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MACROSCOPIC PHENOMENA IN QUANTUM FIELD THEORY

BY

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ABSTRACT

In this work we present and discuss a set of theoretical problems in condensed matter and relativistic high energy physics which belong to the class of physically observable macroscopic phenomena.

After briefly reviewing standard formalisms used for the theoretical treatment of such macroscopic phenomena problems in quantum field theory, we first consider the semiclassical problem of finding the soliton solution in the adiabatic approximation of a quasirealistic continuum model of interacting electron, optical phonon and acoustic phonon fields, describing the dynamics of the linear trans-polyacetylene molecule. We find that acoustic interaction effects induce and control the dynamics of the kink which is now constrained to move across the linear molecular chain at constant velocity. The so-called charge fractionalization mechanism also remains an observable consequence of the quasirealistic model.

In a second step, we consider the thermodynamical problem of understanding the thermally induced spontaneous supersymmetry breaking of the relativistic N=3 O'Raifeartaigh model. Because this model allows for spontaneous supersymmetry breaking at zero temperature, it shows a mixed mechanism of symmetry breaking when temperature is switched on. We then identify the Goldstone zero-energy modes of the model from the infrared structure of the (diagonalized) fermion propagator matrix and show that while the original zero temperature Goldstone fermion survives temperature effects, a new Goldstone fermion appears in the channel where mass degeneracy is removed among supermultiplets and a thermal superpair shows up in

the remaining mass degenerate channel. Comparison with the Wess-Zumino model is also made.

Finally we present an original approach to the computation of the critical indices in thermo-field dynamics by making use of the temperature-dependent renormalization group. The indices are obtained to the one-loop approximation in the context of a $\lambda\phi^4$-theory.

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CHAPTER I

INTRODUCTION

1. The Adaptability of the Quantum Field Theoretical Language

In this work, we shall describe new contributions by the author and co-workers to the physics and mathematics of macroscopic phenomena occurring in quantum field theoretic systems originating from various branches of theoretical physics such as condensed matter physics and high energy physics.

Ever since its appearance in physics in the late thirties, quantum field theory (QFT) has shown remarkable adaptability to the description and modeling of physical processes from all areas. This powerful quality renders QFT a universal formalism capable of explaining and predicting phenomena from condensed matter physics, atomic physics and nuclear to subnuclear high energy physics.

If it is clear today, from the accumulated successes over the past decades, that QFT or some natural extension (string theories?) is the correct formalism to describe a large area of physical processes; it has not always been so in the past.

The first problem which has been haunting the second quantized theory of fields was the emergence of infinities. However, the works of Schwinger and others [1-7] showed that the formalism can accomodate a consistent theory of removal of infinities known as renormalization theory. The latter theory has been, since then, extensively elaborated to include the so-called renormalization group (RG)[8-16] which has been shown to contain non-trivial physical content with respect to the phase structure (phase transitions, critical phenomena) of physical systems.

Therefore QFT enlarged with renormalization theory is a very powerful formalism indeed. One can say that renormalization theory provided the first test of adaptability of the QFT formalism as well as the first step toward self-consistency (e.g. the mass equation).

A second test of adaptability occurred some time later (around 1955), with the development of the so-called Matsubara technique[17-18] to evaluate statistical averages. Although quite limited, the Matsubara method represents the first (successful) attempt to unify the formalisms of QFT and statistical mechanics (SM). The imaginary-time formalism of Matsubara also exposed for the first time the deep analogy between QFT and SM. Although real-time formalisms (complex-time path method) have been known for some time,[19-22] the complete unification of SM and QFT can be dated around 1974 with the birth of the so-called thermo-field dynamics (TFD) formalism constructed by Umezawa and co-workers.[23-37] In this formalism, it is recognized that statistical averages are equivalent to vacuum expectation values (VEV) of the corresponding zero-temperature theory with a doubling of the original number of degrees of freedom. We shall briefly describe the TFD formalism in chapter II. This unified formalism makes contact[24,27-31] with several (all) earlier constructions such as the C^*-algebra[38] and the Kubo-Martin-Schwinger (KMS) condition[39-40] as well as the complex-time path method.[19-22] Furthermore it allows a renormalization theory[35] completely analogous to zero-temperature QFT. However, because temperature enters the renormalization conditions, the corresponding renormalization group (RG)[36-37] becomes a two-parameter abelian group. The added parameter now leads to a second set of renormalization group equations (RGE), the solutions of which play a central role in the theory of phase transitions and critical phenomena through the computation of the critical exponents. In chapter V, making use of the RG at finite temperature, we will discuss the critical behavior of the $\lambda\phi^4$-theory and perform a one-loop determination of the critical exponents.

Although the spontaneous symmetry breaking (SSB) mechanism of symmetries (internal, external, supersymmetry) by temperature effects[41-54] is still in an un-matured state, we shall see in chapter IV how TFD yields a satisfactory quantum field theoretic interpretation of this statistical mechanical mechanism.

Also in an embryonic state is the extension of TFD to states of matter away from thermodynamical equilibrium. Here we simply mention that such a formalism has been proposed recently.[55-62]

It is somewhat remarkable that ~~the~~ unified formalism of QFT and SM still has adaptive capability. It is a well known physical fact that bosonic matter is not constrained by the Pauli exclusion principle and therefore can condense in the lowest available state of the system at zero temperature. This phenomenon is called the Bose-Einstein condensation and is a direct result of quantum statistical mechanics. Such a condensation appears in zero-temperature quantum field theory in the form of an order parameter defined as the vacuum expectation value of (bosonic) field operators.

The appearance of an order parameter, which can be promoted to the status of thermodynamical variable, is most often associated with spontaneous symmetry breaking and phase transitions. When it happens that the condensation is local, the order parameter may then become space-time dependent and one formally enters the domain of extended objects in QFT.

The study of extended objects in QFT is a fascinating subject still extremely popular among researchers. Its early developments took place in the seventies and underwent tremendous activity especially because of the successes of gauge theory in high energy physics. Although the existence of extended objects is still problematic in high energy experimental physics, it is not in doubt in condensed matter physics where solid states physicists have known them for a long time. Very good reviews have been written.[63-69] Perhaps one of the most useful methods to deal with extended objects in canonical QFT is the so-called boson method[23,26,68-74] developed by Umezawa and co-workers. This method is intimately related to the physical particle representation of QFT systems and is therefore in accordance with the concept of a dynamical map in which Heisenberg operators are formally expressed as sums of normal products of physical fields. The local condensation of a physical field in the vacuum takes the shape of a space-time dependent c-number function obeying the Heisenberg equation for free physical fields. This function has been called the boson function. It can be shown from the boson transformation theorem[26] that the knowledge of the boson function is sufficient to determine the entire extended object (soliton) structure which satisfies an Euler-type (classical) field equation. A description of the boson transformation method for self-interacting scalar systems will be presented in chapter II.

If the boson function is regular, i.e. it carries no topological singularities, then it can be shown that the boson transformation defines the vacuum state for which the annihilation operators are eigenoperators. This construction leads to the well-known coherent states introduced by Glauber[75] some time ago. Such states are minimum uncertainty wave packets from which classical mechanics (CM) can be extracted. We therefore arrive at the remarkable result that classical mechanics can be obtained from the unification of quantum field theory and statistical mechanics through the Bose-Einstein condensation phenomenon.

When the boson function is not single-valued however, the topology of the underlying classical background field is no longer trivial and boundary effects can now be treated consistently. Classification of the "boundaries" or topological singularities is obtained from usual methods dealing with topological invariants (Chern, Euler, Pontryagin numbers) on appropriate manifolds.

Particularly interesting among systems with topological extended objects is the case of relativistic fermion-soliton systems.[76-99] Such systems have been extensively studied in any number of dimensions and have been shown to be the continuous limit of various discrete electron-phonon interacting models[79-98] of condensed matter physics in the mean-field approximation. Relativistic fermion-soliton continuous models can therefore simulate, to acceptable accuracy, the physics of polymers in some special cases. The dimerized trans-polyacetylene molecule is such a well-known case and we shall concentrate on improving the accuracy of the continuous description of its dynamics in chapter III.

Fermion-soliton systems are also well-known to exhibit the so-called fermion number fractionization mechanism[76,80-84,98-99] when fermion number is conserved. A good review of such a mechanism in the general case and its relation to topological invariants such as the spectral asymmetry is given in reference (98).

When extended objects are static, they in general break translational invariance. Canonical quantization in the soliton sector, when certain normalization conditions are satisfied, usually implies the existence of quantum mechanical position and momentum operators.[26,100-110] Through the so-called c-q transmutation condition, the soliton becomes quantized as it now carries a position operator. In the no-particle sector, the vacuum of the Foch space of the theory is then realized as a quantum mechanical state of the quantum soliton. Furthermore it can be shown that in a fermion-quantum soliton system, the fermionic zero-energy mode associated with the fractionization mechanism together with the quantum mechanical operators of the soliton create a supersymmetry algebra at the level of the physical operators.[105-106]

The above discussion clearly shows that, through the elaboration and unification of the QFT language with other formalisms, quantum field theory is an invaluable formalism when dealing with systems exhibiting macroscopic phenomena.

2. Macroscopic Phenomena as Limits of Quantum Field Theory

Macroscopic phenomena can arise in QFT as a result of two independent limiting procedures.

The first limiting procedure is usually known as the thermodynamic limit in which both the number of particles of an ensemble (in our case the grand canonical ensemble) and the volume of the system go to infinity while their ratio N/V remains finite. Because of the great number of particles involved, a statistical approach has to be devised and the particle-states are no longer pure states but rather mixtures characterized by macroscopically defined thermodynamical parameters such as pressure and temperature. In QFT at finite temperature, with SSB, macroscopically observed phase transitions are non-trivially related to the microscopic quantum symmetries. In chapter IV we shall analyze in more details a spontaneously broken quantum symmetry (supersymmetry) both at zero and finite temperature and point out the corresponding rearrangement mechanisms. The relevant models for such an analysis are the Wess-Zumino model[111-113] and the $N=3$ O'Raifeartaigh model.[114] Comparison of results for both models will clarify the physics of supersymmetry breaking by temperature effects.

The other limiting procedure, the so-called classical mechanical or in short classical limit, provided room, surprisingly, for controversy. Historically in the development of quantum mechanics, it has been argued from the Wigner-Kramer-Brillouin (WKB) approximation method that the limit of vanishing Planck constant ($\hbar \to 0$) reduces to the classical mechanical case. One has also argued that classical mechanics can be obtained in the limit of large quantum numbers ($n \gg 1$). A discussion on such arguments can be found in reference (115). From the viewpoint of QFT, however, it is clear that the correct classical limit is obtained when the ratio of the quantum fluctuations $\hbar \Delta n$ with the corresponding state $\hbar n$ is small, that is $\Delta n/n \ll 1$. In this way the classical limit is totally independent of the Planck constant.[26,69]

As mentioned earlier, fermion-soliton systems are ideal cases exhibiting non-trivial interplay between the classical and quantum field theoretical degrees of freedom. In chapter III, we shall provide an improvement in the understanding of the dynamics of a continuous model of the one dimensional trans-polyacetylene molecule obtained from a discrete model by keeping next to leading order terms in lattice spacing expansion. Introducing a new computational method for finding soliton solutions in such models,[95,97] we will solve the mean field equations using perturbation theory around small lattice spacing (acoustic phonon effects) and will indicate the modifications to the fermion number fractionization mechanism.[96-97]

In the next chapter, we will summarize the formalisms relevant for both macroscopic limits as a preparation to the detailed treatments of the specific problems analyzed in chapters III, IV and V.

In this work, we use natural units setting the Planck constant $\hbar$ and the speed of light c as well as the Boltzmann constant k_B to unity. Since some conventions, such as the metric tensor, differ from chapter to chapter, they will be specified in the main text at the beginning of each chapter.

CHAPTER II

MACROSCOPIC PHENOMENA FORMALISMS IN QUANTUM FIELD THEORY

1. The Boson Transformation Method

In this section we present a calculational method, the boson transformation method,[23,26,68-74] which deals with the problem of finding classical extended object solutions to a given Heisenberg equation for boson field operators. This method is particularly well suited for self-interacting scalar boson systems, although it is equally applicable, in principle, to any type of system with boson field operators. For simplicity, we only consider here self-interacting scalar systems. This method is also quite powerful in finding multi-soliton solutions.

Let us start for convenience with the following Heisenberg equation for the real scalar field operator ϕ,

$$\Lambda(\partial)\phi(x) = F[\phi(x)] \quad , \tag{2.1}$$

where $F[\phi]$ is a functional of the field ϕ.

A formal solution to (2.1) is the following Yang-Feldman equation,

$$\phi(x) = \phi_o(x) + \Lambda^{-1}(\partial)F[\phi(x)] \quad , \tag{2.2}$$

where the field $\phi_o(x)$ describes the physical free field operator satisfying the homogeneous differential equation,

$$\Lambda(\partial)\phi_o(x) = 0 \quad . \tag{2.3}$$

Here we put aside the renormalization problem since we will be mainly concerned with solutions to equation (2.1) in the tree approximation. All fields considered here are therefore bare fields.

Iterating the formal Yang-Feldman equation (2.2) gives a solution for the Heisenberg field $\phi(x)$ as a linear combination of normal products of the physical field ϕ_o,

$$\phi(x) = \phi(x;\phi_o) \quad , \tag{2.4}$$

where one can write in general,

$$\phi(x;\phi_o) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{-\infty}^{\infty} d^4x_1 \ldots d^4x_n c(x;x_1 \ldots x_n) : \phi_o(x_1) \ldots \phi_o(x_n) : . \tag{2.5}$$

The above relation is called a dynamical map[23,26] and must be understood as a weak relation. The Hilbert space of the theory is the usual Fock space built on some vacuum state by the application of the creation operators for the physical field $\phi_o(x)$. This theory of Heisenberg fields is then said to be in the physical particle representation.[26]

If one allows the physical scalar boson $\phi_o(x)$ to locally Bose condense in the vacuum, then the vacuum expectation value (VEV) of ϕ_o leaves the normal state to enter the domain of extended object phase. This condensation is best described by a space-time dependent c-number shift of the physical field,

$$\phi_o(x) \rightarrow \phi_o(x) + f(x) \quad , \tag{2.6}$$

where the c-number function $f(x)$ is called the boson function and the transformation (2.6) called the boson transformation. Therefore,

$$\langle 0|\phi_o(x)|0\rangle = f(x) \quad , \tag{2.7}$$

describes the local condensation of a single particle in the vacuum. Note that the function $f(x)$ must satisfy the same homogeneous equation as for $\phi_o(x)$,

$$\Lambda(\partial) f(x) = 0 \quad . \tag{2.8}$$

The effect of the condensation of physical particles on the Heisenberg field $\phi(x)$ is readily seen by re-writing the dynamical map (2.4) as,

$$\phi^f(x) = \phi(x;\phi_o + f) \quad . \tag{2.9}$$

This substitution changes the coefficients $c(x;x_1 \ldots x_n)$ in (2.5) as,

$$c(x;x_1\ldots x_n) \to c^f(x;x_1\ldots x_n) = \sum_{m=0}^{\infty} \frac{1}{m!} \int_{-\infty}^{\infty} dy_1 \ldots dy_m \bar{c}(x;x_1\ldots x_n;y_1\ldots y_m) \times f(y_1)\ldots f(y_m) \quad . \tag{2.10}$$

Taking the VEV of the Heisenberg operator $\phi^f(x)$ leads to,

$$\langle 0|\phi^f(x)|0\rangle = c^f(x) \quad . \tag{2.11}$$

From (2.6) one finally obtains,

$$\langle 0|\phi^f(x)|0\rangle = \sum_{m=0}^{\infty} \frac{1}{m!} \int_{-\infty}^{\infty} dy_1 \ldots dy_m c(x;y_1\ldots y_m) f(y_1)\ldots f(y_m) \quad . \tag{2.12}$$

The formal solution (2.12) indicates that the classical extended object described by the VEV of $\phi^f(x)$ is completely built from the knowledge of the boson function $f(x)$. The extended object is then seen as a condensation of large numbers of physical particles in the vacuum. In such a case, quantum fluctuations $\hbar\Delta n$ are much smaller than the occupational number $\hbar n$,

$$\frac{\Delta n}{n} << 1 \quad , \tag{2.13}$$

which indicates that this classical result is totally independent of the actual value of the Planck constant $\hbar$.[26,69]

Before discussing a practical example, we now state a very important theorem (without proving it) related to the boson transformation method. This is the so-called boson transformation theorem[23,26,70-71] which states that the boson transformed Heisenberg operator $\phi^f(x)$ in (2.9) satisfies the same Heisenberg equation as for $\phi(x)$,

$$\Lambda(\partial)\phi^f(x) = F[\phi^f(x)] \quad . \tag{2.14}$$

An immediate consequence of the boson transformation theorem (2.14) is the fact that, denoting by $\varphi(x)$ the VEV (2.11) in the tree approximation, the classical extended object (2.11) satisfies the following Euler equation in the tree approximation,

$$\Lambda(\partial)\varphi(x) = F[\varphi(x)] \quad . \tag{2.15}$$

The above equation together with the expansion of the VEV (2.12) in terms of the boson function f(x), in the tree approximation, is enough to determine completely the solution for $\varphi(x)$,

$$\varphi(x) = f(x)+\Lambda^{-1}(\partial)F[\varphi(x)] \quad . \tag{2.16}$$

Iteration of (2.16) determines the unknown coefficients $\bar{c}(x;y_1\ldots y_m)$ in equation (2.12), in the tree approximation.

As a practical example, let us consider the problem of finding the well-known single soliton solution of the Sine-Gordon theory in the tree approximation.[74]

The Heisenberg equation of the real scalar field $\phi(x)$ in 1+1 dimensions for the Sine-Gordon theory is given as,

$$(\partial_0^2-\partial_1^2)\phi(x)+\frac{m^3}{g}\sin\left(\frac{g\phi(x)}{m}\right) = 0 \quad . \tag{2.17}$$

Expanding the sine term, equation (2.17) can be re-written as,

$$(\partial^2+m^2)\phi(x)=\frac{m^3}{g}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{(2k+1)!}\left[\frac{g\phi(x)}{m}\right]^{2k+1} \quad . \tag{2.18}$$

The physical field $\phi_o(x)$ satisfies the following homogeneous equation,

$$(\partial^2+m^2)\phi_o(x) = 0 \quad . \tag{2.19}$$

Allowing for the condensation of physical fields in the vacuum, we perform the boson transformation (2.6) for which the boson function now satisfies,

$$(\partial^2+m^2)\, f(x) = 0 \quad . \tag{2.20}$$

In the tree approximation and according to (2.15), the corresponding soliton solution $\varphi(x)$ thus built from such a condensation now satisfies the following Euler equation,

$$(\partial^2+m^2)\varphi(x)=\frac{m^3}{g}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{(2k+1)!}\left[\frac{g\varphi(x)}{m}\right]^{2k+1} \quad , \tag{2.21}$$

the solution of which is formally written as,

$$\varphi(x)=f(x)+(\partial^2+m^2)^{-1}\frac{m^3}{g}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{(2k+1)!}\left[\frac{g\varphi(x)}{m}\right]^{2k+1} \quad . \tag{2.22}$$

In order to solve equation (2.22) recursively, we now follow the steps of Oberlechner et al..[74] First, remembering the expansion (2.12) for the soliton, we expand $\varphi(x)$ as,

$$\varphi(x) = {\sum_{n=1}^{\infty}}{}' \varphi^{(n)}(x) \quad , \tag{2.23}$$

where $\varphi^{(1)}(x)=f(x)$ and where the prime indicates that the sum is carried over odd integers only. This is suggested by the form of the interaction.

Insertion of the expansion (2.23) into both sides of (2.22) yields the following set of equations for individual orders $\varphi^{(n)}(x)$,

$$\varphi^{(n)}(x)=(\partial^2+m^2)^{-1}\frac{m^3}{g}\sum_{k=1}^{(n-1)/2}\left[\frac{(-1)^{k+1}}{(2k+1)!}\left(\frac{g}{m}\right)^{2k+1}\sum_{\substack{i_1,i_2\ldots\geq 1\\ i_1+i_2+\ldots+i_{2k+1}=n}}{}^{'}\times\varphi^{(i_1)}(x)\varphi^{(i_2)}(x)\ldots\varphi^{(i_{2k+1})}(x)\right]\quad;\quad n>1\quad. \tag{2.24}$$

Rescaling the (classical) fields and expressing derivatives in units of the mass equation (2.24) is re-written as,

$$\varphi^{(n)}(x)=(\partial^2+1)^{-1}\sum_{k=1}^{(n-1)/2}\left[\frac{(-1)^{k+1}}{(2k+1)!}\sum_{\substack{i_1,i_2\ldots\geq 1\\ i_1+i_2+\ldots+i_{2k+1}=n}}{}^{'}\varphi^{(i_1)}\varphi^{(i_2)}\ldots\varphi^{(i_{2k+1})}\right];n>1. \tag{2.25}$$

The rescaled boson function now satisfies,

$$(\partial^2+1)f(x) = 0 \quad. \tag{2.26}$$

Boosted solutions to (2.26) exist:

$$f(X) = \exp[X] \quad, \tag{2.27}$$

where,

$$X \equiv \gamma x_1-\beta x_0+\delta' \quad, \tag{2.28}$$

in which the boost parameters are constrained by special relativity to the condition,

$$\gamma^2-\beta^2 = 1 \quad. \tag{2.29}$$

The operator $(\partial^2+1)^{-1}$ in (2.25) is now well defined,

$$(\partial^2+1)^{-1}\exp[\alpha X] \equiv (-\alpha^2+1)^{-1}\exp[\alpha X] \quad. \tag{2.30}$$

Individual orders $\varphi^{(n)}$'s can now be expressed as monomials of the boson function,

$$\varphi^{(n)}(x) = A_n f^n(X) \quad ; \quad n \geq 1 \quad , \tag{2.31}$$

in which $A_1 = 1$.

Insertion of the latter equation on both sides of (2.25) yields the following recurrence relation for the coefficients A_n,

$$(1-n^2)A_n = \sum_{k=1}^{(n-1)/2} \left[\frac{(-1)^{k+1}}{(2k+1)!} \sum_{\substack{i_1, i_2 \ldots \geq 1 \\ i_1+i_2+\ldots+i_{2k+1}=n}}' A_{i_1} A_{i_2} \ldots A_{i_{2k+1}}\right] \quad ; \quad n>1 \quad . \tag{2.32}$$

Such a recurrence relation has been solved and yields the following solution for A_n,

$$A_n = (-1)^{(n-1)/2} \frac{1}{n} \left[\frac{1}{4}\right]^{n-1} \quad ; \quad n \geq 1 \quad . \tag{2.33}$$

Remembering equations (2.23) and (2.31), the final soliton solution can be summed up to yield the following closed form,

$$\varphi(x) = 4 \tan^{-1} \exp [X] \quad . \tag{2.34}$$

This is the well-known single soliton of the Sine-Gordon equation.

The above technique can also easily accommodate the computation of N-soliton solutions. In this case the boson function is chosen as,

$$f(x_0, x_1) = \sum_{n=1}^{N} \exp [X_n] \quad , \tag{2.35}$$

which also satisfies equation (2.26). Here the X_n's are defined as,

$$X_n \equiv \gamma_n x_1 - \beta_n x_0 + \delta_n \quad ; \quad \gamma_n^2 - \beta_n^2 = 1 \quad . \tag{2.36}$$

Explicit computations for the two-soliton case have been carried out and the final solution is given as,

$$\varphi(x_0,x_1) = 4\tan^{-1} \frac{\exp(X_1)+\exp(X_2)}{1+a\ \exp(X_1+X_2)} \tag{2.37}$$

in which the constant a is defined as,

$$a \equiv \frac{1-\gamma_1\gamma_2+\beta_1\beta_2}{1+\gamma_1\gamma_2-\beta_1\beta_2} \quad . \tag{2.38}$$

The boson transformation method has also been used to derive other well-known solutions such as the single soliton solution of the $\lambda\phi^4$-theory.[73]

As a final comment before ending this section, when the boson function is Fourier transformable, the boson transformation is equivalent to a c-number shift of the annihilation operators for the physical field and therefore induces the so-called coherent states introduced some time ago by Glauber.[75] When the boson function is not Fourier transformable, however, topological objects may occur in any number of dimensions. One should also note that when the boson function is not required to be single-valued everywhere, one enters the study of extended objects carrying topological singularities in higher dimensions.[26,68-69,71,115]

2. Thermo-Field Dynamics

In this section we will briefly review a most elegant and most convenient formalism dealing with the statistical average of Heisenberg operators in the physical particle representation, the so-called thermo-field dynamics (TFD) formalism[23-37] originally proposed by Leplae, Mancini and Umezawa[23] in 1974, followed by Takahashi and Umezawa[24] in 1975.

The main idea behind this formalism is to find a way to express statistical averages as vacuum expectation values (VEV's). Once we obtain the representation of the "thermal vacuum", all the mathematical apparatus of quantum field theory at zero temperature will be readily available to the study of dynamical problems at finite temperature. These include the usual Feynman diagrams technique, the spectral representation for Heisenberg operators, the reduction and Lehmann-Symanzik-Zimmerman (LSZ) formulae,[26,117-120] so important in the analysis of dynamical maps for field operators and the S-matrix, the Ward-Takahashi identities (WT)[26,121-122] as well as the Nambu-Goldstone theorem[26,123-125] and perhaps most importantly, renormalization theory[1-7,35] and the renormalization group.[8-16,36-37] Previous formalisms of QFT at finite temperature, such as the imaginary time Matsubara method,[17-18] usually have shortcomings in one or more of the above available tools of analysis.

The thermo-field dynamics formalism also makes contact with previous statistical mechanical constructions such as the Kubo-Martin-Schwinger (KMS) condition[39-40] in the axiomatic C*-algebra,[28,38] closely related to the thermal state condition,[23-26,28,34] and the complex time path method under a specific choice of path in the complex time plane.[19-22,29-30]

The basic problem of TFD is therefore to find a vacuum state $|0(\beta)\rangle$ such that the VEV of an arbitrary dynamical variable A is formally equivalent to its statistical average, that is,

$$\langle 0(\beta)|A|0(\beta)\rangle = Z^{-1}(\beta)\sum_n \langle n|A|n\rangle e^{-\beta\omega_n} \quad , \qquad (2.39)$$

where β is the inverse temperature , $Z(\beta)$ the partition function and in which ω_n is the eigenvalue of the Hamiltonian H acting on the eigenstate $|n\rangle$,

$$H|n\rangle = \omega_n|n\rangle \quad ; \quad \langle n|m\rangle = \delta_{nm} \quad . \qquad (2.40)$$

In order to realize such a representation for $|0(\beta)\rangle$, one must first implement an effective doubling of the number of degrees of freedom of the original zero temperature Foch space described by (2.40). This is done by introducing unphysical states (denoted by a tilde) into the original space of states. For this tilde subsystem, we can define a Hamiltonian $\tilde{H}$ and states $|\tilde{n}\rangle$ such that,

$$\tilde{H}|\tilde{n}\rangle = \omega_n|\tilde{n}\rangle \quad ; \quad \langle\tilde{n}|\tilde{m}\rangle = \delta_{nm} \quad . \tag{2.41}$$

The total Fock space is then spanned by the direct product of the states $|n\rangle$ and $|\tilde{m}\rangle$. Such an enlarged state is denoted by $|n,\tilde{m}\rangle$. Therefore, in TFD, to any operator A, there exists a corresponding tilde conjugate operator $\tilde{A}$. If the thermal vacuum state $|0(\beta)\rangle$ is now defined as follows,

$$|0(\beta)\rangle \equiv Z^{-\frac{1}{2}}(\beta) \sum_n e^{-\frac{1}{2}\beta\omega_n}|n,\tilde{n}\rangle \quad , \tag{2.42}$$

and since tilde and untilde subspaces are independent, that is,

$$\langle\tilde{m},n|A|n',\tilde{m}'\rangle = \langle n|A|n'\rangle\,\delta_{mm'} \quad , \tag{2.43}$$

and,

$$\langle\tilde{m},n|\tilde{A}|n',\tilde{m}'\rangle = \langle\tilde{m}|\tilde{A}|\tilde{m}'\rangle\,\delta_{nn'} \quad , \tag{2.44}$$

it is then straightforward to check that equation (2.39) is satisfied.

The thermo-field dynamics formalism is best constructed when use is made of the following axioms.[23-28,33-34]

<u>Axiom 1</u> :

At equal time, for boson (fermion) fields A and $\tilde{B}$, the following (anti-) commutation relation is satisfied,

$$[A,\tilde{B}]_{\pm} = 0 \quad . \tag{2.45}$$

Axiom 2 :

The mapping between the two independent subspaces of the theory is called the tilde conjugation and is defined by the following tilde conjugation rules,

$$\text{a)} \quad (AB)^{\sim} = \tilde{A}\tilde{B} \quad , \tag{2.46}$$

$$\text{b)} \quad (c_1A+c_2B)^{\sim} = c_1^*\tilde{A}+c_2^*\tilde{B} \quad , \tag{2.47}$$

$$\text{c)} \quad (\tilde{A})^{+} = (A^{+})^{\sim} \quad , \tag{2.48}$$

in which c_1 and c_2 are c-numbers.

Axiom 3 :

The thermal vacuum is invariant under tilde conjugation,

$$\widetilde{|0(\beta)>} = |0(\beta)> \quad . \tag{2.49}$$

Axiom 4 :

The thermal vacuum satisfies the following thermal state condition,

$$A(t,\vec{x})|0(\beta)>=\sigma\tilde{A}^{+}(t-i\beta/2,\vec{x})|0(\beta)> \quad , \tag{2.50}$$

$$<0(\beta)|A(t,\vec{x})=<0(\beta)|\tilde{A}^{+}(t+i\beta/2,\vec{x})\sigma^{*} \quad , \tag{2.51}$$

where $|\sigma|=1$.

Axiom 5 :

The double tilde conjugation is defined as follows,

$$\tilde{\tilde{A}} = A \quad . \tag{2.52}$$

Note that in general, there is some freedom with the choice of σ in (2.50)-(2.51). The choice of the phase factors has been shown to be conditioned by the presence of particle number conservation laws.[34] For a Majorana fermion, $\sigma=\pm i$. This is due to the fact that no fermion number can be defined in a theory including Majorana fermions. Note also that the choice of phase factors affects the double tilde conjugation rule (2.52). The rule (2.52) corresponds to the choice $\sigma=1(i)$ for bosonic (fermionic) operator $A(t,\vec{x})$.

The thermal state conditions (2.50)-(2.51) have also been shown to reproduce the so-called Kubo-Martin-Schwinger (KMS) conditions of the axiomatic C^*-algebra.

The tilde conjugation rules (2.46)~(2.48) can help us in turn to determine the Heisenberg equation for the tilde field. From the Heisenberg equation,

$$i\partial_\mu A(x) = [A(x),P_\mu] \quad , \tag{2.53}$$

where P_μ is the four-momentum generator of space-time translation, we obtain the corresponding equation for the tilde field,

$$i\partial_\mu \tilde{A}(x) = -[\tilde{A}(x),\tilde{P}_\mu] \quad . \tag{2.54}$$

Equations (2.53)-(2.54) imply that the total generator of space-time translation in TFD[24,26] is given as,

$$\hat{P}_\mu \equiv P_\mu - \tilde{P}_\mu \quad , \tag{2.55}$$

which in turn implies that the total Lagrangean is obtained in a similar way,

$$\hat{\mathcal{L}} \equiv \mathcal{L} - \tilde{\mathcal{L}} \quad . \tag{2.56}$$

As we will see shortly, the Feynman rules for the computation of n-point functions in TFD are those obtained from the total Lagrangean $\hat{\mathcal{L}}$.

The thermal state conditions (2.50)-(2.51) also allow for the existence of two types of annihilation operators.[26,32-33] Let the operators $a(t)$ and $\tilde{a}(t)$ stand for annihilation operators of the $|0,\tilde{0}\rangle$ vacuum. Then the above thermal state conditions imply the existence of operators $a_\beta(t)$ and $\tilde{a}_\beta(t)$ defined as,

$$a_\beta(t) \equiv n^{\frac{1}{2}}(-i\partial_t)\,[a(t+i\beta/2)-\sigma\tilde{a}^\dagger(t)] \quad , \tag{2.57}$$

and,
$$\tilde{a}_\beta(t) \equiv n^{\frac{1}{2}}(-i\partial_t)^*[\tilde{a}(t-i\beta/2)-\sigma^* a^\dagger(t)] \quad , \tag{2.58}$$

such that,

$$a_\beta(t)|0(\beta)\rangle=\tilde{a}_\beta(t)|0(\beta)\rangle=\langle 0(\beta)|a_\beta(t)=\langle 0(\beta)|\tilde{a}^\dagger_\beta(t)=0 \quad . \tag{2.59}$$

Introducing the thermal doublet notation A^α,

$$A^\alpha \equiv \begin{cases} A & ;\ \alpha = 1 \\ \tilde{A} & ;\ \alpha = 2 \end{cases} \quad , \tag{2.60}$$

the set of linear Bogoliubov (canonical) transformations (2.57)-(2.58) is re-written in the compact form,

$$a^\alpha_\beta(t) = U^{-1}(-i\partial_t)^{\alpha\gamma}a^\gamma(t) \quad , \tag{2.61}$$

where the Bogoliubov transformation matrix U^{-1} is defined as,

$$U^{-1}(\varepsilon) \equiv n^{\frac{1}{2}}(\varepsilon) \begin{bmatrix} e^{\frac{1}{2}\beta\varepsilon} & -\sigma \\ -\sigma & e^{\frac{1}{2}\beta\varepsilon} \end{bmatrix} \quad , \tag{2.62}$$

and normalized as,

$$\tilde{U}(\varepsilon)U^{-1}(\varepsilon) = 1 \quad , \tag{2.63}$$

for fermions and,

$$U(\varepsilon)\tau U(\varepsilon) = \tau \quad , \tag{2.64}$$

for bosons. Note that the matrix τ is defined as,

$$\tau \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad . \tag{2.65}$$

The normalization $n^{\frac{1}{2}}(\varepsilon)$ has been chosen to reproduce the correct statistical distribution for the particle number at thermal equilibrium,that is,

$$n(\varepsilon) = \frac{1}{e^{\beta\varepsilon}-\rho} \quad , \tag{2.66}$$

where ρ is +1(-1) for bosons (fermions). The inverse of the transformation (2.61) is obtained as,

$$a^{\alpha}(t) = U(-i\partial_t)^{\alpha\gamma} a^{\gamma}_{\beta}(t) \quad . \tag{2.67}$$

The transformation (2.67) together with equations (2.62)∿(2.66) enable us to systematically obtain expressions for the retarded, advanced and causal propagators at finite temperature. Note that equation (2.67) can also be written in the form,[24]

$$\overset{(\sim)}{a} = e^{iG(\beta)} \overset{(\sim)}{a_\beta} e^{-iG(\beta)} \quad , \tag{2.68}$$

where the generator $G(\beta)$ satisfies,

$$G(\beta) = G(\beta)^{\dagger} = -\tilde{G}(\beta) \quad . \tag{2.69}$$

The thermal vacuum can then be obtained in the canonical form,[24]

$$|0(\beta)\rangle = e^{-iG(\beta)}|0,\tilde{0}\rangle \quad . \tag{2.70}$$

It is relatively easy to show that equation (2.70) yields the expression (2.42) for $|0(\beta)\rangle$ by explicit computation of the generator $G(\beta)$ for bosons or fermions.

The formalism developed so far has the great advantage that it accomodates a perturbation theory completely analogous to the zero temperature theory. In particular, the computation of n-point functions in the Heisenberg representation is performed by the use of the Gell-Mann-Low formula,[25,126]

$$\langle 0(\beta)|TA_1^{\alpha_1}(x_1)\ldots A_n^{\alpha_n}(x_n)|0(\beta)\rangle = \frac{\langle\Phi(\beta)|TA_1^{\alpha_1}(x_1)\ldots A_n^{\alpha_n}(x_n)\ \exp[i\int_{-\infty}^{\infty} d^4x\,\hat{\mathcal{L}}_I(x)]|\Phi(\beta)\rangle}{\langle\Phi(\beta)|T\ \exp[i\int_{-\infty}^{\infty} d^4x\,\hat{\mathcal{L}}_I(x)]|\Phi(\beta)\rangle}, \tag{2.71}$$

where the rhs is in the interaction representation with interacting total Lagrangean $\hat{\mathcal{L}}_I$. The state $|\Phi(\beta)\rangle$ is the (free) thermal vacuum in the interaction representation. The relation (2.71) supplemented with Wick's theorem for time ordered products yield the Feynman rules of the theory. The rules are similar to the ones obtained at zero temperature except for the fact that vertex diagrams now carry a thermal index which should be summed over and that, since the vacuum state is the thermal state $|\Phi(\beta)\rangle$, the normal ordering prescription applies to the thermal operators $a_\beta(t)$, $\tilde{a}_\beta(t)$ defined in (2.57)-(2.58) and their hermitean conjugates, instead of the annihilation and creation operators of the zero temperature theory. This in turn implies that internal (or external) lines now represent finite temperature causal propagators. The perturbation theory given by (2.71) has also been shown to be completely analogous to the complex time or path ordering method of Mills[19] under a suitable choice of the complex time integration contour. Finite temperature causal propagators appearing as internal (external) lines in the perturbative expansion have the following matrix form in TFD,

$$G^{\alpha\beta}(x,y) \equiv \langle\Phi(\beta)|T\phi^{\alpha}(x)\phi^{+\beta}(y)|\Phi(\beta)\rangle$$

$$= \frac{i}{(2\pi)^4}\int d^4p\, e^{ipx}\, G^{\alpha\beta}(p) \quad , \tag{2.72}$$

in which,

$$G^{\alpha\beta}(p) = \int_{-\infty}^{\infty} d\kappa\,\rho_B(\kappa,\vec{p}) \left[U_B(p_0)\frac{\tau}{p_0-\kappa+i\tau\delta}U_B(p_0) \right]^{\alpha\beta}, \tag{2.73}$$

for bosons and,

$$G^{\alpha\beta}(p) = \int_{-\infty}^{\infty} d\kappa\,\rho_F(\kappa,\vec{p}) \left[U_F(p_0)\frac{1}{p_0-\kappa+i\tau\delta}U_F^{-1}(p_0) \right]^{\alpha\beta}, \tag{2.74}$$

for fermions with positive definite spectral functions $\rho_B(\kappa,\vec{p})$ and $\rho_F(\kappa,\vec{p})$ for bosons and fermions respectively. The matrices $U_{B,F}(p_0)$ have been defined through (2.62)~(2.66).

As a comment, one may add that functional methods have been constructed in the context of TFD for the computation of n-point functions as well as for the effective potential at finite temperature.[31,127] Path integral quantization is also possible in TFD and the usual integration measure must now be enlarged tc [illegible] the summation over unphysical tilde fields.[49-50,52]

Since renormalizat[illegible], the renormalization group at finite temperature will be pres[illegible] detail in chapter V in connection with the computation of the [illegible] exponents of a $\lambda\phi^4$-theory, we do not discuss them here.

Finally, we mention that an extension of the TFD formalism to non-equilibrium processes[55-62] is in an early stage of development at the present time.

CHAPTER III

CLASSICAL PHENOMENA: SOLITON DYNAMICS IN POLYMER MODELS

1. General Considerations

In this chapter, we first discuss briefly a class of one dimensional polymer models[79-83,98,128-129] and compare them to corresponding models in high-energy physics.[76-78,84,90,98] We then specialize to the case of the quasi-linear (1+1) -dimensional trans-polyacetylene molecule t-$(CH)_x$ and discuss the physical properties of such a system as well as the discrete Su-Schrieffer-Heeger (SSH) model[79-82,84,91,98] describing it.

Going to the continuum limit[83,95-98] of the SSH model, and keeping next to leading order terms in lattice constant expansion, we obtain a continuous model for polyacetylene including acoustic phonon effects.[95-97] When such effects are neglected, the Takayama-Lin Liu-Maki (TLM) model[83,92-93] is recovered.

Next we proceed to find topological (kink) solutions to the continuous model with acoustic effects. However, as will be seen, such a model is rather intractable as far as perturbation theory is concerned. We are thus motivated to introduce our quasirealistic model[96-97] which is believed to yield sensible physical results at least in the adiabatic approximation where quantum phonon modes and their quantum corrections are ignored. One important feature of the mean-field theory obtained from the quasirealistic model, is the fact that it yields the same Bardeen-Cooper-Schrieffer (BCS) -type gap equation[130] as for the TLM model in the homogeneous sector. This therefore enables us to apply perturbation theory about small acoustic phonon velocity to the quasi-realistic model for which acoustic effects are treated as perturbations on the TLM theory.

After discussing briefly some of the solutions[83-90,92-95,98] of the TLM theory in the mean-field approximation, we then concentrate, as stated earlier, on the problem of finding acoustic phonon corrections to the single TLM kink solution. To that purpose, we formulate a mathematical method to obtain soliton solutions in fermion-boson systems (asymptotic expansion method[95,97]), which is very similar to the so-called boson transformation method[23,26,68-74] discussed in the previous chapter and applicable to self-interacting boson systems.

The major finding of this chapter is the fact that acoustic effects induce and control the dynamics of the soliton.[96,97] Comparison with the results obtained from numerical treatment of the discrete SSH model by other authors[82,91] is also made.

Finally we close this chapter by presenting a self-consistent proof of our solutions obtained from the asymptotic expansion method by an explicit computation of fermion wavefunctions.[92,96]

2. 1+1 Dimensional Polymer Models

Linear one-dimensional molecular chains are frequent structures in polymers. A very popular candidate exhibiting such a structure is the well known semiconductor trans-polyacetylene which was first studied some time ago by Su, Schrieffer and Heeger. The low dimension of such molecules makes them very attractive to theoreticians since computations become somewhat easier.

