University of Alberta

Application of the boundary integral equation method to the problems of linear piezoelectricity

by

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Abstract

Problems involving the behavior of piezoelectric solids receive a considerable amount of attention in the current scientific and engineering literature as the need for developments in micro electro-mechanical systems and their applications increases. It is well known that all piezoelectric materials are anisotropic which makes any analysis of the equations of piezoelectricity (equilibrium or motion) very difficult. This leads to the lack of the development of the corresponding mechanical-mathematical models that are present in classical isotropic elasticity.

The objective of this work is to study the boundary value problems for two cases of deformation in linear piezoelectricity: anti-plane shear and generalized plane strain states. Although both anti-plane shear and generalized plane strain are two-dimensional models they have a variety of practical applications. For example, the problems of static torsion, or torsional vibrations, reduce to the state of anti-plane shear; the case of general loading applied to the generators of a cylindrical body with arbitrary cross-section can be described by equations of the generalized plane strain. It is shown, however, that not all the piezoelectric materials can obey the state of anti-plane shear. The conditions on material properties necessary for the anti-plane shear to exist are mentioned.

The present work gives complete treatment of three fundamental boundary value problems: Dirichlet, Neumann and mixed the case of equations of equilibrium and Dirichlet and Neumann for the equations of steady state vibrations of a piezoelectric body. Our tool for the analysis is an analytical technique known as the boundary integral

equation method: boundary value problems stated in bounded or unbounded domains are reduced to corresponding systems of singular integral equations stated on the boundary. We show that Fredholm theorems on existence of solutions can be applied to the systems of singular integral equations. The question of uniqueness of solutions of the boundary value problems is also examined for the case of an unbounded do main (problems of equilibrium and steady state vibrations) and the conditions for which the uniqueness of solutions is guaranteed are derived. The analytical solutions for the boundary value problems are given in the form of the single and double layer potentials.

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Elizaveta O. Lyubimova

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Dedication

To my mother Natalia Lyubimova

and my father Oleg Lyubimov

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List of Symbols and Notations

Symbol/Notation	Description
$\alpha, \beta, \gamma, \eta, \delta, \kappa, \lambda, \mu, \nu, \rho$	-Greek indices when used as subscripts
$A_{\alpha\beta\delta\gamma}$	-real numbers
B	-magnetic flux vector
$B_{\alpha\beta\delta\gamma}$	-real numbers
C , c_i	-arbitrary constants
C_{ijkl}	-elastic constants
C_{ij}	-elastic constants in compressed notation
$C^{\alpha}(S)$	-space if α -times continuously differentiable functions defined on S
$C^{0,\beta}(S)$	-space of Hölder continuous functions defined on S
$C^{1,\beta}(S)$	-space of functions whose first derivatives are Hölder continuous functions defined on S
D_i	-components of electric displacement
D^*	- prescribed boundary value of the surface charge
$\tilde{D}$	-constant
$D(x, y), D(x, y, \omega)$	-matrices of singular solutions
$\delta(x-y)$	Dirac distribution
$\delta_{\alpha\beta}$	-Kronecker's delta, alternating tensor
E_i	-components of electric field
e_{ijk}	-piezoelectric constants
e_{ij}	-piezoelectric constants in compressed notation
ε_{ijk}	-alternating tensor
ε_{ij}	-strain tensor components

Symbol/Notation	Description
ϵ_{ij}	-electric permittivity
f_i	-components of external forces
$\{f^i\}$	-a set of vectors $f^{(1)}, f^{(2)}$, etc.
$F^{(i)}$	- i -th column of a matrix F
$F_{(i)}$	- i -th row of a matrix F
$\mathcal{F}_{\xi \to x}$	-Fourier transformation from domain ξ to domain x
$G,(x,y)$, $G^D(x,y)$, $G^N(x,y)$	-Green's tensors for Dirichlet and Neumann problems
ϕ	-electric potential
ϕ^*	-prescribed boundary value of the electric potential
φ (x)	-density function
φ^k	-eigensolutions of the homogeneous external Dirichlet problem
$\Gamma(x,y), \Gamma(x,y,\omega)$	-matrices of fundamental solutions
H	-enthalpy
$H^*(S)$	-space of Hölder continuous functions with weak singularities defined on S
$H_0^{(1)}$	-Hankel function of the first kind of order zero
I_ω, I_c	-matrices of coefficients
I_{x_α}	-moments of inertia about an axis x_α
i	-imaginary unit
i,j,k,l,l,n	-Latin indices when used as subscripts
J	-electric current vector
K^s, K^ω	-integral operators
k, k_1, k_2	-constants
$L(\partial x)$	-differential operator

Symbol/Notation	Description
$L^{*}(\partial x)$	-transposed matrix of cofactors of operator $L(\partial x)$
M	-magnetic field vector
$M_{m\times n}$	-space of $(m\times n)$ matrices
n_i	-components of the unit normal
ρ	-material density
$P(x)$	-vector function prescribing boundary condition for the interior Dirichlet problem
$Q(x)$	-vector function prescribing boundary condition for the exterior Neumann problem
$R(x)$	- vector function prescribing boundary condition for the exterior Dirichlet problem
R^{j}	-coefficient matrix
r	-distance between points x and y
S	-domain on the plane of real numbers
$S(x)$	-vector function prescribing boundary condition for the interior Neumann problem
∂S	-boundary of the region S
t_i	-stress vector components
t_i^{*}	-prescribed boundary values of stress vector components
τ_{ij}	-components of stress tensor
U	-internal electro-magnetic energy
$U(u,u)$	-internal piezoelectric energy as a quadratic form of u
u_i	-components of the displacement vector
u^{k}	-eigensolutions of the homogeneous internal Dirichlet problem
u_b^{k}	-boundary values of u^{k}
$u^{*}, u_{+}^{*}, u_{-}^{*}$	-prescribed values for vector u

Symbol/Notation	Description
V	-volume
∂V	-surface enclosing the volume V
$V\varphi(x)$	-single layer potential
υ, υ_k	-vector functions
υ^k	-boundary values of eigensolutions of the homogeneous internal Neumann problem
$W\varphi(x)$	-double layer potential
w^k	-eigensolutions of the homogeneous internal Neumann problem
ω	-frequency
x_i	-coordinates
y	-integration variable, point lying on the boundary ∂S
ψ^k	-eigensolutions of the homogeneous exterior Neumann problem
ξ, ζ	-complex variables
ξ_α	-first order partial derivative with respect to x_α
ξ_α^β	-partial derivative of order β with respect to x_α
z	-complex variable

Chapter 1

Introduction

The phenomenon of piezoelectricity can be described as a generation of electric polarization due to applied pressure. It was discovered in 1880 by brothers Pierre and Jacques Curie in the laboratory of mineralogy, at the University of Paris-Sorbonne. In 1881 W.G. Hankel introduced the term "piezoelectricity" that was readily accepted in scientific circles. The word is derived from the Greek piezein, which means to squeeze or press [12].

After discovery of piezoelectric phenomenon by the brothers Curie the equations of piezoelectricity became available in Voigt's book "Lehrbuch der Kristallphysik" first published in 1910. W. Voigt formulated thermodynamic potentials for mechanical, electric and thermal interactions for crystals and derived the constitutive relations for direct and converse piezoelectric effects. The theory of linear piezoelectricity is based on the idea of coupling of the quazi-static electric field and the dynamic mechanical motion in polarizable dielectrics (not magnetizable). In the development of the theory it was assumed that the relations between strain and stress are linear for electromechanical interactions. This assumption was confirmed experimentally for a variety of crystals except ferroelectrics [12]. The first publications on the applications of piezoelectricity and the development of the theory of vibrations in piezoelectric solids began to appear in the early part of the twentieth century [10, 11].

Among the publications on the theory of piezoelectricity the book "Piezoelectricity" by W.G. Cady should be noted. The author gave the development of the theory of the piezo resonator, the investigation of physical properties of Rochelle Salt and Seignette electrics and their applications, also, some results on crystal vibrations. Another work on the theory of vibrations in piezoelectricity is H.F. Tiersten's book "Linear piezoelectric plate vibrations" published in 1969. In this work systematic development of the general form of differential equations of motion and boundary conditions from fundamental continuum concepts are given and some problems on plate vibrations are considered.

Modern developments of micro-electric-mechanical systems, miniaturized power sources and other devices, such as piezomotors, renewed the interest in the fundamental theory of linear piezoelectric materials, their applications [9, 17, 19, 22, 47, 50] and the development of new piezoelectric materials [8, 21, 25, 28]. The use of piezoelectric materials in micro power systems produces significant advantages due to their light weight, superior energy conversion efficiency and energy density [66]. One class of problems that are currently being investigated is related to the fracture of piezoelectric solids, including functionally graded piezoelectric materials [2, 39, 40, 56, 57, 65]. In [4]-[7] the problems of the effect of electro-mechanical coupling on the decay of Saint-Venant end effects in static linear piezoelectricity for the state of anti-plane shear are studied for different classes of piezoelectric materials. It should be noted that the majority of publications on the subject of piezoelectricity are written from the practical point of view, using both analytical and numerical techniques. Very few publications deal with the mathematical formulations and solutions of boundary value problems arising in the theory of piezoelectricity. It is well known that all piezoelectric materials are necessarily anisotropic which makes any analysis of the governing equations (equilibrium or motion) extremely difficult. This leads to the lack of the development of the corresponding mechanical-mathematical models that are available in the classical isotropic elasticity [44, 55, 80]. In most works the boundary value problems in classical elasticity deal with either two-dimensional or simple three-dimensional models. The work of V.D. Kupradze "Three-Dimensional Problems of Mathematical Theory of Elasticity and Thermoelasticity" provided a complete treatment of the three-dimensional boundary value problems arising in statics, problems of stationary oscillations, dynamic problems in the classical elasticity, thermoelasticity and couple-stress elasticity. Here boundary value problems are solved by means of the theory of singular integral equations, employing the method of potentials. In [68]-[71], [15] this approach was applied to some problems in the classical plane elasticity and micropolar elasticity. The same method was used by D. Iesan to formulate uniqueness and existence theorems for the generalized plane strain state in the linear static piezoelectricity for the piezoelectric material possessing cubic $\bar{4}3m$ symmetry [27]. This analysis though overlooks some differentiability properties of the solutions of the integral equations.

Anti-plane shear deformation is one of the simplest classes of deformations that a solid body can undergo. In the classical theory of elasticity the state of anti-plane shear in an arbitrary cylindrical body corresponds to the case when the only non-zero component of mechanical displacement is the anti-plane component, parallel to the generators of the cylinder, which is independent of the out-of-plane coordinate. In the linear theory of piezoelectricity the anti-plane shear is the state when the mechanical displacement vector takes the same form as in the purely elastic case and is accompanied by the presence of two in-plane components of an electric field vector, independent of the out-of-plane coordinate whereas the third, anti-plane component of the electric field vector, vanishes. Let us now assume that the cylindrical body is subjected to the surface stresses that act in the plane normal to the generators of the cylinder, i.e. the stress vector component t_3 is zero with x_3 being the axis parallel to the generators, and do not vary along the generators. In this case, an isotropic body would experience plane deformation, i.e. the displacement along the generators vanishes or is constant, while two in-plane components of the displacement are independent of the out-of-plane coordinate. For an anisotropic body, however, we cannot assume that the third component of the displacement vanishes as we would have overdetermined system of governing equations. Therefore we can only expect all the displacement components to be independent of the third coordinate due to the form of the boundary conditions and the deformation is then called generalized plane deformation [37]. In the context of piezoelectricity, the generalized plane strain state corresponds to the case when all the field quantities: three displacement components and electric potential, are independent of the out-of-plane coordinate [27]. Although both anti-plane shear and generalized plane strain mathematically are two-dimensional models they have a variety of applications. For example, the problems that reduce to the state of anti-plane shear are those of static torsion, or torsional vibrations of the cylindrical piezoelectric body [30]; the case of a general load applied to the generators of a cylindrical body with arbitrary cross-section can be described by equations of the generalized plane strain. The governing equations for the state of anti-plane shear were established in different constitutive theories, for example, in nonlinear elasticity [24], static piezoelectricity [4]-[7], micropolar elasticity [61, 62]. The state of the generalized plane strain was only studied for the cubic $\bar{4}3m$ in the context of the static linear piezoelectricity [27]. Here, the governing equations decouple into in-plane and anti-plane systems. Consequently, no attention is paid to purely anti-plane deformations for piezoelectric materials with more general symmetry properties. Since a variety of piezoelectric devices operate on resonant frequencies such as piezoelectric transformers, actuators, resonators and etc [12, 20, 72] the investigation of the nature of steady-state vibrations is also of great importance.

The objective of the present work is to formulate and solve rigorously the fundamental Dirichlet, Neumann and mixed boundary value problems for the anti-plane shear state in the static theory of piezoelectricity, and Dirichlet and Neumann boundary value problems for steady-state vibrations in piezoelectric generalized plane strain. Our tool for the analysis is an analytical technique known as the boundary integral equation method. The advantage of this technique is that it allows us to establish the existence of the solutions of the boundary value problems, so that numerical procedures can be applied to solve particular problems arising in the context of the anti-plane and generalized plane strain states in linear piezoelectricity. Boundary value problems for the static anti-plane shear state are considered for the piezoelectric materials of rather general symmetry, tetragonal $\bar{4}$ class. The choice of this more general symmetry type is motivated by the variety of newly developed piezoelectric materials that exhibit tetragonal symmetry [8, 21, 25, 28, 88]. The solutions of these problems have applications in different areas where the models of anti-plane shear are applicable [24]. The problems of steady-state vibrations are considered for hexagonal $6mm$ class (for the case of the unbounded domain) and tetragonal $\bar{4}$ class (for bounded domain).

The thesis is organized as follows. In Chapter 2 we give a brief description of the quasi-static approach and derivation of the general form of the equations of motion for arbitrary piezoelectric solids. In Chapter 3 we derive the equations for the static anti-plane shear state of linear piezoelectricity and formulate fundamental boundary value problems. Also, we mention the conditions on material properties of the piezoelectric body that must be met for the body to undergo the state of anti-plane shear. Using the boundary integral equation method, we prove existence and uniqueness theorems for the solutions of boundary value problems. We give closed form solutions of Dirichlet and Neumann boundary value problems in the form of single and double layer potentials.

In Chapter 4 we derive the equations of steady state vibrations of piezoelectric solids. The Dirichlet and Neumann boundary value problems for the unbounded domain are considered for the hexagonal $6mm$ symmetry class. In vibration problems for the unbounded domain we establish the radiation, or so-called Sommerfeld conditions, to guarantee the uniqueness of the solution. The case of the bounded domain is considered for the tetragonal $\bar{4}$ piezoelectric material. We prove the existence of the solutions for the resonance and non-resonance cases.

In Chapter 5 we give a brief summary of the obtained results and make suggestions for possible directions of future work.

Chapter 2

The basic equations of linear piezoelectricity

In this chapter we will give the fundamental equations of motion for the linear piezoelectric continuum and corresponding boundary conditions. We will also define anti-plane shear and generalized plane strain states in piezoelectricity and formulate the boundary value problems for certain symmetry classes that will be studied in the present work.

In what follows Greek indices take the values 1, 2, the convention of the summation over repeated indices is understood, $\mathcal{M}_{m\times n}$ is the space of $(m \times n)$-matrices, I is the identity element in $\mathcal{M}_{n\times n}$, a superscript T indicates matrix transposition and $(...)_{,\alpha} \equiv \partial(...)/\partial x_\alpha$. For $A \in \mathcal{M}_{m\times n}$ we denote the m-th row as $A_{(m)}$ and n-th column as $A^{(n)}$. Also, if X is a space of scalar functions and v is a matrix, $v \in X$ means that every component of v belongs to X. The notation $v \in C^{\alpha}(S)$ indicates that a function v belongs to the space of α-times continuously differentiable functions on S. If a function v defined on S is such that

$$|v(x) - v(y)| \leq C|x - y|^{\alpha} \qquad \text{for all } x, y \in S$$

for arbitrary $C = const$ and $\alpha \in (0, 1]$, then it is said to be Hölder continuous on S [14]; . We will denote a space of Hölder continuous on S functions with an index α by $C^{0,\alpha}(S)$ and by $C^{1,\alpha}(S)$ a space of functions on S whose first derivatives belong to $C^{0,\alpha}(S)$.

The derivation of the equations for the theory of linear piezoelectricity is based on the idea of coupling the quazi-static electric field and the dynamic mechanical motion in polarizable dielectrics (not magnetizable). To give an idea of the quasi-static approach used to derive governing equations for the theory of linear piezoelectricity we will first briefly mention several important concepts from the purely electromagnetic considerations.

One of the consequences of Maxwell's equations is Poynting's theorem [79] which gives us that for an arbitrary volume V bounded by a surface ∂V

$$\frac{1}{4\pi} \int_V \left(E \cdot \dot{D} + M \cdot \dot{B} \right) dV = - \int_{\partial V} n \cdot h ds - \int_V E \cdot J dV. \qquad (2.1)$$

Here E, D, M, B, n, $h = \frac{c}{4\pi} E \times M$, $J \in \mathcal{M}_{3\times 1}$ denote respectively, the electric field, electric displacement, magnetic field, magnetic flux vector, outward normal to the surface ∂V, Poynting's energy flux and current; c denotes the speed of light in vacuum. We will define the term

$$\dot{U}\frac{1}{4\pi}\left(E \cdot \dot{D} + M \cdot \dot{B}\right)$$

as the time rate of change of electromagnetic energy. The Poynting's theorem now can be interpreted as follows: the time rate of change of internal electromagnetic energy in an arbitrary volume is equal to negative of the rate of flow of electromagnetic energy through the surface enclosing the volume minus the rate of dissipation of electric energy by thermal means inside the volume. If we introduce the vector potential $A \in \mathcal{M}_{3\times 1}$ such that

$$B_i = \varepsilon_{ijk} A_{k,j}$$

and scalar potential ϕ such that

$$E_k = -\phi_{,k} - \frac{1}{c}\dot{A}_k,$$

Poynting's energy flux vector components become (for details see [79])

$$h_i = \phi\left(\frac{\dot{D}_i}{4\pi} + J_i\right) - \frac{1}{4\pi} e_{ijk} \dot{A}_j M_k. \tag{2.2}$$

The assumption on the electric potential ϕ made to make a quasi-static approximation is that for each component $\phi_{,i}$ the following condition is satisfied:

$$\left\| \frac{\dot{A}}{c} \right\| \ll \|\phi_{,i}\| \,. \tag{2.3}$$

The assumption (2.3) is valid when the electromagnetic waves uncouple from the elastic waves, and also when acoustic wavelengths are shorter than the electromagnetic wavelengths of the same frequency. Also, since we consider only polarizable dielectrics we can set

$$M = J = 0.$$

This will allow us to assume that the magnetic portion of Poynting's energy flux vector is negligible and, therefore, we can assume h in the form

$$h = \frac{\phi \dot{D}}{4\pi}. \tag{2.4}$$

Also, components of the electric field E_i are now defined simply by

$$E_i = -\phi_{,i}. \tag{2.5}$$

The equations (2.4) and (2.5) are the major consequences of the electrostatic approximation and will be used to derive the constitutive relations for polarizable dielectric medium. Now, with the use of (2.4), we can formulate the principle of energy conservation in an arbitrary volume occupied by a piezoelectric material

$$\frac{\partial}{\partial t} \int_V \left(\frac{1}{2} \rho \dot{u}_j \dot{u}_j + U \right) dV = \int_{\partial V} \left(t_j \dot{u}_j - n_j \phi \dot{D}_j \right) dS, \tag{2.6}$$

where u_j, t_j are, respectively, components of displacement and stress vector. The equality (2.6) states that change in total energy, kinetic and internal, is the rate at which work is done by the traction forces acting across ∂V minus the flux of electric energy out through ∂V.

By applying the divergence theorem to (2.6) we obtain the rate of change of internal energy $\dot{U}$

$$\dot{U} = \left(\tau_{ij,i} - \rho \ddot{u}_j \right) \dot{u}_j + \tau_{ij} \dot{u}_{j,i} - \phi \dot{D}_{i,i} - \phi_{,i} \dot{D}_i.$$

Here τ_{ij} are components of the stress tensor. Now, from the equations of motion from linear elasticity (in the absence of external forces)

$$\tau_{ij,i} - \rho \ddot{u}_j = 0$$

and the charge conservation equation (in the absence of external charge)

$$D_{i,i} = 0$$

we obtain the rate of change of internal energy

$$\dot{U} = \tau_{ij}\dot{u}_{i,j} - \phi_{,i}\dot{D}_i. \tag{2.7}$$

Next, we introduce the electric enthalpy function H

$$H = U + \phi_{,i}D_i. \tag{2.8}$$

If we differentiate (2.8) with respect to time, together with (2.7), we have

$$\dot{H} = \tau_{ij}\dot{u}_{i,j} + D_i\dot{\phi}_{,i}, \tag{2.9}$$

so we can assume that enthalpy $H = H(u_{i,j}, \phi_{,i})$. Now, for the $\dot{H}$ we obtain

$$\dot{H} = \frac{\partial H}{\partial u_{i,j}}\dot{u}_{i,j} + \frac{\partial H}{\partial \phi_{,i}}\dot{\phi}_{,i},$$

which together with (2.9) produces the identity

$$\left(\tau_{ij} - \frac{\partial H}{\partial u_{i,j}}\right)\dot{u}_{i,j} + \left(D_i - \frac{\partial H}{\partial \phi_{,i}}\right)\dot{\phi}_{,i} = 0$$

Since the above identity must hold for arbitrary $\dot{u}_{i,j}$ and $\dot{\phi}_{,i}$ we have that

$$\tau_{ij} = \frac{\partial H}{\partial u_{i,j}}, \qquad D_i = \frac{\partial H}{\partial \phi_{,i}}. \tag{2.10}$$

We assume the enthalpy function H in the form [79]

$$H = \frac{1}{2} C_{ijkl} u_{i,j} u_{k,l} + e_{kij} \phi_{,k} u_{i,j} - \frac{1}{2} \epsilon_{ij} \phi_{,i} \phi_{,j}, \tag{2.11}$$

where C_{ijkl}, e_{kij}, ϵ_{ij} are, respectively, elastic, piezoelectric and dielectric constants such that

$$\begin{aligned} C_{ijkl} &= C_{klij} = C_{lkij}, \\ e_{kij} &= e_{kji}, \\ \epsilon_{ij} &= \epsilon_{ji}. \end{aligned}$$

In a similar way as in [79], from (2.10) and (2.11) we obtain the constitutive relations of linear piezoelectricity

$$\begin{aligned} \tau_{ij} &= C_{ijkl} u_{k,l} + e_{kij} \phi_{,k}, \\ D_i &= e_{ikl} u_{k,l} - \epsilon_{ik} \phi_{,k}. \end{aligned} \tag{2.12}$$

Now, substitution of τ_{ij} from (2.12) into the equation of motion for a linearly elastic medium

$$\tau_{ij,j} = \rho \ddot{u}_i - f_i$$

and D_i from (2.12) into the charge equation

$$D_{i,i} = q$$

produces the equations that govern the behavior of a linearly piezoelectric continuum

$$\begin{aligned} C_{ijkl} u_{k,jl} + e_{kij} \phi_{,kj} &= \rho \ddot{u}_i - f_i, \\ e_{ikl} u_{k,il} - \epsilon_{ik} \phi_{,ik} &= q. \end{aligned} \tag{2.13}$$

By f_i we denote the components of the body force and by q the external charge. The equations (2.13) are accompanied by the set of mechanical and electrical boundary conditions. In the present work we will be dealing with several types of boundary conditions. First, we consider the boundary conditions when the displacement vector components and electric potential are prescribed on the boundary. The boundary value problem for the equations (2.13) accompanied by the this type of boundary conditions we will refer to as the Dirichlet boundary value problem. If the traction vector and surface charge are prescribed on the boundary we will have the Neumann boundary value problem. The last case of boundary conditions that we will consider is when Dirichlet boundary conditions are given on one portion of the boundary and Neumann conditions on the remaining portion of the boundary. The equations (2.13) together with this type of boundary conditions we will refer to as the mixed boundary value problem.

