UNIVERSITY OF ALBERTA

THE THEORY OF ADMISSIBLE REPRESENTATIONS OF GL(2,F)

BY

TERRY JASON JACKSON ©

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Table of Contents

Notation

F non-archimedean local field
x, y, a, b elements of F
F^+ denotes the abelian group $(F, +)$
$|\cdot|$ normalized absolute value on F
$F^\times$ group of units of F namely $F \setminus \{0\}$
$\mathcal{O}$ ring of integers of F namely $\{x \in F : |x| \leq 1\}$
$\mathcal{U}$ group of units of $\mathcal{O}$ namely $\{x \in F : |x| = 1\}$
ϵ elements of $\mathcal{U}$
$\mathcal{P}$ the unique prime ideal of $\mathcal{O}$ namely $\{x \in F : |x| < 1\}$
$q := |\mathcal{O}/\mathcal{P}|$
$G = GL_2(F)$
$K = GL_2(\mathcal{O})$
M denotes arbitrary compact or open subsets of a topological space
val denotes the valuation homomorphism $val : F^\times \longrightarrow \mathbb{Z}$
f will denote functions
ρ will denote right translation
g, h denote elements of G
c, z denote complex numbers
ψ non-trivial unitary character of F^+
l denotes the conductor of ψ
$F^*, \mathcal{U}^*, \ldots$ denotes the Pontryagin duals of $F^+, \mathcal{U}, \ldots$
X, Y, V are for complex vector spaces
V^* is the space of linear functions on V
(π, V) denotes a representation of a group
u denotes vectors in X
χ character of $F^\times$
$\mathcal{F}(\chi)$
Φ for functions in $\mathcal{F}(\chi)$
$\Pi(G)$ is the set of equivalence classes of irreducible admissible representations of G.
$L(s, \pi)$ is the Euler factor of the representation $\pi \in \Pi(G)$
$\gamma(s, \pi, \psi)$ is the gamma factor of $\pi \in \Pi(G)$ with respect to ψ
$\epsilon(s, \pi, \psi)$ is the epsilon factor of $\pi \in \Pi(G)$ with respect to ψ
$\mathcal{Z}(s, \chi, \psi)$ is Tates zeta function of χ with respect to ψ
$\mathcal{Z}(s, \varphi, \pi)$ is the zeta function of an infinite dimensional class in $\Pi(G)$ at $\varphi \in \mathcal{K}(\pi)$
$\mathcal{Z}(\pi)$ the fractional ideal corresponding to $\pi \in \Pi(G)$
ω_π is the central character of $\pi \in \Pi(G)$
χ_0 denotes the restriction of the character χ of $F^\times$ to $\mathcal{U}$.
$(\tilde{\pi}, \tilde{V})$ denotes the contragredient representation of (π, V)
$(\pi_\psi, \mathcal{K}(\pi)$ denotes the Kirillov model of π with respect to ψ
$\chi := \chi_1 \chi_2^{-1}$ where $\chi_1, \chi_2 \in \Pi(GL_1(F))$
$\pi(\chi_1, \chi_2)$ denotes a principal series representation
$\sigma(\chi_1, \chi_2)$ denotes a special representation
$(\rho, \mathcal{B}(\chi_1, \chi_2))$
μ and ν denote unitary characters of $\mathcal{U}$, i.e. elements of $\mathcal{U}^*$
$\mathcal{S}_X(\mathcal{T})$ denotes the Schwartz-Bruhat space of $\mathcal{T}$ and X
$\mathcal{S}_X$ abbreviates $\mathcal{S}_X(F^\times)$
$\mathcal{S}(\mathcal{T})$ abbreviates $\mathcal{S}_{\mathbb{C}}(\mathcal{T})$
L denotes linear functionals
φ, η for functions from $F^\times$ to a complex vector space
R denotes linear operators and intertwining operators
n, m, k denote integers

ϖ denotes uniformizing parameters of F

$Z = \{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \in G : a \in F^\times \}$ is the center of G

$B = \{ \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \in G : a, b \in F^\times\ x \in F \}$ is the standard Borel subgroup

$U = \{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \in G : x \in F \}$ is the unipotent radical of B

$T = \{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in G : a, b \in F^\times \}$ is the torus subgroup of diagonal matrices in G

$A = \{ \begin{pmatrix} a & x \\ 0 & 1 \end{pmatrix} \in G : a \in F^\times\ x \in F \}$ is the affine subgroup of G.

λ_G is the Haar measure on G such that $\lambda_G(K) = 1$

λ^+ is the Haar measure on F^+ such that $\lambda^+(\mathcal{O}) = 1$

λ_ψ is the Haar measure self dual with respect to $1 \neq \psi \in F^*$

$\lambda^\times$ is the Haar measure on $F^\times$ such that $\lambda^\times(\mathcal{U}) = 1$

d^+x abbreviates $d\lambda^+(x)$

$d^\times x$ abbreviates $d\lambda^\times(x)$

$d_\psi x$ abbreviates $d\lambda_\psi(x)$

$w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Introduction

Let F be a non-archimedean local field. The theory of admissible representations of the group $GL_2(F)$ was first introduced in the second and third paragraphs of [1]. In the end they achieve what they need but polished routes to these goals is not a priority. The result is a first and fundamental step towards the understanding of what type of results one should expect from this new theory, but unfortunately due to the terse exposition the ideas are clouded and difficult to get in to. The next step is a much easier one and it involves cleaning the proofs up and finding more direct and natural arguments. This is the purpose of this thesis. Our two main references are [1] and [2]. We take the best approaches from both, combine them, and the proper definitions and formulations are discovered. Our goal was to give a relatively detailed and friendly exposition of this theory, something which did not exist before. In search of this goal we obtained three minor improvements over the previous theory.

Firstly the dependence of the non-trivial unitary character ψ on the theory is hidden in the background in [1]. This causes some confusion. We ammend this problem by clearly stating the role ψ plays in the results. In [2] the character ψ is fixed "once and for all". We disagree with this approach feeling that results in the theory should be phrased for arbitrary ψ. It is important to understand how objects defined with respect to ψ will change when ψ changes.

Secondly theorem 1 in chapter 7 is a little more general than in [1] or [2] since it is stated for arbitrary X. A close examination of the methods of [1] and [2] shows that these techniques can be carried over to prove this seemingly more general result.

Finally our development and exposition of the functional equation considerably improves upon [1] and [2]. In [1] the proof of the functional equation is only completed in the third paragraph and involves the Weil representation. This complication is completely unnecesssary as the approach of [2] shows. Unfortunately [2] uses the strange choice of $2s-1$ in the definition of the zeta function of π where $s-\frac{1}{2}$ should really be used. This has the consequence of a loss of elegance in certain formulas in the theory. Moreover they both use a group element g of G and a character χ of $F^\times$ as parameters in thier definition of the zeta function of π. This again is not necessary and we give the proper definitions. The naturality of the results attest to the correctness of the definitions found here.

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Terry Jackson
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Chapter 1 Background from the General Theory

Some Remarks on Topological Groups

For background in the theory of topological groups the reader is may consult [5]. Any finite group can be considered as a compact group by providing it with the discrete topology. We recall the simple fact that if G is a topological group and H is an open subgroup of G then H is necessarily closed. Indeed every coset of H in G is open and so the set defined by $\cup\{gH : g \in G\ g \notin H\}$ is open in G. It is clear that the complement of this set is H. and hence H is closed. An open subgroup of a compact group is necessarily of finite index as the cosets form a mutually disjoint open cover of the group.

Proposition 1 Let G be a compact group and H a closed subgroup of G. Then H is open iff H has finite index in G.

Proof We have just remarked that H open implies that the index of H in G is finite. Conversely let S be a complete set of representatives of the left (or right) cosets of H in G and suppose that S is finite. We may suppose $1 \in S$. If $H = G$ of course H is open so we may suppose $H \neq G$. The finite union $\cup\{gH : g \in S, g \neq 1\}$ is closed. and hence its complement in G. which is clearly H, is open.

□

Characters of Topological Groups

Let G be a topological group. We define a *character* of G to be a continuous homomorphism from G to $\mathbb{C}^\times$ and a *unitary character* of G to be a continuous homomorphism from G to $\mathbb{T}$. the group of complex numbers of absolute value 1. A character is said to be *positive* if it takes values in the subgroup of positive real numbers $(0, \infty)$ of $\mathbb{C}^\times$. Observe that if χ is a homomorphism from G to $\mathbb{C}^\times$ then to show that χ is continuous it suffices to show that χ is continuous at the identity of G. If χ and ω are characters of G we define their product. which will be denoted by $\chi\omega$, to be the character $(\chi\omega)(g) := \chi(g)\omega(g)$ where $g \in G$. With this law of composition the set of all characters of G becomes an abelian group. and the set of unitary characters of G is a subgroup. The unitary character $: G \longrightarrow \mathbb{T} \quad g \longmapsto 1$ is called the *trivial character* of G, and it is the identity element of the group of characters of G. If χ is a character of G we denote by χ^{-1} the inverse of χ in the group of all characters. namely $\chi^{-1}(g) := \chi(g)^{-1}$ where $g \in G$. Hence for every $g \in G$ we have $\chi^{-1}(g) = \chi(g)^{-1} = \chi(g^{-1})$. If χ is unitary we could add the complex conjugate of $\chi(g)$ to this string of equalities.

Proposition 2 If G is a topological group which is the union of its compact subgroups then any character of G is necessarily a unitary character. In particular this holds for compact groups.

Proof Let $\chi : G \longrightarrow \mathbb{C}^\times$ be a character of G. Let M be a compact subgroup of G. Then as χ is continuous the image of M in $\mathbb{C}^\times$ is a compact subgroup of $\mathbb{C}^\times$. If $c \in \mathbb{C}^\times$ and $|c| \neq 1$ then $\{c^n : n \in \mathbb{Z}\}$ is an unbounded subset of $\mathbb{C}^\times$ and hence c cannot be contained in a compact subgroup. This shows that the restriction of χ to M is a unitary character of M. Since G is the union of its compact subgroups we see that χ is in fact a unitary character of G.

□

Haar Groups

The theory we review in this section is developed in detail in [5]. We define a *Haar group* to be a locally compact Hausdorff topological group. It is a fundamental result that there exists a left invariant measure on the Borel σ-algebra of any Haar group. Here by a *left invariant* measure we mean a positive measure λ on the group such that for any Borel measurable set E and element g of the group we have $\lambda(gE) = \lambda(E)$. Any such measure is called a *left Haar measure*. Similarly we say a positive measure is *right invariant* if $\lambda(Eg) = \lambda(E)$ for all g and E, and it is proven that there exists a right invariant measure on any Haar group. We call any such measure a *right Haar measure*. A Haar measure always assigns to any compact set a finite measure. and assigns to any open set a positive. possibly infinite, measure. In general there does not exist a measure on a Haar group which is both a left and a right Haar measure. A Haar group is said to be *unimodular* if there is a measure that is both a left and a right Haar measure. There are important classes of Haar groups which are unimodular. For example any abelian Haar group. or any compact Haar group is unimodular. When dealing with unimodular groups we consequently need not make a distinction between left and right Haar measure. Any two left Haar measures (or right Haar measures) are a positive multiple of one another. Consequently when we have a compact Haar group it has a unique Haar measure which assigns the group a total mass of 1. We call this the *normalized* Haar measure. Similarly if we are given some Haar group with a compact open subgroup then there is a unique Haar measure which assigns this subgroup a measure of 1.

Let G be any Haar group, and let α be a left Haar measure on G. Denote the complex vector space of continuous compactly supported functions on G by $C_c(G)$. We have a positive linear functional

$$I := \int_G \cdot \, d\alpha : C_c(G) \longmapsto \mathbb{C}$$

corresponding to the measure α. We denote the image of $f \in C_c(G)$ under this map as

$$\int_G f \, d\alpha \quad \text{or} \quad \int_G f(x) \, d\alpha(x).$$

Since the measure α is left invariant this functional has the property that $I(f) = I(f(a\,\cdot))$ for all $f \in C_c(G)$ and $a \in G$. This is equivalently expressed as

$$\int_G f(ax) \, d\alpha(x) = \int_G f(x) \, d\alpha(x).$$

Similar remarks apply for right Haar measures.

When G is an abelian Haar group we denote the group of unitary characters of G by G^*, and we call G^* the *Pontryagin dual* of G. We can provide G^* with the structure of an abelian Haar group. A basis of open neighborhoods at the identity in G^* is given by the sets

$$N(K,\epsilon) := \{\nu \in G^* : |\nu(x) - 1| < \epsilon \quad \text{for all} \quad x \in K\}$$

where K is any compact subset of G and $\epsilon > 0$. This is the topology of uniform convergence on compact sets.

The Pontryagin duality theorem asserts that G and $G^{**} := (G^*)^*$ are isomorphic topological groups under the canonical map

$$: G \longrightarrow G^{**} \qquad g \longmapsto g^{**}$$

where we define $g^{**}(\mu) := \mu(g)$ for $\mu \in G^*$. One proves that G is compact iff G^* is discrete. Hence by Pontryagin duality G is discrete iff G^* is compact.

Let M be a compact Haar group and λ the normalized left, or right, Haar measure of M. Since M is compact any character of M is necessarily unitary. For a unitary character

χ of M define $\delta(\chi) = 0$ if $\chi \neq 1$ and define $\delta(\chi) = 1$ when $\chi = 1$. The invariance of λ easily implies that for any unitary character χ of M we have

$$\int_M \chi(g)\, d\lambda(g) = \delta(\chi).$$

Fourier Analysis on Abelian Haar Groups

For the material covered in this section we refer the reader to [6] and [9]. Let G be an abelian Haar group, which will be written additively, and let α be a Haar measure on G. For any function $f \in L^1(G)$ we can define its *Fourier transform* with respect to α by

$$\hat{f}_\alpha : G^* \longrightarrow \mathbb{C} \qquad \hat{f}_\alpha(\mu) := \int_G f(a)\mu(a)\, d\alpha(a).$$

When it is clear which Haar measure α is being used we will write $\hat{f}$ instead of $\hat{f}_\alpha$. One can prove that if f is continuous then $\hat{f}$ is continuous. The Fourier inversion formula states that there exists a unique Haar measure α^* on G^*, called the *dual measure* of α, such that whenever $\hat{f} \in L^1(G^*)$, so that $\hat{\hat{f}} := (\hat{f})_{\alpha^*}$ is defined, then for all $a \in G$,

$$f(-a) = \hat{\hat{f}}(a^{**}).$$

Or equivalently for every $a \in G$,

$$f(-a) = \int_{G^*} \hat{f}(\mu)\mu(a)\, d\alpha^*(\mu).$$

This is of course the same as

$$f(a) = \int_{G^*} \hat{f}(\mu)\mu(-a)\, d\alpha^*(\mu) = \int_{G^*} \hat{f}(\mu)\mu^{-1}(a)\, d\alpha^*(\mu).$$

We will not have to be concerned with any L^1-convergence properties of Fourier transforms in these notes. The application of this theory to our needs will not require the generality it is available in. For example we will mainly use it on functions which have compact support.

For the proof of the following refer to the sources quoted at the beginning of this section.

Proposition 3 Let G be a compact abelian Haar group, and let α be the normalized Haar measure on G. Then the dual measure α^* of α on the discrete abelian Haar group G^* is the counting measure. By Pontryagin duality we have the converse statement as well.

Basic Notions in the Theory of Group Representations

In this section we review some elementary aspects of the theory of group representations. The subject matter of these notes in fact does not require much background from representation theory beyond the basic definitions given below. The reader may consult [5] for results on the representation theory of compact and locally compact groups.

Let G be an arbitrary group. A *representation* of G is a pair (π, V) where V is a non-zero complex vector space, of finite or infinite dimension, and $\pi : G \longrightarrow GL(V)$ is a homomorphism of groups. Here $GL(V)$ denotes the group of invertible linear operators on V. The vector space V is called the *representation space*. The *dimension* of a representation is the dimension of its representation space. Note that if (π, V) is a representation of G, and M is a subgroup of G, then by restricting π to M we obtain a representation $(\pi|_M, V)$ of M on the same space V.

For any subgroup, or even for any subset, M of G we define the subspace of V,

$$V^M = \{v \in V : \pi(g)v = v \quad \text{for all} \quad g \in M\}.$$

Suppose moreover V is a topological vector space and G is a topological group. We say that a representation (π, V) of G is *continuous* if the map : $G \times V \longrightarrow V \quad (g, v) \longmapsto \pi(g)v$ is continuous. Note that characters of G are the same as continuous one dimensional representations of G.

Let (π, V) be a representation of G. We say that a subspace W of V is *π-invariant* iff for every $g \in G$ we have that W is an invariant subspace of the operator $\pi(g)$, that is $\pi(g)W \subseteq W$. The subspaces $\{0\}$ and V of V are always π-invariant. Let W be π-invariant. We claim that for any $g \in G$ we have $\pi(g) \in GL(W)$. At this point we only know that $\pi(g) \in End(W)$. Since $\pi(g)$ is an injective map we only need to prove $\pi(g)$ as an operator in $End(W)$ is surjective. Well let $w \in W$ and define $v := \pi(g^{-1})w$. Then $v \in W$ and $\pi(g)(v) = w$, which proves the surjectivity assertion. Hence if $W \neq 0$ we obtain a representation of G on W. We call this representation of G on W a *subrepresentation* of (π, V).

We say that (π, V) is an *irreducible* representation of G if there are no non-trivial π-invariant subspaces of V. More precisely the only π-invariant subspaces of V are $\{0\}$ and V.

Given $v \in V$ clearly the smallest π-invariant subspace of V containing v is $span\{\pi(g)v : g \in G\}$.

Let (π_1, V_1) and (π_2, V_2) be two representations of G, and let $R : V_1 \longrightarrow V_2$ be a linear operator. We say R is an *intertwining operator* of the two representations if for every $g \in G$ and $v \in V_1$ we have $R(\pi_1(g)(v)) = \pi_2(g)(R(v))$. If moreover R is an isomorphism of the vector spaces V_1 and V_2 then we call R an *intertwining isomorphism* of the two representations. We say that the representations (π_1, V_1) and (π_2, V_2) are *equivalent* if there exists an intertwining isomorphism between them. We say that (π_1, V_1) *occurs* in (π_2, V_2) iff (π_1, V_1) is equivalent to a subrepresentation of (π_2, V_2). Equivalence of representations is an equivalence relation on the category of all representations of G. If (π_1, V_1) and (π_2, V_2) are equivalent then of course their dimensions as representations are the same, as the existence of an intertwining isomorphism shows that V_1 and V_2 have the same dimension. Hence we may speak about the *dimension* of an equivalence class of representations.

We adopt the convention that when we speak of (π, V) we refer to a representation as defined above, but when we speak of π alone then we are referring to the equivalence class that (π, V) belongs to. Hence we write π to denote both a homomorphism of G to $GL(V)$ and for the unique equivalence class that contains the representation (π, V). It will be clear which interpretation is intended whenever it is used.

Proposition 4 Let G be a compact group and let (π, V) be a representation of G such that $ker(\pi)$ is closed in G. Then $ker(\pi)$ is open in G iff $im(\pi)$ is a finite subgroup of $GL(V)$.

Proof By Proposition 1.1 we know $ker(\pi)$ is open iff $(G : ker(\pi)) < \infty$ iff $im(\pi)$ is finite.

□

Definition 5 Let (π, V) be a representation of a group G. Suppose X and Y are π-invariant subspaces of V such that $Y \subseteq X$. If we define $\pi' : G \longrightarrow GL(X/Y)$ by

$$\pi'(g)(x + Y) := \pi(g)(x) + Y$$

then π' is a well defined group homomorphism. Hence $(\pi', X/Y)$ is a representation of G. We call $(\pi', X/Y)$ a *constituent* or a *subquotient* of (π, V). We will prefer the term constituent even though subquotient is more descriptive. Observe that any subrepresentation X of (π, V) is a constituent of (π, V) by taking $Y = \{0\}$.

Chapter 2 Finite Dimensional Smooth Representations

Locally Profinite Topologies

Recall that a topological space is said to be *connected* iff the only subsets of it which are both open and closed are the empty set and the whole space. A subset of a topological space is said to be a *connected* subset of the space iff it is connected in the induced topology.

Let X be a topological space. For $x \in X$ we define

$$C_x := \cup\{E \subseteq X : E \text{ is connected and } x \in E\}.$$

We call C_x the *connected component* of x in X. C_x is the largest connected subset of X having x as an element.

Definition 1 A topological space X is said to be *totally disconnected* iff for every $x \in X$ we have $C_x = \{x\}$.

Observe that a discrete topological space is totally disconnected.

Proposition 2 Let Y be any non-empty subset of a totally disconnected topological space X. Then Y is totally disconnected in the induced topology from X.

Proof Let W be any non-empty subset of Y. It is simple to verify that the induced topology on W from X coincides with the induced topology on W from Y. Hence W is a connected subset of X iff W is a connected subset of Y. Now let $y \in Y$ and define C_y to be the connected component of y in Y. We wish to show that $C_y = \{y\}$. Well by what we have just said C_y being a connected subset of Y implies that C_y is a connected subset of X. Since X is totally disconnected we must have $y \in C_y \subseteq \{y\}$. Therefore $C_y = \{y\}$ as desired.

□

The next result is simple and can be found in any introductory book on general topology.

Proposition 3 The product of totally disconnected topological spaces is totally disconnected.

Definition 4

(1) A totally disconnected, compact, Hausdorff space is said to be *profinite*.

(2) A topological space is said to be *locally profinite* iff every point of the space has a neighborhood basis consisting of profinite sets.

(3) A topological group is said to be *profinite* (*locally profinite*) iff the underlying topological space is profinite (locally profinite).

Of course a profinite space or group is also locally profinite. The proof of the next results can be found in Bourbaki or the book of Montgomery and Zippen titled "Topological Transformation Groups".

Proposition 5

(1) A topological space is locally profinite iff it is locally compact, Hausdorff, and totally disconnected.

(2) Let G be a locally profinite group. Then the set of compact open subgroups of G form a basis of topology at the identity of G.

(3) Let G be a profinite group. Then the set of open normal subgroups of G form a basis of topology at the identity of G. Any such subgroup is necessarily closed and hence compact.

The Schwartz-Bruhat Space

Let $\mathcal{T}$ be a topological space. If X is a set and $f : \mathcal{T} \longrightarrow X$ is a function then we say that f is *locally constant* iff for any point in $\mathcal{T}$ there is an open neighborhood in $\mathcal{T}$ about that point on which f is constant. Obviously under any topology on X a locally constant function $f : \mathcal{T} \longrightarrow X$ is necessarily continuous. Of course there are continuous functions that are not locally constant. We will usually consider locally constant functions with values in a complex vector space which will usually be $\mathbb{C}$. Let X be a complex vector space. If $f : \mathcal{T} \longrightarrow X$ is any function we define the *support* of f to be the closure in $\mathcal{T}$ of the set $\{x \in \mathcal{T} : f(x) \neq 0\}$. The support of f is denoted by $supp(f)$. If $supp(f)$ is a compact subset of $\mathcal{T}$ we say that f is *compactly supported*. It is trivial that if $\mathcal{T}$ is a compact Hausdorff space then any function $f : \mathcal{T} \longrightarrow X$ is compactly supported.

Now let $f : \mathcal{T} \longrightarrow X$ be a locally constant function. If f is compactly supported then the image of f in X is finite, and $supp(f) = \{x \in \mathcal{T} : f(x) \neq 0\}$, namely there is no need to take the closure in this case. We conclude that $supp(f)$ is an open and closed subset of $\mathcal{T}$. For similar reasons the preimage of any vector in X is an open and closed subset of $\mathcal{T}$. We conclude that the only locally constant functions on a connected topological space are the constant functions.

We can now give the following definition.

Definition 6 Let $\mathcal{T}$ be a locally profinite topological space and let X be a complex vector space. We define the *Schwartz-Bruhat space* of $\mathcal{T}$ and X to be the complex vector space consisting of all locally constant, compactly supported functions from $\mathcal{T}$ to X. It will be denoted by $\mathcal{S}_X(\mathcal{T})$. Since the case $X = \mathbb{C}$ arises so often we abbreviate $\mathcal{S}_{\mathbb{C}}(\mathcal{T})$ by $\mathcal{S}(\mathcal{T})$.

By what we have said above any function in $\mathcal{S}_X(\mathcal{T})$ takes only finitely many values in X. Therefore if furthermore $\mathcal{T}$ is a measure space such that any compact subset has finite measure then we may integrate any function in $\mathcal{S}_X(\mathcal{T})$, the value of the integral being a vector in X.

Finite Dimensional Smooth Representations

Definition 7 Let G be a locally profinite group, and let (π, V) be a representation of G. We say that (π, V) is a *smooth representation* iff for every $v \in V$ the stabilizer of v, namely $\{ g \in G : \pi(g)v = v \}$, is an open subgroup of G.

Note that the stabilizer of v is trivially a group. The content of the condition is that this stabilizer is open in the topology.

If V is a finite dimensional complex vector space we can give it the unique non-discrete locally compact topology making it into a topological vector space. In this case V and $\mathbb{C}^n$ are isomorphic topological vector spaces, where $n = dim(V)$. If (π, V) is a finite dimensional representation of G and we speak of (π, V) being continuous we always mean with respect to this topology on V.

Let V be a non-zero finite dimensional complex vector space. Let $|\cdot|$ be any norm on V. We define the usual operator norm $||\cdot||$ on $End(V)$ by

$$||\alpha|| := sup_{v \neq 0} \frac{|\alpha(v)|}{|v|}$$

where $\alpha \in End(V)$.

Lemma 8 If $\alpha \in GL(V)$ is such that $||\alpha^n - I|| < \sqrt{3}$ for all $n \in \mathbb{Z}$ then $\alpha = I$.

Proof We first prove that every eigenvalue of α is equal to 1. Let $a \in \mathbb{C}$ be any eigenvalue of α and let $v \in V$, $v \neq 0$ be a corresponding eigenvector, so that $\alpha(v) = av$. Since α is injective we have that $a \neq 0$. For any $n \in \mathbb{Z}$ we have

$$|a^n - 1| = \frac{|(\alpha^n - I)v|}{|v|} \leq ||\alpha^n - I|| < \sqrt{3},$$

and therefore $|a|^n \leq 1 + |a^n - 1| < 1 + \sqrt{3}$. Passing $n \to -\infty$ shows we cannot have $0 < |a| < 1$ and passing $n \to \infty$ shows we cannot have $|a| > 1$. Hence $|a| = 1$. Let $\theta \in (-\frac{1}{2}, \frac{1}{2}]$ satisfy $a = e^{2\pi i\theta}$. Observe that for $n \in \mathbb{Z}$

$$|sin(\pi n|\theta|)| = |sin(\pi n\theta)| = \frac{1}{2}|a^n - 1| < \frac{\sqrt{3}}{2}.$$

Hence $|sin(\pi n|\theta|)| < \frac{\sqrt{3}}{2}$ for all $n \in \mathbb{Z}$. We show that this implies $\theta = 0$. To see this we distinguish between three cases. First if $\frac{1}{3} \leq |\theta| \leq \frac{1}{2}$ then $\frac{\pi}{3} \leq \pi|\theta| \leq \frac{\pi}{2}$ and so $|sin(\pi|\theta|)| \geq \frac{\sqrt{3}}{2}$, which is not true. If $\frac{1}{6} \leq |\theta| < \frac{1}{3}$ then $\frac{\pi}{3} \leq 2\pi|\theta| < \frac{2\pi}{3}$ and so $|sin(2\pi|\theta|)| \geq \frac{\sqrt{3}}{2}$ which again is not true. Finally if $0 < |\theta| < \frac{1}{6}$ then choose the least positive integer $n \in \mathbb{N}$ such that $n|\theta| \geq \frac{1}{3}$. Then $n \geq 3$ and $(n-1)|\theta| < \frac{1}{3}$. Thus $\frac{1}{3} \leq n|\theta| < \frac{1}{3}\frac{n}{n-1} \leq \frac{1}{3}\frac{3}{2} = \frac{1}{2}$ and so $\frac{\pi}{3} \leq \pi n|\theta| \leq \frac{\pi}{2}$. This implies $\frac{\sqrt{3}}{2} \leq |sin(\pi n|\theta|)|$ which is not so. Hence we must have $\theta = 0$, that is $a = 1$. This proves that every eigenvalue of α is 1. By the Cayley-Hamilton theorem $\alpha - I$ is nilpotent. Define $\beta := \alpha - I$ and let $r \in \mathbb{N}$ be the smallest positive integer such that $\beta^r = 0$. Suppose $r \geq 2$ so that $\beta^{r-1} \neq 0$. For $m \geq r - 1$ we have

$$\alpha^m - I = (\beta + I)^m - I = \sum_{k=0}^{m} \binom{m}{k} \beta^k - I = \sum_{k=1}^{r-1} \binom{m}{k} ||\beta^k||.$$

This identity is easily seen to imply

$$||\beta^{r-1}|| \leq \binom{m}{r-1}^{-1} \left(\sqrt{3} + \sum_{k=1}^{r-2} \binom{m}{k} \beta^k\right)$$

where the summation here from 1 to $r - 2$ is defined to be 0 when $r = 2$. Passing $m \to \infty$ implies $||\beta^{r-1}|| = 0$ and so $\beta^{r-1} = 0$, a contradiction. Thus $r = 1$ and so $\alpha = I$ as desired.

□

Theorem 9 If (π, V) is a finite dimensional continuous representation of a profinite group G then $ker(\pi)$ is an open subgroup of G, and the image of π in $GL(V)$ is finite.

Proof The second assertion follows from the first. Indeed if $ker(\pi)$ was open then as it is also closed Proposition 1.1 implies that $ker(\pi)$ is of finite index in G and thus the image of π is finite. Now to show that $ker(\pi)$ is open it suffices to show that $ker(\pi)$ contains an open subgroup of G. Define

$$X := \{\alpha \in GL(V) : ||\alpha - I|| < \sqrt{3}\}.$$

Lemma 2.8 shows that the only subgroup of $GL(V)$ contained in X is the trivial subgroup. Therefore X being open in $GL(V)$ implies that $\pi^{-1}(X)$ is open in G. By Proposition 2.5 there is an open subgroup H of G such that $H \subseteq \pi^{-1}X$, or rather $\pi(H) \subseteq X$. So $\pi(H)$ is a subgroup of $GL(V)$ contained in X which implies $H \subseteq ker(\pi)$. Thus $ker(\pi)$ contains the open subgroup H and thus $ker(\pi)$ is open as desired.

□

Corollary 10 If (π, V) is a finite dimensional continuous representation of a locally profinite group G then $ker(\pi)$ is an open subgroup of G.

Proof To show that $ker(\pi)$ is open in G it suffices to prove that $ker(\pi)$ contains a subgroup that is open in G. Let M be a compact open subgroup of G. Then M is a profinite and the representation π of G on V yeilds a representation π_M of M on V by restriction. By the above theorem $ker(\pi_M)$ is open in M. Since M is open in G we conclude that $ker(\pi_M)$ is a subgroup of $ker(\pi)$ that is open in G.

□

Proposition 11 Let G be a locally profinite group and let (π, V) be a finite dimensional representation of G. Then (π, V) is continuous iff (π, V) is smooth.

Proof Suppose that (π, V) is smooth. To prove that (π, V) is continuous we must show that the map : $G \times V \longrightarrow V \quad (g, v) \longmapsto \pi(g)v$ is continuous. To do this fix $(g_0, v_0) \in G \times V$ and let U be an open neighborhood of $\pi(g_0)v_0$ in V. Let $v_1, ..., v_m$ be a basis of V and for $j = 1, ..., m$ define M_j to be the stabilizer of v_j in G. Then M_j is open for each j and hence if we define $M := \cap_{j=1}^{m} M_j$ then M is an open subgroup of G that is a subgroup of $ker(\pi)$. Define $U' := \pi(g_0)^{-1}U$. Since $\pi(g_0)$ is continuous we know U' is open in V. Then $g_0 M \times U'$ is an open neighborhood of (g_0, v_0) in $G \times V$ whose image under the map $(g, v) \longmapsto \pi(g)v$ is contained in U. Thus (π, V) is continuous.

Suppose now that (π, V) is continuous. By Corollary 2.10 we know $ker(\pi)$ is open in G. Thus the stabilizer of any vector in V contains the open subgroup $ker(\pi)$ and hence itself is open. Thus (π, V) is smooth.

□

Corollary 12 If G is a locally profinite group and (π, V) is a finite dimensional smooth representation of G then $ker(\pi)$ is an open subgroup of G.

Proof By Proposition 2.11 (π, V) is necessarily continuous and hence $ker(\pi)$ is open by Corollary 2.10.

□

Corollary 13 A character of a locally profinite group has an open kernel and hence is necessarily locally constant.

Proof Let G be a locally profinite group and let χ be a character of G, that is $\chi : G \longrightarrow \mathbb{C}^\times$ is a continuous homomorphism. Then χ can be considered as a continuous one-dimensional representation of G under the identification of $\mathbb{C}^\times$ with $GL_1(\mathbb{C})$. Hence by Corollary 2.12 if $M := ker(\chi)$ then M is an open subgroup of G. For any $g \in G$ we see that χ is constant on the open neighborhood gM of g.

□

Chapter 3 Admissible Representations of Locally Profinite Groups

The Definition of an Admissible Representation

Definition 1 Let G be a locally profinite group. We say that a smooth representation (π, V) of G is an *admissible* representation of G iff for every open subgroup M of G the subspace

$$V^M = \{v \in V : \pi(g)v = v \quad \text{for all} \quad g \in M\}$$

of V is finite dimensional.

Observe that the definition of admissibility does not assume that V has a topology, and no conditions of continuity are present. The condition of admissibility is trivially equivalent to demanding that V^M is finite dimensional only for open subgroups M of K where K is a fixed compact open subgroup of G.

If (π, V) is a finite dimensional representation of G then by Proposition 2.1 it is trivial that (π, V) is continuous iff (π, V) is smooth iff (π, V) is admissible.

Of course if two representations are equivalent, that is there exists an intertwining isomorphism between them, then one is smooth or admissible iff the other is as well. Consequently we may consider the set of equivalence classes of irreducible admissible representations of G, and it is denoted by $\Pi(G)$. We call $\Pi(G)$ the *admissible dual* of G.

Isotypic Subspaces and Admissible Representations

The material in this section is from [4].

Definition 2 Suppose G is a group, M is a subgroup of G, and (π, V) is a representation of G. Let (π_M, V) denote the representation of M obtained by restricting π to M. Let σ be an equivalence class of representations of M. We define the *σ-isotypic subspace* of the representation (π, V), which is denoted by $V(\sigma)$, to be the sum of all π_M-invariant subspaces of V that are in the equivalence class σ. If there are no such subspaces we define $V(\sigma)$ to be the zero subspace of V.

We will use this definition for G being a locally profinite group and M a compact open subgroup of G. In the case when G is a finite group and $M = G$ it is a standard result that for any representation (π, V) of G we have

$$V = \bigoplus_{\sigma \in \Pi(G)} V(\sigma)$$

where this is a finite direct sum as $\Pi(G)$ is a finite set.

Lemma 3 Suppose (π_1, V_1) and (π_2, V_2) are equivalent representations of a group G. Then $ker(\pi_1) = ker(\pi_2)$.

Proof Let $R : V_1 \longrightarrow V_2$ be an intertwining isomorphism of (π_1, V_1) and (π_2, V_2). Thus for $g \in G$ we have $g \in ker(\pi_1)$ iff $\pi_1(g) = I_{V_1}$ iff $R = \pi_2(g)R$ iff $\pi_2(g) = I_{V_2}$ iff $g \in ker(\pi_2)$.

□

Hence given any equivalence class π of representations of G we may define a subgroup of G, denoted $ker(\pi)$, and called the *kernel* of π, to be $ker(\pi_0) = \{g \in G : \pi_0(g) = I_V\}$ where (π_0, V) is any representation of G that is in the equivalence class π. The particular choice of representation (π_0, V) does not matter due to the above lemma.

Now let G be a locally profinite group and let K be a compact open subgroup of G. Suppose K_0 is an open normal subgroup of K. Since the index of K_0 in K is necessarily finite the group K/K_0 is finite. The quotient map from K to K/K_0 induces an injective map of $\Pi(K/K_0)$ into $\Pi(K)$. The image of this map is $\{\sigma \in \Pi(K) : K_0 \subseteq ker(\sigma)\}$. We will identify elements of $\Pi(K/K_0)$ with their image in $\Pi(K)$.

Lemma 4 Let G be a locally profinite group, let K be a compact open subgroup of G, and let K_0 be an open normal subgroup of K. Suppose (π, V) is a representation of G and let (π_K, V) denote the representation of K obtained by restricting π to K.

(1) The space V^{K_0} is π_K-invariant. Therefore the representation (π_K, V) of K gives rise to the representation $(\pi_{K/K_0}, V^{K_0})$ of K/K_0.

(2) For any $\sigma \in \Pi(K/K_0)$ we have $V^{K_0}(\sigma) = V(\sigma)$. Here $V^{K_0}(\sigma)$ is the σ-isotypic subspace of the representation $(\pi_{K/K_0}, V^{K_0})$ and $V(\sigma)$ is the σ-isotypic subspace of (π_0, V).

Proof

(1) Let $v \in V^{K_0}$ and let $g \in K$. We must show that $\pi_K(g)v \in V^{K_0}$. Well if $g_0 \in K_0$ then as $g^{-1}g_0g \in K_0$ we know $\pi_K(g^{-1}g_0g)v = v$ and hence $\pi_K(g_0)(\pi_K(g)v) = \pi_K(g)v$ as desired. Thus the representation (π_K, V) of K gives rise to the representation $(\pi_{K/K_0}, V^{K_0})$ of K/K_0.

(2) Note that any π_{K/K_0}-invariant subspace of V^{K_0} that is in σ is also a π_K-invariant subspace of V that is in σ. Thus $V^{K_0}(\sigma) \subseteq V(\sigma)$. Therefore to prove $V^{K_0}(\sigma) = V(\sigma)$ we need only show that any π_K-invariant subspace of V that is in σ is in fact a subspace of V^{K_0}. Well if Y is any such subspace let (π_Y, Y) denote the associated representation of K on Y. Since (π_Y, Y) is in σ we know that if $y \in Y$ and $g \in K_0 \subseteq ker(\sigma) = ker(\pi_Y)$ then $\pi_Y(g)y = y$. Hence $Y \subseteq V^{K_0}$ and this establishes our claim.

□

We now come to the main result of this section.

Theorem 5 Let G be a locally profinite group and let K be a compact open subgroup of G. Suppose (π, V) is a smooth representation of G. Then

$$V = \bigoplus_{\sigma \in \Pi(K)} V(\sigma).$$

Moreover the representation (π, V) is admissible iff for each $\sigma \in \Pi(K)$ the space $V(\sigma)$ is finite dimensional.

Proof Let $v \in V$. Since the π-stabilizer of v in G is open there is an open normal subgroup K_0 of K that is contained in this stabilizer. As in Lemma 3.4 we have the representation $(\pi_{K/K_0}, V^{K_0})$ of the finite group K/K_0. By what we have shown above

$$v \in V^{K_0} = \bigoplus_{\sigma \in \Pi(K/K_0)} V^{K_0}(\sigma) = \bigoplus_{\sigma \in \Pi(K/K_0)} V(\sigma).$$

This proves $V = \sum_{\sigma \in \Pi(K)} V(\sigma)$.

Now we prove that this sum of subspaces is direct. Let S be a finite non-empty subset of $\Pi(K)$. Suppose for each $\sigma \in S$ we are given $v_\sigma \in V(\sigma)$ such that

$$\sum_{\sigma \in S} v_\sigma = 0.$$

Define $K_0 = \cap\{ker(\sigma) : \sigma \in S\}$. K_0 is an open normal subgroup of K. Since $\sigma \in S$ implies $K_0 \subseteq ker(\sigma)$ we may consider S as a subset of $\Pi(K/K_0)$. Again using the identity

$$V^{K_0} = \bigoplus_{\sigma\in\Pi(K/K_0)} V^{K_0}(\sigma) = \bigoplus_{\sigma\in\Pi(K/K_0)} V(\sigma)$$

we see that we must have $v_\sigma = 0$ for every $\sigma \in S$. Hence $V = \bigoplus_{\sigma\in\Pi(K)} V(\sigma)$.

It remains to prove the second assertion. Suppose first that the representation (π, V) is admissible. Let $\sigma \in \Pi(K)$ and suppose W is a π_K-invariant subspace of V such that the resulting representation of K on W is in σ. Let us denote this representation of K by (π_W, W). By definition $ker(\sigma) = ker(\pi_W)$. Thus if $g \in ker(\sigma)$ and $v \in W$ then $\pi(g)v = \pi_W(g)v = I_W(v) = v$, and hence $W \subseteq V^{ker(\sigma)}$. Since W was arbitrary this proves that $V(\sigma) \subseteq V^{ker(\sigma)}$. Theorem 2.9 implies that $ker(\sigma)$ is open in K. Since (π, V) is admissible we must have that the space $V^{ker(\sigma)}$ is finite dimensional. Thus $V(\sigma)$ is finite dimensional as well.

Conversely suppose that (π, V) is a smooth representation that is not admissible. There is an open subgroup K_0 of K such that V^{K_0} is infinite dimensional. By Proposition 2.5.3 we may assume that K_0 is normal in K. As above we know

$$V^{K_0} = \bigoplus_{\sigma\in\Pi(K/K_0)} V(\sigma).$$

Since $\Pi(K/K_0)$ is a finite set this is a finite direct sum. Therefore as V^{K_0} is infinite dimensional there must be a $\sigma \in \Pi(K/K_0) \subset \Pi(K)$ such that $V(\sigma)$ is infinite dimensional.

□

The Schur Lemma for Irreducible Admissible Representations

Again fix a locally profinite group G. The following result is Schur's lemma for irreducible admissible representations of G.

Theorem 6 Suppose (π, V) is an irreducible admissible representation of G and $R \in End(V)$ is an intertwining isomorphism of (π, V) with itself. Then necessarily R is a scalar operator, namely there is a $c \in \mathbb{C}$ such that $R = cI_V$ where I_V is the identity operator on V.

Proof We first show that there exists a compact open subgroup K of G such that V^K is non-zero. Indeed we can fix any non-zero vector in V and define K to be any compact open subgroup of G which is contained in the open π-stabilizer of this vector in G. Fix such a subgroup K. Observe that V^K is an invariant subspace of R. Indeed if $v \in V^K$ then for $g \in K$ we have $\pi(g)(R(v)) = R(\pi(g)(v)) = R(v)$ and hence $R(v) \in V^K$. Thus the restriction of R to V^K yields an element of $End(V^K)$. Since (π, V) is admissible V^K is a finite dimensional space. Thus R restricted to V^K has an eigenvalue. Let $c \in \mathbb{C}$ be such an eigenvalue and $0 \neq v \in V^K$ a corresponding eigenvector so that $R(v) = cv$. Consider the subspace $W := ker(R - cI)$. Note that W is a π-invariant subspace of V. Indeed if $w \in W$ then $(R - cI)(\pi(g)w) = R\pi(g)w - c\pi(g)w = \pi(g)((R - cI)(w)) = \pi(g)(0) = 0$ and hence $\pi(g)w \in W$. By the irreducibility of (π, V) we must have $W = 0$ or $W = V$. Since $0 \neq v \in W$ we conclude $W = V$ and thus $R = cI$ as claimed.

□

Corollary 7 Let G be an abelian locally profinite group. Then any irreducible admissible representation of G is necessarily one dimensional. Hence $\Pi(G)$ is the group of characters of G.

The Contragredient Representation

This section is based on [2] and [4] where proofs of the results quoted can be found. Throughout let G be a locally profinite group. We will use V to denote complex vector spaces. The vector space of linear functionals on V is denoted by V^*. This is not to be confused with the Pontryagin dual of an abelian Haar group which also uses a star as a superscript. Since V has no topology we know V^* can not denote the Pontryagin dual so there is no chance of confusion.

Let (π, V) be a representation of G. We can associate to (π, V) a representation $(\tilde{\pi}, V^*)$ of G on V^* by defining

$$(\tilde{\pi}(g)v^*)(v) := v^*(\pi(g^{-1})v)$$

where $g \in G$, $v^* \in V^*$, $v \in V$.

Definition 8 Let (π, V) be a representation of G and let $v^* \in V^*$ be a linear functional on V. We say that v^* is a *smooth functional* with respect to π iff there exists an open subgroup M of G such that $v^*(\pi(g)v) = v^*(v)$ for all $g \in M$ and $v \in V$. The set of functionals that are smooth with respect to π form a subspace of V^* which we denote by $\tilde{V}$.