Models of linear polymers are usually considered in the tight binding approximation where the lattice pattern created by the strong covalent bonds is described by a displacement field with values on the lattice points. The electrons participating in weak covalent bonding are then considered as free to hop from site to site. The resulting model is a discrete interacting boson-fermion model for which the interaction is determined by the hopping matrix elements. In a linear approximation, the interaction is of Yukawa-type. Such a model can accomodate the description of polyatomic molecules.[98,128-129] In that case one needs to introduce as many displacement fields as there are atomic or group species in the chain. A diatomic model has been studied by Kivelson[128] as well as Rice and Mele.[129] Also, a general discussion on the fermion number in the context of such a model has been given by Niemi and Semenoff.[98]

The polyacetylene model is the simplest of the class of models described above, which is why it has attracted so much attention. In this case there is only one group specie, the CH-group, which forms covalent bonding with neighboring groups involving four valence electrons. Three of these valence electrons create strong σ-type molecular bonds. The remaining fourth electron is a weak π-electron and is assumed to be free to move across the molecular chain by hopping from site to site. Note that in such one dimensional systems, the electron spin degree of freedom is assumed not to play any role in the dynamics. The monoatomic Su, Schrieffer and Heeger model is given by the following Lagrangean,[79-82, 95-97]

$$L = \sum_n C_n^+ \left[\frac{i\partial}{\partial t} - \mu \right] C_n + \sum_n \frac{1}{2} \left[\rho \left(\frac{\partial u_n}{\partial t} \right)^2 - K(u_{n+1} - u_n)^2 \right]$$

$$+ \sum_n t_{n,n+1} \left(C_n^+ C_{n+1} + C_{n+1}^+ C_n \right) , \qquad (3.1)$$

where C_n is the annihilation operator of the π-electron at lattice point r_n, u_n is the displacement field at site n, ρ is the mass of the CH-group, K is the spring constant and μ is the electron chemical potential. The hopping matrix elements $t_{n,n+1}$ are given in a linear approximation as,

$$t_{n,n+1} = t_0 - \alpha (u_{n+1} - u_n) . \tag{3.2}$$

The operators C_n and u_n obey the following canonical commutation relations (CCR),

$$[C_n, C_m^{+}]_{+} = \delta_{nm} , \tag{3.3}$$

and,

$$[u_n, \frac{\partial}{\partial t} u_m]_{-} = \frac{i}{\rho} \delta_{nm} . \tag{3.4}$$

The model given by the Lagrangean (3.1) describes a one-dimensional metal. However, it is known that such a system is unstable against the creation of an electronic gap $2\Delta_0$ (Peierls gap) at the Fermi surface. Such an instability creates a Peierls distortion[131] of the lattice in which the bond lengths between CH-groups follow the alternative pattern of short double bonds and long single bonds. This dimerization of the chain effectively doubles the size of the unit cell and creates a charge density wave (CDW) commensurate with twice the original lattice length. The CDW originates from the fact that there is one π-electron per CH-group. The gap opened at the Fermi surface between the valence and conduction bands makes trans-polyacetylene a semiconductor.

Given the physical characteristics of polyacetylene, one may be justified to use the so-called adiabatic approximation in the treatment of the model, where quantum phonon modes can be neglected. In t-$(CH)_x$, the Peierls gap is $2\Delta_o \approx 1.4$ eV[132] and is about ten times as large as the optical phonon energy. In this approximation, the displacement field u_n is treated as a c-number.

As a result of the dimerization, the ground state has a twofold degeneracy. The two corresponding phases of the system are related by spatial reflexion. They are the single-double and double-single sequences of bond pattern. However, a third phase can also exist and is the phase with a topological soliton interpolating between both previous phases now coexisting in a stable pattern. Although the soliton mass is finite, decay of one phase into the other is forbidden as an infinite energy barrier separates them. The antisoliton is the configuration interpolating between the same phases, but interchanged.

In the one soliton sector, however, the SSH model is full of non trivial content . It can be shown that the interaction between the π-electron and the single soliton allows for the existence of a normalizable zero-energy fermionic mode. Such a mode is a bound-state of the electron with the soliton and is therefore situated at the center of the gap (Fermi surface). Second quantization by expansion in eigenmodes in the single soliton sector requires the existence of a new anticommuting operator , a ,associated with the zero-energy eigenmode satisfying,[80,84,98-99,105-106]

$$[a,a^{\dagger}]_+ = 1 \; ; \; [a,a]_+ = [a^{\dagger},a^{\dagger}]_+ = 0 \; . \tag{3.5}$$

Because the number of degrees of freedom associated with the above operator is unity, the algebra (3.5) implies that it has a two-dimensional representation. The ground state of the system in the single soliton sector is therefore twice degenerate.

To understand what this degeneracy implies physically, one must now discuss the symmetries of the SSH model.[84,98] The two important symmetries to consider are the phase symmetry (fermion number conservation) and the charge conjugation symmetry C.

In general, polymer systems exhibit fermion number conservation. Although a fermion pair condensation is responsible for the creation of a mass gap similar to the case of BCS superconductivity where the fermionic pair is an electron-electron Cooper pair,[133-134] the situation in polymer systems is different in the sense that the fermionic pair is of electron-hole type. Therefore, contrary to superconductivity, fermion number is here a good quantum number. Note that a pair condensation originates from electron-phonon (displacement field) interaction, which is one way to understand the Peierls instability. A further symmetry, the charge conjugation symmetry, is also sometimes realized. The charge conjugation symmetric case is usually obtained when the masses of the different group species are identical, as in the case of trans-polyacetylene. The diatomic models earlier mentioned are not in general charge conjugation symmetric.

Fermion number is odd under C or CP. When C is a good symmetry, fermion number is always integer or half-integer. Explicit computation of the fermion number in trans-polyacetylene in the single soliton sector, taking into account the existence of the zero-energy mode satisfying (3.5), leads to a twofold degenerate ground state carrying electric charge $\pm \frac{1}{2}$ per spin degree of freedom.[80,84,98-99,105-106] Because the soliton carries fractional electric charge, electrical conductivity is enhanced in trans-polyacetylene.

On an experimental basis however, no fractional charge has been seen in t-$(CH)_x$. That this is so can be understood from the fact that real electrons have two spin degrees of freedom. In this case the operator a in (3.5) carries a spin index. The soliton now has four degenerate states, each carrying fractional spin $\pm \frac{1}{2}$. The possible charges are zero and ± 1. The quantum numbers for (continuum) fermion-soliton systems with charge conjugation symmetry have been obtained from group theoretical considerations in reference (99).

In high-energy physics, (1+1) -dimensional models such as the one studied by Jackiw and Rebbi[76] (JR) also exhibits topological non trivial soliton solutions as well as fractional charge.

Taking the continuous limit of the SSH model, Takayama, Lin Liu and Maki obtained, neglecting higher orders in lattice constant expansion, a relativistic continuum fermion-boson interacting model similar to the JR model and also allowing for the existence of topological solitons with fractional charge.

In the above models, charge conjugation symmetry is a good symmetry. When it happens that this latter symmetry is not realized, fermion number may become irrational.[78,98,135] The computation of the fermion number in such a case is much more involved and requires sophisticated mathematical treatments such as the use of index theorems on appropriate spaces.[98,136-142]

In the rest of this chapter, from now on, we concentrate our analysis on the dynamics of the polyacetylene molecular system in the one soliton sector for which next to leading order terms in lattice constant expansion have been kept. The model is studied in the continuum limit and these next to leading order terms, the so-called acoustic effects, will be shown to induce and control the soliton dynamics. This is an improvement on the TLM theory. The complete derivation of the continuum model from the SSH model will be given in appendix A. The modification to the continuum model leading to the analytically solvable quasirealistic model will also be indicated in the next subsection.

2.1 - The Continuum Polyacetylene Lagrangean Model

The continuum model for the trans-polyacetylene molecule was first obtained by Takayama, Lin Liu and Maki (TLM) in early 1980. However this model fails to consider acoustic phonon effects since it totally neglects higher order terms in lattice constant expansion. In appendix A, however, we present a complete derivation of the continuum polyacetylene Lagrangean from the discrete Su-Schrieffer-Heeger (SSH) model (3.1). Acoustic effects are now considered by keeping the previously neglected terms up to second order in lattice constant. The result is a continuum (1+1)-dimensional model with Lagrangean density described as,[95-97]

$$\mathcal{L} = \psi^{\dagger}\left[i\frac{\partial}{\partial t} - \mu + iv_F\tau_3\frac{\partial}{\partial x}\right]\psi + \frac{1}{2}\left[\dot{\xi}^2 - v^2\left(\frac{\partial\xi}{\partial x}\right)^2\right] + \frac{1}{2}\left[\dot{\phi}^2 + v^2\left(\frac{\partial\phi}{\partial x}\right)^2 - m^2\phi^2\right] + g\psi^{\dagger}\tau_1\psi\phi + \frac{gv^2}{m^2}\left[-i\left(\psi^{\dagger}\tau_3\frac{\partial\psi}{\partial x} - \frac{\partial\psi^{\dagger}}{\partial x}\tau_3\psi\right)\frac{\partial\xi}{\partial x} + \left(\psi^{\dagger}\tau_1\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi^{\dagger}}{\partial x^2}\tau_1\psi\right)\phi\right] \tag{3.6}$$

The Fermi velocity v_F, the acoustic phonon velocity v, the optical phonon mass m and the electron-phonon coupling g are obtained in terms of the electron hopping amplitude t_0, the electron chemical potential μ, the mass of the lattice atom ρ, the electron-phonon coupling α, the spring constant K and the lattice constant a of the discrete SSH model in the following way,

$$v_F \equiv 2at_0; \; v^2 \equiv (K/\rho)a^2; \; m^2 \equiv 4K/\rho; \; g \equiv 4\alpha\sqrt{a/\rho} \; . \qquad (3.7)$$

The fields ψ, ξ and Φ describe two-components electron fields, acoustic phonon and optical phonon fields respectively while the τ_i-matrices are the usual classical generators of SU(2) in the fundamental representation. It is straightforward to see that neglecting higher order terms in acoustic phonon velocity expansion (lattice constant expansion in the discrete model), one recovers the Yukawa-type TLM theory of interacting optical phonon and quasielectron fields. Note that when acoustic effects are so neglected, the Lagrangean (3.6) takes a relativistic form and that the TLM theory is consequently Lorentz invariant in the homogeneous sector. When the optical phonon is allowed to Bose-condense in the vacuum, and if we restrict ourselves to the homogeneous sector, the TLM theory is known to generate a gap Δ_o (Peiarls gap) on both sides of the Fermi surface thereby creating an effective mass for the quasielectrons. The resulting gap equation of the TLM model is very similar to the BCS gap equation of superconductivity. Since it can be shown that the ground state energy $E(\Delta_o)$ has two minima corresponding to the states $\pm\Delta_o$, the polyacetylene molecule appears in two phases, as shown in figure 1. When the condensate becomes space-time dependent, both phases may then coexist with an interpolating soliton squeezed between them. This situation is also shown in figure 1. The configuration is physically stable for topological reasons. It was first shown by Takayama et al.[83], using a one-parameter variational technique, that the solution of the field equations (Bogoliubov-de Gennes equations) obtained from the Lagrangean (3.6) in the mean-field approximation while neglecting acoustic effects, is exact and coincides with the so-called kink solution of the $\lambda\phi^4$- theory.[85-90,92-94] The crucial property of the kink is, of course, that one should recover the homogeneous theory at spatial infinity($x\to\pm\infty$).

When one considers acoustic effects, however, the model becomes much more complicated. Already in the homogeneous sector, a derivation of the gap-equation from the complete Lagrangean (3.6) exhibits a deviation from the BCS-type equation of the TLM approximation. Furthermore acoustic effects increase the degree of divergence of the gap equation because of derivatives in the optical phonon-electron coupling. These deviations are proportional to the square of the acoustic phonon velocity. Because the v^2-order derivative coupling term between electrons and optical phonons modifies the homogeneous theory, it therefore prohibits the use of perturbation theory for small v^2 around the one-soliton sector of the TLM theory in the problem of computing acoustic effects on the TLM kink.

Assuming that the deviation from the BCS gap equation due to acoustic effects can be neglected, we modify the Lagrangean (3.6) in a somewhat <u>ad hoc</u> manner[96-97] by replacing the optical phonon-electron derivative coupling term with the following effective interaction,

$$\mathcal{L}_{corr.} = \lambda \frac{gv^2}{m^2} \frac{\partial}{\partial x} (\psi^\dagger \tau_1 \psi) \frac{\partial \Phi}{\partial x} \qquad (3.8)$$

To justify the correction (3.8), we note that integrating by parts the optical phonon-electron derivative interaction appearing in (3.6), we obtain the form (3.8) with $\lambda = -1$ together with a term which modifies the BCS-type gap equation. Neglecting the latter, we introduce the dimensionless constant λ in order to correct roughly the change of effective coupling caused by the term ignored. The correction (3.8) is the only one which respects both the total degree of derivative of the original interaction term as well as the BCS-type condition imposed on the gap equation. The model for which the interaction (3.8) replaces the real one will be called quasirealistic. Such a model, in the following sections of this chapter, will be shown to be analytically solvable. Note that the parameter λ of the quasirealistic model remains unknown and has to be chosen so as to fit experimental (or numerical) data . However, the correction (3.8) is not expected to yield a realistic description beyond the mean-field or adiabatic approximation where quantum phonon modes become important.

The field equations obtained from the bare Lagrangean (3.6) together with the modification (3.8) are written as follows,

$$\left[\frac{i\partial}{\partial t} - \mu + iv_F\tau_3\frac{\partial}{\partial x}\right]\psi = g\left[\left(-1 + \frac{\lambda v^2}{m^2}\frac{\partial^2}{\partial x^2}\right)\Phi\tau_1 + \frac{iv^2}{m^2}\left(\xi'' + 2\xi'\frac{\partial}{\partial x}\right)\tau_3\right]\psi \quad , \quad (3.9)$$

$$\left[-\frac{\partial^2}{\partial t^2} - v^2\frac{\partial^2}{\partial x^2} - m^2\right]\Phi = -g\left(1 - \frac{\lambda v^2}{m^2}\frac{\partial^2}{\partial x^2}\right)(\psi^+\tau_1\psi) \quad , \quad (3.10)$$

and,

$$\left[-\frac{\partial^2}{\partial t^2} + v^2\frac{\partial^2}{\partial x^2}\right]\xi = -\frac{igv^2}{m^2}\left[\psi^+\tau_3\frac{\partial^2\psi}{\partial x^2} - \frac{\partial^2\psi^+}{\partial x^2}\tau_3\psi\right] \quad . \quad (3.11)$$

Writing the vacuum expectation value for the acoustic and optical phonon fields as,

$$<0|\xi|0> \equiv <\xi>(x,t) \quad , \tag{3.12}$$

and,
$$<0|\Phi|0> \equiv (\Delta_o/g)\ \phi(x,t) \quad , \tag{3.13}$$

where $2\Delta_o$ is the Peierls gap,the set of equations (3.9)∼(3.11), when neglecting the boson excitation modes as well as their quantum corrections, leads to the following mean-field equations,

$$[\ i\frac{\partial}{\partial x_0} - \mu + iv_F\tau_3\frac{\partial}{\partial x_1}\ ]\ \psi = [\left(-1 + \frac{\lambda v^2}{m^2}\frac{\partial^2}{\partial x_1^2}\right)\Delta_o\phi\tau_1$$

$$+ \frac{igv^2}{m^2}\left(<\xi>'' + 2<\xi>'\frac{\partial}{\partial x_1}\right)\circ\tau_3\ ]\ \psi \quad , \tag{3.14}$$

$$[\ \frac{\partial^2}{\partial x_0^2} + v^2\frac{\partial^2}{\partial x_1^2} + m^2\ ]\ \frac{\Delta_o\phi}{g^2} = \left(1 - \frac{\lambda v^2}{m^2}\frac{\partial^2}{\partial x_1^2}\right)<0|\psi^+\tau_1\psi|0> \quad , \tag{3.15}$$

and,
$$[\ -\frac{\partial^2}{\partial x_0^2} + v^2\frac{\partial^2}{\partial x_1^2}\ ]<\xi> = -\frac{igv^2}{m^2}\ <0|\ [\ \psi^+\tau_3\frac{\partial^2\psi}{\partial x_1^2} - \frac{\partial^2\psi^+}{\partial x_1^2}\tau_3\psi\ ]\ |0> \ , \tag{3.16}$$

where (x_0,x_1) stands for (t,x) and x denotes from now on space-time dependency in general. Note that the normalization Δ_o/g in equation

(3.13) is chosen to satisfy $|\phi|\to 1$ at spatial infinity. Furthermore, the soliton is assumed to be topological, that is at any given time, it interpolates between the two degenerate ground states $\pm\Delta_o$. This is the case when $\phi(0,x_1) = -\phi(0,-x_1)$ in which the soliton center is taken at the origin at initial time.

The set of coupled equations (3.14)∿(3.16) is highly non-linear. To obtain a solution for the space-time-dependent order parameters ϕ and $\langle\xi\rangle$ in (3.15)-(3.16), we first compute explicitly the fermion two-point functions appearing as source terms on the right-hand side (rhs) of these equations by solving equation (3.14) in the way of a Schwinger-Dyson expansion. Because such an expansion is non-local, local source terms can be obtained from a suitable limiting procedure. This is the so-called point splitting method.

The quasi-electron two-point function is defined as,

$$iG_{\alpha\beta}(x,y) \equiv \langle 0|T\psi_\alpha(x)\psi^+_\beta(y)|0\rangle \quad . \tag{3.17}$$

Since the source terms are given as,

$$\langle 0|\psi^+\tau_1\psi|0\rangle = \mathrm{tr}\;\; \tau_I\langle 0|\psi^+(x)\psi(x)|0\rangle \quad , \tag{3.18}$$

and,

$$\langle 0|[\psi^+\tau_3\frac{\partial^2\psi}{\partial x_1^2} - \frac{\partial^2\psi^+}{\partial x_1^2}\tau_3\psi]|0\rangle = \mathrm{tr}\;\; \tau_3\langle 0|[\psi^+\left(\frac{\partial^2\psi}{\partial x_1^2}\right) - \left(\frac{\partial^2\psi^+}{\partial x_1^2}\right)\psi]|0\rangle \; , \tag{3.19}$$

one can rewrite them in terms of the two-point function (3.17) as follows,

$$\langle 0|\psi^+(x)\psi(x)|0\rangle = \lim_{\substack{y\to x\\(y_0>x_0)}} -iG(x,y) \;\equiv\; -iG(x,x) \quad , \tag{3.20}$$

and, $$\langle 0|\left[\psi^{\dagger}\left(\frac{\partial^2\psi}{\partial x_1^2}\right)-\left(\frac{\partial^2\psi^{\dagger}}{\partial x_1^2}\right)\psi\right]|0\rangle = \lim_{\substack{y\to x\\(y_0>x_0)}} -i\left(\frac{\partial^2}{\partial x_1^2}-\frac{\partial^2}{\partial y_1^2}\right)G(x,y) \quad . \tag{3.21}$$

Now since the order parameters are assumed to be topological, we need only consider half the space coordinate, say the positive interval, because the other half is obtained by mere reflexion about the soliton center. Restricting ourselves from now on to the positive interval starting at the soliton center and choosing the $+\Delta_0$ phase as the asymptotic homogeneous theory $(x_1\to+\infty)$, one defines on this interval,

$$\varphi \equiv \phi - 1 \quad . \tag{3.22}$$

Defining also,

$$S(x,y) \equiv \tau_1 G(x,y) \quad , \tag{3.23}$$

and, $$\square(x,y) \equiv \tau_3 G(x,y) \quad , \tag{3.24}$$

the mean-field eqations (3.14)∿(3.16) are re-written as,

$$\left[\frac{i\partial}{\partial x_0} - \mu + iv_F\tau_3\frac{\partial}{\partial x_1} + \Delta_0\tau_1\right]\psi = -\left[\left(1 - \frac{v^2\partial^2}{m^2\partial x_1^2}\right)\Delta_0\varphi\tau_1 - ig\frac{v^2}{m^2}\left(\langle\xi\rangle'' + 2\langle\xi\rangle'\frac{\partial}{\partial x_1}\right)\tau_3\right]\psi \quad , \tag{3.25}$$

$$\left[\frac{\partial^2}{\partial x_0^2} + v^2\frac{\partial^2}{\partial x_1^2} + m^2\right]\frac{\Delta_0}{g^2}\varphi = -i\left(1 - \frac{\lambda v^2\partial^2}{m^2\partial x_1^2}\right)\mathrm{tr}\, S(x,x) - \frac{m^2}{g^2}\Delta_0 \quad , \tag{3.26}$$

and,
$$\left[-\frac{\partial^2}{\partial x_0^2} + v^2\frac{\partial^2}{\partial x_1^2}\right]<\xi> = -\frac{gv^2}{m^2} \lim_{y\to x}\left(\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial y_1^2}\right) \text{ tr } \Box(x,y) . \quad (3.27)$$

Now the equation (3.25) yields the following Schwinger-Dyson expansion for the two-point function of the quasi-electron field,

$$\begin{aligned} G(12) = \; & G_o(12) - G_o(11')\Sigma(1')G_o(1'2) \\ & + G_o(11')\Sigma(1')G_o(1'2')\Sigma(2')G_o(2'2) \\ & - G_o(11')\Sigma(1')G_o(1'2')\Sigma(2')G_o(2'3')\Sigma(3')G_o(3'2) \\ & + - \ldots \quad , \end{aligned} \quad (3.28)$$

where G(12) stands for G(x,y) and primed integers stand for internal space-time coordinates being integrated over. The self-energy Σ is given as,

$$\begin{aligned} \Sigma(x;\partial_x) \equiv \; & \Delta_o\left(1 - \frac{\lambda v^2}{m^2}\frac{\partial^2}{\partial x_1^2}\right)\phi\tau_1 \\ & - i\frac{gv^2}{m^2}\left(<\xi>'' + 2<\xi>'\frac{\partial}{\partial x_1}\right)\tau_3 \quad . \end{aligned} \quad (3.29)$$

The "free" propagator $G_o(x,y)$ in (3.28) satisfies the following relation,

$$\left[\frac{i\partial}{\partial x_0} - \mu + iv_F\tau_3\frac{\partial}{\partial x_1} + \Delta_o\tau_1\right]G_o(x,y) = \delta^{(2)}(x-y) \quad , \quad (3.30)$$

and is obtained explicitly as,

$$G_o(x,y) = \frac{1}{(2\pi)^2}\int d^2p \; e^{-ip(x-y)} \; G_o(p_0,p_1) \quad , \quad (3.31)$$

where, $$G_o(p_0,p_1) = \frac{(p_0 - \mu) - \Delta_o \tau_1 + v_F p_1 \tau_3}{[(p_0 - \mu)^2 - E_p^2] + i\epsilon} . \tag{3.32}$$

The fermion energy E_p has been defined as,

$$E_p^2 \equiv v_F^2 p_1^2 + \Delta_o^2 . \tag{3.33}$$

Making use of the identities,

$$\frac{\partial^2}{\partial x_1^2} S_o(x,x) = 0 , \tag{3.34}$$

and, $$\lim_{y \to x} \left(\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial y_1^2} \right) \Box_o(x,y) = 0 , \tag{3.35}$$

and upon defining,

$$\hat{G}(x,y) \equiv G(x,y) - G_o(x,y) , \tag{3.36}$$

the set of equations (3.26) and (3.27) for the order parameters of the model can be cast into the following forms,

$$[\frac{\partial^2}{\partial x_0^2} + v^2 \frac{\partial^2}{\partial x_1^2} + m^2] \frac{\Delta_o}{g^2} \phi = -i \left(1 - \frac{\lambda v^2}{m^2} \frac{\partial^2}{\partial x_1^2} \right) \mathrm{tr}\, \hat{S}(x,x)$$

$$- \left(\frac{m^2}{g^2} \Delta_o + i\, \underline{\mathrm{tr}}\, S_o(x,x) \right) , \tag{3.37}$$

and, $$[-\frac{\partial^2}{\partial x_0^2} + v^2\frac{\partial^2}{\partial x_1^2}]\langle\xi\rangle = -\frac{gv^2}{m^2}\lim_{y\to x}\left(\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial y_1^2}\right) \mathrm{tr}\ \hat{G}(x,y) \quad . \qquad (3.38)$$

Equations (3.37) and (3.38) together with the expansion given by (3.28) - (3.29) for the two-point function of the quasi-electron constitute a set of mixed equations for the optical and acoustic phonon order parameters. We shall show later how to solve such equations by making use of perturbation theory for small acoustic phonon velocity. But even in the context of perturbation theory, one must recognize that individual orders in the perturbation expansion will be given by highly non-linear equations for which a practical method for finding analytical solutions will be needed. In the case of self-interacting scalar models such as the sine-Gordon equation[74] or the $\lambda\phi^4$-theory[73], we already know how to solve the corresponding classical Euler equation in the tree approximation by the use of the so-called boson transformation method where the physical field condensate builds up the entire soliton structure. For fermion-scalar models such as our quasirealistic polyacetylene model, a somewhat different, although closely related approach must be used. In the following subsection, we introduce such a method, the asymptotic expansion method, by computing explicitly the kink solution of the TLM model.

2.2 - The Asymptotic Expansion Method and the TLM Soliton

In this subsection, we present a practical description of the so-called asymptotic expansion method by computing explicitly the single kink solution of the TLM model.

The TLM theory can be recovered in the context of our quasirealistic model by simply taking the limit of vanishing acoustic phonon velocity v and restricting ourselves to static cases. In this approximation,[95] only the optical phonon order parameter survives and the equations (3.28), (3.29) and (3.37) now become,

$$\hat{S}(11) = - S_o(11')\Sigma(1')\tau_1 S_o(1'1) + S_o(11')\Sigma(1')\tau_1 S_o(1'2')\Sigma(2')\tau_1 S_o(2'1) + \ldots , \qquad (3.39)$$

in which,

$$\Sigma(x_1)\tau_1 = \Delta_o \phi(x_1) \qquad (3.40)$$

and,

$$\frac{m^2}{g^2}\Delta_o \phi(x_1) = -i \,\mathrm{tr}\, \hat{S}(x_1,x_1) - \left(\frac{m^2}{g^2}\Delta_o + i \,\mathrm{tr}\, S_o(x_1,x_1) \right). \qquad (3.41)$$

As we pointed out earlier, Takayama et al.[83] first showed that the single kink solution of the above equations coincides with the one of the $\lambda\phi^4$-theory. They used a variational approach assuming that the soliton had the same form as the one for the $\lambda\phi^4$-theory. With this assumption, a self-consistency of their analysis was shown for a certain value of the variational parameter by minimizing the soliton energy under the assumed shape. Then, making explicit use of the variational solution, they computed the electron wavefunctions and substituted them in the source terms of the original equation (3.41) for the kink. The results showed that (3.41) was satisfied thereby implying that the variational solution is indeed exact. In a later section, we shall also follow this fermion wavefunctions computational approach in order to check our soliton solutions in the context of the quasirealistic model including acoustic effects.

As is well known, the TLM model contains other types of soliton solutions[90,94] although these are not topological extended structures. Campbell and Bishop first realized that the TLM model coincides with a previously well-known and much studied model, the so-called N=2 Gross-Neveu model.[90] Their analysis of the solutions to the TLM model, such as the kink and the polaron state, was therefore carried out by comparison with the previously known solutions of the Gross-Neveu model. The question asking how far the Yukawa-type TLM model is similar to the $\lambda\phi^4$-theory was answered, on the other hand, by Rella et al.,[94] who showed that a set of solutions for the static optical phonon order parameter $\Delta(x)$ can be classified by the following equation parametrized by the real number n:

$$[\Delta_o^2 - \Delta^2(x_1)]\Delta(x_1) + \frac{1}{2} v_F^2 \left[\frac{\partial^2\Delta(x_1)}{\partial x_1^2} - \frac{n}{\Delta(x_1)}\left(\frac{\partial\Delta(x_1)}{\partial x_1}\right)^2\right] = 0 \quad (3.42)$$

The solution for which n=0 yields the kink solution since equation (3.42) with n=0 coincides with the static Euler equation of the $\lambda\phi^4$-theory. A solution with n=3/2 yields the polaron solution. Other values of n correspond to the various solutions of the model. The latter authors also presented an integral form for solutions of equation (3.42) with arbitrary n. The topological kink solution is however of more physical interest to us since it leads to the so-called charge fractionalization mechanism.

In this section, as in the rest of this chapter, we concentrate our studies on the topologically non-trivial kink solution for which the asymptotic expansion scheme is particularly well-suited.

The boundary and asymptotic conditions for a static topological kink solution to equation (3.41) are respectively,

$$\phi(x_1) = -\phi(-x_1) \quad , \tag{3.43}$$

and,

$$\phi(x_1) \sim K \exp(-Mx_1) \; ; \; (x_1 \to +\infty) \quad . \tag{3.44}$$

The constants K and M are to be determined self-consistently through the calculation. Our computational technique is called an asymptotic expansion because, as we shall see, the soliton will be expressed as a power series of the asymptotic form (3.44). When dealing with self-interacting scalar theories such as $\lambda\phi^4$-theory, the entire soliton profile can also be computed, through the boson transformation method, as a power series of the so-called boson function which is obtained from a c-number space-time dependent shift of the physical (asymptotic) boson field. The boson function satisfies the same homogeneous equation as the free scalar field operator does.

In our case, however, the combined system (3.39)∿(3.41) is an integral rather than a differential system of equations. Looking for a corresponding differential equation for the soliton, let us introduce the differential operator $\lambda_o(-i\partial)$ and $\lambda(-i\partial_1)$ through the following relations,

$$\lambda_o(-i\partial)\, e^{-ikx} = \lambda_o(k)\, e^{-ikx} \quad , \tag{3.45}$$

and,

$$\lambda(k_1) = \lambda_o(k) \Big|_{k_0 = 0} \quad , \tag{3.46}$$

where,

$$\lambda_o(k) \equiv \frac{1}{(2\pi)^2} \int d^2p \; S_o(p_0,p_1)S_o(p_0-k_0,p_1-k_1) \quad . \tag{3.47}$$

Now since equations (3.39)-(3.40), when using the Fourier representation and integrating over internal time coordinates, lead to the following expansion,

$$\hat{S}(x_1,x_1) = \frac{-1}{(2\pi)^3} \int dz_1 \int d^2p dq_1 S_o(p_0p_1)S_o(p_0q_1)e^{i(p_1-q_1)(x_1-z_1)}\Delta_o\phi(z_1)$$

$$+ \frac{1}{(2\pi)^4} \int dz_1 dw_1 \int d^2p dq_1 d\ell_1 S_o(p_0p_1)S_o(p_0q_1)S_o(p_0\ell_1)$$

$$\times e^{i(p_1-q_1)(x_1-z_1)+i(q_1-\ell_1)(x_1-w_1)}\Delta_o^2\phi(z_1)\phi(w_1)$$

$$- \frac{1}{(2\pi)^5} \int dz_1 dw_1 dy_1 \int d^2p dq_1 d\ell_1 ds_1 S_o(p_0p_1)S_o(p_0q_1)S_o(p_0\ell_1)S_o(p_0s_1)$$

$$\times e^{i(p_1-q_1)(x_1-z_1)+i(q_1-\ell_1)(x_1-w_1)+i(\ell_1-s_1)(x_1-y_1)}\Delta_o^3\phi(z_1)\phi(w_1)\phi(y_1)$$

$$+ - \ldots \quad , \tag{3.48}$$

and since the linear term in ϕ in the latter expansion is given by $\lambda(-i\partial_1)\Delta_o\phi(x_1)$, it is then possible to re-write the combined system (3.39)$\sim$(3.41) in the following differential form,

$$D_o(-i\partial_1)\Delta_\rho\phi(x_1) = i \; \mathrm{tr} \; S_o(x_1,x_1) + \frac{m^2}{g^2}\Delta_o + F[x_1;\phi] \quad , \tag{3.49}$$

where we defined,

$$D_o(-i\partial_1) \equiv i\,\mathrm{tr}\,\lambda(-i\partial_1) - \frac{m^2}{g^2} \quad , \tag{3.50}$$

and,

$$\begin{aligned} F[x;\phi] \equiv i\,\mathrm{tr}\Big\{ & \frac{1}{(2\pi)^4}\int dz_1 dw_1 \int d^2p\,dq_1\,d\ell_1\,S_o(p_0p_1)S_o(p_0q_1)S_o(p_0\ell_1) \\ & \times e^{i(p_1-q_1)(x_1-z_1)+i(q_1-\ell_1)(x_1-w_1)}\Delta^2\phi(z_1)\phi(w_1) \\ & - \frac{1}{(2\pi)^5}\int dz_1 dw_1 dy_1 \int d^2p\,dq_1\,d\ell_1\,ds_1\,S_o(p_0p_1)S_o(p_0q_1)S_o(p_0\ell_1)S_o(p_0s_1) \\ & \times e^{i(p_1-q_1)(x_1-z_1)+i(q_1-\ell_1)(x_1-w_1)+i(\ell_1-s_1)(x_1-y_1)} [illegible] \phi(w_1)\phi(y_1) \\ & + - \ldots \Big\} \quad . \end{aligned} \tag{3.51}$$

One then solves equation (3.49) as,

$$\begin{aligned} \Delta_o\phi(x_1) = {} & D_o^{-1}(-i\partial_1)\left[i\,\mathrm{tr}\,S_o(x_1,x_1) + \frac{m^2\Delta_o}{g^2} \right] + \Delta_o f(x_1) \\ & + D_o^{-1}(-i\partial_1)\,F[x_1;\phi] \quad , \end{aligned} \tag{3.52}$$

where $f(x_1)$ satisfies the homogeneous equation,

$$D_o(-i\partial_1)\,f(x_1) = 0 \quad . \tag{3.53}$$

Again, $D_o^{-1}(-i\partial_1)$ in (3.52) is defined as,

$$D_o^{-1}(-i\partial_1)\; e^{-ikx} \equiv D_o^{-1}(k_1)\; e^{-ikx} \quad . \tag{3.54}$$

From the asymptotic condition (3.44), one finds immediately,

$$f(x_1) = K\, e^{-Mx_1} \quad , \tag{3.55}$$

and,

$$\frac{m^2}{g^2}\, \Delta_o = -i\, \mathrm{tr}\, S_o(x_1,x_1) \quad . \tag{3.56}$$

Equation (3.55) is an exponentially damping solution with mass scale M indicating how fast the soliton solution approaches the order parameter of the homogeneous theory at spatial infinity, while equation (3.56) is nothing but the BCS-type gap equation of the TLM model. The function $f(x_1)$, therefore, plays the same role as the boson function of the boson transformation method. The mass parameter M in (3.55) is determined from the equation,

$$D_o(iM) = 0 \quad . \tag{3.57}$$

To understand even more how the asymptotic expansion scheme and the boson transformation technique are very closely related, one should note that, if we define,

$$\hat{D}_o(k) \equiv i\, \mathrm{tr}\, \lambda_o(k) - \frac{m^2}{g^2} \quad , \tag{3.58}$$

which becomes $D_o(k_1)$ for $k_0 = 0$; then equation (3.10) for the optical phonon field with v=0 indicates that the equation,

$$k_0^2 + g^2\hat{D}_o(k) = 0 \quad , \tag{3.59}$$

yields the energy spectrum $\omega^2(k_1)$ of the optical phonon field in the homogeneous sector of the TLM model. Shifting the field by the time-independent c-number function $f(x_1)$ (boson transformation) shows that equation (3.53) is equivalent to,

$$\omega^2(-i\partial_1)f(x_1) = 0 \quad , \tag{3.60}$$

In self-interacting scalar theories such as the sine-Gordon or $\lambda\phi^4$-theory, the energy spectrum takes the relativistic form,

$$\omega^2(k_1) = k_1^2 + M^2 \quad . \tag{3.61}$$

The condition (3.60) also applies to such theories and becomes,

$$(\partial_1^2 - M^2)f(x_1) = 0 \quad , \tag{3.62}$$

which yields an exponentially damping solution for the boson function $f(x_1)$, as in the TLM model. The asymptotic expansion is then equivalent to the boson transformation method applied to the optical phonon field in the homogeneous sector.

Equation (3.52) can now be solved iteratively as an expansion in terms of powers of the boson function. Remembering the gap equation (3.56) as well as (3.51), equation (3.52) can now be re-written up to third order as,

$$\Delta_o\hat{\phi}(x_1) = \Delta_o f(x_1) + i\,tr\{ \frac{1}{(2\pi)^4}\int dz_1 dw_1 \int d^2p dq_1 d\ell_1\, D_o^{-1}(p_1-\ell_1)$$

$$\times S_o(p_0p_1)S_o(p_0q_1)S_o(p_0\ell_1)e^{i(p_1-q_1)(x_1-z_1)+i(q_1-\ell_1)(x_1-w_1)}\Delta_o^2 f(z_1)f(w_1)$$

$$+\frac{2}{(2\pi)^4}\int dz_1 dw_1 \int d^2p dq_1 d\ell_1 D_o^{-1}(p_1-\ell_1)S_o(p_0p_1)S_o(p_0q_1)S_o(p_0\ell_1)$$

$$\times e^{i(p_1-q_1)(x_1-z_1)+i(q_1-\ell_1)(x_1-w_1)}\Delta_o f(z_1)[\frac{i}{(2\pi)^4}tr\int d\bar{z}_1 d\bar{w}_1\int d^2\bar{p} d\bar{q}_1 d\bar{\ell}_1 D_o^{-1}(\bar{p}_1-\bar{\ell}_1)$$

$$\times S_o(\bar{p}_0\bar{p}_1)S_o(\bar{p}_0\bar{q}_1)S_o(\bar{p}_0\bar{\ell}_1)e^{i(\bar{p}_1-\bar{q}_1)(\bar{x}_1-\bar{z}_1)+i(\bar{q}_1-\bar{\ell}_1)(\bar{x}_1-\bar{w}_1)}\Delta_o^2 f(\bar{z}_1)f(\bar{w}_1)]$$

$$-\frac{1}{(2\pi)^5}\int dz_1 dw_1 dy_1 \int d^2p dq_1 d\ell_1 ds_1 D_o^{-1}(p_1-s_1)S_o(p_0p_1)S_o(p_0q_1)S_o(p_0\ell_1)S_o(p_0s_1)$$

$$\times e^{i(p_1-q_1)(x_1-z_1)+i(q_1-\ell_1)(x_1-w_1)+i(\ell_1-s_1)(x_1-y_1)}\Delta_o^3 f(z_1)f(w_1)f(y_1)$$

$$+ \ldots \quad \} \qquad , \qquad (3.63)$$

where dots stand for higher powers of the boson function f. A convenient form for the boson function is the following rotated expression into the complex plane,

$$f(x_1) = Ke^{i\beta x_1} \quad , \qquad (3.64)$$

with,

$$\beta \equiv iM \quad . \qquad (3.65)$$

The function $f(x_1)$ is rotated back to the real axis at the end of the calculation. This method is a computational trick enabling us to treat $f(x_1)$ as an external leg in the Feynman diagrams appearing in the expansion (3.63) for the soliton.

