Amenability properties of certain Banach algebras of operators on Banach spaces

by

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Abstract

In this thesis, we prove the non-amenability of the Banach algebra $\mathcal{B}(E)$, the Banach algebra of all operators on an infinite dimensional Banach space E, where, for $p \in [1, \infty)$, E is an infinite dimensional $\mathfrak{L}^p$-space in the sense of Lindenstrauss and Pełczyński. In addition, we prove that $\mathcal{SS}(E)$, the Banach algebra of all strictly singular operators on E, is not weakly amenable if $E = C[0,1]$ or $E = L^p[0,1]$, where $p \in [1, \infty)$. Then we generalize this last result to all infinite dimensional separable $\mathfrak{L}^p$-space E such that $E \not\simeq \ell^p$, where $p \in (1, \infty)$.

To my father, for all the sacrifices he had made on our behalf

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بسم الله الرحمن الرحيم

In the name of Allah the most Merciful and Beneficent

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Contents

Chapter 1

Introduction and Preliminaries

The existence of a finitely additive set function which is invariant under a certain group action was studied extensively in the days of Banach and Tarski. It turns out that groups that possess a left invariant finitely additive measure have nice properties and were called amenable by Mahlon M. Day in 1949. In a totally different area of Mathematics, a certain type of cohomology, known as Hochschild cohomology, was introduced in [Hoc45] by G. Hochschild in 1945 to study properties in abstract algebra, and was extended to the theory of Banach algebras by H. Kamowitz in 1962, [Kam62].

In 1972, B. E. Johnson characterized amenable locally compact groups G through the first cohomology group of the Banach algebra $L^1(G)$. In his memoir, Johnson proved that a locally compact group G is amenable if, and only if, the Banach algebra $L^1(G)$ has a trivial first cohomology group $\mathcal{H}^1(L^1(G), X^*)$. Since then, the Banach algebras that satisfy the the last property, having trivial first cohomology groups with coefficients in dual Banach bimodules, are to be amenable . Due to Johnson's work, the notion of amenability has been transferred from groups to Banach algebras, and the theory of amenable Banach algebras has been developed ever since.

Over the years, several generalizations of amenability have been introduced by numerous authors, among those is the concept of weak amenability. Weak amenability of Banach algebras was first introduced for commutative Banach algebras by W. Bade, P. C. Curtis and H. Dales in [BCD87], and was then generalized by B. E. Johnson to general Banach algebras [Joh87]. A Banach algebra $\mathfrak{A}$ is weakly amenable if the first cohomology group $\mathcal{H}^1(\mathfrak{A}, \mathfrak{A}^*)$ is trivial. Clearly, every amenable Banach algebra is weakly amenable.

Characterizing the amenable members of a certain class of Banach algebras is interesting under many aspects. For instance, a C^* algebra $\mathfrak{A}$ is always weakly amenable but $\mathfrak{A}$ is amenable if, and only if, $\mathfrak{A}$ is nuclear, [Con78] and [Haa83]. The group algebra $L^1(G)$ is always weakly amenable but is amenable if, and only if, the group G is amenable, [Joh72] and [Joh87]. Another Banach algebra that so many researchers were interested in is the Banach algebra of all approximable operators on a Banach space X, $\mathcal{A}(X)$, which is the norm closure of the linear space of all finite rank operators on X. Under some conditions, $\mathcal{A}(X)$ is nothing but $\mathcal{K}(X)$, the Banach algebra of all compact operators on X. Extensive research has been done to characterize the weak amenability and the amenability of the Banach algebra $\mathcal{A}(X)$ in terms of properties of the underlying Banach space X. On the other hand, less work has been done to investigate the weak amenability and the amenability of the Banach algebra $\mathcal{B}(X)$ of all operators on a Banach space X, and, to the author's knowledge, no work has been done so far to investigate that of the Banach algebra $\mathcal{SS}(X)$ of all strictly singular operators on X.

Investigating the amenability of the Banach algebra $\mathcal{B}(X)$ for an infinite dimensional Banach space X was first motivated by the following questions that were raised by B. E. Johnson in [Joh72]:

Question 1. Does there exist an infinite dimensional Banach space X such that $\mathcal{B}(X)$ is amenable?

Question 2. Does there exist a Hilbert space $\mathcal{H}$ such that $\mathcal{B}(\mathcal{H})$ is amenable?

Since for any Hilbert space $\mathcal{H}$, the Banach algebra $\mathcal{B}(\mathcal{H})$ is a von Neumann algebra, i.e., a C^*-algebra such that $\mathcal{B}(\mathcal{H}) = Y^*$ for some Banach space Y, the amenability of $\mathcal{B}(\mathcal{H})$ was fully characterized due to a result by S. Wassermann in [Was76] and the equivalence between nuclearity and amenability for C^*-algebras, [Con78] and [Haa83]. Combining these results, it has been shown that $\mathcal{B}(\mathcal{H})$ is amenable if, and only if, $dim\mathcal{H} < \infty$, so in particular $\mathcal{B}(\ell^2)$ is not amenable. One expects that such a condition still holds in the Banach space case. Surprisingly, an infinite dimensional Banach space E_{AH} such that $\mathcal{B}(E_{AH})$ is amenable exists due to S. Argyros and R. Haydon in [AH11].

For several years, it has not been known whether $\mathcal{B}(\ell^p)$ is amenable for any $p \neq 2$, till 2004 when C. Read proved the non-amenability of $\mathcal{B}(\ell^1)$. Read's proof was simplified by Pisier in [Pis04] and eventually was simplified even more by N. Ozawa in [Oza04]. In [Oza04], N. Ozawa managed to provide a proof that simultaneously establishes the non-amenability of several Banach algebras, including $\mathcal{B}(\ell^1)$ and $\mathcal{B}(\ell^\infty)$. Another contribution in this direction was made by M. Daws and V. Runde in [DR07]. Their approach was to investigate the consequences of the hypothetical amenability of the Banach algebra $\mathcal{B}(\ell^p)$, and they proved, among other results, that the amenability of the Banach algebra $\mathcal{B}(\ell^p)$ forces the Banach algebra $\ell^\infty(\mathcal{K}(\ell^2 \oplus \ell^p))$ to be amenable. This last implication was the start of Runde's work in 2010, where V. Runde provided a proof that establishes the non-amenability of the Banach algebra $\ell^\infty(\mathcal{K}(\ell^2 \oplus \ell^p))$, and hence that of the Banach algebra $\mathcal{B}(\ell^p)$. In fact, Runde's result states that for any $p \in (1, \infty)$, the Banach algebra $\mathcal{B}(E)$ is not amenable if $E \simeq \ell^p(E)$ and E is an $\mathfrak{L}^p$-space in the sense of Lindenstrauss and

Pełczyński.

The purpose of this thesis is to generalize this last result by V. Runde. We first prove the non amenability of $\mathcal{B}(E)$ if E is an infinite dimensional $\mathfrak{L}^1$-space such that $E \simeq \ell^1(E)$ or if E is an infinite dimensional $\mathfrak{L}^\infty$-space such that $E \simeq c_0(E)$. Then, we prove that the Banach algebra $\mathcal{B}(E)$ is not amenable for any infinite dimensional $\mathfrak{L}^p$-space E such that $E \not\simeq \ell^p(E)$, $p \in [1, \infty)$. Finally, we prove that for a special class of Banach spaces, the Banach algebra $\mathcal{SS}(X)$ is not weakly amenable, and hence not amenable. In particular, if $p \in [1, \infty)$ the Banach algebras $\mathcal{SS}(L^p[0,1])$, and $\mathcal{SS}(C[0,1])$ are not amenable.

In Chapter One, we introduce some basic definitions and standard results that have been used throughout this thesis. As a general guide, the results we quote in Section One can be found in [FHH$^+$10], [Rya13] and [Dal00]. Material of Section Two and that of Section Four can be found in [Run02], whereas most of the results we quote in Section Three can be found in [LP68] and [LR69]. In Chapter Two, we generalize Runde's result and prove the non-amenability of the Banach algebra $\mathcal{B}(E)$ for any infinite dimensional $\mathfrak{L}^p$-space E ($p \in [1, \infty)$). Chapter Three is dedicated to discussing the weak amenability of the Banach algebra $\mathcal{SS}(E)$ for an infinite dimensional separable $\mathfrak{L}^p$-space E, and we conclude this thesis with some remarks and open problems in Chapter Four.

1.1 Banach Space Theory

Let X and Y be Banach spaces. By an operator $T : X \longrightarrow Y$ we mean a bounded linear map from X into Y, and the Banach space of all operators from X into Y will be denoted by $\mathcal{B}(X, Y)$, and If $X = Y$, then we write $\mathcal{B}(X)$. By $I_X \in \mathcal{B}(X)$ we mean the identity operator on X and by a projection

$P \in \mathcal{B}(X)$ we mean an operator $P : X \longrightarrow X$ such that $P(P(x)) = P(x)$ for each $x \in X$, i.e., $P^2 = P$. If Z is a closed subspace of X, we will write $Z \subseteq X$. All subspaces will be be assumed to be closed unless otherwise stated. A subspace $Z \subseteq X$ is said to be complemented, and will be denoted by $Z \overset{c}{\hookrightarrow} X$, if there exists a projection $P \in \mathcal{B}(X)$ such that $P(X) = Z$. The dual space of X is denoted by X^*, and the canonical pairing of X and X^* is denoted by $\langle ., . \rangle$. Two Banach spaces X and Y are said to be isomorphic, in symbols $X \simeq Y$, if there exists an isomorphism operator $T \in \mathcal{B}(X, Y)$, that is, a bijection operator T. If $X \simeq Y$, then the Banach-Mazur distance between X and Y is defined by $d(X, Y) \coloneqq \inf\{\|T\|\|T^{-1}\| \,; T \text{ is an isomorphism}\}$.

An operator $T \in \mathcal{B}(X, Y)$ is called compact if $T(B_1(X))$ is relatively compact in Y, where $B_1(X)$ is the unit ball of X, and T is called strictly singular if there is no infinite dimensional subspace $Z \subseteq X$ such that $T_{|_Z}$ is an isomorphism into Y. The closed subspace of all compact operators from X into Y, respectively all strictly singular operators from X into Y, is denoted by $\mathcal{K}(X, Y)$, respectively $\mathcal{SS}(X, Y)$. If $X = Y$, then we write $\mathcal{K}(X)$ and $\mathcal{SS}(X)$.

Let $(X_i)_{i \in \Lambda}$ be a family of Banach spaces. We write $\Pi_{i \in \Lambda} X_i$ for its Cartesian product. For $p \in [1, \infty)$, we set

$$\ell^p\text{-}\bigoplus_{i \in \Lambda} X_i \coloneqq \left\{ (x_i) \in \Pi_{i \in \Lambda} X_n \,;\, \sum_{i \in \Lambda} \|x_i\|^p < \infty \right\},$$

it is a normed space that becomes a Banach space if equipped with the norm

$$\|x\|_p = \left(\sum_{i=1}^{\infty} \|x_i\|^p \right)^{1/p}, \quad x = (x_i) \in \ell^p\text{-}\bigoplus_{i \in \Lambda} X_i.$$

Furthermore, we define

$$\ell^\infty\text{-}\bigoplus_{i\in\Lambda} X_i := \left\{ (x_i) \in \Pi_{i\in\Lambda} X_i \,;\; \sup_{i\in\Lambda} \|x_i\| < \infty \right\},$$

which is, also, a Banach space under the norm

$$\|x\|_\infty = \sup_{i\in\Lambda} \|x_i\|, \qquad x = (x_i) \in \ell^\infty\text{-}\bigoplus_{i\in\Lambda} X_i.$$

By c_0-$\bigoplus_{i\in\Lambda} X_i$ we mean the closure of those $(x_i) \in \ell^\infty$-$\bigoplus_{i\in\Lambda} X_i$ for which $x_i = 0$ for all but finitely many $i \in \Lambda$. If $X_i = X$ for all $i \in \Lambda$, then we write $\ell^p(\Lambda, X), \ell^\infty(\Lambda, X)$ and $c_0(\Lambda, X)$ for ℓ^p-$\bigoplus_{i\in\Lambda} X_i$, ℓ^∞-$\bigoplus_{i\in\Lambda} X_i$ and c_0-$\bigoplus_{i\in\Lambda} X_i$, respectively, and if $\Lambda = \mathbb{N}$ and $X_i = X$ for all $i \in \mathbb{N}$, then we write $\ell^p(X), \ell^\infty(X)$ and $c_0(X)$ for $\ell^p(\mathbb{N}, X)$, $\ell^\infty(\mathbb{N}, X)$ and $c_0(\mathbb{N}, X)$, respectively. If $X = \mathbb{C}$, then $\ell^p(X) = \ell^p$ for all $p \in [1, \infty]$, and $c_0(X) = c_0$. A well-known fact in functional analysis is that $\ell^p(\ell^p) \simeq \ell^p(\mathbb{N}^2, \mathbb{C}) \simeq \ell^p(\mathbb{C}) = \ell^p$ for all $p \in [1, \infty]$, similarly $c_0(c_0) \simeq c_0(\mathbb{N}^2, \mathbb{C}) \simeq c_0(\mathbb{C}) = c_0$. For $p \in [1, \infty]$, let q be the conjugate of p, i.e., $\dfrac{1}{p} + \dfrac{1}{q} = 1$. We denote by (δ_n) (respectively, (δ_n^*)) the standard basis of ℓ^p or c_0 (respectively, ℓ^q).