Chapter 3

Anti-plane shear state in linear piezoelectricity

3.1 Governing equations and boundary conditions

The anti-plane shear state will be considered in the frame of the static theory, i.e. all the field quantities are independent of time. Let Ω be an infinite cylinder $\Omega = \{\mathbf{x} \in \mathbb{R}^3 : (x_1, x_2) \in S\}$ where S is a simply-connected domain of $\mathbb{R}^2$ such that its boundary ∂S is C^2-curve. Let Ω be occupied by a homogeneous linearly piezoelectric material. The lateral surfaces of the cylinder are loaded as shown in the Figure 3.1. As in [4] we define a state of anti-plane shear for the piezoelectric body by requiring that the components of the displacement vector and the electric potential take the form:

$$\begin{cases} u_1 &= 0, \\ u_2 &= 0, \\ u_3 &= u_3(x_1, x_2), \\ \phi &= \phi(x_1, x_2), \qquad \forall \quad x_1, x_2 \in S. \end{cases} \tag{3.1.1}$$

If we now substitute (3.1.1) back into the governing equations (2.13) we obtain

$$\begin{cases} C_{i\alpha 3\beta} u_{3,\alpha\beta} + e_{\nu i\alpha}\phi_{,\nu\alpha} &= -f_3, \\ -e_{\gamma i\alpha} u_{3,\gamma\alpha} + \epsilon_{\alpha\beta}\phi_{,\alpha\beta} &= q \quad \text{in } S. \end{cases} \tag{3.1.2}$$

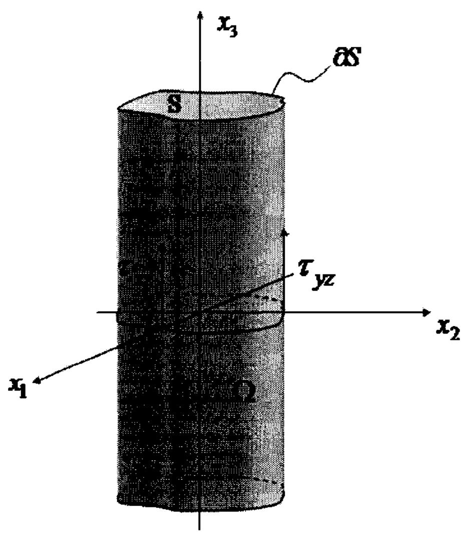

Figure 3.1: Piezoelectric cylinder under an anti-plane shear loading

We note here that now we have two unknowns but the number of the equations remains four, i.e. the system (3.1.2) is overdetermined, in general. Consequently, an arbitrary piezoelectric body will not, in general, sustain an anti-plane shear state (u_3, ϕ). The same is true for the purely mechanical problem [24]. However, as in [4], it can be shown that a sufficient condition for a non-trivial state of anti-plane shear to exist is given by

$$C_{\gamma\alpha 3\beta} = 0, \quad e_{\nu\gamma\alpha} = 0. \tag{3.1.3}$$

The conditions (3.1.3) imply that we consider only those piezoelectric materials in which there is no in-plane stress produced, i.e. τ_{11}, τ_{12}, τ_{22} are zero, by anti-plane shear strains ε_{13}, ε_{23} and in-plane components of the electric field E_1 and E_2.

In this case the equations of equilibrium become

$$\begin{cases} C_{3\alpha 3\beta} u_{3,\alpha\beta} + e_{\nu 3\alpha} \phi_{,\nu\alpha} &= -f_3, \\ -e_{\gamma 3\alpha} u_{3,\gamma\alpha} + \epsilon_{\alpha\beta} \phi_{,\alpha\beta} &= q, \text{ in } S. \end{cases} \tag{3.1.4}$$

The appropriate boundary conditions for (3.1.4) are given by [4]

$$\begin{aligned} (C_{3\alpha 3\beta} u_{3,\beta} + e_{\nu 3\alpha} \phi_{,\nu}) n_\alpha &= t_3^* (x_1, x_2), \\ -(e_{\gamma 3\alpha} u_{3,\gamma} - \epsilon_{\alpha\beta} \phi_{,\beta}) n_\alpha &= D^*(x_1, x_2) \quad \text{on } \partial S, \end{aligned} \tag{3.1.5}$$

in the case of the Neumann problem, and by

$$\begin{cases} u_3 &= u_3^*(x_1, x_2) \\ \phi &= \phi^*(x_1, x_2) \quad \text{on } \partial S, \end{cases} \tag{3.1.6}$$

in the case of the Dirichlet problem. Here D^*, t_3^*, u_3^*, ϕ^* are prescribed functions on ∂S and n_α are the components of the outward unit normal n to ∂S.

To write the mixed boundary conditions we divide ∂S into two (for simplicity) arcs ∂S_1 and ∂S_2 with common endpoints a and b. The case when ∂S is divided into more than two parts is relatively straightforward and proceeds as in [33]. As in [33], the set $\chi = \{a, b\}$ is included in ∂S_1, so that ∂S_2 is taken as an open arc and ∂S_1 as a closed one. Let us introduce the differential "stress" operator $T = T(\partial x, n)$ which describes Neumann type boundary conditions:

$$T = T(\partial x, n) = \begin{bmatrix} C_{3\alpha 3\beta}\xi_\beta n_\alpha & e_{\nu 3\alpha}\xi_\nu n_\alpha \\ -e_{\gamma 3\alpha}\xi_\gamma n_\alpha & \epsilon_{\alpha\beta}\xi_\beta n_\alpha \end{bmatrix}.$$

By ξ_α we denote $\frac{\partial}{\partial x_\alpha^\beta}$. Let us denote by $u = u(x) \in \mathcal{M}_{2\times 1}$ such that $u(x) = (u_3(x), \phi(x))^T$. Now mixed boundary conditions become:

$$\begin{cases} u(x) &= u^*(x) \qquad x \in \partial S_1 \\ Tu(x) &= t^*(x) \qquad x \in \partial S_2 \end{cases} \tag{3.1.7}$$

Let us introduce the differential operator $L = L(\partial x)$ corresponding to the equations (3.1.4):

$$L = L(\partial x) = \begin{bmatrix} C_{3\alpha 3\beta}\xi_\alpha\xi_\beta & e_{\nu 3\alpha}\xi_\nu\xi_\alpha \\ -e_{\gamma 3\alpha}\xi_\gamma\xi_\alpha & \epsilon_{\alpha\beta}\xi_\alpha\xi_\beta \end{bmatrix} \tag{3.1.8}$$

In the particular case of tetragonal crystal symmetry $\bar{4}$ the differential operator

L becomes

$$L = L_{\bar{4}}(\partial x) = \begin{bmatrix} C_{44}(\xi_1^2+\xi_2^2) & e_{15}(\xi_1^2-\xi_2^2)+2e_{14}\xi_1\xi_2 \\ -e_{15}(\xi_1^2-\xi_2^2)-2e_{14}\xi_1\xi_2 & \epsilon(\xi_1^2+\xi_2^2) \end{bmatrix}, \tag{3.1.9}$$

and the "stress" operator T takes the form

$$T = T_{\bar{4}}(\partial x, n) =$$

$$= \begin{bmatrix} C_{44}(n_1\xi_1+n_2\xi_2) & e_{15}(n_1\xi_1-n_2\xi_2)+e_{14}(n_2\xi_1+n_1\xi_2) \\ -e_{15}(n_1\xi_1-n_2\xi_2)-e_{14}(n_2\xi_1+n_2\xi_2) & \epsilon(n_1\xi_1+n_2\xi_2) \end{bmatrix}, \tag{3.1.10}$$

where

$$\begin{aligned} C_{44} &= C_{3131} = C_{3232}, \\ e_{14} &= e_{132} = e_{231}, \quad e_{15} = e_{131} = -e_{232}, \\ \epsilon &= \epsilon_{11} = \epsilon_{22}. \end{aligned}$$

For the cubic $\bar{4}3m$ crystal class the operators L and T take the form [27]:

$$L = L_{\bar{4}3m}(\partial x) = \begin{bmatrix} C_{44}(\xi_1^2+\xi_2^2) & 2e_{14}\xi_1\xi_2 \\ -2e_{14}\xi_1\xi_2 & \epsilon(\xi_1^2+\xi_2^2) \end{bmatrix}, \tag{3.1.11}$$

$$T = T_{\bar{4}3m}(\partial x, n) = \begin{bmatrix} C_{44}(n_1\xi_1+n_2\xi_2) & e_{14}(n_2\xi_1+n_1\xi_2) \\ -e_{14}(n_2\xi_1+n_2\xi_2) & \epsilon(n_1\xi_1+n_2\xi_2) \end{bmatrix}, \tag{3.1.12}$$

where

$$\begin{aligned} C_{44} &= C_{3131} = C_{3232}, \\ e_{14} &= e_{132} = e_{231}, \quad e_{131} = e_{232} = 0, \\ \epsilon &= \epsilon_{11} = \epsilon_{22}. \end{aligned}$$

For convenience, we introduce the following notations for the system of equations (3.1.4):

$$\begin{aligned} A_{11\alpha\beta} &= C_{3\alpha 3\beta} \\ A_{12\alpha\beta} &= e_{\alpha 3\beta} \\ A_{21\alpha\beta} &= -e_{\alpha 3\beta} \\ A_{22\alpha\beta} &= \epsilon_{\alpha\beta} \\ u_1(x_1, x_2) &= u_3(x_1, x_2) \\ u_2(x_1, x_2) &= \phi(x_1, x_2) \end{aligned} \tag{3.1.13}$$

Now we can write the operator $L(\partial x)$ for both cases of material symmetry as:

$$L(\partial x) = \{A_{\lambda\mu\alpha\beta}\xi_\alpha\xi_\beta\}.$$

Similarly, for the stress operator $T(\partial x, n)$ we have

$$T(\partial x, n) = \{A_{\lambda\mu\alpha\beta}\xi_\beta n_\alpha\}.$$

Let S^+ be the bounded domain enclosed by ∂S. Also, we define S^- as following

$$S^- = \lim_{R\to\infty} \left(\Gamma_R \setminus S^+\right),$$

where Γ_R is a circle in $\mathbb{R}^2$ of sufficiently large raduis R. By $\bar{S}^+$ and $\bar{S}^-$ we will denote $S^+ \bigcup \partial S$ and $S^- \bigcup \partial S$, respectively, i.e. regions containing S^+ or S^-, respectively, together with the boundary ∂S.

Remark 1 *The following assertions are true for both tetragonal* $\bar{4}$ *and cubic* $\bar{4}3m$ *classes:*

(1) $L(\partial x)$ *is elliptic if* C_{44} *and* ϵ *are of the same sign. Henceforth, we assume that* C_{44}, ϵ *are both positive.*

(2) The internal energy takes the form [79]:

$$U(u,u) = C_{3\alpha 3\beta} u_{,\beta} u_{,\alpha} + \epsilon_{\alpha\beta} \phi_{,\beta} \phi_{,\alpha}, \tag{3.1.14}$$

and, specifically for both classes, we have

$$U(u,u) = \frac{1}{2} C_{44}(u_{,1}^2 + u_{,2}^2) + \frac{1}{2}\epsilon(\phi_{,1}^2 + \phi_{,2}^2),$$

and $U(u,u) = 0$ *if and only if*

$$u = (u, \phi)^T = (u_1, u_2)^T = (c_1, c_2)^T, \tag{3.1.15}$$

where c_α *are arbitrary constants.*

(3) As in [14], we can show that (Betti Formula) if $u \in C^2(S^+) \bigcap C^1(\overline{S}^+) \bigcap \mathcal{M}_{2\times 1}$ *is a solution of the system* (3.1.4) *(with the absence of body force and external charge) in* S^+ *then:*

$$2\int_{S^+} U(u,u)dA = \int_{\partial S} u^T T u ds,$$

and

$$\int_{S^+} L_{\alpha\beta} u_\beta dA = \int_{\partial S} T_{\alpha\beta} u_\beta ds.$$

We can also establish the Reciprocity Relation: if $u,\, v \in C^2(S^+) \bigcap C^1(\overline{S}^+) \bigcap \mathcal{M}_{2\times 1}$ *then*

$$\int_{S^+} (v^T L u - u^T L v) dA = \int_{\partial S} (v^T T u - u^T T v) ds.$$

3.2 Fundamental and singular solutions

In this section we will construct matrices of fundamental and singular solutions. In the classical elasticity theory the fundamental solutions correspond to the displacement field due to the force applied at a point. These are known as Kelvin point-load solutions. In piezoelectricity, where the field quantities are the displacement vector and the electric potential, the fundamental solutions correspond to the displacement field and electric potential due to the point-wise applied force and the electric charge. One of the important properties of fundamental solutions for the application of the boundary integral equation method is that they satisfy the governing equations. This will be used to construct the single and double layer potentials in Section 3.5. To find the matrix of fundamental solutions we will use the Galerkin representation of the solution.

We introduce $L^*(\partial x) \in \mathcal{M}_{2\times 2}$, the transposed matrix of cofactors of $L(\partial x)$, such that

$$L^*_{\lambda,\mu} = B_{\lambda\mu\gamma\delta}\xi_\gamma\xi_\delta,$$

where the coefficients $B_{\lambda\mu\gamma\delta}$ are

$$\begin{aligned} B_{11\gamma\delta} &= \epsilon_{\gamma\delta}, \\ B_{12\gamma\delta} &= -e_{\gamma 3\delta}, \\ B_{21\gamma\delta} &= e_{\gamma 3\delta}, \\ B_{22\gamma\delta} &= C_{3\gamma 3\delta}. \end{aligned} \tag{3.2.1}$$

Clearly, we can write

$$LL^* = \sum_{\mu=1}^{2} A_{\kappa\mu\alpha\beta} B_{\lambda\mu\gamma\delta} \xi_\alpha \xi_\beta \xi_\gamma \xi_\delta = \delta_{\kappa\lambda} \det L. \tag{3.2.2}$$

We are looking for the Galerkin representation of the solution of the system (3.1.4) in the form

$$u(x) = L^* g,$$

where $g \in \mathcal{M}_{2\times 1}$. Substitute now $u(x)$ into the equation

$$L(\partial x) u(x) = -d,$$

where $d \in \mathcal{M}_{2\times 1}$ is such that first we take d_1 to be equal $\delta(x-y)$, Dirac distribution, and d_2 equal to 0, and then $d_1 = 0$ and $d_2 = \delta(x-y)$. We obtain

$$L(\partial x) L^*(\partial x) g = \det L(\partial x) g = -d.$$

Clearly, first we can take g_1 to be equal to some function $\psi = \psi(x,y)$ and $g_2 = 0$, and then take $g_1 = 0$, $g_2 = \delta(x-y)$. Thus we now have to solve one equation:

$$\det L(\partial x) \psi(x,y) = -\delta(x-y),$$

which will give us the matrix of fundamental solutions $\Gamma(x, y)$ such that

$$\Gamma(x,y) = L^*(\partial x)\psi(x,y). \tag{3.2.3}$$

From [38] we find that the unknown function $\psi(x, y)$ is given by

$$\psi(x,y) = -ia \sum_{j=1}^{4} (-1)^j d_j \sigma_j^2 \log \sigma_j, \tag{3.2.4}$$

where

$$a = \frac{1}{4\pi\epsilon C_{44}}, \qquad \sigma_j = \alpha_j(x_1 - y_1) + (x_2 - y_2), \qquad j = 1, ..., 4;$$

d_j is the cofactor of α_j^3 in the determinant $\tilde{D}$, divided by $\tilde{D}$, where $\tilde{D}$ is given by

$$\tilde{D} = \begin{vmatrix} \alpha_1^3 & \alpha_1^2 & \alpha_1 & 1 \\ \alpha_2^3 & \alpha_2^2 & \alpha_2 & 1 \\ \alpha_3^3 & \alpha_3^2 & \alpha_3 & 1 \\ \alpha_4^3 & \alpha_4^2 & \alpha_4 & 1 \end{vmatrix},$$

and α_j are the roots of the characteristic equation corresponding to the differential operator $L(\partial x)$. In the case of cubic symmetry $\bar{4}3m$ we have the characteristic equation for the operator $L(\partial x)$

$$\epsilon C_{44}\alpha^4 + (2\epsilon C_{44} + 4e^2)\alpha^2 + \epsilon C_{44} = 0$$

and the roots are found to be

$$\alpha_{1,2} = i\sqrt{\frac{b \pm \sqrt{b^2 - 4}}{2}}, \quad \alpha_{3,4} = -i\sqrt{\frac{b \pm \sqrt{b^2 - 4}}{2}}, \quad \text{where} \quad b = \frac{2\epsilon C_{44} + 4e_{14}^2}{\epsilon C_{44}}.$$

The corresponding characteristic equation for $L(\partial x)$ in the case of tetragonal $\bar{4}$ symmetry becomes

$$\epsilon C_{44}(\alpha^2+1)^2 + (e_{15}(\alpha^2-1) - 2e_{14}\alpha)^2 = 0$$

with roots

$$\alpha_{1,2} = \frac{-2i\mu \pm \sqrt{-4\mu^2 - 4(l^2+1)}}{2(l-i)}, \qquad \alpha_{3,4} = \frac{2i\mu \pm \sqrt{-4\mu^2 - 4(l^2+1)}}{2(l+i)},$$

where

$$l = \sqrt{C_{44}\epsilon}, \qquad \mu = \frac{e_{14}}{e_{15}}.$$

We note here that in both cases $\alpha_{1,2}$ are complex conjugate of $\alpha_{3,4}$; the function $\psi(x,y)$ is a real-valued function. Now, the substitution of the function $\psi(x,y)$ given by (3.2.4) into the equation (3.2.3) gives us the components of the matrix of fundamental solutions

$$\begin{aligned}\Gamma_{\lambda\mu}(x,y) &= \textstyle\sum_{\gamma,\delta=1}^{2} B_{\lambda\mu\gamma\delta}\frac{\partial^2\psi}{\partial x_\gamma \partial x_\delta} = \\ &= \textstyle\sum_{j=1}^{4}\sum_{\gamma,\delta=1}^{2} ai(-1)^j d_j B_{\lambda\mu\gamma\delta}\alpha_j^{4-(\gamma+\delta)}(2\log\sigma_j + 3).\end{aligned} \tag{3.2.5}$$

Next, we introduce the matrix of singular solutions $D = D(x,y)$

$$D(x,y) = (T(\partial x, n)\Gamma(y,x))^T.$$

Following the procedure similar to the one used [27] we will make some transformations to write $D(x,y)$ in the more convenient form. First, we write the components of the matrix $T(\partial x, n)$ in the form

$$T_{\kappa\mu}(\partial x, n) = \sum_{\mu=1}^{2} A_{\kappa\mu\alpha\beta}\frac{\partial}{\partial x_\beta}n_\alpha.$$

Now we can write

$$T_\kappa(\partial x, n)\Gamma^{(\lambda)}(x,y) = \sum_{\mu=1}^{2} A_{\kappa\mu\alpha\beta}\Gamma_{\mu\lambda,\beta}n_\alpha = \tag{3.2.6}$$

$$\sum_{j=1}^{4}\sum_{\mu=1}^{2}\sum_{\gamma,\delta=1}^{2} 2ai(-1)^j d_j B_{\lambda\mu\gamma\delta}\alpha^{6-(\beta+\gamma+\delta)}\frac{1}{\sigma_j}A_{\kappa\mu\alpha\beta}n_\alpha.$$

Bearing in mind (3.2.2) we recall that the characteristic equation can be written

(λ is not summed) as follows

$$\sum_{\mu=1}^{2}\sum_{\beta,\nu,\gamma,\delta=1}^{2} A_{\lambda mu\nu\beta}B_{\lambda\mu\gamma\delta}\alpha^{8-(\nu+\beta+\gamma+\delta)} = 0,$$

or, we can write

$$\sum_{\mu=1}^{2}\sum_{\beta,\nu,\gamma,\delta=1}^{2} A_{1\mu1\beta}B_{1\mu\gamma\delta}\alpha^{7-(\nu+\beta+\gamma+\delta)} +$$

$$+\sum_{\mu=1}^{2}\sum_{\beta,\nu,\gamma,\delta=1}^{2} A_{1\mu2\beta}B_{1\mu\gamma\delta}\alpha^{6-(\nu+\beta+\gamma+\delta)} = 0.$$

So we have

$$\sum_{\mu=1}^{2}\sum_{\beta,\nu,\gamma,\delta=1}^{2} A_{1\mu1\beta}B_{1\mu\gamma\delta}\alpha^{7-(\nu+\beta+\gamma+\delta)} =$$

$$-\sum_{\mu=1}^{2}\sum_{\beta,\nu,\gamma,\delta=1}^{2} A_{1\mu2\beta}B_{1\mu\gamma\delta}\alpha^{6-(\nu+\beta+\gamma+\delta)}. \tag{3.2.7}$$

Substitution of (3.2.7) into (3.2.6) produces

$$T_\kappa(\partial x, n)\Gamma^{(\lambda)}(x,y) = \sum_{j=1}^{4}\sum_{\mu=1}^{2}\sum_{\gamma,\delta=1}^{2}\Big(A_{\kappa\mu1\beta}B_{\lambda\mu\gamma\delta}\alpha^{6-(\nu+\beta+\gamma+\delta)}+$$

$$+A_{\kappa\mu2\beta}B_{\lambda\mu\gamma\delta}\alpha^{6-(\nu+\beta+\gamma+\delta)}\Big)\, 2ai(-1)^j d_j \frac{1}{\sigma_j} n_\alpha =$$

$$= \sum_{j=1}^{4}\sum_{\mu=1}^{2}\sum_{\gamma,\delta=1}^{2}\Big(A_{\kappa\mu1\beta}B_{\lambda\mu\gamma\delta}\alpha^{6-(\nu+\beta+\gamma+\delta)}-$$

$$-\alpha_j A_{\kappa\mu1\beta}B_{\lambda\mu\gamma\delta}\alpha^{6-(\nu+\beta+\gamma+\delta)}\Big)\, 2ai(-1)^j d_j \frac{1}{\sigma_j} n_\alpha =$$

$$= \sum_{j=1}^{4}\sum_{\mu=1}^{2}\sum_{\gamma,\delta=1}^{2} A_{\kappa\mu1\beta}B_{\lambda\mu\gamma\delta}\alpha^{6-(\nu+\beta+\gamma+\delta)}(n_1-\alpha_j n_2)2ai(-1)^j d_j \frac{1}{\sigma_j}. \tag{3.2.8}$$

