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THE INFLATIONARY PROCESS AND STABILISATION POLICIES IN A DYNAMIC MACRO-MODEL INVOLVING THE GOVERNMENT BUDGET CONSTRAINT

by

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled **THE INFLATIONARY PROCESS AND STABILISATION POLICIES IN A DYNAMIC MACRO-MODEL INVOLVING THE GOVERNMENT BUDGET CONSTRAINT** submitted by Ioannis Tirkides in partial fulfilment of the requirements for the degree of Master of Arts.

Brian L. Scarfe
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Date 22 September, 1986

ABSTRACT

This study examines the behaviour of a dynamic model of the inflationary process which allows for expectational adjustments, incorporates a government budget constraint, and in some cases a monetary policy reaction function. The specific objective is one of exploring the conditions that are both necessary and sufficient for the dynamic stability of the inflationary process in the neighbourhood of a steady inflation equilibrium path under alternative policy regimes.

The main results can be stated as follows:

(1) The system cannot be stable under both policy regimes of endogenous monetary growth and endogenous debt issuance via the government budget constraint. If it is stable under either of these two policy regimes then it must be unstable under the other. This result is termed the 'dual instability theorem'.

(2) In that case in which the system is unstable when debt issuance is the policy instrument endogenised via the government budget constraint, if there is to be hope for stability when a monetary policy reaction function is integrated into the system, monetary policy must at least be completely accommodative.

(3) In general, monetary accommodation enters the system with stabilising repercussions in those cases in which debt issuance is the residual instrument of financing, and can

potentially become destabilising in those cases in which government spending is the policy instrument endogenised via the budget constraint.

(4) Any flexibility with respect to adjusting the level of government expenditures can significantly enhance stability.

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NOMENCLATURE

B	Nominal value of the stock of bonds outstanding.
BP^{-1}	Real value of the stock of bonds outstanding.
b	$= B^{-1}DB$, the rate of growth of the nominal stock of bonds.
g	The level of real government expenditures.
M	Nominal stock of money balances.
MP^{-1}	Real stock of money balances.
m	$= M^{-1}DM$, the growth rate of the nominal stock of money balances.
P	Aggregate price index.
p	$= P^{-1}DP$, the rate of price inflation.
p^*	Expected inflation.
q	Trend growth rate of labour productivity.
r	The nominal rate of interest.
W	Money wage rate.
WP^{-1}	Real wage rate.
w	$= W^{-1}DW$, the rate of wage inflation.
y	Real output.

Note that a bar over a variable denotes its long-run equilibrium value, and that

$D = d/dt$ is the time differential operator.

parameters

$a > 0$

$a_1 > 0$

$0 < c' \leq 1$

$0 < \tau < 1$

$l_1 > 0$

$l_2 < 0$

$l_3 < 0$

$l_4 > 0$

$c_1 = c'(1-\tau)$

$c_2 = i' < 0$

$c_3 = -c_2 > 0$

$z_1 \leq 0$

$0 \leq z_2 \leq 1$

$a_2 = (1-a_1)^{-1}\Phi_y > 0$

I. INTRODUCTION

The endogeneity/exogeneity of policy in general and of monetary policy in particular is a major area of controversy in the economics literature.[1] In the Monetarist models, the exogeneity of monetary policy is usually assumed. The line of causation runs from the monetary authorities, to the monetary base, to the money supply and money income, and to output and prices. This is a precondition necessary to sustain the proposition that inflation is above all a monetary phenomenon.

Keynesians, in general, tend to emphasize the impact of increased money incomes on output rather than prices and to close the line of causation with a fourth linkage running from prices and output to monetary policy. With a budget constraint in place at least one policy instrument has to be endogenous. With a government deficit putting upward pressures on interest rates and with the Central Bank monetising the debt to ease the pressures on interest rates, fiscal and monetary policies become interdependent and the money supply is endogenous.

This interdependence of policy instruments via the government budget constraint and the consequent endogeneity of at least one policy instrument generated a substantial body of literature. In early work, including the

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[1]More will be said on this theme in the context of chapter two.

contributions of Christ (1968) and of Blinder and Solow (1973, 1974), the concern was with fix-price models. Generally speaking, these models tended to supplement a static IS-LM system with a government budget constraint.

Certainly, the assumption that the price level is fixed is unrealistic, and in later work the emphasis shifted to inflationary models. The aggregate price level was usually endogenised by means of a Phillips-curve relationship of some sort. In general, inflation models which incorporate a government budget constraint fall into two main categories. Firstly, there are those which treat productive capacity as fixed (see for instance the Christ 1978 article). Secondly, there are those models which allow for growth of productive capacity and which are more associated with the money and growth literature. The contributions of J. Niehans (1974), Turnovsky (1977 chapter 8), Feldstein (1980), and Infante and Stein (1980) among others, have mainly concentrated on this latter facet of the issue.

The study at hand falls into the first category. The purpose is one of analysing the inflationary process in the context of a dynamic macro-model that incorporates a government budget constraint, thus endogenising one policy instrument.

However, the causal role of governments in inflation is different in different circumstances. When the government increases the money supply in an ad hoc fashion it directly causes inflation. But when it responds to exogenous shocks the government is faced with the problem of whether or not to validate cost-push price increases. The result of validation would be a once-and-for-all rise in the general price level. How much the price level will rise would depend on whether in the bargaining process labour and capital recognize the shock. If the cost consequences of exogenous shocks get into expectations, then validating would entail accommodating the subsequent wage-wage and wage-price consequences. If, on the other hand, the government chooses not to validate, then this would have the consequence of higher unemployment with associated deflationary effects offsetting the inflationary effects of the exogenous cost-push shock. In this particular study, the role which monetary authorities can potentially play in the inflationary process will be analysed by means of a monetary policy reaction function.

The central issue is that of dynamic stability and, in this sense, this study attempts to extend the seminal work of Carl Christ (1978). In that 1978 article, Christ examines the issue of dynamic stability in the context of an economic model which consists of the familiar IS-LM apparatus, a Phillips-curve relationship, and a government budget constraint. Inflationary expectations are treated exogenous and there are five potential policy instruments: the high powered money stock, the amount of government debt outstanding, government expenditures, the marginal tax-transfer rate, and the level of autonomous nominal taxes less transfers.

Based on certain assumptions regarding the parameters of the model, Carl Christ demonstrates that the necessary and sufficient conditions for stability are satisfied in those cases in which either the tax-transfer variable, or government spending, or the high powered money stock is the variable endogenised via the government budget constraint. When the amount of government debt outstanding is the policy instrument endogenised via the government budget constraint, the system becomes unstable.

There are two basic ways in which the present study extends the work of Carl Christ in particular. First, by getting out of a world of static expectations, and second, by considering the influence of policy reaction functions.

However, before proceeding with such an analysis, the issue that will concern us is that of model specification. This task is explicitly dealt with, in the context of chapter two. The issues that are then raised involve the long-run steady-state properties of the system, and the short-run macro-dynamics involved in-between such long-run steady states. The first issue is dealt with in the context of chapter three. The latter issue, on the other hand, occupies chapters four to seven and is dealt with by means of exploring qualitatively the short-run dynamics of the system, in the neighbourhood of its long-run equilibrium time path, under alternative policy regimes. The system in question considers three policy instruments: the rates of monetary growth and debt issuance, and the level of real government expenditures. Six distinctly different policy regimes are here examined:

(a) Endogenous monetary growth via the government budget constraint with the rate of debt issuance and the level of government spending exogenous.

(b) Endogenous debt issuance via the government budget constraint with monetary growth and government spending exogenous.

(c) Endogenous debt issuance and monetary growth, the former via the government budget constraint and the latter via a monetary policy reaction function, when again government spending is exogenous.

(d) Endogenous government expenditures via the budget constraint with the rates of monetary growth and debt issuance both exogenous.

(e) Endogenous government spending and monetary growth the former via the budget constraint and the latter via the monetary policy reaction function, when debt issuance is exogenous.

(f) All three policy instruments are simultaneously endogenous, government spending and monetary growth as in (e), and the rate of debt issuance constrained to follow the time path of monetary growth.

The first of these policy regimes is explored in chapter four. Chapter five deals with the policy regimes (b) and (c), chapter six with (d) and (e), and chapter seven with the policy regime (f).

II. THE GENERAL MODEL: SPECIFICATION AND OVERVIEW

The immediate question that is being addressed in the present chapter is that of model specification, an issue that is fundamentally a matter of art, one may say. A model has to meet the objectives of a particular study, involve the minimum amount of complexity, and at the same time allow us to draw conclusions which are of some use to policy makers.

The inflationary process constitutes the general theme of this study. The specific objective, however, is one of exploring the conditions that are both necessary and sufficient for the dynamic stability of the inflationary process, at least in the neighbourhood of a steady inflation equilibrium path.

The imported inflation thesis is not the issue here and consequently there is no reason to open up our model economy to foreign trade. A closed economy context would serve the purpose of this study. In addition, and in order to avoid the complexities associated with relative prices, it is assumed that the model economy in question is also a one-good economy. There are, thus, three principal economic agents: households, that sell their labour services at some wage rate; firms, that produce the economy's aggregate output; and a government, that finances its budget deficits by the sale of bonds to the household sector, and/or the creation of new money.

A. The Wage-Price Sector.

The wage-price sector will here be modelled on the basis of a purely bargaining model for wage setting, and a combination of the normal-cost-pricing and expectations-excess-demand hypotheses for price setting.

the wage equation

The mere fact that most wages in the economy are the result of a bargaining procedure has led to the development of two lines of thought with respect to wage determination. First, wage setting is explained via explicit bargaining models, and second, variables that are proxies for trade union bargaining strength such as for example, the percentage of the labour force that is unionised, are introduced into wage equations.

The main conclusions associated with bargaining models of wage determination are, firstly, that inflationary expectations are important, and secondly, that nominal price movements affect wage setting in a one-to-one fashion. Both of these conclusions are incorporated into the wage equation here specified.

In the context of a bargaining model, wage setting is primarily an institutional procedure. Wage inflation, hence, is most importantly a function of the trend growth rate of labour productivity (q) and price inflation (p) as a kind of a proxy variable for cost of living changes. The adjustment to inflation, however, can be decomposed into an ex-ante factor represented by inflationary expectations (p^*), and an ex-post factor represented by the actual rate of price inflation. To allow for a one hundred percent ex-ante and ex-post adjustment to inflation, the weights on actual and expected inflation must add-up to unity.

With all the above in mind one can write the wage equation in the following way:

$$w = q + a_1 p + (1-a_1)p^*, \qquad (1.0)$$

where a_1 is positive but strictly less than unity. This equation, however, can be augmented by means of excess demand in the labour market as an argument which can have an independent impact on wage setting. Indeed, in empirical work, it does turn out to be the case that excess demand enters the wage-price sector via wage adjustments which in turn feed into price adjustments by way of changes in the unit costs of production.

Such being the case, the most appropriate model for wage setting behaviour seems to be a mix of the bargaining and expectations-excess-demand hypotheses. However, our intention is to retain expression (1.0) as the basic wage equation in the system, and instead allow demand pressure to enter the wage-price sector via price adjustments. Doing so will not have any qualitative significant impact upon the workings of the wage-price sector, as will become apparent late in this section, while at the same time it will spare us the additional complexity of having to explain the rate of unemployment as a proxy for demand pressure in the labour market.

the price equation

Many models of the inflationary process seem to treat price changes as being passive on the basis that prices adjust to unit cost increases. The 'pure mark-up' and the 'normal cost' models are precisely two such models widely used in the literature. Their main underlying idea is that, in an uncertain world firms seek to coordinate their price changes for mutual benefits. One way to coordinate price changes is to base them on unit costs which are thought to change fairly uniformly across firms in each industry. In this sense, costs are more important determinants of price setting behaviour than demand fluctuations, a conclusion consistent with the prevalence of unit cost pricing in manufacturing industries. Moreover, unit cost pricing is consistent with sluggish price adjustment to shifts in demand and catch-up induced increases even after demand has fallen.[2]

The normal cost pricing model postulates that prices are fixed on the basis of the costs of producing some

[2]A dissatisfaction with such a theory of price determination is that it is not derived from any behavioural assumptions coming from established economic theory. Rather, it is empirically based.

'normal' level of output rather than the costs of producing current output levels. The basic idea of normal cost pricing is that, because price changes impose certain costs on the firm in question, that particular firm will avoid doing so very often. Rather, price changes are more likely to occur at discrete time intervals. Prices are thus thought to move along with long run costs and to be generally unresponsive to temporary cost variations. That is, prices will not be adjusted in response to output and cost fluctuations due to cyclical or other short-run factors. Actual costs are, hence, rejected as direct determinants of prices.

The most significant source of changes in normal costs is likely to be the trend growth rate of unit labour costs which, for analytical purposes, can be represented by the trend growth rate of nominal wages deflated by the trend growth rate of labour productivity (w-q). In addition, such things as the rental cost of capital and price inflation on imported goods can also be supposed to be sources of changes in normal costs, but are here omitted for simplicity.

In such models, demand factors are usually treated as insignificant and subsumed away. However, in both theoretical and empirical work, it is usually found that demand factors do play a role in price determination as for example in Scarfe (1972). For this reason, and because it has been suppressed as an independent argument in the wage equation, demand pressure will here be allowed to enter the basic price equation, which can hence be written in the following way:

$$p = w - q + \Phi(y), \tag{2.0}$$

where y is the level of real output. This equation carries the connotation that product prices move with changes in unit labour costs at normal levels of capacity utilisation. In essence, it constitutes a normal cost pricing equation augmented for demand pressure influences captured in the term $\Phi(y)$. That is to say, the term $\Phi(y)$ captures competitive deviations from normal cost pricing practices, measuring the pressure of demand with respect to productive capacity. If real output increases relative to productive capacity, the pressure of demand increases leading to higher inflation. Consequently, the partial derivative Φ_y is positive in sign.

To have a coefficient of unity attached to the growth rate of money wages is to argue that wage changes are fully reflected in prices. To have the same unitary coefficient attached to the trend growth rate of labour productivity is to argue that changes in input prices (w) and productivity (q) have a symmetrical effect on product prices. Furthermore, with this unitary coefficient attached to the growth rate of unit labour costs (w-q) the above equation (2.0) allows for a one hundred percent pass through of costs in the same way as wage inflation incorporates a one hundred percent ex-ante and ex-post adjustment to inflation.

Hence, in this wage-price sub-model given by equations (1.0) and (2.0), there is no asymmetry in terms of the responses of the goods and labour markets. That is to say, wage changes are fully reflected in prices, and price changes are fully reflected in wages. This symmetric response is precisely what makes the output-inflation trade-off function vanish in the long-run. This can be more clearly seen when one eliminates w-q across equations (1.0) and (2.0) to get:

$$p = (1-a_1)^{-1}\Phi(y) + p^*. \qquad (3.0)$$

What this equation says is that price expectations enter the inflation process with full power such that, when the system settles to a steady state and expectations are realised, output will be at its natural level given by $\Phi(y)=0$, and there is no trade-off between output and inflation.

In the short-run, however, a trade-off between output and inflation does exist as the manifestation of the institutional, contractual, and expectational arrangements that characterise a modern economy. Such arrangements call for a lag to be imposed upon equation (3.0) something which will not be done here exclusively in order to keep the model within manageable proportions. The Monetarist view that expectational errors constitute the only source of rigidities in the economy will hence be the view to be adopted here.

B. The Expectations Formation Equation.

Expectations are non-observable, and non-quantifiable in any direct way. Hence, to make the concept operational, it is necessary to impose a-priori restrictions on expectations formation. Present economic theory, however, does not offer a wholly satisfactory mechanism of expectations formation in the sense that there is not one such mechanism upon which the profession seems to have reached a concensus. Analyses thus have to rely on hypotheses that have proved to be analytically useful. The most prevalent of these hypotheses are those of the adaptive and rational expectations.

The rational expectations hypothesis holds that, in forming their expectations about the relevant variable, economic agents tend to use all existing relevant information such that all systematic forecasting errors are eliminated. All systematic elements influencing the rate of price inflation will become known and will be incorporated into the expectations formation mechanism, such that only random shocks can cause deviations of actual from expected inflation.

The hypothesis of rational expectations is really nothing more than an application of marginal economics to economic forecasting. That is to say, it is a proposition that says that in collecting and utilising information in a forecast, market participants will continue to obtain information and use it up to the point at which the benefits from using that additional information equal the marginal cost for obtaining that information.

The central part of the hypothesis, however, is the translation of that information into economic forecasts. A collection of facts is meaningless unless translated into forecasts of market outcomes. So, implicit in the use of information must be the use of some model which explains how the markets function and thus enables one to predict market prices and quantities.

All this was spelled out by John F. Muth in a 1961 article. Muth assumes that all market participants know the model which correctly describes the way in which the markets operate so that expected prices and quantities would be exactly equal to the solution of the model. This is an extreme version of the rational expectations hypothesis, one that rests on two critical assumptions. First, that private agents have knowledge of the structure of the economy and adjust their expectations to that knowledge, and second that all exogenous variables to the private sector of the economy can be accurately predicted except for the impact of random shocks.

Under the Muth view of the world, everyone knows the model and the probability distributions associated with alternative values of variables. The future is known in a conditional sense. Given knowledge about the way in which policy makers react to economic situations, the actual changes that take place in variables will have no real effects. People would have forecasted that those changes would take place and then markets would have reacted accordingly to those forecasts. Hence, as long as changes in monetary and fiscal policies are anticipated by economic agents, there is no uncertainty, and markets respond accordingly until actual and expected p[illegible] are equal and no further opportunities exist for making profits. Consider for example the stock market. If people expect an increase in the price of a particular equity, then they will go in and buy quantities of that equity. In so doing they would be pushing the price up until it becomes equal to its expected value and the opportunities for further profits are eliminated.

The rational expectations hypothesis in its strict version would imply that inflation is always fully anticipated except for the influence of random shocks. Expectations, in turn, have full power in the inflation process such that systematic monetary policy is neutral. Systematic policy actions (those based on feedback control rules) cannot influence real variables in the economy at any point in time. Inflation would influence real variables to the extent that it is unanticipated. Hence, in such a configuration, and in the context of the expectations-excess-demand model, unemployment would always be at its natural level, except for random disturbances.

In reality, policy actions do have an effect on real variables at least in the short-run. The inflation rate is not very flexible in response to restrictive policy. These facts alone suggest that there must be something wrong with the rational expectations hypothesis, despite its seemingly powerful logic.

In the context of inflation, the hypothesis of adaptive expectations in contrast, holds that individual economic agents form expectations about future rates of price inflation on the basis of past experience. The operationality of the concept is enhanced by taking the basic idea one step further. Expected inflation adapts to past inflation rates via a distributed lag. Using the geometrically distributed lag, the easiest one to employ, the expected rate of price inflation (p^*) is a weighted average of all past rates of inflation with the weights declining geometrically.

One may thus write:

$$p_t^* = ap_{t-1} + (1 - a)ap_{t-2} + (1 - a)^2ap_{t-3} + \ldots$$

where a is greater than zero but less than unity. When the parameter a takes on the value of unity, expected inflation is simply last period's inflation rate. The restriction that a lies between the values of one and zero implies a finite limit and, hence, that the geometric progression converges.

An application of the Koyck transformation would produce:

$$p^*_t - p^*_{t-1} = a(p_{t-1} - p^*_{t-1}).$$

That is, expectations will be adjusted upwards or downwards when, and only when, last period's expected inflation deviates from last period's actual inflation. If it turns out for example that $p > p^*$ in any time period then expectations will be adjusted upwards in the subsequent time period by a fraction of the error given by a.

However, the question that arises with respect to the mechanism of adaptive expectations is what is the speed of response of expected inflation to a disequilibrium situation?[3] This amounts to determining the value of the parameter a, an empirical question. The closer to zero a is, the slower would the weights be declining and the stronger the influence of past observations of actual inflation on expected inflation would be. In other words, the closer to zero a is, the longer the expectations lag will be.[4]

The continuous time counterpart of the adaptive expectations mechanism developed above can be written in the

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[3]In this context, a disequilibrium situation will be generated when actual and expected inflation do not coincide.

[4]The variance of actual inflation experience around trend also plays a role in expectations formation. The bigger this variance is, the bigger the incentive for a correct forecast will be, and the more weight will be assigned to recent experience in which case the value of the parameter a increases and consequently the lag becomes shorter.

following way:

$$Dp^* = a(p - p^*), \tag{4.0}$$

where $0 < a < \infty$. This expression simply says that inflationary expectations will adjust to past errors with speed of response a. The bigger a is, the faster expectations adjust, and vice-versa.

The adaptive expectations mechanism, critics argue, is not accurate because it has a narrow information set, namely past data. Distributed lag functions are a convenient method to employ in econometric estimation, but certainly such functions are unlikely to tell a full story about expectations formation. Current observations on a variety of relevant developments, such as changes in the monetary growth rate or policy pronouncements, are likely to have an influence on inflationary expectations. However, for practical reasons, the adaptive expectations mechanism as given in expression (4.0) will here be employed.

C. The Aggregate Demand Relationship.

Expectations formation and wage and price behaviour interact with other behavioural relationships in the system, and particularly with excess demand, in determining the rate of price inflation. The relationship between excess demand and inflation is nothing but a component of a whole system of relationships that together generate the inflationary process. A demand pressure equation is therefore needed if the system is to be complete.

In addition, financial markets and their interactions with other markets in the macro-economy also need to be integrated into the basic model under construction. Changes in interest rates, for example, can affect the prices of different assets and via them expenditures. Where interest rates have a large impact on stock market valuation, then the accrued marginal gains, via the marginal propensity to consume attached to them, can make consumption responsive to monetary policy.

Demand pressure will here be proxied by real output (y) in relation to productive capacity $\bar{y}$, and there will be assumed to exist one rate of interest (r). With respect to the determination of real output and the interest rate, the all familiar IS-LM apparatus will be employed, within which both these variables are simultaneousely determined. That is to say, real output and the nominal interest rate are determined by the interaction between the commodity and money markets.

the commodity market.

The reduced form equation for the commodity market is given in the following expression:[5]

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[5]Bonds in this model are not perpetuities but rather one period bonds with short time to maturity. They can be seen as instruments for short-term borrowing by the government. Hence, rB is the nominal bond interest.

$$y = c[y - \tau y + (1-\tau)rBP^{-1}] + i(r - p^*) + g. \qquad (5.0)$$

In our context of a closed economy, real output is given as the sum of consumption, investment, and government expenditures. Consumption is a function of real disposable income which here is the sum of national income (y) and interest income (rBP^{-1}), less income tax payments (τy) and tax payments on interest income (τrBP^{-1}). Note that taxes are modelled to be functioned on income with a single uniform marginal income tax rate ($0 < \tau < 1$). The tax system is in addition assumed to be fully indexed such that inflation has a neutral effect on tax revenue or payments. What this amounts to is that there is no fiscal drag. That is to say, the partial derivative of real tax revenue with respect to inflation is zero. Finally, investment is simply an inverse function of the real interest rate, ($r-p^*$), and government expenditures can be endogenous or exogenous depending on the policy regime under consideration.

the money market.

With respect to the money market, we will employ the liquidity preference approach and write the corresponding equilibrium condition in the following way:

$$MP^{-1} = L(\ y,\ r,\ p^*,\ BP^{-1}\ ). \qquad (6.0)$$

This expression, however, can be inverted making r the subject to write:

$$r = l(y, p^*, MP^{-1}, BP^{-1}). \qquad (6.0)'$$

That is, the demand for real money balances (MP^{-1}) is a positive function of real income (y) and the real value of the stock of bonds (BP^{-1}), and an inverse function of the interest rate (r) and expected inflation (p^*). Equivalently, the nominal interest rate is a positive function of real income and the stock of bonds outstanding, and an inverse function of the real money supply and inflationary expectations.

Real income is here used to model the transactions demand for real money balances, the interest rate to represent the trade-off between money and bonds, and expected inflation to represent the trade-off between money and real goods. The stock of bonds, on the other hand, can be seen as a proxy for wealth effects even though wealth effects are not really considered in any direct w[illegible]s model.

However, the inclusion of inflationary expectations in the money market equation is ad hoc. If for example [illegible] s variable, p^*, represents the trade-off between money and real goods as argued above, then it is nothing else than the relative price of money with respect to real goods. In such a case inflationary expectations ought to be in the consumption function as well. Furthermore, if real goods are to be treated as an asset then the model is really mispecified since only the demand for real money balances is here explained. That is, if the model comprises three assets (real money balances, real bonds, and real goods), then by virtue of Walras' law two of them have to be explained. In any case, with these qualifications in mind, we will proceed with the money market equation as specified in expression (6.0)'. Doing so will not distort the analysis as the implications of adding inflationary expectations in the money market equation can easily be isolated.

In summary, the demand pressure variable that interacts with wage and price behaviour is being determined within an IS-LM framework which, in essence, defines an aggregate demand relationship. The presence of four policy instruments (the money supply, debt issuance, taxes, and government spending) in this framework raises the issue of what policy regime to employ in this analysis. In particular, the issue of the endogeneity/exogeneity of monetary policy is central to the Keynesian-Monetarist controversy as already noted in chapter one.

