THE UNIVERSITY OF ALBERTA

Relation Between Procedural Skill and Understanding in Multiplication

by

(C) Eliza Krzanowska

A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES AND RESEARCH

IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE

OF Master of Science

Department of Psychology

EDMONTON, ALBERTA

Spring, 1988

ISBN 0-315-42887-2

THE UNIVERSITY OF ALBERTA

RELEASE FORM

NAME OF AUTHOR Eliza Krzanowska

TITLE OF THESIS Relation Between Procedural Skill and Understanding in Multiplication

DEGREE FOR WHICH THESIS WAS PRESENTED Master of Science

YEAR THIS DEGREE GRANTED Spring, 1988

(SIGNED) Eliza Krzanowska

PERMANENT ADDRESS:

220 Hillington Court

Edmonton, AB, T5R 5X5

DATED 24.03 1988

THE UNIVERSITY OF ALBERTA

FACULTY OF GRADUATE STUDIES AND RESEARCH

The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled Relation Between Procedural Skill and Understanding in Multiplication submitted by Eliza Krzanowska in partial fulfilment of the requirements for the degree of Master of Science.

....
Supervisor

....

....

Date 24.03.1988

Abstract

The basic issue addressed in this study was the relation between skill and understanding in multiplication. Of primary interest was the hypothesis that automatization of the algorithmic procedure is related to greater operational and verbal understanding of multiplication. An additional objective was to examine possible differences in verbal and operational aspects of understanding principles of multiplication.

The study was carried out with grade 4 and grade 6 children and consisted of three tasks. Task 1 was designed to assess the level of automatization of multi-digit multiplication algorithm. Automatization was assessed in terms of speed of several types of multiplication problems that differed in complexity of the solution algorithm. Task 2 tested operational understanding of multiplication. The measure of operational understanding was the ability to employ short-cut strategies or heuristics that were based on conceptual knowledge about the selected principles of multiplication. In Task 3 children's verbal understanding was assessed in terms of explicit, verbal explanation of the principles of multiplication.

With respect to the relation between skill and understanding, results were supportive of the automatization hypothesis for grade 6, in which speed of the algorithm was related to greater ability to use and explain principles of multiplication. The hypothesis was not confirmed, however, for grade 4. This pattern of results may indicate that some critical level of computational skill is necessary for the relation with conceptual knowledge to be established.

With respect to the relation between operational and verbal understanding, children in both grades were much more advanced in explaining principles of multiplication than in using them in problem solving. This pattern of performance is characterized as a type of production deficiency, that is, an inability to produce most efficient strategies despite the presence of relevant knowledge.

Table of Contents

List of Tables

I. Introduction

A. The Purpose and Overview of the Study

Conceptual and procedural knowledge often have been described as if they were somewhat incompatible and unrelated. In the present study an attempt was made to investigate the relation between procedural and conceptual knowledge in the acquisition of mathematical competence. Procedural knowledge is defined here as mechanical knowledge of facts and algorithms. Conceptual knowledge or understanding is equated with explicit or implicit knowledge of concepts and principles. The primary purpose of the study was to provide an explanation for how these two types of knowledge might interact in the process of attaining multiplication skill. Of particular interest was the hypothesis that automatization of procedural skills promotes (a) the use of certain conceptually based short-cut strategies (heuristics) that reflect understanding of the number system and multiplication, and (b) children's ability to explain these strategies. An additional objective was to compare children's performance on two measures of understanding multiplication, active use of short-cut strategies versus verbal explanation of the underlying principles. The purpose of this comparison was to examine possible differences in verbal and operational understanding of principles of multiplication. In the following sections I first present the rationale for studying multi-digit multiplication. Next a brief review is presented of research on multiplication and the relation between procedural and conceptual knowledge. Of particular interest are current models of cognitive development in which the relation between procedural and conceptual knowledge is described in terms of processing resources and increasing automatization of basic procedures. Finally, proposed tests of the relation between procedural and conceptual knowledge in multiplication are described.

