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# Groups With Permutable Subgroups And Infinite Metabelian Groups

BY

**Pan Soo Kim** ©

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To my parents

for the life

long support

## ABSTRACT

The first part of this thesis is the study of the following type. Let $u(x_1, \ldots, x_n) = x_{11} \cdots x_{1m}$ be a word in the alphabet $x_1, \ldots, x_n$ such that $x_{1i} \neq x_{1i+1}$ for all $i = 1, \ldots, m-1$. If $(H_1, \ldots H_n)$ is an $n$-tuple of subgroups of a group $G$ then denote by $u(H_1, \ldots, H_n)$ the set $\{u(h_1, \ldots, h_n) \mid h_i \in H_i\}$. If $\sigma \in S_n$ then denote by $u_\sigma(H_1, \ldots, H_n)$ the set $u(H_{\sigma(1)}, \ldots, H_{\sigma(n)})$. We study groups $G$ with the property that for each $n$-tuple $(H_1, \ldots, H_n)$ of subgroups of $G$, there is some $\sigma \in S_n$, $\sigma \neq 1$ such that $u(H_1, \ldots, H_n) = u_\sigma(H_1, \ldots H_n)$. In chapter 2, we show that if $G$ is finitely generated soluble groups, then $G$ has this property for some word $u$ if and only if $G$ is nilpotent-by-finite. We also look at some specific words $u$ and study the properties of the associated groups.

For the middle parts, we consider two kinds of group properties and will see easily the one includes the other. Suppose $G$ is an infinite group with the property that whenever $X_1, X_2, X_3, X_4$ are infinite subsets of $G$ there exist $x_i \in X_i$ $(i = 1, 2, 3, 4)$ such that

(i) $\langle x_1, x_2, x_3, x_4 \rangle$ *is metabelian* or,

(ii) $[\,[x_1, x_2], [x_3, x_4]\,] = 1$

We show any infinite group satisfying (i) is metabelian and any locally soluble group satisfying (ii) is also metabelian.

For the final part, a finitely generated infinite group $G$ is proved to be quasi-Hamiltonian if any two infinite sets of subgroups of $G$ contain pair H, K that permute. This is not true for arbitrary infinite groups, a counter example is provided.

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# CHAPTER 1

# INTRODUCTION

The work done in this thesis is concerned about some soluble groups satisfing certain group properties. Chapter 2 is independent of the rest . The next chapters are closely related and have the same origin; a question of Paul Erdös.

Chapter 2 is initiated by several authors' papers on rewriting products of group elements and permutable subgroups in [$BL_1$], [$BL_2$], [$LMR_1$] and [$RW_1$]. It is based on the paper by P.S. Kim and A.H. Rhemtulla in [KR].
We will introduce notations and examples before the main results.

Let $n$ be a fixed positive integer, $X = \{x_1, \dots, x_n\}$ a set of $n$ symbols and $F = F(X)$ the free group on $X$. Let $U = \{u_1, u_2, \dots\}$ and $V = \{v_1, v_2, \dots\}$ be non-empty sets of elements in $F$. Define the class $P(U, V)$ to consist of groups $G$ such that given an $n$-tuple $(g_1, \dots, g_n)$ of elements in $G$, $u(g_1, \dots, g_n) = v(g_1, \dots, g_n)$ for some $u \in U$ and some $v \in V$, $v \neq u$. Some examples;

**(1.1)** Let $U = \{u\}$ where $u(x_1, \dots, x_n) = x_1 x_2 \cdots x_n$ and $V = \{u_\sigma \mid \sigma \in S_n \backslash 1\}$ where $S_n$ is the symmetric group of degree $n$ and

$$u_\sigma(x_1, \dots, x_n) = u(x_{\sigma(1)}, \dots, x_{\sigma(n)}) = x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(n)}.$$

Then every group in $P(U, V)$ is finite-by-abelian-by-finite. Conversely every finite-by-abelian-by-finite group is in $P(U, V)$ for some suitable $n$. This was shown by Curzio, Longobardi, Maj and Robinson in [CLMR]. These groups are more commonly referred to as $P_n$-groups.

**(1.2)** Let $u = u(x_1, \ldots, x_n) = x_1 x_2 \cdots x_n$, $U = \{u_\sigma \mid \sigma \in S_n\}$ and $V = U$. Then $P(U,V)$-groups, more commonly referred to as $Q_n$-groups or rewritable groups, are again finite-by-abelian-by-finite groups as shown by Blyth in [BL$_1$]. That an abelian-by-finite group is in $Q_n$ for some $n$ is implicit in Theorem 1 of Kaplansky [KA].

**(1.3)** If we take $n = 2$, $u = u(x_1, x_2) = (x_1 x_2)^r$, $v = v(x_1, x_2) = (x_2 x_1)^r$ where $r > 0$ is fixed and $U = \{u\}$, $V = \{v\}$ then $G \in P(U,V)$ if and only if $G/Z(G)$ is of exponent $r$. If $G \in P(U,V)$, $x_1 = y^{-1}x$ and $x_2 = y$, then $y^{-1}x^r y = (y^{-1}xy)^r = x^r$ for all $y$ in $G$ and hence $x^r \in Z(G)$. Conversely suppose $G/Z(G)$ is of exponent $r$, then $(x_1 x_2)^r \in Z(G)$ and $(x_1 x_2)^r = x_1^{-1}(x_1 x_2)^r x_1 = (x_2 x_1)^r$.

In general, the classes $P(U,V)$ may be viewed as generalising varieties and, except for some specific sets $U$ and $V$, it is very difficult to describe them. We now turn to related classes of groups.

Let $n > 0$ be fixed, $X = \{x_1, \ldots, x_n\}$ be a set of idempotent variables and $S = S(X)$ the free semigroup generated by $X$. Thus for any $u \in S$, $u = u(x_1, \ldots, x_n) = x_{11} x_{12} \cdots x_{1m}$ where $x_{1i} \in X$ and $x_{1i} \neq x_{1i+1}$ for all $i = 1, \ldots, m-1$. If $(H_1, \ldots, H_n)$ is an $n$-tuple of subgroups of a group $G$ then denote by $u(H_1, \ldots, H_n)$ the set $\{u(h_1, \ldots, h_n) \mid h_i \in H_i\}$. Thus if $u(x_1, \ldots, x_n)$ is as above, then

$$u(H_1, \ldots, H_n) = H_{11} H_{12} \ldots H_{1m}.$$

If $U$ and $V$ are sets of elements in $S$ then define the class $SP(U,V)$ to consist of groups $G$ such that for any $n$-tuple $(H_1, \ldots, H_n)$ of subgroup of $G$,

$u(H_1,\dots,H_n) = v(H_1,\dots,H_n)$ for some $u \in U$ and some $v \in V$, $v \neq u$. Some examples:

**(1.4)** Let $U = \{u\}$ where $u(x_1,\dots,x_n) = x_1x_2\cdots x_n$ and $V = \{u_\sigma \mid \sigma \in S_n\backslash 1\}$. As in (1.1), $u_\sigma(x_1,\dots,x_n) = u(x_{\sigma(1)},\dots,x_{\sigma(n)})$. Finitely generated soluble $SP(U,V)$ groups are finite-by-abelian. Conversely every finite-by-abelian group is an $SP(U,V)$ group for some integer $n$. These results are contained in [$RW_1$]. From [$LMR_1$] we know that periodic $SP(U,V)$-groups are locally finite.

**(1.5)** For each positive integer $r$, let $u_r = u_r(x,y) = (xy)^r$, and $v_r = v_r(x,y) = (yx)^r$. Let $U = \{u_r,\ r = 1,2,\dots\}$ and $V = \{v_r,\ r = 1,2,\dots\}$. Then the class $SP(U,V)$ is precisely the class of groups in which every subgroup is elliptically embedded. Groups with this property are considered in [$RW_2$] and [SM]. It is known that a finitely generated soluble group $G$ is in this class if and only if it is finite-by-nilpotent. The same is true if we replace "soluble groups" by "residually finite $p$-group" in the above statement. We show the two main results as follows in chapter 2,

**(1.6)Theorem.** *Let* $U = \{u\}$ *where* $u$ *is a word in idempotent variables* $x_1,\dots,x_n$ *for* $n > 1$ *and let* $V = \{u_\sigma \mid \sigma \in S_n\}$. *If* $G$ *is a finitely generated soluble group in* $SP(U,V)$, *then* $G$ *is nilpotent-by-finite.*

**(1.7)Theorem.** *Let* $G$ *be a finitely generated soluble group,* $U = \{u\}$ *where* $u = u(x_1,\dots,x_n) = (x_1\cdots x_n)^r$ *and* $V = \{u_\sigma \mid \sigma \in S_n\}$ *where* $u_\sigma(x_1,\dots,x_n) = u(x_{\sigma(1)},\dots,x_{\sigma(n)})$. *Then* $G$ *is an* $SP(U,V)$*-group for some* $n > 1$, $r > 0$ *if and only if* $G$ *is finite-by-nilpotent.*

Chapter 3 is motivated by a question of P. Erdös: If $G$ is a group such that subsets consisting of mutually noncommuting elements are all finite, then are they boundedly finite? B.H. Neumann [$NE_1$] answered this question affirmatively by proving that $G$ has the property stated above if and only if $G$ is centre-by-finite. Extensions of problems of this type are to be found in [GR] and [LW].

In addition, J.C. Lennox [LE] has studied bigenetic properties of finitely generated groups, that is, when all two-generator subgroups of a group $G$ has a group property $\wp$, $G$ also has $\wp$ for some classes of hyper-(abelian-by-finite) groups.

Contrarily, it is kmown that M.F. Newman has constructed an example (Unpublished. See [LE]) of a three-generator infinite $p$-group ($p$ a prime) with all of its two-generator subgroups nilpotent. Therefore it is necessary for $G$ to be hyper-(abelian-by-finite) in the study of bigenetic properties.

In some classes of infinite groups, $G$ has $\wp$ if whenever we choose infinite sets $X_1, \ldots, X_n$ $(n \geq 1)$ of $G$, there exist $x_i \in X_i$ such that $\langle x_1, \ldots, x_n \rangle$ satisfies $\wp$. Some examples;

**(1.8)** Let $G$ be an infinite group. If for every pair $(X, Y)$ of infinite subsets of $G$, there exists $x$ in $X$ and $y$ in $Y$ such that $xy = yx$, then $G$ is abelian. From a result of B.H. Neumann [$BN_1$], it follows that $G$ is centre-by-finite so that the centre $Z$ of $G$ is infinite. For any $x, y$ in $G$ we take the infinite sets $Zx, Zy$ and by hypothesis, $[xz_1, yz_2] = [x, y] = 1$ for some $z_1, z_2$ in $Z$.

**(1.9)** Let $n > 1$ be a fixed integer and $G$ an infinite group. If every infinite subset $X$ of $G$ contains an element of order dividing $n$, then $G$ is of exponent dividing $n$. Let $F$ denote the $FC$-centre of $G$. This is the set of all elements of $G$ having finitely many conjugates. If $x$ is in $G$ and $x^n \neq 1$, then the number of conjugates of $x$ must be finite and hence $x \in F$. Thus $C = C_G(x)$ is infinite since $|G : C|$ is finite. Let $D \subseteq C$ be the set of those elements in $C$ of order dividing $n$. Clearly $D$ is an infinite set and $(dx)^n = x^n \neq 1$ for all $d \in D$, contradicting the hypothesis.

With above examples in mind, it is proper to ask the extent to which a property for an infinite group $G$ is determined in the above fashion: Let $\mathcal{V}$ be a variety defined by the law $w(x_1, \dots, x_n) = 1$ and assume that $n$ is the least number of variables required to determine $\mathcal{V}$. If $G$ is an infinite group such that whenever $X_1, \dots, X_n$ are infinite subsets of $G$ there exists $x_i$ in $X_i$, $i = 1, \dots, n$ such that $\langle x_1, \dots, x_n \rangle$ is a $\mathcal{V}$-group. Does it follow that $G$ is a $\mathcal{V}$-group? It is too much to expect this to be true for all varieties $\mathcal{V}$ but we have no counter example.

In chapter 3, based on the paper by P.S. Kim, A.H. Rhemtulla and H. Smith [KRS], we proved the result when $\mathcal{V}$ is the class of metabelian groups.
P. Longobardi, M. Maj and A.H. Rhemtulla [LMR$_2$] proved the result when $\mathcal{V}$ is the class of nilpotent groups of class $n-1$. Their result is slightly stronger for they do not assume that $\langle x_1, \dots, x_n \rangle$ is nilpotent of class $n-1$ but only that $[x_1, \dots, x_n] = 1$.

From the point stated above, it is natural to ask a similiar type of a question: if $G$ is an infinite group such that whenever $X_1, X_2, X_3, X_4$ are any infinite subsets of $G$ there exists $x_i \in X_i$, $i = 1, 2, 3, 4$ such that $[[x_1, x_2], [x_3, x_4]] = 1$, then is $G$ metabelian?

At present we don't know the answer but we have affirmative answer for locally soluble groups in chapter 4.

Recently several authors in [LMRS] and [CLRW] have studied the group property when every infinite set of subgroups contains a pair that permute as subgroups. One of main results is that a finitely generated group $G$ is centre-by-finite if and only if every infinite set of subgroups of $G$ contains a pair that permute .

Using this result, we prove the following in chapter 5; A finitely generated group $G$ is quasi-Hamiltonian if every two infinite sets $\mathsf{X}$, $\mathsf{Y}$ of subgroups of $G$ contains $H \in \mathsf{X}$, $K \in \mathsf{Y}$ such that $HK = KH$.

# CHAPTER 2

## PERMUTABLE WORD PRODUCTS IN GROUPS

In this chapter, we shall not look at $P(U,V)$-groups, but concentrate our attention on $SP(U,V)$-groups. At present little is known about the various classes $SP(U,V)$ and $P(U,V)$. The reduction from soluble to nilpotent-by-finite in (1.6)Theorem will be achieved using several Lemmas. Most of Lemmas contain complicated computations. All results are from the paper by [KR].