D. The Exogeneity/Endogeneity of Monetary Policy.

The exogeneity of monetary policy is a fundamental Monetarist proposition necessary to generate the basic Monetarist results.[6] With monetary policy exogenous, the

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[6]Specifically, Monetarists argue that the real money supply is endogenous to the system but that the nominal stock of

line of causation would run from policy changes, to excess demand, to prices. Monetary policy, that is, is the active independent variable that precedes and causes shifts in demand. If monetary policy is exogenous, then the Central Bank through its control over policy instruments can affect the 'monetary base', which in turn via the money markets will affect the money supply and nominal income. Lastly, via its effects on aggregate demand, changes in nominal income will affect output and the general price level.

That is roughly the line of causation when monetary policy is exogenous. With monetary policy endogenous, determined within the system as a response to changing economic conditions, the line of causation runs from excess demand and prices, to policy variables. In such a case, inflation models would include policy reaction functions explaining how authorities change their monetary and/or fiscal policies in response to fluctuations in aggregate demand and the rate of price inflation.

For Monetarists, inflation is always the consequence of excessive monetary growth in relation to the growth of real output because this condition is the most important of the necessary conditions for a continuing price inflation. Temporary disturbances, such as crop failures, oil price shocks, or an increase in the minimum wage rate, are seen as sporadic events capable of moving the inflation rate upwards temporarily, but cannot explain a sustained increase in

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'(cont'd) money is exogenous.

prices. Cost-push shocks, the argument goes, affect relative prices and not the general price level.

That the level of the money supply in the long-run cannot have any sustained effects on real output but only on inflation, and that cost-push shocks hitting the economy cannot affect the inflation rate in any sustained way unless accommodated by monetary policy, are propositions upon which economists seem to have reached a general concensus.

True, monetary growth is a necessary condition for continuing inflation to occur, but this does not mean that money is the cause of inflation in any fundamental way. Monetary growth can be a symptom of the operation of the economic system. Monetarists concentrate on the relationship between money and prices but do not offer any complete explanation of what causes money supply variations. They unanimously agree that monetary growth is the proximate cause of inflation but do not take any clear position on the issue of what is it that determines monetary expansion. It hence becomes necessary to explain why the authorities expand the money supply thus driving the economy into inflationary periods. Is monetary expansion "...caused by governments for their own benefit" as Parkin (1975) once argued?

Surely, there exist political constraints to maintain output and employment. Authorities expand the money supply in line with money demand so as not to depress aggregate demand. If, however, an economy is depressed, then to maintain output and employment the authorities would have to expand the money supply faster than money demand. In such a case, monetary policy has an inflationary bias.

A change in monetary growth, if maintained long enough, will initially affect output positively with some lag. Monetary policy will also affect the price level with some lag. The impact on prices, however, will initially be relatively small because of output effects. But in the long-run any sustained change in monetary growth will be primarily reflected in prices.

In addition, changes in the rate of growth of the money supply will also affect inflationary expectations and in this way will reduce the demand for real cash balances creating additional inflationary pressures. Expectations, thus, come to prolong the inflationary process. When demand comes down and inflation still continues to mount, it is because of inflationary expectations which take time to get out of the system. Monetary expansion affects prices with longer lags than it affects output. Hence, with short-run problems pressing, the authorities expand the money supply leading to inflation later on.

Money financed government expenditures are bound to lead to faster monetary growth, fueling inflation. Bond or tax financed government expenditures, on the other hand, are less inflationary. Higher taxes shift demand from the public to the government; bond issues shift demand from investors to the government. However, with tax increases in response to budget imbalances being politically unpopular, and with bond financed deficits raising interest rates, it is argued that financing deficits by printing money becomes politically preferable at least for the short-run. In this way government deficits cause money supply changes.

The possibility also exists for the monetary authorities to expand the money supply so as to accommodate cost-push shocks. For Monetarists, cost-push forces can only affect costs and prices of particular industries relative to other industries but do not have any effect on the general price level. Cost-push theorists, on the other hand, see cost-push forces as the most important determinant of excess demand. Given a constant monetary growth rate, cost push shocks will raise prices reducing real purchasing power and real spending. Output and employment will thus decrease. They advocate an accommodative monetary policy to offset the contractionary influences of cost-push factors on the economy.

In other words, the argument is that cost-push shocks that raise the costs of production in various industries force upon the monetary authorities an accommodative response to avoid output and employment loses that would otherwise follow. In this way, monetary policy becomes endogenous, 'validating' price increases.

E. A Monetary Policy Reaction Function.

If cost factors and government deficits cause monetary accommodation, then the implication is that monetary policy must be endogenous. There will be many occasions in the analysis at hand in which the concern will be with the endogeneity of monetary policy. One way to endogenise monetary policy is via a reaction function which, for practical purposes again, one would like to be as simple as possible.

The hypothesis is that the monetary authorities are concerned with two basic things in setting monetary policy. Firstly, they are concerned with where real output is relative to their view of capacity or target output (y'), and secondly, where the rate of price inflation is relative to some target rate (p'). Given these two targets, one can further hypothesize that current deviations from them will both influence monetary policy with some weights, say z_1 and z_2, respectively. One can thus write a monetary policy reaction function in the following way:[1]

$$m - p' = z_1(y - y') + z_2(p - p'). \qquad (7.0)$$

This reaction function simply says that monetary growth will be determined by the deviations of real output from target with weight z_1, and by the deviations of actual inflation from target with weight z_2.

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[1]The monetary policy reaction function here employed as given in expression (7.0), is reproduced from Scarfe (1983).

In a model economy that incorporates a 'natural rate' proposition in the sense that expectations enter the inflation process with full power, it will turn out to be the case that the targets y' and p' cannot arbitrarily be chosen. In actual fact, the only sensible target output level that the monetary authorities can have is that level of output consistent with full capacity utilisation. If, for example, the monetary authorities were to base their policy on an unattainable target output level, then the reaction function would in fact be destabilising, leading to accelerating inflation.

A negative z_1 coefficient in the above reaction function would signify a countercyclical monetary policy. That is to say, if real output exceeds target, then monetary policy becomes tighter. A z_2 coefficient, on the other hand, that is positive but strictly less than unity, makes the reaction function partly accommodative. That is, the weight z_2 may well convey an intrinsic dislike of inflation per se, but nevertheless imply a gradualist approach to its elimination. Thus, a monetary policy that follows a strategy of gradualism has z_1 negative and z_2 positive but less than unity.

A unitary z_2, coupled with a z_1 that equals zero, signifies a completely accommodative monetary policy. That is to say, the money supply in such a scenario, grows at the same rate as price inflation. In contrast, monetary policy can be purely countercyclical in nature when the coefficient

z_1 remains negative and z_2 attains a value of zero. In such a case, the policy concerns about output and employment completely outweigh the concerns about inflation. Finally, when both these coefficients attain the value of zero, one ends up with a Monetarist 'cold shower approach', or a constant monetary growth rule.

However, no matter what the precise nature of the monetary policy reaction function is, monetary policy will, in reality, respond to changes in economic conditions with some lag. Associated with monetary policy, one can identify a 'recognition' lag, an 'implementation' lag, and also an 'effectiveness' lag. The presence of all these lags can very well have important implications with respect to the usefulness and effectiveness of a policy reaction function. But again for practical reasons, that is in order to keep the analysis within manageable proportions, we will abstract from reality once more and subsume these lags away.

F. Interdependence of Policy Instruments and the Government Budget Constraint.

A closely related issue to that of the endogeneity/exogeneity of policy instruments is that of their interdependence. Are the monetary and fiscal policy variables independent of one another or are they interrelated? Many economists would argue that the domestic monetary and fiscal policies are interdependent via the government budget constraint which relates the budget deficit to its sources of financing. This is precisely the view that is being employed in this study. The budget constraint, in effect, reduces the degrees of freedom in setting the fiscal and monetary policy variables by one, and hence, not all policy instruments can be exogenous. At least one must be endogenous via—this government budget constraint.

The government budget constraint equates total government expenditures with the total flow of financing from all sources. The expenditure side would be the sum of such things as wage and non-wage government spending, public investments, unemployment insurance compensation and welfare payments, interest payments on debt outstanding, and transfer payments. To keep things simple, this expenditure side will here be modelled by way of aggregating together in a single variable (g), all kinds of government expenditures in real terms. Interest payments on the public debt, however, will be treated separately. The term BP^{-1}, as previously defined, represents in real terms the outstanding stock of the public debt held by households. Thus, interest payments on the debt will be given by the term rBP^{-1}, where r, as defined before, is the nominal rate of interest.

The revenue side, on the other hand, can be thought to be composed of a whole tax structure ranging from property taxes, direct and indirect taxes, tariff revenues and royalties, to revenues from crown corporations. Again to keep things simple, one can model the revenue side of the government budget constraint in terms of a uniform marginal income tax rate (τ). That is, as already indicated in the context of the consumption function earlier in this chapter, the model contains a fully indexed tax system with real income and interest payments being the tax base.• In other words, total tax revenue in this system denoted T, will be given by:

$$T = \tau(y + rBP^{-1}).$$

Given the level of government expenditures and tax revenues, any deficit will have to be financed in one or both of two ways: money creation, and debt issuance. There is no banking sector in this model. The government operates a Central Bank which buys bonds from domestic households and pays for them with fiat money. In a closed economy context then, the Central Bank's balance sheet would look as follows:

$$B^C = M,$$

where M refers to high powered money or base money, and B^C to the nominal value of bonds held by the Central Bank. Since there are no commercial banks, M is also the money supply in this model economy.

•Certainly, a fully indexed tax system is an unrealistic assumption to make. The effective tax rates may increase under inflation in the sense that increased money wages may in fact bring people into higher income tax brackets irrespective of how their real incomes change. To allow for such a fiscal drag, the rate of price inflation has to enter as an argument in a net tax revenue function. Doing so, however, would complicate the analysis unnecessarily.

The total value of the stock of bonds outstanding, denote it by B^t, would be the sum of the value of the stock of bonds held by the Central bank (B^c), and that held by domestic households (B). That is,

$$B^t = B + B^c$$

in algebraic terms. However, since B^c=M, it follows that

$$B^t = B + M.$$

The value of the total stock of bonds in the economy must be equal to the sum of the value of the stock of bonds held by domestic households and money balances in the economy. Hence, when the Central Bank makes an open market purchase of bonds from households, the money supply increases, and the value of the stock of bonds held by households decreases. That is to say, open market operations have the effect of altering the composition of the public debt as the following relationship,

$$dM + dB = 0,$$

must always hold true in this model when open market operations occur.

Bringing all the above together, the government budget constraint can, in real terms, be written as the following expression:

$$P^{-1}DB + P^{-1}DM = \bar{g} + (1-\tau)rBP^{-1} - \tau y. \qquad (8.0)$$

This expression says nothing else than that any positive financing requirement must result in bond sales to the public, and/or new issues of base money.

G. Summing-up: the General Model.

Bringing all the preceeding together, the general model can be specified to be composed of the following expressions:'

$$p = (1-a_1)^{-1}\Phi(y) + p^* \qquad (3.0)$$
$$Dp^* = a(p - p^*) \qquad (4.0)$$
$$y = c[y - \tau y + (1-\tau)rBP^{-1}] + i(r - p^*) + \bar{g} \qquad (5.0)$$
$$r = l(y, p^*, MP^{-1}, BP^{-1}) \qquad (6.0)$$
$$m - p' = z_1(y - y') + z_2(p - p') \qquad (7.0)$$
$$P^{-1}DM + P^{-1}DB = g + (1-\tau)rBP^{-1} - \tau y \qquad (8.0)$$

where,

$0 < a_1 < 1$; $a > 0$; $z_1 \leq 0$; $0 \leq z_2 \leq 1$; $0 < \tau < 1$;

$y' = \bar{y}$ and $p' = \bar{p}$.

Expressions (3.0) to (8.0) constitute the base equations of the system. Expression (3.0) represents the wage-price sector in the model, and (4.0) the adaptive expectations mechanism. The IS relationship is given by

'Note that expression (3.0) was obtained by eliminating w-q across expressions (1.0) and (2.0).

expression (5.0) and the LM by (6.0). The government budget constraint is given by (8.0) and has the effect of endogenising one policy instrument. Expression (7.0), the monetary policy reaction function, endogenises monetary growth but it will not be used in all cases analysed.

The system above embodies four potential policy instruments: the money supply, debt issuance, government spending, and the marginal tax rate. This last instrument however, the marginal tax rate, will be treated as a constant throughout the analysis. The emphasis thus will be on the endogeneity of the remaining three policy instruments.

the total differential version of the system

Totally differentiating the system given in expressions (3.0) to (8.0) with respect to time, one obtains:

$$Dp = a_2Dy + Dp^* \qquad (3.1)$$

$$Dp^* = \alpha(p - p^*) \qquad (4.0)$$

$$(1-c_1)Dy = c_1BP^{-1}r(b-p) + (c_1BP^{-1}+c_2)Dr + c_3Dp^* + Dg \qquad (5.1)$$

$$Dr = l_1Dy + l_2Dp^* + l_3MP^{-1}(m-p) + l_4BP^{-1}(b-p) \qquad (6.1)$$

$$Dm = z_1Dy + z_2Dp \qquad (7.1)$$

$$MP^{-1}Dm + MP^{-1}m(m-p) + BP^{-1}Db + BP^{-1}b(b-p) = Dg - \tau Dy + (1-\tau)BP^{-1}r(b-p) + (1-\tau)BP^{-1}Dr \qquad (8.1)$$

where,

$$a_2 = (1 - a_1)^{-1}\Phi_y > 0$$

$$c_1 = c'(1 - \tau) > 0 \qquad 0 < c_1 < 1$$

$$c_2 = l < 0$$

$$c_3 = -c_2 > 0$$

$$l_1 > 0$$

$$l_2 < 0$$

$$l_3 < 0$$

$$l > 0$$

$$z_1 \leq 0$$

$$0 \leq z_2 \leq 1$$

$b = B^{-1}DB$, the growth rate of debt issuance,

$m = M^{-1}DM$, the rate of monetary growth,

and,

$p = P^{-1}DP$, the rate of price inflation.

The coefficient a_2 captures the influence of demand pressure on price inflation operating independently and via the wage-price nexus. The coefficient c_1 is the net marginal propensity to consume and is less than the gross marginal propensity to consume given by c'. The coefficients c_2 and c_3 are arithmetically equal and are attached to investment behaviour in the model. An increase in the real interest rate, induced by an increase in the nominal interest rate, will crowd out investment expenditures and, hence, c_2 is negative. On the other hand, an upward expectational adjustment reduces the real interest rate leading to a positive investment effect and, hence, c_3 is positive.

Lastly, the coefficients l_1, l_2, l_3, and l_4, are directly related to the elasticities of the interest rate with respect to real income, expectational adjustments, money supply variations, and debt issuance, respectively. An increase in income would push interest rates up via the transactions demand for money. Upward expectational adjustments would generate asset substitution effects out of money and into real goods thus putting downward pressure on the interest rate. Monetary expansion will also reduce the interest rate, while debt issuance would increase it via increasing the pressures on the demand for loanable funds.

To eliminate p^* and Dp^* across the system, invert expression (3.0) making $(p-p^*)$ the subject, then substitute the result into expression (4.0) and subsequently eliminate Dp^* across the system to obtain the following expressions:[10]

$$Dp = a_2Dy + \alpha a_2(y - \bar{y}) \qquad (3.2)$$

$$(1-c_1)Dy = c_1BP^{-1}r(b-p) + (c_1BP^{-1}+c_2)Dr + \alpha a_2c_3(y-\bar{y}) + Dg \qquad (5.2)$$

$$Dr = l_1Dy + \alpha a_2l_2(y-\bar{y}) + l_3MP^{-1}(m-p) + l_4BP^{-1}(b-p) \qquad (6.2)$$

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[10] From (3.0) obtain,

$$p - p^* = (1-a_1)^{-1}\Phi(y) \qquad (3.0)'$$

Eliminate $(p - p^*)$ across (3.0)' above, and (4.0), to obtain:

$$Dp^* = \alpha(1-a_1)^{-1}\Phi(y) \qquad (4.1)$$

$\Phi(y)$ upon linearisation around the equilibrium real output $\bar{y}$, yields $\Phi_y(y - \bar{y})$. Hence, (4.1) becomes:

$$Dp^* = \alpha a_2(y - \bar{y}) \qquad (4.2)$$

Substituting Dp^* given in (4.2) into expressions (3.1), (5.1), and (6.1), then one obtains the system given by expressions (3.2), (5.2), (6.2), (7.1), and (8.1).

$$Dm = z_1 Dy + z_2 Dp \tag{7.1}$$

$$MP^{-1}Dm + MP^{-1}m(m-p) + BP^{-1}Db + BP^{-1}b(b-p) = Dg - \tau Dy + (1-\tau)BP^{-1}r(b-p) + (1-\tau)BP^{-1}Dr \tag{8.1}$$

With the monetary policy reaction function in place, and with the tax rate (τ) a constant, then one of b and g, must be exogenous and the other endogenous via the budget contraint. In such a scenario, the model comprises five endogenous variables: price inflation, real output, monetary growth, the interest rate, and one of real government spending and the growth rate of debt issuance. When the monetary policy reaction function is subsumed away then the system comprises four endogenous variables, the inflation rate, real output, the interest rate, and one policy instrument.

H. An overview of the model: a Sketch of the Inflationary Process

To trace out the inflationary process embodied in the general model, assume that initially an expansionary fiscal policy increases aggregate demand beyond the economy's capacity to produce. How will the model economy respond to this shock? What can be said at the very outset is that such an increase in the rate of spending, however caused, can only continue if it is accompanied by sustained monetary expansion.

From the price equation in expression (2.0), it is seen that increased demand pressure will have an independent effect on product prices. This inflationary consequence of excess demand will set off what one may call a wage-price spiral. Given the purely bargaining model of wage setting in the system, the inflation induced by increased demand pressure will raise nominal wages. This increase in money wages will subsequently feed back into prices on a one-to-one basis given the normal-cost-pricing incorporated into the general model. This development will generate further wage adjustments, which in turn will generate further price adjustments, and so on.

Increased demand pressure in the general model leads directly into wage and price adjustments as the consequence of no lag being imposed on the wage-price nexus. With a lag in operation in the wage-price subsystem, the initial adjustment due to increased demand pressure would fall on quantities. That is, the business sector would respond to increased spending initially, not by raising prices, but by reducing inventories and expanding their outputs. Increases in production would eventually lead to increased wages via an increased demand for inputs.

Increased wage costs raise production costs through the inter-industry structure and firms would subsequently raise prices. Price increases would then feed into the wage bargaining process with labour demands for higher wages to compensate for cost of living increases. Increased labour costs induce further price increases, and the process continues spirally in the form of rounds of wage and price increases.

However, the only lag in our system comes from expectational adjustments which are slow because past actual inflation experience is allowed to influence expectations formation. Induced inflation, by increasing the expectations-inflation gap, will result in upward expectational adjustments with speed of response a. The bigger a is, the faster the adjustment will be.

Inflationary expectations, once under way, will feed back into the wage-price mechanism as money wages adjust in an ex-ante sense to changes in inflationary expectations. The model, hence, includes a feedback mechanism between wage and price adjustments, on the one hand, and expectational adjustments on the other. That is to say, wage and price adjustments feed into expectational adjustments which in turn feed back into further wage and price adjustments, and so on.

Increased demand pressure and its consequences with respect to inflation and inflationary expectations will have repercussions in the goods and money markets. In the first place, increased demand pressure will feed directly into interest rate increases via the transactions demand for money, and into further demand increases via the marginal propensity to consume.

Inflation erodes the real values of money balances and bonds outstanding. Reduced real money balances lead to higher interest rates. At the same time the reduced real value of the stock of bonds leads to lower interest rates in the money market and generates a negative consumption effect via reduced interest income in the goods market.

Upward expectational adjustments induced by higher inflation will also have repercussions in the goods and money markets. In the goods market, upward expectational adjustments in effect put downward pressure on the real interest rate, thus adding to demand pressure via a positive investment effect thus generated. In the money market on the other hand, upward expectational adjustments generate asset substitution effects out of money and into real goods thus driving the interest rate down.

Changes in the interest rate, however caused, will feed into the goods market affecting both consumption and investment expenditures. If the interest rate increases, then it will generate a positive consumption effect via increased interest income and a negative investment effect via a higher real interest rate. The net impact upon output will be determined by the relative magnitudes of these two effects. To the extent that the investment effect dominates, then the interest rate mechanism has a dampening effect on the economy in the sense that it dampens the fluctuations in output.

At this point it can be noted that the interactions between real output and the interest rate and between real output and price inflation are simultaneous. Because expectational adjustments are slow to occur, the interest rate mechanism, to the extent that it dampens the fluctuations in output, will have a dampening effect on expectational adjustments as well.

Thus far, we have talked about the relationships and feedback mechanisms that exist with respect to the wage-price sector, expectational adjustments, and aggregate demand. That is to say, we have been ignoring policy responses. Considering first the budget constraint, which in effect endogenises one policy instrument, changes in real output, inflation, and the interest rate will generate a policy response which will subsequently feed back into the whole system.

The initial increase in demand pressure, for example, will generate more tax revenue thus allowing the money supply to contract, or government expenditures to increase, or debt issuance to decrease, depending on policy regime.'' An interest rate increase will have repercussions for the endogenous policy instrument via increased interest payments. Inflation, on the other hand, by eroding the real values of money balances and of the stock of bonds

''The kind of initial shock that sets off the inflationary process will also have repercussions for the behaviour of the model in general. With fiscal expansion as the policy shock that initiates inflation, either monetary growth or the rate of debt issuance will have to increase endogenously via the government budget constraint.

outstanding, leads to less interest payments, easing the pressures on monetary expansion or debt issuance, depending again on policy regime. At the same time however, since bonds and money serve as instruments of financing budget deficits, they would have to be replaced or government expenditures to be reduced.[12]

When a monetary policy reaction function is incorporated into the system, the budget constraint will be determining how debt issuance or government expenditures will be affected by developments elsewhere in the economy. The system is simultaneous in nature and, hence, any change originating in any particular market will have repercussions for the entire system. This is not the case in long-run steady-state equilibrium because along a steady-state equilibrium path the system is decomposable, as is shown in the next chapter.

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[12]Note that such policy responses, as the case is with every single coefficient in the system, constitute partial effects.

III. THE LONG-RUN PROPERTIES OF THE MODEL AND OVERSHOOTING OF EQUILIBRIUM VALUES

The long-run will here be used to refer to the steady-state equilibrium that will eventually be realised if the system is stable. The short-run, by contrast, will be used to refer to a transitional phase in-between long-run steady-states, during which the system adjusts to exogenous or policy shocks.

This chapter deals primarily with the long-run properties of the system, and with overshooting of equilibrium values as can be inferred by analytically comparing corresponding impact and long-run multipliers. However, impact multipliers for endogenous policy instruments under alternative policy regimes will not here be considered. The question that is raised is, what are the consequences for the level of real output, the nominal interest rate, and the rate of price inflation, of exogenous increases in the rates of monetary growth and debt issuance, the level of government spending, and inflationary expectations? That is, what will here be considered is the IS-LM subsystem extended by the Phillips-curve relationship.

A. Short-Run Multipliers in the Static Submodel

The IS-LM equations are the static equations of the model, and in linear form with the autonomous components suppressed can be written in the following expressions:

$$\text{IS:} \quad (1-c_1)y = g + (c_1BP^{-1}+c_2)r + c_3p^* \tag{9}$$

$$\text{LM:} \quad r = l_1y + l_2p^* + l_3MP^{-1} + l_4BP^{-1} \tag{10}$$

To obtain short-run multipliers one only has to solve these static equations of the system treating all variables, except the ones explained by the static equations, as exogenous. In the present context, the rate of price inflation, inflationary expectations, the level of government spending, and the rates of monetary growth and debt issuance, are all assumed to be exogenous.

The corresponding reduced form equations for real output and the nominal interest rate can be readily obtained by eliminating each of these variables across the static subsystem given in expressions (9) and (10) above.

Eliminating the nominal interest rate across these two equations one obtains the following reduced form equation for real output,

$$\begin{aligned}[1-c_1-l_1(c_1BP^{-1}+c_2)]y = g &+ [c_3+l_2(c_1BP^{-1}+c_2)]p^* \\ &+ l_3MP^{-1}(c_1BP^{-1}+c_2) \\ &+ l_4BP^{-1}(c_1BP^{-1}+c_2),\end{aligned} \tag{11}$$

which defines the short-run aggregate demand relationship.

