1} + k_i(p^{d_i} - p^{d_i - 1}).$$

Also let $q = q(K) = \sum_{i=1}^{r} k_i + k$. Then

$$p^k \cdot \Theta_{m,n,q} \cdot z_1^{s_1} \cdot z_2^{s_2} \cdots z_r^{s_r}$$

lies in Δ^l, where

$$\Theta_{m,n,q}(G) = \Theta_{m,n,q} = y^{(m-n)(p^{n-q}-p^{n-q-1})}(x^{p^{n-q}} y^{p^{n-q-1}} - x^{p^{n-q-1}} y^{p^{n-q}}) \text{ and}$$

$$l(G,k) = l = (m-n)(p^{n-q} - p^{n-q-1}) + p^{n-q} + p^{n-q+1} + \sum_{i=1}^{r} s_i + 1.$$

3.3 Remark. Lemma 3.2 is the multivariable equivalent of lemma 3.1.

These two lemmas give identities which hold in the augmentation quotients $Q_N(G)$ for N large enough. The way these lemmas will be used is as follows.

3.4 Lemma. *Let G be defined as in lemma 3.2 (except we require $e_r > 1$). Suppose $K' = \{k, k_1, \ldots, k_r\}$ exists such that G^p and K' satisfy the conditions of lemma 3.2. If $K = \{k + 1, k_1, \ldots, k_r\}$ then G and K satisfy lemma 3.2 and the identity satisfied by G and K is p times the identity satisfied by G^p and K'.*

Proof. First since G^p and K' satisfy 3.2 we have $0 \leq k < e_r - 1$ implies $0 \leq k + 1 < e_r$. Also for $i = 1, \ldots, r$

$$(e_i - 1) + k_i \leq (e_{i-1} - 1) \text{ where } o(a_i^p) = p^{e_i - 1}$$

implies

$$e_i + k_i \leq e_{i-1} \text{ where } o(a_i) = p^{e_i}.$$

Hence G and K satisfy the conditions. Also for $i = 1, \ldots, r$

$$d'_i = d_i(G^p, K') = (e_i - 1) - (\sum_{j=i+1}^{r} k_j) - k = e_i - (\sum_{j=i+1}^{r} k_j) - (k+1) = d_i(G, K) = d_i,$$

and

$$s'_i = s_i(G^p, K') = p^{d'_i - 1} + k_i(p^{d'_i} - p^{d'_i - 1}) = s_i(G, K) = s_i.$$

Finally $q(K') = \sum_{i=1}^{r} k_i + k = \sum_{i=1}^{r} k_i + (k+1) - 1 = q(K) - 1$.
Note $(m-1) + (n-1) = m - n$ and $(n-1) - q(K') = n - 1 - (q(K) - 1) = n - q(K)$. Thus the powers in $\Theta_{m-1,n-1,q(K')}(G^p)$ are the same as those powers in $\Theta_{m,n,q(K)}(G)$, and $l(G^p, K') = l(G, K)$. Since $p \cdot p^k = p^{k+1}$ we have the result. ∎

3.5 Lemma. *Let G be defined as in lemma 3.2. Suppose $K' = \{k, k_1, \ldots, k_r\}$ and $0 \le k_{r+1}$ exists such that $G^{p^{k_{r+1}}}$ and K' satisfy the conditions of lemma 3.2. Also let $\bar{G} = G \times < a_{r+1} >$ where $o(a_{r+1}) = p^{e_{r+1}}$, $z_{r+1} = a_{r+1} - 1$, and $\bar{K} = \{k, k_1, \ldots, k_r, k_{r+1}\}$. If all cyclic factors of $G^{p^{k_{r+1}}}$ have order $\ge p^{e_{r+1}}$ then $\bar{G}$ and $\bar{K}$ satisfy lemma 3.2 and the identity satisfied by $\bar{G}$ and $\bar{K}$ is*

$$z_{r+1}^{p^{e_{r+1}-1-k} + k_{r+1}(p^{e_{r+1}-k} - p^{e_{r+1}-1-k})}$$

times the identity satisified by $G^{p^{k_{r+1}}}$ and K'.

<u>*Proof.*</u> Since $G^{p^{k_{r+1}}}$ and K' satisfy lemma 3.2 we have for $i = 1, \ldots, r$

$$(e_i - k_{r+1}) + k_i \le (e_{i-1} - k_{r+1}) \Rightarrow e_i + k_i \le e_{i-1}.$$

Since all factors of $G^{p^{k_{r+1}}}$ have order $\ge p^{e_{r+1}}$, we also have $e_r - k_{r+1} \ge e_{r+1}$ implies $e_{r+1} + k_{r+1} \le e_r$. Thus $\bar{G}$ and $\bar{K}$ satisfy lemma 3.2. Also

$$d'_i = d_i(G^{p^{k_{r+1}}}, K') = (e_i - k_{r+1}) - \sum_{j=i+1}^{r} k_j - k = e_i - \sum_{j=i+1}^{r+1} k_j - k = d_i(\bar{G}, \bar{K}) = \bar{d}_i$$

for $i = 1, \ldots, r$ and $\bar{d}_{r+1} = e_{r+1} - k$. Thus

$$s'_i = s_i(G^{p^{k_{r+1}}}, K') = s_i(\bar{G}, \bar{K}) = \bar{s}_i \text{ for } i = 1 \ldots, r, \text{ and}$$

$$\bar{s}_{r+1} = z_{r+1}^{p^{e_{r+1}-1-k} + k_{r+1}(p^{e_{r+1}-k} - p^{e_{r+1}-1-k})}.$$

Since

$$\bar{q} = q(\bar{G}) = \sum_{i=1}^{r+1} k_i + k = (\sum_{i=1}^{r} k_i + k) + k_{r+1} = q(G') + k_{r+1} = q' + k_{r+1},$$

this gives the result (as in the proof of lemma 3.4). ∎

Let G be a finite abelian p-group. In order to show $Q_\infty \cong Q_G$, the first thing we need to show is that $| Q_\infty | = | Q_G |$.

3.6 Theorem. *Let G be a finite abelian p-group. Then $| Q_\infty(G) | = | Q_G |$.*

Proof. Singer showed in [Singer '77b] that the order of $Q_\infty(G)$ is p^{c-1}, where c is the number of components of the rational group algebra QG. But c is also the number of cyclic subgroups of G, including $< 1 >$. (See [Curtis and Riener '62 §39].) Finally by the definition of $Q_\infty(G)$ it is easy to see that its order is p^{c-1}. This is because Q_G has one generator x_H for every cyclic subgroup H of G, and the relations $px_H = x_K$ whenever H covers K in P_G. Hence there are $c-1$ extensions $(H \in P_G - \{< 1 >\}) < x_H > \supset < x_K >$ of index p. Thus $| Q_\infty(G) | = | Q_G | = p^{c-1}$ where c is the number of cyclic subgroups of G. ∎

This next lemma provides the stepping stone to be used in the proof of the conjecture.

3.7 Lemma. *Let $G = \prod_{i=1}^m < g_i >$ where $o(g_i) = p^{e_i}$ with $e_1 \geq e_2 \geq \cdots \geq e_m \geq 1$. Suppose for $k = 0, 1, \ldots, e_1 - 1$ that $p^k Q_N(G)$ (for large N) can be generated by the number of elements required to generate $p^k Q_G$. Then $Q_\infty \cong Q_G$.*

Proof. Recall by proposition 2.3 that any maximal cyclic subgroup of G has order $\leq p^{e_1}$. Hence any cyclic subgroup of G has order $\leq p^{e_1}$ and so by definition 1.3 of Q_G it is easy to see that the exponent of Q_G is p^{e_1}. Also for $i = 1, \ldots, m$

$$p^{e_1}(g_i - 1) = p^{e_1 - e_i}(p^{e_i}(g_i - 1)) \in p^{e_1 - e_i}\Delta^2(G)$$

(as $p^{e_i}(g_i - 1) = -\sum_{j=2}^{p^{e_i}} \binom{p^{e_i}}{j}(g_i - 1)^j$).
Hence $p^{e_1}\Delta^n \subseteq \Delta^{n+1}$ implies the exponent of $Q_\infty \leq p^{e_1}$. As both Q_G and Q_∞ are abelian and have exponent $\leq p^{e_1}$ we may write

$$Q_\infty \cong \prod_{i=1}^{e_1} (Z_{p^i})^{a_i}, \text{ and } Q_G \cong \prod_{i=1}^{e_1} (Z_{p^i})^{b_i}$$

for some a_i's and b_i's non-negative integers. Note that $p^k Q_\infty$ and $p^k Q_G$ can be generated by $a_{k+1} + a_{k+2} + \cdots + a_{e_1}$ and $b_{k+1} + b_{k+2} + \cdots + b_{e_1}$ elements respectively.
By hypothesis, $p^k Q_N(G)$ (for large N) can be generated by the number of elements needed to generate $p^k Q_G$. But $p^k Q_G$ is generated by $b_{k+1} + b_{k+2} + \cdots + b_{e_1}$ elements (the remaining cyclic components under the map $\alpha \mapsto p^k\alpha$). Hence

(1) $$a_{k+1} + \cdots + a_{e_1} \leq b_{k+1} + \cdots + b_{e_1}, \text{ for } k = 0, 1, \ldots, e_1 - 1.$$

But

$$| Q_\infty | = p^{a_1} \cdot (p^2)^{a_2} \cdot \cdots \cdot (p^{e_1})^{a_{e_1}} = p^{a_1 + 2a_2 + \cdots + e_1 a_{e_1}}.$$

Likewise

$$| Q_G | = p^{b_1 + 2b_2 + \cdots + e_1 b_{e_1}}.$$

Writing

$$a_1 + 2a_2 + \cdots + e_1 a_{e_1} = (a_1 + \cdots + a_{e_1}) + (a_2 + \cdots + a_{e_1}) + \cdots + (a_{e_1})$$

and similiarly for $b_1 + 2b_2 + \cdots + e_1 b_{e_1}$, then it is clear that $| Q_\infty | = | Q_G |$ (theorem 3.6) and (1), forces all the inequalities to be equalities. Hence $Q_\infty \cong \prod_{i=1}^{e_1} (Z_{p^i})^{a_i} \cong Q_G$. ∎

At last, we are ready to show that $Q_G \cong Q_\infty(G)$.

3.8 Theorem. Hales' Conjecture Proved.
Let G be a finite abelian p-group. Then $Q_G \cong Q_\infty(G)$.

Proof. Let $G = \prod_{i=1}^{m} < g_i >$ where $o(g_i) = p^{e_i}$, and $e_1 \geq e_2 \geq \cdots \geq e_m$. For each i let $x_i = g_i - 1 \in ZG$. By lemma 3.7 it is sufficient to show that, for $k = 0, 1, \ldots, e_1 - 1$ and for large N, $p^k Q_N$ can be generated by the number of elements required to generate $p^k Q_G$. Note that from definition 1.3 it is easy to see that the number of elements required to generate $p^k Q_G$ is the number of maximal cyclic subgroups of G^{p^k}.

Let $k \in \{0, 1, \ldots, e_1 - 1\}$ be given. Then, for any N, $p^k Q_N$ is generated by (the images of) all monomials $p^k \prod_{j=1}^{m} x_j^{\alpha_j}$ with $\sum \alpha_j = N$ (proposition $I.1.6$ and $I.1.7$). We will show that, for large N, there is a subcollection of these monomials which also generates $p^k Q_N$ and whose cardinality is the number of maximal cyclic subgroups of G^{p^k}. We also show that the identities of lemma 3.2 suffice to give the appropriate collection of monomials which generates $p^k Q_N$. We proceed by induction on m.

If $m = 1$ then $G = < g_1 >$ and $\Delta^n(G) = ZG(g_1 - 1)^n$ (by corollary $I.1.8$) for all $n \in N$. Hence $p^k Q_N$ is generated by the one element $p^k(g_1 - 1)^N + Q_{N+1}$.
But $MCS(G^{p^k}) = \{< g_1^{p^k} >\}$ has only one element and so the $m = 1$ case is true.

Let $m > 1$ and assume that our induction hypothesis holds for all finite abelian p-groups G' with rank $< m$. That is, for large N, $p^k Q_N(G')$ can be generated by the number of elements required to generate $p^k Q_{G'}$, and the appropriate set of monomials which generates $p^k Q_N(G')$ has been found through the identities of lemma 3.2.

First, recall definition 2.7. We have that

$$MCS(G^{p^k}) = S(G^{p^k}) \cup \bigcup_{i=0}^{e_1 - e_m} T_i(G^{p^k})$$

where $m_i < m$ is chosen maximal so that $e_{m_i} \geq e_m + i$ and $G_i = \prod_{i=1}^{m_i} < g_i >$ (for $i = 0, 1, \ldots, e_1 - e_m$).

Now for $0 \leq k < e_m$, construct sets $C_0(N), C_1(N), \ldots C_{e_1 - e_m}(N)$, $D_1(N)$ of monomials representing elements of $\Delta^N(G)$ in such a way that $| C_i(N) | = | T_i(G^{p^k}) |$ (for all i) and $| D(N) | = | S(G^{p^k}) |$.

Let $C_0(N)$ be the set of all monomials $p^k \prod_{j=1}^{m} x_j^{\alpha_j}$ where $0 \leq \alpha_m < p^{e_m - k}$, $\sum_{j=1}^{m} \alpha_j = N$, and where $p^k \prod_{j=1}^{m-1} x_j^{\alpha_j}$ is a monomial which is a member of a previously chosen set of generators for $p^k Q_{N - \alpha_m}(G_0)$. Note $G_0 = \prod_{i=1}^{m_0} < g_i >$ where $m_0 < m$ and so G_0 has rank less than m. By our induction hypothesis the result holds for G_0 and so this definition is valid.