Following [33, 27] we write

$$\log\sigma_j = \log\frac{\sigma_j}{\sigma} - \frac{1}{2}\log\frac{\bar\sigma}{\sigma} + \ln r, \tag{3.2.9}$$

where

$$\sigma = i(x_1-y_1)+(x_2-y_2), \qquad r^2 = (x_1-y_1)^2+(x_2-y_2)^2.$$

Having noticed that

$$(n_1-\alpha_j n_2)\frac{1}{\sigma_j} = \frac{\partial\log\sigma_j}{\partial s_x}$$

we can write

$$\frac{\partial}{\partial s_x}\log\frac{\sigma_j}{\sigma} = \frac{(i-\alpha_j)r^2}{\sigma\sigma_j}\frac{\partial\ln r}{\partial n_x}.$$

Finally, (3.2.6) becomes

$$D_{\lambda\kappa}(x,y) = T_{(\kappa)}(\partial x, n)\Gamma^{(\lambda)}(x,y) = \sum_{j=1}^{4} R^{j}_{\kappa\lambda}\left[\frac{\partial \ln r}{\partial s_x} + \left(\frac{(i-\alpha_j)r^2}{\sigma\sigma_j} - i\right)\frac{\partial \ln r}{\partial n_x}\right], \tag{3.2.10}$$

where coefficients $R^{j}_{\kappa\lambda}$ are given by

$$R^{j}_{\kappa\lambda} = 2ai(-1)^j d_j \sum_{\mu=1}^{2}\sum_{\beta,\gamma,\delta=1}^{2} A_{\kappa\mu 1\beta}B_{\lambda\mu 1\beta}\alpha_j^{6-(\beta+\gamma+\delta)}.$$

Remark 2 *We note here that the columns of* $\Gamma(x,y)$ *and* $D(x,y)$ *are solutions of the system* (3.1.4) *for all* $x \in \mathbb{R}^2 : x \neq y$ *and for the normal vector* $n \in \mathcal{M}_{2\times 1}$ *independent of* x. *To see this, first we recall that*

$$L(\partial x)\Gamma(x,y) = L(\partial x)L^{*}(\partial x)\psi(x,y) = I\det L(\partial x)\psi(x,y) = 0.$$

Using the definition of the matrix $D(x,y)$ *we find*

$$L(\partial x)D(x,y) = L(\partial x)\left(T(\partial y)\Gamma(x,y)\right)^T = L(\partial x)\Gamma^T(x,y)T^T(\partial y,n) =$$

$$= L_{\alpha\beta}(\partial x)\left((-1)^{(\beta+\gamma)}\Gamma_{\beta\gamma}(x,y)\right)(-1)^{(\gamma+\delta)}T_{\gamma\delta}(\partial y,n) =$$

$$= (-1)^{(\alpha+2\gamma+1)}L_{\alpha\beta}(\partial x)\Gamma_{\beta\gamma}(x,y)(-1)^{(\gamma+\delta)}T_{\gamma\delta}(\partial y,n) =$$

$$= (-1)^{(\alpha+2\gamma+1)}L_{\alpha\beta}(\partial x)L^{*}_{\beta\gamma}(\partial x)\psi(x,y)(-1)^{(\gamma+\delta)}T_{\gamma\delta}(\partial y,n) =$$

$$= (-1)^{(\alpha+2\gamma+1)}\delta_{\alpha\gamma}\det L(\partial x)\psi(x,y)(-1)^{(\gamma+\delta)}T_{\gamma\delta}(\partial y,n) = 0.$$

3.3 Representation formulae for the bounded domain

Now we are in a position to derive the representation formulae for the solutions of (3.1.4) for the interior domain. We will use these results in Section 3.5 to investigate the behavior of the single and double layer potentials in S^+ and its boundary ∂S.

Theorem 3 *(Representation formulae) If* $u \in C^2(S^+) \bigcap C^1\left(\bar{S}^+\right)$ *is a solution of the homogeneous system* (3.1.4), *then*

$$\int_{\partial S} [\Gamma(x,y)T(\partial y)u(y) - D(x,y)u(y)]\,ds(y) = \begin{cases} I_\omega u(x) \quad , & x \in S^+, \\ \frac{1}{2} I_\omega u(x) \quad , & x \in \partial S, \\ 0, & x \in S^-, \end{cases}$$

where

$$I_\omega = -2i\pi \sum_{j=1}^{4} R^j.$$

Proof. Let $\sigma_{x,\delta}$ be a disk lying inside S^+ with the centre at $x = (x_1, x_2)$ and with sufficiently small radius δ. Applying Reciprocity relation (Remark 1) in $S^+ \setminus \sigma_{x,\delta}$ taking each time v be equal to the columns of the matrix $\Gamma(x,y)$ and bearing in mind Remark 2 we obtain

$$\begin{aligned} 0 = &\int_{\partial S} \left[\Gamma^{(\mu)}(x,y)T(\partial y, n)u(y) - D_{(\mu)}(x,y)u(y)\right] ds(y) - \\ &- \int_{\partial \sigma_{x,\delta}} \left[\Gamma^{(\mu)}(x,y)T(\partial y, n)u(y) - D_{(\mu)}(x,y)u(y)\right] ds(y), \end{aligned}$$

where $\partial \sigma_{x,\delta}$ is the boundary of $\sigma_{x,\delta}$. From (3.2.5) we conclude that

$$\lim_{\delta \to 0} \int_{\partial \sigma_{x,\delta}} \Gamma^{(\mu)}(x,y)T(\partial y, n)u(y)ds(y) = \lim_{\delta \to 0} \int_0^{2\pi} \ln \delta M_1 \delta d\theta = M_2 \lim_{\delta \to 0} \delta \ln \delta = 0.$$

Let us now consider the second integral over $\partial\sigma_{x,\delta}$. We can write

$$\int_{\partial\sigma_{x,\delta}} D_{(\mu)}(x,y)u(y)ds(y) = \int_{\partial\sigma_{x,\delta}} D_{(\mu)}(x,y)(u(y)-u(x))ds(y) +$$

$$+u(x)\int_{\partial\sigma_{x,\delta}} D_{(\mu)}(x,y)ds(y).$$

From (3.2.8) we see that each component of $D(x,y)$ is $O(\frac{1}{\delta})$ for $y \in \partial\sigma_{x,\delta}$.

Since $u \in C^{0,\alpha}(S^+ \bigcup \partial S)$ and $ds(y)$ is $O(\delta)$ then for $y \in \partial\sigma_{x,\delta}$ we have

$$\int_{\partial\sigma_{x,\delta}} D_{(\mu)}(x,y)(u(y)-u(x))ds(y) = \lim_{\delta\to 0} M_3 \int_{\partial\sigma_{x,\delta}} \frac{1}{\delta}\delta ds(y) = M_4 \lim_{\delta\to 0}\delta = 0,$$

here M_j, $j = 1,...,4$ are arbitrary constants. Thus we obtain that

$$\lim_{\delta\to 0}\int_{\partial\sigma_{x,\delta}} D_{(\mu)}(x,y)u(y)ds(y) = u(x)\lim_{\delta\to 0}\int_{\partial\sigma_{x,\delta}} D_{(\mu)}(x,y)ds(y) = I_{\omega(\mu)}u(x).$$

Using the results established in [14], the components of the matrix I_ω are calculated to be

$$I_{\omega(\mu\kappa)} = \sum_{j=1}^{4} R^j_{\kappa\mu}\int_{\partial\sigma_{x,\delta}} \left[\frac{\partial \ln r}{\partial s_x} + \left(\frac{(i-\alpha_j)r^2}{\sigma\sigma_j} - i\right)\frac{\partial \ln r}{\partial n_x}\right] ds(y) = -2i\pi\sum_{j=1}^{4} R^j_{\kappa\mu}.$$

For the case when $x \in \partial S$ the same procedure is applied for the domain $S^+ \backslash \tilde{\sigma}_{x,\delta}$, where $\tilde{\sigma}_{x,\delta}$ is the part of $\sigma_{x,\delta}$ lying inside S^+. Thus, instead of $\partial\sigma_{x,\delta}$ we will have its part $\partial\tilde{\sigma}_{x,\delta}$ in S^+. It was shown [23] that in the case of a Lyapunov curve ∂S the length of $\partial\tilde{\sigma}_{x,\delta}$ is $\pi\delta$.

For $x \in S^-$ the result follows directly with the use of Remark 2.

3.4 Representation formulae for the unbounded domain

We recall that we define S^- as follows

$$S^- = \lim_{R \to \infty} \left(\Gamma_R \setminus S^+\right).$$

To establish the representation formulae in an unbounded domain, as in Theorem 3, we will apply the Reciprocity relation in S^-. However, in this case, instead of the boundary ∂S we will have $\partial S \bigcup \partial \Gamma_R$ and the integral over $\partial \Gamma_R$ becomes divergent as $R \to \infty$. Thus, the representation of the solution of boundary value problems for (3.1.4) in the unbounded domain requires the restriction of the behavior of the solution when $x = (x_1, x_2)^T$ approaches infinity. To this end we want to define a special class of functions $\mathcal{A}$ of vectors $u(y) \in \mathcal{M}_{2\times 1}$ whose asymptotic expansion in polar coordinates (r, θ) has a form

$$\begin{aligned} u_\alpha(r,\theta) &= r^{-1}(a_2^{(\alpha)} \cos\theta - a_3^{(\alpha)} \sin\theta - M_j^2 b_2^{(\alpha)}(\sin\theta + p_j \cos\theta) + \\ &+ M_j^2 b_3^{(\alpha)} q_j^2 \cos\theta) + O(r^{-2}) \end{aligned}$$

where $a_i^{(\alpha)}, b_i^{(\alpha)}$, $i = 1..3$, are arbitrary constants; p_j, q_j are real and imaginary parts, respectively, of α_j and

$$M_j = \frac{1}{\sin\theta^2 + 2p_j \sin\theta\cos\theta + (p_j^2 + q_j^2)\cos\theta^2}.$$

Let $\tilde{\mathcal{A}}$ be the set of vectors $u(y) \in \mathcal{M}_{2\times 1}$ defined as

$$\tilde{\mathcal{A}} = \{\tilde{u} : \tilde{u} = u + u_0; \quad u \in \mathcal{A}, \quad u_0 : \quad E(u_0, u_0) = 0\}.$$

Let y be a fixed point and let $|x| \to \infty$. In this case

$$\ln|x-y| = \ln\left(\frac{|x-y||x|}{|x|}\right) = \ln|x| + \ln\frac{|x-y|}{|x|} =$$

$$\ln|x| + \frac{1}{2}\ln\frac{|x|^2+|y|^2-2(x,y)}{|x|^2} = \ln|x| + \ln(1+\varepsilon),$$

$$\text{where} \quad \varepsilon = \frac{|y|^2+2(x,y)}{|x|^2} = O(\frac{1}{|x|}).$$

From (3.2.9) we conclude that $\log\sigma_j = O(\ln|x-y|)$. Similarly, for $|x| \to \infty$ we write

$$\frac{1}{|\sigma_j|} = O\left(\frac{1}{|x-y|}\right) = O\left(\frac{1}{|x|}\right). \tag{3.4.1}$$

By making use of these results we can write asymptotic properties for the components of matrices $\Gamma(x,y)$ and $D(x,y)$:

$$\Gamma_{\alpha\beta}(x,y) = O\left(\ln|x|\right), \qquad D_{\alpha\beta}(x,y) = O\left(\frac{1}{|x|}\right). \tag{3.4.2}$$

The following assertions hold for the solutions of boundary value problems for (3.1.4) in the case of an unbounded domain.

Remark 4 *As in Remark 1, it can be shown that the Betti formula holds in the case of an unbounded domain:*

If $u \in C^2 \bigcap (S^-) C^1 \left(\bar{S}^-\right) \bigcap \tilde{\mathcal{A}}$ *is a solution of* (3.1.4) *then*

$$\int_{S^-} U(u,u)dA = -\int_{\partial S} u^T Tu \quad ds.$$

Proof. We can write

$$0 = \int_{S^-} u^T L u dA = \lim_{R\to\infty}\left\{\int_{\Gamma_R} u^T L u dA - \int_{S^+} u^T L u dA\right\} =$$

$$= \lim_{R\to\infty} \left\{ \int_{\partial\Gamma_R} u^T T u ds - 2\int_{\Gamma_R} U(u,u) dA \right\} - \int_{\partial S} u^T T u ds + 2\int_{S+} U(u,u) dA =$$

$$= -2\int_{S_-} U(u,u) dA - \int_{\partial S} u^T T u ds,$$

since $u \in \tilde{\mathcal{A}}$ and therefore the integral

$$\int_{\partial\Gamma_R} u^T T u ds$$

vanishes as $R \to \infty$ which completes the proof.

Theorem 5 *(Representation formula for an unbounded domain)*

If $u \in C^2 \bigcap (S^-) \, C^1(\overline{S}^-) \bigcap \mathcal{A}$ *is a solution of the equations (3.1.4) then*

$$-\int_{\partial S} \left[\Gamma^{(\mu)}(x,y) T(\partial y) u(y) - D_{(\mu)}(x,y) u(y) \right] ds(y) = \begin{cases} I_{\omega(\mu)} u(x), & x \in S^-, \\ \frac{1}{2} I_{\omega(\mu)} u(x), & x \in \partial S, \\ 0, & x \in S^+. \end{cases}$$

Proof. Following the classical procedure [14], for $x \in S^-$ we apply the reciprocity relation in $\Gamma_R \setminus S^+$:

$$\begin{aligned} 0 = \; & - \; \textstyle\int_{\partial S} \left[\Gamma^{(\mu)}(x,y) T(\partial y, n) u(y) - D_{(\mu)}(x,y) u(y) \right] ds(y) + \\ & + \; \textstyle\int_{\partial \Gamma_R} \left[\Gamma^{(\mu)}(x,y) T(\partial y, n) u(y) - D_{(\mu)}(x,y) u(y) \right] ds(y). \end{aligned}$$

Using the relations (3.4.2) we conclude that for the second intergal an asymptotic estimate

$$O\left(\frac{1}{R^2} \ln R\right)$$

holds and therefore, as $R \to \infty$, the integral over $\partial\Gamma_R$ vanishes. We must note here that the first integral changes its sign since the normal $n = (n_1, n_2)^T$ is an outward normal. Now the results follow by utilizing calculations and arguments given in the proof of Theorem 3.

3.5 Single and double layer potentials

In this section we define the single and the double layer potentials and investigate their behavior in domains S^+, S^- and on the boundary ∂S. The properties of the single and the double layer potentials are essential as they are the key concept of the boundary integral equation method. We will see that the definitions of the single and the double layer potentials and their behavior on the boundary ∂S make it possible to reduce the boundary value problems in S^+ and S^- to the corresponding systems of singular integral equations on the boundary ∂S. We construct the single layer potential $V\varphi$ in the form

$$V\varphi(x) = \int_{\partial S} \Gamma(x,y)\varphi(y)ds(y)$$

and double layer potential $W\varphi$

$$W\varphi(x) = \int_{\partial S} D(x,y)\varphi(y)ds(y)$$

with the density $\varphi \in \mathcal{M}_{2\times 1}$.

Theorem 6 *If $\varphi \in C(\partial S)$ then $V\varphi(x)$ and $W\varphi(x)$ are analytic and satisfy* (3.1.4) *in $S^+(S^-)$.*

Proof. To prove the second assertion we recall Remark 2. Analyticity is examined in the classical way as in [51].

Theorem 7 *If* $\varphi \in C(\partial S)$ *then*

1. $W\varphi \in \mathcal{A}$

2. $V\varphi \in \mathcal{A}$ *if and only if*

$$\int_{\partial S} \varphi ds = 0. \tag{3.5.1}$$

Proof. The proof of the first part of the assertion is obtained using Remark 2 and the definitions of the single and double layer potentials. For the proof of the second part, first, we write

$$\log \sigma_j = \log\left(x_{\alpha_j} - y_{\alpha_j}\right) = \frac{1}{2}\ln|x_{\alpha_j} - y_{\alpha_j}|^2 + i\mathrm{Arg}\sigma_j =$$

$$= \frac{1}{2}\ln\left(\frac{|x_{\alpha_j} - y_{\alpha_j}|^2|x_{\alpha_j}|^2}{|x_{\alpha_j}|^2}\right) + i\mathrm{Arg}\sigma_j = \frac{1}{2}\ln|x_{\alpha_j}|^2 + \frac{1}{2}\ln(1+\varepsilon_{\alpha_j}) + i\mathrm{Arg}\sigma_j, \tag{3.5.2}$$

where

$$x_{\alpha_j} = \alpha_j x_1 + x_2, \qquad y_{\alpha_j} = \alpha_j y_1 + y_2, \qquad \varepsilon_{\alpha_j} = \frac{|y_{\alpha_j}|^2 - 2\left(x_{\alpha_j}, y_{\alpha_j}\right)}{|x_{\alpha_j}|^2}.$$

We see that when $|x_{\alpha_j}| \sim |x| \to \infty$ the value of ε_{α_j} is sufficiently small, so we can apply the well known formula (for small ε)

$$\ln(1+\varepsilon) = \varepsilon - \frac{1}{2}\varepsilon^2 + \frac{1}{3}\varepsilon^3 + O(\varepsilon^4).$$

to obtain the expansion for $\ln(1+\varepsilon_{\alpha_j})$ (for small ε_ε, i.e. when $|x_{\alpha_j}| \sim |x| \to \infty$). When $|x_{\alpha_j}| \to \infty$ the value of ε_{α_j} tends to zero. For the λ-component of the single layer potential we now have:

$$V\varphi_\lambda(x) = \int_{\partial S} \Gamma_{\lambda\mu}(x,y)\phi_\mu(y)ds(y) =$$

$$= \sum_{j=1}^{4}\sum_{\gamma,\beta=1}^{2} 2ai(-1)^j d_j B_{\lambda\mu\gamma\delta}\alpha_j^{4-(\gamma+\delta)}\left[\frac{1}{2}\ln|x_{\alpha_j}|^2\int_{\partial S}\varphi_\mu(y)ds(y)\right] + \tilde{V}\varphi_\lambda(x)$$

where $\tilde{V}\varphi_\lambda(x) \in \mathcal{A}$. Thus we see that $V\varphi(x)$ is in the class $\mathcal{A}$ if and only if we require

$$\int_{\partial S}\varphi(y)ds(y) = 0.$$

The result for the double layer potential follows from expansion of $W\varphi(x)$ when $|x_{\alpha_j}| \sim |x| \to \infty$ using the relation (3.4.1).

Theorem 8 *1. If $\varphi \in C(\partial S)$ then $V\varphi \in C^{0,\alpha}(\mathbb{R}^2)$.*

2. If $\varphi \in C^{1,\alpha}(\partial S)$ for $\alpha \in (0,1]$, then $W\varphi$ has $C^{1,\beta}$-extensions $(W\varphi)^+$ and $(W\varphi)^-$ to $\overline{S}^+$ and $\overline{S}^-$, respectively, with $\beta \in (0,1)$.

$$(W\varphi)^+ = \begin{cases} W\varphi, & x \in S^+, \\ -\frac{1}{2}I_\omega\varphi + W_0\varphi, & x \in \partial S, \end{cases}$$

$$(W\varphi)^- = \begin{cases} W\varphi, & x \in S^-, \\ \frac{1}{2}I_\omega\varphi + W_0\varphi, & x \in \partial S. \end{cases}$$

3. If $\varphi \in C^{0,\alpha}(\partial S)$ then $V\varphi \in C^{1,\alpha}(\mathbb{R}^2)$ for $\alpha \in (0,1)$ and

$$T\,(V\varphi)^+ = \begin{cases} T\,(V\varphi), & x \in S^+, \\ \frac{1}{2}I_\omega\varphi + T\,(V_0\varphi), & x \in \partial S, \end{cases}$$

$$T\left(V\varphi\right)^{-}=\begin{cases}T\left(V\varphi\right), & x\in S^{-},\\ -\frac{1}{2}I_{\omega}\varphi+T\left(V_{0}\varphi\right), & x\in\partial S,\end{cases}$$

4. $T\left(W\varphi\right)^{+}=T\left(W\varphi\right)^{-}$ *on* ∂S.

Here $W_0\varphi$ and $V_0\varphi$ denote the double layer potential and single layer potential, respectively, for $x \in \partial S$, τ_β denote components of the unit tangent to ∂S.

Proof. To prove the assertion 1 from (3.2.5) we conclude that the kernel of the potential $V\varphi$

$$k(x,y)\sim O(\ln r).$$

The required result now follows from [14]. For parts 2,4, bearing in mind (3.2.10) we write

$$W\varphi=\sum_{j=4}^{4}R^{j}\left\{v^{f}(x)+(i-\alpha_{j})p(x)+iw(x)\right\},$$

where

$$v^{f}(x)=\int_{\partial S}\frac{\partial\ln r}{\partial s_{y}}\varphi(y)ds(y),$$
$$w(x)=-\int_{\partial S}\frac{\partial\ln r}{\partial n_{y}}\varphi(y)ds(y),$$
$$p(x)=\int_{\partial S}\left(\frac{r^{2}}{\sigma\sigma_{j}}\right)\frac{\partial\ln r}{\partial n_{y}}\varphi(y)ds(y).$$

The proof now follows from [14]. To prove 3 we can write [33]

$$T\left(V\varphi\right)=T(\partial x,n)\left(V\varphi\right)=T(\partial x,n)\int_{\partial S}\Gamma(x,y)\varphi(y)ds(y)=$$

$$= \int_{\partial S} T(\partial x, n)\Gamma(x,y)\varphi(y)ds(y) =$$

$$= \int_{\partial S} \varphi^T(y)\left[T(\partial x, n)\Gamma(x,y) + T(\partial y, n)\Gamma(x,y)\right] ds(y) -$$

$$- \int_{\partial S} \varphi^T(y) T(\partial y, n)\Gamma(x,y)ds(y) =$$

$$= \int_{\partial S} \varphi(y)^T \left[T(\partial x, n) + T(\partial y, n)\right]\Gamma(x,y)ds(y) -$$

$$- \int_{\partial S} \left[T(\partial y, n)\Gamma(x,y)\right]^T \varphi(y)ds(y).$$

We can see that the last integral is a double-layer potential, so we can write

$$T\left(V\varphi\right) =$$

$$= \int_{\partial S} \varphi^T(y)\left[T(\partial x, n) + T(\partial y, n)\right]\Gamma(x,y)ds(y) - W\varphi(x). \tag{3.5.3}$$

We can show that the first integral is a continuous term. We recall that

$$\frac{\partial \log \sigma_j}{\partial s(x)} = (n_1(x) - \alpha_j n_2(x)).$$

Similarly,

$$\frac{\partial \log \sigma_j}{\partial s(y)} = -(n_1(y) - \alpha_j n_2(y)).$$

Thus, using (3.2.8) we can write that

$$\left[T_\kappa(\partial x, n) + T_\kappa(\partial x, n)\right]\Gamma^{(\lambda)}(x,y) =$$

$$= \sum_{j=1}^{4}\sum_{\mu=1}^{2}\sum_{\gamma,\delta=1}^{2} A_{\kappa\mu 1\beta} B_{\lambda\mu\gamma\delta}\alpha^{6-(\nu+\beta+\gamma+\delta)} 2ai(-1)^j d_j \frac{1}{\sigma_j} \times$$

$$\times \left[n_1(x) - n_1(y) + (n_2(y) - n_2(x))\alpha_j\right].$$

Since ∂S is a C^2-curve we can conclude that the first term in (3.5.3) is continuous, which completes the proof.