We recall that a soluble group $G$ is said to be *minimax* if and only if it has a series $1 = G_0 \lhd G_1 \lhd \ldots \lhd G_n = G$ in which the factors are cyclic or quasicyclic. We shall call a group *constrained* if and only if there is no prime $p$ for which it has a section isomorphic to $C_p \wr C_\infty$, the standard restricted wreath product of a cyclic group of order $p$ by an infinite cyclic group. This terminology is due to P.H. Kropholler [KO], who gave a celebrated Theorem; *Every finitely generated constrained soluble group is minimax and hence it has finite rank.*

Since $SP(U,V)$ is subgroup and quotient closed, (2.1)Lemma implies that finitely generated soluble groups in the class of $SP(U,V)$ have finite rank.

**(2.1)Lemma.** *The wreath product of a cyclic group of order $p$ with the infinite cyclic group is not in the class $SP(U,V)$ where $U$, $V$ are as in the statement of (1.6)Theorem.*

**Proof.** Let $G$ be the wreath product of a cyclic group of order $p$ and an infinite cyclic group $\langle t\rangle$. Then we can identify each element of $G$ by a pair $(f(t),\ t^\alpha)$ where $f(t) \in \mathbb{F}_p\langle t\rangle$, the additive group of the group ring of the infinite cyclic group $\langle t\rangle$ over the field $\mathbb{F}_p$ of $p$ elements, and $\alpha \in \mathbb{Z}$. The product of two such elements is then given by the rule: $(f(t),\ t^{\alpha})(g(t),\ t^{\beta}) = (f(t)+t^{\alpha}\cdot g(t),\ t^{\alpha+\beta})$. The elements of the base group correpond to those pairs where $\alpha = 0$ and the elements of the top group correspond to those pairs where $f(t) = 0$.

We are given $X = \{x_1, \ldots, x_n\}$, $u = u(X) = x_{11}x_{12}\ldots x_{1m}$; $x_{1i} \in X$, $x_{1i} \neq x_{1i+1}$, $i = 1, \ldots, m-1$ and we are required to show that there exist subgroups $H_1, \ldots, H_n$ of $G$ such that $H_{11}H_{12}\ldots H_{1m} \neq H_{\phi(11)}H_{\phi(12)}\cdots H_{\phi(1m)}$ for any $\phi \neq 1$ in the symmetric group of degree $n$.

Take $H_i = \langle h_i \rangle$ where $h_i = ((1-t^{\alpha_i})f_i,\ t^{\alpha_i})$; $f_i = f_i(t)$ and $\alpha_i$ are to be chosen appropriately. Note that $h_i^k = ((1-t^{k\alpha_i})f_i,\ t^{k\alpha_i})$ and a general element of $u(H_1, \ldots, H_n)$ is $h_{11}^{k_1}\ldots h_{1m}^{k_m} =$

$$((1-t^{k_1\alpha_{11}})f_{11},\ t^{k_1\alpha_{11}})\cdots((1-t^{k_m\alpha_{1m}})f_{1m},\ t^{k_m\alpha_{1m}}) =$$
$$((1-t^{k_1\alpha_{11}})f_{11} + t^{\lambda_1}(1-t^{k_2\alpha_{12}})f_{12} + \cdots + t^{\lambda_{m-1}}(1-t^{k_m\alpha_{1m}})f_{1m},\ t^{\lambda_m})$$

where $\lambda_i = k_1\alpha_{11} + \cdots + k_i\alpha_{1i}$, $i = 1, \ldots, m$. Partition the set $\{1, \ldots, m\}$ as the union $S_1 \cup \cdots \cup S_n$ where $S_i = \{j \mid x_{1j} = x_i\}$. Then a general element of $u(H_1, \ldots, H_n)$ is of the form

$$(\sum_{i=1}^{n}(f_i \sum_{j\in S_i}(t^{\lambda_{j-1}} - t^{\lambda_j})),\ t^{\lambda_m})$$

with the understanding that $\lambda_0 = 0$. Likewise $u(H_{\phi(1)}, \ldots, H_{\phi(n)})$ consists of elements of the form

$$(\sum_{i=1}^{n}(f_{\phi(i)} \sum_{j\in S_i}(t^{\mu_{j-1}} - t^{\mu_j})),\ t^{\mu_m})$$

where $\mu_i = \ell_1\alpha_{\phi(11)} + \cdots + \ell_i\alpha_{\phi(1i)}$, $i = 1,\ldots,m$ and $\mu_0 = 0$. If $\sigma$ denotes the inverse of $\phi$, then we may write these elements as

$$(\sum_{i=1}^{n}(f_i \sum_{j\in S_{\sigma(i)}} (t^{\mu_{j-1}} - t^{\mu_j})),\ t^{\mu_m}).$$

Now

$$(\sum_{i=1}^{n}(f_i \sum_{j\in S_i}(t^{\lambda_{j-1}} - t^{\lambda_j})),\ t^{\lambda_m}) = (\sum_{i=1}^{n}(f_i \sum_{j\in S_{\sigma(i)}} (t^{\mu_{j-1}} - t^{\mu_j})),\ t^{\mu_m})$$

implies $\lambda_m = \mu_m$ and

$$\sum_{i=1}^{n} f_i(\sum_{j\in S_i}(t^{\lambda_{j-1}} - t^{\lambda_j})) = \sum_{i=1}^{n} f_i(\sum_{j\in S_{\sigma(i)}} (t^{\mu_{j-1}} - t^{\mu_j})). \tag{2.2}$$

Let $p_1,\ldots,p_n$ be distinct primes, each greater than $m$. Put $p = p_1\ldots p_n$, $k_i = 1$ and $\alpha_i = p/p_i$, $i = 1,\ldots,n$. Then for each $i > 0$, $\lambda_i = \alpha_{11} + \cdots + \alpha_{1i}$. Let $f_i = t^{p^i}$. Note that $\lambda_j < p$ for all $j$ and they are all distinct. Also note that $f_i t^{\lambda_j} = f_{i'} t^{\mu_{j'}}$ implies $p^i + \lambda_j = p^{i'} + \mu_{j'}$. Hence $\mu_{j'} \equiv \lambda_j \mod p$ so that $\mu_{j'} \not\equiv \lambda_i \mod p$ for any $i \neq j$. Thus each $\lambda_j$ is congruent modulo $p$ to precisely one $\mu_{j'}$.

Now $1 \in S_{\sigma(k)}$ for some $k$. Thus $f_k(t^{\mu_0} - t^{\mu_1})$ is a term on the right hand side of (2.2). Since $\mu_0 = 0$ and the only $\lambda_i$ equal to zero is $\lambda_0$, $f_k(t^{\lambda_0} - t^{\lambda_1})$ appears on the left hand side of (2.2). In particular $1 \in S_k$. Since $S_1,\ldots,S_n$ partition the set $\{1,\ldots,m\}$ and $1 \in S_k \cap S_{\sigma(k)}$, it follows that $\sigma(k) = k$. Hence $\mu_1$ and $\lambda_1$ are both congruent to zero mod $p/p_k$; it follows that $\mu_1 = \lambda_1$.

Suppose, by way of induction, that we have established that $\mu_j = \lambda_j$ for all $j < e$. Then $\mu_e - \mu_{e-1} = \ell_e\alpha_{\phi(1e)}$ which is congruent to zero mod all primes $p_i$ except possibly one namely, $p_{\phi(1e)}$. Now we look at $\{\lambda_j - \lambda_{e-1},\ j = e,\ldots,m\}$. $\lambda_e - \lambda_{e-1} = \alpha_{1e}$ is congruent to zero mod all primes $p_i$, $p_i \neq p_{1e}$.

For each of the other $\lambda_j - \lambda_{e-1}$, we can find at least two primes amongst $\{p_1, \ldots, p_n\}$ such that $\lambda_j - \lambda_{e-1}$ is not congruent to zero mod either of them. Hence $0 \not\equiv \alpha_{1e} \equiv \ell_e \alpha_{\phi(1e)}$ mod $p_{1e}$. But $\alpha_{\phi(1e)} \neq \alpha_{1e}$ implies $\alpha_{\phi(1e)} \equiv 0$ mod $p_{1e}$. Thus $\alpha_{\phi(1e)} = \alpha_{1e}$, and $x_{1e} = x_{\phi(1e)} = x_{k'}$, say. Thus $\sigma(k') = k'$ and $e \in S_{k'}$. Thus $f_{k'}(t^{\mu_{e-1}} - t^{\mu_e}) = f_{k'}(t^{\lambda_{e-1}} - t^{\lambda_e})$ and $\lambda_e = \mu_e$.

It is now clear that $\phi(j) = j$ for all $j = 1, \ldots, n$ and hence $\phi$ is the identity permutation of the set $\{1, \ldots, n\}$ as required. ♠

Let $C$ be a torsion-free abelian group and let $A \leq Aut(C)$, we extend the action of $A$ to the rational vector space $V = C \otimes_{\mathbf{Z}} \mathbf{Q}$ as natural way, where $\mathbf{Q}$ is the field of rational number.

$A$ is said to be *rationally irreducible* on $C$ if $C/B$ is periodic whenever $B$ is nontrivial $A$-admissible subgroup of $C$. Then it is easy to see that $A$ is rationally irreducible if and only if $A$ is irreducible as a group of linear transformation of $V$.

**(2.3)Lemma.** *Let $G = \langle A, t \rangle$ where $A$ is a torsion-free abelian group of finite rank on which $\langle t \rangle$ acts rationally irreducibly. If $G \in SP(U, V)$ where $U$, $V$ are as in the statement of (1.6) Theorem, then for some positive integer $k$, $\langle t^k \rangle$ acts trivially on $A$.*

**Proof.** We are given a group $G = \langle A, t \rangle$ where $A$ is torsion-free abelian of finite rank on which $\langle t \rangle$ acts rationally irreducibly. Let us assume, if possible, that $[A, t] \neq 1$. Then $V = A \otimes_{\mathbf{Z}} \mathbf{Q}$ is an irreducible $\mathbf{Q}\langle t \rangle$- module and by Schur's Lemma, the centralizer ring $\Gamma = End_{\mathbf{Q}\langle t \rangle} V$ is a division ring of finite dimension over $\mathbf{Q}$. The image of $\langle t \rangle$ in $End_{\mathbf{Q}} V$ clearly lies in and spans $\Gamma$ so that $\Gamma$ is an algebraic number field. Moreover, regarded as a $\Gamma$-space, $V$ is one dimensional. Thus we may consider $A$ to be an additive subgroup of $\mathbf{Q}(\tau)$ for some algebraic number $\tau$ and the action of conjugation by $t$ as multiplication by $\tau$.

Let $h_i = b_i(1-\tau^{\alpha_i})t^{\alpha_i}$ for suitable integer $\alpha_i$ and $b_i(1-\tau^{\alpha_i}) \in A$. Let $H_i = \langle h_i \rangle$. Note that $h_i^k = b_i(1-\tau^{k\alpha_i})t^{k\alpha_i}$.

As in (2.1)Lemma, we are given $u = u(x_1, \dots, x_n) = x_{11}x_{12}\dots x_{1m};\ x_{1i} \in \{x_1, \dots, x_n\},\ x_{1i} \neq x_{1i+1},\ i = 1, \dots, m-1;$ and we need to show that with proper choice of $b_i$ and $\alpha_i$, the subgroups $H_1, \dots, H_n$ can be found such that $H_{11}\dots H_{1m} \neq H_{\phi(11)}\dots H_{\phi(1m)}$ for any $\phi \neq 1$ in $S_n$. Now

$$\begin{aligned} h_{11}^{k_1}\dots h_{1m}^{k_m} &= b_{11}(1-\tau^{k_1\alpha_{11}})t^{k_1\alpha_{11}}\dots b_{1m}(1-\tau^{k_m\alpha_{1m}})t^{k_m\alpha_{1m}} \\ &= b_{11}(1-\tau^{\lambda_1}) + b_{12}(\tau^{\lambda_1}-\tau^{\lambda_2}) + \dots + b_{1m}(\tau^{\lambda_{m-1}}-\tau^{\lambda_m})t^{\lambda_m} \end{aligned}$$

where $\lambda_i = k_1\alpha_{11} + \dots + k_i\alpha_{1i},\ i = 1, \dots, m.$

We shall put $\lambda_0 = 0$ and write $1 = \tau^0 = \tau^{\lambda_0}$.

Thus a general element of $u(H_1, \dots, H_n)$ has the form

$$\sum_{i=1}^{n}(b_i \sum_{j\in S_i}(\tau^{\lambda_{j-1}} - \tau^{\lambda_j}))t^{\lambda_m}$$

where $S_i = \{j;\ x_{1j} = x_i\}$ so that $\{1, \dots, m\}$ is the disjoint union of $S_1, \dots, S_n$. Likewise the general element of $u(H_{\phi(1)}\dots H_{\phi(n)})$ has the form

$$\sum_{i=1}^{n}(b_i \sum_{j\in S_{\sigma(i)}}(\tau^{\mu_{j-1}} - \tau^{\mu_j}))t^{\mu_m}$$

where $\mu_i = \ell_1\alpha_{\phi(1)} + \dots + \ell_i\alpha_{\phi(i)},\ i = 1, \dots, m;\ \mu_0 = 0$ and $\sigma = \phi^{-1}$. This is shown in the same way as in the proof of (2.1)Lemma.

In particular $\mu_m = \lambda_m$ and

$$(2.4) \qquad \sum_{i=1}^{n} b_i\Big(\sum_{j\in S_i}(\tau^{\lambda_{j-1}} - \tau^{\lambda_j}) - \sum_{j\in S_{\sigma(i)}}(\tau^{\mu_{j-1}} - \tau^{\mu_j})\Big) = 0.$$

Now we return to pick $b_i$ and $\alpha_i$ appropriately. For each integer $r > 1$, pick primes $p_{r1}, \ldots, p_{rn}$ to satisfy $2^r < p_{r1}$ and $p_{ri}^2 < p_{ri+1}$, $i = 1, \ldots, n-1$. Put $q_r = p_{r1} \cdots p_{rn}$, $b_{ri}(y) = y^{q_r^i}$ and $\alpha_{ri} = q_r/p_{ri}$, $i = 1, \ldots, n$. To make the notation simpler, we shall write $b_i$ for $b_{ri}$ and $\alpha_i$ for $\alpha_{ri}$, where there is no ambiguity. Since there are infinitely many choices of $q_r$ and each choice of $q_r$ determines the sequence $H_1, \ldots, H_n$ of subgroups which in turn corresponds to some permutation $\phi \neq 1$ such that $u(H_1, \ldots, H_n) = u(H_{\phi(1)}, \ldots, H_{\phi(n)})$. there is an infinite number of choices of $r$ such that $q_r$ correspond to the same permutation $\phi$.