Definition 9 Let (π, V) be a smooth representation tof G. Then $\tilde{V}$ is a $\tilde{\pi}$-invariant subspace of V^*, and hence we obtain a representation $(\tilde{\pi}, \tilde{V})$ of G. The representation $(\tilde{\pi}, \tilde{V})$ is called the *contragredient representation* of (π, V). It is easily checked that $(\tilde{\pi}, \tilde{V})$ is smooth.

See [4] for more on the next lemma.

Lemma 10 If (π, V) is an irreducible admissible representation of G then $v^* \in V^*$ is smooth iff v^* is zero on the subspace $V(\sigma)$ for almost every $\sigma \in \Pi(K)$.

See [2] and [4] regarding the next proposition.

Proposition 11 Let (π, V) be an admissible representation of G so that

$$V = \bigoplus_{\sigma \in \Pi(K)} V(\sigma)$$

and each $V(\sigma)$ is finite dimensional. Then we have $\tilde{V} \simeq \bigoplus_{\sigma \in \Pi(K)} V(\sigma)^*$. Moreover for any σ the spaces $\tilde{V}(\sigma)$ and $V(\tilde{\sigma})^*$ are isomorphic. Consequently $(\tilde{\pi}, \tilde{V})$ is admissible.

Define the non-degenerate bilinear form $< \cdot , \cdot >: V \times V^* \longrightarrow \mathbb{C}$ by $< v, v^* > := v^*(v)$. Then by definition we have

$$< \pi(g)v, \tilde{\pi}(g)\tilde{v} > = < v, \tilde{v} >$$

for all $g \in G$, $v \in V$, and $\tilde{v} \in \tilde{V}$. Irreducibility and the existence of such a non-degenerate bilinear form characterizes the form as the next result shows. Its proof is an easy consequence of Theorem 3.6. The details are in [1].

Lemma 12 Let (π_1, V_1) and (π_2, V_2) be irreducible admissible representations of G. Suppose $A, B : V_1 \times V_2 \longrightarrow \mathbb{C}$ are two maps which are linear in the first variable and either both linear or both conjugate linear in the second variable. Suppose moreover that A and B are non-degenerate and for all $g \in G$, $v_1 \in V_1$, $v_2 \in V_2$ we have

$$A(\pi_1(g)v_1, \pi_2(g)v_2) = A(v_1, v_2)$$

and

$$B(\pi_1(g)v_1, \pi_2(g)v_2) = B(v_1, v_2).$$

Then there is a $c \in \mathbb{C}^\times$ such that $B = cA$.

Definition 13 Let (π, V) be an admissible representation of G. If W is a subspace of V we define

$$W^\perp := \{\tilde{v} \in \tilde{V} : \quad \tilde{v}(w) = 0 \quad \text{for all} \quad w \in W\}.$$

The following is easy to prove.

Proposition 14 Let (π, V) be an admissible representation of G. If W is a π-invariant subspace of V then $W^\perp$ is a $\tilde{\pi}$-invariant subspace of $\tilde{V}$. We obtain a bijection between the subrepresentations of (π, V) and the subrepresentations of $(\tilde{\pi}, \tilde{V})$. In particular if (π, V) is irreducible then $(\tilde{\pi}, \tilde{V})$ is also irreducible.

If (π_1, V_1) and (π_2, V_2) are equivalent smooth representations then $(\tilde{\pi}_1, \tilde{V}_1)$ and $(\tilde{\pi}_2, \tilde{V}_2)$ are equivalent smooth representations. Hence to any equivalence class π of smooth representations there is an associated equivalence class $\tilde{\pi}$ which we call the *contragredient class.* In particular $\tilde{\pi}$ is defined for every $\pi \in \Pi(G)$ and moreover $\tilde{\pi} \in \Pi(G)$. Hence we obtain a map

$$\tilde{} : \Pi(G) \longrightarrow \Pi(G) \qquad \pi \longmapsto \tilde{\pi}.$$

The next result follows from Proposition 3.11.

Proposition 15 If $\pi \in \Pi(G)$ then $\tilde{\tilde{\pi}} = \pi$.

Chapter 4 Background in the Theory of Local Fields

Some Structure Theory of Local Fields

It is not our intention to develop the theory of local fields. Rather we wish only to mention the parts of the theory that will play a role in what we wish to do. Proofs of the results stated in this section can be found in [3] and [9].

Any absolute value on a field provides a metric on the field, and so we may speak of Cauchy sequences and completeness. The topology given by the metric makes the field into a topological field. We say that two absolute values on a field are *equivalent* iff they generate the same topology.

Recall that an absolute value $|\cdot|$ on a field F is said to satisfy the *ultrametric inequality* iff for all $x, y \in F$ we have $|x+y| \leq min\{|x|, |y|\}$. An absolute value is said to be *non-archimedean* or *archimedean* according to whether it satisfies or does not satisfy the ultrametric inequality.

Definition 1 A *local field* F is a field with an absolute value $|\cdot|$ that gives F a non-discrete, complete, locally compact topology. We say that a local field is *archimedean* or *non-archimedean* according to whether the absolute value $|\cdot|$ is archimedean or non-archimedean.

In these notes we shall deal only with non-archimedean local fields. The theory we develop later on can also be developed for archimedean local fields.

Let $(F, |\cdot|)$ be a non-archimedean local field. We will often write F^+ when we wish to emphasize that we are only considering the abelian group structure of the field F. The topology on F is totally disconnected. So F^+ and $F^\times$ are totally disconnected abelian Haar groups, or equivalently they are abelian locally profinite groups. We define

$$\begin{aligned} \mathcal{O}_F &:= \{x \in F : |x| \leq 1\} \\ \mathcal{P}_F &:= \{x \in F : |x| < 1\} \\ \mathcal{U}_F &:= \{x \in F : |x| = 1\}. \end{aligned}$$

If there is no chance of confusion we suppress the subscript F and just write $\mathcal{O}, \mathcal{P}, \mathcal{U}$. The sets $\mathcal{O}$, $\mathcal{P}$, and $\mathcal{U}$ only depend on the equivalence class of the absolute value $|\cdot|$. $\mathcal{O}$ is the maximal compact subring of F and $\mathcal{O}$ is open in F. $\mathcal{O}$ is an integral domain and has F for its field of fractions. $\mathcal{P}$ is the unique non-zero prime ideal of $\mathcal{O}$, and so $\mathcal{O}$ is a discrete valuation ring. $\mathcal{P}$ is open and compact in F. The group of units of the ring $\mathcal{O}$ is $\mathcal{U}$, that is $\mathcal{O}^\times = \mathcal{U}$. $\mathcal{U}$ is an open compact subgroup of $F^\times$. The relative topologies on each of these subgroups are totally disconnected. $\mathcal{O}$ and $\mathcal{U}$ are abelian profinite groups, and so in particular are compact abelian Haar groups. Every ideal of the ring $\mathcal{O}$ is of the from $\mathcal{P}^n$ for some $n \geq 0$.

A *fractional ideal* I of $\mathcal{O}$ is a non-zero $\mathcal{O}$-submodule of F such that for some $x \in F^\times$ we have $xI \subseteq \mathcal{O}$. When we say *ideals* of F we mean fractional ideals of $\mathcal{O}$. The ideals of F are precisely the $\mathcal{O}$-submodules $\mathcal{P}^n$ for $n \in \mathbb{Z}$. The ideals of F that are contained in $\mathcal{O}$ are precisely the non-zero ideals of $\mathcal{O}$. Since $F = \cup_{n\in\mathbb{Z}}\mathcal{P}^n$ we see that F^+ is the union of its compact subgroups. If a subgroup of $F^\times$ is not contained in $\mathcal{U}$ then it is unbounded in $F^\times$ and hence not compact. Thus $F^\times$ is not the union of its compact subgroups.

Since $\mathcal{O}$ is a discrete valuation ring we must have that $\mathcal{P}$ is a maximal ideal of $\mathcal{O}$ and hence $\mathcal{O}/\mathcal{P}$ is a field. Since the $|\cdot|$-topology on F is locally compact it follows that $\mathcal{O}/\mathcal{P}$ is a finite field. We call $\mathcal{O}/\mathcal{P}$ the *residue field* of F and its cardinality will always be denoted by q.

Any element of $\mathcal{O}$ which generates the prime ideal $\mathcal{P}$ of $\mathcal{O}$ is called a *uniformizing parameter* of F. Such elements will always be denoted by $\tilde{\omega}$. Hence we have the identity $\mathcal{P} = (\tilde{\omega}) = \tilde{\omega}\mathcal{O}$, where the second equality is by definition. If we fix a uniformizing parameter $\tilde{\omega}$ then any element a of $F^\times$ is uniquely expressible in the form $\epsilon\tilde{\omega}^k$ for some $\epsilon \in \mathcal{U}$ and $k \in \mathbb{Z}$.

The only archimedean local fields up to valued field isometry are $\mathbb{R}$ and $\mathbb{C}$. Hence archimedean local fields are necessarily of characteristic 0. To classify non-archimedean fields we need to distinguish between the cases of characteristic 0 and positive characteristic. We have that F is a non-archimedean local field of characteristic 0 iff F is isomorphic to a finite extension of $\mathbb{Q}_p$ for some prime $p \neq \infty$ of $\mathbb{Q}$. Moreover in this case $\mathcal{O}$ is the integral closure of $\mathbb{Z}_p$ in F. In the other case we have that F is a non-archimedean local field of characteristic p iff F is isomorphic to the field of fractions $\mathbb{F}_q((T))$ of the integral domain $\mathcal{O} = \mathbb{F}_q[[T]]$ of formal power series in the indeterminate T with coefficients in the finite field $\mathbb{F}_q$ of q elements, where q is a power of p.

The locally constant complex valued functions on a non-archimedean local field are the proper analogue in the non-archimedean setting of the notion of C^∞ functions on $\mathbb{R}$ or $\mathbb{C}$ in the case of the archimedean fields. Consequently the analysis in the non-archimedean case is considerably simpler than in the archimedean case.

For a complex vector space X we will find it convenient to abbreviate $\mathcal{S}_X(F^\times)$ by $\mathcal{S}_X$. So in our notation $\mathcal{S}_\mathbb{C} = \mathcal{S}(F^\times)$ and we shall usually prefer the latter.

Let $\tilde{\omega}$ be a unifomizing parameter of F. Define the surjective homomorphism of groups $val : F^\times \longrightarrow \mathbb{Z}$ by $val(\epsilon\tilde{\omega}^k) = k$ where $k \in \mathbb{Z}$ and $\epsilon \in \mathcal{U}$. This homomorphism is independent of the choice of $\tilde{\omega}$ used to define it, and is called the *valuation* of F. Note that $val(0)$ has not been defined and we will keep it that way. We have the short exact sequence

$$1 \longrightarrow \mathcal{U} \longrightarrow F^\times \longrightarrow \mathbb{Z} \longrightarrow 0,$$

where the map from $F^\times$ to $\mathbb{Z}$ is the valuation homomorphism val.

Given a local field F there is a whole equivalence class of absolute values on F which all realize the topology of the local field. Amongst these absolute values there is a canonical choice. This absolute value, which we will always use, is the unique one defined by the condition that $\lambda(aE) = |a|\lambda(E)$ where λ is any Haar measure of F^+, E is any Borel subset of F, and $a \in F^\times$. This is the same as to say that for any $f \in \mathcal{S}(F)$ we have

$$\int_E f(x)\,d\lambda(x) = |a| \int_{a^{-1}E} f(ax)\,d\lambda(x).$$

We call this the *normalized* or *canonical* absolute value of F, and it will be denoted by $|\cdot|_F$. When there is no chance of confusion we abbreviate $|\cdot|_F$ to $|\cdot|$. The absolute value $|\cdot|_\mathbb{R}$ is the usual absolute value on $\mathbb{R}$, and $|\cdot|_\mathbb{C}$ is the square of the usual absolute value on $\mathbb{C}$. When F is non-archimedean this canonical absolute value is given by the formula

$$|\cdot| = q^{-val(\cdot)}.$$

A proof of this result can be found in **[9]**.

Throughout the rest of these notes F will denote a non-archimedean local field and $|\cdot|$ will denote the normalized absolute value of F. We now state some basic properties of F, the proofs being found in **[3]** or **[9]**.

(1) $F^\times \simeq \mathbb{Z} \times \mathcal{U}$

(2) For $k \in \mathbb{N}$ we have $(1+\mathcal{P}^k)/(1+\mathcal{P}^{k+1}) \simeq \mathcal{P}^k/\mathcal{P}^{k+1}$. The isomorphism is induced from the surjective homomorphism : $1+\mathcal{P}^k \longrightarrow \mathcal{P}^k/\mathcal{P}^{k+1} \quad 1+x \longmapsto x+\mathcal{P}^{k+1}$.

(3) For $k \in \mathbb{N}$ we have $\mathcal{U}/(1+\mathcal{P}^k) \simeq (\mathcal{O}/\mathcal{P}^k)^\times$. The isomorphism is induced by the surjective homomorphism : $\mathcal{U} \longrightarrow (\mathcal{O}/\mathcal{P}^k)^\times \quad \epsilon \longmapsto \epsilon + \mathcal{P}^k$.

(4) For any $k \in \mathbb{Z}$ we have $\mathcal{O}/\mathcal{P} \simeq \mathcal{P}^k/\mathcal{P}^{k+1}$. The isomorphism is induced by the surjective homomorphism : $\mathcal{O} \longrightarrow \mathcal{P}^k/\mathcal{P}^{k+1} \quad x \longmapsto x\tilde{\omega}^k + \mathcal{P}^{k+1}$.

(5) For any $r, k \in \mathbb{N}$

$$|\mathcal{P}^k/\mathcal{P}^{k+r}| = q^r = |(1+\mathcal{P}^k)/(1+\mathcal{P}^{k+r})|.$$

(6) $(\mathcal{U} : 1+\mathcal{P}^m) = |(\mathcal{O}/\mathcal{P}^m)^\times| = |\mathcal{O}/\mathcal{P}^m| - |\mathcal{P}/\mathcal{P}^m| = q^m - q^{m-1} = q^{m-1}(q-1)$.

(7) For every $n \in \mathbb{N}$ the quotient group $F^\times/(F^\times)^n$ is finite.

The Haar measure on F^+ that assigns $\mathcal{O}$ measure 1 is called the normalized Haar measure of F, and it will be denoted by λ^+. The Haar measure on $F^\times$ that assigns $\mathcal{U}$ measure 1 is called the normalized Haar measure of $F^\times$, and it will be denoted by $\lambda^\times$. Thus $\lambda^+(\mathcal{O}) = 1$ and $\lambda^\times(\mathcal{U}) = 1$. When integrating functions with respect to these measures we abbreviate $d\lambda^+$ to d^+ and $d\lambda^\times$ to $d^\times$. Some identities which may be useful are

(1) $\lambda^\times(1+\mathcal{P}^k) = (\mathcal{U} : 1+\mathcal{P}^k)^{-1} = (q^{k-1}(q-1))^{-1}$ for $k \in \mathbb{N}$
(2) $\lambda^\times(val^{-1}(k)) = \lambda^\times(\mathcal{U}) = 1$ for $k \in \mathbb{Z}$
(3) $\lambda^+(\mathcal{P}^k) = q^{-k}$ for $k \in \mathbb{Z}$
(4) $\lambda^+(val^{-1}(k)) = \lambda^+(\mathcal{P}^k) - \lambda^+(\mathcal{P}^{k+1}) = q^{-k} - q^{-(k+1)} = q^{-k}(1-q^{-1})$ for $k \in \mathbb{Z}$
(5) $\lambda^+(1+\mathcal{P}^k) = \lambda^+(\mathcal{P}^k) = q^{-k}$ for $k \in \mathbb{N}$

For example we can prove $\lambda^+(\mathcal{U}) = 1 - q^{-1}$ by noting

$$\frac{1}{q} = \frac{1}{(\mathcal{O} : \mathcal{P})} = \int_{\mathcal{P}} 1\, d^+x = \int_{\mathcal{O}} 1\, d^+x - \int_{\mathcal{U}} 1\, d^+x = 1 - \lambda^+(\mathcal{U}).$$

Proposition 2 For any Haar measure α of F^+ we have the identity

$$d^\times x = \alpha(\mathcal{U})^{-1}|x|^{-1}d\alpha(x).$$

This means for all $f \in \mathcal{S}(F^\times)$ we have

$$\int_{F^\times} f(x)\, d^\times x = \alpha(\mathcal{U})^{-1} \int_{F^\times} f(x)|x|^{-1}\, d\alpha(x).$$

Proof Define a linear functional $I : \mathcal{S}(F^\times) \longrightarrow \mathbb{C}$ by

$$I(f) := \int_{F^\times} f(x)|x|^{-1}\, d\alpha(x)$$

Then for any $a \in F^\times$ and $f \in \mathcal{S}(F^\times)$ we have

$$I(f(a\,\cdot)) = \int_{F^\times} f(ax)|x|^{-1}\, d\alpha(x) = |a|^{-1}\int_{F^\times} f(x)|a^{-1}x|^{-1}\, d\alpha(x) = I(f).$$

Thus $|x|^{-1}d\alpha(x)$ provides a Haar measure on $F^\times$. Hence there is a $c \in (0, \infty)$ such that $d^\times x = c|x|^{-1}d\alpha(x)$. We can explicitly determine c by the calculation

$$1 = \int_{\mathcal{U}} 1\, d^\times x = c \int_{\mathcal{U}} 1\, d\alpha(x) = c\alpha(\mathcal{U}).$$

Thus $c = \alpha(\mathcal{U})^{-1}$ and the proof is complete.

□

Character Theory of $\mathbf{F}^+$ and $\mathbf{F}^\times$

By Corollary 2.13 we know that a character of any of the groups $\mathcal{U}$, F^+ or of $F^\times$ has an open kernel and hence is locally constant. Since $\mathcal{U}$ is a compact group any character of $\mathcal{U}$ is necessarily unitary. Since F^+ is the union of its compact subgroups $\mathcal{P}^n$ for $n \in \mathbb{Z}$, we know any character of F^+ is also necessarily unitary. In spite of this fact we will still be explicit and say "unitary character of F^+" even though the "unitary" is redundant. On the contrary there certainly exist non-unitary characters of $F^\times$. Indeed the restriction of the absolute value $|\cdot|$ of F to $F^\times$ provides an example of such a character. If χ is a character of $F^\times$ then clearly χ and χ^{-1} take the same value at $-1 \in F^\times$.

For every $n \in \mathbb{N}$ the group $\mathcal{U}^*$ contains an isomorphic copy of $(\mathcal{U}/(1+\mathcal{P}^n))^*$. Hence $\mathcal{U}^*$ is an infinite group. Also since $\mathcal{U}$ is compact we know by Pontryagin duality that $\mathcal{U}^*$ is discrete.

Definition 3

(1) Let $\psi \in F^*$ and $\psi \neq 1$. Since $ker(\psi)$ is open in F there exists an integer l such that $\mathcal{P}^l \subseteq ker(\psi)$. Since $\psi \neq 1$ there is a smallest integer l with this property, and it is called the *conductor* of ψ. We interchangeably refer to the conductor as both the integer l and the ideal $\mathcal{P}^l$. We do not define the conductor of the trivial character of F^+.

(2) Let $\mu \in \mathcal{U}^*$ and $\mu \neq 1$. Since $ker(\mu)$ is open in $\mathcal{U}$ there exists a $n \in \mathbb{N}$ such that $1+\mathcal{P}^n \subseteq ker(\mu)$. The smallest $n \in \mathbb{N}$ with this property is called the *conductor* of μ. When $\mu = 1$ we define the conductor of μ to be 0. When $n > 0$ we will think of the conductor both as the integer n and as the subgroup $1+\mathcal{P}^n$.

(3) If χ is a character of $F^\times$ then the restriction of χ to $\mathcal{U}$ will be denoted by χ_0, so that $\chi_0 \in \mathcal{U}^*$. We define the *conductor* of χ to be the conductor of χ_0.

We will denote the conductor of μ, χ or ψ by $cond(\mu)$, $cond(\chi)$, or $cond(\psi)$ respectively. When we use the notation $cond(\cdot)$ we are thinking of the conductor as an integer. Even though $l = cond(\psi)$ is the smallest integer with the property that $\mathcal{P}^l \subseteq ker(\psi)$, it is not true in general that $\mathcal{P}^l = ker(\psi)$. A similar remark applies to characters of $\mathcal{U}$, and hence also to characters of $F^\times$. Note that if $\chi \neq 1$, $\psi \neq 1$ and we define $l := cond(\psi)$ and $n := cond(\chi)$ then ψ is constant on the cosets of $\mathcal{P}^l$ in F and χ is constant on the cosets of $1+\mathcal{P}^n$ in $F^\times$. When μ is a character of $\mathcal{U}$ the same reason shows μ is constant on the cosets of $1+\mathcal{P}^n$ in $\mathcal{U}$, where $n := cond(\mu)$, and hence μ descends to a (unitary) character of the finite group $\mathcal{U}/1+\mathcal{P}^n$. Observe that for $b \in F^\times$ and $1 \neq \psi \in F^*$ we have the identity

$$cond(\psi(b\,\cdot)) = cond(\psi) - val(b).$$

Definition 4 A non-trivial unitary character ψ of F is said to be *unramified* iff the conductor of ψ is $\mathcal{O} = \mathcal{P}^0$. Otherwise we say ψ is *ramified*. A character χ of $F^\times$ is said to be *unramified* iff it is trivial on $\mathcal{U}$, and otherwise we say χ is *ramified*. Hence ψ or χ is unramified ifff its conductor is 0.

The absolute value $|\cdot|$ of F when restricted to $F^\times$ gives an example of an unramified non-unitary character of $F^\times$. Observe that if χ is an unramified character of $F^\times$ then $\chi(x) = \chi(y)$ whenever $val(x) = val(y)$. In particular the value $\chi(\tilde{\omega})$ does not depend on the choice of uniformizing parameter $\tilde{\omega}$. We will denote the value of an unramified character χ on the subset $val^{-1}(1)$ by $\chi(\mathcal{P})$, namely the value of χ at any uniformizing parameter.

Note that if χ is an unramified character of $F^\times$ and $\tilde{\omega}$ is a uniformizing parameter of F then for all $x \in F^\times$ we have $\chi(\tilde{\omega})^{val(x)} = \chi(x)$. Hence in particular an unramified character of $F^\times$ is determined by its value on a uniformizing parameter.

See [3] for details on the next result.

Proposition 5 For any fixed $\psi \in F^*$, $\psi \neq 1$ the map

$$: F \longrightarrow F^* \quad b \longmapsto \psi(b\cdot)$$

is an isomorphism of topological groups.

The groups $GL_1(F)$ and $F^\times$ are isomorphic locally profinite groups. Therefore both of $\Pi(GL_1(F))$ and $\Pi(F^\times)$ are defined and they are equal. Since $F^\times$ is abelian $\Pi(F^\times)$ is the group of characters of $F^\times$ and has $(F^\times)^*$ as a subgroup.

Definition 6 Let χ be a character of $F^\times$. We first define the positive character $|\chi|$ of $F^\times$ by

$$|\chi| : F^\times \longrightarrow (0, \infty) \qquad |\chi|(a) := |\chi(a)|_0$$

where $|\cdot|_0$ denotes the usual absolute value on $\mathbb{C}$. Secondly define the unitary character χ_u of $F^\times$ by $\chi_u := \frac{\chi}{|\chi|} = \chi|\chi|^{-1}$. Finally define

$$\alpha(\chi) := \frac{log|\chi(\tilde{\omega})|_0}{log|\tilde{\omega}|} \in \mathbb{R}$$

where $\tilde{\omega}$ is any uniformizing parameter of F, the particular choice not influencing the value of $\alpha(\chi)$.

Proposition 7 If χ is a character of $F^\times$ then $\chi = \chi_u|\cdot|^{\alpha(\chi)}$. χ is unitary iff $\alpha(\chi) = 0$. χ is unramified iff χ_u is unramified.

Proof We are required to prove that $|\chi| = |\cdot|^{\alpha(\chi)}$. Let $\tilde{\omega}$ be a uniformizing parameter of F^+. The definition of $\alpha(\chi)$ is equivalent to $|\chi(\tilde{\omega})|_0 = |\tilde{\omega}|^{\alpha(\chi)}$. Hence if $a \in F^\times$ and we write $a = \epsilon\tilde{\omega}^k$ for some $\epsilon \in \mathcal{U}$, $k \in \mathbb{Z}$ then $|\chi|(a) = |\chi(a)|_0 = |\chi(\epsilon)\chi(\tilde{\omega})^k|_0 = |\tilde{\omega}|^{k\alpha(\chi)} = |a|^{\alpha(\chi)}$. The assertions in the last two sentences are obvious.

□

The next result is trivial.

Proposition 8

(1) The map : $\Pi(F^\times) \longrightarrow (F^\times)^* \times \mathbb{R}^+$ defined by $\chi \longrightarrow (\chi_u, \alpha(\chi))$ is an isomorphism of groups.

(2) For any uniformizing parameter $\tilde{\omega}$ of F the map : $\Pi(F^\times) \longrightarrow \mathcal{U}^* \times \mathbb{C}^\times$ defined by $\chi \longmapsto (\chi_0, \chi(\tilde{\omega}))$ is an isomorphism of groups.

Proposition 9

(1) Every unramified character of $F^\times$ is of the form $|\cdot|^s$ for some $s \in \mathbb{C}$. Hence any unramified unitary character of $F^\times$ is of the form $|\cdot|^{it}$ for some $t \in \mathbb{R}$.

(2) Every unramified character of $F^\times$ is of the form $exp(s\, val(\cdot))$ for some $s \in \mathbb{C}$.

Proof Let $\tilde{\omega}$ be a uniformizing parameter of F. Choose $s \in \mathbb{C}$ such that $\chi(\tilde{\omega}) = |\tilde{\omega}|^s$. Then for any $x = \epsilon\tilde{\omega}^k$ in $F^\times$ we have $\chi(x) = \chi(\epsilon\tilde{\omega}^k) = |\tilde{\omega}|^{ks} = |x|^s$. Enough said about (1). To prove (2) choose $s \in \mathbb{C}$ such that $\chi(\tilde{\omega}) = exp(s)$. Then if $x = \epsilon\tilde{\omega}^k \in F^\times$ we have $\chi(x) = \chi(\tilde{\omega})^k = exp(ks) = exp(s\, val(x))$.

□

Fourier Analysis in Non-Archimedean Local Fields

For this section the reader may consult [9] for more details. Let α be a Haar measure on F^+. Then for $f \in \mathcal{S}(F)$ the Fourier transform of f with respect to α is the function

$$\hat{f}_\alpha : F^* \longrightarrow \mathbb{C} \qquad \hat{f}_\alpha(\psi) = \int_F f(x)\psi(x)\, d\alpha(x).$$

The Fourier inversion formula states that if α is any Haar measure of F^+, and α^* is the Haar measure on F^* that is dual to α then for $x \in F$

$$f(-x) = \int_{F^*} \hat{f}_\alpha(\psi)\psi(x)\, d\alpha^*(\psi).$$

By Proposition 4.5 if $1 \neq \psi \in F^*$ then the map $: F \longrightarrow F^* \quad b \longmapsto \psi(b\,\cdot)$ is an isomorphism of topological groups. This map transforms any Haar measure of F^+ to a Haar measure on F^*. It allows us to replace the integral over F^* in the Fourier inversion formula with an integral over F. This is done as follows. To each $1 \neq \psi \in F^*$ we can associate a unique Haar measure λ_ψ on F^+, which is transformed under $b \longmapsto \psi(b\,\cdot)$ to the dual measure $(\lambda_\psi)^*$ of F^*. We will abbreviate $d\lambda_\psi(x)$ to $d_\psi x$. The Fourier inversion formula now says that if for $f \in \mathcal{S}(F)$ we define

$$\hat{f}_\psi : F \longrightarrow \mathbb{C} \qquad \hat{f}_\psi(y) = \int_F f(x)\psi(xy)\, d_\psi(x)$$

then for $x \in F$ we have

$$f(-x) = \int_F \hat{f}_\psi(y)\psi(xy)\, d_\psi(y).$$

The next result is not difficult to prove.

Proposition 10 For any $1 \neq \psi \in F^*$ the map $: \mathcal{S}(F) \longrightarrow \mathcal{S}(F) \quad f \longmapsto \hat{f}_\psi$ is in $GL(\mathcal{S}(F))$.

See [4] regarding the next result.

Proposition 11 Let ψ be a non-trivial unitary character of F^+. Then λ_ψ is the Haar measure of F^+ determined by the property that it assigns $\mathcal{O}$ measure $q^{l/2}$ where $l := cond(\psi)$. Thus $\lambda_\psi = q^{l/2}\lambda^+$.

Suppose X is a non-zero complex vector space. We need to define the Fourier transform of functions in $\mathcal{S}_X(F)$. Let ψ be a non-trivial unitary character of F^+. For $f \in \mathcal{S}_X(F)$ the Fourier transform of f with respect to ψ is denoted by $\hat{f}_\psi$, and is defined analogously by

$$\hat{f}_\psi : F \longrightarrow X \quad a \longmapsto \int_F f(x)\psi(ax)\, d_\psi x.$$

If $f \in \mathcal{S}_X(F)$ then $\hat{f}_\psi \in \mathcal{S}_X(F)$, and the map $f \longmapsto \hat{f}_\psi$ is in $GL(\mathcal{S}_X(F))$.

Gaussian Sums In Non-Archimedean Local Fields

For $a \in F$, $\mu \in \mathcal{U}^*$, and $\psi \in F^*$ define the *Gaussian sum* by

$$\gamma(a, \mu, \psi) = \int_{\mathcal{U}} \mu(\epsilon)\psi(a\epsilon)\, d^\times \epsilon.$$

Since $\epsilon \longmapsto \mu(\epsilon)\psi(a\epsilon)$ is a locally constant map on the compact space $\mathcal{U}$ this integral is really a finite sum. Observe that $\gamma(a, \mu, \psi) = \gamma(1, \mu, \psi(a\,\cdot))$, and that for $\epsilon \in \mathcal{U}$ we have $\gamma(\epsilon a, \mu, \psi) = \mu^{-1}(\epsilon)\gamma(a, \mu, \psi)$. The purpose of this section is to explicitly determine $\gamma(a, \mu, \psi)$, or at least determine its absolute value. We are going to need some prepatory lemmas.

Let $l \in \mathbb{Z}$ be arbitrary. If $\alpha \in F/\mathcal{P}^l$ and $\alpha \neq \mathcal{P}^l$ then there exists a unique $m \in \mathbb{Z}$ with $m < l$ such that $\alpha \subset val^{-1}(m)$. Therefore if $\alpha \in F/\mathcal{P}^l$ and $\alpha \neq \mathcal{P}^l$ then we may define $val(\alpha)$ to be this unique integer.

Suppose ψ is a non-trivial unitary character of F^+. Define $l := cond(\psi)$. Since ψ is constant on the cosets of $\mathcal{P}^l$ in F we have that ψ descends to a non-trivial unitary character of $F/\mathcal{P}^l$ or of $\mathcal{P}^m/\mathcal{P}^l$ for any $m < l$. We will use ψ again to denote any of these characters. So for any $\alpha \in F/\mathcal{P}^l$ we write $\psi(\alpha)$ to denote the constant value that ψ takes on the subset α of F.

Having made these remarks we may now state the first lemma.

Lemma 12 Let $\psi \in F^*$, $\psi \neq 1$ and define $l := cond(\psi) \in \mathbb{Z}$.

(1) For any $m \in \mathbb{Z}$ with $m < l$ we have

$$\begin{aligned}\sum_\alpha \psi(\alpha) &= 0 \quad \text{if} \quad m < l-1 \\ &= -1 \quad \text{if} \quad m = l-1\end{aligned}$$

where the summation is over all $\alpha \in F/\mathcal{P}^l$ with $val(\alpha) = m$.

(2) Suppose $n \in \mathbb{N}$ satisfies $n \geq l$. Let S be a set of representatives for the cosets of $1+\mathcal{P}^n$ in $\mathcal{U}$. For $s \in S$, $s \notin 1+\mathcal{P}^n$ we have

$$\begin{aligned}\sum_{t \in S} \psi((s-1)t) &= q^{n-1}(q-1) \quad \text{if} \quad l-1 < val(s-1) < n \\ &= -q^{n-1} \quad \text{if} \quad val(s-1) = l-1 \\ &= 0 \quad \text{if} \quad 0 \leq val(s-1) < l-1\end{aligned}$$

Proof **(1)** First note that

$$\sum_\alpha \psi(\alpha) = \sum_{\alpha \in \mathcal{P}^m/\mathcal{P}^l} \psi(\alpha) - \sum_{\alpha \in \mathcal{P}^{m+1}/\mathcal{P}^l} \psi(\alpha).$$

If $m < l-1$ then ψ is a non-trivial character of both $\mathcal{P}^m/\mathcal{P}^l$ and of $\mathcal{P}^{m+1}/\mathcal{P}^l$. Hence in this case both the sums on the right side of the above equality are equal to zero. This proves the assertion when $m < l-1$. Suppose now that $m = l-1$. In this case ψ is non-trivial on $\mathcal{P}^m/\mathcal{P}^l = \mathcal{P}^{l-1}/\mathcal{P}^l$ but of course ψ is trivial on $\mathcal{P}^{m+1}/\mathcal{P}^l = \mathcal{P}^l/\mathcal{P}^l$. Hence in this case $\sum_\alpha \psi(\alpha) = 0 - 1 = -1$, as desired.

(2) Fix $s \in S$, $s \notin 1 + \mathcal{P}^n$. For brevity define $k := val(s-1)$. Note that $0 \le k < n$. First suppose that $l - 1 < k < n$. In this case for any $t \in S$ we have $(s-1)t \in \mathcal{P}^k \subseteq \mathcal{P}^l$ and so $\psi((s-1)t) = 1$. Hence

$$\sum_{t \in S} \psi((s-1)t) = |S| = (\mathcal{U} : 1 + \mathcal{P}^n) = q^{n-1}(q-1).$$

Having dealt with this case we suppose now that $k \le l - 1$. Since $1 \le l - k \le n$ we see that $1 + \mathcal{P}^n$ is a subgroup of $1 + \mathcal{P}^{l-k}$. Define $r := (1 + \mathcal{P}^{l-k} : 1 + \mathcal{P}^n) = q^{n-(l-k)}$. We may partition S into r disjoint subsets $S_1, \ldots, S_r$ such that for each $j = 1, \ldots, r$ we have that S_j is a set of representatives of the cosets of $1 + \mathcal{P}^{l-k}$ in $\mathcal{U}$. Observe that for any $j = 1, \ldots, r$ we have

$$\{\alpha \in F/\mathcal{P}^l : val(\alpha) = k\} = \{(s-1)t + \mathcal{P}^l : t \in S_j\}.$$

Hence by part (1) of the lemma for any $j = 1, \ldots, r$ we have

$$\begin{aligned} \sum_{t \in S_j} \psi((s-1)t) &= 0 \quad \text{if} \quad k < l - 1 \\ &= -1 \quad \text{if} \quad k = l - 1 \end{aligned}$$

Therefore

$$\begin{aligned} \sum_{t \in S} \psi((s-1)t) = \sum_{j=1}^{r} \sum_{t \in S_j} \psi((s-1)t) &= 0 \quad \text{if} \quad k < l - 1 \\ &= -r \quad \text{if} \quad k = l - 1. \end{aligned}$$

The lemma now follows after noting that when $k = l - 1$ we have $-r = -q^{n-(l-k)} = -q^{n-1}$.

□

Lemma 13 Let $\mu \in \mathcal{U}^*$, $\mu \ne 1$ and define $n := cond(\mu) \in \mathbb{N}$. Let S be a set of representatives for the cosets of $1 + \mathcal{P}^n$ in $\mathcal{U}$. For $0 \le k < n$ we have

$$\begin{aligned} \sum_{s \in S \; val(s-1)=k} \mu(s) &= 0 \quad \text{if} \quad k < n - 1 \\ &= -1 \quad \text{if} \quad k = n - 1 \end{aligned}$$

Proof First suppose $n = 1$. Then necessarily $k = 0$. We have

$$0 = \sum_{s \in S} \mu(s) = 1 + \sum_{s \in S \; s \notin 1 + \mathcal{P}} \mu(s) = 1 + \sum_{s \in S \; val(s-1)=0} \mu(s).$$

This proves the lemma in the case $n = 1$ so we may assume that $n > 1$. Suppose first that $0 < k < n$. Observe that for any $s \in F$ we have $val(s-1) = k$ iff $s \in 1 + \mathcal{P}^k$ and $s \notin 1 + \mathcal{P}^{k+1}$. Hence

$$\begin{aligned} \sum_{s \in S \; val(s-1)=k} \mu(s) &= \sum_{s \in S \; s \in 1 + \mathcal{P}^k} \mu(s) - \sum_{s \in S \; s \in 1 + \mathcal{P}^{k+1}} \mu(s) \\ &= - \sum_{s \in S \; s \in 1 + \mathcal{P}^{k+1}} \mu(s) \\ &= 0 \quad \text{if} \quad k < n - 1 \\ &= -1 \quad \text{if} \quad k = n - 1 \end{aligned}$$

Now we deal with the case $k = 0$. Well by what we have just shown we find

$$\begin{aligned} 0 = \sum_{s \in S} \mu(s) &= 1 + \sum_{k=0}^{n-1} \sum_{s \in S\, val(s-1)=k} \mu(s) \\ &= 1 + \sum_{s \in S\, val(s-1)=0} \mu(s) - 1 \\ &= \sum_{s \in S\, val(s-1)=0} \mu(s) \end{aligned}$$

□

The following is the main result of this section.

Theorem 14 Let $a \in F$, $\psi \in F^{*}$, $\mu \in \mathcal{U}^{*}$.

(1) If $a = 0$ or $\psi = 1$ then $\gamma(a, \mu, \psi) = \delta(\mu)$ where $\delta(\mu)$ is defined to be 1 when $\mu = 1$ and to be 0 otherwise.

(2) If $a \neq 0$, $\psi \neq 1$ and $\mu = 1$ then

$$\begin{aligned} \gamma(a, 1, \psi) &= 1 \quad \text{if} \quad val(a) > l - 1 \\ &= (1-q)^{-1} \quad \text{if} \quad val(a) = l - 1 \\ &= 0 \quad \text{if} \quad val(a) < l - 1 \end{aligned}$$

where $l = cond(\psi)$.

(3) If $a \neq 0$, $\psi \neq 1$. and $\mu \neq 1$ then

$$\begin{aligned} |\gamma(a, \mu, \psi)| &= (1-q^{-1})^{-1} q^{-\frac{n}{2}} \quad \text{if} \quad val(a) = l - n \\ &= 0 \quad \text{if} \quad val(a) \neq l - n \end{aligned}$$

where $l = cond(\psi)$ and $n = cond(\mu)$.

Proof **(1)** If $a = 0$ or $\psi = 1$ then $\psi(a\epsilon) = 1$ for all $\epsilon \in \mathcal{U}$. Hence

$$\gamma(a, \mu, \psi) = \int_{\mathcal{U}} \mu(\epsilon)\, d^{\times}\epsilon = \delta(\mu).$$

(2) Suppose now that $a \neq 0$, $\psi \neq 1$ and $\mu = 1$. Define $k := val(a) \in \mathbb{Z}$. Then

$$\begin{aligned} \gamma(a, 1, \psi) = \int_{\mathcal{U}} \psi(a\epsilon)\, d^{\times}\epsilon = \int_{a\mathcal{U}} \psi(y)\, d^{\times} y \\ &= (1-q^{-1})^{-1} \int_{a\mathcal{U}} \psi(y) |y|^{-1}\, d^{+}y \\ &= (1-q^{-1})^{-1} |a|^{-1} \int_{a\mathcal{U}} \psi(y)\, d^{+}y \\ &= (1-q^{-1})^{-1} q^{k} \left(\int_{\mathcal{P}^{k}} \psi(y)\, d^{+}y - \int_{\mathcal{P}^{k+1}} \psi(y)\, d^{+}y \right) \end{aligned}$$

The first case is $k > l - 1$. We then have $\mathcal{P}^{k+1} \subset \mathcal{P}^{k} \subseteq \mathcal{P}^{l}$ and so

$$\gamma(a, 1, \psi) = (1-q^{-1})^{-1} q^{k} (q^{-k} - q^{-(k+1)}) = 1.$$

The next case is $k = l - 1$. Now we have $\mathcal{P}^l = \mathcal{P}^{k+1} \subset \mathcal{P}^k$ and so

$$\gamma(a, 1, \psi) = (1 - q^{-1})^{-1} q^k (0 - q^{-(k+1)}) = (1 - q)^{-1}.$$

The final case is $k < l - 1$. This time $\mathcal{P}^l \subset \mathcal{P}^{k+1} \subset \mathcal{P}^k$ and so

$$\gamma(a, 1, \psi) = (1 - q^{-1})^{-1} q^k (0 - 0) = 0.$$

(3) Suppose now that $a \neq 0$, $\psi \neq 1$, and $\mu \neq 1$. Since $cond(\psi(a \cdot)) = cond(\psi) - val(a)$ and $\gamma(a, \mu, \psi) = \gamma(1, \mu, \psi(a \cdot))$ we may suppose $a = 1$. So we need to prove that $l \neq n$ implies $\gamma(1, \mu, \psi) = 0$ and $l = n$ implies $|\gamma(1, \mu, \psi)| = (1 - q^{-1})^{-1} q^{-\frac{n}{2}}$. Let S be a set of representatives for the cosets of $1 + \mathcal{P}^n$ in $\mathcal{U}$. We may, and do, assume that $1 \in S$. Consider the calculation,

$$\begin{aligned}
\gamma(1, \mu, \psi) = \int_{\mathcal{U}} \mu(\epsilon)\psi(\epsilon)\, d^\times \epsilon &= \sum_{s \in S} \int_{1+\mathcal{P}^n} \mu(\epsilon s)\psi(\epsilon s)\, d^\times \epsilon \\
&= (1 - q^{-1})^{-1} \sum_{s \in S} \left(\mu(s) \int_{1+\mathcal{P}^n} \psi(sy)\, d^+ y \right) \\
&= (1 - q^{-1})^{-1} \sum_{s \in S} \left(\mu(s) \int_{\mathcal{P}^n} \psi(s(y+1))\, d^+ y \right) \\
&= (1 - q^{-1})^{-1} \sum_{s \in S} \left(\mu(s)\psi(s) \int_{\mathcal{P}^n} \psi(sy)\, d^+ y \right) \\
&= (1 - q^{-1})^{-1} \left(\int_{\mathcal{P}^n} \psi(y)\, d^+ y \right) \sum_{s \in S} \mu(s)\psi(s).
\end{aligned}$$

If $l > n$ then ψ is non-trivial on $\mathcal{P}^n$ and so

$$\int_{\mathcal{P}^n} \psi(y)\, d^+ y = 0.$$

Thus $l > n$ implies $\gamma(1, \mu, \psi) = 0$. So we may assume $l \leq n$. With this assumption in place ψ will then be trivial on $\mathcal{P}^n$, and consequently

$$\gamma(1, \mu, \psi) = (1 - q^{-1})^{-1} q^{-n} \sum_{s \in S} \mu(s)\psi(s).$$

Also if $l \leq 0$ then $\psi(\epsilon) = 1$ for all $\epsilon \in \mathcal{U}$ and so

$$\gamma(1, \mu, \psi) = \int_{\mathcal{U}} \mu(\epsilon)\, d^\times \epsilon = \delta(\mu) = 0.$$

Therefore we may assume $1 \leq l \leq n$. Now consider the calculation.

$$\begin{aligned}
|\sum_{s \in S} \mu(s)\psi(s)|^2 &= \left(\sum_{s \in S} \mu(s)\psi(s) \right) \left(\sum_{t \in S} \mu(t^{-1})\psi(-t) \right) \\
&= \sum_{t \in S} \sum_{s \in S} \mu(st^{-1})\psi(s - t) \\
&= \sum_{t \in S} \sum_{s \in S} \mu(s)\psi(st - t).
\end{aligned}$$

We must justify the last equality. First observe that if $t \in \mathcal{U}$ and $s, s' \in \mathcal{U}$ satisfy $s(1+\mathcal{P}^n) = s'(1+\mathcal{P}^n)$ then $\mu(st^{-1})\psi(s-t) = \mu(s't^{-1})\psi(s'-t)$. This shows that if S' is any other set of representatives of the cosets of $1+\mathcal{P}^n$ in $\mathcal{U}$ then for any $t \in \mathcal{U}$,

$$\sum_{s\in S} \mu(st^{-1})\psi(s-t) = \sum_{s'\in S'} \mu(s't^{-1})\psi(s'-t).$$

It is trivial to see that if $t \in \mathcal{U}$ then $S' := \{st : s \in S\}$ is a set of representatives of the cosets of $1+\mathcal{P}^n$ in $\mathcal{U}$. Hence for any $t \in S$,

$$\sum_{s\in S} \mu(st^{-1})\psi(s-t) = \sum_{s\in S} \mu(s)\psi(st-t).$$

This justifies the equality. Now we continue the above calculation.

$$\begin{aligned}
|\sum_{s\in S}\mu(s)\psi(s)|^2 &= \sum_{s\in S}\sum_{t\in S}\mu(s)\psi((s-1)t) \\
&= \sum_{t\in S}\mu(1)\psi((1-1)t) + \sum_{s\in S\, s\neq 1}\sum_{t\in S}\mu(s)\psi((s-1)t) \\
&= q^{n-1}(q-1) + \sum_{s\in S\, s\neq 1}\mu(s)\left(\sum_{t\in S}\psi((s-1)t)\right).
\end{aligned}$$

Here the last equality is seen by noting

$$\sum_{t\in S}\mu(1)\psi(0) = \sum_{t\in S} 1 = |S| = (\mathcal{U} : 1+\mathcal{P}^n) = q^{n-1}(q-1).$$

By the lemma proven at the beginning we know

$$|\sum_{s\in S}\mu(s)\psi(s)|^2 = q^{n-1}(q-1) + \sum_{k=l-1}^{n-1}\left(\sum_{s\in S\; val(s-1)=k}\mu(s)\left(\sum_{t\in S}\psi((s-1)t)\right)\right).$$

First suppose $l = n$. We then have

$$|\sum_{s\in S}\mu(s)\psi(s)|^2 = q^{n-1}(q-1) + \sum_{s\in S\; val(s-1)=l-1}\mu(s)(-q^{n-1}) = q^{n-1}(q-1) - q^{n-1}(-1) = q^n.$$

Hence $|\gamma(1,\mu,\psi)| = (1-q^{-1})^{-1}q^{-n}q^{\frac{n}{2}} = (1-q^{-1})^{-1}q^{-\frac{n}{2}}$ as claimed in the theorem. Now suppose that $l < n$. We finally conclude that the expression on the left equals

$$\begin{aligned}
& q^{n-1}(q-1) + \sum_{s\in S\; val(s-1)=l-1}\mu(s)(-q^{n-1}) + \sum_{k=l}^{n-1}\sum_{s\in S\; val(s-1)=k}\mu(s)(q^{n-1}(q-1)) \\
&= q^{n-1}(q-1) - q^{n-1}(0) + q^{n-1}(q-1)(-1) \\
&= 0
\end{aligned}$$

as desired.