As a first step toward the actual solution, one must evaluate explicitly the differential operator $D_o(-i\partial_1)$ at some finite momentum. Taking the trace of equation (3.47) at $k_0=0$, one obtains for $\lambda(k_1)$,

$$\mathrm{tr}\,\lambda(k_1) = \frac{2}{(2\pi)^2}\int d^2p \frac{[(p_0-\mu)^2+\Delta_o^2 - v_F^2 p_1(p_1-k_1)]}{[(p_0-\mu)^2-E_p^2][(p_0-\mu)^2-E_{p-k}^2]} , \qquad (3.66)$$

where we made use of the propagator (3.32)-(3.33). Integrating over p_0, equation (3.66) becomes,

$$\mathrm{tr}\,\lambda(k_1) = \frac{2i}{(2\pi)^2}\int_{-\Lambda}^{\Lambda} dp_1 \left\{ \frac{[E_{p-k}^2+\Delta_o^2 - v_F^2 p_1(p_1-k_1)]}{2E_{p-k}(E_p+E_{p-k})(E_p-E_{p-k})} - \frac{[E_p^2+\Delta_o^2 - v_F^2 p_1(p_1-k_1)]}{2E_p(E_p+E_{p-k})(E_p-E_{p-k})} \right\} , \qquad (3.67)$$

where Λ is a high-momentum cut-off. After some manipulations one finds,

$$i\,\mathrm{tr}\,\lambda(k_1) = \frac{1}{\pi v_F}\left[-\frac{\sqrt{v_F^2 k_1^2+4\Delta_o^2}}{v_F k_1}\sinh^{-1}\left(\frac{v_F k_1}{2\Delta_o}\right) + \sinh^{-1}\left(\frac{\Lambda v_F}{\Delta_o}\right)\right] . \qquad (3.68)$$

The second term in (3.68) is logarithmic divergent. However, let us compute explicitly the gap equation (3.56) as,

$$\frac{m^2}{g^2} = -i\,\frac{\mathrm{tr}\,S_o(x_1,x_1)}{\Delta_o} = \frac{2i}{(2\pi)^2}\int d^2p \frac{1}{[(p_0-\mu)^2-E_p^2]} , \qquad (3.69)$$

which leads to,

$$\frac{m^2}{g^2} = \frac{1}{\pi v_F}\sinh^{-1}\left(\frac{v_F\Lambda}{\Delta_o}\right) . \qquad (3.70)$$

Therefore, from the definition (3.50) for $D\,(k_1)$ as well as equations (3.68) and (3.70), one gets the finite result,

$$D_o(k_1) = \frac{-\sqrt{v_F^2k_1^2+4\Delta_o^2}}{\pi v_F^2 k_1} \sinh^{-1}\left(\frac{v_Fk_1}{2\Delta_o}\right) \quad . \qquad (3.71)$$

Remembering equation (3.57) for the mass parameter M, the explicit form (3.71) for $D_o(k_1)$ leads to,

$$v_F^2 \beta^2 + 4\Delta_o^2 \equiv 0 \quad \rightarrow \quad M = \frac{2\Delta_o}{v_F} \quad . \qquad (3.72)$$

Wishing now to compute the coefficients up to third order in the asymptotic expansion (3.63) for the soliton, we define the operators $B_n(-i\partial_1)$ (n=1,2,3) as follows,

$$B_1(-i\partial_1)f^2(x_1) \equiv \frac{i}{(2\pi)^4} \operatorname{tr} \int dz_1 dw_1 \int d^2p dq_1 d\ell_1 D_o^{-1}(p_1-\ell_1)S_o(p_0p_1)$$
$$\times S_o(p_0q_1)S_o(p_0\ell_1)e^{i(p_1-q_1)(x_1-z_1)+i(q_1-\ell_1)(x_1-w_1)}f(z_1)f(w_1) \quad , \qquad (3.73)$$

$$B_2(-i\partial_1)f^3(x_1) \equiv \frac{2iB_1(2\beta)}{(2\pi)^4} \operatorname{tr} \int dz_1 dw_1 \int d^2p dq_1 d\ell_1 D_o^{-1}(p_1-\ell_1)S_o(p_0p_1)$$
$$\times S_o(p_0q_1)S_o(p_0\ell_1)e^{i(p_1-q_1)(x_1-z_1)+i(q_1-\ell_1)(x_1-w_1)}f(z_1)f^2(w_1) \quad , \qquad (3.74)$$

and,
$$B_3(-i\partial_1)f^3(x_1) \equiv \frac{i \operatorname{tr}}{(2\pi)^5} \int dz_1 dw_1 dy_1 \int d^2p dq_1 d\ell_1 ds_1 D_o^{-1}(p_1-s_1)S_o(p_0p_1)$$
$$\times S_o(p_0q_1)S_o(p_0\ell_1)S_o(p_0s_1)e^{i(p_1-q_1)(x_1-z_1)+i(q_1-\ell_1)(x_1-w_1)+i(\ell_1-s_1)(x_1-y_1)}$$
$$\times f(z_1)f(w_1)f(y_1) \quad , \qquad (3.75)$$

where,
$$B_n(-i\partial_1)\, e^{-ikx} = B_n(k_1)\, e^{-ikx} \quad . \qquad (3.76)$$

Inserting the expression (3.64) for the boson function into the relations (3.73)-(3.75) and integrating over spatial coordinates yields,

$$B_1(2\beta) = \frac{D_o^{-1}(2\beta)}{(2\pi)^2}\, i\, \mathrm{tr} \int d^2p S_o(p_0p_1) S_o(p_0p_1-\beta) S_o(p_0p_1-2\beta) \quad , \tag{3.77}$$

$$B_2(3\beta) = \frac{2B_1(2\beta) D_o^{-1}(3\beta)}{(2\pi)^2}\, i\, \mathrm{tr} \int d^2p S_o(p_0p_1) S_o(p_0p_1-\beta) S_o(p_0p_1-3\beta) \quad , \tag{3.78}$$

and,

$$B_3(3\beta) = \frac{D_o^{-1}(3\beta)}{(2\pi)^2}\, i\, \mathrm{tr} \int d^2p S_o(p_0p_1) S_o(p_0p_1-\beta) S_o(p_0p_1-2\beta) S_o(p_0p_1-3\beta) \quad . \tag{3.79}$$

A lengthy calculation yields the following expressions,

$$B_1(2\beta) = \frac{8\Delta_o D_o^{-1}(2\beta)}{(2v_F\beta)^2} \frac{\sqrt{4\Delta_o^2+4v_F^2\beta^2}}{2\pi v_F^2\beta} \sinh^{-1}\left(\frac{v_F\beta}{\Delta_o}\right) \quad , \tag{3.80}$$

$$B_2(3\beta) = \frac{18 D_o^{-1}(3\beta)}{\pi v_F^4 (3\beta)^3} \left[\frac{1}{2}\sqrt{4\Delta_o^2+9v_F^2\beta^2} \sinh^{-1}\left(\frac{3v_F\beta}{2\Delta_o}\right) - \sqrt{\Delta_o^2+v_F^2\beta^2} \sinh^{-1}\left(\frac{v_F\beta}{\Delta_o}\right) \right] \quad , \tag{3.81}$$

and,

$$B_3(3\beta) = \frac{648\Delta_o^2 D_o^{-1}(3\beta)}{\pi v_F^6 (3\beta)^5} \sqrt{\Delta_o^2+v_F^2\beta^2} \sinh^{-1}\left(\frac{v_F\beta}{\Delta_o}\right) \quad . \tag{3.82}$$

From the expansion (3.63) as well as the definitions (3.74) and (3.75), the complete coefficient of f^3 is given by B_2' defined as,

$$B_2'(3\beta) = B_2(3\beta) - B_3(3\beta) \quad . \tag{3.83}$$

Inserting (3.81) and (3.82) into (3.83) yields,

$$B_2'(3\beta) = \frac{18D_o^{-1}(3\beta)}{\pi v_F^4(3\beta)^3} \left[\frac{1}{2}\sqrt{4\Delta_o^2+9v_F^2\beta^2} \sinh^{-1}\left(\frac{3v_F\beta}{2\Delta_o}\right) - \sqrt{\Delta_o^2+v_F^2\beta^2}\left(1 + \frac{4\Delta_o^2}{v_F^2\beta^2}\right) \sinh^{-1}\left(\frac{v_F\beta}{\Delta_o}\right) \right] . \tag{3.84}$$

Remembering the mass shell identity (3.72), equation (3.84) simplifies as,

$$B_2'(3\beta) = \frac{9D_o^{-1}(3\beta)}{\pi v_F^4(3\beta)^3} \sqrt{4\Delta_o^2+9v_F^2\beta^2} \sinh^{-1}\left(\frac{3v_F\beta}{2\Delta_o}\right) . \tag{3.85}$$

Finally, inserting the expression (3.71) for $D_o(k_1)$ into (3.80) and (3.85), and making use of equations (3.65) and (3.72), we get the following coefficients,

$$B_1(2\beta) = \frac{1}{2\Delta_o} , \tag{3.86}$$

and,

$$B_2'(3\beta) = \frac{1}{4\Delta_o^2} . \tag{3.87}$$

The expression (3.63) now reads as,

$$\Delta_o\phi(x_1) = \Delta_o f(x_1) + \Delta_o^2 B_1(2\beta)f^2(x_1) + \Delta_o^3 B_2'(3\beta)f^3(x_1) + \ldots , \tag{3.88}$$

which becomes, when inserting (3.86)-(3.87) as well as (3.55) and (3.72),

$$\phi(x_1) = K\, e^{-(2\Delta_o/v_F)x_1} + \tfrac{1}{2} K^2\, e^{-(4\Delta_o/v_F)x_1} + \tfrac{1}{4} K^3\, e^{-(6\Delta_o/v_F)x_1} + \ldots . \tag{3.89}$$

Upon the following re-definition of the parameter K,

$$K \equiv -2\, e^{(2\Delta_o/v_F)\bar{x}_1} , \tag{3.90}$$

where $\bar{x}_1$ is some positive undetermined position parameter (the soliton center), ϕ becomes now a function of $x_1-\bar{x}_1$ and the coefficients b_n of the expansion (3.89) are defined as,

$$b_n \equiv 2(-1)^n \quad ; \quad (n=1,2,3) \quad . \tag{3.91}$$

Remembering the equation (3.22) and assuming that the relation (3.91) remains valid for all n, the static soliton solution of the TLM model is therefore given as,

$$\phi(x_1-\bar{x}_1) = 1 + \delta(x_1-\bar{x}_1) = 1 + 2\sum_{n=1}^{\infty}(-1)^n e^{-n(2\Delta_o/v_F)(x_1-\bar{x}_1)} \quad . \tag{3.92}$$

Now since,

$$\tanh\frac{z}{2} = 1 + 2\sum_{n=1}^{\infty}(-1)^n e^{-nz} \quad ; \quad (z>0) \quad , \tag{3.93}$$

then,

$$\phi(x_1-\bar{x}_1) = \tanh\frac{\Delta_o}{v_F}(x_1-\bar{x}_1) \quad . \tag{3.94}$$

One readily checks that the boundary condition (3.43) is satisfied for the specific choice $\bar{x}_1=0$.

The drawback of the asymptotic expansion method is, of course, the increasingly tedious algebra as we go to higher orders. In the case of the TLM model, the solution (3.94) can be obtained by more straightforward techniques since the model is relatively simple. The purpose of this subsection was therefore pedagogical. When acoustic phonon effects will be taken into account in the next section with respect to the full quasi-realistic model for polyacetylene, the asymptotic expansion method, although leading to very lengthy algebra, will become an invaluable tool in the discovery of analytical solutions. Since the computation of higher order terms is prohibitive, the knowledge of the first few orders will be sufficient to determine a general expression for the coefficients of the expansion. A self-consistency check of the solution will then be needed. Such a proof will rely on the explicit computation of the fermion wavefunctions in the inhomogeneous sector of our quasirealistic theory.

3. Acoustic Effects and The Asymptotic Expansion Scheme

As we showed in the previous section, a completely realistic continuum model of the trans-polyacetylene molecule including acoustic phonon interaction effects can be obtained from the discrete SSH Lagrangean model by keeping terms up to second order of the molecular lattice spacing while going to the continuum limit (see appendix A). The limit of vanishing lattice spacing (or acoustic phonon velocity in the terminology of the continuum model) yields the well known TLM model.

In this section, we are mainly interested in the acoustic phonon interaction effects on the dynamics and profile of the TLM kink solution. Note that other kinds of corrections to the TLM model, such as quantum corrections, have also been evaluated.[92-93] When acoustic effects are taken into account, however, the homogeneous theory itself deviates from the corresponding homogeneous sector of the TLM model by increasing the degree of divergence of the gap equation. This is due to the existence of derivatives in the optical phonon-quasielectron interaction. The deviations are proportional to the square of the acoustic phonon velocity. An analysis of such a realistic continuum model is therefore very challenging. We made, however, a preliminary study of this realistic model in the inhomogeneous sector and, although no analytical form for the single kink solution was obtained, it was found that a static kink yields inconsistency. The only solution, assuming its existence, must be time-dependent. This preliminary analysis also showed that the soliton motion may be more complicated than a simple uniform translation and that oscillatory behavior may be present. Such a non-linear dynamics seems to be in accordance with the results obtained recently from numerical computations applied to the SSH model by Bishop et al.[91] It was already shown by Su and Schrieffer[82] that the propagation velocity of the single soliton was of the same order of magnitude as the acoustic phonon velocity and Bishop et al.[91] further found an approximate maximum soliton speed of about 2.7 v. In high kinetic regions, above a threshold of 0.1 Δ_o , the latter authors found that when a soliton-antisoliton pair is created, uniform translation is not observed for the separating solitons moving away from each other. An oscillatory tail develops between them and the result is a localized neutral oscillatory persistent breather.

As explained in the previous section however, acoustic effects can be successfully analyzed in the context of the quasirealistic model for the trans-polyacetylene molecule. In such a model, the gap equation is the same as the BCS-type gap equation of the TLM model and the acoustic phonon-electron coupling is same as the one in the realistic model. Therefore acoustic effects, when treated perturbatively, modify the shape and the dynamics of the TLM soliton in the mean-field approximation. Our perturbative analysis of the quasirealistic model is going to show that, although acoustic effects only slightly modify the shape of the TLM soliton, the single kink becomes time-dependent.[96-97] This situation may be understood by recalling that the Lagrangean (3.6) with the modification (3.8) is not Lorentz invariant and that the continuum model it describes has been obtained from the SSH model for the reference system in which the lattice points are at rest.

The time dependency of our solution coincides with that of a constant uniform motion of the soliton in the lattice reference system. The velocity of the kink is controlled by that of the acoustic phonon. We find further that the acoustic phonon soliton, which is a companion excitation to the optical phonon order parameter, is also moving with the same velocity as the one for the optical phonon soliton. The existence of this companion soliton is not surprising when one intuitively realizes that the original kink deforms the displacement field (the acoustic phonon field) and induces lattice deformations around it.

The computation of acoustic phonon effects proceeds as follows. We first apply perturbation theory to the mean field equations (3.37)-(3.38) and the expansion (3.28)-(3.29) for the quasielectron two-point function. The order parameters of the quasirealistic model are assumed to be expressed as expansions about the square of the acoustic phonon velocity (v^2) since the additional coupling in the interaction is proportional to v^2. At each order of the expansion, a set of differential equations is obtained and solved by means of the asymptotic expansion method. Each order of the perturbation expansion is therefore displayed as a power series of the asymptotic form (the boson function) of the unperturbed part of the soliton (the TLM kink). In this computation, however, we restrict ourselves to first order acoustic effects.

Perturbative modifications to soliton systems, however, raise the serious question of the proper choice for the unperturbed state. In the context of the quasirealistic model, as discussed above, we expect nontrivial dynamics to show up as a result of acoustic effects. A natural choice for a time-dependent unperturbed state is the boosted TLM kink because the TLM model is Lorentz invariant. This implies, in turn, that space and time coordinates always appear in a special linear combination called a generalized coordinate X.[96-97] Self-consistency of the perturbative expansion then determines the soliton velocity. Our calculation is going to show that the latter velocity is proportional to the acoustic phonon velocity and that its actual value is given as a function of the λ-parameter of the effective derivative coupling of optical phonon and quasielectron of our quasirealistic model.

The existence of the special configuration X in the class of soliton systems treated perturbatively is not restricted to our model. Other systems,[143-147] such as the modified sine-Gordon equation[146] which is used to model the Josephson junction, also have a special configuration X, although it needs not be linear in space and time as is the case for our model. In the modified sine-Gordon model, however, the perturbational interaction dissipates energy, while in our case the acoustic phonon interaction does not violate energy conservation. The determination of a suitable X is therefore closely related to the problem of choosing the unperturbed state for the perturbative calculations. The explicit choice for X is determined by the physical properties of the system and later checked self-consistently in the computations. The boosted-like form for our non-dissipative system will be shown to yield a self-consistent solution.

3.1 - Self-Consistent Perturbation Theory

Before proceeding to solve the combined equations (3.28)-(3.29) and (3.37)-(3.38) by the use of perturbation theory, we wish to re-write equation (3.37) as,

$$m^2\Delta_o\Phi + ig^2 \mathrm{tr}\, \hat{S}(x,x) = v^2\left[\frac{ig^2}{m^2}\frac{\partial^2}{\partial x_1^2}\mathrm{tr}\, \hat{S}(x,x) - \Delta_o\frac{\partial^2}{\partial x_1^2}\Phi\right] - \frac{\partial^2}{\partial x_o^2}\Delta_o\Phi \, , \tag{3.95}$$

in which use was made of the gap equation (3.56). The hyperbolic tangent profile corresponds to the static solution of (3.95) with $v=0$. Equation (3.95) then reduces to equation (3.41). A time-dependent solution to (3.41) is obtained from the static one by a Lorentz boost since the equation with $v=0$ can be considered as a static situation of a Lorentz invariant system with fermions obeying a Dirac-type equation in (1+1) dimensions. This suggests that a reasonable choice for the unperturbed state will be given by the boosted TLM solution and the perturbation by the right-hand side (rhs) of equation (3.95) including the apparently v^2-independent time derivative "anomalous" term. This choice implies that the latter time derivative term is of order $O(v^2)$, suggesting that the soliton velocity is proportional to the acoustic phonon velocity. As will be pointed out later in this subsection, static solutions for finite λ yield inconsistency.

Defining the boosted configuration (generalized coordinate),

$$X \equiv M \left[1 - \frac{v_{sol}^2}{v_F^2}\right]^{-\frac{1}{2}} [x_1 \pm v_{sol}x_0 - \bar{x}] \quad , \tag{3.96}$$

where M is same as (3.72) and v_{sol} is the soliton velocity given by,

$$v_{sol} \equiv \sigma v \quad , \tag{3.97}$$

we now assume the following expansions,

$$\Phi = \Phi_o + v^2\Phi_1 + \ldots \quad , \tag{3.98}$$

$$\langle\xi\rangle = \xi_o + v^2\xi_1 + \ldots \quad , \tag{3.99}$$

and,

$$\Sigma = \Sigma_o + v^2 \Sigma_1 + \ldots \quad , \tag{3.100}$$

where explicit space-time dependency appears in the configuration (3.96). Note that the dimensionless constant σ in (3.97) will be determined later self-consistently. The expansions (3.98) $\sim$ (3.100) enable us to expand ,

$$\hat{G}(x,y) = \hat{G}^{(0)}(x,y) + v^2 \hat{G}^{(1)}(x,y) + \ldots \quad . \tag{3.101}$$

The dots stand for higher order terms in the acoustic phonon velocity expansion. Inserting the expansions (3.98)-(3.100) into equation (3.29) for the self-energy yields the following relationships (to first order in perturbation theory),

$$\Sigma_o = \Delta_o \tau_1 \phi_o \quad , \tag{3.102}$$

and,

$$\Sigma_1 = \Delta_o \tau_1 \phi_1 - \frac{\lambda \Delta_o \tau_1}{m^2} \phi_o'' + \frac{ig\tau_3}{m^2} \left(\xi_o'' + 2 \xi_o' \frac{\partial}{\partial x_1} \right) \quad . \tag{3.103}$$

Inserting in turn (3.102) and (3.103) into (3.28) and making use of (3.101) yields,

$$\begin{aligned} \hat{G}^{(0)}(12) = & - G_o(11')\Sigma_o(1')G_o(1'2) \\ & + G_o(11')\Sigma_o(1')G_o(1'2')\Sigma_o(2')G_o(2'2) \\ & - G_o(11')\Sigma_o(1')G_o(1'2')\Sigma_o(2')G_o(2'3')\Sigma_o(3')G_o(3'2) \\ & + - \ldots \quad , \end{aligned} \tag{3.104}$$

and,

$$\hat{G}^{(1)}(12) = -G_o(11')\,\Sigma_1(1')\,G_o(1'2) + G_o(11')\,\Sigma_o(1')\,G_o(1'2')\,\Sigma_1(2')\,G_o(2'2)$$

$$+G_o(11')\,\Sigma_1(1')\,G_o(1'2')\,\Sigma_o(2')\,G_o(2'2)$$

$$-G_o(11')\,\Sigma_o(1')\,G_o(1'2')\,\Sigma_o(2')\,G_o(2'3')\,\Sigma_1(3')\,G_o(3'2)$$

$$-G_o(11')\,\Sigma_o(1')\,G_o(1'2')\,\Sigma_1(2')\,G_o(2'3')\,\Sigma_o(3')\,G_o(3'2)$$

$$-G_o(11')\,\Sigma_1(1')\,G_o(1'2')\,\Sigma_o(2')\,G_o(2'3')\,\Sigma_o(3')\,G_o(3'2)$$

$$+-\;\ldots \qquad (3.105)$$

The field equations (3.37)-(3.38) then become, finally,

$$\Delta_o\phi_o(x) = -i\frac{g^2}{m^2}\,\mathrm{tr}\,\hat{S}^{(0)}(x,x) \quad , \qquad (3.106)$$

$$\Delta_o\phi_1(x) = -i\frac{g^2}{m^2}\,\mathrm{tr}\,\hat{S}^{(1)}(x,x)$$

$$-\frac{\Delta_o}{m^2}\left(\frac{\partial^2}{\partial(vx_0)^2} + (1+\lambda)\frac{\partial^2}{\partial x_1^2}\right)\phi_o(x) \quad , \qquad (3.107)$$

and,

$$\xi_o(x) = -\frac{g}{m^2}\left(\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial(vx_0)^2}\right)^{-1}\lim_{y\to x}\left(\frac{\partial^2}{\partial x_1^2}-\frac{\partial^2}{\partial y_1^2}\right)\mathrm{tr}\,\hat{\Box}^{(0)}(x,y) \qquad (3.108)$$

for the optical and acoustic phonon order parameters respectively. Note that the "anomalous" term in (3.95) has led to the appearance of v-dependent time derivative operators in (3.107) and (3.108). When the soliton velocity is given by (3.97), however, any v-dependence will disappear.

The time-dependent solution to (3.106) is nothing but the unperturbed boosted TLM kink profile obtained from (3.94) as,

$$\phi_o = \tanh \frac{X}{2} - 1 \quad , \tag{3.109}$$

where X is the generalized coordinate (3.96). In order to solve the combined system of equations (3.102) $\sim$ (3.108), one now makes use of the asymptotic expansion method described in the previous section. Expanding ϕ_o, ϕ_1 and ξ_o as,

$$\phi_o(X) = \sum_{n=1}^{\infty} b_n f^n(X) \quad ; \quad X>0 \quad , \tag{3.110}$$

$$\phi_1(X) = \sum_{n=0}^{\infty} a_n f^n(X) \quad ; \quad X>0 \quad , \tag{3.111}$$

and,

$$\xi_o(X) = \sum_{n=0}^{\infty} c_n f^n(X) \quad ; \quad X>0 \quad , \tag{3.112}$$

in which the boson function f(X) is given as,

$$f(X) \equiv e^{-X} \quad , \tag{3.113}$$

we can obtain explicit solutions through the determination of the coefficients a_n and c_n. The coefficients b_n for the unperturbed part have already been determined in the last section. They were obtained as,

$$b_n = 2(-1)^n = (-1)^{n+1} b_1 \quad . \tag{3.114}$$

Since the expansions (3.110)∿(3.112) are defined only for positive X, the complete solutions for the entire range of X will be determined from topological considerations at the end of the computation. Again, a convenient expression for the boson function is the following complex form,

$$f(x) = Ke^{-ikx} = Ke^{ik_1x_1 - ik_0x_0} \quad , \tag{3.115}$$

where,

$$k_1 \equiv \frac{\beta}{\sqrt{1 - \frac{\sigma^2 v^2}{v_F^2}}} \simeq \beta + O(v^2) \quad , \tag{3.116}$$

and,

$$k_0 = \pm \sigma v k_1 \simeq \pm \sigma v \beta + O(v^2) \quad , \tag{3.117}$$

in which we made explicit use of equation (3.97) for the soliton velocity and where,

$$\beta \equiv iM \quad , \tag{3.118}$$

and,

$$K \equiv \exp \frac{M\bar{x}}{\sqrt{1 - \frac{\sigma^2 v^2}{v_F^2}}} \quad . \tag{3.119}$$

As previously discussed, this method is a computational trick enabling us to treat the boson function as an external leg with momentum β in the Feynman diagrams.

We can now proceed to solve equations (3.107) and (3.108) for the order parameters by insertion of the expansions (3.110)∿(3.112). Since ξ_0 appears explicitly in the perturbed part (3.103) for the self-energy, one must first [illegible] the coefficients c_n of the expansion (3.112) before attempting to find the perturbation ϕ_1 for the optical phonon order parameter.

Making use of equations (3.24), (3.102), (3.104) as well as the expansion (3.110) for Φ_o, one finds explicitly,

$$\hat{\Box}^{(0)}(12) = - \Box_o(11')\tau_1\tau_3\, \Box_o(1'2)\{\Delta_o b_1 f(1')+\Delta_o b_2 f^2(1')+\Delta_o b_3 f^3(1')+\ldots\}$$

$$+ \Box_o(11')\tau_1\tau_3\, \Box_o(1'2')\tau_1\tau_3\, \Box_o(2'2)\{\Delta_o^2 b_1^2 f(1')f(2')$$

$$+\Delta_o^2 b_1 b_2\, [f(1')f^2(2')+f^2(1')f(2')]+\ldots\}$$

$$- \Box_o(11')\tau_1\tau_3\, \Box_o(1'2')\tau_1\tau_3\, \Box_o(2'3')\tau_1\tau_3\, \Box_o(3'2)\{\Delta_o^3 b_1^3 f(1')f(2')f(3')+\ldots\}$$

$$+- \ldots \qquad (3.120)$$

Evaluation for the rhs of (3.108) when making use of (3.120) gives an expansion in powers of f which determines through (3.112) each coefficient c_n and therefore the ξ_o order parameter. In the following, we compute c_n up to n=2 and find a generic form by extrapolation. To that purpose, in the same spirit as for the computation of the TLM soliton in the previous section, we define the following differential operators,

$$I_1(\partial)f^n(x) \equiv \left(\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial(vx_0)^2}\right)^{-1} \lim_{y\to x}\left(\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial y_1^2}\right)$$

$$\times\, \mathrm{tr}\int d^2z\, \Box_o(xz)\tau_1\tau_3\, \Box_o(zy)f^n(z) \quad , \qquad (3.121)$$

and,

$$I_2(\partial)f^2(x) \equiv \left(\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial(\text{[illegible]}}\right)^{-1} \lim_{y\to x}\left(\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial y_1^2}\right)$$

$$\times\, \mathrm{tr}\int d^2z\, d^2w\, \Box_o(xz)\tau_1 \text{[illegible]} \tau_1\tau_3\, \Box_o(wy)f(z)f(w), \qquad (3.122)$$

where, as usual,

$$I_i(\partial)e^{-inkx} \equiv I_i(nk)e^{-inkx} \quad , \tag{3.123}$$

Going to the Fourier representation (3.31) and making use of the form (3.115) for the boson function, equations (3.121)-(3.122) become,

$$I_i(\partial)f^n(x) = \frac{K^n}{(2\pi)^4}\left(\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial(vx_0)^2}\right)^{-1}\lim_{y\to x}\left(\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial y_1^2}\right)$$

$$\times \int d^2z \int d^2p d^2q \, e^{-ip(x-z)-iq(z-y)-inkz}$$

$$\times \, \mathrm{tr} \, \Box_o(p_0p_1)\tau_1\tau_3 \, \Box_o(q_0q_1) \quad , \tag{3.124}$$

and,

$$I_2(\partial)f^2(x) = \frac{K^2}{(2\pi)^6}\left(\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial(vx_0)^2}\right)^{-1}\lim_{y\to x}\left(\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial y_1^2}\right)$$

$$\times \int d^2zd^2w \int d^2pd^2qd^2\ell \, e^{-ip(x-z)-iq(z-w)-i\ell(w-y)-ik(z+w)}$$

$$\times \, \mathrm{tr} \, \Box_o(p_0p_1)\tau_1\tau_3 \, \Box_o(q_0q_1)\tau_1\tau_3 \, \Box_o(\ell_0\ell_1) \quad . \tag{3.125}$$

Taking derivatives and limits in the displayed order as well as integrating over internal coordinates, equations (3.124)-(3.125) yield,

$$I_1(nk) = (1-\sigma^2)^{-1}\frac{1}{nk_1(2\pi)^2}\mathrm{tr}\int d^2p(2p_1-nk_1) \, \Box_o(p_0p_1)\tau_1\tau_3$$

$$\times \, \Box_o(p_0-nk_0,p_1-nk_1) \quad , \tag{3.126}$$

and,

$$I_2(2k) = (1-\sigma^2)^{-1}\frac{1}{k_1(2\pi)^2}\mathrm{tr}\int d^2p(p_1-k_1) \, \Box_o(p_0p_1)\tau_1\tau_3$$

$$\times \, \Box_o(p_0-k_0,p_1-k_1)\tau_1\tau_3\Box_o(p_0-2k_0,p_1-2k_1) \quad , \tag{3.127}$$

Because of the v-dependency of k_0 and k_1 through (3.116)-(3.117), the above expressions give higher-order corrections to the soliton shape. Neglecting such v-dependent terms at this stage, equations (3.126) and (3.127) are evaluated in the static limit, that is,

$$k_1 \to \beta \quad ; \quad k_0 \to 0 \quad . \tag{3.128}$$

One must be careful, however, to perform any time derivative before taking this limit. Such a prescription leads to,

$$I_1(n\beta) = \frac{-2v_F\Delta_o(1-\sigma^2)^{-1}}{n\beta(2\pi)^2}\int d^2p \frac{(2p_1-n\beta)^2}{(p_0^2-E_p^2)(p_0^2-E_{p-n\beta}^2)} \quad , \tag{3.129}$$

and,

$$I_2(2\beta) = \frac{2v_F(1-\sigma^2)^{-1}}{\beta(2\pi)^2}\int d^2p \frac{(p_1-\beta)^2[p_0^2+3\Delta_o^2-v_F^2p_1(p_1-2\beta)]}{(p_0^2-E_p^2)(p_0^2-E_{p-\beta}^2)(p_0^2-E_{p-2\beta}^2)} \quad , \tag{3.130}$$

where (3.32) has been used and the trace taken. Integrating and making use of the identity (3.72) as well as the gap equation (3.70), finally yield ,

$$I_1(n\beta) = \frac{1}{n}\left(S-\frac{m^2}{g^2}\right)(1-\sigma^2)^{-1} \quad ; \tag{3.131}$$

and,

$$I_2(2\beta) = \frac{1}{4\Delta_o}\left(S-\frac{m^2}{g^2}\right)(1-\sigma^2)^{-1} \quad . \tag{3.132}$$

While integrating, a surface term denoted by S ($S=1/\pi v_F$) has appeared. It originates from the high momentum cut-off regularization scheme and is shown to vanish by a suitable symmetrization procedure. Such surface terms will be deliberately kept in the remaining part of this subsection as well as the following one since they will help discover the structure of the computation when we attack the problem of finding the optical phonon soliton in closed form.

Remembering the expansion (3.120) one easily finds the following expressions for the coefficients c_n (n=0,1,2),

$$c_0 = \text{undetermined} \quad , \tag{3.133}$$

$$c_1 = \frac{g}{m^2}\left[\Delta_o b_1 I_1(\beta)\right] \quad , \tag{3.134}$$

and,

$$c_2 = \frac{g}{m^2}\left[\Delta_o b_2 I_1(2\beta)+\Delta_o^2 b_1^2 I_2(2\beta)\right] \quad . \tag{3.135}$$

Inserting (3.131)-(3.132), one gets,

$$c_1 = \frac{g\Delta_o}{m^2}\left(S-\frac{m^2}{g^2}\right)(1-\sigma^2)^{-1}\, b_1 \quad , \tag{3.136}$$

and,

$$c_2 = \frac{g\Delta_o}{m^2}\left(S-\frac{m^2}{g^2}\right)(1-\sigma^2)^{-1}\, b_2 \quad , \tag{3.137}$$

which suggests that, in general,

$$c_n = \frac{g\Delta_o}{m^2}\left(S-\frac{m^2}{g^2}\right)(1-\sigma^2)^{-1}\, b_n \quad ; \quad n\geq 1 \quad . \tag{3.138}$$

This result, together with the expansions (3.110) and (3.112), implies that,

$$\xi_o(X) = \frac{g\Delta_o}{m^2}\left(S-\frac{m^2}{g^2}\right)(1-\sigma^2)^{-1}\,\Phi_o(X) + \text{constant} \quad . \tag{3.139}$$

A complete proof for this solution will be given in the next section when we present a self-consistent computation involving quasielectron wavefunctions.

Having determined ξ_o we can now attack the more difficult problem of finding the perturbation Φ_1 in a closed form. As a first step toward the solution , one wishes to find a differential equation for the perturbation similar to equation (3.49) for the unperturbed state. To that purpose we write the following expansion for $S^{(1)}(x,x)$,

$$S^{(1)}(11) = -S_o(11')S_o(1'1)\{\Delta_o\phi_o(1')+\nabla(1')\}-S_o(11')[\Omega(1')]\Box_o(1'1)$$

$$+S_o(11')S_o(1'2')S_o(2'1)\{\Delta_o^2[\phi_o(1')\phi_1(2')+\phi_1(1')\phi_o(2')]$$

$$+\Delta_o[\phi_o(1')\nabla(2')+\nabla(1')\phi_o(2')]\}$$

$$+S_o(11')[\Delta_o\phi_o(1')]S_o(1'2')[\Omega(2')]\Box_o(2'1)$$

$$+S_o(11')[\Omega(1')]\Box_o(1'2')[\Delta_o\phi_o(2')]S_o(2'1)$$

$$-S_o(11')S_o(1'2')S_o(2'3')S_o(3'1)\{\Delta_o^3[\phi_o(1')\phi_o(2')\phi_1(3')+\phi_o(1')\phi_1(2')\phi_o(3')$$

$$+\phi_1(1')\phi_o(2')\phi_o(3')]+\Delta_o^2[\phi_o(1')\phi_o(2')\nabla(3')$$

$$+\phi_o(1')\nabla(2')\phi_o(3')+\nabla(1')\phi_o(2')\phi_o(3')]\}$$

$$-S_o(11')[\Delta_o\phi_o(1')]S_o(1'2')[\Delta_o\phi_o(2')]S_o(2'3')[\Omega(3')]\Box_o(3'1)$$

$$-S_o(11')[\Delta_o\phi_o(1')]S_o(1'2')[\Omega(2')]\Box_o(2'3')[\Delta_o\phi_o(3')]S_o(3'1)$$

$$-S_o(11')[\Omega(1')]\Box_o(1'2')[\Delta_o\phi_o(2')]S_o(2'3')[\Delta_o\phi_o(3')]S_o(3'1)$$

$$+-\ \ldots\ , \qquad (3.140)$$

where,

$$\nabla(x) \equiv -\frac{\lambda\Delta_o\phi_o''}{m^2}\ , \qquad (3.141)$$

and,

$$\Omega(x;\partial) \equiv -\frac{ig}{m^2}\left(\xi_o'' + 2\xi_o'\frac{\partial}{\partial x_1}\right)\ . \qquad (3.142)$$

In (3.140), use has been made of (3.23), (3.24), (3.102), (3.103) and (3.105). Defining the following differential operators,

$$D_o(\partial)\Delta_o\phi_1(1) \equiv i\,tr\, S_o(11')S_o(1'1)[\Delta_o\phi_1(1')] - \frac{m^2}{g^2}\Delta_o\phi_1(1) \quad ,(3.143)$$

$$D_1(\partial)\Delta_o\phi_o(1) \equiv -i\,tr\, S_o(11')S_o(1'1)[\nabla(1')] + \frac{\Delta_o}{g^2}\left[(1+\lambda)\phi_o''(1) + \frac{\ddot{\phi}_o(1)}{v^2}\right] \quad , \quad (3.144)$$

and,
$$D_2(\partial)\xi_o(1) \equiv -i\,tr\, S_o(11')\,[\Omega(1')]\,\Box_o(1'1) \quad , \quad (3.145)$$

where,
$$D_i(\partial)\, e^{-inkx} = D_i(nk)\, e^{-inkx} \quad , \quad (3.146)$$

and inserting equation (3.140) into (3.107) for ϕ_1, one gets the following differential equation,

$$\begin{aligned}
D_o(\partial)\Delta_o\phi_1(1) &= D_1(\partial)\Delta_o\phi_o(1) + D_2(\partial)\xi_o(1) + i\,tr\{S_o(11')S_o(1'2')S_o(2'1) \\
&\times \{\Delta_o^2[\phi_o(1')\phi_1(2')+\phi_1(1')\phi_o(2')]+\Delta_o[\phi_o(1')\nabla(2')+\nabla(1')\phi_o(2')]\} \\
&+ S_o(11')[\Omega(1')]\Box_o(1'2')[\Delta_o\phi_o(2')]S_o(2'1) \\
&+ S_o(11')[\Delta_o\phi_o(1')]\Box_o(1'2')[\Omega(2')]\Box_o(2'1) \\
&- S_o(11')S_o(1'2')S_o(2'3')S_o(3'1)\ \{\Delta_o^3[\phi_o(1')\phi_o(2')\phi_1(3') \\
&\qquad + \phi_o(1')\phi_1(2')\phi_o(3')+\phi_1(1')\phi_o(2')\phi_o(3')] \\
&\qquad + \Delta_o^2[\phi_o(1')\phi_o(2')\nabla(3')+\phi_o(1')\nabla(2')\phi_o(3') \\
&\qquad + \nabla(1')\phi_o(2')\phi_o(3')]\} \\
&- S_o(11')[\Delta_o\phi_o(1')]S_o(1'2')[\Delta_o\phi_o(2')]S_o(2'3')[\Omega(3')]\Box_o(3'1) \\
&- S_o(11')[\Delta_o\phi_o(1')]S_o(1'2')[\Omega(2')]\Box_o(2'3')[\Delta_o\phi_o(3')]S_o(3'1) \\
&- S_o(11')[\Omega(1')]\Box_o(1'2')[\Delta_o\phi_o(2')]S_o(2'3')[\Delta_o\phi_o(3')]S_o(3'1) \\
&+ - \ldots\ \} \quad . \quad (3.147)
\end{aligned}$$

In the static limit, the operator $D_o(\partial)$ is obtained from equation (3.71). One should be careful, when rotating back to the real axis, however, to choose the proper branch for the latter expression. As will be shown in later computations, this problem is of no concern to us since $D_o(\partial)$ will disappear altogether throughout the determination of the coefficients of the asymptotic expansion for the perturbation ϕ_1. The operator $D_1(\partial)$ is now easily obtained. Making use of (3.141) as well as the following rules for derivatives of the boson function,

$$f^{n'}(x) = ink_1 f^n(x) \quad , \tag{3.148}$$

$$f^{n''}(x) = -n^2k_1^2 f^n(x) \quad , \tag{3.149}$$

and,

$$\ddot{f}^n(x) = -n^2k_1^2\sigma^2v^2 f^n(x) \quad , \tag{3.150}$$

equation (3.144) yields,

$$D_1(nk)e^{-inkx} = \frac{-n^2k_1^2}{m^2} \{ i\lambda \text{ tr } S_o(xz)S_o(zx)\, e^{-inkz} + \frac{m^2}{g^2}[(1+\lambda)+\sigma^2]\, e^{-inkx} \} \quad . \tag{3.151}$$

Remembering equation (3.143) for $D_\sigma(\partial)$, equation (3.151) leads to, in the static limit,

$$D_1(n\beta) = \frac{4m^2\Delta_o^2}{v_F^2m^2}\left[\frac{m^2}{g^2}[(2\lambda+1)+\sigma^2] + \lambda D_o(n\beta)\right] \quad . \tag{3.152}$$

Now making use of (3.142) as well as (3.148)$\sim$(3.150), equation (3.145) for $D_2(\partial)$ yields,

$$D_2(nk)\, e^{-inkx} = \frac{g^2}{m^2}\, \mathrm{tr}\; S_o(xz) \left(n^2 k_1^2 - 2ink_1 \frac{\partial}{\partial z_1}\right) \square_o(zx) e^{-inkz} \quad (3.153)$$

Going to the Fourier representation and integrating over internal coordinates yields ,

$$D_2(nk) = \frac{nk_1 g}{(2\pi)^2 m^2} \int d^2p (2p_1 - nk_1)\, \mathrm{tr}\, S_o(p_0 p_1) \square_o(p_0 - nk_0 p_1 - nk_1). \quad (3.154)$$

Explicitly, equation (3.154) is re-written as, in the static limit,

$$D_2(n\beta) = \frac{-2n\Delta_o \beta g}{m^2 (2\pi)^2} \int d^2p \frac{(2p_1 - n\beta)^2}{(p_0^2 - E_p^2)(p_0^2 - E_{p-n\beta}^2)} \quad (3.155)$$

Comparison with expressions (3.129) and (3.131) finally yields,

$$D_2(n\beta) = \frac{-4n\Delta_o g}{v_F^2 m^2} \left(S - \frac{m^2}{g^2}\right) . \quad (3.156)$$

Having obtained the explicit expressions (3.71),(3.152) and (3.156) for the differential operators $D_o(\partial)$,$D_1(\partial)$ and $D_2(\partial)$ respectively at given external momentum, we are now ready to calculate the remaining coefficients a_n by inserting the asymptotic expansions (3.110)$\sim$(3.112) into both sides of the equation (3.147) for ϕ_1. The determination of the a_n-coefficients proceeds by comparing on both sides of the latter equation equal powers of the boson function f(x). The a_n-coefficients are then obtained in terms of the known coefficients a_{n-1} as well as b_n and c_n given by equations (3.114) and (3.138) respectively. Linear f-terms in the equation (3.147) will yield a constraint determining the soliton velocity σ. Such a constraint will also appear in higher order terms so as to render the perturbative analysis of this section self-consistent. In the next subsection, we therefore apply the asymptotic expansion method to solve

(3.147) for the perturbation ϕ_1. Computations will be restricted to third order and an algorithm will be developed putting the complete solution into an elegant closed form.

3.2 - A Closed Form for Soliton Solutions

Inserting the asymptotic expansions (3.110)∿(3.112) into the differential equation (3.147) for the perturbation yields, up to the linear f-terms,

$$a_0 = 0 \quad , \tag{3.157}$$

and,

$$D_o(\beta)\Delta_o a_1 = D_1(\beta)\Delta_o b_1 + D_2(\beta)c_1 \quad . \tag{3.158}$$

Since the left-hand side (lhs) of (3.158) contains $D_o(\beta)$ which satisfies the on-shell condition (3.57), it therefore vanishes while leaving the coefficient a_1 undetermined. Insertion of (3.114) and (3.138) for the coefficients b_1 and c_1 as well as (3.152) and (3.156) into (3.158) yields the following condition on the soliton velocity,

$$\frac{m^2}{g^2}\left[(2\lambda+1)+\sigma^2\right] - \frac{g^2}{m^2}\left[S-\frac{m^2}{g^2}\right]^2 (1-\sigma^2)\Big|^{1} = 0 \quad . \tag{3.159}$$

Setting the surface term to zero, the condition (3.159) is re-written as,

$$\sigma^4 + 2\lambda\sigma^2 - 2\lambda = 0 \quad , \tag{3.160}$$

which is solved easily as,[97]

$$\sigma^2 = \begin{cases} (\lambda^2+2\lambda)^{\frac{1}{2}}-\lambda & ; \quad \lambda \geq 0 \\ \pm(\lambda^2+2\lambda)^{\frac{1}{2}}-\lambda & ; \quad \lambda \leq -2 \end{cases} \quad . \tag{3.161}$$

The condition (3.159) will re-appear in higher orders of the asymptotic expansion in such a way as to render the perturbative analysis self-consistent. As is obvious from (3.161), it is easy to see how perturbation theory developed around the static TLM soliton leads to inconsistency for finite value of λ. This is the case for which $\sigma = 0$ and such a case yields no solution to (3.160) for non-vanishing λ.

Going now to the computation of the second order coefficient a_2 in the asymptotic expansion, we find, by equating second powers of the boson function on both sides of equation (3.147), the following relationship,

$$D_o(2\beta)\Delta_o a_2 f^2(1) = D_1(2\beta)\Delta_o b_2 f^2(1) + D_2(2\beta)c_2 f^2(1)$$

$$+ itrS_o(11')S_o(1'2')S_o(2'1)f(1')f(2')\{2\Delta_o^2 b_1 a_1 + 2\Delta_o^2 \lambda(\beta^2/m^2)b_1^2\}$$

$$-i\frac{g}{m^2}\Delta_o b_1 c_1 \{itrS_o(11')S_o(1'2')\left(-\beta^2+2i\beta\frac{\partial}{\partial 2'}\right)\Box_o(2'1)f(1')f(2')$$

$$+ itrS_o(11')\left(-\beta^2+2i\beta\frac{\partial}{\partial 1'}\right)\Box_o(1'2')S_o(2'1)f(1')f(2')\} \quad , \qquad (3.162)$$

where $\partial/\partial 1'$ and $\partial/\partial 2'$ mean the space derivatives with respect to internal coordinates 1' and 2' respectively. Note that the above equation is evaluated in the static limit and that we made use of (3.141), (3.142) as well as (3.148)~(3.152) in its derivation.