Similiarly, for each i with $1 \leq i \leq e_1 - e_m$, let $C_i(N)$ be the set of monomials $p^k \prod_{j=1}^{m} x_j^{\alpha_j}$ where

$$p^{e_m - k - 1} + i(p^{e_m - k} - p^{e_m - k - 1}) \leq \alpha_m < p^{e_m - k - 1} + (i + 1)(p^{e_m - k} - p^{e_m - k - 1}),$$

$\alpha_j < p^{e_m - k - 1}$, for $m_i < j < m$, $\sum_{j=1}^{m} \alpha_j = N$, and $p^k \prod_{j=1}^{m_i} x_j^{\alpha_j}$ is a monomial which is a member of a previously chosen set of generators for $p^k Q_{N - \sum_{j=m_i+1}^{m} \alpha_j}(G_i^{p^i})$. Since

$rank(G_i^{p^i}) = m_i < m$ we again have this is well defined by our induction hypothesis.

Finally, let $D(N)$ be the set of monomials $p^k \prod_{j=1}^{m} x_j^{\alpha_j}$ where $\alpha_j < p^{e_m-k-1}$ for $j < m$, and $\sum_{j=1}^{m} \alpha_j = N$.

We now need to show that $| C_i(N) |=| T_i(G^{p^k}) |$ (for all i) and $| D(N) |=| S(G^{p^k}) |$, and that the sets $C_i(N)$ and $D(N)$ are pairwise disjoint for N sufficiently large. This is also by induction on m.

If $m = 2$ then $G_i = < g_1 >$ for $0 \leq i \leq e_1 - e_2$. Hence $| MCS(G_i^{p^k}) |= 1$ for $0 \leq i \leq e_1 - e_2 = (e_1 - k) - (e_2 - k)$ and by corollary 2.9 we have

$$\begin{aligned}
| T_0(G_i^{p^k}) | &= p^{e_2-k} \cdot 1 = p^{e_2-k}, \\
| T_i(G_i^{p^k}) | &= (p^{e_2-k} - p^{e_2-k-1}) \cdot (p^{e_2-k-1})^0 \cdot 1 \\
&= p^{e_2-k} - p^{e_2-k-1}, \text{ for } 1 \leq i \leq e_1 - e_2, \text{ and} \\
| S(G_i^{p^k}) | &= p^{e_2-k-1}.
\end{aligned}$$

Also, since G_0 is cyclic we have one generator for $p^k Q_{N-\alpha_2}(G_0)$ which is $\{p^k x_1^{N-\alpha_2}\}$. Thus,

$$\begin{aligned}
C_0(N) &= \{p^k x_1^{\alpha_1} x_2^{\alpha_2} \in \Delta^N(G) \mid 0 \leq \alpha_2 < p^{e_2-k}, \alpha_1 + \alpha_2 = N\} \\
&= \{p^k (x_1^{N-\alpha_2} \cdot x_2^{\alpha_2}) \mid 0 \leq \alpha_2 < p^{e_2-k}\}.
\end{aligned}$$

Also for $i = 1, \ldots, e_1 - e_2$,

$$\begin{aligned}
C_i(N) = \{p^k x_1^{N-\alpha_2} x_2^{\alpha_2} \mid\ & p^{e_2-k-1} + i(p^{e_2-k} - p^{e_2-k-1}) \leq \alpha_2, \\
& \alpha_2 < p^{e_2-k-1} + (i+1)(p^{e_2-k} - p^{e_2-k-1})\} \qquad \text{and} \\
D(N) = \{p^k x_1^{\alpha_1} x_2^{N-\alpha_2} \mid\ & 0 \leq \alpha_1 < p^{e_2-k-1}\}.
\end{aligned}$$

Note

$$\begin{aligned}
| C_0(N) | &= p^{e_2-k} =| T_0(G^{p^k}) |, \\
| C_i(N) | &= p^{e_2-k-1}[(1 + (i+1)(p-1)) - (1 + (p-1))] = p^{e_2-k-1}(p-1) \\
&=| T_i(G^{p^k}) | \text{ for } i = 1, \ldots, e_1 - e_2, \text{ and} \\
| D(N) | &= p^{e_2-k-1} =| S(G^{p^k}) |.
\end{aligned}$$

Finally note for any $N \geq p^{e_2-k-1} + (e_1 - e_2 + 1)(p^{e_2-k} - p^{e_2-k-1})$, and any $i \neq j$, $0 \leq i, j \leq e_1 - e_2$, that $C_i(N) \cap C_j(N) = 0$. This is because the range of the α_i's are completely disjoint. Also, for $N \in \{1, 2, \ldots\}$ such that

$$N \geq p^{e_2-k-1} + (e_1 - e_2 + 1)(p^{e_2-k} - p^{e_2-k-1})$$

then $D(N) \cap C_i(N) = 0$ for all $i = 0, \ldots, e_1 - e_m$ as the exponent of x_2 for any element in $D(N)$ is greater than any exponent of x_2 in $\bigcup_{i=0}^{e_1-e_2} C_i(N)$. Hence the claim is true for

$m = 2$.

Assume $m > 2$ and the claim is true for all appropriate groups with rank $< m$. Let N_0 be large enough so that $C_i(N_0)$ $(i = 0, \ldots, e_1 - e_{m-1})$ and $D(N_0)$ are pairwise disjoint for the group G_0. Then

$$\begin{aligned} \mid C_0(N_0) \mid &= p^{e_m-k} \cdot (\text{number of generators for } p^k Q_{N-\alpha_m}(G_0)) \\ &= p^{e_m-k} \cdot \mid MCS(G_0^{p^k}) \mid \\ &\quad (\text{ by our main induction and since } rank(G_0) < rank(G)) \\ &= \mid T_0(G^{p^k}) \mid, \\ \mid C_i(N_0) \mid &= p^{e_m-k-1}[(1+(i+1)(p-1)) - (1+(p-1))] \cdot (p^{(e_m-k-1)})^{m-m_i-1} \\ &\quad \cdot (\text{number of generaters for } p^k Q_{N_0 - \sum_{j=m_i+1}^{m} \alpha_j}(G_i^{p^i})) \\ &= (p^{e_m-k} - p^{e_m-k-1}) \cdot (p^{(e_m-k-1)})^{m-m_i-1} \cdot \mid MCS((G_i^{p^i})^{p^k}) \mid \\ &\quad (\text{since } rank(G_i^{p^i}) < rank(G^{p^i})) \\ &= \mid T_i(G^{p^k}) \mid \text{ for } i = 1, \ldots, e_1 - e_m, \text{ and} \\ \mid D(N_0) \mid &= (p^{e_m-k-1})^{m-1} = \mid S(G^{p^k}) \mid. \end{aligned}$$

Now let $N = (p^{e_m-k} - 1) + N_0$. Then $N - (p^{e_m-k} - 1) \geq N_0$ and so as above all sets must be pairwise disjoint. i.e. We have finished showing that for large enough N the sets $C_i(N)$ and $D(N)$ are disjoint and of the correct size.

We now continue on with the proof of our main induction.

To show that the (images of the) monomials in the $C_i(N)$ and $D(N)$ generate $p^k Q_N$, where N is sufficiently large, suppose to the contrary that some monomial $p^k \prod_{j=1}^m x_j^{\alpha_j}$ with $\sum_{j=1}^m \alpha_j = N$ cannot be expressed as a linear combination (modulo Δ^{N+1}) of monomials in the $C_i(N)$ and $D(N)$. Let the m-tuple $(\alpha_1, \ldots, \alpha_m)$ be lexiographically greatest with this property.

Consider the exponent α_m and first suppose $\alpha_m < p^{e_m}$. By induction the monomial $p^k \prod_{j=1}^{m-1} x_j^{\alpha_j}$ is (modulo $\Delta(G_0)^{N+1-\alpha_m}$) a linear combination of a previously chosen set of generators for $p^k Q_{N-\alpha_m}(G_0)$. Multiplying by $x_m^{\alpha_m}$, we obtain an expression for $p^k \prod_{j=1}^m x_j^{\alpha_j}$ as a linear combination (modulo Δ^{N+1}) of elements of $C_0(N)$.

Suppose $\alpha_m \geq p^{e_m}$. Note since $o(g_m) = p^{e_m}$ we have $p^{e_m} x_m = p^{e_m}(g_m - 1) \in \Delta^2$ (see the proof of $I.1.9$). Now if $k \geq e_m$ then $p^k x_m^{\alpha_m} = p^{k-e_m}(p^{e_m} x_m) x_m^{\alpha_m - 1}$ is in the ideal $\Delta^2 \cdot \Delta^{\alpha_m - 1} = \Delta^{\alpha_m + 1}$. Thus $p^k \prod_{j=1}^m x_j^{\alpha_j} \in \Delta^{N+1}$ or $p^k \prod_{j=1}^m x_j^{\alpha_j} = 0$ in $\Delta^{N+1}(G)$. Hence we need only consider the case $\alpha_m \geq p^{e_m}$ and $k < e_m$.

Suppose for some i with $1 \leq i \leq e_1 - e_m$ that

$$p^{e_m-k-1} + i(p^{e_m-k} - p^{e_m-k-1}) \leq \alpha_m < p^{e_m-k-1} + (i+1)(p^{e_m-k} - p^{e_m-k-1}).$$

If α_j with $m_i < j < m$ satisfies $\alpha_j \geq p^{e_m-k-1}$ then by the definition of m_i we have $e_j < e_m + i$ or $e_j - e_m < i$. Therefore

$$\begin{aligned} (e_j - e_m)(p^{e_m-k} - p^{e_m-k-1}) + p^{e_m-k} &\leq (i-1)(p^{e_m-k} - p^{e_m-k-1}) + p^{e_m-k} \\ &= i(p^{e_m-k} - p^{e_m-k-1}) + p^{e_m-k-1}. \end{aligned}$$

Therefore lemma 3.1 (or lemma 3.2) allows us to simultaneously replace α_m by $\alpha_m - (p^{e_m-k} - p^{e_m-k-1})$, and α_j by $\alpha_j + (p^{e_m-k} - p^{e_m-k-1})$ without changing the monomial modulo Δ^{N+1}. This contradicts the lexicographic maximality of $(\alpha_1, \ldots, \alpha_m)$. Hence we conclude $\alpha_j < p^{e_m-k-1}$ for $m_i < j < m$. Now the monomial $p^k \prod_{j=1}^{m_i} x_j^{\alpha_j}$ (considered as an element of $p^k Q_{N-\sum_{j=m_i+1}^{m} \alpha_j}(G_i^{p^i})$) can be written as a linear combination of the (images of the) appropriate set of proviously chosen generators by our induction hypothesis and by applying instances of lemma 3.2 to $G_i^{p^i}$ (since $m_i < m$). Note since $e_{m_i} \geq e_m + i$ we have that $e_j - i \geq e_m$ for all $1 \leq j \leq e_{m_i}$. i.e. All cyclic factors of $G_i^{p^i}$ have order $\geq p^{e_m}$. Hence if $G_i^{p^i}$ and $\{\bar{k}, k_1, \ldots, k_{m_i}\}$ satisfies the conditions of lemma 3.2 then this identity holds for $G_i \times < g_m >$ and $\{\bar{k}, k_1, \ldots, k_{m_i}, e_m\}$ by lemma 3.5. Therefore multiplication by $\prod_{j=m_i+1}^{m_i} x_j^{\alpha_j}$ converts $p^k \prod_{j=1}^{m_i} x_j^{\alpha_j}$ to an expression for $p^k \prod_{j=1}^{m} x_j^{\alpha_j}$ as a linear combination (mod Δ^{N+1}) of elements of $C_i(N)$, converts monomials from the appropriate set for $G_i^{p^i}$ to those for G, and converts instances of lemma 3.2 for $G_i^{p^i}$ to instances of lemma 3.2 for G.

Finally suppose $\alpha_m \geq p^{e_m-k-1} + (e_1 - e_m + 1)(p^{e_m-k} - p^{e_m-k-1})$. Then as above, lemma 3.1 allows us to conclude that $\alpha_j < p^{e_m-k-1}$ for all $j < m$, so $p^k \prod_{j=1}^{m} x_j^{\alpha_j}$ lies in $D(N)$.

This proves that for large N, $p^k Q_N(G)$ can be generated by the number of maximal cyclic subgroups of G^{p^k}, and by lemma 3.7 we have the result. ∎

Using this result we will now present a formula for the calculation of $Q_\infty(G) \cong Q_G$ in the following chapter.

CHAPTER 3

A General Formula

In this chapter we examine the result of Chapter 2 and actually determine a formula for calculating the augmentation terminal of a finite abelian p-group. In Section 1 in order to establish definitions and also for illustrative purposes, we first give a formula for a certain class of groups through the method suggested by the proof of Hales' theorem. In Section 2 we prove the main formula for all groups. Finally in Section 3 we illustrate some results that were previously known.

Section 1 *Some Examples.*

If we follow the proof of theorem $II.5.1$ in order to determine the augmentation quotient of the finite abelian p-group G, we must examine the group Q_G and determine its invariants. In order to illustrate how this is done we have the following example.