If, for some value of $r$, we have the following stronger version of (2.4):

$$\sum_{i=1}^{n} b_i(y)L_i(y) - \sum_{i=1}^{n} b_i(y)M_i(y) = 0$$

where

$$L_i(y) = \sum_{j\in S_i}(y^{\lambda_{j-1}} - y^{\lambda_j}), \quad M_i(y) = \sum_{j\in S_{\sigma(i)}}(y^{\mu_{j-1}} - y^{\mu_j})$$

and $y$ is an indeterminant; then $\mu_j = \lambda_j$ for all $j$ and $\phi = 1$. This is seen using arguments similiar to those in the proof of (2.1)Lemma. Thus we may suppose that for every $r$,

$$P(y) = \sum_{i=1}^{n} b_i(y)L_i(y) - \sum_{i=1}^{n} b_i(y)M_i(y)$$

is not zero but $P(\tau) = 0$. If $Q(y)$ is any non-trivial segment of $P(y)$ such that $Q(\tau) = 0$, then $Q'(y) = P(y) - Q(y)$ is a segment of $P(y)$ with $Q'(\tau) = 0$.

Moreover one or both of $Q(y)$ or $Q'(y)$ contains at least as many monomials from $\sum_{i=1}^{n} b_i(y)L_i(y)$ as from $\sum_{i=1}^{n} b_i(y)M_i(y)$. Let $Q(y)$ be such a segment of $P(y)$ of shortest length. Thus

I. $Q(y) \neq 0$

II. $Q(\tau) = 0$

III. $Q(y)$ contains at least as many monomials from $\sum_{i=1}^{n} b_i(y)L_i(y)$ as from $\sum_{i=1}^{n} b_i(y)M_i(y)$ and

IV. No proper segment has properties I and II.

We write $Q(y) = Q_1(y) - Q_2(y)$ where $Q_1(y)$ is a segment $\sum_{i=1}^{u} \pm y^{\lambda_i'}$ of $\sum_{i=1}^{n} b_i(y)L_i(y)$, $Q_2(y)$ a segment $\sum_{i=1}^{v} \pm y^{\mu_i'}$ of $\sum_{i=1}^{n} b_i(y)M_i(y)$ and we may suppose that there is no term in $Q_1(y)$ equal to any term in $Q_2(y)$. If $Q_1(y)$ has only one term in it, then $Q(y) = \pm y^{\lambda}$ or $\pm y^{\lambda} \pm y^{\mu}$ where $0 \neq \lambda$ and $\mu \neq \lambda$. In both cases $Q(\tau) = 0$ implies $\tau$ is a root of unity and $[t^k, A] = 1$ for some $k > 0$, as required.

We may therefore assume that $Q_1(y) = \sum_{i=1}^{u} \pm y^{\lambda_i'}$ has more than one term; $\lambda_i' = q^{i'} + \lambda_{ji}$ and $0 \leq \lambda_1' < \cdots < \lambda_u'$. Similiarly $Q_2(y) = \sum_{i=1}^{v} \pm y^{\mu_i'}$ where $\mu_1' \leq \cdots \leq \mu_v'$. Let

$$\nu_1 = \min\{\lambda_1', \mu_1'\} \qquad \text{and} \qquad \nu_2 = \max\{\lambda_u', \mu_v'\}$$

Then $y^{-\nu_1} \cdot Q(y)$ is a polynomial of degree $\nu_2 - \nu_1$ with non-zero constant term. Moreover $\nu_2 - \nu_1 \geq \lambda_u' - \lambda_1' > 2^r$. Thus the degree of the polynomial increases with $r$ and hence there are infinitely many expressions

$$1 = \sum_{i=1}^{u+v} \varepsilon_i \tau^{\gamma_i}$$

where $\gamma_i \geq 0$, $\varepsilon_i \in \{-1,0,1\}$ and no subsum of the right hand side of the equation is zero. But this is not possible by Theorem 1 of [PO] which we state below for convenience. This completes the proof. ♠

**(Van Der Poorten)Theorem.** *Let $K$ be a field of characteristic zero and $H$ a finitely generated subgroup of the multiplicative group of $K$. Then for each integer $m > 0$ there are only finitely many relations $u_1 + \cdots + u_m = 1$ with each $u_i \in H$ and no subsum of the left hand side is zero.*

**(2.5)Lemma.** *If $G = \langle A, t \rangle$, where $A \trianglelefteq G$ and is abelian of finite rank, and $G \in SP(U, V)$ where $U$, $V$ are as in the statement of (1.6)Theorem, then for some $\ell > 0$, $\langle A, t^\ell \rangle$ has a non-trivial centre.*

**Proof.** If the torsion subgroup of $A$ is non-trivial then it has a non-trivial subgroup $A_1$ of exponent $p$ for some prime $p$. This is finite since $A$ has finite rank, hence it is centralized by $t^\ell$ for some $\ell > 0$ and $A_1$ lies in the centre of $\langle A, t^\ell \rangle$. We may thus assume that $A$ is torsion-free. Let $D$ be a non-trivial subgroup of $A$ of least rank subject to $D \trianglelefteq G$. (2.3)Lemma applies to $\langle D, t \rangle$ and we conclude that $\langle D, t^k \rangle$ is abelian for some $k > 0$. Hence $D$ lies in the centre of $\langle A, t^k \rangle$. ♠

**(2.6)Lemma.** *Let $G = \langle A, t \rangle$, where $A$ is a torsion-free abelian group of finite rank on which $\langle t \rangle$ acts rationally irreducibly. If $G \in SP(U, V)$ where $U$, $V$ are as in the statement of (1.7)Theorem, then $\langle t \rangle$ acts trivially on $A$.*

**proof.** As the hypothesis of (1.7)Theorem is stronger than that of (1.6)Theorem , (2.3)Lemma and its proof applies. We follow the proof of (2.3)Lemma and reach the situation where we may assume $A$ to be an additive subgroup of $Q(\tau)$

for some algebraic number $\tau$ and the action of $t$ under conjugation is that of multiplication by $\tau$. Furthermore we may assume $\tau$ to a primitive $k$th root of unity and we need to show that $\tau = 1$.

Let $h_i = (k^i(1-\tau),\ t^{-1})$ and $H_i = \langle h_i \rangle$. Observe that $h_i^\lambda = (k^i(1-\tau^\lambda),\ t^{-\lambda})$ and $h_i^k = t^{-k}$. Let $X = (H_1 \cdots H_n)^r$ and suppose that for some $\phi \neq 1$ in $S_n$, $X = (H_{\phi(1)} \dots H_{\phi(n)})^r$. If $\phi(1) \neq 1$, then $\langle H_1, H_{\phi(1)} \rangle \subseteq X$. But this is not possible since $X$ is the union of a finite number of cosets of $\langle t^k \rangle$ whereas $\langle H_1, H_{\phi(1)} \rangle$ contains the subgroup generated by $(k^{\phi(1)} - k)(1-\tau)$, an infinite cyclic subgroup of $Q(\tau)$, not contained in any finite union of cosets of $\langle t^k \rangle$. Hence $\phi(1) = 1$ and similiarly $\phi(n) = n$.

For any permutation $\pi$ in $S_n$, a typical element $x$ of $H_{\pi(1)} \dots H_{\pi(n)}$ has the form

$$\begin{aligned} x &= h_{\pi(1)}^{\lambda_1} \cdots h_{\pi(n)}^{\lambda_n} \\ &= (k^{\pi(1)}(1-\tau^{\lambda_1}) + k^{\pi(2)}(1-\tau^{\lambda_2})\tau^{\lambda_1} + \cdots + k^{\pi(n)}(1-\tau^{\lambda_n})\tau^{\lambda_1+\cdots+\lambda_{n-1}},\ t^{-\mu}) \end{aligned}$$

where $\mu = \lambda_1 + \cdots + \lambda_n$ and $\lambda_i$ are arbitrary integers. In turn these elements may be written as

$$(k^{\pi(1)}(1-\tau^{\alpha_1}) + k^{\pi(2)}(\tau^{\alpha_1} - \tau^{\alpha_2}) + \cdots + k^{\pi(n)}(\tau^{\alpha_{n-1}} - \tau^{\alpha_n}),\ t^{-\alpha_n})$$

where $\alpha_i$ are arbitrary integers.

In particular $t^{-\alpha_0}x$ has the form $(h,\ t^{-\alpha_n})$ where

$$\begin{aligned} h &= k^{\pi(1)}(\tau^{\alpha_0} - \tau^{\alpha_1}) + k^{\pi(2)}(\tau^{\alpha_1} - \tau^{\alpha_2}) + \cdots + k^{\pi(n)}(\tau^{\alpha_{n-1}} - \tau^{\alpha_n}) \\ &= k^{\pi(1)}\tau^{\alpha_0} + \tau^{\alpha_1}(k^{\pi(2)} - k^{\pi(1)}) + \cdots + \tau^{\alpha_{n-1}}(k^{\pi(n)} - k^{\pi(n-1)}) - \tau^{\alpha_n}k^{\pi(n)}. \end{aligned}$$

Thus, if $\alpha_0$ is a given fixed integer, the real part of $h$ is maximised by choosing $\alpha_i \equiv 0 \mod k$ if $\pi(i+1) > \pi(i)$, $\alpha_i \equiv q \mod k$ where $q = [k/2]$ if $\pi(i+1) < \pi(i)$; $i = 1, \ldots, n-1$ and $\alpha_n \equiv q \mod k$. If $\pi$ is the identity permutation then this value is

$$k \cos(2\pi\alpha_0/k) + (k^n - k) - k^n \cos(2\pi q/k).$$

On the other hand if $\pi \neq 1$ and $\pi(1) = 1, \pi(n) = n$, then the maximum real part of the value of $h$ is

$$k \cos\frac{2\pi\alpha_0}{k} + (k^n - k) - k^n \cos\frac{2\pi q}{k} + \sum (k^{\pi(i)} - k^{\pi(i+1)})(1 - \cos\frac{2\pi q}{k})$$

where the sum is over all values of $i$ such that $\pi(i) > \pi(i+1)$. This value is clearly greater than the value obtained for the identity permutation $\pi$.

Now the general element of $(H_{\pi(1)} \cdots H_{\pi(n)})^r$ is $(h, t^{-\alpha})$ where $h$ is expressible in the form

$$\sum_{i=1}^{r} k^{\pi(1)} \tau^{\alpha_{i0}} + \tau^{\alpha_{i1}}(k^{\pi(2)} - k^{\pi(1)}) + \cdots + \tau^{\alpha_{in-1}}(k^{\pi(n)} - k^{\pi(n-1)}) - \tau^{\alpha_{in}} k^{\pi(n)}$$

where $\alpha_{10} = 0$, $\alpha_{in} = \alpha_{i+10}$, $i = 1, \ldots, r-1$, $\alpha_{rn} = \alpha$. By picking the values for $\alpha_{ij}$ to maximise the real part of $h$ as above, it is clear that the value achieved when $\pi \neq 1$ is greater than for $\pi = 1$. Thus $(H_{\phi(1)} \cdots H_{\phi(n)})^r$, $\phi(1) = 1$, $\phi(n) = n$, $\phi \neq 1$, contains elements not contained in $(H_1 \ldots H_n)^r$. This completes the proof. ♠

For convenience, we write main Theorems from Introduction.

**(1.6)Theorem.** *Let* $U = \{u\}$ *where* $u$ *is a word in idempotent variables* $x_1, \ldots, x_n, n > 1$; *and let* $V = \{u_\sigma \mid \sigma \in S_n\}$. *If* $G$ *is a finitely generated soluble group in* $SP(U,V)$, *then* $G$ *is nilpotent-by-finite.*

**Proof.** By hypothesis, $X = \{x_1, \ldots, x_n\}$ $u = u(X) = x_{11}x_{12}\ldots x_{1m}$ where $x_{1i} \in X$ for all $i = 1, \ldots, m$ and $x_{1i} \neq x_{1i+1}$ for all $i = 1, \ldots, m-1$. Let $G$ be a finitely generated soluble group such that for any $n$-tuple $(H_1, \ldots, H_n)$ of subgroups of $G$, there is a permutation $\sigma \neq 1$ in $S_n$ such that

$$u(H_1, \ldots, H_n) = H_{11}H_{12}\ldots H_{1m} = u(H_{\sigma(1)}, \ldots, H_{\sigma(n)}) = H_{\sigma(11)}\ldots H_{\sigma(1m)}.$$

We need to show that $G$ is nilpotent-by-finite, and we proceed by induction on the solubility length of $G$. If $G$ is abelian then there is nothing to prove. Let $G$ be soluble of length $d$ and assume that the result holds for soluble groups of smaller length. Since the class $SP(U,V)$ is subgroup and quotient closed, we may suppose that $G$ has a normal abelian subgroup $A$ such that $G/A$ is nilpotent-by-finite. In particular $G$ is abelian-by-polycyclic. By (2.1)Lemma, $G$ has finite rank.

As $G$ is finitely generated abelian-by-polycyclic, it satisfies the maximal condition for normal subgroups. If $G$ is not nilpotent-by-finite, then let $B$ be a maximal normal subgroup of $G$ such that $G/B$ is not nilpotent-by-finite. Now we replace $G$ by $G/B$ and hence assume that every proper quotient of $G$ is nilpotent-by-finite.

Let $T$ be the torsion subgroup of $A$. Then $T$ has finite rank and is of bounded exponent since $C$ satisfies the maximal condition for normal subgroups. Thus $T$ is finite, and $C = C_G(T)$, the centraliser of $T$ in $G$, is of finite index in $G$. If $T \neq 1$ then $G/T$ is nilpotent-by-finite and hence $C/T$ is nilpotent-by-finite. Since $T \leq Z(C)$, the centre of $C$, then $C$ and hence $G$ would be nilpotent-by-finite. Thus we assume $T = 1$ and hence $A$ is torsion-free, and by passing to a suitable subgroup of finite index in $G$, if necessary, we may assume further that $G/A$ is a finitely generated torsion-free nilpotent group. Thus there exists a finite set $T = \{t_1, \ldots, t_r\}$ of elements in $G$ such that $G = \langle A, T\rangle$ and

$$A = G_0 \leq \langle G_0, t_1\rangle = G_1 \leq \ldots \leq \langle G_{r-1}, t_r\rangle = G_r = G$$

is a central series from $A$ to $G$ wi'' torsion-free factors.