Similarly, eliminating real output across the same equations, (9) and (10), one obtains the following reduced form equation for the nominal rate of interest:

$$[1-c_1-l_1(c_1BP^{-1}+c_2)]r = [l_1c_3+l_2(1-c_1)]p^* + l_3MP^{-1}(1-c_1) + l_4BP^{-1}(1-c_1) + l_1g. \quad (12)$$

From these reduced form equations one can easily obtain the corresponding short-run multipliers. Note, however, that in this context of the static IS-LM subsystem, the term,

$$[1-c_1-l_1(c_1BP^{-1}+c_2)],$$

is the denominator of the afore-mentioned multipliers. Its sign comes to be ambiguous under the influence of the component c_1BP^{-1}, which represents a consumption effect induced by interest rate changes through their impact upon interest income, and which enters the denominator in question negatively, thus reducing its positivity.

Assume that c_1BP^{-1} is constant, and furthermore, that $(c_1BP^{-1}+c_2)$ is negative. The denominator of the IS-LM multiplier, under these assumptions, will be constant and unambiguously positive.[13]

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[13]Note that without the assumption that c_1BP^{-1} is a constant, all the corresponding multipliers will be variable. Treating this term as constant is equivalent to assuming that BP^{-1} is constant, which would be the case if B and P are growing at approximately the same rate. This can be valid in the neighbourhood of long-run equilibrium.

Note that $c_1BP^{-1}+c_2$ being positive would imply that the IS curve is positively sloped. From expression (9) one can readily obtain that the slope of the IS curve is given by the term

$$(1-c_1)[c_1BP^{-1}+c_2]^{-1}.$$

Since c_1 is positive but strictly less than unity, the slope of the static IS curve will be negative if, and only if, the term $c_1BP^{-1}+c_2$ is negative.

Denoting the denominator of the static IS-LM multiplier by Ω, the corresponding short-run multipliers can then be written as the following expressions:

$$\frac{dy}{dg} = \Omega^{-1} > 0$$

$$\frac{dr}{dg} = l_1\Omega^{-1} > 0$$

$$\frac{dy}{dm} = l_3(c_1BP^{-1}+c_2)\Omega^{-1}\frac{dMP^{-1}}{dm} > 0$$

$$\frac{dr}{dm} = l_3(1-c_1)\Omega^{-1}\frac{dMP^{-1}}{dm} < 0$$

$$\frac{dy}{db} = l_4(c_1BP^{-1}+c_2)\Omega^{-1}\frac{dBP^{-1}}{db} < 0$$

$$\frac{dr}{db} = l_4(1-c_1)\Omega^{-1}\frac{dBP^{-1}}{db} > 0$$

$$\frac{dy}{dp} = \{(c_1BP^{-1}+c_2)[l_3\frac{dMP^{-1}}{dp} + l_4\frac{dBP^{-1}}{dp}]\}\Omega^{-1} \ ?$$

$$\frac{dr}{dp} = \{(1-c_1)[l_3\frac{dMP^{-1}}{dp} + l_4\frac{dBP^{-1}}{dp}]\}\Omega^{-1} \ ?$$

$$\frac{dy}{dp^*} = [c_3+l_2(c_1BP^{-1}+c_2)]\Omega^{-1} > 0$$

$$\frac{dr}{dp^*} = [l_1c_3+l_2(1-c_1)]\Omega^{-1} \;?$$

It must be noted, however, that the standard textbook multipliers involve the derivatives $\frac{dy}{dMP^{-T}}$, $\frac{dr}{dMP^{-T}}$, $\frac{dy}{dBP^{-T}}$, and $\frac{dr}{dBP^{-T}}$ instead of the derivatives $\frac{dy}{dm}$, $\frac{dr}{dm}$, $\frac{dy}{db}$, and $\frac{dr}{db}$ respectively. The multipliers here and the standard textbook multipliers are exactly the same conditioned on the derivatives $\frac{dM}{dm}$ and $\frac{dB}{db}$ both being positive in the short-run.

Given the assumption that $(c_1BP^{-1}+c_2)$ is negative, which means that an increase, say, in the nominal rate of interest will lead to a reduction in real output precisely because the IS curve is negatively sloped, an increase in the level of government spending will be expansionary, and will also push the nominal rate of interest up via the transactions demand for real money balances. A monetary expansion will also be expansionary with respect to the level of real output by way of reducing the rate of interest, which will feed into the goods market with a positive investment effect and a negative consumption effect due to reduced interest income. A reduction in the rate of interest is here expansionary by virtue of the assumed negativity of the term $(c_1BP^{-1}+c_2)$.

An increase in the rate of debt issuance, on the other hand, will generate effects that parallel those associated with monetary expansion but which work in the opposite direction. In the money market, increased debt issuance will have the effect of raising the nominal rate of interest which in turn will feed into the goods market generating a negative output effect.

Inflation will feed into the money market with ambiguous repercussions for the nominal rate of interest. That is to say, the partial derivative, $\frac{dr}{dp}$, is not signable. Inflation will have the effects of reducing the real stocks of money balances and bonds outstanding. Reduced real money balances push the interest rate up while reduced real public debt pushes it down. The net effect would depend on relative elasticities in the money market. The interest rate will increase, ceteris paribus, to the extent that the money market is more sensitive to money supply variations than it is to changes in the real public debt.[14]

The non-signability of the partial derivative, $\frac{dy}{dp}$, is a direct consequence of the non-signability of the partial derivative, $\frac{dr}{dp}$. Observe that,

$$l_3 \frac{dMP^{-1}}{dp} + l_4 \frac{dBP^{-1}}{dp} = \frac{1}{1-c_1} \frac{dr}{dp}.$$

Substituting this result in the derivative, $\frac{dy}{dp}$, one can write:

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[14] Note that, if BP^{-1} was not an argument in the money market equation then, induced inflation would unambiguously generate the usual cost-push effects. That is to say, the nominal rate of interest would unambiguously increase and the level of real output would decrease assuming a negatively sloped IS curve.

$$\frac{dy}{dp} = \frac{c_1 BP^{-1} + c_2}{1 - c_1} \frac{dr}{dp}$$

From this expression it is apparent that the sign of $\frac{dy}{dp}$ would directly depend on the sign of $\frac{dr}{dp}$. Furthermore, the coefficient, $\frac{c_1 BP^{-1} + c_2}{1 - c_1}$, is the inverse of the slope of the IS curve and, hence, it is negative in sign to the extent that the IS is negatively sloped. Consequently, $\frac{dy}{dp}$ and $\frac{dr}{dp}$, must have opposite signs.

Upward expectational adjustments will have the consequence of increasing the level of real output but their effect on the rate of interest comes to be ambiguous, manifested in the non-signability of the derivative $\frac{dr}{dp^*}$. Upward expectational adjustments will generate asset substitution effects out of money and into real goods thus forcing the rate of interest down. But the same asset substitution effects have the effect of raising the level of real output, thus driving the interest rate up via the transactions demand for money. The net effect of expectational adjustments on the rate of interest is not-signable.

B. The Phillips-Curve Relationship and Short-Run Multipliers

Having discussed the short-run multipliers in the context of the IS-LM subsystem, the question to be raised is one that involves the impact or short-run effects of exogenous and policy changes in the system. In particular, the concern will be with the Phillips-curve relationship while continuing to treat inflationary expectations, government expenditures, and the rates of monetary growth and debt issuance as exogenous.

The Phillips-curve relationship developed in the preceeding chapter has the following form:

$$p = (1-a_1)^{-1}\Phi(y) + p^*. \tag{3.0}$$

Linearising the function $\Phi(y)$ around the long-run equilibrium value of its argument $\bar{y}$, the above expression can be rewritten in the following way:

$$p = a_2(y-\bar{y}) + p^* \tag{13}$$

where, $a_2 = (1-a_1)^{-1}\Phi_y > 0$.

Short-run multipliers with respect to changes in the policy variables g, m, and b, and inflationary expectations, can be easily obtained from expression (13).

$$\frac{dp}{dg} = a_2 \frac{dy}{dg} > 0$$

$$\frac{dp}{dm} = a_2 \frac{dy}{dm} > 0$$

$$\frac{dp}{db} = a_2 \frac{dy}{db} < 0$$

$$\frac{dp}{dp^*} = a_2 \frac{dy}{dp^*} + 1 > 1$$

The signs of the short-run multipliers were obtained on the basis of the positivity of the parameter a_2, and the signs of the derivatives $\frac{dy}{dg}$, $\frac{dy}{dm}$, $\frac{dy}{db}$, and $\frac{dy}{dp^*}$.

An increase in the level of government spending and the rate of monetary expansion both have inflationary consequences that come about via increased demand pressure operating independently and via the wage-price nexus. Increased debt issuance, on the other hand, by reducing demand pressure via increasing the nominal rate of interest, nets out deflationary. Expectational adjustments, lastly, will be inflationary directly via the wage-price nexus, and indirectly via increased demand pressure.

However, whether or not such effects can be sustained in the long-run, is a different question.

C. The Long-Run Steady-State Equilibrium Conditions of the System.

In a static steady-state equilibrium position everything is constant and, hence, to obtain the long-run properties of the system, one only has to set all growth rates equal to zero. A stationary steady state equilibrium position, however, is not very interesting and we are not considering one here. Instead, we do consider a dynamic steady state equilibrium path in which all real variables are constant and all nominal magnitudes are growing at a common constant rate. That is, real output, the real money supply, the real value of the stock of bonds outstanding, and real government expenditures, are all constant. The nominal money stock on the other hand, the value of the stock of bonds in nominal terms, product prices and associated expectations, all grow at a common constant rate.

In fact, by setting Dp, Dp^*, Dy, Dm and Db equal to zero in the system specified above, one obtains the result that in the long-run the following conditions,

$$\bar{m} = \bar{p} = \bar{p}^* = \bar{b}$$

must hold.

By substituting the above conditions in the Phillips-curve relationship, given in expression (13) in linear form, it can be readily established that in the long-run excess demand will be zero. That is to say, in the long-run, actual and expected inflation coincide such that,

$$a_2(y - \bar{y}) = 0$$

and, hence,

$$y = \bar{y}.$$

In the long-run, output is at full capacity utilisation and excess demand consequently is zero.

Likewise, by substituting the same equilibrium conditions into the expression for wage setting behaviour specified in chapter three,

$$w = q + a_1 p + (1-a_1)p^*, \qquad (1.0)$$

it can be easily established that,

$$\bar{w} = \bar{p} + q,$$

or equivalently,

$$\bar{p} = \bar{w} - q.$$

That is, in the long-run product prices grow at the same rate as unit labour costs.

Summing up, in long-run steady-state equilibria the system will be characterised by the following conditions:

$$\bar{m} = \bar{p} = \bar{p}^* = \bar{b} \qquad (14)$$

$$y = \bar{y} \qquad (15)$$

$$\bar{w} = \bar{p} + q$$

In such a long-run scenario, the system is recursive or decomposable in nature. This results from the fact that expectations enter the inflationary process with full power. That is to say, with a given production technology, the conditions in the labour market determine the equilibrium level of utilisation and money is neutral. Hence, in a steady state the economy operates as if output is a supply determined phenomenon. Money is neutral in the long-run and inflation is purely a monetary phenomenon. Monetary expansion will feed into inflation in a one-for-one fashion and will have no real effects.

D. Long-Run Multipliers

In the long-run, therefore, the system is decomposable in that output is a supply side phenomenon, and inflation a purely monetary phenomenon. Since output is constant at the full capacity utilisation rate, it follows that all the corresponding multipliers must be zero. That is,

$$\frac{d\bar{y}}{dg} = \frac{d\bar{y}}{dm} = \frac{d\bar{y}}{db} = \frac{d\bar{y}}{dp^*} = 0$$

Given the long-run decomposability of the system, both fiscal and monetary policies become totally ineffective to move output and employment around in steady-state equilibria. This is a result that is borne out of the natural rate proposition that is being incorporated into the model.

From the condition that,

$$\bar{m} = \bar{p} = \bar{p}^* = \bar{b}$$

one obtains the result that,

$$\frac{d\bar{p}}{dm} = \frac{d\bar{p}}{db} = \frac{d\bar{p}}{dp^*} = 1.$$

That is, there is one-to-one correspondence between the rate of price inflation, inflationary expectations, and the rates of monetary growth and debt issuance. An increase in government spending, on the other hand, will have no effect on the inflation rate precisely because it will have no effect upon the level of real output. That is to say,

$$\frac{d\bar{p}}{dg} = 0 \quad \text{because} \quad \frac{d\bar{y}}{dg} = 0.$$

With the above in mind, consider again the IS-LM subsystem in long-run equilibrium which is given in the following expressions:

$$\text{IS:} \quad (1-c_1)\bar{y} = g + (c_1BP^{-1}+c_2)\bar{r} + c_3\bar{p}^* \qquad (9)'$$

$$\text{LM:} \quad \bar{r} = l_1\bar{y} + l_2\bar{p}^* + l_3MP^{-1} + l_4BP^{-1} \qquad (10)'$$

Totally differentiating the expression for the long-run IS relationship in (9)' one obtains:

$$(1-c_1)d\bar{y} = dg + (c_1BP^{-1}+c_2)d\bar{r} + c_3d\bar{p}^* \qquad (9)''$$

Treating m and b constant and allowing g to vary yields the following result:

$$(1-c_1)\frac{d\bar{y}}{dg} = (c_1BP^{-1}+c_2)\frac{d\bar{r}}{dg} + 1$$

Utilising the result found earlier in this section that the government spending multiplier is zero in the long-run, it can be established that,

$$\frac{d\bar{r}}{dg} = -\frac{1}{c_1BP^{-1}+c_2},$$

which is positive to the extent that the term $c_1BP^{-1}+c_2$ is negative.

Allowing m to vary while holding b and g constant, from expression (9)'' one can obtain:

$$(1-c_1)\frac{d\bar{y}}{dm} = (c_1BP^{-1}+c_2)\frac{d\bar{r}}{dm} + c_3\frac{dp^*}{dm}.$$

Since monetary policy is neutral in the long-run and $\frac{dp^*}{dm}=1$, it follows that,

$$\frac{d\bar{r}}{dm} = -\frac{c_3}{c_1BP^{-1}+c_2}.$$

If the above derivative is positive, which will be the case if $c_1BP^{-1}+c_2$ is negative, then $\frac{dMP^{-1}}{dm}$ must be negative.

Consider expression (6.0) specified in chapter three which is here rewritten:

$$MP^{-1} = L(\ y,\ r,\ p^*,\ BP^{-1}\ ). \qquad (6.0)$$

Its linearised version in long-run equilibrium with the autonomous component suppressed would be:

$$MP^{-1} = L_1\bar{y} + L_2\bar{p}^* + L_3\bar{r} + L_4BP^{-1}$$

where, $L_1 > 0$

$L_2 < 0$

$L_3 < 0$

and, $L_4 > 0$.

Consequently,

$$\frac{dMP^{-1}}{dm} = L_1\frac{d\bar{y}}{dm} + L_2\frac{dp^*}{dm} + L_3\frac{d\bar{r}}{dm}.$$

Since, $\frac{d\bar{y}}{dm} = 0$ and $\frac{dp^*}{dm} = 1$, it follows that,

$$\frac{dMP^{-1}}{dm} = L_2 + L_3 \frac{d\bar{r}}{dm}.$$

With $\frac{d\bar{r}}{dm}$ positive and the coefficients L_2 and L_3 both negative, then $\frac{dMP^{-1}}{dm}$ will be negative.

That is, a sustained higher rate of monetary growth will lead to a lower equilibrium stock of real balances because people economise on money holdings in real terms. The sustained higher rate of monetary growth in the long-run generates a higher rate of price inflation and higher inflationary expectations, inducing people to substitute away from real balances and into real goods as a way of holding their wealth. In other words, higher inflationary expectations generate asset substitution effects that lead to a portfolio reallocation out of real money balances. With a lower stock of real money balances, the nominal rate of interest would increase and, hence, the following two derivatives,

$$\frac{d\bar{r}}{dm} > 0 \quad \text{and} \quad \frac{dMP^{-1}}{dm} < 0,$$

imply each other.

Allowing now the rate of debt issuance (b) to vary, from the total differential in expression (9)'' we can obtain the following result:

$$(1-c_1)\frac{d\bar{y}}{db} = (c_1BP^{-1}+c_2)\frac{d\bar{r}}{db} + c_3\frac{dp^*}{db}.$$

With $\frac{d\bar{y}}{db} = 0$ and $\frac{dp^*}{db} = 1$ then,

$$\frac{d\bar{r}}{db} = -\frac{c_3}{c_1BP^{-1}+c_2},$$

which is exactly the same as $\frac{d\bar{r}}{dm}$. This is not surprising. Since in the long-run

$$\bar{m} = \bar{p} = \bar{p}^* = \bar{b},$$

then,

$$\frac{d\bar{r}}{dm} = \frac{d\bar{r}}{db} = \frac{d\bar{r}}{dp} = -\frac{c_3}{c_1BP^{-1}+c_2} > 0.$$

Note that unlike the derivative, $\frac{dMP^{-1}}{dm}$,

$$\frac{dBP^{-1}}{db} = \frac{dBP^{-1}}{db} = \frac{dBP^{-1}}{dp} = 0,$$

in the long-run. This is the case because in the present chapter we abstract from the influence of policy regimes by treating all policy instruments as exogenous. The stock of real money balances will be affected in the long-run by a change in either of the rates of monetary growth, or debt issuance, or price inflation. This is the case because a change in either one of these rates will have repercussions for inflationary expectations and the nominal rate of interest both of which would affect the demand for money balances. That is, although the supply of nominal money is constant in the present context, the demand is variable and, hence, the real stock of money balances will also be variable. Such a relationship does not exist with respect to the real stock of bonds outstanding in the present context.[15]

E. Fiscal and Monetary Policies in the Short- and Long-Runs

Some of the multipliers, as just discussed, vanish in the long-run because the system is decomposable in such long-run steady states.

Fiscal policy is both expansionary and inflationary in the short-run but not in the long-run. That is to say, an increase in the level of government expenditures in the short-run, nets out to be expansionary for as long as $c_1BP^{-1}+c_2$ is negative and, thus, the denominator of the IS-LM multiplier positive. The interest rate will also increase, thus crowding out investment expenditures. This crowding out, however, will not be complete in the short-run.

In the long-run fiscal policy is totally ineffective precisely because of complete crowding out. This result comes about via the interest rate mechanism. Observe that an increase in the level of government spending will cause the

[15]It is here implicitly assumed that inflationary expectations induce a portfolio reallocation out of real money balances and into real goods such that the demand for bonds is not affected. As argued in section C of chapter two, with p^* an argument in the money market equation, then for the model to be complete it is necessary to explain either the demand for real capital goods or the demand for real bonds. Not having done so, the model violates an adding up restriction that would be imposed upon by a wealth constraint.

interest rate to increase in the long-run to a greater extent compared to the short-run. That is to say,

$$\frac{d\bar{r}}{dg} > \frac{dr}{dg}.$$

This result can be shown algebraically in that,

$$\frac{d\bar{r}/dg}{dr/dg} = -\frac{\Omega}{I_1(c_1BP^{-1}+c_2)} > 1.$$

Hence, in response to an increase in the level of government spending, the interest rate mechanism in the long-run works simply to shift demand from the private sector to the government on a one-to-one basis. Real output does not change and neither does the inflation rate.

With respect to monetary policy, consider an increase in the rate of monetary growth however caused. That the monetary policy multiplier is positive in the short-run but zero in the long-run indicates, as already argued, that monetary policy is neutral in the long-run as a monetary expansion feeds one-to-one into inflation.

In the short-run, a monetary expansion would increase the stock of real money balances, thus lowering the rate of interest and generating a positive output effect. But in the long-run, a monetary expansion has the effect of reducing the stock of real money balances, hence pushing the rate of interest up. That is, the long-run effects of a sustained increase in the rate of monetary growth are in the opposite direction than one would normally expect the effects of monetary expansion to be. This result was rationalised in the preceeding section by arguing that a monetary expansion in the long-run generates higher inflation and higher inflationary expectations, inducing people to economise on money holdings in real terms. This fact alone would give rise to an overshooting proposition.

F. Overshooting of Equilibrium Values

In the short-run, a monetary expansion lowers the rate of interest by way of a larger money stock. If this higher rate of monetary growth is sustained, it will eventually build into inflation pulling the rate of interest up in a new equilibrium with a reduced stock of real money balances. For this to be the case, the rate of price inflation in this model must overshoot its long-run equilibrium value. That is, precisely because the system economises on real money balances by way of asset substitution effects in response to higher inflation and expected inflation, a monetary expansion will eventually reduce the real stock of money. It will thus generate in transition an inflation rate is larger than the one which it eventually ends up with as a necessary condition for reducing real money balances. Overshooting, hence, is rooted in the stock equilibrium requirements of the model. However, lead-lag relationships in the system can also generate overshooting of equilibrium values.

Consider a situation in which increased demand pressure sets off an inflationary process. In the model at hand, with an expectational lag, the inflation rate will peak after the level of real output. There is, in other words, a lead-lag relationship in this model between output and the rate of price inflation. The causal factor in this relationship is the inflationary expectations that are built into the system from actual inflation experience and which take time to adjust downwards.

If this is the case, then in a context of monetary expansion, initially the stock of real money balances increases as the rate of price inflation is slow to respond. Output, hence, expands because of a low rate of interest. As time unfolds, however, and inflationary expectations fall behind actual inflation experience, an inflationary psychology is gradually being built into the system, which ultimately becomes responsible for high inflation rates even when aggregate demand begins to slacken.

Output and prices thus will overshoot their equilibrium values, and the lead-lag relationship that exists between them may well take the economy into a stagflationary phase.

In all that has been said thus far, policy was, in effect, treated to be exogenous and the analysis [illegible] comparative static in nature. Nevertheless, the [illegible] allowed us to make certain inferences about the short-run disequilibrium behaviour of the model. Asset substitution effects, and the lead-lag relationship between output and the rate of price inflation that is rooted in the expectational lag, will lead to overshooting of both real output and price inflation, and may potentially take the system through an inflationary recession. The question that will now be raised is one that concerns the dynamic stability of the system under alternative policy regimes.

IV. DYNAMICS WITH MONETARY GROWTH ENDOGENOUS

This study employs the view that not all policy instruments can be independent of one another at each point in time. At least one of them must be endogenously determined via the government budget constraint. This particular chapter deals with the endogeneity of monetary growth. Government spending and the rate of growth of debt issuance are both assumed to be held constant. The money supply thus becomes the variable that will adjust so as to equate total government expenditures with the total flow of financing from all sources.

A. The Model and its Characteristic Equation

When the rate of monetary growth is assumed to be endogenous via the government budget constraint, the monetary policy reaction function is, in effect, subsumed away. Government spending being constant, it washes out of the differential version of the system. Debt issuance on the other hand, is here being exogenised by assuming that its growth rate is constant and has the value it is to have in the long-run steady inflation equilibrium path. That is to say, $b=\bar{b}$ and $Db=0$.

With the above stated assumptions, the total differential version of the general model specified in chapter three in expressions (3.2), (5.2), (6.2), (7.1), and

(8.1), takes the following form:

$$Dp = a_2Dy + \alpha a_2(y-\bar{y}) \qquad (3.2)$$

$$(1-c_1)Dy = c_1BP^{-1}r(\bar{b}-p) + [c_1BP^{-1}+c_2]Dr + \alpha a_2c_3(y-\bar{y}) \qquad (5.3)$$

$$Dr = l_1Dy + \alpha a_2l_2(y-\bar{y}) + l_3MP^{-1}(m-p) + l_4BP^{-1}(\bar{b}-p) \qquad (6.3)$$

$$MP^{-1}Dm + MP^{-1}m(m-p) + BP^{-1}\bar{b}(\bar{b}-p) = -\tau Dy + (1-\tau)BP^{-1}r(\bar{b}-p) + (1-\tau)BP^{-1}Dr \qquad (8.2)$$

This is a dynamic simultaneous system in four endogenous variables: price inflation (p), real output (y), the nominal interest rate (r), and monetary growth (m).

To study the dynamic properties of this system, the above four equations may be linearised around the corresponding steady-inflation equilibrium path. Let,

$$x(t) = \bar{x} + [x(0) - \bar{x}]e^{\rho t}$$

where, x = p, y, r, and m.

Note that $\bar{x}$ represents the value the variable in question is to have in the long-run, and $x(0)-\bar{x}$ the initial deviation from that long-run value.

By evaluating the corresponding partial derivatives at the long-run steady-inflation solution one obtains the following matrix expression:[14]

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[14] The model at hand is in actual fact non-linear with the non-linearities taking the form of initial deviations from equilibrium appearing multiplicatively. Such non-linearities are here and throughout the analysis eliminated by assuming that the product of two initial deviations from equilibrium is, in the neighbourhood of that equilibrium, so small as to be legitimately set equal to zero. For more details on the linearisation procedures employed in this study, refer to appendix 1.

$$
\begin{bmatrix}
\rho a_2+\alpha a_2 & 0 & -\rho & 0 \\
\rho(1-c_1)-\alpha a_2 c_3 & -\rho(c_1 BP^{-1}+c_2) & \bar{r}c_1 BP^{-1} & 0 \\
\rho l_1+\alpha a_2 l_2 & -\rho & -(l_3 MP^{-1}+l_4 BP^{-1}) & l_3 MP^{-1} \\
\rho\tau & -\rho(1-\tau)BP^{-1} & \xi_1 & (\rho+\bar{m})MP^{-1}
\end{bmatrix}
\begin{bmatrix}
y(0)-\bar{y} \\
r(0)-\bar{r} \\
p(0)-\bar{p} \\
m(0)-\bar{m}
\end{bmatrix}
=
\begin{bmatrix}
0 \\ 0 \\ 0 \\ 0
\end{bmatrix}
$$

where, $\xi_1 = \bar{r}(1-\tau)BP^{-1}-\bar{b}BP^{-1}-\bar{m}MP^{-1}$.