□

Chapter 5 The Theory for $GL_2(F)$

The Group $GL_2(F)$

If A is a commutative ring with identity we write $M_2(A)$ for the ring of 2×2 matrices with coefficients in A. We define $GL_2(A) := (M_2(A))^\times$ to be the group of units of the ring $M_2(A)$. More explicitly,

$$GL_2(A) = \{g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(F) : det(g) = ad - bc \neq 0\}.$$

Consider $M_2(F)$ where F is a non-archimedean local field. Given $g, h \in M_2(F)$ and $n \in \mathbb{Z}$ we write $g \equiv h(mod\,\mathcal{P}^n)$ to mean that the matrix $g - h$ has its coefficients in $\mathcal{P}^n$. We now define a topology on $M_2(F)$. Given $g \in M_2(F)$ a basis of the topology at g will consist of the sets $\{h \in M_2(F) : g \equiv h(mod\,\mathcal{P}^n)\}$ for $n \in \mathbb{N}$. Under this topology $M_2(F)$ is a topological ring. This topology is such that if we consider $M_2(F)$ only as an additive group then it is isomorphic, as a topological group, to the additive group F^4 under the product topology. Hence $M_2(F)$ is a locally profinite group. Any locally closed subset of $M_2(F)$ will be locally profinite in the relative topology.

Throughout the rest of these notes G will denote the group $GL_2(F)$ and K will denote the subgroup $GL_2(\mathcal{O})$ of G. Regarding G and K as subsets of the topological space $M_2(F)$ they inherit the relative topology and become topological groups. In fact since they are locally closed subsets of $M_2(F)$ they are locally profinite. Since $\mathcal{O}$ is open in F we have that K is open in G.

For more on the next result see **[4]**.

Proposition 1 K is a compact open subgroup of G. In fact K is a maximal compact subgroup of G, namely if M is any compact subgroup of G such that $K \subseteq M$ then necessairly $M = K$. Moreover the set of maximal compact subgroups of G are precisely the conjugates of K in G, namely the subgroups of the form $g^{-1}Kg$ where $g \in G$. Thus every maximal compact subgroup of G is necessarily open.

Note that the determinant map $det : G \longrightarrow F^\times$ is continuous but not locally constant. The subset of G defined by

$$\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G : c \neq 0\}$$

is called the *big cell*. It is a dense subset of G. Hence any locally constant function on G is determined by its restriction to the big cell.

The elements

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

of G will be denoted by e and w respectively. Hence $w^{-1} = -w$. Some identities in G that will arise in the theory and proofs are

(1) For all $a \in F^\times$,

$$w \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix} w.$$

(2) If $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $c \neq 0$, namely if g is in the big cell, then

$$g = \begin{pmatrix} c^{-1}det(g) & a \\ 0 & c \end{pmatrix} w^{-1} \begin{pmatrix} 1 & c^{-1}d \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} c^{-1}det(g) & -a \\ 0 & c \end{pmatrix} w \begin{pmatrix} 1 & -c^{-1}d \\ 0 & 1 \end{pmatrix}.$$

(3) For $a, d \in F^\times$ and $b \in F$ we have

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} ad^{-1} & bd^{-1} \\ 0 & 1 \end{pmatrix}.$$

(4) For any $x \in F^\times$,

$$w \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x^{-1} & 1 \\ 0 & x \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -x^{-1} & 1 \end{pmatrix}$$

(5) For any $x \in F^\times$,

$$w \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} w^{-1} = \begin{pmatrix} 1 & -x^{-1} \\ 0 & 1 \end{pmatrix} w \begin{pmatrix} x & -1 \\ 0 & x^{-1} \end{pmatrix}$$

Some Subgroups of G

For $n = 0, 1, \ldots$ define

$$K(n) := \{g \in K : g \equiv e \, (mod \, \mathcal{P}^n)\}.$$

Note that $K(0) = K$. Observe that

$$K(n) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(F) : a, d \in 1 + \mathcal{P}^n \;\; b, c \in \mathcal{P}^n\}.$$

Indeed suppose $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(F)$ is such that $a, d \in 1 + \mathcal{P}^n$ and $b, c \in \mathcal{P}^n$. It is clear that $g \in M_2(\mathcal{O})$. We have $ad \in (1 + \mathcal{P}^n)^2 \subseteq 1 + \mathcal{P}^n$ and $bc \in \mathcal{P}^{2n} \subset \mathcal{P}^n$. Therefore $det(g) = ad - bc \in 1 + \mathcal{P}^n$ and so in particular $g \in K$. By supposition $g \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (mod \, \mathcal{P}^n)$ and so we conclude $g \in K(n)$.

Lemma 2 For any $n \in \mathbb{N}$ we have that $K(n)$ is an open normal subgroup of K and $K/K(n) \simeq GL_2(\mathcal{O}/\mathcal{P}^n)$. Moreover if $g \in K(n)$ then $det(g) \in 1 + \mathcal{P}^n$.

Proof We just proved the last assertion in the preceeding paragraph. Since $\mathcal{P}^n$ is open in $\mathcal{O}$ it follows that $K(n)$ is open in K and hence in G also. Now define the homomorphism of groups

$$: K \longrightarrow GL_2(\mathcal{O}/\mathcal{P}^n) \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} a + \mathcal{P}^n & b + \mathcal{P}^n \\ c + \mathcal{P}^n & d + \mathcal{P}^n \end{pmatrix}.$$

The kernel of this homomorphism is $K(n)$ and hence $K(n)$ is a normal subgroup of K. In order to prove $K/K(n) \simeq GL_2(\mathcal{O}/\mathcal{P}^n)$ we need only prove that this homomorphism is surjective. Well choose an arbitrary $g = \begin{pmatrix} a + \mathcal{P}^n & b + \mathcal{P}^n \\ c + \mathcal{P}^n & d + \mathcal{P}^n \end{pmatrix} \in GL_2(\mathcal{O}/\mathcal{P}^n)$ where of course $a, b, c, d \in \mathcal{O}$. We know that $(ad - bc) + \mathcal{P}^n \in (\mathcal{O}/\mathcal{P}^n)^\times$. This easily implies that necessarily $ad - bc \in \mathcal{U}$. Hence the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is in K and its image under the homomorphism is g. Thus the homomorphism is surjective.

□

Proposition 3 Each $K(n)$ is a compact open normal subgroup of K and hence a compact open subgroup of G. Moreover the collection of all the subgroups $K(n)$ form a neighborhood basis of the identity in K, and hence also in G.

Proof By Lemma 5.2 we know each $K(n)$ is an open normal subgroup of K. Thus $K(n)$ is closed in K and hence compact. All that remains to prove is that the collection of subgroups $K(n)$ form a neighborhood basis of the identity of K. Let M be any neighborhood of the identity in K. Since the family of open sets $\mathcal{P}^n$ for $n = 1, 2, \ldots$ is a fundamental system of neighborhoods of 0 in F there is a $m \in \mathbb{N}$ such that whenever $a, d \in 1 + \mathcal{P}^m$ and $b, c \in \mathcal{P}^m$ then we have $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M$. Hence $K(m) \subseteq M$.

□

If (π, V) is a representation of G then for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ we write $\pi\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ instead of the pedantic $\pi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)$.

For the rest of these notes we fix our notation for certain subgroups of G which will arise often. The center of G will be denoted by Z,

$$Z = \{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \in G \, : \, a \in F^\times \}.$$

The *standard Borel subgroup* of G, namely the upper triangular matrices in G, will be denoted by B,

$$B = \{ \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \in G \, : \, a, b \in F^\times \quad x \in F \}.$$

Note that the big cell defined above is the complement of B in G. The unipotent radical of B will be denoted by U,

$$U = \{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \in G \, : \, x \in F \}.$$

We do not require the notion of unipotent radical in these notes. The mention of it here is merely to motivate the choice of notation. We may just take the above to be the definition of U. The *torus* subgroup of G will be denoted by T,

$$T = \{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in G \, : \, a, b \in F^\times \}.$$

Finally we denote the *affine* subgroup of G by A,

$$A = \{ \begin{pmatrix} a & x \\ 0 & 1 \end{pmatrix} \in G \, : \, a \in F^\times \quad x \in F \}.$$

All of these subgroups inherit the relative topology from G and they then become locally profinite groups. Observe that as topological groups we have the obvious isomorphisms $U \simeq F^+$ and $Z \simeq F^\times$. Also observe that the subgroup $\{ \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \in G : x \in F^\times \}$ of G is isomorphic to $F^\times$. Hence the structure of the groups F^+ and $F^\times$ and their subgroups is contained within the structure of G and its subgroups. We clearly have $B = AZ = ZA = UT = TU$.

U is a normal subgroup of B and B/U is isomorphic to T by the map

$$: T \longrightarrow B/U \quad \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \longmapsto \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} U = \{ \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} : x \in F \}.$$

Proposition 4 Any open subgroup of G which contains Z and $SL_2(F)$ is necessarily of finite index in G.

Proof The map

$$: G/ZSL_2(F) \longrightarrow F^\times/(F^\times)^2 \quad g(mod\, ZSL_2(F)) \longmapsto det(g)(mod\,(F^\times)^2)$$

is a well defined injective homomorphism into the finite group $F^\times/(F^\times)^2$. The result follows.

□

The identity $G = BK$ of the next result is called the *Iwasawa decomposition* of G.

Proposition 5 We have the identity $G = BK$.

Proof Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ be arbitrary. If $|c| \leq |d|$ holds then $d \neq 0$. $cd^{-1} \in \mathcal{O}$ and we define $h = \begin{pmatrix} 1 & 0 \\ -cd^{-1} & 1 \end{pmatrix}$. If $|d| < |c|$ holds then $c \neq 0$, $c^{-1}d \in \mathcal{O}$ and we define $h = \begin{pmatrix} -c^{-1}d & 1 \\ 1 & 0 \end{pmatrix}$. Then regardless of which case holds we have that $h \in K$ and $gh \in B$. Therefore $g \in Bh^{-1} \subset BK$.

□

Corollary 6 The space $B \setminus G$ of right cosets of B in G is a compact topological space under the quotient topology.

Proof If $i : K \longrightarrow G$ is the inclusion map and $q : G \longrightarrow B \setminus G$ is the quotient map then both i and q are continuous. Therefore $q \circ i$ is a continuous map from K to $B \setminus G$ and it is surjective by the Iwasawa decomposition of G. Hence as K is compact so is $B \setminus G$.

□

Proposition 7 The group G is generated by the subgroup B and the matrix w. More precisely we have the identity $BwU \cup B = G$ where this is a disjoint union.

Proof Since U is a subgroup of B it is clear that the second assertion implies the first. Note that

$$Bw = \{\begin{pmatrix} a & b \\ c & 0 \end{pmatrix} \in G : a \in F \quad b, c \in F^\times\}.$$

If $\begin{pmatrix} a & b \\ c & 0 \end{pmatrix} \in Bw$ and $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \in U$ then their product equals the matrix $\begin{pmatrix} a & ax+b \\ c & cx \end{pmatrix}$. Hence if this product was in B then necessarily $c = 0$ which is impossible. Thus $BwU \cap B$ equals the empty set. Finally observe that if $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is in G but not in B. so that $c \neq 0$, then we have

$$g = \begin{pmatrix} -c^{-1}det(g) & -a \\ 0 & -c \end{pmatrix} w \begin{pmatrix} 1 & c^{-1}d \\ 0 & 1 \end{pmatrix} \in BwU.$$

This proves the result.

□

Note that the big cell equals $BwU = Bw^{-1}U$ since we have the partition $G = BwU \cup B$ proven above.

Lemma 8

(1) The matrices in $SL_2(F)$ of the form $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$, and $\begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix}$, where $x \in F$, $y \in F^\times$, generate $SL_2(F)$.

(2) The matrices in $SL_2(F)$ of the form $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}$ where $x, y \in F$, generate $SL_2(F)$.

Proof Clearly all of the matrices listed are in $SL_2(F)$.

(1) Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(F)$. If $c = 0$ then $ad = 1$ and we have the identity.

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & -a^{-1}b \\ 0 & 1 \end{pmatrix}^{-1}.$$

If $c \neq 0$ we have the identity,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & -c^{-1}a \\ 0 & 1 \end{pmatrix}^{-1} w \begin{pmatrix} -c & 0 \\ 0 & -c^{-1} \end{pmatrix} \begin{pmatrix} 1 & -c^{-1}d \\ 0 & 1 \end{pmatrix}^{-1}.$$

(2) We use part (1). For $x \in F^\times$ we have the identity,

$$\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ x^{-1} - 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ x - 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -x^{-1} \\ 0 & 1 \end{pmatrix}$$

and also

$$w = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

□

Proposition 9 Let H be an open subgroup of G.

(1) If H contains U then H contains $SL_2(F)$.

(2) If H is normal in G then H contains $SL_2(F)$.

Proof

(1) By Lemma 5.8 we need only show that for every $y \in F$, $\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} \in H$. Since H is open it contains a matrix of the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $c \neq 0$ (we could even assume $a = d = 1$ and $b = 0$). Hence

$$\begin{pmatrix} 0 & b' \\ c & 0 \end{pmatrix} = \begin{pmatrix} 1 & -ac^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & -dc^{-1} \\ 0 & 1 \end{pmatrix} \in H$$

where $b' = b - ac^{-1}d$. Let $x \in F$ be given and define $y = b'xc^{-1}$. We have

$$\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} = \begin{pmatrix} 0 & b' \\ c & 0 \end{pmatrix} \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & b' \\ c & 0 \end{pmatrix}^{-1} \in H.$$

(2) Since H is open there is a $n \in \mathbb{N}$ such that if $y \in \mathcal{P}^n$ then

$$\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \in H \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} \in H.$$

Let $x \in F$ be arbitrary. Choose $a \in F^\times$ only so that $ax \in \mathcal{P}^n$. Then as H is normal.

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & ax \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \in H$$

and

$$\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ ax & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} \in H.$$

Now use the Lemma 5.8.

□

The Central Character

Proposition 10 Suppose (π, V) is an irreducible admissible representation of G. Then there exists a unique character ω of $F^\times$ such that for all $a \in F^\times$,

$$\pi \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = \omega(a) I_V.$$

Proof Uniqueness is obvious. For any fixed $a \in F^\times$ the operator $\pi \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ commutes with π. Therefore the irreducibility of π implies by Theorem 3.6 that there exists a function $\omega : F^\times \longrightarrow \mathbb{C}^\times$ such that for all $a \in F^\times$,

$$\pi \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = \omega(a) I_V.$$

It is easily seen that ω is a homomorphism. It remains to prove the continuity of ω, and to do this it suffices to prove continuity at $1 \in F^\times$. We will even prove that ω is constant in a neighborhood of 1. Fix $v \in V$, $v \neq 0$ and define $H = \{g \in G : \pi(g)v = v\}$. Since H is open there is a neighborhood of 1 in $F^\times$ such that for any element a of this neighborhood we have $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \in H$. Hence $\omega(a) = 1$ for any element a of this neighborhood.

□

Definition 11 If (π, V) is an irreducible admissible representation of G the unique character of $F^\times$ associated to π, which we will denote by ω_π, that satisfies

$$\pi \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = \omega_\pi(a) I_V,$$

for all $a \in F^\times$ will be called the *central character* of (π, V).

Lemma 12 Suppose (π_1, V_1) and (π_2, V_2) are equivalent irreducible admissible representations of G. Then $\omega_{\pi_1} = \omega_{\pi_2}$.

Proof Let $R : V_1 \longrightarrow V_2$ be an intertwining isomorphism of the equivalent representations (π_1, V_1) and (π_2, V_2). Fix a non-zero $v \in V_1$. Let $a \in F^\times$ be arbitrary. Then $\omega_{\pi_1}(a)R(v) = R(\omega_{\pi_1}(a)v) = \omega_{\pi_2}(a)R(v)$ and thus as $R(v) \neq 0$ this implies $\omega_{\pi_1}(a) = \omega_{\pi_2}(a)$.

□

Due to the above lemma for any $\pi \in \Pi(G)$ we have an associated character ω_π of $F^\times$, which we call the *central character* of the class π.

Proposition 13 If $\pi \in \Pi(G)$ then $\omega_{\tilde{\pi}} = \omega_\pi^{-1}$.

Proof Let (π, V) be a representation in the equivalence class π. For any $a \in F^\times$, $\tilde{v} \in \tilde{V}$, and $v \in V$ we have

$$\left(\tilde{\pi}\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\tilde{v}\right)(v) = \tilde{v}\left(\pi\begin{pmatrix} a^{-1} & 0 \\ 0 & a^{-1} \end{pmatrix}v\right) = (\omega_\pi^{-1}(a)\tilde{v})(v).$$

Hence $\omega_{\tilde{\pi}} = \omega_\pi^{-1}$.

□

The proof of the next theorem can be found in [2] or [4].

Theorem 14 Suppose (π, V) is an irreducible admissible representation of G. Then the representations $(\omega_\pi^{-1} \otimes \pi, V)$ and $(\tilde{\pi}, \tilde{V})$ of G are equivalent.

Classification of the Finite Dimensional Classes in $\Pi(G)$

Lemma 15 Suppose $\mathcal{A}$,$\mathcal{B}$,$\mathcal{C}$ are groups, and $\alpha : \mathcal{A} \longrightarrow \mathcal{B}$, $\beta : \mathcal{A} \longrightarrow \mathcal{C}$ are homomorphims with α being surjective. Then $ker(\alpha) \subseteq ker(\beta)$ iff there exists a homomorphism $\gamma : \mathcal{B} \longrightarrow \mathcal{C}$ such that $\beta = \gamma\alpha$.

Proof This is trivial. First suppose such a γ is given. Then for $g \in ker(\alpha)$ we have $\beta(g) = (\gamma\alpha)(g) = \gamma(1_\mathcal{B}) = 1_\mathcal{C}$, and so $g \in ker(\beta)$. Conversely suppose $ker(\alpha) \subseteq ker(\beta)$. If $g_1, g_2 \in \mathcal{A}$ are such that $\alpha(g_1) = \alpha(g_2)$ then $\beta(g_1) = \beta(g_2)$ holds. Therefore we may define $\gamma : \mathcal{B} \longrightarrow \mathcal{C}$ by $\gamma(h) = \beta(g)$ where $g \in \mathcal{A}$ is any element such that $\alpha(g) = h$. (such a g exists since α is surjective). Then by definition γ is a homomorphism such that $\beta = \gamma\alpha$.

□

Corollary 16 If $\pi : G \longrightarrow \mathbb{C}^\times$ is a homomorphism then $SL_2(F) \subseteq ker\,\pi$ iff there exists a homomorphism $\chi : F^\times \longrightarrow \mathbb{C}^\times$ such that $\pi(g) = \chi(det\, g)$ for all $g \in G$.

Proof This follows from the above lemma after noting that the homomorphism $det : G \longrightarrow F^\times$ is surjective and has kernel $SL_2(F)$.

□

Theorem 17 There is a bijection between the set of characters of $F^\times$ and the set of finite dimensional classes in $\Pi(G)$. This bijection associates a character χ of $F^\times$ to the equivalence class which contains the representation $(\pi_\chi, \mathbb{C})$ of G defined by

$$\pi_\chi : G \longrightarrow GL_1(\mathbb{C}) \qquad \pi_\chi(g) = \chi(detg).$$

In particular any finite dimensional class in $\Pi(G)$ is necessarily one dimensional.

Proof We first need to prove that π_χ is admissible, which will show that the above map, which is claimed to be a bijection, is at least well defined. To show that π_χ is a smooth representation we must establish that for every $v \in V$ the subgroup $\{g \in G : \chi(det\ g)v = v\}$ is open in G. This is obvious for $v = 0$ so we suppose that $v \neq 0$. In this case the above subgroup is $det^{-1}(ker(\chi))$ which, as a continuous preimage of an open set, is open. Thus π_χ is smooth. Since $\mathbb{C}$ is one dimensional it is trivial that π_χ is admissible.

Let us first show injectivity of the map. Let χ_1 and χ_2 be characters of $F^\times$ such that π_{χ_1} and π_{χ_2} are equivalent representations. It follows that for all $g \in G$ we have $\chi_1(det\ g) = \chi_2(det\ g)$. Since $det : G \longrightarrow F^\times$ is surjective this implies $\chi_1 = \chi_2$.

Now we show surjectivity. Let $(\pi, \mathbb{C}^n)$ be any finite dimensional irreducible admissible representation of G. By Corollary 2.12 we know that $ker(\pi)$ is open. Hence $ker(\pi)$ is an open normal subgroup of G and thus by Proposition 5.9.2 it contains $SL_2(F)$. Hence for any $g_1, g_2 \in G$ we have $g_1 g_2 g_1^{-1} g_2^{-1} \in ker(\pi)$ and so $\pi(g_1)\pi(g_2) = \pi(g_2)\pi(g_1)$. Thus for every $g \in G$, $\pi(g)$ is a nonzero scalar multiple of I_n. This proves that $n = 1$, and so π is necessarily one dimensional. Since $ker(det) = SL_2(F) \subseteq ker(\pi)$ by Lemma 5.15 there is a homomorphism $\chi : F^\times \longrightarrow \mathbb{C}^\times$ such that $\pi(g) = \chi(det\ g)$ for all $g \in G$. Since for $a \in F^\times$, $\chi(a) = \chi(det \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}) = \pi \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$ we see χ is continuous and hence a character. Thus $\pi = \pi_\chi$.

□

Hence the classes in $\Pi(G)$ are either one dimensional or infinite dimensional. Having completely understood the one dimensional representations our focus turns to the structure of the infinite dimensional representations. We conclude this chapter with some results that will be needed later.

Proposition 18 Let $\pi \in \Pi(G)$ be one dimensional. If χ is the character of $F^\times$ that corresponds to π then $\omega_\pi = \chi^2$.

Proof As above the one dimensional representation $(\pi_\chi, \mathbb{C})$ defined by

$$\pi_\chi(g)z := \chi(det(g))z$$

is in π. Hence for $a \in F^\times$ we have

$$\omega_\pi(a) = \pi_\chi \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} (1) = \chi(a^2) = \chi^2(a).$$

□

If χ is a character of $F^\times$ and (π, V) is a representation of G then we define the representation $(\chi \otimes \pi, V)$ of G by $(\chi \otimes \pi)(g)v := \chi(det(g))\pi(g)v$. If (π, V) is smooth, admissible, or irreducible then $(\chi \otimes \pi, V)$ also has this property. If $(\pi_\chi, \mathbb{C})$ is the one dimensional representation of G corresponding to χ then $\chi \otimes \pi$ is just the tensor product of the representations π_χ and π. Note that $V \otimes \mathbb{C} = V$. In the obvious way if $\pi \in \Pi(G)$ we consider $\chi \otimes \pi$ as a class in $\Pi(G)$.

The following is easy to prove.

Proposition 19 If $\pi \in \Pi(G)$ and χ is a character of $F^\times$ then the contragredient class of $\pi \otimes \chi$ is $\tilde{\pi} \otimes \chi^{-1}$.

Chapter 6 Technical Preparation

Fourier Theory on $\mathcal{U}$

Since $\mathcal{U}$ is a (compact) abelian Haar group we may consider its theory of Fourier analysis. For any locally constant function $f : \mathcal{U} \longrightarrow \mathbb{C}$, or in other words any element of $\mathcal{S}(\mathcal{U})$, we define its Fourier transform with respect to the normalized Haar measure of $\mathcal{U}$ in the usual way,

$$\hat{f} : \mathcal{U}^* \longrightarrow \mathbb{C} \qquad \hat{f}(\mu) := \int_{\mathcal{U}} \mu(\epsilon) f(\epsilon)\, d^\times \epsilon.$$

Proposition 1 Let $f \in \mathcal{S}(\mathcal{U})$.

(1) For almost every $\mu \in \mathcal{U}^*$ we have $\hat{f}(\mu) = 0$. So the function $\hat{f} : \mathcal{U}^* \longrightarrow \mathbb{C}$ has finite support.

(2) For every $\epsilon \in \mathcal{U}$ we have the identity,

$$f(\epsilon) = \sum_{\mu \in \mathcal{U}^*} \mu(\epsilon) \hat{f}(\mu^{-1})$$

where this is a finite summation.

Proof

(1) Since f is locally constant there is a $n \in \mathbb{N}$ such that f is constant on the cosets of $1+\mathcal{P}^n$ in $\mathcal{U}$. Let S be a set of representatives for the cosets of $1+\mathcal{P}^n$ in $\mathcal{U}$. Then for $\mu \in \mathcal{U}^*$,

$$\hat{f}(\mu) = \sum_{s \in S} \int_{1+\mathcal{P}^n} \mu(\epsilon s) f(\epsilon s)\, d^\times \epsilon = \sum_{s \in S} \mu(s) f(s) \int_{1+\mathcal{P}^n} \mu(\epsilon)\, d^\times \epsilon.$$

Hence $1+\mathcal{P}^n \not\subseteq ker(\mu)$ implies $\hat{f}(\mu) = 0$. Since the groups $\{\mu \in \mathcal{U}^* : 1+\mathcal{P}^n \subseteq ker(\mu)\}$ and $(\mathcal{U}/1+\mathcal{P}^n)^*$ are isomorphic we conclude that there are only finitely many $\mu \in \mathcal{U}^*$ such that $1+\mathcal{P}^n \subseteq ker(\mu)$. This proves the first part.

(2) Just for now let α denote the normalized Haar measure on $\mathcal{U}$, so that for $\mu \in \mathcal{U}^*$

$$\hat{f}(\mu) = \int_{\mathcal{U}} \mu(\epsilon) f(\epsilon)\, d\alpha(\epsilon).$$

Let α^* denote the Haar measure on $\mathcal{U}^*$ that is the dual measure to α. The Fourier inversion theorem implies that for $\epsilon \in \mathcal{U}$,

$$f(\epsilon) = \int_{\mathcal{U}^*} \mu(\epsilon^{-1}) \hat{f}(\mu)\, d\alpha^*(\mu).$$

Since α^* is the counting measure on $\mathcal{U}^*$ the above can be written as

$$f(\epsilon) = \sum_{\mu \in \mathcal{U}^*} \mu(\epsilon^{-1}) \hat{f}(\mu) = \sum_{\mu \in \mathcal{U}^*} \mu(\epsilon) \hat{f}(\mu^{-1}).$$

By part (1) of the proposition we know this is a finite summation as $\hat{f}(\mu^{-1}) = 0$ for almost every $\mu \in \mathcal{U}^*$.

□

We will need to generalize this result to functions which take values in vector spaces. Let X be a non-zero complex vector space and let $f : \mathcal{U} \longrightarrow X$ be locally constant, or in other words let $f \in \mathcal{S}_X(\mathcal{U})$. We define the Fourier transform of f with respect to the normalized Haar measure of $\mathcal{U}$ by

$$\hat{f} : \mathcal{U}^* \longrightarrow X \qquad \hat{f}(\mu) := \int_{\mathcal{U}} \mu(\epsilon) f(\epsilon)\, d^\times \epsilon.$$

As f assumes only finitely many values on $\mathcal{U}$ this integral is well defined, being a linear combination of the finite number of vectors in the image of f. We can easily generalize the above result.

Proposition 2 Let $f \in \mathcal{S}_X(\mathcal{U})$.

(1) For almost every $\mu \in \mathcal{U}^*$ we have $\hat{f}(\mu) = 0 \in X$. So the function $\hat{f} : \mathcal{U}^* \longrightarrow X$ has finite support.

(2) For every $\epsilon \in \mathcal{U}$ we have the identity,

$$f(\epsilon) = \sum_{\mu \in \mathcal{U}^*} \mu(\epsilon) \hat{f}(\mu^{-1})$$

where this a finite summation.

Proof The proof of the first part of the proposition is completely identical to the first part of the proof of this result in the case $X = \mathbb{C}$. Since we have just provided the proof of this case in the previous proposition we shall move on to the proof of the second part of the proposition. Let $u_1, \ldots, u_m$ be linearly independent vectors in X whose span contains the image of f. Define the functions $\gamma_1, \ldots, \gamma_m : \mathcal{U} \longrightarrow \mathbb{C}$ by the requirement that for every $\epsilon \in \mathcal{U}$

$$f(\epsilon) = \sum_{k=1}^{m} \gamma_k(\epsilon) u_k.$$

Since f is locally constant each γ_k is also locally constant. By the Fourier inversion formula for each $k = 1, \ldots, m$ we have the identity,

$$\gamma_k = \sum_{\mu \in \mathcal{U}^*} \mu \hat{\gamma}_k(\mu^{-1})$$

where the summation is finite. Hence

$$f = \sum_{k=1}^{m} \left(\sum_{\mu \in \mathcal{U}^*} \mu \hat{\gamma}_k(\mu^{-1}) \right) u_k = \sum_{\mu \in \mathcal{U}^*} \mu \left(\sum_{k=1}^{m} \hat{\gamma}_k(\mu^{-1}) u_k \right).$$

Observe that

$$\hat{f}(\mu^{-1}) = \int_{\mathcal{U}} \mu^{-1}(\epsilon) \left(\sum_{k=1}^{m} \gamma_k(\epsilon) u_k \right) d^\times \epsilon = \sum_{k=1}^{m} \left(\int_{\mathcal{U}} \mu^{-1}(\epsilon) \gamma_k(\epsilon)\, d^\times \epsilon \right) u_k = \sum_{k=1}^{m} \hat{\gamma}_k(\mu^{-1}) u_k.$$

Hence $f = \sum_{\mu \in \mathcal{U}^*} \mu \hat{f}(\mu^{-1})$. This is really a finite sum since $\hat{f}(\mu^{-1}) = 0 \in X$ for almost every $\mu \in \mathcal{U}^*$.

□

The Space $\mathcal{S}_X$

In what follows X will denote a non-zero complex vector space. and $\mathcal{S}_X$ will be an abbreviation for $\mathcal{S}_X(F^\times)$. Therefore in our notation $\mathcal{S}(F^\times) = \mathcal{S}_\mathbb{C}$, and we shall use them interchangeably, although we shall prefer $\mathcal{S}(F^\times)$. For $\mu \in \mathcal{U}^*$ define $\mu^\times$ in $\mathcal{S}(F^\times)$ by $\mu^\times(y) = \mu(y)$ if $y \in \mathcal{U}$ and $\mu^\times(y) = 0 \in \mathbb{C}$ when $y \in F^\times$, $y \notin \mathcal{U}$. If χ is a character of $F^\times$ we denote the restriction of χ to $\mathcal{U}$ by χ_0. Note that χ and $\chi_0^\times$ agree only on $\mathcal{U}$.

If $f \in \mathcal{S}(F^\times)$ and $u \in X$ it is clear that the function : $F^\times \longrightarrow X$ $\quad y \longmapsto f(y)u$ is in $\mathcal{S}_X$. This function will be denoted by $f \otimes u$. As $\mathcal{S}_X = \mathcal{S}_\mathbb{C} \bigotimes X$ this notation is motivated. Since $\mathcal{S}(F^\times)$ is translation invariant, for any $a \in F^\times$ the function $f(a\cdot) \otimes u$ is also in $\mathcal{S}_X$. Note that $\mu^\times(\epsilon\cdot) \otimes u = \mu(\epsilon)(\mu^\times \otimes u)$ for $\epsilon \in \mathcal{U}$, and we have for any $a \in F^\times$ that $\mu^\times(a\cdot) \otimes u = (\mu^\times \otimes u)(a\cdot)$.

We introduce some notation. Let $\tilde{\omega}$ be a uniformizing parameter of F. For any locally constant function $\varphi : F^\times \longrightarrow X$ we define for each $n \in \mathbb{Z}$ the function

$$\varphi_n : \mathcal{U} \longrightarrow X \qquad \varphi_n(\epsilon) := \varphi(\epsilon\tilde{\omega}^n).$$

Observe that the definition of the functions φ_n depends on the choice of $\tilde{\omega}$ although this is not reflected in the notation.

We first wish to generalize the results we developed in the previous section on the Fourier expansions of functions in $\mathcal{S}_X(\mathcal{U})$ to an appropriate analogue for functions in $\mathcal{S}_X(F^\times)$. This is accomplished in the following result.

Proposition 3 Let $\tilde{\omega}$ be a uniformizing parameter of F and let $\varphi : F^\times \longrightarrow X$ be a locally constant function.

(1) For any $a \in F^\times$ we have the identity

$$\varphi(a) = \sum_{n\in\mathbb{Z}} \sum_{\mu\in\mathcal{U}^*} \mu^\times(\tilde{\omega}^{-n}a)\hat{\varphi}_n(\mu^{-1})$$

where for any given $a \in F^\times$ the sums over $\mathbb{Z}$ and $\mathcal{U}^*$ are both finite summations.

(2) If moreover $\varphi \in \mathcal{S}_X$ then we have the identity

$$\varphi = \sum_{n\in\mathbb{Z}} \sum_{\mu\in\mathcal{U}^*} \mu^\times(\tilde{\omega}^{-n}\cdot) \otimes \hat{\varphi}_n(\mu^{-1})$$

where the sums over $\mathbb{Z}$ and $\mathcal{U}^*$ are both finite summations. Each of the terms in this summation are in $\mathcal{S}_X$.

Proof Since $\varphi_n \in \mathcal{S}_X(\mathcal{U})$ by Proposition 6.2.2 we can write

$$\varphi_n = \sum_{\mu\in\mathcal{U}^*} \mu\hat{\varphi}_n(\mu^{-1})$$

where this is a finite sum. Thus for $\epsilon \in \mathcal{U}$,

$$\varphi(\epsilon\tilde{\omega}^n) = \varphi_n(\epsilon) = \sum_{\mu\in\mathcal{U}^*} \mu(\epsilon)\hat{\varphi}_n(\mu^{-1}).$$

For $n \in \mathbb{Z}$ define $\chi_n : F^\times \longrightarrow \{0,1\}$ to be the characteristic function of $val^{-1}(n)$. Namely if $y \in F^\times$ then $\chi_n(y) = 1$ when $val(y) = n$ and $\chi_n(y) = 0$ otherwise. Let $\chi_n\varphi : F^\times \longrightarrow X$ denote the function in $\mathcal{S}_X$ given by $y \longmapsto \chi_n(y)\varphi(y)$. The above identity implies.

$$\chi_n\varphi = \sum_{\mu\in\mathcal{U}^*} \mu^\times(\tilde{\omega}^{-n}\cdot) \otimes \hat{\varphi}_n(\mu^{-1}).$$

Observe that $\varphi = \sum_{n\in\mathbb{Z}} \chi_n\varphi$ in the sense that for any $a \in F^\times$ we have $\varphi(a) = \sum_{n\in\mathbb{Z}} \chi_n\varphi(a)$. Hence

$$\varphi(a) = \sum_{n\in\mathbb{Z}} \chi_n(a)\varphi(a) = \sum_{n\in\mathbb{Z}} \sum_{\mu\in\mathcal{U}^*} \mu^\times(\tilde{\omega}^{-n}a)\hat{\varphi}_n(\mu^{-1}).$$

When $\varphi \in \mathcal{S}_X$ the identity $\varphi = \sum_{n\in\mathbb{Z}} \chi_n\varphi$ makes sense as an identity in $\mathcal{S}_X$. Indeed in this case it is only a finite summation over $\mathbb{Z}$ since φ is compactly supported on $F^\times$. Thus in this case we have the identity of functions

$$\varphi = \sum_{n\in\mathbb{Z}} \chi_n\varphi = \sum_{n\in\mathbb{Z}} \sum_{\mu\in\mathcal{U}^*} \mu^\times(\tilde{\omega}^{-n}\,\cdot\,) \otimes \hat{\varphi}_n(\mu^{-1}).$$

□

The following corollary is a trivial consequence of the previous proposition. We therefore do not provide anything more than its statement.

Corollary 4 Let $\tilde{\omega}$ be a uniformizing parameter of F.

(1) The vector space $\mathcal{S}_X$ is spanned by the functions of the form $\mu^\times(\tilde{\omega}^k\,\cdot\,) \otimes u$ where $\mu \in \mathcal{U}^*$, $k \in \mathbb{Z}$, and $u \in X$.

(2) The vector space $\mathcal{S}(F^\times)$ is spanned by the functions of the form $\mu^\times(\tilde{\omega}^k\,\cdot\,)$ where $\mu \in \mathcal{U}^*$ and $k \in \mathbb{Z}$.

For $\mu \in \mathcal{U}^*$ we define $\mathcal{S}_X(\mu)$ to be the set of all $\varphi \in \mathcal{S}_X$ such that $\varphi(\epsilon a) = \mu(\epsilon)\varphi(a)$ whenever $\epsilon \in \mathcal{U}$ and $a \in F^\times$. $\mathcal{S}_X(\mu)$ is a translation invariant subspace of $\mathcal{S}_X$. Also for $\mu \in \mathcal{U}^*$ define the linear operator $P_\mu^X : \mathcal{S}_X \longrightarrow \mathcal{S}_X$ by

$$(P_\mu^X\varphi)(a) = \int_{\mathcal{U}} \mu^{-1}(\epsilon)\varphi(\epsilon a)\, d^\times\epsilon.$$

Of course this integral is defined as the map $\mu^{-1} \otimes \varphi(\,\cdot\, a)$ is in $\mathcal{S}_X(\mathcal{U})$. We shall write P_μ to abbreviate $P_\mu^{\mathbb{C}}$.

Proposition 5

(1) P_μ^X is a projection operator of $\mathcal{S}_X$ onto $\mathcal{S}_X(\mu)$.

(2) For each $\mu \in \mathcal{U}^*$, $k \in \mathbb{Z}$, $u \in X$ the function $\mu^\times(\tilde{\omega}^k\,\cdot\,) \otimes u$ is in $\mathcal{S}_X(\mu)$.

(3) $\mathcal{S}_X = \bigoplus_{\mu\in\mathcal{U}^*} \mathcal{S}_X(\mu)$.

Proof

(1) Let $\varphi \in \mathcal{S}_X$, $\epsilon \in \mathcal{U}$, and $a \in F^\times$. An easy computation shows $(P_\mu^X\varphi)(\epsilon a) = \mu(\epsilon)(P_\mu^X\varphi)(a)$, and therefore the range of P_μ^X is contained in $\mathcal{S}_X(\mu)$. We also have,

$$(P_\mu^X\varphi)(a) = \int_{\mathcal{U}} \mu^{-1}(\epsilon)\varphi(\epsilon a)\, d^\times\epsilon = \int_{\mathcal{U}} \mu^{-1}(\epsilon)\mu(\epsilon)\varphi(a)\, d^\times\epsilon = \varphi(a)$$

which shows that P_μ^X is the identity on $\mathcal{S}_X(\mu)$ and hence the first assertion is proved.

(2) Since $\mu^\times \in \mathcal{S}_{\mathbb{C}}(\mu)$ it is trivially seen that $\mu^\times(\tilde{\omega}^k\,\cdot\,) \otimes u$ is in $\mathcal{S}_X(\mu)$.

(3) Since the functions of the form $\mu^\times(\tilde{\omega}^k\,\cdot\,) \otimes u$ span $\mathcal{S}_X(\mu)$ we conclude from part (2) that $\mathcal{S}_X$ is the sum of the subspaces $\mathcal{S}_X(\mu)$. We need only to show that this sum is direct. Well let $\mu, \nu \in \mathcal{U}^*$ and suppose $\varphi \in \mathcal{S}_X(\mu) \cap \mathcal{S}_X(\nu)$. Then for all $a \in F^\times$ and $\epsilon \in \mathcal{U}$ we have $\varphi(a)(\mu(\epsilon) - \nu(\epsilon)) = 0$. Hence $\mu \neq \nu$ implies $\varphi = 0$.

□

The Representation ξ_ψ^X of A

Again X denotes a non-zero complex vector space. Given $\psi \in F^\bullet$ we define a representation ξ_ψ^X of A on the space of all functions from $F^\times$ to X by

$$(\xi_\psi^X(g)\varphi)(y) = \psi(by)\varphi(ay)$$

where $g = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in A$, $y \in F^\times$, and $\varphi : F^\times \longrightarrow X$ is any function. We shall write ξ_ψ to abbreviate $\xi_\psi^{\mathbb{C}}$.

It is not hard to show that $supp(\xi_\psi^X(b)\varphi) \subseteq a^{-1}supp(\varphi)$ and hence the space of all compactly supported functions from $F^\times$ to X is ξ_ψ^X-invariant. Also as $\psi(b\,\cdot)$ is locally constant we see that the space of all locally constant functions from $F^\times$ to X is ξ_ψ^X-invariant. Combining these two facts we find that $\mathcal{S}_X$ is ξ_ψ^X-invariant.

Lemma 6 Suppose $\psi \in F^\bullet$ and $\mu, \nu \in \mathcal{U}^\bullet$ satisfy $\psi \neq 1$ and $\mu \neq \nu$. Define $l \in \mathbb{Z}$ and $n \in \mathbb{N}$ by $l = cond(\psi)$ and $n = cond(\mu^{-1}\nu)$. Then for any $\varphi \in \mathcal{S}_X(\nu)$ and $x \in F$ with $val(x) = l - n$ we have

$$P_\mu^X\left(\xi_\psi^X\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\varphi\right) = \gamma(x, \mu^{-1}\nu, \psi)(\mu^\times \otimes \varphi(1)).$$

Proof Define $\eta = P_\mu^X\left(\xi_\psi^X\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\varphi\right) \in \mathcal{S}_X(\mu)$. For $a \in F^\times$ we have

$$\begin{aligned}
\eta(a) &= \int_{\mathcal{U}} \mu^{-1}(\epsilon)\left(\xi_\psi^X\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\varphi\right)(\epsilon a)\, d^\times\epsilon \\
&= \int_{\mathcal{U}} \mu^{-1}(\epsilon)\psi(\epsilon a x)\varphi(\epsilon a)\, d^\times\epsilon \\
&= \left(\int_{\mathcal{U}} (\mu^{-1}\nu)(\epsilon)\psi(\epsilon a x)\, d^\times\epsilon\right)\varphi(a) \\
&= \gamma(ax, \mu^{-1}\nu, \psi)\varphi(a).
\end{aligned}$$

If $a \in F$, $a \notin \mathcal{U}$ then $val(xa) \neq l - n$ and so $\gamma(xa, \mu^{-1}\nu, \psi) = 0$. Hence if $a \notin \mathcal{U}$ then as $\mu^\times(a) = 0$ we find $\eta(a) = 0 = \gamma(x, \mu^{-1}\nu, \psi)(\mu^\times \otimes \varphi(1))(a)$. Now if $a \in \mathcal{U}$ then as $\varphi \in \mathcal{S}_X(\mu)$ we see $\eta(a) = \mu(a)\eta(1) = \gamma(x, \mu^{-1}\nu, \psi)\mu^\times(a)\varphi(1)\gamma(x, \mu^{-1}\nu, \psi)(\mu^\times \otimes \varphi(1))(a)$. Hence $\eta = \gamma(x, \mu^{-1}\nu, \psi)(\mu^\times \otimes \varphi(1))$ as desired.