Let X and Y be Banach spaces. For $x \in X$ and $y \in Y$, consider the operator $T : X^* \longrightarrow Y$ defined by: $T(f) = \langle f, x\rangle y$, for all $f \in X^*$. Then $T \in \mathcal{B}(X^*, Y)$ with $\|T\| = \|x\|\|y\|$. We write $x \otimes y$ for T.

Let $X \otimes Y = \text{span}\{x \otimes y : x \in X, y \in Y\}$. Then $X \otimes Y$ is a linear subspace of $\mathcal{B}(X^*, Y)$. One can define many norms on $X \otimes Y$, however, we are interested in the following two norms:

1. The injective norm: For $u = \sum_{i=1}^{n} x_i \otimes y_i \in X \otimes Y$, define

$$\|u\|_\epsilon = \sup_{\|f\|=1} \left\| \sum_{i=1}^{n} \langle f, x_i \rangle y_i \right\|_Y, \qquad f \in X^*.$$

   Then $\|u\|_\epsilon$ is just the operator norm of $u : X^* \longrightarrow Y$. The normed space $(X \otimes Y, \|.\|_\epsilon)$ need not be complete. By $X \otimes_\epsilon Y$ we mean the completion of the normed space $(X \otimes Y, \|.\|_\epsilon)$, and we call it the injective tensor product of X and Y.

2. The projective norm: For $u = \sum_{i=1}^{n} x_i \otimes y_i \in X \otimes Y$, define

$$\|u\|_\pi = \inf \sum_{i=1}^{n} \|x_i\| \|y_i\|,$$

   where the infimum is taken over all representations of u in $X \otimes Y$. Then $(X \otimes Y, \|.\|_\pi)$ is a normed space that need not be complete. Let $X \otimes_\pi Y$ denote the completion of $(X \otimes Y, \|.\|_\pi)$. Then $X \otimes_\pi Y$ is called the projective tensor product of X and Y.

For further information about norms on tensor product of Banach spaces we refer the reader to [Rya13].

Definition 1.1. [Dal00] A Banach algebra $\mathfrak{A}$ is a Banach space with an associative and distributive multiplication such that for all $a, b \in \mathfrak{A}$ and for all $\lambda \in \mathbb{C}$ $\lambda(ab) = (\lambda a)b = a(\lambda b)$ and such that $\|ab\| \leqslant \|a\| \|b\|$.
An involution on a Banach algebra $\mathfrak{A}$ is a continuous map $a \to a^*$ from $\mathfrak{A}$ into $\mathfrak{A}$ such that for all $a, b \in \mathfrak{A}$ and all $\lambda \in \mathbb{C}$

- $(a + b)^* = a^* + b^*$.

- $(\lambda a)^* = \overline{\lambda} a^*$.

- $a^{**} = a$.

- $(ab)^* = b^* a^*$.

If in addition, $\|a^*a\| = \|a\|^2$ for all $a \in \mathfrak{A}$, then $\mathfrak{A}$ is called a C^*-algebra.

Let $\mathfrak{A}$ be a Banach algebra and let $J \subset \mathfrak{A}$ be a closed subspace. If for all $a \in \mathfrak{A}$ and for all $x \in J$: $ax \in J$ and $xa \in J$, then J is called an ideal. A Banach algebra $\mathfrak{A}$ is called unital if there exists an element $\mathbf{1} \in \mathfrak{A}$ such that $a\mathbf{1} = \mathbf{1}a = a$ for all $a \in \mathfrak{A}$. The element $\mathbf{1}$ is called a unit. If $\mathfrak{A}$ is not unital, then we define the unitization of $\mathfrak{A}$, $\mathfrak{A}^\sharp = \mathfrak{A} \times \mathbb{C}$. Under the multiplication $(a, \lambda).(b, \gamma) = (ab + \lambda b + \gamma a, \lambda\gamma)$, $\mathfrak{A}^\sharp$ becomes a unital Banach algebra with unit $\mathbf{1} = (0, 1)$ that contains $\mathfrak{A}$ as a closed ideal.
By a left (right) bounded approximate identity in $\mathfrak{A}$ we mean a bounded net (e_α) such that $\lim\limits_{\alpha} e_\alpha a = a$ ($\lim\limits_{\alpha} ae_\alpha = a$) for each $a \in \mathfrak{A}$. A bounded approximate identity for $\mathfrak{A}$ (BAI) is a bounded net (e_α) which is both left and a right bounded approximate identity. By a homomorphism ϕ between two Banach algebra $\mathfrak{A}$ and $\mathfrak{B}$ we mean an operator $\varphi : \mathfrak{A} \longrightarrow \mathfrak{B}$ such that $\varphi(ab) = \varphi(a)\varphi(b)$ for all $a, b \in \mathfrak{A}$.

Example 1.1. *Let X be a Banach space. Then $\mathcal{B}(X)$ is a unital Banach algebra. Both $\mathcal{K}(X)$ and $\mathcal{SS}(X)$ are closed ideals in $\mathcal{B}(X)$, and $\mathcal{K}(X) \subseteq \mathcal{SS}(X)$.*

We should point out here that a Banach space isomorphism $\psi \in \mathcal{B}(X, Y)$ between two Banach spaces X and Y induces a Banach algebra homomorphism ϕ from the Banach algebra $\mathcal{B}(X)$ onto the Banach algebra $\mathcal{B}(Y)$ defined as follows: $\phi(T) = \psi T \psi^{-1}$.
Some important results regarding the ideals $\mathcal{K}(X)$ and $\mathcal{SS}(X)$ for the classical sequence spaces are the following:

Theorem 1.2 (Pitt's Theorem)**.** *[FHH⁺10, Proposition 4.49] Let* $1 \leq p < r < \infty$ *and let* $X = \ell^r$ *or* c_0*. Then every operator* $T : X \longrightarrow \ell^p$ *is compact.*

On the other hand, we have the following:

Theorem 1.3. *[AK06, Theorem 2.1.9] Let* $1 \leq p \neq r < \infty$*, then every operator* $T : \ell^p \longrightarrow \ell^r$ *is strictly singular.*

Example 1.2. *Let* $\mathfrak{A}$ *be a Banach algebra. Then* $c_0(\mathfrak{A})$ *and* $\ell^p(\mathfrak{A})$*, for* $p \in [1, \infty]$*, are also Banach algebras under point-wise multiplication.*

Theorem 1.4. *[FGM67] Let* $p \in [1, \infty)$ *and* $X = \ell^p$ *or* c_0*. Then* $\mathcal{K}(X)$ *is the only non-trivial proper ideal in* $\mathcal{B}(X)$*.*

For two Banach algebras $\mathfrak{A}$ and $\mathfrak{B}$, the $\mathfrak{A} \otimes_\pi \mathfrak{B}$ is always a Banach algebra under the point-wise multiplication; i.e.,

$$(a \otimes b)(c \otimes d) = ac \otimes bd \qquad (a, c \in \mathfrak{A}, b, d \in \mathfrak{B}).$$

Definition 1.5. [Run02, Section 2.2] If $\mathfrak{A}$ is a Banach algebra, then the corresponding diagonal operator $\Delta : \mathfrak{A} \otimes_\pi \mathfrak{A} \longrightarrow \mathfrak{A}$ is defined through

$$\Delta(a \otimes b) = ab.$$

The following result is a well-known fact in operator theory, the proof of which can be obtained from [Dal00, Corollary 2.9.15].

Theorem 1.6. *Let* $\mathfrak{A}$ *be a Banach algebra with a bounded approximate identity. Then* $\ell^\infty(\mathfrak{A})$ *also has a bounded approximate identity.*

1.2 Amenable Banach Algebras

In this section, we will present different, but equivalent, definitions of amenable Banach algebras that are being used in this thesis, together with some main properties and examples.

Definition 1.7. [Run02, Definition 2.1.1] Let $\mathfrak{A}$ be a Banach algebra and X be Banach space. Then X is called a Banach left $\mathfrak{A}$-module if a mapping $(a, x) \mapsto a.x$ from $\mathfrak{A} \times X$ into X is bilinear and satisfies the following conditions:

1. for all $a, b \in \mathfrak{A}$, and for all $x \in X$: $a.(b.x) = (ab).x$.
2. there exists a positive constant C such that

$$\|a.x\| \leq C\|a\|\|x\|, \qquad (a \in \mathfrak{A}, x \in X).$$

If a mapping $(a, x) \mapsto x.a$ is bilinear and satisfies the conditions:

1. for all $a, b \in \mathfrak{A}$, $\forall x \in X$: $(x.a).b = x.(ab)$.
2. there exists a positive constant C such that

$$\|x.a\| \leq C\|a\|\|x\|, \qquad (a \in \mathfrak{A}, x \in X),$$

then X is called a Banach right $\mathfrak{A}$-module. X is called a Banach $\mathfrak{A}$-bimodule if it is both a left $\mathfrak{A}$-module and a right $\mathfrak{A}$-module. such that

$$a.(x.b) = (a.x).b \quad (a, b \in \mathfrak{A}, x \in X).$$

Let $\mathfrak{A}$ be a Banach algebra and X and Y be Banach $\mathfrak{A}$-bimodules. A homomorphism $\varphi : X \longrightarrow Y$ is said to be a bimodule homomorphism if φ

preserves the module multiplication, i.e.,

$$\varphi(a.x.b) = a.\varphi(x).b \qquad (a, b \in \mathfrak{A}, x \in X).$$

For any Banach algebra $\mathfrak{A}$, $\mathfrak{A} \otimes_\pi \mathfrak{A}$ is a Banach $\mathfrak{A}-$bimodule through

$$a.(b \otimes c) = ab \otimes c \quad \text{and} \quad (\mathrm{b} \otimes \mathrm{c}).\mathrm{a} = \mathrm{b} \otimes \mathrm{ca} \qquad (\mathrm{a}, \mathrm{b}, \mathrm{c} \in \mathfrak{A}).$$

With respect to this module structure and the point-wise multiplication on $\mathfrak{A} \otimes_\pi \mathfrak{A}$, the diagonal operator Δ is a bimodule homomorphism.

Definition 1.8. [Run02, Section 2.1] Let $\mathfrak{A}$ be a Banach algebra and X be a Banach $\mathfrak{A}$-bimodule. A bounded linear map $D : \mathfrak{A} \to X$ is called a derivation if for all $a, b \in \mathfrak{A}$:

$$D(ab) = a.D(b) + D(a).b$$

If in addition, there exists $x \in X$ such that for all $a \in \mathfrak{A}$:

$$D(a) = a.x - x.a,$$

then D is called an inner derivation.

Let $\mathcal{Z}^1(\mathfrak{A}, X) \subseteq \mathcal{B}(\mathfrak{A}, X)$ denote the closed subspace of all derivations from $\mathfrak{A}$ into X, and $\mathcal{B}^1(\mathfrak{A}, X)$ denote the space of all inner derivations in $\mathcal{Z}^1(\mathfrak{A}, X)$. Then $\mathcal{B}^1(\mathfrak{A}, X)$ is a subspace of $\mathcal{Z}^1(\mathfrak{A}, X)$ that need not to be closed.

Definition 1.9. [Run02, Section 2.1] Let $\mathfrak{A}$ be a Banach algebra and X be a Banach $\mathfrak{A}$-bimodule. Then

$$\mathcal{H}^1(\mathfrak{A}, X) = \mathcal{Z}^1(\mathfrak{A}, X)/\mathcal{B}^1(\mathfrak{A}, X)$$

is called the first Hochschild cohomology group of $\mathfrak{A}$ with coefficients in X

It turns out that the module action of $\mathfrak{A}$ on X is preserved by duality, and hence we get:

Theorem 1.10. *[Run02, Exercise 2.1.1] Let $\mathfrak{A}$ be a Banach algebra and X be a Banach $\mathfrak{A}$-bimodule. Then X^* becomes a Banach $\mathfrak{A}$-bimodule through*

$$\langle x, \phi.a \rangle = \langle a.x, \phi \rangle \qquad \textit{and} \qquad \langle x, a.\phi \rangle = \langle x.a, \phi \rangle,$$

where $a \in \mathfrak{A}, x \in X$, and $\phi \in X^*$. In this case, X^* is called a dual Banach bimodule.

Now we are ready to define the amenability and the weak amenability of Banach algebras as defined by B. E. Johnson in [Joh72] and [Joh87].

Definition 1.11. [Run02, Definition 2.1.9 and Definition 4.2.1] A Banach algebra $\mathfrak{A}$ is called amenable if $\mathcal{H}^1(\mathfrak{A}, X^*) = 0$ for every dual Banach $\mathfrak{A}$-bimodule X^*. If $\mathcal{H}^1(\mathfrak{A}, \mathfrak{A}^*) = 0$, then $\mathfrak{A}$ is called weakly amenable.

Example 1.3.

