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Extensions and Isomorphisms for the Generalized Fourier Algebras of a Locally Compact Group

by

Mehdi Sangani Monfared ©

A thesis submitted to the Faculty of Graduate Studies and Research in partial fulfilment of the requirements for the degree of Doctor of Philosophy

in

Mathematics

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled **Extensions and Isomorphisms for the Generalized Fourier Algebras of a Locally Compact Group** submitted by **Mehdi Sangani-Monfared** in partial fulfillment of the requirements for the degree of **Doctor of Philosophy** in Mathematics.

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January 25, 2002

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To My Family

My Parents: Vajihallah and Fatemeh

My Wife and Son: Elizabeth and Armand

My Brothers and Sisters: Taghi, Marzieh, Hamid, Razieh, and Said

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Abstract

In this thesis we study certain properties of the generalized Fourier algebras $A_p(G)$, $(1 < p < \infty)$, and their relations with the properties of the underlying group G.

It is shown that for amenable groups, all finite-dimensional extensions of $A_p(G)$ split strongly. Furthermore, each extension of $A_p(G)$ which splits algebraically also splits strongly. We also show that if G is an almost connected locally compact group, or a subgroup of $GL_n(V)$ (V being a finite-dimensional vector space), and if for a fixed $p \in (1, \infty)$, all finite-dimensional singular extensions of $A_p(G)$ split strongly, then G is amenable.

Continuous order isomorphisms for the pointwise order of $A_p(G)$ algebras, are characterized as weighted composition maps. Similarly, order isomorphisms for the pointwise order of $B_p(G)$ algebras, are characterized as $*$-algebra isomorphisms followed by multiplication by an invertible positive multiplier. In addition, it is shown that for amenable groups, an order isomorphism for the pointwise order between $A_p(G)$ algebras that preserve cozero sets is necessarily continuous, and hence induces an algebra isomorphism.

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Dual, bidual, and spectrum of Lau direct sums of $A_p(G)$ algebras are characterized. The existence of topological invariant means, and approximate zeros for such direct sums are verified.

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Chapter 1

Introduction

The group algebra $L_1(G)$ and the measure algebra $M(G)$ have been the subject of investigation by analysts from the early inception of harmonic analysis. In 1964, P. Eymard introduced [22] the Fourier algebra $A(G)$ and the Fourier-Stieljes algebra $B(G)$ for an arbitrary locally compact group G. In addition to generalizing some of the classical results of the commutative theory to non-commutative groups, Eymard also showed the deep and fruitful interaction between the properties of the underlying group G and the properties of the Banach algebras $A(G)$ and $B(G)$. It should be mentioned that, in the case of a commutative group G, $A(G)$ and $B(G)$ coincide with the Fourier transform of $L_1(\hat{G})$ and $M(\hat{G})$, respectively.

In 1965, C. Herz [37] in a note on a work of N. Th. Varoupolous [69] on the applications of the techniques of tensor products to harmonic analysis, obtained a new method of showing that $A(G)$ is closed under pointwise multiplications. In 1970, it was realized by Herz [38] that the same technique can be used to show that a certain Banach space of functions $A_p(G)$, $1 < p < \infty$, used by A. Figà-Talamanca [25] in connection with his studies of multipliers of L_p-spaces, is in fact a Banach algebra under pointwise multiplication, and furthermore, when $p = 2$, $A_p(G)$ coincides with the Fourier algebra $A(G)$, introduced by Eymard.

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The generalized Fourier algebras $A_p(G)$ (also called the Figà-Talamanca-Herz algebras) and their multipliers algebra $B_p(G)$ (especially for the particular case of $p = 2$) have since been studied by several authors including A. Lau [45], [46], [47], [48], Lau and Losert [50], Lau and Loy [51], Lau and Ülger [49], Lau and Wong [52], E. Granirer [32], Bade, Dales, and Lykova [12], A. Derighetti [19], [20], [21], Delaporte and Derighetti [17], [18], B. Forrest [29], [30], [31], N. Lohouè [54], [56], [55], [58], [57], V. Runde [66], V. Losert [59], G. Xu [74], J. Font [27], [28], and many others.

In this thesis we study several properties of these algebras with emphasis on their connection with the properties of the underlying group G.

In the second chapter we give a brief introduction to the basic notions and terminologies of the Banach algebras and harmonic analysis that are used constantly throughout the rest of the thesis. Section 1 is devoted to locally compact groups and amenability. Section 2 is a quick review of some terminologies from Banach algebra theory. And section 3 deals with the basic terminologies related to the extensions of Banach algebras.

In chapter 3, we study Banach algebra extensions of $A_p(G)$ algebras with emphasize on finite-dimensional extensions. In section 1 we show that if G is amenable, then all finite-dimensional extensions of $A_p(G)$ split strongly. Furthermore, we show that in this case, each extension of $A_p(G)$ which splits algebraically also splits strongly. Section 2 is devoted to the converse of our main result of section 1 in the case of almost connected locally compact groups. More precisely, we will prove that if G is an almost connected locally compact group, or a subgroup of $GL(V)$ for a finite- dimensional vector space V, and if all finite-dimensional singular extensions of $A_p(G)$ ($p \in (1, \infty)$ fixed) split strongly, then G is amenable.

In Chapter 4, we study order isomorphisms of the Banach algebras $A_p(G)$ and $B_p(G)$. In section 1 we introduce and study the pointwise and positive definite orders on $A_p(G)$ algebras. Here among other things we prove that a

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continuous order isomorphism for the pointwise order is a weighted composition map. In section 2 we obtain a characterization of order isomorphisms for the pointwise order of $B_p(G)$ algebras. These are $*$-algebra isomorphisms followed by multiplication with an invertible element in $B_p(G)_+$. As a corollary we show that a biorder isomorphism between $B_p(G)$ algebras is a multiple of an $*$-algebra isomorphism. In section 3, we briefly study the disjointness preserving mappings between $A_p(G)$ algebras. As an application, we show that for amenable groups, an order isomorphism for the pointwise order between $A_p(G)$ algebras that preserve cozero sets is necessarily continuous, and consequently the two algebras are isomorphic.

In chapter 5, we study Lau direct sums of $A_p(G)$ algebras. In section 1, we characterize the dual, bidual, and the spectrum of such direct sums. In section 2, we prove the existence of topological invariant means on these spaces, and finally in section 3, we prove the existence of approximate zeros.

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Chapter 2

Preliminaries and Notations

2.1 Locally Compact Groups, Haar Measures, and Amenability

We assume throughout this thesis that all groups under consideration are locally compact groups. In particular, we assume free groups have discrete topology. The free group on r generators is denoted by $\mathbf{F}_r$

We assume that every locally compact group is equipped with an arbitrary, but fixed left Haar measure (in the case of discrete groups we assume that the Haar measure is the counting measure, and for the case of compact groups we assume the Haar measure is normalized). The integral of a measurable function f on G with respect to such a measure is denoted by

$$\int_G f(x)\,dx$$

If E is a measurable subset of G, $|E|$ denotes the measure of E. The modular function of a left Haar measure is denoted by Δ.

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If f is a function on G and $a \in G$, we define

$$\begin{aligned}
{}_af(x) &= L_{a^{-1}}f(x) = f(ax), \\
f_a(x) &= R_af(x) = f(xa), \\
\check{f}(x) &= f(x^{-1}), \\
\tilde{f}(x) &= \overline{f(x^{-1})}, \\
f^*(x) &= \overline{f(x^{-1})}/\Delta(x),
\end{aligned}$$

for all $x \in G$.

As usual, $C_{00}(G)$, $C_0(G)$, and $C(G)$ are respectively, the collection of all continuous functions on G whose support is compact, the collection of all continuous functions on G that vanish at infinity, and the collection of all bounded continuous functions on G.

A function $f \in C(G)$ is said to be *positive-definite* if for every $x_1, \ldots, x_n \in G$, and $r_1, \ldots, r_n \in \mathbf{C}$

$$\sum_{i,j} r_i\overline{r_j}f(x_i^{-1}x_j) \geq 0.$$

The collection of all continuous positive-definite functions is denoted by $P(G)$. The collection of all positive definite functions f such that $f(e) = 1$ is denoted by $P_1(G)$.

If $1 \leq p \leq \infty$, $f \in L_1(G)$, and $g \in L_p(G)$, then the function $f * g$ in $L_p(G)$ defined by the integral

$$f * g(x) = \int_G f(y)g(y^{-1}x)\,dy$$

is called the convolution of f and g.

Let G be a locally compact group. A *left invariant mean* for G is a continuous linear functional $m \in L_\infty(G)^*$ such that

1. $\|m\| = m(1) = 1$.

2. $m({}_af) = m(f)$ for all $f \in L_\infty(G)$ and all $a \in G$.

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A locally compact group that has a left invariant mean is called *amenable.* The study of amenable groups was first started with the work of von Neumann in 1929 [70]. Since then, many properties of G as well as various properties of algebraic structures on G are shown to be equivalent to the amenability. Such studies are still a dominant trend in harmonic analysis of locally compact groups. The book of Paterson [63] gives a detailed study of these developments. Compact groups and abelian groups are amenable. It is known that any group with a closed subgroup isomorphic to the free group on two generators, is not amenable. However, A. Yu. Ol'shanskii [62], has constructed a *non-amenable* group such that every proper subgroup is cyclic (and hence none are free on two generators). So in general, amenability is not equivalent to the non-existence of non abelian free subgroups. The following two classes of groups are exceptions. If G is a locally compact group and G_0 is the connected component of the identity of G, then G is called *almost connected* if G/G_0 is compact. Such a group is amenable if and only if it does not contain $\mathbf{F}_2$ as a closed subgroup [65, Theorem 5.5]. The other class of groups for which amenability is equivalent to the non-existence of closed subgroups isomorphic to $\mathbf{F}_2$ is the class of linear groups over finite-dimensional vector spaces (with arbitrary field characteristics), (Tits Dichotomy Theorem, [68, pp. 250–251, Theorem 1 and Corollary 1]).

2.2 Banach Algebras

Let A be a Banach algebra and let E be a Banach space which is a left and a right A-module. If the module multiplications are continuous, then E is called a *Banach A-bimodule.* The Banach spaces A^* and A^{**} are Banach A-bimodules in the usual way. When A is commutative, the left and right module actions on A^* (and on A^{**}) coincide.

When $a \cdot x = x \cdot a = 0$, for all $a \in A$ and all $x \in E$, we call E an *annihilator*

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A-bimodule. We define

$$A \cdot E = \{a \cdot e : a \in A,\ e \in E\}.$$

When $\overline{\operatorname{span}(A \cdot E)} = E$, (respectively, $\overline{\operatorname{span}(E \cdot A)} = E$), we call E *essential as a left* (respectively, *essential as a right*) *module.* If S is a subset of A, and $n \in \mathbf{N}$, then we define $S^{[n]} = \{a_1 \cdots a_n : a_1, \ldots, a_n \in S\}$, and $S^n = \operatorname{Span}\{a_1 \cdots a_n : a_1, \ldots, a_n \in S\}$, the linear span of $S^{[n]}$.

An *approximate identity* for a Banach algebra A is a net $(e_\alpha)_{\alpha \in I}$ in A such that for every $a \in A$

$$\begin{aligned} \|e_\alpha a - a\| &\longrightarrow 0, \\ \|a e_\alpha - a\| &\longrightarrow 0. \end{aligned} \tag{$*$}$$

If in addition, this net is bounded, it is called *a bounded approximate identity.* If a bounded net $(e_\alpha)_{\alpha \in I}$ satisfies only the first (respectively, the second) condition of $(*)$, it is called a *bounded left* (respectively *right*) *approximate identity* for A.

Let X be a topological space. We say that $(A, \|\cdot\|_A)$ is a *Banach algebra in* $C_0(X)$ if A is a subalgebra of $C_0(X)$ and if A is a Banach algebra under pointwise operations and its own norm $\|\cdot\|_A$. For each $x \in X$, let $\delta_x(f) = f(x)$ for all $f \in A$. This is a one-to-one, bi-continuous map from X into $\sigma(A)$, the spectrum of A. If it is also onto, we say A is a Banach algebra in $C_0(X)$, where X is the spectrum of A [40, p. 489]. Such a Banach algebra is called a *strong Ditkin algebra* if

1. A has a bounded approximate identity consisting of functions with compact support.

2. For each $x \in X$, the set $M_x = \{u \in A : u(x) = 0\}$ has a bounded approximate identity consisting of functions whose supports are compact subsets of $X - \{x\}$.

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(See [12, p. 41], compare also with [40, p. 497], where an algebra *satisfying Ditkin's conditions* is defined).

For $1 < p < \infty$, the Banach algebra $A_p(G)$, is defined as the collection of all functions $u \in C_0(G)$ such that

$$u = \sum_{i=1}^{\infty} g_i * \check{f}_i,$$

where $f_i \in L_p(G)$, $g_i \in L_{p'}(G)$, $1/p + 1/p' = 1$, and $\sum_{i=1}^{\infty} \|f_i\|_p \, \|g_i\|_{p'} < \infty$. The norm on $A_p(G)$ is defined by

$$\|u\|_{A_p} = \inf \left\{ \sum_{i=1}^{\infty} \|f_i\|_p \, \|g_i\|_{p'} : u = \sum_{i=1}^{\infty} g_i * \check{f}_i, \ f_i \in L_p(G), \ g_i \in L_{p'}(G) \right\}.$$

Using the above terminology, $A_p(G)$ is a Banach algebra in $C_0(G)$, where G is the spectrum of $A_p(G)$ (also denoted by $\sigma(A_p(G))$). In the special case of $p = 2$, $A_p(G)$ is simply the Fourier algebra $A(G)$ introduced by Eymard in [22]. It is well known that $A_p(G)$ has a bounded approximate identity if and only if G is amenable [39].

For every $x \in G$, $\{x\}$ is a *set of spectral synthesis* for $A_p(G)$, in the sense that the set

$$J_x = \{u \in A_p(G) : u \text{ has a compact support disjoint from } \{x\}\},$$

is dense in $M_x = \{u \in A_p(G) : u(x) = 0\}$ [39, Theorem 2, Proposition 2, and Proposition 1, pp. 92–94] (see also [17, Corollary 4]).

Let $A_p(G)^*$ be the dual of $A_p(G)$. Suppose $\{\lambda_p, L_p(G)\}$ is the left regular representation of $L_1(G)$ on $L_p(G)$ defined by

$$\lambda_p(f)g = f * g \qquad (f \in L_1(G), \ g \in L_p(G)),$$

then $A_p(G)^* = \overline{\lambda_p(L_1(G))} \subset \mathcal{B}(L_p(G))$—where the closure is with respect to the weak*-topology of $\mathcal{B}(L_p(G))$, the space of all bounded linear operators on $L_p(G)$ [23, p. 60]. The dual norm on $A_p(G)^*$ coincides with the operator norm

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induced from $\mathcal{B}(L_p(G))$ [39, p. 116], and for $f \in L_1(G)$, $u \in A_p(G)$ we have [23, p. 60]:

$$\langle \lambda_p(f), u \rangle = \int f(x)u(x)\,dx.$$

If $\phi, \psi \in \sigma(A_p(G)) \cup \{0\}$, we define an $A_p(G)$-bimodule action on the set of complex numbers $\mathbf{C}$ by

$$u \cdot z = \phi(u)z, \quad z \cdot u = \psi(u)z, \qquad (\text{for all } u \in A_p(G),\ z \in \mathbf{C}).$$

Following the notation in [12], we denote the resulting Banach $A_p(G)$-bimodule by $\mathbf{C}_{\phi,\psi}$. If $\phi = \delta_x$ is the evaluation functional at x, we use the slightly modified notation $\mathbf{C}_{x,\psi}$, instead of $\mathbf{C}_{\phi,\psi}$. Similarly, when ϕ and ψ are both the zero functional, we denote the resulting Banach $A_p(G)$-bimodule by $\mathbf{C}_{0,0}$.