1.1 Example. Let $G = < g > \times < h >$ where $o(g) = o(h) = 3^2$. We wish to calculate Q_G. First recall the definition of Q_G $(II.1.3)$. We let P_G denote the poset of cyclic subgroups H of G. We then take one generator x_H for each $H \in P_G$ and relations $px_H = x_K$ whenever H covers K in P_G. We also define $H_0 = \{1\}$ and $x_{H_0} = 0$. Now, in order to determine the cyclic subgroups of G it suffices to generate all the maximal cyclic subgroups of G. By example $II.2.10$, $MCS(G) = T_0(G) \cup S(G)$ where

$$T_0(G) = \{ < (g, h^j) > \mid 0 \leq j < 9\} \quad \text{and} \quad S(G) = \{ < (g^{3j}, h) > \mid 0 \leq j < 3\}.$$

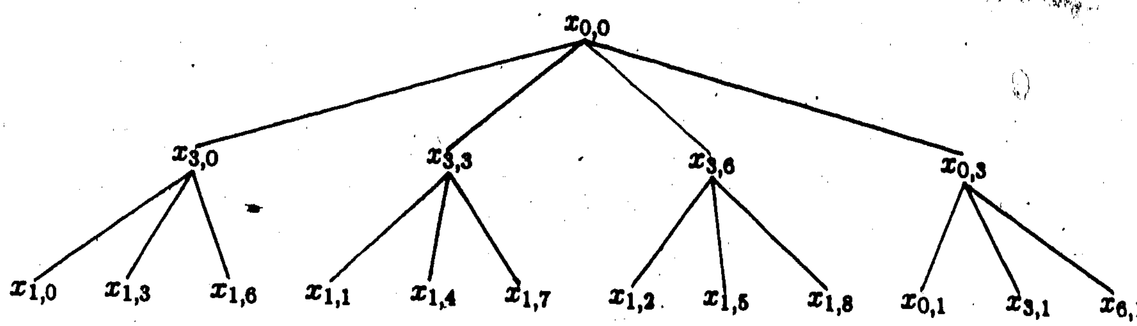
They are also in standardized form as we only work with the standardized M.C.S.'s. Now note that if we have $< (g, h^j) >$ for some $0 \leq j < 9$ then $< (g, h^j) >$ covers $< (g^3, h^{j3}) >$. Also $< (g^{3j}, h) >$ only covers $< (1, h^3) >$. Hence all the cyclic subgroups are

$$\mathcal{H} = MCS(G) \cup [\bigcup_{i=0}^{2} < (g^3, h^{3i}) >] \cup < (1, h^3) > \cup < (1,1) > .$$

Also

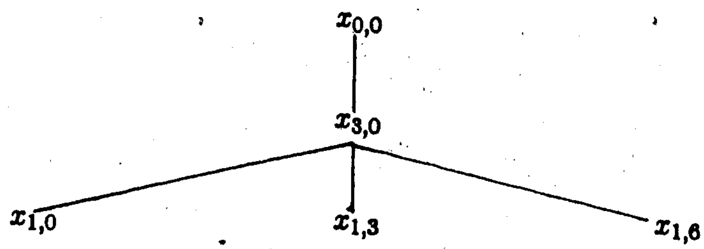
and $$3x_{<(g,h^j)>} = x_{<(g^3,h^{3j})>} \; ; \; 3x_{<(g^{3j},h)>} = x_{<(1,h^3)>} \, ,$$
$$3x_H = x_{<(1,1)>} \text{ for all other } H \in \mathcal{H}.$$

Therefore we get the following poset P_G:



where $x_{i,j} = x_{<(g^i, h^j)>}$.

So what are the invariants of this group? Consider the part of P_G:



This subgroup is

$$< x_{1,0}, x_{1,3}, x_{1,6} \mid o(x_{1,3i}) = 3^2 \text{ for } i = 0,1,2 \text{ and } 3(x_{1,0}) = 3(x_{1,3}) = 3(x_{1,6}) >$$

$= < a, b, c \mid o(a) = 9,\ o(b) = o(c) = 3 >$ where $a = x_{1,0}$, $b = x_{1,3} - a$, $c = x_{1,6} - a$.
$\cong (Z_{3^2}) \times (Z_3)^2$

Hence $G \cong \left[Z_{3^2} \times (Z_3)^2\right]^4 \cong (Z_{3^2})^4 \times (Z_3)^8$.

Thus the invariants are read off the poset P_G. First we need this definition.

1.2 Definition. *Given the poset P_G as defined in II.1.3 we say that*

$$p^n x_{H_1} = p^{n-1} x_{H_2} = \cdots = p x_{H_n} = x_{H_{n+1}}$$

forms a chain of n nodes. i.e. Geometrically we have

$$x_1 —— x_2 —— x_3 \cdots\cdots x_n —— x_{n+1}$$

Now continuing with the example, note the repetition of $x_{<(\ ,\ )>}$. If one always works with standardized subgroups (as we do) then we can drop the x notation and represent

$x_{<(g^i,h^j)>}$ by (g^i, h^j) itself. When doing this we will also go back to multiplicative notation (as it now makes sense). i.e. If we have

$$3x_{<(g^1,h^4)>} = px_{<(g^1,h^4)>} = x_{<(g^p,h^{4p})>} = x_{<(g^3,h^3)>}$$

this is now represented by

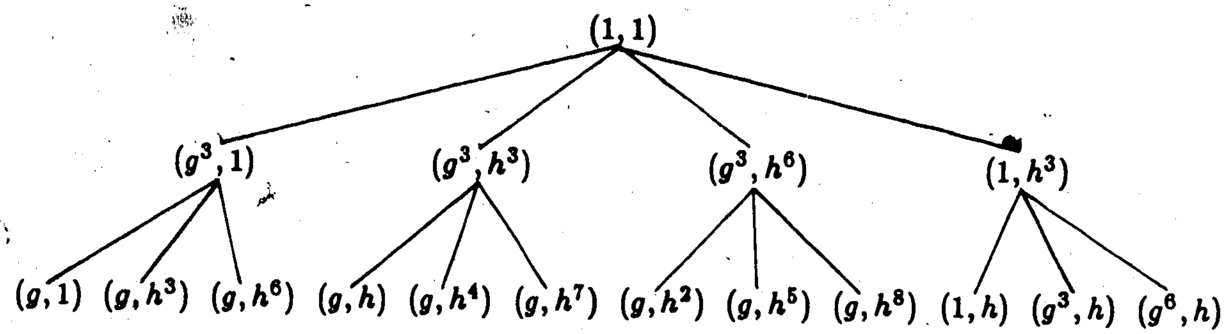
$$(g^1, h^4)^3 = (g^1, h^4)^p = (g^p, h^{4p}) = (g^3, h^3).$$

The main thing to remember is that (g^i, h^j) is the standardized representative of the cyclic subgroup $< (g^i, h^j) >$ of G.

Note $< (g, h^4) > = < (g^2, h^8) >$. So if we did not work with standardized representatives we would have $(g^3, h^3) = (g, h^4)^3 = (g^2, h^8)^3 = (g^6, h^6)$ and this notation would not be valid.

It is hoped this notation will not cause the reader any problems.

Thus we have that Q_G is isomorphic to the abelian p-group:



To determine the invariants we need only count the number of chains of each length beginning at a M.C.S. of G and never reusing a node except at the end of a chain. In the above we have four chains of length two,

$$(g, 1) \longrightarrow (g^3, 1) \longrightarrow (1, 1) \qquad (g, h^2) \longrightarrow (g^3, h^6) \longrightarrow (1, 1)$$
$$(g, h) \longrightarrow (g^3, h^3) \longrightarrow (1, 1) \qquad (1, h) \longrightarrow (1, h^3) \longrightarrow (1, 1)$$

Also we have eight chains of length one,

$$(g, h^3) \longrightarrow (g^3, 1) \quad (g, h^4) \longrightarrow (g^3, h^3) \quad (g, h^5) \longrightarrow (g^3, h^6) \quad (g^3, h) \longrightarrow (1, h^3)$$
$$(g, h^6) \longrightarrow (g^3, 1) \quad (g, h^7) \longrightarrow (g^3, h^3) \quad (g, h^8) \longrightarrow (g^3, h^6) \quad (g^6, h) \longrightarrow (1, h^3)$$

Then a chain of length n means Q_G has a factor Z_{p^n} in it. Hence from above we see $Q_G \cong (Z_{3^2})^4 \times (Z_3)^8$.

As we can see from the above example, determining the invariants from Q_G as it is presented is time-consuming and tedious. For each group G we must first calculate out maximal cyclic subgroups and then determine what the poset P_G looks like. Then we have to count the chains and their lengths. This method does not make it very likely that we can find a general formula. Even with certain classes of groups with high symmetry this is not very easy. To illustrate we will take the class of groups $G \cong (Z_{p^n})^m$ and exhibit a general formula for Q_G. It should be noted that this is an extension of [Hales '82] in which he verified that the augmentation terminal of G was Q_G.

1.3 Definition. *Let $G_j = \prod_{i=1}^{j} < g_i > \cong (Z_{p^n})^j$. Define the sets $L_l(G_j)$ for $l = 1, \ldots, j$ as follows:*

$$L_l(G_j) = \{(g_1^{p \cdot i_1}, g_2^{p \cdot i_2}, \ldots, g_{l-1}^{p \cdot i_{l-1}}, g_l, g_{l+1}^{i_{l+1}}, \ldots, g_j^{i_j}) \mid 0 \le i_1, i_2, \ldots, i_{l-1} < p^{n-1}, \\ 0 \le i_{l+1}, \ldots, i_j < p^n\}$$

1.4 Remarks.
1) Note if $l_1 \neq l_2$ then $L_{l_1}(G_j) \cap L_{l_2}(G_j) = \emptyset$ (This is because $o(g_{l_1}) = o(g_{l_2}) = p^n$.)
2) $\mid L_l(G_j) \mid = p^{(l-1)(n-1)+(j-l)n}$.
3) Each element of $L_l(G_j)$ is a standardized representative of a M.C.S. of G_j. (See prop. $II.2.3$.)

1.5 Lemma. *Let $G_m = \prod_{i=1}^{m} < g_i > \cong (Z_{p^n})^m$. Then $MCS(G_m) = \bigcup_{l=1}^{m} L_l(G_m)$.*

Proof. By induction on m.
Let $m = 1$. Then $G_1 = < g_1 > \cong Z_{p^n}$. Thus $MCS(G_1) = \{ < g_1 > \}$. Also $L_1(G_1) = \{ < g_1 > \}$. Hence $MCS(G_1) = \bigcup_{l=1}^{1} L_l(G_1) = L_1(G_1)$.
Thus we will assume the result is true for G_l where $1 \le l < m$. In the notation of $II.2$ we have $e_1 = e_2 = \ldots e_m = n$. Thus $e_1 - e_m = 0$ and $m_0 = m - 1$. Then

$$T_0(G_m) = \{ < (g_1^{\beta_1}, g_2^{\beta_2}, \ldots, g_{m-1}^{\beta_{m-1}}, g_m^{\beta_m}) > \mid < (g_1^{\beta_1}, \ldots, g_{m-1}^{\beta_{m-1}}) > \in MCS(G_{m-1}) \\ \text{and } 0 \le \beta_m < p^n\}$$

By induction $MCS(G_{m-1}) = \bigcup_{l=1}^{m-1} L_l(G_{m-1})$.
Note for $l = 1, \ldots, m-1$

$$L_l(G_m) = \{ < (h, g_m^{i_m}) > \mid 0 \le i_m < p^n \text{ and } h \in L_l(G_{m-1})\}.$$

(i.e. if $h = < (g_1^{\beta_1}, \ldots, g_{m-1}^{\beta_{m-1}}) >$ then $(h, g_m^{i_m}) = (g_1^{\beta_1}, \ldots, g_{m-1}^{\beta_{m-1}}, g_m^{i_m})$). Hence

$$T_0(G_m) = \bigcup_{l=1}^{m-1} L_l(G_m).$$

Also

$$S(G_m) = \{ < (g_1^{\beta_1}, \ldots, g_{m-1}^{\beta_{m-1}}, g_m^{\beta_m}) > \mid 0 \le \beta_i < p^n \text{ and } o(g_i^{\beta_i}) \le p^{n-1} \\ \text{for } i = 1, \ldots, m-1\}.$$

But $< (g_1^{\beta_1}, \ldots, g_m^{\beta_m}) > \in MCS(G_m)$ so $o(g_m^{\beta_m}) = p^n$ (by $II.2.3$) and $\beta_m = 1$ when this element is standardized ($II.2.4$).
i.e.

$$\begin{aligned} S(G_m) &= \{ < (g_1^{\beta_1}, \ldots, g_{m-1}^{\beta_{m-1}}, g_m) > \mid \text{ where } 0 \le \beta_i < p^n \text{ and } o(g_i^{\beta_i}) \le p^{n-1} \\ &\qquad \text{for } i = 1, \ldots, m-1\} \\ &= \{ < (g_1^{p \cdot i_1}, \ldots, g_{m-1}^{p \cdot i_{m-1}}, g_m) > \mid 0 \le i_j < p^{n-1} \text{ for } j = 1, \ldots, m-1\} \\ &= L_m(G_m) \end{aligned}$$

Thus $MCS(G_m) = T_0(G_m) \cup S(G_m) = \bigcup_{l=1}^m L_l(G_m)$. ∎

Note that the hard part in the above proof is coming up with the general form of $MCS(G_m)$. This must be done by examing the lower cases and hoping that we can see the general result. This is what makes determing the invariants of Q_G extremely difficult when the group G can not be broken down into parts of some symmetry.