Let us now define the following differential operators,

$$A_o(\partial)f^2(1) \equiv itrS_o(11')S_o(1'2')S_o(2'1)f(1')f(2') \quad , \qquad (3.163)$$

$$A_1(\partial)f^2(1) \equiv itrS_o(11')S_o(1'2')\left(-\beta^2+2i\beta\frac{\partial}{\partial 2'}\right)\Box_o(1'2')S_o(2'1)f(1')f(2'), \qquad (3.164)$$

and,

$$A_2(\partial)f^2(1) \equiv itrS_o(11')\left(-\beta^2+2i\beta\frac{\partial}{\partial 1'}\right)\Box_o(1'2')S_o(2'1)f(1')f(2') \quad , \qquad (3.165)$$

where again we defined,

$$A_i(\partial)f^2(1) \doteq A_i(2\beta)f^2(1) \quad . \tag{3.166}$$

Making use of the Fourier representation for the two-point function and integrating over internal coordinates as usual, we can get the following expressions from (3.163)~(3.165),

$$A_o(2\beta) = B_1(2\beta)D_o(2\beta) \quad , \tag{3.167}$$

$$A_1(2\beta) = \frac{-i\beta}{(2\pi)^2} \, \mathrm{tr}\int d^2p(2p_1-3\beta)S_o(p_0p_1)S_o(p_0p_1-\beta)\square_o(p_0p_1-2\beta) \quad , \tag{3.168}$$

and,

$$A_2(2\beta) = \frac{-i\beta}{(2\pi)^2} \, \mathrm{tr}\int d^2p(2p_1-\beta)S_o(p_0p_1)\square_o(p_0p_1-\beta)S_o(p_0p_1-2\beta) \quad , \tag{3.169}$$

where $B_1(2\beta)$ was obtained in equation (3.86). Since $A_1(\partial)$ and $A_2(\partial)$ appear in the linear combination $A_1(\partial)+A_2(\partial)$ in equation (3.162), taking the trace and summing up (3.168) and (3.169) give explicitly,

$$A_1(2\beta)+A_2(2\beta) = \frac{-8iv_F}{(2\pi)^2}\int d^2p\left[\frac{p_0^2[p_1^2-2\beta(p_1-\beta)]+(p_1-\beta)^2[3\Delta_o^2-v_F^2p_1(p_1-2\beta)]}{(p_0^2-E_p^2)(p_0^2-E_{p-\beta}^2)(p_0^2-E_{p-2\beta}^2)}\right] . \tag{3.170}$$

Carrying out the integration yields, after tedious algebra,

$$A_1(2\beta)+A_2(2\beta) = \frac{-4i\Delta_o}{v_F^2}\left[D_o(2\beta) + \left(S - \frac{m^2}{g^2}\right)\right] \quad . \tag{3.171}$$

Insertion of the $A_i(2\beta)$'s $(i=0,1,2)$ into equation (3.162) leads to the following relation among the coefficients a_i, b_i and c_i ,

$$D_o(2\beta)\Delta_o a_2 = \frac{16\Delta_o^2}{v_F^2 m^2}\left[\frac{m^2}{g^2}[(2\lambda+1)+\sigma^2] + \lambda D_o(2\beta)\right]\Delta_o b_2$$

$$- \frac{8\Delta_o^2 g}{v_F^2 m^2}\left(S-\frac{m^2}{g^2}\right)c_2 + \Delta_o\left(a_1 b_1 - \frac{4\lambda\Delta_o^2 b_1^2}{v_F^2 m^2}\right)D_o(2\beta) -$$

$$- \frac{4\Delta_o^2 g}{v_F^2 m^2} b_1 c_1\left[D_o(2\beta) + \left(S-\frac{m^2}{g^2}\right)\right] \quad . \qquad (3.172)$$

Plugging the explicit expressions for b_i and c_i finally yields,

$$D_o(2\beta)a_2 = \frac{32\Delta_o^2}{v_F^2 m^2}\left[\frac{m^2}{g^2}[(2\lambda+1)+\sigma^2] - \frac{g^2}{m^2}\left(S-\frac{m^2}{g^2}\right)^2 (1-\sigma^2)^{-1}\right]$$

$$+\left\{\frac{16\Delta_o^2}{v_F^2 m^2}\left[\lambda - \frac{g^2}{m^2}\left(S-\frac{m^2}{g^2}\right)(1-\sigma^2)^{-1}\right] - 2a_1\right\} D_o(2\beta) \quad . \qquad (3.173)$$

Remembering equation (3.159) for the soliton velocity, one determines the coefficient a_2 as,

$$a_2 = 8Q - 2a_1 \quad , \qquad (3.174)$$

where one has defined,

$$Q \equiv \frac{2\Delta_o^2}{v_F^2 m^2}\left[\lambda - \frac{g^2}{m^2}\left(S-\frac{m^2}{g^2}\right)(1-\sigma^2)^{-1}\right] \quad . \qquad (3.175)$$

To shed more light on the solution for ϕ_1, one obviously needs a computation of the third order coefficient a_3. The following calculation indeed shows that a_3 is obtained as a linear combination of a_1 and Q.

Equating third powers of the boson function on both sides of the differential equation (3.147) for the perturbation ϕ_1, one gets the relationship (note that $b_1c_2 = b_2c_1$),

$$D_0(3\beta)\Delta_0 a_3 f^3(1) = D_1(3\beta)\Delta_0 b_3 f^3(1) + D_2(3\beta) c_3 f^3(1) + i\,tr\,\{$$

$$+ S_0(11')S_0(1'2')S_0(2'1)[f^2(1')f(2') + f(1')f^2(2')]$$

$$\times [\Delta_0^2(a_1 b_2 + a_2 b_1) + 5\lambda \frac{\Delta_0^2 \beta^2}{m^2} b_1 b_2]$$

$$- \frac{ig\Delta_0 c_2}{m^2} [S_0(11')S_0(1'2')\left(-\beta^2 + 2i\beta\frac{\partial}{\partial 2'}\right) \Box_0(2'1) f^2(1') f(2')$$

$$+ S_0(11')S_0(1'2')\left(-4\beta^2 + 4i\beta\frac{\partial}{\partial 2'}\right)\Box_0(2'1) f(1') f^2(2')$$

$$+ S_0(11')\left(-4\beta^2 + 4i\beta\frac{\partial}{\partial 1'}\right)\Box_0(1'2') S_0(2'1) f^2(1') f(2')$$

$$+ S_0(11')\left(-\beta^2 + 2i\beta\frac{\partial}{\partial 1'}\right)\Box_0(1'2') S_0(2'1) f(1') f^2(2')]$$

$$- S_0(11')S_0(1'2')S_0(2'3')S_0(3'1) f(1') f(2') f(3') [\,3\Delta_0^3\left(b_1^2 a_1 + \lambda\frac{\beta^2 b_1^3}{m^2}\right)]$$

$$+ i\frac{g}{m^2}\Delta_0^2 b_1^2 c_1 f(1') f(2') f(3')\,[S_0(11')S_0(1'2')S_0(2'3')\left(-\beta^2 + 2i\beta\frac{\partial}{\partial 3'}\right)\Box_0(3'1)$$

$$+ S_0(11')S_0(1'2')\left(-\beta^2 + 2i\beta\frac{\partial}{\partial 2'}\right)\Box_0(2'3') S_0(3'1)$$

$$+ S_0(11')\left(-\beta^2 + 2i\beta\frac{\partial}{\partial 1'}\right)\Box_0(1'2') S_0(2'3') S_0(3'1)]\,\}, \quad (3.176)$$

in which we used again (3.141)-(3.142) and (3.148)~(3.152). The above relationship is also evaluated in the static limit. In order to obtain an equation for a_3 from (3.176), one needs now to define the following differential operators for which the boson function is an eigenvector,

$$J_1(\partial)f^3(1) \equiv i\,\mathrm{tr}\,S_o(11')S_o(1'2')S_o(2'1)[f^2(1')f(2')+f(1')f^2(2')], \quad (3.177)$$

$$J_2(\partial)f^3(1) \equiv i\,\mathrm{tr}\,S_o(11')S_o(1'2')S_o(2'3')S_o(3'1)f(1')f(2')f(3') \quad , \quad (3.178)$$

$$\begin{aligned} L_o(\partial)f^3(1) \equiv i\,\mathrm{tr}\{ & S_o(11')S_o(1'2')S_o(2'3')\left(-\beta^2+2i\beta\frac{\partial}{\partial 3'}\right)\Box_o(3'1) \\ & + S_o(11')S_o(1'2')\left(-\beta^2+2i\beta\frac{\partial}{\partial 2'}\right)\Box_o(2'3')S_o(3'1) \\ & + S_o(11')\left(-\beta^2+2i\beta\frac{\partial}{\partial 1'}\right)\Box_o(1'2')S_o(2'3')S_o(3'1)\ \} \\ & \times f(1')f(2')f(3') \quad , \end{aligned} \quad (3.179)$$

$$\begin{aligned} L_1(\partial)f^3(1) \equiv i\,\mathrm{tr}\{ & S_o(11')S_o(1'2')\left(-\beta^2+2i\beta\frac{\partial}{\partial 2'}\right)\Box_o(2'1)f^2(1')f(2') \\ & + S_o(11')\left(-\beta^2+2i\beta\frac{\partial}{\partial 1'}\right)\Box_o(1'2')S_o(2'1)f(1')f^2(2')\ \} \quad , \end{aligned} \quad (3.180)$$

and,

$$\begin{aligned} L_2(\partial)f^3(1) \equiv i\,\mathrm{tr}\{ & S_o(11')S_o(1'2')\left(-4\beta^2+4i\beta\frac{\partial}{\partial 2'}\right)\Box_o(2'1)f(1')f^2(2') \\ & + S_o(11')\left(-4\beta^2+4i\beta\frac{\partial}{\partial 1'}\right)\Box_o(1'2')S_o(2'1)f^2(1')f(2')\ \} \quad . \end{aligned} \quad (3.181)$$

Again we defined,

$$J_i(\partial)f^n \equiv J_i(n\beta)f^n \quad , \quad (3.182)$$

and,

$$L_i(\partial)f^n \equiv L_i(n\beta)f^n \quad , \quad (3.183)$$

in the static limit. Solving (3.177) and (3.178), one obtains the following relationships for J_1 and J_2,

$$J_1(3\beta) = \frac{B_2(3\beta)}{B_1(2\beta)} D_o(3\beta) \qquad (3.184)$$

and,

$$J_2(3\beta) = B_3(3\beta) D_o(3\beta) \qquad (3.185)$$

where B_1, B_2, B_3 and D_o are given by equations (3.80), (3.81), (3.82) and (3.71) respectively. Further simplifications yield,

$$J_1(3\beta) = \frac{1}{\Delta_o} [(1/2)D_o(3\beta) - (1/3)D_o(2\beta)] \qquad (3.186)$$

and,

$$J_2(3\beta) = \frac{-1}{6\Delta_o} D_o(2\beta) \qquad (3.187)$$

Going now to the Fourier representation and carrying out differentiations and integrations over internal coordinates, equations (3.179)~(3.181) for the L_i's yield the following relations in the static limit,

$$L_o(3\beta) = \frac{-i\beta}{(2\pi)^2} \mathrm{tr}\int d^2p\{ (2p_1-\beta)S_o(p_0p_1)\Box_o(p_0p_1-\beta)S_o(p_0p_1-2\beta) \times S_o(p_0p_1-3\beta) + (2p_1-3\beta)S_o(p_0p_1)S_o(p_0p_1-\beta)\Box_o(p_0p_1-2\beta)S_o(p_0p_1-3\beta) + (2p_1-5\beta)S_o(p_0p_1)S_o(p_0p_1-\beta)S_o(p_0p_1-2\beta)\Box_o(p_0p_1-3\beta) \}, \qquad (3.188)$$

$$L_1(3\beta) = \frac{-i\beta}{(2\pi)^2} \mathrm{tr}\int d^2p\{ (2p_1-\beta)S_o(p_0p_1)\Box_o(p_0p_1-\beta)S_o(p_0p_1-3\beta) + (2p_1-5\beta)S_o(p_0p_1)S_o(p_0p_1-2\beta)\Box_o(p_0p_1-3\beta) \}, \qquad (3.189)$$

$$L_2(3\beta) = \frac{-4i\beta}{(2\pi)^2} \mathrm{tr} \int d^2p\{ (p_1-\beta)S_o(p_0p_1)\Box_o(p_0p_1-2\beta)S_o(p_0p_1-3\beta) + (p_1-2\beta)S_o(p_0p_1)S_o(p_0p_1-\beta)\Box_o(p_0p_1-3\beta) \}. \qquad (3.190)$$

Explicit computation of the above results yields finally, after very tedious algebra,

$$L_o(3\beta) = \frac{2i}{v_F^2} \left[(11/3)\, D_o(2\beta) - D_o(3\beta) \right] , \qquad (3.191)$$

and,

$$L_1(3\beta)+L_2(3\beta) = -\frac{12i\Delta_o}{v_F^2} \left[D_o(3\beta) - D_o(2\beta) + \left(S-\frac{m^2}{g^2}\right)\right] . \qquad (3.192)$$

Insertion of the expressions (3.186), (3.187), (3.191) and (3.192) for the L_i's and L_i's as well as the relations (3.152) and (3.156) for D_1 and D_2 into equation (3.176) for a_3 gives the following,

$$\begin{aligned}
D_o(3\beta)\Delta_o a_3 = {} & \frac{36\Delta_o^3}{v_F^2 m^2} \left[\frac{m^2}{g^2}[(2\lambda+1)+\sigma^2] + \lambda D_o(3\beta)\right] b_3 \\
& - \frac{12g^2\Delta_o^3}{v_F^2 m^4} \left(S - \frac{m^2}{g^2}\right)^2 (1-\sigma^2)^{-1}\, b_3 \\
& + \left[\Delta_o(a_1b_2+a_2b_1) - \frac{20\lambda\Delta_o^3}{v_F^2 m^2} b_1 b_2\right]\left[(1/2)D_o(3\beta)-(1/3)D_o(2\beta)\right] \\
& - \frac{12g^2\Delta_o^3}{v_F^2 m^4} b_1 b_2 \left(S-\frac{m^2}{g^2}\right)(1-\sigma^2)^{-1}\left[D_o(3\beta)-D_o(2\beta)+\left(S-\frac{m^2}{g^2}\right)\right] \\
& + \frac{\Delta_o}{2}\left[b_1^2 a_1 - \frac{4\lambda\Delta_o^2}{v_F^2 m^2} b_1^3 \right] D_o(2\beta) \\
& - \frac{4g^2\Delta_o^3}{v_F^2 m^4} b_1^3 \left(S-\frac{m^2}{g^2}\right)(1-\sigma^2)^{-1}\left[\frac{11}{3}D_o(2\beta)-D_o(3\beta)\right] , \qquad (3.193)
\end{aligned}$$

in which we used relations (3.138)-(3.139). The above equation for a_3 simplifies greatly when use is made of equations (3.114), (3.174) and (3.175). It becomes,

$$D_o(3\beta)a_3 = \frac{-72\Delta_o^2}{v_F^2 m^2}\left[\frac{m^2}{g^2}\left[(2\lambda+1)+\sigma^2\right] - \frac{g^2}{m^2}\left(S-\frac{m^2}{g^2}\right)^2(1-\sigma^2)^{-1}\right]$$

$$+ \left\{3a_1 - \frac{48\Delta_o^2}{v_F^2 m^2}\left[\lambda - \frac{g^2}{m^2}\left(S-\frac{m^2}{g^2}\right)(1-\sigma^2)^{-1}\right]\right\} D_o(3\beta) \quad . \qquad (3.194)$$

The first term on the rhs of (3.194) is nothing but the condition (3.159) for the soliton velocity. It therefore vanishes. The third order coefficient a_3 is therefore determined as,

$$a_3 = 3a_1 - 24Q \quad , \qquad (3.195)$$

with Q given by (3.175).

The results (3.174) and (3.195) for the coefficients a_2 and a_3 of the asymptotic expansion for the perturbation ϕ_1 seem to indicate that the generic form for the coefficients a_n can be written as,

$$a_n = A_n Q - B_n a_1 \quad ; \quad n \geq 1 \qquad (3.196)$$

The following solutions,

$$B_n = (-1)^n n \quad ; \quad n=1,2,\ldots \qquad (3.197)$$

and,

$$A_1 = 0 \quad , \qquad (3.198)$$

are consistent with (3.174) and (3.195). Finding the sequence $\{A_n\}$ is less trivial. In order to discover the correct sequence, we need to inspect the structure of Q as well as that of the steps (3.172) and (3.193) in the computation for a_2 and a_3. The key quantity is the surface term S which always shows up as powers of the expression $(S-m^2/g^2)$. From (3.173) and (3.194) it is easy to recognize that the calculation of a_n is displayed in the following characteristic form,

$$D_o(n\beta)a_n = \frac{4n^2\Delta_o^2}{v_F^2m^2}\,[\ \text{Eq.}(3.159)\ ]\ b_n$$

$$+\ [\ A_nQ - B_na_1\ ]\ D_o(n\beta) \quad ; \quad n=1,2,3 \quad . \qquad (3.199)$$

which is a mere restatement of (3.196) since equation (3.159) for the soliton velocity gives no contribution. The coefficient $4n^2$ appearing in the first term of (3.199) originates from the explicit form (3.152) for $D_1(n\beta)$. In an earlier stage of the calculation, equations (3.172) and (3.193) indicate that equation (3.199) takes the form,

$$D_o(n\beta)a_n = \frac{4n^2\Delta_o^2}{v_F^2m^2}\left[\frac{m^2}{g^2}[(2\lambda+1)+\sigma^2]\right]b_n - \frac{4n\Delta_o g}{v_F^2m^2}\left(S-\frac{m^2}{g^2}\right)c_n$$

$$-\frac{\Delta_o^2}{m^2v_F^2}\left[\delta_n^{(1)}\frac{g^2}{m^2}\left(S-\frac{m^2}{g^2}\right)(1-\sigma^2)^{-1}D_o(n\beta)\right.$$

$$\left.+\ \delta_n^{(2)}\frac{g^2}{m^2}\left(S-\frac{m^2}{g^2}\right)^2(1-q^2)^{-1}\right]$$

$$+\ \gamma_n\ D_o(n\beta) \quad ; \quad n=1,2,3 \qquad (3.200)$$

The coefficients of the first two terms in the above expression are obtained from (3.152) and (3.156) for $D_1(n\beta)$ and $D_2(n\beta)$ respectively. The third term is a linear combination of linear and quadratic forms for $(S-m^2/g^2)$ with coefficients $\delta_n^{(1)}$ and $\delta_n^{(2)}$. It originates from computations involving ξ_o. The remaining term is denoted by γ_n and plays no role in our analysis. Insertion of (3.138) for c_n into (3.200) yields,

$$\begin{aligned} D_o(n\beta)a_n = \frac{\Delta_o^2}{v_F^2 m^2} \Big\{ & 4n^2 \left[\frac{m^2}{g^2}[(2\lambda+1)+\sigma^2]\right] b_n \\ & -4n \left[\frac{g^2}{m^2} \left(S - \frac{m^2}{g^2}\right)^2 (1-\sigma^2)^{-1}\right] b_n \\ & -\delta_n^{(1)}\left[\frac{g^2}{m^2} \left(S - \frac{m^2}{g^2}\right) (1-\sigma^2)^{-1}\right] D_o(n\beta) \\ & -\delta_n^{(2)}\left[\frac{g^2}{m^2} \left(S - \frac{m^2}{g^2}\right)^2 (1-\sigma^2)^{-1}\right] \Big\} \\ & +\gamma_n D_o(n\beta) \qquad ; \; n=1,2,3 \end{aligned} \tag{3.201}$$

Since the factor $(S-m^2/g^2)$ appears linearly in the expression (3.175) for Q, comparison between (3.201) and the form (3.199) identifies A_n as,

$$A_n = \tfrac{1}{2}\, \delta_n^{(1)} \; ; \; n=1,2,3 \; . \tag{3.202}$$

Now since the same factor $(S-m^2/g^2)$ appears quadratically in the condition (3.159) for the soliton velocity, again comparison between (3.201) and (3.199) yields,

$$\delta_n^{(2)} = (4n^2-4n)\, b_n \qquad ; \quad n=1,2,3 \; . \tag{3.203}$$

From the critical observation,

$$\delta_n^{(1)} = \delta_n^{(2)} \qquad ; \quad n=1,2,3 \; , \tag{3.204}$$

we finally obtain the following form for A_n by comparing (3.202) and (3.203),

$$A_n = 2n(n-1)b_n \quad ; \quad n=1,2,3 \quad . \tag{3.205}$$

Insertion of (3.114) for b_n gives,

$$A_n = (-1)^n \, 4n(n-1) \quad ; \; n=1,2,3 \quad . \tag{3.206}$$

which agrees with earlier results. Upon the assumption that the generic forms (3.197) and (3.206) for B_n and A_n remain valid for all n, equation (3.196) for the coefficients a_n now becomes,

$$a_n = (-1)^n \left[4n(n-1)Q - na_1\right] \quad ; \quad n \geq 1 \quad . \tag{3.207}$$

Remembering the expansion (3.111), $\phi_1(X)$ is now determined as,

$$\phi_1(X) = 4Q \sum_{n=1}^{\infty} (-1)^n \, n(n-1) \, e^{-nX}$$

$$-a_1 \sum_{n=1}^{\infty} (-1)^n \, n \, e^{-nX} \quad ; \quad X>0 \quad . \tag{3.208}$$

Now since,

$$\tanh \frac{X}{2} = 1 + 2 \sum_{n=1}^{\infty} (-1)^n e \text{ [illegible]} \tag{3.209}$$

and,

$$\operatorname{sech} \frac{X}{2} = 2 \sum_{n=0}^{\infty} (-1)^n e^{-(2n+1)X/2} \quad ; \quad X>0 \tag{3.210}$$

one can show that,

$$\operatorname{sech}^2\frac{X}{2} = -4\sum_{n=1}^{\infty}(-1)^n n e^{[illegible]} \quad ; \quad X>0 \quad , \tag{3.211}$$

and,

$$\operatorname{sech}^2\frac{X}{2}\left[\tanh\frac{X}{2} - 1\right] = -4\sum_{n=1}^{\infty}(-1)^n n(n-1)e^{-nX} \quad ; \quad X>0 \quad . \tag{3.212}$$

Insertion of the latter results into the expansion (3.208) for the perturbation ϕ_1 finally gives,

$$\phi_1(X) = \frac{a_1}{4}\operatorname{sech}^2\frac{X}{2} - Q\operatorname{sech}^2\frac{X}{2}\left[\tanh\frac{X}{2} - 1\right] \quad ; \quad X>0 \quad , \tag{3.213}$$

with X given by (3.96)-(3.97) in which the velocity σ is constrained by equation (3.161).

Remembering the definition (3.22) as well as the perturbation expansions (3.98) and (3.99) for the optical and acoustic phonon solitons on the positive X interval, the solution (3.213) and (3.139) yield,

$$\phi(X) \simeq \tanh\frac{X}{2} + v^2\left[\left(\frac{a_1}{4}+Q\right)\operatorname{sech}^2\frac{X}{2} - Q\operatorname{sech}^2\frac{X}{2}\tanh\frac{X}{2}\right], \tag{3.214}$$

and,

$$\langle\xi\rangle(X) = R\tanh\frac{X}{2} + \text{constant} \quad , \tag{3.215}$$

where, setting the surface term to zero, Q and R are given as,

$$\tilde{Q} \equiv \frac{2\Delta_o^2}{v_F^2 m^2}\left[\lambda+(1-\sigma^2)^{-1}\right] \quad , \tag{3.216}$$

and,

$$R \equiv -\frac{\Delta_o}{g}(1-\sigma^2)^{-1} \qquad (3.217)$$

Assuming the following topological property for $\phi(X)$ at any finite time,

$$\phi(+\infty) = -\phi(-\infty) = 1 \qquad , \qquad (3.218)$$

as well as a similar relation for the acoustic phonon order parameter with proper normalization, one should have, for the same time, the following condition at the soliton center,

$$\phi(0) = \langle\xi\rangle(0) = 0 \qquad . \qquad (3.219)$$

These boundary conditions imply that the constant appearing in the expression (3.215) for $\langle\xi\rangle$ should vanish and that the constant a_1 in (3.214) is now determined as,

$$a_1 = -4Q \qquad . \qquad (3.220)$$

Therefore, for all X, one gets the final result,

$$\phi(X) = [1 - v^2 Q \operatorname{sech}^2 \frac{X}{2}] \tanh \frac{X}{2} + \ldots \qquad , \qquad (3.221)$$

and,

$$\langle\xi\rangle(X) = R \tanh \frac{X}{2} + \ldots \qquad , \qquad (3.222)$$

where dots stand for higher order terms in acoustic phonon velocity perturbative expansion.

Our results therefore suggest that, although the modification to the optical phonon soliton shape due to acoustic effects is small, acoustic effects induce the motion of the soliton across the molecular chain. For the allowed range of λ given by equation (3.161), the motion is that of a uniform translation. On an experimental basis, this implies that, far from impurities where pinning effects may be neglected, the soliton translation should be observable, if acoustic effects could be observed. Furthermore, because the motion is uniform, the fermionic zero-energy mode at the Fermi surface survives in the frame moving with the soliton. Therefore the so-called charge fractionalization mechanism is still operative, at least in the context of our quasirealistic model.

Note that a value $\lambda=-4.22$ for the effective phonon-quasielectron coupling yields the maximum soliton velocity $v_{sol}=2.7v$ obtained by Bishop et al,[91] from numerical integration of the discrete SSH model. For positive λ, the acoustic phonon velocity v plays the role of a speed of light since the soliton velocity can never be greater than v. The region $-2<\lambda<0$ may well correspond to a forbidden kinetic zone for constant translational motion.

Although we succeeded, in this section, in finding closed form analytic solutions to our quasirealistic model for trans-polyacetylene through self-consistent perturbation theory and the asymptotic expansion method, use of algorithms and extrapolations were needed because of the increasingly ferocious algebra encountered when going to higher order computations in the asymptotic expansion.

In the next section, we will present a self-consistent proof of the results obtained in this section by computing explicitly the quasielectron wavefunctions assuming that our soliton solutions are valid.

Inserting back the latter wavefunctions into the field equations for the solitons will show that the results are indeed exact. We also compute the corrections, due to acoustic effects, to the zero-energy fermionic mode of the TLM model.

4. Self-Consistent Proof for Soliton Solutions

In the last section, we showed how to solve analytically the quasi-realistic model for the polyacetylene molecule which includes acoustic phonon interaction effects by use of a self-consistent perturbation theory for small acoustic phonon velocity and the asymptotic expansion method. The results for soliton solutions are summarized as follows,

$$\phi(X) = \phi_o(X) - v^2\phi_1(X) + \ldots \qquad (3.223)$$

and

$$\langle\xi\rangle(X) = \xi_o(X) + \ldots \quad , \qquad (3.224)$$

for the optical and acoustic phonon order parameters respectively in which space and time coordinates always appear in the special configuration X (generalized coordinate) defined as,

$$X = \frac{2\Delta_o}{v_F}\,\frac{1}{(1-v_{sol}^2/v_F^2)^{1/2}}\,(x_1 \pm v_{sol}\,x_0 - \bar{x}) \quad , \qquad (3.225)$$

and restricted by the following condition on the soliton velocity,

$$\sigma^2 \equiv \frac{v_{sol}^2}{v^2} = \begin{cases} (\lambda^2+2\lambda)^{1/2} - \lambda & ; \quad \lambda \geq 0 \\ \pm(\lambda^2+2\lambda)^{1/2} - \lambda & ; \quad \lambda \leq -2 \end{cases} \quad . \qquad (3.226)$$

The solutions for ϕ_0, ϕ_1 and ξ_0 have been obtained explicitly as,

$$\phi_0(X) = \tanh \frac{X}{2} \quad , \tag{3.227}$$

$$\phi_1(X) = Q \operatorname{sech}^2 \frac{X}{2} \tanh \frac{X}{2} = Q(\phi_0 - \phi_0^3) \quad , \tag{3.228}$$

and

$$\xi_0(X) = R\phi_0(X) \quad , \tag{3.229}$$

in which the constants Q and R are given as,

$$Q = \frac{2\Delta_o^2}{v_F^2 m^2} \left[\lambda + (1-\sigma^2)^{-1}\right] \quad , \tag{3.230}$$

and,

$$R = -\frac{\Delta_o}{g} (1-\sigma^2)^{-1} \quad . \tag{3.231}$$

Although the configuration X given by (3.225) has the form of a boosted coordinate, Lorentz invariance is broken by acoustic effects because of the constraint (3.226) for the soliton velocity which chooses a preferred frame moving uniformly with respect to the reference frame at which the lattice points are at rest.

In this section, we propose to present a self-consistent proof for the above soliton solutions by explicit computation of the fermion wavefunctions.[96] To this purpose, the form of the relevant set of field equations most easily manageable is given by equations (3.14), (3.15) and (3.16). The strategy of our proof consists then in obtaining the quasi-electron wavefunctions from the mean-field equation (3.14) by assuming from the start the validity of the solutions given by (3.223)∿(3.231).

Having obtained an expression for the latter wavefunctions to correct order in perturbation theory for small acoustic phonon velocity v, we then insert them back into the rhs of equations (3.15)-(3.16) for the optical and acoustic phonon solitons, thereby checking the exactness of the original assumption. In this way, we are led to a relatively simple and quite elegant self-consistent proof.

4.1 - Computation of the Quasielectron Wavefunctions

In order to bring our proof to success, one first has to apply perturbation theory to the set of mean-field equations (3.14)∿(3.16) as is naturally suggested by (3.223) and (3.224). This, of course, requires a corresponding perturbative expansion of the quasielectron fields for small acoustic phonon velocity. In our perturbative analysis, as in the previous section, some care is required when expanding the electron fields. The difficulty here is that equation (3.14) for the electron holds in the frame which is at rest with respect to the lattice points. Since our soliton solutions (3.223)∿(3.231) indicate that the unperturbed state is the Lorentz invariant part obtained in the boosted frame, the unperturbed quasielectron fields must then transform as spinors under Lorentz transformation to the boosted frame. However, equation (3.14) is not Lorentz invariant. We therefore implement a Lorentz boost on the full equation (3.14) and then extract the invariant part as the unperturbed state. This procedure yields a consistent perturbation expansion of the quasielectron fields in the boosted frame. Implementing the above procedure, let us multiply equation (3.14) by the 2x2 matrix τ_1 and re-write it as (to v^2-order),

$$[i\Gamma_\mu \partial^\mu + \Delta_o \phi_o(X)]\psi(x_0,x_1) = v^2\Big[\Delta_o \phi_1(X) + \frac{\lambda\Delta_o}{m^2}\frac{\partial^2}{\partial x_1^2}\phi_o(X)$$

$$+ i\gamma_1 \frac{g}{m^2}\Big[\frac{\partial^2 \xi_o(X)}{\partial x_1^2} + 2\frac{\partial \xi_o(X)}{\partial x_1}\frac{\partial}{\partial x_1}\Big]\psi(x_0,x_1) \, , \quad (3.232)$$

in which we used expansion (3.223)-(3.224) and where we switched to the convention,

$$x_\mu \to (v_f x_0, x_1) \quad . \qquad (3.233)$$

Also we defined,

$$\Gamma_\mu \equiv v_F \gamma_\mu = v_F(\tau_1, \tau_1\tau_3) \quad , \qquad (3.234)$$

in which the γ-matrices satisfy the usual Clifford algebra,

$$\{\gamma_\mu, \gamma_\nu\} = 2\, g_{\mu\nu} \quad , \qquad (3.235)$$

and where diag $g_{\mu\nu}$ is given by (1,-1). In the remainder of this section, the convention (3.233) will always be implied. The generalized coordinate (3.225) is now re-written as,

$$X = \frac{2\Delta_o}{v_F}\frac{1}{(1 - v_{sol}^2/v_F^2)^{1/2}}[x_1 \pm (v_{sol}/v_F)x_0 - \bar{x}] \quad . \qquad (3.236)$$

Note that the chemical potential μ has been set equal to zero in (3.232). This is merely a redefinition for the zero point of the energy.

Boosted coordinates X_μ are obtained from the rest-frame coordinates (3.233) by the following Lorentz transformation,

$$X_\mu = \Lambda_\mu{}^\nu x_\nu \quad , \tag{3.237}$$

where,

$$\Lambda_\mu{}^\nu = \begin{bmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{bmatrix} \quad , \tag{3.238}$$

θ being the boost parameter. It is related to the soliton velocity through the following relations,

$$\cosh\theta = \frac{1}{(1-v_{sol}^2/v_F^2)^{1/2}} \quad , \tag{3.239}$$

and,

$$\sinh\theta = \frac{v_{sol}/v_F}{(1-v_{sol}^2/v_F^2)^{1/2}} \quad . \tag{3.240}$$

It is easy to see that the generalized coordinate X is related to the boosted coordinate X_1 as follows,

$$X = \frac{2\Delta_0}{v_F}(X_1-\bar{X}_1) \quad , \tag{3.241}$$

where,

$$\bar{X}_1 = \cosh\theta \; \bar{x}_1 \quad , \tag{3.242}$$

in which the initial time $\bar{x}_0$ has been set equal to zero.

We now define,

$$\psi(x_0,x_1) \equiv S^{-1}\psi(X_0,X_1) \quad , \tag{3.243}$$

where the Lorentz transformation matrix S is related to the Lorentz boost matrix (3.238) by the following basic relation,

$$S\gamma_\mu S^{-1} = (\Lambda^{-1})_\mu{}^\nu \gamma_\nu \quad . \tag{3.244}$$

From the relations (3.234) and (3.238) we obtain explicitly,

$$S = \begin{bmatrix} \sqrt{\cosh\theta + \sinh\theta} & 0 \\ 0 & \sqrt{\cosh\theta + \sinh\theta} \end{bmatrix} , \tag{3.245}$$

and,

$$S^{-1} = \begin{bmatrix} \sqrt{\cosh\theta - \sinh\theta} & 0 \\ 0 & \sqrt{\cosh\theta - \sinh\theta} \end{bmatrix} . \tag{3.246}$$

These matrices enjoy the following properties,

$$S^{+} = S \quad ; \quad S^{-1} = \gamma_0 S \gamma_0 \quad . \tag{3.247}$$

From the equations (3.239)-(3.240) for the boost as well as the constraint (3.226) imposed on the soliton velocity, one easily shows that,

$$S = 1 + O(v) \quad , \tag{3.248}$$

and,

$$\Lambda_\mu{}^\nu = \delta_\mu{}^\nu + O(v) \quad . \tag{3.249}$$

We are now ready to re-write equation (3.232) for the quasielectron fields in the frame moving with the soliton. Inserting equations (3.241), (3.243), (3.244) as well as the approximations (3.248)-(3.249) into (3.232), one obtains the following field equation,

$$[i\Gamma_\mu D^\mu + \Delta_o \phi_o(X_1)]\psi(X_0,X_1) = v^2\Big[\Delta_o\phi_1(X_1) + \frac{\lambda\Delta_o}{m^2}\frac{\partial^2}{\partial X_1^2}\phi_o(X_1) + i\gamma_1\frac{g}{m^2}\Big[\frac{\partial^2\xi_o(X_1)}{\partial X_1^2} + 2\frac{\partial\xi_o(X_1)}{\partial X_1}\cdot\frac{\partial}{\partial X_1}\Big]\Big]\psi(X_0,X_1) + \ldots , \tag{3.250}$$

where the dots stand for higher order terms in acoustic phonon velocity and in which we defined,

$$D^{\mu} \equiv \frac{\partial}{\partial X_{\mu}} \qquad (3.251)$$

The lhs of e[illegible].250) is Lorentz invariant and therefore the breakdown of this invariance is of order v^2.

We are now justified in expanding $\psi(X_0,X_1)$ perturbatively,

$$\psi(X_0,X_1) = \psi_o(X_0,X_1) + v^2\psi_1(X_0,X_1) + \ldots \qquad (3.252)$$

Insertion of the latter expansion into (3.250) yields separate equations for ψ_o and ψ_1,

$$[i\Gamma_{\mu}D^{\mu} + \Delta_o\phi_o(X_1)]\psi_o(X_0,X_1) = 0 \quad , \qquad (3.253)$$

and,

$$[i\Gamma_{\mu}D^{\mu} + \Delta_o\phi_o(X_1)]\psi_1(X_0,X_1) = \Big[\Delta_o\phi_1(X_1) + \frac{\lambda\Delta_o}{m^2}\phi_o''(X_1) + \frac{ig}{m^2}\gamma_1[\xi_o''(X_1) + 2\xi_o'(X_1)\frac{\partial}{\partial X_1}]\Big]\psi_o(X_0,X_1) . \qquad (3.254)$$

Making use of (3.234), the above equations are finally re-written as follows,

$$[iv_F\frac{\partial}{\partial X_o} + iv_F\tau_3\frac{\partial}{\partial X_1} + \Delta_o\phi_o(X_1)\tau_1]\psi_o(X_0,X_1) = 0 \quad , \qquad (3.255)$$

and,

$$[iv_F\frac{\partial}{\partial X_o} + iv_F\tau_3\frac{\partial}{\partial X_1} + \Delta_o\phi_o(X_1)\tau_1]\psi_1(X_0,X_1) = \Big[\Delta_o\tau_1\phi_1(X_1) + \frac{\lambda\tau_1\Delta_o\phi_o''(X_1)}{m^2} + \frac{ig\tau_3}{m^2}[\xi_o''(X_1) + 2\xi_o'(X_1)\frac{\partial}{\partial X_1}]\Big]\psi_o(X_0,X_1) . \qquad (3.256)$$

Obviously, prior to solving (3.256) for the perturbation ψ_1, one must first determine the unperturbed spinor wavefunctions ψ_o satisfying the field equation (3.255). Such wavefunctions have been computed some time ago and a detailed review of these computations can be found in references (92) and (93) ..

Looking for stationary solutions to equation (3.255), we write, following Nakahara,[92]

$$\psi(X_0,X_1) = \begin{bmatrix} \bar{U}(X_1) \\ \bar{V}(X_1) \end{bmatrix} \exp\left(-i\frac{\omega}{v_F} X_0\right) \quad , \tag{3.257}$$

where the functions $\bar{U}$ and $\bar{V}$ are expanded as,

$$\bar{U}(X_1) = \bar{U}_o(X_1) + v^2\bar{U}_1(X_1) + \ldots \quad , \tag{3.258}$$

and,

$$\bar{V}(X_1) = \bar{V}_o(X_1) + v^2\bar{V}_1(X_1) + \ldots \quad , \tag{3.259}$$

in accordance with (3.252). Equation (3.255) now yields coupled differential equations for $\bar{U}_o$ and $\bar{V}_o$,

$$\omega\bar{U}_o + iv_F\bar{U}'_o + \Delta_o\phi_o\bar{V}_o = 0 \quad , \tag{3.260}$$

and,

$$\omega\bar{V}_o - iv_F\bar{V}'_o + \Delta_o\phi_o\bar{U}_o = 0 \quad . \tag{3.261}$$

Defining,

$$\bar{f}_{\pm} \equiv \bar{U} \pm i\bar{V} = \bar{f}^{(0)}_{\pm} + v^2\bar{f}^{(1)}_{\pm} = (\bar{U}_o \pm i\bar{V}_o) + v^2(U_1 \pm i\bar{V}_1) , \tag{3.262}$$

equations (3.260)-(3.261) translate into the following,

$$\omega\tilde{f}_{\pm}^{(0)} + iv_F\tilde{f}_{\mp}^{(0)'} \pm i\Delta_o\phi_o\tilde{f}_{\mp}^{(0)} = 0 \quad . \tag{3.263}$$

Equations (3.263), after some manipulations, can be written as the following uncoupled second order differential equations,

$$v_F^2\tilde{f}_{\pm}^{(0)''} + (\omega^2 - \Delta_o^2\phi_o^2 \pm v_F\Delta_o\phi_o')\tilde{f}_{\pm}^{(0)} = 0 \quad . \tag{3.264}$$

Upon a suitable change of variables, the latter equation is recognized as the associated Legendre differential equation. The unperturbed spinor wavefunctions are then obtained as associated Legendre polynomials. Further defining,

$$\bar{U} \equiv Ue^{ikX_1} \quad , \tag{3.265}$$

$$\bar{V} \equiv Ve^{ikX_1} \quad , \tag{3.266}$$

and

$$\bar{f}_{\pm} \equiv f_{\pm}e^{ikX_1} \quad , \tag{3.267}$$

equations (3.264) have been shown to yield the following unperturbed scattered wavefunctions,

$$(U_o)_k = \frac{A_k}{2}\left[\frac{[\omega_k + v_F k]}{\omega_k} + \frac{i\Delta_o\phi_o}{\omega_k}\right] \quad , \tag{3.268}$$

and,

$$(V_o)_k = \frac{A_k}{2}\left[\frac{-i[\omega_k - v_F k]}{\omega_k} + \frac{\Delta_o\phi_o}{\omega_k}\right] \quad , \tag{3.269}$$

where the fermion energy is given as,

$$\omega_k = \pm\sqrt{v_F^2k^2 + \Delta_o^2} \quad . \tag{3.270}$$

Besides scattered solutions, there is also a zero-energy mode bound state obtained as,

$$(U_o)_B = -i(V_o)_B = \frac{1}{2}\sqrt{\frac{\Delta_o}{2v_F}}\,\sqrt{1-\phi_o^2} \quad . \tag{3.271}$$

Note that the normalization factor A_k, which we leave undetermined, can be obtained from orthonormalization of the eigenfunctions $f_{\pm}^{(0)}$ of (3.264). The normalization of the bound state (3.271) is obtained from,

$$\int_{-\infty}^{\infty} dX_1\ (f_-^{(0)})_B (f_-^{(0)})_B = 1 \quad . \tag{3.272}$$

Since $(f_+^{(0)})_B$ vanishes, the zero-energy mode therefore turns out to be a Majorana Spinor (charge self-conjugate). Having determined the unperturbed spinor wavefunctions we now turn to the problem of solving the perturbation satisfying equation (3.256). This turns out to be a relatively easy task. Inserting (3.257), (3.258), (3.259) as well as taking into account the definitions (3.265)∿(3.267), equation (3.256) for the perturbation yields the following coupled differential equations for U_1 and V_1,

$$(\omega - v_F k)U_1 + iv_F U_1' + \Delta_o \phi_o V_1 = \Delta_o\left(\phi_1 + \frac{\lambda\phi_o''}{m^2}\right)V_o + \frac{ig}{m^2}(\xi_o'' + 2ik\xi_o')U_o + \frac{2ig\xi_o' U_o'}{m^2} \ , \tag{3.273}$$

and,

$$(\omega + v_F k)V_1 - iv_F V_1' + \Delta_o \phi_o U_1 = \Delta_o\left(\phi_1 + \frac{\lambda\phi_o''}{m^2}\right)U_o - \frac{ig}{m^2}(\xi_o'' + 2ik\xi_o')V_o - \frac{2ig\xi_o' V_o'}{m^2} \ , \tag{3.274}$$

in which the subscript k or B, according to whether one deals with scattered wavefunctions with momentum $\vec{k}$ or bound states, have been omitted for notational convenience. Note that the rhs of (3.273)-(3.274) are completely known and act as source terms. The lhs for the perturbation

U_1 and V_1 is same as equations (3.260)-(3.261) for the unperturbed wavefunctions U_o and V_o. We therefore expect the appearance of a homogeneous term linear in U_o and V_o in the final expressions for U_1 and V_1. The coefficient of such a linear term should be determined from an orthogonality condition between the unperturbed and perturbed wavefunctions $f^{(0)}_{\pm}$ and $f^{(1)}_{\pm}$ respectively.