B. Rationale for Studying Multiplication

The concern with multiplication was motivated by the significance of this operation for both instruction and for psychological research. With respect to instruction, mastery of multiplication is a very important step in learning elementary school mathematics. It is a necessary prerequisite for learning division and fractions, operations that follow multiplication in school curricula. Multi-digit multiplication is also a good indicator of previous learning because it is not possible without proficiency in addition. An additional educational consideration for studying multiplication is that, for a large number of children, this operation is one of the most serious and persistent sources of difficulties in learning arithmetic. Deficits in multiplication have also been found to be the most common problem of math-disabled children (Cohn, 1968; Ginsburg, 1977; McLeod, 1982; Ross, 1964; Weinstein, 1978).

With respect to psychological research, multiplication appears to be a particularly informative domain for identifying developmental changes in cognitive processes and strategies. Being an example of a complex, multi-step problem-solving process, multi-digit multiplication is a rich domain for studying the nature and development of memory retrieval, for investigating the allocation of processing resources among competing demands, for analyzing children's understanding of problem-solving procedures, and for studying the relation between knowledge of the domain and the procedures in complex multi-step tasks.

C. Research on Multiplication

Until quite recently it has been traditional in mathematics education and psychology to treat conceptual and procedural knowledge as distinct domains, each having a separate and independent course of development. For example, in the area of multiplication a growing amount of research has been directed toward understanding the development of computational skill. According to extant research (Campbell & Graham, 1985; Miller, Perlmutter, & Keating, 1984; Parkman, 1972; Siegler, 1987; Stazyk, Ashcraft, & Hamann, 1981) knowledge of single-digit multiplication is represented in memory as a network of problem-answer associations. Retrieval of an answer from this network is accomplished through the spread of activation, or, if the association is not sufficiently strong, through some backup strategy. The model posits that learning the basic combinations entails gradually strengthening problem answer associations by means of extensive practice. One aspect of multiplication knowledge not addressed by the model is the relation between the acquisition of computational skill and understanding of underlying principles of multiplication. Moreover, some researchers explicitly suggested that acquisition of single digit multiplication proceeds independent of whatever knowledge of multiplication principles children have (Siegler, 1987).

Consistent with this view are also the accounts of more complex computational behavior, e.g., of multi digit computational skill. For example, Brown and Van Lehn (1980, 1982) presented a model of how children solve multi digit computational tasks. According to this model, children's behavior on arithmetic computational tasks can be described in purely syntactic (procedural) terms. Problems are solved by retrieving and applying the subsequent steps of appropriate, previously memorized algorithms. The model postulates that, when the algorithmic procedure cannot be retrieved or is obviously distorted, a child will attempt to "repair" it with self invented procedures. The suitability and correctness of these procedures are believed to be checked against a set of rules that are derived from knowledge about the algorithmic procedure itself. These rules specify certain constraints that the algorithm must obey, such as acting at least once on each column, not writing more than one digit in one column, and so on. All these rules are essentially syntactic ones. No rules are included that are based on conceptual knowledge about the operation. Analogous to accounts of single-digit computation, the problem of possible relations between algorithmic skill and understanding of the domain was not even addressed.

Is development of procedural knowledge related at all to the acquisition of the related concepts? If it is, what is the basis of this relationship? Do concepts and procedures gradually build upon each other or do they initially develop in isolation and interconnect at some later point in development? How is this connection or link established? Some insights into these issues have been offerred recently by cognitive models that describe acquisition of knowledge in the framework of modern attentional theories.