If $r = 1$ then $G = \langle A, t_1\rangle$. By (2.5)Lemma $Z(\langle A, t_1^{\ell_1}\rangle) \neq 1$ for some $\ell_1 > 0$ and hence $D = A \cap Z(\langle A, t_1^{\ell_1}\rangle)$ is a non-trivial normal subgroup of $G$. By our choice of $G, G/D$ is nilpotent-by-finite and hence $G$ is nilpotent-by-finite.

Now suppose we have established the result for the case $r < d$ and suppose $r = d$. By the induction hypothesis, $G_{d-1}$ is nilpotent-by-finite and $G = \langle G_{d-1}, t_d\rangle$. Let $H = \langle A, G_{d-1}^{\ell}\rangle$ for some suitable $\ell > 0$ so that $H$ is nilpotent. Let $Y = A \cap Z(H)$ then $Y$ is normal in $\langle H, t_d\rangle$ which is of finite index in $G$. Moreover $Z(\langle Y, t_d^{\ell_1}\rangle) \neq 1$ for some $\ell_1 > 0$ by (2.5)Lemma, so that $D_1 = Y \cap Z(\langle Y, t_d^{\ell_1}\rangle)$ is a non-trivial subgroup of $G$ contained in the centre of $\langle H, t_d^{\ell_1}\rangle$ which is of finite index in $G$. We may replace $\langle H, t_d^{\ell_1}\rangle$ by its normal interior in $G$, if necessary; it still contains $A$ and hence $D_1$. Now $\langle H, t_d^{\ell_1}\rangle/D_1$ is nilpotent-by-finite, $D_1 \leq Z(\langle H, t_d^{\ell_1}\rangle)$ and $\langle H, t_d^{\ell_1}\rangle$ is of finite index in $G$. Thus $G$ is nilpotent-by-finite, as required. ♠

In the case of finitely generated linear $SP(U, V)$-group, (1.6)Theorem follow from Tits'Theorem (See[WE]); *A finitely generated linear group either is soluble-by-finite or contains a non-cyclic free subgroup.*

The latter does not occur in $SP(U,V)$-groups and hence we have the following immediately.

**(2.7)Corollary.** *A finitely generated linear $SP(U,V)$-group is nilpotent-by-finite where $U$, $V$ are as in the statement of (1.6)Theorem.*

**(1.7)Theorem.** *Let $G$ be a finitely generated soluble group, $U = \{u\}$ where $u = u(x_1, \ldots, x_n) = (x_1 \cdots x_n)^r$ and $V = \{u_\sigma \mid \sigma \in S_n\}$ where $u_\sigma(x_1, \ldots, x_n) = u(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$. Then $G$ is an $SP(U,V)$-group for some $n > 1$, $r > 0$ if and only if $G$ is finite-by-nilpotent.*

**Poof.** Since the hypothesis of (1.6)Theorem are satisfied by the group of (1.7)Theorem, we may assume $G$ to be finitely generated nilpotent-by-finite. Let $T$ be the maximal finite normal subgroup of $G$. Since we wish to show that $G$ is finite-by-nilpotent we may look at $G/T$, if necessary, and hence assume that $G$ has no non-trivial finite normal subgroup. Let $F$ be the Fitting subgroup of $G$. If $F \neq G$ then pick any $t \in G \backslash F$ such that $t^p \in F$. Clearly it is sufficient to show that $\langle F, t\rangle$ is nilpotent for $G/F$ is finite and soluble, we can reach $G$ from $F$ by a subnormal series with factors of prime order. Thus we assume $G = \langle F, t\rangle$, $t^p \in F$ and $F$ is torsion-free.

Let $H$ be the hypercentre of $G$. Then $H \cap F$ is isolated in $F$. This may be seen by first checking it for $Z(G) \cap F$ and then by taking the quotient of $G$ by this subgroup, and using induction. Observe that if $H \not\leq F$ then $G = HF$ and $G$ is nilpotent. So assume $H \leq F$. Next we look at $G/H$. If $G/H$ is nilpotent then so is $G$. So we assume $H = 1$. Let $A$ be a non-trivial normal subgroup of $G$ of least Hirsch length and $A \leq Z(F)$. Since $\langle A, t\rangle \in SP(U,V)$, $\langle t\rangle$ acts trivially on $A$ by (2.6)Lemma. Thus $A \leq Z(G)$ contradicting the assumption that $Z(G) = 1$. This concludes the proof that if $G \in SP(U,V)$ then $G$ is finite-by-nilpotent.

Now suppose that $G$ is a finitely generated finite-by-nilpotent group. For any subgroup $L$ of $G$ let $\gamma(L)$ denote the nilpotent residual of $L$. Thus $\gamma(L)$ is the intersection of the terms of the lower central series of $L$. Let $F = \gamma(G)$. It is finite by hypothesis and $G/F$ is nilpotent of class $c_1$ for some $c_1 > 0$. Thus $\gamma(L) = \gamma_c(L)$ for all $L \leq G$ where $c = |F| + c_1$. We show, by induction on $|\gamma\langle H,K\rangle| = s$, that $(HK)^{d_s} = (KH)^{d_s} = \langle H,K\rangle$ for all subgroups $H$, $K$ of $G$ where $d_1 = (4r)^c$, $r =$ rank of $G$; $d_i = d_{i-1} + 2i(i+d_1)$, $i > 1$. In particular $(HK)^d = (KH)^d = \langle H,K\rangle$ for all $H$, $K$ where $d = d_f$ and $f = |F|$.

By Proposition 2 of [RW$_2$] $\Gamma(HK)^t = \Gamma(KH)^t = \langle K,H\rangle$ where $\Gamma = \gamma\langle K,H\rangle, t = (4r)^c$, and $r$ is the rank of $G$. Thus if $\Gamma = 1$ then $d_1 = t$ will suffice.

For any $a \in (HK)^t$, $a = gb$ for some $g \in \Gamma$ and $b \in (KH)^t$ so that $ab^{-1} = g \in \Gamma \cap (HK)^{2t}$. If $\Gamma \cap (KH)^{2t} = 1$, then $a = b$ and $(HK)^t = (KH)^t$. This implies $\langle H,K\rangle = (HK)^t$, and again $d_1 = t$ suffices.

If $\Gamma_1 = \Gamma \cap (HK)^{2t} \neq 1$, then for each integer $m \geq 1$ let $\Gamma_{m+1} = \Gamma_m \cap \Gamma_m^{HK}$ so that $\Gamma_m \subseteq (HK)^{2t+2m}$. Observe that $\Gamma_m = \Gamma_{m+1}$ implies $\langle\Gamma_m\rangle = \langle\Gamma_m^H\rangle = \langle\Gamma_m^K\rangle$. Since $\Gamma_m \subseteq \Gamma$ and $|\Gamma| = s$, $\Gamma_s = \Gamma_{s+1}$. Also note that $\langle\Gamma_m\rangle \subseteq \Gamma_m^s$. Thus the normal closure $N$ of $\Gamma_1$ in $\langle H,K\rangle$ lies in $(HK)^\lambda$ where $\lambda = \lambda_s = 2(s^2 + ts)$.

Now $NH$ and $NK$ both lie in $(HK)^\lambda$ and $(NHNK)^m \subseteq (HK)^{\lambda+m}$ for all $m > 0$. Rank of $\langle H,K\rangle/N$ is no greater than $r$, $\gamma(\langle H,K\rangle/N) =$

$\gamma_c(\langle H,K\rangle/N)$ and $|\gamma(\langle H,K\rangle/N)| < |\gamma(\langle H,K\rangle)|$. Thus by the induction hypothesis, $N(HK)^{d'} = N(KH)^{d'} = \langle H,K\rangle$ where $d' = t+\lambda_2+\cdots+\lambda_{s-1} = d_{s-1}$. Since $(HK)^{\lambda} \geq N$, we obtain $(HK)^{d_s} = (KH)^{d_s} = \langle H,K\rangle$ where $d_s = d_{s-1} + \lambda_s$.

Now that we have shown that for a finitely generated finite-by-nilpotent group $G$ there is an integer $d$ such that $(HK)^d = (KH)^d$ for all subgroups $H,K$ of $G$, we let $u = u(x,y) = (xy)^d$, $v = v(x,y) = (yx)^d$ then $G \in SP(U,V)$ where $U = \{u\}$, $V = \{u_\sigma \mid \sigma \in S_2\}$. This completes the proof of the second part of the theorem. ♠

We can not replace "nilpotent-by-finite" in (1.6)theorem by the stronger condition "finite-by-nilpotent" of (1.7)Theorem. It is tedious, but we will show that the infinite dihedral group $D_\infty$ lies in $SP(U,V)$ where $U = \{u\}$, $V = \{u_\sigma | \sigma \in S_n\}$ and $u = u(x_1,x_2,x_3,x_4) = x_1x_4x_2x_3x_2x_3x_4x_1$ and it is well-known that $D_\infty$ is not finite-by-nilpotent.

**(2.8)Example.** Let $U = \{u\}$ where $u = u(x_1,\ldots,x_4) = x_1x_4x_2x_3x_2x_3x_4x_1$ and $V = \{u_\sigma \mid \sigma \in S_4\}$. Then the infinite dihedral group $G$ is in $SP(U,V)$.

Consider $u(H_1,H_2,H_3,H_4)$ for given subgroups $H_1,H_2,H_3,H_4$ of $G$. If $H_1$ or $H_4$ is normal in $G$ then $u(H_1,H_2,H_3,H_4) = u(H_4,H_2,H_3,H_1)$. If $H_2$ or $H_3$ is normal in $G$ then $u(H_1,H_2,H_3,H_4) = u(H_1,H_3,H_2,H_4)$. So assume none of the $H_i$'s is normal in $G$. If for some $i$, $H_i$ is not of order two, then it contains a subgroup $K_i$ normal in $G$ and of index two in $H_i$. Moreover $u(H_{\sigma(1)}\ldots H_{\sigma(4)}) = K_i u(H_{\sigma(1)}\ldots H_{\sigma(4)})$ for all $\sigma \in S_n$, and we may replace $G$ by $G/K_i$ and each of $H_j$ by $H_jK_i/K_i$. Thus the essential case to be considered is one where each $H_i$ is of order two.

Now $G = \langle a, t \rangle$ where $a^t = a^{-1}$ and $t^2 = 1$. $H_i = \langle a^{\lambda_i} t \rangle$, $i = 1, 2, 3, 4$. We will show that the set $L = H_4H_2H_3H_2H_3H_4$ equals the set $R = H_4H_3H_2H_3H_2H_4$. From this it follows that $u(H_1, H_2, H_3, H_4) = H_1LH_1 = H_1RH_1 = u(H_1, H_3, H_2, H_4)$.

Now $a^\lambda \in H_2H_3H_2H_3$ if and only if $\lambda = 0$, $\lambda_2 - \lambda_3$, $\lambda_3 - \lambda_2$ or $2\lambda_2 - 2\lambda_3$. $a^\lambda t \in H_2H_3H_2H_3$ if and only if $\lambda = \lambda_2$, $\lambda_3$, $2\lambda_2 - \lambda_3$ or $2\lambda_3 - \lambda_2$. Hence $H_2H_3H_2H_3 \backslash H_3H_2H_3H_2$ consists of $a^{2\lambda_2 - 2\lambda_3}$ only and $H_3H_2H_3H_2 \backslash H_2H_3H_2H_3$ consists of $a^{2\lambda_3 - 2\lambda_2}$ only.
But $H_4a^{2\lambda_2 - 2\lambda_3}H_4$ consists of $a^\lambda$ where $\lambda \in \{2\lambda_2 - 2\lambda_3,\ 2\lambda_3 - 2\lambda_2\}$ and $a^\lambda t$ where $\lambda \in \{\lambda_4 - 2\lambda_2 + 2\lambda_3,\ \lambda_4 + 2\lambda_2 - 2\lambda_3\}$. From the symmetry between $\lambda_2$ and $\lambda_3$ above it is clear that $H_4a^{2\lambda_2 - 2\lambda_3}H_4 = H_4a^{2\lambda_3 - 2\lambda_2}H_4$. Thus the sets $L$ and $R$ are equal and $G \in SP(U, V)$.

We have not tried to analyse conditions on words $u$ for which $D_\infty \notin SP(U, V)$ where $U = \{u\}$ and $V = \{u_\sigma \mid \sigma \in S_n\}$.

# CHAPTER 3

# A CHARACTERIZATION OF INFINITE METABELIAN GROUPS

Throughout this chapter, $G$ is an infinite group . We write

$G \in \mathfrak{A}^2$ when $G$ is metabelian.

$G \in \mathfrak{A}^2_*$ whenever $X_1, X_2, X_3, X_4$ are infinite subsets of $G$ there exist $x_i \in X_i$ for $i = 1,\ 2,\ 3,\ 4$ such that $\langle x_1, x_2, x_3, x_4 \rangle$ is metabelian.