3.6 Fundamental boundary value problems: uniqueness and existence of regular solutions

In this section we state the fundamental boundary value problems for the equations of equilibrium (3.1.4). Then we prove the uniqueness theorem for solutions of the Dirichlet, Neumann and mixed boundary value problems. Using the properties of the single and the double layer potentials (see Theorem 8) we reduce boundary value problems to the corresponding systems of singular integral equations for which the Fredholm Alternative holds. This allows us to prove the existence of the solutions of systems of singular integral equations and thus to show that the fundamental boundary value problems are well posed.

Let functions $P(x)$, $Q(x)$, $R(x)$, $S(x) \in C(\partial S) \bigcap \mathcal{M}_{2\times 1}$ be prescribed on the boundary ∂S. The first component of the vector $P(x)$ (and $R(x)$) gives the value of the displacement component u_3 on the boundary ∂S. The second component of $P(x)$ (and $R(x)$) gives the electric potential ϕ on ∂S. The first component of $Q(x)$ (and $S(x)$) prescribes the stress vector component t_3 on the boundary ∂S and the second component of $Q(x)$ (and $S(x)$) gives the value of the electric charge on ∂S.

We shall state the interior and exterior Dirichlet and Neumann boundary value problems:

Find $u \in C^2(S^+) \bigcap C^1(\overline{S}^+) \bigcap \mathcal{M}_{2\times 1}$ satisfying (3.1.4) in S^+ such that

$$u\mid_{\partial S} = P(x). \tag{D^+}$$

Find $u \in C^2(S^+) \bigcap C^1(\overline{S}^+) \bigcap \mathcal{M}_{2\times 1}$ satisfying (3.1.4) in S^+ such that

$$Tu\mid_{\partial S} = Q(x). \tag{N^+}$$

Find $u \in C^2(S^-) \bigcap C^1(\overline{S}^-) \bigcap \tilde{\mathcal{A}} \bigcap \mathcal{M}_{2\times 1}$ satisfying (3.1.4) in S^- such that

$$u\mid_{\partial S} = R(x). \tag{D^-}$$

Find $u \in C^2(S^-) \bigcap C^1(\overline{S}^-) \bigcap \mathcal{A} \bigcap \mathcal{M}_{2\times 1}$ satisfying (3.1.4) in S^- such that

$$Tu\mid_{\partial S} = S(x). \tag{N^-}$$

For the mixed boundary value problems we have the following formulations for the bounded domain:

Find $v \in C^2(S^+) \cap C^1(\bar{S}^+ \setminus \chi)$ satisfying (3.1.4) in S^+ such that

$$\begin{aligned} L(\partial x)v(x) &= 0, & x &\in S^+ \\ v(x) &= B^+(x), & x &\in \partial S_1 \\ T(\partial x)v(x) &= C^+(x), & x &\in \partial S_2, \end{aligned} \tag{M^+}$$

where $B, C \in \mathcal{M}_{2\times 1}$ are prescribed on ∂S_1 and ∂S_2, respectively. In the similar way we formulate mixed boundary value problems for the unbounded domain:

Find $v \in C^2(S^-) \cap C^1(\bar{S}^- \setminus \chi) \cap \mathcal{A}^*$ satisfying (3.1.4) in S^- such that

$$\begin{aligned} v(x) &= B^-(x), & x &\in \partial S_1, \\ T(\partial x)v(x) &= C^-(x), & x &\in \partial S_2, \end{aligned} \tag{M^-}$$

where functions $B^{\pm}(x)$ and $C^{\pm}(x)$ are prescribed on ∂S_1 and ∂S_2 respectively.

It would be more convenient for us to have homogeneous Dirichlet conditions. Later in this section we will assume the solutions of the boundary value problems (M^+) and (M^-) of the certain form such that the homogeneous Dirichlet conditions will be satisfied automatically. Here we will follow the procedure used in [33] to reduce problems (M^+) and (M^-) to the problems with simpler boundary conditions. Let $\Phi(x) \in C^2(S^+) \cap C^1(\bar{S}^+)$ be the (known) solution of a related Dirichlet problem for the equation (3.1.4) such that on ∂S_1 its values coincide with $B^+(x)$. Let $u(x) = v(x) - \Phi(x)$. Then for the new unknown function $u(x)$ we have the following problem:

(M^+) Find $u \in C^2(S^+) \cap C^1(\bar{S}^+ \setminus \chi)$ satisfying (3.1.4) in S^+ such that

$$u(x) = 0, \qquad x \in \partial S_1, \tag{3.6.1}$$

$$T(\partial x)u(x) = f(x), \qquad x \in \partial S_2, \tag{3.6.2}$$

where $f = C - T\Phi \in \mathcal{M}_{2\times 1}$.

Similarly, for the exterior problem we have:

(M^-) Find $u \in C^2(S^-) \cap C^1(\bar{S}^- \setminus \chi) \cap \mathcal{A}^*$ satisfying (3.1.4) in S^- such that

$$u(x) = 0, \qquad x \in \partial S_1, \tag{3.6.3}$$

$$T(\partial x)u(x) = q(x), \qquad x \in \partial S_2, \tag{3.6.4}$$

where $q \in \mathcal{M}_{2\times 1}$ is prescribed on ∂S_2. In view of the asymptotic behavior of the matrix $\Gamma(x,y)$ we pose the exterior problem $(\mathcal{M}^-)$ in $\mathcal{A}^*$ to allow as large a set of admissible matrix functions as possible.

Theorem 9 *Each of the problems (D^+), (D^-), (N^-), (M^+), (M^-) has at most one solution. Any two solutions of (N^+) differ by a (2×1)-matrix of the form (3.1.15).*

Proof. The proof is conducted using classical techniques [14] by making use of the Betti formulae established above. Let us first show the proof for the Dirichlet problem for the bounded domain. We assume that there are two solutions of (D^+) $u^{(1)}(x)$ and $u^{(2)}(x)$:

$$\begin{aligned} L(\partial x)u^{(1)}(x) &= 0, & \qquad L(\partial x)u^{(2)}(x) &= 0, \\ u^{(1)}(x) &= P(x), & \qquad u^{(2)}(x) &= P(x), \end{aligned}$$

here by $L(\partial x)$ we denote the differential operator for (3.1.4) for both tetragonal $\bar{4}$ and cubic $\bar{4}3m$ symmetry. Next, we introduce some function $w(x) \in \mathcal{M}_{2\times 1}$ such that $w(x) = u^{(1)} - u^{(2)}$. Obviously, $w(x)$ solves the homogeneous Dirichlet problem for $L(\partial x)$, i.e.

$$\begin{aligned} L(\partial x)w(x) &= 0 \qquad \text{in } S^+ \\ w(x) &= 0 \qquad \text{on } \partial S. \end{aligned}$$

From Betti formula (see Remark 1) we find

$$\int_{S^+} U(w,w)dV = 0.$$

Since the internal energy $U(w, w)$ is a positive quadratic form we conclude that

$$U(w, w) = 0$$

and, consequently,

$$w(x) = (c_1, c_2)^T = c,$$

where $c \in \mathcal{M}_{2\times 1}$ and c_α are arbitrary constants. Since $w(x) \in C^1(\bar{S}^+)$

$$\lim_{x \in S^+ \to y \in \partial S} w(x) = w(y) = 0,$$

or,

$$\lim_{x \in S^+ \to y \in \partial S} c = 0,$$

which is possible only if $w(x) = 0$, $x \in S^+$. Thus we obtained that $u^{(1)}(x) = u^{(2)}(x)$. Using the same procedure we can prove the uniqueness of the solution for mixed problem (M^+). For the case of (N^+) we have

$$w(x) = c$$

and the boundary condition

$$T(\partial x, n)w(x) = 0$$

from which we conclude that solutions $u^{(1)}(x)$ and $u^{(2)}(x)$ can differ by arbitrary constant vector $c = (c_1, c_2)^T$. In other words, the solution of (3.1.4) for the bounded domain with Neumann boundary conditions is determined up to some arbitrary constant vector c which denotes free body translation in the

x_3-direction and constant electric potential. However, it should be noted that in the case of the Neumann boundary value problem shear strain components ε_{13}, ε_{23} and components of the electric field E_1 and E_2 are uniquely determined even though the displacement u_3 and the electric potential are not unique.

Solutions $u(x)$ of (D^-), (N^-), (M^-) are unique as long as they are, respectively, in classes $\tilde{\mathcal{A}}$, $\mathcal{A}$, $\tilde{\mathcal{A}}$ (this will allow us to use Betti formula for an unbounded domain).

Now we are in the position to reduce boundary value problems to the corresponding integral equations. Since the single layer potential $V\varphi(x)$ and double layer potential $W\varphi(x)$ solve the equations (3.1.4) we may seek solutions of (D^+) and (D^-) in the form of extensions $(W\varphi(x))^+$ and $(W\varphi(x))^-$, respectively, with $\varphi \in C^{1,\alpha}(\partial S)$, the solutions of (N^+) and (N^-) in the form of single layer potential with the density $\phi \in C^{0,\alpha}(\partial S)$. Using the results of Theorem 8, Dirichlet and Neumann boundary value problems (for bounded and unbounded domains) are now reduced to the corresponding singular integral equations:

$$-\frac{1}{2}I_\omega\varphi + \int_{\partial S} D(x,y)\varphi(y)ds(y) = P(x), \qquad (\mathcal{D}^+)$$

$$\frac{1}{2}I_\omega\varphi + \int_{\partial S} D(x,y)\varphi(y)ds(y) = R(x) - u_0, \qquad (\mathcal{D}^-)$$

$$\frac{1}{2}I_\omega\varphi + \int_{\partial S} T(\partial x)\Gamma(x,y)\varphi(y)ds(y) = Q(x), \qquad (\mathcal{N}^+)$$

$$-\frac{1}{2}I_\omega\varphi + \int_{\partial S} T(\partial x)\Gamma(x,y)\varphi(y)ds(y) = P(x), \qquad (\mathcal{N}^-)$$

for $x \in \partial S$ with unknown density function φ.The vector $u_0 \in \mathcal{M}_{2\times 1}$ is of the form (3.1.15). We denote the corresponding homogeneous equations by $(\mathcal{D}_0^+)$, $(\mathcal{D}_0^-)$, $(\mathcal{N}_0^+)$, $(\mathcal{N}_0^-)$.

Theorem 10 *If $P(x) \in C^{1,\alpha}(\partial S)$, $\alpha \in (0,1)$, then any solution $\varphi \in C^{0,\alpha}(\partial S)$ of $(\mathcal{D}^+)$ belongs to class $C^{1,\alpha}(\partial S)$. A similar result holds for $(\mathcal{D}^-)$ if $R(x) \in C^{1,\alpha}(\partial S)$.*

Proof. We recall that integral equation $(\mathcal{D}^+)$ is

$$\sum_{j=1}^{4} R^j \int_{\partial S} \left[\frac{\partial \ln r}{\partial s_y} + \left(\frac{(i-\alpha_j)r^2}{\sigma\sigma_j} - i\right)\frac{\partial \ln r}{\partial n_y}\right]\varphi(y)ds(y) + \frac{1}{2}I_\omega \varphi(x) = P(x). \tag{3.6.5}$$

Introducing complex variables $z = x_1 + ix_2$ and $\zeta = y_1 + iy_2$ we write

$$\frac{\partial}{\partial s_y}\log(\zeta - z)ds_y = \frac{d\zeta}{\zeta - z} = \left\{\frac{\partial}{\partial s_y}\ln|\zeta - z| + i\frac{\partial}{\partial s_y}\theta\right\}ds_y =$$
$$= \left\{\frac{\partial}{\partial s_y}\ln|x-y| + i\frac{\partial}{\partial n_y}\ln|x-y|\right\}ds_y,$$

thus, we have

$$\frac{\partial}{\partial s_y}\ln|x-y|ds_y = \frac{d\zeta}{\zeta - z} - i\frac{\partial}{\partial n_y}\ln|x-y|. \tag{3.6.6}$$

On the basis of the relation (3.6.6) we can rewrite the integral operator of (3.6.5) as the sum of a singular part and a weakly singular part

$$\sum_{j=1}^{4} R^j \left[K^s + K^w\right]\varphi(z) + \frac{1}{2}I_\omega\varphi(z) = P(z),$$

where $K^s\varphi$ and $K^w\varphi$ are given by

$$K^s\varphi = \int_{\partial S}\frac{\varphi(\zeta)d\zeta}{\zeta - z}, \quad K^w\varphi = \int_{\partial S}\left(\frac{(i-\alpha_j)r^2}{\sigma\sigma_j} - 2i\right)\frac{\partial \ln r}{\partial n_y}\varphi(y)ds(y),$$

We conduct the regularization process [54] by applying another singular operator, say, $\tilde{M}$ given by

$$\tilde{M} = K^s + \frac{i\pi}{2} I,$$

such that the reduced integral equation becomes of Fredholm type

$$\sum_{j=1}^{4} R^j \left\{ K^s K^w + \frac{i\pi}{2} K_w + (\frac{\pi^2}{4} - \pi^2) \right\} \varphi(z) = \tilde{M} P(z).$$

Bearing in mind results established in Theorem 8, as in [14], we conclude that K^w maps $C^{1,\alpha}(\partial S)$ into $C^{1,\alpha}(\partial S)$. Since $C^{1,\alpha}(\partial S)$ is invariant under K^s [14], we obtain the required result.

Theorem 11 *The Fredholm Alternative holds for* $(\mathcal{D}^+)$, $(\mathcal{N}^-)$ *and for* $(\mathcal{N}^+)$, $(\mathcal{D}^-)$ *in the real dual system* $(C^{0,\alpha}, C^{0,\alpha})$, $\alpha \in (0,1)$, *with the bilinear form*

$$(\varphi, \psi) = \int_{\partial S} \varphi^T \psi \quad ds_y \tag{3.6.7}$$

Proof. We denote by $\mathcal{D}$, $\mathcal{N}$ the integral operators from the corresponding integral equations. Recalling that $D(x,y) = (T(\partial y)\Gamma(y,x))^T$ we can write for any $\varphi, \psi \in C^{0,\alpha}(\partial S)$

$$
\begin{aligned}
(\mathcal{D}\varphi, \psi) &= \int_{\partial S} \left[\int_{\partial S} D(x,y)\varphi(y)ds(y)\right]^T \psi(x)ds(x) = \\
&= \int_{\partial S} \left[\int_{\partial S}(T(\partial y)\Gamma(y,x))^T \varphi(y)ds(y)\right]^T \psi(x)ds(x) = \\
&= \int_{\partial S} \varphi^T \left[\int_{\partial S} T(\partial y)\Gamma(y,x)\psi(x)ds(x)\right] ds(y) = \\
&= (\varphi, \mathcal{N}\psi).
\end{aligned}
$$

Due to the symmetry of the bilinear form (3.6.7), we have

$$
(\mathcal{N}\varphi, \psi) = (\varphi, \mathcal{D}\psi),
$$

which means that $\mathcal{N}$ and $\mathcal{D}$ are mutually adjoint in the given dual system. We shall now show that the index ρ [54, 14]

$$
\rho = \frac{1}{2\pi}\left[\arg\frac{\det(\frac{1}{2}I_\omega I - \pi i\hat{k}(z,z))}{\det(\frac{1}{2}I_\omega I + \pi i\hat{k}(z,z))}\right]_{\partial S}
$$

of the equation $\mathcal{D}^+$ written in terms of complex variables is zero. In our case

$$
\hat{k}(z,\zeta) = \sum_{j=1}^{4} \left(R^j\right)\left(I + (z-\zeta)k^w(z,\zeta)\right),
$$

consequently, $\hat{k}(z,z) = \sum_{j=1}^{4} \left(R^j\right) I$ and

$$
\det\left(-\frac{1}{2}I_\omega I - i\pi\sum_{j=1}^{4}\left(R^j\right) I\right) = \det\left(\sum_{j=1}^{4}\left(R^j\right)(i\pi I - i\pi I)\right) = 0.
$$

Thus, for $\mathcal{D}^+$ we find that $\rho = 0$. Hence we deduce that the Fredholm Alternative holds for the operator $\mathcal{D}$ in the complex dual system $(C^{0,\alpha}, C^{0,\alpha})$ with bilinear form (3.6.7), and hence it holds [54] in the real dual system $(C^{0,\alpha}, C^{0,\alpha})$. Similarly, we obtain corresponding results for $\mathcal{D}^-$ and $\mathcal{N}^+$.

Theorem 12 *1. The problem (D_0^-) has precisely two linearly independent $C^{0,\alpha}$-solutions.*

2. *The interior Dirichlet problem (D^+) has a unique solution for any $P(x) \in C^{1,\alpha}$, $\alpha \in (0,1)$. This solution can be represented in the form of $W\varphi$ with the density $\varphi \in C^{1,\alpha}(\partial S)$.*

3. *The exterior Neumann problem (N^-) has a unique solution for any $S(x) \in C^{0,\alpha}(\partial S)$, $\alpha \in (0,1)$, if and only if*

$$\int_{\partial S} u_0^T S ds = 0, \tag{3.6.8}$$

where u_0 is of the form (3.1.15). This solution can be represented in the form of $V\varphi$ with $\varphi \in C^{0,\alpha}(\partial S)$.

4. *The interior Neumann problem (N^+) is soluble for any $Q \in C^{0,\alpha}(\partial S)$, $\alpha \in (0,1)$, if and only if*

$$\int_{\partial S} u_0^T Q ds = 0, \tag{3.6.9}$$

where u_0 is of the form (3.1.15). The solution, unique up to a 2×1-matrix u of the form (3.1.15), can be represented in the form of $V\varphi$ with the density $\varphi \in C^{0,\alpha}(\partial S)$.

5. *The exterior Dirichlet problem (D^-) has a unique solution $R(x) \in C^{1,\alpha}(\partial S)$.*

This solution can be represented as the sum of $W\varphi$ with the density $\varphi \in C^{1,\alpha}(\partial S)$ and a particular vector u_0 of the form (3.1.15).

Proof. The proof of assertions 1, 2, 3, 4 follows the procedure used in [14]. To prove the assertion 5 let $\{f^\alpha\}$ and $\{g^\beta\}$ be systems of linearly independent solutions of $\mathcal{D}^-$ and $\mathcal{N}^+$, respectively. We assume that $\{f^\alpha\}$ and $\{g^\beta\}$ are biorthonormalized [34], i.e.

$$(f^\alpha, g^\beta) = \delta_{\alpha\beta},$$

where $\delta_{\alpha\beta}$ is Kronecker's delta. To satisfy the condition of solubility

$$(g^\beta, R - u_0) = 0$$

we express u_0 as a linear combination of f^α

$$u_0 = c_\alpha f^\alpha$$

with coefficients c_α in the form

$$c_\alpha = \int_{\partial S} (g^\beta)^T R ds.$$

In this case we will have

$$(g^\beta, R - u_0) = \int_{\partial S} g^{\beta^T} (R - c_\alpha f^\alpha) ds = \int_{\partial S} (c_\alpha - c_\alpha) ds = 0.$$

Remark 13 *The conditions* (3.6.8), (3.6.9) *express zero resultant force and charge acting on ∂S.*

We turn our attention now to the boundary value problems with mixed boundary conditions. Mixed boundary conditions appear in the problems related to the modeling of cracks in piezoelectric materials [39, 56]. Consider first the interior mixed problem $(\mathcal{M}^+)$. We seek the solution in the form

$$u(x) = V\varphi(x) = \int_{\partial S_2} [\Gamma(x,y) - H(x,y)]\,\varphi(y)ds(y), \tag{3.6.10}$$

where $\varphi \in \mathcal{M}_{2\times1}$ is some unknown matrix density and the matrix $H(x,y) \in \mathcal{M}_{2\times2}$ is constructed as follows. Let Ω_1 be a bounded domain with C^2-boundary $\partial\Omega_1$ such that

$$\text{(i)} \quad S^+ \subset \Omega_1 \qquad \text{(ii)} \quad \partial S_2 \subset \Omega_1 \qquad \text{(iii)} \quad \partial S_1 \subset \partial\Omega_1.$$

The columns of the matrix $H(x,y)$ are such that

$$\begin{aligned} L(\partial x)H^{(\alpha)}(x,y) &= 0, \qquad & x &\in \Omega_1, \\ H^{(\alpha)}(x,y) &= \Gamma^{(\alpha)}(x,y), \qquad & x &\in \partial\Omega_1. \end{aligned}$$

From the existence result for the interior Dirichlet problem, it is clear that $H^{(\alpha)}(x,y)$ exist uniquely for each $y \in \partial S_2$ in the class $C^2(\Omega_1) \cap C^1(\bar{\Omega}_1)$. In fact, for each y, $H^{(\alpha)}$ take form of a double layer potential. Suppose that the unknown density φ from (3.6.10) is of the class $H^*(\partial S_2)$ [84], i.e. φ is Hölder-continuous on ∂S_2 but may admit 'weak singularities' near the endpoints χ. Proceeding as in [84], using properties of the single layer potential for the anti-plane shear state and the definition of the matrix $H(x,y)$, we find that $u(x)$ from (3.6.10) satisfies the continuity conditions of the problem $(\mathcal{M}^+)$, equations

(3.1.4) in S^+ and the displacement condition (3.6.1) on ∂S_1. The remaining boundary condition (3.6.2) leads to the following system of singular integral equations over the open arc ∂S_2:

$$-\frac{1}{2}I_\omega\varphi(x) + \int_{\partial S_2} T(\partial x)\Gamma(x,y)\varphi ds(y) -$$

$$-\int_{\partial S_2} T(\partial x)H(x,y)\varphi(y)ds(y) = f(x). \tag{3.6.11}$$

The first integral on the left-hand side of (3.6.11) must be interpreted in the sense of principal value while the second is a Fredholm integral. It is clear that $u(x)$ from (3.6.10) will be the unique solution of $(\mathcal{M}^+)$ provided (3.6.11) yields a solution $\varphi \in H^*(\partial S_2)$ for sufficiently smooth boundary data f.

Lemma 14 *The homogeneous system* $(3.6.11)^0$ *from* (3.6.11) *has only the trivial solution in the space* $H^*(\partial S_2)$.

Proof. Let $\varphi_0 \in H^*(\partial S_2)$ be a solution of $(3.6.11)^0$. Then $V\varphi_0(x)$ from (3.6.10) solves the homogeneous problem $(\mathcal{M}^+)^0$. Theorem 2 now yields $V\varphi_0(x) = 0$, $x \in S^+$. The continuity of a single layer potential (Theorem 8) gives that $V\varphi_0(x) = 0$, $x \in \partial S$. Furthermore, using the boundary value of $H(x,y)$, $V\varphi_0(x) = 0$, $x \in \partial\Omega_1$. Hence $V\varphi_0 = 0$ on the boundary of the bounded domain $\Omega_1 \setminus S^+$. By the uniqueness result for the interior Dirichlet problem, $V\varphi_0(x) = 0$, $x \in \Omega_1 \setminus S^+$ so that the jump relations arising from the application of the T-operator to a single layer potential yield

$$(TV\varphi_0)^+ - (TV\varphi_0)^- = \varphi_0 = 0 \quad \text{on } \partial S_2$$

which completes the proof.