The determinant of the characteristic matrix of the system above, when evaluated and set equal to zero, will yield the following third degree characteristic equation:

$$k_0\rho^3 + k_1\rho^2 + k_2\rho + k_3 = 0$$

where,''

$$k_0 = (1-c_1) - l_1(c_1BP^{-1}+c_2)$$

$$k_1 = \bar{m}(1-c_1) - \bar{m}l_1(c_1BP^{-1}+c_2) - (1-c_1)(1-\tau)BP^{-1}l_3 + \tau(c_1BP^{-1}+c_2)l_3 + a_2(c_1BP^{-1}+c_2)(l_3MP^{-1}+l_4BP^{-1}) + a_2\bar{r}c_1BP^{-1} - \alpha a_2[c_3+l_2(c_1BP^{-1}+c_2)]$$

$$k_2 = a_2(c_1BP^{-1}+c_2)\bar{m}(l_3MP^{-1}+l_4BP^{-1}) + a_2\bar{m}\bar{r}c_1BP^{-1} + a_2(c_1BP^{-1}+c_2)l_3[\bar{r}(1-\tau)BP^{-1}-\bar{b}BP^{-1}-\bar{m}MP^{-1}]$$

''Note that the term for the real stock of money balances, MP^{-1}, factors out of the characteristic equation as it is common in all of its coefficients and, consequently, drops out. In addition, note that with p^* not an argument in the money market equation, the coefficient l_2 takes on the value of zero and consequently the term $l_2(c_1BP^{-1}+c_2)$ vanishes.

$$- a_2(1-\tau)BP^{-1}l_3\bar{r}c_1BP^{-1} + \alpha a_2\bar{r}c_1BP^{-1}$$

$$+ \alpha a_2(c_1BP^{-1}+c_2)(l_3MP^{-1}+l_4BP^{-1})$$

$$+ \alpha a_2c_3(1-\tau)BP^{-1}l_3 - \alpha a_2\bar{m}[c_3+l_2(c_1BP^{-1}+c_2)]$$

$$k_3 = \alpha a_2(c_1BP^{-1}+c_2)\bar{m}(l_3MP^{-1}+l_4BP^{-1}) + \alpha a_2\bar{m}\bar{r}c_1BP^{-1}$$

$$+ \alpha a_2(c_1BP^{-1}+c_2)l_3[\bar{r}(1-\tau)BP^{-1}-\bar{b}BP^{-1}-\bar{m}MP^{-1}]$$

$$- \alpha a_2(1-\tau)BP^{-1}l_3\bar{r}c_1BP^{-1}$$

The linearised system is of third order, comprising three first order differential equations: the Phillips-curve relationship, the aggregate demand relationship implicit in the IS-LM apparatus, and the equation for the government budget constraint. The initial system, however, is of fourth order because the aggregate demand relationship is a second order differential equation in the nominal rate of interest. It reduces to one of first order and, hence, the system to one of third order, due to the linearisation procedures employed here-in, and particularly the removal of the inherent non-linearities by suppressing the products of initial deviations from equilibrium (see appendix 1).

The equilibrium of the dynamic system in question will be stable if, and only if, all of the characteristic roots are either real negative numbers, or conjugate complex pairs with negative real parts. A necessary condition for this to hold true is that all the coefficients of the characteristic equation have the same sign. This comes from the Descartes rule of signs which states that the number of positive real roots to a polynomial equals the number of sign changes associated with it less an even integer. If this rule is applied to a situation in which there are no sign changes, then one would know from this that there can be no positive real roots, thus ruling out the possibility of non-cyclical instability.

While this would be both necessary and sufficient for the stability of first and second order systems, it will not be sufficient for third and higher order systems because it cannot rule out the possibility of conjugate complex roots with positive real parts.

Hence, in the context of this chapter with a third order system, the conditions that will be necessary and sufficient for stability are the Routh-Hurwitz conditions. All the characteristic roots of the cubic characteristic equation of the system will have negative real parts if, and only if, the three principal minors of the matrix,

$$\begin{bmatrix} k_1 & k_0 & 0 \\ k_3 & k_2 & k_1 \\ 0 & 0 & k_3 \end{bmatrix}$$

are all positive when normalising on the positivity of k_0.

Thus, while k_0, k_1, k_2, and k_3 being positive is a necessary condition for the stability of the system, it is not by itself sufficient. For sufficiency it is further required that the inequality,

$$k_1 k_2 - k_0 k_3 > 0$$

holds true.[18]

B. Necessary Conditions: the Positivity of k_0.

By inspecting the components of k_0, it can be seen that the only factor that may potentially cause it to be negative is the mechanism embodied in the term $c_1BP^{-1}+c_2$. The term c_1BP^{-1} captures a consumption effect that operates via interest income variations resulting from interest rate changes. The term c_2, on the other hand, captures an investment effect induced by the same interest rate changes.

These two effects, however, work in opposite directions. That is to say, an interest rate increase puts upward pressure on aggregate demand via a positive consumption effect, and downward pressure on aggregate demand via a negative investment effect. Hence, the positivity of $c_1BP^{-1}+c_2$ entails a potentially explosive feedback mechanism into the system that runs from interest rate increases, to output increases, to further interest rate increases via the transactions demand for money in this context.

If, however, the investment effect induced by interest rate changes dominates the consumption effect similarly induced in the opposite direction such that $c_1BP^{-1}+c_2$ is negative, then an increase in the interest rate will subsequently lead to a decrease in output. The negativity of $c_1BP^{-1}+c_2$, hence, constitutes a dampening element in the

[18]See appendix 2 for a more detailed account of the Routh-Hurwitz stability conditions.

system and it is here sufficient to ensure the positivity of k_0. But sufficient though it may be, it is not necessary. It is not necessary because k_0 embodies another dampening mechanism that works through the leakage into savings given by $1-c_1$ positive. That is to say, part of the initial increase in demand pressure will leak out of the circular flow of product and income in the form of savings, thus reducing demand pressure.

What is required for k_0 to be positive is that this leakage effect coupled with the investment effect dominate the destabilising consequences of the associated consumption effect. That is, for k_0 to be positive, the inequality

$$1 - c_1 - l_1 c_2 > l_1 c_1 BP^{-1}$$

must hold.

It was found earlier in chapter four that, the positivity of the term $c_1 BP^{-1} + c_2$ implies a positively sloped IS curve. It was also found that

$$1 - c_1 - l_1 (c_1 BP^{-1} + c_2),$$

is the denominator of the static IS-LM multiplier. Using these results we can say, on the basis of the above discussion, that a positively sloped IS curve does not necessarily lead to the negativity of k_0 provided that the denominator of the static IS-LM multiplier is positive.

C. **Necessary Conditions: the Positivity of k_1.**

The mechanisms that are embodied in k_0 just discussed do not actually operate via inflation adjustments. They are rather initiated by increased demand pressure per se. Besides these, there will be further repercussions in the system that also do not operate via inflation adjustments. There will, for example, be stabilising consequences in response to demand pressure, occurring via the budget constraint, as indicated by the positivity of the term $\tau c_2 l_3$. Tax revenues would increase, hence, allowing for an endogenous monetary contraction which will put upward pressures on the interest rate thus leading to investment crowding out. But this same interest rate increase, would generate a destabilising positive consumption effect via increased interest income which will be dominated, by the afore-mentioned negative investment effect, if $c_1 BP^{-1} + c_2$ is indeed negative as discussed in the preceeding section.

Nevertheless, increased demand pressure will have an inflationary consequence via the wage-price nexus. One consequence of this induced inflation will be to erode the real stocks of both money and bonds. Reduced real money balances and real public debt will subsequently feed into the money market affecting interest rates in a counteracting way: reduced real money balances pushing it up, and reduced real debt pushing it down. What the net effect on interest rates would be is a matter of relative elasticities in the money market. The interest rate will increase to the extent that it is relatively more responsive to changes in real money balances than to changes in the real value of bonds outstanding, in which case $l_3MP^{-1}+l_4BP^{-1}$ will be negative.

But if the above condition holds true and the 'real balance' effect on the interest rate dominates what one may call the 'real debt' effect on the interest rate, it will not be stabilising unless the interest rate induced investment effect dominates the similarly induced consumption effect. That is to say, if $l_3MP^{-1}+l_4BP^{-1}$ is negative, then $c_1BP^{-1}+c_2$ being negative, enhances the stability of the system as the term,

$$a_2(c_1BP^{-1}+c_2)(l_3MP^{-1}+l_4BP^{-1}),$$

will be positive.

However, the above mechanisms will be augmented by inflation induced expectational adjustments. Inflationary expectations once under way will lead into further inflation via the wage-price nexus and will also feed directly into the goods and money markets by means of asset substitution effects.'' In the money market, upward expectational adjustments will be generating asset substitution effects out of money balances thus driving the interest rate down. This economising on real money balances will feed into the goods market in the form of an increased demand for real goods. Furthermore, the reduction in the interest rate would also be adding to demand pressure via investment

''Note that these asset substitution effects associated with the interest rate mechanism, that is, with the coefficient l_2, are present only because of the inclusion of p as an argument in the money market equation.

crowding-in.

Hence, even if $l_3MP^{-1}+l_4BP^{-1}$ is negative, asset substitution effects induced by inflationary expectations may be sufficiently strong to cause an interest rate decrease. What would now be a sufficient condition for the interest rate to increase, though not necessary for stability, is the negativity of the term,

$$l_3MP^{-1} + l_4BP^{-1} - \alpha l_2.$$

That is, we would like the 'real balance' effect on the interest rate, to dominate the 'real debt' and asset substitution effects in the money market combined together. If this holds true, then the interest rate mechanism will be stabilising if $c_1BP^{-1}+c_2$ is also negative.

In summary, in the context of the coefficient k_1, what will ensure that the interest rate will increase, in response to demand pressure, is the following inequality

$$\bar{m}l_1 - a_2l_3MP^{-1} - \pi l_3 > a_2l_4BP^{-1} - \alpha a_2l_2,$$

holding true. That is to say, the sum of the demand pressure effect via the transactions demand for money, and the real balance effects induced by inflation and monetary contraction via the budget constraint, should dominate the sum of the real debt and the expectations induced interest rate effects.[20]

Coupling the above condition with a negative $c_1BP^{-1}+c_2$, the only source of negativity in the coefficient, k_1, would

[20]Note that the terms monetary contraction and monetary expansion, are, here and throughout the analysis, used to refer to a decrease or an increase in the rate of monetary growth respectively.

remain to be the influence of upward expectational adjustments on investment expenditures via a reduced real interest rate in the context of the asset substitution effects manifested in the negativity of the term $-aa_2c_3$.

D. Necessary Conditions: the positivity of k_3.

Inflationary expectations, once under way, will feed into further inflation via the wage-price nexus, and the reduction in real money balances and in the real debt will be enhanced. These consequences, besides feeding into the money market as previously discussed with real balance and real debt effects, will also feed into the budget constraint causing an endogenous monetary expansion. Since both money balances and bonds are instruments of financing, erosion of their real values will lead to what one may call a replacement monetary expansion. This monetary expansion will in turn feed into the money market lowering the rate of interest.

To the extent that the interest rate-induced investment effect dominates the corresponding consumption effect, the above drop in the rate of interest will be expansionary adding to the demand pressure already in place. Thus, the negativity of the term $c_1BP^{-1}+c_2$ does here create a potential source of instability embedded in the term

$$-aa_2(c_1BP^{-1}+c_2)l_3(\bar{b}BP^{-1}+\bar{m}MP^{-1}).$$

If $c_1BP^{-1}+c_2$ is indeed negative, then the destabilising trends generated by the mechanisms embedded in the immediately preceeding term will, for stability, have to be dominated by the dampening trends embodied in the terms,

$$\alpha a_2(c_1BP^{-1}+c_2)\bar{m}(l_3MP^{-1}+ \ldots BP \ldots),$$

and

$$\alpha a_2[c_2 l_3 \bar{r}(1-\tau)BP^{-1} \ldots$$

The reduction [illegible] real debt due to inflation, while leading to mone[illegible]ansion by virtue of its role in deficit financing, [illegible]so leads to monetary contraction endogenously by way of reduced interest payments. This latter monetary contraction is, in and of itself, stabilising as it pushes interest rates up with a subsequent crowding out of investment expenditures. In addition, reduced interest payments imply a reduced interest income which has an associated stabilising negative consumption effect.

How does the endogeneity of monetary growth affect the positivity of k_3? A reduction in real money balances leads to monetary expansion via the budget constraint. A reduction in the real debt due to inflation, on the other hand, also causes monetary expansion, but it does at the same time cause monetary contraction as interest payments on the public debt are reduced. To the extent that an interest rate increase is dampening, a monetary contraction becomes desireable in the sense that it dampens output. This will come about if the inequality

$$\bar{r}(1-\tau)BP^{-1} > \bar{b}BP^{-1} + \bar{m}MP^{-1}$$

holds true.

This inequality simply says that, the net flow of payments between the government and the private sector on bond account, must be larger than the increase in the stock of real money balances. Otherwise being the case, the endogeneity of monetary growth via the budget constraint adds to the negativity of k_3 and hence introduces a destabilising element into the system.

E. Necessary Conditions; the Positivity of k_2.

Before expectational adjustments take effect, the inflation induced by increased demand pressure operating through the wage-price nexus will generate precisely the same effects, qualitatively speaking, that were just discussed in the context of the coefficient k_3. This can be very clearly seen by rewriting the coefficient k_2 in the following way:

$$\begin{aligned} k_2 = {} & (1/\alpha)k_3 + \alpha a_2\bar{r}c_1BP^{-1} \\ & + \alpha a_2(c_1BP^{-1}+c_2)(l_3MP^{-1}+l_4BP^{-1}) \\ & + \alpha a_2c_3(1-\tau)BP^{-1}l_3 - \alpha a_2\bar{m}[c_3+l_2(c_1BP^{-1}+c_2)]. \end{aligned}$$

Once inflationary expectations get under way, however, they will generate the same destabilising asset substitution effects that were previously discussed in the context of k_1. That is, inflationary expectations generate destabilising repercussions for the system by means of asset substitution effects that reduce the demand for money balances thereby lowering the interest rate, and increase the demand for real capital goods.

Nothing new is added qualitatively by this coefficient of the characteristic equation. It is quite obvious why the same mechanisms embodied in k_3 are found operating in k_2 as well. These mechanisms are based on inflation adjustments. An initial increase in demand pressure will have an inflationary consequence both independently and via the wage-price nexus. But inflation will, with a lag, feed into upward expectational adjustments which in turn will feed back into further inflation adjustments again via the wage-price nexus.

One conclusion which one may reach from the preceeding discussion is that, to the extent that $c_1BP^{-1}+c_2$ is negative, the endogeneity of monetary growth will, in one regard, be stabilising. The initial increase in demand pressure and its inflationary consequences will be dampened by monetary contraction occurring through increased tax revenues. Given the lag in expectational adjustments, the destabilising forces associated with inflationary expectations will be weakened.

F. Sufficiency: the Positivity of $k_1k_2 - k_0k_3$

The coefficients k_0, k_1, k_2, and k_3, being positive can only guarantee that the system has no positive real roots. While this is necessary for stability, it is not sufficient as previously argued because the possibility of conjugate complex roots with positive real parts cannot be ruled out. Given that the four coefficients of the characteristic equation are all positive, then $k_1k_2-k_0k_3$ also being positive would be sufficient for stability. Embodied in this last term, is a statement of relative strengths. But in fact, when expanded, it becomes so complicated a term that it raises doubts as to whether anything can be learned by exploring it qualitatively.

The analysis done thus far, was centred on the positivity of k_0, k_1, k_2, and k_3. Such a result can only rule out the possibility of non-cyclical instability, but not that of explosive oscillations. This latter possibility is ruled out by the condition that $k_1k_2-k_0k_3$ must be positive.[21]

The positivity of each and every coefficient in the characteristic equation would imply that, starting with increased demand pressure, that is to say, an increase in real output, then the system will operate in such a way so as to reduce real output. Similarly, starting with a situation of slack demand, the system will work to stimulate production. If, however, the system responds to increased

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[21] For a mathematical proof of this statement, see appendix 3.

demand pressure by reducing output to a greater extent than it initially increased, then the result would be explosive oscillations.

In the case of cyclical motions, stability requires that the amplitude of the fluctuations must be decreasing over time. To put it differently, the limit of the amplitude of cyclical fluctuations must be approaching the value of zero over time for the system to be stable. This will indeed be the case if the dampening mechanisms in the system work in such a way so as to reduce real output when demand pressure is increased, but to a lesser extent in absolute terms, than the initial increase in output.

What factors can be helpful in bringing about such a result? If expectations are static and, hence, $a=0$, then the coefficient k_3 would attain the value of zero and the order of the system would be reduced by one.

In this case of static expectations the characteristic equation of the system can be written in the following way:

$$k_0'\rho^2 + k_1'\rho + k_2' = 0,$$

where

$$k_0' = (1-c_1) - l_1(c_1BP^{-1}+c_2)$$

$$\begin{aligned} k_1' = {} & \bar{m}(1-c_1) - \bar{m}l_1(c_1BP^{-1}+c_2) \\ & - (1-c_1)(1-\tau)BP^{-1}l_3 + \tau(c_1BP^{-1}+c_2)l_3 \\ & + a_2\bar{r}c_1BP^{-1} + a_2(c_1BP^{-1}+c_2)(l_3MP^{-1}+l_4BP^{-1}) \end{aligned}$$

$$\begin{aligned} k_2' = {} & a_2(c_1BP^{-1}+c_2)\bar{m}(l_3MP^{-1}+l_4BP^{-1}) + a_2\bar{m}\bar{r}c_1BP^{-1} \\ & + a_2(c_1BP^{-1}+c_2)l_3[\bar{r}(1-\tau)BP^{-1}-\bar{b}BP^{-1}-\bar{m}MP^{-1}] \end{aligned}$$

$$- a_2(1-\tau)BP^{-1}l_3\bar{r}c_1BP^{-1}.$$

The corresponding coefficients k_0', k_1', and k_2' were obtained by setting $a=0$ in the coefficients k_0, k_1, and k_2 respectively.

Observe that, given all the above, the coefficients k_0, k_1, k_2, and k_3 can be rewritten in the following way:

$$k_0 = 1 - c_1 - l_1(c_1BP^{-1}+c_2) = k_0'$$

$$k_1 = k_1' - aa_2[c_3+l_2(c_1BP^{-1}+c_2)]$$

$$k_2 = (1/a)k_3 + aa_2(c_1BP^{-1}+c_2)(l_3MP^{-1}+l_4BP^{-1})$$
$$+ aa_2\bar{r}c_1BP^{-1} + aa_2c_3(1-\tau)BP^{-1}l_3$$
$$- aa_2\bar{m}[c_3+l_2(c_1BP^{-1}+c_2)]$$

$$k_3 = ak_2'$$

The coefficients k_0 and k_0' are identical and, hence, have the same sign. k_1' is unambiguously positive if k_1 is positive. In addition, if k_3 is positive and since a is positive, k_2' will also be positive. Hence, the argument can be made that if k_0, k_1, k_2, and k_3 are positive, then k_0', k_1' and k_2' must also be positive.

Assuming that,

$$c_1BP^{-1} + c_2,$$

and

$$l_3MP^{-1} + l_4BP^{-1},$$

are both negative, it can be observed that, taking the system involving static expectations as the numeraire, expectational adjustments have the consequence that k_1 and k_2 lose in positivity in comparison with k_1' and k_2' respectively under the influence of asset substitution effects. Hence, what can lead $k_1k_2-k_0k_3$ to negativity are precisely these expectations induced asset substitution effects. These must be the factors that can potentially lead the system to an explosive oscillatory time path.

It can also be observed that, the smaller a is, the bigger k_1k_2 and the smaller k_0k_3 will be. Hence, a large a also pushes the system in the direction of an explosive oscillatory time path. Consequently, the argument can be made that to the extent that rational expectations increase the speed with which expectational adjustments occur, they will be increasing the possibility of explosive oscillations in those cases of oscillatory time paths.

G. Conclusions.

The system at hand comprises such a complex network of inter-relationships that it is impossible to establish that any one specific condition has to hold for stability or instability. It can, however, be said that such factors the negativity of both $c_1BP^{-1}+c_2$ and $l_3MP^{-1}+l_4BP^{-1}$ work in the direction of the stability of the system. That is, the interest rate mechanism will have a stabilising impact upon the system to the extent that the investment demand is more sensitive to interest rate fluctuations than the consumption demand is. This condition would ensure that an increase in the nominal interest rate, however caused, will have the consequence of dampening output.

Whether or not the interest rate will increase in response to demand pressure is something which ultimately depends on the relative sensitivity of the money market to variations in the arguments that determine its behaviour. Abstracting from the influence of expectational adjustments, the money market will be stabilising to the extent that the 'real balance' effect referred to earlier dominates the 'real debt' effect, a condition that can realistically be expected to hold.

Upward expectational adjustments, on the other hand, would generate destabilising repercussions directly via both the goods and money markets. In the goods market, they will take the form of a reduced real interest rate, thus leading to a positive investment effect. In the money market, they will generate asset substitution effects out of money balances pushing interest rates down. These are precisely the mechanisms that are ultimately responsible for the possibility of explosive oscillations in the system.

Certainly, the endogeneity of monetary growth via the government budget constraint will also have repercussions for the stability of the system. In the context of higher demand pressure, what will be working in the direction of stability would be monetary contraction. Monetary contraction will indeed be the consequence of increased demand pressure via increased tax revenues. In addition, an erosion in the real value of bonds outstanding due to inflation would also have a contractionary impact upon monetary growth via reduced interest payments.

The same erosion in the real debt, however, coupled with the similarly induced erosion in real money balances, would lead to a replacement monetary expansion. Interest rate increases, desired for the stability of the system, would also be generating monetary expansion via increased interest payments.

It cannot possibly be established qualitatively in which direction monetary growth will move in response to an increase in demand pressure. Consequently, it cannot be argued that the endogeneity of monetary growth via the government budget constraint will be adding to the stability or the instability of the system in general.

V. DYNAMICS WITH THE RATES OF MONETARY GROWTH AND DEBT ISSUANCE ENDOGENOUS

Continuing on the same theme, that is the endogenisation of one policy instrument via the government budget constraint, this chapter first examines the stability of the system under the assumption that the rate of debt issuance is endogenous. The analysis is subsequently extended to allow for monetary growth to be endogenous as well by integrating into the model the monetary policy reaction function specified in expression (7.1) in chapter three. Two basic policy regimes will thus be examined in the context of this chapter. The first deals with the endogeneity of the rate of debt issuance via the government budget constraint. The second deals with the simultaneous endogeneity of both the rates of debt issuance and monetary growth, the former via the government budget constraint and the latter via the monetary policy reaction function.

A. The Model with the Rate of Debt Issuance Endogenous

Debt issuance will here be the variable to adjust in response to budget imbalances. The marginal tax rate, government expenditures, and the rate of monetary growth, are all assumed to be held constant. Like debt issuance in the preceeding chapter, monetary growth is here being exogenised by assuming that its rate is constant and has the value it is to have on the long-run steady-inflation equilibrium path ($\bar{m}$).

By setting Dg=0, $m=\bar{m}$, and $D\bar{m}=0$ in the general model specified in expressions (3.2), (5.2), (6.2), and (8.1) in chapter three, one obtains the following system in time differential form:

$$Dp = a_2 Dy + \alpha a_2(y-\bar{y}) \qquad (3.2)$$

$$(1-c_1)Dy = c_1 BP^{-1}r(b-p) + (c_1 BP^{-1}+c_2)Dr + \alpha a_2 c_3(y-\bar{y}) \qquad (5.4)$$

$$Dr = l_1 Dy + \alpha a_2 l_2(y-\bar{y}) + l_3 MP^{-1}(\bar{m}-p) + l_4 BP^{-1}(b-p) \qquad (6.4)$$

$$BP^{-1}Db + BP^{-1}b(b-p) = (1-\tau)BP^{-1}r(b-p) + (1-\tau)BP^{-1}Dr - \bar{m}MP^{-1}(\bar{m}-p) - \tau Dy \qquad (8.3)$$

This is again a third order system in four endogenous variables: price inflation (p), real output (y), the nominal rate of interest (r), and the growth rate of debt issuance (b). To study the dynamic properties of the above system, its four constituent equations may be linearised around the corresponding steady inflation equilibrium path, by following exactly the same procedures as in chapter five, to obtain the following matrix expression:

$$\begin{bmatrix} \rho a_2+\alpha a_2 & 0 & -\rho & 0 \\ \rho(1-c_1)-\alpha a_2 c_3 & -\rho(c_1 BP^{-1}+c_2) & \bar{r}c_1 BP^{-1} & -\bar{r}c_1 BP^{-1} \\ \rho l_1+\alpha a_2 l_2 & -\rho & -(l_3 MP^{-1}+l_4 BP^{-1}) & l_4 BP^{-1} \\ \rho\tau & -\rho(1-\tau)BP^{-1} & \xi_1 & \rho BP^{-1}+\xi_2 \end{bmatrix}$$

$$\begin{bmatrix} y(0)-\bar{y} \\ r(0)-\bar{r} \\ p(0)-\bar{p} \\ b(0)-\bar{b} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

where, $\xi_1 = \bar{r}(1-\tau)BP^{-1}-\bar{b}BP^{-1}-\bar{m}MP^{-1}$.

and, $\xi_2 = \bar{b}BP^{-1}-\bar{r}(1-\tau)BP^{-1}$.