1. *Let G be an amenable locally compact group. Then the Banach algebra $L^1(G)$ is amenable.*

2. *Every C^*-algebra is weakly amenable, but a C^*-algebra is amenable if, and only if, it is nuclear.*

3. *$\mathbb{C}$ is an amenable Banach algebra with the usual product and norm.*

4. *Let $M_n(\mathfrak{A})$ denote the algebra of all $n \times n$ matrices with entries from the Banach algebra $\mathfrak{A}$. Then $M_n = M_n(\mathbb{C})$ is amenable. Moreover, if $\mathfrak{A}$ is amenable, then so is $M_n(\mathfrak{A})$.*

A question naturally arises: from the amenability of a Banach algebra, can we determine whether another Banach algebra is amenable or not? Thankfully in some cases, the answer is, yes we can.

Theorem 1.12. *[Run02, Proposition 2.3.1] Let $\mathfrak{A}$ and $\mathfrak{B}$ be Banach algebras, and let $\varphi : \mathfrak{A} \longrightarrow \mathfrak{B}$ be a continuous homomorphism with a dense range in $\mathfrak{B}$. If $\mathfrak{A}$ is amenable, then so is $\mathfrak{B}$. In particular, if $J \subseteq \mathfrak{A}$ is a closed ideal and $\mathfrak{A}$ is amenable, then $\mathfrak{A}/J$ is amenable.*

An immediate consequence of Theorem 1.12 is the fact that amenability of Banach algebras is preserved by Banach algebras isomorphisms. Unfortunately, the amenability of a Banach algebra $\mathfrak{A}$ is not inherited by arbitrary closed ideals in $\mathfrak{A}$. On the other hand, we have the following:

Theorem 1.13. *[Run02, Theorem 2.3.7] Let $\mathfrak{A}$ be an amenable Banach algebra, and J be a closed ideal of $\mathfrak{A}$. The the following are equivalent:*

(i) J is amenable.

(ii) J has a bounded approximate identity.

(iii) J is weakly complemented.

Theorem 1.14. *[Run02, Theorem 2.3.10] Let $\mathfrak{A}$ be an amenable Banach algebra, and J be a closed ideal of $\mathfrak{A}$ such that both J and $\mathfrak{A}/J$ are amenable. Then $\mathfrak{A}$ is amenable.*

Corollary 1.15. *[Run02, Corollary 2.3.11] A Banach algebra $\mathfrak{A}$ is amenable if, and only if, its unitization $\mathfrak{A}^\sharp$ is amenable.*

Remark 1.16. The hereditary properties of weak amenability are not as nice as those for amenability. For example, while the amenability of $\mathfrak{A}^\sharp$ forces $\mathfrak{A}$ to be amenable, this is not the case for weak amenability as shown by B. E. Johnson and M. White in [JW] in which the authors proved that the augmentation ideal J of the Banach algebra $L^1(SL(2,\mathbb{R}))$ is not weakly amenable but its unitization $J^\sharp$ is weakly amenable. Where for a locally compact group G with a Haar measure h, the augmentation ideal J of the Banach algebra $L^1(G)$ is defined as follows:

$$J = \{f \in L^1(G) \ : \ \int_G f \ dh = 0\}.$$

We now turn our attention to an equivalent characterization for the amenability of Banach algebras.

Theorem 1.17. *[Oza04, Definition 1.2] Let $\mathfrak{A}$ be a unital Banach algebra. Then $\mathfrak{A}$ is amenable if there exists a constant $C > 0$ such that: for any finite set $F \subset \mathfrak{A}$ and $\epsilon > 0$, there exists $T = \sum_{k=1}^{r} a_k \otimes b_k \in \mathfrak{A} \otimes \mathfrak{A}$ such that*

(i) $\Delta(T) = \mathbf{1}$.

(ii) $\|x.T - T.x\|_\pi \leq \epsilon$ *for all* $x \in \mathfrak{F}$.

(iii) $\|T\|_\pi \leq C$.

The following Theorem is an elementary fact, we will use the version from [HR13].

Theorem 1.18 (Cohen's Factorization Theorem)**.** *Let $\mathfrak{A}$ be a Banach algebra with norm $\|.\|$ having a bounded left approximate identity [bounded by d]. If X is a Banach left $\mathfrak{A}$-module with norm $|||.|||$, then $\mathfrak{A}.X = \{a.y : \ a \in \mathfrak{A}, y \in X\}$ is a closed subspace of X. More precisely, let z be an element in the closed linear span S of $\mathfrak{A}.X$, and suppose that $\delta > 0$. Then there exists an element $a \in \mathfrak{A}$ and an element $y \in X$ such that* *(i)* $z = a.y$.

(ii) $\|a\| \leq d$.

(iii) $|||y - z||| \leq \delta$.

One implication of Theorem 1.18 is that for a Banach algebra with a bounded left approximate identity, $\mathfrak{A}^2 = \text{linear span}\{ab : a, b \in \mathfrak{A}\}$ is dense in $\mathfrak{A}$, [HR13, Corollary 32.26]. In fact, a stronger statement is also true:

Theorem 1.19. *[Run02, Exercise 4.2.1(i)] If $\mathfrak{A}$ is weakly amenable, then $\mathfrak{A}^2$ is dense in $\mathfrak{A}$.*

One last property of amenable Banach algebras is the following:

Proposition 1.20. *[Run10a, Lemma 1.2] Let $\mathfrak{A}$ be a unital amenable Banach algebra. Then for any $\epsilon > 0$ and any finite set $F \subset \mathfrak{A}$, there are $a_1, b_1, ..., a_r, b_r \in \mathfrak{A}$ such that the following holds:*

$$\sum_{k=1}^{r} a_k b_k = \mathbf{1} \tag{1.1}$$

$$\left\| \sum_{k=1}^{r} x a_k \otimes b_k - a_k \otimes b_k x \right\|_{\mathfrak{A} \otimes_\pi \mathfrak{A}} < \epsilon \qquad (x \in F). \tag{1.2}$$

1.3 $\mathfrak{L}^p$-Spaces

In [LP68], J. Lindenstrauss and A. Pełczyński introduced a new class of Banach spaces known as the $\mathfrak{L}^p$- spaces. These are spaces which locally look like the ℓ^p-spaces. In this section we recall the definition of these spaces from [LP68] and list some of their properties.

Definition 1.21. [LP68, Definition 3.1] Let $1 \leq p \leq \infty$ and $1 \leq \lambda < \infty$. A Banach space E is said to be a $\mathfrak{L}^p_\lambda$-space if for every finite dimensional

subspace $Y \subseteq E$, there are $n \in \mathbb{N}$ and an n-dimensional subspace $Z \subseteq E$ such that $Y \subseteq Z$ and $d(Z, \ell_n^p) \leq \lambda$, where d is the Banach-Mazur distance.
A Banach space E is said to be an $\mathfrak{L}^p$-space if it is an $\mathfrak{L}_\lambda^p$-space for some $\lambda < \infty$.

Example 1.4. *[LP68]*

- *Let $p \in [1, \infty]$. Then $L^p(\mu)$-spaces, i.e., spaces of p-integrable functions on some measure space, ℓ^p are $\mathfrak{L}^p$-spaces.*
- *If $p \in (1, \infty)$, then $\ell^2 \oplus \ell^p$ and $\ell^p(\ell^2)$ are mutually non-isomorphic $\mathfrak{L}^p$-spaces.*
- *Let K be a compact Hausdorff space. The Banach space $C(K)$ (the space of continuous functions on K) and c_0 are $\mathfrak{L}^\infty$-spaces.*
- *The class of $\mathfrak{L}^2$-spaces coincides with the class of spaces isomorphic to Hilbert spaces.*

In 1972, H. Rosenthal proved the existence of other $\mathfrak{L}^p$-spaces. Rosenthal's result in [Ros70] led to the construction of infinitely many non-isomorphic separable infinite dimensional examples in [Sch75], and eventually to the construction of uncountable many isomorphically distinct separable infinite dimensional $\mathfrak{L}^p$-spaces in [BRS81].

Some properties of the $\mathfrak{L}^p$-spaces are:

Proposition 1.22. *[LR69, Theorem III] Let E be a Banach space and $p \in [1, \infty]$. Then*

(i) E is a $\mathfrak{L}^p$-space if, and only if, its dual E^* is a $\mathfrak{L}^q$-space; where q is the conjugate of p.

(ii) If E is a $\mathfrak{L}^p$-space, then any complemented subspace $M \subseteq E$ which is not isomorphic to a Hilbert space is a $\mathfrak{L}^p$-space. If $p = 1$ or ∞, then M cannot be isomorphic to an infinite dimensional Hilbert space.

(iii) If E is a $\mathfrak{L}^p$-space, then E has the bounded approximation property.

The main properties of $\mathfrak{L}^p$-spaces that turn out to be useful in our study of the amenability of the Banach algebras $\mathcal{B}(E)$ and $\mathcal{SS}(E)$ are the following:

Theorem 1.23. *[LR69, Theorem 1.1] Let $p \in [1, \infty]$ and E be an $\mathfrak{L}^p$-space. Then E is isomorphic to a subspace of $L^p(\mu)$ for some measure μ. Moreover, if $p \in (1, \infty)$, then E is isomorphic to a complemented subspace of $L^p(\mu)$, and if E is separable, then E is isomorphic to a complemented subspace of $L^p[0, 1]$.*

Theorem 1.24. *[LP68, Proposition 7.3] Let $p \in [1, \infty)$ and let E be an infinite dimensional $\mathfrak{L}^p$-space. Then E contains a complemented subspace isomorphic to ℓ^p.*

Theorem 1.25. *[Rya13, Exercise 3.9] Let E and F be $\mathfrak{L}^\infty$-spaces. Then $E \otimes_\epsilon F$ is also an $\mathfrak{L}^\infty$-space.*

Theorem 1.26. *[GJW94, Theorem 6.4] Let $p \in [1, \infty]$ and let E be an $\mathfrak{L}^p$-space. Then $\mathcal{K}(E)$ is amenable.*

A distinguished result regarding the amenability of the Banach algebra $\ell^\infty(\mathcal{K}(E))$ if E is an $\mathfrak{L}^p$-space for some p is due to M.Daws and V.Runde in [DR07], in which the authors proved the following:

Theorem 1.27. *[DR07, Theorem 4.3] Let $p \in [1, \infty]$ and let Λ be an index set. Then one of the following assertions is true:*

- *$\ell^\infty(\Lambda, \mathcal{K}(E))$ is amenable for every infinite dimensional $\mathfrak{L}^p$-space E.*

- $\ell^\infty(\Lambda, \mathcal{K}(E))$ *is not amenable for any infinite dimensional* $\mathfrak{L}^p$*-space* E.

Theorem 1.28. *[Run10a, Theorem 3.2] Let* $p \in (1, \infty)$. *Then* $\ell^\infty(\mathcal{K}(\ell^2 \oplus \ell^p))$ *is not amenable.*

Consequently, one has the following:

Corollary 1.29. *[Run10a, Proposition 4.3] Let* $p \in (1, \infty)$ *and let* E *be an infinite dimensional* $\mathfrak{L}^p$*-space. Then* $\ell^\infty(\mathcal{K}(E))$ *is not amenable.*

1.4 Non-Amenability of $\mathcal{B}(\ell^p)$

In his memoir [Joh72], B. E. Johnson asked:

- Is $\mathcal{B}(X)$ ever amenable for an infinite dimensional Banach space X? [Joh72, 10.4].
- Is $\mathcal{B}(\mathcal{H})$ amenable for an infinite dimensional Hilbert space $\mathcal{H}$? [Joh72, 10.2].

The Hilbert space case was solved shortly afterwards. In [Was76], S. Wassermann showed that a nuclear von Neumann algebra had to be subhomogeneous. This result together with the equivalence between nuclearity and amenability for C^*-algebras means that $\mathcal{B}(\mathcal{H})$ can not be amenable unless $dim\mathcal{H} < \infty$. Ever since, some progress has been done to solve the general Banach space case. In [Rea06], C. J. Read proved the following:

Theorem 1.30. *[Rea06, Theorem 1.1] The Banach algebra* $\mathcal{B}(\ell^1)$ *is not amenable.*

Moreover, he showed that:

Theorem 1.31. *[Rea06, Corollary 5.1] For* $p \in [1, \infty] \backslash \{2\}$ *the Banach algebra* $\ell^\infty - \bigoplus_{n=1}^{\infty} \mathcal{B}(\ell^p_n)$ *is not amenable.*

Read's proof was simplified by Pisier in [Pis04], and eventually, N. Ozawa simplified Pisier's proof even further and provided a proof that simultaneously established the non-amenability of several Banach algebras.

Theorem 1.32. *[Oza04, Theorem 1] The Banach algebra $\mathcal{B}(\ell^p)$ is not amenable for $p \in \{1, 2, \infty\}$, and for any $p \in [1, \infty] \setminus \{2\}$ the Banach algebra $\ell^\infty - \bigoplus_{n=1}^{\infty} \mathcal{B}(\ell_n^p)$ is not amenable.*

Even though it is not explicitly stated in [Oza04], the proof of Theorem 1.32 works also for $\mathcal{B}(c_0)$.

In [DR07], M. Daws and V. Runde investigated the consequences the hypothetical amenability of $\mathcal{B}(\ell^p)$ for $p \in (1, \infty) \setminus \{2\}$ would have, and they proved the following:

Proposition 1.33. *[DR07, Theorem 2.1] Let $p \in [1, \infty)$ and X be a Banach space. Then*

(i) the Banach algebra $\mathcal{B}(\ell^p(X))$ is amenable if, and only if, $\ell^\infty(\mathcal{B}(\ell^p(X)))$ is amenable.