A function u on G is called a *multiplier* of $A_p(G)$ if

$$uv \in A_p(G) \qquad (\text{ for all } v \in A_p(G)).$$

The set of all multipliers of $A_p(G)$ is denoted by $B_p(G)$. It is easy to verify that with the usual operations of pointwise addition and multiplication, and with the multiplier norm

$$\|u\|_{B_p} = \inf\{\|uv\|_{A_p} : v \in A_p(G), \|v\|_{A_p} = 1\},$$

$B_p(G)$ is a Banach algebra.

2.3 Extensions of Banach Algebras

For a basic reference on algebraic and strong splittings of extensions of Banach algebras we refer to [12]. Our terminologies and notations are mainly those used in the above reference.

Suppose B and A are Banach algebras, and I is a closed two sided ideal of B. An *extension* $\Sigma(B; I)$ (or simply Σ) *of* A *by* I is a short exact sequence

$$0 \longrightarrow I \xrightarrow{\iota} B \xrightarrow{\pi} A \longrightarrow 0,$$

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where ι is the inclusion map and π is a continuous algebra epimorphism with $\ker \pi = I$.

We call the extension $\Sigma(B; I)$

1. *finite-dimensional*, when I is finite-dimensional;

2. *singular*, if $I^2 = \{0\}$;

3. *admissible*, if there exists a continuous linear map $d : A \longrightarrow B$, such that $\pi \circ d = id_A$ (the identity map on A).

We say that $\Sigma(B; I)$ *splits algebraically* if there exists an algebra homomorphism $h : A \longrightarrow B$ such that $\pi \circ h = id_A$; if in addition, such a homomorphism is continuous, we say that the extension *splits strongly*. It is easy to verify that an extension $\Sigma(B; I)$ splits algebraically if and only if there exists a subalgebra C of B such that $B = I \odot C$. The symbol $\odot$ means that $B = I + C$ and $I \cap C = \{0\}$. Similarly, an extension $\Sigma(B; I)$ splits strongly if and only if there exists a closed subalgebra C of B such that $B = I \oplus C$. The symbol $\oplus$ means that I and C are closed subspaces of B, and $B = I \odot C$.

In a singular extension

$$0 \longrightarrow I \xrightarrow{\iota} B \xrightarrow{\pi} A \longrightarrow 0,$$

we regard I as a Banach A-bimodule with the following module operations: for $a \in A$, and $x \in I$, if $b \in B$ is such that $\pi(b) = a$, we define $a \cdot x = bx$, and $x \cdot a = xb$.

Suppose E is a Banach A-bimodule, and Σ is a singular extension of A by I. If I is isomorphic to E as a Banach A-bimodule, then Σ is also called a *singular extension of A by E*.

For a Banach algebra A and a Banach A-bimodule E, let $\mathcal{B}^n(A, E)$ denote the set of all continuous n-linear maps from $A \times \cdots \times A$ into E, and $\mathcal{L}^n(A, E)$ denote the set of all n-linear maps from $A \times \cdots \times A$ into E. If $S \in \mathcal{L}^1(A, E)$,

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then $\delta^1 S \in \mathcal{L}^2(A, E)$ is defined by

$$(\delta^1 S)(a, b) = a \cdot S(b) - S(ab) + S(a) \cdot b \qquad (a, b \in A).$$

We also define

$$\mathcal{N}^2(A, E) = \{\delta^1 S : S \in \mathcal{B}(A, E)\},$$
$$N^2(A, E) = \{\delta^1 S : S \in \mathcal{L}(A, E)\}.$$

A bounded bilinear map $T \in \mathcal{B}^2(A, E)$ is called a *continuous 2-cocycle*, if for all a, b, and $c \in A$

$$a \cdot T(b, c) - T(ab, c) + T(a, bc) - T(a, b) \cdot c = 0.$$

The set of all continuous 2-cocycles is a closed subspace of $\mathcal{B}^2(A, E)$, denoted by $\mathcal{Z}^2(A, E)$. It is easy to verify that $\mathcal{N}^2(A, E) \subset \mathcal{Z}^2(A, E)$, but in general, $N^2(A, E)$ is not a subspace of $\mathcal{Z}^2(A, E)$. We put $\tilde{N}^2(A, E) = \mathcal{Z}^2(A, E) \cap N^2(A, E)$. The *second continuous cohomology group of A with coefficients in E* is defined as

$$\mathcal{H}^2(A, E) = \mathcal{Z}^2(A, E)/\mathcal{N}^2(A, E).$$

An analogue of the above group is defined as

$$\tilde{H}^2(A, E) = \mathcal{Z}^2(A, E)/\tilde{N}^2(A, E).$$

The latter group is used in the study of the algebraic splitting of extensions of A by E.

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Chapter 3

Algebraic and Strong Splittings of Extensions of the Banach Algebras $A_p(G)$

3.1 Introduction

A classical theorem of Wedderburn [72, Theorem 28] states that when B is a finite-dimensional algebra and $\operatorname{rad} B$ is its Jacobson radical, then there exists a subalgebra A of B such that $B = \operatorname{rad} B \odot A$.

By definition, a (non semi-simple) Banach algebra B has a *Wedderburn decomposition* if there exists a subalgebra A of B such that $B = \operatorname{rad} B \odot A$. We say that B has a *strong Wedderburn decomposition* if there exists a closed subalgebra A of B such that $B = \operatorname{rad} B \oplus A$.

In 1951, C. Feldman [24, p. 771] found an example of a Banach algebra without a *strong* Wedderburn decomposition. Later W.G. Bade and P.C. Curtis [8, Theorem 6.1] showed that the example of Feldman has a Wedderburn decomposition, and therefore the two kinds of decompositions are distinct. The question of the existence of Wedderburn decompositions for Banach algebras is

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a special case of a rather more general question of splittings of Banach algebra extensions (see Section 2.3). Suppose A is a specified Banach algebra. We want to know under what conditions on A, every Banach algebra extension $\Sigma(B; I)$ of A splits algebraically or strongly. Of course, when $I = \mathrm{rad}\, B$, this question is essentially about the existence of Wedderburn decompositions of B.

The algebraic and strong splittings of Banach algebra extensions have been studied by many authors, among them Bade and Curtis [8], [9], Bade, Curtis, and Sinclair [10], Bade and Dales [11], Feldman [24], Helemskii [34], [35], and Johnson [43], [44]. The most comprehensive of such studies is the work by Bade, Dales, and Lykova [12].

Our study of extensions of $A_p(G)$ algebras was motivated by the results of H. Steiniger in [67] for the special case of $p = 2$.

The main result in Section 2 is Theorem 3.2.7, which states that when G is amenable, all finite-dimensional extensions of $A_p(G)$ split strongly. We also show that when G is amenable, each extension of $A_p(G)$ which splits algebraically also splits strongly (Proposition 3.2.10).

The main result in Section 3 (Theorem 3.3.11), shows that the converse of Theorem 3.2.7 is true if the group G is a linear group on a finite-dimensional vector space or an almost connected locally compact group.

3.2 Extensions of the Banach Algebras $A_p(G)$

In [32, Theorem 5, p. 123] Granirer showed the existence of a topological invariant mean Ψ on $A_p(G)^*$, the dual of $A_p(G)$. In other words, he showed the existence of an element $\Psi \in A_p(G)^{**}$ such that $\|\Psi\| = \Psi(\delta_e) = 1$, and $u \cdot \Psi = u(e)\Psi$ for every $u \in A_p(G)$. Now if $x \in G$ and $L_{x^{-1}}$ is the left translation by x^{-1} on $A_p(G)$, and if we set $\Psi_x = L_{x^{-1}}^{**}\Psi$, then it is easy to see that $\|\Psi_x\| = \Psi_x(\delta_x) = 1$, and for every $u \in A_p(G)$, $u \cdot \Psi_x = u(x)\Psi_x$. So we

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have the following result:

3.2.1 Lemma. For every $x \in G$, there exists an $A_p(G)$-bimodule homomorphism $\Psi_x : A_p(G)^* \longrightarrow \mathbf{C}_{x,x}$ such that $\|\Psi_x\| = \Psi_x(\delta_x) = 1$.

The first part of the next result is proved in [44, Proposition 1.5, p. 10], and its second part can be proved in a similar way.

3.2.2 Proposition. Let A be a Banach algebra, and let E be an annihilator Banach A-bimodule.

1. If A has a left (or right) bounded approximate identity, then $\mathcal{H}^2(A, E) = \{0\}$.

2. If A has a left (or right) approximate identity, then $\widetilde{H}^2(A, E) = \{0\}$.

For the convenience of reference, we mention the following result due to Bade, Dales, and Lykova. This result is proved in Proposition 2.5(i), p. 28, and Proposition 2.10(i), p. 32 of [12].

3.2.3 Proposition. Let A be a Banach algebra and let E be a Banach A-bimodule.

1. Every singular admissible extension of A by E splits strongly if and only if $\mathcal{H}^2(A, E) = \{0\}$.

2. Every singular admissible extension of A by E splits algebraically if and only if $\widetilde{H}^2(A, E) = \{0\}$.

3.2.4 Lemma. Suppose E is a finite-dimensional Banach $A_p(G)$-bimodule, and let E be essential as a left (or as a right) module. Then every singular extension of $A_p(G)$ by E splits strongly.

Proof. We assume that E is essential as a left module. By Proposition 3.2.3, it suffices to show that $\mathcal{H}^2(A_p(G), E) = \{0\}$. If $x \in G$, from the fact that $\{x\}$

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is a set of spectral synthesis for $A_p(G)$, we can easily deduce that $\overline{M_x^2} = M_x$ (where, as we defined before $M_x = \{u \in A_p(G) : u(x) = 0\}$). Therefore by [12, pp. 56–57], $E = \bigoplus_{i=1}^{n} \mathbf{C}_{\phi_i,\psi_i}$, where $n = \dim E$, and $\phi_i, \psi_i \in \sigma(A_p(G)) \cup \{0\}$, for every i. Hence $\mathcal{H}^2(A_p(G), E) = \bigoplus_{i=1}^{n} \mathcal{H}^2(A_p(G), \mathbf{C}_{\phi_i,\psi_i})$. Since by assumption E is essential as a left module, $\phi_i \neq 0$ for every $i = 1, 2, \ldots n$. To complete the proof, it suffices to show that $\mathcal{H}^2(A_p(G), \mathbf{C}_{x,\psi}) = \{0\}$, whenever $x \in G$, and $\psi \in \sigma(A_p(G)) \cup \{0\}$ (note that $\sigma(A_p(G)) = \{\delta_x : x \in G\}$).

Let $T \in \mathcal{Z}^2(A_p(G), \mathbf{C}_{x,\psi})$, and let T' be the unique element in $(A_p(G)\widehat{\otimes}A_p(G))^*$ canonically corresponding T (that is, $T'(a \otimes b) = T(a, b)$)—here $\widehat{\otimes}$ denotes the projective tensor product operation. Now consider the canonical isometric isomorphism

$$(A_p(G)\widehat{\otimes}A_p(G))^* \simeq \mathcal{B}(A_p(G), A_p(G)^*)$$

$$R \longrightarrow \Phi_R$$

where $\langle \Phi_R(a), b\rangle = R(b \otimes a)$, for every $a, b \in A_p(G)$. Then the 2-cocycle identity for T', that is,

$$a(x)T'(b \otimes c) - T'(ab \otimes c) + T'(a \otimes bc) - T'(a \otimes b)\psi(c) = 0,$$

can be written as

$$\langle T'(b \otimes c)\delta_x, a\rangle - \langle \Phi_{T'}(c), ab\rangle + \langle \Phi_{T'}(bc), a\rangle - \langle \psi(c)\Phi_{T'}(b), a\rangle = 0,$$

or

$$\langle T'(b \otimes c)\delta_x, a\rangle - \langle b \cdot \Phi_{T'}(c), a\rangle + \langle \Phi_{T'}(bc), a\rangle - \langle \psi(c)\Phi_{T'}(b), a\rangle = 0.$$

As $a \in A_p(G)$ is arbitrary, it follows that

$$T'(b \otimes c)\delta_x - b \cdot \Phi_{T'}(c) + \Phi_{T'}(bc) - \psi(c)\Phi_{T'}(b) = 0.$$

Suppose Ψ_x is the $A_p(G)$-bimodule homomorphism in Lemma 3.2.1. If we define $S \in \mathcal{B}(A_p(G), \mathbf{C}_{x,\psi})$ to be $\Psi_x \circ \Phi_{T'}$, then by applying Ψ_x from the left to the last equality we obtain

$$T'(b \otimes c) - b(x)S(c) + S(bc) - \psi(c)S(b) = 0,$$

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or

$$T(b,c) = b \cdot S(c) - S(bc) + S(b) \cdot c.$$

This means that $T = \delta^1 S \in \mathcal{N}^2(A_p(G), \mathbf{C}_{x,\psi})$, and hence $\mathcal{H}^2(A_p(G), \mathbf{C}_{x,\psi}) = \{0\}$. **Q.E.D.**

3.2.5 Remark. The above proof shows that all finite-dimensional $A_p(G)$-bimodules are isomorphic to direct sums of the form $\bigoplus_{i=1}^{n} \mathbf{C}_{\phi_i,\psi_i}$, where $n \in \mathbf{N}$, and $\phi_i, \psi_i \in \sigma(A_p(G)) \cup \{0\}$.

3.2.6 Proposition. All singular finite-dimensional extensions of $A_p(G)$ split algebraically if the algebra has an approximate identity.

Proof. By Proposition 3.2.3, it suffices to show $\widetilde{H}^2(A_p(G), E) = \{0\}$ for each finite-dimensional Banach $A_p(G)$-bimodule E. Our proof of Lemma 3.2.4 shows that if $\dim E = n$, then $\widetilde{H}^2(A_p(G), E) = \bigoplus_{i=1}^{n} \widetilde{H}^2(A_p(G), \mathbf{C}_{\phi_i,\psi_i})$, where $\phi_i, \psi_i \in \sigma(A_p(G)) \cup \{0\}$. By the same lemma and the fact that $\mathcal{N}^2(A_p(G), \mathbf{C}_{\phi_i,\psi_i})$ is a subset of $\widetilde{N}^2(A_p(G), \mathbf{C}_{\phi_i,\psi_i})$, it follows that $\widetilde{H}^2(A_p(G), \mathbf{C}_{\phi_i,\psi_i}) = \{0\}$, whenever ϕ_i or ψ_i is nonzero. But by Proposition 3.2.2, $\widetilde{H}^2(A_p(G), \mathbf{C}_{0,0}) = \{0\}$ as well. Thus $\widetilde{H}^2(A_p(G), E) = \{0\}$. **Q.E.D.**

3.2.7 Theorem. If G is amenable, all finite-dimensional extensions of $A_p(G)$ split strongly.

Proof. By [12, Theorem 1.8, p. 13] it suffices to show that all finite-dimensional singular extensions of $A_p(G)$ split strongly. Suppose E is a finite-dimensional Banach $A_p(G)$-bimodule. By Remark 3.2.5, $\mathcal{H}^2(A_p(G), E) = \bigoplus_{i=1}^{n} \mathcal{H}^2(A_p(G), \mathbf{C}_{\phi_i,\psi_i})$. Now the result follows from Proposition 3.2.3, Proposition 3.2.2 (and the amenability G), and Lemma 3.2.4. **Q.E.D.**

3.2.8 Remark. An alternative proof for Theorem 3.2.7 is as follows. By [12, Theorem 4.18, p. 72], if A is a Banach algebra such that every closed ideal of finite codimension in A has a bounded left approximate identity, then

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every finite-dimensional extension of A splits strongly. The fact that when G is amenable, $A_p(G)$ is such a Banach algebra is a result of B. Forrest in [31, Theorem 4.2, p. 239].

The following result for the case of $p = 2$, is due to A. Lau [48, Corollary 4.4]. For a detailed study of the Ditkin sets of $A_p(G)$ algebras see [21].