□

Proposition 7 If Y is a ξ_ψ^X-invariant subspace of $\mathcal{S}_X$ then Y is an invariant subspace for all of the operators P_μ^X where $\mu \in \mathcal{U}^\bullet$.

Proof Let $\varphi \in Y$ and let $\mu \in \mathcal{U}^\bullet$. The function

$$f : \mathcal{U} \longrightarrow Y \qquad \epsilon \longmapsto \mu^{-1}(\epsilon)\xi_\psi\begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix}\varphi$$

is in $\mathcal{S}_Y(\mathcal{U})$. Hence $\eta := \int_{\mathcal{U}} f(\epsilon)\, d^\times\epsilon$ is a well defined function in Y. Now

$$(P_\mu^X\varphi)(a) = \int_{\mathcal{U}} \mu^{-1}(\epsilon)\left(\xi_\psi^X\begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix}\varphi\right)(a)\, d^\times\epsilon = \eta(a).$$

Thus $P_\mu^X \varphi = \eta \in Y$.

□

Proposition 8 The representation $(\xi_\psi, \mathcal{S}(F^\times))$ of A is irreducible.

Proof Let Y be a non-zero ξ_ψ-invariant subspace of $\mathcal{S}(F^\times)$. Observe that Y is translation invariant since if $a \in F^\times$ and $\varphi \in Y$ then

$$\varphi(a\cdot) = \xi_\psi \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \varphi \in Y.$$

Thus by Corollary 6.4.2 to prove the proposition it suffices to show that $\mu^\times \in Y$ for all $\mu \in \mathcal{U}^*$.

We claim that if for some $\nu \in \mathcal{U}^*$ we have $Y \cap \mathcal{S}_\mathbb{C}(\nu)$ being non-zero then $\mu^\times \in Y$ for all $\mu \neq \nu$. Let us first show how this claim implies the proposition. Take any non-zero $f \in Y$. By Proposition 6.5 there is a $\nu \in \mathcal{U}^*$ such that $P_\nu f \neq 0$. Since Y is ξ_ψ-invariant by Proposition 6.7 it is also invariant for P_ν and hence $P_\nu f$ is a non-zero element of $Y \cap \mathcal{S}_\mathbb{C}(\nu)$. Now if our claim held then $\mu^\times \in Y$ for all $\mu \neq \nu$. Since $\mathcal{U}^*$ is an infinite group there certainly exists a $\mu \in \mathcal{U}^*$ with $\mu \neq \nu$. If we fix such a μ then $\mu^\times$ is a non-zero element of $Y \cup \mathcal{S}_\mathbb{C}(\mu)$ and hence by using the claim we conclude in particular that $\nu^\times \in Y$. Hence the above claim implies the proposition.

We now prove the claim. Suppose f is a non-zero element of $Y \cap \mathcal{S}_\mathbb{C}(\nu)$. Since $Y \cap \mathcal{S}_\mathbb{C}(\nu)$ is translation invariant we may assume $f(1) \neq 0$. Let $\mu \in \mathcal{U}^*$ satisfy $\mu \neq \nu$. Define $l \in \mathbb{Z}$ and $n \in \mathbb{N}$ so that $\mathcal{P}^l$ is the conductor of ψ and $1+\mathcal{P}^n$ is the conductor of $\mu^{-1}\nu \neq 1$. Choose any element $x \in F$ with $val(x) = l - n$. Then $P_\mu \left(\xi_\psi \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} f\right)$ is an element of Y and by Lemma 6.6 it is a non-zero scalar multiple of $\mu^\times$. Hence $\mu^\times \in Y$ which proves the claim. and therefore also the proposition.

□

Lemma 9 For a compact open subset Ω of $F^\times$ we define $\varphi_\Omega \in \mathcal{S}(F^\times)$ to be the characteristic function of Ω in $F^\times$. Let $\varphi \in \mathcal{S}(F^\times)$ and $\psi \in F^*$, $\psi \neq 1$ be given. Choose $k \in \mathbb{Z}$ so that $supp(\varphi) \cup supp(\hat{\varphi}) \subseteq \mathcal{P}^k$, where $\hat{\varphi}$ is the Fourier transform of φ with respect to ψ. Then there exist $b_1, ..., b_n \in F$ and $c_1, ..., c_n \in \mathbb{C}$ such that for any compact open set Ω in $F^\times$ with $supp(\varphi) \subseteq \Omega \subseteq \mathcal{P}^k$ we have the identity

$$\varphi = \sum_{j=1}^{n} c_j \xi_\psi \begin{pmatrix} 1 & b_j \\ 0 & 1 \end{pmatrix} \varphi_\Omega.$$

Moreover we may choose the c_j so that $\sum_{j=1}^n c_j = 0$.

Proof We regard φ as an element of $\mathcal{S}(F)$ by defining $\varphi(0) = 0$. Let $m \in \mathbb{Z}$ be sufficiently large so that ψ is constant on $\mathcal{P}^m$ and $\mathcal{P}^{m-k} \subseteq \mathcal{P}^k$. Observe that for any $a \in \mathcal{P}^k$ the function $: F \longrightarrow \mathbb{C} \quad x \longmapsto \psi(ax)\hat{\varphi}(-x)$ is constant on the cosets of $\mathcal{P}^{m-k}$ in F since each of the functions $x \longmapsto \psi(ax)$ and $x \longmapsto \hat{\varphi}(-x)$ have this property. Let S be a set of representatives for the cosets of $\mathcal{P}^{m-k}$ in $\mathcal{P}^k$. For $b \in S$ define $c_b = \lambda_\psi(\mathcal{P}^{m-k})\hat{\varphi}(-b) \in \mathbb{C}$. Then for any $a \in \mathcal{P}^k$ we have,

$$\begin{aligned}
\varphi(a) &= \int_{\mathcal{P}^k} \psi(ax)\hat{\varphi}(-x)\, d_\psi(x) \\
&= \sum_{b\in S} \int_{\mathcal{P}^{m-k}} \psi(a(x+b))\hat{\varphi}(-(x+b))\, d_\psi(x) \\
&= \sum_{b\in S} \lambda_\psi(\mathcal{P}^{m-k})\psi(ab)\hat{\varphi}(-b).
\end{aligned}$$

This identity shows two things. Firstly it immediately implies

$$\varphi = \sum_{b \in S} c_b \xi_\psi \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \varphi_\Omega$$

whenever Ω is a compact open set in $F^\times$ with $supp(\varphi) \subseteq \Omega \subseteq \mathcal{P}^k$. Secondly by putting $a = 0$ we find $\sum_{b \in S} c_b = 0$.

□

Corollary 10 Let ψ be a non-trivial unitary character of F^+. Then the collection of functions of the form

$$\xi_\psi \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \varphi - \varphi$$

where $x \in F$ and $\varphi \in \mathcal{S}(F^\times)$ form a spanning set for the space $\mathcal{S}(F^\times)$.

Proof Let $\varphi \in \mathcal{S}(F^\times)$, and define Ω to be the compact open set $supp(\varphi)$. Let $\varphi_\Omega \in \mathcal{S}(F^\times)$ denote the characteristic function of Ω in $F^\times$. By the Lemma 6.9 there are $b_1, \dots, b_n \in F$ and $c_1, \dots, c_n \in \mathbb{C}$ such that $\sum_{j=1}^n c_j = 0$ and

$$\varphi = \sum_{j=1}^{n} c_j \xi_\psi \begin{pmatrix} 1 & b_j \\ 0 & 1 \end{pmatrix} \varphi_\Omega.$$

Thus,

$$\varphi = \sum_{j=1}^{n} c_j (\xi_\psi \begin{pmatrix} 1 & b_j \\ 0 & 1 \end{pmatrix} \varphi_\Omega - \varphi_\Omega)$$

and we are done.

□

The next result is a trivail consequence of Corollary 6.10.

Corllary 11 Suppose $L : \mathcal{S}(F^\times) \longrightarrow \mathbb{C}$ is a linear map and $\psi \in F^*$, $\psi \neq 1$. If for all $\varphi \in \mathcal{S}(F^\times)$ and $x \in F$ we have

$$L(\xi_\psi \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \varphi) = L(\varphi),$$

then $L = 0$.

Lemma 12 Let ψ be a non-trivial unitary character of F^+. Then the subspace of linear functionals L in $\mathcal{S}(F^\times)^*$ which satisfy

$$L\left(\xi_\psi \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \varphi\right) = \psi(a) L(\varphi)$$

for all $\varphi \in \mathcal{S}(F^\times)$ and $a \in F$ is one dimensional. This one dimensional space has as a basis the linear functional : $\mathcal{S}(F^\times) \longrightarrow \mathbb{C}$ $\varphi \longmapsto \varphi(1)$.

Proof Clearly the property of the lemma is satisfied for the non-zero functional $\varphi \longmapsto \varphi(1)$. So all we have to show is if L is any functional satisfying the property of the lemma then there is a $c \in \mathbb{C}$ such that $L(\varphi) = c\varphi(1)$ for all $\varphi \in \mathcal{S}(F^\times)$. First note that it suffices to show that if $\varphi \in \mathcal{S}(F^\times)$ and $\varphi(1) = 0$ then $L(\varphi) = 0$. Indeed if $\varphi_1, \varphi_2 \in \mathcal{S}(F^\times)$ satisfy $\varphi_1(1) \neq 0$ and $\varphi_2(1) \neq 0$ then as $(\varphi_1(1)^{-1}\varphi_1 - \varphi_2(1)^{-1}\varphi_2)(1) = 0$ the claim would imply

$\varphi_1(1)^{-1}L(\varphi_1) = \varphi_2(1)^{-1}L(\varphi_2)$ and the result would follow by defining c to be this common value.

Well suppose $\varphi \in \mathcal{S}(F^\times)$ and $\varphi(1) = 0$. Let Ω be a compact open set in $F^\times$ which contains $1 \in F^\times$ and $supp(\varphi)$. Let $\varphi_\Omega \in \mathcal{S}(F^\times)$ denote the characteristic function of Ω in $F^\times$. By the Lemma 6.9 there are $b_1, \ldots, b_n \in F$ and $c_1, \ldots, c_n \in \mathbb{C}$ such that

$$\varphi = \sum_{j=1}^{n} c_j \xi_\psi \begin{pmatrix} 1 & b_j \\ 0 & 1 \end{pmatrix} \varphi_\Omega.$$

Since $\varphi(1) = 0$ and $\varphi_\Omega(1) = 1$ we have $\sum_{j=1}^{n} c_j \psi(b_j) = 0$. Thus

$$L(\varphi) = \sum_{j=1}^{n} c_j (L(\xi_\psi \begin{pmatrix} 1 & b_j \\ 0 & 1 \end{pmatrix} \varphi_\Omega) - \psi(b_j) L(\varphi_\Omega)).$$

By hypothesis for any j we know $L(\xi_\psi \begin{pmatrix} 1 & b_j \\ 0 & 1 \end{pmatrix} \varphi_\Omega) = \psi(b_j) L(\varphi_\Omega)$ and so $L(\varphi) = 0$.

□

The Key Theorem

We remind the reader that $G = GL_2(F)$ and what we do in the rest of the notes has this assumption in place. For this section our goal is to prove the following theorem.

Theorem 1 Let $\psi \in F^*$, $\psi \neq 1$ and let X be a non-zero complex vector space. Suppose $(\pi, \mathcal{K})$ is an infinite dimensional, irreducible, admissible representation of G such that

(1) $\mathcal{K}$ is a space of functions from $F^\times$ to X.
(2) For every $g \in A$ and $\varphi \in \mathcal{K}$ we have $\pi(g)\varphi = \xi_\psi^X(g)\varphi$.
(3) $span(\{\varphi(F^\times) : \varphi \in \mathcal{K}\}) = X$.

Then it is necessary that the dimension of X is one, every function in $\mathcal{K}$ is locally constant and vanishes outside a compact subset of F, $\mathcal{S}_X$ is a subspace of $\mathcal{K}$ of finite codimension, $\mathcal{K} = \mathcal{S}_X + \pi(w)\mathcal{S}_X$, and for any $g \in U$ and $\varphi \in \mathcal{K}$ we have $\varphi - \pi(g)\varphi \in \mathcal{S}_X$.

For the rest of this section we fix a representation $(\pi, \mathcal{K})$ of G which satisfies the hypothesis of the theorem. Our goal for this section is to prove Theorem 7.1.

Since Z acts on $\mathcal{K}$ as ω_π and since we know how A acts on $\mathcal{K}$ we therefore know how $B = AZ$ acts on $\mathcal{K}$. Namely if $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in B$ and $\varphi \in \mathcal{K}$ then

$$\pi\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\varphi = \pi\begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix}\pi\begin{pmatrix} ad^{-1} & bd^{-1} \\ 0 & 1 \end{pmatrix}\varphi = \omega_\pi(d)\psi(bd^{-1}\cdot)\varphi(ad^{-1}\cdot).$$

Let us quickly make some obvious remarks. Suppose φ is a function with domain $F^\times$ and whose range is some vector space. We will say that φ vanishes outside a compact subset of F if there is a compact set M in F such that if $x \in F^\times$ and $x \notin M$ then $\varphi(x) = 0$. Of course this is equivalent to requiring that there is a $n \in \mathbb{Z}$ such that if $x \in F^\times$ and $x \notin \mathcal{P}^n$ then $\varphi(x) = 0$. Observe that if for example φ is non-zero on $\mathcal{O}\backslash\{0\}$ and zero elsewhere then φ does not vanish outside of a compact set of $F^\times$, but it does vanish outside a compact set of F.

Proposition 2 Every function in $\mathcal{K}$ is locally constant and vanishes off of a compact subset of F.

Proof Let $\varphi \in \mathcal{K}$. Choose a $n \in \mathbb{N}$ so that if $g \in G$, $g \equiv e\,(mod\,\mathcal{P}^n)$ then $\pi(g)\varphi = \varphi$. Then for $a \in 1 + \mathcal{P}^n$ and $y \in F^\times$ we have

$$\begin{aligned} \varphi(ay) &= \left(\pi\begin{pmatrix} ay & 0 \\ 0 & 1 \end{pmatrix}\varphi\right)(1) \\ &= \left(\pi\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\pi\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\varphi\right)(1) \\ &= \left(\pi\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\varphi\right)(1) \\ &= \varphi(y). \end{aligned}$$

This proves that φ is locally constant.

Now let $\mathcal{P}^l$ be the conductor of ψ. We claim that if $x \in F$ and $val(x) < l - n$ then $\varphi(x) = 0$. Having proven this claim we will know that φ vanishes outside of the compact set $\mathcal{P}^{l-n}$ of F and we will be done. Well let $x \in F$ satisfy $val(x) < l - n$. First observe that $x\mathcal{P}^n \not\subseteq \mathcal{P}^l$. Indeed if $y \in \mathcal{P}^n$ satisfies $val(y) = n$ then $val(xy) < (l-n) + n = l$. and so $xy \notin \mathcal{P}^l$. Therefore we must have $\mathcal{P}^l \subset x\mathcal{P}^n$. Hence there is a $y \in \mathcal{P}^n$ such that $xy \notin ker(\psi)$. Note that

$$\varphi(x) = \left(\pi \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \varphi\right)(x) = \psi(xy)\varphi(x),$$

or rather $(1 - \psi(xy))\varphi(x) = 0$. Since $\psi(xy) \neq 1$ this implies $\varphi(x) = 0$ as desired.

□

Proposition 3 For any $a \in F$ and $\varphi \in \mathcal{K}$ the function

$$\varphi - \pi \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \varphi$$

is in $\mathcal{S}_X$.

Proof For the sake of brevity define $\eta = \varphi - \pi \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \varphi$. The proposition is trivial when $a = 0$ so we suppose $a \neq 0$. By Proposition 7.2 we know η is locally constant and vanishes off of a compact subset of F. Observe that for any $y \in F^\times$ we have

$$\eta(y) = \varphi(y) - \left(\xi_\psi^X \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\right)(y) = (1 - \psi(ay))\varphi(y).$$

Let $\mathcal{P}^l$ be the conductor of ψ. Then for $y \in a^{-1}\mathcal{P}^l$, $y \neq 0$ we see $\eta(y) = 0$. This proves $\varphi \in \mathcal{S}_X$.

□

Lemma 4 For every $u \in X$ there is a $\varphi \in \mathcal{K} \cap \mathcal{S}_X$ such that $\varphi(1) = u$.

Proof Let us first show that for any $u \in X$ there is a $\varphi \in \mathcal{K}$ such that $\varphi(1) = u$. Our assumption that

$$span\left(\cup_{\varphi \in \mathcal{K}} \varphi(F^\times)\right) = X$$

implies the existence of $\varphi_1, ..., \varphi_m \in \mathcal{K}$, $a_1, ..., a_m \in F^\times$, and $c_1, ..., c_m \in \mathbb{C}$ such that $u = \sum_{k=1}^m c_k \varphi_k(a_k)$. For $k = 1, ..., m$ define $\eta_k := \pi \begin{pmatrix} a_k & 0 \\ 0 & 1 \end{pmatrix} \varphi_k = \varphi_k(a_k \cdot)$. Then $\eta_k \in \mathcal{K}$ and $\eta_k(1) = \varphi_k(a_k)$. Thus if $\varphi := \sum_{k=1}^m c_k \eta_k \in \mathcal{K}$ then $\varphi(1) = u$ as desired.

Now let $u \in X$ be given and choose $\varphi \in \mathcal{K}$, $x \in F$ such that $\varphi(1) = u$ and $\psi(x) \neq 1$. If we define $\eta := \varphi - \pi \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \varphi$ then $\eta \in \mathcal{K} \cap \mathcal{S}_X$ and $\eta(1) = (1 - \psi(x))\varphi(1) = (1 - \psi(x))u$. Hence $\varphi := (1 - \psi(x))^{-1}\eta$ is in $\mathcal{K} \cap \mathcal{S}_X$ and has value u at $1 \in F^\times$.

□

Observe how the above lemma is an apparently stronger form of our third hypothesis in the definition of a Kirillov representation. Also note that it is certainly necessary for us to have this hypothesis in order for the next proposition to hold.

Proposition 5 $\mathcal{S}_X$ is a subspace of $\mathcal{K}$.

Proof It will suffice to prove that for every $f \in \mathcal{S}(F^\times)$ and $u \in X$ the element $f \otimes u$ of $\mathcal{S}_X$ is in $\mathcal{K}$. By Lemma 7.4 if $u \in X$ is given then there is a $\varphi \in \mathcal{K} \cap \mathcal{S}_X$ such that $\varphi(1) = u$. Then $u = \varphi(1) = \sum_{\mu \in \mathcal{U}^*} (P_\mu^X \varphi)(1)$ and so for $f \in \mathcal{S}(F^\times)$ we see $f \otimes u = \sum_{\mu \in \mathcal{U}^*} f \otimes ((P_\mu^X \varphi)(1))$.

Hence to prove the proposition we must show for every $f \in \mathcal{S}(F^\times)$, $\nu \in \mathcal{U}^*$ and $\varphi \in \mathcal{K} \cap \mathcal{S}_X(\nu)$ that $f \otimes \varphi(1)$ is in $\mathcal{K}$.

Well let $\nu \in \mathcal{U}^*$ and $\varphi \in \mathcal{K} \cap \mathcal{S}_X(\nu)$ be given. Put $u = \varphi(1)$ and define Y to be the space of all $f \in \mathcal{S}(F^\times)$ such that $f \otimes u \in \mathcal{K}$. Our goal is to show $Y = \mathcal{S}(F^\times)$. Observe that Y is ξ_ψ-invariant since if $f \in Y$ and $g \in B$ then

$$(\xi_\psi(g)f)u = \xi_\psi^X(g)(f \otimes u) = \pi(g)(f \otimes u) \in \mathcal{K}.$$

By Proposition 6.8 the representation $(\xi_\psi, \mathcal{S}(F^\times))$ of A is irreducible. Thus to show $Y = \mathcal{S}(F^\times)$ it now suffices to prove $Y \neq 0$. Let $\mu \in \mathcal{U}^*$ be so that $\mu \neq \nu$. Define $l \in \mathbb{Z}$ and $n \in \mathbb{N}$ to be the conductors of ψ and $\mu^{-1}\nu \neq 1$ respectively. Take $x \in F$ only to satisfy $val(x) = l - n$ and define $\eta = P_\mu^X \left(\xi_\psi^X \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \varphi \right)$. Then $\eta \in \mathcal{K} \cap \mathcal{S}_X$ and $\eta = \gamma(x, \mu^{-1}\nu, \psi)\nu^\times \otimes u$ by Lemma 6.6. Therefore $0 \neq \gamma(x, \mu^{-1}\nu, \psi)\nu^\times \in Y$ and we are done.

□

Proposition 6 The subspace $\mathcal{S}_X$ of $\mathcal{K}$ is invariant under B.

Proof Let $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in B$ and $\varphi \in \mathcal{S}_X$ be arbitrary. Then $d \neq 0$ and so

$$\pi \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \varphi = \omega_\pi(d)\xi_\psi^X \begin{pmatrix} ad^{-1} & bd^{-1} \\ 0 & 1 \end{pmatrix} \varphi = \omega_\pi(d)\psi(bd^{-1}\cdot)\varphi(ad^{-1}\cdot).$$

Since $\psi(bd^{-1}\cdot) \in \mathcal{S}(F^\times)$ and $\varphi(ad^{-1}\cdot) \in \mathcal{S}_X$ it is evident that $\pi \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \varphi$ is in $\mathcal{S}_X$.

□

Proposition 7 $\mathcal{K}$ is the sum of the subspaces $\mathcal{S}_X$ and $\pi(w)\mathcal{S}_X$.

Proof Fix a nonzero $\varphi \in \mathcal{S}_X$. Since $(\pi, \mathcal{K})$ is irreducible, $\mathcal{K}$ is spanned by the set $\{\pi(g)\varphi \, : \, g \in G\}$. If $g \in B$ then as $\mathcal{S}_X$ is invariant under B we know $\pi(g)\varphi \in \mathcal{S}_X$. Now let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be in G, but not in B, so that $c \neq 0$. Define $h = \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix}$ and $\eta = \pi \begin{pmatrix} -c & -d \\ 0 & b - adc^{-1} \end{pmatrix} \varphi$, so that $\eta \in \mathcal{S}_X$. Since $h \in U$ we know that $\pi(h)\pi(w)\eta - \pi(w)\eta \in \mathcal{S}(F^\times)$. Thus we have

$$\pi(g)\varphi = \pi(h)\pi(w)\eta = (\pi(h)\pi(w)\eta - \pi(w)\eta) + \pi(w)\eta \in \mathcal{S}_X + \pi(w)\mathcal{S}_X.$$

□

We have already explicitly described how B acts on $\mathcal{K}$. Since $G = BwU \cup B$ to understand the action π of G on $\mathcal{K}$ it suffices to understand how w acts on $\mathcal{K}$. Moreover as $\mathcal{K} = \mathcal{S}_X + \pi(w)\mathcal{S}_X$ it suffices to understand how w acts on $\mathcal{S}_X$. We now consider this problem.

For $a \in F^\times$ and $\mu \in \mathcal{U}^*$ we define the linear operator

$$J(a, \mu) : X \longrightarrow X \qquad J(a, \mu)u := (\pi(w)(\mu^\times(a^{-1}\cdot) \otimes u))(1).$$

So if $L : \mathcal{K} \longrightarrow X$ is the canonical linear map defined by $L(\varphi) := \varphi(1)$ then

$$J(a, \mu)u := L(\pi(w)(\mu^\times(a^{-1}\cdot) \otimes u)).$$

Proposition 8 For any $a \in F^\times$, $\epsilon \in \mathcal{U}$, and $\mu \in \mathcal{U}^*$ we have $J(\epsilon a, \mu) = \mu^{-1}(\epsilon) J(a, \mu)$.

Proof For $u \in X$ we find

$$\begin{aligned} J(\epsilon a, \mu)u &= (\pi(w)(\mu^\times(\epsilon^{-1}a^{-1}\cdot) \otimes u))(1) \\ &= \mu^{-1}(\epsilon)\pi(w)(\mu^\times(a^{-1}\cdot) \otimes u)(1) \\ &= \mu^{-1}(\epsilon)J(a, \mu)u. \end{aligned}$$

□

Proposition 9 For $a \in F^\times$, $\mu \in \mathcal{U}^*$ and $u \in X$ we have

$$J(a, \mu)u = (\pi(w)(\mu^\times(a^{-1}\cdot) \otimes u))(1) = \omega_\pi^{-1}(a)(\pi(w)(\mu^\times \otimes u))(a).$$

Hence for any fixed $\mu \in \mathcal{U}^*$ and $u \in X$ the map

$$: F^\times \longrightarrow X \qquad a \longmapsto J(a, \mu)u$$

is locally constant and vanishes off of a compact subset of F.

Proof

$$\begin{aligned} \pi(w)(\mu^\times(a^{-1}\cdot) \otimes u))(1) &= \left(\pi(w)\pi\begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix}(\mu^\times \otimes u)\right)(1) \\ &= \left(\pi\begin{pmatrix} a^{-1} & 0 \\ 0 & a^{-1} \end{pmatrix}\pi\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\pi(w)(\mu^\times \otimes u)\right)(1) \\ &= \omega_\pi^{-1}(a)\pi(w)(\mu^\times \otimes u)(a). \end{aligned}$$

The second assertion is an immediate consequence of the first assertion and Proposition 7.2.

□

Proposition 10 For $a, x \in F^\times$, $\mu \in \mathcal{U}^*$, and $u \in X$ we have

$$(\pi(w)(\mu^\times(a^{-1}\cdot) \otimes u))(x) = \omega_\pi(x)(\pi(w)(\mu^\times((ax)^{-1}\cdot) \otimes u))(1) = \omega_\pi(x)J(ax, \mu)u.$$

Proof The following calculation will prove the result.

$$\begin{aligned} (\pi(w)(\mu^\times(a^{-1}\cdot) \otimes u))(x) &= \left(\pi\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}\pi(w)(\mu^\times(a^{-1}\cdot) \otimes u)\right)(1) \\ &= \left(\pi\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}\pi(w)\pi\begin{pmatrix} x^{-1} & 0 \\ 0 & 1 \end{pmatrix}(\mu^\times(a^{-1}\cdot) \otimes u)\right)(1) \\ &= \omega_\pi(x)\pi(w)(\mu^\times((ax)^{-1}\cdot) \otimes u)(1) \\ &= \omega_\pi(x)J(ax, \mu)u. \end{aligned}$$

□

Proposition 11 If $\varphi \in \mathcal{S}_X$ and $b \in F^\times$ then

$$(\pi(w)\varphi)(b) = \omega_\pi(b)\sum_{\mu \in \mathcal{U}^*}\int_{F^\times} J(ab, \mu)\varphi(a)\, d^\times a.$$

Proof We first prove the formula when $b = 1$. By Proposition 6.3.2 we have the formula

$$\varphi = \sum_{n \in \mathbb{Z}} \sum_{\mu \in \mathcal{U}^*} \mu^\times(\tilde{\omega}^{-n} \cdot) \otimes \hat{\varphi}_n(\mu^{-1})$$

where the summations over $\mathbb{Z}$ and $\mathcal{U}^*$ are finite. By Proposition 7.8 we have

$$\begin{aligned}
(\pi(w)\varphi)(1) &= \sum_{n \in \mathbb{Z}} \sum_{\mu \in \mathcal{U}^*} \left(\pi(w)(\mu^\times(\tilde{\omega}^{-n} \cdot) \otimes \hat{\varphi}_n(\mu^{-1})\right)(1) \\
&= \sum_{n \in \mathbb{Z}} \sum_{\mu \in \mathcal{U}^*} J(\tilde{\omega}^n, \mu)\hat{\varphi}_n(\mu^{-1}) \\
&= \sum_{n \in \mathbb{Z}} \sum_{\mu \in \mathcal{U}^*} J(\tilde{\omega}^n, \mu) \left(\int_{\mathcal{U}} \mu^{-1}(\epsilon)\varphi(\epsilon\tilde{\omega}^n)\, d^\times \epsilon \right) \\
&= \sum_{n \in \mathbb{Z}} \sum_{\mu \in \mathcal{U}^*} \int_{\mathcal{U}} J(\epsilon\tilde{\omega}^n, \mu)\varphi(\epsilon\tilde{\omega}^n)\, d^\times \epsilon \\
&= \sum_{\mu \in \mathcal{U}^*} \int_{F^\times} J(a, \mu)\varphi(a)\, d^\times a.
\end{aligned}$$

Therefore for $b \in F^\times$ we have

$$\begin{aligned}
(\pi(w)\varphi)(b) &= \left(\pi \begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix} \pi(w)\varphi \right)(1) \\
&= \left(\pi \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} \pi(w)\pi \begin{pmatrix} b^{-1} & 0 \\ 0 & 1 \end{pmatrix} \varphi \right)(1) \\
&= \omega_\pi(b)(\pi(w)\varphi(b^{-1} \cdot))(1) \\
&= \omega_\pi(b) \sum_{\mu \in \mathcal{U}^*} \int_{F^\times} J(a, \mu)\varphi(b^{-1}a)\, d^\times a \\
&= \omega_\pi(b) \sum_{\mu \in \mathcal{U}^*} \int_{F^\times} J(ab, \mu)\varphi(a)\, d^\times a.
\end{aligned}$$

□

The proof of the following is lemma 5 in the first section of [2].

Lemma 12 The family of operators $\{J(a, \mu) : a \in F^\times\ \mu \in \mathcal{U}^*\}$ is a commutative family.

Proposition 13 If $R \in End(X)$ commutes with each of the operators $J(a, \mu)$ where $a \in F^\times$, $\mu \in \mathcal{U}^*$ then $R = cI_X$ for some $c \in \mathbb{C}$.

Proof Define Y to be the space of all functions from $F^\times$ to X. Note that $\mathcal{K}$ is a subspace of Y. Define the linear operator

$$T' : Y \longrightarrow Y \qquad (T'\varphi)(x) := R(\varphi(x)).$$

Clearly $\mathcal{S}_X$ is an invariant subspace of T'. Thus if $\varphi \in \mathcal{S}_X$ and $x \in F^\times$ then we have

$$\begin{aligned}
(\pi(w)(T'\varphi))(x) &= \omega_\pi(x) \sum_{\mu \in \mathcal{U}^*} \int_{F^\times} J(xy, \mu)R(\varphi(y))\, d^\times y \\
&= R\left(\omega_\pi(x) \sum_{\mu \in \mathcal{U}^*} \int_{F^\times} J(xy, \mu)\varphi(y)\, d^\times y \right) \\
&= R((\pi(w)\varphi)(x)) \\
&= (T'(\pi(w)\varphi))(x).
\end{aligned}$$

Hence $\pi(w)T'\varphi = T'\pi(w)\varphi$ for $\varphi \in \mathcal{S}_X$. This immediately implies that $\mathcal{K}$ is an invariant subspace of T'. Indeed if $\varphi \in \mathcal{K}$ we can write $\varphi = \varphi_1 + \pi(w)\varphi_2$ where $\varphi_1, \varphi_2 \in \mathcal{S}_X$ and hence

$$T'\varphi = T'\varphi_1 + T'\pi(w)\varphi_2 = T'\varphi_1 + \pi(w)T'\varphi_2 \in \mathcal{S}_X + \pi(w)\mathcal{S}_X = \mathcal{K}.$$

Let T denote the restriction of T' to $\mathcal{K}$ so that $T \in End(\mathcal{K})$. We have proven that $\pi(w)T\varphi = T\pi(w)\varphi$ for $\varphi \in \mathcal{S}_X$. We can easily show that this holds for all $\varphi \in \mathcal{K}$ be writing $\varphi = \varphi_1 + \pi(w)\varphi_2$ where $\varphi_1, \varphi_2 \in \mathcal{S}_X$ and then noting

$$\begin{aligned}\pi(w)T\varphi &= \pi(w)(T\varphi_1 + \pi(w)T\varphi_2)\\ &= T\pi(w)\varphi_1 + \omega_\pi(-1)T\varphi_2\\ &= T(\pi(w)\varphi_1 + \omega_\pi(-1)\varphi_2)\\ &= T\pi(w)\varphi\end{aligned}$$

Thus the operators T and $\pi(w)$ commute. Next we show that T and $\pi(g)$ commute for any $g \in B$. Indeed if $g = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in B$ and $\varphi \in \mathcal{K}$ then we have noted that

$$\pi(g)\varphi = \omega_\pi(d)\psi(bd^{-1}\cdot)\varphi(ad^{-1}\cdot).$$

Hence for $x \in F^\times$,

$$\begin{aligned}(T(\pi(g)\varphi))(x) &= T(\omega_\pi(d)\psi(bd^{-1}x)\varphi(ad^{-1}x))\\ &= \omega_\pi(d)\psi(bd^{-1}x)T(\varphi(ad^{-1}x))\\ &= \omega_\pi(d)\psi(bd^{-1}x)(T\varphi)(ad^{-1}x)\\ &= (\pi(g)(T\varphi))(x),\end{aligned}$$

and thus $T\pi(g) = \pi(g)T$. So $\pi(g)$ and T commute for $g \in B$ and $g = w$. By Proposition 5.7 G is generated by B and w thus we conclude that T commutes with π. Thus for some $c \in \mathbb{C}$ we have $T = cI_\mathcal{K}$ and this obviously implies that $R = cI_X$.

□

Corollary 14 The dimension of X is 1.

Proof Since the family of operators $\{J(a,\mu) : a \in F^\times \mu \in \mathcal{U}^*\}$ is commutative, the above proposition implies that each $J(a,\mu)$ is a scalar operator. Hence any $R \in End(X)$ commutes with every $J(a,\mu)$. Thus again by the above proposition we conclude that every $R \in End(X)$ is a scalar operator. This forces X to be one-dimensional.

□

The lemma tells us that even though initially X was an arbitrary complex vector space we lose nothing in assuming that $X = \mathbb{C}$. The following result is lemma 7 and 8 of [2].

Lemma 15

(1) For every $\mu \in \mathcal{U}^*$ the space $\pi(w)\mathcal{S}_X \cap \mathcal{S}_X(\mu)$ has finite codimension in $\mathcal{S}_X(\mu)$.

(2) For almost every $\mu \in \mathcal{U}^*$ the space $\pi(w)\mathcal{S}_X$ contains $\mathcal{S}_X(\mu)$.

With this lemma the next result follows easily.

Proposition 16 The space $\mathcal{S}_X$ has finite codimension in $\mathcal{K}$.

Proof Since $\pi(w) \in GL(\mathcal{K})$ we have that $\mathcal{S}_X$ and $\pi(w)\mathcal{S}_X$ are isomorphic vector spaces. Hence

$$\mathcal{K}/\mathcal{S}_X \simeq \mathcal{K}/\pi(w)\mathcal{S}_X = (\mathcal{S}_X + \pi(w)\mathcal{S}_X)/\pi(w)\mathcal{S}_X \simeq \mathcal{S}_X/(\mathcal{S}_X \cap \pi(w)\mathcal{S}_X).$$

Thus we need only prove that the space $\mathcal{S}_X/(\mathcal{S}_X \cap \pi(w)\mathcal{S}_X)$ is finite dimensional. The direct sum $\mathcal{S}_X = \bigoplus_{\mu\in\mathcal{U}^*} \mathcal{S}_X(\mu)$ implies that

$$\mathcal{K}/\mathcal{S}_X \simeq \mathcal{S}_X/(\mathcal{S}_X \cap \pi(w)\mathcal{S}_X) = \bigoplus_{\mu\in\mathcal{U}^*} \mathcal{S}_X(\mu)/(\pi(w)\mathcal{S}_X \cap \mathcal{S}_X(\mu)).$$

By Lemma 7.15 each summand in this direct sum is finite dimensional and moreover only finitely many of them are non-zero. The proposition is proven.

□

Uniqueness of Kirillov Models

Definition 17 Let ψ be a non-trivial unitary character of F. A representation $(\pi, \mathcal{K})$ of G is called a *Kirillov model* with respect to ψ iff $(\pi, \mathcal{K})$ is an infinite dimensional, irreducible, admissible representation of G which satisfies the following two properties:

(1) $\mathcal{K}$ is a vector space of complex valued functions on $F^\times$.

(2) For all $g \in A$ and $\varphi \in \mathcal{K}$ we have that $\pi(g)\varphi = \xi_\psi(g)\varphi$.

When we have a need to be explicit in the notation about the character ψ then we will denote the representation by $(\pi_\psi, \mathcal{K})$.

By Theorem 7.1 we obtain the following corollary.

Corollary 18 Suppose $(\pi, \mathcal{K})$ is a Kirillov model with respect to ψ. Then every function in $\mathcal{K}$ is locally constant and vanishes outside a compact subset of F. $\mathcal{S}(F^\times)$ is a subspace of $\mathcal{K}$ of finite codimension, $\mathcal{K} = \mathcal{S}(F^\times) + \pi(w)\mathcal{S}(F^\times)$, and for any $g \in U$ and $\varphi \in \mathcal{K}$ we have $\varphi - \pi(g)\varphi \in \mathcal{S}(F^\times)$.

Proposition 19 Let $(\pi, \mathcal{K})$ be a Kirillov model with respect to ψ. Then the space of linear functionals L in $\mathcal{K}^*$ which satisfy

$$L\left(\pi\begin{pmatrix}1 & a\\ 0 & 1\end{pmatrix}\varphi\right) = \psi(a)L(\varphi)$$

for all $a \in F$ and $\varphi \in \mathcal{K}$ is one dimensional. Moreover this one dimensional subspace of $\mathcal{K}^*$ is spanned by the functional : $\mathcal{K} \longrightarrow \mathbb{C}$ $\varphi \longmapsto \varphi(1)$.

Proof Let $L \in \mathcal{K}^*$ be any functional satisfying the property of the proposition. By restricting L to the subspace $\mathcal{S}(F^\times)$ of $\mathcal{K}$ Lemma 6.12 implies that there is a $c \in \mathbb{C}$ such that $L(\varphi) = c\varphi(1)$ for all $\varphi \in \mathcal{S}(F^\times)$. Let $\varphi \in \mathcal{K}$ be arbitrary. Choose any $x \in F$ so that $\psi(x) \neq 1$. Then as $\varphi - \pi\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\varphi \in \mathcal{S}(F^\times)$ we have

$$L(\varphi) = L(\varphi - \pi\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\varphi) + L(\pi\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\varphi) = c(\varphi(1) - \psi(x)\varphi(1)) + \psi(x)L(\varphi).$$

This implies $(1 - \psi(x))L(\varphi) = c(1 - \psi(x))\varphi(1)$. Thus $L(\varphi) = c\varphi(1)$ for all $\varphi \in \mathcal{K}$.

□

Theorem 20 If $(\pi, \mathcal{K})$ and $(\pi', \mathcal{K}')$ are Kirillov models with respect to ψ which are equivalent representations then $(\pi, \mathcal{K}) = (\pi', \mathcal{K}')$. Namely $\mathcal{K} = \mathcal{K}'$ as sets and $\pi = \pi'$ as functions.

Proof Let $R : \mathcal{K} \longrightarrow \mathcal{K}'$ be an intertwining isomorphism of the equivalent representations $(\pi, \mathcal{K})$ and $(\pi', \mathcal{K}')$. So if $g \in G$ and $\varphi \in \mathcal{K}$ we have $R(\pi(g)\varphi) = \pi'(g)R(\varphi)$. Define the linear functional $L : \mathcal{K} \longrightarrow \mathbb{C}$ by $L(\varphi) = R(\varphi)(1)$. Observe that for $x \in F$ and $\varphi \in \mathcal{K}$ we have

$$L(\pi\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\varphi) = R(\pi\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\varphi)(1) = (\pi'\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}R(\varphi))(1) = \psi(x)L(\varphi).$$

Hence by Propsition 7.19 there is a $c \in \mathbb{C}$ such that $L(\varphi) = c\varphi(1)$ for all $\varphi \in \mathcal{K}$. Thus for $a \in F^\times$ and $\varphi \in \mathcal{K}$

$$R(\varphi)(a) = (\pi'\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}R(\varphi))(1) = L(\pi\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\varphi) = c(\pi\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\varphi)(1) = c\varphi(a).$$

Thus $R(\varphi) = c\varphi$ and as R is injective we see $c \neq 0$. It now follows immediately that $(\pi, \mathcal{K}) = (\pi', \mathcal{K}')$.

□

Existence of Kirillov Models

In this section we prove that every infinite dimensional class of $\Pi(G)$ has a Kirillov model, which is necessarily unique by Theorem 7.20.

Lemma 21 Suppose (π, V) is a smooth representation of G. Fix a $v \in V$ and define the function $f : F \longrightarrow V$ by

$$f(x) = \pi\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}v.$$

The following hold.

(1) If for some $m \in \mathbb{Z}$ we have that f is constant on $\mathcal{P}^m$ then f is constant on the cosets of $\mathcal{P}^m$ in F.

(2) f is locally constant.

(3) Suppose $m \in \mathbb{Z}$, $a \in F$ and $\psi \in F^*$ satisfy

$$\int_{\mathcal{P}^m} \psi(ax)f(x)\, d_\psi x = 0.$$

Then for any $k \leq m$ this integral also vanishes when evaluated over $\mathcal{P}^k$.

Proof Suppose f is constant on $\mathcal{P}^m$. Let $x, y \in F$ be in the same coset of $\mathcal{P}^m$ in F so that $x - y \in \mathcal{P}^m$. Then as f is constant on $\mathcal{P}^m$ we find

$$\pi\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}^{-1}\pi\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}v = \pi\begin{pmatrix} 1 & x-y \\ 0 & 1 \end{pmatrix}v = f(x-y) = f(0) = v.$$

We therefore have that

$$f(x) = \pi\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}v = \pi\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}v = f(y).$$

This proves (1). To prove (2) let H be the open subgroup of G that stabilizes v. There is a $m \in \mathbb{N}$ so that $y \in \mathcal{P}^m$ implies $\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \in H$. Thus f is constant on $\mathcal{P}^m$. Now by part (1) for every $x \in F$ we know that f is constant on the open neighborhood $x + \mathcal{P}^m$ of x. Now we prove (3). First note that by (2) the integral is well defined. Observe that as $\psi(a\,\cdot)$ is in F^* we could assume that $a = 1$. Since the proof is the same as when $a \neq 1$ we will not bother making this assumption. Suppose $k \leq m$ and S is a set of representatives for the cosets of $\mathcal{P}^m$ in $\mathcal{P}^k$. Thus

$$\begin{aligned}
\int_{\mathcal{P}^k} \psi(ax) f(x)\, d_\psi x &= \sum_{s \in S} \int_{\mathcal{P}^m} \psi(a(s+x)) f(s+x)\, d_\psi x \\
&= \sum_{s \in S} \psi(as) \int_{\mathcal{P}^m} \psi(ax) \pi \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \pi \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} v\, d_\psi x \\
&= \sum_{s \in S} \psi(as) \pi \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \left(\int_{\mathcal{P}^m} \psi(ax) f(x)\, d_\psi x \right) \\
&= 0.
\end{aligned}$$

□

Lemma 22 Let ψ be a non-trivial unitary character of F^+ and define $l := cond(\psi) \in \mathbb{Z}$. Suppose Y is a complex vector space, $m \in \mathbb{Z}$, and $f \in \mathcal{S}_Y(\mathcal{P}^m)$. For any $n \geq m$ the following are equivalent

(1) f is constant on the cosets of $\mathcal{P}^n$ in $\mathcal{P}^m$.