Once we have the maximal cyclic subgroups we can then determine the invariants of Q_G as follows:

1.6 Theorem. *Let* $G = \prod_{i=1}^m < g_i > \cong (Z_{p^n})^m$. *Then*

$$Q_G \cong (Z_{p^n})^{\left(\frac{p^m-1}{p-1}\right)} \times \prod_{j=1}^{n-1} (Z_{p^{n-j}})^{\left(\frac{p^m-1}{p-1}\right) p^{(m-1)(j-1)}(p^{m-1}-1)}.$$

Proof. In order to count the chains in P_G we need to know what the poset of cyclic subgroups looks like. To do this we need to know how many cyclic subgroups of order p there are. Note

$$(L_l(G_m))^{p^{n-1}} = \{< (1, 1, \ldots, 1, g_l^{p^{n-1}}, g_{l+1}^{p^{n-1} i_{l+1}}, \ldots, g_m^{p^{n-1} i_m}) > \}$$

$$= \{< (1, 1, \ldots, 1, g_l^{p^{n-1}}, g_{l+1}^{p^{n-1} j_{l+1}}, \ldots, g_m^{p^{n-1} j_m}) > \mid 0 \le j_{l+1}, j_{l+2}, \ldots, j_m < p\}$$

and that every p-order subgroup must come from one of these $(L_l(G_m))^{p^{n-1}}$ (using 1.5.). Hence the set of all subgroups of order p is $\bigcup_{l=1}^m (L_l(G_m))^{p^{n-1}}$. Also note that $(L_i(G_m))^{p^{n-1}} \cap (L_j(G_m))^{p^{n-1}} = \emptyset$ if $i \ne j$ by using the fact that all elements were standardized. Also note $|(L_l(G_m))^{p^{n-1}}| = p^{m-l}$ for $l = 1, \ldots, m$ and so the number of p-subgroups is

$$\sum_{l=1}^{m} p^{m-l} = \sum_{i=0}^{m-1} p^i = (\frac{p^m - 1}{p-1}).$$

Now consider an arbitrary element x from some $(L_l(G_m))^{p^{n-1}}$. i.e.
$x = (1, 1, \ldots, 1, g_l^{p^{n-1}}, g_{l+1}^{j_{l+1}(0)p^{n-1}}, \ldots g_m^{j_m(0)p^{n-1}})$ for some $0 \le j_i(0) < p$, $i = l+1, \ldots, m$.
Then the subgroups of order p^2 which lie over x are

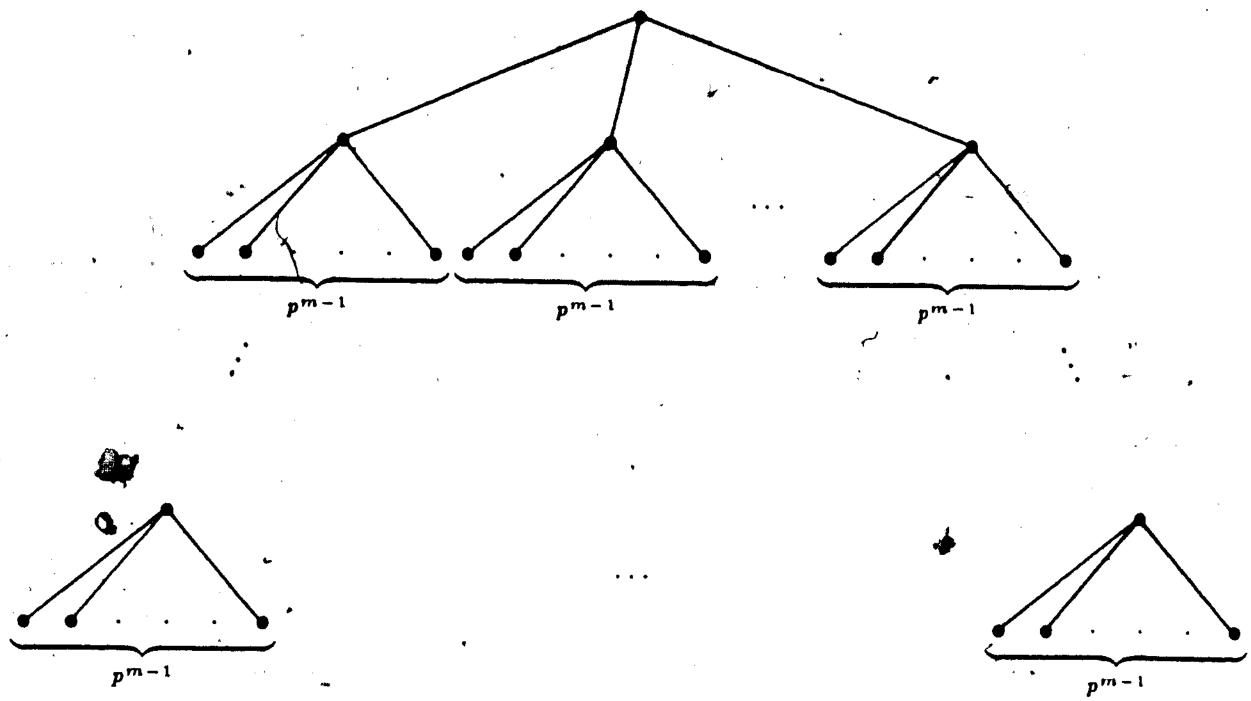
$$(g_1^{j_1(1)p^{n-1}}, g_2^{j_2(1)p^{n-1}}, \ldots, g_{l-1}^{j_{l-1}(1)p^{n-1}}, g_l^{p^{n-2}}, \\ g_{l+1}^{j_{l+1}(0)p^{n-2} + j_{l+1}(1)p^{n-1}}, \ldots, g_m^{j_m(0)p^{n-2} + j_m(1)p^{n-1}})$$

where $0 \leq j_k(1) < p$, $k = 1, \ldots, m$, $k \neq l$.
That is because the only element that can lie over x is from the set $L_l(G_m)$ (by standardization) and the elements of order p^2 come form $(L_l(G_m))^{p^{n-2}}$. Hence there are p^{m-1} subgroups of order p^2 which lie over each subgroup of order p.

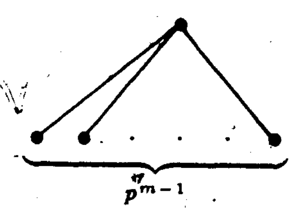
(Note that only $g_l^{p^{n-2}}$ is mapped into $g_l^{p^{n-1}}$ as it is standardized.)

In a like manner we find that each subgroup of order p^i ($i = 1, \ldots, n-1$) has p^{m-1} subgroups of order p^{i+1} that lie over it. Also the maximum order of a cyclic subgroup is p^n. Therefore we have that the dot diagram of G_m (or the poset P_G) is as in diagram 1.7.

1.7 Diagram.

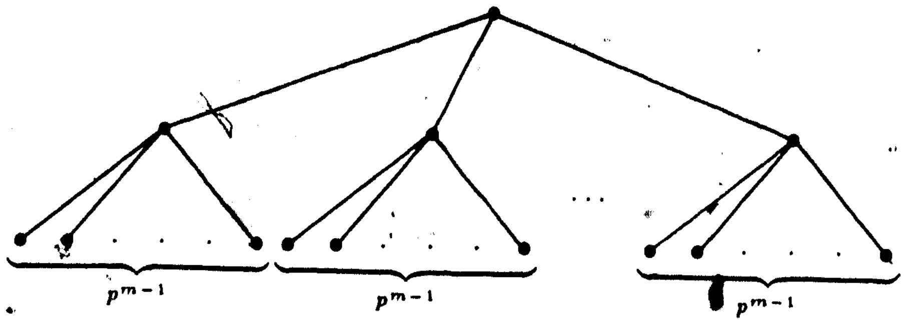


We need only count the number of chains of each length to get our result. Let us say the top node is at level 0, the next row of nodes at level 1, etc. Then we have n levels of nodes in diagram 1.7 and $(\frac{p^m-1}{p-1})$ nodes in level 1 (the number of p-subgroups). Now starting at the nodes at level $n-1$ we see each piece

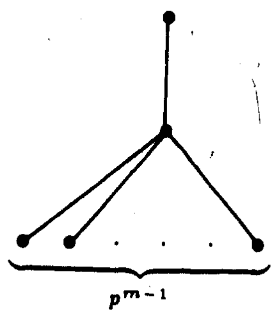


has p^{m-1} chains of length 1.

Then all the nodes at level $n-2$ look like:
(*)



Hence there are p^{m-1} parts



which each have

$$1 \text{ chain of length } 2,$$

and

$$p^{m-1}-1 \text{ chains of length } 1.$$

Thus each structure (*) has

$$p^{m-1} \text{ chains of length } 2,$$

and

$$p^{m-1}(p^{m-1}-1) \text{ chains of length } 1.$$

Now continuing in this manner we see that each node at level 1 has a subtree beneath it with

$$p^{m-1} \text{ chains of length } n-1$$

and

$$(p^{m-1}-1)(p^{m-1})^j \text{ chains of length } n-j-1$$

$$\text{for } j=1,\ldots,n-2.$$

Thus placing all this information together we get that Q_G has

$$\left(\frac{p^m-1}{p-1}\right) \text{ chains of length } n,$$

$$\left(\frac{p^m-1}{p-1}\right)(p^{m-1}-1) \text{ chains of length } n-1, \text{ and}$$

$$\left(\frac{p^m-1}{p-1}\right)(p^{m-1}-1)(p^{m-1})^j \text{ chains of length } n-j-1$$

$$\text{for } j=1,\ldots,n-2$$

which gives our result. ∎

Thus we have calculated the invariants of this particular class of groups which Hales first worked with. But as one can see, this method does not help in establishing a general formula applicable to all finite abelian p-groups. Thus we search for a different method.

Section 2 *The General Formula.*

As stated earlier, in this section we derive the general formula. The method depends on two results which allow us to proceed in steps. Each one of these two results comes from looking geometrically at the posets $P_{G'}$ of some specific smaller group G', and then examining the effect which occurs in $P_{G'}$ from changing G' into G. These two steps are amazing in their simplicity. However because of the way they work we can not derive a general formula which is pleasing to the eye. But this formula was expected to be complicated by our knowledge of previous results.

It is interesting in the way these steps were discovered. When one tackles the problem of finding the invariants of Q_G by building up the maximal cyclic subgroups, the geometrical simplicity is lost in the actual calculations. It is very hard indeed to know what is happening to the chains and nodes within the poset when you have the sets $T_0, \ldots, T_{e_1-e_m}$, S all being defined in different manners. Also because of the tedious calculations, one naturally gets lost. Because of this, a fairly complex computer program was developed which for a given prime p and a given group G, calculated the maximal cyclic subgroups as in $II.2$. It then used these M.C.S.'s and built the poset P_G finally giving the invariants of Q_G and outputting the picture of P_G to the user. By looking at the structure of P_G (for different primes when needed) we could generalize the poset for an arbitrary prime and this G. We then could calculate the chains and nodes and come out with the result. However even this is not as easy as it sounds, as these posets Q_G get large fairly rapidly. For instance, the group $G = Z_{3^3} \times (Z_3)^4$ has 201 maximal cyclic subgroups and the group $G = Z_{3^3} \times Z_{3^2} \times (Z_3)^4$ has 1881 maximal cyclic subgroups. Hence the program was not helpful for any modest sized group. However the program gave a picture of the posets for G and by looking at these posets the manner in which to proceed became quite clear.

Before we start, we need some notations which will make the result easier to use.

2.1 Notation. Let G be a finite abelian p-group. Given $f_i \geq 0$ for $i = 1, \ldots, n$, let $G \cong (Z_{p^n})^{f_1} \times (Z_{p^{n-1}})^{f_2} \times \cdots \times (Z_p)^{f_n}$. Then we denote G by $G = (f_1, f_2, \ldots, f_n)$.

It should be noted that this notation will only be applied to Q_G, and G itself will be written in the normal fashion. Hence this notation also makes it easier to know which group we are referring to.

We also have the following easy results.

2.2 Proposition. *Let $G \cong Z_{p^{e_1}} \times Z_{p^{e_2}} \times \cdots Z_{p^{e_n}}$ where $e_1 \geq e_2 \geq \cdots \geq e_n \geq 1$. Then*

$$Q_G \cong (f_1, f_2, \ldots, f_l)$$

where $f_1 \neq 0$ and $l = e_1$. (i.e. Q_G has at least one factor $Z_{p^{e_1}}$.)

Proof. Note if $G \cong \prod_{i=1}^{n} < x_i >$ where $o(x_i) = p^{e_i}$, then the subgroup $H = < (x_1, 1, 1, \ldots, 1) > \in MCS(G)$ by proposition $II.2.3$ and H is standardized. Hence $H \in P_G$ and H is in the n long chain

$$H \supseteq H^p \supseteq \cdots \supseteq H^{p^{n-1}} \supseteq H^{p^n} = < 1 > .$$

Hence Q_G has the factor $Z_{p^{e_1}}$ in it (see 1.1). Also it can not have any M.C.S. of order $\geq p^{e_1}$ by $II.2.3$. ∎

2.3 Proposition. *If $y \in G^p$ has index of standardization i_0 and $x^p = y$ then x has index of standardization i_0.*

Proof. Follows directly from the definition of index of standardization and the fact that G is an abelian group of direct factors. ∎

We are now ready for our first result.