First dwelling upon the problem of finding scattered solutions to (3.273)-(3.274), one realizes that the latter equations are nothing but polynomials in ϕ_o when we make use of the following properties for ϕ_o,

$$\phi_o' = \frac{\Delta_o}{v_F}(1-\phi_o^2) \quad ; \quad \phi_o'' = -\frac{2\Delta_o}{v_F^2}(\phi_o-\phi_o^3) \quad , \tag{3.275}$$

as well as the expressions (3.228)-(3.229) for ϕ_1 and ξ_o, and also equations (3.268)-(3.269) for unperturbed scattered wavefunctions.

Assuming that the perturbations $(U_1)_k$ and $(V_1)_k$ are also given as polynomials in ϕ_o, one easily finds from power-counting arguments that they can be at most third degree polynomials. Therefore one writes,

$$(U_1)_k = \frac{A_k}{2}\left[(a_0)_k+(a_1)_k\phi_o+(a_2)_k\phi_o^2+(a_3)_k\phi_o^3\right] \quad , \tag{3.276}$$

and,

$$(V_1)_k = \frac{A_k}{2}\left[(b_0)_k+(b_1)_k\phi_o+(b_2)_k\phi_o^2+(b_3)_k\phi_o^3\right] \quad . \tag{3.277}$$

Again making use of the properties (3.275), insertion of the above polynomials into (3.273) and (3.274) reduces the problem to a simple algebraic determination of the coefficients a_i's and b_i's. Straightforward algebra yields the following solutions,

$$(U_1)_k = (U_o)_k\delta + A_k\left(\frac{g\Delta_o R}{v_F m^2}\right)\left[\frac{i(\omega_k+v_Fk)}{\Delta_o}\phi_o-\frac{(\omega_k+3v_Fk)}{2\omega_k}\phi_o^2-\frac{3i\Delta_o}{2\omega_k}\phi_o^3\right] \tag{3.278}$$

and, $$(V_1)_k = (V_o)_k\delta + A_k\left(\frac{g\Delta_o R}{v_F^2 m^2}\right)\left[-\frac{(\omega_k - v_F k)}{\Delta_o}\phi_o' + i\frac{(\omega_k - 3v_F k)}{2\omega_k}\phi_o^2 + \frac{3\Delta_o}{2\omega_k}\phi_o^3\right] \quad (3.279)$$

As expected, a homogeneous term has appeared and the coefficient δ must be evaluated from an orthogonality condition between unperturbed and perturbed wavefunctions. Carrying out such an orthogonality condition determines δ as,

$$\delta = \frac{g\Delta_o R}{v_F^2 m^2} \quad (3.280)$$

Finally, turning to the computation of the perturbation to the zero-mode bound state (3.271), we notice the following properties,

$$(U_o')_B = -\frac{\Delta_o}{v_F}\phi_o (U_o)_B \quad , \quad (3.281)$$

and, $$(V_o')_B = -\frac{\Delta_o}{v_F}\phi_o (V_o)_B \quad , \quad (3.282)$$

which are easily obtained from (3.275). Because of the latter relations, one recognizes that the set of equations (3.273)-(3,274), when applied to the bound state problem with ω=0, suggests that the ratio of perturbed and unperturbed parts for the fermion wavefunctions is again a polynomial in ϕ_o. Power-counting arguments show that it is at most a second degree polynomial in ϕ_o. Therefore one writes,

$$(U_1)_B = (a_0 + a_1\phi_o + a_2\phi_o^2)(U_o)_B \quad , \quad (3.283)$$

and,

$$(V_1)_B = (b_0 + b_1\phi_o + b_2\phi_o^2)(V_o)_B \quad , \quad (3.284)$$

Insertion of the above expansion, together with (3.281)-(3.282), into the differential equations (3.273) and (3.274) with $\omega=0$ finally yields,

$$b_i = a_i \quad (i=0,1,2) \quad , \tag{3.285}$$

where

$$a_1 = 0 \quad ; \quad a_2 = \frac{-3\Delta_o gR}{v_F^2 m^2} \quad . \tag{3.286}$$

The remaining a_0-coefficient plays a role similar to δ in the computation for the scattered wavefunctions. It is obtained as,

$$a_0 = \frac{\Delta_o gR}{v_F^2 m^2} \quad , \tag{3.287}$$

when implementing the following orthogonality condition,

$$\int_{-\infty}^{\infty} dX_1 \; (f_-^{(0)})_B (f_-^{(1)})_B \;=\; 0 \quad . \tag{3.288}$$

Having determined the perturbations (3.278) and (3.279) for scattered wavefunctions, we are now ready to check that equations (3.15) and (3.16) for the optical and acoustic phonon solitons are satisfied for the set of solutions (3.223)∿(3.231) by insertion of the former wavefunctions into the source terms of the latter equations for the solitons. This is done in the next subsection.

4.2 - Verification of Soliton Solutions

Turning to the optical and acoustic phonon equations (3.15)-(3.16), we now insert the perturbation expansions (3.223), (3.224) and (3.252) into them. Remembering (3.243) for Lorentz transformation on spinors as well as the properties (3.247) for the Lorentz tranformation matrix S, equations (3.15)-(3.16) lead to,

$$\frac{m^2}{g^2} \Delta_o \phi_o'(X_1) = \langle 0|\psi_o^{+}(X_0,X_1)\tau_1\psi_o(X_0,X_1)|0\rangle \quad , \qquad (3.289)$$

$$\left[\frac{\partial^2}{\partial(\frac{v}{v_F}x_0)^2} - \frac{\partial^2}{\partial x_1^2}\right]\frac{\Delta_o}{g^2}\phi_o'(X_1) - \frac{m^2}{g^2}\Delta_o\phi_o(X_1) =$$
$$\langle 0|\psi_o^{+}(X_0,X_1)\tau_1\psi_1(X_0,X_1) + \psi_1^{+}(X_0,X_1)\tau_1\psi_0(X_0,X_1)|0\rangle$$
$$- \frac{\lambda}{m^2}\frac{\partial^2}{\partial x_1^2} \langle 0|\psi_o^{+}(X_0,X_1)\tau_1\psi_o(X_0,X_1)|0\rangle \quad , \qquad (3.290)$$

and,

$$\left[\frac{-\partial^2}{\partial(\frac{v}{v_F}x_0)^2} + \frac{\partial^2}{\partial x_1^2}\right]\xi_o(X_1) = -\frac{ig}{m^2} \langle 0| \left[\psi_o^{+}(X_0,X_1)(S^{-1})^{+}\tau_3 S^{-1}\frac{\partial^2}{\partial x_1^2}\psi_o(X_0,X_1)\right.$$
$$\left. -\frac{\partial^2}{\partial x_1^2}\psi_o^{+}(X_0,X_1)(S^{-1})^{+}\tau_3 S^{-1}\psi_o(X_0,X_1)\right]|0\rangle \quad . \qquad (3.291)$$

Equation (3.289) is the unperturbed boosted TLM equation for the optical phonon soliton. Carrying out the time-derivative operations in (3.290) and (3.291) and inserting (3.289) into the rhs of (3.290) while remembering the approximations (3.248) and (3.249), equations (3.290) and (3.291) are re-written as,

$$(1-\sigma^2)\xi_o''(X_1) = -\frac{ig}{m^2} \langle 0|\psi_o^{+}(X_0,X_1)\tau_3\psi_o''(X_0,X_1)|0\rangle + c.c., \quad (3.292)$$

and,

$$(\lambda+1+\sigma^2)\frac{\Delta_o}{g^2}\phi_o''(X_1) - \frac{m^2}{g^2}\Delta_o\phi_1(X_1) = \langle 0|\psi_o^{+}(X_0,X_1)\tau_1\psi_1(X_0,X_1)|0\rangle + c.c., \quad (3.293)$$

where the primes indicate space-derivatives with respect to X_1.

Inserting equation (3.25‾) into the rhs of equations (3.289), (3.292) and (3.293) while remembering the relations (3.258) - (3.259) and (3.265) - (3.266) and summing over the momenta yields the following relations,

$$\frac{m^2}{g^2}\Delta_o\phi_o = \sum_k [(U_o)^*_k(V_o)_k + (V_o)^*_k(U_o)_k] \quad , \tag{3.294}$$

for the unperturbed TLM soliton,

$$(1-\sigma^2)\xi''_o = -\frac{ig}{m^2}\sum_k \{[(U_o)^*_k(U''_o)_k - (V_o)^*_k(V''_o)_k] - [(U''_o)^*_k(U_o)_k - (V''_o)^*_k(V_o)_k]\} \quad , \tag{3.295}$$

for the acoustic phonon soliton and,

$$(\lambda+1+\sigma^2)\frac{\Delta_o\phi''_o}{m^2} - \frac{m^2}{g^2}\Delta_o\phi_1 = \sum_k [(U_o)^*_k(V_1)_k + (V_o)^*_k(U_1)_k + (U_1)^*_k(V_o)_k + (V_1)^*_k(U_o)_k] \ , \tag{3.296}$$

for the perturbation to the TLM soliton. Making use of the unperturbed fermion wavefunctions (3.268)-(3.269) as well as (3.270), equation (3.294) for the unperturbed TLM soliton gives the following,

$$\frac{m^2}{g^2}\Delta_o\phi_o = -\Delta_o\phi_o\sum_k \frac{A_k^2}{\omega_k} \quad , \tag{3.297}$$

which is nothing but the gap equation,

$$\frac{m^2}{g^2} = -\sum_k \frac{A_k^2}{\omega_k} \quad . \tag{3.298}$$

Again, use of the same unperturbed fermion wavefunctions (3.268)-(3.269) together with (3.270) enables us to re-write equation (3.295) for the acoustic phonon order parameter in the following way,

$$(1-\sigma^2)\xi_o'' = \frac{g}{m^2}\Delta_o\phi_o'' \sum_k \frac{A_k^2}{\omega_k} . \tag{3.299}$$

Use of the gap equation (3.298) together with the form (3.229) for ξ_o determines the constant R as,

$$R = \frac{-\Delta_o}{g}(1-\sigma^2)^{-1} , \tag{3.300}$$

which readily agrees with the result (3.231) obtained from the asymptotic expansion method.

Finally, let us check the most critical part, that is the solution (3.228) for the perturbation to the TLM kink. From the explicit forms (3.268)-(3.269) as well as (3.278)-(3.279) for the fermion wavefunctions, one gets the following relationship,

$$[(U_o)_k^*(V_1)_k + (V_o)_k^*(U_1)_k + (U_1)_k^*(V_o)_k + (V_1)_k^*(U_o)_k] =$$

$$A_k^2 \left[\left(\frac{g\Delta_o R}{v_F^2 m^2}\right)\left(\frac{2\Delta_o}{\omega_k}\right)(2\phi_o^3-\phi_o) - \frac{2\Delta_o\phi_o\delta}{\omega_k}\right] , \tag{3.301}$$

where use was made of (3.270). Insertion of the above relation into the rhs of equation (3.296) for the perturbation ϕ_1, with δ given by (3.280), and again making use of the gap equation (3.298) finally yield,

$$(\lambda+1+\sigma^2)\frac{\Delta_o}{g^2}\phi_o'' - \frac{m^2}{g^2}\Delta_o\phi_1 = \frac{4\Delta_o^2 R}{v_F^2 g}(\phi_o-\phi_o^3) . \tag{3.302}$$

Inserting (3.228), (3.230), (3.231) as well as the relation (3.275) into (3.302) leads us to the following condition for the soliton velocity,

$$2\lambda + 1 + \sigma^2 - (1-\sigma^2)^{-1} = 0 \quad . \tag{3.303}$$

The physical roots of the above equation are easily obtained as,

$$\sigma^2 = \begin{cases} (\lambda^2+2\lambda)^{1/2} - \lambda & ; \quad \lambda \geq 0 \\ \pm(\lambda^2+2\lambda)^{1/2} - \lambda & ; \quad \lambda \leq -2 \end{cases} , \tag{3.304}$$

which readily agrees with the constraint (3.226) obtained from the asymptotic expansion method. This completes our proof.

The set of solutions (3.223)∼(3.231) therefore satisfies the mean-field equations (3.14)∼(3.16) if and only if the solitons are time-dependent and constrained to move uniformly across the linear molecular chain with the velocity controlled by the acoustic phonon velocity and specified by equation (3.304). Note that in the above computations, only scattered fermion wavefunctions contribute in shaping the solitons. Furthermore, in the boosted frame moving with the solitons, the zero-energy mode bound state survives acoustic effects and consequently the so-called charge fractionalization mechanism remains an observable in the context of the quasirealistic model. If the completely realistic model admits more complicated time-dependency, as suggested by recent numerical calculations[91] on the SSH model, such a mechanism is expected to break down.

CHAPTER IV

THERMODYNAMICAL PHENOMENA: THERMALLY INDUCED SUPERSYMMETRY BREAKING

1. General Considerations

In this chapter, we intend to discuss a second aspect of macroscopic phenomena arising in quantum systems through the study of the spontaneous symmetry breaking (SSB) of quantum symmetries by temperature effects.[41-54] Although we discuss on general grounds the SSB of internal and external[53-54] (space-time) symmetries, we will concentrate on the particular case of supersymmetry[41-52] (SUSY) by first reviewing briefly the Wess-Zumino model[49-51,111-113] (WZ) and then by giving a more detailed analysis of the O'Raifeartaigh model[52,114] (ORF). While the WZ model[111-113] is the supersymmetric theory of interacting one-component scalar superfields, the ORF model[114] is a more general (extended) supersymmetric theory of interacting multi-components scalar superfields.

In section 2, we discuss what happens to the Goldstone theorem at finite temperature for the Lorentz symmetry and the Wess-Zumino supersymmetric model. Section 3 will be devoted to the analysis of the N=3 O'Raifeartaigh model for which supersymmetry is already spontaneously broken at zero temperature and which exhibits a mixed mechanism of symmetry breaking when temperature effects are switched on. In each case we discuss the Goldstone mode phase structure and identify such (zero-energy) modes for both the free and interacting theories.

In section 3, though we do not go into a detailed calculation of the Ward-Takahashi identities of the ORF model, we speculate on the physical spectrum through the computation of the fermion propagator matrix obtained in the infrared region. The results are compared with the corresponding physical spectrum of the WZ model in the Goldstone phase. Because of the added internal degrees of freedom, the ORF model has richer symmetry content and will be shown to allow for different phases to coexist.[52]

In this chapter, we adopt the convention of Wess and Zumino for which the γ-matrices are taken in the real Majorana representation, $(\gamma^0)^2=(\gamma^5)^2=-1$, and the metric chosen as $g_{00}=-g_{ii}=-1$.

2. Review of Thermally Induced Symmetry Breaking

When one wants to answer the question asking what happens to the symmetries of a system when it is brought into contact with a thermal reservoir, one first has to specify the computational formalism for calculating statistical averages. Here and for the rest of this chapter, the elegant thermo-field dynamics (TFD) formalism[23-37] will be used. The main advantage of the TFD formalism is its deep resemblance with ordinary QFT at zero temperature. Temperature comes in the theory by making a canonical (Bogoliubov) transformation on a zero temperature system with twice the original number of degrees of freedom. The "second field" or tilde field is an unphysical ghost-like particle with negative energy. This "thermal hole" consequently never appears as an external line in the expressions for N-point functions. The above Bogoliubov transformation is in turn used to define the thermal vacuum of the theory. Expectation values of zero temperature field operators, making use of the thermal vacuum, are then shown to be equivalent to usual statistical averages. Because temperature dependent vacuum expectation values replace the trace operation, the TFD formalism therefore allows for a field theoretic interpretation of any spontaneous symmetry breaking by temperature effects. This is then the ideal formalism to elucidate the Nambu-Goldstone phase[26,123-125] structure of a given system at finite temperature.

At zero temperature, a given symmetry is said to be spontaneously broken if the vacuum does not carry the symmetry of the Heisenberg fields. Given a conserved charge Q, which represents the generator(s) of the symmetry transformation, as well as the corresponding current $J^\mu(x)$, the usual condition for SSB is written as,

$$Q|0> \neq 0 \quad . \tag{4.1}$$

A general expression for the dynamical map of the charge Q is the following expansion in terms of physical particles creation and annihilation operators,

$$Q = \sum_i (c_i b_i + c_i^* b_i^\dagger) + \sum_{ij} c_{ij} \alpha_i^\dagger \alpha_j + \ldots \quad , \tag{4.2}$$

where the ellipses stand for higher orders in normal products of physical field operators. The $\alpha_i^{(\dagger)}$'s are annihilation (creation) operators of the physical particles of the theory while the $b_i^{(\dagger)}$'s operators are the corresponding annihilation (creation) operators of the Goldstone particles. In the normal phase, the c_i's vanish and the dynamical map of Q starts at the bilinear term. However, if the condition (4.1) is satisfied, the linear term of (4.2) cannot vanish. The corresponding dynamical map of the current J^μ can be written as,

$$J^\mu(x) = c\partial^\mu b(x) + \ldots \quad , \tag{4.3}$$

Current conservation $\partial^\mu J_\mu = 0$ requires that the Goldstone particle b(x) be massless. Note that the bilinear term does not give any contribution to (4.1).

When temperature is switched on, however, the situation is drastically changed since the bilinear term $\alpha_i^\dagger \alpha_j$ fails to annihilate the temperature-dependent vacuum. In fact we now have,

$$\alpha_i^\dagger \alpha_j |0(\beta)\rangle = \tanh \theta_B^{(j)} \; \alpha_i^\dagger \tilde{\alpha}_j^\dagger |0(\beta)\rangle \neq 0 \quad , \qquad (4.4)$$

for bosons and,

$$\alpha_i^\dagger \alpha_j |0(\beta)\rangle = -\tan \theta_F^{(j)} \; \alpha_i^\dagger \tilde{\alpha}_j^\dagger |0(\beta)\rangle \neq 0 \quad , \qquad (4.5)$$

for fermions. In the above expressions we defined $\theta_B^{(i)}$ and $\theta_F^{(i)}$ as,

$$\sinh^2 \theta_B^{(i)} \equiv \frac{1}{e^{\beta\omega_i}-1} \quad ; \quad \sin^2 \theta_F^{(i)} \equiv \frac{1}{e^{\beta\omega_i}+1} \quad . \qquad (4.6)$$

The pair $\alpha_i^\dagger \tilde{\alpha}_j^\dagger$ appearing in equations (4.4)-(4.5) is called a thermal pair.[49-54] In the center of mass frame at rest, this is a zero energy mode since $\alpha_i^\dagger$ is a particle with positive energy ω_i while $\tilde{\alpha}_j^\dagger$ represents a hole with negative energy $-\omega_j = -\omega_i$. The fact that the multiplets are mass degenerate is the condition for the coefficients c_{ij} not to vanish in the dynamical map[26] (4.2).

In the finite temperature theory, however, the total generator for the symmetry under consideration is the following charge,[24,26]

$$\hat{Q} = Q - \tilde{Q} \quad , \qquad (4.7)$$

where $\tilde{Q}$ is the tilde conjugate of Q. Therefore the condition (4.1) for SSB is now replaced by,

$$\hat{Q} \, |0(\beta)\rangle \neq 0 \quad , \qquad (4.8)$$

at finite temperature. We are now ready to consider the possible cases (free and interacting theories) where a given symmetry is thermally broken.

A first example is a free theory with global internal symmetries. At zero temperature, the condition (4.1) is never realized for a free theory. At finite temperature, however, under special circumstances, the condition (4.8) can be realized even though no linear term shows up in the dynamical map for the generator. That this is so is due to the existence of thermal pairs $\alpha_i^{\dagger}\tilde{\alpha}_j^{\dagger}$ with zero energy. On the other hand because the symmetry generator is bosonic, it is easy to show from relations such as (4.4) and (4.5) that, although both Q and $\tilde{Q}$ fail to annihilate the vacuum $|0(\beta)\rangle$, they give the same contributions which eventually cancel each other when one considers the total generator $\hat{Q}$. Therefore, for internal symmetries, since the symmetry generators are always bosonic, thermally induced spontaneous symmetry breaking does not occur in the free case. No linear term appears in the dynamical map of the generators. The above cancellation occurs for bosonic generators because individual members of a thermal pair obey the same statistics.[53]

When interaction is switched on, however, the situation becomes quite different. In the interacting case, the formation of a temperature-dependent gap (order parameter) may then remove the mass degeneracy among multiplets and the usual Goldstone boson is then needed for the rearrangement mechanism. No zero-energy mode thermal pair exists in this case and the theory is quite similar to the SSB at zero temperature.

Going back to the free theory, a notable exception to the rule of no SSB for bosonic generators is the case of the Lorentz symmetry. Again, at finite temperature, the total generator of Lorentz boost is given as,

$$\hat{L}_{\mu\nu} = L_{\mu\nu} - \tilde{L}_{\mu\nu} \quad . \tag{4.9}$$

It can be shown, as recently discussed by Umezawa et al.[53] through the commutator of $\hat{L}_{\mu\nu}$ with the canonical energy-momentum tensor $T_{\mu\nu}$ that the following components for $\hat{L}_{\mu\nu}$ obey the non-vanishing condition for SSB,

$$\hat{L}_{oi}|0(\beta)\rangle \neq 0 \quad ; \quad i=1,2,3 \quad . \tag{4.10}$$

That this is so may be traced back to the fact that L_{oi} carries explicit t and x dependency,

$$\hat{L}_{oi} = t\hat{T}_{oi} - \int d^3x \; x_i \hat{T}_{oo}(x) \quad . \tag{4.11}$$

Because the explicit x_i inside the space integration acts as a momentum derivative $\partial/\partial p_i$, it can be shown by making use of spectral representations that the contributions from thermal pairs in this case fail to cancel, thereby producing the result (4.10). The existence of the zero-energy mode thermal pairs also accounts for the non-violation of the Ward-Takahashi identities (WT) for the broken Lorentz symmetry. They therefore play the rôle of Goldstone-type zero-energy modes. These, however, are not particle modes. It has also been argued that the same mechanism survives in the interacting case since the multiplets likely carry the same energy.

We now consider the problem of thermally induced symmetry breaking for which the symmetry generators are of fermionic type. Examples of such symmetries are given by the Becchi-Rouet-Stora (BRS) symmetry[148] for quantum gauge theories and the graded Poincaré symmetry or supersymmetry[111-114] of high-energy physics. Thermally induced supersymmetry breaking was first studied by Das and Kaku.[41] Other authors[42-48] also contributed, making use of different formalisms such as the so-called graded-trace operations[42-43] and the imaginary time Matsubara technique.[44-47]

Here we concentrate on the case of supersymmetric theories involving scalar superfields only, that is scalar boson and Majorana fermion component fields. The simplest of such theories is the one-component scalar superfield Wess-Zumino (WZ) model.

In the auxiliary field formulation of the WZ model, in which supersymmetric transformations take a simple linear form, the finite temperature Lagrangean is given as,[49-51,111-113]

$$\hat{\mathcal{L}} = \mathcal{L} - \tilde{\mathcal{L}} \quad , \tag{4.12}$$

where,

$$\begin{aligned}\mathcal{L} = & -\frac{1}{2}(\partial_\mu A)^2 - \frac{1}{2}(\partial_\mu B)^2 - \frac{i}{2}\bar{\psi}\partial\!\!\!/\psi + \frac{1}{2}F^2 + \frac{1}{2}G^2 \\ & + m(FA + GB - \frac{i}{2}\bar{\psi}\psi) + \frac{1}{2}g[F(A^2 - B^2) + 2GAB - i\bar{\psi}(A - \gamma^5 B)\psi] \\ & - \frac{1}{2}(Z-1)[(\partial_\mu A)^2 + (\partial_\mu B)^2 + i\psi\partial\!\!\!/\psi - F^2 - G^2] \quad .\end{aligned} \tag{4.13}$$

The supermultiplet (A,B,ψ,F,G) contains a scalar A-component field, a pseudoscalar B-field, a Majorana fermion ψ and the scalar and pseudoscalar auxiliary fields F and G. In the above Lagrangean, m is the mass of the supermultiplet, g is the coupling constant and Z is the supermultiplet wave function renormalization factor. Note that all fields and parameters appearing in (4.13) are the renormalized ones.

The generator of SUSY transformations is the following fermionic charge,

$$\hat{Q} \equiv Q - \tilde{Q} = \int d^3x[J^0(x) - \tilde{J}^0(x)] \quad , \tag{4.14}$$

where the current $J^\mu(x)$ is defined as,

$$J^\mu(x) \equiv [\partial\!\!\!/(A - \gamma^5 B) - (F + \gamma^5 G)]\gamma^\mu\psi \quad . \tag{4.15}$$

Making use of the following thermal doublet notation,

$$\phi^{\alpha} \equiv \begin{bmatrix} \phi \\ \tilde{\phi} \end{bmatrix} \quad ; \quad \phi = A, B, \psi, F, G \quad , \tag{4.16}$$

as well as,

$$Q^{\alpha} = \begin{bmatrix} Q \\ \tilde{Q} \end{bmatrix} \quad , \tag{4.17}$$

the Lagrangean density (4.13) is invariant, up to a four-divergence, under the following set of transformations,

$$[Q^{\alpha}, A^{\beta}]_{-} = i\tau^{\alpha\beta}\psi^{\beta} \quad , \tag{4.18}$$

$$[Q^{\alpha}, B^{\beta}]_{-} = i\tau^{\alpha\beta}\gamma^{5}\psi^{\beta} \quad , \tag{4.19}$$

$$[Q^{\alpha}, F^{\beta}]_{-} = i\tau^{\alpha\beta}\not{\partial}\psi^{\beta} \quad , \tag{4.20}$$

$$[Q^{\alpha}, G^{\beta}]_{-} = i\tau^{\alpha\beta}\gamma^{5}\not{\partial}\psi^{\beta} \quad , \tag{4.21}$$

and,

$$[Q^{\alpha}, \bar{\psi}^{\beta}]_{+} = \delta^{\alpha\beta}[-\not{\partial}(A^{\alpha} - \gamma^{5}B^{\alpha}) + F^{\alpha} + \gamma^{5}G^{\alpha}] \quad , \tag{4.22}$$

where,

$$\tau^{\alpha\beta} \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad . \tag{4.23}$$

Three important WT identities can also be obtained,

$$\partial^{x}_{\mu}\langle 0(\beta)|TJ^{\alpha\mu}(x)\bar{\psi}^{\beta}(y)|0(\beta)\rangle = \langle 0(\beta)|F^{\alpha}(x)|0(\beta)\rangle\delta^{\alpha\beta}\delta^{(4)}(x-y), \tag{4.24}$$

$$\begin{aligned} &\partial^{x}_{\mu}\langle 0(\beta)|TJ^{\alpha\mu}(x)A^{\beta}(y)\bar{\psi}^{\gamma}(z)|0(\beta)\rangle \\ &= i\langle 0(\beta)|T\psi^{\beta}(x)\bar{\psi}^{\gamma}(z)|0(\beta)\rangle\tau^{\alpha\beta}\delta^{(4)}(x-y) \\ &- \langle 0(\beta)|T[-\not{\partial}A^{\alpha}(x) + F^{\alpha}(x)]A^{\beta}(y)|0(\beta)\rangle\delta^{\alpha\gamma}\delta^{(4)}(x-z) \quad , \end{aligned} \tag{4.25}$$

and,

$$\begin{aligned} &\partial^{x}_{\mu}\langle 0(\beta)|TJ^{\alpha\mu}(x)F^{\beta}(y)\bar{\psi}^{\gamma}(z)|0(\beta)\rangle \\ &= -i[\not{\partial}^{x}\delta^{(4)}(x-y)]\langle 0(\beta)|T\psi^{\alpha}(x)\bar{\psi}^{\gamma}(z)|0(\beta)\rangle\tau^{\alpha\beta} \\ &- \langle 0(\beta)|T[-\not{\partial}A^{\alpha}(x) + F^{\alpha}(x)]F^{\beta}(y)|0(\beta)\rangle\delta^{\alpha\gamma}\delta^{(4)}(x-z) \quad . \end{aligned} \tag{4.26}$$

Allowing for the condensation of the A^{α} and F^{α} fields in the thermal vacuum, one re-writes the latter fields as,

$$A^{\alpha} = v + A'^{\alpha} \quad , \tag{4.27}$$

and,

$$F^{\alpha} = f + F'^{\alpha} \quad , \tag{4.28}$$

where the conditions,

$$\langle 0(\beta)|A'^{\alpha}|0(\beta)\rangle = \langle 0(\beta)|F'^{\alpha}|0(\beta)\rangle = 0 \quad , \tag{4.29}$$

yield self-consistent equations for the vacuum expectation values (VEV) v and f. A substitution of equations (4.27) and (4.28) into the Lagrangean (4.13) yields the following,

$$\begin{aligned} \mathcal{L} = & -\frac{Z}{2}\left[(\partial_{\mu}A')^2+(\partial_{\mu}B)^2+i\bar{\psi}\not\partial\psi-F'^2-G^2\right] \\ & + (m+gv)\left[F'A'+GB-\frac{i}{2}\bar{\psi}\psi\right] + \frac{1}{2}gf(A'^2-B^2) \\ & + \frac{1}{2}g\left[F'(A'^2-B^2)+2GA'B-i\bar{\psi}(A'-\gamma^5B)\psi\right] \\ & + \left[Zf+mv+\tfrac{1}{2}gv^2\right]F'+(m+gv)fA' \quad . \end{aligned} \tag{4.30}$$

Extracting unperturbed propagators for A,B and ψ-fields from the bilinear terms of the Lagrangean (4.30), one realizes that the mass degeneracy among these fields has been effectively removed. The mass spectrum is shown to be $(m+gv)^2 \pm gf$ for boson fields and $(m+gv)^2$ for fermion fields. It is clear that the VEV of the auxiliary F-field alone is responsible for the removal of the mass degeneracy. SUSY is then spontaneously broken for the interacting theory. Note that the order parameter f vanishes at zero temperature implying that there is no SSB of SUSY at zero temperature. SUSY is indeed a very difficult symmetry to break at zero temperature and to obtain SSB one has to introduce theories with higher number of component-fields, such as the O'Raifeartaigh model[114] discussed in the next section.

Although the WT relation (4.24) is satisfied to one-loop order, explicit computations show that the loop expansion is not consistent with the relations (4.25)-(4.26). That this is so can be understood from the fact that unperturbed propagators obtained from the Lagrangean (4.30) already exhibit the mass shifts. Since the mass shifts, obtained from the self-consistent equations (4.29) are already of one-loop order, it is clear that some of the WT identities are not satisfied by a naive loop expansion. A proper perturbation expansion, the so-called modified loop expansion,[51] has been devised to remedy this situation. An analysis of the WT identities (4.24)∿(4.26) has then shown that a massless fermion pole appears in a channel which is the linear combination $A\psi+c_1\psi$ and that a massive fermion pole appears in the channel $\psi+c_2A\psi$. The coefficients c_1 and c_2 vanish in the limit of no interaction. The former massless fermion pole is interpreted as the Goldstone fermion responsible for the rearrangement of the SSB of SUSY at finite temperature. At zero temperature, of course, such a pole disappears.

In the limit of vanishing coupling constant (free theory) however, although the VEV's v and f vanish, it can be shown that the SSB condition (4.8) still holds. This phenomenon can be traced back to the existence of thermal pairs similar to those occurring in (4.4) and (4.5). However, since the generator of SUSY is of fermionic type, each member of the pair obeys different statistics. Such pairs are called thermal superpairs.[49-52] Because in the free theory supermultiplets are mass degenerate, thermal superpairs are zero-energy modes in the center of mass frame at rest, that is $\phi_B(\vec{k},\omega(\vec{k}))\tilde{\phi}_F(-\vec{k},-\omega(\vec{k}))$ or $\phi_F(\vec{k},\omega(\vec{k}))\tilde{\phi}_B(-\vec{k},-\omega(\vec{k}))$, where the suffixes B and F stand for boson and fermion respectively. Because individual members of thermal superpairs obey different statistics, their contributions from Q and $\tilde{Q}$ fail to cancel contrary to the case where the symmetry generators are bosonic; SUSY is therefore spontaneously broken in the free theory at finite temperature. Note that thermal superpairs are necessary in order for the WT identities (4.24)$\sim$(4.26) to be satisfied. There, they appear as δ-function singularities at zero energy. They therefore play the role of Goldstone modes for the free theory. These are n[illegible] particle modes however.

[illegible] superpairs creation can be interpreted as a single-particle reaction in which the fields change statistics as well as tilde character. The existence of such zero-energy modes in the free theory can also help us identify the nature of the Goldstone fermion which appears in the interacting case. Since we know that a massive pole survives in the ψ channel of the interacting theory, it is therefore impossible to characterize the Goldstone fermion of the interacting theory as an elementary particle obtained from the original massive elementary fermion for which the bare mass is totally compensated by the mass shift. It has therefore been argued that the massless pole in the channel $A\psi+c_1\psi$ is a bound state of more elementary excitations whose binding energy vanishes at the limit of no interaction.[51,53] Such elementary excitations are thermal superpairs for which the binding energy compensates the mass difference caused by the interaction.

In order to understand the interplay between the SSB mechanisms at zero and finite temperature in supersymmetric theories, one must consider more complicated models than the WZ model. To that purpose, the O'Raifeartaigh model seems appropriate.

In the following section, we concentrate on the study of thermally induced supersymmetry breaking by presenting an analysis of an extended SUSY model, the O'Raifeartaigh model, which allows for SSB of SUSY already at zero temperature. We then identify the Goldstone mode phase structure of this mixed SSB mechanism by explicit computation of the inverse fermion propagator. Both the interacting and the free theory are considered.

3. The N=3 O'Raifeartaigh Model

In the previous section, we briefly reviewed the theory of thermally induced broken supersymmetry in the context of the simple one-component scalar superfield Wess-Zumino model.[49-51] We saw that in order to satisfy the Ward-Takahashi identities for the interacting system, a massless Goldstone-type particle mode must appear in the linear combination of the elementary fermion Green's function and the channel of the fermion-scalar boson composite of the model.

Since the removal of mass degeneracy between supermultiplets by temperature effects occurs only beyond the tree approximation and therefore only for the interacting case, it was argued that the Goldstone fermion is a bound state of thermal superpairs which dissociate in the limit of vanishing coupling constant. The mass difference must therefore account for the binding energy of the thermal superpair's bound state when interaction is switched on. When interaction is switched off, the Goldstone particle disappears and the zero energy thermal superpair mode takes over the role of a Goldstone mode for the free theory, as required from the Ward-Takahashi identities. Because of this most probable mechanism, the physical spectrum for both the free and interacting phases is quite different.

We now extend the previous analysis to the case of the N-components scalar superfield model, which is known as the O'Raifeartaigh model. The main motivation for the study of higher components supersymmetric models, early in the development of SUSY, came from the fact that supersymmetry is not an easy symmetry to break at zero temperature, as opposed to the case of internal symmetries. O'Raifeartaigh,[114] in the mid-seventies, however, succeeded in presenting a set of necessary (but not sufficient) conditions for which CP-invariant N-components interacting scalar superfields models may allow for SSB of supersymmetry. He showed that models for which $N<3$, such as the Wess-Zumino model, can never exhibit SSB of SUSY at zero temperature. He also presented an explicit $N=3$ model which allows for SSB of SUSY. This is the model we will now be interested in since it is the simplest one allowing for supersymmetry breaking at zero temperature.

The main interest of the model in the finite temperature context, is that it allows for a mixed mechanism of supersymmetry breaking: the zero temperature and the thermally induced SSB of SUSY. The interplay between two a priori independent mechanisms is by itself an interesting problem. As we shall see, both mechanisms effectively operate independently of one another and some competition may occur according to which effect (zero or finite temperature mass shift) is predominant in the removal of mass degeneracy among supermultiplets. The main feature of our solution of the $N=3$ O'Raifeartaigh model is the fact that two Goldstone fermions are now required for the symmetry rearrangement mechanism at finite temperature. More generally, our solution for the interacting case has the following features:[52]

(1) The Goldstone fermion responsible for the rearrangement of the SSB of SUSY at T=0 appears in the 0-0 channel of the fermion propagator matrix and survives at T≠0.

(2) The mass degeneracy among supermultiplets is removed only in the 1-1 channel at T=0.

(3) At T≠0 and after re-diagonalization of the fermion propagator matrix, a further temperature-dependent mass shift appears in the 1-1 channel only, thereby still leaving the 0-0 and 2-2 channels mass degenerate.

(4) As a consequence of (3), a Goldstone fermion particle mode similar to the one observed in the Wess-Zumino model appears in the mass shifted channel and a thermal superpair mode shows up in the remaining degenera[illegible] 2-2 channel.

Note that the T=0 Goldstone fermion appears in a channel orthogonal to the mass shifted channel (transversal Goldstone fermion), in agreement with the Goldstone theorem, while the T≠0 WZ-type Goldstone fermion appears in the mass-shifted channel itself (longitudinal Goldstone fermion).

In the following subsections, we present a detailed analysis of the N=3 O'Raifeartaigh model at finite temperature.

?.1 - The Lagrangean Model

At zero temperature, the Lagrangean density of the N-components O'Raifeartaigh model is more generally written as,[52,114]

$$\mathcal{L} = \mathcal{L}_o + \mathcal{L}_m + \mathcal{L}_g + \mathcal{L}_\lambda + \mathcal{L}_z + \mathcal{L}_c \quad , \tag{4.31}$$

where,

$$\mathcal{L}_o \equiv -\frac{1}{2}(\partial_\mu A_c)^2-\frac{1}{2}(\partial_\mu B_c)^2-\frac{i}{2}\bar{\psi}_c\partial\!\!\!/\psi_c+\frac{1}{2}F_c^{\ 2}+\frac{1}{2}G_c^{\ 2} \quad , \tag{4.32}$$

$$\mathcal{L}_m \equiv -\frac{i}{2}\bar{\psi}_c m_{ca}(v_0)\psi_a+m_{ca}(v_0)[\ A_aF_c+B_aG_c\ ]$$
$$+\frac{1}{2}f_c^{(0)}g_{cab}[\ A_aA_b-B_aB_b\ ] \quad , \tag{4.33}$$

$$\mathcal{L}_g \equiv \frac{1}{2}g_{cab}[\ -i\bar{\psi}_c(A_a-\gamma^5B_a)\psi_b+(A_aA_b-B_aB_b)F_c+2G_cA_aB_b\ ] \quad , \tag{4.34}$$

$$\mathcal{L}_\lambda \equiv \frac{1}{2}f_c^{\ (0)}[\ F_c+m_{ca}(v_0)A_a\ ]+\lambda_c(v_0)F_c \quad , \tag{4.35}$$

$$\mathcal{L}_z \equiv -\frac{1}{2}\delta Z_{ab}[\ (\partial_\mu A_a)(\partial^\mu A_b)+(\partial_\mu B_a)(\partial^\mu B_b)+i\bar{\psi}_a\partial\!\!\!/\psi_b-F_aF_b-G_aG_b\ ]$$
$$+\ \delta Z_{ab}f_a^{\ (0)}F_b \quad , \tag{4.36}$$

and,
$$\mathcal{L}_c \equiv \frac{1}{2}f_c^{\ (0)}[\ f_c^{\ (0)}+2\lambda_c(v_0)\ ] \quad . \tag{4.37}$$

In the above expressions, the A and B-fields represent renormalized scalar and pseudoscalar fields while F and G represent renormalized auxiliary scalar and pseudoscalar fields respectively. The fields ψ stand for renormalized Majorana fermions. Note that the "internal symmetry" indices run from 0 to N-1 and are being summed over when repeated. The parameters $v_a^{\ (0)}$ and $f_a^{\ (0)}$ are defined as the zero temperature order parameters of unshifted renormalized A'_a and F'_a - fields respectively. Because the Lagrangean described by equations (4.31)∿(4.37) exhibits the shifts, one has obviously,

$$\langle 0|A_a|0\rangle=\langle 0|F_a|0\rangle=0 \quad , \tag{4.38}$$

if one writes unshifted fields as,

$$A'_a=A_a+v_a^{\ (0)} \quad ; \quad F'_a=F_a+f_a^{\ (0)} \quad . \tag{4.39}$$

The mass matrix and "tadpole constants" appearing in the Lagrangean (4.31)∿(4.37) are the shifted parameters $m_{ca}(v_0)$ and $\lambda_c(v_0)$, since the

Lagrangean is written in terms of shifted fields. They are defined in terms of the unshifted m_{ca} and λ_c parameters as well as the coupling constant g_{cab} and A-fields vacuum expectation values $v_a^{(0)}$ as,

$$m_{ca}(v_0) \equiv m_{ca}+g_{cab}v_b^{(0)} \quad , \tag{4.40}$$

and,

$$\lambda_c(v_0) \equiv \lambda_c+m_{ca}v_a^{(0)}+\frac{1}{2}g_{cab}v_a^{(0)}v_b^{(0)} \quad . \tag{4.41}$$

The reason why only the A and F-fields need be shifted in this model is the fact that the unshifted Lagrangean is invariant under a rotation in the A-γ^5B and F-γ^5G planes. We therefore have always the freedom to re-define the fields in such a way that shifts occur only as in equation (4.39).