(2) For all $a \in F$, $a \notin \mathcal{P}^{l-n}$ we have

$$\int_{\mathcal{P}^m} \psi(ax) f(x)\, d_\psi x = 0.$$

Proof First suppose (1) holds. Let S be a set of representatives for the cosets of $\mathcal{P}^n$ in $\mathcal{P}^m$. Then for $a \in F$, $a \notin \mathcal{P}^{l-n}$ we have

$$\int_{\mathcal{P}^m} \psi(ax) f(x)\, d_\psi x = \sum_{s \in S} \int_{\mathcal{P}^n} \psi(a(s+x)) f(s+x)\, d_\psi x = \sum_{s \in S} \psi(as) \left(\int_{\mathcal{P}^n} \psi(ax)\, d_\psi x \right) f(s).$$

Hence it suffices to show that $\psi(a\,\cdot)$ is non-trivial on $\mathcal{P}^n$. Well there is a $y \in F$ with $val(y) = l - 1$ such that $\psi(y) \neq 1$. We need only show that $a^{-1}y \in \mathcal{P}^n$. This is clear since $val(a^{-1}y) = val(y) - val(a) > (l-1) - (l-n) = n - 1$. We have now proved that (1) implies (2).

Suppose now that (2) holds. Consider f as an element of $\mathcal{S}_Y(F)$ by defining f to be equal to $0 \in Y$ at elements of F not in $\mathcal{P}^m$. Then (2) implies that the Fourier transform of f with respect to ψ, which as usual is defined for $y \in F$ by

$$\hat{f}(y) := \hat{f}_\psi(y) = \int_F f(x)\psi(xy)\, d_\psi x = \int_{\mathcal{P}^m} f(x)\psi(xy)\, d_\psi x,$$

has its support in $\mathcal{P}^{l-n}$. Therefore by the Fourier inversion formula if $x \in F$,

$$f(x) = \int_{\mathcal{P}^{l-n}} \psi(-xy)\hat{f}(y)\, d_\psi(y).$$

Let $a \in \mathcal{P}^m$ and $b \in \mathcal{P}^n$ be given. Note that for any $y \in \mathcal{P}^{l-n}$ we have $-by \in \mathcal{P}^l$ and so $\psi(-by) = 1$. Hence,

$$f(a+b) = \int_{\mathcal{P}^{l-n}} \psi(-(a+b)y)\hat{f}(y)\, d_\psi(y) = \int_{\mathcal{P}^{l-n}} \psi(-ay)\hat{f}(y)\, d_\psi(y) = f(a).$$

Thus f is constant on the coset $a + \mathcal{P}^n$ of $\mathcal{P}^m$.

□

Lemma 23 If (π, V) is an infinite dimensional irreducible admissible representation of G then the only $v \in V$ for which

$$\pi \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} v = v$$

for all $x \in F$ is $v = 0$.

Proof Of course this holds for $v = 0$. Suppose that the condition is satisfied for some $v \in V$. Define $H' = \{g \in G : \pi(g)v = v\}$ and $H = \{g \in G : \pi(g)(\mathbb{C}v) = \mathbb{C}v\}$. Clearly $H' \subseteq H$. Note that as H' is open and contains the matrices $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ for all $x \in F$ we must have by Lemma 5.9.1 that $SL_2(F)$ is contained in H' and hence also in H. Also for any $a \in F^\times$, since $\pi \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} v = \omega_\pi(a)v$ we see H contains the center of G. Thus by Proposition 5.4 the index of H in G is finite. Let $g_1, ..., g_m \in G$ be a complete set of representatives of the left cosets of H in G. For $j = 1, ..., m$ define $w_j = \pi(g_j)v$ and then define $W = span\{w_1, ..., w_m\}$. Let $g \in G$ and $j \in \{1, ..., m\}$ be given. Take $h \in H$ and $i \in \{1, ..., m\}$ so that $gg_j = g_ih$. Then $\pi(g)w_j = \pi(g_i)\pi(h)v \in \pi(g_i)(\mathbb{C}v) \in \mathbb{C}w_i$. Hence W is π-invariant and thus $W = 0$ or $W = V$. Since V is infinite dimensional and W is finite dimensional we conclude that $W = 0$. This implies $v = 0$.

□

We are now ready to construct the Kirillov model of an arbitrary infinite dimensional $\pi \in \Pi(G)$. For what follows we fix an infinite dimensional, irreducible, admissible representation (π, V) of G, and a non-trivial unitary character ψ of F. We always use l to denote the conductor of ψ. Define V_0 to be the set of all $v \in V$ such that for some $m \in \mathbb{Z}$

$$\int_{\mathcal{P}^m} \psi(-x)\pi \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} v\, d_\psi x = 0.$$

By Lemma 7.21 if $v \in V_0$ and $m \in \mathbb{Z}$ is such that

$$\int_{\mathcal{P}^m} \psi(-x)\pi \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} v\, d_\psi x = 0$$

then for any $k \in \mathbb{Z}$ with $k \leq m$ we again have that the integral vanishes

$$\int_{\mathcal{P}^k} \psi(-x)\pi \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} v\, d_\psi x = 0.$$

It follows that V_0 is a subspace of V.

Define $X = V/V_0$ and let $L : V \longrightarrow X$ be the quotient map. For $v \in V$ define the function $\varphi_v : F^\times \longrightarrow X$ by

$$\varphi_v(a) = L\left(\pi \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} v\right).$$

Let $\mathcal{K}$ denote the set of all the functions φ_v for $v \in V$. It is easy to verify that for $v_1, v_2 \in V$ and $c \in \mathbb{C}$ we have $\varphi_{cv_1+v_2} = c\varphi_{v_1} + \varphi_{v_2}$. Thus $\mathcal{K}$ is a complex vector space and the map $: V \longrightarrow \mathcal{K} \quad v \longmapsto \varphi_v$ is linear. By the definition of $\mathcal{K}$ this map is also surjective.

Proposition 24 The map $: V \longrightarrow \mathcal{K} \quad v \longmapsto \varphi_v$ is an isomorphism of vector spaces.

Proof We need only to show that the map is injective. Well let $v \in V$ be so that $\varphi_v = 0$. We wish to show that $v = 0$. Define $f : F \longrightarrow V$ by $f(x) = \pi \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} v$. The idea of the proof is to show that f is constant. For if this held then v would be fixed by all of the operators $\pi \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ for $x \in F$, and by Lemma 7.23 this implies $v = 0$.

Let $t \in \mathbb{N}$ be so that if $g \in G$ and $g \equiv e \, (mod \mathcal{P}^t)$ then $\pi(g)v = v$. Note that f is constant on $\mathcal{P}^t$. We proceed by induction. Suppose for some $n \in \mathbb{Z}$ we have proven that f is constant on $\mathcal{P}^n$. We will show that this implies that f is also constant on $\mathcal{P}^{n-1}$ which will finish the proof. Let us first observe that for any $a \in F^\times$ there is a $n(a) \in \mathbb{Z}$ such that

$$\int_{\mathcal{P}^{n(a)}} \psi(-ax) f(x) \, d_\psi x = 0.$$

Indeed, since $\varphi_v = 0$ we know $\pi \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} v \in V_0$. Thus there is a $k \in \mathbb{Z}$ such that

$$\int_{\mathcal{P}^k} \psi(-x) \pi \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \pi \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} v \, d_\psi x = 0.$$

We have the relation

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & a^{-1}x \\ 0 & 1 \end{pmatrix}.$$

Using this and performing the transformation $x \longmapsto ax$ we find

$$|a| \pi \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \left(\int_{a^{-1}\mathcal{P}^k} \psi(-ax) f(x) \, d_\psi x \right) = 0.$$

Therefore our observation holds by choosing $n(a) \in \mathbb{Z}$ to satisfy $\mathcal{P}^{n(a)} = a^{-1}\mathcal{P}^k$.

Now choose $a_1, ..., a_r \in F$ such that $val(a_j) = l - n$ for each j and so that if $a \in F$ is any element with $val(a) = l - n$ then there is a j and $\epsilon \in 1 + \mathcal{P}^t$ so that $a = \epsilon a_j$. Fix $m \in \mathbb{Z}$ with the only requirement being $m < min\{n, n(a_1), ..., n(a_r)\}$. By Lemma 7.21.3 we have for any j,

$$\int_{\mathcal{P}^m} \psi(-a_j x) f(x) \, d^+ x = 0.$$

Let $a \in F$ be any element with $val(a) = l - n$. Find a j and $\epsilon \in 1 + \mathcal{P}^t$ such that $a = \epsilon a_j$. We have the relation

$$\begin{pmatrix} \epsilon^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \epsilon^{-1}x \\ 0 & 1 \end{pmatrix}.$$

Transforming $x \longmapsto \epsilon^{-1}x$, using the above relation, and noting $\pi \begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix} v = v$ we find

$$\begin{aligned} \int_{\mathcal{P}^m} \psi(-ax) f(x) \, d_\psi x &= \int_{\mathcal{P}^m} \psi(-a_j x) f(\epsilon^{-1}x) \, d_\psi x \\ &= \pi \begin{pmatrix} \epsilon^{-1} & 0 \\ 0 & 1 \end{pmatrix} \left(\int_{\mathcal{P}^m} \psi(-a_j x) f(x) \, d_\psi x \right) \\ &= 0. \end{aligned}$$

Since f is constant on $\mathcal{P}^n$ by Lemma 7.21.1 it is also constant on the cosets of $\mathcal{P}^n$ in F. In particular f is constant on the cosets of $\mathcal{P}^n$ in $\mathcal{P}^m$. Thus by Lemma 7.22 and by the calculation above we know for any $a \in F$ with $a \notin \mathcal{P}^{l-(n-1)}$,

$$\int_{\mathcal{P}^m} \psi(ax)f(x)\, d_\psi x = 0.$$

The other direction of Lemma 7.22 implies that f is constant on the cosets of $\mathcal{P}^{n-1}$ in $\mathcal{P}^m$. In particular f is constant on $\mathcal{P}^{n-1}$ and this completes the proof.

□

Having established the above proposition we may now define the map $\pi_{\mathcal{K}} : G \longrightarrow GL(\mathcal{K})$ by

$$\pi_{\mathcal{K}}(g)\varphi_v = \varphi_{\pi(g)v}$$

where $g \in G$ and $\varphi \in \mathcal{K}$. It is easy to see that $(\pi_{\mathcal{K}}, \mathcal{K})$ is a representation of G.

Theorem 25 The map : $V \longrightarrow \mathcal{K}$ $v \longmapsto \varphi_v$ is an intertwining isomorphism of the representations (π, V) and $(\pi_{\mathcal{K}}, \mathcal{K})$ of G. Thus $(\pi_{\mathcal{K}}, \mathcal{K})$ is a Kirillov with respect to ψ that is equivalent to (π, V)

Proof That the isomorphism $v \longmapsto \varphi_v$ intertwines the two representations follows directly from the definition of $(\pi_{\mathcal{K}}, \mathcal{K})$. We have remarked that if there is an intertwining isomorphism between two representations and if one of the representations has any of the properties of being irreducible, infinite dimensional, or admissible then the respective property must hold as well for the other representation. Hence we need only verify that $(\pi_{\mathcal{K}}, \mathcal{K})$ is a Kirillov model, and this requires us to prove that $\pi_{\mathcal{K}}$ and ξ_ψ are equal on A. Well let $g = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in A$, $v \in V$, and $y \in F^\times$. We have

$$(\pi_{\mathcal{K}}(g)\varphi_v)(y) = \varphi_{\pi(g)v}(y) = L\left(\pi\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\pi\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}v\right).$$

and

$$(\xi_\psi(g)\varphi)(y) = \psi(by)\varphi_v(ay) = \psi(by)L\left(\pi\begin{pmatrix} ay & 0 \\ 0 & 1 \end{pmatrix}v\right).$$

Therefore we must show $\pi\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\pi\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}v - \psi(by)\pi\begin{pmatrix} ay & 0 \\ 0 & 1 \end{pmatrix}v \in ker(L) = V_0$. Observe that

$$\pi\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\pi\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}v - \psi(by)\pi\begin{pmatrix} ay & 0 \\ 0 & 1 \end{pmatrix}v$$

and

$$\pi\begin{pmatrix} 1 & by \\ 0 & 1 \end{pmatrix}\pi\begin{pmatrix} ay & 0 \\ 0 & 1 \end{pmatrix}v - \psi(by)\pi\begin{pmatrix} ay & 0 \\ 0 & 1 \end{pmatrix}v$$

are equal. Therefore all we have to prove is that if $b \in F$ and $v \in V$ then

$$\pi\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}v - \psi(b)v \in V_0.$$

To show this choose $n \in \mathbb{Z}$ so that $b \in \mathcal{P}^n$. Then

$$\int_{\mathcal{P}^n} \psi(-x)\pi\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\left(\pi\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}v - \psi(b)v\right)\, d^+x$$

$$= \int_{\mathcal{P}^n} \psi(-x)\pi \begin{pmatrix} 1 & b+x \\ 0 & 1 \end{pmatrix} v\, d^+x - \int_{\mathcal{P}^n} \psi(-x+b)\pi \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} v\, d^+x = 0.$$

□

We have proven the following theorem.

Theorem 26 Let an infinite dimensional class in $\Pi(G)$ be given. Then for each $\psi \in F^*$, $\psi \neq 1$ there exists a unique Kirillov model $(\pi, \mathcal{K})$ with respect to ψ which is in the class π.

So any result we prove about Kirillov models is really a result on the infinite dimensional classes in $\Pi(G)$.

Proposition 27 Suppose $\pi \in \Pi(G)$ is infinite dimensional. Let ψ and ψ' be non-trivial unitary characters of F^+, and let $b \in F^\times$ be the unique element of $F^\times$ which satisfies $\psi' = \psi(b\,\cdot)$. Let (π_ψ, V_ψ) and $(\pi_{\psi'}, V_{\psi'})$ be the Kirillov representations with respect to ψ and ψ' that are in π. Then (π_ψ, V_ψ) and $(\pi_{\psi'}, V_{\psi'})$ are equivalent representations. Moreover $V_\psi = V_{\psi'}$ and for $g \in G$, $\varphi \in V_{\psi'}$ we have

$$\pi_{\psi'}(g)\varphi = (\pi_\psi(g)\varphi(b^{-1}\,\cdot))(b\,\cdot).$$

Proof Define $\sigma : G \longrightarrow GL(V_\psi)$ by

$$\sigma(g)\varphi := (\pi_\psi(g)\varphi(b^{-1}\,\cdot))(b\,\cdot).$$

We easily check that (σ, V_ψ) is a well defined representation of G. The map : $V_\psi \longrightarrow V_\psi$ $\varphi \longmapsto \varphi(b^{-1}\,\cdot)$ is an intertwining isomorphism of the representations (σ, V_ψ) and (π_ψ, V_ψ). Observe that if $g = \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \in A$ and $\varphi \in V_\psi$ we have

$$\sigma(g)\varphi = (\xi_\psi(g)\varphi(b^{-1}\,\cdot))(b\,\cdot) = \psi(by\,\cdot)\varphi(x\,\cdot) = \psi'(y\,\cdot)\varphi(x\,\cdot) = \xi_{\psi'}(g)\varphi.$$

Thus (σ, V_ψ) is a Kirillov representation with respect to ψ' that is in π. By uniqueness we have $(\sigma, V_\psi) = (\pi_{\psi'}, V_{\psi'})$ as desired.

□

The above result implies that to any infinite dimensional class in $\Pi(G)$ there is associated a unique space of functions on $F^\times$, which does not depend on ψ. We formaly introduce a notation for this space.

Definition 28 Let π be an infinite dimensional class in $\Pi(G)$. We will denote by $\mathcal{K}(\pi)$ the unique space of complex valued functions on $F^\times$ which is the space of the Kirillov models that are in π. We call $\mathcal{K}(\pi)$ the *Kirillov space* of π. Given a non-trivial unitary character ψ of F^+ we denote by $(\pi_\psi, \mathcal{K}(\pi))$ the unique Kirillov representation with respect to ψ that is in π. We will call $(\pi_\psi, \mathcal{K}(\pi))$ the *Kirillov model* of π with respect to ψ.

When ψ is understood we will abbreviate $(\pi_\psi, \mathcal{K}(\pi))$ to $(\pi, \mathcal{K}(\pi))$. This notation is abusive as now π denotes both an equivalence class of representations and a specific action of G on $\mathcal{K}(\pi)$, this action depending on ψ. This will not cause any confusion.

Some Results on Kirillov Models

Corollary 29 Let (π, V) be an infinite dimensional irreducible admissible representation of G. Fix $\psi \in F^*$, $\psi \neq 1$. Then the space of linear functionals $L \in V^*$ such that for all $a \in F$ and $v \in V$

$$L\left(\pi\begin{pmatrix}1 & a\\ 0 & 1\end{pmatrix} v\right) = \psi(a)L(v)$$

is one dimensional.

Proof Since this result was already proven for Kirillov models in Proposition 7.19 the result follows from Theorem 7.26.

□

Proposition 30 Let π and π' be infinite dimensional classes in $\Pi(G)$ and let ψ be a non-trivial unitary character of F^+. If $\omega_\pi = \omega_{\pi'}$ and $\pi_\psi(w)(\mu^\times) = \pi'_\psi(w)(\mu^\times)$ for all $\mu \in \mathcal{U}^*$ then $\pi = \pi'$.

Proof Here when we write $\pi_\psi(w)(\mu^\times) = \pi'_\psi(w)(\mu^\times)$ we mean equality as functions from $F^\times$ to $\mathbb{C}$. We will prove that $(\pi_\psi, \mathcal{K}(\pi)) = (\pi'_\psi, \mathcal{K}(\pi'))$ which will certainly show that $\pi = \pi'$. First note if $\mu \in \mathcal{U}^*$ and $b \in F^\times$ then

$$\begin{aligned}\pi_\psi(w)(\mu^\times(b\cdot)) &= \pi_\psi(w)\xi_\psi\begin{pmatrix}b & 0\\ 0 & 1\end{pmatrix}\mu^\times\\ &= \omega_\pi(b)\xi_\psi\begin{pmatrix}b^{-1} & 0\\ 0 & 1\end{pmatrix}\pi_\psi(w)\mu^\times\\ &= \omega_{\pi'}(b)\xi_\psi\begin{pmatrix}b^{-1} & 0\\ 0 & 1\end{pmatrix}\pi'_\psi(w)\mu^\times\\ &= \pi'_\psi(w)(\mu^\times(b\cdot))\end{aligned}$$

Since the set of functions $\{\mu^\times(b\cdot) : \mu \in \mathcal{U}^*. b \in F^\times\}$ span $\mathcal{S}(F^\times)$ this shows that the operators $\pi_\psi(w)$ and $\pi'_\psi(w)$ agree on $\mathcal{S}(F^\times) \subseteq \mathcal{K}(\pi) \cap \mathcal{K}(\pi')$. Hence

$$\mathcal{K}(\pi) = \mathcal{S}(F^\times) + \pi_\psi(w)\mathcal{S}(F^\times) = \mathcal{S}(F^\times) + \pi'_\psi(w)\mathcal{S}(F^\times) = \mathcal{K}(\pi').$$

It remains to show that $\pi_\psi = \pi'_\psi$. Since $\omega_\pi = \omega_{\pi'}$ we know $\pi_\psi(g) = \pi'_\psi(g)$ whenever $g \in Z$. Also we have of course $\pi_\psi(g) = \xi_\psi(g) = \pi'_\psi(g)$ for $g \in A$. Hence $\pi_\psi(g) = \pi'_\psi(g)$ for all $g \in B = AZ$. By Proposition 5.7 it will now suffice to prove that $\pi_\psi(w) = \pi'_\psi(w)$. Let $\varphi \in \mathcal{K}(\pi) = \mathcal{K}(\pi')$ and write $\varphi = \varphi_1 + \pi_\psi(w)\varphi_2$ where $\varphi_1, \varphi_2 \in \mathcal{S}(F^\times)$. Then by above

$$\begin{aligned}\pi_\psi(w)\varphi = \pi_\psi(w)\varphi_1 + \omega_\pi(-1)\varphi_2 &= \pi'_\psi(w)\varphi_1 + \omega_{\pi'}(-1)\varphi_2\\ &= \pi'_\psi(w)(\varphi_1 + \pi'_\psi(w)\varphi_2)\\ &= \pi'_\psi(w)(\varphi_1 + \pi_\psi(w)\varphi_2)\\ &= \pi'_\psi(w)\varphi.\end{aligned}$$

□

Proposition 31 Let $\pi \in \Pi(G)$ be infinite dimensional and let χ be a character of $F^\times$. Then the map

$$: \mathcal{K}(\pi) \longrightarrow \mathcal{K}(\pi \otimes \chi) \quad \varphi \longmapsto \chi\varphi$$

is an isomorphism of vector spaces, where $\chi\varphi$ is the function on $F^\times$ defined by pointwise multiplicaltion. We have the formula

$$(\pi \otimes \chi)_\psi(g)(\chi\varphi) = \chi(det(g))\chi\pi_\psi(g)\varphi.$$

Proof Define the space $V := \{\chi\varphi : \varphi \in \mathcal{K}(\pi)\}$. Define a representation (σ, V) of G by

$$\sigma(g)(\chi\varphi) := \chi(det(g))\chi\pi_\psi(g)\varphi.$$

It is simple to check that the map $\varphi \longmapsto \chi\varphi$ is an intertwining isomorphism of the representations $(\pi_\psi \otimes \chi, \mathcal{K}(\pi))$ and (σ, V). Hence σ is in the equivalence class $\pi \otimes \chi$. One easily checks that (σ, V) is a Kirillov model with respect to ψ and hence $V = \mathcal{K}(\pi \otimes \chi)$.

□

Given $\varphi \in \mathcal{K}(\pi)$ we define $\tilde{\varphi} : F^\times \longrightarrow \mathbb{C}$ by $\tilde{\varphi}(a) := \omega_\pi^{-1}(a)\varphi(a)$.

Corollary 32 Let $\pi \in \Pi(G)$ be infinite dimensional. For every $\varphi \in \mathcal{K}(\pi)$ we have $\tilde{\varphi} \in \mathcal{K}(\tilde{\pi})$. The linear operator

$$: \mathcal{K}(\pi) \longrightarrow \mathcal{K}(\tilde{\pi}) \quad \varphi \longmapsto \tilde{\varphi}$$

is an isomorphism of vector spaces. It is an intertwining isomorphism iff $\omega_\pi = 1$. Hence $\omega_\pi = 1$ implies $\pi = \tilde{\pi}$. Moreover the action $\tilde{\pi}$ of G on $\mathcal{K}(\tilde{\pi})$ in terms of the action π of G on $\mathcal{K}(\pi)$ is given by the formula

$$\tilde{\pi}_\psi(g)\tilde{\varphi} = \omega_\pi^{-1}(det\, g)\omega_\pi^{-1}\pi_\psi(g)\varphi.$$

Proof This follows from the above proposition and the theorem that $\tilde{\pi} = \omega_\pi^{-1} \otimes \pi$.

□

Chapter 8 Classification of the Infinite Dimensional Classes in $\Pi(G)$

Cuspidal Representations

Lemma 1 Let (π, V) be a smooth representation of G. Suppose $v \in V$ and $n \in \mathbb{Z}$ satisfy

$$\int_{\mathcal{P}^n} \pi \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} v \, d^+x = 0.$$

Then for any $m \in \mathbb{Z}$ with $m < n$ the integral also vanishes when evaluated over $\mathcal{P}^m$.

$$\int_{\mathcal{P}^m} \pi \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} v \, d^+x = 0.$$

Proof Note that since the representation is smooth the integrand is a locally constant function. Hence the integral is well defined as it is taken over a compact set. Let S be a set of representatives of the cosets of $\mathcal{P}^n$ in $\mathcal{P}^m$. We have

$$\begin{aligned}\int_{\mathcal{P}^m} \pi \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} v \, d^+x &= \sum_{s \in S} \int_{\mathcal{P}^n} \pi \begin{pmatrix} 1 & s+x \\ 0 & 1 \end{pmatrix} v \, d^+x \\ &= \sum_{s \in S} \pi \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \int_{\mathcal{P}^n} \pi \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} v \, d^+x \\ &= 0\end{aligned}$$

□

Definition 2 Let (π, V) be an irreducible admissible representation of G. Then we say that (π, V) is a *cuspidal* representation of G iff for every $v \in V$ we have for some $n \in \mathbb{Z}$ that

$$\int_{\mathcal{P}^n} \pi \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} v \, d^+x = 0.$$

By the above lemma we know that if (π, V) is a cuspidal representation then for any $v \in V$ we have for sufficiently small $n \in \mathbb{Z}$ that

$$\int_{\mathcal{P}^n} \pi \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} v \, d^+x = 0.$$

Proposition 3 If (π, V) is a cuspidal representation then it is necessary that (π, V) be infinite dimensional.

Proof Let χ be a character of $F^\times$ and define the representation $(\pi_\chi, \mathbb{C})$ of G by $\pi_\chi(g)z := \chi(det(g))z$. We must show that $(\pi_\chi, \mathbb{C})$ is not cuspidal. This is clear since for $n \in \mathbb{Z}$, $z \in \mathbb{C}^\times$ we have

$$\int_{\mathcal{P}^n} \pi_\chi \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} z \, d^+x = \int_{\mathcal{P}^n} \chi(1) z \, d^+x = z\lambda^+(\mathcal{P}^n) \neq 0.$$

Hence every non-zero element z of $\mathbb{C}$ fails the condition of cuspidality.

□

Proposition 4 Let π be an infinite dimensional class in $\Pi(G)$ and let ψ be a non-trivial unitary character of F^+. If $(\pi, \mathcal{K}(\pi))$ is the Kirillov model of π with respect to ψ then $(\pi, \mathcal{K}(\pi))$ is cuspidal iff $\mathcal{K}(\pi) = \mathcal{S}(F^\times)$.

Proof Define $l := cond(\psi)$. For $n \in \mathbb{Z}$ and $\varphi \in \mathcal{K}(\pi)$ define

$$\varphi_n = \int_{\mathcal{P}^n} \pi \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \varphi \, d^+x.$$

Since the integrand is a locally constant function, and since $\mathcal{P}^n$ is compact we see that φ_n is a finite linear combination of functions in $\mathcal{K}(\pi)$. Therefore $\varphi_n \in \mathcal{K}(\pi)$ for every $n \in \mathbb{Z}$. For $y \in F^\times$ we have

$$\varphi_n(y) = \int_{\mathcal{P}^n} \psi(xy)\varphi(y) \, d^+x = \varphi(y) \int_{\mathcal{P}^n} \psi(xy) \, d^+x.$$

Observe that $\int_{\mathcal{P}^n} \psi(xy) \, d^+x \neq 0$ iff $\psi(\,\cdot\, y)$ is trivial on $\mathcal{P}^n$ iff $y\mathcal{P}^n \subseteq \mathcal{P}^l$ iff $y \in \mathcal{P}^{l-n}$.

Suppose $(\pi, \mathcal{K}(\pi))$ is cuspidal. Given $\varphi \in \mathcal{K}(\pi)$ choose a $n \in \mathbb{Z}$ so that $\varphi_n = 0$. Thus for $y \in \mathcal{P}^{l-n}$, $y \neq 0$ we have $\varphi(y) = 0$ and hence $\varphi \in \mathcal{S}(F^\times)$. Hence $\mathcal{K}(\pi) = \mathcal{S}(F^\times)$. Conversely suppose that $\mathcal{S}(F^\times) = \mathcal{K}(\pi)$, and let $\varphi \in \mathcal{K}(\pi)$ be given. For sufficiently small $n \in \mathbb{Z}$ we have $\mathcal{P}^{l-n} \cap supp(\varphi) = \emptyset$ and for such n we see $\varphi_n = 0$. Thus $(\pi, \mathcal{K}(\pi))$ is cuspidal.

□

The Representation $(\rho, \mathcal{B}(\chi_1, \chi_2))$

Let χ_1 and χ_2 be two characters of $F^\times$. Define $\mathcal{B}(\chi_1, \chi_2)$ to be the space of all locally constant functions $f : G \longrightarrow \mathbb{C}$ which satisfy

$$f\left(\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} g\right) = \chi_1(a)\chi_2(b)|\frac{a}{b}|^{\frac{1}{2}} f(g)$$

for all $a, b \in F^\times$, $x \in F$, $g \in G$.

Clearly any function in $\mathcal{B}(\chi_1, \chi_2)$ is invariant under left translations by elements of U. Also by putting $g = e$ in the above we see how functions in $\mathcal{B}(\chi_1, \chi_2)$ behave on B. Since $G = BK$ any two functions in $\mathcal{B}(\chi_1, \chi_2)$ which agree on K must be identical.

For any $g \in G$ and any function $f : G \longrightarrow \mathbb{C}$ we define $\rho(g)f = f(\,\cdot\, g)$ to be the right translate of f by g. Clearly if $g \in G$ and $f \in \mathcal{B}(\chi_1, \chi_2)$ then $\rho(g)f \in \mathcal{B}(\chi_1, \chi_2)$. Thus we obtain a representation $(\rho, \mathcal{B}(\chi_1, \chi_2))$ of G.

Proposition 5 Let $f : K \longrightarrow \mathbb{C}$ be a locally constant function which satisfies

$$f\left(\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} g\right) = \chi_1(a)\chi_2(b)f(g)$$

for all $a, b \in \mathcal{U}$, $x \in \mathcal{O}$, $g \in K$. Then there exists a unique extension of f to a locally constant function on G such that the extension is an element of $\mathcal{B}(\chi_1, \chi_2)$. Conversely the restriction of any element of $\mathcal{B}(\chi_1, \chi_2)$ to K satisfies this condition.

Proof This is simple but we write the details down anyway. The converse statement is trivial so the main point will be to prove the first assertion. Let f be satisfy the condition of the proposition. Since elements of $\mathcal{B}(\chi_1, \chi_2)$ are determined by their values on K we see that there is at most one extension of f to G such that the extension is in $\mathcal{B}(\chi_1, \chi_2)$. Hence the main thing to prove is the existence of such an extension. For this let $g \in G$ be given and write $g = \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} h$ where $\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \in B$ and $h \in K$. We then define

$$f(g) := \chi_1(a)\chi_2(b)|\frac{a}{b}|^{\frac{1}{2}} f(h).$$

We must verify that this provides a well defined extension of f to G. Suppose that

$$g = \begin{pmatrix} a_1 & x_1 \\ 0 & b_1 \end{pmatrix} h_1 = \begin{pmatrix} a_2 & x_2 \\ 0 & b_2 \end{pmatrix} h_2$$

are two decompositions of g arising from the Iwasawa decomposition $G = BK$. Define $t \in F$ by the relation

$$\begin{pmatrix} a_1^{-1}a_2 & t \\ 0 & b_1^{-1}b_2 \end{pmatrix} = \begin{pmatrix} a_1 & x_1 \\ 0 & b_1 \end{pmatrix}^{-1} \begin{pmatrix} a_2 & x_2 \\ 0 & b_2 \end{pmatrix}.$$

Thus $\begin{pmatrix} a_1^{-1}a_2 & t \\ 0 & b_1^{-1}b_2 \end{pmatrix} = h_1 h_2^{-1} \in K$. Hence $t \in \mathcal{O}$ and $a_1^{-1}a_2$ and $b_1^{-1}b_2$ are in $\mathcal{U}$. or what is the same $|a_1| = |a_2|$ and $|b_1| = |b_2|$. Now by our assumption on f we know

$$f(h_1) = f\left(\begin{pmatrix} a_1^{-1}a_2 & t \\ 0 & b_1^{-1}b_2 \end{pmatrix} h_2\right) = \chi_1(a_1^{-1}a_2)\chi_2(b_1^{-1}b_2) f(h_2).$$

Hence $\chi_1(a_1)\chi_2(b_1)f(h_1) = \chi_1(a_2)\chi_2(b_2)f(h_2)$. Combining this with our observation above that $|a_1| = |a_2|$ and $|b_1| = |b_2|$ yields

$$\chi_1(a_1)\chi_2(b_1)|\frac{a_1}{b_1}|^{\frac{1}{2}} f(h_1) = \chi_1(a_2)\chi_2(b_2)|\frac{a_2}{b_2}|^{\frac{1}{2}} f(h_2).$$

Thus our definition extends f to a well defined function on G which we also denote by f. We now have to prove $f \in \mathcal{B}(\chi_1, \chi_2)$. Well let $g \in G$ and $\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \in B$ be arbitrary. Write $g = \begin{pmatrix} a_1 & x_1 \\ 0 & b_1 \end{pmatrix} h$ where $h \in K$. Then by the definition of f we have

$$\begin{aligned} f\left(\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} g\right) &= f\left(\begin{pmatrix} aa_1 & * \\ 0 & bb_1 \end{pmatrix} h\right) \\ &= \chi_1(aa_1)\chi_2(bb_1)|\frac{aa_1}{bb_1}|^{\frac{1}{2}} f(h) \\ &= \left(\chi_1(a)\chi_2(b)|\frac{a}{b}|^{\frac{1}{2}}\right)\left(\chi_1(a_1)\chi_2(b_1)|\frac{a_1}{b_1}|^{\frac{1}{2}} f(h)\right) \\ &= \chi_1(a)\chi_2(b)|\frac{a}{b}|^{\frac{1}{2}} f(g). \end{aligned}$$

Now f is easily seen to be locally constant using the formula just proven and the fact that the restriction of f to K is locally constant. Thus $f \in \mathcal{B}(\chi_1, \chi_2)$ as desired.

□

See [4] regarding the next result.

Proposition 6 The representation $(\rho, \mathcal{B}(\chi_1, \chi_2))$ of G is admissible.

Theorem 7 If $\pi \in \Pi(G)$ is infinite dimensional and not cuspidal then π is equivalent to a constituent of $(\rho, \mathcal{B}(\chi_1, \chi_2))$ for some characters χ_1 and χ_2 of $F^\times$.

Proof Fix $1 \neq \psi \in F^*$. Define the finite dimensional space $V := \mathcal{K}(\pi)/\mathcal{S}(F^\times)$. Since π is not cuspidal V is non-zero. For $\varphi \in \mathcal{K}(\pi)$ denote $\bar{\varphi} := \varphi + \mathcal{S}(F^\times) \in V$. If we restrict the representation $(\pi_\psi, \mathcal{K}(\pi))$ to B then $\mathcal{S}(F^\times)$ is B-invariant. Hence we obtain a representation of B on V. Since $g \in U$ and $\varphi \in \mathcal{K}(\pi)$ imply that $\pi_\psi(g)\varphi - \varphi \in \mathcal{S}(F^\times)$ we see that the subgroup U of B acts trivially on V. Thus we get a representation $B/U \simeq T$ on V. More explicitly we have the well defined representation (τ, V) of T defined by

$$\tau(g)(\varphi + \mathcal{S}(F^\times)) := \pi_\psi(g)\varphi + \mathcal{S}(F^\times)$$

where $g \in T$ and $\varphi \in \mathcal{K}(\pi)$. Since T is abelian V decomposes into the direct sum of one dimensional τ-invariant subspaces. Hence there is a $\varphi_0 \in \mathcal{K}(\pi)$ and a subspace W of V such that $V = \mathbb{C}\bar{\varphi}_0 \oplus W$ and both of the subspaces $\mathbb{C}\bar{\varphi}_0$ and W are τ-invariant. There exist characters χ_1 and χ_2 of $F^\times$ such that

$$\tau\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\bar{\varphi}_0 = \chi_1(a)\chi_2(b)\bar{\varphi}_0$$

for all $a, b \in F^\times$. Define the linear functional

$$\bar{L} : V \longrightarrow \mathbb{C} \quad \bar{L}(\alpha\bar{\varphi}_0 + \eta) := \alpha$$

where $\alpha \in \mathbb{C}$ and $\eta \in W$. Extend $\bar{L}$ to $\mathcal{K}(\pi)$ by defining

$$L : \mathcal{K}(\pi) \longrightarrow \mathbb{C} \quad L(\varphi) := \bar{L}(\bar{\varphi}).$$

Let $\varphi \in \mathcal{K}(\pi)$ and $\begin{pmatrix} a & y \\ 0 & b \end{pmatrix} \in B$. We are going to prove that

$$L\left(\pi_\psi\begin{pmatrix} a & y \\ 0 & b \end{pmatrix}\varphi\right) = \chi_1(a)\chi_2(b)L(\varphi).$$

Let $\alpha \in \mathbb{C}$ and $\eta \in W$ satisfy $\bar{\varphi} = \alpha\bar{\varphi}_0 + \eta$. Thus

$$\tau\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\bar{\varphi} = \alpha\chi_1(a)\chi_2(b)\bar{\varphi}_0 + \eta'$$

where $\eta' \in W$. By Corollary 7.18 we see that

$$\pi_\psi\begin{pmatrix} a & y \\ 0 & b \end{pmatrix}\varphi - \pi_\psi\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\varphi = \pi_\psi\begin{pmatrix} 1 & b^{-1}y \\ 0 & 1 \end{pmatrix}\pi_\psi\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\varphi - \pi_\psi\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\varphi$$

is in $\mathcal{S}(F^\times)$. Therefore

$$\begin{aligned} L\left(\pi_\psi\begin{pmatrix} a & y \\ 0 & b \end{pmatrix}\varphi\right) &= \bar{L}\left(\tau\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\bar{\varphi}\right) \\ &= \alpha\chi_1(a)\chi_2(b) \\ &= \chi_1(a)\chi_2(b)L(\varphi) \end{aligned}$$

as desired.

Define the characters χ_1' and χ_2' of $F^\times$ by $\chi_1' := \chi_1|\cdot|^{-\frac{1}{2}}$ and $\chi_2' := \chi_2|\cdot|^{\frac{1}{2}}$. Hence for any $\begin{pmatrix} a & y \\ 0 & b \end{pmatrix} \in B$ and $\varphi \in \mathcal{K}(\pi)$ we have

$$L\left(\pi_\psi\begin{pmatrix} a & y \\ 0 & b \end{pmatrix}\varphi\right) = \chi_1'(a)\chi_2'(b)\left|\frac{a}{b}\right|^{\frac{1}{2}}L(\varphi).$$

For $\varphi \in \mathcal{K}(\pi)$ define $f_\varphi : G \longrightarrow \mathbb{C}$ by

$$f_\varphi(g) := L(\pi_\psi(g)\varphi).$$

Since the map : $G \longrightarrow V \quad g \longmapsto \pi_\psi(g)\varphi$ is locally constant we see that f_φ is locally constant. Also by what we have just proven we know that for any $\begin{pmatrix} a & y \\ 0 & b \end{pmatrix} \in B$ and $g \in G$ we have

$$f_\varphi\left(\begin{pmatrix} a & y \\ 0 & b \end{pmatrix} g\right) = \chi_1'(a)\chi_2'(b)|\frac{a}{b}|^{\frac{1}{2}} f_\varphi(g).$$

Hence $f_\varphi \in \mathcal{B}(\chi_1', \chi_2')$. Define the linear operator

$$R : \mathcal{K}(\pi) \longrightarrow \mathcal{B}(\chi_1', \chi_2') \quad \varphi \longmapsto f_\varphi.$$

We first note that R is injective. Indeed suppose $\varphi \in \mathcal{K}(\pi)$ is such that $R(\varphi) = f_\varphi = 0$. Hence for every $g \in G$ we have $L(\pi_\psi(g)\varphi) = 0$. If $\varphi \neq 0$ then $span\{\pi_\psi(g)\varphi \,:\, g \in G\} = \mathcal{K}(\pi)$. Since $L \neq 0$ we conclude $\varphi = 0$. Thus R is injective as desired.

To finish the proof it suffices to show that R intertwines the actions π_ψ and ρ. Let $g \in G$ and $\varphi \in \mathcal{K}(\pi)$. We must show that $R(\pi_\psi(g)\varphi) = \rho(g)R(\varphi)$. This is clear as $R(\pi_\psi(g)\varphi) = f_{\pi_\psi(g)\varphi} = f_\varphi(\cdot\, g) = \rho(g) f_\varphi = \rho(g)R(\varphi)$.

□

The Space $\mathcal{F}(\chi)$ and $\hat{\mathcal{F}}(\chi)$

Definition 8 Let χ be a character of $F^\times$. Define $\mathcal{F}(\chi)$ to be the space of all locally constant functions $\Phi : F \longrightarrow \mathbb{C}$ such that $\Phi(x)\chi(x)|x|$ is constant for $|x|$ sufficiently large.

Clearly $\mathcal{S}(F)$ is a subspace of $\mathcal{F}(\chi)$. Note that $\mathcal{F}(\chi)$ is translation invariant by elements of $F^\times$. Also note that the space $\mathcal{F}(|\cdot|^{-1})$ consists of all locally constant functions on F which are constant on the complement of a compact subset of F.

Given a character χ of $F^\times$ define $\Phi_\chi : F \longrightarrow \mathbb{C}$ by $\Phi_\chi(x) := \chi^{-1}(x)|x|^{-1}$ when $val(x) \leq 0$ and $\Phi_\chi(x) := 0$ when $x = 0$ or $val(x) > 0$. It is clear that $\Phi_\chi \in \mathcal{F}(\chi)$.

Proposition 9 We have the direct sum decomposition $\mathcal{F}(\chi) = \mathcal{S}(F) \oplus \mathbb{C}\Phi_\chi$. In particular $\mathcal{S}(F)$ is a subspace of $\mathcal{F}(\chi)$ of codimension 1.

Proof Let $\Phi \in \mathcal{F}(\chi)$. There is a $n \in \mathbb{Z}$, $n < 0$ and $c \in \mathbb{C}$ such that if $x \in F$, $val(x) < n$ then $\Phi(x)\chi(x)|x| = c$, or equivalently $\Phi(x) = c\Phi_\chi(x)$. Define $f \in \mathcal{S}(F)$ by $f(x) = 0$ when $val(x) < n$ and $f(x) = \Phi(x) - c\Phi_\chi(x)$ when $x = 0$ or $val(x) \geq n$. Then $\Phi = f + c\Phi_\chi$ and so $\mathcal{F}(\chi) = \mathcal{S}(F) + \mathbb{C}\Phi_\chi$. It remains to show that the sum of these subspaces is direct. Well suppose for some $c \in \mathbb{C}$ we have $c\Phi_\chi \in \mathcal{S}(F)$. Then for $x \in F^\times$ with $val(x)$ sufficiently small we have $0 = (c\Phi_\chi)(x) = c\chi^{-1}(x)|x|^{-1}$. This implies that $c = 0$ and hence $\mathcal{S}(F) \cap \mathbb{C}\Phi_\chi = \{0\}$.

□

What we wish to do now is to examine the Fourier transform $\hat{\Phi}_\chi$ of the function $\Phi_\chi \in \mathcal{F}(\chi)$ defined above. We need a lemma.

Lemma 10 Let χ be a character of $F^\times$ and let ψ be a non-trivial unitary character of F^+. For brevity define $\hat{\Phi}_\chi = (\hat{\Phi}_\chi)_\psi$. For $x \in F$ we have the identity

$$\hat{\Phi}_\chi(x) = \int_F \Phi_\chi(y)\psi(xy)\, d_\psi y = \lambda_\psi(\mathcal{U}) \sum_{k=0}^{\infty} \chi(\bar{\omega})^k \gamma(x\bar{\omega}^{-k}, \chi_0^{-1}, \psi)$$

where one side of this equality converges iff the other side converges. Here $\bar{\omega}$ is any uniformizing parameter of F, the choice being unimportant as the expression $\chi(\bar{\omega})^k\gamma(x\bar{\omega}^{-k},\chi_0^{-1},\psi)$ does not depend on $\bar{\omega}$.

Proof For $x \in F$ we have

$$\begin{aligned}\hat{\Phi}_\chi(x) = \int_F \Phi_\chi(y)\psi(xy)\,d_\psi y &= \int_{F\setminus\mathcal{P}} \chi^{-1}(y)|y|^{-1}\psi(xy)\,d_\psi y \\ &= \lambda_\psi(\mathcal{U})\int_{F\setminus\mathcal{P}} \chi^{-1}(y)\psi(xy)\,d^\times y \\ &= \lambda_\psi(\mathcal{U})\sum_{k=0}^{\infty}\int_{\mathcal{U}} \chi^{-1}(\epsilon\bar{\omega}^{-k})\psi(\epsilon\bar{\omega}^{-k}x)\,d^\times\epsilon \\ &= \lambda_\psi(\mathcal{U})\sum_{k=0}^{\infty}\chi(\bar{\omega})^k\gamma(x\bar{\omega}^{-k},\chi_0^{-1},\psi).\end{aligned}$$

□

Proposition 11 Let χ be a character of $F^\times$, and let ψ be a non-trivial unitary character of F. Define $l := cond(\psi)$ and for brevity denote $\hat{\Phi}_\chi = (\hat{\Phi}_\chi)_\psi$.