2.4 Lemma. *(The Moving Up Lemma)*
Let $G \cong Z_{p^{e_1}} \times Z_{p^{e_2}} \times \cdots Z_{p^{e_m}}$ where $e_1 \geq e_2 \geq \cdots \geq e_m \geq 1$ and suppose $Q_G = (f_1, f_2, \ldots, f_{e_1})$. Let $G' \cong Z_{p^{e_1+1}} \times Z_{p^{e_2+1}} \times \cdots \times Z_{p^{e_m+1}}$. Then

$$Q_{G'} = (f_1, f_2, \ldots, f_{e_1}, f_{e_1+1}) \text{ where } f_{e_1+1} = (\sum_{i=1}^{e_1} f_i)(p^{m-1} - 1).$$

Proof. Let $G = \prod_{i=1}^{m} < g_i >$ and let $G' = \prod_{i=1}^{m} < h_i >$ where, without loss of generality, we may assume $h_i^p = g_i$. Let $x = (h_1^{\beta_1}, \ldots, h_m^{\beta_m})$ be a standardized maximal cyclic subgroup of G'. (Note we are using the notation of 1.1, i.e. actually $< x > \in MCS(G')$.) Then

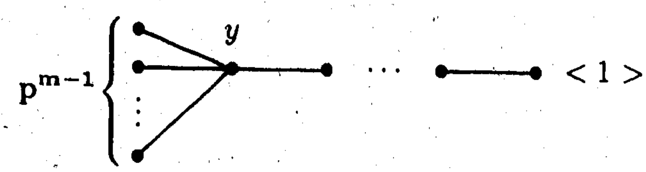
$$\begin{aligned} x^p &= ((h_1^{\beta_1})^p, \ldots, (h_m^{\beta_m})^p) = ((h_1^p)^{\beta_1}, \ldots, (h_m^p)^{\beta_m}) \\ &= (g_1^{\beta_1}, \ldots, g_m^{\beta_m}). \end{aligned}$$

We have that $0 \leq \beta_i < p^{e_i+1}$, but by proposition $II.2.2$ for $i = 1, \ldots, m$, there exists a $1 \leq j \leq m$ such that p does not divide β_j. Let $\beta_i' = \beta_i \bmod p^{e_i}$ for $i = 1, \ldots, m$. Then $0 \leq \beta_i' < p^{e_i}$ and p does not divide β_j implies p does not divide β_j'. Thus by $II.2.3$ we have $x^p \in MCS(G)$. (Note x^p is also standardized.) Therefore for every $x \in MCS(G')$ we have $x^p \in MCS(G)$. Now consider $y \in MCS(G)$ and suppose $y = (g_1^{\alpha_1}, \ldots, g_m^{\alpha_m})$. Thus $0 \leq \alpha_i < p^{e_i}$ for $i \leq i \leq m$ and there exists $1 \leq j \leq m$ such that p does not divide α_j. Suppose $x \in MCS(G')$ and $x^p = y$. Then obviously

$$x = (h_1^{\alpha_1+\beta_1 p^{e_1}}, h_2^{\alpha_2+\beta_2 p^{e_2}}, \ldots, h_m^{\alpha_m+\beta_m p^{e_m}})$$

for some $0 \leq \beta_i < p$ for $i = 1, \ldots, m$. Also since p does not divide α_j we have that p does not divide $(\alpha_j + \beta_j p^{e_j})$ for any $0 \leq \beta_j < p$. Hence x will be in $MCS(G')$ if it is standardized.

But the index of standardization of y is i_0 (say) which says that the index of standardization of x is also i_0. (By proposition 2.3.) Hence $\alpha_{j_0} = 1$ and so must $\alpha_{j_0} + \beta_{j_0} p^{e_{j_0}} = 1$ for x to be standardized. i.e. If $y \in MCS(G)$ then there are exactly p^{m-1} x's in $MCS(G')$ such that $x^p = y$. Since every $x \in MCS(G')$ is such that $x^p \in MCS(G)$ we have $P_{G'}$ is the poset P_G with every M.C.S. (or every external node) of P_G now having p^{m-1} fathers. Hence every chain of n nodes has now become 1 chain of $n+1$ nodes and $(p^{m-1}-1)$ chains of length 1. (As each chain begins at a M.C.S. for its group.)



But $Q_G = (f_1, \ldots, f_{e_1})$ which means there are f_i chains of length $e_1 - i + 1$ for $i = 1, \ldots, e_1$. By the above this becomes

$$f_i \text{ chains of length } (e_1 - i + 1) + 1 = e_1 - i + 2$$

and

$$f_i \cdot (p^{m-1} - 1) \text{ chains of length } 1$$

for $i = 1, \ldots, e_1$.

Therefore altogether we get there are

$$f_i \text{ chains of length } e_1 - i + 2 \text{ for } i = 1, \ldots, e_1$$

and

$$\left(\sum_{i=1}^{e_1} f_i\right)(p^{m-1} - 1) \text{ chains of length } 1.$$

Hence define f_{e_1+1} as above and we have the result. ∎

2.5 Remark.

1) Because of the notation we adopted for Q_G we have an easy to remember result as all one does is tack on to the previous result.

2) Geometrically, what has happened is that every M.C.S. (or external node) of G (within P_G) sprouts p^{m-1} fathers. For an example where $G = Z_{3^2} \times Z_3$ and $G' = Z_{3^3} \times Z_{3^2}$ see diagrams 2.7a) and 2.7b).

We now get the following easy corollary.

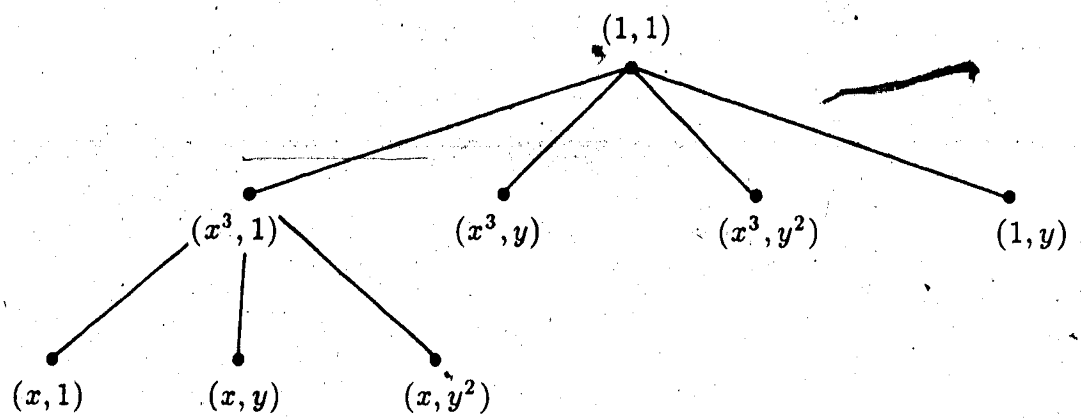
2.6 Proposition. *Let G be as in 2.4. Let $H_n = Z_{p^{e_1+n}} \times Z_{p^{e_2+n}} \times \cdots \times Z_{p^{e_m+n}}$ for $n \geq 1$. Then*

$$Q_{H_n} \cong (f_1, \ldots, f_{e_1}, f_{e_1+1}, \ldots, f_{e_1+n}) \text{ where } f_{e_1+j} = \left(\sum_{i=1}^{e_1} f_i\right) p^{(j-1)(m-1)} (p^{m-1} - 1)$$

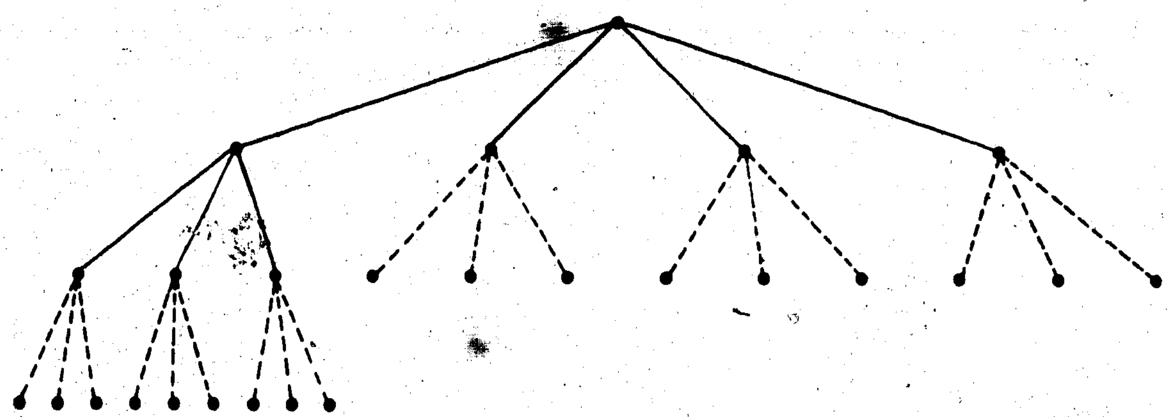
for $j = 1, \ldots, n$.

2.7 Diagram. Illustration of how Q_G changes as the group G does.

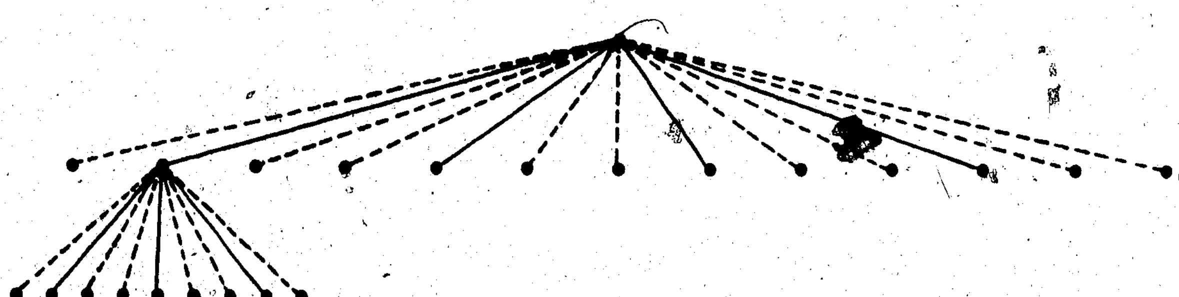
a) Let $G = Z_{3^2} \times Z_3$. Then $Q_G \cong \langle x \rangle \times \langle y \rangle$ as follows:



b) Let $G' = Z_{3^3} \times Z_{3^2}$. Then $Q_{G'}$ is as follows: (The solid lines represent Q_G and the dashed lines indicate the addition of $Q_{G'}$).



c) Let $G'' = G \times Z_3$. Then $Q_{G''}$ is as follows: (The solid lines represent Q_G and the dashed lines indicate the addition of $Q_{G''}$).



Proof. By induction on n.
If $n = 1$ then this is just Lemma 2.4.
Suppose the result holds true for H_{n-1}. Then

$$Q_{H_{n-1}} = (f_1, f_2, \ldots, f_{e_1}, f_{e_1+1}, \ldots, f_{e_1+n-1}) \text{ where } f_{e_1+l} = (\sum_{i=1}^{e_1} f_i) p^{(l-1)(m-1)} \, (p^{m-1} - 1)$$

for $l = 1, \ldots, n-1$.
To move up to H_n we apply Lemma 2.4 to H_{n-1} which gives
$Q_{H_n} = (f_1, f_2, \ldots, f_{e_1+n-1}, f_{e_1+n})$ where f_i $(i = 1, \ldots, e_1 + n - 1)$ are defined as before and

$$\begin{aligned}
f_{e_1+n} &= (\sum_{i=1}^{e_1+n-1} f_i)(p^{m-1} - 1) \\
&= (\sum_{i=1}^{e_1} f_i)(p^{m-1} - 1) + (\sum_{l=1}^{n-1} f_{e_1+l})(p^{m-1} - 1) \\
&= (\sum_{i=1}^{e_1} f_i)(p^{m-1} - 1) + [\sum_{l=1}^{n-1} (\sum_{i=1}^{e_1} f_i) p^{(l-1)(m-1)} (p^{m-1} - 1)](p^{m-1} - 1) \\
&= (\sum_{i=1}^{e_1} f_i)(p^{m-1} - 1)[1 + (p^{m-1} - 1) \sum_{l=1}^{n-1} p^{(l-1)(m-1)}] \\
&= (\sum_{i=1}^{e_1} f_i)(p^{m-1} - 1)[1 + p^{(n-1)(m-1)} - 1] \\
&= (\sum_{i=1}^{e_1} f_i)(p^{m-1} - 1) p^{(n-1)(m-1)}
\end{aligned}$$

which gives the result. ∎

Therefore we can now move up to higher order groups by knowing the structures of lower order groups (with the same basic structure). What is needed now is to be able to tack on to a group. For this we have the following lemma.

2.8 Lemma. *(The Tacking On Lemma)*
Let G and Q_G be as in lemma 2.4. Let $G' \cong G \times Z_p$. Then

$$Q_{G'} = (f_1, f_2, \ldots, f_{e_1-1}, F + 1 + pf_{e_1}) \text{ where } F = (\sum_{i=1}^{e_1-1} (e_1 - i + 1) f_i)(p - 1).$$

Proof. Let $G = \prod_{i=1}^{m} < g_i > \times < 1 >$ and we can let $H = \prod_{i=1}^{m+1} < g_i >$. (We will adopt this non standard notation for G for ease of use.) Let $x = (g_1^{\beta_1}, \ldots, g_m^{\beta_m}, 1)$ be a standardized M.C.S. of G. Then $< x > \supseteq < x^p > \supseteq \ldots \supseteq < x^{p^{n-1}} > \supseteq < 1 >$ generates a chain of length n in Q_G. Now x is standardized and $x \in MCS(G')$. Hence for any $0 \leq j < p$ we have that

$$x_{(1,j)} = (g_1^{\beta_1}, \ldots, g_m^{\beta_m}, g_{m+1}^j) \in MCS(G)$$

and is standardized. (By prop $II.2.3$, and the fact that the index of standardization is the same for x and $x_{(1,j)}$ by definition $II.2.4$.). Also for $j = 0, \ldots, p-1$ we have $(x_{(1,j)})^p = x^p$. Now for $i = 1, \ldots, n-1$ consider $x^{p^i} = (g_1^{p^i\beta_1}, \ldots, g_m^{p^i\beta_m}, 1)$ of order p^{n-i}. Then for $1 \leq i \leq n-1$ and $1 \leq j < p$ consider the element $x_{(p^i,j)}$ of G' defined by

$$x_{(p^i,j)} = (g_1^{p^i\beta_1}, \ldots, g_m^{p^i\beta_m}, g_{m+1}^j).$$

Because p does not divide j and $o(x_{(p^i,j)}) = o(x^{p^i}) = p^{n-i} \geq p$ we have that $x_{(p^i,j)}$ is in $MCS(G')$ by $II.2.3$. Also as $x \in MCS(G)$ we must have the index of standardization of $x_{(p^i,j)}$ $(j = 1, \ldots, p-1;\ i = 1, \ldots, n-1)$ is the same as x itself. (Apply 2.3 several times.) By $II.2.3$ we have that p does not divide β_j for some j. Let

$$S_1 = \{x_{(1,j)} \mid x \in MCS(G), 0 \leq j < p\} \text{ and}$$
$$S_2 = \{x_{(p^i,j)} \mid x \in MCS(G), 1 \leq i < n \text{ where } o(x) = p^n, 1 \leq j < p\}.$$