In [84], Vekua developed a theory of solvability for systems of singular integral equations with discontinuous coefficients. To see that this theory applies to our system, we rewrite (3.6.11) as a system with discontinuous coefficients over the closed curve ∂S. For $x \in \partial S_2$

$$\int_{\partial S_2} T(\partial x)\Gamma(x,y)\varphi ds(y) =$$
$$= \sum_{j=1}^{4} R^j \int_{\partial S_2} \left[\frac{\partial \ln r}{\partial s_x} + \left(\frac{(i-\alpha_j)r^2}{\sigma\sigma_j} - i\right)\frac{\partial \ln r}{\partial n_x}\right] ds(y) =$$
$$= \sum_{j=1}^{4} R^j \left[\int_{\partial S_2} \frac{\varphi(\xi)}{\xi - z} d\xi + \int_{\partial S_2} k(z,\xi)\varphi(\xi)d\xi\right],$$

where $\xi = y_1 + iy_2 \in \partial S_2$, $z = x_1 + ix_2 \in \partial S_2$, $k(z,\xi)$ is a Fredholm kernel. Now we can rewrite (3.6.11) in the form

$$A(z)\varphi(z) + \frac{B(z)}{\pi i}\int_{\partial S} \frac{\varphi(\xi)}{\xi - z}d\xi + \int_{\partial S} C(z,\xi)\varphi(\xi)d\xi = f(z), \quad z \in \partial S, \quad (3.6.12)$$

or

$$A(z)\varphi(z) + \frac{1}{\pi i}\int_{\partial S} \frac{K(z,\xi)}{\xi - z}d\xi = g(z), \qquad z \in \partial S,$$

where

$$K(z,\xi) = B(z,\xi) + \pi i(\xi - z)C(z,\xi), \qquad z \in \partial S,$$

$$A(z) = \begin{cases} I_3, & z \in \partial S_1, \\ -\frac{1}{2}I_\omega, & z \in \partial S_2; \end{cases} \qquad B(z) = \begin{cases} 0, & z \in \partial S_1, \\ R^j, & z \in \partial S_2; \end{cases}$$

$$C(z,\xi) = \begin{cases} 0, & z \text{ or } \xi \in \partial S_1 \\ k(z,\xi) - [T(\partial x)\Gamma(x,y)](z,\xi), & z \text{ or } \xi \in \partial S_2, \end{cases}$$

and

$$g(z) = \begin{cases} 0, & z \in \partial S_1, \\ f(z), & z \in \partial S_2. \end{cases}$$

It is clear that A and K are Hölder continuous everywhere on ∂S except perhaps at the points of χ where they have discontinuities of the first kind [84].

Lemma 15 *The Fredholm alternative holds for the system* (3.6.12) *and its adjoint or associated system in the space* $H^*(\partial S)$.

Proof. According to Vekua [84] Nöether's theorems are valid for the system (3.6.12). Furthermore, proceeding as in [14], calculation shows that the index of the singular integral operator from (3.6.12) is zero so that Nöether's theorems reduce to Fredholm theorems. Finally, the endpoints a and b are shown to be "special" [84] so that any solution of equation (3.6.12) with $g \in H^*(\partial S)$ is necessarily of the same class [84].

Theorem 16 *The mixed problem* $(\mathcal{M}^+)$ *is uniquely solvable for any* $f \in H^*(\partial S_2)$. *The solution is given by* (3.6.10) *with* $\varphi \in H^*(\partial S_2)$ *obtained from the system* (3.6.11).

Proof. Using Lemma 14, we see that the homogeneous system $(3.6.12)^0$ has only the trivial solution in $H^*(\partial S)$. Hence, since Fredholm theorems apply (by Lemma 15), the associated homogeneous system has also only the trivial solution in $H^*(\partial S)$ and (3.6.12) is uniquely solvable in $H^*(\partial S)$ for any $g \in H^*(\partial S)$.

This means that the system (3.6.11) is uniquely solvable in $H^*(\partial S_2)$ whenever the boundary data $f \in H^*(\partial S_2)$. Consequently, the unique (by Theorem 9) solution of $(\mathcal{M}^+)$ with $f \in H^*(\partial S_2)$ is given by (3.6.10) with $\varphi \in H^*(\partial S_2)$ obtained from the system (3.6.11).

In case of the exterior mixed problem, the asymptotic behavior of the matrix $\Gamma(x,y)$ requires that we seek the solution in the form [68]

$$u(x) = V_E\varphi(x) = \int_{\partial S_2} \left[\Gamma(x,y) - M^\infty(x)\mathcal{F}^T - \Psi(x,y)\right]\varphi(y)ds(y), \quad (3.6.13)$$

where $\varphi \in \mathcal{M}_{3\times 1}$ is unknown density function and $M^\infty \in \mathcal{M}_{2\times 2}$ is given by

$$M^\infty_{\alpha\beta} = M^\infty_{\alpha\beta}(x) = 2ai\sum_{j=1}^{4}\sum_{\gamma,\delta=1}^{2}(-1)^j d_j B_{\alpha\beta\gamma\delta}\alpha_j^{4-(\gamma+\delta)}\ln r,$$

where $r = |x|$. The matrix $\mathcal{F}^T$ is given by

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

It can be verified that $LM^\infty = 0$ in $\mathbb{R}^2 \setminus \{0\}$. The matrix $\Psi = \Psi(x,y)$ is constructed using a procedure similar to that used to construct the matrix H for $(\mathcal{M}^+)$. That is, let Ω_2 be an infinite domain with closed C^2-boundary $\partial\Omega_2$ such that

$$\text{(i)}\quad S^- \subset \Omega_2 \qquad \text{(ii)}\quad \partial S_2 \subset \Omega_2 \qquad \text{(iii)}\quad \partial S_1 \subset \partial\Omega_2 \qquad \text{(iv)}\quad \{0\} \notin \bar{\Omega}_2.$$

The columns $\Psi^{(\alpha)}(x,y)$ are such that

$$\begin{aligned} L(\partial x)\Psi^{(\alpha)}(x,y) &= 0, \qquad & x &\in \Omega_2, \\ \Psi^{(\alpha)}(x,y) &= \mathcal{G}^{(\alpha)}(x,y), \qquad & x &\in \partial\Omega_2, \end{aligned}$$

where $\mathcal{G}^{(\alpha)} \in \mathcal{M}_{2\times 2}$ is given by $\mathcal{G}^{(\alpha)}(x,y) = \Gamma(x,y) - M^{\infty}(x)\mathcal{F}^T$. The existence result for the exterior Dirichlet problem guarantees that columns $\Psi^{(\alpha)}(x,y)$ exist uniquely for each $y \in \partial S_2$ in the class $C^2(\Omega_2) \cap C^1(\bar{\Omega}_2) \cap \mathcal{A}^*$. In fact, for each y, $\Psi^{(\alpha)}(x,y)$ take the form of the sum of an double layer potential and a matrix of the form (3.1.15).

The fact that with $\varphi \in H^*(\partial S_2)$, u from (3.6.13) satisfies the continuity conditions of the problem $(\mathcal{M}^-)$, equations (3.1.4) in S^- and the displacement condition (3.6.3) on ∂S_1 follows as in the case of $(\mathcal{M}^+)$ using the properties of M^{∞} and Ψ described above and the smoothness properties of the single layer potential (Section 3.5). The fact that $u(x) = V_E\varphi(x) \in \mathcal{A}^*$ follows from the fact that, as $|x| \to \infty$ [15],

$$\int_{\partial S_2} [\Gamma(x,y) - M^{\infty}(x)\mathcal{F}^T]\varphi(y)ds(y) =$$

$$= M^{\infty}(x)\int_{\partial S_2} \mathcal{F}^T\varphi(y)ds(y) + u_0 - M^{\infty}(x)\int_{\partial S_2} \mathcal{F}^T\varphi(y)ds(y) = u_0 \in \mathcal{A}.$$

Also,

$$\int_{\partial S_2} \Psi(x,y)\varphi(y)ds(y) = \int_{\partial S_2} \Psi^{(\alpha)}(x,y)\varphi_\alpha(y)ds(y) \in \mathcal{A}^*,$$

since, as noted above, $\Psi^{(\alpha)}(x,y) \in \mathcal{A}^*$ for each y. Hence, $u(x) = V_E\varphi(x) \in \mathcal{A}^*$. As in the case of the problem $(\mathcal{M}^+)$, the remaining boundary condition (3.6.4) leads to the following system of singular integral equations over the open arc ∂S_2:

$$\frac{1}{2}I_\omega\varphi + \int_{\partial S_2} T(\partial x)\Gamma(x,y)\varphi(y)ds(y) -$$

$$-\int_{\partial S_2} T(\partial x)[\Psi(x,y)+M^{\infty}(x)\mathcal{F}^T]\varphi(y)ds(y)=q(x), \quad x\in\partial S_2. \qquad (3.6.14)$$

Consequently, $u(x)$ from (3.6.13) will be the unique solution of $(\mathcal{M}^-)$ provided (3.6.14) yields a solution $\varphi \in H^*(\partial S_2)$ whenever $q \in H^*(\partial S_2)$.

Lemma 17 *The homogeneous system* $(3.6.14)^0$ *from* (3.6.14) *has only the trivial solution in the space* $H^*(\partial S_2)$.

Proof. Let $\varphi_0 \in H^*(\partial S_2)$ be a solution of $(3.6.14)^0$. Then $V_E\varphi(x)$ from (3.6.13) solves the homogeneous problem $(\mathcal{M}^-)^0$. Theorem 9 now yields $V_E\varphi_0(x) = 0$, $x \in S^-$. Proceeding as in the proof of Lemma 14, we find that $V_E\varphi_0(x) = 0$, $x \in \Omega_2 \setminus S^-$ so that

$$(TV_E\varphi_0)^+ - (TV_E\varphi_0)^- = \varphi_0 = 0 \text{ on } \partial S_2,$$

which completes the proof.

Theorem 18 *The mixed problem* $(\mathcal{M}^-)$ *is uniquely solvable for any* $q \in H^*(\partial S_2)$. *The solution is given by* (3.6.13) *with* $\varphi \in H^*(\partial S_2)$ *obtained from the system* (3.6.14).

Proof. The system (3.6.14) is similar to the system (3.6.11). Following the steps leading to the system (3.6.12), we can rewrite (3.6.14) as a system with discontinuous coefficients over the closed curve ∂S. As in the proof of Lemma 15, the index of the resulting system over ∂S is shown to be zero and the end points a and b to be special [84]. Vekua's theory again shows that

Nöether's theorems reduce to the Fredholm alternative in the space $H^*(\partial S)$. Using Lemma 6 and the Fredholm alternative, the associated homogeneous system has only the trivial solution in $H^*(\partial S)$. Hence, the system (3.6.14) is uniquely (Theorem 9) solvable in $H^*(\partial S)$ for any $q \in H^*(\partial S_2)$ and this solution is given by (3.6.13) with $\varphi \in H^*(\partial S_2)$.

Thus, in this chapter we investigated problem of static equilibrium of a prismatic piezoelectric body for different types of boundary conditions. We established the uniqueness and existence of the solutions and found analytical solutions in the form of single and double layer potentials.

Chapter 4

Steady-state vibrations in the case of generalized plane-strain piezoelectricity

For the piezoelectric applications the problems of steady state vibrations, i.e. when all the transient considerations can be ignored, are of great interest [12, 79]. In this chapter we look at the problems of steady-state vibrations for a more general class of deformation, namely, generalized plane strain. Using the boundary integral equation method we investigate uniqueness and existence of the solutions for both bounded and unbounded domains. For the case of an unbounded domain we will derive radiation conditions, or so-called Sommerfeld conditions, to guarantee the uniqueness of the solution. As the solution is not unique for the case of a bounded domain we consider only the solubility of the problems for different frequencies.

4.1 Governing equations for the generalized plane strain state in linear piezoelectricity

Suppose, the cylindrical body Ω is subjected to the stress and electric field on its lateral surface lying in the plane perpendicular to the axis of the cylinder, as shown in the Figure 4.1.

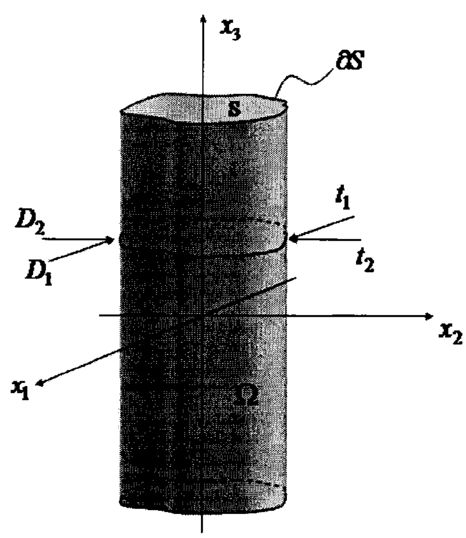

Figure 4.1: Piezoelectric cylinder in the generalized plane strain state

In the state of generalized plane strain the displacement vector components and electric potential are assumed in the form [27, 37]:

$$u_i = u_i(x_1, x_2, t), \qquad \phi = \phi(x_1, x_2, t).$$

The equations of motion and charge equation in the case of plane piezoelectricity are given by [27]:

$$\begin{aligned} C_{i\alpha k\beta} u_{k,\alpha\beta} + e_{\nu i\alpha}\phi_{,\nu\alpha} &= \rho\ddot{u}_i - f_i, \\ -e_{\gamma i\alpha} u_{i,\gamma\alpha} + \epsilon_{\alpha\beta}\phi_{,\alpha\beta} &= -q, \end{aligned} \tag{4.1.1}$$

Here we note that in the charge equation there are no terms containing derivatives with respect to time due to the quasi-static approximation. So, as in [29], we conclude that time in the electric potential $\phi(x)$ is just a parameter, the consequence of a time-dependent displacement field. The boundary conditions are specified similarly as in the case of anti-plane shear. Thus we have

$$\begin{cases} (C_{i\alpha k\beta} u_{k,\beta} + e_{\nu i\alpha}\phi_{,\nu}) n_\alpha = t_i^*(x), \\ -(e_{\gamma i\alpha} u_{,\gamma} - \epsilon_{\alpha\beta}\phi_{,\beta}) n_\alpha = D^*(x) \quad \text{on } \partial S, \end{cases} \tag{4.1.2}$$

in the case of the Neumann problem, and

$$\begin{cases} u_i(x) = u_i^*(x), \\ \phi(x) = \phi^*(x) \quad \text{on } \partial S, \end{cases} \tag{4.1.3}$$

in the case of the Dirichlet problem. Here D^*, t_i^*, u_i^*, ϕ^* are prescribed functions on ∂S. We assume that the vector $F = (f_1, f_2, f_3, q)^T$ of external force and charge is a periodic function of time, i.e.

$$F = F^{(1)}\cos\omega t + F^{(2)}\sin\omega t,$$

where ω, the frequency of oscillation, is real number. Then it is natural to expect the displacement components and electric potential to be of the form

$$\begin{aligned} u_i(x,t) &= \mathrm{Re}(u_i(x)e^{-i\omega t}), \quad u_i(x) = u_i^{(1)}(x) + iu_i^{(2)}(x), \\ \phi(x,t) &= \mathrm{Re}(\phi(x)e^{-i\omega t}), \quad \phi(x) = \phi^{(1)}(x) + i\phi^{(2)}(x), \quad x = (x_1, x_2). \end{aligned} \tag{4.1.4}$$

Substitution of (4.1.4) into (4.1.1) gives

$$\begin{aligned} C_{i\alpha k\beta}u_{k,\alpha\beta} + e_{\nu i\alpha}\phi_{,\nu\alpha} + \rho\omega^2 u_i &= 0, \\ -e_{\gamma i\alpha}u_{i,\gamma\alpha} + \epsilon_{\alpha\beta}\phi_{,\alpha\beta} &= 0. \end{aligned} \tag{4.1.5}$$

Equations (4.1.5) represent the governing system of equations describing steady-state vibrations in the context of (generalized) plane strain state in the theory of linear piezoelectricity. We will consider first the hexagonal piezoelectric material as in this case the analysis is simpler. In the particular case of a hexagonal material ($6mm$) the system (4.1.5) becomes

$$\begin{aligned} C_{11}\tfrac{\partial^2 u_1}{\partial x_1^2} + C_{66}\tfrac{\partial^2 u_1}{\partial x_2^2} + (C_{11} - C_{66})\tfrac{\partial^2 u_2}{\partial x_1 \partial x_2} + \rho\omega^2 u_1 &= 0, \\ (C_{11} - C_{66})\tfrac{\partial^2 u_1}{\partial x_1 \partial x_2} + C_{11}\tfrac{\partial^2 u_2}{\partial x_1^2} + C_{66}\tfrac{\partial^2 u_2}{\partial x_2^2} + \rho\omega^2 u_2 &= 0, \\ C_{44}\triangle u_3 + e_{15}\triangle\phi + \rho\omega^2 u_3 &= 0, \\ e_{15}\triangle u_3 - \epsilon\triangle\phi &= 0, \end{aligned} \tag{4.1.6}$$

where

$$\begin{aligned} C_{11} &= C_{1111}, \quad C_{44} = C_{3131} = C_{3232}, \\ C_{66} &= C_{1212} = C_{1221} = C_{2112} = C_{2121}, \\ e_{15} &= e_{231} = e_{132}, \quad \epsilon = \epsilon_{11} = \epsilon_{22}. \end{aligned}$$

We see that system (4.1.6) consists of a purely elastic part with respect to the in-plane displacement components u_1 and u_2 and a second part describing piezoelectric coupling between the out-of-plane displacement component u_3 and the electric potential ϕ. This means we can decouple the system (4.1.6) and examine each part separately. In fact, using simple transformations the first two equations of (4.1.6) can be written in a form similar to that describing the equations for steady-state vibrations of a classical isotropic elastic medium:

$$(C_{11} - C_{66})\text{grad div } u(x) + C_{66}\Delta u(x) + \rho\omega^2 u(x) = 0, \tag{4.1.7}$$

where $u(x) = (u_1(x), u_2(x))^T$. The equations of steady-state vibrations (4.1.7) can be also written in the form

$$L^{\omega}_{6mm}(\partial x)u(x) = 0,$$

where $L^{\omega}_{6mm}(\partial x)$ is differential operator given by

$$L^{\omega}_{6mm}(\partial x) = \begin{bmatrix} C_{11}\xi_1^2 + C_{66}\xi_2^2 + \rho\omega^2 & -(C_{11} - C_{66})\xi_1\xi_2 \\ -(C_{11} - C_{66})\xi_1\xi_2 & C_{11}\xi_2^2 + C_{66}\xi_1^2 + \rho\omega^2 \end{bmatrix}. \tag{4.1.8}$$

The first two boundary conditions from (4.1.2) which accompany equations (4.1.7), for the particular case of a transversely isotropic material, can also be written in a form similar to that of classical isotropic elasticity:

$$2C_{66}\frac{\partial u(x)}{\partial n} + n \cdot (C_{11} - 2C_{66})\nabla \cdot u(x) + C_{66} n \times (\nabla \times u(x)) = t^*(x), \tag{4.1.9}$$

or

$$T(\partial x, n)u(x) = t^*(x),$$

where

$$T(\partial x, n) = \begin{bmatrix} C_{11}\xi_1 n_1 + C_{66}\xi_2 n_2 & -(C_{11} - C_{66})\xi_2 n_1 \\ -(C_{11} - C_{66})\xi_1 n_2 & C_{11}\xi_2 n_2 + C_{66}\xi_1 n_1 \end{bmatrix}. \tag{4.1.10}$$

Here $n = (n_1(x), n_2(x))^T$ is an outward normal to the boundary ∂S. The remaining part of (4.1.6), simply reduces to the Helmholtz equation for the displacement $u_3(x)$ and Poisson's equation for electric potential $\phi(x)$:

$$\begin{aligned} \triangle u_3(x) + k^2 u_3(x) &= 0, \\ \triangle \phi(x) &= f, \end{aligned} \tag{4.1.11}$$

where

$$k^2 = \frac{\epsilon\rho\omega^2}{C_{44}\epsilon + e_{15}^2}, \quad f = -\frac{e_{15}\rho\omega^2}{\epsilon C_{44} + e_{15}^2} u_3(x).$$

Let us now consider the equations of steady state vibrations (4.1.5) for the materials with tetragonal $\bar{4}$ symmetry:

$$\begin{aligned} C_{11}\frac{\partial^2 u_1}{\partial x_1^2} + C_{16}\left(\frac{\partial^2 u_2}{\partial x_1^2} + 2\frac{\partial^2 u_1}{\partial x_1 \partial x_2} - \frac{\partial^2 u_2}{\partial x_2^2}\right) + C_{12}\frac{\partial^2 u_1}{\partial x_2^2} + 2C_{66}\frac{\partial^2 u_2}{\partial x_1 \partial x_2} + \rho\omega^2 u_1 &= 0, \\ C_{16}\left(\frac{\partial^2 u_1}{\partial x_1^2} - 2\frac{\partial^2 u_2}{\partial x_1 \partial x_2} - \frac{\partial^2 u_1}{\partial x_2^2}\right) + 2C_{66}\frac{\partial^2 u_1}{\partial x_1 \partial x_2} + C_{12}\frac{\partial^2 u_2}{\partial x_1^2} + C_{11}\frac{\partial^2 u_2}{\partial x_2^2} + \rho\omega^2 u_2 &= 0, \\ C_{44}\left(\frac{\partial^2 u_3}{\partial x_1^2} + \frac{\partial^2 u_3}{\partial x_2^2}\right) + e_{15}\left(\frac{\partial^2 \phi}{\partial x_1^2} - \frac{\partial^2 \phi}{\partial x_2^2}\right) + 2e_{14}\frac{\partial^2 \phi}{\partial x_1 \partial x_2} + \rho\omega^2 u_3 &= 0, \\ -e_{15}\left(\frac{\partial^2 u_3}{\partial x_1^2} - \frac{\partial^2 u_3}{\partial x_2^2}\right) - 2e_{14}\frac{\partial^2 u_3}{\partial x_1 \partial x_2} + \epsilon\left(\frac{\partial^2 \phi}{\partial x_1^2} + \frac{\partial^2 \phi}{\partial x_2^2}\right) &= 0. \end{aligned} \tag{4.1.12}$$

We also note that equations (4.1.12) decouple into two systems which can be studied separately. So, we write the equations using the differential operator L:

$$L_{\bar{4}}^{\omega}(\partial x) = \begin{bmatrix} L_{1,\bar{4}}(\partial x) & 0 \\ 0 & L_{2,\bar{4}}(\partial x) \end{bmatrix},$$

where $L_{1,\bar{4}}(\partial x)$ is given by

$$L_{1,\bar{4}}(\partial x) =$$

$$= \begin{bmatrix} C_{11}\xi_1^2 + 2C_{16}\xi_1\xi_2 + C_{12}\xi_2^2 + \rho\omega^2 & C_{16}\xi_1^2 + 2C_{66}\xi_1\xi_2 - C_{16}\xi_2^2 \\ C_{16}\xi_1^2 + 2C_{66}\xi_1\xi_2 - C_{16}\xi_2^2 & C_{12}\xi_1^2 - 2C_{16}\xi_1\xi_2 + C_{11}\xi_2^2 + \rho\omega^2 \end{bmatrix}, \tag{4.1.13}$$

and $L_{2,\bar{4}}(\partial x)$ is given by

$$L_{2,\bar{4}}(\partial x) = \begin{bmatrix} C_{44}(\xi_1^2 + \xi_2^2) + \rho\omega^2 & e_{15}(\xi_1^2 - \xi_2^2) + 2e_{14}\xi_1\xi_2 \\ -e_{15}(\xi_1^2 - \xi_2^2) - 2e_{14}\xi_1\xi_2 & \epsilon(\xi_1^2 + \xi_2^2) \end{bmatrix}. \tag{4.1.14}$$

Remark 19 *Let $u(x)$ denote a vector $u(x) = (u_1(x), u_2(x), u_3(x), \phi(x))^T$. Similarly as in the Section 3.1 it can be shown that the following assertions are valid. Let $\overline{S}^+ = S^+ \cup \partial S$.*

1. *For $u, v \in C^2(S) \cap C^1(\bar{S}) \cap \mathcal{M}_{4\times 1}$ the following equality holds*

$$\int_{S+} \left[v^T L u - u^T L v\right] dA = \int_{\partial S} \left[v^T T u - u^T T v\right] ds. \tag{4.1.15}$$

   Here $L = L(\partial x)$ is a differential operator for equations (4.1.6) or (4.1.12), $T = T(\partial x)$ is a "stress"-operator for the boundary value problems with Neumann-type boundary conditions, defined by (4.1.2).