(1) Suppose $\chi = 1$. Then $\hat{\Phi}_1(0) = \infty$ and for $x \in F^\times$ we have

$$\begin{aligned}\hat{\Phi}_1(x) &= 0 \quad \text{if} \quad val(x) < l-1 \\ &= a\,val(x) + b \quad \text{if} \quad val(x) \geq l-1\end{aligned}$$

where $a := \lambda_\psi(\mathcal{U}) \neq 0$ and $b \in \mathbb{C}$.

(2) Suppose χ is unramified and $\chi \neq 1$. Then $\hat{\Phi}_\chi(0) = \infty$ when $|\chi(\mathcal{P})| \geq 1$ and $\hat{\Phi}_\chi(0) = \lambda_\psi(\mathcal{U})(1-\chi(\mathcal{P}))^{-1}$ when $|\chi(\mathcal{P})| < 1$. Also for $x \in F^\times$ we have

$$\begin{aligned}\hat{\Phi}_\chi(x) &= 0 \quad \text{if} \quad val(x) < l-1 \\ &= \lambda_\psi(\mathcal{U})(1-q)^{-1} \quad \text{if} \quad val(x) = l-1 \\ &= a\chi(x) + b \quad \text{if} \quad val(x) > l-1\end{aligned}$$

where $a := \lambda_\psi(\mathcal{U})\chi(\mathcal{P})^{1-l}(\chi(\mathcal{P})-q)(\chi(\mathcal{P})-1)^{-1}(1-q)^{-1}$ and $b := \lambda_\psi(\mathcal{U})(1-\chi(\mathcal{P}))^{-1}$. We moreover have that $a \neq 0$ iff $\chi \neq |\cdot|^{-1}$.

(3) Suppose χ is ramified and $n := cond(\chi)$. Then $\hat{\Phi}_\chi(0) = 0$ and for $x \in F^\times$ we have

$$\begin{aligned}\hat{\Phi}_\chi(x) &= 0 \quad \text{if} \quad val(x) < l-n \\ &= a\chi(x) \quad \text{if} \quad val(x) \geq l-n\end{aligned}$$

where $a := \lambda_\psi(\mathcal{U})\chi(\bar{\omega})^{n-l}\gamma(\bar{\omega}^{l-n},\chi_0^{-1},\psi) \neq 0$. Here $\bar{\omega}$ is any uniformizing parameter of F. the choice being unimportant as the expression $\chi(\bar{\omega})^{n-l}\gamma(\bar{\omega}^{l-n},\chi_0^{-1},\psi)$ does not depend on $\bar{\omega}$.

Hence in all cases $\hat{\Phi}_\chi$ converges on $F^\times$, and vanishes off of a compact subset of F.

Proof The basic formula we will use to prove the proposition is from the previous lemma

$$\hat{\Phi}_\chi(x) = \lambda_\psi(\mathcal{U})\sum_{k=0}^{\infty}\chi(\bar{\omega})^k\gamma(x\bar{\omega}^{-k},\chi_0^{-1},\psi)$$

(1) When $\chi = 1$ the formula becomes

$$\hat{\Phi}_1(x) = \lambda_\psi(\mathcal{U}) \sum_{k=0}^{\infty} \gamma(x\bar{\omega}^{-k}, 1, \psi).$$

When $x = 0$ we see $\gamma(x\bar{\omega}^{-k}, 1, \psi) = \gamma(0, 1, \psi) = 1$, and so $\hat{\Phi}_1$ does not converge at $0 \in F$. Suppose now that $x \in F^\times$. Then by Theorem 4.14 we know that

$$\begin{aligned}\gamma(x\bar{\omega}^{-k}, 1, \psi) &= 1 \quad \text{if} \quad k < val(x) - (l-1) \\ &= (1-q)^{-1} \quad \text{if} \quad k = val(x) - (l-1) \\ &= 0 \quad \text{if} \quad k > val(x) - (l-1)\end{aligned}$$

Thus $val(x) < l - 1$ implies $\gamma(x\bar{\omega}^{-k}, 1, \psi) = 0$ for all $k \geq 0$. Hence if $val(x) < l - 1$ we have $\hat{\Phi}_1(x) = 0$. When $val(x) = l - 1$ we get $\hat{\Phi}_1(x) = \lambda_\psi(\mathcal{U})(1-q)^{-1}$. Now suppose $val(x) > l - 1$. Then $\hat{\Phi}_1(x) = \lambda_\psi(\mathcal{U})(val(x) - (l-1) + (1-q)^{-1})$ and the desired conclusion follows.

(2) Now the formula becomes

$$\hat{\Phi}_\chi(x) = \lambda_\psi(\mathcal{U}) \sum_{k=0}^{\infty} \chi(\mathcal{P})^k \gamma(x\bar{\omega}^{-k}, 1, \psi).$$

When $x = 0$ we have $\hat{\Phi}_\chi(0) = \lambda_\psi(\mathcal{U}) \sum_{k=0}^{\infty} \chi(\mathcal{P})^k$. This series converges iff $|\chi(\mathcal{P})| < 1$, and when this is the case the sum is $\lambda_\psi(\mathcal{U})(1 - \chi(\mathcal{P}))^{-1}$. Suppose now that $x \in F^\times$. As above we know

$$\begin{aligned}\gamma(x\bar{\omega}^{-k}, 1, \psi) &= 1 \quad \text{if} \quad k < val(x) - (l-1) \\ &= (1-q^{-1})^{-1} \quad \text{if} \quad k = val(x) - (l-1) \\ &= 0 \quad \text{if} \quad k > val(x) - (l-1)\end{aligned}$$

The same reasons as in the first part show that when $val(x) < l - 1$ we have $\hat{\Phi}_\chi(x) = 0$ and when $val(x) = l - 1$ we have $\hat{\Phi}_\chi(x) = \lambda_\psi(\mathcal{U})(1-q)^{-1}$. Suppose that $val(x) > l - 1$. Since $\chi(\mathcal{P}) \neq 1$ we find

$$\begin{aligned}\hat{\Phi}_\chi(x) &= \lambda_\psi(\mathcal{U}) \left(\sum_{k=0}^{val(x)-l} \chi(\mathcal{P})^k + \chi(\mathcal{P})^{val(x)-l+1}(1-q)^{-1} \right) \\ &= \lambda_\psi(\mathcal{U}) \left(\frac{\chi(\mathcal{P})^{val(x)-l+1} - 1}{\chi(\mathcal{P}) - 1} \right) - q^{-1}\chi(\mathcal{P})^{val(x)-l+1}.\end{aligned}$$

Since χ is unramified we know that $\chi(\mathcal{P})^{val(x)} = \chi(x)$. Hence our calculation becomes $\hat{\Phi}_\chi(x) = a\chi(x) + b$ where a and b are defined as in the statement of the proposition. Now $a \neq 0$ iff $\chi(\mathcal{P}) \neq q$. Since χ is unramified by Proposition 4.9.1 there is a $s \in \mathbb{C}$ such that $\chi = |\cdot|^s$. Then $\chi(\mathcal{P}) = q$ iff $q^{-s} = q$ iff $s = -1$. Hence $a = 0$ iff $\chi = |\cdot|^{-1}$.

(3) In this case

$$\hat{\Phi}_\chi(0) = \lambda_\psi(\mathcal{U}) \sum_{k=0}^{\infty} \chi(\bar{\omega})^k \gamma(0, \chi_0^{-1}, \psi) = \lambda_\psi(\mathcal{U}) \sum_{k=0}^{\infty} \chi(\bar{\omega})^k \delta(\chi_0) = 0.$$

Suppose now that $x \in F^\times$. We know that $\gamma(x\bar{\omega}^{-k}, \chi_0^{-1}, \psi) \neq 0$ iff $val(x\bar{\omega}^{-k}) = l - n$ iff $k = val(x) - (l - n)$. Hence $val(x) < l - n$ implies $\hat{\Phi}_\chi(x) = 0$. When $val(x) \geq l - n$ then

$$\hat{\Phi}_\chi(x) = \lambda_\psi(\mathcal{U})\chi(\bar{\omega})^{val(x)-(l-n)}\gamma(x\bar{\omega}^{(l-n)-val(x)}, \chi_0^{-1}, \psi).$$

Write $x = \epsilon\varpi^m$ where $\epsilon \in U$ and $m \in \mathbb{Z}$. We then have $\gamma(x\varpi^{(l-n)-val(x)}, \chi_0^{-1}, \psi) = \chi(\epsilon)\gamma(\varpi^{l-n}, \chi_0^{-1}, \psi)$. Thus $\hat{\Phi}_\chi(x) = a\chi(x)$ where a is defined as in the statement of the proposition.

□

If M is a neighborhood of 0 in F then we call the subset $M \setminus \{0\}$ of $F^\times$ a *punctured neighborhood* of 0.

Corollary 12 Let χ be a character of $F^\times$, and let ψ be a non-trivial unitary character of F^+. If $\Phi \in \mathcal{F}(\chi)$ then $\hat{\Phi} := \hat{\Phi}_\psi$ is convergent on $F^\times$. Considering $\hat{\Phi}$ as a function on $F^\times$ it is locally constant, vanishes off a compact subset of F, and in a punctured neighborhood of $0 \in F$ is given by

$$\begin{aligned}\hat{\Phi}(x) &= a\chi(x) + b \quad \text{if} \quad \chi \neq 1, |\cdot|^{-1}\\ &= a\, val(x) + b \quad \text{if} \quad \chi = 1\\ &= b \quad \text{if} \quad \chi = |\cdot|^{-1}\end{aligned}$$

where $a, b \in \mathbb{C}$ depend on Φ.

Proof This follows from the above result, the decomposition $\mathcal{F}(\chi) = \mathcal{S}(F) \oplus \mathbb{C}\Phi_\chi$, and the fact that $\Phi \in \mathcal{S}(F)$ implies $\hat{\Phi} \in \mathcal{S}(F)$.

□

Let χ be a character of $F^\times$ and let $1 \neq \psi \in F^*$. We have shown that if $\Phi \in \mathcal{F}(\chi)$ then the Fourier transform $\hat{\Phi}_\psi$ of Φ converges on $F^\times$, but not necessarily at $0 \in F$. We wish to introduce the convention, which only applies to this situation, that we use $\hat{\Phi}_\psi$ to denote the restiction to $F^\times$. So from now on when $\Phi \in \mathcal{F}(\chi)$ and we speak of $\hat{\Phi}_\psi$ we mean the function : $F^\times \longrightarrow \mathbb{C} \quad a \longmapsto \hat{\Phi}_\psi(a)$.

Lemma 13 Let ψ and ψ' be two non-trivial unitary characters of F^+. Then for any character χ of $F^\times$ we have

$$\{\hat{\Phi}_\psi : \Phi \in \mathcal{F}(\chi)\} = \{\hat{\Phi}_{\psi'} : \Phi \in \mathcal{F}(\chi)\}.$$

Proof Let $b \in F^\times$ be such that $\psi' = \psi(b\,\cdot)$. For $\Phi \in \mathcal{F}(\chi)$ we easily prove the identity $\hat{\Phi}_\psi = |b|(\hat{\Phi}(b\,\cdot))_{\psi'}$. The lemma now follows by recalling that the space $\mathcal{F}(\chi)$ is invariant under translations by elements of $F^\times$.

□

Definition 14 Let χ be a character of $F^\times$. Choose a non-trivial unitary character ψ of F^+. Define the complex vector space

$$\hat{\mathcal{F}}(\chi) := \{\hat{\Phi}_\psi \ : \ \Phi \in \mathcal{F}(\chi)\}.$$

By the lemma above this space does not depend on the particular choice of ψ.

So $\mathcal{F}(\chi)$ is a space of functions on F and $\hat{\mathcal{F}}(\chi)$ is a space of functions on $F^\times$.

Corollary 15 Let χ be a character of $F^\times$. Then $\hat{\mathcal{F}}(\chi)$ is the space of functions $\eta : F^\times \longrightarrow \mathbb{C}$ of the form

$$\begin{aligned}\eta(x) &= \chi(x)f_1(x) + f_2(x) \quad \text{if} \quad \chi \neq 1, |\cdot|^{-1}\\ &= val(x)f_1(x) + f_2(x) \quad \text{if} \quad \chi = 1\\ &= f_1(x) \quad \text{if} \quad \chi = |\cdot|^{-1}.\end{aligned}$$

where $f_1, f_2 \in \mathcal{S}(F)$ are arbitrary. Hence $\mathcal{S}(F^\times)$ is a subspace of $\dot{\mathcal{F}}(\chi)$ with codimension 2 when $\chi \neq |\cdot|^{-1}$. and with codimension 1 when $\chi = |\cdot|^{-1}$.

Proof We have already seen in Corallary 8.12 that any function in $\dot{\mathcal{F}}(\chi)$ is of the claimed form. Conversely it is easily seen that any function satisfying these conditions is in $\dot{\mathcal{F}}(\chi)$. This is done by using Proposition 8.11 where we explicitly computed $\dot{\Phi}_\chi$. and by recalling that $\dot{} : \mathcal{S}(F) \longrightarrow \mathcal{S}(F) \quad f \longmapsto \dot{f}$ is an isomorphism. The last assertion is easy to prove.

□

Theorem 16 Let χ be a character of $F^\times$ and let ψ be a non-trivial unitary character of F^+. The surjective linear operator

$$\dot{} : \mathcal{F}(\chi) \longrightarrow \dot{\mathcal{F}}(\chi) \qquad \Phi \longmapsto \dot{\Phi}_\psi$$

is an isomorphism of vector spaces iff $\chi \neq |\cdot|^{-1}$. If $\chi = |\cdot|^{-1}$ then the kernel of this operator is the subspace of $\mathcal{F}(|\cdot|^{-1})$ consisting of the constant functions on F. Moreover if $\chi \neq |\cdot|^{-1}$ and $\Phi \in \mathcal{F}(\chi)$ is such that $\dot{\Phi} \in \mathcal{S}(F^\times)$ then $\Phi \in \mathcal{S}(F)$.

Proof For any $\Phi \in \mathcal{F}(\chi)$ we make the abbreviation $\dot{\Phi} := \dot{\Phi}_\psi$. The map $\Phi \longmapsto \dot{\Phi}$ is easily seen to be linear and by the definition of $\dot{\mathcal{F}}(\chi)$ it is also surjective. We need to determine the kernel of this operator.

First consider the case $\chi \neq |\cdot|^{-1}$. Let $\Phi \in \mathcal{F}(\chi)$ be in the kernel and write $\Phi = f + c\Phi_\chi$ where $f \in \mathcal{S}(F)$ and $c \in \mathbb{C}$. Then $0 = \dot{\Phi} = \dot{f} + c\dot{\Phi}_\chi$ where here we are considering $\dot{f} \in \mathcal{S}(F)$ as a function on $F^\times$. Hence $c\dot{\Phi}_\chi$, which is a function on $F^\times$, can be defined at $0 \in F$ such that the resulting function on F is in $\mathcal{S}(F)$. Since $\chi \neq |\cdot|^{-1}$ Proposition 8.11 implies that $c = 0$. Hence $\dot{f} = 0$ and so $f = 0$. We conclude that $\Phi = 0$ and thus when $\chi \neq |\cdot|^{-1}$ the map is an isomorphism.

Suppose now that $\chi = |\cdot|^{-1}$. Recall that the space of constant functions on F is a subspace of $\mathcal{F}(\chi)$. Let us first observe that if $\Phi : F \longrightarrow \mathbb{C}$ is a constant function then Φ is in the kernel. This is trivial since for any $x \in F^\times$ we have

$$\begin{aligned}\dot{\Phi}(x) = \int_F \Phi(y)\psi(xy)\, d_\psi y &= \Phi(0) \int_F \psi(xy)\, d_\psi y \\ &= \Phi(0) lim_{m \to -\infty} \int_{\mathcal{P}^m} \psi(xy)\, d_\psi y \\ &= 0(\Phi(0)) \\ &= 0.\end{aligned}$$

Define $l := cond(\psi)$. Recall that $\dot{\Phi}_\chi(x)$ equals 0 when $val(x) < l - 1$ and equals $\lambda_\psi(\mathcal{U})(1 - q)^{-1}$ when $val(x) \geq l - 1$. Since $\dot{} : \mathcal{S}(F) \longrightarrow \mathcal{S}(F) \quad \eta \longmapsto \dot{\eta}$ is an isomorphism and since $\dot{\Phi}_\chi \in \mathcal{S}(F)$ we know that there is a unique $\eta \in \mathcal{S}(F)$ such that $\dot{\eta} = \dot{\Phi}_\chi$. It is easy to check that $\eta(x)$ equals -1 when $val(x) > 0$ and equals 0 when $val(x) \leq 0$. Now let $\Phi \in \mathcal{F}(\chi)$ be a function in the kernel. Write $\Phi = f + c\Phi_\chi$ where $f \in \mathcal{S}(F)$ and $c \in \mathbb{C}$. Hence $0 = \dot{f} + c\dot{\Phi}_\chi = \dot{f} + c\dot{\eta}$. Now $c = 0$ implies $\dot{f} = 0$, and so $f = 0$, and thus $\Phi = 0$. When $c \neq 0$ we have $-c^{-1}\dot{f} = \dot{\eta}$ and hence $-c^{-1}f = \eta$. Since $\Phi_\chi - \eta = 1$ this implies $\Phi = f + c\Phi_\chi = -c\eta + c\Phi_\chi = c(\Phi_\chi - \eta) = c$. Thus Φ is constant. Hence Φ is in the kernel iff Φ is constant.

It remains to prove the final assertion. Suppose that $\Phi \in \mathcal{F}(\chi)$ is such that $\dot{\Phi} \in \mathcal{S}(F^\times)$. Write $\Phi = f + c\Phi_\chi$ where $f \in \mathcal{S}(F)$ and $c \in \mathbb{C}$. We conclude that $c\dot{\Phi}_\chi$ is in $\mathcal{S}(F^\times)$. Since $\chi \neq |\cdot|^{-1}$ the same reason as above implies that $c = 0$. Thus $\Phi = f \in \mathcal{S}(F)$ as desired.

□

Whenever χ_1 and χ_2 are characters of $F^\times$ we define $\chi := \chi_1\chi_2^{-1}$.

Definition 17 Let χ_1 and χ_2 be characters of $F^\times$. For $f \in \mathcal{B}(\chi_1, \chi_2)$ we define the function

$$\Phi_f : F \longrightarrow \mathbb{C} \qquad \Phi_f(x) := f\left(w\begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix}\right) = f\begin{pmatrix} 0 & 1 \\ -1 & x \end{pmatrix}.$$

Recall that if $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is in the big cell, namely if $c \neq 0$, then we have the identity

$$g = \begin{pmatrix} c^{-1}det(g) & -a \\ 0 & c \end{pmatrix} w \begin{pmatrix} 1 & -c^{-1}d \\ 0 & 1 \end{pmatrix}.$$

Therefore for $f \in \mathcal{B}(\chi_1, \chi_2)$ and g in the big cell we see

$$f(g) = \chi_1(c^{-1}det(g))\chi_2(c)|\frac{c^{-1}det(g)}{c}|^{\frac{1}{2}}\Phi_f(\frac{d}{c}).$$

Simplifying this gives the identity

$$f(g) = \chi_1(det(g))|det(g)|^{\frac{1}{2}}|c|^{-1}\chi^{-1}(c)\Phi_f(c^{-1}d).$$

Proposition 18 Let χ_1 and χ_2 be characters of $F^\times$ and define $\chi := \chi_1\chi_2^{-1}$. If $f \in \mathcal{B}(\chi_1, \chi_2)$ then Φ_f is locally constant and for large $|x|$ we have

$$\Phi_f(x) = \chi^{-1}(x)|x|^{-1}f(e).$$

Hence $\Phi_f \in \mathcal{F}(\chi)$.

Proof Since f is locally constant on G it is obvious that Φ_f is locally constant on F. For $x \in F^\times$ we have the identity

$$w\begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x^{-1} & 1 \\ 0 & x \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -x^{-1} & 1 \end{pmatrix}.$$

It follows that

$$\Phi_f(x) = f\left(w\begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix}\right) = \chi^{-1}(x)|x|^{-1}f\begin{pmatrix} 1 & 0 \\ -x^{-1} & 1 \end{pmatrix}.$$

For $|x|$ sufficiently large we have $f\begin{pmatrix} 1 & 0 \\ -x^{-1} & 1 \end{pmatrix} = f(e)$. The proposition now follows.

□

Proposition 19 Let χ_1 and χ_2 be characters of $F^\times$ and define $\chi := \chi_1\chi_2^{-1}$. The map

$$: \mathcal{B}(\chi_1, \chi_2) \longrightarrow \mathcal{F}(\chi) \qquad f \longmapsto \Phi_f$$

is an isomorphism of vector spaces.

Proof The map is well defined by Proposition 8.18 and it is clearly linear. To show it is injective suppose that $f \in \mathcal{B}(\chi_1, \chi_2)$ is such that $\Phi_f = 0$. If $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is in the big cell, namely if $c \neq 0$, then

$$f(g) = \chi_1(det(g))|det(g)|^{\frac{1}{2}}|c|^{-1}\chi^{-1}(c)\Phi_f(c^{-1}d) = 0.$$

So f is zero on the big cell. Since f is locally constant on G and since the big cell is dense in G this implies that $f = 0$. Hence the map is injective.

Now we show that the map is surjective. Let $\Phi \in \mathcal{F}(\chi)$ be arbitrary. There is a $z \in \mathbb{C}$ such that $\Phi(x) = z\chi^{-1}(x)|x|^{-1}$ when $|x|$ is sufficiently large. Define $f : G \longrightarrow \mathbb{C}$ by

$$f(g) := z\chi_1(a)\chi_2(d)|\frac{a}{d}|^{\frac{1}{2}},$$

for $g = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in B$, and when $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c^{-1}det(g) & -a \\ 0 & c \end{pmatrix} w \begin{pmatrix} 1 & -c^{-1}d \\ 0 & 1 \end{pmatrix}$ is in the big cell define

$$f(g) := \chi_1(det(g))\chi^{-1}(c)|c|^{-1}|det(g)|^{\frac{1}{2}}\Phi(\frac{d}{c}).$$

Then $f \in \mathcal{B}(\chi_1, \chi_2)$ and $\Phi_f = \Phi$.

□

Definition 20 Let ψ be a non-trivial unitary character of F^+. For $f \in \mathcal{B}(\chi_1, \chi_2)$ define

$$\varphi_f^{\psi} : F^{\times} \longrightarrow \mathbb{C} \qquad \varphi_f^{\psi} := \chi_2 | \cdot |^{\frac{1}{2}}(\hat{\Phi}_f)_{\psi}.$$

When there is no need to emphasize the dependence on ψ we make the abbreviation $\varphi_f = \chi_2| \cdot |^{\frac{1}{2}}\hat{\Phi}_f$. We also define the complex vector space

$$\mathcal{K}(\chi_1, \chi_2) := \{\varphi_f^{\psi} : f \in \mathcal{B}(\chi_1, \chi_2)\}.$$

This space does not depend on ψ, since $\hat{\mathcal{F}}(\chi)$ does not depend on ψ. The map

$$: \hat{\mathcal{F}}(\chi) \longrightarrow \mathcal{K}(\chi_1, \chi_2) \quad \eta \longmapsto \chi_2| \cdot |^{\frac{1}{2}}\eta$$

is an isomorphism of vector spaces. Clearly $\eta \in \mathcal{S}(F^{\times})$ iff $\chi_2| \cdot |^{\frac{1}{2}}\eta \in \mathcal{S}(F^{\times})$. Therefore as $\mathcal{S}(F^{\times})$ is a subspace of $\hat{\mathcal{F}}(\chi)$ we conclude that $\mathcal{S}(F^{\times})$ is also a subspace of $\mathcal{K}(\chi_1, \chi_2)$. Hence for $f \in \mathcal{B}(\chi_1, \chi_2)$ we have $\varphi_f^{\psi} \in \mathcal{S}(F^{\times})$ iff $(\hat{\Phi}_f)_{\psi} \in \mathcal{S}(F^{\times})$. Also by the previous theorem we know that $\varphi_f^{\psi} \in \mathcal{S}(F^{\times})$ implies that $\Phi_f \in \mathcal{S}(F)$.

Corollary 21 Let χ_1 and χ_2 be characters of $F^{\times}$. Then $\mathcal{K}(\chi_1, \chi_2)$ is the space of functions $\varphi : F^{\times} \longrightarrow \mathbb{C}$ which are of the form

$$\begin{aligned} \varphi(x) &= |x|^{\frac{1}{2}}(\chi_1(x)f_1(x) + \chi_2(x)f_2(x)) \quad \text{if} \quad \chi \neq 1, | \cdot |^{-1} \\ &= |x|^{\frac{1}{2}}\chi_2(x)(val(x)f_1(x) + f_2(x)) \quad \text{if} \quad \chi = 1 \\ &= |x|^{\frac{1}{2}}\chi_2(x)f_2(x) \quad \text{if} \quad \chi = | \cdot |^{-1} \end{aligned}$$

where $f_1, f_2 \in \mathcal{S}(F)$ are arbitrary. Hence $\mathcal{S}(F^{\times})$ is a subspace of $\mathcal{K}(\chi_1, \chi_2)$ of codimension 2 when $\chi \neq | \cdot |^{-1}$ and when $\chi = | \cdot |$ it is of codimension 1.

Proof This follows immediately from Corollary 8.15.

□

Proposition 22 Let ψ be a non-trivial unitary character of F^+, let χ_1 and χ_2 be characters of $F^{\times}$, and define $\chi := \chi_1\chi_2^{-1}$. Consider the surjective linear operator

$$: \mathcal{B}(\chi_1, \chi_2) \longrightarrow \mathcal{K}(\chi_1, \chi_2) \qquad f \longmapsto \varphi_f^{\psi}.$$

If $\chi \neq |\cdot|^{-1}$ then this operator is an isomorphism of vector spaces. If $\chi = |\cdot|^{-1}$ then the kernel of this operator is a one dimensional ρ-invariant subspace of $\mathcal{B}(\chi_1, \chi_2)$ spanned by the function

$$f_0 : G \longrightarrow \mathbb{C} \qquad g \longmapsto \chi_1(det(g))|det(g)|^{\frac{1}{2}}.$$

Consequently we obtain an isomorphism of vector spaces

$$: \mathcal{B}(\chi_1, \chi_2)/\mathbb{C}f_0 \longrightarrow \mathcal{K}(\chi_1, \chi_2).$$

Proof A function $f \in \mathcal{B}(\chi_1, \chi_2)$ is in the kernel iff $\varphi_f^{\psi} = 0$ iff $(\dot{\Phi}_f)_{\psi} = 0$. In Theorem 8.16 we determined the kernel of the map

$$: \mathcal{F}(\chi) \longrightarrow \hat{\mathcal{F}}(\chi) \qquad \Phi \longmapsto \dot{\Phi}_{\psi}.$$

Recall that the kernel of this map is trivial when $\chi \neq |\cdot|^{-1}$, and when $\chi = |\cdot|^{-1}$ it consists precisely of the constant functions in $\mathcal{F}(|\cdot|^{-1})$. Therefore if f is in the kernel then $\chi = |\cdot|^{-1}$ implies that Φ_f is constant, and $\chi \neq |\cdot|^{-1}$ implies that $\Phi_f = 0$ and hence $f = 0$. This proves that when $\chi \neq |\cdot|^{-1}$ the map $f \longmapsto \varphi_f^{\psi}$ is an isomorphism, and when $\chi = |\cdot|^{-1}$ the kernel of $f \longmapsto \varphi_f^{\psi}$ is the set of $f \in \mathcal{B}(\chi_1, \chi_2)$ such that Φ_f is constant. The first assertion of the proposition is now proven. To prove the second suppose $\chi = |\cdot|^{-1}$. Define Y to be the kernel of $f \longmapsto \varphi_f^{\psi}$,

$$Y := \{f \in \mathcal{B}(\chi_1, \chi_2) : \varphi_f^{\psi} = 0\} = \{f \in \mathcal{B}(\chi_1, \chi_2) : \Phi_f \quad \text{is constant}\}.$$

We are now going to prove that there exists a unique function $f_0 \in \mathcal{B}(\chi_1, \chi_2)$ such that $\Phi_{f_0} = 1$ and moreover we have the formula

$$f_0(g) = \chi_1(det(g))|det(g)|^{\frac{1}{2}}.$$

To prove this first recall that if $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is in the big cell then for any $f \in \mathcal{B}(\chi_1, \chi_2)$ we have $f(g) = \chi_1(det(g))|det(g)|^{\frac{1}{2}}\Phi_f(\frac{d}{c})$. Therefore if $f_0 \in \mathcal{B}(\chi_1, \chi_2)$ satisfies $\Phi_{f_0} = 1$ then $f_0(g) = \chi_1(det(g))|det(g)|^{\frac{1}{2}}$ for g in the big cell. Since the big cell is dense in G and f_0 is locally constant this forces the formula to hold for all $g \in G$. Uniqueness is now established. It remains to check that if we define f_0 as above then $f_0 \in \mathcal{B}(\chi_1, \chi_2)$ and $\Phi_{f_0} = 0$. This is easily done.

Having established this we conclude that $Y = \mathbb{C}f_0$. It remains to show that Y is ρ-invariant. This is obvious since for $g \in G$ we have $\rho(g)f_0 = f_0(g)f_0 \in \mathbb{C}f_0 = Y$.

□

Recall that if $\varphi : F^{\times} \longrightarrow \mathbb{C}$ is any function and g is any element of A then $\xi_{\psi}(g)\varphi$ is defined.

Proposition 23 Let χ_1 and χ_2 be characters of $F^{\times}$ and let $1 \neq \psi \in F^{*}$. If $f \in \mathcal{B}(\chi_1, \chi_2)$ and $g = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in A$ then

$$\varphi_{\rho(g)f}^{\psi} = \xi_{\psi}(g)\varphi_f^{\psi}.$$

Proof For $x \in F^\times$ we have

$$\begin{aligned}(\hat{\Phi}_{\rho(g)f})_\psi(x) &= \int_F \Phi_{\rho(g)f}(y)\psi(xy)\,d_\psi y \\ &= \int_F f\left(w\begin{pmatrix} a & b-y \\ 0 & 1\end{pmatrix}\right)\psi(xy)\,d_\psi y \\ &= \int_F f\left(w\begin{pmatrix} a & -y \\ 0 & 1\end{pmatrix}\right)\psi(xy+bx)\,d_\psi y \\ &= \psi(bx)|a|\int_F f\left(w\begin{pmatrix} a & -ay \\ 0 & 1\end{pmatrix}\right)\psi(axy)\,d_\psi y.\end{aligned}$$

The identity

$$w\begin{pmatrix} a & -ay \\ 0 & 1\end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & a\end{pmatrix} w \begin{pmatrix} 1 & -y \\ 0 & 1\end{pmatrix}$$

implies that

$$f\left(w\begin{pmatrix} a & -ay \\ 0 & 1\end{pmatrix}\right) = \chi_2(a)|a|^{-\frac{1}{2}}\Phi_f(y).$$

Thus $(\hat{\Phi}_{\rho(g)f})_\psi(x) = \psi(bx)\chi_2(a)|a|^{\frac{1}{2}}(\hat{\Phi}_f)_\psi(ax)$ and so

$$\begin{aligned}\varphi^\psi_{\rho(g)f}(x) &= \chi_2(x)|x|^{\frac{1}{2}}(\hat{\Phi}_{\rho(g)f})_\psi(x) \\ &= \psi(bx)\chi_2(ax)|ax|^{\frac{1}{2}}(\hat{\Phi}_f)_\psi(ax) = \psi(bx)\varphi^\psi_f(ax) \\ &= (\xi_\psi(g)\varphi^\psi_f)(x).\end{aligned}$$

□

Since the kernel of the surjective linear operator

$$R_\psi : \mathcal{B}(\chi_1,\chi_2) \longrightarrow \mathcal{K}(\chi_1,\chi_2) \quad f \longmapsto \varphi^\psi_f$$

is ρ-invariant the representation of G on $\mathcal{B}(\chi_1,\chi_2)/ker(R_\psi)$ gives rise to a representation of G on $\mathcal{K}(\chi_1,\chi_2)$. If π denotes this action of G on $\mathcal{K}(\chi_1,\chi_2)$ then the above result shows that $\pi(g)\varphi = \xi_\psi(g)\varphi$ for all $g \in A$ and $\varphi \in \mathcal{K}(\chi_1,\chi_2)$.

The next result is Theorem 6 in the first section of [2].

Theorem 24 Let χ_1 and χ_2 be two characters of $F^\times$.

(1) The representation $(\rho, \mathcal{B}(\chi_1,\chi_2))$ of G is irreducible except when $\chi = |\cdot|$ or $\chi = |\cdot|^{-1}$.

(2) If $\chi = |\cdot|$ then $\mathcal{B}(\chi_1,\chi_2)$ contains an irreducible invariant subspace of codimension one. This subspace consists of all functions $f \in \mathcal{B}(\chi_1,\chi_2)$ which satisfy the condition

$$\int_F \Phi(x)\,d^+x = 0.$$

(3) If $\chi = |\cdot|^{-1}$ then $\mathcal{B}(\chi_1,\chi_2)$ contains a one-dimensional invariant subspace, spanned by the function $g \longmapsto \chi_1(det(g))|det(g)|^{\frac{1}{2}}$. The representation of G obtained on the quotient of $\mathcal{B}(\chi_1,\chi_2)$ by this one-dimensional subspace is irreducible.

Definition 25 Let χ_1 and χ_2 be characters of $F^\times$ and define $\chi := \chi_1\chi_2^{-1}$.

(1) Suppose χ does not equal $|\cdot|$ or $|\cdot|^{-1}$. The class in $\Pi(G)$ which is the class of the irreducible representation $(\rho, \mathcal{B}(\chi_1, \chi_2))$ is denoted by $\pi(\chi_1, \chi_2)$ and is called a *principal series representation.*

(2) Suppose $\chi = |\cdot|$. The class in $\Pi(G)$ which is the class of the irreducible representation of G obtained by restricting to the irreducible invariant subspace of codimension one is denoted by $\sigma(\chi_1, \chi_2)$, and is called a *special representation.* The class in $\Pi(G)$ which is the class of the one dimensional representation of G on the quotient space of $\mathcal{B}(\chi_1, \chi_2)$ with the invariant hyperplane is denoted by $\pi(\chi_1, \chi_2)$.

(3) Suppose $\chi = |\cdot|^{-1}$. The class in $\Pi(G)$ which is the class of the irreducible representation of G on the the quotient space of $\mathcal{B}(\chi_1, \chi_2)$ with the one dimensional space spanned by the function $g \longmapsto \chi_1(det(g))|det(g)|^{\frac{1}{2}}$ is also called a *special representation*, and is again denoted by $\sigma(\chi_1, \chi_2)$. The class in $\Pi(G)$ which is the class of the one dimensional representation of G on the invariant space spanned by the function $g \longmapsto \chi_1(det(g))|det(g)|^{\frac{1}{2}}$ is denoted by $\pi(\chi_1, \chi_2)$.

Thus whenever $(\rho, \mathcal{B}(\chi_1, \chi_2))$ is irreducible its class in $\Pi(G)$ is denoted by $\pi(\chi_1, \chi_2)$, and is said to be a principal series representation. Also note that for any two characters χ_1 and χ_2 of $F^\times$ that $\pi(\chi_1, \chi_2)$ is defined, but only when χ is distinct from both $|\cdot|$ and $|\cdot|^{-1}$ is it true that $\pi(\chi_1, \chi_2)$ is infinite dimensional. We have $\sigma(\chi_1, \chi_2)$ being defined only when $\chi_1\chi_2^{-1}$ equals $|\cdot|$ or $|\cdot|^{-1}$ and it is always infinite dimensional. Observe that if $\sigma(\chi_1, \chi_2)$ is a special representation then $cond(\chi_1) = cond(\chi_2)$, and so the characters χ_1, χ_2 are either both ramified or both unramified.

See pages 34 and 35 of [2] for a proof of the next result.

Proposition 26 Let $\pi \in \Pi(G)$ be an infinite dimensional class that is not cuspidal, and let $1 \neq \psi \in F^*$.

(1) Suppose $\pi = \pi(\chi_1, \chi_2)$ is a principal series representation. Then the map

$$: \mathcal{B}(\chi_1, \chi_2) \longmapsto \mathcal{K}(\pi) \qquad f \longmapsto \varphi_f^\psi$$

is an intertwining isomorphism of $(\rho, \mathcal{B}(\chi_1, \chi_2))$ and $(\pi_\psi, \mathcal{K}(\pi))$.

(2) Suppose $\pi = \sigma(\chi_1, \chi_2)$ is a special representation with $\chi = |\cdot|$. Define Y to be the irreducible ρ-invariant subspace of $\mathcal{B}(\chi_1, \chi_2)$ of codimension 1, so that (ρ, Y) is in $\sigma(\chi_1, \chi_2)$. Then the map

$$: Y \longmapsto \mathcal{K}(\pi) \qquad f \longmapsto \varphi_f^\psi$$

is an intertwining isomorphism of (ρ, Y) and $(\pi_\psi, \mathcal{K}(\pi))$.

(3) Suppose $\pi = \sigma(\chi_1, \chi_2)$ is a special representation with $\chi = |\cdot|^{-1}$. The function $: G \longrightarrow \mathbb{C} \quad g \longmapsto \chi_1(det(g))|det(g)|^{\frac{1}{2}}$ is in $\mathcal{B}(\chi_1, \chi_2)$. Define Y to be the one dimensional ρ-invariant subspace of $\mathcal{B}(\chi_1, \chi_2)$ that is spanned by this function. The resulting representation $(\rho, \mathcal{B}(\chi_1, \chi_2)$ is in $\sigma(\chi_1, \chi_2)$. If $f \in \mathcal{B}(\chi_1, \chi_2)$ and $f' \in Y$ then $\varphi_{f+f'}^\psi = \varphi_f^\psi$. The map

$$: \mathcal{B}(\chi_1, \chi_2)/Y \longrightarrow \mathcal{K}(\pi) \qquad f + Y \longmapsto \varphi_f^\psi$$

is well defined and is an intertwining isomorphism of $(\rho, \mathcal{B}(\chi_1, \chi_2)/Y)$ and $(\pi_\psi, \mathcal{K}(\pi))$.

The following result completely describes the functions in the Kirillov space $\mathcal{K}(\pi)$.

Theorem 27 Let $\pi \in \Pi(G)$ be infinite dimensional.

(1) If π is cuspidal then $\mathcal{K}(\pi) = \mathcal{S}(F^\times)$.

(2) If $\pi = \pi(\chi_1, \chi_2)$ is a principal series representation with $\chi_1 \neq \chi_2$ then the functions in $\mathcal{K}(\pi)$ are those of the form

$$x \longmapsto |x|^{\frac{1}{2}}(\chi_1(x)f_1(x) + \chi_2(x)f_2(x))$$

where $x \in F^\times$ and $f_1, f_2 \in \mathcal{S}(F)$ are arbitrary.

(3) If $\pi = \pi(\chi_1, \chi_2)$ is a principal series representation with $\omega := \chi_1 = \chi_2$ then the functions in $\mathcal{K}(\pi)$ are precisely those of the form

$$x \longmapsto |x|^{\frac{1}{2}}\omega(x)(f_1(x) + val(x)f_2(x))$$

where $x \in F^\times$ and $f_1, f_2 \in \mathcal{S}(F)$ are arbitrary.

(4) If $\pi = \sigma(\chi_1, \chi_2)$ is a special representation with $\chi = |\cdot|$ then the functions in $\mathcal{K}(\pi)$ are precisely those of the form

$$x \longmapsto |x|^{\frac{1}{2}}\chi_1(x)f(x)$$

where $x \in F^\times$ and $f \in \mathcal{S}(F)$ are arbitrary.

(5) If $\pi = \sigma(\chi_1, \chi_2)$ is a special representation with $\chi = |\cdot|^{-1}$ then the functions in $\mathcal{K}(\pi)$ are precisely those of the form

$$x \longmapsto |x|^{\frac{1}{2}}\chi_2(x)f(x)$$

where $x \in F^\times$ and $f \in \mathcal{S}(F)$ are arbitrary.

Proof This follows directly from Corollary 8.21 and Proposition 8.26.

□

The next result follows easily from Theorem 8.27.

Corollary 28 Let π be an infinite dimensional class in $\Pi(G)$. Then the codimension of $\mathcal{S}(F^\times)$ in $\mathcal{K}(\pi)$ equals 0 if π is cuspidal, equals 1 if π is a special representation. and equals 2 if π is a principal series representation.

Corollary 29 $\Pi(G)$ can be partitioned into four disjoint subsets consisting of the one dimensional classes, the principal series representations, the special representations. and the cuspidal representations.

Proof Theorem 5.17, Theorem 8.7, and Theorem 8.24.

□

The following result is theorem 7 in the first section of **[2]**.

Theorem 30 Let $\chi_1, \chi_2, \omega_1, \omega_2$ be characters of $F^\times$.

(1) $\pi(\chi_1, \chi_2)$ and $\pi(\omega_1, \omega_2)$ are equivalent representations iff $(\chi_1, \chi_2) = (\omega_1, \omega_2)$ or $(\chi_1, \chi_2) = (\omega_2, \omega_1)$.

(2) If $\sigma(\chi_1, \chi_2)$ and $\sigma(\omega_1, \omega_2)$ are defined then they are equivalent representations iff $(\chi_1, \chi_2) = (\omega_1, \omega_2)$ or $(\chi_1, \chi_2) = (\omega_2, \omega_1)$.

Proposition 31 If $\pi \in \Pi(G)$ is equal to a principal series representation $\pi(\chi_1, \chi_2)$ or a special representation $\sigma(\chi_1, \chi_2)$ then $\omega_\pi = \chi_1\chi_2$.

Proof First suppose that $\pi = \pi(\chi_1, \chi_2)$. In this case the representation $(\rho, \mathcal{B}(\chi_1, \chi_2)$ is in π. If $a \in F^\times$, $f \in \mathcal{B}(\chi_1, \chi_2)$, and $g \in G$ then

$$\left(\rho\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} f\right)(g) = \chi_1(a)\chi_2(a)|\frac{a}{a}|^{\frac{1}{2}}f(g) = (\chi_1\chi_2)(a)f(g).$$

This proves $\omega_{\pi(\chi_1,\chi_2)} = \chi_1\chi_2$. Now suppose that $\pi = \sigma(\chi_1, \chi_2)$ is a special representation. Since $\sigma(\chi_1, \chi_2) = \sigma(\chi_2, \chi_1)$ we may assume that $\chi := \chi_1\chi_2^{-1} = |\cdot|$. Let Y denote the ρ-invariant hyperplane of $\mathcal{B}(\chi_1, \chi_2)$ so that the representation (ρ, Y) is in π. Then the same calculation as above shows that for any $a \in F^\times$ and $f \in Y$ we have $\rho \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} f = \chi_1\chi_2 f$. This shows $\omega_{\sigma(\chi_1,\chi_2)} = \chi_1\chi_2$.

□

The following is not difficult to prove.

Proposition 32

(1) The contragredient of $\pi(\chi_1, \chi_2)$ is $\pi(\chi_1^{-1}, \chi_2^{-1})$.

(2) The contragredient of $\sigma(\chi_1, \chi_2)$ is $\sigma(\chi_1^{-1}, \chi_2^{-1})$.

(3) If π is cuspidal then so is $\tilde{\pi}$.

(4) If π is a one-dimensional class corresponding to the character χ of $F^\times$, then $\tilde{\pi}$ is the one-dimensional class corresponding to the character χ^{-1}.

Unramified Representations

If $\pi \in \Pi(G)$ then as π is admissible we know that $V(\sigma)$ is finite dimensional for any $\sigma \in \Pi(K)$ and hence σ will occur in π at most finitely many times. It is a more delicate question to ask exactly how many times it occurs. In particular we may consider this question for the trivial irreducible representation of K.

Definition 33 Let $\pi \in \Pi(G)$. We say that π is *unramified* iff the restriction of π to K contains the identity representation of K.

Other authors use the terms *class 1* or *spherical* for the type of representations we are calling unramified. We can easily characterize the finite dimensional classes in $\Pi(G)$ that are unramified.

Proposition 34 Let $\pi \in \Pi(G)$ be a one dimensional class and let χ be the character of $F^\times$ which corresponds to π. Then π is unramified iff χ is unramified.