The determinant of the characteristic matrix of the system above, evaluated and set equal to zero, yields the following cubic characteristic equation:

$$h_0\rho^3 + h_1\rho^2 + h_2\rho + h_3 = 0$$

where

$$h_0 = (1-c_1) - l_1(c_1 BP^{-1}+c_2)$$

$$\begin{aligned} h_1 = {} & [1-c_1-l_1(c_1 BP^{-1}+c_2)][\bar{b}-\bar{r}(1-\tau)] \\ & - (1-c_1)(1-\tau)l_4 BP^{-1} - l_1(1-\tau)\bar{r}c_1 BP^{-1} \\ & + \tau(c_1 BP^{-1}+c_2)l_4 + \tau\bar{r}c_1 \\ & + a_2(c_1 BP^{-1}+c_2)(l_3 MP^{-1}+l_4 BP^{-1}) + a_2\bar{r}c_1 BP^{-1} \\ & - \alpha a_2[c_3+l_2(c_1 BP^{-1}+c_2)] \end{aligned}$$

$$\begin{aligned} h_2 = {} & a_2(c_1 BP^{-1}+c_2)[l_3-l_4]\bar{m}MP^{-1} \\ & - a_2 c_2\bar{r}(1-\tau)l_3 MP^{-1} - a_2\bar{r}c_1\bar{m}MP^{-1} \end{aligned}$$

$$+ \alpha a_2(c_1BP^{-1}+c_2)(l_3MP^{-1}+l_4BP^{-1}) + \alpha a_2\bar{r}c_1BP^{-1}$$

$$- \alpha a_2\bar{b}[c_3+l_2(c_1BP^{-1}+c_2)]$$

$$+ \alpha a_2c_3(1-\tau)(\bar{r}-l_4BP^{-1}) - \alpha a_2l_2c_2\bar{r}(1-\tau)$$

$$h_3 = \alpha a_2(c_1BP^{-1}+c_2)[l_3-l_4]\bar{m}MP^{-1}$$

$$- \alpha a_2c_2\bar{r}(1-\tau)l_3MP^{-1} - \alpha a_2\bar{r}c_1\bar{m}MP^{-1}$$

Given this third degree characteristic equation, the stability of the system can be discussed by applying the Routh-Hurwitz conditions. However, casual inspection of the coefficients of this characteristic equation would readily reveal that in many ways they are very similar to those corresponding to the characteristic equation associated with the endogeneity of monetary growth. The analysis, thus, can be facilitated and simplified by using the discussion in the immediately preceeding chapter.

B. Stability Conditions

The coefficient attached to ρ^3, h_0, is exactly the same as k_0.[22] No new mechanisms are embodied therein. There will be stabilising trends coming from the savings leakage and interest rate induced investment effects, and destabilising consequences coming from the interest rate induced consumption effect.

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[22]Note that the h_is refer to the coefficients of the characteristic equation associated with the policy regime in which debt issuance is endogenous via the government budget constraint. The k_is on the other hand, refer to the coefficients of the characteristic equation associated with the policy regime in which monetary growth is endogenous via the government budget constraint.

the positivity of h_1

Considering the coefficient h_1, one can observe that the initial increase in demand pressure that raises tax revenues, contrary to what the case was under the endogeneity of monetary growth, will here have an ambiguous impact upon the stability of the system. This increase in tax revenues will allow for an endogenous contraction in the rate of debt issuance. This development will feed directly into the goods market with a contractionary effect on real output, and indirectly via the money market, with a destabilising expansionary effect. That is, interest income will be reduced generating a negative consumption effect manifested in the positivity of the term, $\tau \bar{r} c_1 BP^{-1}$. At the same time, the reduction in the real public debt outstanding would allow the nominal interest rate to come down, hence causing output to expand to the extent that $c_1 BP^{-1}+c_2$ is negative, and therefore the term,

$$\tau(c_1 BP^{-1}+c_2)l_1,$$

is also negative.

Hence, the endogenous reduction in the rate of debt issuance, effected via increased tax revenues, cannot be established to have a stabilising effect on the system by way of increasing the positivity of h_1, as the case was under the endogeneity of monetary growth.

However, while increased demand pressure feeds into the budget constraint by means of increasing tax revenues causing the rate of debt issuance to decrease, it also pushes the interest rate up via the transactions demand for money, hence increasing interest payments on the debt and forcing an endogenous increase in the rate of debt issuance. This latter development becomes destabilising in that it induces expansionary consumption effects as interest income increases.

Increased debt issuance, in turn, generates a destabilising momentum of its own as it feeds back onto itself. That is, an increase in the rate of interest leads to increased debt issuance by means of increased interest payments. This increase in the rate of debt issuance will feed into the money market raising the nominal interest rate, which in turn will feed back into the budget constraint causing further increases in debt issuance. In addition, increases in the rate of debt issuance also increase interest payments on the debt, generating a feedback mechanism that runs from debt issuance to increased interest payments to further debt issuance without going through interest rate increases.

The endogeneity of debt issuance via the budget constraint creates strong destabilising feedback mechanisms in the system which grow stronger as the stock of bonds increases. An increase in the rate of interest, for example, proves stabilising for as long as $c_1BP^{-1}+c_2$ is negative. As debt issuance grows adding to the stock of bonds in the economy, the consumption effect associated with interest income gains momentum, and it may eventually come to dominate the investment effect, which goes in the opposite direction.

When inflation gets under way, it will lead to the same consequences, qualitatively, as under the regime of endogenous monetary growth via the budget constraint. The real values of money and bonds will be eroded leading to interest rate increases if $l_3MP^{-1}+l_4BP^{-1}$ is negative, and subsequently to a reduction in demand pressure if $c_1BP^{-1}+c_2$ is also negative. But it will also lead to upward expectational adjustments with destabilising consequences via the asset substitution effects previously discussed.

However, many similarities can be noted between h_1 and its counterpart under the regime of endogenous monetary growth, k_1. Indeed, on subtracting k_1 from h_1, one obtains:

$$h_1 - k_1 = [(1-c_1-\tau)BP^{-1}-\tau c_2](l_3-l_4) - \bar{r}(1-c_1-\tau) + l_1c_2\bar{r}(1-\tau).$$

The above term is unambiguously negative. Consequently, h_1 is unambiguously smaller than k_1.

In addition, note that the two terms

$$+l_1c_2\bar{r}(1-\tau) \quad \text{and} \quad -\bar{r}(1-c_1-\tau)$$

that appear in h_1 but not in k_1, are both negative and incorporate mechanisms that are generated by the endogeneity of debt issuance. That is, compared to the regime of endogenous monetary growth via the government budget constraint, the endogeneity of debt issuance adds to the instability of the system by reducing the positivity of the coefficient attached to ρ^2 [illegible] corresponding characteristic equation..

The endogeneity of debt issuance via the government budget constraint seems to add momentum to the unstable trends embedded in the system. This is primarily due to the links that exist between debt issuance and further debt issuance, and their feedbacks on real output and the interest rate. This can be more clearly seen in the coefficient h_3.

the positivity of h_3.

With respect to the constant coefficient of the characteristic equation, h_3, and its relationship to the corresponding k_3 under the endogeneity of monetary growth, a very interesting result can be established.

Recall that

$$
\begin{aligned}
k_3 = {} & \alpha a_2(c_1BP^{-1}+c_2)\bar{m}(l_3MP^{-1}+l_4BP^{-1}) + \alpha a_2\bar{m}\bar{r}c_1BP^{-1} \\
& + \alpha a_2(c_1BP^{-1}+c_2)l_3[\bar{r}(1-\tau)BP^{-1}-\bar{b}BP^{-1}-\bar{m}MP^{-1}] \\
& - \alpha a_2(1-\tau)BP^{-1}l_3\bar{r}c_1BP^{-1}
\end{aligned}
$$

Observe that k_3 can be simplified and rewritten in the following way:

$$k_3 = -\alpha a_2(c_1BP^{-1}+c_2)(l_3-l_8)\bar{b}BP^{-1}$$
$$+ \alpha a_2c_2l_3\bar{r}(1-\tau)BP^{-1}$$
$$+ \alpha a_2\bar{m}\bar{r}c_1BP^{-1}.$$

Since $\bar{m}=\bar{b}$, h_3 and k_3 can be related in the following way:

$$h_3[MP^{-1}]^{-1} = -k_3[BP^{-1}]^{-1} = \alpha a_2(c_1BP^{-1}+c_2)(l_3-l_8)\bar{b}$$
$$- \alpha a_2c_2l_3\bar{r}(1-\tau)$$
$$- \alpha a_2\bar{m}\bar{r}c_1.$$

That is,

$$h_3 = -k_3\ (M/B).$$

Both these coefficients involve exactly the same mechanisms (discussed in chapter five) working in opposite directions. Hence, whatever is stabilising with k_3, it is destabilising with h_3. Consequently, if k_3 is positive and the system stable under a regime of endogenous monetary growth, then h_3 must be negative and, hence, under a regime of endogenous debt issuance via the government budget constraint, the system is unstable.

This allows us to state a 'dual instability' theorem in the following way:

dual instability theorem

If the system given by expressions (3.2), (5.2), (6.2), and (8.1) is stable under a policy regime of endogenous monetary growth via the government budget constraint, it will be unstable under a policy regime of endogenous debt issuance via the government budget constraint. Similarly, if the system is stable under a policy regime of endogenous debt issuance via the government budget constraint, it will be unstable under a policy regime of endogenous monetary growth via the government budget constraint. However, endogenous monetary growth is more likely to lead to stability than endogenous debt issuance. Both regimes cannot be stable.

It is important to point out, however, that the above theorem is not specific to situations with endogenous expectations. It will still hold true in a situation of exogenous expectations where a takes on the value of zero as in the Christ (1978) model. In this scenario of static expectations, one finds that,[23]

$$h_2' = (1/a)h_3$$

and $$k_2' = (1/a)k_3.$$

From these relationships it can easily be seen that the 'dual instability theorem' is not altered when expectations are static. Hence, it is not dependent upon the nature of

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[23]Note that h_2' and k_2' are the coefficients of the corresponding characteristic equations under endogenous debt issuance and monetary growth respectively when expectations are exogenous.

expectations formation.

the positivity of h_2

Note that the same elements that appear in h_3 appear in h_2 as well, so that the negativity of h_3 pushes h_2 in the same direction. With h_3 negative, the corresponding characteristic equation would involve at least one sign change and, hence, at least one positive real root. If, in addition, h_0, h_1, and h_2, are positive, the remaining two roots of the system will either be real negative numbers or a conjugate complex pair with negative real parts.[24]

The implications of the result that the system will be unstable with debt issuance endogenous via the budget constraint whenever it is stable with monetary growth endogenous via the same budget constraint, are that the authorities are constrained in setting policy variables. They are constrained not to use debt issuance as the

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[24] This can be shown in the following way:
Let β be the positive real root of the system in question. Factoring this root out of the corresponding characteristic equation one obtains that

$$(\rho-\beta)[h_0\rho^2 + \epsilon\rho + \theta] = 0.$$

Upon expanding the above expression one may obtain the following:

$$h_0\rho^3 + (\epsilon-\beta h_0)\rho^2 + (\theta-\beta\epsilon)\rho - \beta\theta = 0.$$

Hence,

$$h_1 = \epsilon - \beta h_0$$
$$h_2 = \theta - \beta\epsilon$$
$$h_3 = -\beta\theta$$

With h_0, h_1, and h_2 positive while h_3 negative, it can be established that ϵ ande θ must both be positive. Consequently, there are no sign changes in the expression

$$h_0\rho^2 + \epsilon\rho + \theta = 0,$$

and, hence, ρ_1 and ρ_2 must be either real negative numbers or a conjugate complex pair with negative real parts (see also appendix 3).

residual instrument of financing, unless something is done to offset the potential instability embedded therein. The natural question to ask comes to be: what impact does the endogenisation of monetary growth via a policy reaction function have upon this system?

C. The Model with Monetary Growth and Debt Issuance Simultaneously Endogenous

To analyse a policy régime in which monetary growth is endogenous via the policy reaction function, and the rate of debt issuance also endogenous as before via the government budget constraint, set Dg=0 in the general model specified in chapter three in expressions (3.2), (5.2), (6.2), (7.1), and (8.1) to obtain the following system of differential equations:

$$Dp = a_2Dy + \alpha a_2(y-\bar{y}) \tag{3.2}$$

$$(1-c_1)Dy = c_1BP^{-1}r(b-p) + (c_1BP^{-1}+c_2)Dr + \alpha a_2c_3(y-\bar{y}) \tag{5.5}$$

$$Dr = l_1Dy + \alpha a_2l_2(y-\bar{y}) + l_3MP^{-1}(m-p) + l_4BP^{-1}(b-p) \tag{6.5}$$

$$Dm = z_1Dy + z_2Dp \tag{7.1}$$

$$BP^{-1}Db + BP^{-1}b(b-p) = -\tau Dy + (1-\tau)BP^{-1}r(b-p) + (1-\tau)BP^{-1}Dr - MP^{-1}Dm - MP^{-1}m(m-p) \tag{8.4}$$

This is again a third order system, but in five endogenous variables: price inflation, real output, the nominal interest rate, and the rates of monetary growth and debt issuance.

Linearising the above five equations around the corresponding steady-inflation equilibrium path following the same procedures as before, one obtains the following matrix expression:

$$\begin{bmatrix} \rho a_2+\alpha a_2 & 0 & -\rho & 0 & 0 \\ \rho(1-c_1)-\alpha a_2 c_3 & -\rho(c_1BP^{-1}+c_2) & \bar{r}c_1BP^{-1} & 0 & -\bar{r}c_1BP^{-1} \\ \rho l_1+\alpha a_2 l_2 & -\rho & \xi_3 & l_3MP^{-1} & l_4BP^{-1} \\ z_1 & 0 & z_2 & -1 & 0 \\ \rho\tau & -\rho(1-\tau)BP^{-1} & \xi_1 & (\rho+\bar{m})MP^{-1} & \rho BP^{-1}+\xi_2 \end{bmatrix} \begin{bmatrix} y(0)-\bar{y} \\ r(0)-\bar{r} \\ p(0)-\bar{p} \\ b(0)-\bar{b} \\ m(0)-\bar{m} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

where, $\xi_1 = \bar{r}(1-\tau)BP^{-1}-\bar{b}BP^{-1}-\bar{m}MP^{-1}$

$\xi_2 = \bar{b}BP^{-1}-\bar{r}(1-\tau)BP^{-1}$

and, $\xi_3 = -(l_3MP^{-1}+l_4BP^{-1})$

The determinant of the characteristic matrix of the system yields the following characteristic equation:

$$d_0\rho^3 + d_1\rho^2 + d_2\rho + d_3 = 0,$$

where

$$d_0 = (1-c_1) - l_1(c_1BP^{-1}+c_2)$$

$$\begin{aligned}
d_1 = {} & [1-c_1-l_1(c_1BP^{-1}+c_2)][\bar{b}-\bar{r}(1-\tau)] \\
& - (1-c_1)(1-\tau)l_4BP^{-1} - l_1(1-\tau)\bar{r}c_1BP^{-1} \\
& + \tau(c_1BP^{-1}+c_2)l_4 + \tau\bar{r}c_1 \\
& + a_2(c_1BP^{-1}+c_2)(l_3MP^{-1}+l_4BP^{-1}) + a_2\bar{r}c_1BP^{-1} \\
& - \alpha a_2[c_3+l_2(c_1BP^{-1}+c_2)] \\
& - a_2(c_1BP^{-1}+c_2)MP^{-1}(l_3-l_4)z_2 \\
& - (c_1BP^{-1}+c_2)MP^{-1}(l_3-l_4)z_1 \\
& + a_2\bar{r}c_1z_2MP^{-1} + \bar{r}c_1z_1MP^{-1}
\end{aligned}$$

$$\begin{aligned}
d_2 = {} & a_2(c_1BP^{-1}+c_2)[l_3-l_4]\bar{m}MP^{-1} \\
& - a_2c_2\bar{r}(1-\tau)l_3MP^{-1} - a_2\bar{r}c_1\bar{m}MP^{-1} \\
& + \alpha a_2(c_1BP^{-1}+c_2)(l_3MP^{-1}+l_4BP^{-1}) + \alpha a_2\bar{r}c_1BP^{-1} \\
& - \alpha a_2\bar{b}[c_3+l_2(c_1BP^{-1}+c_2)] \\
& + \alpha a_2c_3(1-\tau)(\bar{r}-l_4BP^{-1}) - \alpha a_2l_2c_2\bar{r}(1-\tau) \\
& - a_2(c_1BP^{-1}+c_2)\bar{b}MP^{-1}(l_3-l_4)z_2 \\
& - (c_1BP^{-1}+c_2)\bar{b}MP^{-1}(l_3-l_4)z_1 \\
& - \alpha a_2(c_1BP^{-1}+c_2)MP^{-1}(l_3-l_4)z_2 \\
& + a_2c_2\bar{r}(1-\tau)l_3MP^{-1}z_2 + a_2\bar{r}c_1\bar{m}MP^{-1}z_2 \\
& + \alpha a_2\bar{r}c_1z_2MP^{-1} + c_2\bar{r}(1-\tau)l_3MP^{-1}z_1 + \bar{m}MP^{-1}\bar{r}c_1z_1
\end{aligned}$$

$$\begin{aligned}
d_3 = {} & \alpha a_2(c_1BP^{-1}+c_2)[l_3-l_4]\bar{m}MP^{-1} \\
& - \alpha a_2c_2\bar{r}(1-\tau)l_3MP^{-1} - \alpha a_2\bar{r}c_1\bar{m}MP^{-1} \\
& - \alpha a_2(c_1BP^{-1}+c_2)\bar{b}MP^{-1}(l_3-l_4)z_2 \\
& + \alpha a_2c_2\bar{r}(1-\tau)l_3MP^{-1}z_2 + \alpha a_2\bar{r}c_1\bar{m}MP^{-1}z_2
\end{aligned}$$

The above coefficients can be readily simplified and written in the following way:

$$d_0 = h_0$$

$$d_1 = h_1 + MP^{-1}[\bar{r}c_1-(l_3-l_4)(c_1BP^{-1}+c_2)](a_2z_2+z_1)$$

$$d_2 = h_2 - (1/\alpha a_2)h_3(a_2z_2+z_1)$$

$$+ \alpha a_2MP^{-1}[\bar{r}c_1-(l_3-l_4)(c_1BP^{-1}+c_2)]z_2$$

$$d_3 = (1-z_2)h_3$$

where, ofcourse, h_0, h_1, h_2, and h_3 are the coefficients of the characteristic equation associated with the endogeneity of debt issuance alone, via the government budget constraint.

D. Stability Conditions

By setting z_1 and z_2 both equal to zero, one ought to obtain the same characteristic equation as the one associated with the policy regime of endogenous debt issuance via the government budget constraint. It is not surprising, therefore, that the coefficients d_0, d_1, d_2, and d_3 completely incorporate their counterparts h_0, h_1, h_2, and h_3 respectively. In fact, d_0 and h_0 are identical. Given this, the question to be asked is, in what ways does the monetary policy reaction function augment the positivity or negativity of each of h_1, h_2, and h_3?

the positivity of d_1

With respect to this coefficient of the characteristic equation, d_1, the way in which the monetary policy reaction function will affect the stability of the system is embodied in the following two terms:

$$a_2z_2 + z_1$$

and,

$$\bar{r}c_1 - (l_3-l_4)(c_1BP^{-1}+c_2)$$

However, the signs of these two terms cannot be established unconditionally. Assuming that $(c_1BP^{-1}+c_2)$ is negative, then one can say that, if $a_2z_2+z_1$ is negative, the policy reaction function will be adding to the stability of the system by increasing the positivity of h_1 to the extent that $\bar{r}c_1-(l_3-l_4)(c_1BP^{-1}+c_2)$ is also negative.

For a strategy of gradualism, z_2 is positive but less than unity, and z_1 is strictly negative. That is to say, the policy reaction function is, in this case, partially accommodative and partially countercyclical in nature. Consequently, an increase in demand pressure will lead to a monetary contraction. This monetary contraction, in turn, will feed into the budget constraint causing increased debt issuance since real money balances serve as an instrument of financing. Increased debt issuance and monetary contraction will subsequently feed into the money market pushing the nominal interest rate up. With $(c_1BP^{-1}+c_2)$ being negative, this development will be stabilising in that it will lead to output being dampened.

The demand induced inflation, on the other hand, will work in much the same way, but in the opposite direction. It will feed into the reaction function causing monetary expansion which in turn will cause an endogenous reduction in the rate of debt issuance. These developments will be destabilising since they push the interest rate down.

These destabilising trends can be neutralised by choosing the weights z_1 and z_2 in such a way so as to make the term $a_2z_2+z_1$ negative. However, changes in the rate of debt issuance will feed into the goods market directly via fluctuations in interest income. In this respect, the accommodative nature of the policy reaction function becomes stabilising as it results in a reduction in the rate of debt issuance, hence dampening output via reduced interest income.

To the extent that such interest income effects generated by debt issuance are dominated by the previously mentioned interest rate effects, then the monetary authorities can increase the positivity of d_1 by choosing the weights z_1 and z_2 in such a way as to make the term $a_2z_2+z_1$ negative. In the extreme, they can follow a strictly countercyclical monetary policy by assigning to z_2 the value of zero.

the coefficients d_2 and d_3

However, the most critical implications for the stability of the system are embedded in the constant coefficient of the characteristic equation, d_3. This coefficient is related to h_3, its counterpart under the regime of endogenous debt issuance alone, via the accommodative nature of the monetary policy reaction function. If h_3 is indeed negative, which would be the case if k_3 is positive, and z_2, while positive, is less than unity, then d_3 will be negative and, hence, the system unstable.

However, a completely accommodative monetary policy, whereby $z_2=1$, will cause d_3 to vanish, while an overaccommodative monetary policy, whereby $z_2>1$, will cause it to be positive. Why is this the case?

The destabilising trends in h_3 are rooted in the endogenous response of debt issuance to demand induced inflation. That is to say, the rate of debt issuance will increase as the real value of money and bonds depreciates because of inflation, leading to an interest rate increase which, to the extent that $(c_1BP^{-1}+c_2)$ is negative, will have a stabilising impact. At the same time, the increase in the rate of debt issuance becomes destabilising in that it adds to demand pressure via increased interest income, and also in that it leads to further debt issuance via increased interest payments. As noted earlier, as the rate of debt issuance increases, the interest income based consumption effect induced by a change in the nominal rate of interest gains momentum, and it may well dominate the similarly induced investment effect. When this happens, the term $(c_1BP^{-1}+c_2)$ will become positive, hence rendering the interest rate mechanism destabilising.

With the monetary policy reaction function in the system, the above destabilising trends can be offset by monetary expansion, which would, in effect, reduce the pressures for continuing debt issuance. In this way, monetary expansion works to counteract the destabilising trends that debt issuance tends to generate. A completely accommodative monetary policy would have the effect of totally offsetting the above destabilising trends.

Such a condition, however, that monetary policy be completely accommodative, does not actually conflict with a desired negative $(a_2z_2+z_1)$, which would increase the positivity of d_1. With z_2 equal to unity, a z_1 that is bigger than a_2 in absolute value would still be working in the direction of the positivity of d_1.

But while the negativity of the term $a_2z_2+z_1$ conditionally improves the positivity of d_1, it can be found to work in the opposite direction with respect to the coefficient d_2.

Given that z_2 being at least unitary is a minimal requirement for the stability of the system, then, to the extent that,

$$\bar{r}c_1 - (l_3-l_4)(c_1BP^{-1}+c_2)$$

is negative, the same condition that z_2 is unitary will be destabilising by adding to the negativity of d_2. In addition, with h_3 negative, the condition that $a_2z_2+z_1$ be negative, while stabilising with respect to d_1, will be destabilising in the context of d_2.