In the study of $\mathfrak{A}^2_*$-groups, **Ramsey's Theorem** occurs. Let $A$ be the family of 4-element sets of $G$ whose elements generate a metabelian subgroup and $B$ be the family of 4-element sets of $G$ whose elements generate a non-metabelian subgroup. Then $A \cup B$ is the family of all 4-element sets of $G$ and then there exists an infinite subset $S$ of $G$ such that either all four element subsets of $S$ belong to $A$ or all four element subsets of $S$ belong to $B$. By the property of $G \in \mathfrak{A}^2_*$, the latter can not happen. Suppose $S = \{s_1, s_2, \ldots\}$ and $Z = \{[s_i, s_j] \mid i \neq j\}$. If $Z$ is infinite then $\langle Z \rangle$ is an infinite abelian subgroup of $G$. Suppose, on the other hand, that $Z = \{z_1, \ldots, z_k\}$. For each $\ell = 1, \ldots, k$, let $U_\ell$ denote the set of all $\{s_i, s_j\}$ such that $[s_i, s_j]$, for $i \leq j$, is equal to $z_\ell$. By Ramsey's Theorem once more, there is an integer $i \leq j$ and an infinite subset $S_1$ of $S$ such that, for all $s_i, s_j$ in $S_1$ with $i \leq j$, $[s_i, s_j] = z_\ell$. We may assume $S = S_1$. Then, for $i \leq j \leq k$, $[s_i, s_j s_k^{-1}] = [s_i, s_k^{-1}][s_i, s_j]^{s_k^{-1}} = [s_i, s_k^{-1}][s_i, s_k]^{s_k^{-1}} = 1$. In particular, for distinct $i, j, k, \ell$, with $i \leq j$ and $k \leq \ell$, $s_i s_j^{-1}$ and $s_k s_\ell^{-1}$ commute. If there is an infinite set of distinct elements $s_i s_j^{-1}$ which commute pairwise then we again have an infinite abelian subgroup. Suppose, on the contrary (by means of a relabelling if necessary), that $T = \{s_1 s_2^{-1}, s_3 s_4^{-1}, \ldots, s_{N-1} s_N^{-1}\}$ is a maximal set of pairwise commuting elements of this type. Choose $i \geq N$. Then the set $Y = \{s_i s_{i+1}^{-1}, s_i s_{i+2}^{-1}, \dots\}$ is certainly infinite and so some element $y$ of $Y$ does not lie in $T$. Clearly $y$ commutes with every element of $T$, and we have a contradiction which allows us to conclude that there is indeed an infinite abelian subgroup of $G$.

**(3.1)Lemma.** *$G \in \mathfrak{A}_*^2$ is metabelian if $G$ is one of the following.*

(1) *$G$ has the infinite centre*

(2) *$G$ is an $FC$-group*

(3) *$G$ is residually finite.*

**Proof.** Let $Z$ be the centre of $G$. For any $a, b, x, y$ in $G$, let $A = aZ = \{az \mid z \in Z\}$, $B = bZ$, $X = xZ$, $Y = yZ$. Then these four infinite subsets give the desired result in (1).

For (2), if $G$ is a periodic $FC$-group then $C_G(a) \cap C_G(b) \cap C_G(x) \cap C_G(y)$ has an infinite abelian subgroup $S$ and four subsets $aS$, $bS$, $xS$, $yS$ yield $[[a,b],[x,y]] = 1$. If $G$ is not periodic then we may assume $G$ is a finitely generated $FC$-group. In that case $G$ has the infinite centre.

For (3), let $N_i$ $(i \in I)$ be subgroups of finite index with $\cap_{i \in I} N_i = 1$, then $[[a,b],[x,y]] \equiv 1 \mod N_i$ for all $i$. So $[[a,b],[x,y]] = 1$ and $G$ is metabelian.

**(3.2)Theorem.** *Any group $G \in \mathfrak{A}_*^2$ having an element $x$ of infinite order is metabelian.*

**Proof.** Clearly we may assume that $G$ is finitely generated.
For any $a, b$ in $G$, consider sets $\{ax^i \mid i \in \mathbb{Z}\}$, $\{bx^i \mid i \in \mathbb{Z}\}$, $\{x^p \mid p : \text{primes}\}$, $\{x^q \mid q : \text{primes}, q \neq p\}$. These infinite sets yield $\langle a, b, x\rangle \in \mathfrak{A}^2$. Hence $[\langle x\rangle, G]$

is abelian. Let $J$ be $[\langle x\rangle, G]$ and $C$ be the centralizer of $\langle x\rangle J$ in $G$. If $J$ is finite, then $\langle x\rangle J$ is in $FC$-centre of $G$. Hence $|G : C|$ is finite and $C$ is finitely generated. Let $c_1, \ldots, c_n$ be generators of $C$. Then for each $c_i$ there exist $n_i$, $\ell_i$ such that $[x^{\ell_i+n_i}, c_i] = [x^{n_i}, c_i]$. Therefore $[x^{\ell_i}, c_i] = 1$ and $[x^{\ell}, C] = 1$ for some $\ell$, for example $\ell = \ell_1 \cdots \ell_n$. Hence the centre of $C$ is infinite and $C \in \mathfrak{A}^2$. Thus $C$ and $G$ are residually finite (See Theorem 1 [PH]) and $G \in \mathfrak{A}^2$.

We may take $J$ to be infinite. Pick any $a, b, c$ in $G$ and let $S = \{s \in J \mid \langle s, ua, vb, wc\rangle \in \mathfrak{A}^2$ for some $u, v, w \in J\}$. Then by $G \in \mathfrak{A}^2_*$, $J \backslash S$ is finite and $\langle S\rangle = J$. Since $J$ is a normal abelian subgroup, $[vb, wc] = [b, c]z$ for some $z \in J$ and $[s, ua] = [s, a]$. Hence $[[b, c], [s, a]] = 1$ for all $s \in S$ and $[[b, c], [J, a]] = 1$. Therefore $[G', [J, G]] = 1$.

If $[J, G]$ is finite, then let $D$ be the centralizer of $[J, G]$ in $G$ so that $[J, D, D] = 1$. Hence $[D', J] = 1$ by the Three Subgroup Lemma (See Lemma 2.13 [$\text{RO}_1$]). For any $a, b, c, d \in D$ and $u, v, w, z \in J$, $[[ua, vb], [wc, zd]] = 1$ implies $[[a, b], [c, d]] = 1$. Thus $D \in \mathfrak{A}^2$ and hence $D$ and $G$ are residually finite, so $G \in \mathfrak{A}^2$.

If $[J, G]$ is infinite, then for any $a, b, c, d$ in $G$ and $u, v, w, z$ in $[J, G]$, $[[ua, vb], [wc, zd]] = [[a, b], [c, d]]$ and hence $G \in \mathfrak{A}^2$. ♠

**Remark.** In the study of a torsion-free group $G \in \mathfrak{A}^2_*$, it is observed that for any $a$, $b$, $c$, in $G$, the sets $\{ac^i \mid i \in \mathbb{Z}\}$, $\{bc^i \mid i \in \mathbb{Z}\}$, $\{c^p \mid p : \text{primes}\}$, $\{c^q \mid q : \text{ primes}, q \neq p\}$ yield $\langle a, b, c\rangle \in \mathfrak{A}^2$ and hence $G$ is 3-metabelian group. This special group has been studied by I.D. MacDonald [MA] and has the properties

(1) $G/Z(G)$ is metabelian (2) $G''$ has exponent 2,

hence $G'' = 1$ and $G$ is metabelian.

From now on, we confine our attention to periodic groups with $\mathfrak{A}^2_*$

**(3.3)Theorem.** *If* $G \in \mathfrak{A}^2_*$ *is periodic and soluble, then* $G$ *is metabelian.*

**Proof.** If $G'$ is finite, then $G$ is an $FC$-group and hence $G \in \mathfrak{A}^2$ by (3.1)Lemma.

We may assume $G'$ is infinite metabelian by induction on the derived length of $G$. If $G \notin \mathfrak{A}^2$, then there exists a finite subgroup $F$ which is not metabelian since $G$ is locally finite. By replacing $G$ by $G'F$, we may take $G = HF$ where $H$ is an infinite metabelian normal subgroup and $F$ is a finite subgroup with $F \notin \mathfrak{A}^2$. We note that if $N$ is any infinite normal subgroup of $G \in \mathfrak{A}^2_*$, then for any $a.b, c, d \in G$, the four sets $Na$, $Nb$, $Nc$, $Nd$ yield $[[a,b],[c,d]] \equiv 1$ mod $N$ and hence $G'' \leq N$.

**(Case 1)** We prove the result when $G$ has an infinite abelian normal subgroup $A$. So we can take $G = AF$.

Suppose $A$ has a non-trivial divisible subgroup. Since $G$ is periodic, we also may take $G = HF$ where $H = (C_{p^\infty})^n$ for some prime $p$, integer $n > 0$ and $n$ is minimal subject to $H \triangleleft G$ and assume that $1 \neq G'' \leq H$, $G''' = 1$. We will prove the case when $n$ is infinite at a later time.

For each $g \in G$, $g$ induces a homomorphism of $H$ to $[g, H]$ by $h \longmapsto [g, h]$ for $h \in H$, hence either $[g, H] = 1$ or $[g, H]$ is infinite. Therefore, if $[F', H]$ is finite, then $[F', H] = 1$. Easy computation shows that for any $g_i \in F$ and $h_i \in H$ $(i = 1, 2, 3, 4)$, $[[h_1g_1, h_2g_2], [h_3g_3, h_4g_4]] = 1$ implies $[[g_1, g_2], [g_3, g_4]] = 1$, which contradicts $F \notin \mathfrak{A}^2$. Hence $[F', H]$ is infinite. By the minimality of $n$ and $[F', H] \triangleleft G = HF$, we have $G'' \leq H = [F', H] = [F', [F', H]] \leq G''$ and hence $[F', H] = H = G''$.

Pick $g_1, g_2$ in $F$ such that $[g_1, g_2]$ does not centralize $[g, H]$ for some $g \in F$. Let $X = \{x \in H \mid [[x, g], [g_1, g_2]] \neq 1\}$, then $X$ is infinite since $[F', H] = [F', [F', H]]$ is infinite. Therefore $[[h_1 g_1, h_2 g_2], [h_3 g, x]] = [[g_1, g_2], [g, x]] \neq 1$ for all $x \in X$, $h_i \in H$ which contradicts $G \in \mathfrak{A}^2_*$.

Suppose $A$ has no non-trivial quasicyclic $p$-group. If every $p$-component of $A$ is finite where $p$ is a prime, then the subgroup $B$ generated by all $p$-components of $A$ for $p$ which does not divide the order of $F$ is an infinite normal subgroup of $G$.
We may take $BF$ as a counter example, but $B \cap F = 1$ and $F \cong BF/B \in \mathfrak{A}^2$. Hence $A$ has an infinite $p$-component for some prime $p$ and hence has an infinite basic subgroup so we can assume $A$ is an elementary abelian $p$-group (the same argument can be applied for $H \cong (C_{p^\infty})^\infty$ ). Thus $A$ and $G$ are residually finite, the proof is complete in this case by (3.1)Lemma.

**(Case 2)** Suppose $G$ has no infinite abelian normal subgroups. We may take $G = HF$ where $G'' \leq H \in \mathfrak{A}^2$, $F \notin \mathfrak{A}^2$, both $H'$ and $F$ are finite, $H$ is infinite normal in $G$.

If $[F', H]$ is finite, then $(F')^G = (F')^{HF} = F'[F', H]$ is finite, the centralizer $C$ of $(F')^G$ in $H$ has finite index and hence it is infinite normal in $G$. Replacing $H$ by $C$, we may assume $[F', H] = 1$.

On the other hand if $[F', H]$ is infinite, then it is not abelian by the assumption and $[F', H] \triangleleft G$ is easily checked. Thus $HF' \notin \mathfrak{A}^2$ and we choose a finite subgroup $F_1 \notin \mathfrak{A}^2$ of $HF'$ such that $HF_1 \notin \mathfrak{A}^2$ but $F_1' \leq H$ for $(HF')' \leq HF'' \leq H$. Hence $[F_1', H]$ is a finite subgroup of $H'$. As above, we can reduce to the case $[F_1', H] = 1$. Therefore, in either case, we may take $G = HF$ with $[F', H] = 1$. Furthermore $H/H'$ is infinite abelian and $G/H' \in \mathfrak{A}^2$ by (Case 1). So $G'' \leq H'$

Now we may assume $G''$ is minimal normal in $G$ and of exponent $p$. Let $C(H')$ be the centralizer of $H'$ in $H$, then $C(H') \triangleleft G$ and $G/C(H')$ is finite. Replacing $H$ by $C(H')$, we have $G'' \leq H' \leq Z(H)$ and hence $H$ is nilpotent. By the minimality of $G''$ and $F'' \triangleleft G$, we also may assume $F'' = G''$.

Write $K = G' \cap H$, then $|G' : K| < \infty$. Hence $K$ is infinite. Otherwise $G$ is an $FC$-group. Put $J = KF$. If $K' = 1$, then replacing $H$ by $K$, it is reduced to the (Case 1). We may assume $K' \neq 1$ and hence $K' = G''$ from $K' \triangleleft G$ and the minimality of $G''$. Also $G'' = F'' \leq J'' \leq G'' = K'$. Replacing $H$ by $K$ and hence $G$ by $J$, we may take $G'' = H'$. $G''$ still remains minimal.

For primes $p$, $q$, $p \neq q$, the $q$-component $H_q$ of $H$ is finite, otherwise consider $L = H_qF$ in which $L'' \cap H_q = F'' \cap H_q = 1$, which contradicts $1 \neq F'', F \notin \mathfrak{A}^2$ for $L/H_q \in \mathfrak{A}^2$ and $F'' \leq H_q$. Since $H$ is periodic nilpotent, $H$ can be chosen a $p$-group.

If $Z(H)$ is infinite, then it is reduced to the (Case 1). If not, then $H/Z(H)$ has a finite exponent, so does $H/H'$. Hence $H/H'$ contains a $G$-invariant infinite elementary $p$-subgroup $K/H'$. As before, we may assume $K' \neq 1$, and hence $K' = G'' = H'$. Replacing $H$ by $K$, we have $H' = G'' \leq Z(H) \leq H$, $H/H'$ is an infinite abelian elementary $p$-group and $H'$ is of exponent $p$.

Now let $A_1/H'$ be a minimal $G$-invariant subgroup of $H/H'$ and $C_1$ be $C_H(A_1)$. Then $C_1 \triangleleft G$ and $|G : C_1|$ is finite.

By the residual finiteness of $G/H'$, we can choose $H_1 \triangleleft G$ such that $G/H_1$ is finite and $A_1 \cap H_1 = H'$. Clearly we can take $H_1 \leq C_1$. Now let $A_2/H'$ be a minimal $G$-invariant subgroup of $H_1/H'$ and $C_2$ be $C_{H_1}(A_2)$.