Therefore we have

$$MCS(G') \supseteq S_1 \cup S_2.$$

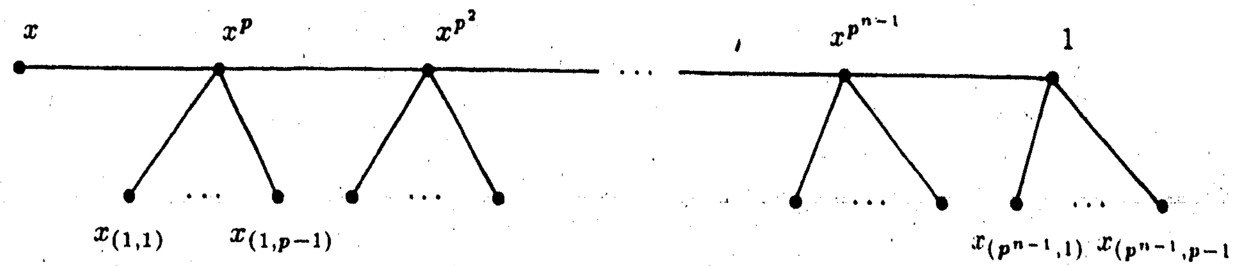
Now let $y = (g_1^{\beta_1}, \ldots, g_m^{\beta_m}, g_{m+1}^{\beta_{m+1}}) \in MCS(G')$ and y standardized, where for $1 \leq i \leq m$ we have $0 \leq \beta_i < p^{e_i}$ and $0 \leq \beta_{m+1} < p$. Suppose that some $\beta_{i_0} \neq 0$ for $1 \leq i_0 \leq m$. Then choose $0 \leq j$ maximal such that $p^j \mid \beta_i$ for all $0 \leq i \leq m$. If $j = 0$ then obviously $y \in S_1$ and if $j > 0$ then $y \in S_2$ (as take $x = (g_1^{\beta_1/p^j}, \ldots, g_m^{\beta_m/p^j}, 1)$ then $y = x_{(p^j,\beta_{m+1})}$). Thus if some $\beta_{i_0} \neq 0$ for $1 \leq i_0 \leq m$ then $y \in S_1 \cup S_2$.
Now suppose that $y = (1, 1, \ldots, 1, g_{m+1}^{\beta_{m+1}})$. Then $y \in MCS(G')$ and y is standardized implies $\beta_{m+1} = 1$. i.e. $y = (1, 1, \ldots, 1, g_{m+1})$. Hence we have

$$(*) \qquad MCS(G') = S_1 \cup S_2 \cup \{(1, 1, \ldots, 1, g_{m+1})\}.$$

We need to consider the chains of $P_{G'}$. By $(*)$ we can look at the chains of P_G. Let

$$x \text{---} x^p \text{---} x^{p^2} \text{---} \cdots \text{---} x^{p^{n-1}} \text{---} 1$$

be a chain of length $n > 0$ in P_G. By $(*)$ we see that this becomes the chain



Hence the chain of length n becomes

and 1 chain of length n
$n(p-1)$ chains of length 1.

Finally the M.C.S. $(1, 1, \ldots, 1, g_{m+1})$ gives a chain of length 1 in $P_{G'}$.
Hence let $Q_G = (f_1, \ldots, f_{e_1})$. i.e. There are f_i chains of length $e_1 - i + 1$, $i = 1, \ldots, e_1$.
Thus these become

and f_i chains of length $e_1 - i + 1$
$f_i(e_1 - i + 1)(p-1)$ chains of length 1

for $i = 1, \ldots, e_1$. Thus

$$
\begin{aligned}
Q_{G'} &= (f_1, \ldots, f_{e_1 - 1}, [\sum_{i=1}^{e_1} f_i(e_1 - i + 1)(p-1)] + 1 + f_{e_1}) \\
&= (f_1, \ldots, f_{e_1 - 1}, [\sum_{i=1}^{e_1 - 1} f_i(e_1 - i + 1)](p-1) + 1 + pf_{e_1}) \\
&= (f_1, \ldots, f_{e_1 - 1}, F + 1 + pf_{e_1}).
\end{aligned}
$$

∎

2.9 Remark. Geometrically, what has happened is that every node of P_G (except $< 1 >$) has acquired $p - 1$ brothers. (Or else these nodes have split into p nodes.) We also have that one new node was created ($(1, 1, \ldots, 1, g_{m+1})$). For an example where $G = Z_{3^2} \times Z_3$ and $G' = Z_{3^2} \times (Z_3)^2$ see diagrams 2.7a) and 2.7c).

We also get the following corollary.

2.10 Corollary. *Let $G \cong Z_{p^{e_1}} \times Z_{p^{e_2}} \times \cdots \times Z_{p^{e_m}}$ where $e_1 \geq e_2 \geq \cdots \geq e_m \geq 1$ and suppose $Q_G = (f_1, f_2, \ldots, f_{e_1})$. Let $G' \cong G \times (Z_p)^k$ where $k \geq 1$. Then*

$$Q_H = (f_1, f_2, \ldots, f_{e_1 - 1}, (\frac{p^k - 1}{p - 1})(F + 1) + p^k f_{e_1})$$

where

$$F = (\sum_{i=1}^{e_1-1}(e_1 - i + 1)f_i)(p-1).$$

Proof. By inductions on k.
If $k = 1$ then this is just lemma 2.8. So suppose the above holds for the groups G_j for $j = 1, \ldots, k-1$ where $G_j \cong G \times (Z_p)^j$. Then

$$Q_{G_{k-1}} = (f_1, f_2, \ldots, f_{e_1-1}, (\frac{p^{k-1}-1}{p-1})(F+1) + p^{k-1}f_{e_1}).$$

By lemma 2.8, since $G' \cong G_{k-1} \times Z_p$ we have that

$$Q_{G'} = (f_1, f_2, \ldots, f_{e_1-1}, F + 1 + p[(\frac{p^{k-1}-1}{p-1})(F+1) + p^{k-1}f_{e_1}]).$$

But

$$\begin{aligned} F + 1 + p[(\frac{p^{k-1}-1}{p-1})(F+1) + p^{k-1}f_{e_1}] &= [p(\frac{p^{k-1}-1}{p-1}) + 1](F+1) + p^k f_{e_1} \\ &= (\frac{p^k - 1}{p-1})(F+1) + p^k f_{e_1}. \end{aligned}$$

∎

We can now state our main result which just uses a combination of The Moving Up Lemma and The Tacking On Lemma. Before we do, it should be noted that in actual use it is easier to use the two lemmas, but for completeness we have the following.

2.11 Theorem. *Let $G \cong (Z_{p^{e_1}})^{m_1} \times (Z_{p^{e_2}})^{m_2} \times \cdots \times (Z_{p^{e_n}})^{m_n}$ where $e_1 > e_2 > \cdots > e_n \geq 1$ and all $m_i \geq 1$. Then*

$$Q_G \cong (Z_{p^{e_1}})^{f_1} \times (Z_{p^{e_1-1}})^{f_2} \times \cdots \times (Z_{p^2})^{f_{e_1-1}} \times (Z_p)^{f_{e_1}}.$$

where for $i = 1, \ldots, e_1$ the f_i are calculated successively as follows:

$$f_i = \begin{cases} (\frac{p^{m_1}-1}{p-1}) & \text{if } i = 1; \\ f^*(j) & \text{if } i = e_1 - e_j + 1 \text{ for some } j = 2, 3, \ldots, n; \\ f^0(j,k) & \text{if } i = e_1 - e_j + 1 + k \\ & \quad \text{where } 1 \leq k \leq e_j - e_{j+1} - 1 \text{ for some } j = 1, \ldots, n-1 \ (e_{n+1} = 0); \end{cases}$$

where

$$f^*(j) = (\frac{p^{m_j}-1}{p-1})[(\sum_{l=1}^{e_1-e_j}(e_1 - e_j + 2 - l)f_l)(p-1) + 1]$$

$$+ p^{m_j}(\sum_{l=1}^{e_1-e_{j-1}+1} f_l)p^{(e_{j-1}-e_j-1)(m_1+m_2+\cdots+m_{j-1}-1)}(p^{m_1+\cdots+m_{j-1}-1} - 1)$$

and

$$f^0(j,k) = (\sum_{l=1}^{e_1-e_j+1} f_l)p^{(k-1)(m_1+m_2+\cdots+m_j-1)}(p^{m_1+\cdots+m_j-1} - 1).$$

Proof. By induction on the number of factors n of G.
Let $n = 1$. Then $G \cong (Z_{p^{e_1}})^{m_1}$ where $e_1 \geq 1$, $m_1 \geq 1$, and we need to show that

$$Q_n \cong (f_1, f_2, \ldots, f_{e_1-1}, f_{e_1})$$

where

$$f_1 = (\frac{p^{m_1}-1}{p-1}) \quad \text{and} \quad f_l = f^0(1, l-1) \text{ for } l = 2, 3, \ldots, e_1.$$

But for $l = 2, 3, \ldots, e_1$ we get

$$\begin{aligned} f_l = f^0(1, l-1) &= (\sum_{t=1}^{e_1-e_1+1} f_t)p^{((l-1)-1)(m_1-1)}(p^{m_1-1} - 1) \\ &= f_1 p^{(l-2)(m_1-1)}(p^{m_1-1} - 1) \\ &= (\frac{p^{m_1}-1}{p-1})p^{(l-2)(m_1-1)}(p^{m_1-1} - 1). \end{aligned}$$

But this is true by theorem 1.6. Hence we may assume the result is true for all groups G with number of factors less than n.
Case 1 Suppose $e_n = 1$.
Let $G' = (Z_{p^{e_1}})^{m_1} \times (Z_{p^{e_2}})^{m_2} \times \cdots \times (Z_{p^{e_{n-1}}})^{m_{n-1}}$. Thus $G \cong G' \times (Z_p)^{m_n}$. By induction $Q_{G'} \cong (h_1, \ldots, h_{e_1})$ where

$$h_i = \begin{cases} (\frac{p^{m_1}-1}{p-1}) & \text{if } i = 1; \\ f^*(j) & \text{if } i = e_1 - e_j + 1 \text{ for some } j = 2, 3, \ldots, n; \\ f^0(j,k) & \text{if } i = e_1 - e_j + 1 + k \\ & \quad \text{where } 1 \leq k \leq e_j - e_{j+1} - 1 \text{ for some } j = 1, \ldots, n-1 \ (e_n = 0); \end{cases}$$

with the corresponding f_i replaced by h_i in the functions f^* and f^0. We need to show $Q_G \cong (f_1, f_2, \ldots, f_{e_1})$ where the f_i's are as stated in the theorem (with $e_n = 1$). Note since $h_1 = f_1$ this says that

$$\begin{aligned} h_i = f_i \quad &\text{for} \quad i = 1, \ldots, (e_1 - e_{n-1}) + 1 + (e_{n-1} - e_n - 1) \\ &\text{or for} \quad i = 1, \ldots, e_1 - 1, \text{ as } e_n = 1. \end{aligned}$$

Note $h_{e_1} = f^0(n-1, e_{n-1} - 1)$ and $f_{e_1} = f^*(n)$. By corollary 2.10 we have

$$Q_G = (f_1, f_2, \ldots, f_{e_1-1}, (\frac{p^{m_n}-1}{p-1})(F+1) + p^{m_n} h_{e_1})$$

where

$$F = (\sum_{i=1}^{e_1-1}(e_1 - i + 1)f_i)(p-1).$$

i.e. By above we need to show that

$$f^*(n) = f_{e_1} = (\frac{p^{m_n}-1}{p-1})[(\sum_{i=1}^{e_1-1}(e_1 - i + 1)f_i)(p-1) + 1] + p^{m_n} f^0(n-1, e_{n-1}-1).$$

i.e. does

$$(\frac{p^{m_n}-1}{p-1})[(\sum_{l=1}^{e_1-1}(e_1 - 1 + 2 - l)f_l)(p-1) + 1]$$

$$+ p^{m_n}(\sum_{l=1}^{e_1-e_{n-1}+1} f_l)p^{(e_{n-1}-1-1)(m_1+\ldots+m_{n-1}-1)}(p^{m_1+\ldots+m_{n-1}-1} - 1)$$

$$= (\frac{p^{m_n}-1}{p-1})[(\sum_{i=1}^{e_1-1}(e_1 - i + 1)f_i)(p-1) + 1] + p^{m_n} f^0(n-1, e_{n-1}-1).$$

Cancelling like terms we need to show that

$$(\sum_{l=1}^{e_1-e_{n-1}+1} f_l)p^{(e_{n-1}-2)(m_1+\ldots+m_{n-1}-1)}(p^{m_1+\ldots+m_{n-1}-1} - 1)$$

$$= f^0(n-1, e_{n-1}-1)$$

$$= (\sum_{l=1}^{e_1-e_{n-1}+1} f_l)p^{(e_{n-1}-1-1)(m_1+\ldots+m_{n-1}-1)}(p^{m_1+\ldots+m_{n-1}-1} - 1)$$

which is obvious. Thus Case 1 is finished.

Case 2 $e_n > 1$.

Let $G' = (Z_{p^{e_1-e_n+1}})^{m_1} \times (Z_{p^{e_2-e_n+1}})^{m_2} \times \cdots \times (Z_{p^{e_{n-1}-e_n+1}})^{m_{n-1}}$.