However, all the above were based on the assumption that,

$$\bar{r}c_1 - (l_3-l_4)(c_1BP^{-1}+c_2)$$

is negative. But there is no reason to argue that such an assumption is a realistic one. With debt issuance on the rise, c_1BP^{-1} also increases so that, starting from a hypothetical situation in which the above term is indeed negative, it will eventually become positive if increasing debt issuance is allowed to continue long enough.

Such a positivity would have important implications with respect to the role of monetary policy in the system. In such a scenario, the above story is reversed. That is, d_1 would increase in positivity if $a_2z_2+z_1$ is positive while the coefficient d_2 would lose in positivity.

In summary, one may say that when the rate of debt issuance is endogenous via the government budget constraint, the system will be unstable to the extent that it is stable under a policy regime in which monetary growth is endogenous via the same government budget constraint. If such is the case, then, if there is to be any hope for stability at all when a monetary policy reaction function is integrated into the system, monetary policy must at least be completely accommodative.

VI. DYNAMICS WITH GOVERNMENT SPENDING AND THE RATE OF MONETARY GROWTH ENDOGENOUS

All the preceeding analysis was based on an assumption that the level of government spending is constant. In the present chapter, the concern will be with what happens with respect to the stability of the system, in particular, when government expenditures are allowed to vary in response to budget imbalances. Two policy regimes will here be examined. The first deals with a situation in which government spending is endogenous via the budget constraint. The second deals with a situation in which government spending and the rate of monetary growth are both endogenous simultaneously, the former via the government budget constraint, and the latter via the monetary policy reaction function.

A. The Model with Government Spending Endogenous

When the level of government expenditures is the instrument endogenised via the budget constraint while monetary growth and debt issuance grow at the constant rates $\bar{m}$ and $\bar{b}$ respectively, the general model takes the following form:

$$Dp = a_2 Dy + \alpha a_2 (y-\bar{y}) \tag{3.2}$$

$$(1-c_1)Dy = c_1 BP^{-1} r(\bar{b}-p) + (c_1 BP^{-1}+c_2)Dr + \alpha a_2 c_3 (y-\bar{y}) + Dg \tag{5.6}$$

$$Dr = l_1 Dy + \alpha a_2 l_2 (y-\bar{y}) + l_3 MP^{-1}(\bar{m}-p) + l_4 BP^{-1}(\bar{b}-p) \qquad (6.6)$$

$$Dg = \tau Dy - (1-\tau)BP^{-1}Dr - (1-\tau)BP^{-1}r(\bar{b}-p) + \bar{b}BP^{-1}(\bar{b}-p) + \bar{m}MP^{-1}(\bar{m}-p) \qquad (8.5)$$

Linearising the above four equation system around the corresponding steady inflation equilibrium path, one obtains the following matrix expression:

$$\begin{bmatrix} \rho a_2+\alpha a_2 & 0 & -\rho & 0 \\ \rho(1-c_1)-\alpha a_2 c_3 & -\rho(c_1 BP^{-1}+c_2) & \bar{r}c_1 BP^{-1} & -\rho \\ \rho l_1+\alpha a_2 l_2 & -\rho & -(l_3 MP^{-1}+l_4 BP^{-1}) & 0 \\ -\rho\tau & -\rho(1-\tau)BP^{-1} & -\bar{r}(1-\tau)BP^{-1}+\bar{b}BP^{-1}+\bar{m}MP^{-1} & \rho \end{bmatrix}$$

$$\begin{bmatrix} y(0)-\bar{y} \\ r(0)-\bar{r} \\ p(0)-\bar{p} \\ g(0)-\bar{g} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Expanding the determinant of the characteristic matrix of the system above and setting it equal to zero, one obtains the following second degree characteristic equation:

$$s_0\rho^2 + s_1\rho + s_2 = 0$$

where,

$$s_0 = (1-c_1-\tau) + l_1[(1-c_1-\tau)BP^{-1}-c_2]$$

$$s_1 = -a_2[(1-c_1-\tau)BP^{-1}-c_2](l_3 MP^{-1}+l_4 BP^{-1}) - a_2\bar{r}BP^{-1}(1-c_1-\tau) + a_2(\bar{b}BP^{-1}+\bar{m}MP^{-1})$$

$$- \alpha a_2 c_3 + \alpha a_2 l_2[(1-c_1-\tau)BP^{-1}-c_2]\}$$

and

$$s_2 = - \alpha a_2[(1-c_1-\tau)BP^{-1}-c_2](l_3 MP^{-1}+l_4 BP^{-1})$$
$$- \alpha a_2 \bar{r} BP^{-1}(1-c_1-\tau) + \alpha a_2(\bar{b}BP^{-1}+\bar{m}MP^{-1})$$

The syste[illegible] second order. This is the case because the budget constraint, when both monetary growth and debt issuance are exogenous, does not add to the order of the system. That is to say, what makes the expression for the government budget constraint a first order differential equation is the endogeneity of either monetary growth or debt issuance. When, for example, monetary growth is endogenous, the budget constraint entails a relationship between the rate of change and the level of monetary growth. This is precisely what makes it a first order differential equation that increases the order of the system by one. No such relationship exists in the budget constraint equation with respect to government spending.

With the system being second order, one can examine the corresponding stability conditions on the basis of Descartes rule of signs. For stability, there must be no sign changes in the characteristic equation, which would be the case if all the coefficients, s_0, s_1, and s_2, have the same sign. We choose to proceed with the stability analysis of this system normalising on the positivity of s_0.

B. Stability Conditions.

Considering first the coefficient s_0, one can readily establish that it is unambiguously positive. There is nothing here which, while destabilising, can potentially dominate. An increase in demand pressure would cause an endogenous expansion in government spending via increased tax revenue. This development, however, will be dominated by the stabilising consequences of the savings leakage effect. This is manifested in the positivity of the term $(1-c_1-\tau)$.[28]

In addition, the interest rate will here increase in response to increased demand pressure via the transactions demand for money. This increase in the nominal rate of interest will be destabilising by way of increased interest income. At the same time, however, it will be adding to stability by way of crowding out private investment expenditures and endogenously causing government spending to contract via increased interest payments. Precisely because $(1-c_1-\tau)$ is positive, these latter two effects would dominate.

the coefficient s_2

With respect to the coefficient s_2 of the characteristic equation, that part of the behaviour of the

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[28] Note that c_1 is the net marginal propensity to consume which is here defined to be equal to $c'(1-\tau)$ where c' is the gross marginal propensity to consume. Hence,

$$(1-c_1-\tau) = [1-c'(1-\tau)-\tau] = (1-\tau)(1-c') > 0$$

since $0 < c' < 1$ and $0 < \tau < 1$.

model that it embodies is associated with inflation induced by upward expectational adjustments operating through the wage-price nexus. Furthermore, the basis of the effects generated by this induced inflation, is the reduction in the real stocks of money balances and bonds resulting from the exogeneity of both debt issuance and monetary growth in an inflationary scenario.

The erosion in the real value of bonds outstanding reduces interest payments, and consequently, interest income at the same time. While this latter development is stabilising in that it generates a negative consumption effect, the former is destabilising because, via reduced interest payments, it leads to an endogenous increase in government expenditures. But, with $(1-c_1-\tau)$ being positive, the dampening negative consumption effect will be dominated by the expansionary effects of government expenditures and, hence, the net impact of these two effects upon the economy will be destabilising.

A reduced real debt, however, coupled with reduced real money balances, in effect limits the government's capacity to spend. Both money and bonds serve as instruments of financing and, being here exogenous, inflation feeds into the budget constraint causing a reduction in the level of government spending.

Such a development will, in and of itself, be stabilising in the face of demand pressure, but whether or not it is powerful enough to dominate the previously mentioned destabilising effects of reduced interest payments remains ambiguous. This is manifested in the non-signability of the following term:

$$\bar{b}BP^{-1} + \bar{m}MP^{-1} - \bar{r}(1-c_1-\tau)BP^{-1}.$$

That is to say, induced inflation has a stabilising effect in that it leads to a fiscal contraction via reduced real debt and real money balances. In addition, by reducing interest payments, and therefore interest income, induced inflation leads to both fiscal expansion and a negative consumption effect. The above term, therefore, is not signable because of the influence of fiscal expansion that results from reduced interest payments.

It is worth noting, however, that, in the context of the above term, whatever is expansionary when government spending is endogenous via the budget constraint, will be contractionary under the policy regimes of endogenous monetary growth or debt issuance via the budget constraint. A reduction in the real stocks of money and bonds, for example, results in monetary expansion when the rate of monetary growth is endogenous, or in an increase in the rate of debt issuance when debt issuance is the residual instrument of financing. However, when government spending is the endogenous policy instrument a reduction in the real stocks of money and bonds deprives the government of spending capacity and leads to a fiscal contraction.

Nevertheless, we cannot here argue that this term must necessarily be positive for stability, as it being negative cannot in and of itself violate the positivity of s_2.

Reduced real money balances and real debt, however, will feed into the money market with counteracting repercussions for the interest rate. How the interest rate will react, however, is important for the stability of the system. In the context of the coefficient, s_2 of the characteristic equation, an interest rate increase will unambiguously have stabilising repercussions manifested in the term,

$$[(1-c_1-\tau)BP^{-1} - c_2].$$

That is, an increase in the rate of interest will generate a negative investment effect, fiscal contraction via increased interest payments, and a positive consumption effect via increased interest income. This last effect will be adding to the negativity of s_2, but it will be dominated by the effects of fiscal expansion as the positivity of the term $(1-c_1-\tau)$ indicates.

How the interest rate will move would depend on its relative responsiveness with respect to variations in real money balances and the real stock of bonds, as already discussed in the preceeding chapters. What will be working in the direction of stability is a relatively sensitive interest rate to variations in the real stock of money, such that the term $l_3MP^{-1}+l_4BP^{-1}$ is negative. Implicit in this term are not only relative elasticities, but also relative stocks. That is to say, a huge public debt can be as damaging to the stability of the system as a very elastic interest rate to real debt changes.

the coefficient s_1

The coefficient s_1, on the other hand, can be rewritten as the following expression:

$$s_1 = (1/a)s_2 + aa_2l_2[(1-c_1-\tau)BP^{-1}-c_2] - aa_2c_3.$$

As with all previous cases, before inflationary expectations get under way, the inflationary consequences of increased demand pressure working through the wage-price nexus will generate the same effects discussed in conjunction with the constant coefficient of the characteristic equation, s_2.

Once inflationary expectations get under way, however, they will feed into the money market generating asset substitution effects out of money, thus pushing the interest rate down. For reasons previously discussed, a reduction in the rate of interest is, in a context of demand pressure, a destabilising consequence. In addition, the same asset substitution effects will also feed into the goods market, ceteris paribus, generating a positive investment effect by reducing the real interest rate.

As the case has been with previous policy regimes, it is desirable for stability that these effects associated with inflationary expectations, and particularly the asset substitution effects, be relatively weak.

C. Expectational Adjustments and Oscillatory Motions

It can easily be seen, in this policy regime of endogenous government expenditures via the budget constraint, that the main sources of destabilising trends in the system are rooted in the implications of expectational adjustments.

With static expectations, that is, with a being zero, the constant coefficient of the characteristic equation, s_2, would vanish. The implication of this would be that the system would be reduced to one of first order. It will, then have one single associated characteristic root whose value would be given by the following relationship:

$$\rho = - s_1'/s_0'$$

where s_1' and s_0' refer to the coefficients of the characteristic equation of the system with static expectations, and which can be obtained by setting $a=0$ in s_1 and s_0, respectively.

Examining the elements of the coefficients s_1' and s_0', it can easily be seen that there exists only one destabilising factor embodied therein. This factor is given by the term,

$$- a_2(1-c_1-\tau)\bar{r}BP^{-1}.$$

Demand induced inflation would lead to an endogenous expansion in the level of government expenditures via reduced interest payments, whose destabilising consequences would dominate the corresponding negative consumption effect induced by reduced interest income. This is the only element that can potentially make the single characteristic root of the system positive.

However, since there is only one characteristic root in such a system, its corresponding time path will be non-cyclical. The argument can, thus, be made that whatever can cause an oscillatory time path must be rooted in the implications expectational adjustments can have in the system. This is something that matches with a result reached in chapter three under the policy regime of endogenous monetary growth via the government budget constraint, namely that what could lead to an explosive oscillatory time path was the destabilising trends generated by expectational adjustments, and particularly the asset substitution effects. This is precisely the case here.

Furthermore, the expectations mechanism employed herein increases the order of the system by one, precisely because it introduces a lag into the system. This lag must ultimately be responsible for lead-lag relationships in the system which can potentially lead to oscillatory time paths.

In any case, the question that will now be raised is again one that involves the monetary policy reaction function. Given that expectational adjustments can potentially lead to oscillatory behaviour, and furthermore to instability, one would want to examine whether or not monetary policy can be used in ways that can potentially enhance the stability of the system, and more to the point, in ways that can impinge upon the type of associated time paths.

D. The Model with Government Spending and Monetary Growth Simultaneously Endogenous

With a monetary policy reaction function in place, the rate of debt issuance held costant at $\bar{b}$, and governemnt spending the instrument which will adjust so as to maintain the equality between the budget deficit and its sources of financing, the general model specified in chapter three will become:

$$Dp = a_2Dy + \alpha a_2(y-\bar{y}) \tag{3.2}$$

$$(1-c_1)Dy = c_1BP^{-1}r(\bar{b}-p) + (c_1BP^{-1}+c_2)Dr + \alpha a_2c_3(y-\bar{y}) + Dg \tag{5.7}$$

$$Dr = l_1Dy + \alpha a_2l_2(y-\bar{y}) + l_3MP^{-1}(m-p) + l_4BP^{-1}(\bar{b}-p) \tag{6.7}$$

$$Dm = z_1Dy + z_2Dp \tag{7.1}$$

$$Dg = \tau Dy - (1-\tau)BP^{-1}Dr - (1-\tau)BP^{-1}r(\bar{b}-p) + \bar{b}BP^{-1}(\bar{b}-p) + MP^{-1}m(m-p) + MP^{-1}Dm \tag{8.6}$$

Linearisation of this system around its steady state equilibrium path applying the usual methods would yield the following matrix expression:

$$\begin{bmatrix} \rho a_2+\alpha a_2 & 0 & -\rho & 0 & 0 \\ \rho(1-c_1)-\alpha a_2 c_3 & -\rho(c_1BP^{-1}+c_2) & \bar{r}c_1BP^{-1} & 0 & -\rho \\ \rho l_1+\alpha a_2 l_2 & -\rho & -(l_3MP^{-1}+l_4BP^{-1}) & l_3MP^{-1} & 0 \\ \rho\tau & -\rho(1-\tau)BP^{-1} & \xi_1 & (\rho+\bar{m})MP^{-1} & -\rho \\ -z_1 & 0 & -z_2 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} y(0)-\bar{y} \\ r(0)-\bar{r} \\ p(0)-\bar{p} \\ m(0)-\bar{m} \\ g(0)-\bar{g} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

where, $\xi_1 = \bar{r}(1-\tau)BP^{-1}-\bar{b}BP^{-1}-\bar{m}MP^{-1}$

The corresponding characteristic equation of this system obtained by expanding the determinant of the characteristic matrix above and setting it equal to zero is given by:

$$u_0\rho^2 + u_1\rho + u_2 = 0$$

where,

$$\begin{aligned} u_Q = {} & (1-c_1-\tau) + l_1[(1-c_1-\tau)BP^{-1}-c_2] \\ & - MP^{-1}(a_2z_2+z_1) \\ u_1 = {} & - a_2[(1-c_1-\tau)BP^{-1}-c_2](l_3MP^{-1}+l_4BP^{-1}) \\ & - a_2\bar{r}BP^{-1}(1-c_1-\tau) + a_2(\bar{b}BP^{-1}+\bar{m}MP^{-1}) \\ & - \alpha a_2c_3 + \alpha a_2l_2[(1-c_1-\tau)BP^{-1}-c_2] \\ & + [(1-c_1-\tau)BP^{-1}-c_2]l_3MP^{-1}(a_2z_2+z_1) \\ & - \bar{m}MP^{-1}(a_2z_2+z_1) - \alpha a_2z_2MP^{-1} \end{aligned}$$

and

$$u_2 = -\alpha a_2[(1-c_1-\tau)BP^{-1}-c_2](l_3MP^{-1}+l_4BP^{-1})$$
$$-\alpha a_2\bar{r}BP^{-1}(1-c_1-\tau)+\alpha a_2(\bar{b}BP^{-1}+\bar{m}MP^{-1})$$
$$+\alpha a_2z_2[(1-c_1-\tau)BP^{-1}-c_2]l_3MP^{-1} - \alpha a_2z_2MP^{-1}$$

The system is again second order. As noted before, when the rates of monetary growth and debt issuance are both exogenous, the budget constraint does not add to the order of the system, and neither does the monetary policy reaction function. The order of the system increases only when monetary growth or debt issuance are endogenised via the government budget constraint. The endogeneity of monetary growth via the reaction function neutralises the impact on the order of the system that its endogeneity would have had should it have been the consequence of the budget constraint.

With the system being second order, the positivity of each and every coefficient of its characteristic equation is a necessary and sufficient condition for stability.

E. Stability Conditions

The coefficients of the characteristic equation in question, u_0, u_1, and u_2, can be rewritten in the following simplified form:

$$u_0 = s_0 - MP^{-1}(a_2z_2+z_1)$$
$$u_1 = s_1 + [(1-c_1-\tau)BP^{-1}-c_2]l_3MP^{-1}(a_2z_2+z_1)$$

$$- \bar{m}MP^{-1}(a_2z_2+z_1) - \alpha a_2z_2MP^{-1}$$

and

$$u_2 = s_2 + \alpha a_2z_2[(1-c_1-\tau)BP^{-1}-c_2]l_3MP^{-1} - \alpha a_2z_2MP^{-1}.$$

Note that s_0, s_1, and s_2 are the coefficients of the characteristic equation corresponding to the policy regime under which government spending is endogenous via the budget constraint. The issue here involves the question, in which direction can a monetary policy reaction function take that model economy?

the coefficient u_0

In the context of the coefficient attached to ρ^2 in the corresponding characteristic equation, u_0, the monetary policy reaction function enters the system in a counteracting way. Its partly accommodative nature is, here, destabilising as the term $-MP^{-1}a_2z_2$ is negative. On the other hand, its partly countercyclical nature is stabilising and is manifested in the positivity of the term $-MP^{-1}z_1$.

In an inflationary scenario, monetary policy would be destabilising in that it leads to an endogenous expansion of government expenditures, thus adding to demand pressure and fueling inflation. By contrast, the monetary contraction that results from the countercyclical nature of the reaction function has the opposite effect as it causes government expenditures to contract endogenously.

The monetary authorities, by choosing the weights z_1 and z_2 in such a way that the term $a_2z_2+z_1$ is negative, would be adding to the positivity of s_0 and, hence, to the stability of the system.

the coefficient u_2

In the context of the constant coefficient of the characteristic equation, u_2, the monetary policy reaction function comes in with destabilising repercussions throughout. What is relevant here is monetary accommodation, which has the effect of driving u_2 in the direction of negativity.

Demand induced inflation has the consequence of reducing the real values of money balances and bonds outstanding. These consequences exert a dampening impact upon the economy in that they lead to an endogenous reduction in government expenditures. Both the stock of money and the stock of bonds serve as instruments of deficit financing and, hence, a reduction in their real lowers the government's capacity to spend.

These pressures to reduce spending will be eased to a certain extent because of inflation induced monetary expansion. In this sense, the monetary policy reaction function becomes destabilising, manifested in the negativity of the term, $-\alpha a_2 z_2 \bar{m} MP^{-1}$. But to the extent that z_2, although positive, is less than unity, this destabilising consequence will be absorbed by the effects of reduced real money balances mentioned above. This, however, does not negate the fact that monetary accommodation reduces the positivity of z_2, thus adding to instability.

In addition, monetary accommodation would also be reducing the positivity of u_2 via the interest rate mechanism. Reduced real money balances feed into the money market raising the interest rate, which in turn, will have a stabilising effect on the economy. By increasing interest payments, this interest rate increase forces an endogenous reduction of government spending whose contractionary impact upon the economy outweighs the expansionary effect of increased consumption via increased interest income. Monetary accommodation generates just the opposite effects, since it enters the money market by reducing the interest rate.

However, as long as z_2 is less than unity, demand induced inflation will net out to be contractionary with respect to the real money supply. Consequently, via the reduction in the real money supply, inflation will be causing the interest rate to increase even with monetary accommodation. However, the same inflation that can lead to this increase in the rate of interest will work via a reduction in the real value of bonds to produce the opposite effect. That is to say, less real debt allows the interest rate to come down. Hence, although monetary accommodation does not, in and of itself, render the interest rate mechanism destabilising, it nevertheless adds to instability by reducing the positivity of u_2.

the coefficient u_1

In the context of the coefficient u_1, as with u_0, the monetary policy reaction function has a counteracting influence. Its accommodative nature is destabilising in that it allows for an endogenous increase in the level of government expenditures, and also in that it exerts downward pressure on the interest rate. Its countercyclical nature, on the other hand, is stabilising for just the opposite reasons. What will here be working in the direction of stability, in the sense of increasing the positivity of u_1, is again the negativity of the term, $a_2z_2+z_1$.

F. The influence of Monetary Policy

Under this policy regime with a monetary policy that follows a strategy of gradualism and endogenised government expenditures via the budget constraint, the destabilising trends in the system have their sources in three factors: the inflation induced erosion in the real value of the stock of bonds outstanding; expectational adjustments; and the partly accommodative nature of the policy reaction function.

In particular, with respect to the policy reaction function, its accommodative nature is everywhere destabilising while its countercyclical nature is everywhere stabilising. Thus, if the monetary authorities follow a purely countercyclical monetary policy, setting $z_2=0$, then the stability of the system will be strengthened vis-a-vis both a policy regime in which government spending alone is endogenous via the budget constraint, and a policy regime which follows a strategy of gradualism with government spending again endogenous as above.

This will precisely be the case as the u_is and the s_is will be related by the following expressions:

$$u_0 > s_0$$

$$u_1 > s_1$$

$$u_2 = s_2$$

That is, a purely countercyclical monetary policy will be increasing the positivity of the coefficients s_0 and s_1 while not affecting s_2.

Since the roots of the characteristic equation,

$$s_0\rho^2 + s_1\rho + s_2 = 0,$$

are given by the expression,

$$[-s_1 +/- (s_1^2 - 4s_0s_2)^{1/2}](1/2s_0),$$

it can be said that the increase in s_1 reduces the possibility of an oscillatory time path, while the increase in s_0 has the opposite effect of increasing the possibility of oscillatory time paths. Consequently, the monetary authorities can potentially influence not only the stability of the system, but also the trajectory it follows over time.

VII. DYNAMICS WITH THE LEVEL OF GOVERNMENT SPENDING AND THE RATES OF MONETARY GROWTH AND DEBT ISSUANCE SIMULTANEOUSLY ENDOGENOUS

A system in which debt issuance constitutes the residual instrument of financing embodies strong destabilising mechanisms that run from debt issuance to further debt issuance via the government budget constraint. These mechanisms can potentially lead to instability. It becomes only natural then to explore qualitatively a situation in which the rate of growth of debt issuance is policy constrained. This is precisely the purpose of the present chapter, namely to explore the dynamic properties of a system which has all three policy instruments, government spending, debt issuance, and monetary growth, endogenously determined.

The rate of monetary growth will be endogenised by means of the monetary policy reaction function. The rate of debt issuance, on the other hand, will here be endogenised by way of constraining it to follow the time path of monetary growth. That is, b=m. The level of government spending, hence, would be the instrument to adjust endogenously in response to the budget constraint.