Then $C_2 \triangleleft G$, $[A_1, A_2] = 1$ and $|G : C_2|$ is finite. Choose $H_2 \leq C_2$, $H_2 \triangleleft G$ such that $|G : H_2|$ is finite with $A_2 \cap H_2 = H'$. Continuing this process, we get a subgroup $N/H'$ of $H/H'$ which is the direct product of infinitely many minimal $G$-invariant finite subgroups $\bar{A}_i = A_i/H'$ where $[A_i, A_j] = 1$ for all $i \neq j$. Set

$$X_1 = \langle A_{4n+1} \mid n = 0,1,2,\dots \rangle \qquad X_2 = \langle A_{4n+2} \mid n = 0,1,2,\dots \rangle$$

$$X_3 = \langle A_{4n+3} \mid n = 0,1,2,\dots \rangle \qquad X_4 = \langle A_{4n} \mid n = 0,1,2,\dots \rangle.$$

Then each $X_i$ is infinite and normal in $G$. Pick $g_i \in G$ for $i = 1,2,3,4$ such that $[[g_1, g_2], [g_3, g_4]] \neq 1$, then for each $x_i \in X_i$, $[[x_1g_1, x_2g_2], [x_3g_3, x_4g_4]] = [[g_1, g_2], [g_3, g_4]]$ since $[F', H] = 1$ and $[x_i, x_j] = 1$ for all $i \neq j$. This contradicts $G \in \mathfrak{A}^2_*$. ♠

**(3.4)Theorem.** *If $G \in \mathfrak{A}^2_*$ is locally soluble then $G$ is metabelian.*

**Proof.** Since $G/K \in \mathfrak{A}^2$ for any infinite normal subgroup $K$, every $G$-invariant proper subgroup of $G''$ is finite. If $G''$ is minimal normal in $G$, then $G''$ is abelian (See Theorem 5.5.1 [RO$_3$]). Therefore $G$ is soluble and $G \in \mathfrak{A}^2$ by the previous Theorem. Therefore we can choose $1 \neq A \triangleleft G$ and $A \lneqq G''$. If $C = C_{G''}(A) \lneqq G''$, then $C$ is finite, also $|G'' : C|$ and $G''$ are finite. Hence $G$ is soluble.

It follows that every proper $G$-invariant subgroup of $G''$ is contained in $Z(G'')$. Hence $G''/Z(G'')$ is minimal normal in $G/Z(G'')$ and so abelian. Therefore $G \in \mathfrak{A}^2$. ♠

**(3.5)Lemma.** *Suppose a periodic group $G \in \mathfrak{A}^2_*$ has an infinite abelian subgroup. If $F$ is any finite subgroup, then there exists an infinite metabelian subgroup containing $F$.*

**Proof.** Let $A$ be an infinite abelian subgroup and $x, y$ be in $F = G_0$. By considering $A, A, Ax, Ay$, we have a metabelian subgroup $\langle a_0, b_0, c_0x, d_0y\rangle$ for some $a_0$, $b_0$, $c_0$, $d_0 \in A$, so that $[[\langle a_0\rangle, x], [\langle a_0\rangle, y]] = 1$. Repeating this with $A\backslash\{a_0\}$ in place of $A$ and so on, there exists a cofinite subset $A_0$ of $A$ such that $[[\langle a\rangle, x], [\langle a\rangle, y]] = 1$ for all $a \in A_0$.

Since $G_0$ is finite, there exists a cofinite subset $A_1$ of $A_0$ such that $[[\langle a\rangle, x], [\langle a\rangle, y]] = 1$ for all $a$ in $A_1$ and all $x, y$ in $G_0$. For an element $1 \neq a_1 \in A_1$, let $N_1 = [\langle a_1\rangle, G_0]$. Then using the identities $[a_1, g]^{a_2} = [a_1a_2, g][a_2, g]^{-1}$ and $[a, g_1]^{g_2} = [a, g_2]^{-1}[a, g_1g_2]$ for any group element $a, g, a_i, g_i$, we can show $N_1 \triangleleft G_1 = \langle a_1, G_0\rangle$. But $G_1/N_1$ is finite and $N_1$ is abelian, hence $G_1$ and $N_1$ are finite.

Similiarly we construct $N_2 = [\langle a_2\rangle, G_1]$, $G_2 = \langle a_2, G_1\rangle$ where $a_2 \in A_2\backslash G_1$ for some cofinite subset $A_2$ of $A_1$ and $[[\langle a_2\rangle, g_1], [\langle a_2\rangle, g_2]] = 1$ for all $g_1$, $g_2 \in G_1$ and all $a_2 \in A_2$. Of course, $N_2$ is an abelian normal subgroup of $G_2$, both groups are finite. Continuing this process , we get an infinite strictly increasing tower of groups, $F = G_0 \subset G_1 \subset G_2 \subset \cdots$. Furthermore, $[G_{i-1}, \langle a_i\rangle] \equiv 1$ mod

$N_i$ for all $i \geq 1$. So we can write

$$G_1 = G_0\langle a_1\rangle N_1$$

$$G_2 = G_1\langle a_2\rangle N_2 = G_0\langle a_1\rangle N_1\langle a_2\rangle N_2$$

$$\vdots$$

$$G_i = G_0\langle a_1\rangle N_1\langle a_2\rangle N_2\cdots\langle a_i\rangle N_i$$

Let $K_n$ be the subgroup generated by the elements of $\{a_i, N_i \mid i = 1, 2, \ldots, n\}$ and $K = \cup_{n=1}^{\infty} K_n$. Clearly $K_1$ is soluble. By induction on $n$, we assume $K_{n-1} = \langle a_i, N_i \mid i \leq n-1\rangle$ is soluble. $N_n$ is abelian and normal in $K_n = \langle K_{n-1}, a_n, N_n\rangle$ and $\langle a_n\rangle$ is central in $K_n$ module $N_n$, which is therefore abelian-by-cyclic-by-soluble and hence soluble. It follows that $K$ is locally soluble and clearly infinite. As $F$ normalizes $K$, $FK/K$ is finite and hence metabelian. Therefore $FK$ is locally soluble and hence metabelian . ♠

From now on, for the further study of $\mathfrak{A}^2_*$-groups, we may assume that every finite subgroup is metabelian by (3.5)Lemma and Ramsey Theorem. Moreover we will show in (3.7) that it is enough to look at simple $\mathfrak{A}^2_*$-groups, in order to prove that $\mathfrak{A}^2_*$-groups are actually metabelian.

**(3.6)Theorem.** *If $G \in \mathfrak{A}^2_*$ is infinite locally finite then $G \in \mathfrak{A}^2$.*

**Proof.** It follows from (3.5)Lemma.

**(3.7)Reduction to Simplicity.** Suppose $G \in \mathfrak{A}^2_*$ but $G \notin \mathfrak{A}^2$. If $G^{(3)}$ is finite, then $G''$ is an $FC$-group where $G^{(3)} = [G'', G'']$. By (3.1)Lemma and (3.3)Theorem, we may assume $G''$ is finite and $G'$ is infinite.

Therefore $G'$ is an $FC$-group and hence $G' \in \mathfrak{A}^2$, it follows that $G$ is soluble and hence metabelian. If $G^{(3)}$ is infinite, then $G/G^{(3)} \in \mathfrak{A}^2$ which implies $G^{(3)} = G''$. Suppose $G''$ has an infinite proper normal subgroup $N$, then $G''/N \in \mathfrak{A}^2$, which implies $G^{(4)} \leq N$, contradicting to $G''$ perfect.

Let $N$ be a finite normal subgroup $N$ of $G''$, then $C_{G''}(N)$ has finite index in $G''$. Since $G''$ has no infinite normal subgroups, $C_{G''}(N) = G''$. Hence the centre $Z$ of $G''$ is the unique maximal normal subgroup of $G''$ so that $G''/Z$ is simple. By (3.6)Theorem, we may assume $G$ is a simple group .

**(3.8)Theorem.** *Any infinite group* $G \in \mathfrak{A}^2_*$ *is metabelian.*

**Proof.** By (3.7)Reduction, (3.6)Theorem and (3.2)Theorem, we may assume $G$ is finitely generated periodic simple. Suppose there exists a subgroup $C$ isomorphic to $C_{p^\infty}$ for some prime $p$, then for each pair $x, y \in G$, by considering sets $C$, $xC$, $C$, $yC$ we can say $\langle [\langle c \rangle, x], [\langle c \rangle, y] \rangle$ is abelian for all but finitely many $c$ in $C$. Hence for all $c \in C$, $\langle [\langle c \rangle, x], [\langle c \rangle, y] \rangle$ is abelian and therefore $[C, G]$. is abelian. Since $G$ is non-abelian, $[C, G] = 1$, therefore $G$ contains no $C_{p^\infty}$-subgroup.

Let $A$ be an infinite abelian subgroup and $S = S(x, y)$ denote the set of all $a \in A$ such that $\langle [\langle a \rangle, x], [\langle a \rangle, y] \rangle$ is abelian, then $S$ is a cofinite set of $A$ and $a \in S$ implies $\langle a \rangle \subseteq S$. If $A$ contains elements of infinitely many distinct prime orders, then choose a prime $p$ and an element $c$ of order $p$ such that no element of the finite set $B = A \backslash S$ has order a multiple of $p$ and such that for all $b \in B$, $bc \notin B$. Since $bc \in S$ and $(bc)^p = b^p \in S$, we have $\langle b \rangle = \langle b^p \rangle$, $b \in S$ and hence $S = A$, which is true for all $x$, $y$ in $G$. So $[\langle c \rangle, G]$ is abelian normal in $G$.

Hence we may assume $A$ has elements of only finitely many prime orders and hence has finitely many $p$-component. Since $A$ contains no $C_{p^\infty}$-subgroup, $A$ contains an infinite elementary $p$-subgroup; $\langle a_1 \rangle \times \langle a_2 \rangle \times \langle a_3 \rangle \cdots$. There exists $a_i \in A$ such that for all $b \in B$, $ba_i \notin B$ and hence $b^p \in S$, it follows that $A^p \subseteq S = S(x,y)$ for all $x$, $y$ in $G$.

If $A^p \neq 1$, then $[\langle a' \rangle, G]$ is abelian normal in $G$ for $1 \neq a' \in A^p$. Therefore every infinite abelian subgroup has prime exponent. If every metabelian subgroup is abelian, then any infinite subset of $G$ contains 4 elements which generate a metabelian subgroup and hence it is abelian. By B.H. Neumann's Theorem [NE$_1$], $G$ is centre-by-finite and $G \in \mathfrak{A}^2$ by (3.1)Lemma. Hence there exists an infinite non-abelian metabelian subgroup $H$ by Ramsey's Theorem and (3.5)Lemma. Let $H_0 = \langle h_1, h_2 \rangle$ with $[h_1, h_2] \neq 1$ for some $h_1, h_2$ in $H$.

**(Case 1)** We prove the theorem when $G$, $H$ are groups stated above and additionally $G$ is a $p$-group.

If $H'$ is finite, then $H$ is an $FC$-group. If $H'$ is infinite, then $L = H'H_0$ is nilpotent ( G. Baumslag, see Lemma 6.34 [RO$_1$] ). Let $M$ be the least term of normal closure series of $H_0$ in $L$ subject to $M$ infinite. Since $H_0$ is subnormal in $L$, $H_0^M$ is finite. Hence $H_0$ has the infinite centralizer in $L$.

In either case, there is an infinite abelian subgroup $D$ centralizing $H_0 = \langle h_1, h_2 \rangle$. For any $x, y \in G$, there exist $d_i \in D$, $i = 1,2,3,4$ such that $T = \langle d_1 h_1, d_2 h_2, d_3 x, d_4 y \rangle \in \mathfrak{A}^2$. So $\alpha = [h_1, h_2] \in T'$, $[\alpha, d_3 x] = [\alpha, x] \in T'$ and

$[\alpha, y] \in T'$, and hence $[[\langle\alpha\rangle, x], [\langle\alpha\rangle, y]] = 1$ for all $x, y \in G$. Therefore $[\langle\alpha\rangle, G]$ is a normal abelian subgroup of $G$. This contradiction induces the other case.

**(Case 2)** When $G$ is not a $p$-group with above statement.

By (Case 1) and (3.5)Lemma, all of the Sylow $p$-subgroup of $G$ are metabelian. If $G$ has an infinite non-abelian Sylow $p$-subgroup of $G$, then the same argument as (Case 1) would yield an abelian normal subgroup of $G$. Hence we can assume every infinite Sylow $p$-subgroup of $G$ is abelian.

Since $G$ contains a non-abelian metabelian subgroup, it contains a finite such subgroup $F$. So $F'$ contains a nontrivial $q$-element $f$ for some prime $q$. Let $A$ be any infinite abelian group so that $A$ is a $p$-group for some $p$. As (3.5)Lemma we construct $H = H(F, A)$ which contains $F$. If $H'$ is finite, then $C_H(F)$ is infinite and hence it contains infinite abelian subgroup centralising $F$. Then similar to last part of (Case 1), $[\langle f\rangle, G]$ is a normal abelian subgroup of $G$. Thus $H'$ is infinite, abelian of prime exponent and contains $F'$, therefore $H'$ is a $q$-group.

It follows that we can choose $A$ as an infinite Sylow $q$-subgroup of $G$ not containing $f$ for $\cap_{g\in G} A^g \lhd G$ and hence $\cap_{g\in G} A^g = 1$. $F$ remains finite as before. Now $[\langle a\rangle, F]$ is abelian for all but finitely many $a \in A$. As (3.5)Lemma we can begin the construction of $H = H(F, A)$ with $N_1 = [\langle a\rangle, F]$ and go on to consider $G_1 = \langle F, a\rangle$ so that $H$ contains $G_1$. As we have seen, $H'$ is a $q$-group. This implies that $[\langle a\rangle, F]$ is an abelian $q$-group for all but finitely many $a \in A$. In particular, $[\langle a\rangle, \langle f\rangle]$ is a $q$-group. Since $\langle a, f\rangle/[\langle a\rangle, \langle f\rangle]$ is $q$-group, $\langle a, f\rangle$ is a $q$-group and is contained in the infinite $q$-group $H'\langle a, f\rangle$, which is abelian for all infinite Sylow subgroups are abelian. Thus $[a, f] = 1$ and $C_A(f)$

contains all but finitely many elements of $A$ and hence all of $A$. Thus $\langle A, f\rangle$ is a $q$-group, which contradicts the choice of $A$ and $f \notin A$.