Let $G'' = G' \times (Z_p)^{m_n}$. Then by case 1 (as we can now assume the result is true for all groups with n factors if $e_n = 1$) $Q_{G''} = (f_1', f_2', \ldots, f_{e_1-e_n+1}')$ where the f_i''s are given by

$$f_i' = \begin{cases} (\frac{p^{m_1}-1}{p-1}) & \text{if } i = 1; \\ \hat{f}^*(j) & \text{if } i = e_1 - e_j + 1 \text{ for some } j = 2, 3, \ldots, n; \\ \hat{f}^0(j,k) & \text{if } i = e_1 - e_j + 1 + k \\ & \quad \text{where } 1 \le k \le e_j - e_{j+1} - 1 \text{ for some } j = 1, \ldots, n \ (e_{n+1} = 0); \end{cases}$$

where

$$\hat{f}^*(j) = f^*(j) \quad \text{and} \quad \hat{f}^0(j,k) = f^0(j,k)$$

(with the corresponding f_l replaced by f_l' in the functions f^*, f^0). We need to show $Q_G = (f_1, f_2, \ldots, f_{e_1})$ where the f_i's are as stated in the theorem. Now by corollory 2.7 we have

$$Q_G = (f_1', \ldots, f_{e_1-e_n+1}', f_{(e_1-e_n+1)+1}', \ldots, f_{(e_1-e_n+1)+(e_n-1)}')$$

where

$$f_{(e_1-e_n+1)+j}' = \left(\sum_{i=1}^{e_1-e_n+1} f_i' \right) p^{(j-1)(m_1+\cdots+m_n-1)}(p^{(m_1+\cdots+m_n-1)} - 1).$$

Note because $f_1' = f_1$ we have by simple induction

$$f_i = f_i' \quad \text{for } i = 1, \ldots, e_1 - e_n + 1.$$

Thus we need to show that

$$f_{(e_1-e_n+1)+j} = f_{(e_1-e_n+1)+j}' \quad \text{for } j = 1, \ldots, e_n - 1.$$

But both are equal to $f^0(n,j)$ by inspection. Hence we are finished with the case $e_n > 1$ and the theorem is proved.

This completes the description of the augmentation terminal of a finite abelian group. Although it is not pretty, it does allow us to prove some earlier known results much easier. This is the subject of the next section.

Section 3 *Illustrations.*

Finally, in this section we illustrate theorem 2.11 by some examples. In particular, we obtain some previously known results, as well as an example using only The Tacking On Lemma and The Moving Up Lemma.

To begin with we obtain the first result due to Passi.

3.1 Example. *[cf. Remark I.3.4c]*
If $G \cong (Z_p)^m$ then $Q_G \cong (Z_p)^{\left(\frac{p^m-1}{p-1}\right)}$ for $m \geq 1$.

Proof. By 2.11 $Q_G \cong (Z_p)^{f_1}$ where $f_1 = \left(\frac{p^m-1}{p-1}\right)$. ∎

Since we used theorem 1.6 in the proof of theorem 2.11 we should be able to recover it easily. It should be noted that we did not need to use 1.6 in the proof of 2.11 (Case $n = 1$). Instead we could have started with theorem *I*.2.4 (i.e. if $G \cong Z_p$ then $Q_G \cong Z_p$) and used The Tacking on Lemma and The Moving Up Lemma to accomplish this task. If we had done this, we would then get:

3.2 Example. *[cf. Theorem 1.6]*
If $G \cong (Z_{p^n})^m$ then

$$Q_G \cong (Z_{p^n})^{\left(\frac{p^m-1}{p-1}\right)} \times \prod_{j=1}^{n-1} (Z_{p^{n-j}})^{\left(\frac{p^m-1}{p-1}\right)p^{(m-1)(j-1)}(p^{m-1}-1)}.$$

Proof. In the notation of theorem 2.11 we have $e_1 = n$ and $m_1 = m$. Hence

$$Q_G \cong (Z_{p^n})^{f_1} \times (Z_{p^{n-1}})^{f_2} \times \cdots \times (Z_p)^{f_n}$$

where

$$f_1 = \left(\frac{p^m-1}{p-1}\right) \quad \text{and} \quad f_{1+j} = f^0(1,j) \text{ for } j = 1, \ldots, n-1.$$

But

$$f^0(1,j) = \left(\sum_{l=1}^{1} f_l\right)p^{(j-1)(m-1)}(p^{m-1}-1) = \left(\frac{p^m-1}{p-1}\right)p^{(j-1)(m-1)}(p^{m-1}-1)$$

which gives the result. ∎

This next example will complete the case for all groups G of exponent p^2.

3.3 Example. *Exponent p^2*
Let $G \cong (Z_{p^2})^{m_1} \times (Z_p)^{m_2}$ for some prime p and $m_1, m_2, \geq 0$. Then

$$Q_G \cong (Z_{p^2})^a \times (Z_p)^b$$

where

$$a = \left(\frac{p^{m_1}-1}{p-1}\right) \quad \text{and} \quad b = p^{m_1}\left(\frac{p^{m_2}-1}{p-1}\right) + \left(\frac{p^{m_1}-1}{p-1}\right)(p^{m_1+m_2-1}-1).$$

Proof. If $m_1 = m_2 = 0$ (i.e. $G = <1>$) then $a = 0 = b$ and so $Q_G = <1>$ and we are trivially done.
If $m_1 = 0,\ m_2 \geq 1$ then $a = 0$ and $b = \left(\frac{p^{m_2}-1}{p-1}\right)$. Hence $G \cong (Z_p)^{m_2}$ and

$$Q_G \cong (Z_{p^2})^0 \times (Z_p)^{\left(\frac{p^{m_2}-1}{p-1}\right)} \cong (Z_p)^{\left(\frac{p^{m_2}-1}{p-1}\right)}$$

which is true by example 3.2.
Likewise, if $m_1 \geq 1$ and $m_2 = 0$ then we easily get our result.
Finally let $m_1, m_2 \geq 1$. By theorem 2.11 we have

$$Q_G \cong (Z_{p^2})^{f_1} \times (Z_p)^{f_2}$$

where

$$
\begin{aligned}
f_1 &= (\frac{p^{m_1}-1}{p-1}) \qquad \text{and} \\
f_2 &= f^*(2) \\
&= (\frac{p^{m_2}-1}{p-1})[(\sum_{l=1}^{e_1-e_2}(e_1-e_2+2-l)f_l)(p-1)+1] \\
&\quad + p^{m_2}(\sum_{l=1}^{e_1-e_1+1} f_l)p^{(e_1-e_2-1)(m_1-1)}(p^{m_1-1}-1)\ . \\
&= (\frac{p^{m_2}-1}{p-1})[2(\frac{p^{m_1}-1}{p-1})(p-1)+1] + p^{m_2}(\frac{p^{m_1}-1}{p-1})(p^{m_1-1}-1) \\
&= (\frac{p^{m_2}-1}{p-1})[(p^{m_1}-1)+p^{m_1}] + p^{m_2}(\frac{p^{m_1}-1}{p-1})(p^{m_1-1}-1) \\
&= p^{m_1}(\frac{p^{m_2}-1}{p-1}) + (\frac{p^{m_1}-1}{p-1})[(p^{m_2}-1)+p^{m_2}(p^{m_1-1}-1)] \\
&= p^{m_1}(\frac{p^{m_2}-1}{p-1}) + (\frac{p^{m_1}-1}{p-1})[p^{m_1+m_2-1}-1]
\end{aligned}
$$

Noticing $a = f_1$ and $b = f_2$ we have our result. ∎

As a final example of the use of 2.11 we give the following result due to Singer.

3.4 Example. *[Singer '77a]*
Let $G \cong (Z_8)^a \times (Z_4)^b \times (Z_2)^c$ where $a, b, c \geq 0$. Then

$$Q_G \cong (Z_8)^A \times (Z_4)^B \times (Z_2)^C$$

where

$$
\begin{aligned}
A &= 2^a - 1 \\
B &= 2^{2a+b-1} + 2^{a+b-1} - 2^{a+1} + 1 \\
C &= 2^{3a+2b+c-2} + 2^{2a+2b+c-2} + 2^{a+b+c-1} - 2^{2a+b} - 2^{a+b} + 2^a.
\end{aligned}
$$

Proof. Note if $a = 0$ then this is example 3.3 with $p = 2$, $m_1 = b$, and $m_2 = c$ as

$$
\begin{aligned}
A &= 2^0 - 1 = 0, \\
B &= 2^{b-1} + 2^{b-1} - 2 + 1 = 2 \cdot 2^{b-1} - 1 = 2^b - 1, \text{ and} \\
C &= 2^{2b+c-2} + 2^{2b+c-2} + 2^{b+c-1} - 2^b - 2^b + 2^0 = 2^{2b+c-1} + 2^{b+c-1} - 2^{b+1} + 1.
\end{aligned}
$$

In example 3.3 we have $Q_G \cong (Z_4)^{f_1} \times (Z_2)^{f_2}$ where

$$\begin{aligned}
f_1 &= \left(\frac{2^{m_1}-1}{2-1}\right) = 2^{m_1} - 1, \text{ and} \\
f_2 &= 2^{m_1}\left(\frac{2^{m_2}-1}{2-1}\right) + \left(\frac{2^{m_1}-1}{2-1}\right)(2^{m_1+m_2-1} - 1) \\
&= 2^{m_1}(2^{m_2}-1) + (2^{m_1}-1)(2^{m_1+m_2-1}-1) \\
&= 2^{m_1+m_2} - 2^{m_1} + 2^{2m_1+m_2-1} - 2^{m_1} - 2^{m_1+m_2-1} + 1 \\
&= 2^{2m_1+m_2-1} + 2^{m_1+m_2-1}(2-1) - 2(2^{m_1}) + 1 \\
&= 2^{2m_1+m_2-1} + 2^{m_1+m_2-1} - 2^{m_1+1} + 1
\end{aligned}$$

which gives us the result.
Hence we may assume $a \geq 1$.
<u>*Case 1*</u> $b = c = 0$
Then $G \cong (Z_8)^a$ which by example 3.2 gives

$$Q_G \cong (Z_8)^{f_1} \times (Z_4)^{f_2} \times (Z_2)^{f_3}$$

where

$$\begin{aligned}
f_1 &= 2^a - 1, (= A) \\
f_2 &= (2^a-1)(2^{a-1}-1) = 2^{2a-1} - 2^a - 2^{a-1} + 1, \\
f_3 &= (2^a-1)2^{(a-1)}(2^{a-1}-1) = 2^{a-1}f_2.
\end{aligned}$$

Now note that

$$-2^a - 2^{a-1} = 2^{a-1}(-2-1) = 2^{a-1}(1-4) = 2^{a-1} - 2^{a+1}$$

which gives $f_2 = B$. Also

$$\begin{aligned}
f_3 = 2^{a-1}f_2 &= 2^{a-1}(2^{2a-1} + 2^{a-1} - 2^{a+1} + 1) \\
&= 2^{3a-2} + 2^{2a-2} - 2^{2a} + 2^{a-1} = C.
\end{aligned}$$

<u>*Case 2*</u> $c \geq 1, b \geq 1$
Then by 2.11

$$Q_G \cong (Z_8)^{f_1} \times (Z_4)^{f_2} \times (Z_2)^{f_3}$$

where

$$
\begin{aligned}
f_1 &= 2^a - 1 = A, \\
f_2 &= f^*(2) = (2^b - 1)[2(2^a - 1)(2 - 1) + 1] + 2^b(2^a - 1)(2^{a-1} - 1) \\
&= (2^b - 1)(2^{a+1} - 1) + 2^b(2^{2a-1} - 2^{a-1} - 2^a + 1) \\
&= 2^{a+b+1} - 2^b - 2^{a+1} + 1 + 2^b(2^{2a-1} + 2^{a-1} - 2^{a+1} + 1) \text{ (see Case 1)} \\
&= 2^{a+b+1} - 2^b - 2^{a+1} + 1 + 2^{2a+b-1} + 2^{a+b-1} - 2^{a+b+1} + 2^b \\
&= 2^{2a+b-1} + 2^{a+b-1} - 2^{a+1} + 1 \\
&= B, \text{ and} \\
f_3 &= f^*(3) = (2^c - 1)[(3f_1 + 2f_2)(2 - 1) + 1] + 2^c(f_1 + f_2)(2^{(a+b-1)} - 1) \\
&= (2^c - 1)[3(2^a - 1) + 2(2^{2a+b-1} + 2^{a+b-1} - 2^{a+1} + 1) + 1] \\
&\quad + (2^{a+b+c-1} - 2^c)[2^{2a+b-1} + 2^{a+b-1} + \text{[illegible]} - 2) + (-1 + 1)] \\
&= (2^c - 1)[2^{2a+b} + 2^{a+b} - 2^a] + (2^{a+b+c-\text{[illegible]}}2^{2a+b-1} + 2^{a+b-1} - 2^a] \\
&= 2^{3a+2b+c-2} + 2^{2a+2b+c-2} + 2^{a+b+c-1}(2\text{[illegible]}{}^{a+b} \\
&\quad - 2^{a+b} + 2^a + 2^{2a+b+c-1}(2 - 1 - 1) + 2^{a\text{[illegible]}}(-1 + 1) \\
&= C.
\end{aligned}
$$

Finally the two cases $b \geq 1, c = 0$, and $b = 0, c \geq 1$ are handled in a similiar manner to case 2. Thus we have proved the result. ∎

It should be noted that we could easily handle exponent p^3 groups as in the above example, except that simplifying the results is no longer so easy as we extensively used the fact that $p - 1 = 2 - 1 = 1$ in the above.

Finally we conclude this section and chapter by finding the augmentation terminal of the group $G = Z_{p^5} \times (Z_{p^3})^2 \times Z_{p^2}$ (of order p^{13}) using lemmas 2.4 and 2.8 and their corollaries.