(ii) the Banach algebra $\mathcal{B}(c_0(X))$ is amenable if, and only if, $\ell^\infty(\mathcal{B}(c_0(X)))$ is amenable.

Since $\forall p \in [1, \infty]$, $\mathcal{K}(\ell^p)$ is amenable and hence has a BAI, then by Theorem 1.6, $\ell^\infty(\mathcal{K}(\ell^p))$ also has a BAI. Consequently, the amenability of $\mathcal{B}(\ell^p)$ forces $\ell^\infty(\mathcal{B}(\ell^p))$ to be amenable by Proposition 1.33. Being a closed ideal in $\ell^\infty(\mathcal{B}(\ell^p))$ with a BAI, $\ell^\infty(\mathcal{K}(\ell^p))$ is an amenable Banach algebra in its own. Together with Theorem 1.27 and Theorem 1.28, one can easily see why the following should hold:

Theorem 1.34. *[Run10a, Corollary 4.5] Let $p \in (1, \infty)$ and E be an $\mathfrak{L}^p$-space such that $E \simeq \ell^p(E)$. Then $\mathcal{B}(E)$ is not amenable. In particular, $\mathcal{B}(\ell^p)$ and $\mathcal{B}(L^p[0,1])$ are not amenable.*

Remark 1.35. It is worth pointing out that not every $\mathfrak{L}^p$-space E is isomorphic to $\ell^p(E)$. For instance; for $p \in (1, \infty)$ the Banach space $E = \ell^2 \oplus \ell^p$ is an $\mathfrak{L}^p$-space and so is the Banach space $\ell^p(\ell^2)$. Thus, $E \not\simeq \ell^p(E)$.

Chapter 2

Non-Amenability of $\mathcal{B}(E)$

In 2004, N. Ozawa provided a proof that simultaneously established the non-amenability of some Banach algebras such as $\mathcal{B}(c_0)$ and $\mathcal{B}(\ell^p)$ if $p = \{1, 2, \infty\}$. Later on, V. Runde proved the non-amenability of the Banach algebra $\mathcal{B}(\ell^p)$ for all $p \in (1, \infty)$. In his paper [Run10a], Runde pointed out that Ozawa's proof could be used to establish the non-amenability of the Banach algebra $\mathcal{B}(\ell^p)$, though Runde's result establishes the non-amenability of the Banach algebra $\mathcal{B}(E)$ for the class of $\mathfrak{L}^p$-spaces E such that $E \simeq \ell^p(E)$, where $p \in (1, \infty)$. The purpose of the present chapter is to generalize Runde's result. We first prove the non-amenability of the Banach algebra $\mathcal{B}(E)$ for any $\mathfrak{L}^1$-space ($\mathfrak{L}^\infty$-space), such that $E \simeq \ell^1(E)$, (respectively, $E \simeq c_0(E)$). The non-amenability of the Banach algebra $\mathcal{B}(E)$ for any $\mathfrak{L}^p$-space E such that $E \not\simeq \ell^p(E)$ for any $p \in [1, \infty)$, is shown in Section Two, and in Section Three we present examples of $\mathfrak{L}^\infty$- spaces E for which the Banach algebra $\mathcal{B}(E)$ is, and is not, amenable.

2.1 Non-Amenability of $\ell^\infty(\mathcal{K}(\ell^p))$

In this section we will prove the non-amenability of the Banach algebras $\ell^\infty(\mathcal{K}(\ell^1))$ and $\ell^\infty(\mathcal{K}(c_0))$ by adopting Ozawa's proof in [Oza04] which mainly uses Lemma 2.3 below.

Before we start the proof, we should point out that an important role in Ozawa's proof has been played by the fact that the group $SL(3,\mathbb{Z})$ is finitely generated, which is due to a property called Kazhdan's property, or property (T).

Recall that a topological group G is a group G endowed with a topology such that the multiplication map $(x,y) \to xy : G \times G \longrightarrow G$ and inversion are continuous. By a unitary representation $(\pi, \mathcal{H})$ of a topological group G we mean a group homomorphism $\pi : G \longrightarrow \mathcal{U}(\mathcal{H})$ such that for all $\zeta \in \mathcal{H}$, the map $g \mapsto \pi(g)\zeta$ is continuous, where $\mathcal{U}(\mathcal{H})$ is the group of all unitary operators on $\mathcal{H}$. For a subset $Q \subset G$ and $\epsilon > 0$, π is said to have a (Q,ϵ)-invariant vector if there exists $\zeta \in \mathcal{H}$ such that $\sup_{x \in Q} \|\pi(x)\zeta - \zeta\| \leq \epsilon \|\zeta\|$.

Definition 2.1. A topological group G has Kazhdan's property, or property (T), if there are a compact subset $Q \subset G$ and an $\epsilon > 0$ such that every unitary representation $(\pi, \mathcal{H})$ of G which has a (Q,ϵ)-invariant vector also has a non-zero invariant vector.

Proposition 2.2. *The group $SL(n,\mathbb{Z})$ of all $n \times n$ matrices with determinant equal to 1 and with entries in $\mathbb{Z}$ has property (T) for all $n \geq 3$, and hence $SL(n,\mathbb{Z})$ is finitely generated.*

For more on Kazhdan's property we refer to [BdLHV08].

Let $\wp$ be the set of all primes and fix $\rho \in \wp$. We write $\mathbb{Z}_\rho$ for the finite field

$\mathbb{Z}/\rho\mathbb{Z}$ and define an equivalence relationship on $\mathbb{Z}_\rho^3$ as follows: for any non-zero points $x, y \in \mathbb{Z}_\rho^3$, x is equivalent to y if there is $\lambda \in \mathbb{Z}_\rho$ such that $y = \lambda x$. The resulting set of equivalence classes is called the *projective plane over* $\mathbb{Z}_\rho$ and will be denoted by $\mathbb{P}_\rho$. Consider now the group $SL(3,\mathbb{Z})$ that acts on $\mathbb{Z}_\rho^3$ through matrix multiplication, which induces an action of $SL(3,\mathbb{Z})$ on $\mathbb{P}_\rho$. Consequently, one has a unitary representation

$$\pi_\rho : SL(3,\mathbb{Z}) \longrightarrow \mathcal{B}(\ell^2_{|\mathbb{P}_\rho|}).$$

Choose a subset S_ρ of $\mathbb{P}_\rho$ such that $|S_\rho| = \dfrac{|\mathbb{P}_\rho| - 1}{2}$ and define an invertible isometry $v_\rho \in \mathcal{B}(\ell^2_{|\mathbb{P}_\rho|}) \setminus \{\pi_\rho(SL(3,\mathbb{Z}))\}$ through

$$v_\rho(e_{\mathbf{x}}) = \begin{cases} e_{\mathbf{x}} & ; \mathbf{x} \in S_\rho \\ -e_{\mathbf{x}} & ; \mathbf{x} \notin S_\rho, \end{cases}$$

where $(e_{\mathbf{x}})$ is the standard basis for $\ell^2_{|\mathbb{P}_\rho|}$. Since the group $SL(3,\mathbb{Z})$ is finitely generated with generators say, $x_1, ..., x_m$, we will write $\pi(x_{m+1})$ for v_ρ.

We can now state Ozawa's Lemma as formulated in [Run10a].

Lemma 2.3. *[Run10a, Ozawa's Lemma] It is impossible to find, for each $\epsilon > 0$, a number $r \in \mathbb{N}$ with the following property:*
for each $\rho \in \wp$, there are $\zeta_{1,\rho}, \eta_{1,\rho}, ..., \zeta_{r,\rho}, \eta_{r,\rho} \in \ell^2_{|\mathbb{P}_\rho|}$ such that $\sum_{k=1}^{r} \zeta_{k,\rho} \otimes \eta_{k,\rho} \neq 0$

and for any $j \in \{1, ..., m+1\}$

$$\| \sum_{k=1}^{r} \zeta_{k,\rho} \otimes \eta_{k,\rho} - (\pi_\rho(x_j) \otimes \pi_\rho(x_j)(\zeta_{k,\rho} \otimes \eta_{k,\rho}) \|_{\ell^2_{|\mathbb{P}_\rho|} \otimes_\pi \ell^2_{|\mathbb{P}_\rho|}} \leq \epsilon \| \sum_{k=1}^{r} \zeta_{k,\rho} \otimes \eta_{k,\rho} \|_{\ell^2_{|\mathbb{P}_\rho|} \otimes_\pi \ell^2_{|\mathbb{P}_\rho|}}$$

To prove the main result, we require two more lemmas.

Lemma 2.4. *[Oza04, Lemma 2.1] Let $p \in \{1, 2, \infty\}$ and q be the conjugate of p, and let (δ_n) and (δ^*_n) be the canonical unit vectors in ℓ^p and ℓ^q respectively. Then for any $N \in \mathbb{N}$ and any operators $S \in \mathcal{B}(\ell^p, \ell^p_N), R \in \mathcal{B}(\ell^q, \ell^q_N)$ the following inequality holds:*

$$\sum_{n=1}^{\infty} \|S(\delta_n)\|_{\ell^2_N} \|R(\delta^*_n)\|_{\ell^2_N} \leqslant N \|S\| \|R\|.$$

Now, let $\mathbb{P}$ denote the disjoint union of $\{\mathbb{P}_\rho \, ; \, \rho \in \wp\}$. Then, one can make the following identification:

$$\ell^1 = \ell^1(\mathbb{P}) = \ell^1 - \bigoplus_{\rho \in \wp} \ell^1_{|\mathbb{P}_\rho|}, \qquad (1)$$

and

$$c_0 = c_0(\mathbb{P}) = c_0 - \bigoplus_{\rho \in \wp} \ell^\infty_{|\mathbb{P}_\rho|}, \qquad (2).$$

Lemma 2.5. *Let $X = \ell^1$ (or $X = c_0$). Then $\ell^\infty - \bigoplus_{\rho \in \wp} \mathcal{B}(\ell^p_{|\mathbb{P}_\rho|}) \subseteq \ell^\infty(\mathcal{K}(X))$ for $p = 1$ (respectively $p = \infty$).*

Proof. We only prove the case $X = \ell^1$.
For each $\rho \in \wp$, let $\mathbf{P}_\rho : \ell^1 \longrightarrow \ell^1_{|\mathbb{P}_\rho|}$ be the projection onto the first $|\mathbb{P}_\rho|-$coordinates, and let $\mathbb{U}_\rho = \{\mathbf{P}_\rho T \mathbf{P}_\rho \, ; \, T \in \mathcal{K}(\ell^1)\}$. Then, each $\mathbb{U}_\rho$ is isomorphic to $\mathcal{B}(\ell^p_{|\mathbb{P}_\rho|})$.

Hence, for each $\rho \in \wp$, $\mathcal{B}(\ell^p_{|\mathbb{P}_\rho|}) \subseteq \mathcal{K}(\ell^1)$. Consequently, $\ell^\infty - \oplus_{\rho\in\wp}\mathcal{B}(\ell^p_{|\mathbb{P}_\rho|}) \subseteq \ell^\infty(\mathcal{K}(\ell^1))$. ☐

Remark 2.6.

- Let $X = \ell^1$ (or $X = c_0$). Then the Banach algebra $\ell^\infty(\mathcal{B}(X))$ can be embedded into $\mathcal{B}(X)$.
- The Banach spaces $\ell^2_{|\mathbb{P}_\rho|}$ and $\ell^1_{|\mathbb{P}_\rho|}$ are finite dimensional and hence are isomorphic.