3.2.9 Proposition. A locally compact group G is amenable if and only if $A_p(G)$ is a strong Ditkin algebra.

Proof. Suppose G is amenable. Then, it is well known that $A_p(G)$ has a bounded approximate identity consisting of functions with compact support [39, Theorem 6, p. 120]. Furthermore, by a result of B. Forrest [31, Proposition 3.4, p. 236] for each $x \in G$, M_x also has a bounded approximate identity consisting of functions with compact support. Now, since $\overline{J_x} = M_x$, we can easily obtain a bounded approximate identity for M_x from elements in J_x, which means that $A_p(G)$ is a strong Ditkin algebra.

The converse follows immediately from the fact that $A_p(G)$ has a bounded approximate identity if and only if G is amenable. However we can prove the converse using a weaker assumption. Suppose M_e has a bounded approximate identity, then by Cohen's factorization theorem, $M_e^{[2]} = M_e$. Now let $v \in A_p(G)$ be arbitrary, and take $u \in A_p(G)$ such that $u(e) = 1$. Then $v - vu \in M_e$, and therefore for some u_1, v_1 in M_e, $v - vu = u_1 v_1$. Thus $v \in A_p(G)^2$. This means that $A_p(G)$ factorizes weakly, and hence G is amenable by [59, Proposition 2, p. 138] and [39, p. 121]. **Q.E.D.**

3.2.10 Proposition. Let G be amenable.

1. If G is infinite, then there exists a singular admissible extension of $A_p(G)$ which does not split algebraically.

2. Each extension of $A_p(G)$ which splits algebraically also splits strongly;

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and $\widetilde{H}^2(A_p(G), E) = \mathcal{H}^2(A_p(G), E)$, for each Banach $A_p(G)$-bimodule E.

Proof. This follows from Proposition 3.2.9 and [12, Theorem 3.11, p. 43, Theorem 3.10, p. 42, and Theorem 3.19, p. 48]. **Q.E.D.**

3.3 Almost Connected Groups

In this section we will prove that when G is an almost connected locally compact group, or a linear group on a finite-dimensional vector space, then the converse of Theorem 3.2.7 is also true.

For an element x of the free group with r generators $\mathbf{F}_r$, $|x|$ denotes the length of the word x (that is, the length of its reduced form, obtained by making the necessary cancellations). We let $W_n = \{x \in \mathbf{F}_r : |x| = n\}$, and we denote the characteristic function of W_n by χ_n. In $\mathbf{F}_r$ there is only one element of length zero (that is, the empty word $(1, 1, \ldots)$), and $2r(2r-1)^{n-1}$ elements of length n ($n \in \mathbf{N}$). We always assume that the free group $\mathbf{F}_r$ is equipped with the discrete topology.

We begin with some lemmas on the estimation of the norms of certain convolution operators on $L_p(G)$, for finitely generated free groups G. Our results generalizes earlier results of M. Leinert [53, pp. 150, and 154], M. Bożejko [14, p. 408], and U. Haagerup [33, Lemmas 1.3 and 1.4]. Our proofs have been inspired by the techniques of the latter.

3.3.1 Lemma. Let k, l, and m be nonnegative integers, and f and g be two functions on $\mathbf{F}_r$, with supports in W_k and W_l respectively. Then

1. $(f * g)\chi_m \neq 0$ implies that $k + l - m$ is even, and $|k - l| \leq m \leq k + l$.

2. If $1 < p \leq 2$, then $\|(f * g)\chi_m\|_p \leq \|f\|_p \|g\|_p$.

3. If $2 \leq p < \infty$, then $\|(f * g)\chi_m\|_p \leq \|f\|_{p'} \|g\|_p$.

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Proof. The first statement is proved in [33, p. 283, Lemma 1.3]. We will prove the second statement. Let us first assume that $m = k + l$. In this case, if $|x| = m$, then the sum $f \star g(x) = \sum_y f(xy)g(y^{-1})$, has only one nonzero term because there is only one word y of length l such that $|xy| = k$. Let us denote this word by y_x. Then

$$\begin{aligned}
\|(f * g)\chi_m\|_p^p &= \sum_{|x|=m} |\sum_y f(xy)g(y^{-1})|^p, \\
&= \sum_{|x|=m} |f(xy_x)|^p |g(y_x^{-1})|^p, \\
&\leq \|g\|_p^p \cdot \sum_{|z|=k} |f(z)|^p, \\
&= \|g\|_p^p \|f\|_p^p.
\end{aligned}$$

Therefore

$$\|(f * g)\chi_m\|_p \leq \|f\|_p \|g\|_p. \tag{3.1}$$

Turning back to the general case, suppose that $m = k + l - 2i$, where $0 \leq i \leq \min(k, l)$ (see statement (1) of the lemma). We define f' and g' with supports in W_{k-i} and W_{l-i} as follows

$$f'(x) = \begin{cases} \left(\sum_{|w|=i} |f(xw)|^p \right)^{1/p} & \text{if } |x| = k - i, \\ 0 & \text{otherwise.} \end{cases}$$

$$g'(x) = \begin{cases} \left(\sum_{|w|=i} |g(w^{-1}x)|^p \right)^{1/p} & \text{if } |x| = l - i, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned}
\|f'\|_p^p &= \sum_{|x|=k-i} \sum_{|w|=i} |f(xw)|^p, \\
&= \sum_{|y|=k} |f(y)|^p, \\
&= \|f\|_p^p.
\end{aligned}$$

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Similarly $\|g'\|_p = \|g\|_p$. Now let $x \in \mathbf{F}_r$, and assume $|x| = k + l - 2i$. We can write $f * g(x) = \sum f(t)g(s)$, where the sum is taken over all $t, s \in \mathbf{F}_r$ such that $x = ts$, $|t| = k$, $|s| = l$. Let t' be the word consisting of the first $k - i$ letters of x, and s' be the word consisting of the last $l - i$ letters of x (in the case that either $k - i$ or $l - i$ is zero, we take the corresponding word to be the empty word $(1, 1, \ldots)$). Since in the sum above, $x = ts$ with $|x| = |t| + |s| - 2i$, there are i cancellations in the product ts. That is, $t = t'w$, $s = w^{-1}s'$, with t' and s' defined as above. Thus

$$\begin{aligned}
|f * g(x)| &= \Big| \sum_{|t|=k,|s|=l,x=ts} f(t)g(s) \Big|, \\
&= \Big| \sum_{|w|=i} f(t'w)g(w^{-1}s') \Big|, \\
&\leq \sum_{|w|=i} |f(t'w)||g(w^{-1}s')|, \\
&\leq \left(\sum_{|w|=i} |f(t'w)|^p \right)^{1/p} \left(\sum_{|w|=i} |g(w^{-1}s')|^{p'} \right)^{1/p'}. \\
&\leq \left(\sum_{|w|=i} |f(t'w)|^p \right)^{1/p} \left(\sum_{|w|=i} |g(w^{-1}s')|^{p} \right)^{1/p}. \\
&= f'(t')g'(s').
\end{aligned}$$

As there is only one pair (t', s') such that $x = t's'$, $|x| = |t'| + |s'|$, $|t'| = k - i$, and $|s'| = l - i$, we have $f' * g'(x) = f'(t')g'(s')$. Hence $|f * g|\chi_m \leq (f' * g')\chi_m$. This implies that

$$\begin{aligned}
\|(f * g)\chi_m\|_p &\leq \|(f' * g')\chi_m\|_p, \\
&\leq \|f'\|_p \|g'\|_p \qquad \text{by inequality (3.1)}, \\
&= \|f\|_p \|g\|_p.
\end{aligned}$$

The proof of the third statement is similar to the one above, once we replace p with p' in the definition of f'. **Q.E.D.**

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3.3.2 Lemma. Let f be a function defined on $\mathbf{F}_r$ such that its support is in W_n for some $n \in \mathbf{N}$.

1. If $1 < p \leq 2$, then $\|f\|_{A_p^*(\mathbf{F}_r)} \leq (n+1)\|f\|_p$.

2. If $2 \leq p < \infty$, then $\|f\|_{A_p^*(\mathbf{F}_r)} \leq (n+1)\|f\|_{p'}$.

Proof. We prove the first inequality; the proof of the second inequality is similar. We know that $\|f\|_{A_p^*(\mathbf{F}_r)} = \sup\{\|f * g\|_p : \ g \in l_p(\mathbf{F}_r), \ \|g\|_p \leq 1\}$. For a function $g \in l_p(\mathbf{F}_r)$, let $g_k = g\chi_k$, for every $k = 0, 1, 2, \ldots$. Hence $\|g\|_p^p = \sum_{k=0}^{\infty} \|g_k\|_p^p$. Let also $h = f * g = \sum_{k=0}^{\infty} f * g_k$, where the convergence of the series is pointwise. Then by Lemma 3.3.1,

$$\begin{aligned} \|h_m\|_p &= \|\sum_{k=0}^{\infty}(f * g_k)\chi_m\|_p, \\ &= \|\sum_{k=|m-n|}^{m+n} (f * g_k)\chi_m\|_p, \\ &\leq \|f\|_p \left(\sum_{k=|m-n|}^{m+n} \|g_k\|_p\right). \end{aligned}$$

By Lemma 3.3.1, for the nonzero terms of the above sum, $n + k - m$ is even which implies that $m + n - k$ is also even. Let us write $k = m + n - 2s$, and define $g_i = 0$ if $i < 0$, then

$$\begin{aligned} \|h_m\|_p &\leq \|f\|_p \left(\sum_{s=0}^{\min(n,m)} \|g_{m+n-2s}\|_p\right), \\ &\leq \|f\|_p \left(\sum_{s=0}^{n} \|g_{m+n-2s}\|_p\right), \\ &\leq (n+1)^{1/p'}\|f\|_p \left(\sum_{s=0}^{n} \|g_{m+n-2s}\|_p^p\right)^{1/p}. \end{aligned}$$

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Therefore

$$
\begin{aligned}
\|h\|_p^p &= \sum_{m=0}^{\infty} \|h_m\|_p^p, \\
&\le (n+1)^{p/p'} \|f\|_p^p \sum_{s=0}^{n} \left(\sum_{m=0}^{\infty} \|g_{m+n-2s}\|_p^p \right), \\
&\le (n+1)^{p/p'} \|f\|_p^p \sum_{s=0}^{n} \left(\sum_{k=0}^{\infty} \|g_k\|_p^p \right), \\
&\le (n+1)^{1+p/p'} \|f\|_p^p \left(\sum_{k=0}^{\infty} \|g_k\|_p^p \right), \\
&= (n+1)^p \|f\|_p^p \|g\|_p^p.
\end{aligned}
$$

Thus we have shown that for every $g \in l_p(\mathbf{F}_r)$, $\|f * g\|_p^p \le (n+1)^p \|f\|_p^p \|g\|_p^p$. This of course implies that

$$\|f\|_{A_p^*(\mathbf{F}_r)} \le (n+1)\|f\|_p.$$

Q.E.D.

3.3.3 Lemma. Let $u \in A_p(\mathbf{F}_r)$, and $n \in \mathbf{N}$.

1. If $1 < p \le 2$, then $\|u\|_{A_p} \ge \frac{1}{n+1}\|u\chi_n\|_{p'}$.
2. If $2 \le p < \infty$, then $\|u\|_{A_p} \ge \frac{1}{n+1}\|u\chi_n\|_p$.

Proof. For $1 < p \le 2$,

$$
\begin{aligned}
\|u\chi_n\|_{p'} &= \sup\left\{ |\langle u, v\rangle| : \operatorname{supp} v \subset W_n,\ \|v\|_p \le 1 \right\}, \\
&\le \sup\left\{ \|u\|_{A_p}\|v\|_{A_p^*} : \operatorname{supp} v \subset W_n,\ \|v\|_p \le 1 \right\}, \\
&\le \sup\left\{ (n+1)\|u\|_{A_p}\|v\|_p : \operatorname{supp} v \subset W_n,\ \|v\|_p \le 1 \right\} \qquad \text{by Lemma 3.3.2,} \\
&= (n+1)\|u\|_{A_p}.
\end{aligned}
$$

The second inequality follows similarly. **Q.E.D.**

3.3.4 Lemma. If $1 < p < \infty$, and M is the maximum of $(2r(2r-1)^{n-1})^{1/p}$ and $(2r(2r-1)^{n-1})^{1/p'}$, then

$$\#W_n \ge M\|\chi_n\|_{A_p},$$

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(where $\#W_n$ denotes the number of elements of W_n).

Proof. We have

$$\begin{aligned}\#W_n = (2r(2r-1)^{n-1}) &= (2r(2r-1)^{n-1})^{1/p}(2r(2r-1)^{n-1})^{1/p'}.\\ &= (2r(2r-1)^{n-1})^{1/p}\|\chi_n\|_{p'}.\\ &\geq (2r(2r-1)^{n-1})^{1/p}\|\chi_n\|_{A_p};\end{aligned}$$

the last inequality follows because $\|\chi_n\|_{A_p} = \|\chi_n * \check{\delta}_e\|_{A_p} \leq \|\delta_e\|_p\|\chi_n\|_{p'} = \|\chi_n\|_{p'}$. Similarly we can show that $\#W_n \geq (2r(2r-1)^{n-1})^{1/p'}\|\chi_n\|_{A_p}$, which completes the proof. **Q.E.D.**

Let A be a Banach algebra and $\pi' : A \times A \longrightarrow A$ be the continuous bilinear map defined by $(u, v) \longmapsto uv$. Suppose $\pi : A\widehat{\otimes}A \longrightarrow A$ is the continuous linear map associated with π'. We say that A has the $\pi-$ *property* if the image of π is equal to $\overline{A^2}$.

3.3.5 Theorem. For $r \geq 2$ and $p \in (1, \infty)$, the map $\pi : A_p(\mathbf{F}_r)\widehat{\otimes}A_p(\mathbf{F}_r) \longrightarrow A_p(\mathbf{F}_r)$ defined as above, is not surjective.