D. Attentional Model of Knowledge Acquisition

Contemporary theories of attention are influenced heavily by the work of Kahneman (1973), who proposed a capacity model of attention in which attention is viewed as a limited resource that can be flexibly deployed. Explicit in Kahneman's theory is the idea that mental operations differ in the amount of attentional capacity they require. More specifically, he proposed that attentional demands increase with the complexity of mental operations. Over the years this idea has been both extended and challenged by a number of researchers who have argued that, by means of extensive practice, even complex operations can be performed with only minimal attentional capacity allocated to them (Posner & Snyder, 1975; Shiffrin & Schneider, 1977). Such mental operations are called automatic and they are characterized as being fast, effortless, unconscious, and autonomous (Shiffrin & Schneider, 1977; Zbrodoff & Logan, 1986). At the other end of the attentional continuum are the operations that require a considerable amount of attentional capacity. These operations, referred to as controlled, are slow, effortful, intentional, and nonautonomous. The significance of the distinction between automatic and controlled processes is that a limited capacity cognitive system operates most efficiently when frequently used basic mental operations are performed automatically, leaving maximal resources available for less often used and more sophisticated mental operations which require controlled processing. The importance of controlled processing lies in its ability to deal with novel kinds of information, to handle unfamiliar tasks, and to flexibly approach changing requirements of the task situation.

The idea of differences between automatic and controlled processes has been used to explain many aspects of cognitive development, one of them being the relation between procedural and conceptual knowledge in learning a domain. For example, Sternberg (1984) proposed a "componential" model of intelligence in which development of intellectual ability was partially explained in terms of automatization of several types of cognitive components in performance. Sternberg postulated that, for each task, there is a set of procedural and knowledge acquisition components that together form task specific functional subsystems. At early stages of learning these subsystems can be activated only by using effortful, controlled processing. With practice they become automatized, and attention is freed to perform other, more global or novel aspects of problem solving.

Case (1984) presented a stage theory of cognitive development in which the mechanism of automatization played a crucial role. Case proposed that development within each major stage can be described as a progression of qualitatively distinct executive strategies. The main factor responsible for this progression is increase in the size of working memory. Case postulated that this increase in capacity of the working memory does not stem from a structural increase in the attentional capacity of the organism but rather from functional growth due to increases in automaticity with which operations are executed. As these operations become automatized, working memory space is freed for additional operations and storage.

Finally, LaBerge and Samuels (1974) proposed a model of reading acquisition in which automatization of basic processes was a central aspect of learning to read. The model predicted that with practice the encoding of words no longer requires conscious attention and thus does not interfere with the deployment of attention to higher level features such as meaning. Consistent with this prediction are the results of research conducted by Perfetti and Lesgold (1977), who found a substantial correlation between text comprehension and speed of word decoding.

E. An Attentional Model of Mathematics Learning

A similar model has been recently advanced for explaining the relation between procedural and conceptual knowledge in learning mathematics (Kaye, 1986; Resnick, 1980, 1983). According to this model, mastery of basic computational skills has significance because it enables the acquisition of higher-level conceptual knowledge. Children who have mastered basic arithmetic facts and algorithms to the level of automatization should have spare capacity to devote to other tasks, such as checking the plausibility of their answers and solutions, looking for numerical relationships, employing novel strategies, and reflecting why these strategies work. Similarly, lack of fluency in the execution of the basic skills may act as a limiting factor on the deployment of attention to these activities. If a simple addition or multiplication is a difficult, capacity-consuming task for a child, he or she may not have any any spare resources available for applying the relevant conceptual knowledge to the solution of the problem in hand. From this perspective, automatization of basic arithmetic facts and procedures is viewed as a necessary condition for building the relation between procedural knowledge and understanding because it provides the necessary attentional resources for this interconnection to be realized (Kaye, 1986).

There are several interesting propositions about the development of mathematical knowledge that stem from the model. First, basic arithmetic skills are viewed as being more than merely rote skills (Steffe & Blake, 1983). Rather, they are postulated to be an essential foundation for the development of more complex abilities and concepts. Second, the model suggests that conceptual and procedural knowledge of mathematics may develop initially in isolation from one another, and that some critical level of efficiency of simple computation is necessary for the connection to be established (Kaye, 1986). Finally, automatization of basic arithmetic skillls is viewed as a triggering mechanism for the continued further development of mathematical knowledge, both procedural and conceptual. That is, mastery of procedural skill results in increasing acquisition of conceptual knowledge, which allows for greater ease and efficiency in the execution of procedures, which in turn allows for more sophisticated forms of later acquisitions of conceptual knowledge.