Note that the parameters and fields have been renormalized according to,

$$\bar{\phi}_b = Z^{\frac{1}{2}}_{ba}\phi_a \quad ; \quad (\phi=A,B,F,G,\psi) \quad , \tag{4.42}$$

$$\bar{m}_{ca} = Z^{-\frac{1}{2}}_{cb}Z^{-\frac{1}{2}}_{ad}m_{bd} \quad , \tag{4.43}$$

$$\bar{g}_{cab} = Z^{-\frac{1}{2}}_{cd}Z^{-\frac{1}{2}}_{ae}Z^{-\frac{1}{2}}_{bf}g_{def} \quad , \tag{4.44}$$

and,

$$\bar{\lambda}_c = Z^{-\frac{1}{2}}_{ca}\lambda_a \quad , \tag{4.45}$$

where the overbars identify bare parameters and fields. One also has the obvious relation,

$$Z_{ab} \equiv Z^{\frac{1}{2}}_{ac}Z^{\frac{1}{2}}_{cb} \quad . \tag{4.46}$$

From the mass terms (4.33) of the Lagrangean, it is apparent that a non-vanishing value for $f_c^{(0)}$ implies a removal of mass degeneracy among supermultiplets, thereby spontaneously breaking supersymmetry.

At zero temperature, the condition for SSB of SUSY can be written as,

$$\lambda_c(v_0) \neq 0 \quad . \tag{4.47}$$

For the N=3 theory, O'Raifeartaigh[114] showed that, in the tree approximation, the choice,

$$m_{ca} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & m_1 & 0 \\ 0 & 0 & -m_2 \end{bmatrix} \quad ; \quad \lambda_c = \begin{bmatrix} \lambda \\ 0 \\ 0 \end{bmatrix} \quad , \tag{4.48}$$

together with,

$$g_{0ab} = \frac{g}{m_1+m_2} \begin{bmatrix} 0 & 0 & 0 \\ 0 & m_1 & m \\ 0 & m & m_2 \end{bmatrix} \quad ; \quad g_{1ab} = \frac{g}{m_1+m_2} \begin{bmatrix} 0 & m_1 & m \\ m_1 & 0 & 0 \\ m & 0 & 0 \end{bmatrix} \quad ;$$

$$g_{2ab} = \frac{g}{m_1+m_2} \begin{bmatrix} 0 & m & m_2 \\ m & 0 & 0 \\ m_2 & 0 & 0 \end{bmatrix} \quad , \tag{4.49}$$

where $m \equiv \sqrt{m_1 m_2}$, yields a consistent theory of SSB of SUSY at zero temperature. Equation (4.48), however, indicates that fermions have negative mass in the 2-2 channel. This difficulty is resolved by making the following redefinition: $\psi_2 \to -\gamma^5 \psi_2$. It is then easily shown that the free propagator in the 2-2 channel carries positive mass. What has been changed, however, is the intrinsic parity of the ψ_2-fields. The intrinsic parity of the ... components remains +i while the parity of the ψ_2 component after ...sformation has become -i. Note that the intrinsic parit... nd B_b-fields is +1 and -1 respectively.

When one c... ...ers ... model at finite temperature, one might expect further temperatu... ...nt shifts of the order parameters since thermal effects are likely to shift the effective potential. As emphasized in previous sections as well as previous chapters of this work, the thermo-field dynamics formalism of QFT at finite temperature seems to us the desired and most practical (as well as elegant) formalism in dealing with statistical averages of both the normal as well as the spontaneously broken symmetry phases of any quantum field theoretical system. Since thermal averages of time-ordered operator products can be written in the following path integral form,

$$<0(\beta)|T(\ldots)|0(\beta)> = \int [dAdBd\psi dFdGd\tilde{A}d\tilde{B}d\tilde{\psi}d\tilde{F}d\tilde{G}](\ldots)\exp[i\int d^4x\hat{\mathcal{L}}(x)] \quad , \quad (4.50)$$

the Feynman rules are obtained from the total Lagrangean $\hat{\mathcal{L}}$ for the fields and their tilde conjugates. Note that $d\bar{\psi}$ does not appear in the path integral measure. This is so since ψ is a Majorana spinor and,therefore, ψ and $\bar{\psi}$ are not independent. If we further supplement the action with boundary terms (Feynman $i\delta$-prescription) specifying the thermal and causal nature of the unperturbed propagators about which one wishes to do perturbation theory, then the total Lagrangean can be written as follows,

$$\hat{\mathcal{L}} = \hat{\mathcal{L}}_o + \hat{\mathcal{L}}_I + \hat{\mathcal{L}}_Z \quad , \quad (4.51)$$

where,

$$\hat{\mathcal{L}}_o \equiv [A_a\tilde{A}_aF_a\tilde{F}_a]\begin{bmatrix}U_B^{-1} & 0\\ 0 & U_B^{-1}\end{bmatrix}\begin{bmatrix}[\delta_{ab}(\Box+i\tau\delta)+f_cg_{cab}] & m_{ab}(v)\\ m_{ab}(v) & \delta_{ab}\end{bmatrix}\begin{bmatrix}\tau U_B^{-1} & 0\\ 0 & \tau U_B^{-1}\end{bmatrix}\begin{bmatrix}A_b\\ \tilde{A}_b\\ F_b\\ \tilde{F}_b\end{bmatrix}$$

$$+ [B_a\tilde{B}_aG_a\tilde{G}_a]\begin{bmatrix}U_B^{-1} & 0\\ 0 & U_B^{-1}\end{bmatrix}\begin{bmatrix}[\delta_{ab}(\Box+i\tau\delta)-f_cg_{cab}] & m_{ab}(v)\\ m_{ab}(v) & \delta_{ab}\end{bmatrix}\begin{bmatrix}\tau U_B^{-1} & 0\\ 0 & \tau U_B^{-1}\end{bmatrix}\begin{bmatrix}B_b\\ \tilde{B}_b\\ G_b\\ \tilde{G}_b\end{bmatrix}$$

$$-\frac{i}{2}\ [\bar{\psi}_a\tilde{\bar{\psi}}_a]\ U_F\ [\delta_{ab}(\not{\partial}-i\tau\delta)+m_{ab}(v)]\ U_F^{-1}\begin{bmatrix}\psi_b\\ \tilde{\psi}_b\end{bmatrix} \quad , \quad (4.52)$$

$$\hat{\mathcal{L}}_I \equiv \frac{1}{2}g_{cab}[F_c(A_aA_b-B_aB_b)+2G_cA_aB_b-i\bar{\psi}_a(A_c-\gamma^5B_c)\psi_b]$$
$$-\frac{1}{2}g_{cab}[\tilde{F}_c(\tilde{A}_a\tilde{A}_b-\tilde{B}_a\tilde{B}_b)+2\tilde{G}_c\tilde{A}_a\tilde{B}_b+i\bar{\tilde{\psi}}_a(\tilde{A}_c-\gamma^5\tilde{B}_c)\tilde{\psi}_b]$$
$$+[f_c+\lambda_c(v)](F_c-\tilde{F}_c) + f_cm_{ca}(v)(A_a-\tilde{A}_a) \quad , \tag{4.53}$$

and,
$$\hat{\mathcal{L}}_Z \equiv -\frac{1}{2}\delta Z_{ab}[(\partial_\mu A_a)(\partial^\mu A_b)+(\partial_\mu B_a)(\partial^\mu B_b)+i\bar{\psi}_a\partial\!\!\!/\psi_b-F_aF_b-G_aG_b]$$
$$+\frac{1}{2}\delta Z_{ab}[(\partial_\mu\tilde{A}_a)(\partial^\mu\tilde{A}_b)+(\partial_\mu\tilde{B}_a)(\partial^\mu\tilde{B}_b)-i\bar{\tilde{\psi}}_a\partial\!\!\!/\tilde{\psi}_b-\tilde{F}_a\tilde{F}_b-\tilde{G}_a\tilde{G}_b]$$
$$+ \delta Z_{ca}f_a(F_c-\tilde{F}_c) \quad . \tag{4.54}$$

In equations (4.51)~(4.54), an overall c-number term similar to (4.37) has been dropped. The factor δ appearing in causal propagators is infinitesimal and positive. Also, the following definitions hold,

$$v_c \equiv v_c^{(0)} + v_c^{(\beta)} \quad ; \quad f_c \equiv f_c^{(0)} + f_c^{(\beta)} \quad , \tag{4.55}$$

$$m_{ca}(v) \equiv m_{ca} + g_{cab}v_b \quad , \tag{4.56}$$

and,
$$\lambda_c(v) \equiv \lambda_c + m_{ca}v_a + \tfrac{1}{2}\, g_{cab}v_av_b \quad . \tag{4.57}$$

The quantities $v_c^{(\beta)}$ and $f_c^{(\beta)}$ in equation (4.55) represent the temperature-dependent shifts of the order parameters corresponding to the A and F fields respectively, while equations (4.56) and (4.57) are the obvious generalization of equations (4.40) and (4.41) to the finite temperature case. The thermal Bogoliubov transformation matrices $U_B(|-i\partial_t|)$ and $U_F(|-i\partial_t|)$ are given in appendix B.

As it is, the above theory at finite temperature is very complicated. We shall see in the following subsection that in order to do perturbation theory in practical computations, a tedious task of suitable diagonalizations must first be carried out as temperature effects render the mass matrix (4.56) non-diagonal.

3.2 - Perturbation Theory

In order to avoid computational difficulties as well as securing a simple spectral representation for the asymptotic (physical) states, it is most convenient to implement perturbation theory about a diagonal unperturbed theory. To do so requires suitable similarity transformations which implement a consistent diagonalization of the unperturbed total Lagrangean density (4.52). If one remembers the O'Raifeartaigh's choice (4.48) and (4.49) for the unshifted parameters of the broken symmetry N=3 model, then it is easy to see that the (symmetric) fermion mass matrix $m_{ca}(v)$ given by equation (4.56) is not diagonal. The first step toward the solution consists then in choosing the asymptotic fermion basis in which the mass matrix is diagonal. Explicitly, equation (4.56) takes the following form,

$$m_{ca}(v) = \begin{bmatrix} 0 & \frac{g}{m_1+m_2}(m_1v_1+mv_2) & \frac{g}{m_1+m_2}(mv_1+m_2v_2) \\ \cdots & m_1\left[1+\frac{gv_0}{m_1+m_2}\right] & \frac{mgv_0}{m_1+m_2} \\ \cdots & \cdots & -m_2\left[1-\frac{gv_0}{m_1+m_2}\right] \end{bmatrix} \quad . \qquad (4.58)$$

Implementing the following similarity transformation,

$$m(v) \to S^{-1}m(v)S \quad , \qquad (4.59)$$

one obtains the following diagonal mass matrix,

$$m'_{ca}(v) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & M_1 & 0 \\ 0 & 0 & -M_2 \end{bmatrix} \quad , \qquad (4.60)$$

where the positive definite masses M_1 and M_2 are given as,

$$M_{\substack{1\\2}} = \pm\sigma + \sqrt{\sigma^2 + m^2 + \frac{g^2}{m_1+m_2}(\sqrt{m}_1 v_1 + \sqrt{m}_2 v_2)^2} \quad , \qquad (4.61)$$

in which

$$2\sigma \equiv m_1 - m_2 + g v_0 \quad . \qquad (4.62)$$

The transformation matrix S appearing in equation (4.59) is defined as,

$$S \equiv \begin{bmatrix} u_0 & p_0 & n_0 \\ u_1 & p_1 & n_1 \\ u_2 & p_2 & n_2 \end{bmatrix} \quad , \qquad (4.63)$$

where,

$$u_0 = m(m_1+m_2)/N_0 \quad ; \quad u_{\substack{1\\2}} = \mp g\sqrt{m}_{\substack{2\\1}}(\sqrt{m}_1 v_1 + \sqrt{m}_2 v_2)/N_0 \quad , \qquad (4.64)$$

$$p_0 = g(\sqrt{m}_1 v_1 + \sqrt{m}_2 v_2)/N_1 \quad ; \quad p_{\substack{1\\2}} = \pm \sqrt{m}_{\substack{1\\2}}(m_{\substack{2\\1}} \pm M_1)/N_1 \quad , \qquad (4.65)$$

$$n_0 = -g(\sqrt{m}_1 v_1 + \sqrt{m}_2 v_2)/N_2 \quad ; \quad n_{\substack{1\\2}} = \mp\sqrt{m}_{\substack{1\\2}}(m_{\substack{2\\1}} \mp M_2)/N_2 \quad , \qquad (4.66)$$

in which,

$$N_0^2 \equiv (m_1+m_2)^2 M_1 M_2 \quad , \qquad (4.67)$$

and,

$$N_{\substack{1\\2}}^2 \equiv (m_1+m_2)(M_1+M_2)M_{\substack{1\\2}} \quad . \qquad (4.68)$$

The transformation matrix (4.63) can now be used to define the new boson basis although diagonalization is not guaranteed in the boson sector. This global change of basis in the theory further implies coupling constant transformations as well as transformations for all other parameters of the Lagrangean. Adopting the convention that primes now indicate parameters or fields in this new basis and defining the mass shifts as,

$$b_{ab} \equiv g'_{abc} f'_c = b_{ba} \quad , \tag{4.69}$$

one can obtain explicitly the following equations for b_{ab} as well as the non-vanishing values of the totally symmetric coupling constant g'_{abc},

$$g'_{011} = g_0\sqrt{\frac{M_1}{M_2}} \; ; \; g'_{022} = g_0\sqrt{\frac{M_2}{M_1}} \; ; \; g'_{012} = g_0 \; ;$$

$$g'_{111} = 3g_1\sqrt{\frac{M_1}{M_1+M_2}} \; ; \; g'_{222} = -3g_1\sqrt{\frac{M_2}{M_1+M_2}} \; ;$$

$$g'_{112} = g_1 \frac{(2M_2-M_1)}{M_2}\sqrt{\frac{M_2}{M_1+M_2}} \; ; \; g'_{122} = -g_1\frac{(2M_1-M_2)}{M_1}\sqrt{\frac{M_1}{M_1+M_2}} \quad , \tag{4.70}$$

where one has defined,

$$g_0 \equiv g\,\frac{Mm}{M_1+M_2} \; ; \; g_1 \equiv \frac{g^2(\sqrt{m_1}v_1+\sqrt{m_2}v_2)}{\sqrt{m_1+m_2}\,(M_1+M_2)} \quad , \tag{4.71}$$

and,

$$b_{00} = 0 \; ; \; b_{01} = g_0\sqrt{\frac{M_1}{M_2}}f'_1+g_0f'_2 \; ; \; b_{02} = g_0f'_1+g_0\sqrt{\frac{M_2}{M_1}}f'_2 \; ;$$

$$b_{12} = g_0f'_0+g_1\frac{(2M_2-M_1)}{M_2}\sqrt{\frac{M_2}{M_1+M_2}}\,f'_1-g_1\frac{(2M_1-M_2)}{M_1}\sqrt{\frac{M_1}{M_1+M_2}}\,f'_2 \; ;$$

$$b_{11} = g_0\sqrt{\frac{M_1}{M_2}}\,f'_0+3g_1\sqrt{\frac{M_1}{M_1+M_2}}\,f'_1+g_1\frac{(2M_2-M_1)}{M_2}\sqrt{\frac{M_1}{M_1+M_2}}\,f'_2 \; ;$$

$$b_{22} = g_0\sqrt{\frac{M_2}{M_1}}\,f'_0-g_1\frac{(2M_1-M_2)}{M_1}\sqrt{\frac{M_1}{M_1+M_2}}\,f'_1-3g_1\sqrt{\frac{M_2}{M_1+M_2}}\,f'_2 \quad . \tag{4.72}$$

Note that in terms of primed parameters and fields, the Lagrangean density described by (4.51)~(4.54) assumes the same general form.

In this new basis, one obtains directly from the unperturbed part (4.52) of the Lagrangean the following inverse (A,F) and (B,G)-boson propagator matrix,

$$\Delta^{-1}_{\substack{(AF)\\(BG)}}(\partial) \propto \begin{bmatrix} \Box & 0 & \pm b_{01} & 0 & \pm b_{02} & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \pm b_{01} & 0 & \Box\pm b_{11} & M_1 & \pm b_{12} & 0 \\ 0 & 0 & M_1 & 1 & 0 & 0 \\ \pm b_{02} & 0 & \pm b_{12} & 0 & \Box\pm b_{22} & -M_2 \\ 0 & 0 & 0 & 0 & -M_2 & 1 \end{bmatrix} , \qquad (4.73)$$

while the fermion propagator matrix elements are determined completely as,

$$S_{00}(p) = -i\not{p}E_0(p) \quad ; \quad S_{\substack{11\\22}}(p) = (-i\not{p}+M_{\substack{1\\2}})E_{\substack{1\\2}}(p) \quad , \qquad (4.74)$$

where,

$$E_0(p) \equiv U_F(|p_0|)[-p^2+i\tau\delta]^{-1}U_F^{-1}(|p_0|) \quad , \qquad (4.75)$$

and,

$$E_{\substack{1\\2}}(p) \equiv U_F(|p_0|)[-p^2-M^2_{\substack{1\\2}}+i\tau\delta]^{-1}U_F^{-1}(|p_0|) \quad . \qquad (4.76)$$

The sign of the mass term in $S_{22}(p)$ is positive since it is the propagator obtained after the transformation $\psi_2' \to \gamma^5\psi_2'$.

As opposed to the fermion case (4.74)~(4.76), where the propagator matrix is diagonal, the boson propagator matrix (4.73) is not yet in a bloc-diagonal form suitable for perturbation theory. This is so since it contains non-vanishing b_{ab}-terms ($a\neq b$).

In the same basis, the interacting part (4.53) of the Lagrangean reads explicitly as,

$$\hat{\mathcal{L}}_I = \mathcal{L}_I - \tilde{\mathcal{L}}_I \quad , \qquad (4.77)$$

where,

$$\mathcal{L}_I = \tfrac{1}{2}g_{011}[F_0(A_1^2-B_1^2)+2G_0A_1B_1+2F_1(A_0A_1-B_0B_1)+2G_1(A_0B_1+A_1B_0)$$

$$-i\bar{\psi}_1(A_0-\gamma^5B_0)\psi_1-i\bar{\psi}_0(A_1-\gamma^5B_1)\psi_1-i\bar{\psi}_1(A_1-\gamma^5B_1)\psi_0]$$

$$+\tfrac{1}{2}g_{022}[F_0(A_2^2-B_2^2)+2G_0A_2B_2+2F_2(A_0A_2-B_0B_2)+2G_2(A_0B_2+A_2B_0)$$

$$+i\bar{\psi}_2(A_0-\gamma^5B_0)\psi_2-i\bar{\psi}_0(B_2+\gamma^5A_2)\psi_2-i\bar{\psi}_2(B_2+\gamma^5A_2)\psi_0]$$

$$+\tfrac{1}{2}g_{012}[2F_0(A_1A_2-B_1B_2)+2G_0(A_1B_2+A_2B_1)+2F_1(A_0A_2-B_0B_2)$$

$$+2G_1(A_0B_2+A_2B_0)$$

$$+2F_2(A_0A_1-B_0B_1)+2G_2(A_0B_1+A_1B_0)-i\bar{\psi}_0(A_2-\gamma^5B_2)\psi_1-i\bar{\psi}_1(A_2-\gamma^5B_2)\psi_0$$

$$-i\bar{\psi}_0(B_1+\gamma^5A_1)\psi_2-i\bar{\psi}_2(B_1+\gamma^5A_1)\psi_0-i\bar{\psi}_1(B_0+\gamma^5A_0)\psi_2-i\bar{\psi}_2(B_0+\gamma^5A_0)\psi_1]$$

$$+\tfrac{1}{2}g_{111}[F_1(A_1^2-B_1^2)+2G_1A_1B_1-i\bar{\psi}_1(A_1-\gamma^5B_1)\psi_1]$$

$$+\tfrac{1}{2}g_{112}[2F_1(A_1A_2-B_1B_2)+2G_1(A_1B_2+A_2B_1)+F_2(A_1^2-B_1^2)+2G_2A_1B_1$$

$$-i\bar{\psi}_1(B_1+\gamma^5A_1)\psi_2-i\bar{\psi}_2(B_1+\gamma^5A_1)\psi_1-i\bar{\psi}_1(A_2-\gamma^5B_2)\psi_1]$$

$$+\frac{1}{2}g_{122}[F_1(A_2^2-B_2^2)+2G_1A_2B_2+2F_2(A_1A_2-B_1B_2)+2G_2(A_1B_2+A_2B_1)$$

$$+i\bar{\psi}_2(A_1-\gamma^5B_1)\psi_2-i\bar{\psi}_1(B_2+\gamma^5A_2)\psi_2-i\bar{\psi}_2(B_2+\gamma^5A_2)\psi_1]$$

$$+\frac{1}{2}g_{222}[F_2(A_2^2-B_2^2)+2G_2A_2B_2+i\bar{\psi}_2(A_2-\gamma^5B_2)\psi_2]$$

$$+(f_0+\lambda_0)F_0+(f_1+\lambda_1)F_1+(f_2+\lambda_2)F_2+M_1f_1A_1-M_2f_2A_2 \quad , \qquad (4.78)$$

in which primes have been omitted for rapidity.

To pursue further a perturbative analysis of the model, we clearly need to diagonalize the boson inverse propagator matrix (3.73).

Before doing so, however, we point out that a significant and self-consistent simplification of the model occurs if the following relations are valid,

$$m_1 = m_2 = m \quad ; \quad v_0 = 0 \quad ;$$
$$v \equiv v_1 = v_2 \quad ; \quad f_1 = -f_2 \quad . \qquad (4.79)$$

As will be shown in the next subsection, a consistent set of "gap" equations is obtained when the constraints (4.79) are implemented on the model. This in turn implies that the solution (4.79) is a minimum of the effective potential. Making practical use of the set of equations (4.61), (4.70), (4.71) and (4.72) yields the following simplifications,

$$M^2 = M_1^2 = M_2^2 = m^2 + 2g^2v^2 \quad , \qquad (4.80)$$

$$g'_{011} = g'_{022} = g'_{012} = g_0 \quad ;$$

$$g'_{111} = -g'_{222} = \frac{3}{\sqrt{2}}\,g_1 \quad ; \quad g'_{112} = -g'_{122} = \frac{1}{\sqrt{2}}\,g_1 \quad , \qquad (4.81)$$

where,

$$g_0 = \frac{gm}{2M} \quad ; \quad g_1 = \frac{g^2 v}{\sqrt{2}M} \tag{4.82}$$

Also one has,

$$b \equiv b_{11} = b_{22} = b_{12} \quad ; \quad b_{01} = b_{02} = 0 \quad , \tag{4.83}$$

where,

$$b = g_0 f_0' + \sqrt{2}\, g_1 f_1' \quad . \tag{4.84}$$

The above simplifications enable us in turn, to diagonalize the propagator matrix (4.73) by making an orthogonal transformation into the 1-2 plane. Defining the following orthogonal transformation matrix,

$$R \equiv \frac{1}{\sqrt{2}} \begin{bmatrix} \begin{matrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{matrix} & 0 & 0 \\ 0 & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} & \begin{matrix} 1 & 0 \\ 0 & -1 \end{matrix} \\ 0 & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} & \begin{matrix} -1 & 0 \\ 0 & 1 \end{matrix} \end{bmatrix} \quad , \tag{4.85}$$

a bloc-diagonalization of (4.73) is performed,

$$\Delta^{-1}_{\substack{(AF)\\(BG)}}(\partial) \to R\Delta^{-1}_{\substack{(AF)\\(BG)}}(\partial)R^{-1} \propto \begin{bmatrix} \begin{matrix} \Box & 0 \\ 0 & 1 \end{matrix} & 0 & 0 \\ 0 & \begin{matrix} \Box \pm 2b & M \\ M & 1 \end{matrix} & 0 \\ 0 & 0 & \begin{matrix} \Box & M \\ M & 1 \end{matrix} \end{bmatrix} \quad , \tag{4.86}$$

Inversion of (4.86) finally yields the set of unperturbed boson propagators. Remembering the $i\delta$-prescription of the unperturbed Lagrangean (4.52), they are obtained explicitly as,

$$\Delta_{(AA)_{00}}(p) = -iU_B(|p_0|) \left[\frac{\tau}{p^2 - i\tau\delta}\right] U_B(|p_0|) \quad ;$$

$$\Delta_{(AF)_{00}}(p) = 0 \quad ; \quad \Delta_{(FF)_{00}}(p) = -p^2\Delta_{(AA)_{00}}(p) \quad , \tag{4.87}$$

$$\Delta_{(AA)_{11}}(p) = -iU_B(|p_0|)\left[\frac{\tau}{p^2+(M^2-2b)-i\tau\delta}\right]U_B(|p_0|) \quad ;$$

$$\Delta_{(AF)_{11}}(p) = -M\Delta_{(AA)_{11}}(p) \quad ; \quad \Delta_{(FF)_{11}}(p) = -(p^2-2b)\Delta_{(AA)_{11}}(p) \quad , \tag{4.88}$$

and,

$$\Delta_{(AA)_{22}}(p) = -iU_B(|p_0|)\left[\frac{\tau}{p^2+M^2-i\tau\delta}\right]U_B(|p_0|) \quad ;$$

$$\Delta_{(AF)_{22}}(p) = -M\Delta_{(AA)_{22}}(p) \quad ; \quad \Delta_{(FF)_{22}}(p) = -p^2\Delta_{(AA)_{22}}(p) \quad . \tag{4.89}$$

Note that the (B,G) propagators are readily obtained from the above relations by changing the sign of the mass shift b-term. Clearly, the removal of mass degeneracy among supermultiplets occurs only in the 1-1 channel.

Taking into account the simplifications (4.79)$\sim$(4.84) as well as the transformation (4.86) on boson field operators, the interaction Lagrangean (4.78) can be re-written in the much shorter form,

$$\begin{aligned}
\mathcal{L}_I = {} & g_0[F_0'(A_1'^2-B_1'^2)+2G_0'A_1'B_1'+2F_2'(A_0'A_1'-B_0'B_1')+2G_2'(A_0'B_1'+A_1'B_0')] \\
& +g_1[F_1'(A_1'^2-B_1'^2)+2G_1'A_1'B_1'+2F_2'(A_1'A_2'-B_1'B_2')+2G_2'(A_1'B_2'+A_2'B_1')] \\
& -i\frac{g_0}{2}[\bar\psi_1'(A_0'-\gamma^5B_0')\psi_1'-\bar\psi_2'(A_0'-\gamma^5B_0')\psi_2'+\bar\psi_1'(B_0'+\gamma^5A_0')\psi_2'\pm\bar\psi_2'(B_0'+\gamma^5A_0')\psi_1' \\
& +\sqrt{2}\,\bar\psi_0'(A_1'-\gamma^5B_1')\psi_1'+\sqrt{2}\,\bar\psi_1'(A_1'-\gamma^5B_1')\psi_0'+\sqrt{2}\,\bar\psi_0'(B_1'+\gamma^5A_1')\psi_2' \\
& +\sqrt{2}\,\bar\psi_2'(B_1'+\gamma^5A_1')\psi_0'] \\
& -i\frac{g_1}{2}[\bar\psi_1'((2A_1'+A_2')-\gamma^5(2B_1'+B_2'))\psi_1'+\bar\psi_2'((2A_1'-A_2')-\gamma^5(2B_1'-B_2'))\psi_2'
\end{aligned}$$

$$+\bar{\psi}_1'(B_2'+\gamma^5 A_2')\psi_2'+\bar{\psi}_2'(B_2'+\gamma^5 A_2')\psi_1']$$

$$+(f_0'+\lambda_0')F_0'+\sqrt{2}(f_1'+\lambda_1')F_1'+\sqrt{2}Mf_1'A_1' \quad . \tag{4.90}$$

The one-loop counterterms (4.54) in the primed basis are in turn written explicitly as,

$$\hat{\mathcal{L}}_Z = \mathcal{L}_Z - \tilde{\mathcal{L}}_Z \quad , \tag{4.91}$$

where,

$$\begin{aligned}\mathcal{L}_Z = & -\tfrac{1}{2}\delta Z_{00}'[(\partial_\mu A_0')^2+(\partial_\mu B_0')^2-F_0'^2-G_0'^2] \\ & -\sqrt{2}\delta Z_{01}'[(\partial_\mu A_0')(\partial^\mu A_2')+(\partial_\mu B_0')(\partial^\mu B_2')-F_0'F_1'-G_0'G_1'] \\ & -\tfrac{1}{2}\delta Z_{11}'[(\partial_\mu A_1')^2+(\partial_\mu B_1')^2+(\partial_\mu A_2')^2+(\partial_\mu B_2')^2-F_1'^2-G_1'^2-F_2'^2-G_2'^2] \\ & -\tfrac{1}{2}\delta Z_{12}'[(\partial_\mu A_1')^2-(\partial_\mu A_2')^2+(\partial_\mu B_1')^2-(\partial_\mu B_2')^2+F_1'^2-F_2'^2+G_1'^2-G_2'^2] \\ & -\tfrac{i}{2}\delta Z_{aa}'\bar{\psi}_a'\partial\!\!\!/\psi_a'-\tfrac{i}{2}\delta Z_{01}'[(\bar{\psi}_0'\partial\!\!\!/\psi_1'+\bar{\psi}_1'\partial\!\!\!/\psi_0')-(\bar{\psi}_0'\partial\!\!\!/\gamma^5\psi_2'+\bar{\psi}_2'\gamma^5\partial\!\!\!/\psi_0')] \\ & -\tfrac{i}{2}\delta Z_{12}'(\bar{\psi}_1'\partial\!\!\!/\gamma^5\psi_2'+\bar{\psi}_2'\gamma^5\partial\!\!\!/\psi_1') \\ & +[\delta Z_{00}'f_0'+2\delta Z_{01}'f_1'][F_0'+\sqrt{2}\tfrac{gv}{m}F_1'] \quad .\end{aligned} \tag{4.92}$$

Note that in the Lagrangean (4.90)∿(4.92), the parameters f_c', λ_c' and $\delta Z_{ab}'$ have been defined as,

$$f_c' \equiv S_{ca}^{-1}f_a \quad , \tag{4.93}$$

$$\lambda_c' \equiv \lambda_c'(v) = S_{ca}^{-1}\lambda_a(v) \quad , \tag{4.94}$$

and,

$$\delta Z_{ab}' \quad [S^{-1}\delta ZS]_{ab} \quad , \tag{4.95}$$

in which the matrix S is given by equations (4.63)∿(4.66) and δZ_{ab} is given as,

$$\delta Z_{ab} = \delta Z \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \quad . \tag{4.96}$$

That δZ_{ab} has the matrix structure (4.96) will be shown in subsection 3.4 in which one computes explicitly the fermion self-energy. Equations (4.93)∿(4.95) yield explicitly,

$$f_0' = \frac{m}{M} f_0 - 2\frac{gv}{M} f_1 \quad ; \quad f_1' = f_2' = \frac{gv}{M} f_0 + \frac{m}{M} f_1 \quad , \tag{4.97}$$

$$\lambda_0' = \frac{m}{M}\lambda - \frac{gm}{M} v^2 \quad ; \quad \lambda_1' = -\lambda_2' = \frac{v}{M}(g\lambda + m^2) + \frac{g^2 v^3}{M} \quad , \tag{4.98}$$

and,

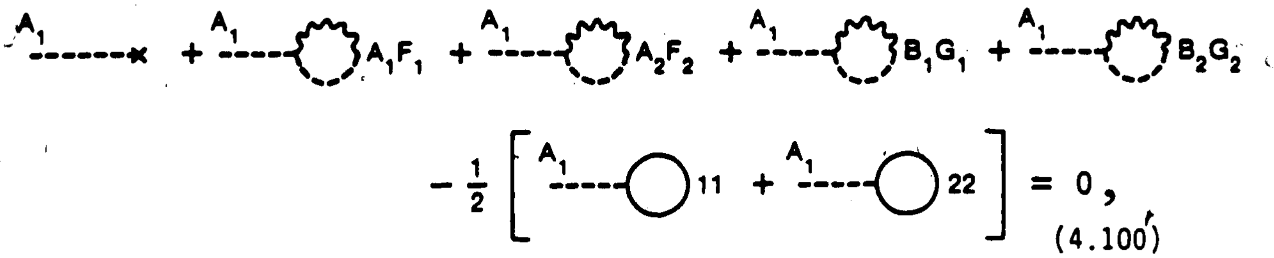
$$\delta Z_{ab}' = \delta Z \begin{bmatrix} \frac{m^2}{M^2} & \frac{mgv}{M^2} & \frac{-mgv}{M^2} \\ \cdots & \frac{(m^2+3g^2v^2)}{M^2} & \frac{(m^2+g^2v^2)}{M^2} \\ \cdots & \cdots & \frac{(m^2+3g^2v^2)}{M^2} \end{bmatrix} = \delta Z_{ba}' \quad . \tag{4.99}$$

The Feynman rules obtained from (4.90) and (4.92) are shown in figure 2. In the next subsection, we derive the self-consistent (gap) equations of our broken symmetry model at finite temperature.

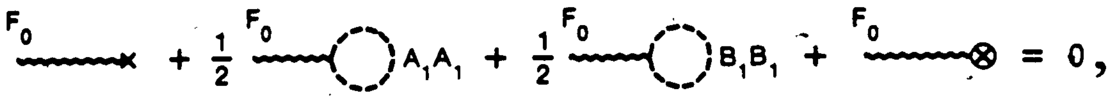
3.3 - The Self-Consistent Equations

In this subsection we derive the set of self-consistent equations for our model to one-loop approximation in perturbation theory.

Minimizing the effective potential at the points $\langle 0(\beta)|A_a|0(\beta)\rangle$ and $\langle 0(\beta)|F_a|0(\beta)\rangle$ in field parameter space, one easily obtains the following set of self-consistent equations,

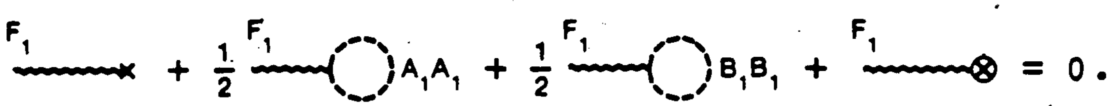


(4.100)



(4.101)

and,



(4.102)

Explicitly, equation (4.100) yields,

$$[gvf_0+mf_1] = -\frac{ig^2v}{2M}\int\frac{d^4p}{(2\pi)^4}\ \mathrm{tr}\ [S^{11}_{11}(p)+S^{11}_{22}(p)]$$

$$-\frac{g^2v}{M}\int\frac{d^4p}{(2\pi)^4}[\Delta^{11}_{(AF)_{11}}(p)+\Delta^{11}_{(AF)_{22}}(p)+\Delta^{11}_{(BG)_{11}}(p)+\Delta^{11}_{(BG)_{22}}(p)]\quad , \tag{4.103}$$

where equations (4.82) and (4.97) have been used. Equation (4.101) together with (4.102) give the following results,

$$f_1 = -mv\quad , \tag{4.104}$$

and, $$(f_0+\lambda+gv^2)+f_0\delta Z = -\frac{ig}{2}\int\frac{d^4p}{(2\pi)^4}[\Delta^{11}_{(AA)_{11}}(p)-\Delta^{11}_{(BB)_{11}}(p)]\quad , \tag{4.105}$$

in which (4.97)-(4.99) have also been used. Inserting (4.104) into (4.103) yields a solution for f_0,

$$f_0 = \frac{m^2}{g} + \frac{g}{M}\int\frac{d^3p}{(2\pi)^3}\Big\{\frac{[p^2+M^2+2b]^{-\frac{1}{2}}}{[e^{\beta\sqrt{p^2+M^2+2b}}-1]} + \frac{[p^2+M^2-2b]^{-\frac{1}{2}}}{[e^{\beta\sqrt{p^2+M^2-2b}}-1]}$$

$$+2[p^2+M^2]^{-\frac{1}{2}}\left[\frac{1}{[e^{\beta\sqrt{p^2+M^2}}-1]} + \frac{2}{[e^{\beta\sqrt{p^2+M^2}}+1]}\right]\Big\} \quad . \qquad (4.106)$$

Note that the shift 2b is computed from equations (3.54) and (3.67) as,

$$2b = gf_0 \quad . \qquad (4.107)$$

As will be shown in the next subsection, the wave-function renormalization factor, when making use of dimensional regularization, is given as,

$$\delta Z = \frac{-g^2}{4}\,\frac{\Gamma(2-D/2)}{(2\pi)^2} \simeq \frac{-g^2}{2(2\pi)^2}\,\frac{1}{(4-D)} \quad ; \quad (D\to 4) \quad . \qquad (4.108)$$

Note that the above result is obtained when a zero temperature renormalization scheme is adopted. Inserting the result (4.108) into (4.105) and making use of the fact that,

$$f_0^{(0)} = \frac{m^2}{g} \quad , \qquad (4.109)$$

as one easily shows from (4.106), the gap equation (4.105) yields in the zero temperature limit.

$$\lambda + g{v^{(0)}}^2 = \frac{-m^2}{g} \quad ; \quad (v^{(0)} \neq 0) \quad . \qquad (4.110)$$

Equation (4.110), which shows that divergence cancellation occurs, is nothing but the statement (4.47) for the condition of SSB of SUSY. It indicates that the order parameter of the F-field is responsible for the symmetry breaking, also consistent with the fact that it creates the shifts removing mass degeneracy among supermultiplets.

Summarizing, equations (4.104)∿(4.107) yield complete solutions for f_0, f_1 and v as functions of temperature and the tadpole parameter λ.

We now turn to the analysis of the analytic structure of the fermion propagator matrix in the infrared region.

3.4 - The Fermion Propagator Matrix

In this subsection we propose to show the existence of interesting Goldstone mode structures in the infrared region of the fermion propagator matrix. We now proceed to compute the complete inverse fermion propagator matrix to one-loop approximation by making use of the spectral representation for Green's functions at finite temperature. Useful product rules for spectral representations are given in appendix B.

The inverse fermion propagator is written as,

$$G^{-1}(p) = S^{-1}(p) - \Sigma(p) \quad , \tag{4.111}$$

where $\Sigma(p)$ is the one-loop renormalized fermion self-energy matrix and S(p) stands for the unperturbed fermion Green's function matrix, the elements of which are given by (4.74)∿(4.76). The Feynman diagrams contributing to the computation of the self-energy matrix elements are shown in figure 3. Given the rules in figure 2, one obtains (D→4),

$$\Sigma_{00}^{\alpha\beta}(p) = 2g_0^2\int \frac{d^Dk}{(2\pi)^D}[\Delta^{\alpha\beta}_{(AA)_{11}}(p+k)+\Delta^{\alpha\beta}_{(BB)_{11}}(p+k)]$$

$$\times[S^{\alpha\beta}_{11}(-k)+\gamma^5 S^{\alpha\beta}_{11}(-k)\gamma^5] - \delta\omega_{00}(p) \quad , \tag{4.112}$$

$$\Sigma_{11}^{\alpha\beta}(p) = \Sigma_{22}^{\alpha\beta}(p) = g_0^2\int \frac{d^Dk}{(2\pi)^D}\{\, [\Delta^{\alpha\beta}_{(AA)_{00}}(p+k)+\Delta^{\alpha\beta}_{(BB)_{00}}(p+k)]$$

$$\times[S^{\alpha\beta}_{11}(-k)+\gamma^5 S^{\alpha\beta}_{11}(-k)\gamma^5] + 2[\Delta^{\alpha\beta}_{(AA)_{11}}(p+k)S^{\alpha\beta}_{00}(-k)+\Delta^{\alpha\beta}_{(BB)_{11}}(p+k)\gamma^5 S^{\alpha\beta}_{00}(-k)\gamma^5]\,\}$$

$$+g_1^2\int\frac{d^Dk}{(2\pi)^D}\{\ 4[\Delta^{\alpha\beta}_{(AA)_{11}}(p+k)S^{\alpha\beta}_{11}(-k)+\Delta^{\alpha\beta}_{(BB)_{11}}(p+k)\gamma^5S^{\alpha\beta}_{11}(-k)\gamma^5]$$

$$+\ [\Delta^{\alpha\beta}_{(AA)_{22}}(p+k)+\Delta^{\alpha\beta}_{(BB)_{22}}(p+k)][S^{\alpha\beta}_{11}(-k)+\gamma^5S^{\alpha\beta}_{11}(-k)\gamma^5]\ \}\ -\ \delta\omega_{\substack{11\\22}}(p)\quad, \quad (4.113)$$

for diagonal matrix elements and,

$$\Sigma^{\alpha\beta}_{01}(p)\ =\ \Sigma^{\alpha\beta}_{10}(p)\ =\ 2\sqrt{2}g_0g_1\int\frac{d^Dk}{(2\pi)^D}[\Delta^{\alpha\beta}_{(AA)_{11}}(p+k)S^{\alpha\beta}_{11}(-k)$$

$$+\ \Delta^{\alpha\beta}_{(BB)_{11}}(p+k)\gamma^5S^{\alpha\beta}_{11}(-k)\gamma^5]\ -\ \delta\omega_{01}(p)\quad, \quad (4.114)$$

$$-\gamma^5\Sigma^{\alpha\beta}_{02}(p)\ =\ -\Sigma^{\alpha\beta}_{20}(p)\gamma^5\ =\ \Sigma^{\alpha\beta}_{01}(p)\quad, \quad (4.115)$$

$$-\Sigma^{\alpha\beta}_{12}(p)\gamma^5\ =\ -\gamma^5\Sigma^{\alpha\beta}_{21}(p)\ =\ g_0^2\int\frac{d^Dk}{(2\pi)^D}\{\ 2[\Delta^{\alpha\beta}_{(AA)_{11}}(p+k)S^{\alpha\beta}_{00}(-k)$$

$$+\Delta^{\alpha\beta}_{(BB)_{11}}(p+k)\gamma^5S^{\alpha\beta}_{00}(-k)\gamma^5]$$

$$+[\Delta^{\alpha\beta}_{(AA)_{00}}(p+k)+\Delta^{\alpha\beta}_{(BB)_{00}}(p+k)][S^{\alpha\beta}_{11}(-k)+\gamma^5S^{\alpha\beta}_{11}(-k)\gamma^5]\ \}$$

$$+\ g_1^2\int\frac{d^Dk}{(2\pi)^D}\{\ [\Delta^{\alpha\beta}_{(AA)_{22}}(p+k)+\Delta^{\alpha\beta}_{(BB)_{22}}(p+k)]$$

$$\times\ [S^{\alpha\beta}_{11}(-k)+\gamma^5S^{\alpha\beta}_{11}(-k)\gamma^5]\ \}\ -\ \gamma^5\delta\omega_{12}(p)\quad, \quad (4.116)$$

for off-diagonal elements. The counterterms are represented by $\delta\omega_{ab}(p)$.