Proof. Assume π is surjective. Then as a consequence of the open mapping theorem, there exists a positive constant C with the property that for every $u \in A_p(\mathbf{F}_r)$, there exists $x \in A_p(\mathbf{F}_r)\widehat{\otimes}A_p(\mathbf{F}_r)$ such that $\pi(x) = u$, and $\|x\| \leq C\|u\|_{A_p}$. Therefore

$$\begin{aligned}C\|u\|_{A_p} \geq \|x\| &= \inf\left\{\sum_{i=1}^{\infty}\|f_i\|_{A_p}\|g_i\|_{A_p} : f_i, g_i \in A_p(\mathbf{F}_r),\ x = \sum_{i=1}^{\infty} f_i \otimes g_i\right\},\\ &\geq \inf\left\{\sum_{i=1}^{\infty}\|f_i\|_{A_p}\|g_i\|_{A_p} : f_i, g_i \in A_p(\mathbf{F}_r),\ u = \sum_{i=1}^{\infty} f_i g_i\right\}.\end{aligned}$$

For an arbitrary $n \in \mathbf{N}$, since $\chi_n \in A_p(\mathbf{F}_r)$ and π is assumed to be surjective, there exists an element $x = \sum_{i=1}^{\infty} f_i \otimes g_i \in A_p(\mathbf{F}_r)\widehat{\otimes}A_p(\mathbf{F}_r)$ such that

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$\pi(x) = \sum_{i=1}^{\infty} f_i g_i = \chi_n$. Now by Lemma 3.3.3 we observe that

$$\begin{aligned}
\sum_{i=1}^{\infty} \|f_i\|_{A_p}\|g_i\|_{A_p} &\geq \sum_{i=1}^{\infty} \frac{\|f_i\chi_n\|_p\|g_i\chi_n\|_{p'}}{(n+1)^2(\#W_n)^{|\frac{1}{p}-\frac{1}{p'}|}}, \\
&\geq \frac{\sum_{i=1}^{\infty}\|(f_i g_i)\chi_n\|_1}{(n+1)^2(\#W_n)^{|\frac{1}{p}-\frac{1}{p'}|}}, \\
&= \frac{\sum_{i=1}^{\infty}\sum_{s\in W_n}|f_i(s)g_i(s)|}{(n+1)^2(\#W_n)^{|\frac{1}{p}-\frac{1}{p'}|}}, \\
&\geq \frac{\sum_{s\in W_n}|\sum_{i=1}^{\infty}f_i(s)g_i(s)|}{(n+1)^2(\#W_n)^{|\frac{1}{p}-\frac{1}{p'}|}}, \\
&= \frac{\#W_n}{(n+1)^2(\#W_n)^{|\frac{1}{p}-\frac{1}{p'}|}}.
\end{aligned}$$

Now let us assume for the moment that $1 < p \leq 2$. Then by Lemma 3.3.4.

$$\begin{aligned}
\frac{\#W_n}{(n+1)^2(\#W_n)^{|\frac{1}{p}-\frac{1}{p'}|}} &\geq \frac{(2r(2r-1)^{n-1})^{1/p}}{(n+1)^2(2r(2r-1)^{n-1})^{\frac{1}{p}-\frac{1}{p'}}}\|\chi_n\|_{A_p}, \\
&= \frac{(2r(2r-1)^{n-1})^{1/p'}}{(n+1)^2}\|\chi_n\|_{A_p}.
\end{aligned}$$

Thus $C\|\chi_n\|_{A_p} \geq \frac{(2r(2r-1)^{n-1})^{1/p'}}{(n+1)^2}\|\chi_n\|_{A_p}$. This is impossible. since $n \in \mathbf{N}$ is arbitrary and $r \geq 2$. The case of $2 \leq p < \infty$ also leads to a contradiction by a similar argument. Therefore π is not surjective. **Q.E.D.**

3.3.6 Corollary. If $r \geq 2$ and $p \in (1, \infty)$, then $A_p(\mathbf{F}_r)$ does not have the π-property.

Proof. Since $A_p(\mathbf{F}_r)$ is regular and its elements with compact support are dense in $A_p(\mathbf{F}_r)$, we have $\overline{A_p(\mathbf{F}_r)^2} = A_p(\mathbf{F}_r)$. Hence the corollary follows from Theorem 3.3.5. **Q.E.D.**

We notice that in the special case of $r = 1$, $\mathbf{F}_r = \mathbf{Z}$ is amenable, and hence $\mathcal{H}^2(A_p(\mathbf{Z}), \mathbf{C}_{0,0}) = \{0\}$. Consequently, it follows from Lemma 3.3.7 (below) that $A_p(\mathbf{Z})$ has the π-property, and in particular π is surjective.

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3.3.7 Lemma. If $(A, \|\cdot\|)$ is a Banach algebra such that $\mathcal{H}^2(A, \mathbf{C}_{0,0}) = \{0\}$, then A has the π property.

Proof. We define the projective norm $\|\cdot\|_{\text{proj}}$ on A^2 by

$$\|u\|_{\text{proj}} = \inf\{\sum_{i=1}^{n} \|a_i\|\|b_i\| : a_i, b_i \in A,\ u = \sum_{i=1}^{n} a_i b_i,\ n \in \mathbf{N}\}.$$

Our assumption implies that $\mathcal{N}^2(A, \mathbf{C}_{0,0}) = \tilde{\mathcal{N}}^2(A, \mathbf{C}_{0,0})$, and hence by [35, Proposition I.1.19, p. 64] the two norms $\|\cdot\|$ and $\|\cdot\|_{\text{proj}}$ are equivalent on A^2. In other words, there exists a positive constant C such that for every $u \in A^2$, $\|u\| \le \|u\|_{\text{proj}} \le C\|u\|$. Now it is not difficult to show that

$$\overline{A^2} = \left\{ \sum_{i=1}^{\infty} a_i b_i : a_i, b_i \in A, \text{ and } \sum_{i=1}^{\infty} \|a_i\|\|b_i\| < \infty \right\}.$$

So if $u = \sum_{i=1}^{\infty} a_i b_i \in \overline{A^2}$ (where $\sum_{i=1}^{\infty} \|a_i\|\|b_i\| < \infty$), then $w = \sum_{i=1}^{\infty} a_i \otimes b_i \in A\hat{\otimes}A$, and $\pi(w) = u$. That is, π is onto $\overline{A^2}$. **Q.E.D.**

3.3.8 Corollary. If $r \ge 2$ and $p \in (1, \infty)$, then there exists a singular extension of $A_p(\mathbf{F}_r)$ by $\mathbf{C}_{0,0}$ which does not split strongly.

Proof. This follows from Proposition 3.2.3, Corollary 3.3.6, and Lemma 3.3.7. **Q.E.D.**

3.3.9 Lemma. Let H be a closed subgroup of a locally compact group G, and let $I(H) = \{u \in A_p(G) : u|_H = 0\}$. Then the quotient algebra $A_p(G)/I(H)$ is isometrically isomorphic to $A_p(H)$.

Proof. By [39, Theorem 1a, p. 92], the restriction of functions $A_p(G) \longrightarrow A_p(H)$, $f \longmapsto f|_H$, is a linear contraction; that is, $\|f|_H\|_{A_p(H)} \le \|f\|_{A_p(G)}$. Also, by [39, Theorem 1b, p. 92], for every $h \in A_p(H)$ and $\epsilon > 0$, there exists a function $g \in A_p(G)$ such that $\|g\|_{A_p(G)} \le \|h\|_{A_p(H)} + \epsilon$, and $g|_H = h$. Therefore for every $h \in A_p(H)$, $\|h\|_{A_p(H)} = \inf\{\|g\|_{A_p(G)} : g|_H = h,\ g \in A_p(G)\}$.

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The natural homomorphism $\Psi : A_p(G)/I(H) \longrightarrow A_p(H)$, $[f] \longmapsto f|_H$ is an isometric algebra isomorphism since

$$\begin{aligned}\|[f]\| &= \inf \{\|f + k\|_{A_p(G)} : \ k \in I(H)\}, \\ &= \inf \{\|g\|_{A_p(G)} : \ g \in A_p(G), \ g|_H = f|_H\}, \\ &= \|f|_H\|_{A_p(H)}.\end{aligned}$$

Q.E.D.

3.3.10 Lemma. Let G be a locally compact group which contains $\mathbf{F}_r$, $r \geq 2$, as a closed subgroup. Then $\mathcal{H}^2(A_p(G), \mathbf{C}_{0,0}) \neq \{0\}$, for any $p \in (1, \infty)$.

Proof. We argue by contradiction. If $\mathcal{H}^2(A_p(G), \mathbf{C}_{0,0}) = \{0\}$, then by Lemma 3.3.7, $A_p(G)$ will have the π-property. And then the quotient algebra $A_p(G)/I(\mathbf{F}_r)$ will also have the π-property [67, Proposition 12, p. 10]. But this is impossible by Corollary 3.3.6 and Lemma 3.3.9. **Q.E.D.**

So as a consequence of Lemma 3.3.10, Rickert's Theorem [65, Theorem 5.5], and Tits' Theorem [68, pp. 250–251, Theorem 1 and Corollary 1]) (see also Section 2.1) we have

3.3.11 Theorem. Suppose $p \in (1, \infty)$, and suppose G is an almost connected group or a subgroup of $GL(V)$ for a finite-dimensional vector space V. Then G is amenable if all finite-dimensional singular extensions of $A_p(G)$ split strongly.

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Chapter 4

Order Isomorphisms of the Banach Algebras $A_p(G)$ and $B_p(G)$

4.1 Introduction

Order isomorphisms of the Fourier algebra $A(G)$ and the Fourier-Stieltjes algebra $B(G)$ are studied by Arendt and De Cannière in [5] and [6]. In this chapter we study order isomorphisms of the Banach algebras $A_p(G)$ and their multiplier algebras $B_p(G)$. The main result of Section 4.2 is Theorem 4.2.8 where continuous order isomorphisms between $A_p(G)$ algebras are characterized as weighted composition maps.

The main result of Section 4.3 is Theorem 4.3.9, where it is shown that order isomorphisms for the pointwise order between $B_p(G)$ algebras are $*$-algebra isomorphisms followed by multiplication with invertible positive multipliers.

Disjointness preserving mappings between various spaces has been studied by many authors including Abramovich [1], Abramovich and Kitover[2], [3], Abramovich, Veksler, and Koldunov[4], Font and Hernandez [26], Jarosz [41],

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de Pagter [16], and most recently by L. G. Brown and N.-C. Wong [15].

In [27] and [28], J. J. Font studied the properties of disjointness preserving mappings between Fourier algebras. Most of this theory can be adopted for $A_p(G)$ algebras without significant changes. However the proof of the main result in [28] uses a Bochner-Schoenberg-Eberline-type characterization of the elements in $B(G)$ [22, p. 202, Corollary 2.24]. Such a characterization is not available for $B_p(G)$ (when G is not commutative). In Section 4.4 we give an alternative proof that does not use such characterization.

The main result of Section 4.4 is Theorem 4.4.4, where it is shown that for amenable groups G_1 and G_2, $A_p(G_1)$ is Banach algebra isomorphic to $A_p(G_2)$ if and only if there exists a disjointness preserving bijection between these algebras. As an application, we show that for amenable groups, an order isomorphism for the pointwise order between $A_p(G)$ algebras that preserve cozero sets is necessarily continuous, and consequently the two algebras are isomorphic.

4.2 Order Isomorphisms of the Banach Algebras $A_p(G)$

4.2.1 Definition. Let $A_p(G)_+$ be the set of all $f \in A_p(G)$ such that $f(x) \geq 0$ for all $x \in G$, and let $P(G)$ be the set of all continuous positive-definite functions on G. The *pointwise order* $\geq$ and the *positive-definite order* $\gg$ on $A_p(G)$ are defined as follows

1. $f \geq g$ if $f - g \in A_p(G)_+$.

2. $f \gg g$ if $f - g \in P(G)$.

One may easily verify that both of these relations are partial orders on $A_p(G)$ (notice that $P(G) \cap (-P(G)) = \{0\}$). Corresponding to the partial orders $\geq$

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and $\gg$ on $A_p(G)$, we may define two order relations on $A_p(G)^*$ which we still denote by $\geq$ and $\gg$. These relations are defined as

1. $T_1 \geq T_2$ if $\langle T_1 - T_2, f\rangle \geq 0$ for all $f \in A_p(G)_+$.

2. $T_1 \gg T_2$ if $\langle T_1 - T_2, f\rangle \geq 0$ for all $f \in A_p(G) \cap P(G)$.

4.2.2 Lemma. The two relations $\geq$ and $\gg$ on $A_p(G)^*$ are partial orders.

Proof. For the case of $\geq$, reflexivity and transitivity are immediate consequences of the definition. To verify the anti-symmetry of $\geq$ suppose $T_1 \geq T_2$ and $T_2 \geq T_1$, then

$$\langle T_1, f\rangle = \langle T_2, f\rangle \qquad \text{(for all } f \in A_p(G)_+).$$

Now let $f \in A_p(G)$ be a real valued function with compact support K, and let $\psi \in A_p(G)$ be such that $\psi \geq 0$ and $\psi|_K = 1$. Then

$$-\|f\|_\infty \psi \leq f \leq \|f\|_\infty \psi,$$

and therefore

$$\langle T_1, \|f\|_\infty \psi - f\rangle = \langle T_2, \|f\|_\infty \psi - f\rangle,$$

and

$$\langle T_1, \|f\|_\infty \psi + f\rangle = \langle T_2, \|f\|_\infty \psi + f\rangle.$$

Subtracting the last two equations we obtain

$$\langle T_1, f\rangle = \langle T_2, f\rangle,$$

for every real valued function $f \in A_p(G) \cap C_{00}(G)$. Since T_1 and T_2 are linear and continuous, and since $A_p(G) \cap C_{00}(G)$ is dense in $A_p(G)$ we conclude that $T_1 = T_2$.

Reflexivity and transitivity of $\gg$ follow from the definition. To check the anti-symmetry suppose $T_1 \gg T_2$ and $T_2 \gg T_1$, then

$$\langle T_1, f\rangle = \langle T_2, f\rangle \qquad \text{(for all } f \in A_p(G) \cap P(G)).$$

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Now let $f, g \in C_{00}(G)$. Then $f * \tilde{g} \in A_p(G)$ and by the polar identity

$$4f * \tilde{g} = (f+g) * (f+g)\tilde{} - (f-g) * (f-g)\tilde{} + \\ + i(f+ig) * (f+ig)\tilde{} - (f-ig) * i(f-ig)\tilde{}.$$

That is, $f * \tilde{g}$ is a linear combination of positive definite functions in $A_p(G)$. Thus $T_1 = T_2$ on the subspace of $A_p(G)$ generated by $\{f * \tilde{g} : f, g \in C_{00}(G)\}$; since this subspace is dense in $A_p(G)$ we have $T_1 = T_2$. **Q.E.D.**

4.2.3 Proposition. Let $s \in G$ and let $\delta_s \in A_p(G)^*$ be the evaluation functional at s. Then

1. $\delta_s \geq 0$.

2. $s = e$ is the only element of G for which $\delta_s \gg 0$.

Proof. The first assertion is an immediate consequence of the definitions. To prove the second assertion we notice that for $f \in A_p(G) \cap P(G)$

$$\langle \delta_e, f \rangle = f(e) = \|f\|_\infty \geq 0.$$

So certainly $\delta_e \gg 0$. Now suppose $s \neq e$, we show that $\delta_s \not\gg 0$. We consider two cases:

Case I $s^2 \neq e$. Let K be a compact neighborhood of e such that $sK \cap s^{-1}K = \emptyset$. Let us choose $\psi, \zeta \in C_{00}(G)$ in such a way that $\psi \geq 0$, $\zeta \geq 0$, $\psi \neq 0$, $\zeta|_{s^{-1}K} = 1$, $\zeta|_{sK} = 0$, and $\operatorname{supp}(\psi) \subset K$. We define

$$\phi = (\psi + i\zeta) * (\psi + i\zeta)\tilde{} = \psi * \tilde{\psi} + \zeta * \tilde{\zeta} + i(\zeta * \tilde{\psi} - \psi * \tilde{\zeta}).$$

Then $\phi \in A_p(G) \cap P(G)$. But from our choices of ψ and ζ it follows that $\phi(s)$ is not even a real number, in fact

$$\begin{aligned} \zeta * \tilde{\psi}(s) - \psi * \tilde{\zeta}(s) &= \int \zeta(t)\tilde{\psi}(t^{-1}s)\,dt - \int \psi(t)\tilde{\zeta}(t^{-1}s)\,dt, \\ &= -\int \psi(st)\zeta(t)\,dt, \\ &= -\int \psi(t)\,dt, \\ &\neq 0. \end{aligned}$$

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Case II $s^2 = e$. Let K be a compact neighborhood of e such that $K \cap sK = \emptyset$. We find ψ and $\zeta \in C_{00}(G)$ such that $\psi \geq 0$, $\zeta \leq 0$, $\psi|_V = 1$, $\zeta|_{sV} = -1$ for some neighborhood V of e contained in K, $\operatorname{supp}(\psi) \subset K$, and $\operatorname{supp}(\zeta) \subset sK$ (in fact, once ψ is found with the above properties, it suffices to take $\zeta = -_{s^{-1}}\psi$). We define $\phi = \psi + \zeta$. Then ϕ is real valued, $\operatorname{supp}(\phi) \subset K \cup sK$, $\phi|_K \geq 0$, $\phi|_{sK} \leq 0$, $\phi|_V = 1$, and $\phi|_{sV} = -1$. Therefore

$$\begin{aligned}
\phi * \tilde{\phi}(s) &= \int \phi(t)\tilde{\phi}(t^{-1}s)\,dt, \\
&= \int \phi(t)\phi(st)\,dt, \\
&= \int_K \phi(t)\phi(st)\,dt + \int_{sK} \phi(t)\phi(st)\,dt, \\
&< 0.
\end{aligned}$$

So $\delta_s \not\gg 0$. **Q.E.D.**

4.2.4 Definition. Let G_1 and G_2 be locally compact groups. A linear map $\Phi : A_p(G_1) \longrightarrow A_p(G_2)$ is called *positive for the pointwise order* (respectively, *positive for the positive-definite order*) if $\Phi A_p(G_1)_+ \subset A_p(G_2)_+$ (respectively, if $\Phi(A_p(G_1) \cap P(G_1)) \subset A_p(G_2) \cap P(G_2))$, in which case we write $\Phi \geq 0$ (respectively, $\Phi \gg 0$). The linear map Φ is called *an order isomorphism for the pointwise order* (respectively, *an order isomorphism for the positive-definite order*) if Φ is bijective and $\Phi A_p(G_1)_+ = A_p(G_2)_+$ (respectively, $\Phi(A_p(G_1) \cap P(G_1)) = A_p(G_2) \cap P(G_2))$. We say that Φ is a *biorder isomorphism* if Φ is order isomorphism for both of pointwise and positive-definite orders.

