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NONLINEAR SIMULTANEOUS DIAMETER AND HEIGHT GROWTH MODELS FOR MAJOR ALBERTA TREE SPECIES

BY

SHONGMING HUANG

A thesis submitted to the Faculty of Graduate Studies and Research in partial fulfillment of the requirements for the degree of DOCTOR OF PHILOSOPHY.

DEPARTMENT OF FOREST SCIENCE

EDMONTON, ALBERTA

FALL 1992

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I dedicate this thesis to my father, may his soul rest in tranquil and eternal peace.

Abstract

Growth and yield models for forest stand dynamics using data from permanent sample plots with the real growth series are often composed of a system of compatible, interdependent, and analytically related equations. Most conceptual frameworks for understanding stand development processes recognize that there is a strong correlation and feedback mechanism among variables that are used to describe various growth and yield relationships. This idea translates into the realization that forest stand dynamics should be described by simultaneous and interdependent systems of equations that involve current, future, and past values of some of the tree and stand characteristics rather than separate and isolated individual equations.

This study described methods for estimating systems of nonlinear simultaneous equations, each developed under the framework of an individual tree distance-independent growth and yield model. The study first compared the relative performance of a variety of potential height-diameter functions on a large, regional data set covering numerous species in order to identify the most appropriate height-diameter functions for major Alberta tree species. A height-diameter function was then selected as the base function, and using the parameter prediction method, individual tree height prediction models expressing tree height as a function of diameter, basal area, stand density, species composition, site productivity, and average size of the trees in the stand were developed. A site productivity measure based on the relationship between total tree height and diameter at breast height of the dominant and codominant trees was presented as a convenient method of quantifying site productivity for uneven-aged and/or mixed-species stands. Biologically based periodic diameter increment and height increment models that relate diameter growth and height growth to other tree and stand characteristics were also developed for selected tree species.

Because of the interdependent nature of the equations developed in this study,

alternative system estimation methods, such as seemingly unrelated regression, two- and three-stage least squares commonly used in econometrics, were applied to estimate the structural parameters simultaneously. Since time series and cross-sectional data were used for fitting individual and systems of equations, parameter estimation under the classical regression structure, where the error terms for individual equations are independent and identically distributed, or the generalized regression structure, where the error terms are heteroskedastic and/or autocorrelated, were considered.

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Chapter 1

Introduction

Growth and yield models describing forest stand dynamics have been widely used in forest management as the *sine qua none* for making appropriate management decisions on forestry operations. Vuokila in 1965 (as referenced in Krumland 1982 p. 1) reported that the first documented effort to relate volume yields to stand age dated back to 17th century China. There are a great many mathematically oriented biological and ecological growth and yield models now available. Most of the models range from those based on individual trees to those for whole stands (Munro 1974, Clutter et al. 1983, Davis and Johnson 1986).

Conventional growth and yield models fitted on the data from permanent sample plots with the real growth series often comprise a system of compatible, interdependent, and analytically related equations (Clutter 1963, Sullivan and Clutter 1972, Burkhart 1986, Daniels and Burkhart 1988). Within such a system of equations, a variable appearing on the left hand side of an equation can also appear on the right hand side of another equation in the system. Understanding the correlation and feedback mechanism among variables that are used to describe various growth and yield relationships is essential in developing rationalized individual equations that make up the growth and yield projection systems. It has been realized that forest stand dynamics should be described by simultaneous and interdependent systems of equations that involve current, future, and past values of some of the tree and stand characteristics rather than separate and isolated individual equations.

The application of simultaneous fitting techniques commonly used in econometrics for systems of growth and yield equations was first reported by Furnival and Wilson (1971).

Many other researchers have also discussed and used two- and three-stage least squares, and seemingly unrelated regression techniques for estimating the structural parameters in systems of forestry equations (Murphy and Sternitzke 1979, Murphy and Beltz 1981, Murphy 1983, Amateis et al. 1984, Burkhart 1986, Borders and Bailey 1986, Reed 1987, Van Deusen 1988, Borders 1989, LeMay 1988, 1990). Previous system modelling techniques in growth and yield studies, however, have mainly concentrated on the linear, stand level volume and basal area equations.

The primary objectives of this study are 1) to develop equations that predict individual tree height, periodic diameter increment, and periodic height increment under the framework of an individual tree distance-independent growth and yield model for selected major Alberta tree species grown in boreal mixed-species stands, 2) to evaluate the effects of species composition, site productivity, and stand density on tree growth and yield relationships, 3) to examine the interdependent nature among the equations and to apply the appropriate statistical procedures for estimating the structural parameters of the system of equations simultaneously, and 4) to evaluate different system estimation methods for nonlinear equations and to compare them with the traditional single-equation based ordinary least squares techniques.

To meet the above mentioned objectives, the theoretical foundation for modelling techniques, especially methods for estimating systems of simultaneous equations were reviewed. Comparison of the relative performance of a variety of potential height-diameter functions on a large, regional data set covering numerous species was made in order to identify the most appropriate height-diameter functions for major Alberta tree species. A site productivity measure for uneven-aged and/or mixed-species stands based on the relationship between total tree height and diameter at breast height of the dominant and codominant trees was developed. Measures that reflect species composition and stand density in mixed-species stands were defined. Using an appropriately selected height-diameter function, and the method of parameter prediction (Clutter et al. 1983), age-independent individual tree height prediction models expressing tree height as a function of diameter, basal area, stand density, species composition, site productivity, and average size of the trees in the stand were developed. A periodic diameter increment model was developed for white spruce, and periodic height increment models were formulated for white spruce and aspen. Because of the interdependent nature among tree height, diameter increment, and height increment equations developed in this study, alternative system estimation methods such as seemingly unrelated regression, and two- and three-stage least squares commonly used in econometrics were applied to estimate the structural parameters of the simultaneous equations.

Since time series and cross-sectional data from permanent sample plots are used for fitting individual equations and system of equations, parameter estimation methods under the classical regression structure, where the error terms for individual equations are independent and identically distributed, or the generalized regression structure, where the error terms for individual equations are heteroskedastic and/or autocorrelated, are considered.

While fundamentals of the nonlinear system modelling techniques were emphasized, the practical implications of such approaches for systems of interdependent forestry equations were also stressed. Methods for model diagnostics and testing, as well as system specification were also discussed to ensure that the models are fitted appropriately. Preliminary results of the simulation based on the fitted and existing equations are also produced to determine whether the equations developed in this study provide appropriate predictions under various conditions.

This thesis is written in a paper-format so each Chapter describes a separate but integrated part of the research. Chapter 2 synthesizes the theoretical foundation for modelling techniques. Chapter 3 compares the relative performance of the potential height-diameter functions. Chapter 4 develops a site productivity measure for uneven-aged and/or mixed-species stands based on the relationship between total tree height and diameter at breast height of the dominant and codominant trees. Chapter 5 first selects an appropriate height-diameter function, and then using the parameter prediction method, develops age-independent individual tree height prediction models. Chapter 6 shows the development of a diameter increment model for white spruce. Chapter 7 presents the periodic height increment models for white spruce and aspen. Chapter 8 describes the techniques for simultaneously fitting a system of three related nonlinear equations for white spruce. Chapter 9 provides the general discussion and conclusions.

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Chapter 2

Theoretical Foundation for Estimating Systems of Equations

2.1 Introduction

This chapter is concerned with the theoretical foundations for the use of the least squares criterion for estimating systems of simultaneous nonlinear equations. Because the theory relies on many analogies with the theory and methods of linear equations, general descriptions of the least squares criterion for estimating a single linear equation and systems of linear equations are given first. Methods for estimating a single nonlinear equation are then provided as the primary step for estimating systems of nonlinear equations. For each single equation and system of equations considered, parameter estimates and their conventional sampling properties such as unbiasedness, efficiency, and consistency are described when the error structure of the individual equation is homoskedastic and uncorrelated, or when the error structure is heteroskedastic and/or autocorrelated. General references that cover the theoretical foundation on statistical models include Judge et al. (1988), Kmenta (1986), and Rawlings (1988) for linear models, and Gallant (1987), Amemiya (1985), Judge et al. (1985), and Seber and Wild (1989) for nonlinear models.

2.2 Generalized linear regression equation

Consider the following general linear regression equation with p unknown parameters on n observations

$$(2.1) \qquad Y = X\beta + \varepsilon$$

where Y is an (n×1) vector of observations on the dependent variable, X is an (n×p) nonstochastic design matrix of known values with linearly independent column vectors of the independent or explanatory variables, β is a (p×1) vector of unknown parameters to be estimated, and ε is an (n×1) vector of unobservable random errors. Assuming the random errors ε in equation (2.1) follow

(2.2) $E[\varepsilon]=0 \qquad V(\varepsilon)=\sigma^2 I \qquad \varepsilon\sim(0,\sigma^2 I)$

that is, the errors ε are statistically independent (uncorrelated) and identically distributed (i.i.d.) with an unknown distribution that has the mean vector $E(\varepsilon)=0$ and the variance-covariance matrix $V(\varepsilon)=E[\varepsilon\varepsilon']=\sigma^2 I$, where σ^2 is the common variance of the random errors, and I is the (n×n) identity matrix. Using the ordinary least squares (OLS) criterion, an estimator for the parameter vector β in (2.1) is obtained by minimizing the sum of squared errors

(2.3) $S=\varepsilon'\varepsilon=(Y-X\beta)'(Y-X\beta)$

Differentiating S with respect to β and setting the resultant matrix equation equal to zero, and replacing β by its OLS estimator b provides the normal equation

(2.4) $(X'X)b=X'Y$

The solution to this normal equation gives the unique OLS estimator of β

(2.5) $b=(X'X)^{-1}X'Y$

The variance-covariance matrix that expresses the sampling variation for b is obtained as

(2.6) $V(b)=E[(b-E[b])(b-E[b])']=(X'X)^{-1}\sigma^2$

According to the Gauss-Markov Theorem (Judge et al. 1985, p. 15), the OLS estimator of β as determined by b in (2.5) is unbiased, consistent, and efficient with respect to the class of linear unbiased estimators. It is the best linear unbiased estimator (BLUE) of β in the sense that it has the minimum variance among all linear unbiased estimators. The error variance σ^2 in (2.6) is usually unknown and estimated by its unbiased estimator (Judge et al. 1988, pp. 205-208):

$$(2.7) \qquad \hat{\sigma}^2 = \frac{(Y-Xb)'(Y-Xb)}{n-p} = \frac{Y'Y-b'X'Y}{n-p}$$

The above classical OLS estimation for the general linear regression model (2.1) relies on the error i.i.d. assumption given in (2.2). It may not be appropriate if the error terms are heteroskedastic and/or autocorrelated. Consider the following more generalized specification for the error terms ε in equation (2.1)

$$(2.8) \qquad E[\varepsilon]=0 \qquad V(\varepsilon)=\sigma^2 V \qquad \varepsilon \sim (0,\sigma^2 V)$$

where V is an (n×n) positive definite symmetric matrix whose diagonal elements are not equal (heteroskedasticity) and the off-diagonal elements are not zero (autocorrelation). Under these conditions, it can be shown (Judge et al. 1988, p. 341) that the OLS estimator b of β is still unbiased and consistent but inefficient. To find the estimator that is also efficient, equation (2.1) can be transformed to

$$(2.9) \qquad Y^*=X^*\beta+\varepsilon^*$$

where $Y^*=PY$, $X^*=PX$, $\varepsilon^*=P\varepsilon$, and P is an (n×n) nonsingular symmetric matrix that satisfies

$$(2.10) \qquad P'P=PP=P^2=V^{-1}$$

It is a relatively simple matter to show that the transformed error terms ε^* in (2.9) follow

$$(2.11) \qquad E[\varepsilon^*]=0 \qquad V[\varepsilon^*]=E[\varepsilon^*\varepsilon^{*\prime}]=\sigma^2 I \qquad \varepsilon^* \sim (0,\sigma^2 I)$$

Hence, OLS technique can be directly applied to equation (2.9) by minimizing

$$(2.12) \qquad S=\varepsilon^{*\prime}\varepsilon^*=\varepsilon' V^{-1}\varepsilon=(Y-X\beta)'V^{-1}(Y-X\beta)$$

This gives the generalized least squares (GLS) or the Aitken estimator

$$(2.13) \qquad \beta_{GLS} = (X^{*\prime}X^{*})^{-1}X^{*\prime}Y^{*} = (X'P'PX)^{-1}X'P'PY = (X'V^{-1}X)^{-1}X'V^{-1}Y$$

The GLS estimator β_{GLS} is BLUE for β in equation (2.1) if the error specification in (2.8) is met and the matrix V is known. If V is unknown, which is usually the case in practice, V is replaced by an estimated matrix $\hat{V}$ and the estimated generalized least squares estimator (EGLS) of β is then given by

$$(2.14) \qquad \beta_{G} = (X'\hat{P}'\hat{P}X)^{-1}X'\hat{P}'\hat{P}Y = (X'\hat{V}^{-1}X)^{-1}X'\hat{V}^{-1}Y$$

where $\hat{P}$ is an (n×n) nonsingular symmetric matrix that satisfies

$$(2.15) \qquad \hat{P}'\hat{P} = \hat{P}\hat{P} = \hat{P}^{2} = \hat{V}^{-1}$$

Judge et al. (1988, pp. 352-6) show that β_G as determined by (2.14) is an asymptotically unbiased, consistent, and efficient estimator of β. The variance-covariance matrix of β_G can be obtained as

$$(2.16) \qquad V(\beta_{G}) = (X'\hat{V}^{-1}X)^{-1}\sigma^{2}$$

A consistent estimator for σ^2 resulted from EGLS is given by

$$(2.17) \qquad \hat{\sigma}^{2}_{G} = \frac{(Y-X\beta_{G})'\hat{V}^{-1}(Y-X\beta_{G})}{n-p}$$

The estimation of the matrix $\hat{V}$ is different, depending on the error structure of the equation under consideration. When the error terms in equation (2.1) have unequal variances but are uncorrelated, the variance-covariance matrix of ε is defined by

$$(2.18) \qquad E[\varepsilon] = 0 \qquad V(\varepsilon) = E[\varepsilon\varepsilon'] = \sigma^{2}V = diag(\sigma^{2}_{1}, \sigma^{2}_{2}, \ldots, \sigma^{2}_{n})$$

With this heteroskedastic error specification, each observation in equation (2.1) is multiplied by the reciprocal of its standard deviation; in other words, both sides of equation (2.1) are multiplied by P

$$(2.19) \qquad PY = PX\beta + P\varepsilon$$

where P is an (n×n) nonsingular symmetric transformation matrix that equals

$$(2.20) \qquad P = diag(\sigma_1^{-1}, \sigma_2^{-1}, \ldots, \sigma_n^{-1})$$

An EGLS estimator β_G as in equation (2.14) for β can be obtained by applying OLS directly to equation (2.19), with the matrix P replaced by its estimator $\hat{P}$, whose elements are generally estimated using the residuals from the OLS fit on the original untransformed data (with heteroskedasticity).

Depending on the form of heteroskedasticity, the variances of errors may be modelled as 1) a linear, power, or an exponential function of some explanatory variables, or 2) proportional to some function of the predicted dependent variable. Respective examples can be found in statistical and econometric literature (Carroll and Ruppert 1988, Seber and Wild 1989, pp. 68-89, Amemiya 1985, pp. 198-207, Judge et al. 1985, pp. 431-41). The occurrence of heteroskedasticity can be detected by examining the plot of OLS residuals (or studentized residuals) against the predicted values of the dependent variable.

Various heteroskedasticity test statistics are also available (Judge et al. 1985, pp. 445-54). The Goldfeld-Quandt test (Judge et al. 1988, p. 371) selected to be used in this study is an exact test that does not rely on asymptotic theory or an explicit known form of heteroskedasticity. It tests the [illegible] hypothesis of homoskedasticity against the alternative hypothesis of heteroskedasticity by 1) ordering the predicted values of the dependent variable from OLS in an ascending sequence according to increasing error variance, 2) dividing the ordered data into three groups and omitting r central observations, 3) performing two separate OLS fits on the first (n-r)/2 observations and the remaining (n-r)/2 observations, and obtaining the mean squared errors MSE_1 and MSE_2 from the first and the second fittings respectively, 4) calculating the test statistic $\lambda = MSE_2/MSE_1$. Under the null hypothesis λ has an F-distribution with [(n-r-2p)/2, (n-r-2p)/2] degrees of freedom. By comparing the calculated value for λ with the critical value from the F-distribution, the null hypothesis of homoskedasticity is accepted or rejected accordingly.

When the error terms in equation (2.1) are homoskedastic but are serially correlated in that the current equation error depends on the values of previous errors, estimation will be different depending on the form and the degree from which the autocorrelation is generated (Box and Jenkins 1976, Judge et al. 1985, pp. 283-318). The most commonly used first-order autoregressive process — AR(1) assumes the ith element of the errors ε in equation (2.1) is given by $\varepsilon_i = \rho\varepsilon_{i-1}+e_i$, and $e=(e_1, e_2, \ldots, e_n)'$ satisfies $E[e]=0$ and $E[ee']=\sigma^2 I$ so that

$$(2.21) \qquad E[\varepsilon]=0 \qquad V(\varepsilon)=E[\varepsilon\varepsilon']=\sigma^2 V=\frac{\sigma^2}{1-\rho^2}\begin{bmatrix} 1 & \rho & \rho^2 & \cdots & \rho^{n-1} \\ \rho & 1 & \rho & \cdots & \rho^{n-2} \\ \rho^2 & \rho & 1 & \cdots & \rho^{n-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \rho^{n-1} & \rho^{n-2} & \rho^{n-3} & \cdots & 1 \end{bmatrix}$$

With this autoregressive error specification, both sides of equation (2.1) can be multiplied by the transformation matrix

$$(2.22) \qquad P=\begin{bmatrix} \sqrt{1-\rho^2} & 0 & 0 & \cdots & 0 & 0 \\ -\rho & 1 & 0 & \cdots & 0 & 0 \\ 0 & -\rho & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & -\rho & 1 \end{bmatrix}$$

where the coefficient of autocorrelation ρ can be estimated by

$$(2.23) \qquad \hat{\rho}=\left(\sum_{i=2}^{n}\hat{\varepsilon}_i\hat{\varepsilon}_{i-1}\right)\Big/\left(\sum_{i=2}^{n}\hat{\varepsilon}_{i-1}^2\right)$$

where $\hat{\varepsilon}_i$ are the residuals obtained from the OLS fitting on the untransformed data. Applying the OLS technique to the transformed equation with the matrix P replaced by its estimator $\hat{P}$ gives the EGLS estimator β_G as determined by equation (2.14). The Durbin-Watson statistic in finite samples is most commonly used to test the presence of first-order autocorrelation. Many other statistics can also be used to test the existence and types of autocorrelation (Judge et al. 1985, pp. 319-30).

If, in addition to autocorrelation, the error terms in equation (2.1) also pose heteroskedasticity, then Engle's (1982) autoregressive conditional heteroskedasticity (ARCH) model or Cragg's (1982) method may be applied. A simple and straightforward two-step transformation procedure is generally applicable in such cases by 1) transforming the original data to correct the autocorrelation first, and then 2) transforming the transformed data to correct the heteroskedasticity.

2.3 Generalized systems of linear regression equations

2.3.1 Seemingly unrelated regression (SUR) equations

Seemingly unrelated regression (SUR) equations are multivariate regression models. Consider the ith equation in a system of M equations on n observations

$$Y_i = X_i \beta_i + \varepsilon_i \qquad (i = 1, 2, \ldots, M) \tag{2.24}$$

where Y_i is an ($n \times 1$) vector of observations on the dependent variable, X_i is an ($n \times p_i$) nonstochastic design matrix of known values with linearly independent column vectors of p_i explanatory variables, β_i is a ($p_i \times 1$) vector of unknown parameters to be estimated, and ε_i is an ($n \times 1$) vector of unobservable random errors. Assuming the error terms ε_i follow

$$E[\varepsilon_i] = 0 \qquad E[\varepsilon_i \varepsilon_j'] = \sigma_{ij} I \qquad (i, j = 1, 2, \ldots, M) \tag{2.25}$$

where σ_{ij} is the covariance of the error terms of the ith and the jth equation. It represents the correlation between the ith and the jth equation. The specification of equation (2.25) reflects that 1) for a given equation the error terms have a zero mean and a constant variance, although each equation may have a different variance, 2) the error terms are uncorrelated both within and across equations in different time periods but are contemporaneously correlated across the equations of the system corresponding to the same time period.

OLS applied to each equation in systems of SUR equations provides unbiased and consistent, but inefficient parameter estimates because it discards the information about the contemporaneous correlation of the error terms (Kmenta 1986, p. 637-9). Using the matrix notation, the system of M equations in (2.24) can be expressed as

$$(2.26) \qquad Y = X\beta + \varepsilon$$

where Y is an (Mn×1) vector of dependent variables, X is an (Mn×p) matrix of explanatory variables, with the total number of parameters in the system $p=\Sigma^{M}_{i=1}p_i$, β is a (p×1) vector of parameters to be estimated, and ε is an (Mn×1) error vector. Given the assumption in (2.25), the variance-covariance matrix of the joint error vector ε can be written as

$$(2.27) \qquad E[\varepsilon]=0 \qquad \omega = E[\varepsilon\varepsilon'] = \begin{bmatrix} \sigma_{11}I & \sigma_{12}I & \cdots & \sigma_{1M}I \\ \sigma_{21}I & \sigma_{22}I & \cdots & \sigma_{2M}I \\ \vdots & \vdots & \vdots & \vdots \\ \sigma_{M1}I & \sigma_{M2}I & \cdots & \sigma_{MM}I \end{bmatrix} = \Sigma \otimes I$$

where ⊗ denotes the Kronecker product, Σ is an (M×M) positive definite symmetric matrix with the typical elements σ_{ij}, and I is an (n×n) identity matrix. It is apparent that the GLS estimator of β in (2.26) can be obtained as

$$(2.28) \qquad \beta_{GLS} = (X'\omega^{-1}X)^{-1}X'\omega^{-1}Y = [X'(\Sigma^{-1}\otimes I)X]^{-1}X'(\Sigma^{-1}\otimes I)Y$$

According to the Aitken Theorem (Theil 1971, pp. 238-9), β_{GLS} is the BLUE estimator for

β in (2.26). It has lower variance than the OLS estimator for β because it takes into account the contemporaneous correlation between the error terms in different equations. Replacing the usually unknown matrix Σ in β_{GLS} by an observable matrix S with its elements consistently estimated by

$$(2.29) \qquad \hat{\sigma}_{ij} = \frac{1}{n}\hat{\varepsilon}_i'\hat{\varepsilon}_j$$

gives the EGLS estimator for β in (2.26) as

$$(2.30) \qquad \beta_G = [X'(S^{-1}\otimes I)X]^{-1}X'(S^{-1}\otimes I)Y$$

which is an asymptotically unbiased, consistent, and efficient estimator of β. The asymptotic variance-covariance matrix of β_G is given by

$$(2.31) \qquad V(\beta_G) = (X'\hat{\Omega}^{-1}X)^{-1} = [X'(S^{-1}\otimes I)X]^{-1}$$

Breusch and Pagan (1980) suggested using the following Lagrange multiplier statistic for testing the significance of the contemporaneous correlation

$$(2.32) \qquad \lambda = n\sum_{i=2}^{M}\sum_{j=1}^{i-1} r_{ij}^2$$

where r_{ij}^2 is the squared correlation and

$$(2.33) \qquad r_{ij}^2 = \frac{\hat{\sigma}_{ij}^2}{\hat{\sigma}_{ii}\hat{\sigma}_{jj}}$$

Under the null hypothesis of contemporaneous covariances are zero, λ has an asymptotic χ^2-distribution with M(M-1)/2 degrees of freedom. By comparing the calculated value for λ with the relevant critical value from the χ^2-distribution, the null hypothesis is accepted or rejected accordingly. If the contemporaneous correlation is not significant, OLS applied separately to each equation is fully efficient and there is no need to employ the SUR estimator (Judge et al. 1988, p. 456).

The SUR procedure can also be applied when the numbers of observations for different equations are not the same (Schmidt 1977). An example of this is shown in Judge et al. (1988, pp. 462-5). In addition to the existence of contemporaneous correlation among equations in the system, the error terms in individual equations may be serially correlated, heteroskedastic, or both. Assuming the error terms in each equation follow a first-order autoregressive process, for example, the assumption stated in equation (2.25) can be replaced by

$$(2.34) \qquad E[\varepsilon_i] = 0 \qquad E[\varepsilon_i \varepsilon_j'] = \sigma_{ij} \begin{bmatrix} 1 & \rho_j & \rho_j^2 & \cdots & \rho_j^{n-1} \\ \rho_i & 1 & \rho_j & \cdots & \rho_j^{n-2} \\ \rho_i^2 & \rho_i & 1 & \cdots & \rho_j^{n-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \rho_i^{n-1} & \rho_i^{n-2} & \rho_i^{n-3} & \cdots & 1 \end{bmatrix}$$

$$(i, j = 1, 2, \ldots, M)$$

where ρ_i is the coefficient of autocorrelation for the ith equation. With this autocorrelated error specification, ρ_i can be estimated separately for each equation using OLS residuals as in equation (2.23). The original observations are then transformed so that the error terms for each transformed equation are independent. The resulting estimates of the parameters by applying GLS to the system of transformed equations have the same asymptotic properties as the GLS estimates (Parks 1967, Kmenta 1986, p. 647). Kmenta and Gilbert (1970), Guilkey and Schmidt (1973), Judge et al. (1985, pp. 483-497) also presented additional autoregressive specifications for SUR equations. Because ρ_is can be readily estimated from OLS residuals for individual equations, GLS estimates can be obtained from the appropriately transformed SUR systems without great difficulties.

When the error terms for individual equations in SUR systems are uncorrelated but are heteroskedastic, each equation can be transformed using the procedures previously described for a single equation to correct the heteroskedasticity. Srivastava and Giles (1987) covered this topic in a rather complete fashion, with the objective of transforming each equation in the system to have an i.i.d. error structure and then applying the GLS estimator directly to the transformed system of equations.

When the error terms in each equation are heteroskedastic and autocorrelated, the two-step transformation procedure can be used by 1) transforming the original data to remove autocorrelation first, and then 2) transforming the transformed data to correct the heteroskedasticity. Applying GLS to the system of transformed equations in which each equation has i.i.d. errors provides asymptotically unbiased, consistent, and efficient estimates.

2.3.2 Systems of simultaneous linear equations

Consider the ith equation in a system of M simultaneous equations on n observations

$$(2.35) \qquad y_i = Y_i\alpha_i + X_i\beta_i + \varepsilon_i \qquad (i=1,2,...M)$$

where y_i is an $(n\times1)$ vector of observations on the ith endogenous variable in the system, Y_i is an $[n\times(m_i-1)]$ matrix of observations for (m_i-1) endogenous variables appearing in the right hand side (RHS) of ith equation, α_i is an $[(m_i-1)\times1]$ vector of structural parameters associated with Y_i, X_i is an $(n\times p_i)$ matrix of observations for p_i exogenous variables appearing in the ith equation, β_i is a $(p_i\times1)$ vector of structural parameters associated with the X_i, and ε_i is an $(n\times1)$ vector of unobservable random errors associated with y_i. Equation (2.35) can be rewritten as

$$(2.36) \qquad y_i = (Y_i \;\; X_i)\begin{pmatrix}\alpha_i\\ \beta_i\end{pmatrix} + \varepsilon_i = Z_i\delta_i + \varepsilon_i$$

where $Z_i = (Y_i \;\; X_i)$. Assuming the error terms ε_i in equation (2.36) follow $\varepsilon_i \sim (0, \sigma_i^2 I)$, Judge et al. (1985, pp. 570-1) showed that the OLS estimator of δ_i is biased and inconsistent because the structural equation contains the stochastic Y_i that is correlated with the error terms ε_i of the equation.

The two-stage least squares (2SLS) estimator (Judge et al. 1985, 597-8) can be obtained by transforming equation (2.36) into

$$(2.37)\qquad X'y_i = X'Z_i\delta_i + X'\varepsilon_i$$

where X is an (n×p) matrix of observations on all the exogenous and predetermined variables in the entire system. The basic idea of this transformation is to remove the endogeneity from the RHS endogenous variables and provide a new set of explanatory variables $X'Y_i$ and $X'X_i$ that when divided by the n, have in probability a nonstochastic limit as the sample size increases (Judge et al. 1988, p. 640), so that $X'Z_i$ can be considered asymptotically uncorrelated with the error terms $X'\varepsilon_i$. Since

$$(2.38)\qquad E[X'\varepsilon_i] = 0 \qquad V(X'\varepsilon_i) = E[X'\varepsilon_i\varepsilon_i'X] = \sigma_{ii}E[X'X] \approx \sigma_{ii}X'X$$

applying the GLS procedure to the transformed equation (2.37) gives the consistent estimator

$$(2.39)\qquad \begin{aligned} \delta_G &= [(X'Z_i)'(\sigma_{ii}X'X)^{-1}(X'Z_i)]^{-1}(X'Z_i)'(\sigma_{ii}X'X)^{-1}X'y_i \\ &= [Z_i'X(X'X)^{-1}X'Z_i]^{-1}Z_i'X(X'X)^{-1}X'y_i \end{aligned}$$

Let $\hat{Z}_i = (\hat{Y}_i \;\; X_i)$, where $\hat{Y}_i$ is an $[n\times(p_i-1)]$ matrix of estimated values of Y_i obtained by regressing Y_i on X, δ_G can be rewritten as (Judge et al. 1988, pp. 644-5):

$$(2.40)\qquad \delta_{2SLS} = (\hat{Z}_i'\hat{Z}_i)^{-1}\hat{Z}_i'y_i$$

The variance-covariance matrix of the two-stage least squares estimator δ_{2SLS} is obtained as

$$(2.41)\qquad V(\delta_{2SLS}) = \hat{\sigma}_{ii}[\hat{Z}_i'X(X'X)^{-1}X'\hat{Z}_i]^{-1}$$

where

$$(2.42)\qquad \hat{\sigma}_{ii}=\frac{1}{n}(y_i-\hat{Z}_i\delta_{2SLS})'(y_i-\hat{Z}_i\delta_{2SLS})$$

The two-stage least squares procedure can be summarized as follows:

Stage 1. Each endogenous variable in the system is regressed on all exogenous variables of the system by OLS to obtain the predicted values for each endogenous variable.

Stage 2. Replacing the endogenous variables on the RHS of each structural equation by the predicted endogenous variables from Stage 1, then applying the OLS rule to each equation separately to obtain the estimates of structure parameters of the equation. Estimates for all structural parameters of the system are obtained by repeating this process for each structural equation.

The 2SLS estimator δ_{2SLS} is often termed a single-equation estimator since it is obtained by applying GLS to individual equations of the system. It is efficient with respect to single-equation estimators but an asymptotically more efficient estimator may be obtained if all the structural equations are jointly estimated by using the SUR procedure. Zellner and Theil (1962) derived the three-stage least squares (3SLS) estimator by writing M simultaneous equations each in the form of (2.37) into

$$(2.43)\qquad \begin{bmatrix} X'y_1 \\ X'y_2 \\ \vdots \\ X'y_M \end{bmatrix} = \begin{bmatrix} X'Z_1 & & & \\ & X'Z_2 & & \\ & & \ddots & \\ & & & X'Z_M \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_M \end{bmatrix} + \begin{bmatrix} X'\varepsilon_1 \\ X'\varepsilon_2 \\ \vdots \\ X'\varepsilon_M \end{bmatrix}$$

or in a more compact matrix form

$$(2.44)\qquad (I\otimes X')\,y=(I\otimes X')\,Z\delta+(I\otimes X')\,\varepsilon$$

where Z is a $[(Mn\times\Sigma(m_i-1+p_i)]$ matrix, y is a $(Mn\times1)$ vector, δ is a $[\Sigma(m_i-1+p_i)\times1]$ vector, and ε is a (mN×1) vector of

$$Z = diag(Z_1, Z_2, \ldots, Z_M) \qquad y' = (y_1, y_2, \ldots, y_M) \tag{2.45}$$

$$\delta' = (\delta_1, \delta_2, \ldots, \delta_M) \qquad \varepsilon' = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_M)$$

Because

$$E[(I \otimes X')\varepsilon] = 0 \qquad E[(I \otimes X')\varepsilon\varepsilon'(I \otimes X)] = \Sigma \otimes E[X'X] = \Sigma \otimes X'X \tag{2.46}$$

Applying the GLS procedure to (2.44) gives the consistent estimator of δ

$$\delta_G = \{Z'[\Sigma^{-1} \otimes X(X'X)^{-1}X']Z\}^{-1} Z'[\Sigma^{-1} \otimes X(X'X)^{-1}X']y \tag{2.47}$$

This can be written in terms of the 3SLS estimator

$$\delta_{3SLS} = \{Z'[S^{-1} \otimes X(X'X)^{-1}X']Z\}^{-1} Z'[S^{-1} \otimes X(X'X)^{-1}X']y \tag{2.48}$$

where S is the consistent estimator of the unknown matrix Σ with its elements computed by

$$\hat{\sigma}_{ij} = \frac{1}{n}(y_i - Z_i\delta_{2SLS})'(y_j - Z_j\delta_{2SLS}) \tag{2.49}$$

The variance-covariance matrix of δ_{3SLS} is consistently estimated by

$$V(\delta_{3SLS}) = [Z'(S^{-1} \otimes X(X'X)^{-1}X')Z]^{-1} \tag{2.50}$$

The three-stage least squares procedure can be summarized as follows:

Stage 1 and Stage 2 are the same as 2SLS.

Stage 3. Use the 2SLS residuals to obtain an estimator S for the error variance-covariance matrix Σ with the elements of S computed by (2.49), then apply the GLS procedure to the whole system of equations.

Judge et al. (1988, pp. 649-51) showed that the 3SLS estimator δ_{3SLS} is consistent and asymptotically more efficient than the 2SLS estimator. The relative inefficiency of the 2SLS estimator arises because it ignores the information contained in the off-diagonal elements of Σ. If the off-diagonal elements of Σ are zeros, which means that the error terms are uncorrelated across equations, or if each equation in the system of equations is just identified, 3SLS reduces to 2SLS.

2.3.3 Systems of simultaneous linear equations with a generalized error structure

The preceding section on systems of simultaneous linear equations depends on the assumption that the error terms for individual equations are homoskedastic and uncorrelated. If the error terms for individual equations are heteroskedastic and/or autocorrelated, modifications to the 2SLS and 3SLS estimators must be made in order to obtain the most appropriate estimates. Kmenta (1986, pp. 704-11) considers a system of simultaneous equations in which the error terms for individual equations are heteroskedastic and/or autocorrelated. Assuming the ith structural equation

$$\begin{aligned} (2.51) \qquad & y_i = Y_i\alpha_i + X_i\beta_i + \varepsilon_i \\ & y_i = Z_i\delta_i + \varepsilon_i \end{aligned}$$

has a generalized error structure

$$(2.52) \qquad E(\varepsilon_i) = 0 \qquad E(\varepsilon_i\varepsilon_i') = \sigma_{ii}\Omega_i$$

where Ω_i is an (n×n) positive definite symmetric matrix whose diagonal elements are not equal (heteroskedasticity) and the off-diagonal elements are not zero (autocorrelation), σ_{ii} is just an alternative way of writing σ_i^2. Under this error specification, the direct application of the two-stage least squares method produces consistent estimates of the structural parameters, but the asymptotic variances are larger than they would be if the autoregressive or heteroskedastic nature of the error terms is taken into account (Kmenta 1986, p. 705). In addition, the estimated variances of the coefficients are inconsistent, and the hypothesis tests and confidence intervals are not valid.

The weighted two-stage least squares (W2SLS) presented by Kmenta (1986, p. 705) corrects for the problem of heteroskedasticity and/or autocorrelation for individual equations in systems of simultaneous linear equations. The reduced form equation (see Judge et al. 1988, pp. 603-6 and Kmenta 1986, 656-7) for Y_i in equation (2.51) can be written as

$$Y_i = X\pi_i + V_i \tag{2.53}$$

Applying GLS to equation (2.53) provides

$$\hat{\pi}_i = (X'\Omega_i^{-1}X)^{-1}X'\Omega_i^{-1}Y_i \tag{2.54}$$

Defining

$$\hat{V}_i = Y_i - \hat{Y}_i = Y_i - X\hat{\pi}_i \tag{2.55}$$

Replacing Y_i by $\hat{Y}_i$ in the structural equation (2.51) gives

$$\begin{aligned} y_i &= \hat{Y}_i\alpha_i + X_i\beta_i + (\hat{V}_i\alpha_i + \varepsilon_i) \\ y_i &= \hat{Z}_i\delta_i + (\hat{V}_i\alpha_i + \varepsilon_i) \end{aligned} \tag{2.56}$$

The W2SLS estimator of δ_i is then obtained by applying the GLS procedure to equation (2.56). This provides

$$\hat{\delta}_i = (\hat{Z}_i'\Omega_i^{-1}\hat{Z}_i)^{-1}\hat{Z}_i'\Omega_i^{-1}y_i \tag{2.57}$$

The variance-covariance matrix of the W2SLS estimator can be consistently estimated by

$$V(\hat{\delta}_i) = \sigma_{ii}(\hat{Z}_i'\Omega_i^{-1}\hat{Z}_i)^{-1} \tag{2.58}$$

where

$$\sigma_{ii} = \frac{1}{n}(y_i - Z_i\hat{\delta}_i)'\Omega_i^{-1}(y_i - Z_i\hat{\delta}_i) \tag{2.59}$$

The W2SLS estimator as expressed by (2.57) can also be obtained by applying OLS procedures directly to the transformed structural equation (Kmenta 1986, p. 706):

$$(2.60) \qquad P_i y_i = P_i \hat{Z}_i \delta_i + P_i (\hat{V}_i \alpha_i + \varepsilon_i)$$

where P_i is an (n×n) positive definite symmetric transformation matrix that satisfies $P_i' P_i = \Omega_i^{-1}$. Kmenta (1986, p. 706-7) also extended W2SLS to systems of equations. Defining $\varepsilon_i^* = P_i(V_i \alpha_i + \varepsilon_i)$, the weighted three-stage least squares (W3SLS) estimator can be obtained by writing the system of M simultaneous linear equations into

$$(2.61) \qquad \begin{aligned} P_1 y_1 &= P_1 \hat{Z}_1 \delta_1 + \varepsilon_1^* \\ P_2 y_2 &= P_2 \hat{Z}_2 \delta_2 + \varepsilon_2^* \\ &\vdots \\ P_M y_M &= P_M \hat{Z}_M \delta_M + \varepsilon_M^* \end{aligned}$$

This system of equations can be simultaneously estimated using the SUR (GLS) procedure, with the variance-covariance matrix of the error terms estimated from the W2SLS residuals.