2. *The internal energy density $U(u, u)$ for the case of transversely isotropic material $6mm$, given by*

$$U(u,u) = \frac{C_{66}}{2}(u_{1,2} + u_{2,1})^2 + \frac{C_{11}}{2}(u_{1,1} + u_{2,2})^2 - \frac{C_{66}}{2}u_{1,1}u_{2,2} +$$

$$+\frac{C_{44}}{2}(u_{3,1}^2+u_{3,2}^2)+\frac{\epsilon}{2}(\phi_{,1}^2+\phi_{,2}^2),$$

is a positive quadratic form if and only if

$$C_{11}, C_{66}, C_{44}, \epsilon > 0.$$

The internal energy density $U(u,u)$ *in the case of tetragonal* $\bar{4}$ *symmetry, written in terms of strain tensor components* $\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ *and electric field components* E_α

$$U(u,u) = \frac{1}{2}C_{11}(\varepsilon_{11}^2+\varepsilon_{22}^2) + 2C_{16}\varepsilon_{11}\varepsilon_{12} + C_{12}\varepsilon_{11}\varepsilon_{22} + 2C_{66}\varepsilon_{12}^2 -$$

$$-2C_{16}\varepsilon_{22}\varepsilon_{12} + 2C_{44}(\varepsilon_{13}^2+\varepsilon_{23}^2) + \frac{1}{2}\epsilon(E_1^2+E_2^2),$$

is a positive quadratic form if the matrix

$$\begin{pmatrix} C_{11} & 2C_{16} & C_{12} & 0 & 0 & 0 & 0 \\ 2C_{16} & C_{66} & -2C_{16} & 0 & 0 & 0 & 0 \\ C_{12} & -2C_{16} & C_{11} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2C_{44} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}\epsilon & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}\epsilon \end{pmatrix}$$

is positive definite. This condition will also guarantee the ellipticity of equations (4.1.6) *and equations* (4.1.12).

In addition, we have that:

3. $U(u,u) = 0$ *if and only if*

$$\begin{aligned} u_1 &= ax_2 + c_1, \\ u_2 &= -ax_1 + c_2, \\ u_3 &= c_3, \\ \phi &= c_4, \end{aligned} \tag{4.1.16}$$

where a, c_j, $j = 1,2,3,4$ *are arbitrary constants. These represent the most general form of the rigid displacement and the constant electric potential.*

4.2 Radiation conditions and uniqueness theorem

For vibrations problems involving an infinite domain we need to establish so-called 'radiation conditions' which prescribe the behavior of the various field quantities at infinity. These conditions are required to establish the necessary uniqueness results for the corresponding boundary value problems. As a result of the decomposition of the system (4.1.6), we can again consider the various components of the solution vector (u_1, u_2, u_3, ϕ) separately. Firstly, since $u_3(x)$ satisfies the plane Helmholtz equation, the radiation condition, or so-called Sommerfeld condition, assumes the form [81]

$$u_3(x) = O\left(\frac{1}{\sqrt{r}}\right), \quad \lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u_3(x)}{\partial r} - iku_3(x)\right) = 0. \tag{4.2.1}$$

For the electric potential $\phi(x)$, we can simply assume that $\phi(x)$ is bounded in the region $S^- = \mathbb{R}^2 \backslash \overline{S}$. To derive radiation conditions of the solution of the system (4.1.9) we follow the procedure for the decomposition of the solution as in [32]. That is, we represent the vector $u(x)$ as a sum of its potential $u^{(p)}(x)$ and solenoidal $u^{(s)}(x)$ components:

$$u^{(p)}(x) = -\frac{1}{k_1^2}\text{grad div } u(x), \qquad u^{(s)}(x) = \frac{1}{k_1^2}\text{grad div } u(x) + u(x)$$

which therefore satisfy the following equations

$$\begin{aligned} \Delta u^{(p)}(x) + k_1^2 u^{(p)}(x) &= 0, \\ \nabla \times u^{(p)}(x) &= 0, \\ \Delta u^{(s)}(x) + k_2^2 u^{(s)}(x) &= 0, \\ \nabla \cdot u^{(s)}(x) &= 0. \end{aligned} \tag{4.2.2}$$

Thus for $u^{(p)}(x)$ and $u^{(s)}(x)$ we have the following estimates

$$u^{(p)}(x) = O\left(\frac{1}{\sqrt{r}}\right), \quad \lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u^{(p)}(x)}{\partial r} - ik_1 u^{(p)}(x)\right) = 0, \tag{4.2.3a}$$

$$u^{(s)}(x) = O\left(\frac{1}{\sqrt{r}}\right), \quad \lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u^{(s)}(x)}{\partial r} - ik_2 u^{(s)}(x)\right) = 0, \tag{4.2.3b}$$

where

$$k_1^2 = \frac{\rho\omega^2}{C_{11}}, \quad k_2^2 = \frac{\rho\omega^2}{C_{66}}, \quad r = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$$

for $(x_1, x_2) \in S^-$, $(y_1, y_2) \in \partial S$. We note here that k^2, k_1^2 and k_2^2 are positive (see Remark 19).

Let R be the cylindrical coordinate of the point $x = (x_1, x_2) \in S^-$, $R_0 \in \mathcal{M}_{2\times 1}$ be the unit vector of the radius vector of x. Following the procedure used in [32] it can be shown that:

If $u = u^{(p)} + u^{(s)} \in C^2(S^-) \cap C^1(\bar{S}^-)$ such that it satisfies (4.1.7) in S and conditions (4.2.3a) and (4.2.3b) then the following estimates are true

$$\begin{aligned} \frac{\partial u^{(p)}(x)}{\partial x_\alpha} &= \frac{\partial u^{(p)}(x)}{\partial R}\frac{\partial R}{\partial x_\alpha} + O(R^{-3/2}), \\ \frac{\partial u^{(s)}(x)}{\partial x_\alpha} &= \frac{\partial u^{(s)}(x)}{\partial R}\frac{\partial R}{\partial x_\alpha} + O(R^{-3/2}), \end{aligned} \tag{4.2.4}$$

$$\begin{aligned} \frac{\partial u^{(p)}(x)}{\partial R} - ik_1 u^{(p)}(x) &= O(R^{-3/2}), \\ \frac{\partial u^{(s)}(x)}{\partial R} - ik_2 u^{(s)}(x) &= O(R^{-3/2}), \end{aligned} \tag{4.2.5}$$

$$\begin{aligned} \nabla \cdot u^{(p)}(x) - ik_1 R_0 \cdot u^{(p)}(x) &= O(R^{-3/2}), \\ \nabla \times u^{(s)}(x) - ik_2 R_0 \times u^{(s)}(x) &= O(R^{-3/2}), \end{aligned} \tag{4.2.6}$$

$$\begin{aligned} Tu^{(p)}(x) - iC_{11}k_1 u^{(p)}(x) &= O(R^{-3/2}), \\ Tu^{(s)}(x) - iC_{66}k_2 u^{(s)}(x) &= O(R^{-3/2}), \end{aligned} \tag{4.2.7}$$

$$u^{(s)} \cdot u^{(p)} = O(R^{-2}), \quad \bar{u}^{(s)} \cdot u^{(p)} = O(R^{-2}). \tag{4.2.8}$$

We can now prove the following result concerning the uniqueness of the solution for the corresponding exterior Dirichlet and Neumann-type boundary value problems for the theory of generalized plane strain of a linear piezoelectric medium:

Theorem 20 *If $u \in C^2(S^-) \cap C^1(\bar{S}^-)$ such that u satisfies* (4.1.7) *in S^-, conditions* (4.2.3a) *and* (4.2.3b) *and either of the homogeneous boundary conditions*

$$\begin{aligned} 1) \quad u(x) &= 0 \text{ on } \partial S \quad \textit{(Dirichlet Problem)}, \\ 2) \quad Tu(x) &= 0 \text{ on } \partial S \quad \textit{(Neumann Problem)}, \end{aligned}$$

then u is identically zero.

Proof. We apply (4.1.15) in the domain $C_R \cap S^-$, where C_R is a circle of sufficiently large radius R, to the solution u and $\bar{u}$, the complex conjugate of u and then consider the limit when $R \to \infty$:

$$\int_{C_R \cap S^-} (u^T L\bar{u} - \bar{u}^T Lu) dA = \int_{\partial C_R} (u^T T\bar{u} - \bar{u}^T Tu) ds - \int_{\partial S} (u^T T\bar{u} - \bar{u}^T Tu) ds.$$

Integrals over S^- and ∂S vanish since u solves (4.1.7) and satisfies one of the homogeneous boundary conditions. As for the integral over ∂C_R we apply the procedure used in [34] and estimates (4.2.4)-(4.2.8) to obtain that as $R \to \infty$

$$\int_{\partial C_R} (u^T T\bar{u} - \bar{u}^T Tu) ds = 2ik_1 C_{11} \int_{\partial C_R} |u^{(p)}|^2 ds +$$

$$+ 2ik_2 C_{66} \int_{\partial C_R} |u^{(s)}|^2 ds + 2i(k_1 C_{11} + k_2 C_{66}) \int_{\partial C_R} \mathrm{Re}(u^{(p)} \bar{u}^{(s)}) ds = 0. \quad (4.2.9)$$

Since $k_1, k_2 > 0$, with the help of Rellich's lemma [73] we conclude that $u^{(p)} = 0$ and $u^{(s)} = 0$ which completes the proof.

Using standard methods [14], Theorem 20 can now be used to establish that under the conditions of Theorem 20, any solution of either the Dirichlet or Neumann problem for the system (4.1.7), and hence for the system (4.1.6), is necessarily unique.

4.3 Fundamental and singular solutions

The fundamental solutions of the governing equations are required for the application of the boundary integral method which is used later to establish results on existence of solution. First, we find the matrix of fundamental solutions for the equations (4.1.6) describing steady-state vibrations for material with hexagonal symmetry $6mm$. Utilizing the decomposition of the system (4.1.6) we note that the fundamental solutions of the two-dimensional Laplace and Helmholtz operators, together with the representation formulae and smoothness properties of the solutions of the corresponding differential equations are well-documented and can be found, for example, in [81, 86]. For the operator $L(\partial x)$ appearing in (4.1.7), the Galerkin representation of the solution of the system

$$L(\partial x)u(x) = -\delta(x - y), \tag{4.3.1}$$

where $L(\partial x)$ is given by (4.1.8), produces the matrix of fundamental solutions in the form

$$\Gamma(x, y) = L^*(\partial x)\psi(x, y). \tag{4.3.2}$$

Here $L^*(\partial x)$ is transposed matrix of cofactors of $L(\partial x)$ and $\psi(x, y)$ is the function which solves the equation

$$L(\partial x)L^*(\partial x)\psi(x, y) = \det L(\partial x)\psi(x, y) = -\delta(x - y). \tag{4.3.3}$$

In our case

$$\det L = C_{11}C_{66}(\triangle + \frac{\rho\omega^2}{C_{11}})(\triangle + \frac{\rho\omega^2}{C_{66}}) = C_{11}C_{66}(\triangle + k_1^2)(\triangle + k_2^2)$$

and $\psi(x, y)$ is found to be

$$\psi(x, y) = \frac{i}{4C_{11}C_{66}(k_2^2 - k_1^2)}(H_0^{(1)}(k_2 r) - H_0^{(1)}(k_1 r)).$$

Here $H_0^{(1)}$ is Hankel function of the first kind of order zero [1]. We need to examine the behavior of matrices $\Gamma(x,y)$ and $D(x,y)$ when $r \to 0$ in order to establish boundary properties of integral potentials which we will consider later. From [1] we have the expansion for $H_0^{(1)}(k_\alpha r)$ for $r \to 0$

$$H_0^{(1)}(k_\alpha r) = 1 + \frac{i2}{\pi} \ln \frac{1}{2} + \frac{i2}{\pi}(1 - \frac{1}{4}k_\alpha^2 r^2) \ln k_\alpha r.$$

Hence, as $r \to 0$, the function $\psi(x,y)$ takes the form

$$\psi(x,y) = \frac{i}{4C_{11}C_{66}(k_1^2 - k_2^2)} \left\{ \frac{i2}{\pi} \ln \frac{k_2}{k_1} + \frac{i}{2\pi} \left(k_1^2 r^2 \ln k_1 r - k_2^2 r^2 \ln k_2 r \right) \right\}.$$

By application of $L^*(\partial x)$ to the function $\psi(x,y)$ we obtain elements of the matrix $\Gamma(x,y)$ as x approaches y:

$$\begin{aligned}
\Gamma_{11}(x,y) &= \left(\tfrac{C_{11}+C_{66}}{4\pi C_{11}C_{66}}\right)\left(k_2^2 \ln k_2 r - k_1^2 \ln k_1 r\right) + \tfrac{1}{4\pi}\left(\tfrac{(x_2-y_2)^2}{C_{66}r^2} + \tfrac{(x_1-y_1)^2}{C_{11}r^2}\right) + \\
&+ \tilde{\Gamma}_{11}, \quad \tilde{\Gamma}_{11} \in C^2(\mathbb{R}^2),
\end{aligned}$$

$$\begin{aligned}
\Gamma_{22}(x,y) &= \left(\tfrac{C_{11}+C_{66}}{4\pi C_{11}C_{66}}\right)\left(k_2^2 \ln k_2 r - k_1^2 \ln k_1 r\right) + \tfrac{1}{4\pi}\left(\tfrac{(x_2-y_2)^2}{C_{11}r^2} + \tfrac{(x_1-y_1)^2}{C_{66}r^2}\right) + \\
&+ \tilde{\Gamma}_{22}, \quad \tilde{\Gamma}_{22} \in C^2(\mathbb{R}^2),
\end{aligned}$$

$$\Gamma_{12}(x,y) = \Gamma_{21}(x,y) = \frac{(C_{11} - C_{66})}{4\pi C_{11}C_{66}} \frac{(x_1 - y_1)(x_2 - y_2)}{r^2} + \tilde{\Gamma}_{12}, \quad \tilde{\Gamma}_{12} \in C^2(\mathbb{R}^2).$$

Together with the matrix of fundamental solutions $\Gamma(x,y)$ we consider the matrix of singular solutions $D(x,y) = (T(\partial y, n)\Gamma(x,y))^T$ with components (as

$x \to y)$ given by

$$
\begin{aligned}
D_{11}(x,y) &= \frac{\partial}{\partial s(y)} \ln r \left(\frac{(C_{66}^2 - C_{11}^2) n_1 n_2}{4\pi C_{66} C_{11}} + \frac{2 n_1^2 (C_{11} n_1^2 + C_{66} n_2^2)}{4\pi C_{11}} + \right. \\
&+ \left. \frac{2 n_1 n_2 (C_{11} n_1^2 + C_{66} n_2^2)}{4\pi C_{66}} + \frac{n_1 n_2 (C_{66} - C_{11})^2 (n_1^2 - n_2^2)}{4\pi C_{11} C_{66}} \right) + \\
&+ \frac{\partial}{\partial n(y)} \ln r \left(\frac{-4 n_1^2 n_2^2 (C_{66}^2 + C_{11}^2) - 2 C_{11} C_{66} (n_1^2 + n_2^2)^2}{4\pi C_{66} C_{11}} \right) + \\
&+ \frac{\partial}{\partial s(y)} \left(\frac{(x_1 - y_1)^2}{r^2} \frac{n_1 n_2 (C_{66} - C_{11})}{4\pi C_{11}} + \frac{(x_2 - y_2)^2}{r^2} \frac{n_1 n_2 (C_{66} - C_{11})}{4\pi C_{66}} \right) + \\
&+ \frac{\partial}{\partial s(y)} \frac{(x_1 - y_1)(x_2 - y_2)}{r^2} \left(\frac{(C_{11} n_1^2 + C_{66} n_2^2)}{4\pi C_{11}} - \frac{(C_{11} n_1^2 + C_{66} n_2^2)}{4\pi C_{66}} + \right. \\
&+ \left. \frac{n_1 n_2 (C_{66} - C_{11})^2}{4\pi C_{11} C_{66}} \right) + \tilde{D}_{11}, \quad \tilde{D}_{11} \in C^1(\mathbb{R}^2),
\end{aligned}
$$

$$
\begin{aligned}
D_{12}(x,y) &= \frac{\partial}{\partial s(y)} \ln r \left(\frac{(C_{11}^2 - C_{66}^2) n_2 n_1}{4\pi C_{66} C_{11}} - \frac{2 n_1^2 n_1 n_2 (C_{11} - C_{66})}{4\pi C_{66}} - \frac{2 n_1^2 n_2^2 (C_{11} - C_{66})}{4\pi C_{11}} + \right. \\
&+ \left. \frac{(C_{66} - C_{11})(C_{66} n_2^2 + C_{11} n_1^2)(n_2^2 - n_1^2)}{4\pi C_{66} C_{11}} \right) + \\
&+ \frac{\partial}{\partial n(y)} \ln r \left(\frac{(C_{66} - C_{11})(C_{66} n_2^2 + C_{11} n_1^2) 2 n_1 n_2}{4\pi C_{66} C_{11}} - \frac{(C_{11}^2 - C_{66}^2) n_2 n_1}{4\pi C_{66} C_{11}} - \right. \\
&- \left. \frac{n_1 n_2 (C_{11} - C_{66})(2 n_2^2 - 1)}{4\pi C_{11}} - \frac{n_1 n_2 (C_{11} - C_{66})(2 n_1^2 - 1)}{4\pi C_{66}} \right) + \\
&+ \frac{\partial}{\partial s(y)} \frac{(x_1 - y_1)(x_2 - y_2)}{r^2} \left(\frac{n_1 n_2 (C_{11} - C_{66})}{4\pi C_{66}} - \frac{n_1 n_2 (C_{11} - C_{66})}{4\pi C_{11}} - \right. \\
&- \left. \frac{(C_{11} - C_{66})^2 n_1 n_2}{4\pi C_{66} C_{11}} \right) + \frac{\partial}{\partial s(y)} \frac{(x_1 - y_1)^2}{r^2} \left(\frac{n_1 n_2 (C_{11} - C_{66})^2}{4\pi C_{66}} \right) + \\
&+ \frac{\partial}{\partial s(y)} \frac{(x_2 - y_2)}{r^2} \left(\frac{n_1 n_2 (C_{11} - C_{66})}{4\pi C_{11}} \right) + \tilde{D}_{12}, \quad \tilde{D}_{12} \in C^1(\mathbb{R}^2),
\end{aligned}
$$

$$
\begin{aligned}
D_{21}(x,y) &= \frac{\partial}{\partial s(y)} \ln r \left(\frac{(C_{66}^2 - C_{11}^2) n_2^2}{4\pi C_{66} C_{11}} + \frac{2 n_1^2 n_1 n_2 (C_{11} - C_{66})}{4\pi C_{11}} - \frac{2 n_1^2 n_2^2 (C_{11} - C_{66})}{4\pi C_{66}} + \right. \\
&+ \left. \frac{(C_{66} - C_{11})(C_{66} n_1^2 + C_{11} n_2^2)(n_2^2 - n_1^2)}{4\pi C_{66} C_{11}} \right) + \\
&+ \frac{\partial}{\partial n(y)} \ln r \left(\frac{(C_{66} - C_{11})(C_{66} n_1^2 + C_{11} n_2^2) 2 n_1 n_2}{4\pi C_{66} C_{11}} - \frac{(C_{11}^2 - C_{66}^2) n_2 n_1}{4\pi C_{66} C_{11}} - \right. \\
&- \left. \frac{n_1 n_2 (C_{11} - C_{66})(2 n_1^2 - 1)}{4\pi C_{11}} - \frac{n_1 n_2 (C_{11} - C_{66})(2 n_2^2 - 1)}{4\pi C_{66}} \right) + \\
&+ \frac{\partial}{\partial s(y)} \frac{(x_1 - y_1)(x_2 - y_2)}{r^2} \left(\frac{n_1 n_2 (C_{11} - C_{66})}{4\pi C_{11}} - \frac{n_1 n_2 (C_{11} - C_{66})}{4\pi C_{66}} - \right. \\
&+ \left. \frac{(C_{11} - C_{66})^2 n_1 n_2}{4\pi C_{66} C_{11}} \right) - \frac{\partial}{\partial s(y)} \frac{(x_1 - y_1)^2}{r^2} \left(\frac{n_2^2 (C_{11} - C_{66})}{4\pi C_{66}} \right) + \\
&- \frac{\partial}{\partial s(y)} \frac{(x_2 - y_2)^2}{r^2} \left(\frac{n_2^2 (C_{11} - C_{66})}{4\pi C_{11}} \right) + \tilde{D}_{21}, \quad \tilde{D}_{21} \in C^1(\mathbb{R}^2),
\end{aligned}
$$

$$
\begin{aligned}
D_{22}(x,y) &= \frac{\partial}{\partial s(y)} \ln r \left(\frac{(C_{66}^2 - C_{11}^2) n_1 n_2}{4\pi C_{66} C_{11}} + \frac{2 n_1 n_2 (C_{11} n_2^2 + C_{66} n_1^2)}{4\pi C_{11}} + \right. \\
&+ \left. \frac{2 n_1^2 (C_{11} n_2^2 + C_{66} n_1^2)}{4\pi C_{66}} + \frac{n_1 n_2 (C_{66} - C_{11})^2 (n_1^2 - n_2^2)}{4\pi C_{11} C_{66}} \right) + \\
&+ \frac{\partial}{\partial n(y)} \ln r \left(\frac{-4 n_1^2 n_2^2 (C_{66}^2 + C_{11}^2) - 2 C_{11} C_{66} (n_1^2 + n_2^2)^2}{4\pi C_{66} C_{11}} \right) + \\
&+ \frac{\partial}{\partial s(y)} \left(-\frac{(x_1 - y_1)^2}{r^2} \frac{n_1 n_2 (C_{66} - C_{11})}{4\pi C_{66}} - \frac{(x_2 - y_2)^2}{r^2} \frac{n_1 n_2 (C_{66} - C_{11})}{4\pi C_{11}} \right) + \\
&+ \frac{\partial}{\partial s(y)} \frac{(x_1 - y_1)(x_2 - y_2)}{r^2} \left(\frac{(C_{11} n_2^2 + C_{66} n_1^2)}{4\pi C_{66}} - \frac{(C_{11} n_2^2 + C_{66} n_1^2)}{4\pi C_{11}} + \right. \\
&- \left. \frac{n_2^2 (C_{66} - C_{11})^2}{4\pi C_{11} C_{66}} \right) + \tilde{D}_{22}, \quad \tilde{D}_{22} \in C^1(\mathbb{R}^2).
\end{aligned}
$$

We also need to consider the behavior of matrices $\Gamma(x,y)$ and $D(x,y)$ when $r \to \infty$ which will be used to establish a representation theorem. So we note here that for the matrix of fundamental solutions $\Gamma(x,y)$ we can similarly define potential and solenoidal components $\Gamma^p(x,y)$ and $\Gamma^s(x,y)$ whose columns satisfy conditions similar to (4.2.3a) and (4.2.3b) and which have representation similar to that used in [34]. That is:

$$
\begin{aligned}
\Gamma^p &= \frac{1}{k_2^2 - k_1^2} (\triangle + k_2^2) \Gamma(x,y) = \frac{1}{k_2^2 - k_1^2} (\triangle + k_2^2) L^*(\partial x) \psi = \frac{L^*(\partial x)}{k_2^2 - k_1^2} (\triangle + k_2^2) \psi, \\
\Gamma^s &= \frac{1}{k_1^2 - k_2^2} (\triangle + k_1^2) \Gamma(x,y) = \frac{1}{k_1^2 - k_2^2} (\triangle + k_1^2) L^*(\partial x) \psi = \frac{L^*(\partial x)}{k_1^2 - k_2^2} (\triangle + k_1^2) \psi.
\end{aligned}
$$

Thus we will have

$$
\Gamma^p(x,y) = \frac{L^*(\partial x) H_0^{(1)}(k_1 r)}{4 C_{11} C_{66} (k_2^2 - k_1^2)}, \quad \Gamma^s(x,y) = \frac{L^*(\partial x) H_0^{(1)}(k_2 r)}{4 C_{11} C_{66} (k_1^2 - k_2^2)}.
$$

For $r \to \infty$ we have the following approximation for $H_0^{(1)}(kr)$ [1]

$$
H_0^{(1)}(kr) = \sqrt{\frac{2}{\pi r}} \exp^{ikr} + \ldots . \tag{4.3.4}
$$

Using (4.3.4) it can be shown that conditions similar to (4.2.4)-(4.2.8) are valid for the columns of the matrices $\Gamma^p(x,y)$ and $\Gamma^s(x,y)$. The method used in

Section 4.3 to find fundamental solutions can not be applied to system (4.1.12) as the roots of the characteristic equation for the differential operator $L_4^{\omega}(\partial x)$ describing governing equations (4.1.12) are not easily found. However, it is possible to establish the behavior of fundamental solutions near the boundary.