We can of course make similar definitions for linear maps between duals of $A_p(G)$ spaces, the details of which are left for the reader.

4.2.5 Proposition. A linear map $\Phi : A_p(G_1) \longrightarrow A_p(G_2)$ is ≥ 0 (respectively, $\gg 0$) if and only if $\Phi^* : A_p(G_2)^* \longrightarrow A_p(G_1)^*$ is ≥ 0 (respectively, $\gg 0$).

Proof. If $\phi \geq 0$, then it is easy to verify that $\phi^* \geq 0$. Conversely suppose

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$\Phi^* \geq 0$ and $f \in A_p(G_1)_+$, then for every $x \in G_2$

$$\Phi f(x) = \langle \delta_x, \Phi f \rangle = \langle \Phi^* \delta_x, f \rangle \geq 0,$$

since $\delta_x \geq 0$ by Proposition 4.2.3. Hence $\Phi \geq 0$.

Similarly for the case of $\gg$, it is easy to show that $\Phi^* \gg 0$ whenever $\Phi \gg 0$. To prove the converse, suppose $\Phi^* \gg 0$ and $f \in A_p(G_1) \cap P(G_1)$, we want to show that $\Phi f \in A_p(G_2) \cap P(G_2)$. It suffices to show that

$$\int (\phi^* * \phi)\Phi f \geq 0 \qquad \text{(for all } \phi \in L_1(G_2)).$$

Let $\{\lambda_p, L_p(G_2)\}$ be the left regular representation of $L_1(G_2)$ on $L_p(G_2)$, then $\lambda_p(\phi^* * \phi) \in A_p(G_2)^*$, and

$$\langle \lambda_p(\phi^* * \phi), g \rangle = \int (\phi^* * \phi) g \geq 0 \qquad \text{(for all } g \in A_p(G_2) \cap P(G_2)).$$

So $\lambda_p(\phi^* * \phi) \gg 0$. Consequently for all $f \in A_p(G_1) \cap P(G_1)$

$$\begin{aligned} \int (\phi^* * \phi)\Phi f &= \langle \lambda_p(\phi^* * \phi), \Phi f \rangle, \\ &= \langle \Phi^* \lambda_p(\phi^* * \phi), f \rangle \geq 0. \end{aligned}$$

Q.E.D.

4.2.6 Corollary. $\Phi : A_p(G_1) \longrightarrow A_p(G_2)$ is an order isomorphism for the pointwise order (respectively, for the positive-definite order) if and only if Φ^* is so.

4.2.7 Lemma. Let $T \in A_p(G)^*$ and $T \geq 0$. Then $s \in \operatorname{supp}(T)$ if and only if for every neighborhood U of s, there exists some $f \in A_p(G)_+$ such that $\operatorname{supp}(f) \subset U$, and $\langle T, f \rangle \neq 0$.

Proof. To prove the necessity of the condition let $s \in \operatorname{supp}(T)$, and let U be a neighborhood of s. Let V be a relatively compact neighborhood of s such that $\overline{V} \subset U$. From the definition of $\operatorname{supp}(T)$ we know that there exists $g \in A_p(G)$ such that $\operatorname{supp}(g) \subset V$, and $\langle T, g \rangle \neq 0$. Since the real and imaginary parts of

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g belong to $A_p(G)$, we may assume that g is real valued. We also notice that since $\overline{V}$ is compact, supp(g) is a compact set. Let $f \in A_p(G)_+$ be such that $f|_{\text{supp}(g)} = 1$ and supp$(f) \subset U$. Since

$$-\|g\|_\infty f \leq g \leq \|g\|_\infty f,$$

we have

$$-\|g\|_\infty \langle T, f\rangle \leq \langle T, g\rangle \leq \|g\|_\infty \langle T, f\rangle.$$

Therefore $\langle T, f\rangle \neq 0$. **Q.E.D.**

4.2.8 Theorem. Let $\Phi : A_p(G_1) \longrightarrow A_p(G_2)$ be a continuous order isomorphism for the pointwise order. Then there exists a homeomorphism $\alpha : G_2 \longrightarrow G_1$ and a continuous function $c : G_2 \longrightarrow (0, \infty)$ such that

$$\Phi f(t) = c(t) f(\alpha(t)) \qquad (t \in G_2).$$

Proof. To define α the idea is to show that for every $t \in G_2$, the support of $\Phi^*\delta_t$ is a single point of G_1 that we denote by $\alpha(t)$. Since Φ^* is injective and $\delta_t \neq 0$, we have $\Phi^*\delta_t \neq 0$, and hence supp $\Phi^*\delta_t \neq \emptyset$. Let us assume that there are at least two different points s_1, s_2 in supp$(\Phi^*\delta_t)$, and let U and V be two disjoint neighborhoods of s_1 and s_2 respectively. By Proposition 4.2.3 and Lemma 4.2.7, there exist $f, g \in A_p(G_1)_+$ such that supp$(f) \subset U$, supp$(g) \subset V$, and

$$\Phi f(t) = \langle \Phi^*\delta_t, f\rangle > 0, \quad \Phi g(t) = \langle \Phi^*\delta_t, g\rangle > 0.$$

Pick an open neighborhood W of t, and a constant $c_0 > 0$, such that for all $x \in W$, $\Phi f(x) > c_0$, and $\Phi g(x) > c_0$. Let $K \subset W$ be a compact neighborhood of t. We can find $u \in A_p(G_2)$ such that $0 \leq u \leq 1$, $u = 1$ on K, and $u = 0$ outside W (to find u let V be a compact symmetric neighborhood of e such that $KV^2 \subset W$, and let χ_{KV} and χ_V be the characteristic functions of KV and V respectively, then $\frac{1}{|V|}\chi_{KV} * \check{\chi}_V$ is the required function). Let $h = c_0 u$. Hence $h \neq 0$, $h \in A_p(G_2)_+$, $0 \leq h \leq \Phi f$, and $0 \leq h \leq \Phi g$. Since by assumption

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$\Phi^{-1} \geq 0$, we obtain $0 \leq \Phi^{-1}h \leq f$, and $0 \leq \Phi^{-1}h \leq g$. But $\Phi^{-1}h \neq 0$, so for some $s \in G_1$,

$$0 < \Phi^{-1}h(s) \leq f(s), \quad \text{and} \quad 0 < \Phi^{-1}h(s) \leq g(s).$$

In other words $s \in U \cap V$, which is impossible. Thus $\operatorname{supp}(\Phi^* \delta_t)$ is a single point in G_1 which we denote by $\alpha(t)$. Thus for some constant $c(t) > 0$,

$$\Phi^* \delta_t = c(t)\delta_{\alpha(t)},$$

(the fact that $c(t) > 0$ is a consequence of $\Phi^*\delta_t \neq 0, \Phi^*\delta_t \geq 0$). Notice that if $f \in A_p(G_1)$, then for every $t \in G_2$,

$$\begin{aligned} \Phi f(t) = \langle \delta_t, \Phi f \rangle &= \langle \Phi^* \delta_t, f \rangle, \\ &= \langle c(t)\delta_{\alpha(t)}, f \rangle, \\ &= c(t) f(\alpha(t)). \end{aligned}$$

To show that α is bijective, we apply the argument above to Φ^{-1} to obtain functions $d : G_1 \longrightarrow (0, \infty)$, and $\beta : G_1 \longrightarrow G_2$ such that for every $s \in G_1$

$$(\Phi^*)^{-1} \delta_s = d(s)\delta_{\beta(s)}.$$

Hence

$$\delta_s = d(s)\Phi^*(\delta_{\beta(s)}) = d(s)c(\beta(s))\delta_{\alpha(\beta(s))} \qquad (\text{for all } s \in G_1).$$

and

$$\delta_t = (\Phi^*)^{-1}(c(t)\delta_{\alpha(t)}) = c(t)d(\alpha(t))\delta_{\beta(\alpha(t))} \qquad (\text{for all } t \in G_2).$$

Therefore

$$\beta(\alpha(t)) = t, \quad \alpha(\beta(s)) = s, \quad \text{and} \quad d(s) = \frac{1}{c(\alpha^{-1}(s))}.$$

It remains to show that α, α^{-1}, and c are continuous functions. Suppose α is not continuous at some $t_0 \in G_2$. Let $(t_j)_{j \in J}$ be a net in G_2 such that

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$t_j \longrightarrow t_0$, but $\alpha(t_j)$ does not converge to $\alpha(t_0)$. So there exists a neighborhood W of $\alpha(t_0)$ such that for any given $j_1 \in J$, there exists some $j_2 \geq j_1$ for which $\alpha(t_{j_2}) \notin W$. Let $f \in A_p(G_1)$ be such that $f = 1$ on some neighborhood U of $\alpha(t_0)$, $U \subset W$, and $f = 0$ outside W. From continuity of Φf we have

$$\Phi f(t_j) \longrightarrow \Phi f(t_0) = c(t_0)f(\alpha(t_0)) = c(t_0) > 0.$$

So we may choose $j_1 \in J$ such that $\Phi f(t_j) > 0$, for all $j \geq j_1$. This is of course impossible in view of the properties of W and f. A similar argument shows that α^{-1} is also continuous.

Finally to show the continuity of $c : G_2 \longrightarrow (0, \infty)$. let $t_0 \in G_2$ and let W be a compact neighborhood of $\alpha(t_0)$. Choose $f \in A_p(G_1)$ such that $f|_W = 1$. Then for every $t \in \alpha^{-1}(W)$,

$$\Phi f(t) = c(t)f(\alpha(t)) = c(t).$$

Since Φf is continuous at t_0, so is c. **Q.E.D.**

4.2.9 Remark. In the above theorem for any $f \in A_p(G_1)$, $f \circ \alpha \in B_p(G_2)$. To see this let $g \in A_p(G_2)$ and let $\Phi h = g$ for some $h \in A_p(G_1)$. Then

$$\begin{aligned} g \cdot f \circ \alpha = \Phi(h) \cdot f \circ \alpha &= c \cdot (h \circ \alpha) \cdot (f \circ \alpha), \\ &= c \cdot (h \cdot f) \circ \alpha, \\ &= \Phi(h \cdot f) \in A_p(G_2). \end{aligned}$$

4.3 Order Isomorphisms of the Banach Algebras $B_p(G)$

4.3.1 Definition. Let $B_p(G)_+$ be the set of all $f \in B_p(G)$ such that $f(x) \geq 0$ for all $x \in G$, and let $P(G)$ be the set of all continuous positive-definite functions on G. The *pointwise order* $\geq$ and the *positive-definite order* $\gg$ on $B_p(G)$ are defined as follows

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1. $f \geq g$ if $f - g \in B_p(G)_+$.

2. $f \gg g$ if $f - g \in P(G)$.

Now let G_1 and G_2 be locally compact groups. A linear map $\Psi : B_p(G_1) \longrightarrow B_p(G_2)$ is called *positive for the pointwise order* (respectively, *positive for the positive-definite order*) if $\Psi B_p(G_1)_+ \subset B_p(G_2)_+$ (respectively, if $\Psi(B_p(G_1) \cap P(G_1)) \subset B_p(G_2) \cap P(G_2))$, in which case we write $\Psi \geq 0$ (respectively, $\Psi \gg 0$). The linear map Ψ is called *an order isomorphism for the pointwise order* (respectively, *an order isomorphism for the positive-definite order*) if Ψ is bijective, and $\Psi B_p(G_1)_+ = B_p(G_2)_+$ (respectively, $\Psi(B_p(G_1) \cap P(G_1)) = B_p(G_2) \cap P(G_2))$. We say that Ψ is a *biorder isomorphism* if Ψ is order isomorphism for both of pointwise and positive-definite orders.

4.3.2 Remark. It is easy to see that an element $u \in B_p(G)$ can be written as a linear combinations of elements in $B_p(G)_+$, in fact since $\overline{u} \in B_p(G)$, without loss of generality we may assume that u is real valued. From $-\|u\|_\infty 1 \leq u \leq \|u\|_\infty 1$, it follows that $u + \|u\|_\infty 1$ and $\|u\|_\infty 1 - u$ belong to $B_p(G)_+$, and hence $u = (u + \|u\|_\infty 1 - (\|u\|_\infty 1 - u))/2$ has the desired form.

4.3.3 Lemma. If $\Psi : B_p(G_1) \longrightarrow B_p(G_2)$ is positive for the pointwise order, then Ψ is continuous.

Proof. We use the closed graph theorem. Let $(f_n)_{n\in\mathbf{N}}$ be a sequence in $B_p(G_1)$ such that $\|f_n\|_{B_p} \longrightarrow 0$, and $\|\Psi f_n - g\|_{B_p} \longrightarrow 0$ for some $g \in B_p(G_2)$. Since $\|\cdot\|_{B_p} \geq \|\cdot\|_\infty$, we obtain $\|f_n\|_\infty \longrightarrow 0$, and $\|\Psi f_n - g\|_\infty \longrightarrow 0$. Furthermore, since

$$-\|f_n\|_\infty 1 \leq \mathrm{Re}\,(f_n) \leq \|f_n\|_\infty 1,$$

$$-\|f_n\|_\infty 1 \leq \mathrm{Im}\,(f_n) \leq \|f_n\|_\infty 1,$$

we have

$$-\|f_n\|_\infty \Psi 1 \leq \Psi(\mathrm{Re}\,(f_n)) \leq \|f_n\|_\infty \Psi 1,$$

$$-\|f_n\|_\infty \Psi 1 \leq \Psi(\mathrm{Im}\,(f_n)) \leq \|f_n\|_\infty \Psi 1.$$

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which implies that $\|\Psi f_n\|_\infty \leq 2\|f_n\|_\infty\|\Psi 1\|_\infty$. So when $n \longrightarrow \infty$, $\|\Psi f_n\|_\infty \longrightarrow 0$, that is, $g = 0$. **Q.E.D.**

Since $B_p(G)$ is unital, if h is a function on G such that $hB_p(G) \subset B_p(G)$, then $h \in B_p(G)$.

4.3.4 Definition. For $h \in B_p(G)$, let $M_h : B_p(G) \longrightarrow B_p(G)$ be defined by $M_h(f) = hf$. We call $\{M_h : h \in B_p(G)\}$, *the set of multipliers of* $B_p(G)$.

4.3.5 Proposition. Let $M : B_p(G) \longrightarrow B_p(G)$ be positive for the pointwise order. Then M is a multiplier of $B_p(G)$ if and only if there exists $c \geq 0$ such that $Mf \leq cf$, for all $f \in B_p(G)_+$.