Because in practice the matrix Ω_i is usually unknown, Kmenta (1986, p. 707) also shows methods for estimating Ω_i. When the error terms in individual equations are heteroskedastic, Ω_i is diagonal. The transformation matrix P_i is then also a diagonal matrix with the diagonal elements equal to the reciprocal of the square root of the corresponding error variance. When the error terms in individual equations are autocorrelated, Kmenta (1986, p. 707-8) assumed a first-order autoregressive process and shows that the coefficient of autocorrelation ρ_i can be estimated by a three-step procedure: 1) apply ordinary two-stage least squares method to each structural equation, 2) compute the residuals, and 3) use the 2SLS residuals to estimate ρ_i according to

$$(2.62) \qquad \hat{\rho}_i = [\sum_{j=2}^{n} \hat{\varepsilon}_{ij} \hat{\varepsilon}_{i(j-1)}] \,/\, [\sum_{j=2}^{n} \hat{\varepsilon}_{i(j-1)}^2]$$

or

$$(2.63) \qquad \hat{\rho}_i = \frac{\sum_{j=2}^{n} \hat{\varepsilon}_{ij}\hat{\varepsilon}_{i(j-1)}}{\sqrt{\sum_{j=2}^{n} \hat{\varepsilon}_{ij}^2}\sqrt{\sum_{j=2}^{n} \hat{\varepsilon}_{i(j-1)}^2}}$$

where $\hat{\varepsilon}_{ij}$ is the residuals obtained from the 2SLS fitting on the untransformed data (Kmenta 1986, p. 619 and p. 708). Based on estimated ρ_i, the transformation matrix P_i can be constructed as

$$(2.64) \qquad P_i = \begin{bmatrix} \sqrt{1-\rho_i^2} & 0 & 0 & \cdots & 0 & 0 \\ -\rho_i & 1 & 0 & \cdots & 0 & 0 \\ 0 & -\rho_i & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & -\rho_i & 1 \end{bmatrix}$$

Individual equations in the system are then transformed to (2.60), and the W2SLS estimator obtained. The system of transformed equations (2.61) can now be simultaneously estimated by applying the GLS method to the entire system, using the W2SLS residuals to estimate the variance-covariance matrix of the transformed error terms across equations (Kmenta 1986, p. 708). The variances and covariances of the transformed error terms can be computed according to

$$(2.65) \qquad \sigma_{ij} = \frac{1}{n}(\hat{P}_i y_i - \hat{P}_i \hat{Z}_i \hat{\delta}_i)'(\hat{P}_j y_j - \hat{P}_j \hat{Z}_j \hat{\delta}_j) \qquad (i,j=1,2,\ldots,M)$$

where $\hat{\delta}_i$ is the W2SLS estimator in (2.57).

Methods for estimating systems of simultaneous linear equations in which the error terms of individual equations are heteroskedastic and/or autocorrelated were also developed independently by LeMay (1988, 1990) in a forestry context and were termed the multistage least squares (MSLS) technique. Several forestry related examples for fitting such systems of equations were demonstrated by LeMay (1988, 1990).

2.4 Generalized nonlinear regression model

Consider the following nonlinear regression model with p unknown parameters and a known functional form f on n observations

$$(2.66) \qquad y_i = f(x_i, \theta) + \varepsilon_i \qquad (i=1,2,...,n)$$

where y_i is the ith observation on the dependent variable, x_i is a (k×1) vector of observations on k explanatory variables, θ is a (p×1) vector of unknown parameters. In matrix notation, equation (2.66) can be written as

$$(2.67) \qquad y = f(\theta) + \varepsilon$$

where $y=(y_1, y_2, ..., y_n)'$, $f(\theta)=[f(x_1, \theta), f(x_2, \theta), ..., f(x_n, \theta)]'$, and $\varepsilon=(\varepsilon_1, \varepsilon_2, ..., \varepsilon_n)'$. Assuming the error terms ε in (2.67) are i.i.d. as specified in (2.2) for the general linear model, the nonlinear least squares (NLS) estimator θ is obtained by minimizing the error sum of squares

$$(2.68) \qquad S(\theta) = \varepsilon'\varepsilon = [y-f(\theta)]'[y-f(\theta)]$$

The partial derivatives of S(θ) with respect to θ provides the nonlinear normal equation

$$(2.69) \qquad \frac{\partial S}{\partial \theta} = -2\frac{\partial f(\theta)'}{\partial \theta}[y-f(\theta)] = 0$$

that is

$$(2.70) \qquad F'(\theta)\,[y-f(\theta)] = 0$$

where

$$(2.71)\qquad F(\theta)=\frac{\partial f(\theta)}{\partial\theta'}=\begin{bmatrix}\frac{\partial f(x_1,\theta)}{\partial\theta_1} & \frac{\partial f(x_1,\theta)}{\partial\theta_2} & \cdots & \frac{\partial f(x_1,\theta)}{\partial\theta_p}\\ \frac{\partial f(x_2,\theta)}{\partial\theta_1} & \frac{\partial f(x_2,\theta)}{\partial\theta_2} & \cdots & \frac{\partial f(x_2,\theta)}{\partial\theta_p}\\ \vdots & \vdots & \vdots & \vdots\\ \frac{\partial f(x_n,\theta)}{\partial\theta_1} & \frac{\partial f(x_n,\theta)}{\partial\theta_2} & \cdots & \frac{\partial f(x_n,\theta)}{\partial\theta_p}\end{bmatrix}$$

Using the Gauss-Newton method with a first-order Taylor series expansion of f(θ) around the starting values θ_0 to obtain a linear approximation of the model

$$(2.72)\qquad f(\theta)\approx f(\theta_0)+F(\theta_0)\,(\theta-\theta_0)$$

Substituting equation (2.72) into the nonlinear normal equation (2.70) with $F(\theta)=\partial f(\theta)/\partial\theta'$ evaluated at $\theta=\theta_0$ gives

$$(2.73)\qquad F'(\theta_0)\,[y-f(\theta_0)-F(\theta_0)\,(\theta-\theta_0)]=0$$

Equation (2.73) can be rewritten as

$$(2.74)\qquad \theta=\theta_0+[F'(\theta_0)F(\theta_0)]^{-1}F'(\theta_0)\,[y-f(\theta_0)]=\theta_0+D$$

where

$$(2.75)\qquad D=[F'(\theta_0)F(\theta_0)]^{-1}F'(\theta_0)\,[y-f(\theta_0)]$$

The nonlinear iterative process begins by using the given starting values θ_0 to compute a D and find a λ between 0 and 1 such that

$$(2.76)\qquad S(\theta_0+\lambda D)<S(\theta_0)$$

The estimated first-round parameters $\theta_1=\theta_0+\lambda D$ is then used as the new starting values and a second-round D is calculated and λ found. This process is continued until the desired convergence criterion is achieved.

Although there are many different methodologies that are available (Gallant 1987, Seber and Wild 1989) for obtaining the nonlinear least squares estimates, in general, one should be aware that the estimates of the parameters are not unbiased, normally distributed, or minimum variance; rather, they achieve these properties only asymptotically. Gallant (1987, pp. 16-25) demonstrated that the NLS estimator θ is asymptotically normally distributed with mean θ and a variance-covariance matrix that is consistently estimated by

$$V(\theta)=\hat{\sigma}^2[\hat{F}(\theta)'\hat{F}(\theta)]^{-1} \tag{2.77}$$

where the estimated error variance is given by

$$\hat{\sigma}^2=\frac{S(\theta)}{n-p} \tag{2.78}$$

If the error terms ε for equation (2.67) are given as in (2.8) for general linear model, equation (2.67) is transformed to

$$Py=Pf(\theta)+P\varepsilon \tag{2.79}$$

where the matrix P satisfies $P'P=V^{-1}$, and V is a known positive definite matrix. Since $E[P\varepsilon] = 0$ and $V[P\varepsilon] = \sigma^2 I$, the NLS procedures can be directly applied to the transformed equation (2.79) by minimizing

$$S(\theta)=[y-f(\theta)]'V^{-1}[y-f(\theta)] \tag{2.80}$$

This gives the generalized nonlinear least squares estimator θ_G, which is asymptotically normally distributed with mean θ and a variance-covariance matrix that is consistently estimated by

$$V(\theta_G)=\hat{\sigma}^2[\hat{F}(\theta)'V^{-1}\hat{F}(\theta)]^{-1} \tag{2.81}$$

where the consistent estimate of the error variance $\hat{\sigma}^2$ is given by

$$\hat{\sigma}^2=\frac{[y-f(\theta_G)]'V^{-1}[y-f(\theta_G)]}{n-p} \tag{2.82}$$

The estimated generalized nonlinear least squares estimator (EGNLS) can be obtained if V is unknown and replaced by its estimator $\hat{V}$. The estimation will of course depend on the structure of $\hat{V}$. The error terms in equation (2.67) may follow an autocorrelated and/or a heteroskedastic structure. EGNLS provides asymptotically unbiased, consistent, and efficient estimator for θ in either case since the appropriate transformation matrices as those shown in linear cases can be consistently estimated from ordinary NLS residuals (Gallant 1987, pp. 123-39, Seber and Wild 1989).

2.5 Generalized systems of nonlinear regression models

2.5.1 Seemingly unrelated nonlinear equations

Consider the ith equation in a system of M nonlinear regression equations on n observations

$$(2.83) \qquad y_i = f_i(\theta) + \varepsilon_i \qquad (i = 1, 2, \ldots, M)$$

where $y_i = (y_{i1}, y_{i2}, \ldots, y_{in})'$, $f_i(\theta) = [f_i(x_1, \theta), f_i(x_2, \theta), \ldots, f_i(x_n, \theta)]'$, and $\varepsilon_i = (\varepsilon_{i1}, \varepsilon_{i2}, \ldots, \varepsilon_{in})'$. In a more compact matrix form, the set of M equations can be written as

$$(2.84) \qquad y = f(\theta) + \varepsilon$$

where $y = (y_1, y_2, \ldots, y_M)'$, $f(\theta) = [f_1(\theta_1), f_2(\theta_2), \ldots, f_M(\theta_M)]'$, and $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_M)'$. Similar to (2.27), it is assumed that

$$(2.85) \qquad E[\varepsilon] = 0 \qquad \omega = E[\varepsilon\varepsilon'] = \Sigma \otimes I$$

where Σ is an (M×M) positive definite symmetric matrix whose elements are σ_{ij}. Gallant (1975) developed a four-step procedure for estimating the parameters in (2.84) by:

1). Treat each equation in the system separately, obtain the NLS estimator $\hat{\theta}_i$ by minimizing

$$(2.86) \qquad S_i(\theta_i) = [y_i - f_i(\theta_i)]'[y_i - f_i(\theta_i)]$$

for each equation respectively.

2). Using the results from step 1, calculate the residual vectors equation by equation

$$(2.87) \qquad \hat{\varepsilon}_i = y_i - f_i(\theta_i) \qquad (i=1,2,...,M)$$

3). Estimate the elements σ_{ij} of the variance-covariance matrix Σ to obtain the estimated Σ, denoted by S

$$(2.88) \qquad \hat{\sigma}_{ij} = \frac{1}{n}\hat{\varepsilon}_i'\hat{\varepsilon}_j \qquad (i,j=1,2,...,M)$$

4). An EGNLS estimator of θ is then obtained by minimizing

$$(2.89) \qquad S(\theta) = [y-f(\theta)]'(S^{-1}\otimes I)[y-f(\theta)]$$

Under fairly general conditions, Gallant (1975) showed that this procedure results in an estimator that is strongly consistent for θ and asymptotically more efficient than the equation-by-equation estimator, unless the variance-covariance matrix Σ is diagonal or the equations have the same functional form and the explanatory variables are the same for all the equations. Procedures for estimating seemingly unrelated nonlinear equations in which the error terms in individual equations are heteroskedastic and/or autocorrelated are the same as those for linear systems.

2.5.2 Systems of simultaneous nonlinear equations

Amemiya (1974, 1977) and Gallant (1977, 1987) developed estimation procedures for nonlinear simultaneous regression equations by expressing the ith equation in a system of M simultaneous nonlinear equations as

$$(2.90) \qquad f_i(y_t,x_t,\theta_i) = \varepsilon_{it} \qquad (i=1,2,...,M;\ t=1,2,...,n)$$

where y_t is an M-vector of endogenous variables, x_t is a k-vector of exogenous variables, and

θ_i is a p_i-vector of unknown parameters to be estimated, and the ε_{it} represent unobservable random errors that are assumed to be independently and identically distributed with zero mean and constant variance. Equation (2.90) can be written more compactly as

$$(2.91) \qquad f_i(\theta_i) = \begin{bmatrix} f_i(y_1, x_1, \theta_i) \\ f_i(y_2, x_2, \theta_i) \\ \vdots \\ f_i(y_n, x_n, \theta_i) \end{bmatrix} = \begin{bmatrix} \varepsilon_{i1} \\ \varepsilon_{i2} \\ \vdots \\ \varepsilon_{in} \end{bmatrix} = \varepsilon_i \qquad (i=1,2,\ldots,M)$$

The nonlinear least squares estimator of θ_i in (2.91) is biased and inconsistent for the same reason that the least squares estimator is biased and inconsistent in simultaneous linear equations. The RHS endogenous variables in the equation are correlated with the error terms of the equation. Using the instrumental variable estimation technique (Amemiya 1985, pp. 245-50, Gallant 1987, pp. 432-9), a set of instrumental variables — Z that are highly correlated with the RHS endogenous variables of the equation but are independent of the error terms was chosen. This set of instruments Z is some matrix of certain constants with rank at least equal to p_i. Amemiya (1974) defined the nonlinear two-stage least squares (N2SLS) estimator, denoted by θ_{N2SLS}, as the value of θ_i in equation (2.91) that minimizes

$$(2.92) \qquad f_i'(\theta_i) Z (Z'Z)^{-1} Z' f_i(\theta_i)$$

Under the conditions specified by Amemiya (1985 pp. 246-7), it is shown that the N2SLS estimator θ_{N2SLS} is a consistent estimator of θ_i and is asymptotically normally distributed.

Jorgenson and Laffont (1974) defined the nonlinear three-stage least squares (N3SLS) estimator θ_{N3SLS} by considering that the system of nonlinear equations each has the form of (2.91) simultaneously. The system of equations can be written as

$$(2.93) \qquad f(\theta)=\begin{bmatrix} f_1(\theta_1) \\ f_2(\theta_2) \\ \vdots \\ f_M(\theta_M) \end{bmatrix}=\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_M \end{bmatrix}=\varepsilon$$

where ε is an (Mn×1) vector of errors for the M equations stacked together. The nonlinear three-stage least squares estimator θ_{N3SLS} is the value of θ that minimizes

$$(2.94) \qquad f'(\theta)\ [S^{-1}\otimes Z(Z'Z)^{-1}Z']\ f(\theta)$$

where S is a (M×M) matrix that estimates the variance-covariance matrix Σ of the error terms across equations. The variances and covariances are estimated using the residuals from the N2SLS

$$(2.95) \qquad \hat{\sigma}_{ij}=\frac{1}{n}f_i'(\theta_{N2SLS})\ f_j(\theta_{N2SLS}) \qquad (i,j=1,2,\ldots,M)$$

Gallant (1987 p. 439) showed that the asymptotic variance-covariance matrix of θ_{N3SLS} can be written as

$$(2.96) \qquad V(\theta_{N3SLS})=\left[\left(\frac{\partial}{\partial\theta'}f(\theta)\right)'(S^{-1}\otimes[Z(Z'Z)^{-1}Z'])\left(\frac{\partial}{\partial\theta'}f(\theta)\right)\right]^{-1}$$

Amemiya (1985) and Judge et al. (1985) described alternative estimators for systems of simultaneous nonlinear equations. Amemiya (1977, 1985) and Gallant (1987) considered the optimal choice of instrumental variables and indicated that the best nonlinear three-stage least squares estimator can be obtained by using the expected values of the parameter derivatives as instruments. It is therefore best to find instruments that, in some linear combination, approximate the expected values (over the errors) of each of the parameter derivatives. In practice, Amemiya (1985 p. 250), Judge et al. (1985 p. 623), and Gallant (1987 p. 439-40) suggested the use of the exogenous variables and their powers and cross products as instrumental variables, making no attempt to find the most efficient set of instruments based on the results on efficiency.

Gallant (1987 p. 433) presented an estimator that affords some protection against the problem of heteroskedasticity in individual equations of a system of simultaneous nonlinear equations. The theory for correcting autocorrelation was also discussed within the framework of dynamic systems of simultaneous nonlinear equations (Gallant 1987, pp. 442-51). In practice, if the error terms in individual equations of the nonlinear system are heteroskedastic and/or autocorrelated, methods for system of simultaneous linear equations with a generalized error structure can be applied.

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Chapter 3

Individual Tree Height-Diameter Relationships[1]

3.1 Introduction

Predicting total tree height based on observed diameter at breast height outside bark is routinely required in practical management and silvicultural research work (Meyer, 1940). The estimation of tree volume, as well as the description of stands and their development over time rely heavily on accurate height-diameter functions (Curtis 1967). Many growth and yield models also require height and diameter as two basic input variables, with all or part of the tree heights predicted from measured diameters (Burkhart et al. 1972, Curtis et al. 1981, Wykoff et al. 1982). In cases where the actual measurements of height growth are not available, height-diameter functions can also be used to indirectly predict height growth (Larsen and Hann, 1987).

Curtis (1967) summarized a large number of available height-diameter functions and used Furnival's index of fit to compare the performance of thirteen linear functions fitted to second-growth Douglas-fir (*Pseudotsuga menziesii* (Mirb.) Franco) data. Since then, many new height-diameter functions have been developed. With the relative ease of fitting nonlinear functions and the nonlinear nature of the height-diameter relationships, nonlinear height-diameter functions have now been widely used in height predictions (Kozak and Yang 1978, Schreuder et al. 1979, Curtis et al. 1981, Wykoff et al. 1982, Wang and Hann 1988, Farr et al. 1989, Arabatzis and Burkhart 1992).

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For sixteen Alberta tree species in nine groups, this study compared eleven published nonlinear height-diameter functions as well as nine nonlinear functions that apparently have not been applied to height-diameter relationships. The primary objectives of the study are to evaluate the relative performance of a variety of potential height-diameter functions on a large, regional data set covering numerous species, and to identify the most appropriate height-diameter functions for major Alberta tree species.

3.2 The data

Alberta Forest Service (AFS) provided felled-tree data for this analysis. Collected over the last two decades, the 13,489 trees were randomly selected throughout the inventory areas of the province to provide representative information for a variety of densities, heights, species composition, stand structure, ages, and site conditions. The data set was initially used for developing individual tree volume equations and includes many different variables for individual trees and qualitative characteristics of their surrounding environment. A detailed description of how the data were collected and recorded can be found in Alberta Phase 3 Forest Inventory: Tree Sectioning Manual (AFS 1988). Two variables available from the records, diameter at breast height (DBH) outside bark and total tree height (H) for each tree, were selected to be used in this analysis.

The 13,489 trees include 16 different species. To facilitate the analysis, species are classified into different species groups according to their similarity, management objectives, and number of observations (Table 3-1). Summary statistics including the mean (Mean), minimum (Min), maximum (Max), and standard deviation (Std) for total tree height and DBH by species group are shown in Table 3-2. The variation in number of sample trees by species group is an indication of relative importance. Lack of consistent quantitative variables for all data prevented using stand characteristics as independent variables.

3.3 Functions selected for comparison

The selection of the height-diameter functions was based on the examination of the height-diameter relationship as revealed by plotting total tree height against DBH for various species groups. Two typical examples for white spruce (*Picea glauca* (Moench) Voss) and aspen (*Populus tremuloides* Michx.) are shown in Figure 3-1 and Figure 3-2. It is clear that the height-diameter relationship for white spruce (Figure 3-1) has a typical sigmoidal shape, with an inflection point occurring in the lower portion of the data points. On the other hand, the shape of the height-diameter relationship for aspen (Figure 3-2) may be regarded as either concave or sigmoidal, with no apparent inflection point. The sigmoidal/concave shape reflects the strong correlation between DBH and age. Both the typical concave functions and the sigmoidal functions were selected for evaluations. Additional nonlinear functions that are common in biological studies were also selected by considering the plots of height versus DBH compared to the typical graphs of the various functions. Table 3-3 provides a complete list of the selected functions. Notice some of the functions (such as 1 and 6) often appear in transformed forms, and the dependent variable may take the form of H-1.3 (Curtis 1967). The quadratic height-diameter functions, first presented by Trorey (1932) and advocated by Ker and Smith (1955) and previously used in the Pacific Northwest (Staebler 1954) and British Columbia (Watts 1983), were not considered because extrapolation of the functions often leads to unrealistic height predictions.

3.4 Methods

A fundamental nonlinear least squares assumption is that the error terms in all 20 height-diameter functions are independent and identically distributed with zero mean and constant variance. However, in many forestry situations there is a common pattern of increasing variation as values of the dependent variable increase. This is clearly evident from the scatter plots of height versus DBH in Figure 3-1 and Figure 3-2, where the values of the error are more likely to be small for small DBH and large for large DBH. When the problem of unequal error variances occurs, weighted nonlinear least squares (WLS) is applied with the weights selected to be inversely proportional to the variance of the error terms.

The WLS estimates of the parameters uses an iterative process with a starting value chosen and continually improved until the weighted error sum of squares is minimized. It should be noted that the use of the WLS changes the estimates of the parameters and the standard errors of the estimates relative to the values obtained in the absence of weighting (Ratkowsky 1990). The interpretations of the weighted statistics are not as straightforward as in cases of without weighting (Carroll and Ruppert 1988). However, comparison of the fit statistics for various functions can be made if the same weight is consistently used in all the function fittings and the same nonlinear least squares iteration procedure is used.

The use of the WLS requires a known weight. In many practical applications, however, this weight may not be readily available so an estimate based on the results of an unweighted least squares fit is often necessary. Although there are many different procedures that are available for approximating the weight or implementing the generalized nonlinear least squares techniques (Gallant 1987, Judge et al. 1988), a simpler procedure that is based on the analysis of the studentized residuals can be equally efficient.

Studentized residuals are the scaled version of residuals obtained by dividing each residual by its standard error. They are designed to take into account that unstandardized residuals have intrinsically unequal variances even though the theoretical error term is assumed to have constant variance (Draper and Smith 1981, Rawlings 1988, Neter et al. 1990). For a correctly identified function, when the assumptions of the regression analysis are met, the studentized residuals have zero mean and constant variance, and the plot of studentized residuals against the predicted values of the dependent variable will show a homogeneous band.

Figure 3-3 shows an example of the plot of studentized residuals against the predicted height for the modified logistic function (Number 19, Table 3-3) fitted to aspen data with unweighted nonlinear least squares. The plot reveals an obvious unequal error variance problem and suggests that a weighting factor in the form of $w_i = 1/DBH_i^k$ should achieve the desired equality of error variance. This function was then fitted with WLS using six alternative values for k (k=0.5, 1.0, 1.5, 2.0, 2.5, 3.0). Among these alternative weights, the most homogeneous band of studentized residuals occurred with $k = 1.0$ (Figure 3-4). In similar comparisons, $w_i = 1/DBH_i$ was also found to be most appropriate for all other species. Accordingly this weighting factor was used in all remaining analysis. This weight also agrees with the weight chosen by Larsen and Hann (1987), Wang and Hann (1988), and Farr et al. (1989) based on different procedures.

The fitting of the height-diameter functions for various species groups was accomplished using the PROC NLIN procedure on SAS software (SAS Institute Inc. 1985). The Gauss-Newton method as described in Gallant (1987) was applied and multiple starting values for parameters were provided to ensure the least squares solution was the global rather than a local minimum.

3.5 Results and discussion

Three different criteria were selected for judging the performance of the height-diameter functions: 1) the asymptotic t-statistics of the estimators, 2) the weighted mean squared error (MSE) of the model, and 3) the plot of studentized residuals against the predicted height. For any appropriate height-diameter function, the asymptotic t-statistic for each coefficient should be significant, and the model MSE should be small. The studentized residual plot should show approximately homogeneous variance over the full range of predicted values. Any other pattern may indicate bias, unequal variation, or other problems such as outliers or poor model specification.

Tables 3-4, 3-5, and 3-6 show the least squares estimates of the parameters. The associated asymptotic *t*-statistics for testing the null hypothesis that each parameter is zero (or in some models, one) are calculated, and the insignificant parameters are marked. The weighted MSE are summarized in Table 3-7. Although not reported here, coefficient of determination (R^2) values for the fitted functions on weighted observations ranged from 0.70 to 0.92, with an average of about 0.85.

Results in Table 3-4 show that for the two-parameter functions 1 to 9, with the exception of the parameter *a* in function 7 for species group 6b, all the t-statistics for the parameters of the functions are significant at $\alpha = 0.05$ level. The weighted MSE results of the two parameter functions shown in Table 3-7 indicate that functions 3, 4, and 5 have lower MSE values when compared the others, with function 4 generally giving the most satisfactory results. Function 8 has very poor performance with large MSE values. Examination of the plots of studentized residuals for function 8 showed biased height estimates for all species groups when DBH is small. The performance of the remaining two parameter functions is roughly the same and can be regarded as intermediate.

Judged from the plots of studentized residuals and the weighted MSE values, the three- parameter functions 10 to 19 generally perform better than the two-parameter functions. Parameter *a* in function 16 shows several insignificant t-statistics (Table 3-5). The parameter estimates for the remaining functions are generally satisfactory, with a few exceptions of insignificant t-statistics in functions 10, 12, 15, and 19 for species group 4a and function 15 parameter *b* for species group 6b. Insignificant t-statistics were generally associated with small data sets. In terms of the weighted MSE values for three-parameter functions 10 to 19 (Table 3-7), functions 12, 13, 15, 18, and 19 generally give lower values. Functions 10 and 14 give rather similar results and can also be regarded as satisfactory. Function 17 has large MSE values and the plots of studentized residuals showed biased estimates when DBH is small. Occasionally, function 11 fit the data well, but in general it performed poorly.

Although the four-parameter function 20 fitted the data well when the sample size is large (such as for species groups 1 and 3), the function failed to converge for species groups 2b and 6b, and in fitting for species group 4a, has resulted in insignificant t-statistics for parameters *b*, *c*, and *d* (Table 3-6). Several additional four-parameter functions (including Bailey's (1980) function) fitted, but not reported here, also suggested that they might perform well for large samples, however, insignificant t-statistics occurred frequently, and in many cases, failed to converge or converged at local rather than at global minimum when the sample size was small. The gain from using the four parameter function may not be substantial. Depending on the choice of the initial values of the parameters and the size of the samples, the fitting of the four parameter functions may also be rather time consuming.

In terms of the fit of the functions for each species group, several functions may give similar results and perform nearly equally well. However, judging from the weighted MSE values, the asymptotic t-statistics of the parameters, and the principle of parsimony, the following functions are most appropriate for each species group taken independently of the others:

1). The Chapman-Richards function 12 for species group 1;

2). The fractional function 16 for species groups 2a;

3). The Gompertz function 14 for species groups 2b and 6b;

4). The Weibull function 13 and the modified Schnute function 15 for species group 3;

5). The two parameter Michaelis-Menten function 3 for species group 4a;

6). The Mitscherlich function 4 for species group 4b;

7). The modified exponential function 10 for species group 5;

8) The modified logistic-type function 19 for species group 6a.

3.6 Conclusions and recommendations

This comparison of nonlinear height-diameter functions shows that, depending on the sample sizes and the species group, many functions perform well in describing the height-diameter relationships for major Alberta tree species. The choice of a particular function may depend on the relative ease of achieving convergence to a solution, the function's mathematical properties and its biological interpretation. Although any function may be considered superior or inferior in a particular situation, *in general*, the functions discussed below are recommended for use since they often give relatively lower MSE values, significant asymptotic t-statistics, and satisfactory plots of studentized residuals against the predicted values of the dependent variable. Any one of these functions could be used when the same model form is desirable for several species. The recommended functions also have the flexibility to assume various shapes with different parameter values and produce satisfactory curves under most circumstances. All the curves assume biologically reasonable shapes that prevent unrealistic height predictions in the cases of extrapolating the functions beyond the range of the original data.

1). Function 12: $H=1.3+a(1-e^{-bD})^c$. This three parameter Chapman-Richards function has been used extensively in describing height-age relationships. The results shown in this analysis indicate that the function is also well suited for modelling height-diameter relationships. One limiting form of the function - equation 14 also gives satisfactory fits, especially when the sample size is relatively small, such as the fits for species group 2b, 4a, and 6b. However, equation 14 may not fit as well as either the Weibull-type function or the Chapman-Richards function when the sample size is large. A cautionary note for the Chapman-Richards function is that it approaches the asymptote too quickly when the dependent variable is only weakly related to the independent variable.

2). Function 13: $H=1.3+a(1-e^{-bD^c})$. This Weibull-type function is consistently among the best height-diameter functions. It is interesting to see that in fitting species group 4a data, the three or four parameter Chapman-Richards function fails to produce a significant t-statistic for the parameter *b*. However, the Weibull function performs better and gives significant t-statistics for all the parameters.

3). Function 19: $H=1.3+a/(1+b^{-1}D^{-c})$. Although termed the modified logistic-type function, this function is quite different from the commonly used logistic function (such as equation 11). It accommodates many shapes that are commonly described by other sigmoidal functions. The function fits the height-diameter relationship well and is consistently among the best height-diameter functions. As examples, the fits of the function for white spruce and aspen are shown in Figure 3-1 and Figure 3-2. The plot of studentized residuals against the predicted height for aspen is shown in Figure 3-4. It is clear that the function appropriately fits the data.

4). Function 18: $H=1.3+a\cdot e^{b/(D+c)}$. This exponential-type function is particularly well suited for deciduous species. However, it might slightly overestimate height for large diameter trees.

5). Function 15: $H=\{y_1{}^b+(c^b-y_1{}^b)[1-e^{-a(D-D_0)}]/[1-e^{-a(D_2-D_0)}]\}^{1/b}$. This modified Schnute function (with origin set at D = 0, H = 1.3) was shown to fit the height-diameter relationships reasonably well. With the versatility of this function and its abilities to describe various biological shapes, and the relatively easy parameter estimations and interpretations, further application and evaluation of the function should prove useful.

It should be straightforward to extend the functions analysized in this study to model other forestry relationships such as volume-age, height-age, and basal area-age functions. The parameter estimates in Tables 3-4, 3-5, and 3-6, if appropriately scaled, might be useful as the initial values in new applications.

3.7 Summary

Twenty nonlinear height-diameter functions were fitted and evaluated for major Alberta species based on a data set consisting of 13,489 felled trees for 16 different species. All functions were fitted using weighted nonlinear least squares regression ($w_i = 1/DBH_i$) because of the problem of unequal error variance. The examination and comparison of the weighted mean squared errors, the asymptotic t-statistics for the parameters, and the plots of studentized residuals against the predicted height show that many concave and sigmoidal functions can be used to describe height-diameter relationships. The sigmoidal functions such as the Weibull-type function, the modified logistic function, the Chapman-Richards function, and the Schnute function generally gave the most satisfactory results.

Table 3-1 Species and species groups

Species group	Species	Scientific name
1	White spruce	*Picea glauca* (Moench) Voss
2a	Lodgepole pine	*Pinus contorta* var. *latifolia* Engelm.
	Whitebark pine	*Pinus albicaulis* Engelm.
	Limber pine	*Pinus flexilis* James
2b	Jack pine	*Pinus banksiana* Lamb.
3	Aspen	*Populus tremuloides* Michx.
4a	White birch	*Betula papyrifera* Marsh.
4b	Balsam poplar	*Populus balsamifera* L.
5	Black spruce	*Picea mariana* (Mill.) B.S.P.
	Engelmann spruce	*Picea engelmannii* Parry
6a	Balsam fir	*Abies balsamea* (L.) Mill.
6b	Douglas fir	*Pseudotsuga menziesii* (Mirb.) Franco.
	Alpine fir	*Abies lasiocarpa* (Hook.) Nutt.
	Alpine larch	*Larix lyallii* Parl.
	Tamarack	*Larix laricina* (Du Roi) K. Koch
	Western larch	*Larix occidentalis* Nutt.

Table 3-2 Species group based tree summary statistics[1]

Species group	Number of sample trees	DBH (cm)				Total tree height (m)			
		Mean	Min	Max	Std	Mean	Min	Max	Std
1	3101	26.41	1.20	89.00	12.19	20.09	1.70	38.40	6.98
2a	3199	22.10	1.10	66.60	8.59	18.11	1.72	37.60	5.18
2b	659	18.01	1.60	45.00	9.81	14.74	2.58	28.20	6.38
3	3647	21.36	1.10	64.40	10.12	18.77	2.23	31.94	5.46
4a	102	12.11	1.60	32.00	5.87	11.88	3.18	21.50	4.13
4b	510	22.75	1.10	52.90	9.79	17.76	2.90	31.95	4.88
5	1628	14.10	1.10	55.30	6.08	12.20	1.76	30.63	4.26
6a	508	21.15	1.30	53.00	9.19	16.11	1.78	31.40	5.50
6b	135	20.60	3.30	48.70	9.72	13.26	3.35	22.33	4.98

[1] see Table 3-1 for species groups, DBH=diameter at breast height.

Table 3-3 Nonlinear height-diameter functions selected for comparison

Number and form[1]	References
1. $H=1.3+aD^b$	Stoffels and Van Soest 1953, Stage 1975, Schreuder et al. 1979
2. $H=1.3+e^{a+b/(D+1)}$	Wykoff et al. 1982
3. $H=1.3+aD/(b+D)$	Bates and Watts 1980, Ratkowsky 1990
4. $H=1.3+a(1-e^{-bD})$	Meyer 1940, Farr et al. 1989, Moffat et al. 1991
5. $H=1.3+D^2/(a+bD)^2$	Loetsch et al. 1973
6. $H=1.3+a\cdot e^{b/D}$	Ek 1973, Burkhart and Strub 1974, Burk and Burkhart 1984, Buford 1986
7. $H=1.3+10^aD^b$	Larson 1986
8. $H=1.3+aD/(D+1)+bD$	Watts 1983
9. $H=1.3+a[D/(1+D)]^b$	Curtis 1967, Pordan 1968
10. $H=1.3+e^{a+bD^c}$	Curtis et al. 1981, Larsen and Hann 1987. Wang and Hann 1988
11. $H=1.3+a/(1+b\cdot e^{-cD})$	Pearl and Reed 1920
12. $H=1.3+a(1-e^{-bD})^c$	Richards 1959
13. $H=1.3+a(1-e^{-bD^c})$	Kozak and Yang 1978, Yang et al. 1978
14. $H=1.3+a\cdot e^{-b\cdot e^{-cD}}$	Winsor 1932
15. $H=\{{y_1}^b+(c^b-{y_1}^b)[1-e^{-a(D-D_0)}]/[1-e^{-a(D_2-D_0)}]\}^{1/b}$	Schnute 1981
16. $H=1.3+D^2/(a+bD+cD^2)$	Curtis 1967, Pordan 1968
17. $H=1.3+aD^{bD^{-c}}$	Sibbesen 1981
18. $H=1.3+a\cdot e^{b/(D+c)}$	Ratkowsky 1990
19. $H=1.3+a/(1+b^{-1}D^{-c})$	Ratkowsky and Reedy 1986
20. $H=1.3+a(1-b\cdot e^{-cD})^d$	Richards 1959

---

[1] H=total tree height (m), D=DBH in (cm), a, b, c, d=parameters to be estimated, e=base of the natural logarithm (≈2.71828), 1.3 is a constant used to account that DBH is measured at 1.3 metres above the ground. For equation 15: $y_1=1.3$, $D_0=0.0$, $D_2=100.0$.

Table 3-4 Parameter estimations for two-parameter height-diameter functions

Function[1]	Parameter	Estimates for various species groups[2]								
		1	2a	2b	3	4a	4b	5	6a	6b
1	a	1.7313	2.0196	1.3150	2.8211	1.9024	2.6947	1.2137	1.2469	1.1000
	b	0.7353	0.6899	0.8126	0.6056	0.6986	0.5871	0.8344	0.8163	0.7954
2	a	3.6042	3.4766	3.3789	3.3910	3.0097	3.3238	3.2087	3.4184	3.2256
	b	-16.1901	-13.8574	-12.6489	-10.1272	-7.5330	-10.9470	-11.3747	-14.3731	-14.1907
3	a	62.9784	51.4152	65.6462	39.9983	33.4618	37.0257	59.4777	58.3695	51.2611
	b	58.0915	43.2873	65.5679	24.7274	24.2608	26.0386	60.7484	59.0756	64.0364
4	a	38.8548	32.4692	37.9810	27.1294	21.3657	25.3302	34.1127	34.1281	29.9225
	b	0.0270	0.0349	0.0260	0.0549	0.0614	0.0512	0.0283	0.0285	0.0263
5	a	1.8737	1.6413	1.6840	1.1800	1.0601	1.3209	1.5986	1.8069	2.0261
	b	0.1519	0.1639	0.1666	0.1753	0.2089	0.1813	0.1814	0.1663	0.1805
6	a	35.2854	30.8991	27.5419	28.2674	18.3182	26.6049	22.7872	29.3762	23.8673
	b	-14.4531	-12.1948	-10.7183	-8.5907	-5.6927	-9.4854	-9.3829	-12.8412	-12.3567
7	a	0.2388	0.3048	0.1189	0.4509	0.2793	0.4305	0.0838	0.0953	0.0413*
	b	0.7350	0.6903	0.8126	0.6053	0.6986	0.5871	0.8347	0.8167	0.7955
8	a	3.8180	4.9317	2.0670	6.4194	3.2636	6.5487	1.5058	1.3123	1.8794
	b	0.5738	0.5487	0.6401	0.5349	0.6306	0.4507	0.6746	0.6418	0.4951
9	a	35.9867	31.6026	28.3882	28.9552	19.2299	27.1752	23.6995	29.9060	24.4681
	b	15.2897	13.0009	11.6357	9.3290	6.5500	10.1979	10.3221	13.5674	13.2207

[1] see Table 3-3 for the form of the function, [2] see Table 3-1 for species groups, * - the asymptotic t-statistic for the parameter is not significant at $\alpha = 0.05$ level.