Passing over to the spectral representation of propagators for real four-momentum simplifies much the computation of the self-energy matrix elements. Let us define the spectral representations as,

$$S(p)\ \equiv\ U_F(p_0)\hat{S}(p)U_F^{-1}(p_0)\quad, \quad (4.117)$$

together with,

$$\Delta(p) \equiv U_B(p_0)\hat{\Delta}(p)U_B(p_0) \quad , \tag{4.118}$$

where,

$$\hat{S}(p)= \int_{-\infty}^{\infty} d\kappa\bar{\sigma}(\kappa,\vec{p};\rho)\ \frac{1}{p_0-\kappa+i\tau\delta}\ (-i\not{\kappa}-i\vec{\not{p}}+\rho) \quad , \tag{4.119}$$

and,

$$\hat{\Delta}(p)=i\tau \int_{-\infty}^{\infty} d\kappa\bar{\sigma}(\kappa,\vec{p};\rho)\ \frac{1}{p_0-\kappa+i\tau\delta} \quad . \tag{4.120}$$

Note that in the above relations, one has,

$$\bar{\sigma}(\kappa,\vec{p};\rho) \equiv \frac{1}{2\omega_p(\rho)}\ [\delta(\kappa-\omega_p(\rho)) - \delta(\kappa+\omega_p(\rho))] \quad , \tag{4.121}$$

and,

$$\omega_p(\rho) \equiv \sqrt{\vec{p}^2+\rho^2} \quad . \tag{4.122}$$

When one makes explicit use of the product rules for spectral representations given in appendix B, a factorization of thermal Bogoliubov transformation matrices similar to (4.117) and (4.118) for free propagators, also holds for the self-energy. One can therefore define,

$$\Sigma(p) \equiv U_F(p_0)\hat{\Sigma}(p)U_F^{-1}(p_0) \quad . \tag{4.123}$$

The complete propagator (4.111) is therefore factorizable as in (4.117) and one can write,

$$\hat{G}^{-1}(p) = \hat{S}^{-1}(p)-\hat{\Sigma}(p) \quad , \tag{4.124}$$

as the inverse propagator suitable for our infrared behavior analysis.

Let us make now the following definitions,

$$E^2_{k_{0,1}} \equiv \vec{k}^2+\rho^2_{0,1} \quad ; \quad \omega^2_{k_{0,1,\pm}} \equiv \vec{k}^2+\rho^2_{0,1,\pm} \quad , \tag{4.125}$$

where,

$$\rho^2_0 \equiv 0 \quad ; \quad \rho^2_1 \equiv M^2 \quad ; \quad \rho^2_\pm \equiv M^2 \pm 2b \quad . \tag{4.126}$$

Also we have,

$$F_\pm(p_0;E,\omega)=F^{(0)}(p_0;E,\omega)+F^{(\beta)}(p_0;E,\omega) \quad , \tag{4.127}$$

where,

$$F^{(0)}_\pm(p_0,E,\omega) \equiv \frac{\pm 1}{[(p_0+i\tau\delta)^2-(\omega_q+E_q)^2]} \quad , \tag{4.128}$$

and,

$$F^{(\beta)}_\pm(p_0;E,\omega) \equiv \frac{f(\omega_q,E_q)}{[(p_0+i\tau\delta)^2-(\omega_q-E_q)^2]} \pm \frac{\bar{f}(\omega_q,E_q)}{[(p_0+i\tau\delta)^2-(\omega_q+E_q)^2]} \quad , \tag{4.129}$$

in which the functions $f(\omega,E)$ and $\bar{f}(\omega,E)$ are given as,

$$f(\omega_q,E_q) \equiv \frac{1}{e^{\beta\omega_q}-1} + \frac{1}{e^{\beta E_q}+1} \quad , \tag{4.130}$$

and,

$$\bar{f}(\omega_q,E_q) \equiv \frac{1}{e^{\beta\omega_q}-1} - \frac{1}{e^{\beta E_q}+1} \quad . \tag{4.131}$$

With the help of the above relations (4.125)$\sim$(4.131), the matrix elements of the self-energy, in the limit of vanishing three-momentum ($\vec{p}\to 0$), can be written as ($D\to 4$),

$$\hat{\Sigma}_{00}(p_0,\vec{0})=-2g_0^2\int\frac{d^{D-1}k}{(2\pi)^{D-1}}\,(i\not{p}_0)\left[\frac{F_+(p_0;\omega_1,\omega_-)}{\omega_-}+\frac{F_+(p_0;\omega_1,\omega_+)}{\omega_+}\right]$$

$$-\,\delta\omega_{00}(p_0)\quad, \tag{4.132}$$

$$\hat{\Sigma}_{11}(p_0,\vec{0})=\hat{\Sigma}_{22}(p_0,\vec{0})=-g_0^2\int\frac{d^{D-1}k}{(2\pi)^{D-1}}\,(i\not{p}_0)$$

$$\times\left[\frac{2F_+(p_0;\omega_1,\omega_0)}{\omega_0}+\frac{F_+(p_0;\omega_0,\omega_-)}{\omega_-}+\frac{F_+(p_0;\omega_0,\omega_+)}{\omega_+}\right]$$

$$-\,2g_1^2\int\frac{d^{D-1}k}{(2\pi)^{D-1}}\Big\{(i\not{p}_0)\left[\frac{F_+(p_0;\omega_1,\omega_1)}{\omega_1}+\frac{F_+(p_0;\omega_1,\omega_-)}{\omega_-}+\frac{F_+(p_0;\omega_1,\omega_+)}{\omega_+}\right]$$

$$+\,M\left[\frac{F_-(p_0;\omega_1,\omega_-)}{\omega_1}-\frac{F_+(p_0;\omega_1,\omega_-)}{\omega_-}-\frac{F_-(p_0;\omega_1,\omega_+)}{\omega_1}+\frac{F_+(p_0;\omega_1,\omega_+)}{\omega_+}\right]\Big\}-\delta\omega_{\substack{11\\22}}(p_0), \tag{4.133}$$

for diagonal elements and,

$$\hat{\Sigma}_{\substack{01\\10}}(p_0,\vec{0})=-\sqrt{2}g_0g_1\int\frac{d^{D-1}k}{(2\pi)^{D-1}}\Big\{(i\not{p}_0)\left[\frac{F_+(p_0;\omega_1,\omega_-)}{\omega_-}+\frac{F_+(p_0;\omega_1,\omega_+)}{\omega_+}\right]$$

$$+\,M\left[\frac{F_-(p_0;\omega_1,\omega_-)}{\omega_1}-\frac{F_+(p_0;\omega_1,\omega_-)}{\omega_-}-\frac{F_-(p_0;\omega_1,\omega_+)}{\omega_1}+\frac{F_+(p_0;\omega_1,\omega_+)}{\omega_+}\right]\Big\}-\delta\omega_{\substack{01\\10}}(p_0), \tag{4.134}$$

$$-\,\gamma^5\hat{\Sigma}_{02}(p_0,\vec{0})=-\hat{\Sigma}_{20}(p_0,\vec{0})\,\gamma^5=\hat{\Sigma}_{01}\,(p_0,0)\quad, \tag{4.135}$$

and,

$$-\,\gamma^5\hat{\Sigma}_{21}(p_0,\vec{0})=-\hat{\Sigma}_{12}(p_0,\vec{0})\,\gamma^5=-g_0^2\int\frac{d^{D-1}k}{(2\pi)^{D-1}}\,(i\not{p}_0)$$

$$\times\left[\frac{F_+(p_0;\omega_0,\omega_-)}{\omega_-}+\frac{F_+(p_0;\omega_0,\omega_+)}{\omega_+}+\frac{2F_+(p_0;\omega_1,\omega_0)}{\omega_0}\right],$$

$$- 2g_1^2 \int \frac{d^{D-1}k}{(2\pi)^{D-1}} (i\not{p}_0) \left[\frac{F_+(p_0;\omega_1,\omega_1)}{\omega_1} \right] + \delta\omega_{12}(p_0)\gamma^5 \quad , \qquad (4.136)$$

for off-diagonal elements. The counterterms matrix elements $\delta\omega_{ab}(p)$ are given as,

$$-\delta\omega_{aa}(p) = -i\not{p}\delta Z'_{aa} \quad ; \quad -\delta\omega_{01}(p) = -i\not{p}\delta Z'_{01} \quad ;$$

$$-\delta\omega_{02}\,(p) = i\not{p}\gamma^5\delta Z'_{01} \quad ; \quad \delta\omega_{12}(p) = -i\not{p}\gamma^5\delta Z'_{12} \quad , \qquad (4.137)$$

where the $\delta Z'_{ab}$ matrix has yet to be determined. In order to evaluate the counterterms, a set of renormalization conditions must now be imposed. Using a zero temperature renormalization scheme, the renormalization conditions are written as,

$$\hat{G}^{-1}_{ab}(p_0)\Big|_{\substack{p_0=0\\T=0}} = -\rho_a \Big|_{T=0} \delta_{ab} \quad , \qquad (4.138)$$

and,

$$\frac{\partial}{\partial i\not{p}_0} \hat{G}^{-1}_{ab}\,(p_0) \Bigg|_{\substack{p_0=0\\T=0}} = -\delta_{ab} \quad , \qquad (4.139)$$

where,

$$\rho_0 \equiv 0 \quad ; \quad \rho_{\substack{1\\2}} \equiv M \quad . \qquad (4.140)$$

It is easy to show that the condition (4.138) is automatically satisfied since the self-energy does not give any contribution at T=0 and p_0=0 to the one-loop correction. Taking the derivatives of the self-energy matrix elements (4.132)~(4.136) with respect to $i\not{p}_0$ at the point p_0=0 and T=0 yields the following relation,

$$\frac{\partial}{\partial i p_0} \hat{\Sigma}'_{ab}(p_0) \Bigg|_{\substack{p_0=0 \\ T=0}} = \left[\frac{-g^2}{4}\int \frac{d^{D-1}k}{(2\pi)^{D-1}} \frac{1}{(k^2+M^2)^{3/2}}\right] S^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} S \Bigg|_{T=0}$$

$$- \delta Z'_{ab}\Big|_{T=0} \quad . \qquad (4.141)$$

In the above relation, the prime superscript for the self-energy means the following,

$$\hat{\Sigma}'_{aa}(p) \equiv \hat{\Sigma}_{aa}(p) \quad ; \quad \hat{\Sigma}'_{01}(p) \equiv \hat{\Sigma}_{01}(p) \quad ;$$

$$\hat{\Sigma}'_{02}(p) \equiv -\hat{\Sigma}'_{02}(p)\gamma^5 \quad ; \quad \hat{\Sigma}'_{12}(p) \equiv -\hat{\Sigma}_{12}(p)\gamma^5 \quad . \qquad (4.142)$$

Making use of the renormalization condition (4.139), equation (4.141) gives directly,

$$\delta Z'_{ab}\Big|_{T=0} = \left[\frac{-g^2}{4}\int \frac{d^{D-1}k}{(2\pi)^{D-1}} \frac{1}{(k^2+M^2)^{3/2}}\right] S^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} S \Bigg|_{T=0}$$

(4.143)

This relation justifies the matrix structure (4.96) and identifies δZ in (4.96) and (4.99) as,

$$\delta Z = \frac{-g^2}{2(2\pi)^2} \frac{1}{(4-D)} \quad ; \quad (D\to 4) \quad , \qquad (4.144)$$

in accordance with our claim (4.108). When temperature is switched on, the one-loop counterterms $\delta Z'_{ab}$ are given explicitly as in equation (4.99) with δZ as in (4.144). In this way, all divergences appearing from internal momentum integration in the expressions (4.132)∿(4.136) for the self-energy are exactly cancelled by the counterterms at finite temperature. The fact that wavefunction renormalization is enough to make the theory finite at zero temperature is a general feature of supersymmetric theories. After divergence cancellations, the contributions to the self-energy originate only from the temperature parts $F_{\pm}^{(\beta)}$ defined by the relations (4.129)~(4.131).

Remembering (4.124) and computing explicitly (4.132)~(4.136), the inverse fermion propagator matrix in the limit $p_0 \to 0$ takes the following form,

$$\lim_{p_0 \to 0} \hat{G}^{-1}(p_0,0) = \begin{bmatrix} 0 & -\Gamma & -\gamma^5\Gamma \\ -\Gamma & -M\gamma + \not{t}_0 & \not{t}_0\gamma^5 \\ -\gamma^5\Gamma & \gamma^5\not{t}_0 & -M\gamma+\not{t}_0 \end{bmatrix} , \tag{4.145}$$

where,

$$\gamma \equiv 1 - \frac{2g_1^2}{b} \int \frac{d^3k}{(2\pi)^3} \left[\frac{1}{\omega_-} \left(\frac{1}{e^{\beta\omega_-}-1} \right) + \frac{1}{\omega_+} \left(\frac{1}{e^{\beta\omega_+}-1} \right) + \frac{2}{\omega_1} \left(\frac{1}{e^{\beta\omega_1}+1} \right) \right] , \tag{4.146}$$

$$\Gamma \equiv \frac{g_0}{\sqrt{2}\, g_1} M(\gamma - 1) \quad , \tag{4.147}$$

and,

$$\not{t}_0 \equiv \lim_{p_0 \to 0} 2g_1^2 \frac{(i\not{p}_0)}{(p_0 + i\tau\delta)^2} \left[\int \frac{d^3k}{(2\pi)^3} \frac{1}{\omega_1} \left(\frac{1}{e^{\beta\omega_1}-1} + \frac{1}{e^{\beta\omega_1}+1} \right) \right] . \tag{4.148}$$

Equation (4.148) is an infrared singluar contribution from the thermal superpair zero-energy mode. It originates in the self-energy from diagrams involving integration over products of fermion and boson lines sharing the same mass shell. These diagrams are indicated by a star in figure 3. Such infrared singularities show up in the self-energy matrix elements because some degeneracy still remains in the unperturbed 2-2 channel. Also the fact that the inverse propagator (4.145) still has non vanishing off-diagonal elements in the infrared region is due to our perturbative calculation which has mixed fermion (as well as boson) channels among each others. To obtain information on the particle spectrum in the infrared region, one has to re-diagonalize the inverse propagator (4.145). To this end, we define the following transformation matrices,

$$T \equiv \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & \gamma^5 \\ 0 & -\gamma^5 & -1 \end{bmatrix} , \tag{4.149}$$

and,

$$\bar{T} \equiv \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & -\gamma^5 \\ 0 & \gamma^5 & -1 \end{bmatrix} = \gamma^0 T (\gamma^0)^T . \tag{4.150}$$

Therefore,

$$G^{-1}(0) \to TG^{-1}(0)\bar{T} = \begin{bmatrix} 0 & 0 & \text{[illegible]}\gamma^5 \\ 0 & 0 & (M\gamma)\gamma^5 \\ \sqrt{2}\Gamma\gamma^5 & (M\gamma)\gamma^5 & 2\not{t}_0 \end{bmatrix} , \tag{4.151}$$

in which the thermal superpair singularity appears in diagonal elements only. Further diagonalization finally yields,

$$\lim_{p_0 \to 0} G^{-1}(p_0,\vec{0}) \to \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2\not{t}_0 \end{bmatrix} . \tag{4.152}$$

The above result clearly shows that two Goldstone fermion particle modes now appear in the fermion propagator matrix. The mode appearing in the 0-0 channel is the original Goldstone fermion already existing at zero temperature while the one showing up in the 1-1 channel is the new Goldstone fermion associated with the SSB of SUSY by temperature effects.

As in the WZ-model, this massless particle mode shows up in the same channel where mass degeneracy is removed among supermultiplets. One therefore has a mixed mechanism of SUSY rearrangement at finite temperature. Also, one notices that the thermal superpair zero-energy mode singularity appears in the 2-2 mass-degenerate channel.

To give more support to the above calculation and its interpretation, it would be an interesting problem to check that this mixed mechanism also satisfies the Ward-Takahashi identities for SUSY at finite temperature. Higher loop corrections may also give further support to the analytic structure (4.152).

Summarizing all the previous calculations, we succeeded in finding a solution to the N=3 O'Raifeartaigh model at finite temperature for which a mixed mechanism of rearrangement of the supersymmetry breaking occurs. Such a solution can be shown to be a minimum of the effective potential through the calculation of the self-consistent equations for the various order parameters of the theory. Through the study of the analytic structure of the inverse fermion propagator matrix, our main result is the fact that a Goldstone fermion particle mode shows up in the channel exhibiting removal of mass degeneracy among supermultiplets. On the other hand, a thermal superpair zero-energy mode shows up in the mass degenerate channel while the Goldstone fermion already existing at zero temperature in the model, survives temperature effects. One can therefore argue that for higher number of supermultiplets (N>3), one should find that at least two Goldstone fermions appear in the fermion propagator matrix.

When we take the limit of vanishing interaction ($g_{abc} \to 0$), naturally all channels become now independent of one another and one has in fact three independent free WZ-models. The internal symmetry index is then a dummy index and although SUSY is still broken by temperature effects, no Goldstone particle should appear. Instead, thermal superpairs now take over the role of Goldstone particles, as is required from the Ward-Takahashi identities at finite temperature for the free WZ-theory.

Note that for the general model with N-component supermultiplets, it may be an interesting problem to understand from a group theoretic viewpoint, the interplay between the SSB of SUSY and of internal symmetries which may be present in a given situation. Internal symmetries are much easier to break than SUSY.

CHAPTER V

CRITICAL PHENOMENA: COMPUTATION OF THE CRITICAL EXPONENTS

1. General Considerations

In the last few years, less than a decade after its conception in the landmark paper by Takahashi and Umezawa,[24] and also Matsumoto,[25] thermo-field dynamics has been developed to a remarkable point. With the formulation of perturbation and renormalization theory[35] within the framework of TFD, it is now automatic to compute finite temperature quantities in a very clear and completely analogous way to corresponding calculations at zero temperature. That this is so is due to the fact that statistical averages are formally equivalent to vacuum expectation values of field operators for which the vacuum state carries the thermal information.

Accomodating the TFD formalism with renormalization theory has led to the recognition that the renormalization points need not be constrained to zero temperature. In fact one can express renormalized parameters at a given temperature in terms of corresponding parameters defined through the renormalization conditions at another temperature. This property leads us inevitably to the renormalization group.[36-37] Because of the added thermal degree of freedom, the renormalization group in the context of TFD is a two-parameter abelian group.

Making use of the TFD renormalization group, a detailed analysis of various asymptotic behaviors has already been carried out in references (36) and (37). Furthermore, the theory of critical behavior has also been treated in the latter references above as well as below the critical temperature T_c. An advantage of the RG in TFD with respect to critical behavior is the fact that scaling in the critical region appears naturally in the formalism.

Although such a formalism offers incomparable clarity and transparency, very few explicit model-dependent calculations have been carried out until now. While the authors of references (149) and (150) have been concentrating on the asymptotic behavior of the running (effective) mass and coupling constant parameters of an $O(N)\times O(N)$ and $O(N)-\lambda\phi^4 (N=1)$ model respectively, the authors of reference (151) have made a similar study of the $O(N)-\lambda\phi^4$ theory in the limit of large N. The results seem to indicate that the temperature behavior in the asymptotic regions as obtained from explicit computation of the RG-coefficients may be different from the naive expectation of the temperature dependency of the renormalized parameters[149-151] or from other computational techniques.[150,152] To shed more light on these calculations, higher loop considerations may be needed.

In this chapter we wish to present an original approach[153] to the critical behavior of the single component $\lambda\phi^4$-theory above T_c by explicitly computing the (finite temperature) RG-coefficients up to the one-loop approximation. We also obtain the infrared (zero mass limit) behavior (to one-loop order) of these coefficients. From the RG-invariant definition of the critical temperature for static quantities as well as the above infrared behavior of the RG-coefficients, it will be possible to extract the critical behavior of the running mass and coupling constant as they approach T_c from above. We also obtain a one-loop determination of the mass critical index carrying the information on how fast the mass attenuates while approaching T_c, as well as two other indices. The results are shown to agree with earlier computations.[154-159]

In this chapter, the metric tensor is chosen as $g_{00}=-g_{ii}=1$.

2. The Renormalization Group in $\lambda\phi^4$-theory

The bare $\lambda\phi^4$ Lagrangean density at finite temperature is given as,[153]

$$\hat{\mathcal{L}} \equiv \mathcal{L}-\tilde{\mathcal{L}} = -\frac{1}{2}\phi^\alpha[U_B^{-1}(\Box-i\tau\delta+\mu^2)\tau U_B^{-1}]^{\alpha\beta}\phi^\beta-\varepsilon^\alpha\frac{\lambda}{4!}\phi^{\alpha 4} \quad , \quad (5.1)$$

where ϕ, μ and λ are the bare hermitean scalar field, bare mass and bare coupling constant respectively. The matrix $U_B^{\alpha\beta}(|i\partial_t|)$ is the Bogoliubov transformation matrix indicating that the unperturbed propagator is the thermally rotated propagator. One has also,

$$\phi^\alpha \equiv \begin{bmatrix}\phi \\ \tilde{\phi}\end{bmatrix} \quad , \quad (5.2)$$

and,

$$\tau^{\alpha\beta} \equiv \begin{bmatrix}1 & 0 \\ 0 & -1\end{bmatrix} \quad ; \quad \varepsilon^\alpha \equiv \tau^{\alpha\alpha} \quad . \quad (5.3)$$

The Lagrangean (5.1) can be re-written in terms of renormalized parameters and fields as well as counterterms in the following way,

$$\hat{\mathcal{L}} = -\frac{1}{2}\varphi^\alpha[U_B^{-1}(\Box-i\tau\delta+m^2)\tau U_B^{-1}]^{\alpha\beta}\varphi^\beta - \varepsilon^\alpha\frac{g}{4!}\varphi^{\alpha 4}$$
$$+\ \varepsilon^\alpha\{-\frac{1}{2}\delta\mu^2\varphi^{\alpha 2}+\frac{1}{2}\delta Z[(\partial\varphi^\alpha)^2-m^2\varphi^{\alpha 2}] - \frac{g}{4!}\delta Z_1\varphi^{\alpha 4}\} \quad , \quad (5.4)$$

where,

$$\phi^\alpha \equiv Z^{\frac{1}{2}}\varphi^\alpha \quad ; \quad \lambda \equiv gZ_1Z^{-2} \quad ; \quad \delta\mu^2 \equiv Z[\mu^2-m^2] \quad , \quad (5.5)$$

and,

$$\delta Z \equiv Z-1 \quad ; \quad \delta Z_1 \equiv Z_1-1 \quad . \quad (5.6)$$

The Feynman rules obtained from (5.4) are given in figure 4. The renormalized parameters as well as wave function and coupling constant renormalization factors appearing in the Lagrangean (5.4) have been obtained from the so-called T_o-renormalization scheme. In this scheme, the renormalization conditions imposed on relevant renormalized proper vertex functions are given as,[36,153]

$$\mathrm{Re}\ i\ \Gamma_R^{(2)^{11}}(k;T;g,m;\kappa,T_o)\Big|_{\substack{k=k(\kappa)\\ T=T_o}} = -\kappa^2 - m^2 \quad , \tag{5.7}$$

$$\frac{\partial}{\partial \vec{k}^2}\ \mathrm{Re}\ i\ \Gamma_R^{(2)^{11}}(k;T;g,m;\kappa,T_o)\Big|_{\substack{k=k(\kappa)\\ T=T_o}} = -1 \quad , \tag{5.8}$$

and,

$$\mathrm{Re}\ i\ \Gamma_R^{(4)^{1111}}(k_i;T;g,m;\kappa,T_o)\Big|_{\substack{k_i=k_i(\kappa)\\ T=T_o}} = g \quad , \tag{5.9}$$

in which $k(\kappa)$ means $k_0=0$ and $|\vec{k}|^2=\kappa^2$. From the scaling of the renormalization point (κ,T_o) to $(s\kappa,tT_o)$ we obtain new renormalized parameters $g(s,t)$ and $m(s,t)$ where $g(1,1)\equiv g$ and $m(1,1)\equiv m$. Using equations relating the bare proper vertex functions to the renormalized ones at different renormalization points, one obtains directly the following renormalization group equation,

$$\Gamma_R^{(N)}(k;T;g,m;\kappa,T_o) = \rho(s,t)^{-N/2}\Gamma_R^{(N)}(k;T;g(s,t),m(s,t);s\kappa,tT_o) \quad , \tag{5.10}$$

where,

$$\rho(s,t) \equiv \lim_{\Lambda\to\infty} \frac{Z(g(s,t),m(s,t);s\kappa,tT_o;\Lambda)}{Z(g,m;\kappa,T_o;\Lambda)} \quad , \tag{5.11}$$

Λ being a high momentum cut-off. Obviously $\rho(1,1)=1$.

Differentiating (5.10)-(5.11) with respect to s and t, one gets the following set of renormalization group Lie differential equations,

$$\left[\kappa\frac{\partial}{\partial\kappa} + \beta_s\frac{\partial}{\partial g} + \Theta_s m^2\frac{\partial}{\partial m^2} - N\gamma_s\right]\Gamma_R^{(N)}(k;T;g,m;\kappa,T_o)=0 \quad , \qquad (5.12)$$

and,

$$\left[T_o\frac{\partial}{\partial T_o} + \beta_t\frac{\partial}{\partial g} + \Theta_t m^2\frac{\partial}{\partial m^2} - N\gamma_t\right]\Gamma_R^{(N)}(k;T;g,m;\kappa,T_o)=0 \quad , \qquad (5.13)$$

where,

$$\beta_s \equiv s\frac{d}{ds}g(s,t)\Big|_{s=t=1} \quad ; \quad \beta_t \equiv t\frac{d}{dt}g(s,t)\Big|_{s=t=1} \quad , \qquad (5.14)$$

$$\Theta_s \equiv \frac{s}{m^2}\frac{d}{ds}m^2(s,t)\Big|_{s=t=1} \quad ; \quad \Theta_t \equiv \frac{t}{m^2}\frac{d}{dt}m^2(s,t)\Big|_{s=t=1} \quad , \qquad (5.15)$$

and,

$$\gamma_s \equiv \frac{1}{2}\, s\,\frac{d}{ds}\ln\rho(s,t)\Big|_{s=t=1} \quad ; \quad \gamma_t \equiv \frac{1}{2}\, t\,\frac{d}{dt}\ln\rho(s,t)\Big|_{s=t=1} \quad . \qquad (5.16)$$

Equations (5.14)∿(5.16) are the so-called renormalization group coefficients. These coefficients are renormalization scheme-dependent.

In terms of scaled quantities, the renormalization scheme (5.7)∿(5.9) can be written as,

$$\rho(s,t)\mathrm{Re}\; i\;\Gamma_R^{(2)^{11}}(k;tT_o;g,m;\kappa,T_o)\Big|_{k=k(s\kappa)} = -s^2\kappa^2 - m^2(s,t), \qquad (5.17)$$

$$\rho(s,t)\frac{\partial}{\partial\vec{k}^2}\,\mathrm{Re}\; i\;\Gamma_R^{(2)^{11}}(k;tT_o;g,m;\kappa,T_o)\Big|_{k=k(s\kappa)} = -1 \quad , \qquad (5.18)$$

and,

$$\rho(s,t)^2\,\mathrm{Re}\; i\;\Gamma_R^{(4)^{1111}}(k_i;tT_o;g,m;\kappa,T_o)\Big|_{k_i=k_i(s\kappa)} = g(s,t). \qquad (5.19)$$

The above set of renormalization conditions enables us to compute explicitly the RG-coefficients. Differentiating (5.17)∿(5.19) with respect to s and t at the point s=t=1 yields,

$$\Theta_s = 2\gamma_s\left(1+\frac{\kappa^2}{m^2}\right) \quad , \tag{5.20}$$

$$\beta_s = 4g\gamma_s + \frac{\kappa\partial k_i(\kappa)}{\partial\kappa}\left[\frac{\partial}{\partial k_i}\mathrm{Re}\ i\ \Gamma_R^{(4)^{1111}}(k_i;tT_o;g,m;\kappa,T_o)\right]\Bigg|_{\substack{k_i=k_i(\kappa)\\ t=1}} \tag{5.21}$$

and,

$$\Theta_t = 2\gamma_t\left(1+\frac{\kappa^2}{m^2}\right) - \frac{1}{m^2}\left[\frac{t\partial}{\partial t}\mathrm{Re}\ i\ \Gamma_R^{(2)^{11}}(k;tT_o;g,m;\kappa,T_o)\right]\Bigg|_{\substack{k=k(\kappa)\\ t=1}} \tag{5.22}$$

$$\beta_t = 4g\gamma_t + \left[\frac{t\partial}{\partial t}\ \mathrm{Re}\ i\ \Gamma_R^{(4)^{1111}}(k_i;tT_o;g,m;\kappa,T_o)\right]\Bigg|_{\substack{k_i=k_i(\kappa)\\ t=1}} \tag{5.23}$$

Note that γ_s and γ_t are obtained directly from (5.18). The latter RG-coefficients are functions of g, m, κ and T_o. The equations for running parameters are obtained by making use of the scaled quantities g(s,t), m(s,t), $s\kappa$ and tT_o Therefore one has,

$$\frac{sdg(s,t)}{ds} = \beta_s(g(s,t),m(s,t);s\kappa,tT_o) \quad , \tag{5.24}$$

$$\frac{sdm^2(s,t)}{ds} = \Theta_s(g(s,t),m(s,t);s\kappa,tT_o)m^2(s,t) \quad , \tag{5.25}$$

and,

$$\frac{tdg(s,t)}{dt} = \beta_t(g(s,t),m(s,t);s\kappa,tT_o) \quad , \tag{5.26}$$

$$\frac{tdm^2(s,t)}{dt} = \Theta_t(g(s,t),m(s,t);s\kappa,tT_o)m^2(s,t) \quad . \tag{5.27}$$

We also mention that momentum and temperature asymptotic behavior can be analyzed most easily when use of the concept of dimensionality is made . A scaling transformation on equation (5.10) yields the following,

$$\Gamma_R^{(N)}(k;T\,\text{[illegible]};\kappa,T\,\text{[illegible]}\,{}^{D_N}\rho(s,t)^{-N/2}\,\Gamma_R^{(N)}\left(\frac{k}{\ell};\frac{T}{\ell};g(st),\frac{m(st)}{\ell};\frac{s\kappa}{\ell};\frac{tT_o}{\ell}\right) , \tag{5.28}$$

where $D_N(\equiv 4-N)$ is the dimensionality of $\Gamma_R^{(N)}$. The study of various asymptotic behaviors making use of (5.28) can be found in reference (36).

Now, in order to obtain information on the behavior of quantities such as the effective mass near the critical point, a more convenient renormalization scheme has to be chosen. Furthermore, the temperature renormalization point is best chosen when measured from the critical temperature T_c. For computational convenience, the renormalization conditions (5.17)∼(5.19) [illegible] now modified as follows,[36,153]

$$\rho(s,t)\ \mathrm{Re}\ i\Gamma_R^{(2)}(k;T_c+tT_o;g,m;\kappa,T_c+T_o)\Big|_{k=0} = -m^2(s,t) , \tag{5.29}$$

$$\rho(s,t)\ \frac{\partial}{\partial \vec{k}^2}\mathrm{Re}\ i\Gamma_R^{(2)^{11}}(k;T_c+tT_o;g,m;\kappa,T_c+T_o)\Big|_{k=0} = -1 , \tag{5.30}$$

and,

$$\rho(s,t)^2\ \mathrm{Re}\ i\Gamma_R^{(4)^{1111}}(k_i;T_c+T_o;g,\underline{m};\kappa,T_c+T_o)\Big|_{k_i=0} = g(s,t) , \tag{5.31}$$

where g and m retain their previous meaning.

In the latter scheme, it is easy to recognize that the renormalization group coefficients with respect to the momentum scale $s\kappa$ vanish, that is $\gamma_s=\Theta_s=\beta_s=0$. The fact that β_s is identically zero also insures us that one sits at a fixed point of the running coupling constant in momentum space. As a result, the "wave-function renormalization factor" ρ as well as the running mass and coupling constant are uniquely dependent upon the temperature scale t measured from above the critical point T_c. Therefore, in this scheme, the only non-trivial RG-coefficients are now obtained as,

$$\gamma_t = \frac{1}{2}\frac{td}{dt} \ln \rho(t)\Big|_{t=1} \quad , \tag{5.32}$$

$$\Theta_t = 2\gamma_t - \frac{1}{m^2}\left[t\frac{\partial}{\partial t} \text{ Re } i\ \Gamma_R^{(2)^{11}}(0;T_c+tT_o;g,m;\kappa,T_c+T_o)\right]\Big|_{t=1} , \tag{5.33}$$

and,

$$\beta_t = 4g\gamma_t + \left[t\frac{\partial}{\partial t} \text{ Re } i\ \Gamma_R^{(4)^{1111}}(0;T_c+tT_o;g,m;\kappa,T_c+T_o)\right]\Big|_{t=1} , \tag{5.34}$$

where,

$$\rho(t) = -\left[\frac{\partial}{\partial k^2} \text{ Re } i\ \Gamma_R^{(2)^{11}}(k;T_c+tT_o;g,m;\kappa,T_c+T_o)\right]^{-1}\Big|_{k=0} , \tag{5.35}$$

In the following section, we compute the coefficients γ_t, Θ_t and β_t to the one loop approximation by making practical use of equations (5.32) $\sim$(5.35). We also obtain the infrared (zero mass limit) behavior of these coefficients which will be found useful when computing the critical indices of the theory.

3. Computation of the RG Coefficients

In order to compute the finite temperature renormalization group coefficients, one first needs to obtain explicit expressions for the inverse boson propagator and the four-point proper vertex function as required from the relations (5.32)$\sim$(5.35). The one-loop diagrams contributing to the self-energy and the four-point vertex function are shown in figure 5. Explicitly, one obtains,

$$i\Gamma_R^{(2)^{11}}(k;T_c+tT_o;g,m;\kappa,T_c+T_o) = k^2 - m^2 - \Sigma_R^{11}(k;T_c+tT_o;g,m;\kappa,T_c+T_o), \tag{5.36}$$

where

$$-i\Sigma_R^{11}(k;T_c+tT_o;g,m;\kappa,T_c+T_o) = \frac{g}{2}\int\frac{d^4p}{(2\pi)^4}\ \Delta^{11}(p) + i[\ -\delta\mu^2 + (k^2-m^2)\delta Z\ ], \tag{5.37}$$

and,

$$i\Gamma_R^{(4)^{1111}}(k_i;T_c+tT_o;g,m;\kappa,T_c+T_o)=g+i\frac{g^2}{2}\int\frac{d^4p}{(2\pi)^4}\Delta^{11}(p)[$$

$$\times\ \Delta^{11}(p-k_1-k_2)+\Delta^{11}(p-k_1-k_3)+\Delta^{11}(p-k_1-k_4)]\ +g\delta Z_1 \quad . \qquad (5.38)$$

In the above formulae, the unperturbed propagator $\Delta(p)$ is given as,

$$\Delta^{\alpha\beta}(p)\equiv\left[U_B(|p_0|)\frac{\tau}{p^2-m^2+i\delta\tau}U_B(|p_0|)\right]^{\alpha\beta}=\Delta_o(p)+\Delta_\beta(p) \quad , \qquad (5.39)$$

where,

$$\Delta_o(p)\equiv\frac{\tau}{p_0^2-\omega_p^2+i\tau\delta} \quad ; \quad \Delta_\beta(p)\equiv\frac{-2\pi i\delta(p_0^2-\omega_p^2)}{(e^{\beta|p_0|}-1)}\begin{bmatrix}1 & e^{\frac{1}{2}\beta|p_0|}\\ e^{\frac{1}{2}\beta|p_0|} & 1\end{bmatrix}, \qquad (5.40)$$

in which the inverse temperature β is taken as,

$$\beta\ \equiv\ (T_c+tT_o)^{-1} \quad , \qquad (5.41)$$

and where,

$$\omega_p^2\ \equiv\ \vec{p}^{\,2}+m^2 \quad . \qquad (5.42)$$

Making use of the renormalization conditions (5.29)∿(5.31) at the point t=1, one obtains for the one-loop temperature-dependent counterterms,

$$\delta Z^{(1)}\ =\ 0 \quad , \qquad (5.43)$$

$$\delta\mu^{2(1)}\ =\ -\ \frac{g}{2}\ \mathrm{Re}\ i\int\frac{d^4p}{(2\pi)^4}\ \bar{\Delta}^{11}(p) \quad , \qquad (5.44)$$

and,

$$\delta Z_1^{(1)}\ =\ -\frac{3}{2}\ g\ \mathrm{Re}\ i\int\frac{d^4p}{(2\pi)^4}\ \bar{\Delta}^{11}(p)\bar{\Delta}^{11}(p) \quad , \qquad (5.45)$$

where the overbars indicate that the propagators are obtained at the inverse temperature β_o given by,

$$\beta_o = (T_c+T_o)^{-1} \quad . \tag{5.46}$$

Insertion of the counterterms (5.43)~(5.45) into equations (5.36)~(5.38) yields,

$$\mathrm{Re}\ i\Gamma_R^{(2)^{11}}(0;T_c+tT_o;g,m;\kappa,T_c+T_o) = -m^2-\frac{g}{2}\ \mathrm{Re}\ i\int\frac{d^4p}{(2\pi)^4}[\Delta^{11}(p)-\bar{\Delta}^{11}(p)], \tag{5.47}$$

$$\frac{\partial}{\partial\vec{k}^2}\mathrm{Re}\ i\Gamma_R^{(2)^{11}}(k;T_c+tT_o;g,m;\kappa,T_c+T_o)\Big|_{k=0} = -1 \tag{5.48}$$

and,

$$\mathrm{Re}\ i\Gamma_R^{(4)^{1111}}(0;T_c+tT_o;g,m;\kappa,T_c+T_o) = g+\frac{3}{2}g^2\mathrm{Re}\ i\int\frac{d^4p}{(2\pi)^4}[$$

$$\times\ \Delta^{11}(p)\Delta^{11}(p)-\bar{\Delta}^{11}(p)\bar{\Delta}^{11}(p)] \quad . \tag{5.49}$$

Defining,

$$I_1(\beta) \equiv \mathrm{Re}\ i\int\frac{d^4p}{(2\pi)^4}\Delta^{11}(p) \quad , \tag{5.50}$$

and,

$$I_2(\beta) \equiv \mathrm{Re}\ i\int\frac{d^4p}{(2\pi)^4}\Delta^{11}(p)\Delta^{11}(p) \quad , \tag{5.51}$$

one obtains explicitly,

$$I_1(\beta) = 2\int_0^\infty\frac{p^2dp}{(2\pi)^2}\frac{1}{\omega_p}\left[\frac{1}{2}+\frac{1}{(e^{\beta\omega_p}-1)}\right] \quad , \tag{5.52}$$

and,

$$I_2(\beta) = -\int_0^\infty\frac{p^2dp}{(2\pi)^2}\left[\frac{1}{2\omega_p^3}+\frac{1}{\omega_p^3}\frac{1}{(e^{\beta\omega_p}-1)}+\frac{1}{\omega_p^2}\frac{1}{(e^{\beta\omega_p}-1)^2}\right] \quad . \tag{5.53}$$

Making use of the formula,

$$t\frac{\partial}{\partial t}\left[\frac{1}{(e^{\beta\omega_p}-1)}\right] = \frac{tT_o}{(T_c+tT_o)}\frac{\beta\omega_p e^{\beta\omega_p}}{(e^{\beta\omega_p}-1)^2} \quad , \tag{5.54}$$

[...]l as,

$$t\frac{\partial}{\partial t}\left[\frac{\beta e^{\beta\omega_p}}{(e^{\beta\omega_p}-1)^2}\right] = \frac{tT_o}{(T_c+tT_o)^2}\frac{e^{\beta\omega_p}}{(e^{\beta\omega_p}-1)^2}\left[\frac{\beta\omega_p(e^{\beta\omega_p}+1)}{(e^{\beta\omega_p}-1)}-1\right], \quad (5.55)$$

and passing over to the dimensionless variables,

$$x \equiv \beta p \quad ; \quad \Delta \equiv \beta m \quad ; \quad \sigma_x^2 \equiv x^2+\Delta^2 \quad , \quad (5.56)$$

the RG-coefficients (5.32)~(5.34), upon insertion of (5.52)-(5.53) into equations (5.47)~(5.49), are finally obtained as (to one-loop order),

$$\gamma_t = 0 \quad , \quad (5.57)$$

$$\Theta_t = (g/\Delta^2)\ \frac{T_o}{(T_c+T_o)}\int_0^\infty \frac{x^2dx}{(2\pi)^2}\frac{e^{\sigma x}}{(e^{\sigma x}-1)^2} \quad , \quad (5.58)$$

and,

$$\beta_t = -\frac{3}{2}\ g^2\ \frac{T_o}{(T_c+T_o)}\int_0^\infty \frac{x^2dx}{(2\pi)^2}\frac{1}{\sigma_x}\frac{e^{\sigma x}(e^{\sigma x}+1)}{(e^{\sigma x}-1)^3} \quad . \quad (5.59)$$

The above coefficients have been evaluated at the point t=1 and are functions of T_o, g and Δ. In order to discover the behavior of the theory near the critical point, one needs expressions for rescaled t-dependent quantities. It is straightforward to re-write the t-dependent coefficients from (5.57)~(5.59). The results are,

$$\gamma_t(g(t),\Delta(t),t) = 0 \quad , \quad (5.60)$$

$$\Theta_t(g(t),\Delta(t),t) = \frac{g(t)}{\Delta^2(t)}\frac{tT_o}{(T_c+tT_o)}\int_0^\infty \frac{x^2dx}{(2\pi)^2}\frac{e^{\sigma x}}{(e^{\sigma x}-1)^2} \quad , \quad (5.61)$$

and,

$$\beta_t(g(t),\Delta(t),t) = -\frac{3}{2}g^2(t)\ \frac{tT_o}{(T_c+tT_o)}\int_0^\infty \frac{x^2dx}{(2\pi)^2}\frac{1}{\sigma_x}\frac{e^{\sigma x}(e^{\sigma x}+1)}{(e^{\sigma x}-1)^3} \quad , \quad (5.62)$$

in which σ_x has been re-defined as,

$$\sigma_x^2 \equiv x^2 + \Delta^2(t) \quad . \quad (5.63)$$

Equations (5.60)∿(5.62) are the desired relations for the study of the critical behavior of the theory.