Proof. First let us assume that $M = M_h$ is a multiplier. Then $h = M_h(1) \in B_p(G)_+$, and hence

$$M_h f = hf \leq \|h\|_\infty f \qquad \text{(for every } f \in B_p(G)_+).$$

The idea for the proof of the converse is to show that $M = M_h$ for $h = M1$. This can be achieved by the following trick. We show that if $v \in B_p(G)$ and $v(t) = 0$ for some $t \in G$, then $Mv(t) = 0$. If $v \in B_p(G)_+$, then $0 \leq Mv(t) \leq cv(t)$ implies that $Mv(t) = 0$. Now if v is arbitrary and $v(t) = 0$, we define a semi-inner product on $B_p(G)$ by

$$(h|k) = M(h\bar{k})(t).$$

Applying the Cauchy-Schwarz inequality we get

$$\begin{aligned} |Mv(t)|^2 = |(v|1)|^2 &\leq (v|v) \cdot (1|1), \\ &= M|v|^2(t) \cdot (1|1). \end{aligned}$$

Since $|v|^2(t) = 0$, the first part of our argument implies that $M|v|^2(t) = 0$, and hence $Mv(t) = 0$.

Now set $h = M1$, and let $t \in G$ be arbitrary. Then for every $f \in B_p(G), (f - f(t)1)(t) = 0$, and hence

$$(Mf - f(t)M1)(t) = 0,$$

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or

$$Mf(t) = h(t)f(t),$$

as we wanted to show. **Q.E.D.**

4.3.6 Lemma. If $\chi : G \longrightarrow \mathbf{T}$ is a continuous homomorphism, then $\chi \in B_p(G)$, $\|\chi\|_{B_p} = 1$, and $M_\chi : A_p(G) \longrightarrow A_p(G)$, $u \longmapsto \chi u$, is an isometric isomorphism.

Proof. Let $f \in L_p(G)$, $g \in L_{p'}(G)$, then for every $t \in G$

$$\begin{aligned}\chi(t)g * \check{f}(t) &= \chi(t)\int g(y)\check{f}(y^{-1}t)\,dy.\\ &= \int \chi(y)g(y)\overline{\chi}(t^{-1}y)\overline{f}(t^{-1}y)\,dy,\\ &= (\chi g) * (\chi f)\check{}(t).\end{aligned}$$

Since $\chi f \in L_p(G)$ and $\chi g \in L_{p'}(G)$, $\chi(g * \check{f}) \in A_p(G)$. Now suppose $u = \sum_{i=1}^{\infty} g_i * \check{f}_i$ is an arbitrary element of $A_p(G)$. Then for every $t \in G$

$$\begin{aligned}\chi(t)u(t) &= \chi(t)\lim_{n\to\infty}\sum_{1}^{n} g_i * \check{f}_i(t).\\ &= \lim_{n\to\infty}\sum_{1}^{n}(\chi g_i) * (\chi f_i)\check{}(t),\\ &= \sum_{1}^{\infty}(\chi g_i) * (\chi f_i)\check{}(t).\end{aligned}$$

Since for every i, $\|f_i\|_p = \|\chi f_i\|_p$ and $\|g_i\|_{p'} = \|\chi g_i\|_{p'}$, the last series is absolutely (and hence uniformly) convergent and therefore $\chi u \in A_p(G)$. We have shown that $\chi \in B_p(G)$, and (by the definition of $\|\cdot\|_{A_p}$) $\|\chi u\|_{A_p} \le \|u\|_{A_p}$; but then

$$\|u\|_{A_p} = \|\overline{\chi}(\chi u)\|_{A_p} \le \|\chi u\|_{A_p} \le \|u\|_{A_p}.$$

So $\|u\|_{A_p} = \|\chi u\|_{A_p}$ for every $u \in A_p(G)$; that is,

$$M_\chi : A_p(G) \longrightarrow A_p(G). \qquad u \longmapsto \chi u,$$

is an isometric isomorphism. **Q.E.D.**

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4.3.7 Proposition. For $h \in B_p(G)$, M_h is an order isomorphism for the positive-definite order if and only if $h = c\chi$, where χ is a continuous character and $c > 0$. Furthermore $M_{c\chi}$ is an isometry if and only if $c = 1$.

Proof. First let us assume that $h = c\chi$. Then $M_{c\chi}$ is certainly injective and since $\overline{\chi} \in B_p(G)$ (Lemma 4.3.6), and $c\chi(\frac{1}{c}\overline{\chi}f) = f$ for every $f \in B_p(G)$, $M_{c\chi}$ is also surjective. In addition, $B_p(G) \cap P(G)$ is invariant under the action of $M_{c\chi}$ and $M_{\frac{1}{c}\overline{\chi}}$, therefore $M_{c\chi}$ is an order isomorphism for positive-definite order.

Conversely, suppose M_h is an order isomorphism for the positive definite order. Since $M_h \gg 0$ and $M_h^{-1} = M_{1/h} \gg 0$, h, and $1/h$ are positive definite functions. Thus

$$|h(t)| \leq h(e), \quad |1/h(t)| \leq 1/h(e) \qquad (\text{for all } t \in G).$$

So $|h(t)| = h(e)$ for all $t \in G$. But on $P(G)$, $\|\cdot\|_\infty = \|\cdot\|_{B(G)}$ [22, (1.19) and (2.5)], hence $\|h\|_{B(G)} = |h(t)| \neq 0$ for all $t \in G$. Now by [6, Lemma 4.3], $h = c\chi$, where $c = h(e)$ and χ is a continuous character on G.

To prove the last part of the theorem, notice that by Lemma 4.3.6, M_χ is an isometry. On the other hand if $M_{c\chi}$, $(c > 0)$, is an isometry then $1 = \|\overline{\chi}\|_{B_p} = \|c\chi \cdot \overline{\chi}\|_{B_p} = c$. **Q.E.D.**

For our next result recall that $B_p(G)$ with pointwise operations and complex conjugation is an involutive Banach algebra.

4.3.8 Lemma. Every $*$-algebra isomorphism $\Psi : B_p(G_1) \longrightarrow B_p(G_2)$ is an order isomorphism for the pointwise order.

Proof. Suppose $\Psi_0 = \Psi|_{A_p(G_1)} : A_p(G_1) \longrightarrow B_p(G_2)$. First we show that Ψ_0 is order preserving. Let $f \in A_p(G_1)_+$ and $t \in G_2$, then

$$\Psi_0 f(t_2) = \langle \delta_{t_2}, \Psi_0 f\rangle = \langle \Psi_0^* \delta_{t_2}, f\rangle.$$

If $\Psi_0^*\delta_{t_2} = 0$ we have nothing to show. Otherwise $\Psi_0^*\delta_{t_2}$ is a nonzero multiplicative linear functional on $A_p(G_1)$ and hence $\Psi_0^*\delta_{t_2} = \delta_{\alpha(t_2)}$, for some $\alpha(t_2) \in G_1$.

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Thus $\Psi_0 f(t_2) = f(\alpha(t_2)) \geq 0$. Similarly one can show that $\Psi^{-1}g \geq 0$ for all $g \in A_p(G_2)_+$.

We now prove that Ψ preserves the pointwise order on $B_p(G_1)$. Suppose $f \in B_p(G_1)_+$ but for some $t_2 \in G_2$, $\Psi f(t_2) < 0$. Since $A_p(G_2)$ is regular we may find a function $g \in A_p(G_2)_+$ such that $g(t_2) \neq 0$, and g is zero outside $\{t \in G_2 : \Psi f(t) < 0\}$ (if the complement of this set is empty, we drop the last condition). Then $g\Psi f \neq 0$, $g\Psi f \leq 0$, and $g\Psi f \in A_p(G_2)$. Therefore by the first part of our proof $(\Psi^{-1}g) \cdot f = \Psi^{-1}(g \cdot \Psi f) \leq 0$ and $\Psi^{-1}g \geq 0$. Since $(\Psi^{-1}g) \cdot f \neq 0$ and $f \geq 0$ we have a contradiction. Similarly we can prove that Ψ^{-1} preserves the pointwise order and thus Ψ is an order isomorphism. **Q.E.D.**

4.3.9 Theorem. In order that a linear map $\Psi : B_p(G_1) \longrightarrow B_p(G_2)$ be an order isomorphism for the pointwise order it is necessary and sufficient that for some $*$-algebra isomorphism $V : B_p(G_1) \longrightarrow B_p(G_2)$,

$$\Psi f = h \cdot Vf \qquad \text{(for all } f \in B_p(G_1)),$$

where $h \in B_p(G_2)_+$ is such that $1/h \in B_p(G_2)_+$.

Proof. First we show the sufficiency of our hypotheses. Since V is linear and injective and $h(t) > 0$ for all $t \in G_2$, Ψ is also an injective linear map. Given $g \in B_p(G_2)$ let $k \in B_p(G_1)$ be such that $Vk = g/h$. Then $\Psi k = h \cdot Vk = g$ and hence Ψ is surjective. By Lemma 4.3.8, V is an order isomorphism for the pointwise order which implies that Ψ is an order isomorphism for the pointwise order as well.

Next we show the necessity of our hypotheses. The idea for obtaining V is as follows: for any given $f \in B_p(G_1)$ we show that $\Psi M_f \Psi^{-1} : B_p(G_2) \longrightarrow B_p(G_2)$ is a multiplier and hence is given by some function in $B_p(G_2)$ which we denote by Vf. First we notice that if $f \in B_p(G_1)_+$, then $\Psi M_f \Psi^{-1} \geq 0$. And since for every $g \in B_p(G_2)_+$, $f \cdot \Psi^{-1}g \leq \|f\|_\infty \Psi^{-1}g$ we have $\Psi M_f \Psi^{-1}g =$

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$\Psi(f\cdot\Psi^{-1}g)\leq\|f\|_\infty g$. So by Proposition 4.3.5, $\Psi M_f\Psi^{-1}$ is a multiplier given by a function in $B_p(G_2)_+$. But $B_p(G_1)_+$ spans $B_p(G_1)$ (Remark 4.3.2), so the above argument shows that for every $f\in B_p(G_1)$, $\Psi M_f\Psi^{-1}$ is a multiplier given by some (necessarily unique) function in $B_p(G_2)$ which we call Vf, that is, $\Psi M_f\Psi^{-1}=M_{Vf}$.

Next we claim that $V: B_p(G_1)\longrightarrow B_p(G_2)$, $f\longmapsto Vf$, is a $*$- algebra isomorphism. The following identities ($r\in\mathbf{C}$, $f,g\in B_p(G_1)$)

$$\begin{aligned}M_{V(f+rg)}=\Psi M_{f+rg}\Psi^{-1}&=\Psi M_f\Psi^{-1}+r\Psi M_g\Psi^{-1},\\&=M_{Vf}+rM_{Vg},\\&=M_{Vf+rVg},\end{aligned}$$

show that V is a linear map. Injectivity of V follows from bijectivity of Ψ. We show that V is onto. For given $g\in B_p(G_2)$, using a similar argument as in above we may show that $\Psi^{-1}M_g\Psi: B_p(G_1)\longrightarrow B_p(G_1)$ is a multiplier, and hence $\Psi^{-1}M_g\Psi=M_f$ for some $f\in B_p(G_1)$. Thus

$$M_{Vf}=\Psi M_f\Psi^{-1}=\Psi(\Psi^{-1}M_g\Psi)\Psi^{-1}=M_g,$$

that is, $Vf=g$. To show that V is a homomorphism we observe that

$$\begin{aligned}M_{V(fg)}=\Psi M_{fg}\Psi^{-1}&=\Psi M_fM_g\Psi^{-1},\\&=(\Psi M_f\Psi^{-1})(\Psi M_g\Psi^{-1}),\\&=M_{Vf}M_{Vg},\\&=M_{VfVg}.\end{aligned}$$

So $V(fg)=VfVg$. We also notice that since Ψ sends real valued functions to real valued functions, so does V. Hence if $f=f_1+if_2$ where f_1,f_2 are real valued functions in $B_p(G_1)$, then from linearity of V we conclude that $V(\overline{f})=V(f_1-if_2)=Vf_1-iVf_2=\overline{Vf}$.

The final stage of the proof is to show that if $h=\Psi 1$, then for every $f\in B_p(G_1), \Psi f=h\cdot Vf$. Since Ψ is order preserving, $h\geq 0$. And since

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$M_{Vf} = \Psi M_f \Psi^{-1}$ we get $M_{Vf}\Psi 1 = \Psi M_f 1$, or $\Psi f = h \cdot Vf$. Furthermore since Ψ is onto, for some $g \in B_p(G_1)$, $h \cdot Vg = \Psi g = 1$. In other words $1/h = Vg \in B_p(G_2)$. **Q.E.D.**

4.3.10 Corollary. Let $\Psi : B_p(G_1) \longrightarrow B_p(G_2)$ be a biorder isomorphism. Then there exists a constant $c > 0$ and a $*$-algebra isomorphism $V : B_p(G_1) \longrightarrow B_p(G_2)$ which is also a biorder isomorphism, such that $\Psi = cV$.

Proof. From Theorem 4.3.9 we know that $\Psi = M_h V$, where $h = \Psi 1$ and $1/h$ both belong to $B_p(G_2)_+$. Since $\Psi \gg 0$, $h \in P(G_2)$. If we can show that $1/h \in P(G_2)$, then $h(t) = h(e)$ for all $t \in G_2$ which is what we need to complete the proof. From the proof of Theorem 4.3.9, we saw that for every $f \in B_p(G_1)$, $M_{Vf} = \Psi M_f \Psi^{-1}$. So if $f \in P(G_1)$, then $M_{Vf} \gg 0$. In particular $Vf = M_{Vf}1 \in P(G_2)$, that is $V \gg 0$. Thus $M_{1/h} = V\Psi^{-1} \gg 0$, which implies that $1/h \in P(G_2)$. **Q.E.D.**

4.4 Disjointness Preserving Mappings Between the Banach Algebras $A_p(G)$

4.4.1 Definition. Let G_1 and G_2 be two locally compact groups. A linear operator $T : A_p(G_1) \longrightarrow A_p(G_2)$ is called disjointness preserving if whenever f_1 and f_2 in $A_p(G_1)$ are such that $f_1 \cdot f_2 = 0$, then $Tf_1 \cdot Tf_2 = 0$.

Clearly every homomorphism between $A_p(G)$ algebras is a disjointness preserving mapping, but the converse is not true. As an example, if ϕ is a multiplier of $A_p(G)$ then multiplication by ϕ is disjointness preserving on $A_p(G)$, but is not a homomorphism.

First we state the following characterization of continuous disjointness preserving bijections that we need in the sequel, due to Font [27]. A rather complicated proof of this theorem can be found scattered through section three of

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[27]. Our proof is quite simple.

4.4.2 Theorem. Let $T : A_p(G_1) \longrightarrow A_p(G_2)$ be a continuous, bijective, disjointness preserving mapping. Then

1. There are continuous maps

$$h : G_2 \longrightarrow G_1,$$
$$\kappa : G_2 \longrightarrow \mathbf{C} \setminus \{0\},$$

such that

$$Tf(y) = \kappa(y) f(h(y)) \qquad \text{for all } f \in A_p(G_1),\ y \in G_2.$$

2. $h(G_2)$ is dense in G_1.

3. T^{-1} is disjointness preserving and for some continuous map $\psi : G_1 \longrightarrow \mathbf{C} \setminus \{0\}$, we have

$$T^{-1}g(x) = \psi(x) g(h^{-1}(x)) \qquad (\text{for all } g \in A_p(G_2), x \in G_1).$$

In particular, h is a homeomorphism and

$$\psi \kappa \circ h^{-1} = 1_{G_1},$$
$$\kappa \psi \circ h = 1_{G_2}.$$

Proof.