A. The Model

With the above assumptions, our general model in differential form will now be composed of the following expressions:

$$Dp = a_2Dy + \alpha a_2(y-\bar{y}) \tag{3.2}$$

$$(1-c_1)Dy = c_1BP^{-1}r(b-p) + (c_1BP^{-1}+c_2)Dr + \alpha a_2c_3(y-\bar{y}) + Dg \tag{5.8}$$

$$Dr = l_1Dy + \alpha a_2l_2(y-\bar{y}) + l_3MP^{-1}(m-p) + l_4BP^{-1}(b-p) \tag{6.8}$$

$$Dm = z_1Dy + z_2Dp \tag{7.1}$$

$$Dg = \tau Dy - (1-\tau)BP^{-1}Dr - (1-\tau)BP^{-1}r(b-p) + BP^{-1}Db + BP^{-1}b(b-p) + MP^{-1}Dm + MP^{-1}m(m-p) \tag{8.7}$$

$$Db = Dm \tag{16}$$

Eliminating b and Db across the system and linearising around the steady inflation equilibrium time path employing the usual methods, one obtains the following matrix expression:

$$\begin{bmatrix} -(\rho a_2+\alpha a_2) & 0 & \rho & 0 & 0 \\ \rho(1-c_1)-\alpha a_2 c_3 & -\rho(c_1 BP^{-1}+c_2) & \bar{r}c_1 BP^{-1} & -\bar{r}c_1 BP^{-1} & -\rho \\ -(\rho l_1+\alpha a_2 l_2) & \rho & \xi_6 & \xi_7 & 0 \\ \rho\tau & -\rho(1-\tau)BP^{-1} & \xi_4 & \xi_5 & \rho \\ -z_1 & 0 & -z_2 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} y(0)-\bar{y} \\ r(0)-\bar{r} \\ p(0)-\bar{p} \\ m(0)-\bar{m} \\ g(0)-\bar{g} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

where, $\xi_4 = \bar{r}(1-\tau)BP^{-1}-\bar{m}(MP^{-1}+BP^{-1})$

$\xi_5 = \rho(MP^{-1}+BP^{-1})-\bar{r}(1-\tau)BP^{-1}+\bar{m}(MP^{-1}+BP^{-1})$

$\xi_6 = l_3 MP^{-1}+l_4 BP^{-1}$

and $\xi_7 = -(l_3 MP^{-1}+l_4 BP^{-1})$

Expanding the determinant of the characteristic matrix of the systen above and setting it equal to zero, one obtains the following second degree characteristic equation:

$$v_0\rho^2 + v_1\rho + v_2 = 0,$$

where

$$v_0 = (1-c_1-\tau) + l_1[(1-c_1-\tau)BP^{-1}-c_2] - (BP^{-1}+MP^{-1})(a_2 z_2+z_1)$$

$$v_1 = - a_2[(1-c_1-\tau)BP^{-1}-c_2](l_3 MP^{-1}+l_4 BP^{-1}) - a_2\bar{r}BP^{-1}(1-c_1-\tau) + a_2(\bar{b}BP^{-1}+\bar{m}MP^{-1})$$

$$+ a_2z_2[(1-c_1-\tau)BP^{-1}-c_2](l_3MP^{-1}+l_4BP^{-1})$$
$$+ a_2z_2\bar{r}BP^{-1}(1-c_1-\tau) - a_2z_2(\bar{b}BP^{-1}+\bar{m}MP^{-1})$$
$$+ z_1[(1-c_1-\tau)BP^{-1}-c_2](l_3MP^{-1}+l_4BP^{-1})$$
$$+ z_1\bar{r}BP^{-1}(1-c_1-\tau) - z_1(\bar{b}BP^{-1}+\bar{m}MP^{-1})$$
$$+ \alpha a_2\{l_2[(1-c_1-\tau)BP^{-1}-c_2]-c_3\}$$
$$- \alpha a_2z_2(MP^{-1}+BP^{-1})$$

and

$$v_2 = - \alpha a_2[(1-c_1-\tau)BP^{-1}-c_2](l_3MP^{-1}+l_4BP^{-1})$$
$$- \alpha a_2\bar{r}BP^{-1}(1-c_1-\tau) + \alpha a_2(\bar{b}BP^{-1}+\bar{m}MP^{-1})$$
$$+ \alpha a_2z_2[(1-c_1-\tau)BP^{-1}-c_2](l_3MP^{-1}+l_4BP^{-1})$$
$$+ \alpha a_2z_2\bar{r}BP^{-1}(1-c_1-\tau) - \alpha a_2z_2(\bar{b}BP^{-1}+\bar{m}MP^{-1})$$

The system is again of second order since neither monetary growth nor debt issuance are being endogenised via the government budget constraint. Hence, the Descartes rule of signs can again provide those conditions necessary and sufficient for stability. As before, we choose to investigate the positiv y of each and every coefficient in the corresponding characteristic equation.

B. Stability Conditions

The coefficients v_0, v_1, and v_2 can be simplified, when related to their counterparts under the two previously discussed policy regimes of endogenous government spending and monetary growth, by rewriting them as the following expressions:[24]

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[24] Recall that the s_i's are the coefficients of the characteristic equation under the policy regime in which

$$v_0 = u_0 - BP^{-1}(a_2z_2+z_1)$$

$$v_1 = (1/\alpha)v_2 - (1/\alpha a_2)s_2z_1 + \alpha a_2\{l_2[(1-c_1-\tau)BP^{-1}-c_2]-c_3\} - \alpha a_2z_2(MP^{-1}+BP^{-1})$$

$$v_2 = (1-z_2)s_2$$

the coefficient v_0

With respect to the coefficient v_0, there is no qualitative difference between it and u_0, its counterpart under the regime of endogenous government spending and monetary growth. The only associated destabilising trends have their roots in monetary accommodation.

What is ambiguous, and may be destabilising, is the way in which increased demand pressure and induced inflation will feed into the monetary policy reaction function. The countercyclical component of this reaction function works in the direction of monetary contraction and consequently reduced debt issuance. On the other hand, its accommodative component, manifested in a positive z_2, works in the opposite direction of monetary expansion and increased debt issuance. For the positivity of v_0, and hence, for the stability of the system, having the sum of z_1 and a_2z_2 being

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[2]'(cont'd) government spending alone is the endogenous policy instrument via the budget constraint. The u_i's, on the other hand, are the coefficients of the characteristic equation associated with the policy regime in which both government spending and monetary growth are simultaneously endogenised, the former via the government budget constraint and the latter via the monetary policy reaction function.

negative is a desirable condition but not a necessary one.

the coefficient v_1

Observe that, by substituting the expression for v_2 into the expression for v_1, one may rewrite this latter coefficient in the following way:

$$v_1 = (1/\alpha a_2)s_2[a_2(1-z_2)-z_1] + \alpha a_2\{l_2[(1-c_1-\tau)BP^{-1}-c_2]-c_3\} - \alpha a_2 z_2(MP^{-1}+BP^{-1}).$$

However, v_1 can be rewritten in an alternative way still, when comparing it directly with s_1, namely

$$v_1 = s_1 - (1/\alpha a_2)s_2[a_2z_2+z_1] - \alpha a_2 z_2(MP^{-1}+BP^{-1}).$$

In all three coeficients, s_1, u_1, and v_1, expectational adjustments enter in much the same way. Once under way, they will feed into the goods market adding to demand pressure via investment crowding in as real interest rates are, ceteris paribus, reduced. They will also feed into the monetary policy reaction function via their inflationary consequences, thus causing monetary expansion, which in turn will feed into the rate of debt issuance on a one-to-one basis. Such induced increases in the real values of money and bonds will, as previously discussed, lead to endogenous increases in government expenditures by increasing the government's financing capacity.

Most importantly, however, upward expectational adjustments will have repercussions for the nominal rate of interest. They will feed into the money market in the form of induced asset substitution effects out of real money balances thus pushing interest rates down with destabilising consequences. In addition, in the goods market asset substitution effects will take the form of increased investment demand, hence, adding to demand pressure in the system with further destabilising consequences.

Before expectational adjustments take effect, however, the initial increase in demand pressure will, among other things, feed into the monetary policy reaction function and by virtue of its countercyclical nature will cause monetary contraction. The rate of debt issuance will at the same time decrease one-to-one with monetary growth. These developments will have stabilising repercussions in the system.

In the first place, they limit the government's capacity to spend, thus countering the respective destabilising effects generated by inflation– induced monetary expansion discussed above. By adding these two effects together, one gets:

$$-(MP^{-1}+BP^{-1})(z_1+\alpha a_2 z_2)$$

which will be positive if and only if,

$$z_1+\alpha a_2 z_2$$

is negative.

The monetary authorities, whose behaviour the reaction function here employed is intended to explain, can be legitimately assumed to have some discretion over the setting of actual policy. This amounts to saying that the monetary authorities can be legitimately assumed to have some discretion over the weights z_1 and z_2 in the monetary policy reaction function. They can thus increase the stability of the system by choosing z_1 and z_2 in such a way that the last term cited above is indeed negative.

However, the monetary contraction via z_1 and the subsequent reduction in debt issuance, will have destabilising effects via the term,

$$\bar{r}(1-\tau-c_1)BP^{-1}.$$

That is, by reducing interest payments they put upward pressure on government spending whose expansionary effects dominate the contractionary effects of reduced interest income.

Like the case above, the fiscal authorities can also be legitimately assumed to be in a position potentially able to exercise discretion. Hence, the destabilising trends generated by increased government spending can be potentially neutralised.

Nevertheless, monetary contraction and reduced debt issuance will in addition feed into the money market, and to the extent that the interest rate is relatively sensitive to variations in real money balances, the net effect will be stabilising as the interest rate will increase.

Monetary accommodation, on the other hand, and the subsequent increase in the rate of debt issuance, enter the system in a purely destabilising way. By increasing the government's capacity to spend they can cause an expansionary endogenous increase in the level of government spending. Such a development, however, can be easily neutralised to the extent that the fiscal authorities apply discretion and simply do not spend.

More importantly, monetary accommodation and increased debt issuance will enter the money market reducing the nominal rate of interest to the extent that this latter variable is relatively sensitive to money supply variations. It is in this way that monetary accommodation can be most damaging to the stability of the system.

the coefficient v_2

With respect to the constant coefficient of the characteristic equation, v_2, no new mechanisms are embodied when compared with s_2, its counterpart under the regime of endogenous government spending. Since v_2 can be related to s_2 linearly,

$$v_2 = (1-z_2)s_2,$$

it can be observed that the strength of the mechanisms embodied in s_2 will be weakened by monetary accommodation.

Recall that the mechanisms embodied in s_2 are based on the inflation induced reduction in the real values of money and bonds outstanding. In the context of v_2, by increasing the money supply and the stock of bonds, monetary accommodation, will have just the opposite effects. Hence, when monetary policy is purely accommodative and $z_2=1$, the coefficient v_2 vanishes.

C. The Influence of the Monetary Policy Reaction Function

When monetary policy is purely accommodative with $z_2=1$ and $z_1=0$, v_2 vanishes and the order of the system is reduced. The system becomes one of first order and its characteristic equation would be:

$$v_0'\rho + v_1' = 0$$

where,

$$v_0' = s_0 - a_2(MP^{-1}+BP^{-1})$$
$$= u_0 - a_2BP^{-1}$$

and,

$$v_1' = s_1 - (1/a)s_2 - aa_2(MP^{-1}+BP^{-1}).$$

For stability the coefficients, v_0', and v_1', must either both be positive or both negative. Recall that

$$s_1 = (1/a)s_2$$
$$+ aa_2\{l_2[(1-c_1-\tau)BP^{-1}-c_2]-c_3\}.$$

By substituting s_1 into v_1' one gets:

$$v_1' = - aa_2(MP^{-1}+BP^{-1})$$

$$+ \alpha a_2\{l_2[(1-c_1-\tau)BP^{-1}-c_2]-c_3\} < 0.$$

That is, v_1' is unambiguously negative, so that for stability v_0' must also be negative.

However, v_0' can be made negative if the term $a_2(MP^{-1}+BP^{-1})$ is big enough to offset the positivity of s_0. That is to say, v_0' can be negative if complete monetary accommodation enters the budget constraint causing a substantial increase in the level of government spending. Otherwise being the case, the system will have an associated non-cyclically divergent time path.

How can the system be affected when a completely accommodative monetary policy is augmented by a countercyclical component? In this case, $z_2=1$ and $z_1<0$. Again, the system is first order with the following characteristic equation:

$$v_0''\rho + v_1'' = 0$$

where,

$$v_0'' = s_0 - (a_2+z_1)(MP^{-1}+BP^{-1})$$

and,

$$v_1'' = s_1 - (1/\alpha)s_2 - (1/\alpha a_2)s_2z_1$$
$$- \alpha a_2(MP^{-1}+BP^{-1}).$$

By substituting s_1 into v_1'', this latter expression becomes:

$$v_1'' = -(1/\alpha a_2)z_1 s_2 - \alpha a_2(MP^{-1}+BP^{-1})$$
$$+ \alpha a_2\{l_2[(1-c_1-\tau)BP^{-1}-c_2]-c_3\}.$$

In this case, monetary policy can be used to potentially take v_1'' to positivity.

In the context of the policy regime examined in the present chapter, when monetary policy is completely accommodative, with $z_2=1$ and $z_1=0$, the system can be stable if government spending is increased by a sufficient amount, in response to inflationary shocks, so that the coefficient v_0' becomes negative. However, countercyclical monetary policy used in conjunction with complete monetary accommodation can, potentially, alleviate this situation.

VIII. CONCLUSIONS

The model employed in this study incorporates a natural rate proposition in the sense that expectations enter the inflation process with full power. With expectational errors or misperceptions being the only source of rigidities in the system, in long run equilibrium the system becomes decomposable. With expectations realised, output is at full capacity utilisation and monetary and fiscal policies become totally ineffective. That is, real output in the long run is supply determined, and inflation is a purely monetary phenomenon.

The neutrality of money is a long run phenomenon. Because of expectational errors, monetary changes can in the short run influence output and employment. In the long run expectations are realized, excess demand is zero, and the Phillips-curve goes to verticality at the natural rate of unemployment. A trade-off function between inflation and unemployment or output is only a short-run phenomenon.

In the short run, a monetary expansion has a real effect by virtue of reducing the rate of interest via increasing the real stock of money balances. In the long-run, however, monetary expansion has the opposite effect of pushing the interest rate up by reducing the real stock of money. This result comes out of the fact that the model economises on real money balances.

That is, induced inflation leads to expectational adjustments that generate asset substitution effects out of money and into real goods. A monetary expansion feeds into higher inflation one-to-one in the long run and, consequently, into higher inflationary expectations, generating asset substitution effects that drive the nominal rate of interest up and reduce the demand for real money balances.

If that is the case, then for stock equilibrium to be established the rate of price inflation must overshoot its long-run equilibrium value. That is to say, if monetary expansion increases the real stock of money in the short-run, but in the long run the system settles at a lower real money stock, then it has to be the case that for some time period inflation must be growing faster than its long-run value for stock equilibrium to be reestablished.

In addition, the lag-structure that the model incorporates can also lead to the same overshooting proposition. The only lag this model incorporates comes from the adaptive nature of expectations formation. Increased demand pressure will feed into inflation both independently and via the wage-price nexus. Inflation will subsequently generate upward expectational adjustments but with a lag. Expectational adjustments, once under way, will feed into further inflation via the wage price nexus. However, because demand induced inflation feeds into expectations with a time lag, and since expectations feed back into inflation, output will peak before the rate of price inflation. That is, the model embodies a lead-lag relationship between the level of real output and the rate of price inflation with output changes preceeding price changes.

In this case, a monetary expansion that expands output will feed into inflation but, because of the expectational lag, price changes will fall behind and, hence, price inflation would be growing at a lower rate than monetary growth. Consequently, the real stock of money would initially increase. Hence, if stock equilibrium is to be re-established eventually, then for some time period, price inflation must be growing faster than the rate of monetary growth and faster than its long-run value.

Inflationary expectations are here responsible for asset substitution effects. Moreover, because they introduce a time lag into the system, they are also responsible for a lead-lag relationship between real output and price inflation. These conditions will cause output and inflation to overshoot their equilibrium values. Furthermore, these same conditions can potentially take the economy through a phase of inflationary recession.

Indeed, it was shown in chapters four and six that what can ultimately lead to explosive oscillations in particular, and oscillatory time paths in general in the system, must be rooted in the implications of expectational adjustments. In particular, the time lag involved in the formation of expectations, and the asset substitution effects generated by expectational adjustments, are precisely the factors that can potentially lead to oscillatory behaviour.

The short-run dynamics of the model turn out to be quite involved. With respect to the question of dynamic stability in particular, it cannot be said that any specific condition must, or must not, hold for stability or instability. However, in all the policy regimes examined, the asset substitution effects associated with expectational adjustments enter the system in a destabilising way. Asset substitution effects, ceteris paribus, have the effect of reducing the interest rate and increasing demand pressure, where demand pressure is already in place in an inflationary process thus started.

To the extent that an interest rate increase generates a negative output effect then the interest rate mechanism will have stabilising repercussions in the system if, and only if, the interest rate increases in the face of increased demand pressure. Two basic questions are implicit here. First, will ongoing inflation in response to increased demand pressure, push the rate of interest up? Secondly, in the context of a demand-induced inflationary process, can an interest rate increase reduce demand pressure?

What is common across all policy regimes examined are three repercussions that work to reduce the stabilising power of the interest rate mechanism namely,

(a) asset substitution effects,

(b) the relative responsiveness of the interest rate to changes in the real stock of money balances and the real debt outstanding, and

(c) interest income induced consumption effects.

Asset substitution effects drive the interest rate down. That is, upward expectational adjustments induce people to economise on real money balances by reallocating wealth into real capital goods. Similarly, by eroding the real stock of bonds, inflation allows the interest rate to come down. Interest income induced consumption effects, on the other hand, can potentially render an interest rate increase to have destabilising consequences for the system. That is to say, an increase in the rate of interest would increase interest income, generating a positive consumption effect that comes to be destabilising in an inflationary context by adding to demand pressure, thereby fueling inflation.

The endogeneity of monetary growth via the government budget constraint enters the system in a counteracting way. By reducing the stocks of money and bonds in real terms, inflation leads to a replacement monetary expansion via the government budget constraint. At the same time, inflation works in the direction of endogenous monetary contraction as well, by reducing interest payments on the public debt. In addition, an increase in real output increases tax revenues also allowing for a monetary contraction.

There is, thus, an ambiguity with respect to the direction in which monetary growth will move in response to increased demand pressure in the system. Consequently, the endogeneity of monetary growth via the government budget constraint cannot be established to have a stabilising or destabilising impact upon the system in the sense of increasing or decreasing the positivity of the relevant coefficients in the corresponding characteristic equation.

The endogeneity of debt issuance via the government budget constraint, on the other hand, creates strong destabilising mechanisms in the system that run from debt issuance to further debt issuance via interest payments. That is to say, an increase in interest payments on the public debt, leads to debt issuance endogenously. Interest payments in turn, would increase in response to debt issuance per se, and increases in the rate of interest. This policy regime, hence, embodies a feedback mechanism between debt issuance and interest rates that can potentially create a strong destabilising force in the system.

With respect to the behaviour of the model under the above two alternative policy regimes of endogenous monetary growth and of debt issuance via the government budget constraint, an important result is established. This is what we termed in chapter five, the 'dual instability' theorem. The system cannot be stable under both these policy regimes of endogenous monetary growth or debt issuance via the government budget constraint. If it is stable when monetary growth is endogenous, it must be unstable when the rate of debt issuance is the instrument endogenised by the budget constraint. Similarly, if the system is stable when the rate of debt issuance is endogenous, it must be unstable when the rate of monetary growth is endogenised via the government budget constraint.

When that is indeed the case and the system is unstable when debt issuance is the instrument adjusting in response to budget imbalances, monetary policy can potentially be used to stabilise the system. But if there is to be any hope for stability at all when a monetary policy reaction function is introduced into the system, monetary policy must be at least completely accommodative. This is so because monetary accommodation, when debt issuance is endogenous via the government budget constraint, weakens the destabilising feedback mechanisms that run from debt issuance to further debt issuance.

When government spending is the policy instrument endogenised through the budget constraint things become much more encouraging. Under this policy regime, the destabilising trends embodied in the system, other than the influence of asset substitution effects, work through endogenous fiscal expansion. For instance, by reducing the real debt outstanding, inflation reduces interest payments and, hence, allows for an endogenous fiscal expansion by increasing the government's capacity to spend.

However, assuming that the fiscal authorities can exercise discretion such destabilising trends can be easily offset. Hence, any degree of flexibility with respect to varying the level of government spending can certainly be helpful in terms of the stability of the system in general.

With government spending endogenous via the government budget constraint, an accommodative monetary policy adds to instability by reducing the positivity of the relevent coefficients of the corresponding characteristic equation. Countercyclical monetary policy, on the other hand, has the opposite effect of strengthening stability throughout. Such being the case, depending on the degree to which the system in question is stable, that is, depending on how strongly positive the coefficients of the corresponding characteristic equation are, the monetary authorites can manipulate the money supply in ways that can impinge upon the time path of the system, potentially evening out over time both costs and benefits associated with exogenous shocks hitting the economy.

The more instruments available and the more effective they are, the more economic policy can achieve in terms of policy targets. It is thus desirable to have as many instruments available as possible. In a situation in which government spending is endogenous via the budget constraint, monetary growth allowed to vary in accordance with a reaction function, and the rate of debt issuance constrained to follow the time path of monetary growth, the result is encouraging. Except for the influence of expectational adjustments and the resulting asset substitution effects, all other destabilising trends in the system work via fiscal expansion. This cannot be seen as a serious problem if the fiscal authorities can be legitimately assumed to be in a position to exercise discretion.

In such a policy regime a purely accommodative monetary policy can be damaging to the stability of the system. In actual fact the only factor that can prevent instability in such a scenario, comes to be endogenous fiscal expansion via the government budget constraint.

Regarding monetary accommodation in general, it can be said that it has stabilising repercussions in the system in those cases where the rate of debt issuance is the residual instrument of financing. This is so because monetary accommodation in those cases weakens the strong destabilising trends that run from debt issuance to further debt issuance both via the increase in the public debt per se and via the interest rate mechanism. However, monetary accommodation becomes destabilising in those cases in which government spending is the policy instrument endogenised via the government budget constraint. Monetary accommodation is destabilising in these cases because it can potentially lead to endogenous fiscal expansion.

In a world in which the monetary and fiscal policy instruments are inter-related via the government budget constraint, both these instruments are effective and can be used in the context of a comprehensive macroeconomic policy potentially to do some good in the economy.

REFERENCES

Addison, J.T., J. Burton, and T. S. Turrance, "On the Causation of Inflation." Manchester Journal of Economics and Social Studies, June 1980, pp. 140-56.

Allen, R.G.D., Mathematical Analysis for Economists, Macmillan, 1938.

Allen, R.G.D., Mathematical Economics, 2nd ed., London, Macmillan, 1960.

Baily, M.N., "The Labour Market in the 1930s", in J.Tobin (ed.), Macroeconomics, Prices and Quantities, Washington, D.C.: The Brookings Institution, 1983, pp. 21-62.

Barro, R.J., "Unanticipated Money Growth and Unemployment in the United States", American Economic Review, March 1977, pp. 101-115.

Barro, R.J., "Unanticipated Money Output and the Price Level in the United States", Journal of political Economy, August 1978, pp. 549-580.

Barro, R.J., and S. Fischer, "Recent Developments in Monetary Theory", Journal of Monetary Economics, April 1976, pp. 133-168.

Batten, D.S., "Money Growth Stability and Inflation: an International Comparison.", Federal Reserve Bank of St. Louis Review, Oct. 1981, 63(8), pp. 7-12.

Blinder, A.S., and D. Holtz-Eakin, "Public Opinion and the Balanced Budget", American Economic Review, May 1984, pp. 144-149.

Blinder, A.S., and R.M. Solow, "Does Fiscal Policy Matter?", Journal of Public Economics, Feb. 1973, pp. 319-338.

Blinder, A.S., and R.M. Solow, "Analytical Foundations of Fiscal Policy", in The Economics of Public Finance, edited by A.S. Blinder et al., Washington: Brookings Institution, 1974.

Bozdarich, M., "The Phenomenon of Inflation and the Prospects for Anti-Inflation Policy", Federal Reserve Bank of San Francisco Economic Review, Fall 1979, pp. 6-22.

Bozdárich, M., "Money, Inflation, and Causality in the U.S. 1959-1979", Federal Reserve Bank of San Francisco

Economic Review, Spring 1980, pp. 50-70.

Brainard, W.C., and R.N. Cooper, "Empirical Monetary Macroeconomics: What Have We Learned in the Last 25 Years?", American Economic Review, May 1975, pp. 167-175.

Branson, W.H., "Monetarist and Keynesian Models of the Transmmission of Inflation", American Economic Review, May 1975, pp. 115-119.

Branson, W.H., Macroeconomic Theory and Policy, 2nd ed., New York, Harper and Row, 1979.

Braun, M., Differential Equations and Their Applications, 2nd edition, Springer-Verlorg, New-York, 1978.

Cagan, P., The Hydra-Headed Monster: The problem of Inflation in the United States, Washington D.C.: American Enterprise Institute for Public Policy Research, 1974.

Cagan, P., "Inflation and Market Structure, 1967-1973", Explorations in Economic Research, Spring 1975, pp. 203-216.

Cagan, P., "Changes in the Recession Behaviour of Wholesale Prices in the 1920's and Post World War 2", Explorations in Economic Research, Winter 1975, pp. 90-96.

Carmichael, J., and P. Stelbing, "Fisher's Paradox and the Theory of Interest", American Economic Review, Sept. 1983, pp. 619-30.

Christ, C.F., "A Short Run Aggregate-Demand Model of the Interdependence and Effects of Monetary and Fiscal Policies with Keynesian and Classical Interest Elasticities", American Economic Review, May 1967, pp. 434-443.

Christ, C.F., "A simple Macroeconomic Model with a Government Budget Restraint", Journal of Political Economy, January/February 1968, pp. 53-67.

Christ, C.F., "Some Dynamic Theory of Macroeconomic Policy Effects on Income and Prices under the Government Budget Restraint", Journal of Monetary Economics, 4(1978), pp. 45-70.