Since we expect the dimensionless running mass $\Delta(t)$ to vanish at the critical point, we are then interested in the infrared behavior of the latter coefficients. Evaluation of the integrals appearing in (5.61) and (5.62) finally yields the following results,

$$\Theta_t(g(t),\Delta(t),t) \simeq \frac{g(t)}{12\Delta^2(t)} \frac{tT_o}{T_c} \quad ; \quad t\to 0 \quad , \tag{5.64}$$

and,

$$\beta_t(g(t),\Delta(t),t) \simeq -\frac{3g^2(t)}{16\pi\Delta(t)} \frac{tT_o}{T_c} \quad ; \quad t\to 0 \quad , \tag{5.65}$$

assuming that,

$$\Delta(t) \simeq \Delta_o t^{\nu} \quad ; \quad t\to 0 \quad ; \quad \nu<1 \quad . \tag{5.66}$$

In the next section, we will determine the index ν as well as other indices describing the critical behavior of the present model from the infrared behavior of the RG-coefficients (5.64)-(5.65).

4. Computation of the Critical Exponents

Following the approach of reference (36) , we define the renormalization group invariant critical temperature in the following way,

$$\text{Re } i\ \Gamma_R^{(2)^{11}}(0;T_c;g(t),m(t);\kappa,T_c+tT_o) = 0 \quad , \tag{5.67}$$

which gives the critical temperature T_c as a function of $g(t)$, $m(t)$ and tT_o.

Also, in addition to the mass critical index ν defined in (5.66), we can further define exponents η and γ as,

$$\mathrm{Re}\ i\Gamma_R^{(2)^{11}}(\vec{k};T_c;g(t),m(t);\kappa,T_c+tT_o)\propto|\vec{k}|^{2-\eta}\ ;\ \vec{k}\to 0\ , \qquad (5.68)$$

and,

$$\mathrm{Re}\ i\Gamma_R^{(2)^{11}}(0;T;g(t),m(t);\kappa,T_c+tT_o)\propto|T-T_c|^{\gamma}\ ;\ T\to T_c\ . \qquad (5.69)$$

Making use of the renormalization group equation (5.28) enriched with dimensionality, analysis of the low-momentum behavior of $\Gamma_R^{(2)}$ at the critical temperature together with similar analysis of the temperature behavior near T_c for static quantities has shown that the following scaling law is satisfied,[36]

$$\nu(2-\eta)=\gamma\ , \qquad (5.70)$$

provided that the following identifications are made,

$$\eta\equiv\gamma_t^*/\Theta_t^*\ , \qquad (5.71)$$

and,

$$\gamma\equiv\Theta_t^*-\frac{1}{2}\gamma_t^*\ , \qquad (5.72)$$

in which,

$$2\nu=\Theta_t^*\ , \qquad (5.73)$$

and,

$$\gamma_t^*\equiv\lim_{t\to 0}\gamma_t(g(t),\Delta(t),t)\ ;\ \Theta_t^*\equiv\lim_{t\to 0}\Theta_t(g(t),\Delta(t),t)\ . \qquad (5.74)$$

It should be emphasized that scaling appears as a natural consequence of the finite temperature RG in the TFD formalism.

In order to obtain the numerical values of the latter critical exponents for the $\lambda\phi^4$ model, one can use the definition (5.67) for the critical temperature to derive a self-consistent equation relating m(t), g(t) and T_c. Remembering equation (5.47) for $\Gamma_R^{(2)}$, equation (5.67) now leads to the following relation,

$$m^2(t) = g(t) \int_0^\infty \frac{p^2 dp}{(2\pi)^2} \frac{1}{\omega_p} \left| \frac{1}{(e^{\beta\omega_p}-1)} - \frac{1}{(e^{\beta_c\omega_p}-1)} \right| \quad , \qquad (5.75)$$

in which,

$$\beta_c \equiv 1/T_c \quad . \qquad (5.76)$$

Re-writing the temperature distribution factors as a Taylor expansion about the critical temperature T_c, one gets,

$$\Delta^2(t) \simeq g(t) \frac{tT_o}{T_c} \int_0^\infty \frac{x^2 dx}{(2\pi)^2} \frac{e^{\sigma_x}}{(e^{\sigma_x}-1)^2} + O(t^2) \quad . \qquad (5.77)$$

where we defined again,

$$x \equiv \beta_c p \quad ; \quad \Delta(t) \equiv \beta_c m(t) \quad ; \quad \sigma_x \equiv \beta_c \omega_p \quad . \qquad (5.78)$$

For small t, the definitions (5.78) become identical to (5.56).

Explicit evaluation of the integral (5.77) in the infrared limit $\sigma_x \to x$ finally yields the self-consistent equation,

$$1 \simeq \frac{g(t)}{12\Delta^2(t)} \frac{tT_o}{T_c} \quad ; \quad t \to 0 \quad . \qquad (5.79)$$

Insertion of the self-consistent equation (5.79) into the expression (5.64) for Θ_t near the critical point identify Θ_t^* as,

$$\Theta_t^* = 1 \quad . \qquad (5.80)$$

One also gets γ_t^* trivially from (5.60),

$$\gamma_t^* = 0 \quad . \qquad (5.81)$$

The critical exponents (5.71)∼(5.73), to the one-loop approximation, are therefore completely determined as,

$$\nu = \tfrac{1}{2} \;\; ; \;\; \gamma = 1 \;\; ; \;\; \eta = 0 \;\; . \tag{5.82}$$

The knowledge of the critical indices (5.82) can now be used to obtain the behavior of the coupling constant $g(t)$ near the critical point. Such a behavior is best analyzed by making use of the β_t function in the infrared limit obtained in (5.65). Insertion of (5.66) and (5.82) into the expression (5.65) for β_t near the critical temperature, we get,

$$\beta_t(g(t),t) \simeq \frac{-3g^2(t)}{16\pi\Delta_o} \frac{T_o}{T_c} t^{\frac{1}{2}} \;\; ; \;\; t \to 0 \;\; . \tag{5.83}$$

Upon the change of variable,

$$\tau(t) \equiv t^{\frac{1}{2}} \;\; , \tag{5.84}$$

as well as the definition,

$$B(g(\tau;\lambda)) \equiv \frac{\beta_\tau(g(\tau;\lambda),\tau)}{\tau} \;\; , \tag{5.85}$$

the differential equation for the running coupling constant $g(\tau;\lambda)$ can now be written as,

$$\frac{dg(\tau;\lambda)}{d\tau} = 2\, B(g(\tau;\lambda)) \;\; , \tag{5.86}$$

for which we assumed the following boundary condition,

$$\lim_{\tau \to 0} g(\tau;\lambda) = \lambda \;\; . \tag{5.87}$$

Note that $B(g)$ has no explicit τ-dependence which makes it particularly easy to integrate. Integration of (5.86) yields,

$$\tau = \frac{1}{2} \int_{\lambda}^{g(\tau;\lambda)} \frac{dz}{B(z)} \;\; . \tag{5.88}$$

Differentiating (5.88) with respect to λ gives,

$$\frac{1}{B(g(\tau;\lambda))}\,\frac{dg(\tau;\lambda)}{d\lambda} - \frac{1}{B(\lambda)} = 0 \quad , \tag{5.89}$$

or,

$$\left[\frac{1}{2}\frac{\partial}{\partial\tau} - B(\lambda)\frac{\partial}{\partial\lambda}\right]g(\tau;\lambda) = 0 \quad , \tag{5.90}$$

where,

$$B(\lambda) = \frac{-3\lambda^2}{16\pi\Delta_o}\,\frac{T_o}{T_c} \quad . \tag{5.91}$$

The solution to (5.90) is now readily obtained as,

$$g(t;\lambda) = \frac{\lambda}{\left[1+\frac{3\lambda}{8\pi\Delta_o}\frac{T_o}{T_c}\,t^{\frac{1}{2}}\right]} \quad ; \quad t\to 0 \quad , \tag{5.92}$$

where use was made of equation (5.84). From the self-consistent equation (5.79), on the other hand, we get,

$$g(0;\lambda) = 12\Delta_o^2\,\frac{T_c}{T_o} \quad , \tag{5.93}$$

which determines λ. Therefore,

$$g(t) = 12\Delta_o^2\,\frac{T_c}{T_o}\left[1+\frac{9\Delta_o}{2\pi}\,t^{\frac{1}{2}}\right]^{-1} \quad ; \quad t\to 0 \tag{5.94}$$

Finally we mention that the exponents (5.82) obtained here are in complete agreement with the so-called ε-expansion method[154,158-159] in the one-loop appro[illegible] as well as with other techniques.[155-157] The method presented [illegible]r can also be generalized, in principle, to higher loop a[illegible] where it can be compared to more accurate estimates[159] of the coefficients.

CHAPTER VI

SUMMARY

In this work we presented and discussed the contributions of the author and co-workers to the physics of macroscopic phenomena in quantum field theoretical systems. These contributions are clearly classified into two important and often connected classes of macroscopic phenomena: the class of extended objects (classical) phenomena and the class of thermodynamical phenomena in quantum systems.

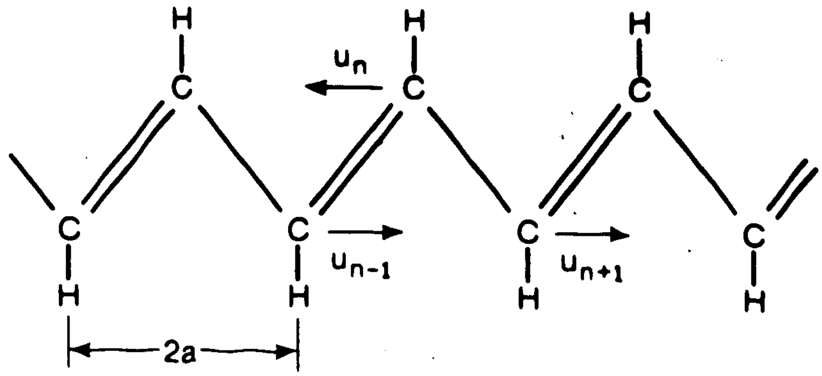
After briefly reviewing in chapter II the formalisms used in this work in connection with the problems at hand, we attacked in chapter III the difficult problem of solving analytically the dynamics of a physically relevant one-dimensional fermion-soliton system with monomial and derivative interactions. Making use of a self-consistent perturbation theory as well as the so-called asymptotic expansion method, we showed how parts of the derivative coupling induce and control the dynamics of the soliton which is now constrained to propagate at fixed velocity. We also succeeded in solving the soliton shape in closed form. The above solution has been obtained in the context of a quasirealistic continuum model of the trans-polyacetylene molecule in which optical and acoustic phonon interaction effects are considered simultaneously. Such a system has been attracting much attention in recent years because it allows for exotic phenomena such as charge or spin fractionization.

In chapter IV, we investigated the Goldstone phase structure of supersymmetric systems for which thermal effects have been responsible for spontaneous supersymmetry breaking. After briefly reviewing the physics of the single scalar superfield Wess-Zumino model, which can be broken only at finite temperature for both the free and interacting cases, we then gave a detailed analysis to the N=3 scalar superfield O'Raifeartaigh model which allows for supersymmetry breaking both at zero and finite temperature, thereby displaying a mixed mechanism of supersymmetry breaking. Because the O'Raifeartaigh model has richer symmetry content due to the added degrees of freedom, the Goldstone phase at finite temperature allows for the coexistence of different types of Nambu-Goldstone zero-energy modes. In mass degenerate channels, such modes are the so-called thermal superpairs while Goldstone fermion particle modes appear in channels where mass degeneracy is removed among supermultiplets. Such modes coexist with the original Goldstone fermion of the zero temperature broken symmetry phase.

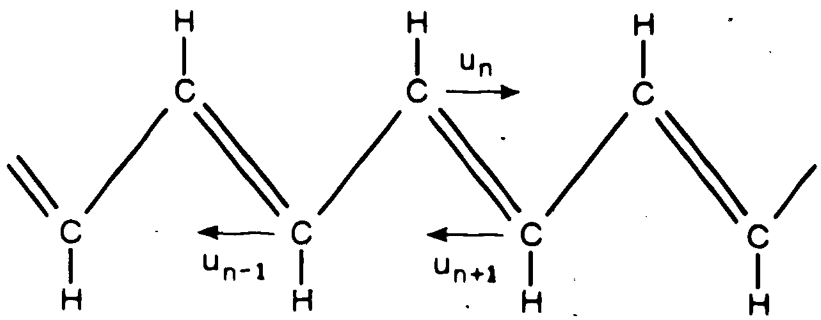
Finally, in chapter V, we reviewed renormalization theory and the renormalization group in the context of thermo-field dynamics and presented a one-loop computation of the renormalization group coefficients together with a discussion of the critical behavior and a one-loop computation of the critical exponents of the $\lambda\phi^4$-theory at finite temperature.

The problems treated in this work of course form a very small subset of the immensely larger set of all macroscopic phenomena encountered in all branches of physics. Although soliton physics enjoys great popularity in condensed matter physics, it is still problematic in high-energy physics. Unless its importance is further clarified in this latter branch of physics the attention of theoreticians is likely to drift increasingly toward the class of macroscopic phenomena involving statistical mechanical concepts. The study of the onset of chaos as well as the transition to the glass phase of matter are some examples of physical problems with great promises. Note that the study of glass systems already overlaps with the domain of topological extended objects.

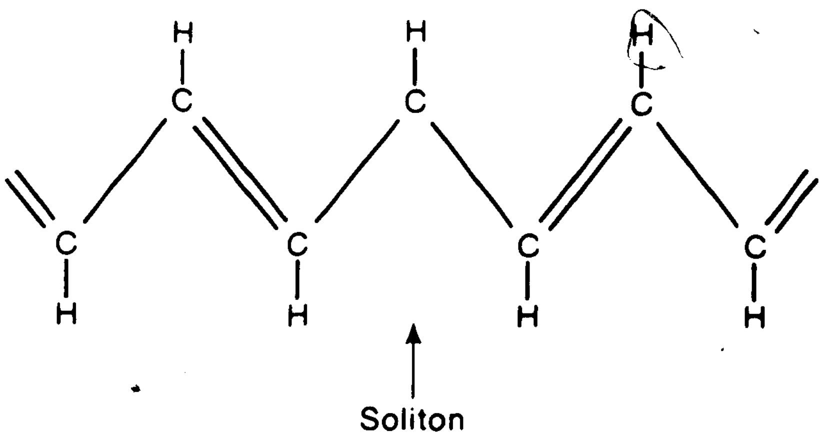
Figure 1. Phases of the dimerized trans-polyacetylene molecule.



A — phase of the trans-polyacetylene molecule.



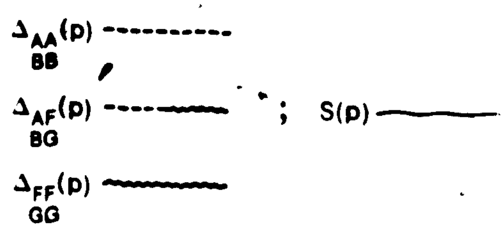
B — phase of the trans-polyacetylene molecule.



Soliton in the chain structure of trans-polyacetylene.

Figure 2. Feynman rules of the N=3 O'Raifeartaigh model at finite temperature.

1] Propagators



2] Vertices

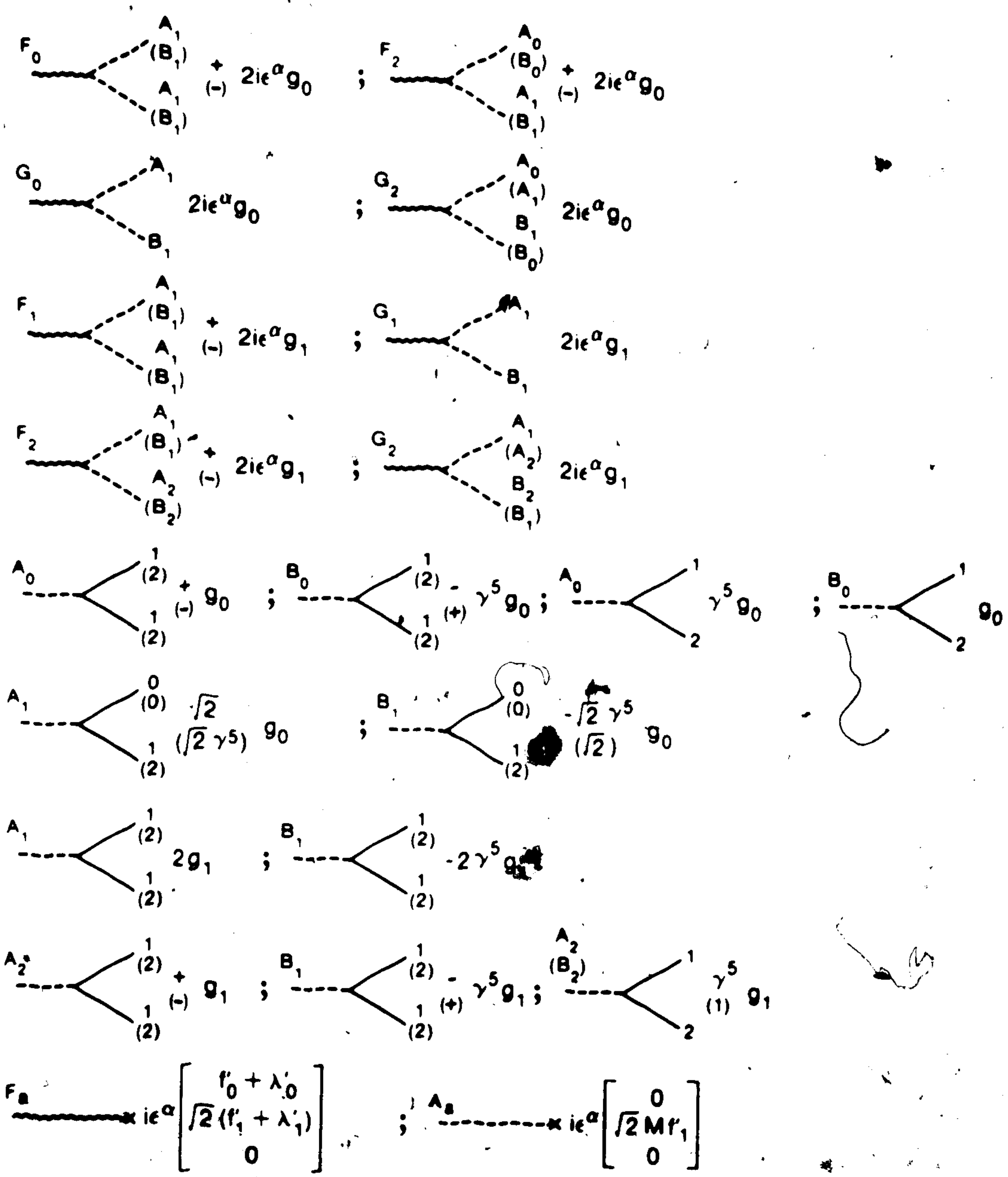


Figure 2. (Continued)

3] Counterterms

0 ∼∼×∼∼ 0	$i\epsilon^{\alpha}\delta Z'_{00}$	0 - - -×- - - 0	$-i\epsilon^{\alpha}p^{2}\delta Z'_{00}$
0 ∼∼×∼∼ 1	$i\epsilon^{\alpha}\sqrt{2}\,\delta Z'_{01}$	0 - - -×- - - 2	$-i\epsilon^{\alpha}\sqrt{2}\,p^{2}\delta Z'_{01}$
1 ∼∼×∼∼ 1	$i\epsilon^{\alpha}[\delta Z'_{11}-\delta Z'_{12}]$	1 - - -×- - - 1	$-i\epsilon^{\alpha}p^{2}[\delta Z'_{11}+\delta Z'_{12}]$
2 ∼∼×∼∼ 2	$i\epsilon^{\alpha}[\delta Z'_{11}+\delta Z'_{12}]$	2 - - -×- - - 2	$-i\epsilon^{\alpha}p^{2}[\delta Z'_{11}-\delta Z'_{12}]$
a ——×—— a	$-i\not{p}\,\delta Z'_{aa}$	0 ——×—— 1	$-i\not{p}\,\delta Z'_{01}$
1 ——×—— 2	$-i\not{p}\gamma^{5}\delta Z'_{12}$	0 ——×—— 2	$+i\not{p}\gamma^{5}\delta Z'_{01}$

F_0 ∼∼∼∼⊗	$i\epsilon^{\alpha}[\delta Z'_{00}f'_{0}+2\delta Z'_{01}f'_{1}]$
F_1 ∼∼∼∼⊗	$i\epsilon^{\alpha}\sqrt{2}\,\frac{gv}{m}\,[\delta Z'_{00}f'_{0}+2\delta Z'_{01}f'_{1}]$

Figure 3. One-loop fermion self-energy diagrams of the N=3 Ô'Raifeartaigh model at finite temperature.

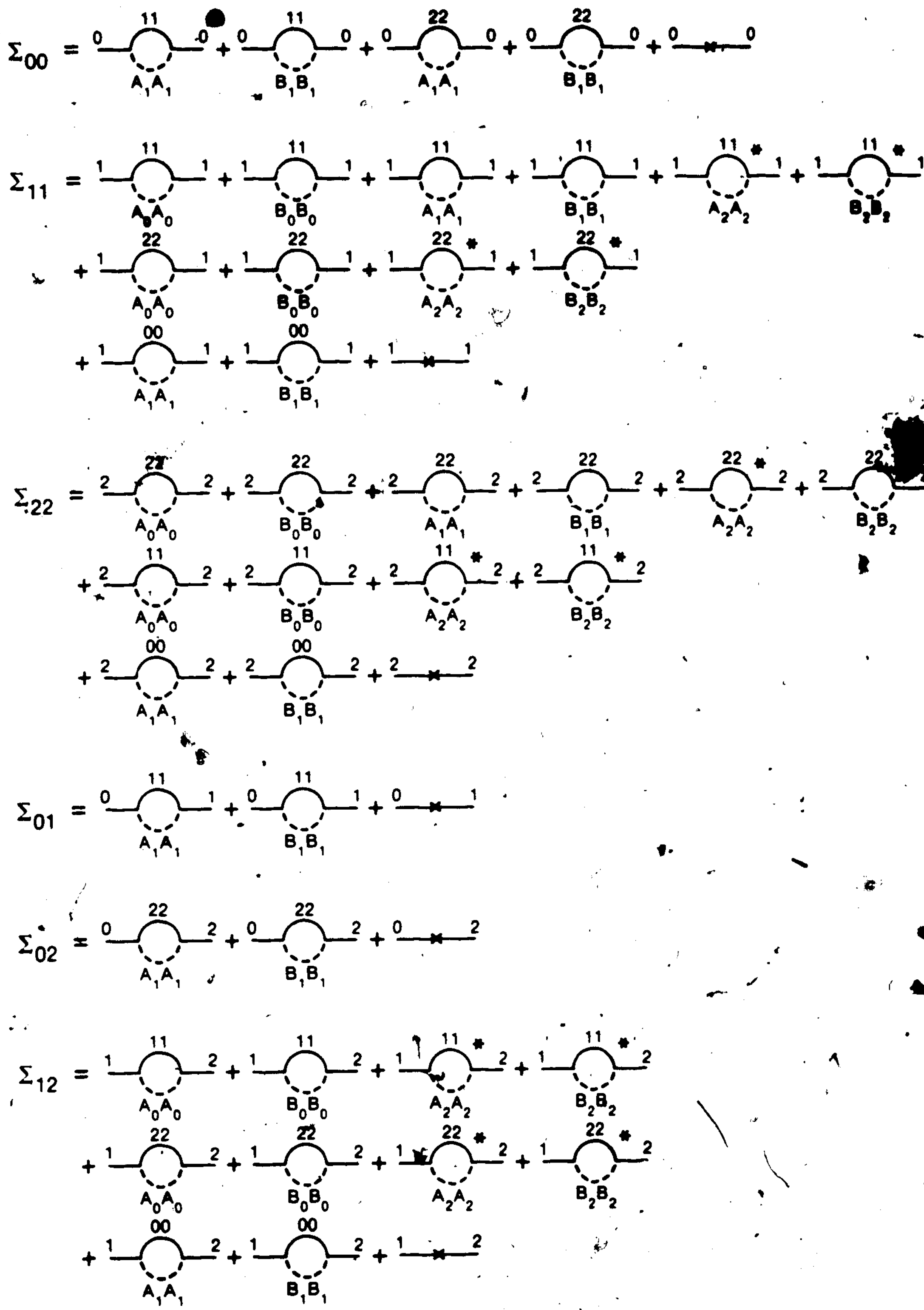
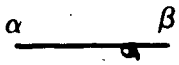
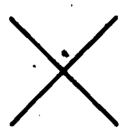

Σ00 =
Σ11 =
Σ22 =
Σ01 =
Σ02 =
Σ12 =

Figure 4. Feynman rules of the $\lambda\phi^4$-theory at finite temperature.

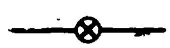
Propagator

$i\Delta^{\alpha\beta}(p)$ 

2] Vertices

 $-i\epsilon^{\alpha}g$

3] Counterterms

 $i\epsilon^{\alpha}[-\delta\mu^2 + (k^2 - m^2)\delta Z]$

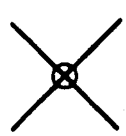
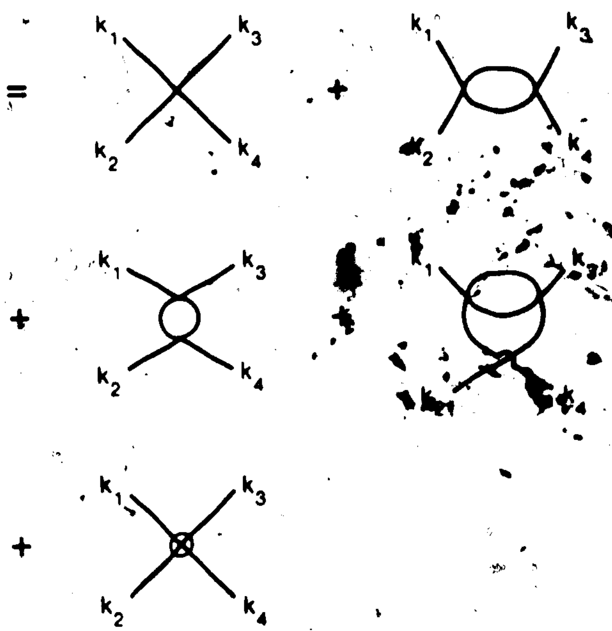
 $-i\epsilon^{\alpha}g\delta Z_1$

Figure 5. One-loop diagrams for the self-energy and the four-point vertex function of the $\lambda\phi^4$-theory.

$$-i\Sigma_R^{11}(k; T_c + tT_o) =$$

$$\Gamma_R^{(4)1111}(k_1, k_2, k_3, k_4; T_c + tT_o) =$$



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APPENDIX A

DERIVATION OF THE CONTINUUM POLYACETYLENE MODEL FROM THE SSH MODEL

In this appendix we derive formally the continuum model (3.6) for the polyacetylene molecule, including acoustic phonon effects, from the discrete SSH model (3.1) to order v^2 in acoustic phonon velocity.[95-97]

The SSH model is given as, in the Lagrangean formalism,

$$L = \sum_n C_n^\dagger \left[i\frac{\partial}{\partial t} - \mu\right] C_n$$

$$+ \sum_n \frac{1}{2}\left[\rho\left(\frac{\partial u_n}{\partial t}\right)^2 - K(u_{n+1}-u_n)^2\right]$$

$$+ \sum_n t_{n,n+1}\,(C_n^\dagger C_{n+1} + C_{n+1}^\dagger C_n) \quad , \qquad \text{(A.1)}$$

where C_n and $C_n^\dagger$ represent the annihilation and creation operators of the π-electron at the lattice site r_n and u_n is the displacement field of the lattice r_n. These operators satisfy the following commutation relations,

$$[C_n, C_m^\dagger]_+ = \delta_{nm} \quad , \qquad \text{(A.2)}$$

and,

$$\left[u_n, \frac{\partial u_m}{\partial t}\right]_- = \frac{i\delta_{nm}}{\rho} \quad . \qquad \text{(A.3)}$$

In the above expressions, ρ is the mass of the CH-group, K is the spring constant for the undimerized lattice, μ is the electron chemical potential and $t_{n,n+1}$ is the hopping matrix obtained in a linear approximation as,

$$t_{n,n+1} = t_0 - \alpha(u_{n+1}-u_n) \quad , \qquad \text{(A.4)}$$

where t_0 is the hopping amplitude and α the electron-phonon coupling.

In order to pass to the continuum description, we now introduce Fourier transforms for operators A_n at the lattice site r_n in the following way,

$$A[k] = \sqrt{\frac{a}{2\pi}} \sum_n e^{-ikr_n} A_n \quad , \tag{A.5}$$

in which a denotes the lattice constant of the model and the momentum k is limited to the first Brillouin zone. The inverse transform is given as,

$$A_n = \sqrt{\frac{a}{2\pi}} \int_{-\pi/a}^{\pi/a} dk\ e^{ikr_n} A[k] \quad . \tag{A.6}$$

If we now make use of the following formulae,

$$\frac{a}{2\pi} \sum_n \exp(ikr_n) = \sum_N \delta(k+2\pi N/a) \quad , \tag{A.7}$$

and,

$$\frac{a}{2\pi} \int_{-\pi/a}^{\pi/a} dk \exp[ik(r_n-r_m)] = \delta_{nm} \quad , \tag{A.8}$$

one can re-write the SSH Langrangean (A.1) in the following form,

$$L = \int dk\ C^{\dagger}[k]\ [i\frac{\partial}{\partial t} - \mu + 2t_0\cos(ka)]\ C[k]$$

$$+ \int dq\ \frac{1}{2} \left[\rho \frac{\partial}{\partial t} u^{\dagger}[q] \frac{\partial}{\partial t} u[q] - 2K[1-\cos(qa)]\, u^{\dagger}[q] u[q]\right]$$

$$- \alpha \sqrt{\frac{a}{2\pi}} \int dk dq\ \{C^{\dagger}[k+q] C[k] u[q] e^{ika}(e^{iqa}-1) + h.c.\ \} \quad . \tag{A.9}$$

Dividing the Brillouin zone into half, we now define two fields $\psi_1[k]$ and $\psi_2[k]$ for the electron as well as the acoustic phonon field $\xi[q]$ and optical phonon field $\Phi[q]$ in the following way,

$$\psi_1[k] \equiv C[k+\pi/2a] \quad ; \quad -\pi/2a<k<\pi/2a \quad , \tag{A.10}$$

$$\psi_2[k] \equiv -iC[k-\pi/2a] \quad ; \quad -\pi/2a<k<\pi/2a \quad , \tag{A.11}$$

$$\xi[q] \equiv u[q] \quad ; \quad -\pi/2a<q<\pi/2a \quad , \tag{A.12}$$

and,

$$\phi[q] \equiv u[q+\pi/a] \quad ; \quad -\pi/2a<q<\pi/2a \quad . \tag{A.13}$$

In terms of the fields (A.10)∿(A.13), the Lagrangean (A.9) is then re-written as,

$$\begin{aligned} L = & \int dk\, \psi^{\dagger}[k]\, [i\frac{\partial}{\partial t}-\mu-2t_0\sin(ka)\tau_3]\, \psi[k] \\ & + \frac{1}{2}\int dq \Big[\rho\frac{\partial}{\partial t}\xi^{\dagger}[q]\frac{\partial}{\partial t}\xi[q] - 2K[1-\cos(qa)]\xi^{\dagger}[q]\xi[q] \\ & \qquad \rho\frac{\partial}{\partial t}\phi^{\dagger}[q]\frac{\partial}{\partial t}\phi[q] - 2K[1+\cos(qa)]\phi^{\dagger}[q]\phi[q] \Big] \\ & + \alpha\sqrt{\frac{a}{2\pi}}\int dkdq\, \{ -ie^{ika}(e^{iqa}-1)\psi^{\dagger}[k+q]\tau_3\psi[k]\xi[q] \\ & \qquad +ie^{ika}(e^{iqa}+1)\psi^{\dagger}[k+q]\tau_1\psi[k]\phi[q] \\ & \qquad + h.c. \; \} \quad . \end{aligned} \tag{A.14}$$

In the above Lagrangean, the τ_i matrices are the generators of SU(2) in the fundamental representation and the $\psi[k]$ field has been defined as,

$$\psi[k] \equiv \begin{bmatrix} \psi_1[k] \\ \psi_2[k] \end{bmatrix} \quad . \tag{A.15}$$

The range of the momentum integrations has been specified in (A.10) ∿(A.13).

Defining fields in configuration space as follows,

$$\psi(x) \equiv \int \frac{dk}{\sqrt{2\pi}}\, e^{ikx}\psi[k] \quad , \qquad (A.16)$$

$$\xi(x) \equiv \sqrt{\rho} \int \frac{dq}{\sqrt{2\pi}}\, e^{iqx}\xi[q] \quad , \qquad (A.17)$$

and,

$$\phi(x) \equiv \sqrt{\rho} \int \frac{dq}{\sqrt{2\pi}}\, e^{iqx}\phi[q] \quad , \qquad (A.18)$$

we can obtain the continuum model by expanding the Lagrangean (A.14) in powers of a small lattice constant a and by inserting the relations (A.16)-(A.18). The resulting Lagrangean density is non-local in the sense that it contains higher spatial derivative coupling terms as we go higher in lattice constant expansion. Restricting ourselves to order a^2, the Lagrangean density is the same as in equation (3.6),

$$\mathcal{L} = \psi^{+}[i\frac{\partial}{\partial x_0} - \mu + iv_F\tau_3\frac{\partial}{\partial x_1}]\psi + \frac{1}{2}\left[\left(\frac{\partial \xi}{\partial x_0}\right)^2 - v^2\left(\frac{\partial \xi}{\partial x_1}\right)^2\right]$$

$$+ \frac{1}{2}\left[\left(\frac{\partial \phi}{\partial x_0}\right)^2 + v^2\left(\frac{\partial \phi}{\partial x_1}\right)^2 - m^2\phi^2\right] + g\psi^{+}\tau_1\psi\phi$$

$$+ \frac{gv^2}{m^2}\left\{ -i[\psi^{+}\tau_3\frac{\partial\psi}{\partial x_1} - \frac{\partial\psi^{+}}{\partial x_1}\tau_3\psi]\frac{\partial \xi}{\partial x_1} + [\psi^{+}\tau_1\frac{\partial^2\psi}{\partial x_1^2} + \frac{\partial^2\psi^{+}}{\partial x_1^2}\tau_1\psi]\phi \right\} , \qquad (A.19)$$

in which v_F denotes the Fermi velocity, v is the acoustic phonon velocity, m is the optical phonon mass and g is the electron-phonon coupling constant. They are given in terms of the parameters of the discrete model as,

$$v_F \equiv 2at_0 \; ; \; v^2 \equiv (K/\rho)a^2 \; ; \; m^2 = 4K/\rho \; ; \; g = 4\alpha\sqrt{a/\rho} \quad . \qquad (A.20)$$

This model is the completely realistic continuum model for polyacetylene when acoustic effects are taken into consideration. In the limit $v^2 \to 0$, the Lagrangean density (A.19) reduces to the well-known TLM model.

APPENDIX B

PRODUCT RULES FOR SPECTRAL REPRESENTATIONS IN TFD

In this appendix, we present useful product rules for spectral representations in the context of TFD.[26,52] Three cases may occur in the [illegible]on of the one-loop self-energy diagrams of fermion-boson systems: t[illegible]uct of two [illegible] may be of fermion-boson type, boson-boson or fermion-fermio[illegible]. [illegible]ote that the conventions we follow here will be given by the ones chosen in chapter IV.

In TFD, the spectral representations for boson and fermion propagators are written respectively as,

$$\Delta^{\alpha\beta}(p) = -i \left[U_B(|p_0|)\frac{\tau}{p^2+M_B^2-i\tau\delta}U_B(|p_0|)\right]^{\alpha\beta}$$

$$= i\int_{-\infty}^{\infty} d\kappa_1 \bar{\sigma}(\kappa_1,\vec{p};M_B)\left[U_B(\kappa_1)\frac{\tau}{p_0-\kappa_1+i\tau\delta}U_B(\kappa_1)\right]^{\alpha\beta}, \qquad (B.1)$$

and,

$$S^{\alpha\beta}(p) = (-i\not{p}+M_F)\left[U_F(|p_0|)\frac{(-1)}{p^2+M_F^2-i\tau\delta}U_F^{-1}(|p_0|)\right]^{\alpha\beta}$$

$$= \int_{-\infty}^{\infty} d\kappa_2\bar{\sigma}(\kappa_2,\vec{p};M_F)(-i\not{\kappa}_2-i\vec{\not{p}}+M_F)\left[U_F(\kappa_2)\frac{1}{p_0-\kappa_2+i\tau\delta}U_F^{-1}(\kappa_2)\right]^{\alpha\beta}, \qquad (B.2)$$

where the Bogoliubov matrices $U_B(\kappa_1)$ and $U_F(\kappa_2)$ are given as,

$$U_B(\kappa_1) = \frac{1}{(e^{\beta\kappa_1}-1)^{\frac{1}{2}}}\begin{bmatrix} e^{\frac{1}{2}\beta\kappa_1} & 1 \\ 1 & e^{\frac{1}{2}\beta\kappa_1}\end{bmatrix}, \qquad (B.3)$$

and,

$$U_F(\kappa_2) = \frac{1}{(e^{\beta\kappa_2}+1)^{\frac{1}{2}}}\begin{bmatrix} e^{\frac{1}{2}\beta\kappa_2} & -i \\ -i & e^{\frac{1}{2}\beta\kappa_2}\end{bmatrix}. \qquad (B.4)$$

The latter matrices satisfy the following relations,

$$U_B \tau U_B = \tau \ ; \ U_F^{-1} = U_F^{*} \quad . \qquad (B.5)$$

Also, we defined in (B.1)-(B.2),

$$\bar{\sigma}(\kappa,\vec{p};M) \equiv \frac{1}{2\omega_p}\,[\delta(\kappa-\omega_p)-\delta(\kappa+\omega_p)] \quad ; \ \omega_p^2 \equiv \vec{p}^{\,2}+M^2 \quad . \qquad (B.6)$$

A product rule for fermion-boson propagators can be obtained as,

$$\int\frac{dk_0}{2\pi}\left[U_B(\kappa_1)\frac{\tau}{k_0+p_0-\kappa_1+i\tau\delta}U_B(\kappa_1)\right]^{\alpha\beta}\left[U_F(\kappa_2)\frac{1}{k_0-\kappa_2+i\tau\delta}U_F^{-1}(\kappa_2)\right]^{\alpha\beta}$$

$$= i\int_{-\infty}^{\infty}d\kappa\,\sigma_{FB}(\kappa;\kappa_1,\kappa_2)\left[U_F(\kappa)\frac{1}{p_0-\kappa+i\tau\delta}U_F^{-1}(\kappa)\right]^{\beta\alpha} \quad , \qquad (B.7)$$

where we defined,

$$\sigma_{FB}(\kappa;\kappa_1,\kappa_2) \equiv \delta(\kappa-\kappa_1+\kappa_2)\left[\frac{1}{e^{\beta\kappa_1}-1}+\frac{1}{e^{\beta\kappa_2}+1}\right] \quad . \qquad (B.8)$$

When p_0 is real in equation (B.7), it follows from the identity,

$$\frac{1}{p_0-\kappa+i\tau\delta} = \frac{P}{p_0-\kappa} - i\pi\tau\delta(p_0-\kappa) \quad , \qquad (B.9)$$

that $U_F(\kappa)$ can be replaced by $U_F(p_0)$ in the rhs of the product rule. When this happens, $U_F(p_0)$ can then be factorized outside the integration, a property which makes product rules convenient computational tools.

We close this appendix by writing explicitly the rules for boson-boson products as well as fermion-fermion products:

$$\int\frac{dk_0}{2\pi}\left[U_B(\kappa_1)\frac{\tau}{k_0+p_0-\kappa_1+i\tau\delta}U_B(\kappa_1)\right]^{\alpha\beta}\left[U_B(\kappa_2)\frac{\tau}{k_0-\kappa_2+i\tau\delta}U_B(\kappa_2)\right]^{\alpha\beta}$$

$$= i\int_{-\infty}^{\infty}d\kappa\,\sigma_{BB}(\kappa;\kappa_1,\kappa_2)\left[U_B(\kappa)\frac{\tau}{p_0-\kappa+i\tau\delta}U_B(\kappa)\right]^{\alpha\beta} \quad , \qquad (B.10)$$

and,

$$\int \frac{dk_0}{2\pi} \left[U_F(\kappa_1)\frac{1}{k_0+p_0-\kappa_1+i\tau\delta}U_F^{-1}(\kappa_1)\right]^{\alpha\beta}\left[U_F(\kappa_2)\frac{1}{k_0-\kappa_2+i\tau\delta}U_F^{-1}(\kappa_2)\right]^{\beta\alpha}$$

$$= i\int_{-\infty}^{\infty} d\kappa\,\sigma_{FF}(\kappa;\kappa_1,\kappa_2)\left[U_B(\kappa)\frac{\tau}{p_0-\kappa+i\tau\delta}U_B(\kappa)\right]^{\beta\alpha}\tau^{\alpha\alpha}\tau^{\beta\beta} , \qquad (B.11)$$

in which we defined,

$$\sigma_{BB}(\kappa;\kappa_1,\kappa_2) \equiv \delta(\kappa-\kappa_1+\kappa_2)\left[\frac{1}{e^{\beta\kappa_1}-1}-\frac{1}{e^{\beta\kappa_2}-1}\right] , \qquad (B.12)$$

and,

$$\sigma_{FF}(\kappa;\kappa_1,\kappa_2) \equiv \delta(\kappa-\kappa_1+\kappa_2)\left[\frac{1}{e^{\beta\kappa_1}+1}-\frac{1}{e^{\beta\kappa_2}+1}\right] . \qquad (B.13)$$