Now, we are ready to prove the main result:

Theorem 2.7. *Let* $X = \ell^1$ *(or* $X = c_0$*). Then the Banach algebra* $\ell^\infty(\mathcal{K}(X))$ *is not amenable.*

Proof. We will only prove the case when $X = \ell^1$, the case $X = c_0$ follows analogously.
For the sake of contradiction, assume that $\ell^\infty(\mathcal{K}(\ell^1))$ is amenable, and so is $\ell^\infty(\mathcal{K}(\ell^1))^\sharp$.
Let $F = \{(\pi_\rho(x_j))_{\rho\in\wp}\ ; j \in \{1, ..., m+1\}\} \subseteq \ell^\infty - \oplus_{\rho\in\wp}\mathcal{B}(\ell^p_{|\mathbb{P}_\rho|}) \subseteq \ell^\infty(\mathcal{K}(\ell^1))$.
Let $\epsilon > 0$, and obtain, by Proposition 1.20, $a_1, b_1, ..., a_r, b_r \in \ell^\infty(\mathcal{K}(\ell^1))^\sharp$ such that $T = \sum_{k=1}^{r} a_k \otimes b_k \in \ell^\infty(\mathcal{K}(\ell^1))^\sharp \otimes_\pi \ell^\infty(\mathcal{K}(\ell^1))^\sharp$ satisfies the following conditions:

$$\Delta(T) = \mathbf{1} \tag{2.1}$$

$$\|(\pi_\rho(x_j)).T - T.(\pi_\rho(x_j))\|_\pi < \frac{\epsilon}{m+1} \qquad (j = 1, ..., m+1). \tag{2.2}$$

Let $\mathbf{P}_\rho$ be as in Lemma 2.5, and define

$$T_\rho(\mathbf{x}) = \sum_{k=1}^{r} \mathbf{P}_\rho a_k e_{\mathbf{x}} \otimes \mathbf{P}^*_\rho b^*_k e^*_{\mathbf{x}} \in \ell^2_{|\mathbb{P}_\rho|} \otimes_\pi \ell^2_{|\mathbb{P}_\rho|}.$$

Then

$$\begin{aligned}\sum_{x\in\mathbb{P}}\|T_\rho(\mathbf{x})\|_{\ell^2_{|\mathbb{P}_\rho|}\otimes_\pi\ell^2_{|\mathbb{P}_\rho|}} &\geq \Big|\sum_{\mathbf{x}\in\mathbb{P}}\sum_{k=1}^{r}\langle \mathbf{P}_\rho a_k e_{\mathbf{x}}, \mathbf{P}_\rho^* b_k^* e_{\mathbf{x}}^*\rangle\Big| \\ &= \Big|\sum_{\mathbf{x}\in\mathbb{P}}\sum_{k=1}^{r}\langle b_k \mathbf{P}_\rho^2 a_k e_{\mathbf{x}}, e_{\mathbf{x}}^*\rangle\Big| \\ &= Tr(\sum_{k=1}^{r} b_k\mathbf{P}_\rho a_k) = Tr(\sum_{k=1}^{r} a_k b_k \mathbf{P}_\rho) \\ &= Tr(\mathbf{P}_\rho) = |\mathbb{P}_\rho|. \end{aligned} \tag{2.3}$$

On the other hand, letting $S = \mathbf{P}_\rho a_k$ and $R = \mathbf{P}_\rho^* b_k^*$ and applying Lemma 2.4, we get

$$\sum_{\mathbf{x}\in\mathbb{P}}\|\mathbf{P}_\rho a_k e_{\mathbf{x}}\|\|\mathbf{P}_\rho^* b_k^* e_{\mathbf{x}}^*\| \leqslant |\mathbb{P}_\rho|\|\mathbf{P}_\rho a_k\|\|\mathbf{P}_\rho^* b_k^*\| \leq |\mathbb{P}_\rho|\|a_k\|\|b_k\|$$

and hence,

$$\sum_{\mathbf{x}\in\mathbb{P}}\sum_{k=1}^{r}\|\mathbf{P}_\rho a_k e_{\mathbf{x}}\|\|\mathbf{P}_\rho^* b_k^* e_{\mathbf{x}}^*\| \leq |\mathbb{P}_\rho|\sum_{k=1}^{r}\|a_k\|\|b_k\|.$$

Thus,

$$\sum_{x\in\mathbb{P}}\|T_\rho(\mathbf{x})\|_{\ell^2_{|\mathbb{P}_\rho|}\otimes_\pi\ell^2_{|\mathbb{P}_\rho|}} \leqslant |\mathbb{P}_\rho|\|T\|.$$

Similarly,

$$
\begin{aligned}
\sum_{\mathbf{x}\in\mathbb{P}} &\|T_\rho(\mathbf{x}) - (\pi_\rho(x_j)\otimes_\pi \pi_\rho(x_j))T_\rho(\mathbf{x})\|_{\ell^2_{|\mathbb{P}_\rho|}\otimes_\pi \ell^2_{|\mathbb{P}_\rho|}} \\
&\leqslant |\mathbb{P}_\rho|\|T - (\pi_\rho(x_j)).T.(\pi_\rho^{-1}(x_j))\|\|(\pi_\rho(x_j))\| \\
&\leqslant |\mathbb{P}_\rho|\|T.(\pi_\rho(x_j)) - (\pi_\rho(x_j)).T\| \qquad \text{(Since each } \pi_\rho(x_j) \text{ is an isometry)} \\
&\leqslant \frac{\epsilon}{m+1}|\mathbb{P}_\rho| \qquad \text{(by (2.2) above).}
\end{aligned}
$$

Thus

$$
\sum_{\mathbf{x}\in\mathbb{P}}\sum_{j=1}^{m+1} \|T_\rho(\mathbf{x}) - (\pi_\rho(x_j)\otimes_\pi \pi_\rho(x_j))T_\rho(\mathbf{x})\|_{\ell^2_{|\mathbb{P}_\rho|}\otimes_\pi \ell^2_{|\mathbb{P}_\rho|}} \leqslant \epsilon|\mathbb{P}_\rho| \tag{2.4}
$$

Combining (2.3) and (2.4) together, we get that there must be an $\hat{x}\in\mathbb{P}$ such that

$$
\sum_{j=1}^{m+1} \|T_\rho(\hat{\mathbf{x}}) - (\pi_\rho(x_j)\otimes_\pi \pi_\rho(x_j))T_\rho(\hat{\mathbf{x}})\|_{\ell^2_{|\mathbb{P}_\rho|}\otimes_\pi \ell^2_{|\mathbb{P}_\rho|}} \leqslant \epsilon\|T_\rho(\hat{\mathbf{x}})\|_{\ell^2_{|\mathbb{P}_\rho|}\otimes_\pi \ell^2_{|\mathbb{P}_\rho|}}
$$

and thus, the following inequality holds for any $j\in\{1,..,m+1\}$

$$
\|T_\rho(\hat{\mathbf{x}}) - (\pi_\rho(x_j)\otimes_\pi \pi_\rho(x_j))T_\rho(\hat{\mathbf{x}})\|_{\ell^2_{|\mathbb{P}_\rho|}\otimes_\pi \ell^2_{|\mathbb{P}_\rho|}} \leqslant \epsilon\|T_\rho(\hat{\mathbf{x}})\|_{\ell^2_{|\mathbb{P}_\rho|}\otimes_\pi \ell^2_{|\mathbb{P}_\rho|}}
$$

which contradicts Lemma 2.3 by letting $\zeta_{k,\rho} = \mathbf{P}_\rho a_k e_{\mathbf{x}}$ and $\eta_{k,\rho} = \mathbf{P}^*_\rho b^*_k e^*_{\mathbf{x}}$. Hence, $\ell^\infty(\mathcal{K}(\ell^1))$ is not amenable.

□

Since the amenability of $\ell^\infty(\mathcal{K}(\ell^1))$ (respectively, $\ell^\infty(\mathcal{K}(c_0))$) is equivalent to the amenability of $\ell^\infty(\mathcal{K}(E))$ for any $\mathfrak{L}^1$-space E (respectively, any $\mathfrak{L}^\infty$-space E), we obtain:

Corollary 2.8. *Let $p\in[1,\infty]$ and let E be an $\mathfrak{L}^p$-space. Then $\ell^\infty(\mathcal{K}(E))$ is not amenable.*

Proof. Since the case $p \in (1, \infty)$ had already been stated in Corollary 1.29, we only need to prove the case $p = 1, \infty$. So let $p = 1$ or ∞ and let E be an $\mathfrak{L}^p$-space. Then by Theorem 1.27, $\ell^\infty(\mathcal{K}(E))$ is amenable if, and only if, $\ell^\infty(\mathcal{K}(\ell^p))$ is amenable. Since $\ell^\infty(\mathcal{K}(\ell^p))$ is not amenable, by Theorem 2.7, then $\ell^\infty(\mathcal{K}(E))$ is not amenable. $\square$

We recall that for any infinite dimensional $\mathfrak{L}^p$-space E such that $E \simeq \ell^p(E)$, the Banach algebra $\mathcal{B}(E)$ is not amenable for any $p \in (1, \infty)$ as shown by V. Runde (Theorem 1.34). We now generalize this result:

Theorem 2.9. *Let $p \in [1, \infty)$ and let E be an infinite dimensional $\mathfrak{L}^p$-space (respectively, an infinite dimensional $\mathfrak{L}^\infty-$space) such that $E \simeq \ell^p(E)$ (respectively, $E \simeq c_0(E)$). Then $\mathcal{B}(E)$ is not amenable.*

Proof. Let E be as above and assume towards a contradiction that $\mathcal{B}(E)$ is amenable. Then, by Proposition 1.33 , $\ell^\infty(\mathcal{B}(E))$ is also amenable. Since $\ell^\infty(\mathcal{K}(E))$ is a closed ideal in $\ell^\infty(\mathcal{B}(E))$ with a bounded approximate identity, the amenability of $\ell^\infty(\mathcal{B}(E))$ forces $\ell^\infty(\mathcal{K}(E))$ to be amenable, which contradicts Corollary 2.8. Hence, $\mathcal{B}(E)$ is not amenable. $\square$

2.2 Non-Amenability of $\mathcal{B}(E)$

In this section, E is assumed to be an infinite dimensional $\mathfrak{L}^p$-space such that $E \not\simeq \ell^p(E)$ with $p \in [1, \infty)$. We recall first, by Theorem 1.24, that E contains a complemented copy of ℓ^p. Thus, $E = X \oplus \ell^p$, where X is either a Hilbert space or an $\mathfrak{L}^p$-space. Consequently, $\mathcal{B}(E)$ has a matrix-like structure:

$$\mathcal{B}(E) = \begin{pmatrix} \mathcal{B}(X) & \mathcal{B}(\ell^p, X) \\ \mathcal{B}(X, \ell^p) & \mathcal{B}(\ell^p) \end{pmatrix}.$$

In order to prove the non amenability of $\mathcal{B}(E)$, we will need the following theorem. Theorem 2.10 below is due to Grønbæk, Willis and Johnson ([GJW94]), though the version we include here is the one mentioned in [DR07].

Theorem 2.10. *[DR07, Theorem 1.2] Let $\mathfrak{A}$ be a Banach algebra and let $P_1 \in \mathcal{M}(\mathfrak{A})$ be a projection. Let $P_2 = I_{\mathfrak{A}} - P_1$. Suppose further that the diagonal map Δ maps $P_2\mathfrak{A}P_1 \otimes_\pi P_1\mathfrak{A}P_2$ onto $P_2\mathfrak{A}P_2$. Then $\mathfrak{A}$ is amenable if, and only if, $P_1\mathfrak{A}P_1$ is amenable.*

The notion $\mathcal{M}(\mathfrak{A})$ stands for the multiplier algebra of $\mathfrak{A}$. A double multiplier on a Banach algebra $\mathfrak{A}$ is a pair of operators, (L, R), on $\mathfrak{A}$ which satisfies: for all a and $b \in \mathfrak{A}$:

$$L(ab) = L(a)b; \quad R(ab) = aR(b); \quad \text{and} \quad aL(b) = R(a)b.$$

The set of all double multipliers on $\mathfrak{A}$ is denoted by $\mathcal{M}(\mathfrak{A})$. Then $\mathcal{M}(\mathfrak{A})$ is a Banach space under the norm $\|(L, R)\| = \max\{\|L\|, \|R\|\}$, which becomes a Banach algebra if endowed with the product $(L_1, R_1)(L_2, R_2) = (L_1L_2, R_2R_1)$. If $\mathfrak{A}$ is unital, then $\mathcal{M}(\mathfrak{A}) = \mathfrak{A}$. For more information about the multiplier algebra we refer the reader to [Pal94, Sec. 1.2].

Now, if X is a Banach space such that $X = Y \oplus Z$, and by setting P_1 to be the projection onto Y and P_2 to be the projection onto Z, Theorem 2.10 can be stated as follows:

Proposition 2.11. *Let X be an infinite dimensional Banach space such that $X = Y \oplus Z$ for some infinite dimensional subspaces Y and Z. Suppose further that every operator on Z factors through Y, i.e., for all $T \in \mathcal{B}(Z)$, there are operators $R \in \mathcal{B}(Y, Z)$ and $S \in \mathcal{B}(Z, Y)$ such that $T = RS$. Then $\mathcal{B}(X)$ is amenable if, and only if, $\mathcal{B}(Y)$ is amenable.*

We will first assume that X is a Hilbert space, and hence $p \in (1, \infty)$. In order to prove the non amenability of $\mathcal{B}(E)$ in this case, the following long-known fact in functional analysis will be needed. We will provide the proof for the reader's convenience.

Proposition 2.12. *Let $\mathcal{H}$ be an infinite dimensional Hilbert space and let $p \in [1, \infty) \setminus \{2\}$. Then $\mathcal{B}(\mathcal{H}, \ell^p) = \mathcal{K}(\mathcal{H}, \ell^p)$ if $p < 2$, and $\mathcal{K}(\mathcal{H}, \ell^p) \subsetneq \mathcal{B}(\mathcal{H}, \ell^p)$ if $p \geq 2$.*

Proof. Assume first that $p < 2$ and let $T \in \mathcal{B}(\mathcal{H}, \ell^p)$. We want to prove that the sequence $(T(x_n)) \subset \ell^p$ has a convergent subsequence for any bounded sequence $(x_n) \subset \mathcal{H}$.

So let $(x_n) \subset \mathcal{H}$ be a bounded sequence and let $Y = [x_n] \subseteq \mathcal{H}$ be the closed linear span of $\{x_1, x_2, x_3, ...\}$. Then $Y \simeq Z \subseteq \ell^2$, being a separable subspace of $\mathcal{H}$. Consider the operator $\tilde{T} = T_{|_Y} : Y \longrightarrow \ell^p$. Then $\tilde{T}$ can be viewed as a bounded operator $\tilde{T} : \ell^2 \longrightarrow \ell^p$. Since $p < 2$, then by Pitt's Theorem, $\tilde{T}$ is compact, and hence $T(x_n)$ has a convergent subsequence in ℓ^p. Since (x_n) is arbitrary, we get that $T \in \mathcal{K}(\mathcal{H}, \ell^p)$.