Table 3-5 Parameter estimations for three-parameter height-diameter functions

Function[1]	Parameter	Estimates for various species groups[2]								
		1	2a	2b	3	4a	4b	5	6a	6b
10	a	4.3207	4.2512	6.1440	3.8984	6.1541	4.3133	4.6202	4.0034	4.4488
	b	-6.5426	-5.7514	-6.6024	-4.7580	-5.8482	-4.5425	-5.6452	-6.4430	-6.0225
	c	-0.4872	-0.4588	-0.2204	-0.5182	-0.1778*	-0.3614	-0.3577	-0.5375	-0.3793
11	a	26.0850	23.7434	21.8863	22.5297	16.9311	21.5241	17.0593	19.2315	17.3308
	b	8.5482	5.9593	8.5656	5.9461	5.7035	5.0012	8.5954	15.9742	9.7975
	c	0.1339	0.1311	0.1612	0.1704	0.1996	0.1404	0.2063	0.2204	0.1703
12	a	32.0363	29.4214	31.7252	25.7461	25.3245	26.0462	25.0216	23.6894	22.3239
	b	0.0456	0.0457	0.0376	0.0669	0.0409*	0.0464	0.0518	0.0724	0.0522
	c	1.2974	1.1381	1.1150	1.1308	0.8779	0.9465	1.2004	1.6232	1.3270
13	a	31.0481	29.0401	29.8908	25.4088	26.2522	26.1321	24.5127	22.4771	20.8982
	b	0.0209	0.0318	0.0269	0.0486	0.0579	0.0535	0.0308	0.0179	0.0219
	c	1.1973	1.0902	1.1061	1.0892	0.9017	0.9659	1.1361	1.3905	1.2490
14	a	27.8725	25.2831	24.1320	23.5467	18.4726	22.6368	18.8367	20.9530	19.0959
	b	2.8490	2.4343	2.7151	2.3800	2.2367	2.1570	2.8446	3.6061	2.9034
	c	0.0848	0.0873	0.0943	0.1152	0.1235	0.0951	0.1247	0.1259	0.0988
15	a	0.0494	0.0466	0.0450	0.0696	0.0382*	0.0464	0.0536	0.0929	0.0685
	b	0.6387	0.8289	0.7717	0.8151	1.2179	1.0716	0.7411	0.2072	0.4335*
	c	32.4840	30.3314	30.5534	26.8357	26.5976	27.0745	26.1924	23.8101	21.9696
16	a	2.6944	1.4431	0.3504*	0.8408	-0.2324*	0.0038*	1.2706	4.4024	2.4627*
	b	0.6514	0.6806	0.9442	0.4951	0.7813	0.7027	0.8044	0.4670	0.9370
	c	0.0214	0.0233	0.0168	0.0284	0.0273	0.0270	0.0246	0.0311	0.0273
17	a	36.8921	28.4645	39.5300	26.1702	22.7752	22.9433	20.8584	27.8154	35.2386
	b	-13.0405	-16.5206	-8.3474	-13.1935	-7.4274	-20.9985	-14.1796	-15.7403	-8.1497
	c	1.3051	1.5168	1.1040	1.5795	1.3156	1.7680	1.5780	1.4637	1.0545
18	a	43.4552	38.6721	43.7438	33.6553	31.0846	33.2971	31.7946	34.2258	33.0533
	b	-24.1871	-21.4197	-28.1548	-14.5592	-18.4473	-18.4014	-18.5302	-18.7186	-25.2112
	c	5.0167	5.0827	7.3227	3.5766	5.8302	5.5088	4.0490	3.1265	5.8787
19	a	39.3710	37.5445	46.1750	31.3194	41.9635*	34.4682	32.8728	27.6307	28.4451
	b	0.0130	0.0203	0.0174	0.0328	0.0365	0.0369	0.0204	0.0109	0.0146
	c	1.3408	1.2169	1.1253	1.2487	0.9155	1.0589	1.2307	1.5829	1.3299

[1]see Table 3-3 for the form of the function, [2]see Table 3-1 for species groups, * - the asymptotic t-statistic for the parameter is not significant at $\alpha = 0.05$ level.

Table 3-6 Parameter estimations for the four-parameter height-diameter function

Function[1]	Parameter	Estimates for various species groups[2]								
		1	2a	2b	3	4a	4b	5	6a	6b
20	a	32.5525	30.8722	24.4874**	25.4676	20.7813	25.2716	31.3035	23.3678	17.8206**
	b	1.0200	1.0413	0.2528**	0.9687	0.8247*	0.9574	1.0334	0.9716	0.0314**
	c	0.0428	0.0383	0.0878**	0.0709	0.0767*	0.0530	0.0298	0.0766	0.1069**
	d	1.2034	0.9570	9.4899**	1.2419	1.5810*	1.1025	0.8964	1.7781	90.9244**

[1]see Table 3-3 for the form of the function, [2]see Table 3-1 for species groups, * - the asymptotic t-statistic for the parameter is not significant at $\alpha = 0.05$ level, ** - convergence is not obtained.

Table 3-7 Weighted mean squared errors of the height-diameter functions

Function[1]	Weighted MSE for various species groups[A]								
	1	2a	2b	3	4a	4b	5	6a	6b
1	0.5082	0.3886	0.2938	0.3863	0.3326	0.3677	0.2685	0.3366	0.3359
2	0.4675	0.3800	0.3564	0.3370	0.3781	0.3687	0.2702	0.2577	0.3373
3	0.4596	0.3702	0.2770	0.3265	0.3257$^{(1)}$	0.3465	0.2571	0.2865	0.3171
4	0.4539	0.3686	0.2748	0.3189$^{(5)}$	0.3261$^{(2)}$	0.3454$^{(1)}$	0.2566	0.2813	0.3147
5	0.4443	0.3685	0.2976	0.3218	0.3557	0.3542	0.2547	0.2466	0.3124
6	0.4832	0.3891	0.4049	0.3599	0.4268	0.3828	0.2876	0.2701	0.3584
7	0.5082	0.3886	0.2938	0.3863	0.3326	0.3677	0.2685	0.3366	0.3359*
8	0.5841	0.4210	0.3170	0.4778	0.3539	0.4112	0.2843	0.3894	0.3588
9	0.4751	0.3842	0.3801	0.3477	0.4009	0.3756	0.2784	0.2642	0.3474
10	0.4459	0.3668$^{(2)}$	0.2804	0.3234	0.3295*	0.3500	0.2502$^{(1)}$	0.2516	0.3158
11	0.4935	0.3877	0.2701$^{(2)}$	0.3400	0.3314	0.3598	0.2915	0.2865	0.3035$^{(2)}$
12	0.4424$^{(2)}$	0.3674$^{(4)}$	0.2729	0.3166$^{(2)}$	0.3268*	0.3458$^{(2)}$	0.2534$^{(5)}$	0.2449$^{(2)}$	0.3089$^{(4)}$
13	0.4426$^{(3)}$	0.3675$^{(5)}$	0.2723$^{(5)}$	0.3165$^{(1)}$	0.3271$^{(4)}$	0.3459$^{(3)}$	0.2539	0.2458$^{(5)}$	0.3080$^{(3)}$
14	0.4597	0.3761	0.2648$^{(1)}$	0.3236	0.3267$^{(3)}$	0.3504	0.2694	0.2533	0.3024$^{(1)}$
15	0.4430$^{(5)}$	0.3676	0.2717$^{(4)}$	0.3165$^{(1)}$	0.3272*	0.3459$^{(3)}$	0.2544	0.2463	0.3069*
16	0.4435	0.3667$^{(1)}$	0.2767*	0.3200	0.3275*	0.3472*	0.2510$^{(3)}$	0.2460	0.3131*
17	0.4658	0.4033	0.2857	0.3795	0.3596	0.4192	0.3014	0.2568	0.3089$^{(4)}$
18	0.4433	0.3677	0.2708$^{(3)}$	0.3181$^{(4)}$	0.3273$^{(5)}$	0.3463$^{(5)}$	0.2540	0.2455$^{(4)}$	0.3090$^{(5)}$
19	0.4427$^{(4)}$	0.3671$^{(3)}$	0.2747	0.3180$^{(3)}$	0.3279*	0.3469	0.2525$^{(4)}$	0.2445$^{(1)}$	0.3100
20	0.4423$^{(1)}$	0.3668$^{(2)}$	0.2662*	0.3165$^{(1)}$	0.3288*	0.3462$^{(4)}$	0.2507$^{(2)}$	0.2453$^{(3)}$	0.3100*

[1] see Table 3-3 for the form of the function, [A] - the smallest five MSE values for each species group with ranks 1 (smallest) to 5 in parentheses, * - the MSE values are not compared because of insignificant t-statistic(s) or the failure of convergence.

Figure 3-1. Plot of total tree height against DBH for white spruce (*Picea glauca* (Moench) Voss). Curve produced by $H=1.3+39.3710/[1+1/(0.0130\times DBH^{1.3408})]$.

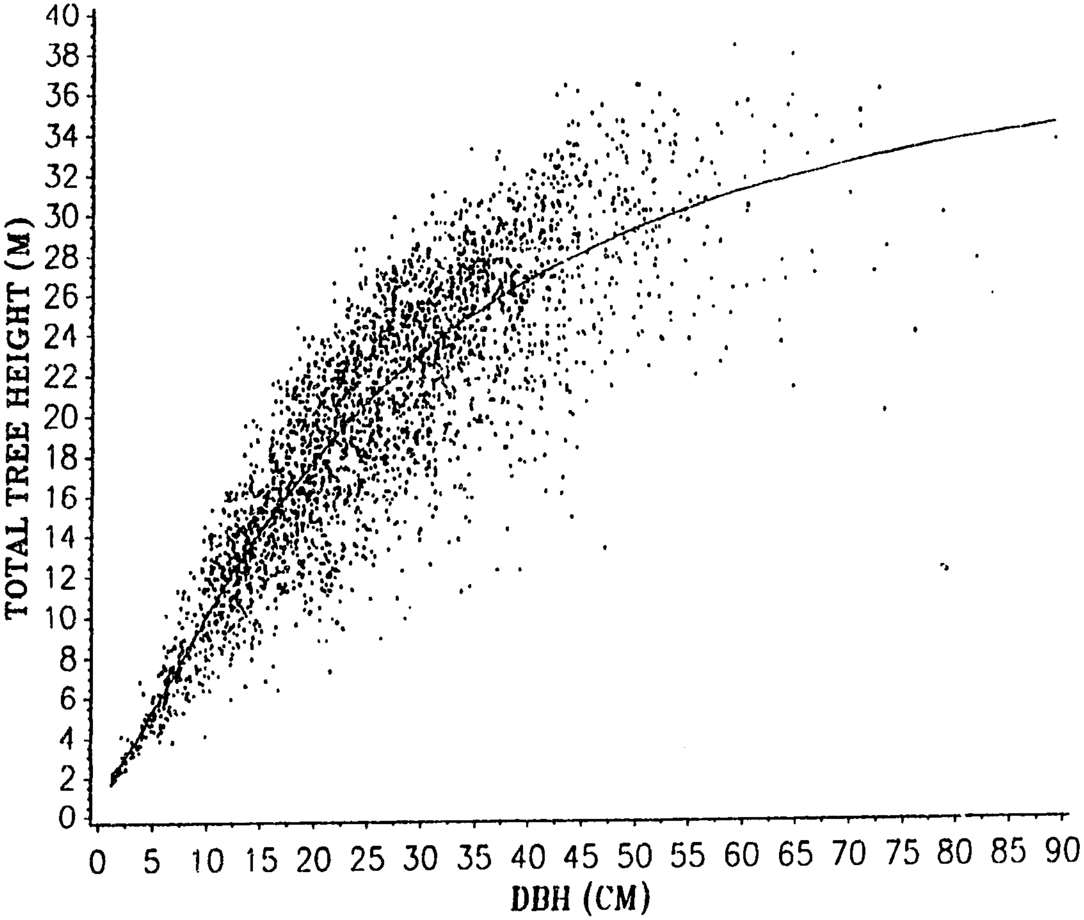
TOTAL TREE HEIGHT (M)
0
2
4
6
8
10
12
14
16
18
20
22
24
26
28
30
32
34
36
38
40
DBH (CM)
0
5
10
15
20
25
30
35
40
45
50
55
60
65
70
75
80
85
90

Figure 3-1.

Figure 3-2. Plot of total tree height against DBH for aspen (*Populus tremuloides* Michx.).

Curve produced by $H=1.3+31.3194/[1+1/(0.0328\times DBH^{1.2487})]$.

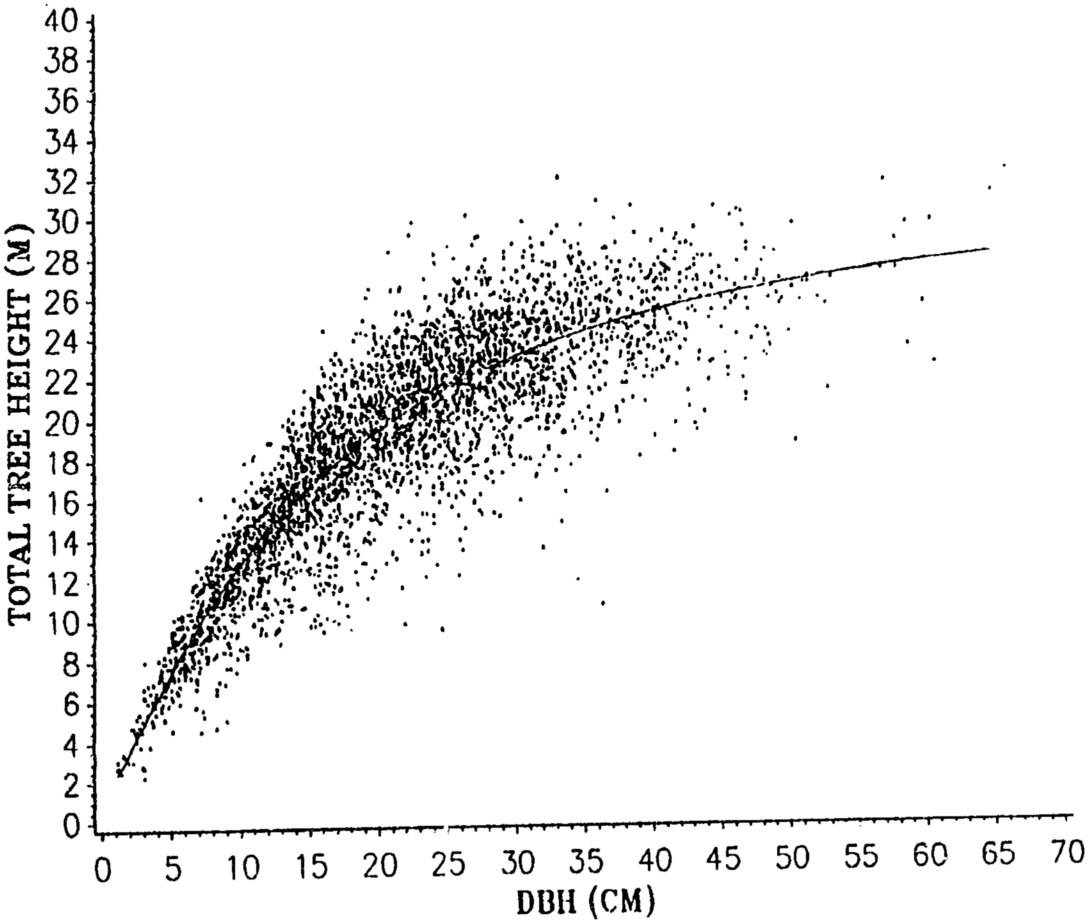
TOTAL TREE HEIGHT (M)
0
2
4
6
8
10
12
14
16
18
20
22
24
26
28
30
32
34
36
38
40
DBH (CM)
0
5
10
15
20
25
30
35
40
45
50
55
60
65
70

Figure 3-2.

Figure 3-3. The plot of studentized residuals against the predicted height for aspen (*Populus tremuloides* Michx.). Studentized residuals are obtained by fitting function 19 without weighting.

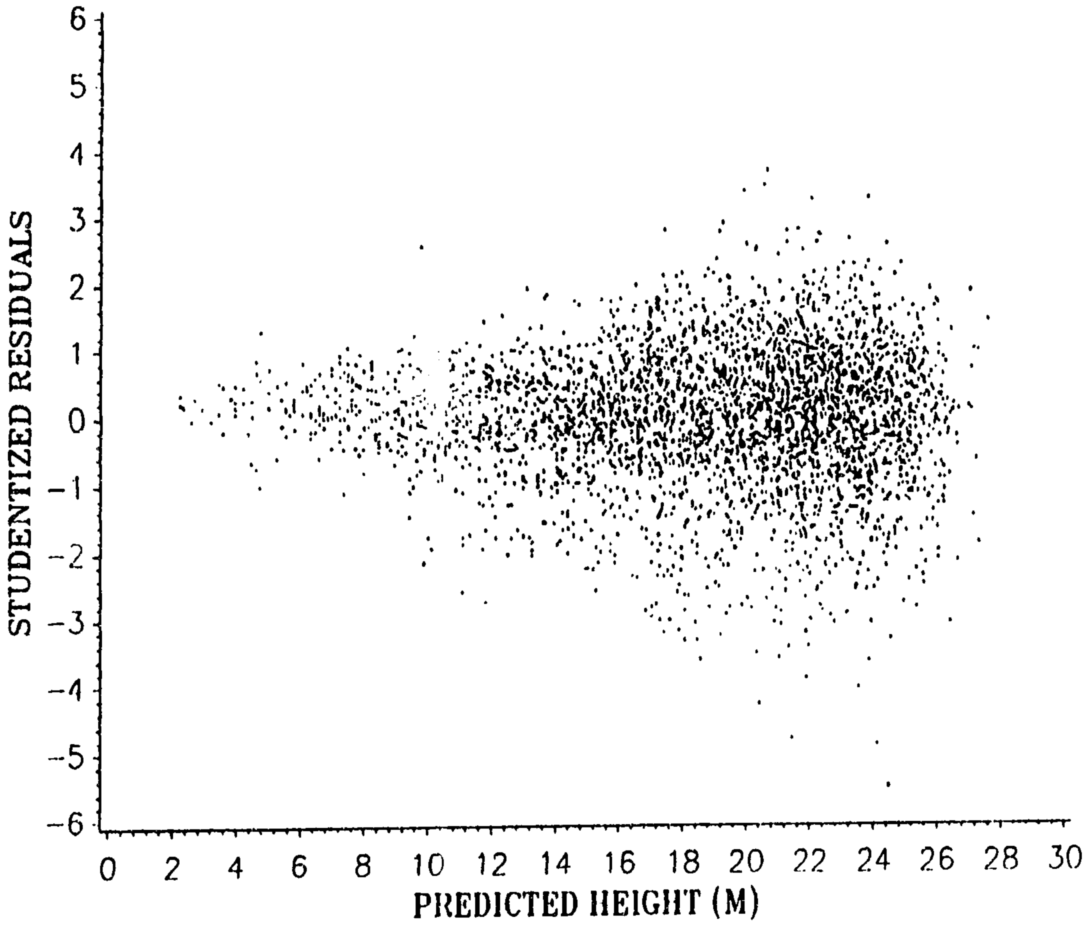
STUDENTIZED RESIDUALS
6
5
4
3
2
1
0
-1
-2
-3
-4
-5
-6
0
2
4
6
8
10
12
14
16
18
20
22
24
26
28
30
PREDICTED HEIGHT (M)

Figure 3-3.

Figure 3-4. The plot of studentized residuals against the predicted height for aspen (*Populus tremuloides* Michx.). Studentized residuals are obtained by fitting function 19 with weight $w_i = 1/DBH_i$.

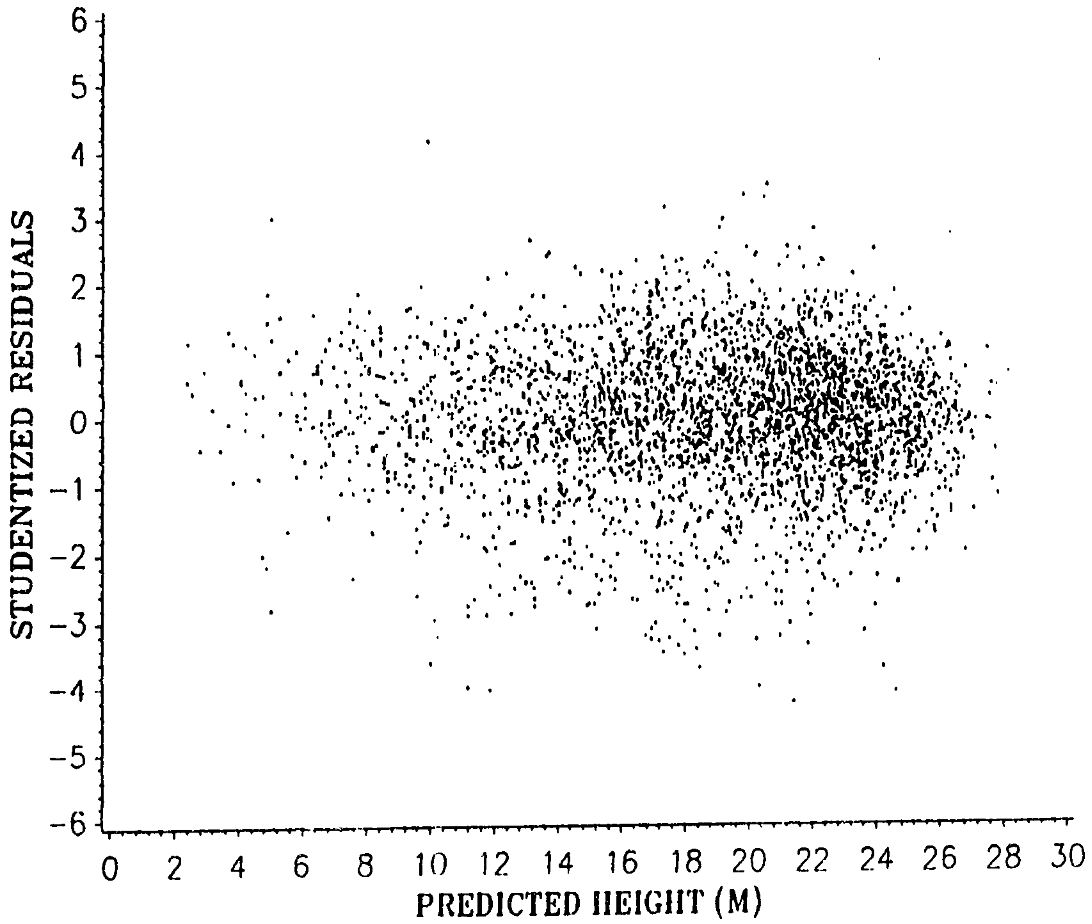
STUDENTIZED RESIDUALS
6
5
4
3
2
1
0
−1
−2
−3
−4
−5
−6
0
2
4
6
8
10
12
14
16
18
20
22
24
26
28
30
PREDICTED HEIGHT (M)

Figure 3-4.

3.8 References

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Chapter 4

An Index of Site Productivity for Mixed Species Stands[1]

4.1 Introduction

Site index, defined as the average height of the dominant (or dominant and codominant) trees in the stand at a specified reference age, is probably the most widely used site productivity measure in North America (Carmean 1975, Alemdag 1991). Methods for constructing site index curves have become increasingly complex and diverse (Devan and Burkhart 1982, Borders et al. 1984, Monserud 1984, Amateis and Burkhart 1985, Biging 1985, Newnham 1988, Lappi and Bailey 1988, Walters et al. 1991). The foundation of site index equations, the height versus age relationship, however, remains unchanged.

The proliferation of site index as a site productivity measure has also been criticized (Madar 1963, Sammi 1965, Jones 1969, Hagglund 1981, Monserud 1988, Verbyla and Fisher 1989). Site index curves came at about the same time as the "normal" yield tables, but were independent of the normality concept. Although the site index method is reasonably stable under thinning and many of the constraints on site trees have been relaxed (such as Monserud 1984), generally, only the dominant trees grown in older, even-aged, well-stocked, free-growing, undisturbed, pure-species stands can be used as suitable site trees for constructing site index curves (Carmean and Lenthall 1989). Because of these restrictions, many researchers have developed alternative site productivity measures that are based on vegetative or habitat types, environmental factors such as soil, climatic conditions,

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and topographic characteristics, and biophysical factors or physiological processes that control productivity. Recent examples of such approaches have been widespread (Wykoff et al. 1982, McLeod and Running 1988, Schmidt and Carmean 1988, Verbyla and Fisher 1989, Klinka and Carter 1990, Wykoff 1990).

Most of the boreal forests in Alberta have a mixed-species composition with irregular age structure. Height-age relationships in such stands are very weak, and Monserud (1988) and Wykoff (1990) suggested that both site index and age are often meaningless concepts in this situation. Alternative approaches based on vegetative or environmental factors provide a long term possibility for developing reasonable and realistic site productivity measures for uneven-aged and mixed-species stands. However, because of the vast complexity involved in mixed-species stands, the cost to develop such approaches can be substantial. Lack of consistent quantitative and qualitative variables for all the data may limit the practical applications of these approaches.

A site productivity measure termed Site Productivity Index (SPI) based on the height-diameter relationship of dominant and codominant trees was formulated in this analysis for boreal mixed-species stands in Alberta. While top height has been suggested as preferable to dominant and codominant height (Arney 1985), most inventory methods still select dominant and codominant trees for height measurement. For that reason this analysis uses the latter. Separate site productivity indices are estimated for each of four tree species – white spruce (*Picea glauca* (Moench) Voss), lodgepole pine (*Pinus contorta* Dougl.), aspen (*Populus tremuloides* Michx.), and black spruce (*Picea mariana* (Mill.) B.S.P.). Use of the height-diameter relationship as a site productivity measure dates back to Trorey (1932). Meyer (1940) and Husch et al. (1982) also suggested that height-diameter relationships can be a good measure of site productivity for uneven-aged and mixed-species stands. McLintock and Bickford (1957) examined several alternatives for evaluating site productivity in uneven-aged stands of red spruce (*Picea rubens* Sarg.) in the northeastern United States, and found that the height-diameter relationship of dominant trees as expressed by the monomolecular function suggested by Meyer (1940) was the most sensitive and reliable measure of site productivity. Stout and Shumway (1982) also found that the height-diameter relationship provided an appropriate site productivity measure for six hardwood species and presented several additional rationales for the use of such a measure from ecological and silvicultural viewpoints. Other examples of height-diameter based site productivity measures have been shown by Reinhardt (1983) for old-growth western larch stands, Lamson (1987) for central Appalachian hardwood stands, and Nicholas and Zedaker (1992) for southern Appalachian red spruce.

4.2 The data

Data from 164 permanent sample plots (PSP) used in this analysis were provided by the Alberta Forest Service. The data were collected over the last three decades and the PSPs were randomly located throughout the inventory areas of the province to provide representative information for a variety of species composition, stand structures, densities, heights, ages, and site conditions. A detailed description of how the data are collected and recorded can be found in the Permanent Sample Plots: Field Procedures Manual (Alberta Forest Service 1990).

Data from dominant and codominant trees for the four most important commercial tree species in Alberta were extracted from the PSP data base. Only live trees with both diameter and height recorded were retained for this analysis. The selected trees have up to five remeasurements, with the time between remeasurements ranging from 3 to 18 years. Each non-overlapping growth period from remeasurements defines a growth interval, that is, the growth intervals are obtained from measurements between first and second, second and third, but not first and third. Summary statistics including the mean, minimum, maximum, and standard deviation for tree diameter at breast height (DBH) and tree height by species at the beginning of the growth interval are shown in Table 4-1.

4.3 Methods

The three-parameter modified Weibull function proposed by Yang et al. (1977) describes the height-diameter relationship well for major Alberta tree species (Huang et al. 1992) and was selected as the base height-diameter function:

$$(4.1)\qquad H=1.3+a[1-\exp(-bDBH^{c})]$$

where H is total tree height in metres (m), DBH is tree diameter at breast height in centimetres (cm), 1.3 is a constant used to account for measurement of DBH at 1.3 metres above the ground, exp raises *e* (≈2.71828) to a specified power, and a, b, c are parameters to be estimated. Using the difference equation method on the repeatedly measured PSP data with real growth series (Clutter et al. 1983, Borders et al. 1984), each site tree is assumed to follow its unique height-diameter curve. For any two succeeding diameters DBH_1 and DBH_2 ($DBH_1<DBH_2$), there are two corresponding tree heights H_1 and H_2 ($H_1<H_2$) on the curve as defined by

$$(4.2)\qquad H_1=1.3+a[1-\exp(-bDBH_1^{c})]$$

$$(4.3)\qquad H_2=1.3+a[1-\exp(-bDBH_2^{c})]$$

Three alternative difference equations can be obtained by isolating each of the three different parameters in equations (4.2) and (4.3). Isolating parameter *a* in both equations, setting them equal, and then solving for H_2 gives

$$(4.4) \qquad H_2 = 1.3 + (H_1 - 1.3)\,\frac{[1-\exp(-bDBH_2^{c})]}{[1-\exp(-bDBH_1^{c})]}$$

Isolating parameter b in equations (4.2) and (4.3), setting them equal, and then solving for H_2 gives

$$(4.5) \qquad H_2 = 1.3 + a\left\{1 - \left[1 - \frac{(H_1 - 1.3)}{a}\right]^{\left(\frac{DBH_2}{DBH_1}\right)^{c}}\right\}$$

Isolating parameter c in equations (4.2) and (4.3), setting them equal, and then solving for H_2 gives

$$(4.6) \qquad H_2 = 1.3 + a\left\{1 - \exp\left(-b \times \left(\frac{1}{-b}\ln\left(1 - \frac{H_1 - 1.3}{a}\right)\right)^{\frac{\ln DBH_2}{\ln DBH_1}}\right)\right\}$$

Site productivity index (SPI) is defined as the tree height (SPI=H_2) at a chosen reference-diameter (DBH_2). A 20 cm reference-diameter was selected for this analysis (DBH_2=20), so SPI actually indicates the tree height at 20 cm reference-diameter. The 20 cm reference-diameter corresponds roughly to the 50 year reference-age in traditional Alberta height/age site curves. For two trees with the same diameter (DBH_1), the taller tree (H_1) has a larger SPI value, and therefore better site productivity.

As with the difference equation curve fitting methods commonly used for site index, any of the three equations (4.4), (4.5), and (4.6) can be used to construct SPI curves. The SPI curves produced from these equations pass through appropriate heights at reference-diameter, and are reference-diameter invariant (Bailey and Clutter 1974). It can also be shown that equation (4.4) produces an anamorphic set of height-diameter curves with varying asymptotes while equations (4.5) and (4.6) produces polymorphic curves with a common asymptote. There seems no common ground for the choice of a particular equation to be the most appropriate. Site index methods described by Burkhart and Tennent (1977), Borders et al. (1984), and Newnham (1988) constrained different parameters and produced different curve forms. It would be most desirable if a single site-specific parameter could be identified. However, because of the different height growth patterns, all three parameters can be different on different sites, so the "most" appropriate equation can only be obtained by comparing the relative fits on the available data. For each species it is necessary to examine all three potential difference equation forms.

The fitting of equations (4.4), (4.5), and (4.6) for each of the four species was accomplished using the PROC MODEL procedure on SAS/ETS software (SAS Institute Inc. 1988). The Gauss-Newton method using the Taylor series expansion as described in Gallant (1987) was applied and multiple starting values for parameters were provided to ensure the least squares solution was the global rather than a local minimum. Candidate equations were judged on the basis of mean squared error (MSE) and coefficient of determination (R^2) of the equations, as well as the plots of studentized residuals against the predicted height. All the R^2 values are calculated according to

$$(4.7) \qquad R^2=1-\Sigma_1^n(H_i-\hat{H}_i)^2/\Sigma_1^n(H_i-\bar{H})^2$$

where H_i is the observed and $\hat{H}_i$ is the predicted height for the *ith* tree (i=1, 2, ..., n), and $\bar{H}$ is the observed average tree height.

4.4 Results and discussion

For all four species, equations (4.5) and (4.6) have larger R^2 and smaller MSE values than those of equation (4.4) (Table 4-2). The difference between equations (4.5) and (4.6) is not substantial. Fit statistics including the nonlinear least squares estimates of the parameters, the asymptotic standard errors and *t*-statistics of the parameters, and the MSE and R^2 for equation (4.5) by species are listed in Table 4-3. It is clear from Tables 4-2 and 4-3 that equation (4.5) fitted the data well. It is also evident that SPI curves similar to traditional site index curves can be constructed based on the estimated parameters, and the resulting SPI curves have properties similar to those of the traditional site index curves obtained from the same difference equation method, namely they are polymorphic in form, reference-diameter invariant, and pass through the appropriate heights at reference-diameter. Graphs illustrating the relationship between dominant height and dominant DBH for a range of SPI values for each species have been shown in Figures 4-1 to 4-4. The equation that is derived from equation (4.5) for computing dominant height from specified dominant DBH and SPI is

$$(4.8) \qquad H=1.3+a\left\{1-\left[1-\frac{(SPI-1.3)}{a}\right]^{\left(\frac{DBH}{20}\right)^{c}}\right\}$$

The dominant height-diameter relationship based SPI provides a simple and quick method of quantifying site productivity for uneven-aged and mixed-species stands in Alberta. It is species-specific but the average site productivity for a mixed-species stand can also be obtained by averaging the species-specific SPI values for a sample of dominant and codominant trees in the stand, although the "right" method for this process involving such questions as the range and pattern of SPI curves of species B found on sites where species A has a given SPI. Estimating SPI requires no time consuming and difficult age measurements, but only tree height and diameter measurements that are readily obtainable from ordinary inventories and are compatible with the existing data-collection process in Alberta.

The validity of the SPI approach requires two important assumptions: 1) decreasing tree taper (DBH divided by tree height) is associated with increasing site productivity, and 2) stand density does not affect the height-diameter relationship of the dominant and codominant trees in uneven-aged or mixed-species stands. Larson's (1963) extensive studies on stem form indicated that the first assumption generally held because increasing site productivity produced increasing tree height for a given diameter. Stout and Shumway (1982) discussed the second assumption in detail and concluded that the influence of stand density on potential site productivity assessment using the height-diameter relationship was minimal. Increasing stand density has been found to reduce both diameter growth and height growth for mixed conifers of the northern Rocky Mountains (Wykoff et al. 1982), also implying that the stand density impact on the height-diameter relationship will probably be minimized, especially if this relationship is only considered for the dominant and codominant trees in the stands. Possibilities for further analysis involving the relationships between dominant height and dominant DBH relative to density (such as Alexander et al. 1967) do exist. An analysis that shows the relationship between SPI and traditional height-age based site index for an appropriate range of stand conditions will also be useful. However, the data used for this study did not include sufficient reliable data for ages and heights of dominant trees of all species.

While the use of SPI as determined by the dominant and codominant height-diameter relationship is by no means the final solution, it provides a simple and reasonable index of site productivity for uneven-aged and mixed-species stands comm[illegible]d in Alberta. SPI uses the two most important components related to volume production – height and diameter, and therefore should also be highly correlated with volume production, another measure of site productivity.

4.5 Summary

A site productivity measure based on the relationship between total tree height and diameter at breast height of dominant and codominant trees is presented for four major tree species — white spruce (*Picea glauca* (Moench) Voss), lodgepole pine (*Pinus contorta* Dougl.), aspen (*Populus tremuloides* Michx.), and black spruce (*Picea mariana* (Mill.) B.S.P.) grown in boreal mixed-species stands in Alberta. The measure is based on a three-parameter modified Weibull function fitted to growth data from permanent sample plots using the difference equation method; R^2 values range from 0.90 to 0.97. The measure has many logical properties similar to those of the site index approach and produces curves that are polymorphic and reference-diameter invariant. It can be used as a simple and quick method of quantifying site productivity for uneven-aged and/or mixed-species stands.

Table 4-1. Tree summary statistics by species[1]

Species	Number of observations	DBH (cm)				Total tree height (m)			
		Mean	Min	Max	Std	Mean	Min	Max	Std
White spruce	1181	30.91	11.20	63.30	8.80	24.17	9.00	37.90	[illegible]
Lodgepole pine	1623	18.51	9.10	47.60	7.28	16.08	7.30	31.40	[illegible]
Aspen	1051	23.37	9.10	56.40	11.17	19.39	7.40	32.90	[illegible]
Black spruce	217	14.77	5.60	39.40	6.09	13.14	5.50	29.30	[illegible]

[1] Note: summary statistics obtained at the begin of the growth interval, number of observations refers to the number of growth intervals, DBH is tree diameter at breast height.