F. Description of the Study

The purpose of the present study was to examine the relation between procedural and conceptual knowledge in multiplication. Of most interest was the hypothesis that automatization of procedural skills is related to greater conceptual understanding of the domain.

Concepts and Measures

Procedural Knowledge

In this study procedural knowledge of multiplication refers to the various processes that are combined to solve multi-digit multiplication problems. Included among these components are retrieval of products for single-digit multiplication problems and the procedures for appropriately sequencing retrieval operations. These components are jointly referred to as the multiplication algorithm.

The level of automatization of the multiplication algorithm was assessed in terms of speed of its performance. I am aware of certain limitations in using speed as a measure of automatization. The measures of trade-off in performance in dual-task situations may be optimal for capturing the complex nature of automatic processing. However, I also believe that speed of processing is a reasonably valid measure of automatization. This claim is based on the Zbrodoff and Logan's (1986) proposal that automacity is a unitary phenomenon, so that all its functional properties (effortfulness, speed, unconsciousness, and autonomity) co-occur in a truly automatic process. Consequently, one would expect faster performance to become less effortful and more autonomous. The opposite would be true for the slower performance.

Conceptual Understanding

Conceptual understanding is defined in the present study as explicit or implicit knowledge of certain selected principles of multiplication. Four principles of multiplication were selected for testing in this study: commutativity, closure, a principle related to multiplying by one (product equal to multiplicand), and a principle specifying the allowable minimum magnitude of the product in multiplying two whole numbers (product less than multiplicand). All these principles capture some important aspect of knowledge of multiplication. The interesting characteristic that differentiates among these principles is the way in which they are learned or discovered. Both the commutativity and the product-equal-to-multiplicand principles are probably learned through practice with solving various multiplication problems. Learning commutativity in this context involves the discovery that the same numbers yield the same products, independently of the order in which they are multiplied. Learning the product-equal-to-multiplicand principle is less direct and probably follows the discovery of the principle of multiplying by one. Discovery of this principle allows for the recognition that only multiplying by one, and not by any other number, yields a product equal to the other multiplied number. The common characteristic of commutativity and product-equal-to multiplicand problems is, therefore, that problems of the same or related type were likely to be already familiar to children and for this reason the underlying principles were probably retrieved rather than discovered during the testing session. Closure and product-less-than multiplicand problems, on the other hand, may represent problems that are not likely to be encountered by children unless some computational error is made. These two principles were more likely to be discovered by children for the first time during the testing situation.

It has been suggested that understanding of a concept is not an all-or-nothing phenomenon (Greeno, Riley, & Gelman, 1984). A child may develop some aspects of conceptual competence but not others. For example, Gelman and Meck (1983) and Baroody and Mason (1984) showed that children's performance on arithmetic tasks was systematically governed by the principles well before these principles could be verbalized. Salatas and Flavell (1976), in contrast, found that verbally demonstrated knowledge about efficient strategies did not guarantee the spontaneous use of these strategies in the task performance. In order to determine whether children were more advanced in verbalizing the principles of multiplication or in actively using them, two measures of understanding were used. First, understanding of the selected principles of multiplication was assessed in terms of the use of procedural heuristics that are based on conceptual knowledge about underlying mathematical principles. This type of knowledge is implicit and is referred to as operational understanding. Second, understanding was assessed in terms of explicit verbal explanation of the principles. Heuristics are defined here as short-cut strategies that shorten the solution of a problem and make it more efficient. This definition is consistent with Webster's Third International Dictionary in which heuristics are defined as "exploratory problem-solving techniques that utilize self-educating techniques to improve performance".