Now, let $p > 2$ and assume towards a contradiction that $\mathcal{K}(\mathcal{H}, \ell^p) = \mathcal{B}(\mathcal{H}, \ell^p)$.

Then, and since the conjugate of any compact operator is again a compact operator, we get $\mathcal{K}(\ell^q, \mathcal{H}) = \mathcal{B}(\ell^q, \mathcal{H})$, where $q < 2$ is the conjugate of p. Which contradicts the fact that the inclusion map $i : \ell^q \longrightarrow \ell^2$ is not compact. Hence, $\mathcal{K}(\mathcal{H}, \ell^p) \subsetneq \mathcal{B}(\mathcal{H}, \ell^p)$ if $p > 2$. If $p = 2$, then $\mathcal{H}$ contains a complemented subspace isomorphic to ℓ^2. Let $P : \mathcal{H} \longrightarrow \ell^2$ be the projection operator onto ℓ^2. Then $P \in \mathcal{B}(\mathcal{H}, \ell^2)/\mathcal{K}(\mathcal{H}, \ell^2)$. □

In fact, Proposition 2.12, establishes the non-amenability of the Banach algebra $\mathcal{B}(\mathcal{H} \oplus \ell^p)$. This is due to the following result by Daws and Runde in [DR07]. We will provide here the version included in [Run10a].

Lemma 2.13. *[Run, Theorem 3.4.3] Let E and F be Banach spaces such that $\mathcal{B}(E, F) = \mathcal{K}(E, F)$ and $\mathcal{K}(F, E) \subsetneq \mathcal{B}(F, E)$. Then $\mathcal{B}(E \oplus F)$ is not amenable.*

In particular, we have:

Corollary 2.14. *Let $p \in (1, \infty) \setminus \{2\}$, and let E be an infinite dimensional $\mathfrak{L}^p$-space such that $E = \mathcal{H} \oplus \ell^p$ for some Hilbert space $\mathcal{H}$. Then $\mathcal{B}(E)$ is not amenable.*

Proof. Let E be an infinite dimensional $\mathfrak{L}^p$-space such that $E = \mathcal{H} \oplus \ell^p$ for some Hilbert space $\mathcal{H}$. Then

$$\mathcal{B}(E) = \begin{pmatrix} \mathcal{B}(\mathcal{H}) & \mathcal{B}(\ell^p, \mathcal{H}) \\ \mathcal{B}(\mathcal{H}, \ell^P) & \mathcal{B}(\ell^p) \end{pmatrix}.$$

If $\mathcal{H}$ is finite dimensional, then by combining Theorem 1.34 and Proposition 2.11, we conclude that the Banach algebra $\mathcal{B}(E)$ is not amenable. So assume that $dim\mathcal{H} = \infty$.

If $p \in (2, \infty)$, then by the separability of ℓ^p and by Pitt's Theorem, we get

$$\mathcal{K}(E) = \begin{pmatrix} \mathcal{K}(\mathcal{H}) & \mathcal{K}(\ell^p, \mathcal{H}) \\ \mathcal{K}(\mathcal{H}, \ell^p) & \mathcal{K}(\ell^p) \end{pmatrix} = \begin{pmatrix} \mathcal{K}(\mathcal{H}) & \mathcal{B}(\ell^p, \mathcal{H}) \\ \mathcal{K}(\mathcal{H}, \ell^p) & \mathcal{K}(\ell^p) \end{pmatrix}.$$

Hence, the Calkin algebra $\mathcal{C}(E)$ has the form

$$\mathcal{C}(E) = \mathcal{B}(E)/\mathcal{K}(E) = \begin{pmatrix} \mathcal{C}(\mathcal{H}) & 0 \\ J & \mathcal{C}(\ell^p) \end{pmatrix} = \begin{pmatrix} \mathcal{C}(\mathcal{H}) & 0 \\ 0 & \mathcal{C}(\ell^p) \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 \\ J & 0 \end{pmatrix},$$

where $J = \mathcal{B}(\mathcal{H}, \ell^p)/\mathcal{K}(\mathcal{H}, \ell^p)$. Then, by Proposition 2.12, $J \neq 0$. Assume now that $\mathcal{B}(E)$ is amenable, then so is the Calkin algebra $\mathcal{C}(E)$. Consequently, the complemented closed ideal

$$\tilde{J} = \begin{pmatrix} 0 & 0 \\ J & 0 \end{pmatrix}$$

is amenable, which contradicts Cohen's Factorization Theorem (Theorem 1.18) since $\tilde{J}$ is nilpotent. Hence, $\mathcal{B}(E)$ is not amenable.

Similarly, if $p \in (1, 2)$, then by Proposition 2.12, we get

$$\mathcal{K}(E) = \begin{pmatrix} \mathcal{K}(\mathcal{H}) & \mathcal{K}(\ell^p, \mathcal{H}) \\ \mathcal{K}(\mathcal{H}, \ell^p) & \mathcal{K}(\ell^p) \end{pmatrix} = \begin{pmatrix} \mathcal{K}(\mathcal{H}) & \mathcal{K}(\ell^p, \mathcal{H}) \\ \mathcal{B}(\mathcal{H}, \ell^p) & \mathcal{K}(\ell^p) \end{pmatrix}.$$

Consequently, the amenability of $\mathcal{B}(E)$ forces the Calkin algebra

$$\mathcal{C}(E) = \begin{pmatrix} \mathcal{C}(\mathcal{H}) & 0 \\ 0 & \mathcal{C}(\ell^p) \end{pmatrix} \oplus \begin{pmatrix} 0 & L \\ 0 & 0 \end{pmatrix}$$

to be amenable, which forces the nilpotent complemented ideal

$$\tilde{L} = \begin{pmatrix} 0 & L \\ 0 & 0 \end{pmatrix}$$

to be amenable, where $L = \mathcal{B}(\ell^p, \mathcal{H})/\mathcal{K}(\ell^p, \mathcal{H})$. Which is again a contradiction. Hence, $\mathcal{B}(E)$ is not amenable. □

Finally, to prove our main result the following will be needed:
For an infinite dimensional $\mathfrak{L}^p$-space $E = X \oplus \ell^p$, let $P_1, P_2 \in \mathcal{B}(E)$ be the projections onto X and ℓ^p respectively. For each $S \otimes R \in \mathcal{B}(E) \otimes \mathcal{B}(E)$ define the product $(P_1 \otimes P_2)(S \otimes R) = P_1 S \otimes R P_2$ and $(S \otimes R)(P_1 \otimes P_2) = SP_1 \otimes P_2 R$.

Lemma 2.15. *Let* $S, R \in \mathcal{B}(E)$ *and let* $P_1 \in \mathcal{B}(E)$ *be the projection on* X. *If* $d = \sum_{i=1}^{n} A_i \otimes B_i \in \mathcal{B}(X) \otimes \mathcal{B}(X)$ *such that* $\Delta(d) = I_X$, *then*

$$\Delta((S \otimes R)d) = \Delta((S \otimes R)(P_1 \otimes P_1))$$

Proof. Since $P_1(X) = I_X$, one has

$$\begin{aligned} \Delta((S \otimes R)d) &= \Delta((S \otimes R)(P_1 \otimes P_1)d) \\ &= \sum_{i=1}^{n} SP_1 A_i B_i P_1 R \\ &= SP_1^2 R = SP_1 R \\ &= \Delta((S \otimes R)(P_1 \otimes P_1)) \end{aligned}$$

□

Now, we are ready to prove our main result:

Theorem 2.16. *Let $p \in [1, \infty)$, and let E be an infinite dimensional $\mathfrak{L}^p$-space. Then $\mathcal{B}(E)$ is not amenable.*

Proof. We only need to prove the non-amenability of $\mathcal{B}(E)$ if $E = X \oplus \ell^p$, where X is an $\mathfrak{L}^p-$space such that $X \not\simeq \ell^p(X)$.

Note first that if X is finite dimensional, then $X \simeq \ell_n^p$ for some $n \in \mathbb{N}$. Hence, $X \overset{c}{\hookrightarrow} \ell^p$ and consequently, every operator on X factors through ℓ^p. Thus, by combining Theorem 1.34 and Proposition 2.11, $\mathcal{B}(E)$ is not amenable. So assume from now on that X is an infinite dimensional $\mathfrak{L}^p$-space such that $X \not\simeq \ell^p(X)$, and let $\lambda > 0$ be such that $\|P_2\| \leq \lambda$.

Assume towards a contradiction that $\mathcal{B}(E)$ is amenable. Since X is also an $\mathfrak{L}^p$-space, then X also contains a complemented copy of ℓ^p. Hence, every operator $T \in \mathcal{B}(\ell^p)$ factors through X. Consequently, and by Proposition 2.11, $\mathcal{B}(X)$ is also amenable.

Since both $\mathcal{B}(E)$ and $\mathcal{B}(X)$ are amenable, then there are positive constants C_1, C_2 such that for any $\epsilon_1 > 0$ and any $\epsilon_2 > 0$ and for any finite sets $F_1 \subseteq \mathcal{B}(E)$ and $F_2 \subseteq \mathcal{B}(X)$, there are $d_1 \in \mathcal{B}(E) \otimes \mathcal{B}(E)$ and $d_2 \in \mathcal{B}(X) \otimes \mathcal{B}(X)$ such that the following conditions are satisfied:

1. $\Delta(d_1) = I_E \quad$ and $\quad \Delta(d_2) = I_X$.

2. $\|(I_E \otimes A - A \otimes I_E)d_1\| < \epsilon_1 \qquad (\forall A \in F_1)$,

   and

   $\|(I_X \otimes B - B \otimes I_X)d_2\| < \epsilon_2 \qquad (\forall B \in F_2)$.

3. $\|d_1\| \leq C_1$ and $\|d_2\| \leq C_2$.

Let $C_3 = \lambda^2 C_1 (C_2 + \lambda)$, and let $\epsilon > 0$.

For any finite set $F = \{T_i : i = 1, 2, \cdots, n\} \subset \mathcal{B}(\ell^p)$, let

$\hat{F}_1 = \left\{ \hat{T}_{1i} = \begin{bmatrix} 0 & 0 \\ 0 & T_i \end{bmatrix} : i = 1, 2, \cdots, n \right\} \subset \mathcal{B}(E)$ be the embedding of F into $\mathcal{B}(E)$. Similarly, let $\hat{F}_2$ be the embedding of F into $\mathcal{B}(X)$. Since $\mathcal{B}(E)$ and $\mathcal{B}(X)$ are amenable, find $d_{\hat{F}_1} \in \mathcal{B}(E) \otimes \mathcal{B}(E)$ and $d_{\hat{F}_2} \in \mathcal{B}(X) \otimes \mathcal{B}(X)$ such that for any $A \in \hat{F}_1$,

$$\|(I_E \otimes A - A \otimes I_E) d_{\hat{F}_1}\| < \frac{\epsilon}{\lambda^2 (C_2 + \lambda^2)},$$

and define

$$d_F = (P_2 \otimes P_2) d_{\hat{F}_1} (d_{\hat{F}_2} + P_2 \otimes P_2).$$

Then

$$\|d_F\| \leq \|P_2^2\| \, \|d_{\hat{F}_1}\| \, (\|d_{\hat{F}_2}\| + \|P_2^2\|) \leq \lambda^2 C_1 (C_2 + \lambda) = C_3.$$

Letting $d = d_{\hat{F}_2}$ in Lemma 2.15, we get

$$\begin{aligned} \Delta(d_F) &= \Delta((P_2 \otimes P_2) d_{\hat{F}_1} (d_{\hat{F}_2} + P_2 \otimes P_2)) \\ &= P_2 \, \Delta(d_{\hat{F}_1} (d_{\hat{F}_2} + P_2 \otimes P_2)) P_2 \\ &= P_2 \, \Delta(d_{\hat{F}_1} (P_1 \otimes P_1 + P_2 \otimes P_2)) P_2 \\ &= P_2 I_E P_2 = P_2^2 = P_2 = I_{\ell^p}. \end{aligned}$$

Finally, let $T \in F$. Then

$$\begin{aligned}
\|(I_E \otimes T - T \otimes I_E)d_F\| &= \|(I_E \otimes T - T \otimes I_E)(P_2 \otimes P_2)d_{\hat{F}_1}(d_{\hat{F}_2} + P_2 \otimes P_2))\| \\
&= \|(P_2 \otimes P_2)(I_E \otimes T - T \otimes I_E)d_{\hat{F}_1}(d_{\hat{F}_2} + P_2 \otimes P_2))\| \\
&\leq \|P_2\|^2 \|(I_E \otimes T - T \otimes I_E)d_{\hat{F}_1}\| \|(d_{\hat{F}_2} + P_2 \otimes P_2))\| \\
&< \frac{\lambda^2 \epsilon}{\lambda^2(C_2 + \lambda^2)}(C_2 + \lambda^2) < \epsilon.
\end{aligned}$$

Equivalently, $\mathcal{B}(\ell^p)$ is amenable, which contradicts Theorem 1.30 and Theorem 1.34. Hence, $\mathcal{B}(E)$ is not amenable. □

Remark 2.17. In [AF96], A. Arias and J. Farmer proved that for a separable $\mathfrak{L}^p$-space E with $p \in (1, \infty)$, the Banach algebra $\mathcal{B}(E) \simeq \mathcal{B}(\ell^p)$. So the non-amenability of $\mathcal{B}(E)$ for such an $\mathfrak{L}^p$-space E follows from Theorem 1.12 and Theorem 1.34. Though, our proofs in Theorem 2.16 and Corollary 2.14 work for any $\mathfrak{L}^p$-space E.