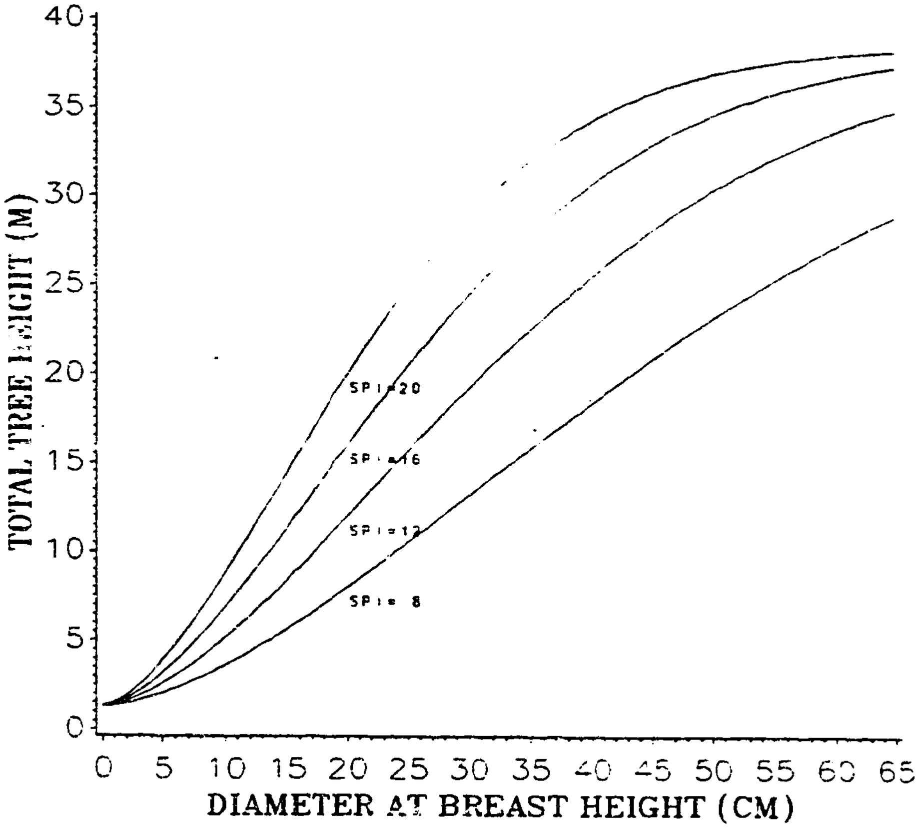
Table 4-2. Mean squared error and R^2 for equation (4.4), (4.5), and (4.6) by species

Species	Equation	Mean squared error	R^2
White spruce	(4.4)	2.00179	0.8889
	(4.5)	1.86023	0.8968
	(4.6)	1.94872	0.8919
Lodgepole pine	(4.4)	0.89962	0.9615
	(4.5)	0.88033	0.9623
	(4.6)	0.89633	0.9616
Aspen	(4.4)	1.84199	0.9357
	(4.5)	1.36359	0.9524
	(4.6)	1.49250	0.9479
Black spruce	(4.4)	0.61300	0.9682
	(4.5)	0.57553	0.9702
	(4.6)	0.55034	0.9715

Table 4-3. Fit statistics for equation (4.5) by species

Species	Parameters	Estimate	Std. error	t-statistic
White spruce	a	37.25707	1.5510	24.04
	c	1.63062	0.0743	21.95
Lodgepole pine	a	41.74221	4.4530	9.37
	c	1.04074	0.0493	21.11
Aspen	a	31.60000	1.2174	25.96
	c	1.50198	0.0627	23.96
Black spruce	a	35.79457	7.8621	4.55
	c	1.32654	0.1387	9.56

Figure 4-1. Site Productivity Index (SPI) curves for white spruce. A 20 cm reference DBH was used.



TOTAL TREE HEIGHT (M)
0
5
10
15
20
25
30
35
40
SPI=20
SPI=16
SPI=12
SPI= 8
0 5 10 15 20 25 30 35 40 45 50 55 60 65
DIAMETER AT BREAST HEIGHT (CM)

Figure 4-3. Site Productivity Index (SPI) curves for lodgepole pine. A 20 cm reference DBH was used.

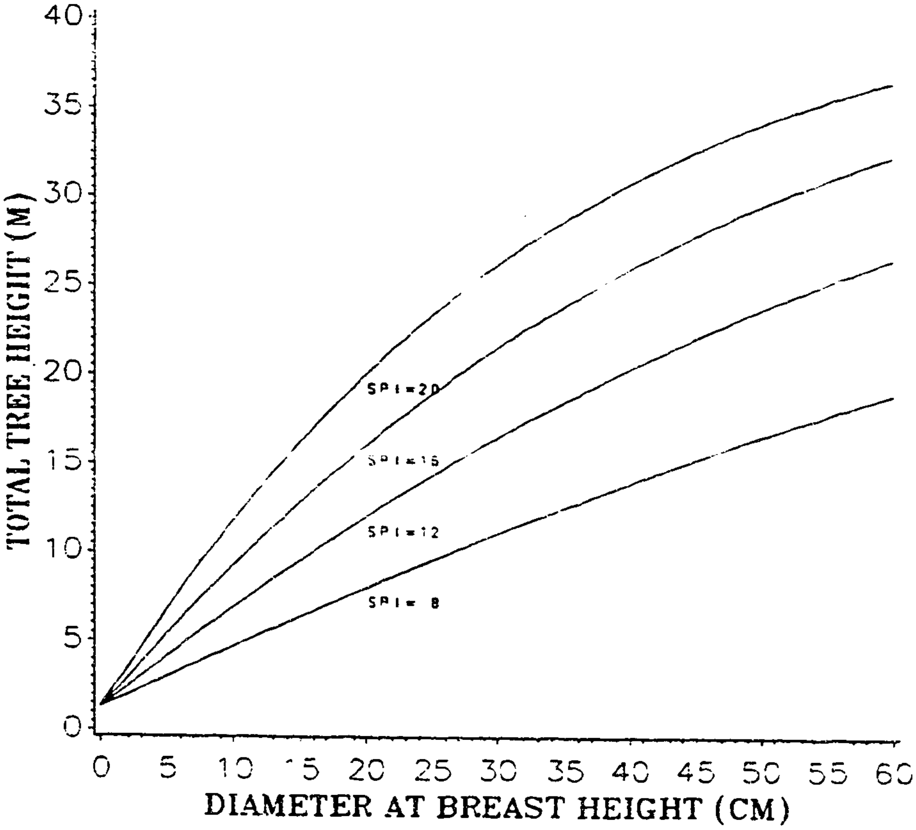

TOTAL TREE HEIGHT (M)
40
35
30
25
20
15
10
5
0
SPI=20
SPI=15
SPI=12
SPI= 8
0
5
10
15
20
25
30
35
40
45
50
55
60
DIAMETER AT BREAST HEIGHT (CM)

Figure 4-3. Site Productivity Index (SPI) curves for aspen. A 20 cm reference DBH was used.

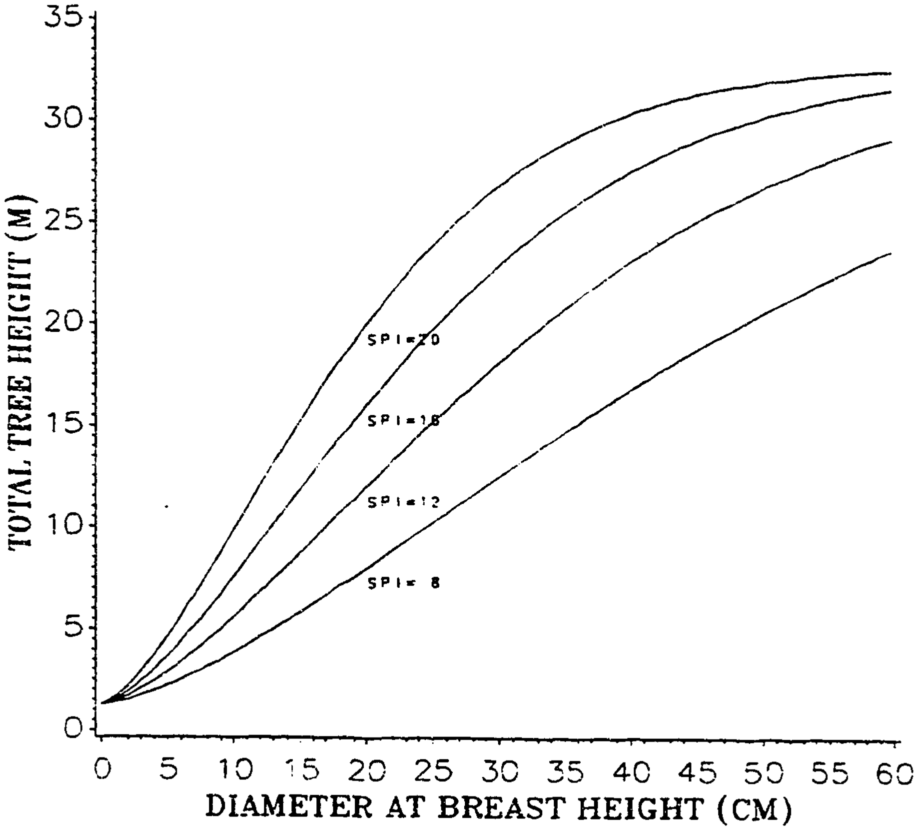
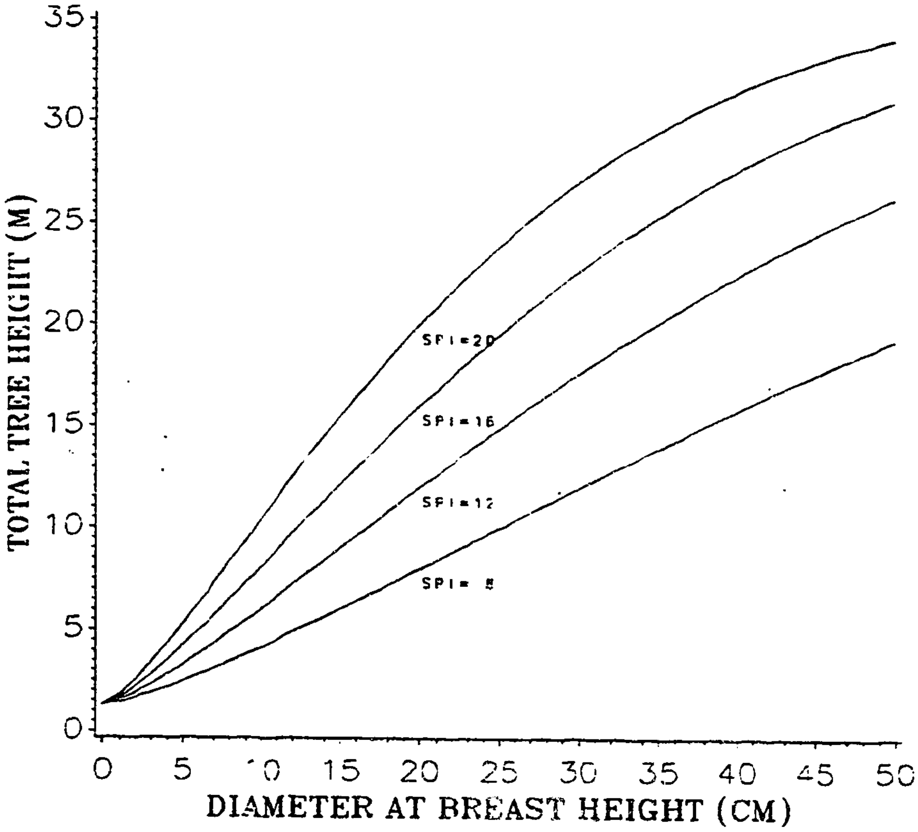

TOTAL TREE HEIGHT (M)
35
30
25
20
15
10
5
0
DIAMETER AT BREAST HEIGHT (CM)
0
5
10
15
20
25
30
35
40
45
50
55
60
SPI=20
SPI=16
SPI=12
SPI= 8

Figure 4-4. Site Productivity Index (SPI) curves for black spruce. A 20 cm reference DBH was used


TOTAL TREE HEIGHT (M)
0
5
10
15
20
25
30
35
DIAMETER AT BREAST HEIGHT (CM)
0
5
10
15
20
25
30
35
40
45
50
SPI=20
SPI=16
SPI=12
SPI= 8

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Chapter 5

Individual Tree Height Prediction Models[1]

5.1 Introduction

Individual tree height prediction models are routinely required in practical management and silvicultural research work (Meyer 1940, Arabatzis and Burkhart 1992). The estimation of tree volume and site index, the description of stands and their development over time, as well as the estimation of growth by stand projection methods rely heavily on accurate height prediction models (Curtis 1967). Larsen and Hann (1987) also suggested that height prediction models may be used to indirectly predict height growth if the actual measurements of height growth are not available.

The most widely used height prediction models are the so-called "height-diameter" equations, which predict tree height as a function of tree diameter at breast height (DBH) (Curtis 1967, Ek 1973, Kozak and Yang 1978, Larsen and Hann 1987, Arabatzis and Burkhart 1992). Many growth and yield projection systems use such equations to predict tree heights (Burkhart et al. 1972, Curtis et al. 1981, Wykoff et al. 1982, Burk and Burkhart 1984, Arney 1985). The predictive capability of the height-diameter equation may be improved if additional tree and stand variables are added into the equation, such as the inclusion of tree age (Curtis 1967), site index (Wang and Hann 1988), and stand basal area and site index (Larsen and Hann 1987). Ek et al. (1984), and Van Deusen and Biging (1985) also incorporated additional stand variables into height-diameter equations to

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[1] A version of this chapter has been submitted for publication in Forest Science. Huang, S., and Titus, S. J. 1992. An age-independent individual tree height prediction model for boreal spruce-aspen stands in Alberta.

provide better height predictions.

This study developed an individual tree height prediction model for white spruce (*Picea glauca* (Moench) Voss) and trembling aspen (*Populus tremuloides* Michx.) grown in boreal mixed-species stands in Alberta. The approach was to first select an appropriate height-diameter model as the base function, and then incorporate other tree and stand level variables such as tree basal area, stand density, species composition, and site productivity using the parameter prediction method (Clutter et al. 1983). The model is age-independent in that age is not explicitly involved in height prediction. Age is not included in height prediction because:

1). The boreal forests in Alberta commonly have a mixed-species composition with irregular age structures. Tree and stand ages in such mixtures have very limited meaning and studies on growth and yield for mixed-species stands rarely involve the explicit use of age as an input variable (Adams and Ek 1974, Lynch and Moser 1986, Wykoff et al. 1982).

2). On average, DBH generally accounts for 70-80% of the height variation from the height-diameter prediction equations for major Alberta tree species (Huang et al. 1992). Using the same data, tree age generally accounts for less than 20% of the height variation from the height-age equation. The strong correlation between height and diameter has been noted in many other studies (Curtis 1967, Larsen and Hann 1987, Wang and Hann 1988, Kozak and Yang 1978, Arabatzis and Burkhart 1992).

3). Age is the most expensive, time consuming, and difficult variable to measure. Because of this, many forest inventories only record tree ages for a very small subsample, resulting in less representation and large variation in age measurements. Approximately 2-3% of the trees from permanent sample plots in Alberta have age recorded.

The height prediction model was fitted using nonlinear least squares regression. If unequal error variance was evident from analysis of residuals, weighted regression was applied to achieve uniform error variance. The model was also tested on an independent data set representing the population on which the model was to be used. Results and discussion of the model's performance and its biological implications are presented here, along with suggestions and recommendations for the most appropriate use of the model.

5.2 Model development

The base function

Richards (1959) derived the following sigmoidal function based on Von Bertalanffy's quantitative laws in organisms:

$$(5.1) \qquad y=a(1-e^{-bt})^{c}$$

where y is the total living biomass, t is the time, a is the asymptote, b is the rate parameter, and c is the shape parameter. The total living biomass y can refer to size or weight in animal growth, or to basal area, volume, diameter, and height in tree growth.

Equation (5.1) is most commonly known as the Chapman-Richards function in forestry. It has been shown on numerous occasions to be very flexible and has been used extensively in growth and yield studies for describing height-age, diameter-age, basal area-age, and volume-age relationships (Pienaar and Turnbull 1973, Burkhart and Tennent 1977, Clutter et al. 1983, Somers and Farrar 1991). The function was derived from basic biological considerations and the parameters of the function have meaningful biological significance.

Because of its flexibility, the Chapman-Richards function has been extended beyond its original application and used to describe other empirical growth phenomena of forest trees and stands in which tree and stand ages were not explicitly involved, such as the basal area-height and height-diameter relationships (Harrison and Daniels 1988, Huang et al. 1992). Arney (1985) also adopted the basic Chapman-Richards form for modelling coastal Douglas-fir diameter increment. In these cases the biological interpretation of the Chapman-Richards function may have limited significance, but the mathematical properties and the physically meaningful parameters of the function remain.

The Chapman-Richards based height-diameter model takes the form

$$(5.2) \qquad H=1.3+a(1-e^{-bDBH})^{c}$$

where H is the total tree height (m) to be predicted, DBH is the diameter (cm) of the tree at breast height, 1.3 is a constant used to account for the use of DBH (measured at 1.3 metres above the ground), and a, b, c are parameters to be estimated. Equation (5.2) has been shown to be one of the most accurate height prediction models for major Alberta tree species (Huang et al. 1992), and was selected as the base function for height prediction in this analysis.

Individual tree height can be most appropriately estimated from equation (5.2) for forest areas that have similar stand conditions. If the variation in stand density and site productivity has significant effects on tree height, they may be incorporated into the equation to provide better height prediction. Generally, a tree of a given DBH is expected to be taller on better sites since height growth should be greater on better sites. It is also accepted that, for a large number of tree species grown in even-aged pure species stands, the average height of the dominant and codominant trees is relatively unaffected by a wide range of stand density. However, the average height of *all* trees in the stand may be affected by stand density, particularly for stands with mixed-species composition and uneven-aged structure commonly found in boreal forest regions of Alberta. It has been observed in boreal mixed-species stands that higher stand density tends to result in taller trees and lower stand density tends to result in shorter trees. The height growth of a tree may also be affected by species composition in boreal mixed-species stands. Definitions and descriptions for stand variables that are used in the height prediction model are described in the following sections.

Stand density

Stand density is a quantitative measure of the degree of crowding within a forested area. It is often expressed either in terms of absolute values such as the number of trees per hectare, basal area per hectare, volume per hectare, percent crown cover, or in relative values such as relative density, stand density index, tree-area ratio, crown competition factor, and spacing index. Detailed descriptions for these measures can be found in Spurr (1952), Clutter et al. (1983), and Davis and Johnson (1986). Most of these density measures are derived and used for even-aged pure species stands. Bredenkamp and Burkhart (1990) provided a comparison for most of the stand density measures.

The mixed-species stand density measure used in this study is the total basal area per hectare (ha) for all species combined. The use of basal area per hectare as a simple and objective measure of stand density has been widely accepted (Clutter et al. 1983), and for reasons described by Spurr (1952), basal area per ha of all species in the stand should be particularly suitable for mixed-species stands with irregular age structures.

Species composition

It is typical that in the early development stage of a mixed white spruce-aspen stand, the shade intolerant aspen has the competitive advantage over white spruce and exhibits faster early height growth and as a result, tends to rapidly establish dominance on the site by occupying the upper layer of the canopy. Shade tolerant white spruce is often established a short but distinct time after the overstorey aspen and exhibits slow juvenile height growth. As time progresses, the competitiveness of aspen is reduced relative to that of white spruce as individual aspen trees start to die at 60 to 80 years of age, giving dominance gradually to the more shade tolerant and long lived white spruce. This dynamic process produces changes both in terms of whole stand density and proportions of the species composing the stand. An appropriate species composition measure is defined to reflect these changes:

$$SC_{sp} = \frac{BASUM_{sp}}{BASUM} \tag{5.3}$$

where SC_{sp} is the species composition of the target species, $BASUM_{sp}$ is basal area per hectare (m^2/ha) for the target species, and BASUM is the total basal area per hectare for all species combined in the stand. The species composition as defined in (5.3) is a number between zero and one, with zero indicating absence of the target species and one indicating a pure species stand of the target species.

At the current state of knowledge concerning the vast complexity of mixed-species stands, the species composition measure as expressed by (5.3) is at best a crude approximation to the "true ratios" as Assmann (1970) termed the species composition in mixed-species stands. Growth and yield models for mixed-species stands concentrate on the species component rather than the whole stand, so explicit species composition measures are rarely seen (Lynch and Moser 1986, Kelty 1989). Assmann (1970) suggested that the true ratio of the species in a mixed stand may depend on the species' relative efficiencies for utilizing the available resources and the so-called natural growth rhythms of the species. Kelty (1989) used the ratio of the number of target species trees over the total number of trees in the stand to compare the differences in species composition for New England hemlock/hardwood stands. The use of a basal area ratio should generally be more advantageous over a tree number ratio because basal area combines number and size of trees.

Site productivity

Site index and age are not used in this study because stands in Alberta commonly have a mixed species composition with an irregular age structure. Both site index and age are often meaningless concepts in this situation (Wykoff 1990). The site productivity index (SPI) based on the height-diameter relationship of dominant and codominant trees (Huang 1992, Huang and Titus 1992) was used as the measure of site productivity for uneven-aged and mixed-species stands in Alberta.

In addition to the three variables representing stand density, species composition, and site productivity, average size of all trees in the stand and the basal area of the tree were also found to significantly affect the height growth of a tree. Using the parameter prediction method (Clutter et al. 1983) and equation (5.2), the final height prediction model expressing tree height as a function of tree DBH, tree basal area, stand density, species composition, site productivity, and average DBH was found to be

$$(5.4)\qquad H=1.3+a(1-e^{-bDBH})^{c}$$

$$a=a_1(1-e^{-a_2BASUM})+a_3SC_{sp}+a_4SPI+a_5BA+a_6AVED$$

$$b=a_7$$

$$c=a_8+a_9DBH/AVED+a_{10}SPI$$

where: H is the tree height to be predicted (m),

BASUM is the basal area per ha for all species in the stand (m^2),

BA is the basal area of the tree (cm^2),

SC_{sp} is the species composition as defined in (5.3), SPI is the site productivity index of the species in mixed-species stands (m),

AVED is the average diameter for all species in the stand (cm),

DBH is the tree diameter at breast height (cm).

Model (5.4) was identified by first plotting tree height versus each explanatory variable and examining the possible linear or nonlinear relationship between them, and then arranging different variable combinations in different parameter prediction equations until the most reasonable residual plot from preliminary nonlinear regressions has been obtained.

5.3 The data

Data from permanent sample plots (PSP) used in this analysis were provided by the Alberta Forest Service. The data were collected over the last three decades and the PSPs were randomly located throughout the inventory areas of the province to provide representative information for a variety of densities, heights, species composition, stand structures, ages, and site conditions. A detailed description of how the data were collected and recorded can be found in the Permanent Sample Plots: Field Procedures Manual (AFS 1990).

For this study, the original PSP data were summarized to provide additional variables such as the number of trees per hectare (ha), basal area per ha, average height, and average diameter, both for all species combined and by individual species in the stand. Summarized data from 164 PSPs were used in this analysis. To remove the possible serial correlation among repeated measurements of the same plot, only the initial measurements were actually retained. Live white spruce and aspen trees with measurements for both DBH and height were used in fitting of the height prediction model, giving a total number of 1612 white spruce and 1138 aspen trees selected from the 164 PSPs. Variables from the selected trees were matched with the whole stand and species summary statistics, so that the height prediction model could be directly fitted on the matched data sets. Descriptive statistics including the mean, minimum, maximum, and standard deviation for tree DBH and tree height by species; total number of trees per ha, average diameter, basal area per ha, and average height, both for the whole stand and by species, are attached in Tables 5-1 and 5-2. Summary statistics for species composition are also given in Tables 5-1 and 5-2. Approximately 80% of trees for each species (1304 for white spruce and 898 for aspen) were randomly selected for model fitting and the remaining 20% (308 for white spruce and 240 for aspen) for model testing.

5.4 Analysis

Preliminary nonlinear least squares fits of the height prediction model (5.4) for white spruce and aspen were accomplished using the PROC MODEL procedure on SAS/ETS software (SAS Institute Inc. 1988). The Gauss-Newton iterative method using the Taylor series expansion as described in Gallant (1987) was applied in model fitting. To ensure the solution was global rather than local least squares estimates, multiple starting values of the parameters were provided for the fits.

Error structure of the model

Residual analyses based on the preliminary nonlinear least squares fits were performed to detect possible model inadequacies and examine the validity of regression assumptions. Because the residuals r_i (calculated as the difference between the observed height and the predicted height) are intrinsically not independent and do not have common variance (Rawlings 1988 p. 249), studentized residuals are used instead to account for the unequal variance problem. The dependency effect among the residuals r_i, according to Neter et al. (1990 p. 116), is relatively unimportant and can be ignored because the sample sizes are large in comparison to the number of parameters appeared in the height prediction model.

Studentized residuals are the scaled version of residuals that are obtained by dividing each residual by its standard error

$$(5.5) \qquad r_i^* = \frac{r_i}{\sqrt{MSE(1-h_{ii})}}$$

where r_i^* is the studentized residual, r_i is the residual, *MSE* is the mean squared error computed by dividing the error sum of squares by error degrees-of-freedom, and h_{ii} is the *i*th diagonal element of the nonlinear "hat matrix" $F(F'F)^{-1}F'$ as described in Gallant (1987) and Rawlings (1988). For a correctly identified model, when the assumptions of the regression analysis are met, the studentized residuals have zero mean and constant variance, and the plot of studentized residuals against the predicted values of the dependent variable will show a homogeneous band. For that reason, the use of studentized residuals has been recommended by Draper and Smith (1981), Montgomery and Peck (1982), and Rawlings (1988).

Figures 5-1 and 5-2 show the plots of studentized residuals against the predicted height for white spruce and aspen respectively. It is clear that the plot for aspen (Figure 5-2) displays an homogeneous band and the zero studentized residuals across the centre of the band, indicating that the height prediction model for aspen is appropriately identified and fitted. On the other hand, the plot for white spruce (Figure 5-1) shows zero studentized residuals across the centre of the band but the band itself displays a clear trend of increasing error variance, indicating that the height prediction model for white spruce is appropriately identified but may not be appropriately fitted because of unequal error variance. The significance of the heteroskedasticity can be tested using the Goldfeld-Quandt test as described in Judge et al. (1988) through a four step procedure:

1). Order the predicted spruce height values from the unweighted nonlinear least squares fit in an ascending sequence according to increasing error variance;

2). Omit r=104 central observations;

3). Perform two separate nonlinear least squares fits on the first 600 observations and the remaining 600 observations. This gives the mean squared errors $MSE_1=3.27667$ and $MSE_2=4.11489$ for the first and the second regressions respectively;

4). Calculate the test statistic $\lambda = MSE_2/MSE_1=1.2558$.

Under the null hypothesis of homoskedasticity, λ has an F-distribution with [(1304-104-2×10)/2, (1304-104-2×10)/2] degrees of freedom. At a 5% significance level the critical value for λ from the F-distribution is $\lambda_{crit.} = 1.00$, and so, using the Goldfeld-Quandt test, the null hypothesis is rejected and we conclude that heteroskedasticity exists for the error terms of the white spruce height prediction model.

Several alternative assumptions about the nature of the heteroskedasticity were then proposed and examined for spruce data:

(1). The variance of error is a linear function of the predicted height.

$$(5.6) \qquad \hat{\sigma}_i^2=\varepsilon_i^2=\alpha_1+\alpha_2\hat{H}_i$$

where $\hat{\sigma}_i^2$ is the estimated variance of error, and ε_i^2 is the estimated squared residual for the *i*th tree (i=1, 2, ..., n) from the height prediction model fitted by unweighted nonlinear least squares, $\hat{H}_i$ is the predicted height for the *i*th tree based on the unweighted model. The estimated coefficients are $\alpha_1=1.14335$ and $\alpha_2=0.118145$, with the MSE=36.88942 and $R^2=0.0108$.

(2). The variance of the error is an exponential function of the predicted height.

$$(5.7) \qquad \hat{\sigma}_i^2=\varepsilon_i^2=\alpha_1 e^{\alpha_2\hat{H}_i}$$

The estimated coefficients are $\alpha_1=1.7461$ and $\alpha_2=0.03391$, with the MSE=36.8804 and

R^2=0.0111.

(3). The variance of the error is a power function of the predicted height.

$$(5.8)\qquad \hat{\sigma}_i^2=\varepsilon_i^2=\alpha_1\hat{H}^{\alpha_2}$$

The estimated coefficients are α_1=0.4931 and α_2=0.6582, with the *MSE*=36.8867 and R^2=0.0109.

(4). The variance of the error is some function of the explanatory variables.

Scatter plots of the predicted height versus each explanatory variable in the height prediction model were first examined to see any possible correlation between the error variances and the variables. Several alternative linear and nonlinear equations expressing the variance of the error as some function of the explanatory variables were then fitted. The multiple linear regression equation was chosen among others

$$(5.9)\qquad \hat{\sigma}_i^2=\varepsilon_i^2=\alpha_1+\alpha_2 DBH_i+\alpha_3\frac{DBH_i}{AVED}$$

where DBH_i is the diameter at breast height of the *i*th tree and AVED is the average diameter of all trees in the stand. The estimated coefficients are α_1=1.4511, α_2=0.03270, and α_3=0.9499, with the MSE=36.4573 and R^2=0.0230.

(5). The variance of error is directly proportional to DBH.

$$(5.10)\qquad \hat{\sigma}_i^2=\varepsilon_i^2=DBH_i$$

This particular situation assumes the error variances increase with increasing DBH of the trees.

Weighted nonlinear least squares techniques, with the weights chosen inversely proportional to the five alternative error variances, were then applied to the white spruce data. The plots of studentized residuals against the predicted height were examined in each case for any possible trend showing heteroskedasticity. The first three assumptions of the error variances resulted in studentized residual plots that were still indicating a trend of increasing error variances. However, the last two error variance assumptions gave satisfactory studentized residual plots, with the error specification in (5) showing the most desirable result. The plot of studentized residuals against the predicted height based on this error specification is shown in Figure 5-3. It is clear that the plot displays an approximately homogenous band of the error variance. Accordingly the weighting factor $w_i = 1/DBH_i$ was chosen to be used in fitting of the weighted white spruce height prediction model.

Final fits based on the model's error structure

Results of the final fits of the height prediction model (5-4) for white spruce and aspen are presented in Table 5-3, showing the unweighted nonlinear least squares estimates of the parameters for aspen and the weighted nonlinear least squares estimates ($w_i=1/DBH_i$) for white spruce. Asymptotic standard errors (Std. err.), *t*-statistics, and *p*-values of the parameters, as well as the model's root mean squares error (RMSE) and R^2 are displayed in Table 5-3. Both R^2 values are calculated according to

$$(5.11) \qquad R^2 = 1 - \frac{\Sigma_1^n (H_i - \hat{H}_i)^2}{\Sigma_1^n (H_i - \bar{H})^2}$$

where H_i is the observed and $\hat{H}_i$ is the predicted height for the *i*th tree (i=1, 2, ..., n), and $\bar{H}$ is the observed average tree height. The large number of observations in this study allows us to interpret the statistics recorded in Table 5-3 as they are in linear regression analysis.

Model validation

The independent test data sets were used for additional evaluation of model performance. Since the spruce model was fitted using weighted regression and aspen was unweighted, different evaluation procedures were required for aspen and spruce:

For aspen, the actual height values from the testing data were compared to these values predicted by model (5-4) using the estimated coefficients in Table 5-3. The bias of the prediction was obtained by subtracting the predicted height from the actual height. The mean and the standard deviation of the prediction bias (δ) are obtained as 0.031647 and 1.51921 respectively, and the standard error of the estimated mean bias is computed as 0.098065. A *t*-test of the null hypothesis that the mean prediction bias was zero was conducted according to the method described by Rawlings (1988). The calculated t = 0.3227, which, with 239 degrees of freedom, is not significant at α = 0.05, and we infer that the mean prediction bias is not significantly different from zero. The mean squared error of prediction (MSEP) for aspen can be obtained as

$$(5.12) \qquad MSEP=\frac{(n-1)\,s_{\delta}^{2}}{n}+\bar{\delta}^{2}=2.29937$$

where the squared prediction bias term $\bar{\delta}^2$ contributes 0.04% of MSEP. The square root of MSEP gives 1.51637, and this is approximately 7.83% of the average observed height.

For white spruce, the weighted actual height values from the testing data were compared to the weighted predicted height values by model (5-4) with the estimated coefficients in Table 5-3. The bias of the prediction was obtained by subtracting the weighted predicted height from the weighted actual height. The mean and the standard deviation of the prediction bias are obtained as 0.039472 and 0.38905 respectively, and the standard error of the estimated mean bias is 0.022168. A *t*-test of the hypothesis that the mean prediction bias was zero gives t = 1.7806, which, with 307 degrees of freedom, is not significant at α = 0.05, and we infer that the mean prediction bias is not significantly different from zero. Similarly, the MSEP for white spruce can be obtained as 0.15243, with the squared prediction bias term δ^2 contributing 1.02% of MSEP. The square root of MSEP gives 0.39042. This value divided by the weighted average observed height (4.19) from the testing data yields 0.093179, indicating an approximately 9.32% average error in white spruce height predictions.

5.5 Discussion

The height prediction model as expressed by (5-4) provides individual tree height predictions for two important tree species in Alberta. It is apparent from the statistics in Table 5-3 that the model is well fitted for both spruce and aspen data. This is also clearly evident from the studentized residual plots shown in Figures 5-2 and 5-3. Except for parameter a_{10} for white spruce, asymptotic *t*-statistics for the parameters of the model are all significant at $\alpha=0.05$ level. The percent of the height variation explained by the model are high (91.92% for white spruce and 90.87% for aspen). The model was developed on a sigmoid base function that can assume various shapes with different parameter values and produce satisfactory curves under most circumstances. Because of the mathematical properties of the base function and the use of the parameter prediction method, all curves produced by the model assume biologically reasonable shapes that not only closely mimic the biological growth process by making accurate height predictions within the range of the observed data, but also provide reasonable and realistic height predictions in cases where the model is extrapolated beyond the range of the original data.

The height prediction model reflects some commonly held beliefs and interesting facts about height growth in mixed spruce-aspen stands. The positive coefficients (a_1, a_2) for the stand density component indicate that increasing stand density has a positive effect on both white spruce and aspen height growth. Given that other factors are approximately the same, the model shows that tree height steadily rises to a limiting value as stand density increases; that is, dense stands have taller trees. This suggests that both white spruce and aspen respond to density stress by growing taller to capture enough light to survive and grow. Apart from the fact that competition in natural stands of mixed white spruce and aspen is largely due to the different natural growth rhythms of the species, this may further indicate that competition in mixed-species stands is also mainly due to the competition for light, as suggested by Weiner and Thomas (1986) for plant monocultures.

For white spruce, the species composition coefficient (a_3) is positive and for aspen it is negative. The positive species composition coefficient for spruce indicates that 1) for two mixed spruce-aspen stands growing under similar conditions, the one with more spruce will also have taller spruce trees. This could be a result of relatively stronger competitiveness for spruce if they occupy more growing space in the stand; 2) over time spruce height increases as the basal area proportion of spruce increases. This can be directly explained by the natural growth rhythms of spruce in mixed spruce-aspen stands, as the juvenile height growth of spruce is slow and the species composition for spruce is generally lower than that of aspen. As time progresses, the competitiveness of spruce is increased relative that of aspen, so that the basal area proportion for spruce increases and the trees grow taller. The spruce height prediction model reflects this process.

For aspen growing in mixed spruce-aspen stands, the negative species composition effect indicates that 1) for two stands growing under similar conditions, the one with more aspen has shorter aspen trees. The reason for this is unclear, but a possible explanation may be that under many circumstances aspen height growth is directly related to the interspecific competition with spruce, and the increasing presence of spruce not only compete with but also stimulate and change the aspen height growth pattern. An example for this explanation is the frequently observed phenomena in which the mixed-species stand may be largely dominated by spruce, but the few remaining aspen trees are healthy, strong, and tall; 2) Over time aspen height increases as the basal area proportion for aspen in the stand decreases. This is also directly compatible with the biological growth process of the white spruce-aspen stands, as the competitiveness of aspen is reduced relative to that of white spruce by reducing its basal area portion in the stands while growing taller over time.

Measuring site productivity and incorporating it into a model is a rather complicated problem, especially for mixed-species stands with uneven-aged structure. This problem is precisely the one that led researchers like McLintock and Bickford (1957), Stout and Shumway (1982), Verbyla and Fisher (1989), and Wykoff (1990) to develop alternatives to traditional site index in attempting to provide accurate and realistic site productivity measures. Based on the estimated coefficient values for a_4 ($a_4 > 0$) and a_{10} ($a_{10} < 0$ means a positive effect on H because a_8, $a_9 > 0$, $0 < (1-e^{-a_7 DBH}) < 1$, and $c > 0$), which reflect the effect of site productivity on height predictions, it is clear that better sites support taller trees. This is true both for white spruce and for aspen.

Because the height prediction model is based on a function that relates tree height to other tree and stand variables, but does not involve individual tree or stand age explicitly, the model can be applied in any stand. Input variables of the model are simple tree and stand variables that are readily obtainable from ordinary inventories, and are compatible with the data-collection process in Alberta. Potential users of the model with limited information or alternative measures for some of the variables appearing in the model may consider re-fitting other forms of the model by replacing the variables that are not available, or by dropping the variables if the measurement cost is too high relative to the amount of additional variation explained by adding such variables to the model. In either case, model development is facilitated by using a base function and the parameter prediction method.

5.6 Summary

This study presents an individual tree height prediction model for white spruce

(*Picea glauca* (Moench) Voss) and trembling aspen (*Populus tremuloides* Michx.) grown in boreal mixed-species stands in Alberta. The model is based on a three-parameter Chapman Richards function fitted to data from 164 permanent sample plots using the parameter prediction method. It is age-independent and expresses tree height as a function of tree diameter, tree basal area, stand density, species composition, site productivity, and stand average diameter. This height prediction model was fitted by weighted nonlinear regression for spruce and unweighted nonlinear regression for aspen. Results show that 1) almost all estimates of parameters are significant at $\alpha=0.05$, 2) plots of studentized residuals against predicted heights show no consistent underestimate or overestimate for tree heights, and 3) model R^2 values are high (0.9192 for white spruce and 0.9087 for aspen). The model was also tested on an independent data set representing the population on which the model is to be used. Results show that the average prediction biases are not significant at $\alpha=0.05$ for both species, indicating that the model appropriately describes the data and performs well when predictions are made.