1. For $y \in G_2$, let $\delta_y \in A_p(G_2)^*$ be the evaluation functional at y, then the map

$$\delta_y T : A_p(G_1) \longrightarrow \mathbf{C}$$
$$f \longmapsto Tf(y)$$

is in $A_p(G_1)^*$. Since T is onto and $A_p(G_2)$ is regular, it follows that $\delta_y T$ is nonzero, and hence $\operatorname{supp} \delta_y T \neq \emptyset$. We show that this support is a

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single point. If $x_1, x_2 \in \operatorname{supp} \delta_y T$, $x_1 \neq x_2$, then we can find disjoint neighborhoods V_1, V_2 of x_1 and x_2 respectively, and functions $f_1, f_2 \in A_p(G_1)$ such that $\operatorname{supp} f_1 \subset V_1$, $\operatorname{supp} f_2 \subset V_2$, and $Tf_1(y) \neq 0$, $Tf_2(y) \neq 0$. But $f_1 \cdot f_1 = 0$ and so by assumption $Tf_1 Tf_2 = 0$, which is a contradiction. Hence $\operatorname{supp} \delta_y T = \{x\}$ for some $x \in G_1$. So we can define a map $h : G_2 \longrightarrow G_1$, $y \longmapsto x$, where x is associated to y as above. It follows that for some $\kappa(y) \in \mathbf{C} \setminus \{0\}$, $\delta_y T = \kappa(y)\, \delta_{h(y)}$, and in particular

$$Tf(y) = \kappa(y) f(h(y)) \qquad (\text{for all } f \in A_p(G_1), \text{ and all } y \in G_2).$$

That is,

$$Tf = \kappa f \circ h.$$

Suppose h is not continuous at some $y_0 \in G_2$. Then for some net $(y_\alpha)_{\alpha \in I}$ in G_2, $y_\alpha \longrightarrow y_0$, and yet $h(y_\alpha)$ does not tend to $h(y_0)$. So there exists a neighborhood U of $h(y_0)$ such that for every $\alpha_0 \in I$, there exists some $\alpha \geq \alpha_0$ such that $h(y_\alpha) \notin U$. Let $f \in A_p(G_1)$ be such that $\operatorname{supp} f \subset U$, and $f(h(y_0)) \neq 0$. Then $Tf(y_0) = \lim_\alpha Tf(y_\alpha) = \lim_\alpha \kappa(y_\alpha) f(h(y_\alpha)) = 0$. Since $Tf(y_0) = \kappa(y_0) f(h(y_0)) \neq 0$ we have a contradiction, which proves the continuity of h. The continuity of κ follows from the continuity of h and that of Tf; since if $y_0 \in G_2$, and if $f \in A_p(G_1)$ is identically equal to 1 on a compact neighborhood U of $h(y_0)$, then for every $y \in h^{-1}(U)$:

$$Tf(y) = \kappa(y) f(h(y)) = \kappa(y).$$

2. We want to show that $h(G_2)$ is dense in G_1. Assuming the contrary, there exists $x_0 \in G_1$ and a neighborhood U of x_0 such that $U \cap h(G_2) = \emptyset$. Let $f \in A_p(G_1)$ be such that $\operatorname{supp} f \subset U$, and $f(x_0) \neq 0$. Then

$$\delta_y T(f) = \kappa(y) f(h(y)) = 0 \qquad (\text{for all } y \in G_2),$$

and thus $Tf = 0$. Since T is one-to-one, $f = 0$, which is a contradiction.

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3. To show that T^{-1} is disjointness preserving, suppose g_1 and g_2 in $A_p(G_2)$ are such that

$$g_1 = Tf_1 = \kappa f_1 \circ h, \; g_2 = Tf_2 = \kappa f_2 \circ h, \text{ and } \; g_1 \cdot g_2 = 0.$$

Then $\kappa^2(f_1 \circ h)(f_2 \circ h) = 0$, and so $(f_1 \cdot f_2) \circ h = 0$. Since $h(G_2)$ is dense in G_1, $f_1 \cdot f_2 = 0$, as we wanted to show. By the Inverse Mapping Theorem, T^{-1} is continuous and therefore by the first part of our theorem

$$T^{-1} g = \psi g \circ k, \qquad (\text{for all } g \in A_p(G_2)),$$

where $\psi : G_2 \longrightarrow \mathbf{C} \setminus \{0\}$, and $k : G_1 \longrightarrow G_2$ are continuous functions, and $k(G_1)$ is dense in G_2 (see part two of the theorem). Now suppose $f \in A_p(G_1)$, then

$$f = T^{-1}(Tf) = T^{-1}(\kappa f \circ h) = \psi \cdot (\kappa \circ k)(f \circ h) \circ k. \qquad (*)$$

Now we claim that $h \circ k = 1_{G_1}$, since otherwise for some $x_0 \in G_1$, $h \circ k(x_0) = x_1 \neq x_0$, and taking $f \in A_p(G_1)$ such that $f(x_0) = 0$, $f(x_1) = 1$ we obtain

$$0 = f(x_0) = \psi(x_0)\kappa(k(x_0))f(x_1) \neq 0,$$

which is of course impossible. Similarly $k \circ h = 1_{G_2}$. It follows from $(*)$ that, $f = \psi(\kappa \circ k)f$, for all $f \in A_p(G_1)$; and hence $\psi\kappa \circ h^{-1} = 1_{G_1}$. Similarly $\kappa\psi \circ h = 1_{G_2}$.

Q.E.D.

4.4.3 Remark. The functions κ and ψ in the above theorem are called the weight functions of T and T^{-1}, respectively.

4.4.4 Theorem. For amenable groups G_1 and G_2, $A_p(G_1)$ is Banach algebra isomorphic to $A_p(G_2)$ if and only if there exists a disjointness preserving bijection between these algebras.

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Proof. We use the terminologies and notations of Theorem 4.4.2. Let $T : A_p(G_1) \longrightarrow A_p(G_2)$ be a disjointness preserving bijection, with weight functions κ and ψ. It follows from [27, p. 339, Theorem 1], the amenability of G_1, and Proposition 3.2.9 that T is continuous. By [28, p. 182, Theorem 3], it suffices to show that the κ and ψ are multipliers of $A_p(G_2)$ and $A_p(G_1)$, respectively. We prove that $\kappa \in B_p(G_2)$ (the proof of $\psi \in B_p(G_1)$ is similar). First we notice that if $g \in A_p(G_2) \cap C_{00}(G)$, and if $f \in A_p(G_1)$ is such that $f|_{h(\mathrm{supp}\,(g))} = 1$, then

$$\kappa \cdot g = \kappa \cdot (f \circ h) \cdot g = (Tf) \cdot g \in A_p(G_2).$$

Next, let $(g_\beta)_{\beta \in J}$ be a bounded approximate identity of bound one for $A_p(G_2)$, consisting of functions with compact support. We claim that $(\kappa g_\beta)_{\beta \in J}$ converges uniformly to κ on compact subsets of G_2. Suppose $E \subset G_2$ is compact. Let $g_E \in A_p(G_2) \cap C_{00}(G_2)$ be such that $g_E|_E = 1$. Then as we saw above, $\kappa g_E \in A_p(G_2)$, and hence

$$\|(\kappa g_E) g_\beta - \kappa g_E\|_{A_p(G_2)} \longrightarrow 0.$$

In particular (since $g_E|_E = 1$)

$$\|\kappa g_\beta|_E - \kappa|_E\,\|_\infty \longrightarrow 0.$$

This proves that $(\kappa g_\beta)_{\beta \in J}$ converges uniformly on compact sets to κ. Since G_2 is amenable, the unit ball of $B_p(G_2)$ is closed in $C(G_2)$ in the topology of uniform convergence on compact sets [23, p. 59, Remark 2, and Proposition 3.2]. So the proof of our theorem is complete if we show that the net $(\kappa g_\beta)_{\beta \in J}$ is bounded. Let $\epsilon > 0$. Using the amenability of G_1, for each $\beta \in J$ we choose $f_\beta \in A_p(G_1) \cap C_{00}(G_1)$ such that $f_\beta|_{h(\mathrm{supp}\, g_\beta)} = 1$, $\|f_\beta\|_{A_p(G_1)} \leq (1 + \epsilon)$. We have

$$(Tf_\beta) \cdot g_\beta = \kappa \cdot (f_\beta \circ h) \cdot g_\beta = \kappa \cdot g_\beta,$$

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and therefore:

$$\begin{aligned}\|\kappa g_\beta\|_{B_p(G_2)} &\leq \|\kappa \cdot g_\beta\|_{A_p(G_2)},\\ &= \|(Tf_\beta)\cdot g_\beta\|_{A_p(G_2)},\\ &\leq \|Tf_\beta\|_{A_p(G_2)}\|g_\beta\|_{A_p(G_2)},\\ &\leq (1+\epsilon)\|T\|.\end{aligned}$$

Q.E.D.

4.4.5 Remark. Under the conditions of the above theorem, if $g \in A_p(G_2)$ we have (see part 3 of Theorem 4.4.2)

$$g \circ h^{-1} = (\psi\kappa \circ h^{-1}) \cdot g \circ h^{-1} = \psi(\kappa g) \circ h^{-1} = T^{-1}(\kappa g) \in A_p(G_1).$$

(since $\kappa \in B_p(G_2)$. Similarly, for $f \in A_p(G_1)$, $f \circ h \in A_p(G_2)$. Now it is not difficult to show that if $f \in B_p(G_1)$, then $\kappa f \circ h \in B_p(G_2)$, and hence a disjointness preserving mapping T between $A_p(G)$ algebras of amenable groups, can be extended in a natural way, to a bijective, disjointness preserving mapping between their multiplier algebras.

For the next result, we recall that the cozero set of a function is the set of all points at which the function is non-zero.

4.4.6 Corollary. Suppose G_1 and G_2 are amenable, and $T : A_p(G_1) \longrightarrow A_p(G_2)$ is an order isomorphism for the pointwise order which preserves the cozero sets. Then T is automatically continuous, and $A_p(G_1)$ and $A_p(G_2)$ are Banach algebra isomorphic.

The proof of this corollary consists of showing that T is disjointness preserving, and is similar to the case of $p = 2$, see [28, p. 184, Corollary 2].

4.4.7 Corollary. Suppose G_1 and G_2 are amenable. If there exists a linear bijection $T : A_p(G_1) \longrightarrow A_p(G_2)$, such that $\|Tf\|_\infty = \|f\|_\infty$, for all $f \in A_p(G_1)$, then $A_p(G_1)$ is Banach algebra isomorphic to $A_p(G_2)$.

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In general for arbitrary locally compact group G, it is easy to see that if $x \in G$ and if V is a neighborhood of x, then there exists $v \in A_p(G)$ such that $\operatorname{supp} v \subset V$, $0 \leq v \leq 1$, and $v(x) = 1$. In other words G is the set of all strong boundary points of $A_p(G)$. Now a proof similar to that of [28, p. 183, Corollary 1] can be used to verify the corollary.

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Chapter 5

Lau Direct Sums of $A_p(G)$ Algebras

5.1 Introduction

In [48], A. T.-M. Lau introduced and studied a new class of Banach algebras, which are called Lau algebras by J. P. Pier (see [64, § 3.A]). Various aspects of these algebras are studied by several authors including Pier [64], Lashkarizadeh Bami [13], and Nasr-Isfahani [60], [61].

A Lau algebra is a pair (A, M), where A is a complex Banach algebra and M is a von Neumann algebra such that A is the predual of M, and the identity of M is in the spectrum of A. These include the algebras $L_1(G)$, $A(G)$, $B(G)$, and $M(G)$, but not $A_p(G)$. However, we will show in this chapter that a few of the important results of [48] can also be obtained for $A_p(G)$ algebras.

5.2 Definition and Basic Properties

Let A be a Banach algebra (as usual over the complex numbers), and let B be a Lau algebra with e the identity of B^*. In [48], Lau defines the following direct sum of these algebras: $A \oplus_L B = \{(a, b) : a \in A,\ b \in B\}$, equipped with

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the coordinatewise addition and scalar multiplication, and the product:

$$(a_1, b_1)(a_2, b_2) = (a_1a_2 + e(b_2)a_1 + e(b_1)a_2,\ b_1b_2) \qquad \text{for all } (a_1, b_1),\ (a_2, b_2) \in A \oplus_L B.$$

The fact that e is in the spectrum of B, guarantees that this product is associative. We notice also that when $B = \mathbf{C}$, the above direct sum coincides with the unitization of A. If we equip this direct sum with the norm $\|(a, b)\| = \|a\| + \|b\|$, then $A \oplus_L B$ is a Banach algebra.

The above definition can be restated for the case of B a commutative Banach algebra, and e an element in the spectrum of B. However, such a definition is not desirable for two reasons. First, this definition is not canonical (as opposed to Lau's definition), due to the non-uniqueness of e as an element in the spectrum of B. And second, as we will see shortly, we require such direct sums for non-commutative Banach algebras as well.

For the case of $A_p(G)$ algebras, we do not face such problems. When $B = A_p(G)$, and e is the identity of G (which can be canonically identified as an element of the spectrum of $A_p(G)$, as well as the identity of $A_p(G)^*$), then we can define the direct sum $A \oplus_L A_p(G)$ for any Banach algebra A, as in the case of Lau algebras.

As usual, in the following we assume all groups involved are locally compact groups. The identity of a group G is denoted by e.

For a Banach algebra A, and locally compact groups $G_2, \ldots, G_n$, we define

$$A \oplus_L A_p(G_2) \oplus_L \cdots \oplus_L A_p(G_n)$$

inductively as

$$(A \oplus_L A_p(G_2) \oplus_L \cdots \oplus_L A_p(G_{n-1})) \oplus_L A_p(G_n).$$

We equip this algebra with the norm $\|(f_1, f_2, \ldots, f_n)\| = \|f_1\| + \|f_2\|_{A_p} + \ldots + \|f_n\|_{A_p}$.

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5.2.1 Lemma. 1. $A \oplus_L A_p(G_2) \oplus_L \cdots \oplus_L A_p(G_n)$ is a Banach algebra. If $F = (f_1, \ldots, f_n)$, and $G = (g_1, \ldots, g_n)$ are in $A \oplus_L A_p(G_2) \oplus_L \cdots \oplus_L A_p(G_n)$, and if $H = (h_1, \cdots, h_n) = FG$, then

$$h_k = f_k g_k + \left[\sum_{i=k+1}^{n} f_i(e_i) \right] g_k + \left[\sum_{i=k+1}^{n} g_i(e_i) \right] f_k \tag{5.1}$$

2. The direct sum is commutative if and only if A is commutative.

3. The dual of $A \oplus_L A_p(G_2) \oplus_L \cdots \oplus_L A_p(G_n)$ can be identified with the Banach space $A^* \times A_p(G_2)^* \times \cdots \times A_p(G_n)^*$, equipped with the maximum norm

$$\|(\phi_1, \ldots, \phi_n)\|_{\max} = \max\{\|\phi_1\|, \ldots, \|\phi_n\|\}.$$

This lemma can be proved by induction.

Our next result describes the spectrum of such direct sums. Suppose A_1 is a commutative Banach algebra, and $A_i = A_p(G_i)$, $(i = 2, \ldots, n)$. Let σ_i denote the spectrum of A_i. Recall that the spectrum of $A_p(G_i)$ is homeomorphic with the group G_i, and consists of all evaluation functionals δ_x, $(x \in G_i)$. Let $0_1, 0_2, \ldots, 0_n$, denote the zero functional on A_1, $A_p(G_2), \ldots, A_p(G_n)$, respectively. We define

$$\hat{\sigma}_i = \{0_1\} \times \cdots \times \{0_{i-1}\} \times \sigma_i \times \{\delta_{i+1}\} \times \cdots \times \{\delta_n\}.$$

We equip $\hat{\sigma}_i$ with the topology induced from σ_i. Now, we define $\sum_1^n \sigma_i = \bigcup_i^n \hat{\sigma}_i$, and put a topology on $\sum_1^n \sigma_i$, by calling a subset U of $\sum_1^n \sigma_i$ to be open, if $U \cap \hat{\sigma}_i$ is open in $\hat{\sigma}_i$, for every $i = 1, \ldots, n$.