2.3 On the non-Amenability of $\mathcal{B}(E)$ for an $\mathfrak{L}^\infty$-space E

We dedicate this section to study the amenability of $\mathcal{B}(E)$ if E is an $\mathfrak{L}^\infty$-space. We will present some $\mathfrak{L}^\infty$-spaces for which the Banach algebra of bounded operators is, and is not, amenable.

Definition 2.18. An infinite dimensional Banach space X is said to have few operators if every operator $T \in \mathcal{B}(X)$ is strictly singular perturbation of the identity operator, that is T is expressible as $\mathcal{S} + \lambda I$ for some strictly singular operator $\mathcal{S}$ and some $\lambda \in \mathbb{C}$. If every operator $T \in \mathcal{B}(X)$ is a compact perturbation of the identity, then X is said to have a very few operators.

The existence of a Banach space with few operators was first proved by Gowers and Maurey [GM93]. A Banach space with very few operators exists due to S. Argyros and R. Haydon in 2009, and a Banach space with few but not very few operators was constructed by M. Tarbard in 2012. Interestingly enough, all three of these spaces are hereditarily indecomposable (HI) $\mathfrak{L}^\infty$-spaces. In fact, it has been shown in [Fer97] that a complex Banach space X is HI if, and only if, every operator from a subspace of X into X is a strictly singular perturbation of a multiple of the identity. We recall that a Banach space X is said to be hereditarily indecomposable (HI) if no infinite dimensional subspace of X can be decomposed into a direct sum of further two infinite dimensional subspaces.

Theorem 2.19. *[GM93] There exists a hereditarily indecomposable infinite dimensional, reflexive, separable real Banach space E_{GM} such that every operator T on a subspace Y of E_{GM} is a strictly singular perturbation of a multiple of the identity operator.*

Theorem 2.20. *[AH11] There exists a hereditarily indecomposable separable $\mathfrak{L}^\infty$-space E_{AH} with $E_{AH}^* \simeq \ell^1$ such that*

$$\mathcal{B}(E_{AH}) = \mathcal{K}(E_{AH}) \oplus \mathbb{C}_I.$$

Theorem 2.21. *[Tar12] Given any $k \in \mathbb{N}, k \geq 2$, there is a hereditarily indecomposable separable $\mathfrak{L}^\infty$-Banach space E_{M_k} with $E_{M_k}^* \simeq \ell^1$ such that:*

(i) There is a strictly singular non-compact operator $\mathcal{S} \in \mathcal{B}(E_{M_k})$ such that $\mathcal{S}$ is nilpotent of degree k, i.e., $\mathcal{S}^j \neq 0, 1 \leq j \leq k-1$ and $\mathcal{S}^k = 0$.

(ii) The set $\{\mathcal{S}^j;\ j \in \{1, ..., k-1\}\}$ is linearly independent in the Calkin algebra $\mathcal{C}(E_{M_k})$.

(iii) Every $T \in \mathcal{B}(E_{M_k})$ can be uniquely represented as $\sum_{j=0}^{k-1} \lambda_j \mathcal{S}^j + K$, where $\lambda_j \in \mathbb{R}$ and $K \in \mathcal{K}(E_{M_k})$.

We will start by presenting an $\mathfrak{L}^\infty$-space E for which $\mathcal{B}(E)$ is amenable. Consider the Banach space E_{AH} constructed by Argyros and Haydon. V. Runde in his expository paper [Run10b] pointed out that any finite direct sum of the Banach space E_{AH} provides a positive answer to the question raised in [Joh72, Question 10.4].

Theorem 2.22. *Let E_{AH} be the Banach space of Argyros and Haydon. Then $\mathcal{B}(E_{AH})$ is amenable. Moreover, for each $n \in \mathbb{N}$, the Banach algebra $\mathcal{B}(E_n)$ is amenable, where $E_n = \bigoplus_{i=1}^{n} E_{AH}$.*

Proof. Since $\mathcal{B}(E_{AH}) = \mathcal{K}(E_{AH}) \oplus \mathbb{C}_I$, it is enough to prove the amenability of $\mathcal{K}(E_{AH})$. Since E_{AH} is an $\mathfrak{L}^\infty$-space, then by Theorem 1.26 , $\mathcal{K}(E_{AH})$ is amenable, and hence by Theorem 1.14, $\mathcal{B}(E_{AH})$ is also amenable. Moreover, since $\mathcal{K}(E_n) \cong M_n(\mathcal{K}(E_{AH}))$, then $\mathcal{K}(E_n)$ is amenable. As $\mathcal{B}(E_n) = \mathcal{K}(E_n) \oplus M_n$, and M_n is amenable, then $\mathcal{B}(E_n)$ is also amenable. □

A natural question arises here: what happens if we take an infinite direct sum of the space E_{AH}? For instance, if $E = c_0 - \bigoplus_{i=1}^{\infty} E_{AH} = c_0(E_{AH})$, is $\mathcal{B}(E)$ still amenable?
In order to answer this question, the following proposition will be needed.

Proposition 2.23. *[Rya13, Example 3.3] For any Banach space X, $c_0 \otimes_\epsilon X \simeq c_0(X)$*

In particular, one gets that $c_0 \otimes_\epsilon c_0 = c_0(c_0) = c_0$, and hence we prove the following:

Theorem 2.24. *Let E_{AH} be the space of Argyros and Haydon, and let $E = c_0(E_{AH})$. Then $\mathcal{B}(E)$ is not amenable.*

Proof. It is enough to prove that E is an $\mathfrak{L}^\infty$-space such that $E \simeq c_0(E)$, as for such Banach spaces Theorem 2.9 establishes the non-amenability of $\mathcal{B}(E)$. Notice first that since $E = c_0(E_{AH})$, then by Proposition 2.23, $E \simeq c_0 \otimes_\epsilon E_{AH}$. Thus by Theorem 1.25, E is a $\mathfrak{L}^\infty$-space. Moreover,

$$c_0(E) \simeq c_0 \otimes_\epsilon E \simeq c_0 \otimes_\epsilon c_0 \otimes_\epsilon E_{AH}.$$

But $c_0 \otimes_\epsilon c_0 \simeq c_0$. Hence, $E \simeq c_0(E)$. □

Similarly, one can prove that $\mathcal{B}(C[0,1])$ is not amenable. To prove this, recall first that the Cantor set $\Omega = \{0,1\}^{\mathbb{N}}$, the topological space that is identified as the countable product of the two-point set $\{0,1\}$, endowed with the product topology. The following is a well-known fact about the space of all real valued, continuous functions on Ω.

Proposition 2.25. *[AK06, Proposition 4.4.5]* $C(\Omega) \simeq c_0(C(\Omega))$.

We will also need the following result by A. Miljutin in [Mil66].

Proposition 2.26 (Miljutin's Theorem)**.** *Suppose* K *is an uncountable compact metric space. Then* $C(K) \simeq C([0,1])$.

Now, we prove

Theorem 2.27. *The Banach algebra* $\mathcal{B}(C(K))$ *is not amenable for any uncountable compact metric space* K*. In particular,* $\mathcal{B}(C[0,1])$ *is not amenable.*

Proof. Let K is an uncountable compact metric space. Then $C(K) \simeq c_0(C(K))$ since $C(K) \simeq C(\Omega)$. Since, by Miljutin's Theorem, $C[0,1] \simeq C(K)$, we conclude that $C[0,1] \simeq c_0(C[0,1])$. Hence, by Theorem 2.9, both $\mathcal{B}(C[0,1])$ and $\mathcal{B}(C(K))$ are not amenable. □

Finally, we will prove the non-amenability of the Banach algebra $\mathcal{B}(E_{M_k})$. To do that, the following lemma is needed.

Lemma 2.28 ([Tar12])**.** *For each* $k \in \mathbb{N}, k \geq 2$*, the Calkin algebra* $\mathcal{C}(E_{M_k})$ *is isomorphic, as a Banach algebra, to the subalgebra* $A \subset M_k$ *of all* $k \times k$ *upper triangular Toeplitz matrices, i.e., the subalgebra* A *generated by*

$$\left\{ \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & 0 \end{pmatrix}^j : 0 \leq j \leq k-1 \right\}$$

Theorem 2.29. *For each* $k \in \mathbb{N}, k \geq 2$*, the Banach algebra* $\mathcal{B}(E_{M_k})$ *is not amenable.*

Proof. It is enough to prove the non-amenability of the Calkin algebra $\mathcal{C}(E_{M_k})$. Since $\mathcal{C}(E_{M_k}) \simeq A$, then it is enough to prove the non-amenability of A.
Since A is the algebra of all $k \times k$ upper triangular Toeplitz matrices, then every matrix $M \in A$ can be expressed as $M = \alpha_0 I_k + \alpha_1 B + ... + \alpha_{k-1} B^{k-1}$, where

$$B = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & 0 \end{pmatrix}.$$

Now, assume towards a contradiction that A is amenable.
Then every complemented ideal is amenable by Theorem 1.13.
Consider now the ideal $J = [B^{k-1}] \subset A$, that is the ideal generated by the operator B^{k-1}. Then J is a nilpotent ideal of degree two. Clearly, J is a closed complemented ideal, and hence by Theorem 1.13, J is amenable, which contradicts Cohen's Factorization theorem. Thus, A , and consequently $\mathcal{C}(E_{M_k})$, is not amenable. □

Chapter 3

On the Weak Amenability of $\mathcal{SS}(X)$

The class of strictly singular operators was first introduced by T. Kato in [Kat58] as an extension of compact operators and in connection with the theory of Fredholm operators. In general, strictly singular operators behave in a different way compared to the way the compact operators do. For instance, in general strictly singular operators are not stable under duality whereas an operator $T \in \mathcal{B}(X,Y)$ is compact if, and only if, $T^* \in \mathcal{B}(Y^*, X^*)$ is compact. However, an operator $T \in \mathcal{B}(L^p[0,1])$ is strictly singular if, and only if, $T^* \in \mathcal{B}(L^q[0,1])$ is strictly singular ([Mil70], [Wei77]).

A Banach space X is called simple if $\mathcal{K}(X)$ is the only non-trivial closed ideal in $\mathcal{B}(X)$, in that case $\mathcal{SS}(X) = \mathcal{K}(X)$. The Banach spaces that are known to be simple so far are $\ell^p, p \in [1,\infty)$, c_0 and the Banach space E_{AH}. On the other hand, it has been shown that $\mathcal{K}(X) \subsetneq \mathcal{SS}(X)$ if $X = L^p[0,1]$ or $X = C[0,1]$ and hence, $L^p[0,1]$ and $C[0,1]$ are not simple, [FGM67]. Moreover, V. Milman proved that the product of any two strictly singular operators on $L^p[0,1]$ or

$C[0,1]$ is in fact a compact operator, [Mil70]. In this chapter, we will first generalize Milman's results for any separable $\mathfrak{L}^p$-space E, with $p \in (1,\infty)$, and then for such $\mathfrak{L}^p$-spaces, the non-amenability of the Banach algebra $\mathcal{SS}(E)$ will be shown in Section Two.

3.1 The Product of Strictly Singular Operators

We start this section with the following fact:

Theorem 3.1. *[FGM67] Let $p \in [1,\infty) \setminus \{2\}$ and let $X = L^p[0,1]$. Then $\mathcal{K}(X) \subsetneq \mathcal{SS}(X)$.*

Later on, V. Milman proved the following:

Theorem 3.2. *[Mil70] Let $p \in [1,\infty)$ and let $X = L^p[0,1]$ or $X = C[0,1]$. Then $TS \in \mathcal{K}(X)$ for all $T, S \in \mathcal{SS}(X)$.*

In order to prove that for any separable $\mathfrak{L}^p$-space such that $E \not\simeq \ell^p$, the Banach algebra $\mathcal{SS}(E)$ enjoys that same properties, the following lemma from [JO74] is needed:

Lemma 3.3. *[JO74, Corollary 1] Suppose that E is a separable $\mathfrak{L}^p$-space $(1 < p < \infty, p \neq 2)$ such that no subspace of E is isomorphic to ℓ^2. Then $E \simeq \ell^p$.*

Now we prove:

Theorem 3.4. *Let $p \in (1,\infty) \setminus \{2\}$ and let E be a separable $\mathfrak{L}^p$-space such that $E \not\simeq \ell^p$. Then $\mathcal{K}(E) \subsetneq \mathcal{SS}(E)$.*

Proof. Let p and E be as above. Then by Lemma 3.3, there exists a subspace $Z \subsetneq E$ such that $Z \simeq \ell^2$. Let $J : \ell^2 \longrightarrow Z$ be the isomorphism. Moreover, by Theorem 1.24, E contains a complemented copy of ℓ^p. Set P to be the projection $P : E \longrightarrow \ell^p$. Now, for $p < 2$, let $i_{p,2} : \ell^p \longrightarrow \ell^2$ be the inclusion map. Then by Theorem 1.3, the map $i_{p,2}$ is a non-compact strictly singular operator. Consequently, the map $T : E \longrightarrow E$

$$T = J\, i_{p,2}\, P$$

is a non-compact strictly singular operator on E.