Table 5-1. Tree and stand summary statistics — white spruce

	Mean	Minimum	Maximum	Std. dev.
Tree DBH (cm)	27.06	2.90	63.30	10.29
Tree height (m)	21.75	3.20	37.90	5.94
Number of trees/ha — all species	1244	148	5580	754
Average DBH (cm) — all species	19.72	5.30	39.70	5.60
Basal area (m^2/ha) — all species	39.09	11.55	83.87	9.39
Average height (m) — all species	21.22	7.90	31.70	3.88
Species number of trees/ha	753	5	4914	673
Species average diameter (cm)	20.60	5.70	54.90	6.47
Species basal area (m^2/ha)	24.63	0.20	56.65	11.98
Species average height (m)	21.57	3.60	31.70	4.25
Site productivity index (m)	16.14	8.22	21.28	2.39
Species composition	0.62	0.0062	1.00	0.26

Table 5-2. Tree and stand summary statistics – aspen

	Mean	Minimum	Maximum	Std. dev.
Tree DBH (cm)	22.49	5.30	56.40	11.06
Tree height (m)	18.99	5.80	32.90	5.33
Number of trees/ha – all species	1522	222	5000	766
Average DBH (cm) – all species	16.76	4.60	39.70	6.55
Basal area (m^2/ha) – all species	31.43	11.97	87.91	11.18
Average height (m) – all species	17.96	7.30	28.80	4.65
Species number of trees/ha	991	10	3515	816
Species average diameter (cm)	19.56	2.80	46.10	9.06
Species basal area (m^2/ha)	19.40	0.47	53.33	9.06
Species average height (m)	18.75	6.30	30.60	5.13
Site productivity index (m)	18.73	8.46	24.52	3.40
Species composition	0.65	0.016	0.99	0.27

Table 5-3. Fit statistics for white spruce and aspen height prediction models

	Parameter	Estimate	Std. err.	t-statistic	p-value	RMSE	R^2
White spruce	a_1	10.683305	1.25170	8.54	0.0001	0.38348	0.9192
	a_2	0.067329	0.0073446	9.17	0.0001		
	a_3	1.221232	0.28764	4.25	0.0051		
	a_4	0.774104	0.04864	15.92	0.0001		
	a_5	0.00350118	0.0004710	7.43	0.0001		
	a_6	0.061654	0.01833	3.36	0.0008		
	a_7	0.090025	0.0078286	11.50	0.0001		
	a_8	2.107447	0.21471	9.82	0.0001		
	a_9	0.277418	0.09439	2.94	0.0033		
	a_{10}	-0.020195	0.01129	-1.79	0.0739		
Aspen	a_1	15.121533	1.58516	9.54	0.0001	1.60907	0.9087
	a_2	0.102744	0.0087079	11.80	0.0001		
	a_3	-1.018396	0.26305	-3.87	0.0001		
	a_4	0.450380	0.04171	10.80	0.0001		
	a_5	0.00162940	0.0005345	3.05	0.0024		
	a_6	0.085430	0.01892	4.52	0.0001		
	a_7	0.101073	0.01157	8.73	0.0001		
	a_8	2.246	0.27676	8.12	0.0001		
	a_9	0.843400	0.19368	4.35	0.0001		
	a_{10}	-0.066804	0.01282	-5.21	0.0001		

Note: white spruce statistics are obtained from weighted nonlinear least squares ($w_i=1/DBH_i$), aspen statistics are obtained from unweighted nonlinear least squares. The model fitted is equation (5-4).

Figure 5-1. Plot of studentized residuals against the predicted height for white spruce from unweighted nonlinear least squares.

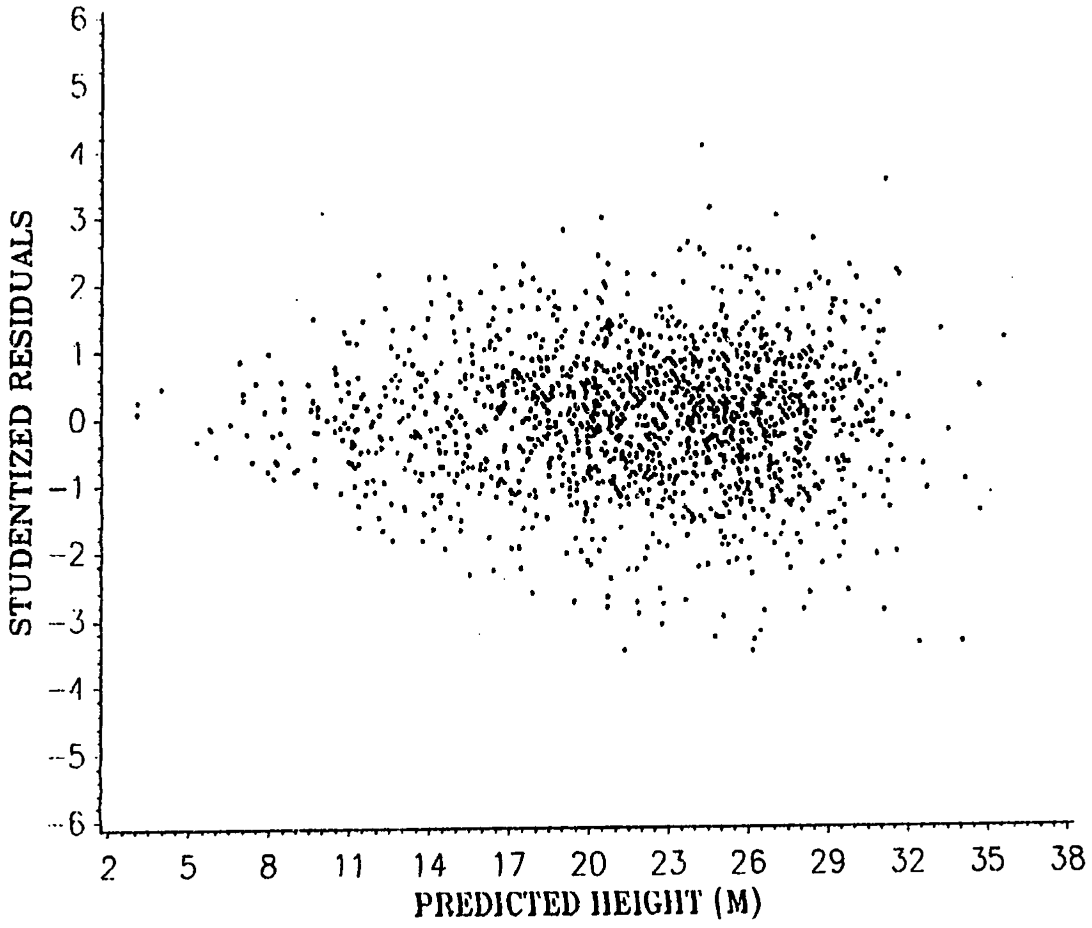

STUDENTIZED RESIDUALS
6
5
4
3
2
1
0
−1
−2
−3
−4
−5
−6
2
5
8
11
14
17
20
23
26
29
32
35
38
PREDICTED HEIGHT (M)

Figure 5-1.

Figure 5-2. Plot of studentized residuals against the predicted height for aspen from unweighted nonlinear least squares.

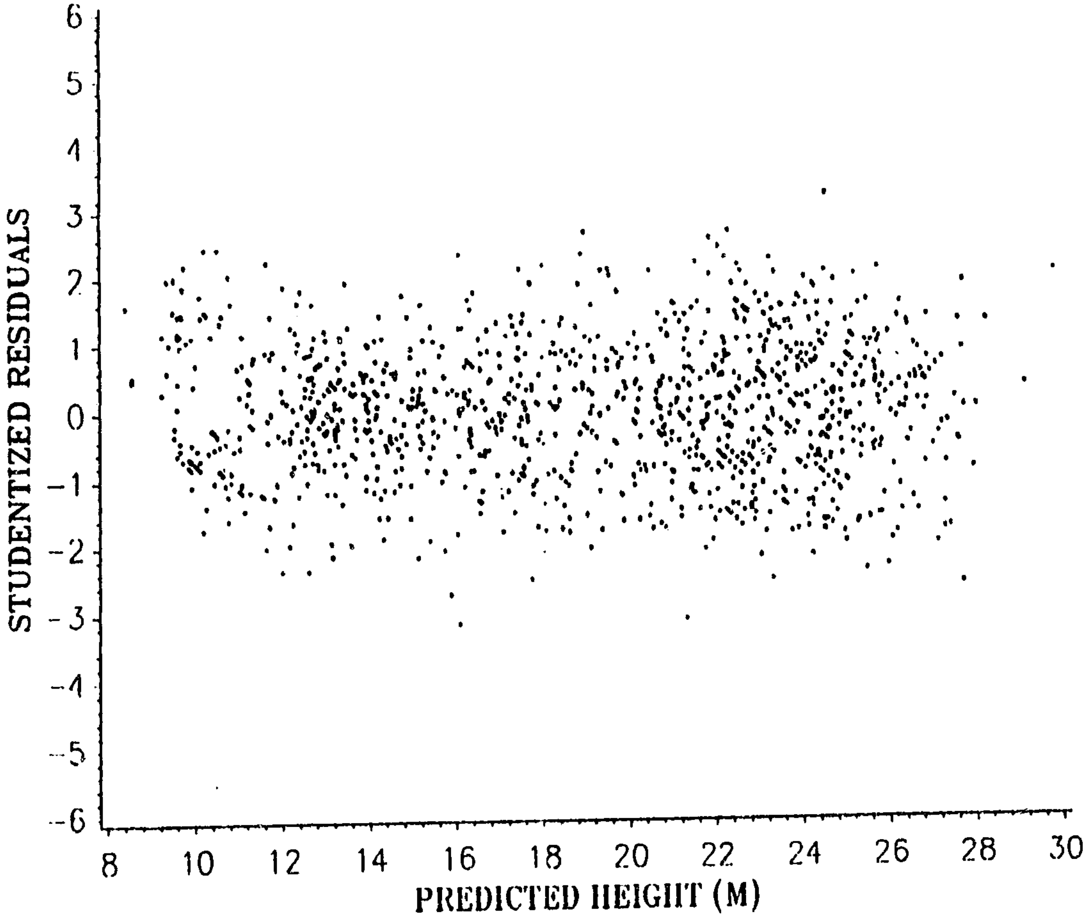
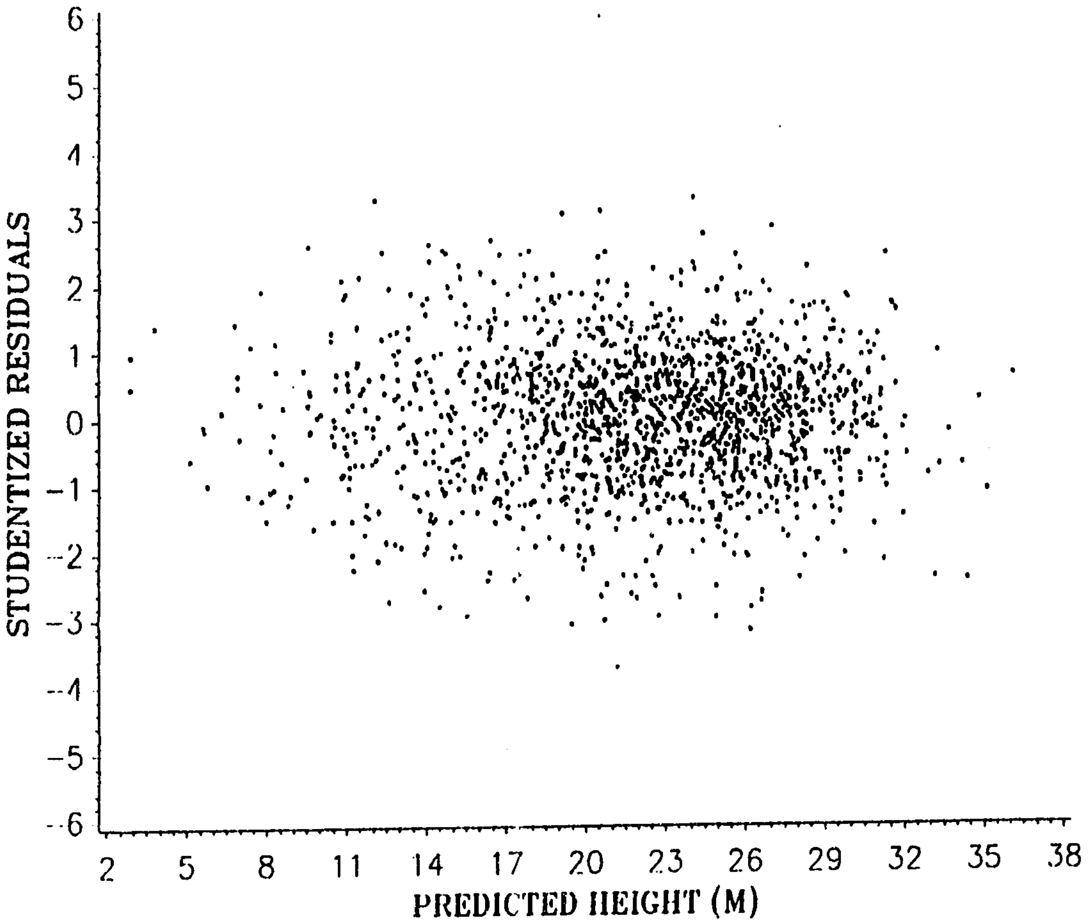

STUDENTIZED RESIDUALS
6
5
4
3
2
1
0
-1
-2
-3
-4
-5
-6
8
10
12
14
16
18
20
22
24
26
28
30
PREDICTED HEIGHT (M)

Figure 5-2.

Figure 5-3. Plot of studentized residuals against the predicted height for white spruce from weighted nonlinear least squares ($w_i = 1/DBH_i$).


STUDENTIZED RESIDUALS
6
5
4
3
2
1
0
-1
-2
-3
-4
-5
-6
2
5
8
11
14
17
20
23
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38
PREDICTED HEIGHT (M)

Figure 5-3.

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Chapter 6

Individual Tree Diameter Increment Model for White Spruce[1]

6.1 Introduction

Individual tree diameter increment models predict periodic diameter growth of each individual tree in the stand. They are one of most basic components for individual tree based growth and yield models such as PTAEDA (Daniels and Burkhart 1975), SPS (Arney 1985), ORGANON (Hester et al. 1989), FOREST (Ek and Monserud 1974), STEMS (Belcher et al. 1982), and PROGNOSIS (Stage 1973, Wykoff et al. 1982, Wykoff 1990). Two different modelling approaches for diameter increment have been commonly used. The growth-potential independent approach develops a regression model that directly relates diameter increment to tree and stand characteristics, including the competitiveness of the tree in the stand (Lemon and Schumacher 1962, Beck 1974, Wykoff et al. 1982, Martin and Ek 1984); The growth-potential dependent approach first selects a function that defines the potential diameter growth of competition-free trees, then a competitive adjustment factor (the modifier) is used to reduce this potential (Daniels and Burkhart 1975, Ek and Monserud 1974). This second approach often involves selecting a base potential diameter increment function and emphasises how the modifier is modelled and how it affects the potential diameter increment. Hahn and Leary (1979), Belcher et al. (1982), Shifley and Brand (1984), and Arney (1985) all implemented this approach. However, Wykoff (1990) regarded the differences in approaches are mostly "semantic" because either approach can produce

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[1] Similar types of diameter increment models for other tree species (aspen, lodgepole pine, black spruce) were also developed but were not reported here in order to facilitate the description of this Chapter.

acceptable predictions. Choice between approaches may simply be a matter of preference or convenience.

Both diameter increment and basal area increment can be used as dependent variables in diameter/basal area increment models. While most researchers choose to use diameter increment, Opie (1968) and Moore et al. (1973) used basal area increment. Krumland (1982) also used basal area increment because basal area was found to be more directly related to volume increment. However, West (1980) reported that the precision of estimates of future diameter is virtually the same, whether diameter or basal area increment equations are used. In a more recent study, Wykoff (1990) also suggested that diameter/basal area increment may be used interchangeably because either variable can be readily transformed into the other.

Diameter increment models can also be derived by taking the first derivative of the cumulative diameter prediction function. For example, Daniels and Burkhart (1975) developed a linear equation expressing the maximum diameter attainable as $D_0=a+bH+cA$, where D_0=open-grown tree diameter at breast height (DBH) (inches), H=total tree height (feet), A=age from seed (years), and a, b, c are coefficients. The first difference of this equation with respect to age gives the maximum annual potential diameter increment $PDIN=bHIN+c$, where PDIN=potential diameter increment (inches), and HIN=observed height increment (feet). Through this process, potential diameter increment is expressed as a function of height increment. The use of this method may simplify the diameter increment model if the height increment is readily available and if the cumulative diameter equation is not overly complicated.

This study developed an individual tree diameter increment model for white spruce (*Picea glauca* (Moench) Voss) grown in boreal mixed-species stands in Alberta. Periodic diameter increment was modelled as a function of tree diameter, tree height, competition, site productivity, and species composition.

6.2 Model development

The base function

Richards (1959) derived a flexible sigmoidal function based on Von Bertalanffy's (1957) quantitative laws in metabolism and growth for organisms:

$$\frac{dy}{dx}=\eta y^{m}-\kappa y \tag{6.1}$$

where the growth rate *dy/dx* of an organism with the total living biomass y is expressed as the difference between the metabolic forces of anabolic rate (constructive metabolism) and catabolic rate (destructive metabolism), x is time, η and κ are constants of anabolism and catabolism respectively, and m is the allometric parameter. The total living biomass y can refer to size or weight in animal growth, or to basal area, volume, diameter, and height in tree growth.

According to Von Bertalanffy (1957), the catabolic component A (defined as $A=-\kappa y$) in equation (6.1) is the continuous loss of building material as the living organism grows over time. Applying this concept to the growth of a tree, this loss may be caused by many physiological processes occurring during growth in which the building material is consumed for the physical enlargement of the tree, such as diameter and height increment, root and crown expansion. Concentrating on the diameter increment only, if the rate at which the catabolic component consumed for diameter increment is assumed to be proportional to the amount of the catabolism available for diameter increment, and the differential equation that governs this process, with respect to the present DBH, D, can be written as:

$$\frac{dA(D)}{dD}=-\theta_1 A(D) \tag{6.2}$$

where θ_1 is the unknown proportional parameter, and A(D) is the amount of the catabolism available for diameter increment. The minus sign on the right-hand side of (6.2) is used to represent the catabolism lost for diameter increment.

Looking from a different prospective, this component, $\theta_1 A(D)$, becomes the constructive metabolic force for diameter increment of the tree. Naturally, the rate and amount of this component consumed will depend on the destructive metabolism used for the diameter increment. Assuming the growth rate of diameter increment (D_I) with respect to the current diameter, dD_I/dD, can be expressed as the difference between the available constructive metabolism for diameter increment and the destructive metabolism which is taken to be proportional to the current diameter increment, the governing differential equation for the growth rate of diameter increment can be written as

$$(6.3) \qquad \frac{dD_I}{dD} = \theta_1 A(D) - \theta_2 D_I$$

where D_I is the diameter increment, D is the current diameter, θ_1 and θ_2 are unknown parameters to be estimated. Note that in equations (6.2) and (6.3), D is serving as a time (such as tree age) surrogate. Equation (6.3) actually expresses the rate of diameter growth with respect to current diameter as the difference between the maximum possible growth potential and the current diameter increment. It is clear from (6.3) that if the current diameter increment is large, $\theta_2 D_I$ is also large so the amount of the potential resource, $\theta_1 A(D)$, consumed for diameter increment will be large, and this in turns reduces the rate of diameter increment.

Mathematically, equations (6.2) and (6.3) constitute a *first-order, linear, non-homogeneous ordinary differential equation problem*. It can be solved if suitable initial and end conditions are specified (see Appendix 1). One solution turns out to be the Box-Lucas function (Box and Lucas 1959):

$$(6.4) \qquad D_I = \frac{\theta_1}{\theta_1-\theta_2}\left(e^{-\theta_2 D}-e^{-\theta_1 D}\right)$$

Equation (6.4) with D_I and D replaced by y and x respectively was originally used to describe an irreversible chemical reaction in which one substance changes into another, and to develop optimal design methodologies in nonlinear situations (Box and Lucas 1959, Hill and Hunter 1974, Hamilton and Watts 1985). It was termed as the two-term exponential function by Rawlings (1988) and classified as an intrinsically nonlinear compartmental model by Seber and Wild (1989).

The biological formulation of equation (6.4) based on (6.2) and (6.3) is readily interpretable but this should not be used to emphasize that equations (6.2) and (6.3) are the fundamentals that underlie the true biological process of diameter growth, rather, they provided a foundation for constructing equation (6.4) in a biologically interpretable way. Several typical graphs of equation (6.4) produced by varying the proportional parameters θ_1 and θ_2 are shown in Figure 6-1. The curves clearly demonstrate the fact that diameter increment begins at a value of zero, increases steadily to reach the maximum, and then decreases smoothly and asymptotically towards zero. These curves are also similar to the cumulative form of the Chapman-Richards model and follow the commonly observed sigmoidal S-shaped yield curves in biology, where the yield starts at the origin, reaching a maximum growth at an inflection point, and then approaches an asymptote as determined by the genetic nature of the living organism and the carrying capacity of the environment. The maximum diameter increment rate of the tree occurs at

$$(6.5) \qquad D = \frac{2\ln(\theta_1/\theta_2)}{\theta_1-\theta_2}$$

which is obtained by taking the second derivatives of equation (6.4) and setting the resultant equation equal to zero, and solving for D.

Yearly individual tree diameter increment can be predicted from equation (6.4) if other factors that affect diameter increment are ignored. If the variation in stand density and site productivity are considered to have significant effects on diameter increment, they may be incorporated into the equation to provide better diameter increment predictions. Naturally, given all other factors are approximately the same, a tree at a given diameter is expected to have larger diameter increment if stand density is lower because the tree has more growing space available. The tree at a given diameter may also be expected to have larger diameter increment on better sites, although this may not be as clear as that of the stand density effect on diameter increment because site productivity is generally considered highly correlated with the height increment but only weakly correlated with or independent of diameter increment. Diameter increment of a tree may also be affected by the species compatibility and their respective proportions in mixed-species stands, such as the dynamic process occurring in typical boreal white spruce-aspen stands. The diameter increment model developed in this analysis attempts to incorporate other tree and stand variables that have significant effects on diameter increment. A description of these variables follows.

Stand density and competition

Both the total number of trees and total basal area per hectare for all species in the stand are used as the mixed-species stand density measures. The use of these two stand level attributes as simple and objective measures of stand density has been widespread (Clutter et al., 1983), and for reasons described by Spurr (1952), they should be particularly suitable for mixed-species stands with irregular age structures. The crown-based measure of competition, crown competition factor, although commonly used as a measure of stand density in growth and yield estimation (Stage 1973, Arney 1985, Wykoff 1990), and is probably more closely related to tree competition in the stand because of its high correlation with the light intensity received by the crown of an individual tree (Hix and Lorimer 1990), is not compatible with the available data since the measurements of crown areas from open-grown trees are not available.

In addition to the use of total number of trees and total basal area per hectare to reflect the degree of overall crowdedness of trees within a mixed-species stand, a diameter-based, distance and age independent individual tree competition index (CI) expressed as the ratio of the target tree diameter (D) to the average diameter of all trees in the stand (AVED), CI=D/AVED, was also used to reflect the competitiveness of an individual tree relative to neighbouring trees. Like the use of crown-related variables such as crown length, crown width, and live crown ratio, competition measures based on relative diameter or basal area are also considered directly related to tree vigour (Lorimer 1983, Martin and Ek 1984, Davis and Johnson 1986). Trees with larger CI values are considered to be stronger competitors and will probably have larger diameter increment.

Species composition and site productivity

It is typical that the boreal mixed-species stands mainly consist of white spruce and aspen. In the early development stage of such a mixed white spruce-aspen stand, the shade intolerant aspen has the competitive advantage over white spruce and exhibits faster early height growth and as a result, tends to rapidly establish dominance on the site by occupying the upper layer of the canopy. Shade tolerant white spruce is often established a short but distinct time after the overstorey aspen and exhibits slow juvenile height growth. As time progresses, the competitiveness of aspen is reduced relative to that of white spruce as individual aspen trees start to die at 60 to 80 years of age, giving dominance gradually to the more shade tolerant and long lived white spruce. This dynamic process results in changes both in the whole stand density and the proportions of the species participated in the stand. An appropriate white spruce species composition measure is defined to reflect these changes:

$$(6.6) \qquad SC=\frac{BASUM_{sp}}{BASUM}$$

where SC is the species composition of white spruce, $BASUM_{sp}$ is basal area per hectare (m^2/ha) for white spruce, and BASUM is the total basal area per hectare for all species combined in the stand. The species composition as defined in (6.6) has a range of zero to one, with zero indicating there is no white spruce species in the stand and one indicating a pure species stand of white spruce.

The white spruce site productivity in boreal mixed-species stands is measured by the site productivity index (SPI) as determined by the dominant and codominant white spruce height-diameter relationship. Details about SPI have been described in Chapter 4.

The growth interval length

Many diameter increment models use a fixed time interval to define projection steps (Daniels and Burkhart 1975, Wykoff 1982). Arney (1985) suggested the use of variable time periods defined as the number of years required to produce 4.5 metres of height growth as projection steps. A variable representing the growth interval (GI) length similar to that proposed by Martin and Ek (1984) is used in this analysis so that diameter increment can be projected by any time interval rather than at some fixed yearly, 5-year, or 10-year interval. The length of the GI is calculated as: $GI=Year_2-Year_1+(Adj_2-Adj_1)$, where $Year_1$ and $Year_2$ are consecutive measurement years, Adj_1 and Adj_2 are the consecutive month adjustments. The month adjustments are used because initial and subsequent remeasurements may not be taken in the same month. They are defined based on the biological growth period for white spruce in boreal mixed-species stands: 1) if month≤4, the adjustment is 0.0, 2) if month=5, the adjustment is 0.2, 3) if month=6, the adjustment is 0.5, 4) if month=7, the adjustment is 0.9, 5) if month≥8, the adjustment is 1.0. For example, if the initial measurement was taken in May ($Adj_1=0.2$), 1965 and the consecutive measurement was taken in August ($Adj_2=1.0$), 1975, the length of GI will be: GI=1975-1965+(1.0-0.2)=10.8 (years).

In addition to the variables representing stand density and competition, species composition, site productivity, and growth interval length, tree height were also found to significantly affect diameter increment. These variables were incorporated into equations that predict the parameters of equation (6.4). This procedure is similar to the method of parameter prediction commonly used for a Chapman-Richards or Weibull-type function in which the parameters of the function were related to other tree and stand characteristics but the form of the original function remains the same (Clutter et al. 1983). The appropriate final diameter increment model was found to take the form of

$$D_I=GI\frac{\theta_1}{\theta_1-\theta_2}(e^{-\theta_2 D}-e^{-\theta_1 D}) \tag{6.7}$$

with

$$\theta_1=a_1+a_2 BASUM+a_3 SC+a_4 H+a_5 D/AVED+a_6 SPI \tag{6.8}$$

$$\theta_2=a_7+a_8\sqrt{TRHAAL} \tag{6.9}$$

where D_I is the periodic diameter increment (cm), BASUM is the basal area per ha for all species in the stand (m^2), H is tree height (m), SC is the species composition as defined in (6.6), SPI is the white spruce site productivity index (m), AVED is the average diameter (cm) for all the trees in the stand, TRHAAL is the total number of trees per ha, D is tree diameter (cm) at breast height, GI is the growth interval length (years), and a_1-a_8 are parameters to be estimated. A positive coefficient in (6.8) indicates a positive effect on diameter increment, and a positive coefficient in (6.9) indicates a negative effect on diameter increment.

6.3 The data

Data from permanent sample plots (PSP) used in this analysis were provided by the Alberta Forest Service. The data were collected over the last three decades and the PSPs were randomly located throughout the inventory areas of the province to provide representative information for a variety of densities, heights, species composition, stand structures, ages, and site conditions. A detailed description of how the data are collected and recorded can be found in the Permanent Sample Plots: Field Procedures Manual (AFS 1990).

The original PSP data were summarized to provide additional variables such as the number of trees per hectare, basal area per ha, average height, and average diameter, both for all species combined and by individual species in the stand. A total number of 164 PSPs were selected to be used in this analysis. These selected plots have up to five remeasurements. Each non-overlapping growth period from the remeasurements defines a growth interval, that is, the growth intervals are obtained from measurements between first and second, second and third, but not first and third. A total number of 1473 growth periods was obtained from 164 PSPs. Periodic diameter increments were obtained as the differences between diameters at the end and beginning of the growth interval.

Descriptive statistics including the mean, minimum, maximum, and standard deviation (Std dev.) of the tree and stand characteristics at the beginning of the growth period are displayed in Table 6-1. Summary statistics for periodic diameter increment and growth interval length are also given in Table 6-1. Approximately 80% of the observations (1194) were randomly selected for model fitting and the remaining 20% (279) were used for model testing.

6.4 Analysis

Preliminary nonlinear least squares (NLS) fits of the diameter increment model (6.7) were accomplished using the PROC MODEL procedure on SAS/ETS software (SAS Institute Inc. 1988). The Gauss-Newton iterative method using the Taylor series expansion as described in Gallant (1987) was applied in model fitting. To ensure the solution is global rather than local least squares estimates, different initial values of the parameters were chosen for the fits.

Asymptotic fit statistics, including the NLS estimates of the parameters, the asymptotic t-statistics, standard errors, p-values of the parameters, the model root mean squared error (RMSE), adjusted coefficient of determination (Adj. R^2), and the Durbin-Watson statistic for testing the first-order autocorrelation of the error terms, from the PROC MODEL procedure, are listed in Table 6-2. The interpretations of the fit statistics in Table 6-2 are similar to those in linear cases because of the relatively large number of observations used in this analysis.

Error structure of the model

First, the Durbin-Watson statistic is used to examine the possible error correlation. Although the Durbin-Watson test procedure is not exactly applicable for nonlinear regression models, the test is approximately valid, especially for large samples (Amemiya 1983 p. 355, Seber and Wild 1988 pp. 318-319). Because the sample size used for this analysis is larger than the maximum sample size (200) shown in the extended Durbin-Watson Table (Judge et al. 1988 pp. 991-994), critical values selected are those recorded for the maximum sample size. The Durbin-Watson statistic of 1.532 is smaller than the lower bound (1.697) of the critical values for the eight parameter diameter increment model, indicating a significant first-order autocorrelation for the error terms of the model. The *ith* observation of the diameter increment model (6.7) with first-order autoregressive errors AR(1) can be written as

$$(6.10)\qquad D_{Ii}=f(x_i,\theta)+\varepsilon_i \qquad \varepsilon_i=\rho\varepsilon_{i-1}+e_i$$

where the diameter increment D_{Ii}, $i=1, 2, \ldots, n$, is expressed as a nonlinear function $f(x_i, \theta)$ of the explanatory variables, ε_i are random errors that follow an AR(1) process, ρ is the autoregressive parameter, and e_i are independent and identically distributed random errors with mean 0 and constant variance σ_e^2. From equation (6.10), the (*i*-1)th observation of the diameter increment model can be written as

$$(6.11)\qquad D_{I(i-1)}=f(x_{i-1},\theta)+\varepsilon_{i-1}$$

Multiplying ρ on both sides of (6.11), then subtracting (6.10) by (6.11) provides a nonlinear equation with uncorrelated errors

$$(6.12)\qquad D_{Ii}=\rho D_{I(i-1)}+f(x_i,\theta)-\rho f(x_{i-1},\theta)+e_i$$

A five step procedure can be used to find the NLS estimates of parameters θ in (6.12):

1). Fit equation (6.10) by ordinary NLS without considering its error structure, find the estimates of the parameters θ_{NLS}.

2). Calculate the residuals $\varepsilon_i=D_{Ii}-f(x_i, \theta_{NLS})$, and estimate ρ by fitting $\varepsilon_i=\rho\varepsilon_{i-1}+e_i$.

3). Find the estimates of θ by applying ordinary NLS to the model

$$(6.13)\qquad D_{Ii}-\rho D_{I(i-1)}=f(x_i,\theta)-\rho f(x_{i-1},\theta)+e_i$$

4). Replace θ_{NLS} in step 2) by θ from 3), then re-estimate ρ in 2).

5). Replace ρ in step 3) by ρ from 4), then re-estimate θ using NLS.

This process is iterated until θ is converged. The resulting fit statistics using the

SAS/ETS software for the diameter increment model with the AR(1) error structure are also shown in Table 6-2. The Durbin-Watson statistic (DW=2.062) is greater than the upper bound (1.841) of the critical value, indicating that autocorrelation for the adjusted error terms is not significant and the AR(1) specification is appropriate for the error terms of the white spruce diameter increment model.

Residual analysis from the preliminary NLS fit (without accounting for the AR(1) error terms) was also performed to detect possible model inadequacies. Because the residuals r_i (calculated as the difference between the observed height and the predicted height) are intrinsically not independent and do not have common variance (Rawlings 1988, p. 249), studentized residuals are used instead to account for the unequal variance problem. The dependency effect among the residuals r_i, according to Neter et al. (1990, p. 116), is relatively unimportant and can be ignored because the sample size is large in comparison to the number of parameters in the diameter increment model. For a correctly identified function, the plot of studentized residuals against the predicted values of the dependent variable will show a homogeneous band of the data points, with the zero studentized residuals across the centre of these points.

Figure 6-2 shows the plots of studentized residuals against the predicted diameter increment. It is clear that the zero studentized residuals across the centre of the data points but the points display a trend of increasing error variances, indicating that the diameter increment model is appropriately identified but the error terms of the model are possibly associated with the heteroskedasticity problem. Similar residual analysis from the NLS fit which accounts for the AR(1) error terms also indicated a increasing error variance pattern.

The significance of the heteroskedasticity can be tested using the Goldfeld-Quandt test as described in Judge et al. (1988). Because autocorrelation reduces the power of Goldfeld-Quandt test, results from NLS fit with the AR(1) error terms included were used.

A four step procedure was implemented to test error heteroskedasticity:

1). Order the predicted diameter increment values from the NLS fit in an ascending sequence according to increasing error variance;

2). Omit r=194 central observations;

3). Perform two separate NLS AR(1) fits on the first 500 observations and the remaining 500 observations. This gives the mean squared errors MSE_1 = 0.01515 and MSE_2 = 0.12228 for the first and the second regressions respectively;

4). Calculate the test statistic $\lambda = MSE_2/MSE_1$=8.0713.

Under the null hypothesis of homoskedasticity λ has an F-distribution with [(1194-194-2×10)/2, (1194-194-2×10)/2] degrees of freedom. At a 5% significance level the critical value for λ from the F-distribution is $\lambda_{crit.}$ = 1.00, and so, using the Goldfeld-Quandt test, the null hypothesis is rejected and we conclude that heteroskedasticity exists for the error terms of the diameter increment model. Several alternative assumptions about the nature of the heteroskedasticity were then proposed and examined:

(1). The variance of error is a linear function of the predicted diameter increment.

$$\sigma_i^2=\varepsilon_i^2=\alpha_1+\alpha_2\hat{D}_{Ii} \tag{6.14}$$

where $\sigma_i{}^2$ is the estimated variance of error and $\varepsilon_i{}^2$ is the estimated squared residuals for the *i*th tree of the diameter increment model based on the NLS AR(1) results, $\hat{D}_{Ii}$ is predicted diameter increment for the *i*th tree based on the fitted model.

(2). The variance of the error is a exponential function of the predicted diameter increment.

$$\sigma_i^2=\varepsilon_i^2=\alpha_1 e^{\alpha_2\hat{D}_{Ii}} \tag{6.15}$$

(3). The variance of the error is a power function of the predicted diameter increment.

$$\sigma_i^2=\varepsilon_i^2=\alpha_1\hat{D}_{Ii}^{\alpha_2} \tag{6.16}$$

(4). The variance of the error is some function of the explanatory variables.

Scatter plots of the predicted diameter increment versus each explanatory variable in the diameter increment model were first examined to see any possible correlation between the error variances and the variables. Several alternative linear and nonlinear equations expressing the variance of the error as some function of the explanatory variables were then fitted. The multiple linear regression equation was chosen among others

$$(6.17) \qquad \sigma_i^2=\varepsilon_i^2=\alpha_1+\alpha_2 D_i+\alpha_3 H_i+\alpha_4 TRHAAL+\alpha_5 D_i/AVED$$

where D_i is the breast height diameter, H_i is the total tree height of the *i*th tree, and AVED is the average diameter of all trees in the stand.

(5). The variance of error is directly proportional to squared diameter.

$$(6.18) \qquad \sigma_i^2=\varepsilon_i^2=D_i^2$$

This particular situation assumes the error variances increase with increasing diameter of the trees.

Weighted NLS techniques with the weights chosen inversely proportional to the five alternative error variances were applied to white spruce diameter increment data. The plots of studentized residuals against the predicted diameter increment were examined in each case for any possible trend showing heteroskedasticity. Models based on the first three assumptions of the error variances resulted in studentized residual plots that were still indicating a trend of increasing error variances. However, models based on the last two error variance assumptions gave satisfactory studentized residual plots. The estimated coefficients for (4) are $\alpha_1=3.08006$, $\alpha_2=0.043277$, $\alpha_3=-0.135819$, $\alpha_4=-0.000406$, and $\alpha_5=0.264314$, with the MSE=6.96071 and $R^2=0.0326$. The plot of studentized residuals against the predicted diameter increment shown in Figure 6-3 based on the error specification in (5) is more satisfactory. This studentized residual plot, although not perfectly homogenous, shows much improvement over Figure 6-2, and displays data points that are approximately equally scattered around the zero studentized residuals. Accordingly the weighting factor $w_i = 1/D_i^2$ was chosen to be used in the final fitting for the diameter increment model.