Christ, C.F., "On Fiscal And Monetary Policies and the Government Budget Restraint", American Economic Review, 4(1979), vol 64, pp. 526-38.

Clower, R., and Axel Leijonhufvud, "The Coordination of Economic Activities: A Keynesian Perspective", American Economic Review, May 1975, pp. 182-188.

Eisner, R., "The Keynesian Revolution Reconsidered", American Economic Review, May 1975, pp. 189-194.

Eisner, R., "Which Budget Deficit? Some Isssues of Measurement and their Implications", American Economic Review, May 1984, pp. 138-143.

Feldstein, M., "Fiscal Policies, Inflation, and Capital Formation", American Economic Review, Sept. 1980, pp. 636-650.

Fischer, S., "Recent Developments in Monetary Theory", American Economic Review, May 1975, pp. 157-65.

Fischer, S., "Long Term Contracts, Rational Expectations, and the Optimal Money Supply Rule", Journal of Political Economy, Feb. 1977, pp. 191-205.

Friedman, B.M., "Lessons from the 1979-1982 Monetary Policy Experiment", American Economic Review, May 1984, pp. 388-91.

Friedman, M., "The Role of Monetary Policy", American Economic Review, March 1968, pp. 1-17.

Friedman, M., "A Theoretical Framework for Monetary Analysis", Journal of Political Economy, March/April 1970, pp. 193-238.

Friedman, M., "Nobel Lecture: Inflation and Unemployment", Journal of Political Economy, June 1971, pp. 451-472.

Friedman, M., "25 Years After the Rediscovery of Money: What Have We Learned-Discussion", American Economic Review , May 1975, pp. 176-179.

Friedman, M., "Lessons from the 1979-1982 Monetary Policy Experiment", American Economic Review, May 1984, pp. 397-400.

Gandolfo, G., Economic Dynamics: Methods and Models, Revised edition, Amsterdam, North Holland, 1980.

Gibson, W.E., "Interest Rates and Inflationary Expectations", American Economic Review, Dec. 1972, pp. 854-65.

Giffin, P.E., J.H. Macomber, and R.E. Berry, "An Empirical Examination of Current Inflation and Deficit Spending", Journal of Post Keynesian Economics, Feb. 1981, pp. 63-67.

Gordon, R.J., "A Century of evidence on Wage and Price Stickiness in the United States, the United Kingdom, and Japan", in J. Tobin (ed.), Macroeconomics Prices and Quantities, Washington D.C., The Brookings Institution, 1983 pp. 85-134.

Gray, M., and M. Parkin, "Discriminating between Alternative Explanations of Inflation", University of Manchester Discussion Paper 7414, Manchester, Dec. 1974.

Haberler, G., "Thoughts on Inflation: The Basic Forces", Business Economics, 10(1), 1975, pp. 12-18.

Hall, R.E., "The Rigidity of Wages and the Persistence of Unemployment", Brookings Papers on Economic Activity, No2, 1975, pp. 301-49.

Hamburger, M.J., and B. Zwick, "Deficits, Money, and Inflation", Journal of Monetary Economics, Jan. 1981, pp. 141-50.

Harris, L., Monetary Theory, New York, McGraw-Hill Book Company, 1981.

Humphrey, T.M., "Some Current Controversies in the Theory of Inflation", Economic Review, July/Aug. 1976.

Humphrey, T.M., "A Monetarist Model of The Inflationary Process", Economic Review , Nov./Dec. 1975.

Humphrey, T.M., "The Persistence of Inflation", The Political Economy of Policy Making, 1979.

Infante, E.F. and J.L. Stein, "Does Fiscal Policy matter?", Journal of Monetary Economics, Nov. 1976, pp. 473-500.

Infante, E.F. and J.L. Stein, "Money-financed Fiscal Policy in a Growing Economy", Journal of Political Economy, vol. 88, No. 21, 1980, pp. 259-287.

Jackson, W.P., "Federal Deficits, Inflation, and Monetary Growth: Can They Predict Interest Rates?", Federal Reserve Bank of Richmond Economic Review, Sept./Oct. 1976.

Johnson, H.G., "The Keynesian Revolution and The Monetarist Counter-Revolution", American Economic Review, May

1971, pp. 1-14.

Lahiri, K., "Inflationary Expectations: their Formation and Interest Rate Effects", American Economic Review, March 1976, pp. 12-21.

Laidler, D., "The Influence of Money on Real Income and Inflation: A Simple Model with some Empirical tests for the United States 1953-1972", Manchester Scool of Economics and Social Studies, Dec. 1973, pp. 367-95.

Laidler, D., "The 1974 Report of the President's Council of Economic Advisors: The Control of Inflation and the Future of the International Monetary System", American Economic Review, Sept. 1974, pp. 535-43.

Lipsey, R.G., "The relation between Unemployment and the Rate of Change of Money Wage Rates in the United Kingdom 1862-1957: A further Analysis", Economica, vol. 27, Feb. 1960, pp. 1-31.

Lipsey, R.G., "The Understanding and Control of Inflation: Is there a Crisis in Macroeconomics?", Canadian Journal of Economics, vol. 14, Nov. 1981, pp. 545-576.

Modigliani, F., and L. Papademos, "Targets for Monetary Policy in the Coming Year", Brookings Papers on Economic Activity, No1, 1975, pp. 141-165.

Modigliani, F., and L. Papademos, "Inflation, Financial and Fiscal Structure, and the Monetary Mechanism", European Economic Review, March/April 1983, pp. 203-250.

Mundell, R., "Inflation and Real Interest", Journal of Political Economy, Vol. XXI, No3, June 1963, pp. 280-83.

Mundell, R., Monetary Theory: Inflation, Interest and Growth in The World Economy., Goodgear Publishing Company, Inc. Santa Monica, California, 1971.

Muth, J.F., "Rational Expectations and the Theory of Price Movements", Econometrica, July 1961, pp. 315-335.

Niehans, J., "Monetary Policy with Full Stock Adjustment", Z. Wirtsch. Socialwiss., No. 1, 1974, pp. 17-43.

Niskanen, W.A., "Deficits, Government Spending, and Inflation", Journal of Monetary Economics, Aug. 1978, pp. 591-602.

Lipsey, R.G., "Government and Inflation", American Economic Review, May 1982, pp. 67-71.

Lindbeck, A., "Budget Equations and Cost Inflation", American Economic Review, May 1983, pp. 285-290.

Lucas, R.E.N., "Econometric Testing of The Natural Rate Hypothesis", Conference sponsored by Board of Governors of the Federal Reserve System and Social Science Research Council, Washington D.C., Oct. 30-31, 1970, pp. 50-59.

McCallum, B.T., "Monetarist Rules in the Light of Recent Experience", American Economic Review, May 1984, pp. 388-91.

Meyer, L., and R. Rasche, "On the Costs and Benefits of Anti-Inflation Policies", Federal Reserve Bank St. Louis Review, Feb. 1980, pp. 3-14.

Modigliani, F., "25 Years after the Rediscovery of Money: What have We Learned - Discussion", American Economic Review, May 1975, pp. 179-181.

Modigliani, F., "The Monetarist Controversy or Should we Forsake Stabilisation Policies?", American Economic Review, March 1977, pp. 1-19.

Okun, A.M.,"Inflation: Its Mechanics and Welfare Costs", Brookings Papers on Economic Activity, No. 2, 1975, pp. 351-90.

Okun, A.M., Prices and Quantities: A Macroeconomic Analysis, Washington, D.C., The Brookings Institution, 1981.

Olson, M., "Beyond Keynesianism and Monetarism", Economic Inquiry, vol. XXII, July 1984, pp. 297-322.

Parkin, M., "The Causes of inflation: Recent Contributions and Current Controversies",in Current Economic Problems, M. Parkin and A.R. Nobay (eds), Cambridge: Cambridge University Press, 1975, pp. 243-91.

Patinkin, Don., Money Interest and Prices: An Integration of Monetary and Value Theory, Harper and Row Publishers, New York, 1965.

Peek, J., "Interest Rates, Income Taxes, and Anticipated Inflation", American Economic Review, Dec. 1982, pp. 980-91.

Perry, G.L., "Determinants of Wage Inflation around the World", Brookings Papers on Economic Activity, No2, 1975, pp. 403-47

Pitchford, J.A., "A Consistent Model of The Expectations-Augmented Phillips Curve and Inflation", Journal of Macroeconomics, Feb. 1981, 3(4), pp. 489-99.

Rau, N., "Simplifying the Theory of the Government Budget Constraint", Oxford Economic Papers, vol. 37, No.2, June 1985, pp. 210-229.

Rivlin, A.M., "Reform of the Budget Process", American Economic Review, May 1984, pp. 133-137.

Robinson, J., "Employment and Inflation", Quarterly Review of Economics and Business, Autumn 1974, pp. 7-16.

Samuelson, P.A., Foundations of Economic Analysis, Cambridge, Harvard University Press, 1947.

Sargent, T.J., and Neil Wallace, "Rational Expectations and the Theory of Economic Policy", Journal of Monetary Economics, April 1976, pp. 169-183.

Sargent, T.J., and Neil Wallace, "Rational Expectations, the Optimal Monetary Instrument and the Optimal Money Supply Rule", Journal of Political Economy, April 1975, pp. 241-254.

Scarfe, B.L., Price Determination and the Process of Inflation in Canada, Prices and Incomes Commission, Ottawa, 1972.

Scarfe, B.L., Cycles, Growth and Inflation: A Survey of Contemporary Macrodynamics, New York, McGraw-Hill, 1977.

Scarfe, B.L., "Optimal Monetary Policy with a Trade-Off Function", Oxford Economic Papers, Vol. 31, March 1979, pp. 20-35.

Scarfe, B.L., "Monetary Policy and the Reflation Problem", in G. Mason (ed.), Macroeconomics: Theory, Policy and Evidence, Winnipeg, Institute for social and Economic Research, University of Manitoba, 1983, pp. 76-104.

Scarfe, B.L., and A.M.C. Waterman, "On applying Ockham's Razor to the Monetarist Controversy", Canadian Journal of Economics, August 1978, pp. 570-583.

Seldon, R., "Monetary Growth and the Long-Run Rate of Inflation", American Economic Review, May 1975, pp. 125-8.

Sheffrin, S.M., *Rational Expectations*, Cambridge, Cambridge University Press, 1983.

Sims, C.A., "Is There a Monetary Business Cycle?", *American Economic Review*, May 1983, pp. 228-233.

Smith, G.W., "Price Determination", in M. Parkin and M.T. Summer (eds), *Inflation in the United Kingdom*, Manchester University Press, 1978, pp. 93-111.

Solow, R.M., *Price Expectations and the Behaviour of the Price level*, Manchester: Manchester University Press, 1969.

Stein, J.L., "Monetarist, Keynesian and New Classical Economics," *American Economic Review*, May 1981, 71(2), pp. 139-44.

Takayama, A., *Mathematical Economics*, Dryden Press, Hinsdale Illinois, 1974.

Tanzi, V., "Inflationary Expectations, Economic Activity, Taxes and Interest Rates", *American Economic Review*, March 1980, pp. 12-21.

Taylor, J.B., "Rational Expectations and the Invisible Handshake", in J. Tobin (ed), *Macroeconomics*, *Prices, and Quantities*, Washington D.C., The Brookings Institution, 1983, pp. 63-84.

Trevithick, J.A., and C. Mulvey, *The Economics of Inflation*, New York: John Wiley and Sons, 1975.

Tobin, J., "The Wage-Price Mechanism: Overview of the Conference", The Econometrics of Price Determination. Conference sponsored by Board of Governors of the Federal Reserve System and Social Science Research Council, Washington, D.C., Oct. 30-31, 1970, pp. 5-15.

Tobin, J., "Keynesian Models of Recession and Depression", *American Economic Review*, May 1975, pp. 195-202.

Tobin, J., "How Dead is Keynes?", *Economic Inquiry*, 15(4), Oct. 1977, pp. 456-68.

Turnovsky, S., *Macroeconomic Analysis and Stabilistion Policy*, Cambridge: Cambridge University Press, 1977.

Webb, R.H., "Interest Rates and Federal Deficits", *Fe[illegible] Reserve Bank of Richmond Economic Review*, July 1982, pp. 16-22.

Wells, D.R., and M.J. Gootzert, "Rational Expectations and the Positively sloped Phillips Curve", Economic Forum, Summer 1981, 12/17, pp. 121-29.

Witteveen, H.J., "Inflation and the International Monetary System", American Economic Review, May 1975, pp. 102-114.

Uspensky, J.V., Theory of Equations, New York, McGraw-Hill, 1948.

APPENDIX 1: LINEARISATION PROCEDURES

The linearisation procedures employed herein will be illustrated by means of one typical example. In particular, the policy regime of endogenous monetary growth via the government budget constraint, analysed in chapter five, will be explicitly considered here.

The model under this policy regime can be given in the following expressions:

$$Dp = a_2Dy + \alpha a_2(y-\bar{y}) \tag{3.2}$$

$$(1-c_1)Dy = c_1(B/P)r(\bar{b}-p) + [c_1(B/P)+c_2]Dr + \alpha a_2c_2(y-\bar{y}) \tag{5.3}$$

$$Dr = l_1Dy + \alpha a_2l_2(y-\bar{y}) + l_3(M/P)(m-p) + l_4(B/P)(\bar{b}-p) \tag{6.3}$$

$$MP^{-1}Dm + MP^{-1}m(m-p) + BP^{-1}\bar{b}(\bar{b}-p) = -\tau Dy + (1-\tau)BP^{-1}r(\bar{b}-p) + (1-\tau)BP^{-1}Dr \tag{8.2}$$

Define

$$\begin{aligned} y(t) &= \bar{y} + [y(0)-\bar{y}]e^{\rho t} \\ r(t) &= \bar{r} + [r(0)-\bar{r}]e^{\rho t} \\ p(t) &= \bar{p} + [p(0)-\bar{p}]e^{\rho t} \\ m(t) &= \bar{m} + [m(0)-\bar{m}]e^{\rho t} \end{aligned} \tag{A1.1}$$

where $\bar{x}$ represents the long run steady inflation equilibrium value of the variable in question, and $[x(0)-\bar{x}]$ is the initial deviation from that equilibrium value. Differentiating the above expressions with respect to time, one gets:

$$
\begin{aligned}
D\bar{y} &= \rho[y(0)-\bar{y}]e^{\rho t} \\
Dr &= \rho[r(0)-\bar{r}]e^{\rho t} \\
Dp &= \rho[p(0)-\bar{p}]e^{\rho t} \\
Dm &= \rho[m(0)-\bar{m}]e^{\rho t}
\end{aligned}
\qquad (A1.2)
$$

Substituting expressions (A1.1) and (A1.2) in the model equations (3.2), (5.3), (6.3), and (8.2), and simplifying, one obtains the following matrix expression (the same as in chapter five):

$$
\begin{bmatrix}
\rho a_2+\alpha a_2 & 0 & -\rho & 0 \\
\rho(1-c_1)-\alpha a_2 c_3 & -\rho(c_1 BP^{-1}+c_2) & \bar{r}c_1 BP^{-1} & 0 \\
\rho l_1+\alpha a_2 l_2 & -\rho & -(l_3 MP^{-1}+l_4 BP^{-1}) & l_3 MP^{-1} \\
\rho\tau & -\rho(1-\tau)BP^{-1} & \xi_1 & (\rho+\bar{m})MP^{-1}
\end{bmatrix}
\begin{bmatrix}
y(0)-\bar{y} \\
r(0)-\bar{r} \\
p(0)-\bar{p} \\
m(0)-\bar{m}
\end{bmatrix}
=
\begin{bmatrix}
0 \\ 0 \\ 0 \\ 0
\end{bmatrix}
$$

where $\xi_1 = \bar{r}(1-\tau)BP^{-1}-\bar{b}BP^{-1}-\bar{m}MP^{-1}$.

Removal of Non-linearities

The system above is inherently non-linear. Non-linearities take the form of initial deviations from equilibrium appearing multiplicatively. Consider for the purpose of illustration the equation for the government budget constraint given in expression (8.2). By substituting in what is relevant from expressions (A1.1) and (A1.2), one ought to obtain:

$$
\begin{aligned}
&\rho MP^{-1}[m(0)-\bar{m}]e^{\rho t} \\
&+ MP^{-1}\{\bar{m}+[m(0)-\bar{m}]e^{\rho t}\}\{[m(0)-\bar{m}]-[p(0)-\bar{p}]\}e^{\rho t} \\
&- BP^{-1}\bar{b}[p(0)-\bar{p}]e^{\rho t} = -\rho\tau[y(0)-\bar{y}]e^{\rho t} \\
&\qquad + \rho(1-\tau)BP^{-1}[r(0)-\bar{r}]e^{\rho t} \\
&\qquad - (1-\tau)BP^{-1}\{\bar{r}+[r(0)-\bar{r}]e^{\rho t}\}[p(0)-\bar{p}]e^{\rho t}
\end{aligned}
$$

Dividing throughout by $e^{\rho t}$, one obtains:

$$
\begin{aligned}
&\rho MP^{-1}[m(0)-\bar{m}] + MP^{-1}\{\bar{m}+[m(0)-\bar{m}]e^{\rho t}\}\{[m(0)-\bar{m}]-[p(0)-\bar{p}]\} \\
&- BP^{-1}\bar{b}[p(0)-\bar{p}] = -\rho\tau[y(0)-\bar{y}] + \rho(1-\tau)BP^{-1}[r(0)-\bar{r}] \\
&\qquad - (1-\tau)BP^{-1}\{\bar{r}+[r(0)-\bar{r}]e^{\rho t}\}[p(0)-\bar{p}]
\end{aligned}
$$

Assuming that the product of two initial deviations from equilibrium, say,

$$[x(0)-\bar{x}][x(0)-\bar{x}]$$

where $x = y$, r p and m in this context, is so small that it can be legitimately set equal to zero, then the non-linearities in question can be removed and the linearised version of the government budget constraint can be written in the following way:

$$\rho MP^{-1}[m(0)-\bar{m}] + \bar{m}MP^{-1}\{[m(0)-\bar{m}]-[p(0)-\bar{p}]\}$$
$$- \bar{b}BP^{-1}[p(0)-\bar{p}] = - \rho\tau[y(0)-\bar{y}] + \rho(1-\tau)BP^{-1}[r(0)-\bar{r}]$$
$$- (1-\tau)BP^{-1}\bar{r}[p(0)-\bar{p}]$$

This approximation can be most valid in the neighbourhood of a long-run steady inflation equilibrium time path.

APPENDIX 2: THE ROUTH HURWITZ STABILITY CONDITIONS

For dynamic systems of order higher than two, the Routh-Hurwitz conditions provide those conditions necessary and sufficient for stability. They can be stated in the following way. Consider an nth order system with an associated nth degree characteristic equation that can be given in the expression below.

$$k_0\rho^n + k_1\rho^{n-1} + k_2\rho^{n-2} + \ldots + k_{n-1}\rho + k_n = 0$$

The system will be stable, that is, its corresponding characteristic roots will all have negative real parts if, and only if, the n principal minors of the matrix,

$$\begin{bmatrix} k_1 & k_0 & 0 & 0 & 0 & \ldots & 0 \\ k_3 & k_2 & k_1 & k_0 & 0 & \ldots & 0 \\ k_5 & k_4 & k_3 & k_2 & k_1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & 0 & \ldots & k_n \end{bmatrix}$$

are positive, when normalising on the positivity of k_0.[27]

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[27] A more complete and detailed account of the Routh-Hurwitz stability conditions can be found in numerous textbooks including Uspensky ([illegible]8), Samuelson (1948), Gandolfo (1984).

The Routh-Hurwitz stability conditions for a third order dynamical system

In the case of a third order system, that is, when n=3, the following principal minors,

$$\begin{vmatrix} k_1 \end{vmatrix}$$

$$\begin{vmatrix} k_1 & k_0 \\ k_3 & k_2 \end{vmatrix}$$

$$\begin{vmatrix} k_1 & k_0 & 0 \\ k_3 & k_2 & k_1 \\ 0 & 0 & k_3 \end{vmatrix}$$

must all be positive for stability when normalising on the positivity of k_0.

By expanding the above determinants, the following conditions can be obtained:

$$k_1 > 0$$

$$k_1 k_2 - k_0 k_3 > 0$$

$$k_3(k_1 k_2 - k_0 k_3) > 0$$

Note that when the second of these three conditions holds the third will also hold if, and only if, $k_3 > 0$. Normalising on the positivity of k_0, the stability conditions corresponding to a third order system can be written as the following inequalities:

$$k_1 > 0.$$

$$k_3 > 0.$$

$$k_1 k_2 - k_0 k_3 > 0$$

Given that k_0 is positive these conditions would imply that k_2 is also positive. Hence, the Descartes rule of signs result is embedded in the Routh-Hurwitz stability conditions.

APPENDIX 3: NECESSARY VERSUS SUFFICIENT CONDITIONS

Consider a third order system with the following characteristic equation:

$$k_0\rho^3 + k_1\rho^2 + k_2\rho + k_3 = 0. \qquad (A3.1)$$

When applied in this context, Descartes rule of signs can only provide the conditions that can rule out the possibility of non-cyclical instability. This is so because, with all the coefficients of the corresponding characteristic equation being positive, the system will have no positive real roots. One would know from this that the system must have at least one negative real root. Let that root be given by $-\beta$ where β is a positive real number.

Factoring this real root out of the characteristic equation given in expression (A3.1) would yield:

$$(\rho+\beta)[k_0\rho^2 + \epsilon\rho + \theta] = 0. \qquad (A3.2)$$

Expanding the above expression would yield:

$$k_0\rho^3 + (\beta k_0+\epsilon)\rho^2 + (\beta\epsilon+\theta)\rho + \beta\theta = 0. \qquad (A3.3)$$

From this expression it can be seen that the coefficients k_1, k_2, and k_3 can be written as the expressions below:

$$k_1 = \beta k_0 + \epsilon \qquad (A3.4)$$

$$k_2 = \beta\epsilon + \theta \qquad (A3.5)$$

$$k_3 = \beta\theta \qquad (A3.6)$$

These expressions constitute a non-linear three equation system in the unknowns β, ϵ, and θ.

Given that $-\beta$ is a negative real root, the remaining two roots of the system, denoted by ρ_2 and ρ_3 respectively, must be given by the relationship,

$$k_0\rho^2 + \epsilon\rho + \theta = 0. \qquad (A3.7)$$

That [illegible],

$$\rho_2, \rho_3 = \{-\epsilon +/- [\epsilon^2 - 4k_0\theta]^{1/2}\}(2k_0)^{-1}. \qquad (A3.8)$$

Define

$$\Delta = [\epsilon^2 - 4k_0\theta]^{1/2}$$

the discriminant of (A3.8). The nature of the roots ρ_1 and ρ_2, would depend on the value of the discriminant Δ. There can be three alternative cases:

(a) $\Delta > 0$

In this case there will be two real and distinct roots which will be negative if, and only if, ϵ is positive.

(b) $\Delta = 0$

In this case there will be two real repeated roots given by, $-\epsilon(2k_0)^{-1}$. Like case (a) above, these roots will be negative if, and only if, ϵ is positive.

(c) $\Delta < 0$

This is the case of complex roots. These roots will have negative real parts if, and only if, ϵ is positive.

Hence, from all the above, it follows that the positivity of ϵ is necessary for the stability of the system. What can it be said about the sign of this coefficient? Invert expressions (A3.4), (A3.5), and (A3.6), making ϵ, θ, and β subjects respectively, to get:

$$\epsilon = k_1 - \beta k_0 \quad (A3.4)'$$

$$\theta = k_2 - \beta\epsilon \quad (A3.5)'$$

$$\beta = k_3/\theta \quad (A3.6)'$$

Substituting (A3.5)' into (A3.6)' and then the result into (A3.4)' yields:

$$\epsilon^2 - (k_2+\beta k_1)\beta^{-1}\epsilon + (k_1k_2-k_0k_3)\beta^{-1} = 0. \quad (A3.9)$$

Consequently, ϵ can have the solutions given by the following expression:

$$\epsilon = \{(k_2+\beta k_1)\beta^{-1} +/- [(k_2+\beta k_1)^2\beta^{-2}-4(k_1k_2-k_0k_3)\beta^{-1}]^{1/2}\}/2$$

If $(k_1k_2-k_0k_3)$ is positive, when the discriminant is also positive then,

$$[(k_2+\beta k_1)^2\beta^{-2}-4(k_1k_2-k_0k_3)\beta^{-1}]^{1/2} < (k_2+\beta k_1)\beta^{-1}$$

and therefore, ϵ will be positive and real. However, if the discriminant in question is negative, then ϵ will be a complex conjugate pair with positive real parts. In these cases, the roots ρ_1 and ρ_2, will be either negative real numbers or complex conjugate with negative real parts.

Hence, this proves that with k_0, k_1, k_2, and k_3 being positive, the positivity of the term,

$$k_1k_2 - k_0k_3 \quad 0$$

is the condition that rules out the possibility of explosive oscillations.