3.5 Example. Let $G = Z_{p^5} \times (Z_{p^3})^2 \times Z_{p^2}$. We want to calculate Q_G using The Tacking On Lemma and The Moving Up Lemma. In order to do this we will return to our notation of section 2. Here we have

$$e_1 = 5,\ m_1 = 1,\ e_2 = 3,\ m_2 = 2, \text{ and } e_3 = 2,\ m_3 = 1.$$

Note $e_1 - e_2 + 1 = 3$.
Thus consider $G_1 \cong Z_{p^3}$. Then $Q_{G_1} \cong (1, 0, 0)$ (example 3.2).
Now let $G_2 \cong Z_{p^3} \times (Z_p)^2$. Then $Q_{G_2} \cong (1, 0, f_3)$ where

$$
\begin{aligned}
f_3 &= \left(\frac{p^2 - 1}{p - 1}\right)\left[(3(1) + 2(0))(p - 1) + 1\right] + p^2(1) \quad \text{(cor. 2.10)} \\
&= 4p^2 + p - 2.
\end{aligned}
$$

Since $e_2 - e_3 + 1 = 2$ we consider $G_3 \cong Z_{p^4} \times (Z_{p^2})^2$. Then by lemma 2.4 we have $Q_{G_3} \cong (1, 0, 4p^2 + p - 2, f_4')$ where

$$f_4' = (1 + 0 + (4p^2 + p - 2))(p^{3-1} - 1) = 4p^4 + p^3 - 5p^2 - p + 1.$$

Now let $G_4 \cong Z_{p^4} \times (Z_{p^2})^2 \times Z_p$. By lemma 2.8 we get $Q_{G_4} \cong (1, 0, 4p^2 + p - 2, f_4)$ where

$$\begin{aligned} f_4 &= [4(1) + 3(0) + 2(4p^2 + p - 2)](p - 1) + 1 + p(4p^4 + p^3 - 5p^2 - p + 1) \\ &= 4p^5 + p^4 + 3p^3 - 7p^2 - p + 1. \end{aligned}$$

Finally let $G_5 \cong G \cong Z_{p^6} \times (Z_{p^3})^2 \times Z_{p^2}$. By lemma 2.4 we have

$$Q_{G_5} \cong (1, 0, 4p^2 + p - 2, 4p^5 + p^4 + 3p^3 - 7p^2 - p + 1, f_5)$$

where

$$\begin{aligned} f_5 &= ((1) + (0) + (4p^2 + p - 2) + (4p^5 + p^4 + 3p^3 - 7p^2 - p + 1))(p^{4-1} - 1) \\ &= 4p^8 + p^7 + 3p^6 - 7p^5 - p^4 - 3p^3 + 3p^2. \end{aligned}$$

i.e. If $G \cong Z_{p^6} \times (Z_{p^3})^2 \times Z_{p^2}$ then

$$Q_G \cong (Z_{p^5})^a \times (Z_{p^4})^b \times (Z_{p^3})^c \times (Z_{p^2})^d \times (Z_p)^e$$

where

$$\begin{aligned} a &= 1, \\ b &= 0, \\ c &= 4p^2 + p - 2, \\ d &= 4p^5 + p^4 + 3p^3 - 7p^2 - p + 1, \text{ and} \\ e &= 4p^8 + p^7 + 3p^6 - 7p^5 - p^4 - 3p^3 + 3p^2. \end{aligned}$$

∎

It should be noted that this method of calculating the augmentation terminal of a given group G can easily be implemented in a computer program.

CHAPTER 4

Summary of the Non-abelian Case.

We now know what $Q_\infty(G)$ is for G a finite abelian group. But what is known about $Q_n(G)$ when G is a finite non-abelian p-group? That is the subject of the following section.

Section 1 *A Brief Account.*

From the study of the abelian case, we know the structure of $Q_n(G)$ can be quite complicated. This is even more so when G is non-abelian. One of the added degrees of difficulty is that $Q_n(G)$ may no longer have an augmentation terminal $Q_\infty(G)$. Bachmann and Grunenfelder's result (II.1.1, which can be stated for all finite groups) only guarantees that the $Q_n(G)$ will become periodic. Hence for a group G we must calculate π groups $(Q_{n_0}(G), Q_{n_0+1}(G), \cdots, Q_{n_0+\pi-1}(G))$ (where π is the period) in order to determine the stable behavior of $Q_n(G)$.

Given this and the complicated nature of the abelian case, it is easy to understand why we do not know what the stable behavior of $Q_n(G)$ is, for all non-abelian finite groups G. Actually, we know relatively little of what occurs in the non-abelian case and it is likely that any complete answer for this will be extremely complicated and detailed. However, there have been several methods developed for calculating $Q_n(G)$ for certain non-abelian p-groups. So far these methods have only been used for groups of order p^m where $m \leq 4$. It should be stated that these methods require very detailed calculations and probably can not be extended for use on all groups.

Gerald Losey and Nora Losey were the first to calculate the stable behavior of $Q_n(G)$ for a non-abelian group G in [Losey and Losey '79a]. They wanted to present some examples of Bachmann and Gruenfelder's result for a group which was not abelian. In order to do this, they considered the class of finite p-groups in which the lower central series is an N_p series. (An N_p series for a group G and prime p is a normal series $\mathcal{H} = \{H_i\}_{i=1}^{\infty}$ in G with $[H_i, H_j] \leq H_{i+j}$ and $H_i^p \leq H_{ip}$ $\forall i,j \geq 1$.)

In this situation, it is true that $Q_n(G)$ is a finite elementary abelian p-group and hence they needed only to calculate the rank of $Q_n(G)$. To do this they first found a special set of basis elements for each Δ^n. From this, a basis for $Q_n(G)$ was found and counting arguements give the results for various groups of this class.

The results Losey and Losey were able to achieve in this manner were as follows:

1.1 Result. (Proposition 3.1 [Losey and Losey '79a].)
Let $G = < x, y, z \mid x^p = y^p = z^p = [x,z] = [y,z] = 1, [x,y] = z >$ where p is an odd prime.

For all $n \geq 2p - 1$, $Q_n(G)$ is an elementary abelian p-group of rank

$$s_n(G) = \begin{cases} \frac{1}{2}(p+1)^2 & \text{if } n \text{ is odd;} \\ \frac{1}{2}(p+1)^2 + 1 & \text{if } n \text{ is even.} \end{cases}$$

1.2 Result. (Proposition 3.3 [Losey and Losey '79a].)
Let G be a finite 2-group of class 2 in which G/G' is elementary abelian of rank t_1 and G' is elementary abelian of rank t_2. For all $n \geq t_1 + 2t_2 - 1$, $Q_n(G)$ is an elementary abelian 2-group of rank $2^{t_1 - 1}(2^{t_2} + 1) - 1$. ($G' = [G, G]$.)

They also presented one more result (Proposition 3.2 [Losey and Losey '79a]) which calculated $Q_n(G)$ for a finite group of order p^4, exponent p, and class 3. However this will be stated later.

Gerald Losey and Nora Losey then calculated the stable behavior of $Q_n(G)$ for all remaining groups of order p^3 in [Losey and Losey '79b]. This meant they had to calculate $Q_n(G)$ for one more non-abelian p-group. (They also had to handle the remaining abelian groups as Hales' result was not yet known.) As this non-abelian group has a lower central series which was not an N_p series, a different method was employed. Again they explicitly constructed a basis for $\Delta^n(G)$ and calculated the structure of $Q_n(G)$ from these. However a different method was involved in calculating the basis. Their result for this final non-abelian group of order p^3 was:

1.3 Result. (Proposition 4.2 [Losey and Losey '79b].)
Let $G = \langle x, y \mid x^p = y^{p^2} = [x, y^p] = 1, [x, y] = y^p \rangle$. For all $n \geq 2p - 1$, $Q_n(G) \cong (Z_p)^{(2p+1)}$.

Finally, the last work known about $Q_n(G)$ where G is non-abelian was seen in a series of three papers by Kazunori Horibe, Ken-Ichi Tahara, and Tsuneyo Yamada in [Horibe and Tahara '84], [Tahara and Yamada '84], and [Tahara and Yamada '85]. In these papers they continued the work of Losey and Losey by calculating the stable behavior of $Q_n(G)$ for all groups G of order p^4.

By extending the methods employed by Losey and Losey and making use of a result of [Tahara '82] they were able to finish the case where G has order p^4. As evidence of the detailed calculations needed it can be noted that these three papers were over seventy pages in length.

The result of Tahara mentioned above was:

1.4 Result. (Theorem 4.3 [Tahara '82].)
If $G = NT$ is the semidirect product of a normal subgroup N and a subgroup T, then

$$\Delta^m(G) \cong \Delta^m(T) \oplus \Lambda_m \oplus \Gamma_m, \quad m \geq 1,$$

where $\{\Lambda_m\}_{m=1}^{\infty}$ is the canonical filtration of $\Delta(N)$ with respect to the N-series

$$N = N_{(1)} \geq N_{(2)} \geq \cdots \geq N_{(d)} \geq N_{(d+1)} = 1;$$

with $N_{(1)} = N$ and $N_{(k)} = [N_{(k-1)}, G]$ $k \geq 2$. Also $\Gamma_1 = \Delta(T)\Lambda_1$ and

$$\Gamma_n = \sum_{i=1}^{n-1} \Delta^{n-i}(T)\Lambda_i \; n \geq 2.$$

This theorem was used in calculating the basis elements for $Q_n(G)$.

For completeness, the results for the non-abelian groups of order p^4 follow.

1.5 Table. $Q_\infty(G)$ for G (non-abelian) of order p^4 where p is an odd prime.

1) [Horibe and Tahara '84 Propostion 3.1]
$G = Z_p \times < x, y, x \mid x^p = y^p = z^p = [z, x] = [z, y] = 1, [y, x] = z >$. Let $n_0 = 3p - 2$.
Then $Q_{n_0+i}(G) = (Z_p)^{(1/2)(p+1)(p^2+p+1)+1}$

2) [Losey and Losey '79a Proposition 3.2]
$G = < x, y, z, w \mid x^p = y^p = z^p = w^p = [w, x] = [w, y] = [w, z] = [z, y] = 1,$
$[y, x] = z, [z, x] = w >$.
Let $n_0 = 4p - 3$. Then $Q_{n_0+i}(G) = (Z_p)^{s_{n_0+i}(G)}$ $(i \in \{0, 1, 2, 3, 4, 5, 6\})$ where:
Case 1. $p \equiv 1 \bmod 3$,
$s_{n_0+i}(G) = K(i = 0, 4),\; K + 1(i = 1, 2, 3),\; K + 2(i = 5)$,
where $K = 1 + p + \frac{1}{2}(p^2 - 1) + \frac{1}{3}(p^3 - 1)$,
Case 2. $p \equiv 2 \bmod 3$,
$s_{n_0+i}(G) =$ [illegible] $0),\; L + 2(i = 1, 5),\; L + 1(i = 2, 3, 4)$,
wher[illegible] $p + \frac{1}{2}(p^2 - 1) + \frac{1}{3}(p^3 - 2)$.

3) [Hor[illegible]ahara '84 Proposition 5.3]
[illegible] $y, z \mid x^p = y^p = z^{p^2} = [y, x] = [z^p, x] = 1, [z, x] = y, [z, y] = z^p >$.
Le[illegible] $4p - 3$. Then $Q_{n_0+i}(G) = (Z_p)^{(\frac{1}{2})(3p^2+2p+1)+i}$, $i = 0, 1$.

4) [Horibe and Tahara '84 Proposition 4.3]
$G = Z_p \times < x, y \mid x^p = y^{p^2} = 1, [y, x] = y^p >$. Let $n_0 = 3p - 2$.
Then $Q_{n_0}(G) = (Z_p)^{2p^2+p+1}$.

5) [Horibe and Tahara '84 Proposition 6.3]
$G = < x, y, z \mid x^p = y^p = z^{p^2} = [y, x] = [z, y] = 1, [z, x] = y >$. Let $n_0 = 4p - 3$.
Then $Q_{n_0+i}(G) = (Z_p)^{(\frac{1}{2})(3p^2+2p-3)+i} \oplus Z_{p^2}$, $i = 0, 1$.

6) [Tahara and Yamada '85 Proposition 4.10]
$G = < x, y \mid x^{p^2} = y^{p^2} = 1, [y, x] = y^p >$. Let $n_0 = p^2 + 2p - 1$.
Then $Q_{n_0}(G) = (Z_p)^{p^2+2p-1} \oplus Z_{p^2}$.

7) [Horibe and Tahara '84 Proposition 4.3]
$G = < x, y, z \mid x^p = y^p = z^{p^2} = [z, x] = [z, y] = 1, [y, x] = z^p >$. Let $n_0 = 3p - 2$.
Then $Q_{n_0}(G) = (Z_p)^{2p^2+p+1}$.

8) [Tahara and Yamada '84 Proposition 4.3]
$G = < x, y, z \mid x^p = y^p = z^{p^2} = [z, y] = 1, [z, x] = y, [y, x] = z^p >$. Let $n_0 = 4p - 3$.
Then $Q_{n_0+i}(G) = (Z_p)^{(\frac{1}{2})(3p^2+2p+1)+i}$ $i = 0, 1$.

9) [Tahara and Yamada '84 Proposition 4.3]
$G = < x, y, z \mid x^p = y^p = z^{p^2} = [z, y] = 1, [z, x] = y, [y, x] = z^{\alpha p} >$. Let $n_0 = 4p - 3$.
Then $Q_{n_0+i}(G) = (Z_p)^{(\frac{1}{2})(3p^2+2p+1)+i}$ $i = 0, 1$.
(α is a quadratic non-residue modulo p.)

10) [Tahara and Yamada '84 Proposition 5.7]
$G = < x, y \mid x^p = y^{p^3} = 1, [y, x] = y^{p^2} >$. Let $n_0 = 3p - 2$.
Then $Q_{n_0}(G) = (Z_p)^{3p-1} \oplus Z_{p^2}$.

11) [Tahara and Yamada '85 Proposition 3.1]
$G = < x, y, z \mid y^3 = z^9 = [z, y] = 1, x^3 = z^3, [y, x] = z^{-3}, [z, x] = y >$. Let $n_0 = 9$.
Then $Q_{n_0+i}(G) = (Z_3)^{17+i}$, $i = 0, 1$.

This concludes what is known for non-abelian groups. As one can see, the problem for non-abelian groups is far from being completed.

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