Final fit based on the model's error structure

The error terms of the white spruce diameter increment model are diagnosed as autocorrelated and heteroskedastic. This may be typical for the data from PSPs where the plots are located over a wide range of forest regions and each plot is repeatedly measured over regular time intervals. An appropriate fit for such a model can be achieved by removing error heteroskedasticity and autocorrelation at the same time. The PROC MODEL procedure on SAS/ETS software is well suited for this purpose. The necessary SAS code for fitting the diameter increment model is presented in Appendix 2. Resulting fit statistics from the weighted NLS with the weights $w_i = 1/D_i^2$ and an AR(1) error specification for model (6.7) are also attached in Table 6-2. Note that the interpretations of the weighted statistics in Table 6-2 may not be as straightforward as those previously obtained without weighting (Carroll and Ruppert 1988).

Diagnostics for multicollinearity

The tree and stand variables appearing in the diameter increment model may somehow be inter-correlated. Multicollinearity could exist in such model because of inter-correlated variables. Collinearity diagnostics using the keyword COLLIN with PROC MODEL procedure were performed to detect the presence, severity, and form of multicollinearity. The eigenvalues of the correlation matrix of the set of explanatory variables were arranged from the largest (6.29841) to smallest (0.02154), and the square root of the ratio of the largest to smallest eigenvalue, the condition index, is obtained as 17.1006, which is less than 30 – the proposed critical value for moderate multicollinearity (Belsley et al. 1980), indicating that multicollinearity is not a serious problem for the white spruce diameter increment model.

While the use of diagnostic methods is recommended for detecting the presence, severity, and form of multicollinearity, use of the biased estimators as remedial measures to correct it may not absolutely be necessary unless extreme multicollinearity is found. Discussion on multicollinearity has been presented in Chapter 5.

Model testing

The independent testing data set was used for testing the model's fit. Because a weighted NLS technique was applied, the weighted actual diameter increment values from the testing data were compared to the weighted predicted diameter increment values using model (6.7) with the estimates of parameters from weighted NLS with AR(1) errors in Table 6-2. The bias of the prediction was obtained by subtracting the weighted predicted diameter increment from the weighted actual diameter increment. The mean (δ) and the standard deviation [$s(\delta)$] of the prediction bias is 0.0027760 and 0.0447787 respectively, so the standard error of the estimated mean bias is $0.0447787/(279)^{1/2} = 0.0026808$. A t-test of the hypothesis that the mean prediction bias was zero gives $t=1.0355$, which, with 278 degrees of freedom, is not significant at $\alpha=0.05$ level, indicating that the mean prediction bias is not significantly different from zero. The mean squared error of prediction (MSEP) can be obtained according to

$$(6.19) \qquad MSEP=\frac{(n-1)\,s^2(\delta)}{n}+\delta^2=0.002006$$

where the squared prediction bias term δ^2 contributes 0.38% of MSEP. The square root of

MSEP gives 0.04478, which is approximately 56.07% of the weighted observed average diameter increment (0.07987) from the testing data.

The model testing procedure demonstrated above on weighted data is appropriate because the model is fitted on weighted data. In practice, however, unweighted diameter increment may also be directly predicted using the estimated coefficients from weighted NLS with AR(1) errors in Table 6-2, and compared to the unweighted observed diameter increment. Using this procedure, the bias of the prediction was obtained by subtracting the unweighted predicted diameter increment from the unweighted actual diameter increment. The mean and the standard deviation of the prediction bias are obtained as 0.056613 and 1.08855 respectively, so the standard error of the estimated mean bias is $1.08855/(279)^{1/2} = 0.06517$. A t-test of the hypothesis that the mean prediction bias was zero gives $t=0.8687$, which, with 278 degrees of freedom, is not significant at $\alpha=0.05$ level, indicating that the mean prediction bias is not significantly different from zero. Similarly, the MSEP for unweighted data is 1.1839, with the squared prediction bias term δ^2 contributing 0.27% of MSEP. The square root of MSEP gives 1.08807, which is approximately 58.13% of the observed average diameter increment (1.87175) from the testing data.

6.5 Discussion

The diameter increment model as expressed by (6.7) provides individual tree diameter increment predictions for white spruce grown in boreal mixed-species stands in Alberta. Asymptotic fit statistics in Table 6-2 show that the model agrees well with the white spruce diameter increment data. The t-statistics of the estimated coefficients of the model are all significant at $\alpha=0.05$ level. Approximately 64.49% percent of the weighted diameter increment variation is explained by the fitted model. The model was developed on an appropriately chosen base function and has the flexibility to assume various shapes with different parameter values and produce satisfactory curves under most circumstances. All curves produced by the model assume biologically reasonable shapes that closely mimic the biological process of diameter growth, and provide realistic diameter increment predictions in the cases where the model is extrapolated beyond the range of the original data.

The diameter increment model affirms some commonly held beliefs and interesting facts about the diameter increment of white spruce in the boreal mixed-species stands in Alberta. The significant negative coefficient $a_2<0$ for total basal area per hectare in (6.8), together with the positive coefficient $a_8>0$ in (6.9) for total number of trees in the stand indicates that the stand density components in the white spruce diameter increment model have a significant negative effect on white spruce diameter growth. Given the other conditions held approximately the same, the model reflects the fact that diameter increment is reduced as stand density increases, and dense stands have smaller diameter trees.

Species composition, as represented by parameter a_3 in model (6.8), has a negative ($a_3<0$) effect on white spruce diameter increment, which indicates that for two mixed-species stands grown in the similar conditions, the one with more white spruce will have small diameter increment. The reason for this is unclear, but a possible explanation may be that under many circumstances the diameter increment of white spruce is directly related to the species compatibility and interspecific competition with other competition tree species, such as the commonly found aspen in mixed spruce-aspen stands. The increasing presence of the aspen not only compete but also stimulate and change the white spruce diameter growth patterns. An example for this explanation is the frequently observed phenomena in which the mixed white spruce-aspen stand may be largely dominated by aspen, but the few white spruce trees in the stand are healthy, strong, and have large diameters.

The negative effects of species composition on white spruce diameter increment also indicate that for the same stand white spruce diameter increment will slow down as the tree grows over an extended time period. It is typical that in the very early stage of development, the diameter growth of white spruce is slow relative to other competing species. As time progresses, the relative competitiveness of white spruce is increased and white spruce diameter growth approaches maximum when the trees reach 10 to 20 cm in diameters. As time progresses further, white spruce becomes pre-dominant in matured and over-matured mixed-species stands in which the diameter growth of white spruce is slowed down. However, the basal area proportion for white spruce may still increase because of its relatively long lived nature and low mortality, resulting in on average a negative species composition effect on diameter increment.

The significant positive coefficient $a_5>0$ of the competition measure on diameter increment reflects the fact that given the other factors constant, stronger competitors as measured by their relatively larger diameter ratios will have larger diameter increment. This may also demonstrate that the commonly observed asymmetric or one-side competition in plant monocultures (Weiner and Thomas 1986) is equally applicable for white spruce grown in uneven-aged and mixed-species stands.

A probably contradictory result was obtained according to the fitted model: site productivity has a negative effect ($a_6<0$) on white spruce diameter increment. Given other conditions approximately the same, increased diameter growth on better sites may be expected for species grown in the even-aged pure species stands. Diameter increment in uneven-aged and mixed-species stands are more complicated but in general might also be expected to have increased diameter growth on better sites. Possible "abnormality" of the site effect on white spruce diameter increment may be explained by:

1). The diameter increment does slow on better sites because the metabolic forces for tree growth on such sites are mainly concentrated on the height growth rather than diameter growth. Trees grown on good sites are relative thinner but taller, and on poor sites are shorter but the diameter tends to be larger. The taper of the trees (diameter divided by total tree height) are different for trees grown on poor sites from those grown on better sites. This has been reported by Larson (1963), who suggested that a decreasing taper was associated with increasing site productivity.

2). The boreal mixed-species regions in Alberta, although covering a wide range of areas, have relative similar site conditions. Using the so-called ecological site classification system or habitat type, these stands could probably be classified into the same biogeoclimatic unit that has similar vegetative, geographic, and climatic conditions. Site variations in such a unit might not be very large, and for this particular sample, a negative site effect resulted. It was clear from Table 6-2 that the t-statistic for site productivity index is the least significant compared to the other factors, although it is statistically significant at 5% level.

3). The problems associated with the use of height-diameter relationships as the site indicator for uneven-aged and mixed-species stands. Measuring site productivity, and consequently incorporating it into a model is certainly a rather complicated problem. This problem is precisely the one that led researchers like McLintock and Bickford (1957), Stout and Shumway (1982), and Wykoff et al. (1982) to develop alternatives to the traditional site index in attempting to provide accurate and realistic site productivity measures. Further research regarding the methodologies for evaluating site productivity in uneven-aged and mixed-species stands may help improve the understanding of the real site effect on diameter growth, although consistent conclusions may be hard to reach.

While the diameter increment model reveals the effects of the stand variables on white spruce diameter increment, and this may provide some useful information about the ecology and silviculture of white spruce grown in boreal mixed stands in Alberta, the model is mainly developed for individual tree diameter increment predictions. Since the model is based on a function that relates diameter increment to tree and stand variables, but does not involve individual tree or stand age explicitly, the model can be applied to any stand. Input variables of the model are simple tree and stand variables that are readily obtainable from ordinary inventories, and are compatible with the data-collection process in Alberta. Potential users of the model with limited information or alternative measures for some of the variables appearing in the model may consider re-fitting other forms of the model by replacing the variables that are not available, or by dropping the variables if the cost to obtain the variables is too high relative to the amount of additional variation explained by adding such variables to the model. In either case, the flexible base diameter increment-diameter model — the Box-Lucas function based on biological principles should be retained, and the method of parameter prediction used in this study is recommended.

6.6 Summary

Based on a data set from 164 permanent sample plots, an age-independent individual tree diameter increment model for white spruce (*Picea glauca* (Moench) Voss) grown in the boreal mixed-species stands in Alberta is presented. The model is age-independent in that it does not explicitly require tree/stand age as input variables. Periodic diameter increment is modelled as a function of tree DBH, tree height, relative competitiveness of the tree in the stand, stand density, species composition, and site productivity. Because data from permanent sample plots are considered time series and cross-sectional, diagnostic techniques were applied to identify the model's error structure. Appropriate fit based on the identified error structure was accomplished using weighted nonlinear least squares with a first-order autoregressive process. Results show that 1) all model parameters are significant at $\alpha=0.05$, 2) the plot of studentized residuals against predicted diameter increment shows no consistent underestimate or overestimate for diameter increment. The model is also tested on an independent data set representing the population on which it is to be used. Results show that the average prediction biases are not significant at $\alpha=0.05$, indicating that the model appropriately describes the data and performs well when predictions are made.

Table 6-1. Summary statistics for white spruce tree and stand characteristics

	Mean	Minimum	Maximum	Std. dev.
Tree DBH (cm)	27.06	2.90	63.30	10.29
Tree height (m)	21.75	3.20	37.90	5.94
Periodic diameter increment (cm)	1.94	0.00	10.50	1.59
Growth interval length (year)	8.94	2.30	18.90	8.94
Number of trees/ha-all species	1244.29	148	5580	754.24
Average DBH (cm)-all species	19.72	5.30	39.70	5.60
Basal area (m^2/ha)-all species	39.09	11.55	83.87	9.39
Average height (m)-all species	21.22	7.90	31.70	3.88
Species number of trees/ha	753.07	5.00	4914.00	673.15
Species average diameter (cm)	20.60	5.70	54.90	6.47
Species basal area (m^2/ha)	24.63	0.20	56.65	11.98
Species average height (m)	21.57	3.60	31.70	4.25
Site productivity index (m)	16.14	8.22	21.28	2.39
Species composition	0.62	0.0062	1.00	0.26

Table 6-2. Fit statistics for white spruce diameter increment model[1]

	Parameter	Estimate	Std. err.	t-statistic	p-value	RMSE	Adj. R^2	D-W
NLS	a_1	0.048739	0.0039218	12.43	0.0001	1.07669	0.5566	1.532
	a_2	-0.00019032	0.00004288	-4.44	0.0001			
	a_3	-0.012572	0.0018750	-6.70	0.0001			
	a_4	-0.00047177	0.00009876	-4.78	0.0001			
	a_5	0.00409558	0.0014325	2.86	0.0043			
	a_6	-0.00057299	0.0002204	-2.60	0.0094			
	a_7	0.015121	0.0039933	3.79	0.0002			
	a_8	0.00077831	0.0001466	5.31	0.0001			
NLS with AR(1) errors	a_1	0.047076	0.0042036	11.20	0.0001	1.04574	0.5817	2.062
	a_2	-0.00022079	0.00004957	-4.45	0.0001			
	a_3	-0.012745	0.0022129	-5.76	0.0001			
	a_4	-0.00027974	0.0001082	-2.59	0.0098			
	a_5	0.00306837	0.0015105	2.03	0.0424			
	a_6	-0.00055843	0.0002558	-2.18	0.0292			
	a_7	0.017453	0.0044620	3.91	0.0001			
	a_8	0.00069438	0.0001675	4.15	0.0001			
	ρ	0.246675	0.02566	9.61	0.0001			
Weighted NLS with AR(1) errors	a_1	0.060596	0.0048120	12.59	0.0001	0.04631	0.6449	2.026
	a_2	-0.00045932	0.00006198	-7.41	0.0001			
	a_3	-0.023597	0.0026078	-9.05	0.0001			
	a_4	-0.00038001	0.0001403	-2.71	0.0068			
	a_5	0.013987	0.0023243	6.02	0.0001			
	a_6	-0.00069998	0.0003149	-2.22	0.0264			
	a_7	0.017087	0.0056630	3.02	0.0026			
	a_8	0.00114770	0.0001864	6.16	0.0001			
	ρ	0.285297	0.02341	12.19	0.0001			

[1] Note: NLS represents nonlinear least squares, AR(1) is the first-order autoregressive process as defined in (6.10), ρ is the AR(1) parameter, the weights chosen are $w_i=1/D_i^2$, D-W is the Durbin-Watson statistic.

Figure 6-1. Typical graphs of equation (6.4) produced by varying parameters θ_1 and θ_2. 1: $\theta_1=0.050$, $\theta_2=0.045$; 2: $\theta_1=0.060$, $\theta_2=0.065$; 3: $\theta_1=0.100$, $\theta_2=0.090$; 4: $\theta_1=0.150$, $\theta_2=0.125$.

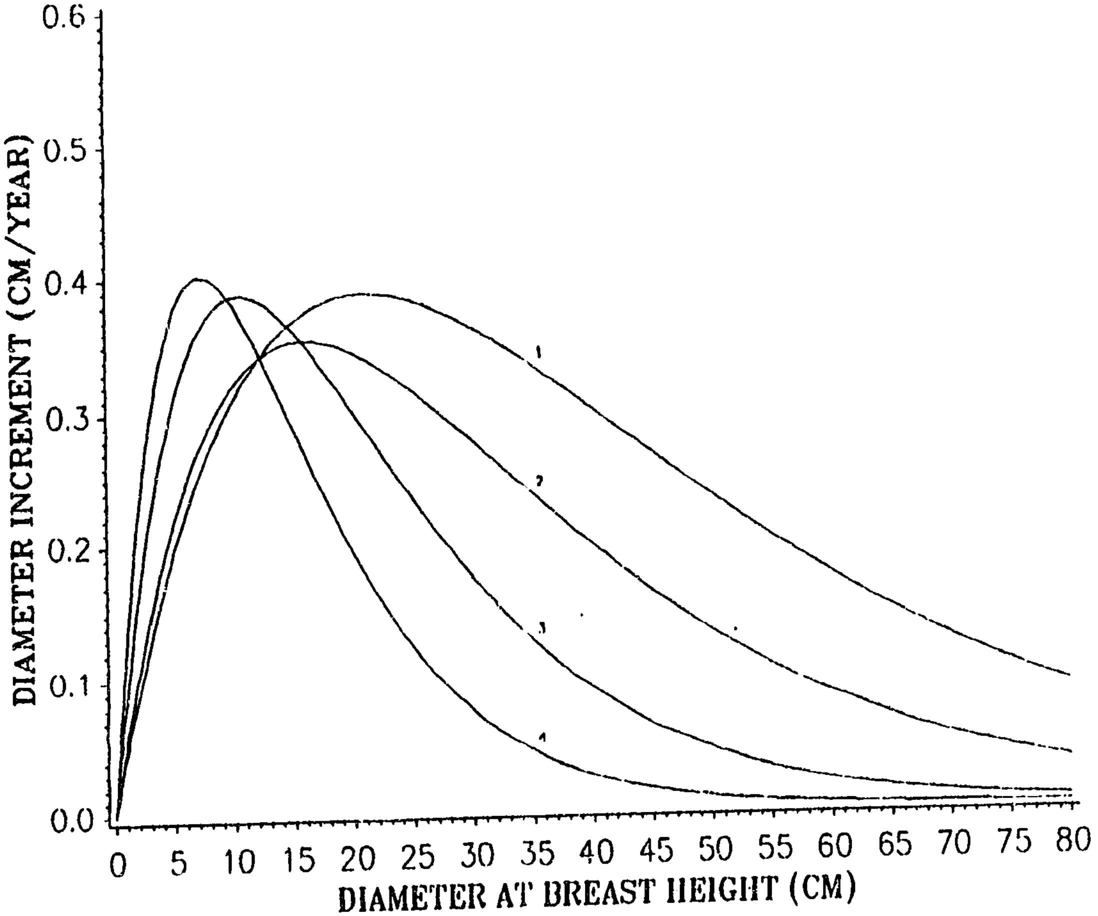
0.6
0.5
0.4
0.3
0.2
0.1
0.0
DIAMETER INCREMENT (CM/YEAR)
1
2
3
4
0
5
10
15
20
25
30
35
40
45
50
55
60
65
70
75
80
DIAMETER AT BREAST HEIGHT (CM)

Figure 6-1.

Figure 6-2. The plots of studentized residuals against the predicted diameter increment from nonlinear least squares without considering model (6.7)'s error structure.

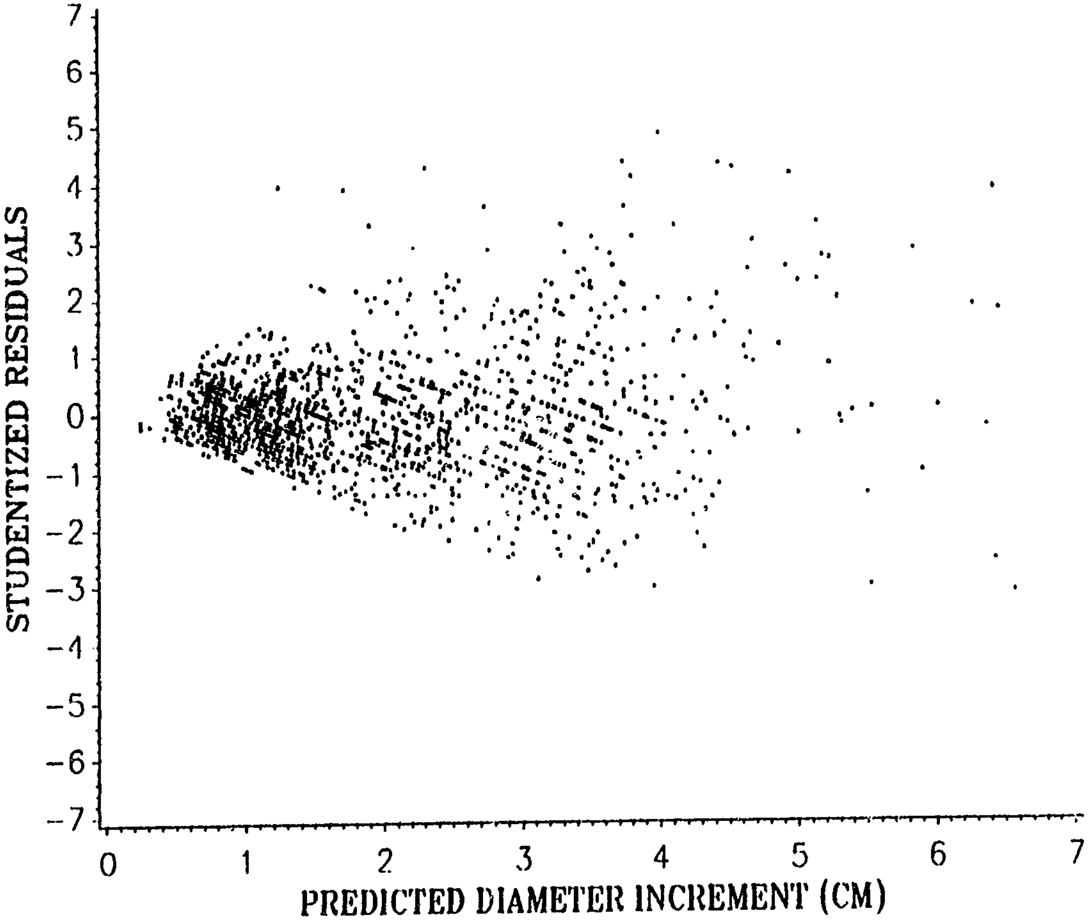
STUDENTIZED RESIDUALS
7
6
5
4
3
2
1
0
−1
−2
−3
−4
−5
−6
−7
0
1
2
3
4
5
6
7
PREDICTED DIAMETER INCREMENT (CM)

Figure 6-2

Figure 6-3. The plots of studentized residuals against the predicted diameter increment from weighted nonlinear least squares with an AR(1) error structure for model (6.7). The chosen weights are $w_i=1/D_i^2$, where D_i is the diameter at breast height of the tree.

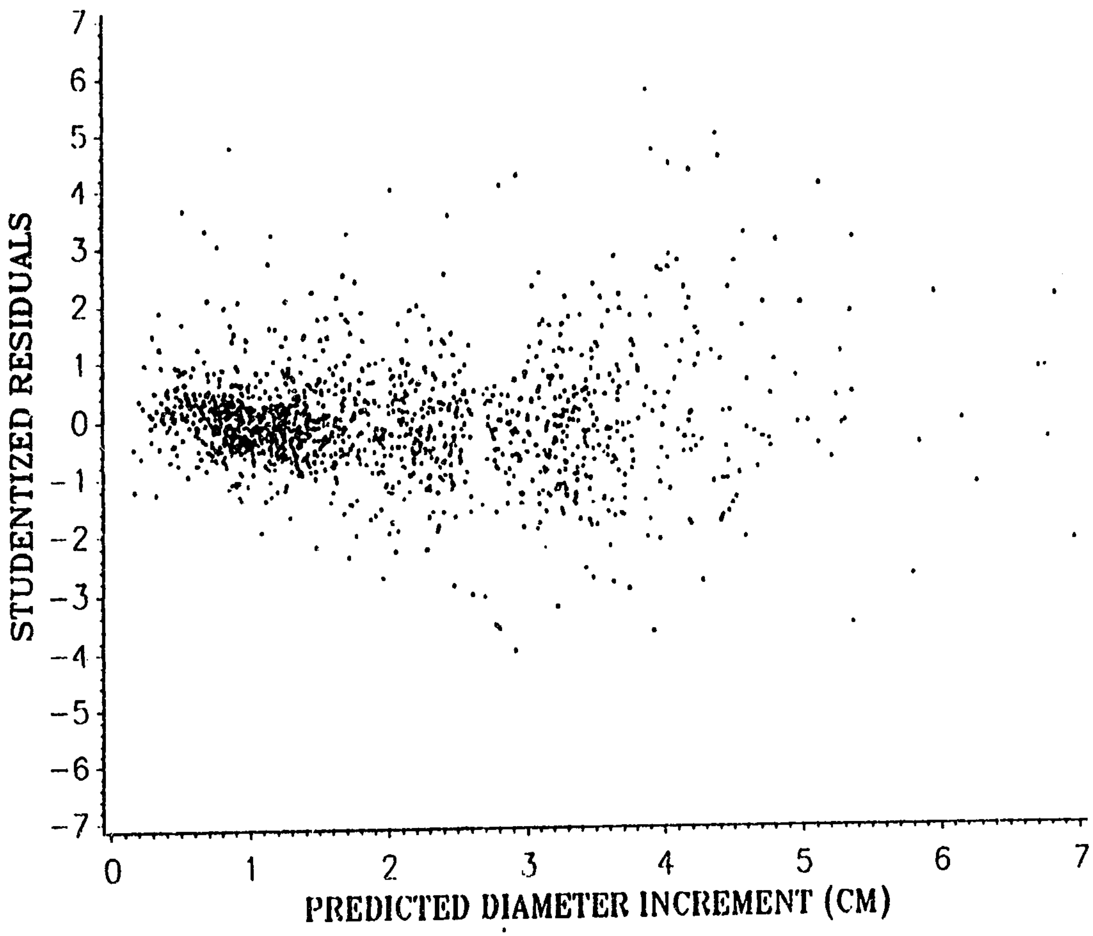
7
6
5
4
3
2
1
0
-1
-2
-3
-4
-5
-6
-7
STUDENTIZED RESIDUALS
0
1
2
3
4
5
6
7
PREDICTED DIAMETER INCREMENT (CM)

Figure 6-3

6.7 References

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Chapter 7

Individual Tree Height Increment Model

7.1 Introduction

Individual tree height increment models predict height growth of each individual tree in the stand. Such models have been used as the "driving force" or main variable for individual tree based growth and yield models such as PTAEDA (Daniels and Burkhart 1975), TASS (Mitchell 1975), and SPS (Arney 1985). Many other individual tree based models such as FOREST (Ek and Monserud 1974), ORGANON (Hester et al. 1989), and PROGNOSIS (Stage 1973, Wykoff et al. 1982, Wykoff 1986) also treated the height increment model as one of the main components. Two different modelling approaches for height increment have been commonly used. The growth-potential independent approach develops a regression model that directly relates height increment to tree and stand characteristics, including the competitiveness of the tree in the stand (Lemmon and Schumacher 1962, Beck 1974, Wykoff et al. 1982); The growth-potential dependent approach first selects a function that defines the potential height growth of competition-free trees, then uses a competitive adjustment factor (the modifier) to reduce this potential (Ek and Monserud 1974, Daniels and Burkhart 1975). This second approach often involves selecting a "biologically meaningful" base potential height increment function and emphasizing how the modifier is modelled and how it affects potential height increment. It is common that the modifier is expressed as a function of the individual tree's attributes such as crown ratio, crown length, tree diameter, tree height, and the competition with other trees in the stand as reflected by a competition index, total number of trees per hectare, or basal area per hectare. Mitchell (1975), Krumland (1982), Arney (1985), and Ritchie and Hann (1986) all used this approach. However, Wykoff (1990) regarded the differences in approaches are mostly "semantic" because either approach can produce acceptable predictions. Choice between the approaches may simply be a matter of preference or convenience.

While most researchers choose to use height increment as the dependent variable and relate it directly to other tree and stand characteristics, height increment models can also be derived by taking the first derivative of the cumulative height function (Hegyi 1974, Daniels and Burkhart 1975). The procedure demonstrated by Daniels and Burkhart (1975) first expressed the average height of dominant stand (HD) as a function of site index (SI) and age (A): $HD=SI\times10^{-5.865(1/A-1/25)}$, then the maximum annual height increment is obtained by taking the first derivative of this equation with respect to age, and a modifier expressed as a function of crown ratio and competition index is used to adjust this maximum height increment. This indirect derivative method has some advantages if the cumulative height equation includes age as an independent variable or if the form of the equation is not overly complicated.

This study developed individual tree height increment models for white spruce (*Picea glauca* (Moench) Voss) and trembling aspen (*Populus tremuloides* Michx.) grown in boreal mixed-species stands in Alberta. Average annual height increment was directly expressed as a function of tree diameter, tree height, diameter increment, stand density and competition, and site productivity. A species composition measure was also incorporated into the model to represent the effects of species proportions on height increment. While the descriptions of the tree and stand variables that affect height increment in mixed-species stands were stressed, the biological implications of the effects of these variables on height increment were also emphasized. Methods for model diagnostics and testing were also applied to ensure that the model was fitted appropriately and is applicable for making predictions.

7.2 Model development

The two parameter Box-Lucas function (Box and Lucas 1959) was selected as the base height increment function:

$$(7.1) \qquad HI=\frac{\theta_1}{\theta_1-\theta_2}\left(e^{-\theta_2 H}-e^{-\theta_1 H}\right)$$

where HI is the average annual height increment (m), H is tree height (m), θ_1 and θ_2 are parameters to be estimated, and e equals 2.71828. Initially, tree diameter (D) at breast height was used in place of H but preliminary analysis indicated that equation (7.1) was more appropriate. This base function of height increment-height is also compatible with the diameter increment-diameter function used in modelling individual tree diameter growth (Wykoff et al. 1982, Wykoff 1990). Several typical graphs of equation (7.1) produced by varying θ_1 and θ_2 are shown in Figure 7-1. The curves clearly demonstrate that height increment begins at a value of zero, increases rapidly to reach a maximum, and then decreases smoothly and asymptotically towards zero. These curves are also directly compatible with the commonly observed sigmoidal S-shaped yield curves in biology where the yield starts at the origin, reaching a maximum growth at an inflection point, and then approaches an asymptote as determined by the genetic nature of the living organism and the carrying capacity of the environment. The maximum annual height increment rate of the tree occurs at

$$(7.2) \qquad H=[2\ln(\theta_1/\theta_2)]/(\theta_1-\theta_2)$$

which is obtained by taking the second derivatives of equation (7.1) and setting the resultant equation equals to zero, and solving for H.

Individual tree height increment can most appropriately be predicted from equation (7.1) if other factors that affect height increment are ignored. If the variations in site productivity and competition are considered to have significant effects on height increment, they may be incorporated into the equation to provide better height increment predictions. It is fairly well established that, given all other factors are approximately the same, a tree at a given height is expected to have larger height increment on better sites. The presence of competition on height growth is also evident and measures that reflect competition have been commonly incorporated into height increment models (Daniels and Burkhart 1975, Wykoff et al. 1982, Arney 1985, Ritchie and Hann 1986). A tree at a given height may be expected to have larger height increment if the tree is more competitive as reflected by its crown dimensions and competition indices. The diameter or basal area of a tree can affect height growth as well since larger diameter trees should be in a more competitive position which should lead to larger height increment. The height increment of a tree may also be affected by the species compatibility and their respective proportions in mixed-species stands. The height increment models developed in this analysis attempt to incorporate other tree and stand variables that have significant effects on height increment. Descriptions of these variables follows.

Competition

Total basal area per hectare for all species combined in the stand is used as the overall mixed-species stand density measure. The use of this stand level attribute as a simple and objective measure of stand density has been widespread (Clutter et al., 1983), and for reasons described by Spurr (1952), is particularly suitable for mixed-species stands with irregular age structures. Other derived stand density measures, such as volume per hectare, spacing index, relative density, tree-area ratio, and stand density index have mainly been used for even-aged pure species stands (Clutter et al. 1983). The crown-based measure of stand density, Crown Competition Factor (CCF), although commonly used as a measure of overall stand density in growth and yield estimation (Stage 1973, Arney 1985, Wykoff 1990), is not compatible with the available data since the measurements of crown dimensions were not available.

In addition to the use of total basal area per hectare to reflect the degree of the overall competitiveness of trees within mixed-species stand, an individual tree competition index (CI) expressed as the ratio of the target tree diameter (D) to the average diameter (AVED) of all trees in the stand (CI=D/AVED), was also used to reflect the competitiveness of an individual tree relative to neighbouring trees. Like the use of crown-related variables such as crown length, crown width, and live crown ratio, competition measures based on relative diameter or basal area are also considered directly related to tree vigour (Lorimer 1983, Davis and Johnson 1986). As the CI gets numerically larger, the tree is in a better competitive position which should result in larger height increment.

Species composition and site productivity

It is typical that the boreal mixed-species stands mainly consist of white spruce and aspen. In the early development stage of such a mixed white spruce-aspen stand, the shade intolerant aspen has the competitive advantage over white spruce and exhibits faster early height growth and as a result, tends to rapidly establish dominance on the site by occupying the upper layer of the canopy. Shade tolerant white spruce is often established a short but distinct time after the overstorey aspen and exhibits slow juvenile height growth. As time progresses, the competitiveness of aspen is reduced relative to that of white spruce as individual aspen trees start to die at 60 to 80 years of age, giving dominance gradually to the more shade tolerant and longer lived white spruce. This dynamic process results in changes to both stand density and the proportions of the species present in the stand. A species composition measure is defined to reflect these changes:

$$(7.3)\qquad SC=\frac{BASUM_{sp}}{BASUM}$$

where SC is the species composition of the target species, $BASUM_{sp}$ is basal area per hectare (m^2/ha) for the target species, and BASUM is the total basal area per hectare for all species combined in the stand. Species composition as defined in (7.3) has a range of zero to one, with zero indicating there is no target species in the stand and one indicating a pure species stand of the target species.

Because the most commonly used site productivity measure, site index, only applies to even-aged pure species stands, a site productivity index (SPI) as determined by the dominant and codominant trees' height-diameter relationships was used as the site productivity measure for boreal mixed-species stands. Details about the development of SPI have been described in Chapter 4.

In addition to the variables representing stand density and competition, species composition, and site productivity, tree diameter was also found to significantly affect the height increment of trees. These variables were incorporated into equations to predict the parameters of equation (7.1). This procedure is similar to the method of parameter prediction commonly used for a Chapman-Richards or Weibull-type function in which the parameters of the function were related to tree and stand characteristics but the form of the original function remains unchanged (Clutter et al. 1983). The appropriate final height increment model for white spruce was found to take the form

$$(7.4)\qquad HI_S=\frac{\alpha_1}{\alpha_1-\alpha_2}\left(e^{-\alpha_2 H_S}-e^{-\alpha_1 H_S}\right)$$

with

$$(7.5) \qquad \alpha_1 = a_1 + a_2 BASUM + a_3 CI$$

$$(7.6) \qquad \alpha_2 = a_4 + a_5 DI + a_6 SC + a_7 D + a_8 \frac{1}{SPI}$$

and for aspen, the appropriate height increment model was found to be

$$(7.7) \qquad HI_A = \frac{\beta_1}{\beta_1 - \beta_2} (e^{-\beta_2 H_A} - e^{-\beta_1 H_A})$$

with

$$(7.8) \qquad \beta_1 = b_1 D + b_2 BASUM + b_3 SPI$$

$$(7.9) \qquad \beta_2 = b_4 + b_5 DI + b_6 SC$$

In equations (7.4) to (7.9), the subscripts S and A indicate white spruce and aspen respectively, HI is the annual height increment (m), H is tree height (m), CI is the tree competition index, DI is the annual diameter increment (cm), BASUM is the basal area per hectare for all species in the stand (m^2), SC is the species composition as defined in (7.3), SPI is the white spruce site productivity index (m), D is tree diameter (cm) at breast height, a_1-a_8 and b_1-b_6 are parameters to be estimated. A positive coefficient in (7.5) or (7.8) indicates a positive effect on height increment, and a positive coefficient in (7.6) or (7.9) indicates a negative effect on height increment.

7.3 The data

Data from permanent sample plots (PSP) used in this analysis were provided by the Alberta Forest Service. The data were collected over the last three decades and the PSPs were randomly located throughout the inventory areas of the province to provide representative information for a variety of densities, heights, species composition, stand structures, ages, and site conditions. A detailed description of how the data are collected and recorded can be found in the Permanent Sample Plots: Field Procedures Manual (AFS, 1990).

The original PSP data were summarized to provide additional variables such as the number of trees per hectare, basal area per ha, average height, and average diameter, both for all species combined and by species in the stand. Summarized data from one of the PSPs established in a group of four were actually retained, giving a total number of 164 PSPs selected to be used in this analysis. These selected plots have up to five remeasurements. Each non-overlapping growth period from the remeasurements defines a growth interval, that is, the growth intervals are obtained from measurements between first and second, second and third, but not first and third. A total number of 1725 growth periods was obtained from 164 PSPs for white spruce, and 1383 for aspen. Periodic height and diameter increments were obtained as the differences between the values at the end and beginning of the growth intervals.

Descriptive statistics including the mean, minimum, maximum, and standard deviation of the tree and stand characteristics at the beginning of the growth period are shown in Table 7-1. Summary statistics for annual height and diameter increment are also given in Table 7-1. For each species, approximately 80% of the observations (1368 for white spruce and 1122 for aspen) were randomly selected for model fitting and the remaining 20% (357 for white spruce and 261 for aspen) were used for model testing.

7.4 Analysis

Model fitting

Preliminary nonlinear least squares (NLS) fits of the height increment models were accomplished using the PROC MODEL procedure on SAS/ETS software (SAS Institute Inc. 1988). The Gauss-Newton iterative method using the Taylor series expansion as described in Gallant (1987) was applied in model fittings. To ensure the solution is global rather than local least squares estimates, different initial values of the parameters were chosen for the fits.

Asymptotic fit statistics including the NLS estimates of the parameters, the asymptotic *t*-statistics, standard errors, *p*-values of the parameters, the model root mean squared error (RMSE), adjusted coefficient of determination (Adj. R^2), and the Durbin-Watson statistic (D-W) for testing the first-order autocorrelation of the error terms are shown in Tables 7-2 and 7-3 for white spruce and aspen respectively. With the large number of observations, direct interpretations of the fit statistics similar to those in linear cases are applicable.

The Durbin-Watson statistic is used to test for possible model error correlations. Although the Durbin-Watson test procedure is not strictly applicable for nonlinear regression models, the test is approximately valid, especially for large samples (Amemiya 1983 p. 355; Seber and Wild 1988 pp. 318-319). Because the sample size used for this analysis is larger than the maximum sample size (200) appearing in the extended Durbin-Watson Table (Judge et al. 1988 pp. 991-994), critical values are those recorded for the maximum sample size.

The Durbin-Watson statistic of 1.641 for white spruce is smaller than the lower bound (1.697) of the critical values for the eight parameter height increment model, indicating a significant first-order autocorrelation for the error terms of the white spruce height increment model. The Durbin-Watson statistic of 1.529 for aspen is also smaller than the lower bound (1.718) of the critical values for the six parameter height increment model, indicating a significant first-order autocorrelation for the error terms of the aspen height increment model.