Specific Aims of the Study

The main purpose of the study was to determine the degree to which automatization of the multiplication algorithm was related to operational and verbal understanding of certain principles of multiplication. Given the attentional hypotheses of Kaye (1986) and Resnick (1983), it was expected that automatization of the algorithmic skill would be related to operational and verbal measures of conceptual knowledge. There might be three possible outcomes of the correlation between speed of performance of the algorithm and both measures of understanding. First, a positive correlation would indicate that slower performance of the algorithm was associated with greater conceptual understanding. This result would be entirely inconsistent with the hypothesis. Second, a zero correlation might indicate that there was no relation between automatization of the algorithm and understanding. Alternatively, it could mean that this relation was not present for the tested age groups, but it might be potentially important earlier or later in development. Third, a negative correlation would be most consistent with the tested hypothesis. This result would suggest that faster, presumably more automatized algorithmic performance was more likely to be related to greater conceptual understanding. Because the correlational method does not permit the inference about the causality of the relation, however, this outcome should be interpreted with caution.

An additional aim of the study was to compare performance on the two selected measures of understanding. Of particular interest in this comparison was whether children were more advanced in using conceptually-based heuristics or in providing explicit verbal explanations of the underlying principles. The first pattern of performance would indicate that ability to explicitly and logically explain the concept lags behind active use of related strategies. This outcome might suggest that conscious awareness of the strategy or concept comes later in development than its intuitive usage. The second outcome would indicate a type of production deficiency (Flavell, 1970), that is, an inability to produce most appropriate and efficient problem-solving strategies despite the demonstrated availability of relevant knowledge. This pattern of performance might suggest that a conscious awareness of the concept or principle is a necessary precursor to spontaneous use of related strategies.

The study consisted of three tasks. Task 1 was designed to assess the level of automatization of a multiplication algorithm. Tasks 2 and 3 were used to assess understanding of the selected principles of multiplication: Operational understanding was assessed in Task 2 and verbal understanding was measured in Task 3. The tasks were administered in a consistent order. Because Tasks 2 and 3 necessarily involved considerable interaction between the experimenter and the children, prior administration of Task 1 made it possible to minimize the effects of this interaction on children's algorithmic performance.

The study was carried out with children in grades 4 and grade 6. The choice of students from these two grades as subjects for the study was motivated by two considerations. First, because I was interested in development of the relation between skill and understanding in learning multiplication, it was necessary to test children from at least two different age and knowledge groups. Second, the children must have sufficient skill and knowledge about multiplication to deal with multi-digit multiplication problems. Students from grades 4 and grade 6 were selected after consultation with the teachers who were familiar with the arithmetic curricula and the requirements of the study. Pilot data also were used to confirm the suitability of these two grades for the study.

Half of children in each age group were girls and half boys. The variable of sex was included in all analyses of variance because there is some evidence for the superiority of boys in mathematical ability (Aiken, 1971). Although the majority of studies show no sex differences in mathematical ability at early grade school ages, later in development boys consistently exceed girls in most aspects of mathematical performance (Maccoby & Jacklin, 1974).

II. TASK 1

The purpose of Task 1 was to determine the level of automatization of a written multiplication algorithm. A complete multi-digit multiplication algorithm consists of several partial steps or components: reading and writing digits, retrieval of basic multiplication combinations, column organization, adding, and carrying. These components are interrelated and each requires application of different knowledge or skills. Depending on the complexity, solution of any multiplication problem involves application of some or all the components of the complete algorithm. In Task 1, speed was measured separately for several algorithmic procedures that differed in the number of components required for execution. The task required writing digits and solving a number of multiplication problems of increasing complexity, from simple single-digit multiplication to 2-digit by 2-digit multiplication with carries. Each increment in complexity involved implementation of a new component, or of some combination of the old components and a new component, to the solution algorithm. Speed of execution at each level of complexity was then correlated with the measures of understanding obtained in Task 2 and Task 3 to determine whether individual differences in speed were related to individual differences in understanding. Also of interest was the question of which components of the algorithm, or which combinations of these components, were most important for understanding.