Proof The representation $(\pi_\chi, \mathbb{C})$ of G defined by $\pi_\chi(g) = \chi(det(g))$ is in π. Thus π is unramified iff $\pi_\chi(g) = 1$ for all $g \in K$ iff $\chi(det(g)) = 1$ for all $g \in K$ iff χ is trivial on $\mathcal{U}$ iff χ is unramified.

□

The following result is Theorem 11 in the first section of [2].

Theorem 35 Let $\pi \in \Pi(G)$ be infinite dimensional. Then π is unramified iff π is a principal series representation $\pi(\chi_1, \chi_2)$ where χ_1 and χ_2 are unramified characters of $F^\times$. Moreover in this case the identity representation of K is contained exactly once in the restriction of π to K.

Hence if (π, V) is an irreducible admissible unramified representation of arbitrary dimension then there is a unique one dimensional subspace of V on which the restriction of π to K acts as the identity. That is there is a non-zero vector $v \in V$ such that $\pi(g)v = v$ for all $g \in K$, and any other vector in V with this property is a multiple of v by a complex scalar.

Chapter 9 The Functional Equation

Fractional Ideals of Meromorphic Functions

Let R denote an integral domain, that is a commutative ring with identity that has no zero divisors. The field of fractions of R will be denoted by L. Consider L as an R-module.

Definition 1 An R-submodule I of L is called a *fractional ideal* of R iff I is non-zero and there exists a non-zero element $a \in L$ such that $aI \subseteq R$.

Clearly if I is a fractional ideal of R then there exists a non-zero element $a \in R$ such that $aI \subseteq R$.

Definition 2 A fractional ideal I of R is said to be *principal* iff there is an element $a \in I$ such that $I = Ra$. Hence a fractional ideal of R is principal iff the fractional ideal is principal as a module.

Obviously if R is a principal ideal domain then every fractional ideal of R is principal.

We recall some standard terminology. A subset D of $\mathbb{C}$ is said to be *discrete* iff for every $x \in D$ there is a neighborhood of x in $\mathbb{C}$ such that the only point of D in this neighborhood is x. Any discrete subset of $\mathbb{C}$ is countable but the converse is certainly not true as the subset $\mathbb{Q}$ of $\mathbb{C}$ illustrates. Clearly if D is discrete then $\mathbb{C} \setminus D$ is open and path connected in $\mathbb{C}$. Suppose now that U is a non-empty open subset of $\mathbb{C}$ and $f : U \longrightarrow \mathbb{C}$ is an analytic function. We say that f is *meromorphic* iff there is a discrete subset D of $\mathbb{C}$ such that $U = \mathbb{C} \setminus D$. We say that f is *entire* iff $U = \mathbb{C}$. Since the empty set is discrete we see that any entire function is meromorphic. The set of entire functions forms an integral domain which has a subfield canonically isomorphic to $\mathbb{C}$. We make the identification. The set of all meromorphic functions forms a field. Moreover the field of fractions of the integral domain of entire functions is the field of meromorphic functions.

As always q denotes the cardinality of the residue field $\mathcal{O}/\mathcal{P}$. For any $k \in \mathbb{Z}$ the function

$$: \mathbb{C} \longrightarrow \mathbb{C} \qquad s \longmapsto q^{ks}$$

is entire. It is standard to denote such a function by just writing q^{ks}. Consider the subring $\mathbb{C}[q^s]$ of the ring of entire functions. As a ring it is isomorphic to the polynomial ring $\mathbb{C}[T]$ where T is an indeterminate. Hence $\mathbb{C}[q^s]$ is a principal ideal domain. The field of fractions of $\mathbb{C}[q^s]$ is $\mathbb{C}(q^s) = \mathbb{C}(q^{-s})$. The field $\mathbb{C}(q^s)$ is a subfield of the field of all meromorphic functions. The localization of $\mathbb{C}[q^s]$ at the multiplicative subset $\{q^{ks} : k \in \mathbb{Z} \quad k \geq 0\}$ is the ring $\mathbb{C}[q^s, q^{-s}]$. Since $\mathbb{C}[q^s]$ is a principal ideal domain the same holds for the localized ring $\mathbb{C}[q^s, q^{-s}]$. Hence every fractional ideal of $\mathbb{C}[q^s, q^{-s}]$ is principal. Observe that

$$(\mathbb{C}[q^s, q^{-s}])^\times = \{cq^{ns} : c \in \mathbb{C}^\times \quad n \in \mathbb{Z}\}.$$

The field of fractions of $\mathbb{C}[q^s, q^{-s}]$ is again the field $\mathbb{C}(q^s)$. We will be interested in certain fractional ideals of the ring $\mathbb{C}[q^s, q^{-s}]$.

Euler Factors

Definition 3 An *Euler factor* of F is a meromorphic function of the form $s \longmapsto P(q^{-s})^{-1}$ where q is the cardinality of the residue field of F, and $P(T) \in \mathbb{C}[T]$ is a polynomial satisfying $P(0) = 1$. So in particular an Euler factor is a function in $\mathbb{C}(q^s)$. The *degree* of the Euler factor is defined to be the degree of $P(T)$. Euler factors of F will often be denoted by $L(s)$.

In these notes only Euler factors of degree less than or equal to two will appear. This is due to the fact that we are only dealing with the $GL(1)$ and $GL(2)$ theory.

Obviously the only degree 0 Euler factor arises when $P(T) = 1$. In this case the Euler factor $P(q^{-s})^{-1}$ is simply the function : $\mathbb{C} \longrightarrow \mathbb{C} \quad s \longmapsto 1$, which is even entire. An Euler factor of degree greater than zero always has poles.

Let us make a simple observation about an Euler factor $L(s) := P(q^{-s})^{-1}$ where $P(T) \neq 1$. Define $n := deg(P(T)) \geq 1$. Since $P(0) = 1$ the n roots $c_1, \ldots, c_n$ of $P(T)$ in $\mathbb{C}$ are non-zero. It is easy to see that $P(T) = (1 - c_1^{-1}T)\ldots(1 - c_n^{-1}T)$. So the Euler factor $L(s)$ can be written as a product

$$L(s) = (1 - c_1{}^{-1}q^{-s})^{-1}\ldots(1 - c_n{}^{-1}q^{-s})^{-1}.$$

Lemma 4 Suppose $P_1(q^{-s})^{-1}$ and $P_2(q^{-s})^{-1}$ are two Euler factors of F such that their quotient is an entire function with no zero. Then $P_1(T) = P_2(T)$.

Proof The image of the map : $\mathbb{C} \longrightarrow \mathbb{C} \quad s \longmapsto q^{-s}$ is $\mathbb{C}^\times$. From this and the fact that $P_1(0) = P_2(0) = 1$ it follows that the function : $\mathbb{C} \longrightarrow \mathbb{C} \quad s \longmapsto \frac{P_1(s)}{P_2(s)}$ is an entire function with no zeros. Clearly if one of $P_1(T)$ or $P_2(T)$ equals 1 then this implies that the other also equals 1. So we may suppose that neither of them equal 1. Define $n = deg(P_1(T)) \in \mathbb{N}$ and $m = deg(P_2(T)) \in \mathbb{N}$. We have just seen that there are $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m \in \mathbb{C}^\times$ such that

$$P_1(T) = (1 - \alpha_1 T)\ldots(1 - \alpha_n T) \quad \text{and} \quad P_2(T) = (1 - \beta_1 T)\ldots(1 - \beta_m T).$$

Hence

$$\frac{P_1(s)}{P_2(s)} = \frac{(1 - \alpha_1 s)\ldots(1 - \alpha_n s)}{(1 - \beta_1 s)\ldots(1 - \beta_m s)}.$$

Now $\frac{P_1(s)}{P_2(s)}$ having no zero implies by setting $s = \alpha_k^{-1}$ that every α_k equals some β_j and hence $m \geq n$. Similarly $\frac{P_1(s)}{P_2(s)}$ being entire implies by setting $s = \beta_j^{-1}$ that every β_j equals some α_k, and so $m \leq n$. Thus $m = n$ and $\alpha_1, \ldots, \alpha_n$ is a permutation of $\beta_1, \ldots, \beta_m$. This proves $P_1(T) = P_2(T)$.

□

Since every fractional ideal of $\mathbb{C}[q^s, q^{-s}]$ is principal we may consider the generators of a given fractional ideal. If $L(s)$ is an Euler factor of F then obviously $(L(s)) := \mathbb{C}[q^s, q^{-s}]L(s)$ is a fractional ideal of $\mathbb{C}[q^s, q^{-s}]$. What we show next is that the only generator of $(L(s))$ which is an Euler factor is $L(s)$. Of course there exist fractional ideals of $\mathbb{C}[q^s, q^{-s}]$ which are not generated by Euler factors. Indeed any non-zero proper ideal of the ring $\mathbb{C}[q^s, q^{-s}]$ is not generated by an Euler factor.

Proposition 5 A fractional ideal of $\mathbb{C}[q^s, q^{-s}]$ has at most one generator that is an Euler factor.

Proof Suppose we are given two Euler factors $L_1(s)$ and $L_2(s)$ of F such that $(L_1(s)) = (L_2(s))$. It follows that the Euler factors $L_1(s)$ and $L_2(s)$ have their quotient being an entire function with no zeros, and hence they are equal by Lemma 9.4.

□

Zeta Functions for $\mathbf{GL_1(F)}$

Lemma 6 Let χ be a character of $F^\times$. Fix $f \in \mathcal{S}(F)$.

(1) If χ is ramified or if $f(0) = 0$ then for every $s \in \mathbb{C}$ the integral

$$\int_{F^\times} f(a)\chi(a)|a|^s \, d^\times a$$

is convergent. Moreover as a function of the complex variable s it equals a function in $\mathbb{C}[q^s, q^{-s}]$, and hence is an entire function on $\mathbb{C}$.

(2) If χ is unramified and $f(0) \neq 0$ then for every $s \in \mathbb{C}$ with $Re(s) > -\alpha(\chi)$ the integral

$$\int_{F^\times} f(a)\chi(a)|a|^s \, d^\times a$$

is convergent. Therefore in the particular case that χ is unitary the integral converges for $Re(s) > 0$. Moreover as a function of the complex variable s the integral equals a function in $(1 - \chi(\mathcal{P})q^{-s})^{-1}\mathbb{C}[q^s, q^{-s}]$ restricted to the right half plane $Re(s) > -\alpha(\chi)$. Hence as a function of s this integral has an analytic continuation from an analytic function on the region $Re(s) > -\alpha(\chi)$ to a meromorphic function in $\mathbb{C}(q^s)$ which has a pole at $s \in \mathbb{C}$ only if $q^s = \chi(\mathcal{P})$.

Proof Choose $n \in \mathbb{N}$ such that $x \in \mathcal{P}^n$ implies $f(x) = f(0)$. Choose $m \in \mathbb{Z}$ with $m < n$ such that $x \in F$, $val(x) < m$ implies $f(x) = 0$. Let $\tilde{\omega}$ be a uniformizing parameter of F. For any $r \in \mathbb{N}$ with $r \geq n$ we have for any $s \in \mathbb{C}$ that

$$\begin{aligned}
\int_{F\setminus\mathcal{P}^{r+1}} f(a)\chi(a)|a|^s \, d^\times a &= \int_{\mathcal{P}^m\setminus\mathcal{P}^{r+1}} f(a)\chi(a)|a|^s \, d^\times a \\
&= \sum_{k=m}^{n-1} \int_{\mathcal{U}} f(\epsilon\tilde{\omega}^k)\chi(\epsilon\tilde{\omega}^k)|\epsilon\tilde{\omega}^k|^s \, d^\times\epsilon + f(0)\sum_{k=n}^{r}\int_{\mathcal{U}} \chi(\epsilon\tilde{\omega}^k)|\epsilon\tilde{\omega}^k|^s \, d^\times\epsilon \\
&= \sum_{k=m}^{n-1} \chi(\tilde{\omega})^k \hat{f}_k(\chi_0)q^{-ks} + f(0)\delta(\chi_0)\sum_{k=n}^{r}(\chi(\tilde{\omega})q^{-s})^k
\end{aligned}$$

When χ is ramified $\delta(\chi_0) = 0$. Hence if χ is ramified or $f(0) = 0$ then by passing to the limit in the above we conclude that for any $s \in \mathbb{C}$,

$$\int_{F^\times} f(a)\chi(a)|a|^s \, d^\times a = \sum_{k=m}^{n-1} \chi(\tilde{\omega})^k \hat{f}_k(\chi_0)q^{-ks}.$$

This is a function in $\mathbb{C}[q^s, q^{-s}]$ and hence the first part of the lemma.

To prove the second part suppose now that χ is unramified and $f(0) \neq 0$. Note that $s \in \mathbb{C}$ satisfies $|\chi(\tilde{\omega})q^{-s}| < 1$ iff $Re(s) > -\alpha(\chi)$. Hence by passing to the limit agian in our initial computation we find that for $Re(s) > -\alpha(\chi)$ we have

$$\int_{F^\times} f(a)\chi(a)|a|^s \, d^\times a = \sum_{k=m}^{n-1} \chi(\tilde{\omega})^k \hat{f}_k(\chi_0)q^{-ks} + f(0)\chi(\tilde{\omega})^n q^{-ns}(1 - \chi(\tilde{\omega})q^{-s})^{-1}$$

This proves the second part of the lemma.

□

The above lemma allows us to make the following definition.

Definition 7 Let χ be a character of $F^\times$. For $f \in \mathcal{S}(F)$ we denote *Tate's zeta function* of χ at f by $\mathcal{Z}(s, f, \chi)$, and define it to be the analytic continuation to $\mathbb{C}$ of the analytic function

$$s \longmapsto \int_{F^\times} \chi(a)f(a)|a|^s \, d^\times a$$

defined on the region $Re(s) > -\alpha(\chi)$. Hence $\mathcal{Z}(s, f, \chi) \in \mathbb{C}(q^s)$ for every $f \in \mathcal{S}(F)$.

Corollary 8 Let χ be a character of $F^\times$, and let $f \in \mathcal{S}(F)$.

(1) If χ is ramified or if $f(0) = 0$ then $\mathcal{Z}(s, f, \chi) \in \mathbb{C}[q^s, q^{-s}]$, and hence in particular is an entire function.

(2) Suppose χ is unramified and $f(0) \neq 0$. Then $\mathcal{Z}(s, f, \chi) \in (1 - \chi(\mathcal{P})q^{-s})^{-1}\mathbb{C}[q^s, q^{-s}]$. Hence $\mathcal{Z}(s, f, \chi)$ has a pole at $s \in \mathbb{C}$ only if $q^s = \chi(\mathcal{P})$. In particular all poles of $\mathcal{Z}(s, f, \chi)$ are on the line $Re(s) = -\alpha(\chi)$.

Proof Lemma 9.6 and Definition 9.7.

□

Proposition 9 Let χ be a character of $F^\times$. Then for every $f \in \mathcal{S}(F)$ and $b \in F^\times$ we have the identity

$$\mathcal{Z}(s, f(b\,\cdot), \chi) = \chi^{-1}(b)|b|^{-s}\mathcal{Z}(s, f, \chi).$$

Proof By the principle of analytic continuation it suffices to show the identity holds for $Re(s)$ sufficiently large. Well for $Re(s)$ sufficiently large the integral defining $\mathcal{Z}(s, \cdot, \chi)$ converges, and thus we may make the computation

$$\begin{aligned}\mathcal{Z}(s, f(b\,\cdot), \chi) &= \int_{F^\times} \chi(a)f(ab)|a|^s \, d^\times a \\ &= \int_{F^\times} \chi(ab^{-1})f(a)|ab^{-1}|^s \, d^\times a \\ &= \chi^{-1}(b)|b|^{-s}\mathcal{Z}(s, f, \chi).\end{aligned}$$

□

Euler Factors for $\mathbf{GL_1(F)}$

Definition 10 Let χ be a character of $F^\times$. Define

$$\mathcal{Z}(\chi) := \{\mathcal{Z}(s, f, \chi) : f \in \mathcal{S}(F)\}.$$

Corollary 11 Let χ be a character of $F^\times$. Then $\mathcal{Z}(\chi)$ is a fractional ideal of $\mathbb{C}[q^s, q^{-s}]$ that is generated by an Euler factor. More precisely $\mathcal{Z}(\chi) = (1)$ when χ is ramified and $\mathcal{Z}(\chi) = ((1 - \chi(\mathcal{P})q^{-s})^{-1})$ when χ is unramified.

Proof This follows from the computation done in Lemma 9.6.

□

Definition 12 Given a character χ of $F^\times$ the unique Euler factor which generates $\mathcal{Z}(\chi)$, will be denoted by $L(s, \chi)$, and will be called the *Euler factor* of χ. Hence we have the identity $\mathcal{Z}(\chi) = (L(s, \chi))$. Consequently $L(s, \chi) = 1$ when χ is ramified, and $L(s, \chi) = (1 - \chi(\mathcal{P})q^{-s})^{-1}$ when χ is unramified.

The following result is now trivial.

Corollary 13 Let χ be a character of $F^\times$. Then for every $f \in \mathcal{S}(F)$ the quotient

$$\frac{\mathcal{Z}(s, f, \chi)}{L(s, \chi)}$$

is in $\mathbb{C}[q^s, q^{-s}]$ and hence is an entire function.

Gamma Factors for $\mathbf{GL_1(F)}$

First we will associate to each character of $F^\times$ a natural function in $\mathcal{S}(F)$, and then calculate its zeta-function, its Fourier transform, and the zeta-function of its Fourier transform.

Lemma 14 Let χ be a character of $F^\times$. Define $f \in \mathcal{S}(F)$ by $f(x) = \chi^{-1}(x)$ if $x \in \mathcal{U}$ and $f(x) = 0$ for $x \in F$, $x \notin \mathcal{U}$. Let ψ be a non-trivial unitary character of F^+ and define $l = cond(\psi)$.

(1) $\mathcal{Z}(s, f, \chi) = 1$.

(2) For every $a \in F$ we have $\hat{f}_\psi(a) = \lambda_\psi(\mathcal{U})\gamma(a, \chi_0^{-1}, \psi)$.

(3) Let $\tilde{\omega}$ be a uniformizing parameter of F. If χ is ramified and $n = cond(\chi_0)$ then

$$\mathcal{Z}(s, \hat{f}_\psi, \chi^{-1}) = \lambda_\psi(\mathcal{U})\chi(\tilde{\omega})^{n-l}\gamma(\tilde{\omega}^{l-n}, \chi_0, \psi)q^{(n-l)s}.$$

This formula does not depend on the choice of $\tilde{\omega}$.

(4) If χ is unramified then

$$\mathcal{Z}(s, \hat{f}_\psi, \chi^{-1}) = \lambda_\psi(\mathcal{U})(1 - q^{-1})^{-1}\chi(\mathcal{P})^{-l}q^{-ls}\frac{1 - \chi(\mathcal{P})q^{s-1}}{1 - \chi^{-1}(\mathcal{P})q^{-s}}.$$

The poles of $\mathcal{Z}(s, \hat{f}_\psi, \chi^{-1})$ are at precisely those $s \in \mathbb{C}$ such that $q^{-s} = \chi(\mathcal{P})$. Hence in particular all poles are located on the line $Re(s) = \alpha$.

Proof The first part of the lemma simply follows from the definitions and the fact that the measure $d^\times\epsilon$ on $\mathcal{U}$ has total mass equal to one. To prove the second part of the lemma observe that for $a \in F$ we have

$$\begin{aligned}
\hat{f}_\psi(a) &= \int_F f(x)\psi(ax)\, d_\psi x \\
&= \int_{\mathcal{U}} \chi^{-1}(x)\psi(ax)\, d_\psi x \\
&= \lambda_\psi(\mathcal{U}) \int_{\mathcal{U}} \chi^{-1}(\epsilon)\psi(a\epsilon)\, d^\times\epsilon \\
&= \lambda_\psi(\mathcal{U})\gamma(a, \chi_0^{-1}, \psi).
\end{aligned}$$

Suppose now that the hypothesis of the third part of the lemma hold. We will use the identity

$$\gamma(\epsilon\tilde{\omega}^{l-n}, \chi_0^{-1}, \psi) = \chi(\epsilon)\gamma(\tilde{\omega}^{l-n}, \chi_0^{-1}, \psi)$$

in the next computation. For any $s \in \mathbb{C}$ we have

$$\begin{aligned}
\mathcal{Z}(s, \hat{f}_\psi, \chi^{-1}) &= \int_{F^\times} \chi^{-1}(a)\hat{f}_\psi(a)|a|^s\, d^\times a \\
&= \lambda_\psi(\mathcal{U}) \int_{F^\times} \chi^{-1}(a)\gamma(a, \chi_0^{-1}, \psi)|a|^s\, d^\times a \\
&= \lambda_\psi(\mathcal{U}) \int_{\mathcal{U}} \chi^{-1}(\tilde{\omega}^{l-n})\gamma(\tilde{\omega}^{l-n}, \chi_0^{-1}, \psi)|\epsilon\tilde{\omega}^{l-n}|^s\, d^\times\epsilon \\
&= \lambda_\psi(\mathcal{U})\chi(\tilde{\omega})^{n-l}\gamma(\tilde{\omega}^{l-n}, \chi_0^{-1}, \psi)q^{(n-l)s}.
\end{aligned}$$

Since for any $\epsilon \in \mathcal{U}$ we have

$$\gamma((\epsilon\tilde{\omega})^{l-n}, \chi_0^{-1}, \psi) = \chi(\epsilon)^{l-n}\gamma(\tilde{\omega}^{l-n}, \chi_0^{-1}, \psi)$$

it is easily seen that the formula does not depend on the choice of $\tilde{\omega}$.

Finally suppose that the hypothesis of the fourth part of the lemma hold. It is simple to show that $s \in \mathbb{C}$ satisfies $|\chi^{-1}(\mathcal{P})q^{-s}| < 1$ iff $Re(s) > \alpha(\chi)$. By Corollary 9.8 when $Re(s) > \alpha(\chi)$ the integral initially defining $\mathcal{Z}(s, \hat{f}_\psi, \chi^{-1})$ is convergent. Using these to facts we find for $Re(s) > \alpha(\chi)$ that $\mathcal{Z}(s, \hat{f}_\psi, \chi^{-1})$ equals

$$\begin{aligned}
&\int_{F^\times} \chi^{-1}(a)\hat{f}_\psi(a)|a|^s\, d^\times a \\
&= \lambda_\psi(\mathcal{U}) \int_{F^\times} \chi^{-1}(a)\gamma(a, 1, \psi)|a|^s\, d^\times a \\
&= \lambda_\psi(\mathcal{U}) \int_{val^{-1}(l-1)} \chi^{-1}(a)(1-q)^{-1}|a|^s\, d^\times a + \lambda_\psi(\mathcal{U}) \sum_{k=l}^{\infty} \int_{val^{-1}(k)} \chi^{-1}(a)|a|^s\, d^\times a \\
&= \lambda_\psi(\mathcal{U})(1-q)^{-1}\chi(\mathcal{P})^{1-l}q^{(1-l)s} + \lambda_\psi(\mathcal{U}) \sum_{k=l}^{\infty} (\chi^{-1}(\mathcal{P})q^{-s})^k \\
&= \lambda_\psi(\mathcal{U})(1-q^{-1})^{-1}\chi(\mathcal{P})^{-l}q^{-ls}\frac{1-\chi(\mathcal{P})q^{s-1}}{1-\chi^{-1}(\mathcal{P})q^{-s}}.
\end{aligned}$$

So $s \in \mathbb{C}$ will be a pole only if $q^{-s} = \chi(\mathcal{P})$.

□

Lemma 15 Let χ be a character of $F^\times$ and let ψ be a non-trivial unitary character of F^+. Then for any $f_1, f_2 \in \mathcal{S}(F)$ we have the identity

$$\mathcal{Z}(1-s, \hat{f}_1, \chi^{-1})\mathcal{Z}(s, f_2, \chi) = \mathcal{Z}(1-s, \hat{f}_2, \chi^{-1})\mathcal{Z}(s, f_1, \chi)$$

in $\mathbb{C}(q^s)$. Here the Fourier transforms of f_1 and of f_2 are taken with respect to ψ.

Proof Define $\alpha := \alpha(\chi)$. The result above shows that the integral defining $\mathcal{Z}(s, \cdot, \chi)$ converges for $Re(s) > -\alpha$ and the integral defining $\mathcal{Z}(1-s, \cdot, \chi^{-1})$ converges for $Re(1-s) > \alpha$. Hence when $-\alpha < Re(s) < 1-\alpha$ the integrals defining $\mathcal{Z}(s, \cdot, \chi)$ and $\mathcal{Z}(1-s, \cdot, \chi^{-1})$ both converge. The region in $\mathbb{C}$ defined by $\{s \in \mathbb{C} : -\alpha < Re(s) < 1-\alpha\}$ is a vertical strip of width 1. By analytic continuation to prove the identity of meromorphic functions claimed in the proposition we need only establish it for s in this region. For $-\alpha < Re(s) < 1-\alpha$ we see that $\mathcal{Z}(1-s, \hat{f}_1, \chi^{-1})\mathcal{Z}(s, f_2, \chi)$ equals

$$\begin{aligned}
&\left(\int_{F^\times} \chi^{-1}(a)\hat{f}_1(a)|a|^{1-s}\, d^\times a\right)\left(\int_{F^\times} \chi(b)f_2(b)|b|^s\, d^\times b\right) \\
&= \left(\int_{F^\times} \chi^{-1}(a)|a|^{1-s}\left(\int_{F^+} f_1(y)\psi(ay)\, d_\psi y\right) d^\times a\right)\left(\int_{F^\times} \chi(b)f_2(b)|b|^s\, d^\times b\right) \\
&= \int_{F^\times}\int_{F^+}\int_{F^\times} f_1(y)f_2(b)\psi(ay)\chi^{-1}(a)\chi(b)|a|^{1-s}|b|^s\, d^\times a\, d_\psi y\, d^\times b \\
&= \int_{F^\times}\int_{F^\times}\int_{F^\times} f_1(y)f_2(b)\psi(ay)\chi^{-1}(a)\chi(b)|a|^{1-s}|b|^s\, d^\times a\, d_\psi y\, d^\times b \\
&= \int_{F^\times}\int_{F^\times}\int_{F^\times} f_1(y)f_2(b)\psi(a)\chi^{-1}(y^{-1}a)\chi(b)|y^{-1}a|^{1-s}|b|^s\, d^\times a\, d_\psi y\, d^\times b.
\end{aligned}$$

Here the second last equality is due to the fact that $\{0\}$ is a subset of F with measure zero. The last equality is arrived at by performing the transformation $a \longmapsto y^{-1}a$ and using the invariance of the Haar measure on $F^\times$.

By Proposition 4.2 we know that $d^+y = \lambda_\psi(\mathcal{U})|y|d^\times y$ holds. Using this to transform from the additive Haar measure to the multiplicative Haar measure in the middle integral above we obtain see that $\mathcal{Z}(1-s, \hat{f}_1, \chi^{-1})\mathcal{Z}(s, f_2, \chi)$ equals

$$\lambda_\psi(\mathcal{U}) \int_{F^\times} \int_{F^\times} \int_{F^\times} f_1(y) f_2(b) \psi(a) \chi(a^{-1}by) |by|^s |a|^{1-s} \, d^\times a \, d^\times y \, d^\times b.$$

Now this reasoning is symmetric in f_1 and f_2 and therefore the same argument shows that $\mathcal{Z}(1-s, \hat{f}_2, \chi^{-1})\mathcal{Z}(s, f_1, \chi)$ equals

$$\lambda_\psi(\mathcal{U}) \int_{F^\times} \int_{F^\times} \int_{F^\times} f_2(y) f_1(b) \psi(a) \chi(a^{-1}by) |by|^s |a|^{1-s} \, d^\times a \, d^\times y \, d^\times b.$$

Comparing these last two formulas and using Fubini's theorem yields the identity of the lemma.

□

These lemmas will serve us first by allowing us to make the following definition.

Definition 16 For any character χ of $F^\times$ and non-trivial unitary character ψ of F^+ we define the meromorphic function $\gamma(s, \chi, \psi) \in \mathbb{C}(q^s)$ to be the quotient

$$\gamma(s, \chi, \psi) = \frac{\mathcal{Z}(1-s, \hat{f}_\psi, \chi^{-1})}{\mathcal{Z}(s, f, \chi)}$$

where f is an arbitrary function in $\mathcal{S}(F)$ such that $\mathcal{Z}(s, f, \chi) \neq 0$. The Lemma 9.14 shows that there certainly exists a $f \in \mathcal{S}(F)$ for which $\mathcal{Z}(s, f, \chi) \neq 0$. The Lemma 9.15 shows that the quotient defining $\gamma(s, \chi, \psi)$ is independent of the choice of $f \in \mathcal{S}(F)$ satisfying $\mathcal{Z}(s, f, \chi) \neq 0$. The function $\gamma(s, \chi, \psi)$ is called the *gamma factor* of χ with respect to ψ.

The following theorem is the local functional equation for $GL_1(F)$. It is a simple consequence of our definitions and lemmas.

Theorem 17 Let ψ be a non-trivial unitary character of F^+. For any character χ of $F^\times$ and $f \in \mathcal{S}(F)$ we have the functional equation of meromorphic functions

$$\mathcal{Z}(1-s, \hat{f}_\psi, \chi^{-1}) = \gamma(s, \chi, \psi)\mathcal{Z}(s, f, \chi).$$

Proof Fix $f \in \mathcal{S}(F)$ and abbreviate $\hat{f}_\psi$ to $\hat{f}$. Define $\eta \in \mathcal{S}(F)$ by $\eta(x) = \chi^{-1}(x)$ when $x \in \mathcal{U}$ and $\eta(x) = 0$ when $x \in F$ and $x \notin \mathcal{U}$, or equivalently define $\eta := (\chi_0^{-1})^\times$. Then $\mathcal{Z}(s, \eta, \chi) = 1$ and so

$$\begin{aligned} \mathcal{Z}(1-s, \hat{f}, \chi^{-1}) &= \mathcal{Z}(1-s, \hat{f}, \chi^{-1})\mathcal{Z}(s, \eta, \chi) \\ &= \mathcal{Z}(1-s, \hat{\eta}, \chi^{-1})\mathcal{Z}(s, f, \chi) \\ &= \gamma(s, \chi, \psi)\mathcal{Z}(s, f, \chi) \end{aligned}$$

□

Proposition 18 Let χ be a character of $F^\times$. Let $1 \neq \psi \in F^*$ and define $l := cond(\psi)$.

(1) Suppose χ is ramified. Define $n := cond(\chi)$, and let $\tilde{\omega}$ be a uniformizing parameter of F. Then $\gamma(s, \chi, \psi)$ is an exponential function explicitly given by

$$\gamma(s, \chi, \psi) = \lambda_\psi(\mathcal{U})\chi(\tilde{\omega})^{n-l}\gamma(\tilde{\omega}^{l-n} \cdot \chi_0, \psi)q^{(n-l)s}.$$

This formula does not depend on the choice of $\tilde{\omega}$.

(2) Suppose χ is unramified. Then $\gamma(s, \chi, \psi)$ has a pole at $s \in \mathbb{C}$ iff $q^s = q\chi(\mathcal{P})$. In particular all poles of $\gamma(s, \chi, \psi)$ are on the line $Re(s) = 1 - \alpha(\chi)$. The function $\gamma(s, \chi, \psi)$ is explicitly given by

$$\gamma(s, \chi, \psi) = \lambda_\psi(\mathcal{U})(1 - q^{-1})^{-1}\chi(\mathcal{P})^{-l}q^{l(s-1)}\frac{L(1 - s, \chi^{-1})}{L(s, \chi)}.$$

Proof Define $f \in \mathcal{S}(F)$ by $f(x) = \chi^{-1}(x)$ when $x \in \mathcal{U}$ and $f(x) = 0$ otherwise. We have proven that $\mathcal{Z}(s, f, \chi) = 1$ and hence

$$\gamma(s, \chi, \psi) = \frac{\mathcal{Z}(1 - s, \hat{f}_\psi, \chi^{-1})}{\mathcal{Z}(s, f, \chi)} = \mathcal{Z}(1 - s, \hat{f}_\psi, \chi^{-1}).$$

All results now follow from the explicit computation of $\mathcal{Z}(s, \hat{f}_\psi, \chi^{-1})$ done in Lemma 9.14.

□

Proposition 19 If χ is a character of $F^\times$ and $1 \neq \psi \in F^*$ then

$$\gamma(s, \chi, \psi)\gamma(1 - s, \chi^{-1}, \psi) = \chi(-1).$$

Proof Fix $f \in \mathcal{S}(F)$ such that $\mathcal{Z}(s, f, \chi) \neq 0$. Define $\eta \in \mathcal{S}(F)$ by $\eta := \hat{f}$. Two applications of the functional equation gives

$$\mathcal{Z}(s, \hat{\eta}, \chi) = \gamma(1 - s, \chi^{-1}, \psi)\mathcal{Z}(1 - s, \hat{f}, \chi^{-1}) = \gamma(1 - s, \chi^{-1}, \psi)\gamma(s, \chi, \psi)\mathcal{Z}(s, f, \chi).$$

Now observe that since $\hat{\eta} = f(- \cdot)$ we have for $Re(s)$ sufficiently large that

$$\mathcal{Z}(s, \hat{\eta}, \chi) = \int_{F^\times} \chi(a)f(-a)|a|^s\, d^\times a = \chi(-1)\mathcal{Z}(s, f, \chi).$$

Hence by analytic continuation we have the identity

$$\chi(-1)\mathcal{Z}(s, f, \chi) = \gamma(1 - s, \chi^{-1}, \psi)\gamma(s, \chi, \psi)\mathcal{Z}(s, f, \chi)$$

in $\mathbb{C}(q^s)$. Since $\mathcal{Z}(s, f, \chi) \neq 0$ the proposition follows.

□

Epsilon Factors for $\mathbf{GL_1(F)}$

Definition 20 Suppose χ is a character of $F^\times$ and $1 \neq \psi \in F^*$. We define the *epsilon factor* of χ with respect to ψ by

$$\epsilon(s, \chi, \psi) := \frac{\gamma(s, \chi, \psi)L(s, \chi)}{L(1 - s, \chi^{-1})}.$$

By definition $\epsilon(s, \chi, \psi)$ is in $\mathbb{C}(q^s)$. Note that if χ is ramified then $\epsilon(s, \chi, \psi) = \gamma(s, \chi, \psi)$.

There is another version of the functional equation that uses the epsilon factors instead of the gamma factors. Both versions play a role in the global theory.

Theorem 21 Let ψ be a non-trivial unitary character of F^+. For any character χ of $F^\times$ and $f \in \mathcal{S}(F)$ we have the functional equation

$$\frac{\mathcal{Z}(1-s, \hat{f}_\psi, \chi^{-1})}{L(1-s, \chi^{-1})} = \epsilon(s, \chi, \psi)\frac{\mathcal{Z}(s, f, \chi)}{L(s, \chi)}.$$

Proof Define $\hat{f} := \hat{f}_\psi$. By Theorem 9.17 for any $f \in \mathcal{S}(F)$ we have

$$\mathcal{Z}(1-s, \hat{f}, \chi^{-1}) = \gamma(s, \chi, \psi)\mathcal{Z}(s, f, \chi).$$

Therefore we find

$$\frac{\mathcal{Z}(1-s, \hat{f}, \chi^{-1})}{L(1-s, \chi^{-1})} = \left(\frac{\gamma(s, \chi, \psi)L(s, \chi)}{L(1-s, \chi^{-1})}\right)\left(\frac{\mathcal{Z}(s, f, \chi)}{L(s, \chi)}\right) = \epsilon(s, \chi, \psi)\frac{\mathcal{Z}(s, f, \chi)}{L(s, \chi)}$$

which proves the functional equation.

□

We say that an entire function is *exponential* iff it is of the form $s \longmapsto ca^s$ where $c \in \mathbb{C}^\times$ and $a \in (0, \infty)$. Observe that an exponential function is entire and has no zeros. In particular any function in $\mathbb{C}(q^s)$ of the form cq^{ks} where $c \in \mathbb{C}^\times$ and $k \in \mathbb{Z}$ is an exponential function. These are the only exponential functions in $\mathbb{C}(q^s)$. The next result is trivial.

Lemma 22 Let $f(s) \in \mathbb{C}(q^s)$. Then $f(s)$ is an exponential function iff $f(s)$ is entire and has no zeros iff $f(s)$ has the from cq^{ks} for some $c \in \mathbb{C}^\times$ and $k \in \mathbb{Z}$ iff $f(s)$ is a unit of the ring $\mathbb{C}[q^s, q^{-s}]$.

Proposition 23 Let χ be a character of $F^\times$ and let ψ be a non-trivial unitary character of F^+. The function $\epsilon(s, \chi, \psi)$ is an exponential function and satisfies the identity

$$\epsilon(s, \chi, \psi)\epsilon(1-s, \chi^{-1}, \psi) = \chi(-1).$$

Proof Choose $f \in \mathcal{S}(F)$ such that $\mathcal{Z}(s, f, \chi) = L(s, \chi)$. By the functional equation we find

$$\epsilon(s, \chi, \psi) = \frac{\mathcal{Z}(1-s, \hat{f}, \chi)}{L(1-s, \chi^{-1})}.$$

Consequently $\epsilon(s, \chi, \psi) \in \mathbb{C}[q^s, q^{-s}]$, and hence in particular it is an entire function. Using the identity $\gamma(s, \chi, \psi)\gamma(1-s, \chi^{-1}, \psi) = \chi(-1)$ we obtain

$$\begin{aligned}\epsilon(s, \chi, \psi)\epsilon(1-s, \chi^{-1}, \psi) &= \left(\frac{\gamma(s, \chi, \psi)L(s, \chi)}{L(1-s, \chi^{-1})}\right)\left(\frac{\gamma(1-s, \chi^{-1}, \psi)L(1-s, \chi^{-1})}{L(s, \chi)}\right)\\ &= \gamma(s, \chi, \psi)\gamma(1-s, \chi^{-1}, \psi)\\ &= \chi(-1).\end{aligned}$$

Since $\epsilon(s, \chi, \psi)$ and $\epsilon(1-s, \chi^{-1}, \psi)$ are entire functions this identity proves that $\epsilon(s, \chi, \psi)$ has no zeros. Thus by the above lemma $\epsilon(s, \chi, \psi)$ is exponential.

□

Proposition 24 If χ is an unramified character of $F^\times$ and ψ is a an unramified character of F^+ then $\epsilon(s,\chi,\psi) = 1$.

Proof In Proposition 9.18.2 we have proven that when χ is unramified we have

$$\gamma(s,\chi,\psi) = \lambda_\psi(\mathcal{U})(1-q^{-1})^{-1}\chi(\mathcal{P})^{-l}q^{l(s-1)}\frac{L(1-s,\chi^{-1})}{L(s,\chi)}$$

where $l = cond(\psi)$. Since we are assuming ψ is unramified we know $l = 0$ and so $\lambda_\psi(\mathcal{U}) = 1 - q^{-1}$. It immediately follows that $\epsilon(s,\chi,\psi) = 1$.

□

Zeta Functions for $\mathbf{GL_2(F)}$

Lemma 25 Let χ be a character of $F^\times$ and fix $f \in \mathcal{S}(F)$.

(1) If χ is ramified or if $f(0) = 0$ then for every $s \in \mathbb{C}$ the integral

$$\int_{F^\times} \chi(a)f(a)val(a)|a|^s\, d^\times a$$

is convergent. Moreover as a function of the complex variable s it equals a function in $\mathbb{C}[q^s, q^{-s}]$.

(2) If χ is unramified and $f(0) \neq 0$ then for every $s \in \mathbb{C}$ with $Re(s) > -\alpha(\chi)$ the integral

$$\int_{F^\times} \chi(a)f(a)val(a)|a|^s\, d^\times a$$

is convergent. As a function of the complex variable s the integral equals a function in the fractional ideal of $\mathbb{C}[q^s, q^{-s}]$ generated by $(1-\chi(\mathcal{P})q^{-s})^{-2}$ restricted to the right half plane $Re(s) > -\alpha(\chi)$. Hence as a function of s this integral has an analytic continuation from an analytic function on the region $Re(s) > -\alpha(\chi)$ to a meromorphic function in $\mathbb{C}(q^s)$, which has a pole at $s \in \mathbb{C}$ only if $q^s = \chi(\mathcal{P})$.

Proof Choose $n \in \mathbb{N}$ such that $x \in \mathcal{P}^n$ implies $f(x) = f(0)$. Choose $m \in \mathbb{Z}$ with $m < n$ such that $x \in F$, $val(x) < m$ implies $f(x) = 0$. Let $\tilde{\omega}$ be a uniformizing parameter of F. For any $r \in \mathbb{N}$ with $r \geq n$ we have for $s \in \mathbb{C}$ that

$$\int_{F\setminus\mathcal{P}^{r+1}} \chi(a)f(a)val(a)|a|^s\, d^\times a = \sum_{k=m}^{n-1} \chi(\tilde{\omega})^k \hat{f}_k(\chi_0)kq^{-ks} + f(0)\delta(\chi_0)\sum_{k=n}^{r} k(\chi(\tilde{\omega})q^{-s})^k.$$

When χ is ramified $\delta(\chi_0) = 0$. Hence if χ is ramified or $f(0) = 0$ then by passing to the limit in the above we conclude that for any $s \in \mathbb{C}$,

$$\int_{F^\times} \chi(a)f(a)val(a)|a|^s\, d^\times a = \sum_{k=m}^{n-1} \chi(\tilde{\omega})^k \hat{f}_k(\chi_0)kq^{-ks}.$$

This is a function in $\mathbb{C}[q^s, q^{-s}]$ and hence the first part of the lemma.

To prove the second part suppose that χ is unramified and $f(0) \neq 0$. It is a simple to prove that for $z \in \mathbb{C}$ with $|z| < 1$ we have for any $n \in \mathbb{N}$ that $\sum_{k=n}^{\infty} kz^k = z^n(n-(n-1)z)(1-z)^{-2}$. Hence in particular $\sum_{k=1}^{\infty} kz^k = z(1-z)^{-2}$. Recall that $|\chi(\mathcal{P})q^{-s}| < 1$ iff $Re(s) > -\alpha(\chi)$. Hence by passing to the limit again in our initial computation we find that for $Re(s) > -\alpha(\chi)$ the integral

$$\int_{F^\times} \chi(a)f(a)val(a)|a|^s\, d^\times a$$

equals

$$\sum_{k=m}^{n-1} \chi(\mathcal{P})^k \hat{f}_k(\chi_0) q^{-ks} + f(0)\chi(\mathcal{P})^n q^{-ns}(n-(n-1)\chi(\mathcal{P})q^{-s})(1-\chi(\mathcal{P})q^{-s})^{-2}.$$

This proves the second part of the lemma.

□

Corollary 26 Let $\pi \in \Pi(G)$ be infinite dimensional. There exists a real number s_0, depending on π, such that if $\varphi \in \mathcal{K}(\pi)$ and $Re(s) > s_0$ then the integral

$$\int_{F^\times} \varphi(a)|a|^{s-\frac{1}{2}}\, d^\times a$$

is convergent. Moreover as a function of the complex variable s it equals a function in $\mathbb{C}(q^s)$ restricted to the right half plane $Re(s) > s_0$. Hence for any $\varphi \in \mathcal{K}(\pi)$ the function of s defined by this integral has an analytic continuation to a meromorphic function in $\mathbb{C}(q^s)$.

Proof Theorem 8.27, Lemma 9.6, and Lemma 9.25.

□

Definition 27 Let π be a class in $\Pi(G)$. We distinguish between the cases of π being one dimensional or infinite dimensional.

(1) Suppose π is finite dimensional. There is a unique character χ of $F^\times$ such that the representation : $G \longmapsto GL_1(\mathbb{C})$ $\quad g \longmapsto \chi(det(g))$ is in π. The *zeta function* of π will be denoted by $\mathcal{Z}(s,f,\pi)$ where s is a complex variable, $f \in \mathcal{S}(F)$, and it will be defined to be the zeta function $\mathcal{Z}(s,f,\chi)$ of χ. Hence $\mathcal{Z}(s,f,\pi) \in \mathbb{C}(q^s)$.

(2) Suppose π is infinite dimensional. In this case the *zeta function* of π will be denoted by $\mathcal{Z}(s,\varphi,\pi)$ where s is a complex variable, $\varphi \in \mathcal{K}(\pi)$, and it is defined to be the meromorphic function in $\mathbb{C}(q^s)$ which is the analytic continuation of the function

$$s \longmapsto \int_{F^\times} \varphi(a)|a|^{s-\frac{1}{2}}\, d^\times a.$$

Note that if $1^\times \in \mathcal{S}(F^\times) \subseteq \mathcal{K}(\pi)$ is the characteristic function of $\mathcal{U}$ in $F^\times$ then clearly $\mathcal{Z}(s,1^\times,\pi) = 1$.