The autocorrelated error terms for both height increment models are typical of data from PSPs where the plots are repeatedly measured over regular time intervals. The parameter estimation for nonlinear models with first-order autoregressive errors AR(1) can be obtained by writing the *i*th observation of the height increment model as

$$(7.10) \qquad HI_i = f(x_i, \theta) + \varepsilon_i \qquad \varepsilon_i = \rho\varepsilon_{i-1} + \mu_i$$

where the height increment HI_i, $i=1, 2, ..., n$, are expressed as a nonlinear function $f(x_i, \theta)$ of the explanatory variables, ε_i are random errors following an AR(1) process, ρ is the autoregressive parameter, and μ_i are independent and identically distributed random errors with mean 0 and constant variance σ_μ^2. From equation (7.10), the (*i*-1)th observation of the height increment model can be written as

$$(7.11) \qquad HI_{(i-1)} = f(x_{i-1}, \theta) + \varepsilon_{i-1}$$

Multiplying ρ on both sides of (7.11), then subtracting (7.10) by (7.11) provides a nonlinear equation with uncorrelated errors

$$(7.12) \qquad HI_i = \rho HI_{(i-1)} + f(x_i, \theta) - \rho f(x_{i-1}, \theta) + \mu_i$$

A five step procedure can be used to find the NLS estimates of parameters θ in (7.12):

1). Fit equation (7.10) by ordinary NLS without considering its error structure, find the estimates of the parameters θ_{NLS}.

2). Calculate the residuals $\varepsilon_i = HI_i - f(x_i, \theta_{NLS})$, and estimate ρ by fitting $\varepsilon_i = \rho\varepsilon_{i-1} + \mu_i$.

3). Find the estimates of θ by applying ordinary NLS to the model

$$(7.13) \qquad HI_i - \rho HI_{(i-1)} = f(x_i, \theta) - \rho f(x_{i-1}, \theta) + e_i$$

4). Replace θ_{NLS} in step 2) by θ from 3), then re-estimate ρ in 2).

5). Replace ρ in step 3) by ρ from 4), then re-estimate θ using NLS.

This process is iterated until θ is converged. Resulting fit statistics using the SAS/ETS software for the height increment models with the AR(1) error structure for white spruce and aspen are also listed in Tables 7-2 and 7-3. The Durbin-Watson statistics (2.018 and 2.040) for both models are greater than their respective upper bounds (1.841 and 1.820) for the critical value, indicating that autocorrelation for the adjusted models is not significant and the AR(1) specification is appropriate for the error terms of the height increment models.

The height increment models were also diagnosed for possible unequal error variances. The plots of studentized residuals against the predicted height increment from the preliminary NLS fits (without accounting for the AR(1) error terms) and the NLS fits accounting for AR(1) error terms showed no consistent underestimate or overestimate for height increments. The error terms of the models were therefore considered to be identically distributed and weighted regressions were deemed unnecessary.

Diagnostics for multicollinearity

The tree and stand variables appearing in the height increment models may somehow be inter-correlated. Multicollinearity could exist in such models because of inter-correlated variables. Collinearity diagnostics using the keyword COLLIN with PROC MODEL procedure were performed to detect the presence, severity, and form of multicollinearity.

For white spruce, the eigenvalues of the correlation matrix for the set of explanatory variables were arranged from the largest (5.00443) to smallest (0.03535), and the square root of the ratio of the largest to smallest eigenvalue, the condition index, was found to be 11.8984, which is less than 30 - the proposed critical value for moderate multicollinearity (Belsley et al. 1980). This indicates that multicollinearity is not a serious problem for the white spruce height increment model.

For aspen, the eigenvalues of the correlation matrix of the set of explanatory variables were also arranged from the largest (3.85428) to smallest (0.08156), and the condition index was found to be 6.8746, which also indicates that multicollinearity is not a serious problem for aspen height increment model.

Model testing

The independent testing data sets were used for testing the height increment models' fit. For white spruce, the actual height increment values from the testing data were compared to the predicted height increment values using model (7.4) with the estimated parameters from NLS with AR(1) errors in Table 7-2. The bias of the prediction was obtained by subtracting the actual height increment from the predicted height increment by the fitted model. The mean and the standard deviation of the prediction bias were 0.0151977 and 0.175266 respectively, so the standard error of the estimated mean bias is $0.175266/(357)^{1/2}=0.009276$. A t-test of the hypothesis that the mean prediction bias was zero gives $t=1.6384$, which, with 356 degrees of freedom, is not significant at an $\alpha=0.05$ level, indicating that the mean prediction bias is not significantly different from zero.

Similarly, using the independent testing data and fitted results from Table 7-3 for aspen, the mean and the standard deviation of the prediction bias was 0.0042523 and 0.2449232 respectively, so the standard error of the estimated mean bias was $0.2449232/(261)^{1/2} = 0.01516$. A t-test of the hypothesis that the mean prediction bias was zero gives $t=0.2805$, which, with 260 degrees of freedom, is not significant at an $\alpha=0.05$ level, indicating that the mean prediction bias for aspen is also not significantly different from zero.

7.5 Discussion

The height increment models as expressed by (7.4) and (7.7) provide individual tree height increment predictions for white spruce and aspen grown in boreal mixed-species stands in Alberta. Most of the asymptotic *t*-statistics (Tables 7-2 and 7-3) for the coefficients of the models are significant at an $\alpha=0.05$ level. However, approximately 12.21% of the height increment variation for white spruce and 21.85% for aspen were explained by the fitted models. This may not be surprising because of the extremely large height increment variation found in the boreal mixed-species stands. Scatter plots of annual height increment versus explanatory variables included in the models showed very weak height increment trends. Because tree height was measured by non-destructive methods over an extended time period, considerable measurement errors might also contribute to why low height increment variation explained by the fitted models. The limited availability of height measurements aggravated this problem because only a very small subset of trees on each PSP were repeatedly measured for height.

In spite of these problems, the height increment models developed in this analysis were based on a biologically appropriate base function so on average realistic height increment predictions are achieved. The models as portrayed by Figure 7-1 have the flexibility to assume various shapes with different parameter values and produce satisfactory curves under most circumstances. All curves produced by the models assume biologically reasonable shapes that closely mimic the biological process of height growth, and provide reasonable height increment estimates in cases where the models are extrapolated beyond the range of the original data.

The fitted height increment models reflected some commonly held beliefs and interesting facts about the height increment of white spruce and aspen in the boreal mixed-species stands. The significant negative coefficients for total basal area per hectare in (7.5) and (7.8) indicate that stand density has a significant negative effect on white spruce and aspen height increment. If other stand conditions remain approximately the same, height increment is reduced as stand density increases, and dense stands result in smaller height increment. This agrees with results found by Wykoff et al. (1982) for mixed conifers of the northern Rocky Mountains and Arney (1985) for coastal Douglas-fir, although a different stand density measure was used in their studies. The reason for reduced height increment with increased stand density may relate to the fact that trees in dense stands have reduced crown development because the available growing space is limited. This in turns restricts the tree's ability to utilize the photosynthetic potential for height increment.

The effects of species composition as represented by parameters a_6 and b_6 in equations (7.6) and (7.9) are positive for both white spruce and aspen height increment, indicating that for two mixed-species stands grown in similar conditions, the one with more white spruce will have larger white spruce height increment, and the one with more aspen will have larger aspen height increment. A possible explanation for this is that under many circumstances the height increment of both species is directly related to the relative abundance of the species in the stands. The ability of a particular tree species to compete with other tree species in the stand is stronger if the relative proportion of this particular species is larger. The increasing presence of white spruce reduces the relative competitiveness of aspen in the stands and the increasing presence of aspen reduces the relative competitiveness of white spruce.

The significant negative coefficients a_5 and b_5 in equations (7.6) and (7.9) indicate that large diameter increment leads to large height increment. This is rather straightforward as increased diameter growth is commonly expected with increased height growth. Because the height increment models are expressed as a function of diameter increment, many variables that directly affect diameter increment can also indirectly affect height increment.

The coefficients a_3 and a_7 in (7.5) and (7.6), and b_1 in (7.8) reflect a positive competition index and diameter effect on height increment. As the diameter, and related CI, get numerically larger, the white spruce trees are in a better competitive position, resulting in larger height increment. The result of increased height increment with increased tree diameter is probably directly related to the increased competitiveness for larger diameter trees. This may also imply that the commonly observed asymmetric or one-side competition in plant monocultures (Weiner and Thomas 1986) is equally applicable for white spruce and aspen grown in uneven-aged mixed-species stands.

Incorporating site productivity into a growth model is a complicated problem, especially for mixed-species stands with an uneven-aged structure. This problem is precisely the one that led researchers like McLintock and Bickford (1957), Stout and Shumway (1982), and Wykoff et al. (1982) to develop alternatives to traditional site index in attempting to provide accurate and realistic site productivity measures. The use of a site productivity index (SPI) as determined by the dominant and codominant height-diameter relationship is by no means an ultimate solution. However, the positive coefficient values for a_8 and b_3 reflect the effect of site productivity on height increment and show that better sites support faster height increment. This is true both for white [illegible] and for aspen, although the SPI effect for white spruce is not significant [illegible] =0.05 level (*p*-value=0.1323).

Since the height increment models are based on functions which relate height increment to tree and stand variables, but do not involve individual tree or stand age explicitly, the models can be applied in any stands. Input variables for the models are simple tree and stand variables that are readily obtainable from most inventories, and are compatible with the data-collection process in Alberta. Potential users of the model with limited information or alternative measures for some of the variables appearing in the models may consider re-fitting other forms of the model by replacing the variables that are not available, or by dropping the variables if the cost to obtain the variables is too high relative to the amount of additional variation explained by adding such variables to the model. In either case, the flexible base height increment-height equation — the Box-Lucas function that closely mimics the biological process of height growth, and the method of parameter prediction for incorporating the effects of other tree and stand variables on height increment, should be retained.

7.6 Summary

Based on a data set from 164 permanent sample plots, age-independent individual tree height increment models for white spruce (*Picea glauca* (Moench) Voss) and trembling aspen (*Populus tremuloides* Michx.) grown in the boreal mixed-species stands in Alberta are presented. The models are age-independent in that they do not explicitly require tree/stand age as an input variable. Height increment is modelled as a function of tree diameter, tree height, diameter increment, relative competitiveness of the tree in the stand, stand density, species composition, and site productivity. Because data from permanent sample plots are considered time series and cross-sectional, diagnostic techniques were applied to identify the model's error structure. Appropriate fits based on the identified error structure were accomplished using the nonlinear least squares technique with a first-order autoregressive error process. The models were also validated on independent data sets representing the population on which the models are to be used. The average prediction biases were not significant at $\alpha=0.05$, indicating that the models appropriately describe the data and perform well when predictions are made.

Table 7-1. Summary statistics for white spruce and aspen tree and stand characteristics

Variable	White spruce				Aspen			
	Mean	Min.	Max.	Std. dev.	Mean	Min.	Max.	Std. dev.
Tree DBH (cm)	27.75	2.90	63.30	10.52	23.67	5.30	64.50	11.21
Tree height (m)	22.38	3.20	37.90	6.11	19.78	5.80	32.90	5.39
Annual diameter increment (cm)	0.21	0.00	0.99	0.13	0.22	0.0	0.72	0.12
Annual height increment (m)	0.16	-0.82	1.15	0.20	0.17	-0.96	1.18	0.26
Number of trees/ha-all species	1251.49	148	5580	744.58	1461.95	222	6864	791.88
Average DBH (cm)-all species	19.78	5.30	39.708	5.60	17.32	4.60	39.70	6.55
Basal area (m^2/ha)-all species	39.81	11.55	83.87	8.92	32.18	11.97	87.91	11.01
Average height (m)-all species	21.57	7.90	31.70	3.90	18.46	7.3	28.80	4.60
Species number of trees/ha	753.87	5.00	4914.00	670.86	919.60	10	3515	785.87
Species average diameter (cm)	20.80	5.70	54.90	6.60	20.38	2.8	46.10	9.02
Species basal area (m^2/ha)	25.10	0.20	56.65	12.11	19.89	0.47	53.33	9.26
Species average height (m)	21.97	3.60	31.70	4.26	19.37	6.30	30.60	5.08
Site productivity index (m)	16.17	8.22	21.28	2.43	18.51	8.46	24.50	3.35
Species composition	0.62	0.0062	1.00	0.25	0.65	0.016	0.99	0.26

Table 7-2. Fit statistics for white spruce height increment model[1]

	Parameter	Estimate	Std. err.	t-statistic	p-value	RMSE	Adj. R^2	D-W
NLS	a_1	0.089485	0.01939	4.61	0.0001	0.19219	0.0932	1.641
	a_2	-0.00172570	0.0003895	-4.43	0.0001			
	a_3	0.048962	0.01343	3.65	0.0003			
	a_4	0.221253	0.01971	11.23	0.0001			
	a_5	-0.161879	0.02178	-7.43	0.0001			
	a_6	-0.034426	0.01266	-2.72	0.0066			
	a_7	-0.00196558	0.0003109	-6.32	0.0001			
	a_8	0.454981	0.24568	1.85	0.0642			
NLS with AR(1) errors	a_1	0.079334	0.01900	4.18	0.0001	0.18911	0.1221	2.018
	a_2	-0.00170770	0.0003888	-4.39	0.0001			
	a_3	0.055419	0.01321	4.20	0.0001			
	a_4	0.219686	0.02167	10.14	0.0001			
	a_5	-0.152172	0.02169	-7.01	0.0001			
	a_6	-0.035352	0.01408	-2.51	0.0121			
	a_7	-0.00193545	0.0003083	-6.28	0.0001			
	a_8	0.400101	0.26573	1.51	0.1323			
	ρ	0.181556	0.02406	7.55	0.0001			

[1] NLS represents nonlinear least squares, D-W is the Durbin-Watson statistic, ρ is the first-order autoregressive AR(1) parameter.

Table 7-3. Fit statistics for aspen height increment model[1]

	Parameter	Estimate	Std. err.	t-statistic	p-value	RMSE	Adj. R^2	D-W
NLS	b_1	0.00292227	0.0012095	2.42	0.0158	0.23549	0.1739	1.529
	b_2	-0.00230944	0.0005943	-3.89	0.0001			
	b_3	0.00847480	0.0017647	4.80	0.0001			
	b_4	0.206881	0.01625	12.73	0.0001			
	b_5	-0.189355	0.02308	-8.20	0.0001			
	b_6	-0.027525	0.01583	-1.74	0.0822			
NLS with AR(1) errors	b_1	0.00242251	0.0013389	1.81	0.0706	0.22906	0.2185	2.040
	b_2	-0.00248709	0.0007048	-3.53	0.0004			
	b_3	0.00987930	0.0023087	4.28	0.0001			
	b_4	0.206976	0.01883	10.99	0.0001			
	b_5	-0.181396	0.02348	-7.72	0.0001			
	b_6	-0.033944	0.01874	-1.81	0.0703			
	ρ	0.235825	0.02658	8.87	0.0001			

---

[1] NLS represents nonlinear least squares, D-W is the Durbin-Watson statistic, ρ is the first-order autoregressive AR(1) parameter.

Figure 7.1. Typical graphs of equation (7.1) produced by varying parameters θ_1 and θ_2.

1: $\theta_1=0.050$, $\theta_2=0.200$;

2: $\theta_1=0.200$, $\theta_2=0.150$;

3: $\theta_1=0.100$, $\theta_2=0.125$;

4: $\theta_1=0.150$, $\theta_2=0.075$.

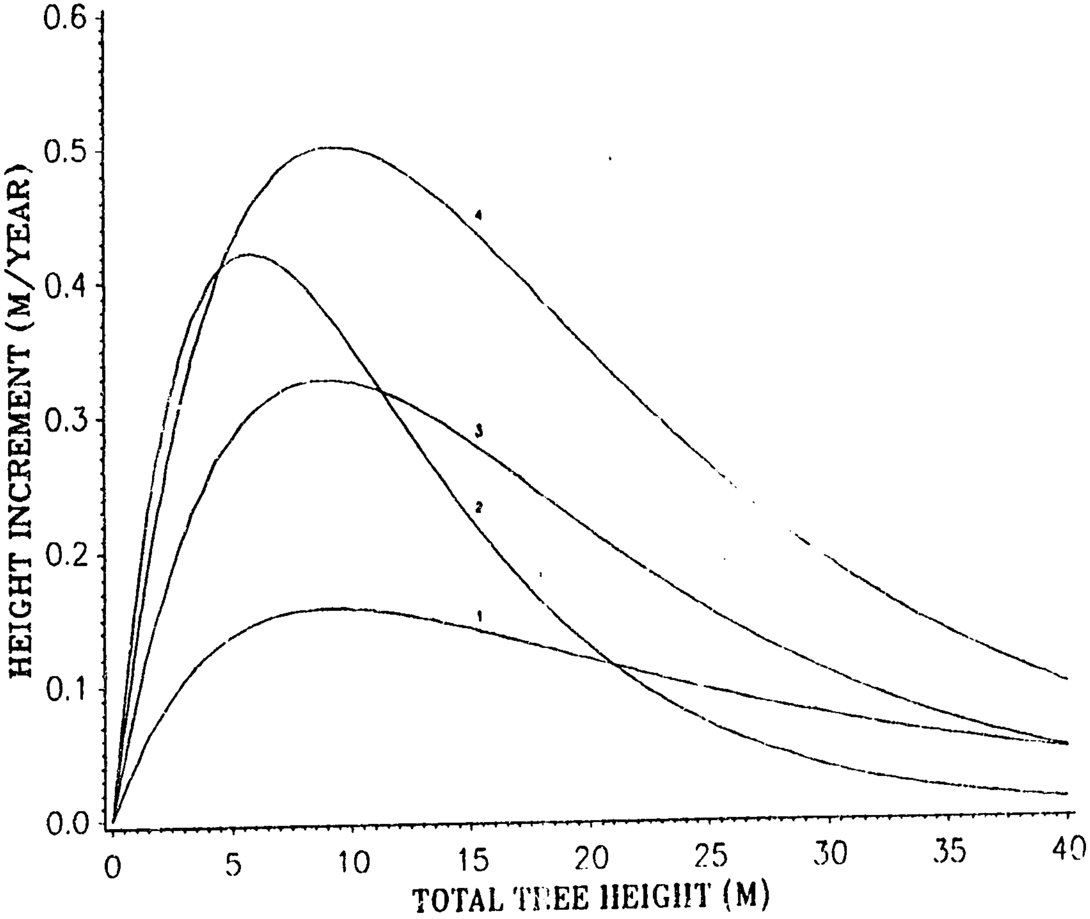
0.6
0.5
0.4
0.3
0.2
0.1
0.0
HEIGHT INCREMENT (M/YEAR)
0
5
10
15
20
25
30
35
40
TOTAL TREE HEIGHT (M)
4
3
2
1

Figure [illegible]-1.

7.7 References

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Chapter 8

Nonlinear Simultaneous Diameter and Height Growth Models

8.1 Introduction

Growth and yield models for forest stand dynamics using data from permanent sample plots with real growth series are often composed of a system of compatible, inter-dependent, and analytically related equations (Clutter 1963, Sullivan and Clutter 1972, Burkhart 1986, Daniels and Burkhart 1988). Most conceptual frameworks for understanding stand development processes recognize that there is a strong correlation and feedback mechanism among variables that are used to describe growth and yield relationships (Turnbull 1978, Munro 1984, Krumland 1982, Walters et al. 1991). This idea translates into the realization that forest stand dynamics should be described by a simultaneous and interdependent system of equations that involve current, future, and past values of some of the tree and stand characteristics rather than separate and isolated individual equations.

One method of estimating parameters in systems of equations is to develop a composite model by algebraic substitution (Sullivan and Clutter 1972). However, Burkhart and Sprinz (1984) pointed out that the estimates obtained using this method are not statistically efficient. Reed (1982), Burkhart and Sprinz (1984), Reed and Green (1984), Reed et al. (1986), Byrne and Reed (1986), and Knoebel et al. (1986) simultaneously estimated the structural parameters by minimizing squared error loss functions. Borders and Bailey (1986), and LeMay (1988, 1990) indicated that estimates obtained by minimizing squared error loss functions may still not be consistent or the most efficient if the equations are simultaneous. Van Deusen (1988) showed that estimation based on minimizing squared error loss functions is very similar to the seemingly unrelated regression (SUR) technique popularized in econometrics (Zellner 1962), and suggested that if the system of equations falls into the SUR framework, estimates of the structural parameters will be consistent and asymptotically efficient.

Application of simultaneous fitting techniques commonly used in econometrics for systems of growth and yield equations dates back to Furnival and Wilson (1971). Many other researchers have also discussed and used two- and three-stage least squares (2SLS and 3SLS), and SUR techniques for estimating the structural parameters in systems of forestry equations (Murphy and Sternitzke 1979, Murphy and Beltz 1981, Murphy 1983, Amateis et al. 1984, Burkhart 1986, Borders and Bailey 1986, Reed 1987, Van Deusen 1988). Borders (1989) presented a detailed discussion on three types of equation systems: 1) seemingly unrelated regression equations, 2) recursive equations, 3) simultaneous equations, and described associated fitting methodologies, as well as an alternative parameter estimation procedure that can be used for any number of sequentially related linear or nonlinear equations. LeMay (1988, 1990) demonstrated applications of a modification of three-stage least squares for several forestry systems of simultaneous linear equations in which the error terms of individual equations may be serially correlated and/or have unequal variances.

This study presents a nonlinear system of three interdependent equations for predicting individual tree height, periodic diameter increment, and periodic height increment for white spruce (*Picea glauca* (Moench) Voss) grown in boreal mixed-species stands in Alberta. The high correlation and simultaneous nature among tree height, diameter increment, and height increment have been reflected in many individual tree based growth and yield models in which diameter increment was expressed as a function of height increment (Hegyi 1974, Daniels and Burkhart 1975), or height increment was expressed as a function of diameter increment and total tree height (Stage 1975, Wykoff et al. 1982). Previous parameter estimation methods, however, were single-equation based. No cognizance is taken of the fact that the error terms from related equations are almost certain to be correlated, and the available information concerning the cross-equation interdependence is not utilized.

The primary objective of this analysis is to treat the nonlinear equations for predicting individual tree height, periodic diameter increment, and periodic height increment for white spruce in mixed-species stands as an integrated system and apply alternative statistical procedures to estimate the structural parameters of the system simultaneously. A second objective is to evaluate different system estimation methods for nonlinear equations and to compare them with the traditional single-equation based ordinary least squares technique. While fundamentals of the nonlinear system modelling methods are emphasized, the practical implications of such approaches for systems of interdependent forestry equations are also stressed.

8.2 The data

Data from 164 permanent sample plots (PSP) used in this analysis were provided by the Alberta Forest Service. The data were collected over the last three decades and the PSPs were randomly located throughout the inventory areas of the province to provide representative information for a variety of densities, heights, species composition, stand structures, ages, and site conditions. A detailed description of how the data are collected and recorded can be found in the Permanent Sample Plots: Field Procedures Manual (AFS 1990).

The original PSP data were summarized to provide additional variables such as the number of trees per hectare, basal area per hectare, average height, and average diameter, both for all species combined and by individual species in the stand. Only live white spruce trees with both diameter and height recorded were retained for this analysis. The selected trees have up to five remeasurements, with the time between the remeasurements ranging from 3 to 18 years. Each non-overlapping growth period from the remeasurements defines a growth interval, that is, the growth intervals are obtained from measurements between first and second, second and third, but not first and third. A total number of 1470 growth intervals was obtained from 164 PSPs for white spruce. Periodic diameter increments and height increments were obtained as the differences between the values at the end and beginning of the growth intervals. Descriptive statistics including the mean, minimum, maximum, and standard deviation of the tree and stand characteristics at the beginning of the growth intervals are displayed in Table 8-1.

8.3 System of equations and its estimations

A system of three related nonlinear equations described in Chapters 5, 6, and 7 for white spruce grown in boreal mixed-species stands were selected for this analysis:

$$(8.1)\qquad H=1.3+\alpha\left(1-e^{-\beta D}\right)^{\delta}+\varepsilon_1$$

$$\alpha=a_1\left(1-e^{-a_2 BASUM}\right)+a_3 SC+a_4 SPI+a_5 BA+a_6 AVED$$

$$\beta=a_7$$

$$\delta=a_8+a_9 D/AVED+a_{10} SPI$$

$$(8.2)\qquad DI=GI\frac{\theta_1}{\theta_1-\theta_2}\left(e^{-\theta_2 D}-e^{-\theta_1 D}\right)+\varepsilon_2$$

$$\theta_1=b_1+b_2 BASUM+b_3 SC+b_4 H+b_5 D/AVED+b_6 SPI$$

$$\theta_2=b_7+b_8\sqrt{TRHAAL}$$

$$(8.3)\qquad HI=GI\frac{\beta_1}{\beta_1-\beta_2}\left(e^{-\beta_2 H}-e^{-\beta_1 H}\right)+\varepsilon_3$$

$$\beta_1=c_1+c_2BASUM+c_3D/AVED$$

$$\beta_2=c_4+c_5DI/GI+c_6SC+c_7D+c_8/SPI$$

where

H = tree height in metres (m),

D = tree diameter at breast height (cm),

BA = basal area of the tree (cm^2),

BASUM = basal area per ha for all species combined in the stand (m^2),

SC = white spruce species composition,

SPI = site productivity index for white spruce in mixed-species stands,

AVED = average diameter for all species in the stand (cm),

DI = periodic diameter increment (cm),

TRHAAL = total number of trees per hectare,

GI = growth interval length (years),

HI = periodic height increment (m),

e = 2.71828,

ε_1, ε_2, and ε_3 = error terms for equations (8.1), (8.2), and (8.3) respectively,

a_1-a_{10}, b_1-b_8, and c_1-c_8 = parameters to be estimated.

White spruce species composition was defined as the ratio of white spruce basal area per hectare over the total basal area per hectare for all species combined in the stand. The site productivity index (SPI) based on the relationship between total tree height and diameter at breast height of the dominant and codominant trees developed for white spruce in uneven-aged and/or mixed-species stands (Chapter 4) was used as the measure of site productivity.

The system of nonlinear equations (8.1)–(8.3) takes into account the simultaneous and interdependent nature of individual tree height, periodic diameter increment, and periodic height increment predictions. It is clear that within this system, variables H and DI appear on the left-hand side (LHS) of the equations as well as on the right-hand side (RHS) of the equations in the system. Together with HI, they are the system outputs and are referred to as endogenous variables that are jointly determined through the joint interaction with other variables within the system. Variables that only appear on the RHS side of the equations in the system are referred to as exogenous or predetermined variables that are determined outside the system.

The presence of endogenous variables in the system violates the independence assumption between the RHS variables and the error terms of the equations (Pindyck and Rubenfeld 1981, Judge et al. 1985). Because DI depends on H, for example, and the observed value of H depends on the error terms ε_1, DI is also dependent on ε_1. The implication of this violation is that the classical ordinary least squares rule will produce parameter estimates that are biased and inconsistent – a phenomenon commonly referred to as simultaneous equation bias.

To eliminate simultaneous equation bias in large samples, the instrumental variable method has been commonly used (Theil 1971, Johnston 1984, Spanos 1986, SAS Institute Inc. 1988). This method is implemented by first selecting a set of instrumental variables (instruments) that are highly correlated with the endogenous variables but are uncorrelated with the error terms of the equations, and then projecting the endogenous variables into the vector space spanned by the chosen instruments and performing regressions on the projections. The instrumental variable method has also been described as the two-stage least squares (2SLS) in systems of simultaneous linear equations, with a first-stage regression of the RHS endogenous variables on the instruments and a second-stage regression on the structural equations with the RHS endogenous variables replaced by the predicted values from the first-stage regression. Amemiya (1974) extended the 2SLS principle to systems of simultaneous nonlinear equations, and derived the nonlinear two-stage least squares (N2SLS) estimator for nonlinear systems.

The estimates of the parameters for each nonlinear equation resulting from N2SLS may still be biased in small samples but are consistent and asymptotically efficient with respect to other single-equation estimators (Amemiya 1985, Judge et al. 1985). A consistent and asymptotically more efficient estimator may be obtained if the system of equations is jointly estimated within the framework of the SUR procedure (Zellner 1962) and its extension for nonlinear systems (Gallant 1975). The nonlinear three-stage least squares (N3SLS) technique described by Jorgenson and Laffont (1974), Gallant (1977), and Gallant and Jorgenson (1979) employed generalized nonlinear least squares procedures to systems of simultaneous nonlinear equations by utilizing the variance-covariance matrix of the structural errors estimated from the residuals obtained with N2SLS. The N3SLS estimator takes into account the correlations of error terms across equations, and makes use of the information that may be available concerning the variance-covariance matrix of the error terms across different structural equations. Resulting estimates of the parameters for the entire system from N3SLS are consistent and asymptotically more efficient than those obtained from the N2SLS if the cross-equation correlations are significant. If the cross-equation covariances are all zero, there is no gain in efficiency for N3SLS over N2SLS.

Two frequently encountered problems in two- and three-stage least squares are identification and the choice of instruments. In order to apply two- and three-stage least squares, each equation in the system of simultaneous equations must be just- or over-identified (Judge et al. 1988). Identification requires that certain rank and order conditions be satisfied, or the structural parameters can not be consistently estimated. For systems of simultaneous linear equations, identification is relatively simple and straightforward (Fisher 1966, Dhrymes 1974, Brown 1991). Amateis et al. (1984) and Borders and Bailey (1986) discussed this problem in forestry situations.

For a system of nonlinear equations such as (8.1)--(8.3), identification can be more complicated (Fisher 1966, Brown 1983, Judge et al. 1985). However, the problem is not rigorous enough, and Gallant (1987) completely ignored identification in systems of simultaneous nonlinear equations. Amemiya (1985) was also critical on the identification problem in nonlinear systems and pointed out that nonlinearity generally helps rather than hampers identification, so that, for example, in a nonlinear simultaneous system the number of excluded exogenous variables in a given equation need not be greater than or equal to the number of included endogenous variables minus one of the same equation. Each equation in the system of nonlinear equations (8.1)—(8.3) is over-identified according to the criteria set by Amemiya (1985).

The problem of finding instrumental variables for simultaneous linear equations is once again relatively simple and straightforward. Asymptotic efficiency of the 2SLS estimator is maximized if all the predetermined variables including the exogenous variables, the lagged exogenous variables, and the lagged endogenous variables for the entire system are chosen as instruments (Schmidt 1976, Judge et al. 1985). Finding an appropriate set of instrument variables in systems of nonlinear simultaneous equations is a very difficult problem, and the theory for doing so is not complete (Amemiya 1985, Judge et al. 1985, Gallant 1987). The implication of this problem in terms of system fitting is also discussed in the SAS/ETS manual (SAS Institute Inc. 1988). The most disturbing aspect of the N2SLS and N3SLS estimators is that they are not invariant with respect to the choice of instruments. Different sets of instruments can lead to quite different parameter estimates even though the model specification and data remain the same (Gallant 1987).

The proper choice of instruments is crucial for obtaining efficient N2SLS and N3SLS estimators. It is necessary that the number of selected instruments at least equals the maximum number of parameters in any equation of the system, or some of the parameters can not be estimated. Using fewer instruments reduces the efficiency of N2SLS and N3SLS estimators, and adding more instruments may improve the efficiency of the estimators, however, after some point this may also reduce the efficiency and increase simultaneous equation bias. Gallant (1987) showed that if too many instruments are used, nonlinear two- and three-stage least squares reduce to ordinary nonlinear least squares.

It is essential that an appropriate set of instrumental variables is chosen in order to maximize the efficiency of instrumental variables and reduce the simultaneous equation bias. Judge et al. (1985) described alternative choices of instruments. Amemiya (1977) suggested that the optimal choice of instrumental variables could be obtained by using the expected values of the partial derivatives of the residuals with respect to the parameters — the parameter derivatives as instruments. Accordingly, it is best to find instruments that approximate the expected values over the errors of each of the parameter derivatives. However, Gallant (1987) noted that this could lead to a large instrumental variable matrix that adds to the small sample bias of the estimator and reduces the small sample variance, and consequently leads to very misleading confidence intervals. One practical choice of instruments suggested by Amemiya (1985) and Gallant (1987) is to use the exogenous variables and their low order monomials such as the squared exogenous variables and the cross products of the exogenous variables as instruments, making no attempt to find the "optimal" set using the results on efficiency.

The system of equations (8.1)—(8.3) and some other examples presented in forestry literature (Amateis et al. 1984, Borders and Bailey 1986, Borders 1989, LeMay 1988, 1990) possess a clear resemblance to a special case of simultaneous equations — the recursive system of equations (Klein 1974, Pindyck and Rubinfeld 1981). Notice that the endogenous variable H in equation (8.1) is not a function of other endogenous variables, but the endogenous variable DI in equation (8.2) is a function of H in equation (8.1), and the endogenous variable HI in equation (8.3) is a function of H in (8.1) and DI in (8.2). Application of the ordinary least squares method to each of the structural equations leads to unbiased, consistent, and asymptotically efficient parameter estimates if the variance-covariance matrix of the structural errors for the recursive system is diagonal (Klein 1974, Kmenta 1986).

If significant cross-equation correlations are present, the RHS endogenous variables will be correlated with the error terms of the equations, and the system is triangular rather than recursive (Klein 1974, Lahiri and Schmidt 1978, Kmenta 1986). The ordinary least squares rule applied to a triangular system will produce simultaneous equation bias. Klein (1974 p. 199) suggested that consistent estimates for triangular systems can be obtained by a modification to the two-stage least squares by regressing, for example,

$$
\begin{aligned}
(8.4) \qquad & y_1 \text{ on } x_1, x_2, \ldots, x_{m_1} \\
& y_2 \text{ on } \hat{y}_1, x_1, x_2, \ldots, x_{m_2} \\
& y_3 \text{ on } \hat{y}_1, \hat{y}_2, x_1, x_2, \ldots, x_{m_3} \qquad \text{etc.}
\end{aligned}
$$

where x's and y's are the exogenous and endogenous variables, the $\hat{y}_1$, $\hat{y}_2$, ... are the predicted endogenous variables obtained from ordinary least squares fits of the previous equations. Klein's (1974) procedure could be combined with the SUR procedure (termed triangular-SUR, or T-SUR) for systems of linear or nonlinear triangular equations (Hausman 1975). Zellner (1971) showed from a Bayesian point of view that the posterior probability density function of the triangular systems was the same as that of the SUR equations, also indicating that SUR procedure could be used for triangular systems. Parameter estimates for triangular systems obtained by the T-SUR method will be consistent and asymptotically efficient (Kmenta 1986). Borders (1989) demonstrated the T-SUR approach for triangular linear and nonlinear equations in forest stand modelling.

For systems of linear equations, the T-SUR approach is asymptotically less efficient than the 3SLS approach. This is because 3SLS uses all exogenous variables in the entire system for the first-stage regression, rather than only those appearing in each equation. As previously described, asymptotic efficiency is maximized if all the exogenous variables in the entire system are chosen as instruments (Schmidt 1976, Judge et al. 1985). For systems of nonlinear equations, however, either the T-SUR or the N3SLS approach can be asymptotically more efficient. Naturally, this will depend on the choice of instruments. Lack of an "optimal" set of instruments for nonlinear systems and ignoring the appropriately identified equation forms in the first-stage of N3SLS can serious undermine the efficiency for using the N3SLS. Borders (1989) suggested that the T-SUR approach has some advantages over N3SLS in practice.

8.4 Methods

Table 8-2 lists all nine exogenous variables appearing in the system of nonlinear equations (8.1)—(8.3). A total number of 54 variables consisting of the exogenous variables and their cross products shown in Table 8-2 were selected as the instrumental variables. Each of the three equations in the system was first fitted by the single-equation method — nonlinear ordinary least squares (NOLS) and N2SLS, and then by the system method — N3SLS and nonlinear T-SUR. All fits were accomplished using the PROC MODEL procedure on SAS/ETS software (SAS Institute Inc., 1988). The Gauss-Newton iterative method using the Taylor series expansion as described in Gallant (1987) was applied for all the fits. The root mean squared error (RMSE) for each equation is calculated according to

$$(8.5) \qquad RMSE=\sqrt{\frac{1}{n-p_j}\Sigma\,(y_i-\hat{y}_i)^2}$$

where n is the number of observations, p_j is the number of parameters in the *j*th equation (j=1, 2, 3), y_i and $\hat{y}_i$ are observed and predicted values of the dependent variable (i=1, 2, ..., n). The coefficient of determination (R^2) for each equation is calculated by

$$(8.6) \qquad R^2=1-\Sigma\,(y_i-\hat{y}_i)^2/\Sigma\,(y_i-\bar{y})^2$$

where $\bar{y}$ is the observed average value of the dependent variable. The mean difference (MD) for each equation is calculated according to

$$(8.7) \qquad MD=\frac{1}{n}\Sigma\,(y_i-\hat{y}_i)$$

Residuals from N2SLS were used to test the significance of the cross-equation correlation. The Lagrange multiplier statistic proposed by Breusch and Pagan (1980) and described in Judge et al. (1988) was used as the appropriate test statistic for testing whether the cross-equation covariances are zero. For the system of three equations in this analysis, the null and alternative hypothesis for this test can be written as

H_0: $\sigma_{12}=\sigma_{13}=\sigma_{23}=0$

H_1: at least one covariance is nonzero

where σ_{12}, σ_{13}, and σ_{23} are the covariances between equations (8.1) and (8.2), (8.1) and (8.3), (8.2) and (8.3). The Breusch and Pagan (1980) test statistic for the three-equation system is given by

$$(8.8) \qquad \lambda=n\,(r_{21}^2+r_{31}^2+r_{32}^2)$$

where r_{ij}^2 is the squared correlation between errors in equations i and j and

$$(8.9) \qquad r_{ij}^2=\frac{\hat{\sigma}_{ij}^2}{\hat{\sigma}_{ii}\hat{\sigma}_{jj}}$$

The notation σ_{ii} is an alternative way of writing the error variance σ_i^2 for the ith equation. Under the null hypothesis H_0, λ has an asymptotic χ^2-distribution with 3 degrees of freedom. The null hypothesis is rejected if λ is greater than the critical value from a $\chi^2_{(3)}$-distribution at a prespecified significance level. Covariance of residuals and correlation of residuals (covariance of the residuals matrix converted to correlational form) are readily available from the PROC MODEL procedure.