We will call matrices $\Gamma^{\alpha}(x, y, \omega) \in \mathcal{M}_{2\times 2}$ satisfying the equation

$$L_{\alpha}(\partial x)\Gamma^{\alpha}(x, y, \omega) = -\delta(x - y)I \tag{4.3.5}$$

the matrices of fundamental solutions for operators $L_{\alpha}(\partial x)$. By $\delta(x - y)$ we denote the Dirac distribution. Applying to (4.3.5) Fourier transformation $\mathcal{F}_{x\to\xi}$ we obtain

$$[L_{\alpha}(\xi)]\tilde{\Gamma}^{\alpha}(\xi, \omega) = I,$$

or, equivalently,

$$\tilde{\Gamma}^{\alpha}(\xi, \omega) = \frac{1}{\det L_{\alpha}(\xi)}\mathrm{adj}(L_{\alpha}(\xi)), \tag{4.3.6}$$

where $\tilde{\Gamma}^{\alpha}(\xi, \omega) = \mathcal{F}_{x\to\xi}[\Gamma^{\alpha}(x, y, \omega)]$. From (4.3.6) it follows now that

$$\Gamma^{\alpha}(x, y, \omega) = \mathcal{F}_{\xi\to x}\left[\frac{1}{\det L_{\alpha}(\xi)}\mathrm{adj}(L_{\alpha}(\xi))\right].$$

In [58] two-dimensional steady-state oscillation problems of anisotropic elasticity are investigated for the most general case of anisotropic material. Since the system described by the differential operator (4.1.13) is a particular case of the equations considered in [58], the results on properties of fundamental solutions from [58] are applied immediately. The equations defined by (4.1.14) have essentially the same properties as (4.1.13) so we can expect that both matrices of fundamental solutions $\Gamma^{\alpha}(x,y,\omega)$ possess the following properties

$$\begin{aligned} \Gamma^{(\alpha)}_{\beta\gamma}(x,y,\omega) &= O(\ln|x-y|), \\ \frac{\partial^{\kappa}}{\partial x_p^{\kappa}}\Gamma^{(\alpha)}_{\beta\gamma}(x,y,\omega) &= O(|x-y|^{|\kappa|}), \quad |\kappa| \geq 1. \end{aligned} \tag{4.3.7}$$

The matrix $\Gamma(x,y,\omega) \in \mathcal{M}_{4\times 4}$ of fundamental solutions for (4.1.12) for the particular case of materials with tetragonal $\bar{4}$ symmetry can be written in the form

$$\Gamma(x,y,\omega) = \begin{bmatrix} \Gamma^1(x,y,\omega) & 0 \\ 0 & \Gamma^2(x,y,\omega) \end{bmatrix}.$$

Together with the matrix of fundamental solutions $\Gamma(x,y,\omega)$ we consider the matrix of singular solutions

$$D(x,y,n) = (T(\partial y,n)\Gamma(y,x,\omega))^T. \tag{4.3.8}$$

4.4 Representation theorems

Here, similarly as in Sections 3.3, 3.4, we formulate representation theorems that will be used later to establish properties of the single and the double layer potentials in the case of steady-state vibrations.

Theorem 21 *If $u \in C^2(S^-) \cap C^1(\bar{S^-})$, satisfying (4.2.3a) and (4.2.3b), is a solution of* (4.1.7) *then $u(x)$ can be represented by the equation*

$$-\int_{\partial S}\left[\Gamma(x,y)T(\partial y,n)u - u^T T(\partial y,n)\Gamma(x,y)\right] ds(y) = \begin{cases} I_c u(x), & x \in S^-, \\ \frac{1}{2}I_c u(x), & x \in \partial S, \\ 0, & x \in S. \end{cases} \tag{4.4.1}$$

Here elements of $I_c \in \mathcal{M}_{2\times 2}$ *are given by*

$$I_{c,11} = I_{c,22} = \frac{-4n_1^2 n_2^2(C_{66}^2 + C_{11}^2) - 2C_{11}C_{66}(n_1^2+n_2^2)^2}{4\pi C_{66}C_{11}},$$

$$\begin{aligned} I_{c,\alpha\beta} &= \tfrac{(C_{66}-C_{11})(C_{66}n_\beta^2 + C_{11}n_\alpha^2)2n_\alpha n_\beta}{4\pi C_{66}C_{11}} - \tfrac{(C_{11}^2 - C_{66}^2)n_\alpha n_\beta}{4\pi C_{66}C_{11}} - \\ &- \tfrac{n_1 n_2(C_{11}-C_{66})(2n_\beta^2-1)}{4\pi C_{11}} - \tfrac{n_1 n_2(C_{11}-C_{66})(2n_\alpha^2-1)}{4\pi C_{66}}, \ \textit{for } \alpha \neq \beta. \end{aligned}$$

Proof. We apply equality (4.1.15) in the domain $S \setminus C(x,\varepsilon)$, where $C(x,\varepsilon)$ is a circular disk with a center $x \in S$ and sufficiently small radius ε, in which we take $v = \Gamma^{(\alpha)}(x,y)$. Using classical techniques [14, 34], for the solution $u(x)$ of the system (4.1.7) we have

$$\int_{\partial S} \left[\Gamma(x,y)T(\partial y,n)u - u^T T(\partial y,n)\Gamma(x,y)\right] ds(y) = \begin{cases} I_c u(x), & x \in S, \\ \frac{1}{2} I_c u(x), & x \in \partial S, \\ 0, & x \in \mathbb{R}^2 \setminus \bar{S}. \end{cases} \tag{4.4.2}$$

We now apply (4.1.15) in the domain $C_R \cap S^-$ and then consider the limit when $R \to \infty$. For $x \in C_R \cap S^-$ we will have

$$\begin{aligned} I_c u(x) = &- \textstyle\int_{\partial S} \left[\Gamma(x,y)T(\partial y,n)u - u^T T(\partial y,n)\Gamma(x,y)\right] ds(y) + \\ &+ \textstyle\int_{\partial C_R} \left[\Gamma(x,y)T(\partial y,n)u - u^T T(\partial y,n)\Gamma(x,y)\right] ds(y). \end{aligned}$$

Next, using the procedure similar to the one used in the proof of Theorem 20, estimates (4.2.4)-(4.2.8) and similar estimates for the columns of matrices $\Gamma^p(x,y)$ and $\Gamma^s(x,y)$ we show that when $R \to \infty$ the integral taken over ∂C_R vanishes. The cases when $x \in \partial S$ and $x \in S$ are treated similarly (see for example the proof of representation theorems for static anti-plane shear in Chapter 3).

We need also to establish the representation theorem for the solutions of the steady-state vibrations for tetragonal $\bar{4}$ symmetry. Using classical techniques [34, 14] and the estimates of matrices of fundamental solutions in case of $\bar{4}$ we have the integral representation of a regular vector $u(x)$ in a bounded domain:

Theorem 22 *If $u \in C^2(S^+) \cap C^1(\bar{S}^+)$ is a solutions of* (4.1.12) *then it can be represented by the equation*

$$\int_{\partial S} \left[\Gamma(x,y,\omega)T(\partial y,n)u - u^T T(\partial y,n)\Gamma(y,x,\omega)\right] ds(y) = \begin{cases} u(x), & x \in S^+, \\ \frac{1}{2}u(x), & x \in \partial S, \\ 0, & x \in S^-. \end{cases}$$

Similarly, the representation formula for the solution of (4.1.12) in an unbounded domain $S^- = \mathbb{R}^2/\bar{S}^+$ can be derived. Using estimates for the behavior of matrices of fundamental solutions at infinity derived for plane anisotropic elasticity in [58] and the fact that both systems defined by operators (4.1.13) and (4.1.14) possess the same properties we have the following representation theorem for the solutions of (4.1.12) in an unbounded domain for the tetragonal symmetry $\bar{4}$:

Theorem 23 *If $u \in C^2(S^-) \cap C^1(\bar{S})$ is a solution of* (4.1.12) *and satisfy the radiation conditions [58] then $u(x)$ can be represented by the equation*

$$-\int_{\partial S} \left[\Gamma(x,y,\omega)T(\partial y,n)u - u^T T(\partial y,n)\Gamma(y,x,\omega)\right] ds(y) = \begin{cases} u(x), & x \in S^+, \\ \frac{1}{2}u(x), & x \in \partial S, \\ 0, & x \in S^-. \end{cases}$$

4.5 Static Green's tensors for Neumann and Dirichlet problems

As in [34] we will use static Green's tensors to establish the existence of a discrete spectrum of eigenfrequencies for Dirichlet and Neumann boundary value problems for the equations (4.1.12). This will be important in the proof of the solvability of homogeneous and inhomogeneous integral equations for the corresponding boundary value problems for the equations (4.1.12) and, therefore, the existence of the solutions of the Dirichlet and Neumann boundary value problems for the equations of steady-state vibrations (4.1.12).

Static Green's tensor of the Dirichlet problem, i.e. Green's tensor for the system (4.1.12) with $\omega = 0$, is a matrix $G^D(x, y)$ such that

$$\begin{aligned} L^0(\partial x)G^D(x,y) &= 0, \quad x \in S^+, \quad x \neq y \\ G^D(x,y) &= 0, \quad x \in \partial S \quad y \in S^+, \\ G^D(x,y) &= \Gamma(x,y) - v_1(x,y), \quad x \in S^+, \end{aligned} \tag{4.5.1}$$

where $v_1(x, y) \in C^2(S^+) \cap C^1(\bar{S}^+)$ is a solution of the equation

$$L^0(\partial x)v_1(x,y) = 0.$$

$L^0(\partial x)$ is obtained by taking $\omega = 0$ in $L(\partial x)$ and $\Gamma(x, y)$ is the matrix of fundamental solutions for $L^0(\partial x)$. Thus, to find $G^D(x, y)$ we have to construct the matrix of fundamental solutions for the operator $L^0(\partial x)$ and solve the following Dirichlet problem to find the appropriate $v_1(x,y)$:

$$\begin{aligned} L^0(\partial x)v_1(x,y) &= 0 \quad x \in S^+, \\ v_1(x,y) &= \Gamma(x,y), \quad x \in \partial S. \end{aligned} \tag{4.5.2}$$

The operator $L^0(\partial x)$ can be written in the form

$$L^0 = \begin{bmatrix} L_1^0 & 0 \\ 0 & L_2^0 \end{bmatrix},$$

where $L_\alpha^0(\partial x)$ denote static operators obtained by taking $\omega = 0$ in $L_\alpha(\partial x)$. Also we can write the matrix $\Gamma(x,y)$ in the form

$$\Gamma(x,y) = \begin{bmatrix} \Gamma^1(x,y) & 0 \\ 0 & \Gamma^2(x,y) \end{bmatrix},$$

where the matrix $\Gamma^1(x,y)$ is the matrix of fundamental solutions for $L_1^0(\partial x)$, given by the following equation (for details see [33, 27]):

$$\Gamma^1(x,y) = L_1^{0*}(\partial x)\psi(x,y). \tag{4.5.3}$$

Here $L_1^{0*}(\partial x)$ is transposed matrix of cofactors of $L_1^0(\partial x)$ and the function $\psi(x,y)$ is given by (3.2.4) with

$$a = -\frac{1}{C_{11}C_{12} - C_{16}^2};$$

and α_j are the roots of the characteristic equation corresponding to the differential operator $L_1^0(\partial x)$ given by

$$\frac{1}{a}\alpha^4 + b_1\alpha^3 + b_2\alpha^2 + b_3\alpha + b_4 = 0,$$

with coefficients

$$\begin{aligned}
b_1 &= 2C_{12}C_{16} - 4C_{16}C_{66} - 2C_{11}C_{16},\\
b_2 &= C_{11}^2 + 2C_{16}^2 + C_{12}^2 + 4C_{66}^2,\\
b_3 &= 2C_{11}C_{16} - 4C_{16}C_{66} - 2C_{12}C_{16},\\
b_4 &= \tfrac{1}{a} = C_{11}C_{12} - C_{16}^2.
\end{aligned}$$

From (4.5.3) we obtain components of the matrix of fundamental solutions $\Gamma^1(x,y)$

$$\begin{aligned}
\Gamma^1_{11}(x,y) &= ia\textstyle\sum_{j=1}^4(-1)^j d_j(C_{12}\alpha_j^2 - 2C_{16}\alpha_j + C_{11})(2\log\sigma_j + 1),\\
\Gamma^1_{22}(x,y) &= ia\textstyle\sum_{j=1}^4(-1)^j d_j(C_{11}\alpha_j^2 + 2C_{16}\alpha_j + C_{12})(2\log\sigma_j + 1),\\
\Gamma^1_{12}(x,y) &= -ia\textstyle\sum_{j=1}^4(-1)^j d_j(C_{16}\alpha_j^2 + 2C_{66}\alpha_j - C_{16})(2\log\sigma_j + 1),\\
\Gamma^1_{21}(x,y) &= \Gamma^1_{12}(x,y).
\end{aligned} \tag{4.5.4}$$

The matrix of fundamental solutions $\Gamma^2(x,y)$ for $L_2^0(\partial x)$ was found in Section 3.2. For the proof of the existence of the solution of (4.5.2) we refer to [33] for the operator $L_1^0(\partial x)$ and to the Section 3.6 for the operator L_2^0. Thus, the existence of the static Green's tensor $G^D(x,y)$ is proved.

The static Green's tensors for the Neumann problem $G^N(x,y)$ for (4.1.12) we will construct as follows [34]:

$$\begin{aligned}
L^0G^N(x,y) &= \textstyle\sum_{j=1}^5 \mathcal{F}^k(x,y), \quad x\in S^+, \quad x\neq y,\\
T(\partial x)G^N(x,y) &= 0, \quad x\in\partial S \quad y\in S^+, \qquad\qquad (4.5.5)\\
G^N(x,y) &= \Pi(x,y) - v_2(x,y), \quad x\in S^+,
\end{aligned}$$

where $v_2(x,y) \in C^2(S^+) \cap C^1(\bar{S}^+)$ is a solution of the equation

$$L^0(\partial x) v_2(x,y) = 0.$$

In (4.5.5) the vector $\mathcal{F}^k(x,y) = f^k(x) * f^k(y)$ is the $*$-product of vectors defined as follows

$$\{f^k f_1^k, f^k f_2^k, f^k f_3^k, f^k f_4^k\}$$

and $\{f^k\}_{k=1}^5$ is a set of linearly independent vectors

$$\begin{aligned} f^1 &= (c_1, 0, 0, 0), \\ f^2 &= (0, c_2, 0, 0), \\ f^3 &= (0, 0, c_3, 0), \\ f^4 &= (0, 0, 0, c_4), \\ f^5 &= (c_5 x_2, -c_5 x_1, 0, 0). \end{aligned} \tag{4.5.6}$$

Every solution $u(x) = (u_1(x), u_2(x), u_3(x), \phi(x))^T$ of the form (4.1.16) of the homogeneous static Neumann problem can be represented as a linear combination of $\{f^k\}_{k=1}^5$. Since the coefficients a, c_j, $j = 1, ..., 4$ are arbitrary they can be chosen so that the set f^k, $k = 1, ..., 5$ is orthonormalized in S^+. As in [34], we can locate the coordinate system in the center of gravity of the cross section and put

$$\begin{aligned} c_j &= \tfrac{1}{\sqrt{m}}, \quad j = 1, ...4, \\ c_5 &= \tfrac{1}{\sqrt{I_{x_1} + I_{x_2}}}, \end{aligned}$$

where m is a "mass" of the cross-section S^+, I_{x_α} are moments of inertia of S^+, then the set (4.5.6) we be orthonormalized, i.e. the following conditions are met:

$$\int_{S^+} f^k f^l dA = \begin{cases} 0 \ , & k \neq l, \\ 1 \ , & k = l, \end{cases} \qquad k, l, = 1, ..., 5.$$

We take matrix $\Pi(x, y)$ in the form [34]:

$$\begin{aligned} \Pi(x,y) = & \ \tfrac{1}{2} \quad \Gamma(x,y) - \tfrac{1}{2}\textstyle\sum_{k=1}^{5} f^k(x) * \int_{S^+} \Gamma(y,\xi) f^k(\xi) d\xi \\ - & \ \tfrac{1}{2}\textstyle\sum_{k=1}^{5} \int_{S^+} \Gamma(x,\xi) f^k(\xi) d\xi * f^k(y) + \\ + & \ \tfrac{1}{4}\textstyle\sum_{k,r=1}^{5} f^k(x) * f^r(y) \int_{S^+}\int_{S^+} f^k(\xi)\Gamma(\xi,\nu) f^r(\nu) d\xi d\nu. \end{aligned} \tag{4.5.7}$$

Since

$$L_0(\partial x) f^k(x) = 0,$$

and

$$L_0(\partial x) \int_{S^+} \Gamma(x,\xi) f^k(\xi) d\xi = -2 f^k(x), \quad x \in S^+$$

it is easy to see that for $x \neq y$

$$L_0(\partial x)\Pi(x,y) = \sum_{k=1}^{5} f^k(x) * f^k(y).$$

To find the columns of the matrix $v_2(x, y)$ we have to solve the following boundary value problem

$$\begin{aligned} L_0(\partial x) v_2^{(j)}(x,y) &= 0, \quad x, y \in S^+, \\ T(\partial x, n) v_2^{(j)}(x,y) &= T(\partial x, n)\Pi^{(j)}(x,y), \quad x \in \partial S. \end{aligned}$$

From the Section 3.6 and [33] for the solubility of this problem it is necessary and sufficient that

$$\int_{\partial S} \left(T(\partial z, n)\Pi^{(\alpha)}(z,y)\right) f^k(z) ds(z) = 0, \quad k = 1, ...5, \quad j = 1, ...4. \tag{4.5.8}$$

We should show that this condition is fulfilled by the choice of the matrix $\Pi(x,y)$. Since $T(\partial x, n)f^k(x) = 0$, $k = 1, \ldots 5$, we can write

$$\begin{aligned} T(\partial x, n)\Pi^{(j)}(x,y) = & \ \tfrac{1}{2} \ T(\partial x, n)\Gamma^{(j)}(x,y) \\ - & \ \tfrac{1}{2} \ \textstyle\sum_{k=1}^{5} T(\partial x, n) \int_{S^+} \Gamma(x,y) f^k(\xi) f_j^k(y) d\xi. \end{aligned} \tag{4.5.9}$$

We can now write (4.5.8) in the following way

$$\begin{aligned} & \textstyle\int_{\partial S} \ \left(T(\partial x, n)\Gamma^{(j)}(x,y)\right) f^r(x) ds(x) - \\ - & \ \textstyle\sum_{k=1}^{5} \int_{S^+} \int_{\partial S} \left(T(\partial x, n)\Gamma(x,y) f^k(\xi)\right) f^r(x) f_j^k(y) ds(x) d\xi = 0, \\ & (r = 1, \ldots 5). \end{aligned} \tag{4.5.10}$$

Since

$$\left(T(\partial x, n)\Gamma(x,y) f^k(\xi)\right) f^r(x) = \left(T(\partial x, n)\Gamma(x,\xi)\right) f^r(x) f^k(\xi)$$

and from representation formula (Section 4.4)

$$f^k(x) = -\frac{1}{2}\int_{\partial S} \left[T(\partial y, n)\Gamma(y,x)\right]^T f^k(y) ds(y), \quad x \in S, \quad k = 1, \ldots 5, \tag{4.5.11}$$

the second term of the equality (4.5.10) assumes the form

$$2\sum_{k=1}^{5}\int_{S^+} f^r(\xi) f^k(\xi) d\xi f_i^k(y).$$

And since the set $\{f^k\}_{k=1}^5$ is orthonormalized in S^+, this integral is equal to $f_i^r(y)$. The substitution of (4.5.11) back into (4.5.10) gives the identity. Thus the condition (4.5.8) is fulfilled. We note here that

$$G^D(x,y) = \left[G^D(y,x)\right]^T, \quad G^N(x,y) = \left[G^N(y,x)\right]^T.$$

Since Green's tensors possess all the properties of the fundamental solutions we can rewrite the representation formulae as follows:

$$
\begin{aligned}
u(x) &= \int_{\partial S}\left[G^D(x,y)\left(T(\partial y,n)u(y)\right)-\left(T(\partial y,n)G^D(y,x)\right)u(y)\right]ds(y)- \\
&- \int_{S^+} G^D(x,y)L_0(\partial y)u(y)dy;
\end{aligned}
\tag{4.5.12}
$$

$$
\begin{aligned}
u(x) &= \int_{\partial S}\left[G^N(x,y)\left(T(\partial y,n)u(y)\right)-\left(T(\partial y,n)G^N(y,x)\right)u(y)\right]ds(y)+ \\
&+ \int_{S^+} u(y)\sum_{k=1}^{5} f^k(y)*f^k(x)dy - \int_{S^+} G^D(x,y)L_0(\partial y)u(y)dy.
\end{aligned}
\tag{4.5.13}
$$

It can be shown [34] that the solutions of the homogeneous Dirichlet and Neumann problems for (4.1.12) can be represented as solutions of integral equations

$$
u(x)-\omega^2\int_{S^+} G(x,y)u(y)dy=0, \quad x\in S^+, \tag{4.5.14}
$$

where $G(x,y)$ denotes $G^D(x,y)$ in the case of the Dirichlet problem and $G^N(x,y)$ in the case of the Neumann problem. These results follow from the representation formulae (4.5.12) and (4.5.13) and the properties of Green's tensors $G^D(x,y)$ and $G^N(x,y)$. The equations (4.5.14) are Fredholm's homogeneous equations with symmetrical kernels in $L^2(S^+)$. In accordance with the Hilbert-Schmidt theorem follows the existence of a discrete spectrum of real eigenvalues of the parameter ω^2 for which the equations (4.5.15) have non-zero solutions [34]. Thus, the following theorems are valid [34]:

Theorem 24 *The homogeneous Dirichlet boundary value problem for* (4.1.12) *has a discrete spectrum of eigenfrequencies which are eigenvalues of the integral* *equation*

$$u(x) - \omega^2 \int_{S^+} G^D(x,y)u(y)dy = 0.$$

These frequencies are positive.