# CHAPTER 4

# LOCALLY SOUBLE GROUPS WITH A METABELIAN LAW ON INFINITE SETS

Throughout this chapter, $G$ denotes an infinite group. We deal with a similiar type of problem as chapter 3 with weaker condition. By $G \in \mathfrak{A}^2_\sharp$ We mean every four infinite sets $X_1, X_2, X_3, X_4$ of $G$ contain $x_i \in X_i$ $(i = 1,2,3,4)$ such that $[[x_1, x_2],[x_3, x_4]] = 1$. Clearly $G \in \mathfrak{A}^2_*$ implies $G \in \mathfrak{A}^2_\sharp$, but we don't know the other inclusion which may actually imply $G \in \mathfrak{A}^2$. Anyhow we prove that if $G \in \mathfrak{A}^2_\sharp$ is locally soluble then $G \in \mathfrak{A}^2$.

Parts of proof mimic (3.2), (3.3)Theorem. We also depend on the properties of $FC$-groups to prove main theorems. Subgroups of this rather special type will play an important role in this chapter. These subgroups were introduced by B.H. Neumann [NE$_2$] and hence we use the term $N$-group ( See [TO]).

An $N$-group $G$ of cardinality $\aleph$ (infinite) is an $FC$-group generated by elements $x_i, y_i$, $i \in I$, where $|I| = \aleph$ and the generators satisfy the conditions

$$[x_i, x_j] = [y_i, y_j] = [x_i, y_j] = 1 \text{ if } i \neq j, \quad [x_i, y_i] = z_i \neq 1.$$

These relations are not to be understood to form a system of defining relations of $G$, there will in general be further relations, not consequences of these relation. But they must not imply $z_i = 1$ for any $i$. In particular, if all $z_i$ are chosen distinct then such an $N$-group is called an $N_1$-group.

The following properties of an $N$-group $G$ are readily verified from the definition. The subgroups $Z_i = \langle x_i, y_i \rangle$ are normal in $G$ and they jointly generate $G$. The centralizer $C(Z_i)$ is generated by all the subgroups $Z_j$ with

$j \neq i$ together with the centre of $Z_i$, and the centre $Z$ of $G$ is generated by the centre of all $Z_i$.

We shall show that any $FC$-group $G$ contains an $N$-group if $G/Z(G)$ is infinite, thus obtaining the main results by considering these groups in more detail. To do this, we quote the following two well known Theorems.

**(4.1)Theorem** (Neumann [NE$_3$]). *If $G$ is an $FC$-group then $G'$ is periodic and hence the elements of finite order form a subgroup of $G$ containing $G'$.*

**(4.2)Theorem** (Černikov [CE]). *A group $G$ is a finitely generated $FC$-group if and only if it is isomorphic to a subgroup of the direct product of a finite group and a free abelian group of finite rank.*

**(4.3)Lemma.** *Let $U$ be a subgroup of finite index in an $FC$-group $G$.*

(1) *If $|U/Z(U)|$ is finite, then $|G/Z(G)|$ is finite*

(2) *If $U'$ is finite, then $G'$ is finite*

**Proof.** There is a finitely generated normal subgroup $F$ of $G$ such that $FU = G$.

Since $|G/C_G(F)|$ is finite and $|G : Z(U)|$ is finite, we have $|G : C_G(F) \cap Z(U)|$ is finite, but $C_G(F) \cap Z(U)$ centralizes $FU = G$ and so $G/Z(G)$ is finite

In (2), we have $G' \leq U'(F \cap G')$. But $F \cap G'$ is contained in the periodic subgroup of $F$ by (4.1)Theorem and so is contained in a finite subgroup of $F$ by (4.2)Theorem. Therefore $G'$ is finite. ♠

**(4.4)Theorem.** *Let $G$ be an $FC$-group.*

(1) *If $|G/Z(G)|$ is infinite, then $G$ contains an infinite $N$-subgroup*

(2) *If $G'$ is infinite then $G$ contains an infinite $N_1$-group*

**Proof.** We define the generators $x_n$, $y_n$ $(n = 1,2,3,\dots)$ inductively. Suppose that we have defined $x_1, y_1, \dots, x_{n-1}, y_{n-1}$ and let $C = C_G(x_1, y_1, \dots, x_{n-1}, y_{n-1})$ so that $|G : C|$ is finite. By (4.3)Lemma, $C/Z(C)$ is infinite and in part (2), $C'$ is infinite.
Therefore $C$ contains elements $x_n, y_n$ such that $[x_n, y_n] \neq 1$, and part (2), we can choose $x_n, y_n$ so that $[x_n, y_n] \notin \{1, z_1, \dots, z_{n-1}\}$. ♠

Same results in chapter 3 hold for $\mathfrak{A}^2_\sharp$ in place of $\mathfrak{A}^2_*$. Proofs require no change. Theses are listed below.

**(4.5)Lemma.** *$G \in \mathfrak{A}^2_\sharp$ is metabelian if*

(1) *$G$ has the infinite centre*

(2) *$G$ is an $FC$-group*

(3) *$G$ is residually finite*

**Proof.** See (3.1)Lemma.

**(4.6)Lemma.** *$G \in \mathfrak{A}^2_\sharp$ is metabelian if $C_G(G')$ has $G$-invariant infinite abelian subgroup.*

**Proof.** Let $C$ be a such subgroup of $G$ and $a, b, c, d$ in $G$. Consider four sets $aC, bC, cC, dC$. By $G \in \mathfrak{A}^2_\sharp$, there exist $z_i \in C$ for $i = 1,2,3,4$ such that $[[az_1, bz_2], [cz_3, dz_4]] = 1$. This implies $[[a,b],[c,d]] = 1$. Hence $G \in \mathfrak{A}^2$. ♠

**(4.7)Corollary.** *$G$ is metabelian if $Z(G')$ is infinite.*

**(4.8)Lemma.** *If $G \in \mathfrak{A}^2_\sharp$, then $[G', [G, N]\,] = 1$ for any infinite normal abelian subgroup $N$ of $G$.*

**Proof.** For any $a, b, c$ in $G$, consider sets $Na, Nb, Nc, N$ and let $S = S(a,b,c) \subset N$ be such that for any $s \in S$ there exist $n_i \in N$ $(i = 1,2,3)$ with

$[[n_1 a, n_2 b], [n_3 c, s]] = 1$, so that $[[a,b]n, [c,s]] = [[a,b],[c,s]] = 1$ for some $n \in N$. Clearly, $N \backslash S$ is finite, $N = \langle S \rangle$. Since $[c, s_1 s_2] = [c, s_2][c, s_1]^{s_2} = [c, s_1][c, s_2]$ and $1 = [c, s_1 {s_1}^{-1}] = [c, {s_1}^{-1}][c, s_1]$ for all $s_1, s_2 \in S$, $[a,b]$ commutes with $[c,n]$ for all $n \in N$. Hence $[[a,b],[c,N]] = 1$, as desired.

**(4.9)Corollary.** *If* $G \in \mathfrak{A}^2_\sharp$ *and* $[G,N]$ *is infinite for some infinite normal abelian subgroup* $N$ *of* $G$, *then* $G \in \mathfrak{A}^2$.

**Proof.** Since $[G,N] \leq G'$ and $[G,N] \leq Z(G')$, $G \in \mathfrak{A}^2$ by (4.7)Corollary.

**(4.10)Theorem.** *If* $G \in \mathfrak{A}^2_\sharp$ *is a finitely generated soluble infinite group, then* $G$ *is metabelian.*

**Proof.** We prove by induction on the solubility length of $G$. We consider two cases;

**(Case 1)** We prove the result when $G$ has an infinite abelian normal subgroup $N$.

By the previous Lemma, we may assume $[N,G]$ is finite. Let $D = C_G[N,G]$. Clearly $[N,D,D] \leq [\,[N,G],D] = 1$, hence $[D',N] = 1$ by the Three Subgroups Lemma (See [RO$_3$]). Therefore, $D$ and $G$ are residually finite, so desired.

**(Case 2)** When $G$ has no infinite abelian normal subgroup.

By induction hypothesis, $G''$ is finite abelian and hence $G'$ is an $FC$-group. Since the centralizer $H$ of $G''$ in $G$ has finite index, it is enough to show $H \in \mathfrak{A}^2$ for being residually finite-by-finite is equivalent to residually finite. Replacing $G$ by $H$, we may assume $[G,G''] = 1$. Since $G$ is finitely generated, choose $\{t_1, \ldots, t_n\}$ as a set of generators of $G/G'$.

Let $t$ be any $t_i$ and $T$ be the subgroup generated by $[G', t]$ and $G''$. First we show $T = [G', \langle t \rangle]\ G'' \lhd G$ and each element of $T$ is of the form $[a, t]g''$ for some $a \in G'$, $g'' \in G''$. The following can be easily checked by using commutator's identities $[xy, z] = [x, z]^y[y, z]$ and $[x, yz] = [x, z][x, y]^z$ for any group elements $x, y, z$.

(1) $[a, t][b, t] = [ab, t]g''$ for some $g'' \in G''$, depending on choice of $a, b \in G'$

(2) $[a, t]^{-1} \in [a^{-1}, t]G''$ for $[a, t][a^{-1}, t] \in G''$ from (1)

(3) $[a, t^2] = [a, t][b, t] = [ab, t]g''$ for some $g'' \in G''$, $b = a^t$

(4) $[a, tt^{-1}] = 1$ implies $[a, t^{-1}] = [b, t]^{-1}$ where $b = a^{t^{-1}}$

(5) $[a, t][t, b] = [t, ba^{-1}]^a = [ba^{-1}, t]^{-1}g''$ for some $g'' \in G''$

(6) $[a, t][t, b] \equiv [t, b][a, t] \mod G''$

(7) $[a, t^g] = [a, t[t, g]\ ] = [a, t]g''$ for some $g'' \in G''$

where $a, b \in G'$, $g \in G$. (1)–(6) tell us the typical form of elements of $T$ and (7) implies $T \lhd G$. If $T_i = [G', t_i]G''$ is finite for all $i$, then $[G', G] \leq T_1T_2 \cdots T_n$. Hence $G$ is finite-by-nilpotent and hence nilpotent-by-finite. (See Theorem 4.25 [RO$_1$]). Therefore $G$ is residually finite ( See P. Hall [PH]) and hence $G \in \mathfrak{A}^2$.

Now we may assume $T = [G', t]G''$ is infinite. By (Case 1), $T$ is non-abelian and has the finite centre. Recall $T \leq G'$ is an $FC$-group and $G''$ is central. By (4.4)Lemma, $T$ has an $N$-group generated by $x_i, y_i$ $(i \in I)$ subject to $[x_i\ y_i] \neq 1$, but $[x_i, y_j] = [x_i, x_j] = [y_i, y_j] = 1$ for all $i \neq j$. Let

$$x_i = [a_i, t]g_i'' \qquad y_j = [b_j, t]f_j''$$

for some $a_i, b_j \in G'$, $g_i'', f_j'' \in G''$. Then clearly for each $i \in I$,

$$[x_i, y_i] \neq 1 \quad \text{implies} \quad [[a_i, t], [b_i, t]] \neq 1,$$

$$[x_i, y_j] = 1 \quad \text{implies} \quad [[a_i, t], [b_j, t]] = 1 \quad \text{and etc.}$$

Let $i_0 \in I$ and $I$ be the disjoint union of $\{i_0\}$, $I_1$ and $J_1$ with $|I_1| = |J_1| = \infty$. Set

$$X_1 = \{a_{i_0} b_{i_1} \mid i_1 \in I_1\} \qquad X_2 = \{t a_{i_1} \mid i_1 \in I_1\}$$

$$X_3 = \{b_{i_0} a_{j_1} \mid j_1 \in J_1\} \qquad X_4 = \{t b_{j_1} \mid j_1 \in J_1\}.$$

Note that $a_i's$ ($b_i's$) are not necessarily distinct, but $G''$ is finite, so $X_i$ is infinite. The set of commutators of elements of $X_1$ and $X_2$, denoted by $[X_1, X_2]$, consists of $[a_{i_0} b_{i_1}, t a_{i_1}] = [a_{i_0}, t][b_{i_1}, t] g''$, and $[X_3, X_4]$ consists of $[b_{i_0}, t][a_{j_1}, t] f''$, for some $g''$, $f'' \in G''$, depending choice of elements of $X_i$. But

$$\begin{aligned}
[[a_{i_0}, t][b_{i_1}, t] g'', \ [b_{i_0}, t][a_{j_1}, t] f''] &= [[a_{i_0}, t][b_{i_1}, t], \ [b_{i_0}, t][a_{j_1}, t]] \\
&= [[a_{i_0}, t][b_{i_1}, t], \ [b_{i_0}, t]]^{g'} \text{ for some } g' \in G' \\
&= [[a_{i_0}, t], [b_{i_0}, t]]^{g} \text{ for some } g \in G
\end{aligned}$$

Therefore $[[x_1, x_2], [x_3, x_4]] \neq 1$ for all $x_i \in X_i$, contradicting to $G \in \mathfrak{A}_{\sharp}^2$. ♠

**(4.11)Theorem.** *If* $G \in \mathfrak{A}_{\sharp}^2$ *is an infinite periodic soluble group then* $G \in \mathfrak{A}^2$.

**Proof.** We use induction on the solubility length of $G$. We may assume $G'$ is metabelian. Also $G'$ is infinite for otherwise $G$ is an $FC$-group.

**(Case 1)** We prove the result when $G$ has an infinite normal abelian subgroup.

Suppose $G$ is a counter example, then there exists a finite subgroup $F \notin \mathfrak{A}^2$. Hence we can take $G = FA$ where $A$ is infinite abelian normal in $G$. Suppose

$A$ has a divisible subgroup and hence a divisible $p$-subgroup $H$. Since $H$ is normal in $G$, $[G',[G,H]] = 1$ by (4.8)Lemma. If $[G,H]$ is infinite then $G \in \mathfrak{A}^2$.