5.2.2 Theorem. With the above terminology,

$$\sigma(A_1 \oplus_L A_p(G_2) \oplus_L \ldots \oplus_L A_p(G_n)) = \sum_1^n \sigma_i.$$

Proof. For simplicity we consider the case of $n = 2$. From lemma above, $(A_1 \oplus_L A_p(G_2))^* = A_1^* \times A_p(G_2)^*$, and hence $\sigma(A_1 \oplus_L A_p(G_2)) \subset A_1^* \times A_p(G_2)^*$.

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Now suppose $(\phi, \psi) \in \sigma(A_1 \oplus_L A_p(G_2))$, then for every (a, f), $(b, g) \in A_1 \oplus_L A_p(G_2)$ we have

$$(\phi, \psi)((a, f)(b, g)) = ((\phi, \psi)(a, f))\,(\phi, \psi)(b, g))$$

$$(\phi, \psi)(ab + g(e_2)a + f(e_2)b, fg) = (\phi(a) + \psi(f))(\phi(b) + \psi(g))$$

$$\phi(ab) + g(e_2)\phi(a) + f(e_2)\phi(b) + \psi(fg) = \phi(a)\phi(b) + \phi(a)\psi(g) + \psi(f)\phi(b) + \psi(f)\psi(g).$$

If we take $f = g = 0$, it follows that $\phi(ab) = \phi(a)\phi(b)$, for all $a, b \in A_1$, and hence $\phi \in \sigma(A_1) \cup \{0_1\}$. Subsequently, if we take $a = b = 0$, it follows that $\psi \in \sigma A_p(G_2) \cup \{0\}$. But, $\psi = 0$ implies that $\phi = 0$ which is impossible, since $(\phi, \psi) \neq (0_1, 0_2)$. Therefore $\psi = \delta_x$, for some $x \in G_2$.

Now, in case that $\phi = 0_1$, we have $(\phi, \psi) = (0_1, \delta_x) \in \sum_1^2 \sigma_i$. Otherwise, we can write the above equality as

$$(g(e_2)-\psi(g))\phi(a)+(f(e_2)-\psi(f))\phi(b) = 0 \qquad (\text{for all } (a, f), (b, g) \in A_1 \oplus_L A_p(G_2)).$$

Setting $b = 0$, and a such that $\phi(a) \neq 0$, it follows that $g(e_2) = \psi(g)$, for all $g \in A_p(G_2)$, and hence $\psi = \delta_{e_2}$. This means that $(\phi, \psi) = (\phi, \delta_{e_2}) \in \sum_1^2 \sigma_i$. The verification that the topology of $\sigma(A_1 \oplus_L A_p(G_2))$ (induced from the product topology of $A_1^* \times A_p(G_2)^*$), is the same as the topology of $\sum_1^2 \sigma_i$, is straight forward. **Q.E.D.**

5.2.3 Proposition. Let A be a Banach algebra and G a locally compact group. Then

1. $A \oplus_L A_p(G)$ has an identity if and only if G is compact.

2. $A \oplus_L A_p(G)$ has a bounded approximate identity if and only if G is amenable.

3. $A \oplus_L A_p(G)$ has an approximate identity if and only if $A_p(G)$ has an approximate identity.

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To proof this lemma, one only need to recall that $A_p(G)$ has an identity if and only if G is compact, and $A_p(G)$ has a bounded approximate identity if and only if G is amenable.

To state our next result, we observe that if $\delta_e \in A_p(G)^*$ is the evaluation functional at e, and if $j : A_p(G)^* \longrightarrow A_p(G)^{***}$ is the canonical injection, then $j(\delta_e)$ is the identity of $A_p(G)^{***}$ and is a multiplicative linear functional on $A_p(G)^{**}$ (where both $A_p(G)^{**}$, and $A_p(G)^{***}$ are equipped with their corresponding first Arens products $\odot$). In fact, for every Ψ, $\zeta \in A_p(G)^{**}$:

$$
\begin{aligned}
\langle j(\delta_e), \Psi \odot \zeta \rangle &= \langle \Psi \odot \zeta, \delta_e \rangle \\
&= \langle \Psi, \zeta \cdot \delta_e \rangle \\
&= \langle \Psi, \langle \zeta, \delta_e \rangle \delta_e \rangle \\
&= \langle \Psi, \delta_e \rangle \langle \zeta, \delta_e \rangle \\
&= \langle j(\delta_e), \psi \rangle \langle j(\delta_e), \zeta \rangle.
\end{aligned}
$$

Therefore we may define $\oplus_L$ direct sums of $A_p(G)^{**}$ algebras with other Banach algebras in the canonical way discussed at the beginning of this section.

5.2.4 Lemma. If A is any Banach algebra, and $G_2, \ldots, G_n$ are locally compact groups, then the Banach algebra $(A \oplus_L A_p(G_2) \oplus_L \cdots \oplus_L A_p(G_n))^{**}$ (equipped with its first Arens product) is isometrically isomorphic to the Banach algebra

$$A^{**} \oplus_L A_p(G_2)^{**} \oplus_L \cdots \oplus_L A_p(G_n)^{**}.$$

It is not difficult to verify that the map

$$
\begin{aligned}
&\Phi : A^{**} \oplus_L A_p(G_2)^{**} \longrightarrow (A \oplus_L A_p(G_2))^{**} \\
&\qquad \Phi(\phi_1, \phi_2)(f_1, f_2) = \phi_1(f_1) + \phi_2(f_2), \qquad (f_1 \in A^*, f_2 \in A_p(G_2)^*)
\end{aligned}
$$

is an isometric algebra isomorphism. The general case follows by induction.

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5.3 Topological Invariant Means

5.3.1 Definition. Let $A = A_p(G_1)\oplus_L \cdots \oplus_L A_p(G_n)$. An element $(\Psi_1, \ldots, \Psi_n) \in A^{**}$ is called a topological invariant mean for A^* if

1. $\|(\Psi_1, \ldots, \Psi_n)\| = \langle(\Psi_1, \ldots, \Psi_n), (\delta_1, \ldots, \delta_n)\rangle = 1$.

2. For every $u_i \in A_p(G_i)$, $(i = 1, \ldots, n)$,

$$(\iota(u_1), \ldots, \iota(u_n))(\Psi_1, \ldots, \Psi_n) = (\sum_1^n u_i(e_i))(\Psi_1, \ldots, \Psi_n),$$

where $\iota(u_j)$ is the canonical image of u_j in $A_p(G_j)^{**}$.

5.3.2 Lemma. If Ψ_1 is a topological invariant mean for $A_p(G_1)^*$, then $(\Psi_1, 0, \ldots, 0)$ is a topological invariant mean for $(A_p(G_1)\oplus_L \cdots \oplus_L A_p(G_n))^*$. Conversely, every topological invariant mean for $(A_p(G_1)\oplus_L \cdots \oplus_L A_p(G_n))^*$ is of this form.

Proof. The first assertion can be easily proved using the equation (5.1). We will prove the converse. If $(\Psi_1, \ldots, \Psi_n)$ is a topological invariant mean, and if $u_1 \in A_p(G_1)$, then on one hand using the definition of the product in (5.1), we have

$$(\iota(u_1), \iota(0), \ldots, \iota(0))(\Psi_1, \ldots, \Psi_n) = (u_1 \odot \Psi_1 + \left[\sum_2^n \Psi_i(\delta_{e_i})\right] \iota(u_1), 0, \ldots, 0).$$

On the other hand, using the fact that $(\Psi_1, \ldots, \Psi_n)$ is a topological invariant mean, we have

$$(\iota(u_1), \iota(0), \ldots, \iota(0))(\Psi_1, \ldots, \Psi_n) = u_1(e_1)(\Psi_1, \ldots, \Psi_n).$$

Since u_1 is arbitrary, we must have $\Psi_2 = 0, \ \ldots, \ \Psi_n = 0$. Subsequently,

$$\iota(u_1) \odot \Psi_1 = u_1 \cdot \Psi_1 = u_1(e)\Psi_1,$$

that is, Ψ_1 is a topological invariant mean. **Q.E.D.**

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5.3.3 Theorem. Let $A = A_p(G_1)\oplus_L \cdots \oplus_L A_p(G_n)$, and let E be a Banach A-bimodule such that for every $f = (f_1, \ldots, f_n) \in A$ and every $x \in E, f \cdot x = (\sum_1^n f_i(e_i))x$. Then every bounded derivation $D : A \longrightarrow E^*$, is an inner derivation.

The proof is similar to the argument in [48, p. 167, Theorem 4.1].

5.3.4 Remark. 1. A Lau algebra A is called *left amenable* if for any two sided Banach A-bimodule E, such that $\psi x = \psi(e)x$ for all $\psi \in A$ and all $x \in E$, every bounded derivation $D : A \longrightarrow E^*$ is an inner derivation. This notion was first introduced and studied by Lau in [48]. Using this terminology, the above theorem might be restated as follows: For any locally compact groups $G_1, \ldots, G_n$, $A_p(G_1)\oplus_L \cdots \oplus_L A_p(G_n)$ is left amenable.

2. Let $\Psi_1 \in A_p(G_1)^{**}$ be a topological invariant mean, and $j : A_p(G_i)^* \longrightarrow A_p(G_i)^{***}$ and $k : A_p(G_i)^{**} \longrightarrow A_p(G_i)^{****}$ be the canonical injections $(i = 1, \ldots, n)$. Then it is easy to see that

   (a) $\langle (k(\Psi_1), 0 \ldots, 0), (j(\delta_{e_1}), \ldots, j(\delta_{e_n}))\rangle = \|(k(\Psi_1), 0, \ldots, 0)\| = 1.$

   (b) For every $\Phi_i \in A_p(G_i)^{**}$, $(i = 1, \ldots, n)$,

$$(\Phi_1, \ldots, \Phi_n)(k(\Psi_1), 0, \ldots, 0) = (\sum_1^n \Phi(\delta_{e_i}))(k(\Psi_1), 0, \ldots, 0).$$

A result similar to the above theorem can be stated for $(A_p(G_1)\oplus_L \cdots \oplus_L A_p(G_n))^{**}$ algebras as well.

5.4 Approximate Zeros

Let $A = A_p(G_1)\oplus_L \cdots \oplus_L A_p(G_n)$, and let $P_1(A_p(G)) = \{f \in A_p(G) : \|f\|_{A_p} = f(e) = 1\}$, and $P_1(A) = \{f = (f_1, \ldots, f_n) \in A : \|f\| = \sum_1^n f_i(e_i) = 1\}$. Suppose Ψ_1 is a topological invariant mean for $A_p(G_1)^*$, and $(g_\alpha)_\alpha \in P_1(A_p(G_1))$,

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is a net such that $g_\alpha \longrightarrow \Psi_1$, in the w*-topology (see [32, pp. 121–123]). Let us set $\Psi = (\Psi_1, 0, \ldots, 0)$, $\hat{g}_\alpha = (g_\alpha, 0, \ldots, 0)$.

5.4.1 Lemma. For every $f = (f_1, \ldots, f_n) \in P_1(A)$,

$$f\hat{g}_\alpha - \hat{g}_\alpha \longrightarrow 0, \qquad \text{(in the w}^*\text{-topology)}.$$

Proof. Using Lemma (5.2.4), for every $\phi = (\phi_1, \ldots, \phi_n) \in A^*$ we have:

$$\begin{aligned}
\lim_\alpha \langle \phi, \hat{g}_\alpha \rangle &= \lim_\alpha \langle \iota(\hat{g}_\alpha), \phi \rangle \\
&= \langle \Psi, \phi \rangle \\
&= \langle \Psi, \phi \cdot f \rangle \\
&= \lim_\alpha \langle \iota(\hat{g}_\alpha), \phi \cdot f \rangle \\
&= \lim_\alpha \langle \phi \cdot f, \hat{g}_\alpha \rangle \\
&= \lim_\alpha \langle \phi, f\hat{g}_\alpha \rangle.
\end{aligned}$$

Since ϕ was arbitrary, this proves our lemma. **Q.E.D.**

5.4.2 Theorem. $A = A_p(G_1) \oplus_L \cdots \oplus_L A_p(G_n)$ has a bounded approximate zero for $P_1(A)$. In other words, there exists a net $(h_\alpha)_\alpha \in P_1(A)$, such that for every $f \in P_1(A)$, $\|fh_\alpha - h_\alpha\| \longrightarrow 0$.

The proof is similar to the proof of Lau in [48, p. 169, Theorem 4.6].

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Chapter 6

Some Open Problems

This section contains some open questions which arise from my study on $A_p(G)$ algebras. We give a short description of the significance of each problem, and a summary of the known partial results in each case.

6.1 Splitting Properties of $A_p(G)$

Problem. Suppose $p \in (1, \infty)$ and G is a locally compact group. Does the strong splitting of all finite dimensional extensions of $A_p(G)$ imply that G is amenable?

We discussed in detail the motivational background of this problem in Section 3.1. We proved in Lemma 3.3.10 that if the above condition holds, then G cannot contain a closed free subgroup of two generators. As a result, we showed in Theorem 3.3.11, that the above question has a positive answer in the cases of almost connected groups, finite dimensional linear groups, and linear groups over fields with characteristic zero.

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6.2 Characterizations of $A_p(G)$

Problem. If $p \in (1, \infty)$ and G_1, G_2 are locally compact groups such that $A_p(G_1)$ is isometrically isomorphic to $A_p(G_2)$, does it follow that G_1 and G_2 are isomorphic as topological groups?

In other words, does $A_p(G)$ uniquely characterize its underlying group? We know that the answer to this question is positive when $p = 2$ [71]. In this case the problem has shown strong connections with theory of C^* and von Neumann algebras. We also know that if G_1 and G_2 are abelian, then the similar question for the case of multiplier algebras $B_p(G)$ has a positive answer [58]. These results extend the earlier results of Johnson [42], Wendel [73], and Helson [36] for the case of group and measure algebras.

Our own investigation of order isomorphisms between $A_p(G)$ algebras in Chapter 4 was motivated by an attempt to answer the above question. In Theorem 4.2.8, we showed that a continuous order isomorphism for the pointwise order between $A_p(G)$ algebras characterizes the underlying groups as topological spaces (that is, induces a homeomorphism between the two groups). An interesting related problem is the following

Problem. Does the existence of a continuous order isomorphism for the pointwise order between $A_p(G_1)$ and $A_p(G_2)$ imply that G_1 is isomorphic to G_2?

For the case of $p = 2$ this question has a positive answer, as shown by Arendt and De Cannière in [5].

6.3 Lau-Ülger Conjecture on Arens Regularity of $A_p(G)$

Conjecture. For $p \in (1, \infty)$, $A_p(G)$ is Arens regular if and only if G is finite.

In 1951, R. Arens showed the existence of two multiplications on the dou-

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ble dual of any Banach algebra A [7]. These products, when restricted to A coincide with the usual product of A. Soon it became clear that certain problems concerning the Banach algebra A can be treated successfully in the context of A^{**} using the Arens multiplications. In our own study of Lau direct sums of $A_p(G)$ algebras, we obtained a characterization of the double dual of $A \oplus_L A_p(G_2) \oplus_L \cdots \oplus_L A_p(G_n)$ equipped with the first Arens product (see Lemma 5.2.4). If A is a Banach algebra, and if the two Arens products of A^{**} coincide with each other, then A is called Arens regular. It was shown by A.T.-M. Lau and J.C.S. Wong [52], that when G is amenable, $A_2(G)$ is Arens regular if and only if G is finite. It is a conjecture of Lau and Ülger [49], that in general, for arbitrary $p \in (1, \infty)$ and an arbitrary locally compact group G, $A_p(G)$ is Arens regular if and only if G is finite. For a large class of locally compact groups (including abelian groups), this conjecture has been proved by B. Forrest [30]. He has also shown that Arens regularity of $A_p(G)$ implies that G is discrete. This is an overwhelming evidence that the above conjecture might in fact be true.

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