Proposition 28 Let $\pi \in \Pi(G)$ be infinite dimensional. Then for any $\varphi \in \mathcal{K}(\pi)$ and $b \in F^\times$ we have the identity

$$\mathcal{Z}(s,\varphi(b\,\cdot),\pi) = |b|^{\frac{1}{2}-s}\mathcal{Z}(s,\varphi,\pi).$$

Proof We need only show that this identity holds for $Re(s)$ sufficiently large. Well for $Re(s)$ sufficiently large the integral defining zeta function $\mathcal{Z}(s,\,\cdot\,,\pi)$ converges, and thus

$$\mathcal{Z}(s,\varphi(b\,\cdot),\pi) = \int_{F^\times} \varphi(ab)|a|^{s-\frac{1}{2}}\, d^\times a = \int_{F^\times} \varphi(a)|b^{-1}a|^{s-\frac{1}{2}}\, d^\times a = |b|^{\frac{1}{2}-s}\mathcal{Z}(s,\varphi,\pi).$$

□

Euler Factors for $\mathbf{GL_2(F)}$

Definition 29 Let $\pi \in \Pi(G)$.

(1) If π is one dimensional, corresponding to the character χ of $F^\times$, we define

$$\mathcal{Z}(\pi) := \mathcal{Z}(\chi) = \{\mathcal{Z}(s,f,\chi) : f \in \mathcal{S}(F)\}.$$

(2) If π is infinite dimensional define

$$\mathcal{Z}(\pi) = \{\mathcal{Z}(s,\varphi,\pi) : \varphi \in \mathcal{K}(\pi)\}$$

Theorem 30 If π is an infinite dimensional class in $\Pi(G)$ then

$$\begin{aligned}\mathcal{Z}(\pi) &= (1) \quad \text{if} \quad \pi \quad \text{is cuspidal}\\ &= (L(s,\chi_1)) \quad \text{if} \quad \pi = \sigma(\chi_1,\chi_2) \quad \text{and} \quad \chi = |\cdot|\\ &= (L(s,\chi_2)) \quad \text{if} \quad \pi = \sigma(\chi_1,\chi_2) \quad \text{and} \quad \chi = |\cdot|^{-1}\\ &= (L(s,\chi_1)L(s,\chi_2)) \quad \text{if} \quad \pi = \pi(\chi_1,\chi_2)\end{aligned}$$

Hence the fractional ideal $\mathcal{Z}(\pi)$ is generated by an Euler factor.

Proof First suppose π is cuspidal. Then $\mathcal{K}(\pi) = \mathcal{S}(F^\times)$. Let $\varphi \in \mathcal{K}(\pi)$. Choose $N \in \mathbb{N}$ such that $\varphi(x) \neq 0$ only if $-N \leq val(x) \leq N$. Thus,

$$\mathcal{Z}(s,\varphi,\pi) = \int_{F^\times} \varphi(a)|a|^{s-\frac{1}{2}}\, d^\times a = \sum_{k=-N}^{N} q^{\frac{k}{2}} \left(\int_{val^{-1}(k)} \varphi(a)\, d^\times a\right) q^{-ks} \in \mathbb{C}[q^s, q^{-s}].$$

Hence $\mathcal{Z}(\pi) \subseteq \mathbb{C}[q^s, q^{-s}]$. If we define $\varphi \in \mathcal{S}(F^\times) = \mathcal{K}(\pi)$ to be the characteristic function of $\mathcal{U}$ in $F^\times$ then clearly $\mathcal{Z}(s,\varphi,\pi) = 1$, and so $1 \in \mathcal{Z}(\pi)$. This proves $\mathcal{Z}(\pi) = \mathbb{C}[q^s, q^{-s}]$.

Now suppose that $\pi = \sigma(\chi_1,\chi_2)$ is a special representation. As usual we define $\chi := \chi_1\chi_2^{-1}$. Therefore $\chi = |\cdot|$ or $\chi = |\cdot|^{-1}$. We first consider the case $\chi = |\cdot|$. Then $\mathcal{K}(\pi)$ is the space of functions $\varphi(x) = |x|^{\frac{1}{2}}\chi_1(x)f(x)$ where $f \in \mathcal{S}(F)$. Note that

$$\mathcal{Z}(s,\varphi,\pi) = \int_{F^\times} \varphi(a)|a|^{s-\frac{1}{2}}\, d^\times a = \int_{F^\times} \chi_1(a)f(a)|a|^s\, d^\times a = \mathcal{Z}(s,f,\chi_1).$$

Hence $\mathcal{Z}(\pi) = \mathcal{Z}(\chi_1) = (L(s,\chi_1))$. Now consider the case $\chi = |\cdot|^{-1}$. Then $\mathcal{K}(\pi)$ is the space of functions $\varphi(x) = |x|^{\frac{1}{2}}\chi_2(x)f(x)$ where $f \in \mathcal{S}(F)$. Similarly as above we prove that

$$\mathcal{Z}(s,\varphi,\pi) = \mathcal{Z}(s,f,\chi_2).$$

Hence $\mathcal{Z}(\pi) = \mathcal{Z}(\chi_2) = (L(s,\chi_2))$.

Suppose now that $\pi = \pi(\chi_1,\chi_2)$ is a principal series representation with $\chi \neq 1$, or equivalently $\chi_1 \neq \chi_2$. Then $\mathcal{K}(\pi)$ is the space of functions

$$\varphi(x) = |x|^{\frac{1}{2}}(\chi_1(x)f_1(x) + \chi_2(x)f_2(x))$$

where $f_1, f_2 \in \mathcal{S}(F)$. We easily see that $\mathcal{Z}(s,\varphi,\pi) = \mathcal{Z}(s,f_1,\chi_1) + \mathcal{Z}(s,f_2,\chi_2)$ and so

$$\frac{\mathcal{Z}(s,\varphi,\pi)}{L(s,\chi_1)L(s,\chi_2)} = \left(\frac{1}{L(s,\chi_2)}\right)\left(\frac{\mathcal{Z}(s,f_1,\chi_1)}{L(s,\chi_1)}\right) + \left(\frac{1}{L(s,\chi_2)}\right)\left(\frac{\mathcal{Z}(s,f_2,\chi_2)}{L(s,\chi_2)}\right).$$

The expressions enclosed in each of the four sets of parenthesis are in $\mathbb{C}[q^s, q^{-s}]$. Hence $\mathcal{Z}(\pi)$ is contained in the principal fractional ideal $(L(s,\chi_1)L(s,\chi_2))$. Now we must show that there is a $\varphi \in \mathcal{K}(\pi)$ such that $\mathcal{Z}(s,\varphi,\pi) = L(s,\chi_1)L(s,\chi_2)$. There are $f_1, f_2 \in \mathcal{S}(F)$ such that $\mathcal{Z}(s,f_1,\chi_1) = L(s,\chi_1)$ and $\mathcal{Z}(s,f_2,\chi_2) = L(s,\chi_2)$. Hence it suffices to show that there are $c_1, c_2 \in \mathbb{C}$ such that $c_1L(s,\chi_1) + c_2L(s,\chi_2) = L(s,\chi_1)L(s,\chi_2)$, for then $\varphi(x) := |x|^{\frac{1}{2}}(\chi_1(x)c_1f_1(x) + \chi_2(x)c_2f_2(x))$ is in $\mathcal{K}(\pi)$ and satisfies our requirement. If either χ_1 or χ_2 is ramified it is then obvious how to define c_1 and c_2 since then either $L(s,\chi_1) = 1$ or $L(s,\chi_2) = 1$. So we may suppose that both χ_1 and χ_2 are unramified. The requisite equations are easily seen to be equivalent to $c_1 + c_2 = 1$ and $c_1\chi_2(\mathcal{P}) + c_2\chi_1(\mathcal{P}) = 0$. Since χ_1 and χ_2 agree on $\mathcal{U}$ but $\chi_1 \neq \chi_2$ we must have $\chi_1(\mathcal{P}) \neq \chi_2(\mathcal{P})$. This fact ensures that the above system of equations in c_1 and c_2 has a solution. Hence $\mathcal{Z}(\pi) = (L(s,\chi_1)L(s,\chi_2))$.

The final case is when $\pi = \pi(\chi_1,\chi_2)$ is a principal series representation and $\chi = 1$, so that $\chi_1 = \chi_2$. Define $\omega := \chi_1 = \chi_2$. The functions in $\mathcal{K}(\pi)$ are those of the form

$$\varphi(x) = |x|^{\frac{1}{2}}\omega(x)(f_1(x) + val(x)f_2(x))$$

where $f_1, f_2 \in \mathcal{S}(F)$. Define

$$I(s) := \int_{F^\times} \omega(a)f_2(a)val(a)|a|^s\, d^\times a.$$

We have shown that $I(s)$ is convergent in a right half plane. Moreover $I(s)$ has an analytic continuation to a function in $\mathbb{C}[q^s, q^{-s}]$ if ω is ramified and to a function in $L(s,\omega)^2\mathbb{C}[q^s, q^{-s}]$ if ω is unramified. We clearly have $\mathcal{Z}(s,\varphi,\pi) = \mathcal{Z}(s,f_1,\chi_1) + I(s)$. Therefore $\mathcal{Z}(s,\varphi,\pi) \in (L(s,\omega)^2)$, and hence $\mathcal{Z}(\pi) \subseteq (L(s,\omega)^2) = (L(s,\chi_1)L(s,\chi_2))$. It remains to prove that there is a $\varphi \in \mathcal{K}(\pi)$ such that $\mathcal{Z}(s,\varphi,\pi) = L(s,\omega)^2$. When ω is ramified we may define $\varphi \in \mathcal{S}(F^\times) \subset \mathcal{K}(\pi)$ to be the characteristic function of $\mathcal{U}$ to obtain $\mathcal{Z}(s,\varphi,\pi) = 1 = L(s,\omega)^2$. So we may suppose that ω is unramified. Define $f \in \mathcal{S}(F)$ to be the characteristic function of $\mathcal{O}$ and define $\eta \in \mathcal{K}(\pi)$ by $\eta(x) := |x|^{\frac{1}{2}}\omega(x)val(x)f(x)$. It is a trivial fact in the theory of power series that for $z \in \mathbb{C}$ with $|z| < 1$ we have $\sum_{k=1}^{\infty} kz^k = z(1-z)^{-2}$. Hence for $Re(s)$ sufficiently large we have

$$\mathcal{Z}(s,\eta,\pi) = \sum_{k=1}^{\infty} k(\omega(\mathcal{P})q^{-s})^k = \omega(\mathcal{P})q^{-s}L(s,\omega)^2.$$

Since $\mathcal{Z}(s,\eta(\tilde{\omega}\,\cdot),\pi) = q^{s-\frac{1}{2}}\mathcal{Z}(s,\eta,\pi) = \omega(\mathcal{P})q^{-s}L(s,\omega)^2$ we see that if we define $\varphi \in \mathcal{K}(\pi)$ by $\varphi := \omega(\mathcal{P})^{-1}q^{\frac{1}{2}}\eta(\tilde{\omega}\,\cdot)$ then $\mathcal{Z}(s,\varphi,\pi) = L(s,\omega)^2 = L(s,\chi_1)L(s,\chi_2)$ as desired.

□

Definition 31 Let $\pi \in \Pi(G)$.

(1) If π is one dimensional corresponding to the character χ of $F^\times$ then we define the *Euler factor* of π to be the Euler factor of χ. It is denoted by $L(s,\pi) := L(s,\chi)$ so that $\mathcal{Z}(\pi) = \mathcal{Z}(\chi) = (L(s,\chi)) = (L(s,\pi))$.

(2) If π is infinite dimensional we define the *Euler factor* of π to be the unique Euler factor which generates $\mathcal{Z}(\pi)$, namely $\mathcal{Z}(\pi) = (L(s,\pi))$.

Corollary 32 Let $\pi \in \Pi(G)$.

(1) If π is one dimensional and associated to the character χ of $F^\times$ then $L(s,\pi) = L(s,\chi)$.
(2) If π is cuspidal then $L(s,\pi) = 1$.
(3) If $\pi = \pi(\chi_1,\chi_2)$ is a principal series then $L(s,\pi(\chi_1,\chi_2)) = L(s,\chi_1)L(s,\chi_2)$
(4) If $\pi = \sigma(\chi_1,\chi_2)$ and $\chi = |\cdot|$ then $L(s,\sigma(\chi_1,\chi_2)) = L(s,\chi_1)$.
(5) If $\pi = \sigma(\chi_1,\chi_2)$ and $\chi = |\cdot|^{-1}$ then $L(s,\sigma(\chi_1,\chi_2)) = L(s,\chi_2)$.

Given an infinite dimensional class $\pi \in \Pi(G)$ we know $\mathcal{Z}(\pi) = (L(s, \pi))$ and so for every $\varphi \in \mathcal{K}(\pi)$ we have

$$\frac{\mathcal{Z}(s, \varphi, \pi)}{L(s, \pi)}$$

being in $\mathbb{C}[q^s, q^{-s}]$ and hence in particular it is entire. Another obvious implication of the identity $\mathcal{Z}(\pi) = (L(s, \pi))$ is that there exists a $\varphi \in \mathcal{K}(\pi)$ such that $\mathcal{Z}(s, \varphi, \pi) = L(s, \pi)$.

Gamma Factors for $\mathbf{GL_2(F)}$

The following lemma can be found in [2] in the section on the functional equation.

Lemma 33 Let $\pi \in \Pi(G)$ be infinite dimensional and let $1 \neq \psi \in F^*$. Then there exists a function $\varphi \in \mathcal{S}(F^\times) \cap \pi_\psi(w)\mathcal{S}(F^\times)$ such that $\mathcal{Z}(s, \varphi, \pi) \neq 0$.

Let $\pi \in \Pi(G)$ be infinite dimensional. Recall that for $\varphi \in \mathcal{K}(\pi)$ we define $\tilde{\varphi} := \omega_\pi^{-1}\varphi$ and the map

$$: \mathcal{K}(\pi) \longrightarrow \mathcal{K}(\tilde{\pi}) \quad \varphi \longmapsto \tilde{\varphi}$$

is an isomorphism of vector spaces. For any $1 \neq \psi \in F^*$ we have the formula

$$\tilde{\pi}_\psi(g)\tilde{\varphi} = \omega_\pi^{-1}(det(g))\omega_\pi^{-1}\pi_\psi(g)\varphi.$$

The following result is the generalization to $GL_2(F)$ of the functional equation for $GL_1(F)$.

Theorem 34 Let ψ be a non-trivial unitary character of F^+.

(1) For each infinite dimensional $\pi \in \Pi(G)$ there exists a unique function in $\mathbb{C}(q^s)$, which we denote by $\gamma(s, \pi, \psi)$, such that for all $\varphi \in \mathcal{K}(\pi)$ we have the functional equation,

$$\mathcal{Z}(1 - s, \tilde{\pi}_\psi(w)\tilde{\varphi}, \tilde{\pi}) = \gamma(s, \pi, \psi)\mathcal{Z}(s, \varphi, \pi).$$

(2) Let $\tilde{\omega}$ be a uniformizing parameter of F. Then we have the formula

$$\gamma(s, \pi, \psi) = \mathcal{Z}(1 - s, \tilde{\pi}_\psi(w)\tilde{1}^\times, \tilde{\pi}) = \sum_{n \in \mathbb{Z}} \omega_\pi(\tilde{\omega})^{-n}\pi_\psi(w)(1^\times)(\tilde{\omega}^n)q^{n(s-\frac{1}{2})}.$$

The second equality only holds for $Re(s)$ sufficiently large. The formula does not depend on $\tilde{\omega}$ since the terms $\omega_\pi(\tilde{\omega})^{-n}\pi_\psi(w)(1^\times)(\tilde{\omega}^n)$ do not depend on $\tilde{\omega}$.

(3) We have the identity

$$\gamma(s, \pi, \psi)\gamma(1 - s, \tilde{\pi}, \psi) = \omega_\pi(-1).$$

Proof Since $\mathcal{Z}(s, 1^\times, \pi) = 1$ it is obviously necessary that

$$\gamma(s, \pi, \psi) = \mathcal{Z}(1 - s, \tilde{\pi}_\psi(w)\tilde{1}^\times, \tilde{\pi}),$$

for the asserted functional equation to hold. This of course implies the uniqueness of $\gamma(s, \pi, \psi)$. So given any π and ψ we define $\gamma(s, \pi, \psi) := \mathcal{Z}(1 - s, \tilde{\pi}_\psi(w)\tilde{1}^\times, \tilde{\pi}) \in \mathbb{C}(q^s)$ and we are required to prove that the above statements hold for this $\gamma(s, \pi, \psi)$.

For an infinite dimensional $\pi \in \Pi(G)$ and $\varphi \in \mathcal{K}(\pi)$ we say that the *functional equation holds for* (φ, π) iff

$$\mathcal{Z}(1 - s, \tilde{\pi}_\psi(w)\tilde{\varphi}, \tilde{\pi}) = \gamma(s, \pi, \psi)\mathcal{Z}(s, \varphi, \pi)$$

holds.

We first show that the functional equation holds for $(\mu^\times, \pi)$ where $\mu \in \mathcal{U}^*$ and π are arbitrary. We easily find $\mathcal{Z}(s, \mu^\times, \pi) = \delta(\mu)$ and for $Re(s)$ sufficiently large

$$\begin{aligned}\mathcal{Z}(s, \tilde{\pi}_\psi(w)\tilde{\varphi}, \tilde{\pi}) &= \int_{F^\times} (\tilde{\pi}_\psi(w)\tilde{\varphi})(a)|a|^{s-\frac{1}{2}}\, d^\times a \\ &= \int_{F^\times} \omega_\pi^{-1}(a)(\pi_\psi(w)\varphi)(a)|a|^{s-\frac{1}{2}}\, d^\times a \\ &= \sum_{n\in\mathbb{Z}} \int_{\mathcal{U}} \omega_\pi^{-1}(\epsilon\tilde{\omega}^n)(\pi_\psi(w)\varphi)(\epsilon\tilde{\omega}^n)|\epsilon\tilde{\omega}^n|^{s-\frac{1}{2}}\, d^\times \epsilon \\ &= \sum_{n\in\mathbb{Z}} \delta(\mu)\omega_\pi(\tilde{\omega})^{-n}(\pi_\psi(w)\mu^\times)(\tilde{\omega}^n)q^{n(s-\frac{1}{2})}.\end{aligned}$$

By putting $\mu = 1$ we obtain the formula for $\gamma(s, \pi, \psi)$ claimed in part (2) of the theorem and by definition of $\gamma(s, \pi, \psi)$ the functional equation holds for $(1^\times, \pi)$. When $\mu \neq 1$ we have $\mathcal{Z}(1-s, \tilde{\pi}_\psi(w)\tilde{\varphi}, \tilde{\pi}) = 0 = \mathcal{Z}(s, \varphi, \psi) = \gamma(s, \pi, \psi)\mathcal{Z}(s, \varphi, \psi)$ and hence the functional equation holds for $(\mu^\times, \pi)$.

Now we show that if the functional equation holds for (φ, π) then for any $b \in F^\times$ it also holds for $(\varphi(b\,\cdot), \pi)$. To do this note that

$$(\tilde{\pi}_\psi(w)\tilde{\varphi}(b\,\cdot))(a) = \omega_\pi^{-1}(a)\left(\pi_\psi(w)\pi_\psi\begin{pmatrix} b & 0 \\ 0 & 1\end{pmatrix}\varphi\right)(a) = \omega_\pi^{-1}(ab^{-1})(\pi_\psi(w)\varphi)(ab^{-1})$$

and hence

$$\begin{aligned}\mathcal{Z}(1-s, \tilde{\pi}_\psi(w)\tilde{\varphi}(b\,\cdot), \tilde{\pi}) &= \int_{F^\times} \omega_\pi^{-1}(ab^{-1})(\pi_\psi(w)\varphi)(ab^{-1})|a|^{(1-s)-\frac{1}{2}}\, d^\times a \\ &= \int_{F^\times} \omega_\pi^{-1}(a)(\pi_\psi(w)\varphi)(a)|ab|^{(1-s)-\frac{1}{2}}\, d^\times a \\ &= |b|^{\frac{1}{2}-s}\mathcal{Z}(1-s, \pi_\psi(\tilde{w})\varphi, \tilde{\pi}) \\ &= |b|^{\frac{1}{2}-s}\gamma(s, \pi, \psi)\mathcal{Z}(s, \varphi, \pi).\end{aligned}$$

Since $\mathcal{Z}(s, \varphi(b\,\cdot), \pi) = |b|^{\frac{1}{2}-s}\mathcal{Z}(s, \varphi, \pi)$ the functional equation holds for $(\varphi(b\,\cdot), \pi)$.

Clearly if the functional equation holds for (φ_1, π) and (φ_2, π) then it also holds for $(c_1\varphi_1 + c_2\varphi_2, \pi)$. Therefore since the functions $\{\mu^\times(b\,\cdot) : \mu \in \mathcal{U}^* \quad b \in F^\times\}$ span $\mathcal{S}(F^\times)$ we conclude that the functional equation holds for (φ, π) for every π and $\varphi \in \mathcal{S}(F^\times)$.

Before we can proceed further we need to prove the identity

$$\gamma(s, \pi, \psi)\gamma(1-s, \tilde{\pi}, \psi) = \omega_\pi(-1).$$

In order to do this fix a $\varphi \in \mathcal{S}(F^\times) \cap \pi_\psi(w)\mathcal{S}(F^\times)$ which satisfies $\mathcal{Z}(s, \varphi, \pi) \neq 0$. Note that $\pi_\psi(w)\varphi \in \pi_\psi(w^2)\mathcal{S}(F^\times) = \omega_\pi(-1)\mathcal{S}(F^\times) = \mathcal{S}(F^\times)$. Define $\eta := \tilde{\pi}_\psi(w)\tilde{\varphi} = \omega_\pi^{-1}\pi_\psi(w)\varphi \in \mathcal{S}(F^\times) \subseteq \mathcal{K}(\tilde{\pi})$. By what we have shown above we know that the functional equation holds for (φ, π) and for $(\eta, \tilde{\pi})$. Hence we find

$$\mathcal{Z}(s, \tilde{\tilde{\pi}}_\psi(w)\tilde{\eta}, \tilde{\tilde{\pi}}) = \gamma(1-s, \tilde{\pi}, \psi)\mathcal{Z}(1-s, \eta, \psi) = \gamma(1-s, \tilde{\pi}, \psi)\gamma(s, \pi, \psi)\mathcal{Z}(s, \varphi, \pi).$$

Recall that $\tilde{\tilde{\pi}} = \pi$ and note that

$$\tilde{\tilde{\pi}}_\psi(w)\tilde{\eta} = \omega_{\tilde{\pi}}^{-1}\omega_{\tilde{\pi}}^{-1}(det(w))\tilde{\pi}_\psi(w)\eta = \omega_{\tilde{\pi}}^{-1}\tilde{\pi}_\psi(w^2)\tilde{\varphi} = \omega_{\tilde{\pi}}(-1)\omega_{\tilde{\pi}}^{-1}\omega_\pi^{-1}\varphi = \omega_\pi(-1)\varphi.$$

Hence $\omega_\pi(-1)\mathcal{Z}(s,\varphi,\pi) = \gamma(1-s,\tilde{\pi},\psi)\gamma(s,\pi,\psi)\mathcal{Z}(s,\varphi,\pi)$. Now we use the fact that $\mathcal{Z}(s,\varphi,\pi) \neq 0$ and we immediately obtain the desired identity.

Using what we have proven above, and the fact that $\mathcal{K}(\pi) = \mathcal{S}(F^\times) + \pi_\psi(w)\mathcal{S}(F^\times)$. we see that the functional equation holds for every (φ,π) provided we show that the functional equation holds for $(\pi_\psi(w)\varphi,\pi)$ for every π and $\varphi \in \mathcal{S}(F^\times)$. After establishing this the proof of the theorem will be complete. Well let π and $\varphi \in \mathcal{S}(F^\times)$ be arbitrary. and define $\eta := \pi_\psi(w)\varphi$. We are required to show that

$$\mathcal{Z}(1-s,\tilde{\pi}_\psi(w)\tilde{\eta},\tilde{\pi}) = \gamma(s,\pi,\psi)\mathcal{Z}(s,\eta,\pi).$$

First note that

$$\tilde{\pi}_\psi(w)\tilde{\eta} = \omega_\pi^{-1}\pi_\psi(w)\eta = \omega_\pi^{-1}\pi_\psi(w^2)\varphi = \omega_\pi\tilde{\varphi}.$$

Since $\tilde{\varphi} \in \mathcal{S}(F^\times) \subseteq \mathcal{K}(\tilde{\pi})$ we know the functional equation holds for $(\tilde{\varphi},\tilde{\pi})$, and so

$$\mathcal{Z}(1-s,\tilde{\tilde{\pi}}_\psi(w)\tilde{\tilde{\varphi}},\tilde{\tilde{\pi}}) = \gamma(s,\tilde{\pi},\psi)\mathcal{Z}(s,\tilde{\varphi},\tilde{\pi}).$$

Recall that $\tilde{\tilde{\pi}} = \pi$ and note that

$$\tilde{\tilde{\pi}}_\psi(w)\tilde{\tilde{\varphi}} = \omega_{\tilde{\pi}}^{-1}\tilde{\pi}_\psi(w)\tilde{\varphi} = \omega_{\tilde{\pi}}^{-1}\omega_\pi^{-1}\pi_\psi(w)\varphi = \pi_\psi(w)\varphi = \eta.$$

Hence by the identity proven above we have

$$\begin{aligned}\mathcal{Z}(1-s,\tilde{\pi}_\psi(w)\tilde{\eta},\tilde{\pi}) &= \omega_\pi(-1)\mathcal{Z}(1-s,\tilde{\varphi},\tilde{\pi}) \\ &= \frac{\omega_\pi(-1)}{\gamma(1-s,\tilde{\pi},\psi)}\mathcal{Z}(1-s,\tilde{\tilde{\pi}}_\psi(w)\tilde{\tilde{\varphi}},\tilde{\tilde{\pi}}) \\ &= \gamma(s,\pi,\psi)\mathcal{Z}(1-s,\eta,\pi)\end{aligned}$$

as desired.

□

Let ψ be a non-trivial unitary character of F^+. Currently we have defined $\gamma(s,\pi,\psi)$ for any infinite dimensional class in $\Pi(G)$. In Tate's theory we defined $\gamma(s,\chi,\psi)$ for any character χ of $F^\times$. If $\pi \in \Pi(G)$ is one dimensional, corresponding to the character χ of $F^\times$, it is natural to define

$$\gamma(s,\pi,\psi) := \gamma(s,\chi,\psi).$$

So now $\gamma(s,\pi,\psi) \in \mathbb{C}(q^s)$ is defined for every class π in $\Pi(G)$.

Theorem 35 Suppose that $\pi \in \Pi(G)$ is either a principal series representation $\pi = \pi(\chi_1,\chi_2)$ or is a special representation $\pi = \sigma(\chi_1,\chi_2)$. Then for any $1 \neq \psi \in F^*$ we have

$$\gamma(s,\pi,\psi) = \gamma(s,\chi_1,\psi)\gamma(s,\chi_2,\psi).$$

Proof If $\pi = \sigma(\chi_1,\chi_2)$ then $\pi = \sigma(\chi_2,\chi_1)$. This shows that we may assume $\chi \neq |\cdot|^{-1}$. If π is a principal series representation define $Y := \mathcal{B}(\chi_1,\chi_2)$. If π is a special representation, so that $\chi = |\cdot|$, then define Y to be the irreducible ρ-invariant subspace of $\mathcal{B}(\chi_1,\chi_2)$ of codimension 1. Let ψ be a non-trivial unitary character of F^+. We have the intertwining isomorphism

$$: Y \longrightarrow \mathcal{K}(\pi) \qquad f \longmapsto \varphi_f^\psi$$

of (ρ, Y) and $(\pi_\psi,\mathcal{K}(\pi))$.

Fix $\eta \in \mathcal{S}(F^\times) \cap \pi_\psi(w)\mathcal{S}(F^\times)$ such that $\mathcal{Z}(s,\eta,\pi) \neq 0$. There is a unique $f \in Y$ such that $\varphi_f^\psi = \eta$. Since the map above intertwines the representations it follows that $\varphi_{\rho(w)f}^\psi =$

$\pi_\psi(w)\eta$. Since $\eta \in \mathcal{S}(F^\times)$ we have $\Phi_f \in \mathcal{S}(F)$. Also $\pi_\psi(w)\eta \in \pi_\psi(w^2)\mathcal{S}(F^\times) = \mathcal{S}(F^\times)$ and so for the same reason we get $\Phi_{\rho(w)f} \in \mathcal{S}(F)$. Using the identity $\eta = \varphi_f^\psi = \chi_2|\cdot|^{\frac{1}{2}}\hat{\Phi}_f$ we find

$$\begin{aligned}
\mathcal{Z}(s,\eta,\pi) &= \int_{F^\times} \varphi_f^\psi(a)|a|^{s-\frac{1}{2}}\,d^\times a \\
&= \int_{F^\times} \chi_2(a)\hat{\Phi}_f(a)|a|^s\,d^\times a \\
&= \mathcal{Z}(s,\hat{\Phi}_f,\chi_2) \\
&= \gamma(1-s,\chi_2^{-1},\psi)\mathcal{Z}(1-s,\Phi_f,\chi_2^{-1}).
\end{aligned}$$

We remark that $\mathcal{Z}(1-s,\Phi_f,\chi_2^{-1})$ is well defined since $\Phi_f \in \mathcal{S}(F)$.

Now we calculate $\mathcal{Z}(1-s,\tilde{\pi}_\psi(w)\tilde{\eta},\tilde{\pi})$. Since $\pi = \pi(\chi_1,\chi_2)$ or $\pi = \sigma(\chi_1,\chi_2)$ we know $\omega_\pi = \chi_1\chi_2$. Therefore

$$\begin{aligned}
\mathcal{Z}(1-s,\tilde{\pi}_\psi(w)\tilde{\eta},\tilde{\pi}) &= \int_{F^\times} (\tilde{\pi}_\psi(w)\tilde{\eta})(a)|a|^{(1-s)-\frac{1}{2}}\,d^\times a \\
&= \int_{F^\times} \omega_\pi^{-1}(a)(\pi_\psi(w)\eta)(a)|a|^{\frac{1}{2}-s}\,d^\times a \\
&= \int_{F^\times} (\chi_1^{-1}\chi_2^{-1})(a)\varphi^\psi_{\rho(w)f}(a)|a|^{\frac{1}{2}-s}\,d^\times a \\
&= \int_{F^\times} \chi_1^{-1}(a)\hat{\Phi}_{\rho(w)f}(a)|a|^{1-s}\,d^\times a \\
&= \mathcal{Z}(1-s,\hat{\Phi}_{\rho(w)f},\chi_1^{-1}) \\
&= \gamma(s,\chi_1,\psi)\mathcal{Z}(s,\Phi_{\rho(w)f},\chi_1).
\end{aligned}$$

Similarly as above we remark that $\mathcal{Z}(s,\Phi_{\rho(w)f},\chi_1)$ is defined since $\Phi_{\rho(w)f} \in \mathcal{S}(F)$. The identity

$$w\begin{pmatrix}1 & -x\\ 0 & 1\end{pmatrix}w = \begin{pmatrix}x^{-1} & 1\\ 0 & x\end{pmatrix}w\begin{pmatrix}1 & x^{-1}\\ 0 & 1\end{pmatrix}$$

implies that

$$\Phi_{\rho(w)f}(x) = f\left(w\begin{pmatrix}1 & -x\\ 0 & 1\end{pmatrix}w\right) = \chi^{-1}(x)|x|^{-1}\Phi_f(-x^{-1}).$$

Therefore

$$\begin{aligned}
\mathcal{Z}(s,\Phi_{\rho(w)f},\chi_1) &= \int_{F^\times} \chi_1(a)\Phi_{\rho(w)f}(a)|a|^s\,d^\times a \\
&= \int_{F^\times} \chi_2(a)\Phi_f(-a^{-1})|a|^{s-1}\,d^\times a \\
&= \int_{F^\times} \chi_2^{-1}(a)\Phi_f(-a)|a|^{1-s}\,d^\times a \\
&= \chi_2(-1)\mathcal{Z}(1-s,\Phi_f,\chi_2^{-1}).
\end{aligned}$$

So our previous computation becomes

$$\mathcal{Z}(1-s,\tilde{\pi}_\psi(w)\tilde{\eta},\tilde{\pi}) = \chi_2(-1)\gamma(s,\chi_1,\psi)\mathcal{Z}(1-s,\Phi_f,\chi_2^{-1}).$$

Since $\mathcal{Z}(s, \eta, \pi) \neq 0$ we obtain

$$\begin{aligned}\gamma(s, \pi, \psi) &= \frac{\mathcal{Z}(1-s, \tilde{\pi}_{\psi}(w)\tilde{\eta}, \tilde{\pi})}{\mathcal{Z}(s, \eta, \pi)} \\ &= \frac{\chi_2(-1)\gamma(s, \chi_1, \psi)\mathcal{Z}(1-s, \Phi_f, \chi_2^{-1})}{\gamma(1-s, \chi_2^{-1}, \psi)\mathcal{Z}(1-s, \Phi_f, \chi_2^{-1})} \\ &= \frac{\chi_2(-1)\gamma(s, \chi_1, \psi)}{\gamma(1-s, \chi_2^{-1}, \psi)}\end{aligned}$$

Now the identity $\gamma(s, \chi_2, \psi)\gamma(1-s, \chi_2^{-1}, \psi) = \chi_2(-1)$ yields the theorem.

□

Epsilon Factors for $\mathbf{GL_2(F)}$

Definition 36 If $\pi \in \Pi(G)$ is infinite dimensional and ψ is a non-trivial unitary character of F^+ then we define the *epsilon factor* of π with respect to ψ by

$$\epsilon(s, \pi, \psi) = \frac{\gamma(s, \pi, \psi)L(s, \pi)}{L(1-s, \tilde{\pi})} \in \mathbb{C}(q^s).$$

The above definition is completely analogous to the definition of the epsilon factor in the one dimensional case. Indeed the above definition includes as a special case the previous epsilon factors provided we allow π to be one dimensional as well. Indeed if $\pi \in \Pi(G)$ is one dimensional, corresponding to the character χ of $F^\times$, then $\tilde{\pi}$ corresponds to χ^{-1}. Thus

$$\epsilon(s, \pi, \psi) = \frac{\gamma(s, \pi, \psi)L(s, \pi)}{L(1-s, \tilde{\pi})} = \frac{\gamma(s, \chi, \psi)L(s, \chi)}{L(1-s, \chi^{-1})} = \epsilon(s, \chi, \psi).$$

Theorem 37 Let $\pi \in \Pi(G)$ be infinite dimensional, and let $1 \neq \psi \in F^*$. For any $\varphi \in \mathcal{K}(\pi)$ we have the functional equation

$$\frac{\mathcal{Z}(1-s, \tilde{\pi}(w)\tilde{\varphi}, \tilde{\pi})}{L(1-s, \tilde{\pi})} = \epsilon(s, \pi, \psi)\frac{\mathcal{Z}(s, \varphi, \pi)}{L(s, \pi)}.$$

The function $\epsilon(s, \pi, \psi)$ is an exponential function and satisfies the identity

$$\epsilon(s, \pi, \psi)\epsilon(1-s, \tilde{\pi}, \psi) = \omega_{\pi}(-1).$$

The above result is proven in exactly the same way as in the one dimensional case so there is no use in repeating it. The only changes are that χ becomes π, χ^{-1} becomes $\tilde{\pi}$, and instead of using the identity

$$\gamma(s, \chi, \psi)\gamma(1-s, \chi^{-1}, \psi) = \chi(-1)$$

we use the identity

$$\gamma(s, \pi, \psi)\gamma(1-s, \tilde{\pi}, \psi) = \omega_{\pi}(-1).$$

Proposition 38 Suppose that $\pi \in \Pi(G)$ is either the principal series representation $\pi(\chi_1, \chi_2)$ or the special representation $\sigma(\chi_1; \chi_2)$. Let ψ be a non-trivial unitary character of F^+.

(1) Suppose π is the principal series representation $\pi(\chi_1, \chi_2)$ or π is the special representation $\sigma(\chi_1, \chi_2)$ with χ_1 and χ_2 being ramified. We have the identity

$$\epsilon(s, \pi, \psi) = \epsilon(s, \chi_1, \psi)\epsilon(s, \chi_2, \psi).$$

(2) Suppose $\pi = \sigma(\chi_1, \chi_2)$ is a special representation with χ_1 and χ_2 being unramified. When $\chi = |\cdot|^{-1}$ we have

$$\epsilon(s, \pi, \psi) = -\chi_1(\mathcal{P})\epsilon(s, \chi_1, \psi)\epsilon(s, \chi_2, \psi)q^{-s}$$

and when $\chi = |\cdot|$ we have

$$\epsilon(s, \pi, \psi) = -\chi_2(\mathcal{P})\epsilon(s, \chi_1, \psi)\epsilon(s, \chi_2, \psi)q^{-s}.$$

Proof Before beginning we recall that if $\sigma(\chi_1, \chi_2)$ is a special representation then χ_1 and χ_2 are either both ramified or both unramified. Hence the above cases encompass all possible principal series and special representations.

Now by definition of $\epsilon(s, \pi, \psi)$ and the result above that

$$\gamma(s, \pi, \psi) = \gamma(s, \chi_1, \psi)\gamma(s, \chi_2, \psi)$$

we find

$$\epsilon(s, \pi, \psi) = \frac{\gamma(s, \pi, \psi)L(s, \pi)}{L(1-s, \tilde{\pi})} = \frac{\gamma(s, \chi_1, \psi)\gamma(s, \chi_2, \psi)L(s, \pi)}{L(1-s, \tilde{\pi})}.$$

Suppose first that $\pi = \pi(\chi_1, \chi_2)$. Then $\tilde{\pi} = \pi(\chi_1^{-1}, \chi_2^{-1})$ and so

$$\begin{aligned}
\epsilon(s, \pi, \psi) &= \frac{\gamma(s, \chi_1, \psi)\gamma(s, \chi_2, \psi)L(s, \chi_1)L(s, \chi_2)}{L(1-s, \chi_1^{-1})L(1-s, \chi_2^{-1})} \\
&= \left(\frac{\gamma(s, \chi_1, \psi)L(s, \chi_1)}{L(1-s, \chi_1^{-1})}\right)\left(\frac{\gamma(s, \chi_2, \psi)L(s, \chi_2)}{L(1-s, \chi_2^{-1})}\right) \\
&= \epsilon(s, \chi_1, \psi)\epsilon(s, \chi_2, \psi).
\end{aligned}$$

Now suppose that $\pi = \sigma(\chi_1, \chi_2)$ with $\chi = |\cdot|$. Then $\tilde{\pi} = \sigma(\chi_1^{-1}, \chi_2^{-1})$ and we see

$$\begin{aligned}
\epsilon(s, \pi, \psi) &= \frac{\gamma(s, \chi_1, \psi)\gamma(s, \chi_2, \psi)L(s, \chi_1)}{L(1-s, \chi_2^{-1})} \\
&= \left(\frac{\gamma(s, \chi_1, \psi)L(s, \chi_1)}{L(1-s, \chi_1^{-1})}\right)\left(\frac{\gamma(s, \chi_2, \psi)L(s, \chi_2)}{L(1-s, \chi_2^{-1})}\right)\left(\frac{L(1-s, \chi_1^{-1})}{L(s, \chi_2)}\right) \\
&= \epsilon(s, \chi_1, \psi)\epsilon(s, \chi_2, \psi)\frac{L(1-s, \chi_1^{-1})}{L(s, \chi_2)}
\end{aligned}$$

In the case that χ_1 and χ_2 are ramified we have $L(s, \chi_2) = L(1-s, \chi_1^{-1}) = 1$ and hence in this case $\epsilon(s, \pi, \psi) = \epsilon(s, \chi_1, \psi)\epsilon(s, \chi_2, \psi)$. Now consider the case where χ_1 and χ_2 are unramified. Since $\chi_1\chi_2^{-1} = |\cdot|$ we find $\chi_1^{-1}(\mathcal{P}) = q\chi_2^{-1}(\mathcal{P})$. Therefore

$$\frac{L(1-s, \chi_1^{-1})}{L(s, \chi_2)} = \frac{(1-\chi_1^{-1}(\mathcal{P})q^{s-1})^{-1}}{(1-\chi_2(\mathcal{P})q^{-s})^{-1}} = \frac{1-\chi_2(\mathcal{P})q^{-s}}{1-\chi_2^{-1}(\mathcal{P})q^{s}} = -\chi_2(\mathcal{P})q^{-s}.$$

Combining this with the previous calculation shows

$$\epsilon(s, \pi, \psi) = -\chi_2(\mathcal{P})\epsilon(s, \chi_1, \psi)\epsilon(s, \chi_2, \psi)q^{-s}.$$

This deals with the case $\pi = \sigma(\chi_1, \chi_2)$ with $\chi = |\cdot|$. The case of $\pi = \sigma(\chi_1, \chi_2)$ with $\chi = |\cdot|^{-1}$ is dealt with in exactly the same way. The proposition follows.

□

Corollary 39 If $\pi \in \Pi(G)$ is an unramified class and ψ is an unramified character of F^+ then

$$\epsilon(s, \pi, \psi) = 1.$$

Proof We have already proven this when π is one dimensional so we suppose that π is infinite dimensional. By Theorem 8.35 we know that $\pi = \pi(\chi_1, \chi_2)$ is a principal series representation with χ_1 and χ_2 being unramified. The result now follows trivially from Proposition 9.38.1 and Proposition 9.24.

□

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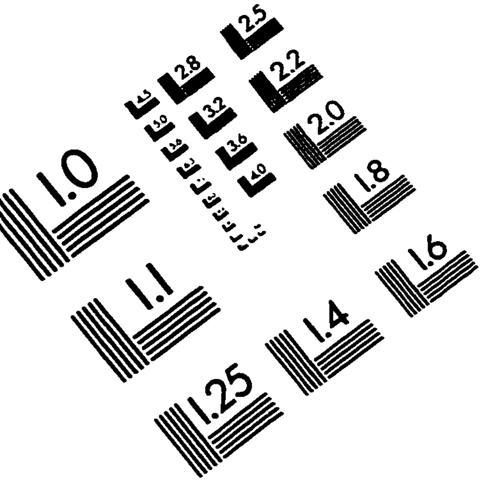

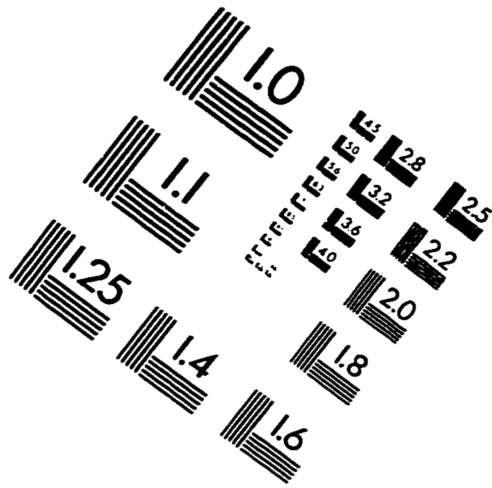

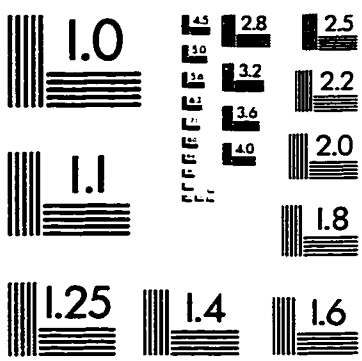

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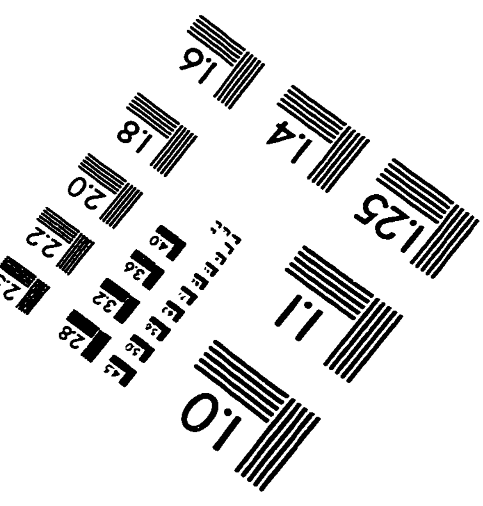

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