If $[G,H]$ is finite then $[g,H]$ is finite for all $g \in G$. Now for each $g \in G$, $g$ induces a homomorphism of $H$ by $h \longmapsto [g,H]$. If $[g,H]$ is finite, then $[g,H] = 1$ and hence $[G,H] = 1$. So done by (4.5)Lemma when $[G,H]$ is finite. Therefore we may assume $A$ has no divisible subgroup.

If $A$ has finite $p$-components of $A$ for infinitely many primes $p$, then the subgroup $B$ generated by infinitely many finite $p$-components of $A$ for prime $p$ which does not divide the order of $F$ is normal in $G$. So we may take $G = BF$ as a counter example with $B \cap F = 1$. Since $B$ is infinite, $G/B \in \mathfrak{A}^2$ and hence $BF/B \cong F/B \cap F \cong F \in \mathfrak{A}^2$, contradicting to the choice of $F$.

It follows that $A$ is an infinite $p$-group and reduced. Hence $A$ has an infinite basic subgroup. (See [FU]) We may assume $A$ is an elementary abelian $p$-group, therefore $A$ is residually finite and hence $G$. So the result follows.

**(Case 2).** We prove the result of the other case.

We may assume $G$ has no infinite normal abelian subgroup and $G''$ is finite. We may take $G = AF$ where $F \notin \mathfrak{A}^2$ is finite, $A \in \mathfrak{A}^2$, $A'$ is finite abelian normal and $G' \leq A$.

Let $D = C_A(A')$ then $G/D$ is finite and $D \triangleleft G$ for $D = A \cap C_G(A')$. For $g \in G$, let $N(g)$ be the subgroup generated by $[g,D]A'$, then clearly

$$[g,D]^h(A')^h = [g^h,D]A' = [g,D][\,[g,h],D]A' = [g,D]A'$$

for all $h \in G$ and hence $N(g) \triangleleft G$. Since $A'$ is finite $N(g) \subset A$ is an $FC$-group. For $d_1, d_2$ in $D$, we have

$$[g,d_1d_2] = [g,d_2][g,d_1]^{d_2} = [g,d_2][g,d_1]a'$$

for some $a' \in A'$. Hence $[g,d_1][g,d_2] \in [g,d_1d_2]A'$. Since $G$ is periodic $[g,d_1]^{-1} = [g,d_1]^\ell \in [g,d]A'$ for some $d \in D$ and $\ell > 1$. Therefore every element of $N(g)$ is of the form $[g,d]a'$ for some $d \in D$ and $a' \in A'$.

We choose a transversal $\{g_1,\dots,g_n\}$ of $D$ in $G$ then $[G,D]A' = [g_1,D]\dots[g_m,D]A'$. If $N(g)$ is finite for all $g \in G$ then $[G,D]$ and $\{g^D\}$ are finite for all $g \in G$, where $\{g^D\}$ denotes the conjugate class of $g$ in $D$. Hence $G$ is an $FC$-group.

Therefore, we may assume $N(g)$ is infinite for a fixed element $g \in G$. By (Case 1), the centre $Z$ of $N(g)$ is finite. Hence an $FC$-group $N(g)$ has an infinite $N$-subgroup generated by $x_i, y_i$ $(i \in I)$ for an infinite $I$, subject to $[x_i,y_i] \neq 1$, $[x_i,y_j] = [x_i,x_j] = [y_i,y_j] = 1$ if $i \neq j$. Set

$$x_i = [g,c_i]a'_i \qquad y_i = [g,d_i]b'_i$$

for $c_i, d_i \in D$ and $a'_i, b'_i \in A'$. This setting is possible for every element of $N(g)$ is of this type.

Recall $A'$ is finite abelian normal, $D \leq A$, $[D,A'] = 1$ and hence $[\,[g,D],A'] = 1$. Choose $i_0 \in I$ and let $I$ be the disjoint union of $I_1$, $J_1$ and $\{i_0\}$ where $|I_1| = |J_1| = \infty$. Let

$$X_1 = X_3 = \{gd \mid d \in D\}$$

$$X_2 = \{c_{i_0}d_{i_1} \mid i_1 \in I_1\}$$

$$X_4 = \{d_{i_0}d_{j_1} \mid j_1 \in J_1\}$$

then, $[X_1,X_2]$ consists of $[gd,c_{i_0}d_{i_1}] = [g,d_{i_1}][g,c_{i_0}]a'$ for some $a' \in A'$, and $[X_3,X_4]$ consists of $[gd,d_{i_0}d_{j_1}] = [g,d_{j_1}][g,d_{i_0}]b'$ for some $b' \in A'$. Hence for $x_i \in X_i$, $[[x_1,x_2],[x_3,x_4]] = [[g,c_{i_0}],[g,d_{i_0}]]^g$ for some $g \in G$, which is not equal to 1, contradicting to $G \in \mathfrak{A}^2_{\mathfrak{f}}$. ♠

**(4.12)Theorem.** *If* $G \in \mathfrak{A}^2_\sharp$ *is locally soluble then* $G$ *is metabelian.*

**Proof.** Since $G/K \in \mathfrak{A}^2$ for any infinite normal subgroup $K$, every $G$-invariant proper subgroup of $G''$ is finite. If $G''$ is minimal normal in $G$, then $G''$ is abelian (See Theorem 5.5.1 [RO$_3$]). Therefore $G$ is soluble and $G \in \mathfrak{A}^2$ by the previous Theorem. Therefore we may assume $1 \neq A \triangleleft G$ and $A \lneqq G''$. If $C = C_{G''}(A) \lneqq G''$, then $C$ is finite, $|G'' : C|$ and $G''$ are finite and hence $G$ is soluble.

It follows that every proper $G$-invariant subgroup of $G''$ is contained in $Z(G'')$. Hence $G''/Z(G'')$ is minimal normal in $G/Z(G'')$ and hence abelian. Therefore $G \in \mathfrak{A}^2$. ♠

(4.12)Theorem tells us it is possible to assume that $G$ is simple for further study of $\mathfrak{A}^2_\sharp$-groups as we have seen in (3.7)Reduction to Simplicity.

# CHAPTER 5

## EXTENSIONS OF A PROBLEM OF PAUL ERDÖS ON QUASI-HAMILTONIAN GROUPS

In this chapter, $G$ denotes an infinite group. A non-abelian group $G$ is said to be ***Hamiltonian*** if every subgroup is normal, and to be ***quasi-Hamiltonian*** if any two subgroups permute as subgroups.

The structure of Hamiltonian groups is well known (See Dedekind and Baer 5.3.7 [$RO_3$]). Quasi-Hamiltonian groups have been studied by K. Iwasawa in [$IW_1$], [$IW_2$]. Joining quasi-Hamiltonian groups with a variation of Erdös problem, P. Longobardi, M. Maj, A.H. Rhemtulla and H. Smith [LMRS] have proved the following, which extends the result of [CLRW].

**(5.1)Theorem** (LMRS). *Suppose every infinite set of subgroups of $G$ contains a pair that permute. Then a finitely generated $G$ is centre-by-finite, conversely a finitely generated centre-by-finite group satisfies the hypothesis.*

Above is not true in general and a counter example was provided in [CLRW] which is based on a group constructed by Iwasawa [$IW_2$] and Napolitani [NA].

We are concerned here about finitely generated groups satisfying (5.2).

**(5.2)** any two infinte sets $\mathbf{X}, \mathbf{Y}$ of subgroups of $G$ contain $H \in \mathbf{X}$, $K \in \mathbf{Y}$ such that $HK = KH$.

**(5.3)Theorem.** *Any finitely generated group $G$ is quasi-Hamiltonian if and only if $G$ satisfies (5.2).*

**Proof.** Clearly it is enough to show $\langle a\rangle\langle b\rangle = \langle b\rangle\langle a\rangle$ for all $a, b \in G$.

(5.2) is a stronger condition than that of (5.1)Theorem and hence $G$ has a torsion-free element $z \in Z(G)$.

**(Case 1)** Suppose $a, b$ are elements of finite order.

Let $(\langle a\rangle\langle b\rangle)^2$ be the set of elements of the form $a^{\ell_1}b^{\ell_2}a^{\ell_3}b^{\ell_4}$ for $\ell_i \in \mathbf{Z}$. Then this set is finite and hence we can choose $0 \neq m \in \mathbf{Z}$ such that $\langle z^m\rangle \cap (\langle a\rangle\langle b\rangle)^2 = 1$. Let

$$X = \{ \langle a\rangle\langle z^m\rangle,\ \langle a\rangle\langle z^{2m}\rangle, \dots \}$$

$$Y = \{ \langle b\rangle\langle z^m\rangle,\ \langle b\rangle\langle z^{2m}\rangle, \dots \}$$

By the hypothesis, $\langle a\rangle\langle b\rangle\langle z^{km}\rangle = \langle b\rangle\langle a\rangle\langle z^{km}\rangle$ for some $k \in \mathbf{Z}$, and hence $\langle a\rangle\langle b\rangle \subset \langle b\rangle\langle a\rangle\langle z^{km}\rangle$. Therefore, $\langle a\rangle\langle b\rangle \subset \langle b\rangle\langle a\rangle$ and similiarly we have $\langle b\rangle\langle a\rangle \subset \langle a\rangle\langle b\rangle$.

**(Case 2)** Suppose $a, b$ are elements of infinite order.

Since $G$ is centre-by-finite, there exists $n$ such that $a^n$, $b^n$ are central in $G$. Set

$$X = \{ \langle a\rangle,\ \langle a^{n+1}\rangle,\ \langle a^{2n+1}\rangle, \dots \}$$

$$Y = \{ \langle b\rangle,\ \langle b^{n+1}\rangle,\ \langle b^{2n+1}\rangle, \dots \}$$

then, $\langle a^{tn+1}\rangle\langle b^{sn+1}\rangle = \langle b^{sn+1}\rangle\langle a^{tn+1}\rangle \subset \langle b\rangle\langle a\rangle$ for some $t$ and $s$. For all $i$, $j$, $a^i b^j a^{tni} b^{snj} \subset \langle b\rangle\langle a\rangle$ and hence $\langle a\rangle\langle b\rangle \subset \langle b\rangle\langle a\rangle$. Similiarly we have $\langle b\rangle\langle a\rangle \subset \langle a\rangle\langle b\rangle$.

**(Case 3)** Suppose the order of $a$ is infinite and the order of $b$ is finite.

Choose $0 < n \in \mathbf{Z}$ such that $a^n \in Z(G)$ and $\langle a^n \rangle \cap \langle b \rangle = 1$. Set

$$X = \{ \langle a \rangle, \langle a^{n+1} \rangle, \langle a^{2n+1} \rangle, \dots \}$$

$$Y = \{ \langle b \rangle, \langle b \rangle \langle a^n \rangle, \langle b \rangle \langle a^{2n} \rangle, \dots \}$$

Then, $X$ and $Y$ are infinite. Hence $\langle a^{tn+1} \rangle \langle b \rangle \langle a^{sn} \rangle = \langle b \rangle \langle a^{sn} \rangle \langle a^{tn+1} \rangle$ for some $s, t$ Therefore, $\langle a \rangle \langle b \rangle = \langle b \rangle \langle a \rangle$, and the result follows. ♠

**(5.4)Corollary.** *Any group $G$ having an element of infinite order is quasi-Hamiltonian if and only if $G$ satisfies (5.2).*

**Proof.** Let $x$ be an element of infinite order. Then a finitely generated infinite group $\langle x, a, b \rangle$ is quasi-Hamiltonian for each choice of $a, b$. Hence $\langle a \rangle \langle b \rangle \subset \langle b \rangle \langle a \rangle$. ♠

Suppose $G$ is an infinite group satisfying (5.2). If $G$ is not periodic, then by (5.4)Corollary $\langle a, b \rangle = \langle a \rangle \langle b \rangle$ for each $a, b \in G$. It is shown in [LMRS] that a periodic group satisfying the hypothesis (5.1)Theorem is locally finite. Hence periodic groups satisfying (5.2) are locally finite. So we have the following;

**(5.5)Corollary.** *Suppose $G$ satisfies (5.2). The set of all elements of finite order in $G$ forms a characteristic subgroup*

(5.3)Theorem is not true in general. Clearly a counter example must be locally finite.

**(5.6)Example.** Let $D_8$ be the dihedral group generated by $a, b$ with relations $a^4 = b^2 = 1$. It is easy to see that $\langle a^2 \rangle = D_8' = Z(D_8)$ and $D_8$ is not quasi-Hamiltonian; for example, $\langle ab \rangle \langle a^2 b \rangle \neq \langle a^2 b \rangle \langle ab \rangle$.

Let $\frac{1}{2}$ be the element of order 2 in the quasi-cyclic 2-group $C_{2^\infty}$, then $N = \langle (a^2, \frac{1}{2}) \rangle$ is normal in $D_8 \oplus C_{2^\infty}$. Let $G$ be the quotient of $D_8 \oplus C_{2^\infty}$ by $N$. Let **X**, **Y** be two infinite sets of subgroups of $G$. Since any subgroup of $C_{2^\infty}$ is considered normal in $D_8 \oplus C_{2^\infty}$, we may assume no subgroups in **X**, **Y** is a subgroup of $C_{2^\infty}/N$ (this notation makes sense since $a^2$ is identified with $\frac{1}{2}$).

Therefore, we can choose $H \in \mathbf{X}$, $K \in \mathbf{Y}$ such that $\overline{x_1 + y_1} \in H$, $\overline{x_2 + y_2} \in K$ with $|y_1| > 4$, $|y_2| > 4$ for $x_i \in D_8, y_i \in C_{2^\infty}$. Then $x_1^4 + y_1^4 \in H$ implies $a^2 \in H$. Similiarly $a^2 \in K$. Now pick any nontrivial elements $d_1 + c_1 \in H$, $d_2 + c_2 \in K$ with $d_i \neq 1$ in $D_8$, then

$$
\begin{aligned}
(d_1 + c_1)(d_2 + c_2) &= d_1 d_2 + c_1 c_2 \\
&= d_2 d_1 [d_1, d_2] + c_2 c_1 \in KH
\end{aligned}
$$

This implies $HK \subset KH$. Similiarly $KH \subset HK$. On the other hand, $G$ contains a subgroup isomorphic to $D_8$. Hence $G$ is not quasi-Hamiltonian but does satisfy (5.2).

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