A. Method

Subjects

Thirty-two grade 4 and 32 grade 6 students participated in the study. Median ages for these age groups were (in years:months) 10:1 for grade 4 (range 8:9—12:0), and 11:11 for grade 6 (range 11:3—13:8).

IQ data were obtained from school files for 28 children in grade 4 and 32 children in grade 6. Median quantitative IQ scores for the two groups, as measured by the quantitative scale of Canadian Cognitive Ability Test (CCAT), were 105 for grade 4 (range 79—132), and

97 for grade 6 (range 75—137). With few exceptions, the IQ data were 18 months old for grade 4 students and 6 months old for grade 6 students.

The Math Survey Test (MST) is a mathematical achievement test developed and administered by the local board of education. The results of most recent administration of this test were available for 29 children in grade 4 and 31 children in grade 6. Median scores for these two grades (in percentages) were 83 for grade 4 (range 55—98), and 65.5 for grade 6 (range 32—93).

Materials

The stimulus set consisted of five types of problems arranged in order of increasing complexity of the solution algorithm. An example of each problem type, together with the components of the algorithm that are required for its solution, are presented in Table 1. Type I items did not involve multiplication and were included in the set to determine whether speed of reading and writing digits was related to speed of performing other components of the algorithm. Type II items required single-digit multiplication and were separated into two subtypes, IIa and IIb, to assess possible differences in retrieval of problems with products less than 10 and products equal or greater than 10. Types III—V required multi-digit multiplication of increasing complexity of the solution algorithm, from 2 digit x 1 digit multiplication without carries to 2 digit x 2 digit multiplication with carries.

Because the amount of time and effort required for solving the problems was not uniform across the types, the number of problems of each type was varied. There were 15 items of type I, 10 problems of types IIa and IIb, 6 problems of type III, and 3 problems each of types IV and V. This manipulation was introduced to assure that, for each problem type, performance was not so rapid that measurement error constituted a substantial part of the latency and not so lengthy as to cause fatigue in the children.

The problems were presented in a booklet, with each problem type on a separate page. For the types I, IIa, IIb, and III latencies were measured for the whole set of problems on the page. For the types IV and V the individual problems within each type were presented on the separate pages and their latencies were also measured separately. This was done in effort to balance the time of solving the problems on each page and therefore to balance the effects of possible interfering variables (e.g. pausing) on children's performance across all problem types.

Subjects were tested with two sets of problems of each type, set A and B (see Appendix 1). In order to balance the difficulty of problems in the two sets, the sets were constructed so that each contained approximately the same number of problems with zeros, ones, and ties (e.g., 4 x 4, 6 x 6).

Procedure

Each child was tested individually with a booklet containing the test problems. Subjects were asked to solve the problems as quickly as possible without making mistakes (see Appendix 1 for the instructions). To familiarize children with the requirements of the task, the test problems were preceded with a set that contained sample problems of each type. In order to decrease the error of measurement, all problem types were tested twice, each time with a different set of of numbers. The order of administration of both sets was counterbalanced across children. Problems for both administrations were presented in the same booklet, but on the separate pages.

B. Results

The analysis of data was carried out in five steps. First, the analysis of errors on each problem type was performed. Second, latencies of each problem type were analysed to to determine the effect of age, sex, and practice on children's performance. Third, the reliabilities of the latencies of each problem type were estimated. Fourth, correlations among the latencies of all problem types were calculated to evaluate the degree of correspondence among the various multiplication algorithms. Finally, children's latencies on all algorithmic procedures were correlated with the scores of Canadian Cognitive Ability Test (CCAT) and Math Survey Test (MST).