8.5 Results and discussion

The upper half estimates of the symmetric asymptotic variance-covariance and correlation matrices of the cross-equation errors resulting from N2SLS are listed in Table 8-3. Fit statistics including the estimates of the parameters, their asymptotic standard errors, the ratio of the estimates to the standard errors (t-ratio), and the significance probability (p-value) of the t-ratio given that the t-ratio has a Student's t-distribution, from NOLS and N2SLS, N3SLS and nonlinear T-SUR are attached in Tables 8-4 and 8-5 respectively. The root mean squared error (RMSE), coefficient of determination (R^2), and mean difference (MD) values calculated according to (8.5), (8.6), and (8.7) for each of the three equations in the system are attached in Table 8-6.

Using the results from Table 8-3 and equation (8.8), the Breusch and Pagan (1980) test statistic is calculated as: $\lambda = 1470[0.0331^2+(-0.1012)^2+(-0.0871)^2] = 27.8175$. The critical value from a χ^2-distribution with 3 degrees of freedom at $\alpha = 0.05$ significance level is 7.81. Hence the null hypothesis of zero covariances is rejected, and the cross-equation correlation is significant.

The significant cross-equation correlation indicates that 1) the system estimation methods (Table 8-5) are asymptotically more efficient than the single-equation methods (Table 8-4), 2) ordinary nonlinear least squares applied to separate equations produces inconsistent or inefficient parameter estimates. Nevertheless the results from NOLS are shown and compared to others because NOLS is commonly used in applied work.

Fit statistics in Tables 8-4 and 8-5 indicate that the estimated coefficients using alternative methods are very similar. The p-values of the coefficients are all significant at $\alpha = 0.05$ level. The standard errors of the coefficients from NOLS are generally smaller than those from N2SLS and N3SLS. However, the NOLS estimates are inconsistent so that the confidence intervals can be very misleading and hypothesis tests can not be appropriately performed (Gallant 1987). The standard errors of the coefficients from N3SLS are generally smaller than those from N2SLS. This is apparent since N3SLS accounts for the significant cross-equation correlation.

The standard errors of the coefficients from nonlinear T-SUR are generally smaller than those from N3SLS (Table 8-5) and NOLS (Table 8-4). Two possible explanations may apply in this case: 1) estimates obtained from N3SLS ignore the original nonlinear equation forms in first-stage regressions, thus allowing for possible large model specification bias at this first stage (Borders 1989), and consequently affect second- and third-stage regressions; 2) lack of an "optimal" set of instruments for N3SLS reduces its ability to achieve the most efficient estimates.

The nonlinear T-SUR procedure used in this analysis can be regarded as a special instrumental method applied to systems of nonlinear equations (Borders 1989). The method emphasizes maintaining the original nonlinear forms of the equations rather than selecting the "optimal" instruments. It is possible that the efficiency of T-SUR may be lost because of ignoring the most efficient choice of instruments. However, lack of "optimal" instruments for nonlinear systems diminishes the importance of choosing instruments and enhances the significance of keeping the appropriately identified equation forms. For this particular example, the nonlinear T-SUR procedure provided more acceptable parameter estimates because of their smaller standard errors than those obtained using N3SLS, or NOLS.

Comparison of the fitting techniques in terms of equation R^2 and RMSE values commonly reported in regression analysis was also made. Table 8-6 shows that for equation (8.1), different methods give almost the same R^2 and RMSE values. For equations (8.2) and (8.3), NOLS gives the largest R^2 values and the smallest RMSE values. The differences, however, are minimal. This is expected because NOLS minimizes the error sum of squares, and the R^2 and RMSE that based on error sum of squares will generally give the most favourable results for NOLS. In many other reported comparisons between system methods and ordinary least squares technique (Murphy 1983, Amaties et al. 1984, Borders and Bailey 1986, LeMay 1988), the R^2 and RMSE values also favoured the ordinary least squares. Gallant (1987) indicated that the system methods are mainly used to inflate the underestimated errors of ordinary least squares. Strictly speaking, the R^2 and RMSE values from NOLS are not directly comparable to those from system methods because estimates obtained from NOLS are inconsistent, and the ordinary least squares tends to give smaller but inappropriate error variances.

Table 8-6 also provides information on the prediction performance of each equation fitted by different methods. The predicted endogenous variables from alternative fits were compared to the observed values. The mean difference calculated according to (8.7) indicates tree height [equation (8.1)] and periodic tree diameter increment [equation (8.2)] are slightly overpredicted by all four methods, while the periodic tree height increment [equation (8.3)] is slightly underpredicted by N2SLS and N3SLS but overpredicted by NOLS and T-SUR. The differences among the methods are somewhat mixed and very small. There is no obvious superiority shown by any particular method for all the equations in terms of predictions. Previous applications of the system methods also indicate very similar results from system methods and ordinary least squares (Murphy 1983, Amateis et al. 1984, Borders and Bailey 1986, LeMay 1988). Nonetheless, the application of system methods is more appealing (Amateis 1984) and intuitively reasonable (Borders 1989) because of their well-grounded theoretical soundness. The small standard errors of the coefficients from the nonlinear T-SUR procedure indicates that this system estimation method is most appropriate.

In practical terms, it should be pointed out that estimators such as N2SLS, N3SLS, and T-SUR rely heavily on asymptotic theory. The small sample properties of these estimators are extremely difficult to derive and in many cases, simply do not exist. Thus, in applying the consistent and asymptotically more efficient estimators, it is worthwhile to note that these estimators may not be superior in a particular case, especially if the sample size is not large enough. However, numerous Monte Carlo experiments have demonstrated that asymptotic theory provides a good approximation to the finite sample estimator performance (Cragg 1967, Judge et al. 1985), and the use of consistent and asymptotically efficient estimators for simultaneous equation problems should provide more reasonable and realistic parameter estimates.

The use of alternative single-equation and system estimators demonstrated in this analysis requires that the error terms for each equation in the system be independent and identically distributed. Many growth and yield related simultaneous equations often involve the use of observations that are made over time, or cross-sectional. In these cases the error terms in each structural equation are likely to be autocorrelated, or the assumption of homoskedasticity is frequently not met. Kmenta (1986, pp. 704-711) described procedures for estimating simultaneous linear equation systems in which the error terms in individual equations are heteroskedastic and/or autocorrelated. LeMay (1988, 1990) also described similar techniques and showed several applications for systems of simultaneous linear equations in forestry growth and yield studies. Gallant (1987 pp. 433-451) described estimators that can be used for nonlinear systems where heteroskedasticity or serial correlation occurs. Although there is a substantial difference in theory between the static and dynamic nonlinear systems of simultaneous equations, there is little difference in terms of practical applications (Gallant 1987). If the correction for the error terms of the equations is deemed necessary, the heteroskedasticity or serial correlation may be modelled along the lines demonstrated in Gallant (1987) for nonlinear systems and in Kmenta (1986) and LeMay (1990) for linear systems. The two-step transformation procedure as described in Chapter 2 is used in this analysis. The necessary SAS code for fitting the system of three simultaneous nonlinear equations using the T-SUR method is attached in Appendix 3. The error terms of the equations are heteroskedastic for the tree height prediction equation (8.1), heteroskedastic and serially correlated for the diameter increment equation (8.2), and serially correlated for the height increment equation (8.3). Table 8-7 shows the resulting fit statistics.

8.6 Summary

A system of three related nonlinear equations for predicting individual tree height, periodic diameter growth, and periodic height growth is presented for white spruce (*Picea glauca* (Moench) Voss) grown in boreal mixed-species stands in Alberta. Because of the interdependent nature of the equations in the system and the significant cross-equation correlation, alternative system estimation methods such as seemingly unrelated regression, two- and three-stage least squares commonly used in econometrics were applied to estimate the structural parameters simultaneously. Resulting fit statistics from system methods were also compared to those obtained from ordinary least squares. While the appropriate system methods are recommended for estimating parameters in interdependent systems of forestry equations, they should not be used without some caution.

Table 8-1. Summary statistics for white spruce tree and stand characteristics

	Mean	Minimum	Maximum	Std. dev.
Tree DBH (cm)	27.06	2.90	63.30	10.29
Tree height (m)	21.75	3.20	37.90	5.94
Periodic diameter increment (cm)	1.94	0.00	10.50	1.59
Periodic height increment (m)	1.68	-0.82	8.6	1.38
Growth interval length (year)	8.94	2.30	18.90	8.94
Number of trees/ha-all species	1244.29	148	5580	754.24
Average DBH (cm)-all species	19.72	5.30	39.70	5.60
Basal area (m^2/ha)-all species	39.09	11.55	83.87	9.39
Average height (m)-all species	21.22	7.90	31.70	3.88
Spruce number of trees/ha	753.07	5.00	4914.00	673.15
Spruce average diameter (cm)	20.60	5.70	54.90	6.47
Spruce basal area (m^2/ha)	24.63	0.20	56.65	11.98
Spruce average height (m)	21.57	3.60	31.70	4.25
Site productivity index (m)	16.14	8.22	21.28	2.39
Species composition	0.62	0.0062	1.00	0.26

Table 8-2. The selection of the instrumental variables

Exogenous variable[1]	V_1	V_2	V_3	V_4	V_5	V_6	V_7	V_8	V_9
V_1	V_1V_1								
V_2	V_2V_1	V_2V_2							
V_3	V_3V_1	V_3V_2	V_3V_3						
V_4	V_4V_1	V_4V_2	V_4V_3	V_4V_4					
V_5	V_5V_1	V_5V_2	V_5V_3	V_5V_4	V_5V_5				
V_6	V_6V_1	V_6V_2	V_6V_3	V_6V_4	V_6V_5	V_6V_6			
V_7	V_7V_1	V_7V_2	V_7V_3	V_7V_4	V_7V_5	V_7V_6	V_7V_7		
V_8	V_8V_1	V_8V_2	V_8V_3	V_8V_4	V_8V_5	V_8V_6	V_8V_7	V_8V_8	
V_9	V_9V_1	V_9V_2	V_9V_3	V_9V_4	V_9V_5	V_9V_6	V_9V_7	V_9V_8	V_9V_9

[1] V_1=D–tree diameter at breast height (cm), V_2=BASUM–basal area per hectare (m^2), V_3=SC–species composition, V_4=SPI–site productivity index, V_5=BA–tree basal area (cm^2), V_6=AVED–average diameter for all species (cm), V_7=D/AVED, V_8=GI–growth interval length (years), V_9=TRHAAL–total number of trees per hectare.

Table 8-3. Variance-covariance and correlation matrices of the errors from N2SLS[1]

	Variance-covariance			Correlation		
	H	DI	HI	H	DI	HI
H	3.7898	0.0694	-0.2283	1.0000	0.0331	-0.1012
DI		1.1640	-0.1088		1.0000	-0.0871
HI			1.3424			1.0000

[1] N2SLS = nonlinear two-stage least squares with the instruments selected according to Table 8-2.

Table 8-4. Parameter estimation for system of equations using NOLS and N2SLS

Equation	Parameter	Nonlinear ordinary least squares				Nonlinear two-stage least squares			
		Estimate	Std. err.	t-ratio	p-value	Estimate	Std. err.	t-ratio	p-value
(8.1)	a_1	11.218036	1.55812	7.20	0.0001	11.482219	1.66568	6.89	0.0001
	a_2	0.089138	0.01041	8.56	0.0001	0.089565	0.01045	8.57	0.0001
	a_3	1.025947	0.30941	3.32	0.0009	1.026607	0.31208	3.29	0.0010
	a_4	0.807880	0.05063	15.96	0.0001	0.808372	0.05202	15.54	0.0001
	a_5	0.00293770	0.0004829	6.08	0.0001	0.00285163	0.0005078	5.62	0.0001
	a_6	0.056449	0.01753	3.22	0.0013	0.055628	0.01791	3.11	0.0019
	a_7	0.083916	0.0090415	9.28	0.0001	0.082115	0.0093184	8.81	0.0001
	a_8	2.262892	0.26566	8.52	0.0001	2.233268	0.26334	8.48	0.0001
	a_9	0.295026	0.10229	2.88	0.0040	0.286391	0.10004	2.86	0.0043
	a_{10}	-0.034489	0.01292	-2.67	0.0077	-0.034389	0.01271	-2.71	0.0069
(8.2)	b_1	0.048690	0.0039207	12.42	0.0001	0.048043	0.0039934	12.03	0.0001
	b_2	-0.00018982	0.00004289	-4.43	0.0001	-0.00016105	0.00004554	-3.54	0.0004
	b_3	-0.012564	0.0018746	-6.70	0.0001	-0.013129	0.0018931	-6.94	0.0001
	b_4	-0.00047230	0.00009875	-4.78	0.0001	-0.00068193	0.0001124	-6.07	0.0001
	b_5	0.00409923	0.0014328	2.86	0.0043	0.00581368	0.0015516	3.75	0.0002
	b_6	-0.00057139	0.0002204	-2.59	0.0096	-0.00044935	0.0002234	-2.01	0.0445
	b_7	0.015078	0.0039950	3.77	0.0002	0.00998818	0.0043096	2.32	0.0206
	b_8	0.00077937	0.0001467	5.31	0.0001	0.00092618	0.0001477	6.27	0.0001
(8.3)	c_1	0.066664	0.0097843	6.81	0.0001	0.069450	0.01001	6.94	0.0001
	c_2	-0.00133895	0.0001965	-6.82	0.0001	-0.00138436	0.0002158	-6.42	0.0001
	c_3	0.050542	0.0081085	6.23	0.0001	0.046988	0.0093336	5.03	0.0001
	c_4	0.186661	0.01032	18.08	0.0001	0.193588	0.01164	16.63	0.0001
	c_5	-0.128393	0.01296	-9.91	0.0001	-0.198156	0.03080	-6.43	0.0001
	c_6	-0.030134	0.0070864	-4.25	0.0001	-0.037298	0.0079266	-4.71	0.0001
	c_7	-0.00185662	0.0002187	-8.49	0.0001	-0.00190990	0.0002550	-7.49	0.0001
	c_8	0.587724	0.13541	4.34	0.0001	0.863626	0.17793	4.85	0.0001

Table 8-5. Parameter estimation for system of equations using N3SLS and T-SUR

Equation	Parameter	Nonlinear three-stage least squares				Nonlinear triangular-SUR			
		Estimate	Std. err.	t-ratio	p-value	Estimate	Std. err.	t-ratio	p-value
(8.1)	a_1	11.172518	1.60629	6.96	0.0001	10.957086	1.51578	7.23	0.0001
	a_2	0.088737	0.01046	8.48	0.0001	0.088256	0.01040	8.48	0.0001
	a_3	1.044479	0.30990	3.37	0.0008	1.046218	0.30780	3.40	0.0007
	a_4	0.816346	0.05125	15.93	0.0001	0.816017	0.05005	16.30	0.0001
	a_5	0.00296158	0.0004943	5.99	0.0001	0.00303570	0.0004728	6.42	0.0001
	a_6	0.054278	0.01761	3.08	0.0021	0.055087	0.01730	3.18	0.0015
	a_7	0.083506	0.0092649	9.01	0.0001	0.085021	0.0089932	9.45	0.0001
	a_8	2.212667	0.26328	8.40	0.0001	2.243809	0.26533	8.46	0.0001
	a_9	0.301613	0.10300	2.93	0.0035	0.310303	0.10500	2.96	0.0032
	a_{10}	-0.032374	0.01291	-2.51	0.0122	-0.032710	0.01308	-2.50	0.0125
(8.2)	b_1	0.048021	0.0039825	12.06	0.0001	0.047812	0.0038425	12.44	0.0001
	b_2	-0.00016160	0.00004542	-3.56	0.0004	-0.00018658	0.00004204	-4.44	0.0001
	b_3	-0.013137	0.0018866	-6.96	0.0001	-0.012258	0.0018327	-6.69	0.0001
	b_4	-0.00068478	0.0001121	-6.11	0.0001	-0.00051681	0.00009618	-5.37	0.0001
	b_5	0.00583859	0.0015481	3.77	0.0002	0.00415442	0.0014085	2.95	0.0032
	b_6	-0.00044465	0.0002227	-2.00	0.0461	-0.00050179	0.0002139	-2.35	0.0191
	b_7	0.00996592	0.0042986	2.32	0.0206	0.014100	0.0040028	3.52	0.0004
	b_8	0.00092625	0.0001472	6.29	0.0001	0.00077972	0.0001464	5.33	0.0001
(8.3)	c_1	0.067373	0.0098981	6.81	0.0001	0.065396	0.0094511	6.92	0.0001
	c_2	-0.00130449	0.0002132	-6.12	0.0001	-0.00120439	0.0001902	-6.33	0.0001
	c_3	0.044512	0.0089701	4.96	0.0001	0.042037	0.0075224	5.59	0.0001
	c_4	0.192634	0.01178	16.36	0.0001	0.186384	0.01100	16.94	0.0001
	c_5	-0.197615	0.03107	-6.36	0.0001	-0.164002	0.01330	-12.33	0.0001
	c_6	-0.036827	0.0079810	-4.61	0.0001	-0.033151	0.0073896	-4.49	0.0001
	c_7	-0.00184118	0.0002664	-6.91	0.0001	-0.00152457	0.0002581	-5.91	0.0001
	c_8	0.847525	0.17933	4.73	0.0001	0.624462	0.14121	4.42	0.0001

Table 8-6. Comparison of the fitting techniques for system of equations[1]

Equation	Method	R^2	RMSE	MO	MP	MD
(8.1)						
	NOLS	0.8878	1.94671	21.75	22.04059	-0.0009486
	N2SLS	0.8878	1.94674	21.75	22.04030	-0.0007576
	N3SLS	0.8878	1.94687	21.75	22.05812	-0.0057314
	T-SUR	0.8878	1.94688	21.75	22.05519	-0.0006030
(8.2)						
	NOLS	0.5587	1.07669	1.94	2.00599	-0.0313315
	N2SLS	0.5569	1.07888	1.94	2.00527	-0.0292220
	N3SLS	0.5568	1.07894	1.94	2.00564	-0.0293372
	T-SUR	0.5586	1.07684	1.94	2.00720	-0.0316797
(8.3)						
	NOLS	0.3325	1.14128	1.68	1.61499	-0.0248991
	N2SLS	0.3121	1.15862	1.68	1.60327	0.0020513
	N3SLS	0.3143	1.15670	1.68	1.59876	0.0112269
	T-SUR	0.3289	1.14434	1.68	1.60700	-0.0031995

[1] Note: R^2, RMSE, and MD are calculated according to equations (8.4), (8.5), and (8.6), MO and MP are the observed and predicted mean values of the endogenous variables.

Table 8-7. Parameter estimation for system of equations with generalized error structures

Equation	Parameter	Nonlinear T-SUR estimation			
		Estimate	Std. err.	t-ratio	p-value
(8.1)	a_1	10.104977	1.26368	8.00	0.0001
	a_2	0.070135	0.008262	8.49	0.0001
	a_3	1.734336	0.29229	5.93	0.0001
	a_4	0.796076	0.04629	17.20	0.0001
	a_5	0.00369014	0.0004478	8.24	0.0001
	a_6	0.056546	0.01735	3.26	0.0011
	a_7	0.091151	0.0076022	11.99	0.0001
	a_8	2.135057	0.20827	10.25	0.0001
	a_9	0.343421	0.09945	3.45	0.0006
	a_{10}	-0.024003	0.01108	-2.17	0.0304
(8.2)	b_1	0.066885	0.0049340	13.56	0.0001
	b_2	-0.00042963	0.00005349	-8.03	0.0001
	b_3	-0.022495	0.0021920	-10.26	0.0001
	b_4	-0.00072404	0.0001311	-5.52	0.0001
	b_5	0.017464	0.0022928	7.62	0.0001
	b_6	-0.00093153	0.0002623	-3.55	0.0004
	b_7	0.013584	0.0045312	3.00	0.0028
	b_8	0.0012556	0.0001513	8.30	0.0001
	ρ	0.052055	0.03304	1.58	0.1154
(8.3)	c_1	0.048029	0.0098842	4.86	0.0001
	c_2	-0.00106107	0.0002094	-5.07	0.0001
	c_3	0.045080	0.0077092	5.85	0.0001
	c_4	0.187962	0.01639	11.47	0.0001
	c_5	-0.131929	0.01858	-7.10	0.0001
	c_6	-0.020676	0.01132	-1.83	0.0679
	c_7	-0.00074692	0.0004239	-1.76	0.0783
	c_8	0.185977	0.20701	0.90	0.3691
	ρ	0.150496	0.02346	6.42	0.0001

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Chapter 9

General Discussion and Conclusions

In developing individual equations under the framework of an individual tree distance-independent growth and yield model, this study emphasized 1) selection of the appropriate base functions, 2) methods for incorporating other tree and stand characteristics into the base functions, and 3) appropriate procedures for estimating parameters of the equations. Simultaneous nonlinear fitting methodologies applied to a system of three related nonlinear equations for predicting individual tree height, periodic diameter increment, and period height increment represented an important step towards a more rational and realistic approach for modelling interdependent systems of simultaneous equations frequently encountered in growth and yield studies. The theoretical advantages for using simultaneous methods are obvious, although in practical terms the gains for using such methods may not be readily evident.

Growth and yield models for stand dynamics using data from permanent sample plots with real growth series often consist of systems of compatible, interdependent, and analytically related equations. Strong correlation and feedback mechanism among variables that are used to describe growth and yield relationships have been recognized in many growth and yield studies as the essential components for understanding stand development processes. Application of simultaneous estimation captures important information about the interdependent nature of the growth and yield relationships, and emphasizes the simultaneously and jointly determined characteristics of stand development patterns.

The four objectives stated in Chapter 1 were accomplished in this study. Equations for predicting individual tree height, periodic diameter increment, and periodic height increment under the framework of an individual tree distance-independent growth and yield model were developed. The effects of species composition, stand density, and site productivity on tree growth and yield relationships have been evaluated. The interdependent nature among the equations has been described and alternative statistical procedures were applied for estimating the structural parameters of the system of equations simultaneously. Comparison of the different estimation methods for nonlinear equations showed that the triangular-seemingly unrelated regression procedure was the most appropriate for the system of three nonlinear equations considered in this study. Comparison of the system methods with the traditional single-equation based ordinary least squares techniques indicated noticeable advantages for the system methods.

The methodological contributions of this study may be interpreted on several different aspects:

1). Comparison of the nonlinear height-diameter functions under the heteroskedastic error structure. Certain items must be noted if a nonlinear regression equation is to be compared to the others. A simple method of finding appropriate weights using the studentized residuals was shown to produce satisfactory results. This comparison also provided guidance for the selection of the base function in developing the age-independent individual tree height prediction model.

2). Development of a site productivity index for uneven-aged and mixed-species stands. Methods for measuring site productivity have always been surrounded by controversy. Traditional site index has been widely used in North America, yet it is often ignored that site index only applies to even-aged and pure species stands. While the use of site productivity index as determined by the dominant and codominant height-diameter relationship is by no means the ultimate solution, it provides a simple and convenient index of site productivity for uneven-aged and mixed-species stands commonly found in the boreal forest regions of Alberta.

3). Using an appropriately selected base function, and the method of parameter prediction, a height prediction model that relates individual tree height to many other tree and stand characteristics was formulated. This height prediction model is unique in that it incorporates additional tree and stand level variables (including site, density, and species composition) to an extent that most of the previous models have not.

4). Development of periodic diameter increment and height increment models for selected major tree species found in boreal mixed-species stands in Alberta. The use of the Box-Lucas equation has apparently never been reported in any studies of growth and yield, yet it poses many desirable properties that are typical of biological growth processes.

5). Use of simultaneous estimation methods for fitting of three related nonlinear equations. Resulting equations have been incorporated into the framework of an individual tree distance-independent growth and yield model.

6). Since time series and cross-sectional data from permanent sample plots are used for fitting individual equations or system of equations, parameter estimation methods under the classical regression structure, where the error terms for individual equations are independent and identically distributed, or the generalized regression structure, where the error terms in individual equations are heteroskedastic and/or autocorrelated are considered. Previous considerations of the error structure have concentrated on linear equations. There has apparently been no reported study on the nonlinear system of equations in which the error terms for individual equations may be heteroskedastic and/or serially correlated.

The selection of an appropriate base function and the method of parameter prediction were consistently used for developing individual equations in this analysis. The abundance of potential variables in growth and yield studies presents an often diverse and confusing array of choices as to the appropriate equation forms. Because models developed in this study are to be used for prediction purposes, additional efforts were put on the mathematical properties and biological realism of the base functions. Attempts were made to ensure that each equation developed in this study was "biologically" appropriate. Use of the method of parameter prediction for developing height prediction models, periodic diameter increment models, and periodic height increment models may not be statistically most favourable. However, such a modelling procedure generally produces equations that are biologically more defensible, while providing the flexibility for additional extensions and some protection against purely statistical modelling of the data. Currently, with the availability of various computer packages, fitting the statistically most favourable regression lines to the existing data poses very few problems, yet they may be biologically less meaningful.

The fitted coefficients from this study were combined with existing mortality and volume functions for the simulation of growth and yield for boreal mixed-species stands. Since the simulation processes involves fairly elaborate procedures in terms of simulation strategies, computer programming, and the output lay-outs, and are beyond the scope of this study, only one example of the resulting simulation for an un-thinned mixed spruce-aspen stand is attached. Figure 9-1 shows the tabulated and the graphical forms for the growth and yield of each species in the stand and the stand as a whole. Additional results consistently showed that the height prediction function, the diameter increment model, and the height increment model behaved well under various simulated stand conditions. However, in a few cases the mortality functions substantially underestimated the mortality rate of aspen trees, causing a noticeable overestimation for aspen volume in mixed-species stands.

It is recommended that another mortality study be conducted for aspen before implementation of the models developed here are used for as an individual tree distance-independent growth and yield projection system. Additional efforts on the evaluation of the models may also be conducted using different data sets in order to detect any possible abnormalities of the models under different conditions. There are numerous possibilities for extensions and improvements to the present study. Nevertheless, the objectives of this study were met and the methodological concerns for developing and fitting systems of simultaneous nonlinear equations in tree growth and yield studies were addressed. These systems of models should be beneficial to the understanding of integrated stand development processes and lead to better management of the forest resources.

Figure 9-1. An example of the simulated stand dynamics for a mixed spruce-aspen stand on a good site (SPI=20). Initial stand density of 10000 stems/ha for aspen and 3000 stems/ha for white spruce.

Green – white spruce, red – aspen.

Age – total age in years.

Density – number of trees/ha.

Diameter – average diameter of the trees (cm).

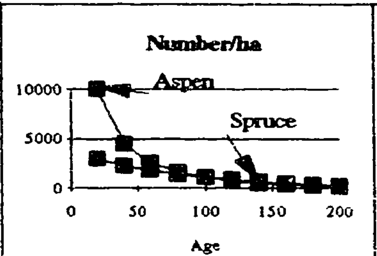
Height – total tree height (m).

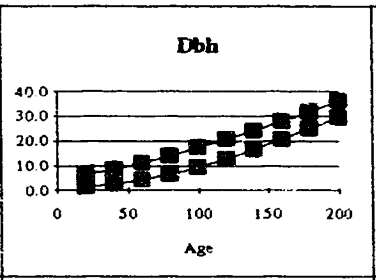
Basal area – basal area/ha (m^2).

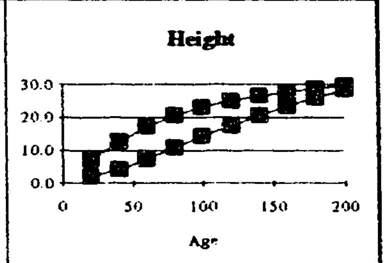
Volume – volume/ha (m^3).

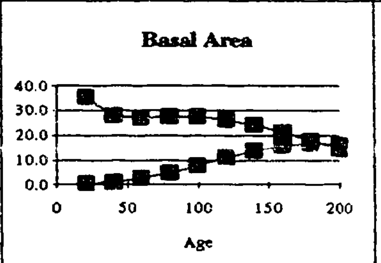
Volume Mai – volume mean annual increment (m^3/year).

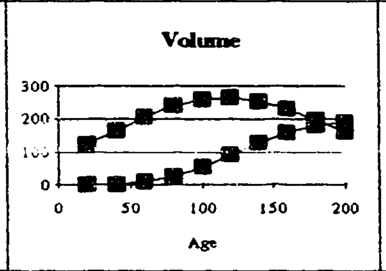
	Density		Diameter		Height		Basal Area		Volume		Volume Mai	
Age	DenAw	DenSw	DiaAw	DiaSw	HtAw	HtSw	BaAw	BaSw	VolAw	VolSw	VmaiAw	VmaiSw
20	10000	3000	6.6	1.5	7.0	2.1	35.4	0.5	125	1	6.2	0.0
40	4452	2286	8.7	2.6	12.8	4.2	27.8	1.2	165	3	4.1	0.1
60	2574	1802	11.3	4.3	17.3	7.4	27.1	2.6	206	11	3.4	0.2
80	1652	1418	14.2	6.6	20.6	10.9	27.4	4.9	240	28	3.0	0.3
100	1107	1101	17.4	9.6	23.1	14.4	27.3	7.9	260	56	2.6	0.6
120	752	841	20.7	12.9	25.1	17.6	26.1	11.1	264	91	2.2	0.8
140	509	632	24.2	16.7	26.6	20.6	24.1	13.8	254	128	1.8	0.9
160	338	467	27.9	20.7	27.9	23.5	21.2	15.7	230	159	1.4	1.0
180	219	338	31.8	24.9	28.8	26.1	17.8	16.5	198	180	1.1	1.0
200	137	240	36.1	29.4	29.6	28.6	14.3	16.3	161	189	0.8	0.9

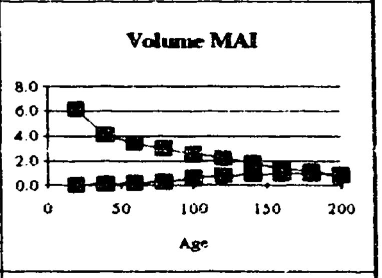


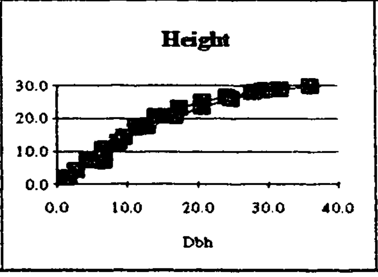


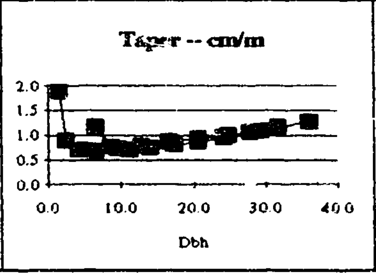


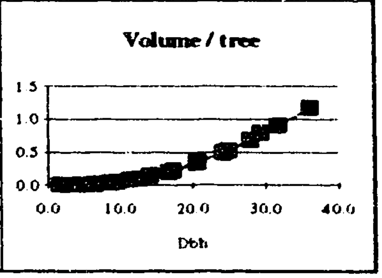












Appendices

Appendix 1. Solutions for equations (6.2) and (6.3)

Starting from the differential equations that govern the diameter increment

$$(6.2) \qquad \frac{dA(D)}{dD} = -\theta_1 A(D)$$

$$(6.3) \qquad \frac{dD_I}{dD} = \theta_1 A(D) - \theta_2 D_I$$

For simplicity, A(D) is written as A. Solving equation (6.2) first by multiplying dD/A on both sides of the equation

$$(6.4a) \qquad \frac{dA}{A} = -\theta_1 dD$$

The integration of equation (6.4a) gives

$$(6.5a) \qquad \int_{A(0)}^{A} \frac{dA}{A} = -\theta_1 \int_0^D dD$$

that is

$$(6.6a) \qquad \ln \frac{A}{A(0)} = -\theta_1 D$$

Assuming A(0)=1, equation (6.6a) can be rewritten to

$$(6.7a) \qquad A = e^{-\theta_1 D}$$

Substituting equation (6.7a) into (6.3) to get

$$(6.8a) \qquad \frac{dD_I}{dD} = \theta_1 e^{-\theta_1 D} - \theta_2 D_I$$

Rearrange equation (6.8a) into

$$(6.9a)\qquad \frac{dD_I}{dD}+\theta_2 D_I=\theta_1 e^{-\theta_1 D}$$

Mathematically, equation (6.9a) is a *first-order, linear, non-homogeneous ordinary differential equation problem* with the general form of

$$(6.10a)\qquad \frac{dy}{dx}+P(x)\,y=Q(x)$$

If the initial condition of the equation is specified by

$$(6.11a)\qquad y(x_0)=y_0$$

Using the method of "variation of coefficients" in ordinary differential equations, the solution formula for (6.10a) is

$$(6.12a)\qquad y(x)=e^{-\int_{x_0}^{x}p(s)\,ds}\left[\int_{x_0}^{x}Q(s)\,e^{\int_{x_0}^{s}p(u)\,du}ds+y_0\right]$$

Applying this formula to equation (6.9a) gives

$$\begin{aligned}
D_I&=e^{-\int_0^D\theta_2 ds}\left[\int_0^D\theta_1 e^{-\theta_1 s}e^{\int_0^s\theta_2 du}ds+0\right]\\
&=e^{-\theta_2 D}\left[\theta_1\int_0^D e^{-\theta_1 s}e^{\theta_2 s}ds\right]\\
&=\theta_1 e^{-\theta_2 D}\left[\int_0^D e^{(\theta_2-\theta_1)s}ds\right]\\
&=\frac{\theta_1}{\theta_2-\theta_1}e^{-\theta_2 D}\left[e^{(\theta_2-\theta_1)D}-1\right]\\
&=\frac{\theta_1}{\theta_2-\theta_1}\left[e^{-\theta_1 D}-e^{-\theta_2 D}\right]
\end{aligned}$$

which is equal to

$$D_I=\frac{\theta_1}{\theta_1-\theta_2}\left(e^{-\theta_2 D}-e^{-\theta_1 D}\right)$$

as shown in (6.4).

Following above procedure, if $\theta_1=\theta_2$ in (6.3), the diameter increment equation can be obtained as

$$D_I=\theta_1 D e^{-\theta_1 D}$$

If in (6.6a) A(0) is assumed as A(0)=k, where k is any constant, the solution for the differential equations provides a diameter increment equation expressed by

$$D_I=k\frac{\theta_1}{\theta_1-\theta_2}\left(e^{-\theta_2 D}-e^{-\theta_1 D}\right)$$

which can be considered as a more generalized form of equation (6.4).

Appendix 2. SAS code for fitting diameter increment model (6.7)

```
*Request the PROC MODEL procedure on the input diameter increment data DINC using
  the Gauss-Newton method;
        PROC MODEL DATA=DINC METHOD=GAUSS;
*Set the initial values, A1-A10 are parameters to be estimated,AR1 is the autoregressive
  parameter;
        PARMS A1-A10 AR1;
*Calculate A and B as defined in (6.8) and (6.9);
        A=A1+A2*BASUM+A3*SC+A4*H+A5*D/AVED+A6*SPI;
        B=A7+A8*TRHAAL**(1/2);
*Compute the predicted diameter increment DI without accounting the error structure and
  assign it to DIHAT, GI is the growth interval length;
        DIHAT=GI*(A/(A-B))*EXP(-B*D)-EXP(-A*D);
*Assign to DI the predicted value DIHAT plus the autoregressive parameter times the lagged
  error process, calculated as the 1st lag of DI minus DIHAT;
        DI=DIHAT+AR1*ZLAG1(DI-DIHAT);
*Fit the DI equation with the assigned weights and output among other statistics, the
  Durbin-Watson statistic, and diagnostics for multicollinearity;
        FIT DI / DW COLLIN;
        _WEIGHT_=1/D**2;
        RUN;
```

Appendix 3. SAS code for fitting system of equations with generalized errors

```
*Request the PROC MODEL on input data SIMDADA using the Gauss-Newton method;
        PROC MODEL DATA=SIMDATA METHOD=GAUSS;
*Set the initial values, A1-A10, B1-B8 are parameters to be estimated, RHO1 and RHO2 are
  the autoregressive parameters;
        PARMS A1-A10 B1-B8 RHO1 C1-C8 RHO2;
*Calculate the parameters as defined in (8.1), (8.2), and (8.3);
        F1=A1*(1-EXP(-A2*BASUM))+A3*SC+A4*SPI+A5*BA+A6*AVED;
        F2=A7;
        F3=A8+A9*D/AVED+A10*SPI;
        A=A1+A2*BASUM+A3*SC+A4*H+A5*D/AVED+A6*SPI;
        B=A7+A8*TRHAAL**(1/2);
        C=C1+C2*BASUM+C3*D/AVED;
        D=C4+C5*DI/GI+C6*SC+C7*D+C8/SPI;
*Assign the weights to observations and computer the dependent variables according to the
  appropriately identified error structures;
        HTW=HT/(DBH**(1/2));
        DBHINCW=DBHINC/DBH;
        HTW=(1.3+F1*(1-EXP(-F2*DBH))**F3)*(1/DBH**(1/2));
        DIHAT=GI*(A/(A-B))*EXP(-B*D)-EXP(-A*D);
        DBHINCW=(DIHAT+RHO1*ZLAG1(DBHINCW-DIHAT))*(1/DBH);
        HIHAT=GI*(C/(C-D))*(EXP(-D*HT)-EXP(-C*HT));
        HTINC=HIHAT+RHO2*ZLAG1(HTINC-HIHAT);
*Fit the height prediction model, the periodic diameter increment model, and the periodic
```

height increment model simultaneously using the T-SUR procedure, output among other statistics, the Durbin-Watson statistic, and diagnostics for multicollinearity;

```
FIT HTW DBHINCW HTINC /
        OUT=RESULT OUTRESID OUTACTUAL OUTPREDICT SUR DW COLLIN;
RUN;
```