Theorem 25 *The homogeneous Neumann boundary value problem for $L(\partial x)$ has a discrete spectrum of real eigenfrequencies which are eigenvalues of the integral equation*

$$u(x) - \omega^2 \int_{S^+} G^N(x,y)u(y)dy = 0.$$

These frequencies are non-negative and $\omega = 0$ is an eigenvalue of the fifth rank and the corresponding eigenvectors are vectors of the form (4.1.16).

4.6 Single and double layer potentials

The question of existence of the solution of the corresponding Dirichlet and Neumann-type boundary value problems is answered using the boundary integral equation method. To this end, we construct the single layer potential $V\varphi \in \mathcal{M}_{2\times1}$ in the form

$$V\varphi(x) = \int_{\partial S} \Gamma(x,y)\varphi(y)ds(y)$$

and the double layer potential $W\varphi \in \mathcal{M}_{2\times1}$

$$W\varphi(x) = \int_{\partial S} D(x,y)\varphi(y)ds(y)$$

with density $\varphi \in \mathcal{M}_{2\times 1}$. The matrices of fundamental and singular solutions are taken accordingly for each class of piezoelectric materials. The potentials have the following properties.

Theorem 26 *1. If $\varphi \in C(\partial S)$ then $V\varphi$ and $W\varphi$ are analytic and satisfy equations (4.1.9) in the case of tetragonal class $6mm$ and equations (4.1.12) in the case of tetragonal class $\bar{4}$ in $S(S^-)$.*

2. *For $W\varphi(x)$ the estimates similar to (4.2.4)-(4.2.8) are valid.*

3. *For $V\varphi(x)$ the estimates similar to (4.2.4)-(4.2.8) are valid if and only if*

$$\int_{\partial S} \varphi ds = 0$$

4. *If $\varphi \in C(\partial S)$ then $V\varphi \in C^{0,\alpha}(R^2)$.*

5. *If $\varphi \in C^{1,\alpha}(\partial S)$ for $\alpha \in (0,1]$, then $W\varphi$ has $C^{1,\beta}$-extensions $(W\varphi)^+$ and $(W\varphi)^-$ to $\overline{S}$ and $\overline{S}^-$, respectively, with $\beta \in (0,1)$.*

$$(W\varphi)^+ = \begin{cases} W\varphi, & x \in S, \\ \frac{1}{2}I_c\varphi + W_0\varphi, & x \in \partial S, \end{cases}$$

$$(W\varphi)^- = \begin{cases} W\varphi, & x \in S^-, \\ -\frac{1}{2}I_c\varphi + W_0\varphi, & x \in \partial S. \end{cases}$$

6. *If $\varphi \in C^{0,\alpha}(\partial S)$ then $V\varphi \in C^{1,\alpha}(R^2)$ for $\alpha \in (0,1)$ and*

$$T\,(V\varphi)^+ = \begin{cases} T\,(V\varphi), & x \in S, \\ -\frac{1}{2}I_c\varphi + T\,(V_0\varphi), & x \in \partial S, \end{cases}$$

$$T\left(V\varphi\right)^{-}=\begin{cases} T\left(V\varphi\right), & x\in S^{-},\\ \frac{1}{2}I_c\varphi+T\left(V_0\varphi\right), & x\in\partial S, \end{cases}$$

$T=T(\partial x,n)$.

7. $T\left(W\varphi\right)^{+}=T\left(W\varphi\right)^{-}$ *on* ∂S.

Here $W_0\varphi$ and $V_0\varphi$ denote the values of the double layer and single layer potentials, respectively, for $x\in\partial S$. The matrix I_c is given in the statement of the Theorem 21 for the case of the hexagonal piezoelectric material $6mm$ and is defined by the particular form of the fundamental solutions for the case of tetragonal piezoelectric material $\bar{4}$. The proof is conducted analogously to the one of the Theorem 8.

4.7 Dirichlet and Neumann boundary value problems: existence of regular solutions

Consider now the exterior Dirichlet and Neumann boundary value problems for the system (4.1.11). For the corresponding existence results, we first note that the existence of solution for the exterior Dirichlet and Neumann problems is considered in [75] for Poisson's equation ($\phi(x)$) and in [86] for the Helmholtz equation ($u_3(x)$). Hence, it remains to prove the existence of solution to the corresponding boundary value problems for the system (4.1.9). Following standard techniques (see, for example, [14, 33]), using Theorem 26, we apply the boundary integral equation method and seek solutions of the exterior Dirichlet problem in the form $(W\varphi)^-$, with $\varphi \in C^{1,\alpha}(\partial S)$ and the solution of the exterior Neumann problem in the form $V\varphi$ with $\varphi \in C^{0,\alpha}(\partial S)$. We can then reduce the Dirichlet boundary value problem to the corresponding system of singular integral equations

$$-\frac{1}{2}I_c\varphi(x) + \int_{\partial S} D(x,y)\varphi(y)ds(y) = u^*(x) - u_0, \qquad (\mathcal{D}^-).$$

Similarly, for the Neumann problem, we obtain the system

$$-\frac{1}{2}I_c\varphi(x) + \int_{\partial S} T(\partial x, n)\Gamma(x,y)\varphi(y)ds(y) = t^*(x), \qquad (\mathcal{N}^-)$$

In each case, $x \in \partial S$, $\varphi(x)$ is an unknown density function and the vector u_0 has components $u_{0\alpha}$ of the form (4.1.16). Thus the problems are reduced to finding solutions of these systems of singular integral equations. As in Section 3.6 we can show that in each case, the systems are each uniquely solvable so that the corresponding potentials (with density function supplied by the corresponding system of integral equations) form the unique solutions of the Dirichlet and Neumann boundary value problems.

Similarly, for the Dirichlet and Neumann boundary value problems for the equations (4.1.12) for the bounded domain we apply the boundary integral equation method and seek solutions of interior and exterior Dirichlet problems in the form of $(W\varphi)^{\pm}$ respectively, with $\varphi \in C^{1,\alpha}(\partial S)$ and the solution of the exterior Neumann problem in the form $V\varphi$ with $\varphi \in C^{0,\alpha}(\partial S)$. Using results from Theorem 26 we reduce the Dirichlet boundary value problems to the corresponding system of singular integral equations:

$$-\frac{1}{2}\varphi + \int_{\partial S} D(x, y, \omega)\varphi(y)ds(y) = u_+^*(x), \tag{$\mathcal{D}^+$}$$

$$\frac{1}{2}\varphi(x) + \int_{\partial S} D(x, y, \omega)\varphi(y)ds(y) = u_-^*(x) - u_0, . \tag{$\mathcal{D}^-$}$$

Here by $u_+^*(x)$ and $u_-^*(x)$ we denote respectively boundary values of $u(x) = (u_1(x), u_2(x), u_3(x), \phi(x))^T$ for interior and exterior Dirichlet problems. The interior and exterior Neumann boundary value problems are reduced to equations:

$$\frac{1}{2}\varphi + \int_{\partial S} T(\partial x, n)\Gamma(x, y, \omega)\varphi(y)ds(y) = t_+^*(x), \tag{$\mathcal{N}^+$}$$

$$-\frac{1}{2}\varphi(x) + \int_{\partial S} T(\partial x, n)\Gamma(x, y, \omega)\varphi(y)ds(y) = t_-^*(x). \tag{$\mathcal{N}^-$}$$

Here by $t_+^*(x)$ and $t_-^*(x)$ we denote boundary values of $T(\partial x, n)u(x)$ for interior and exterior Neumann problems for (4.1.12) respectively. In each case, $x \in \partial S$, $\varphi(x)$ is an unknown density function and the vector u_0 has components of the form (4.1.16). Thus the problems are reduced to finding solutions of these systems of singular integral equations. Due to the properties of the matrices of fundamental and singular solutions described the Section 4.3 we conclude that integral equations ($\mathcal{D}^+$), ($\mathcal{D}^-$), ($\mathcal{N}^+$), ($\mathcal{N}^-$) are of the kind considered in Section 3.6, therefore the Fredholm theorems are valid.

We now will study the integral equations for homogeneous external Dirichlet

$(\mathcal{D}_0^-)$ and Neumann $(\mathcal{N}_0^-)$ problems for which the following theorems are valid [34]:

Theorem 27 *The necessary and sufficient condition for the equation*

$$\frac{1}{2}\varphi(x) + \int_{\partial S} D(x, y, \omega)\varphi(y)ds(y) = 0, \quad x \in \partial S, \tag{$\mathcal{D}_0^-$}$$

to have a nontrivial solution is that the parameter ω^2 coincides with one of the eigenfrequencies of the homogeneous Neumann boundary value problem for (4.1.12) (N_0^+). If ω^2 is a ν-fold eigenfrequency of this problem, then the integral equation $(\mathcal{D}_0^-)$ has ν linearly independent solutions, coinciding with the boundary values of the eigenfunctions of (N_0^+).

Proof. First we will prove the necessity: let $(\mathcal{D}_0^-)$ have a nontrivial solution. We must show that in this case ω^2 is an eigenfrequency of the problem N_0^+.

Let us assume the opposite: ω^2 is not an eigenfrequency of N_0^+. Since, by our assumption, equation $(\mathcal{D}_0^-)$ has a nontrivial solution, its associate equation

$$\frac{1}{2}\varphi + \int_{\partial S} T(\partial x, n)\Gamma(x, y, \omega)\varphi(y)ds(y) = 0, \tag{$\mathcal{N}_0^+$}$$

also admits a nontrivial solution $\varphi(x)$. If we now consider a single-layer potential $V\varphi(x)$, $x \in S^+$ we see that it solves (N_0^+) (see Theorem 26). But since ω^2 is not, by assumption, an eigenfrequency of this problem we obtain that

$$V\varphi(x) \equiv 0, \quad x \in \bar{S}^+. \tag{4.7.1}$$

Due to continuity of the single-layer potential (see Theorem 26) and the behavior at infinity, by uniqueness theorem, we obtain that

$$V\varphi(x) \equiv 0, \quad x \in \bar{S}^-. \tag{4.7.2}$$

From (4.7.1) and (4.7.2) it follows that $\varphi(y) = 0$, $y \in \partial S$, which contradicts with the assumption made above.

Now we will prove sufficiency: let ω be a ν-fold eigenfrequency of N_0^+ and w^k, $k = 1, ..., \nu$, be the corresponding linearly independent solutions. It will be shown that the boundary values v^k, $k = 1, ..., \nu$ are linearly independent. Suppose that they are not, i.e.

$$\sum_{k=1}^{\nu} c_k v^k(y) = 0, \quad y \in \partial S.$$

Consider

$$w(x) = \sum_{k=1}^{\nu} c_k w^k(x), \quad x \in S^+.$$

According to the assumption, $w(x)$ solves $L(\partial x)w(x) = 0$ and $T(\partial x)w(x) = 0$, $u = 0$ for $x \in \partial S$. But then by representation theorem (see Section 3) we have that $w(x) = 0$ for $x \in S^+$, which is the contradiction with linear independence of w^k, $k = 1, ..., 5$. From representation theorem for $x \in \partial S$ we obtain

$$\frac{1}{2}v^k(x) = -\int_{\partial S} [T(\partial y, n)\Gamma(y, x, \omega)]^T v^k(y) ds(y).$$

Therefore integral equation $(\mathcal{D}_0^-)$ has at least ν linearly independent solutions $v^k(x)$. We will show that $(\mathcal{D}_0^-)$ has only ν linearly independent solutions: we will assume the opposite and let the number of solutions μ be greater than ν. Then its adjoint equation $\mathcal{N}_0^+$ also has μ solutions w^k, $k = 1, ..., \mu$. Let us construct single layer potentials Vw^k, $k = 1, ..., \mu$. Clearly, they are linearly independent as w^k, $k = 1, ..., \mu$ and are solutions of $\mathcal{N}_0^+$ which, however, admits only ν linearly independent solutions. Consequently, $\mu = \nu$ which completes the proof.

Theorem 28 *The necessary and sufficient condition for the equation*

$$-\frac{1}{2}\varphi(x) + \int_{\partial S} T(\partial x, n)\Gamma(x, y, \omega)\varphi(y)ds(y) = 0 \qquad (\mathcal{N}_0^-)$$

to have a nontrivial solution is that the parameter ω^2 coincides with one of the eigenfrequencies of homogeneous Dirichlet boundary value problem (D_0^+). If ω^2 is a ν-fold eigenfrequency of this problem, then the internal equation $(\mathcal{N}_0^-)$ has ν linearly independent solutions, coinciding with the boundary values of the vectors obtained by application of operator $T(\partial x, n)$ to the solutions of the problem (D_0^+).

The proof is analogous to the one of the Theorem 27.

Let us now turn our attention to the investigation of inhomogeneous internal problems. In case of Dirichlet problem for the equations (4.1.12), according to Theorem 26, we look the solution in the form:

$$u(x) = \int_{\partial S} D(x, y, \omega)\varphi(y)ds(y) + \frac{1}{2}\int_{S^+} \Gamma(x, y, \omega)F(y)dy. \qquad (4.7.3)$$

The corresponding integral equation for $\varphi(x)$ will be

$$\frac{1}{2}\varphi + \int_{\partial S} D(x,y,\omega)\varphi(y)ds(y) = u_+^*(x) - \frac{1}{2}\int_{S^+} \Gamma(x,y,\omega)F(y)dy. \qquad (\mathcal{D}^+)$$

According to Theorem 28, the discrete set of the values of ω^2 will be characteristic for $(\mathcal{D}^+)$. For all other values of ω^2 the problem is solved directly and solution is in the form (4.7.3). If ω^2 is an eigenvalue then, according to Theorem 28, it is also an eigenfrequency of the homogeneous Dirichlet boundary value problem (D_0^+). The conditions of solubility of $(\mathcal{D}^+)$ take the form (see Section 3.6):

$$\int_{\partial S} u_+^*(\xi)\psi^k(\xi)ds(y) - \frac{1}{2}\int_{\partial S}\int_{S^+} \Gamma(\xi,y,\omega)F(y)\psi^k(\xi)dyds(\xi) = 0, \qquad (4.7.4)$$

where ψ^k, $k = 1,...,\nu$, is a complete system of solutions of the associated homogeneous equation $(\mathcal{N}_0^-)$. Now, according to Theorem 28, ψ^k coincide with the boundary values of the application of the operator $T(\partial x, n)$ to the eigenvectors which are solutions u^k, $k = 1,...,\nu$, of the homogeneous Dirichlet problem (D_0^+). So, we can rewrite conditions (4.7.4) as follows:

$$\int_{\partial S} u_+^*(\xi)\left(Tu^k(\xi)\right)_b ds(y) - \frac{1}{2}\int_{\partial S}\int_{S^+} \Gamma(\xi,y,\omega)F(y)\left(Tu^k(\xi)\right)_b dyds(\xi) = 0. \qquad (4.7.5)$$

We also have that

$$\begin{aligned} &\tfrac{1}{2}\int_{\partial S}\{ \int_{S^+} \Gamma(x,y,\omega)F(y)dy\}(Tu(\xi))_b ds(\xi) = \\ &= \tfrac{1}{2}\int_{S^+}\{\int_{\partial S}\Gamma(x,y,\omega)(Tu(\xi))_b ds(\xi)\}F(y)dy. \end{aligned} \qquad (4.7.6)$$

From representation formula, since $u_b^k(x) = 0$ (see the proof of Theorem 27), we find that

$$u^k(y) = \int_{\partial S} \Gamma(\xi, y, \omega) \left(Tu(\xi)\right)_b ds(\xi), \quad y \in S^+.$$

This together with (4.7.5) gives

$$\int_{\partial S} u_+^* \left(Tu^k\right) ds - \int_{S^+} F(y) u^k(y) dy = 0, \quad k = 1, ..., \nu. \tag{4.7.7}$$

Setting each time either $F(y) = 0$ or $u_+^* = 0$ the conditions (4.7.7) become

$$\int_{\partial S} u_+^* \left(Tu^k\right) ds = 0, \tag{4.7.8}$$

and

$$\int_{S^+} F(y) u^k(y) dy = 0, \quad k = 1, ..., \nu. \tag{4.7.9}$$

In case of inhomogeneous Neumann boundary value problem, from Theorem 26, we seek the solution in the form

$$u(x) = \int_{\partial S} \Gamma(x, y, \omega) \psi(y) ds(y) + \frac{1}{2} \int_{S^+} \Gamma(x, y, \omega) F(y) dy.$$

For the density function $\psi(x)$, from Theorem 26, we obtain integral equation

$$\begin{aligned} \tfrac{1}{2}\psi(x) + & \int_{\partial S} \; T(\partial x, n)\Gamma(x, y, \omega)\psi(y) ds(y) = t_+^*(x) \\ -\tfrac{1}{2} & \int_{S^+} \; T(\partial x, n)\Gamma(x, y, \omega) F(y) dy. \end{aligned} \tag{$\mathcal{N}^+$}$$

If ω^2 is not an eigenfrequency than the problem is solved directly for an arbitrary right-hand side. If ω^2 is an eigenfrequency then the following solubility conditions must be satisfied:

$$\begin{aligned} & \int_{\partial S} \; t_+^*(y) \varphi^k(y) ds(y) - \\ -\tfrac{1}{2} & \int_{\partial S} \; \left\{ \int_{S^+} T(\partial \xi, n) \Gamma(\xi, y, \omega) F(y) dy \right\} \varphi^k(z) ds(\xi) = 0, \quad k = 1, ..., \nu, \end{aligned}$$

where φ^k, $k = 1, ..., \nu$, is a complete system of solutions of the associated homogeneous equation

$$\frac{1}{2}\varphi(x) + \int_{\partial S} D(x, y, \omega)\varphi(y)ds(y) = 0.$$

In the similar way as in the case of inhomogeneous Dirichlet problem we arrive to following resonance conditions

$$\int_{\partial S} f(\xi)v^k(\xi)ds(\xi) + \int_{S^+} F(y)v^k(y)dy = 0, \quad k = 1, ..., \nu. \tag{4.7.10}$$

Here, by v^k, $k = 1, ..., \nu$, we denote the boundary values of eigensolutions of internal homogeneous Neumann problem (N_0^+).

The condition (4.7.8) can be interpreted as following: in case of internal inhomogeneous Dirichlet boundary value problem the mechanical displacement with components u_i, $i = 1, 2, 3$ may have a critical value only if on the boundary mechanical displacement is orthogonal to the stresses induced by eigenoscillations of the same frequency and the integral taken over ∂S of the product of the electric potential given on the boundary and the surface charge produced by eigenoscillations vanishes.

The conditions (4.7.9), similarly, state that the external forces may have resonant oscillation frequency only if they are orthogonal to the displacements produced in the body by the eigenoscillations of the same eigenfrequency and the integral taken over the cross-section S^+ of the product of applied external charge and electric potential produced by eigenoscillations vanishes.

The conditions (4.7.10) are interpreted similarly.

Chapter 5

Conclusions and suggestions for future work

The rigorous mathematical analysis of boundary value problems in the theory of piezoelectricity has remained absent form the literature with the exception of the work by D. Iesan [27], in which the author used the boundary integral equation method to prove the existence of the solutions of boundary value problems for the state of generalized plane strain. Here, however, certain differential properties of the solutions were overlooked. The rigorous analysis of the fundamental boundary value problems arising from the anti-plane shear state in linear piezoelectricity has remained absent until recently. The objective of the present work has been to give the complete treatment of several boundary value problems in the theory of linear piezoelectricity. We considered two types of deformations in piezoelectric solids: anti-plane shear and generalized plane strain. Anti-plane shear state was considered in the frame of static theory only, for rather general class of piezoelectric materials with tetragonal $\bar{4}$ symmetry. The problems of steady-state vibrations were considered for the state of generalized plane strain state for hexagonal $6mm$ and tetragonal $\bar{4}$ piezoelectric materials. The following are the results that have been obtained:

1. We formulated Dirichlet, Neumann and mixed boundary value problems. We proved the uniqueness theorem and using boundary integral equation method showed that solutions exist and gave the analytical solutions in the form of integral potentials. Also, we gave the analytical expressions for the fundamental solutions.

2. For the problem of steady-state vibrations of the unbounded domain we derived the radiation conditions to provide the uniqueness of the solutions of Dirichlet and Neumann boundary value problems. The existence of the solutions of the problems of steady state vibrations is established using the boundary integral equation method in the same way as for the static problems. We also provided the fundamental solutions for the system of governing equations for the hexagonal $6mm$ class of piezoelectric material.

3. For the steady-state vibrations in the bounded domain we showed that the discrete spectrum of eigenfrequencies exists for homogeneous Dirichlet and Neumann boundary value problems. The conditions of solubility

of the non-homogeneous Dirichlet and Neumann boundary value problems are given for the case when the parameter ω^2 coincides with one of the eigenfrequencies of the homogeneous boundary value problems. The solutions are given in the form of integral potentials.

Thus, we showed that the fundamental boundary value problems arising from the states of anti-plane shear and generalized plane strain are well posed so that numerical procedures can be applied to solve particular problems in the context of anti-plane shear and generalized plane strain states.

As an extension of the present work the method of generalized Fourier series [34] can be used to find the unknown density of the integral potentials to solve several problems arising in the theory of piezoelectricity, for example, the problems of static torsion of cylindrical piezoelectric body for different types of cross-section. The objective of this analysis is to investigate the effects of the anisotropy and the electro-mechanical coupling and compare with the solutions obtained in the classical isotropic elasticity.

Another rapidly growing direction in piezoelectricity and its applications is the development of piezoelectric composites [52, 74, 82] and functionally graded piezoelectric materials [39, 65]. The problems arising in this area would include nonhomogeneous piezoelectric media, contact and transmission problems. In [34] several contact problems for nonhomogeneous elastic medium are investigated. The anisotropy of piezoelectric materials makes the analysis of the analogous problems in piezoelectricity extremely challenging. Thus another possible direction of future work would be to attempt to apply the method used in [34] to the investigation of the contact problems arising in the context of anti-plane and plane piezoelectricity.

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