Now, let $p > 2$ and let $i_{2,p} : \ell^2 \longrightarrow \ell^p$ and $j_p : \ell^p \longrightarrow E$ be the embedding of ℓ^2 into ℓ^p and the embedding of ℓ^p into E respectively. Then the map $S : E \longrightarrow E$

$$S = j_p\, i_{2,p}\, J^{-1}$$

is a non-compact strictly singular operator on E since the map $i_{2,p}$ is a non-compact strictly singular operator. $\square$

Theorem 3.5. *Let* $p \in (1, \infty)$ *and let* E *be a separable* $\mathfrak{L}^p$*-space. Then* $TS \in \mathcal{K}(E)$ *for all* $T, S \in \mathcal{SS}(E)$.

Proof. Let E be as above. Then by Theorem 1.23, E is isomorphic to a complemented subspace of $L^p[0,1]$. Thus, there exists a subspace $Y \subseteq L^p[0,1]$ such that $L^p[0,1] = Y \oplus E$. Consequently,

$$\mathcal{B}(L^p[0,1]) = \begin{pmatrix} \mathcal{B}(Y) & \mathcal{B}(E,Y) \\ \mathcal{B}(Y,E) & \mathcal{B}(E) \end{pmatrix}.$$

Now, let $T, S \in \mathcal{SS}(E)$. Clearly, the operators $T = \begin{pmatrix} 0 & 0 \\ 0 & T \end{pmatrix}$ and $S = \begin{pmatrix} 0 & 0 \\ 0 & S \end{pmatrix} \in \mathcal{SS}(L^p[0,1])$. Consequently, by Theorem 3.2,

$$TS = \begin{pmatrix} 0 & 0 \\ 0 & TS \end{pmatrix} \in \mathcal{K}(L^p[0,1]) = \begin{pmatrix} \mathcal{K}(Y) & \mathcal{K}(E,Y) \\ \mathcal{K}(Y,E) & \mathcal{K}(E) \end{pmatrix}.$$

Thus, $TS \in \mathcal{K}(E)$. Since T and S were arbitrary, we get $\mathcal{SS}^2(E) \subseteq \mathcal{K}(E)$. $\square$

3.2 The Non-Amenability of $\mathcal{SS}(X)$

In this section, we will prove that $\mathcal{SS}(X)$ is not weakly amenable and hence not amenable for a specific class of Banach spaces.

Theorem 3.6. *Let X be an infinite dimensional Banach space such that $\mathcal{K}(X) \subsetneq \mathcal{SS}(X)$ and such that $TS \in \mathcal{K}(X)$ for all $T, S \in \mathcal{SS}(X)$. Then $\mathcal{SS}(X)$ is not weakly amenable.*

Proof. Let X be as above and assume towards a contradiction that $\mathcal{SS}(X)$ is weakly amenable. Then by Theorem 1.19, $\mathcal{SS}^2(X)$ is dense in $\mathcal{SS}(X)$. Thus, we get

$$\mathcal{SS}(X) = \overline{\mathcal{SS}^2(X)} \subseteq \mathcal{K}(X) \subsetneq \mathcal{SS}(X),$$

which is a contradiction. Hence, $\mathcal{SS}(X)$ is not weakly amenable. □

Remark 3.7. Using the same argument as in the proof of Theorem 3.6, one can show that $\mathcal{SS}(X)$ does not have a BAI whenever X is a Banach space that satisfies the conditions of Theorem 3.6.

Corollary 3.8. *Let X be an infinite dimensional Banach space such that $\mathcal{K}(X) \subsetneq \mathcal{SS}(X)$ and such that $TS \in \mathcal{K}(X)$ for all $T, S \in \mathcal{SS}(X)$. Then $\mathcal{SS}(X)$ is not amenable.*

Proof. Let X be as above and assume that $\mathcal{SS}(X)$ is amenable. Then $\mathcal{SS}(X)$ is weakly amenable which contradicts Theorem 3.6. □

In particular, we have

Corollary 3.9. *Let $p \in (1, \infty)$ and let E be an infinite dimensional separable $\mathfrak{L}^p$-space such that $E \not\simeq \ell^p$, $E = L^1[0,1]$ or $E = C[0,1]$. Then $\mathcal{SS}(E)$ is not weakly amenable.*

Remark 3.10. A Banach algebra $\mathfrak{A}$ is called self-induced $\mathfrak{A} \otimes_{\mathfrak{A}} \mathfrak{A} \simeq \mathfrak{A}$, where $\mathfrak{A} \otimes_{\mathfrak{A}} \mathfrak{A} = \mathfrak{A} \otimes_{\lambda} \mathfrak{A}/N$, where $N = \overline{\text{span}}\{a.x \otimes y - x \otimes y.a\,;\ a, x, y \in \mathfrak{A}\}$. The concept of self-induced Banach algebras was introduced in [Grø95] as a generalization of unital Banach algebras. It has been shown that every unital Banach algebra is self-induced, and in fact every Banach algebra with a BAI is self-induced. Another characterization for self-inducedness for Banach algebra is the following: a Banach algebra $\mathfrak{A}$ is self-induced if, and only if, $\mathfrak{A}^2$ is dense in $\mathfrak{A}$ and every balanced bilinear map $\phi : \mathfrak{A} \times \mathfrak{A} \longrightarrow \mathbb{C}$ is of the form $\phi(a, b) = f(ab)$ for some $f \in \mathfrak{A}^*$. The proof of Theorem 3.6 also means that $\mathcal{SS}(L^p[0, 1])$ for instance is not self-induced, even though both $\mathcal{B}(L^p[0, 1])$ and $\mathcal{K}(L^p[0, 1])$ are self-induced.

Chapter 4

Remarks and Open Problems

In this chapter, we make some remarks on our work and also some natural questions related to some results obtained in this thesis.

4.1 Remarks On Chapter 2 and Related Problems

The problem of characterizing the amenability of the Banach algebra $\ell^\infty(\mathfrak{A})$ in terms of properties of the underlying Banach algebra $\mathfrak{A}$ is still open. In [SG02], F. Sánchez and R. García claimed that under certain conditions, the Banach algebra $\mathfrak{A}^{**}$ can be considered as a quotient of $\ell^\infty(\mathfrak{A})$; hence, if $\mathfrak{A}^{**}$ is not amenable, then neither is $\ell^\infty(\mathfrak{A})$. Unfortunately, M. Daws and V. Runde pointed out a mistake in the proof of Sánchez and García, and so far it is not known whether Sánchez and García's claim is true or not. As we focus on the Banach algebra of all operators on infinite dimensional Banach spaces, we consider the Banach algebra $\ell^\infty(\mathcal{B}(X))$, which is not amenable whenever $\mathcal{B}(X)$ is not amenable. In fact, even when $\mathcal{B}(E_{AH})$ is amenable, the Banach algebra

$\ell^\infty(\mathcal{B}(E_{AH}))$ turns out not to be amenable. This follows from the fact that $\ell^\infty(\mathcal{B}(E_{AH})) = \ell^\infty(\mathcal{K}(E_{AH})) \oplus \ell^\infty$ and that the Banach algebra $\ell^\infty(\mathcal{K}(E_{AH}))$ is not amenable by Corollary 2.8.

Conjecture 1. For a Banach space X, the Banach algebra $\ell^\infty(\mathcal{B}(X))$ is amenable if, and only if, X is finite dimensional.

On the other hand, we think that the Banach algebra $\ell^\infty(\mathcal{K}(E))$ is weakly amenable for any $\mathfrak{L}^p$-space, $p \in [1, \infty]$, but we do not have a proof for that.

Open Problem 1. Let $p \in [1, \infty] \setminus \{2\}$, and let E be an infinite dimensional $\mathfrak{L}^p$-space. Is $\ell^\infty(\mathcal{K}(E))$ weakly amenable? In particular, is $\ell^\infty(\mathcal{K}(\ell^p))$ weakly amenable?

In his expository paper, V. Runde concluded his paper with the following question, [Run10b, Question 3]:

Question. Is there an infinite dimensional Banach space X such that $\mathcal{C}(X)$ is amenable and infinite dimensional?

We doubt that there is a positive answer for this question. As we saw in Section 2.3, even finite dimensional Calkin algebras need not be amenable. We think that the amenability of the Banach algebra $\mathcal{B}(X)$ forces any quotient of $\mathcal{B}(X)$ to be finite dimensional, so we are wondering whether the following conjecture is true or not:

Conjecture 2. For an infinite dimensional Banach space X, if $\mathcal{B}(X)$ is amenable, then the Calkin algebra $\mathcal{C}(X)$ is finite dimensional. Moreover, if $J \subseteq \mathcal{B}(X)$ is a closed ideal, and $\mathcal{B}(X)$ is amenable, then the quotient algebra $\mathcal{B}(X)/J$ is finite dimensional.

Unfortunately, the Banach space E_{AH} is simple, and hence we cannot test whether this claim is true or not using this Banach space. On the other hand, since the Banach algebra $\mathcal{B}(E_{GM})$ is amenable if, and only if, the Banach algebra $\mathcal{SS}(E_{GM})$ is amenable, then investigating the amenability of the Banach algebra $\mathcal{SS}(E_{GM})$ for the Banach space of Gowers and Maurey will shed light whether our claim is correct or not.

Open Problem 2. Is the Banach algebra $\mathcal{SS}(E_{GM})$ amenable?

Using Theorem 1.13, if $\mathcal{SS}(E_{GM})$ is amenable, then the closed ideal $\mathcal{K}(E_{GM})$ is amenable if, and only if, it has a BAI. It is not known whether $\mathcal{K}(E_{GM})$ is amenable or not.

Another approach to investigating the amenability of the $\mathcal{SS}(E_{GM})$ is to study whether it has a BAI or not.

Finally, in Section 2.3, we proved that the Banach algebra $\mathcal{B}(X)$ is not amenable for the classical $\mathfrak{L}^\infty$-spaces, and we provided examples of HI-spaces for which the Banach algebra of all operators is, and is not amenable. So we ask:

Open Problem 3. For which $\mathfrak{L}^\infty$-spaces E are the Banach algebra $\mathcal{B}(E)$ amenable?

4.2 Remarks on Chapter 3 and Related Problems

Definition 4.1. Let Y be a Banach space and let $A \subset Y$ be a subset. Then A is called spaceable if $A \cup \{0\}$ contains an infinite dimensional closed subspace.

Spaceability of subsets in vector spaces was first introduced in [AGS05]. If Y is taken to be the Banach algebra of all operators on a Banach space X, and A is taken to be the quotient algebra of two closed ideals in $\mathcal{B}(X)$, then the spaceability of A becomes a tool to measure how large is the quotient algebra A. In [HRS15], Hernández et al. investigated the spaceability of the set $A = \mathcal{I}_1(X, Y)/\mathcal{I}_2(X, Y)$ for some operators ideals $\mathcal{I}_2(X, Y) \subseteq \mathcal{I}_1(X, Y)$, and gave sufficient conditions on the Banach spaces X and Y to obtain the spaceability of A. Among other results, the authors proved the following:

Theorem 4.2. *[HRS15] Let X be a Banach space such that $X \simeq \ell^p(X)$ for some $p \in [1,\infty)$ or $X \simeq c_0(X)$. Then the Banach algebra $\mathcal{SS}(X)/\mathcal{K}(X)$ is spaceable if and only $\mathcal{SS}(X)/\mathcal{K}(X) \neq \emptyset$.*

The fact that $\mathcal{SS}(X)/\mathcal{K}(X) \neq \emptyset$ plays a role in proving the non-amenability of the Banach algebra $\mathcal{SS}(X)$ if $X = L^p[0,1]$ or $X = C[0,1]$. Since the spaceability of the quotient algebra $\mathcal{SS}(X)/\mathcal{K}(X)$ is equivalent to the existence of a non-compact strictly singular operator on those Banach spaces, we ask the following question:

Open Problem 4. Let $p \in [1,\infty)$ and let E be an infinite dimensional $\mathfrak{L}^p$-space such that $E \not\simeq \ell^p(E)$. Is $\mathcal{SS}(X)/\mathcal{K}(X)$ spaceable?

Keeping in mind that the fact the product of strictly singular operators is compact on $X = L^p[0,1]$ or $X = C[0,1]$ also plays a role in the proof of Theorem 3.6 , we are wondering if Milman's result can be generalized to general $\mathfrak{L}^p$-spaces.

Open Problem 5. Let $p \in [1,\infty \setminus \{2\}$ and let E be an infinite dimensional $\mathfrak{L}^p$-space such that $\mathcal{SS}(X)/\mathcal{K}(X) \neq \emptyset$. Is the product of any two strictly singular operators compact?

We do not know if the spaceability of the quotient algebra $\mathfrak{A}/J$ is sufficient for the non-amenability of the Banach algebra $\mathfrak{A}$, so we generalize Conjecture two:

Conjecture 3. Let $\mathfrak{A}$ be an infinite dimensional Banach algebra, and let $J \subsetneq \mathfrak{A}$ be a non-trivial closed ideal. Then if $\mathfrak{A}$ is amenable, then $\mathfrak{A}/J$ is not spaceable.

We do not know whether the converse of Conjecture 3 is true.

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