Errors

Mean percentages of errors are presented in Table 2 as a function of age, problem type, and administration. A response was counted as a single error if it was incorrect, even if more than one incorrect calculation was involved. In both grades error rates for the types I, IIa, IIb, and III were lower than 6%. Because it was unlikely that interpretation of children's latencies would be affected as a result of such a small number of errors, these error data were not analysed further. Error rates for the types IV and V were higher (see Table 2).

Latencies

The mean solution time for problems of each type was calculated by averaging the solution times from both administrations. For types IV and V the mean solution times were calculated by averaging the median solution times from both administrations. Because high error rates on the types IV and V may reflect the use of some nonalgorithmic solution strategies that could affect the latency data, only latencies for correct solutions were used for calculating the medians for these two types of problems. Because there were 5 children in grade 4 and 2 in grade 6 with no correct latencies on one or both administrations of type IV or V, it was necessary to estimate these missing latencies in order to calculate the median and mean latencies for these children. For the children who did not have correct latencies on only one administration of type IV or V, estimation of the missing latencies could be made with existing correct data from the other administration of the same problem type, providing that the effect of administration on children's performance was nonsignificant. Because analyses of variance performed on the correct latencies of children with full data sets revealed that there was a significant effect of administration for type IV, $\underline{F}(1, 57) = 5.09$, $\underline{p} < .05$, but not for type V problems, only the missing latencies on the latter type could be estimated with the correct data from the other administration. For the children who did not have correct latencies on either administration of type IV, or on both administrations of type V, the possibility was checked that the missing correct data might be estimated with the incorrect latencies, providing that correct and incorrect latencies were not significantly different. The t tests performed on the latencies of children with both correct and incorrect latencies revealed that there was no significant difference between correct and incorrect latencies on either type or administration for Grade 6, and therefore all missing correct latencies of children in this grade could be estimated with their incorrect latencies. For Grade 4, there was no significant difference between correct and incorrect problem latencies on any administration of type IV, but there was a significant difference on type V, $t(23) = 3.24$, $p < .05$. Therefore, the data of one child in Grade 4 who was missing correct latencies on both administrations of Type V could not be replaced. The data from this child were excluded from all further analyses.

Because variances for solution times differed widely as a function of problem type, separate 2(Age) x 2(Sex) x 2(Administration) analyses of variance (ANOVA) with repeated measures on the last factor were performed for each problem type. Mean solution latencies for each grade and problem type are presented in Table 3.

Older children were faster than younger children on problem types I, $F(1, 59) = 10.78$, $p < .05$; IIa, $F(1, 59) = 4.79$, $p < .05$; III, $F(1, 59) = 5.19$, $p < .05$; IV, $F(1, 59) = 18.14$, $p < .001$; and V, $F(1, 59) = 7.74$, $p < .01$. On type IIb older children were also faster, but the difference was nonsignificant. Because grade 6 children presumably had more practice in solving multiplication problems than grade 4 children, one would expect grade differences in speed of performance on this type of task. The effects of grade on almost all types of problems partially confirms the validity of latencies as a measure of automatization.

Performance improved across administrations only for the types IIa and IV, $Fs(1, 59) > 4.40$, $p < .05$, from 14.12 to 13.20 s for type IIa, and from 10.42 to 9.73 s for type IV. For the type IIb, administration interacted with age, $F(1, 59) = 4.04$, $p < .05$. Tests of simple main effects showed that the effect of administration was significant only for grade 4 children, $F(1, 118) = 5.51$, $p < .05$, who were faster on the first than on the second administration of this problem type. Examination of means of the problem types for which effect of administration was present revealed that performance improved across administrations for the types that were least demanding in terms of solution time (types IIa and IV) and deteriorated on the type that required longest time for solution (type IIb).

There was no significant effect of sex on performance on any problem type. This finding seems to suggest that, at least in the tested age groups, there are no consistent differences between girls and boys in arithmetical computation.

Reliabilities of Latency Measures

Because correlations are limited by the reliability of the measures involved, it was necessary to determine whether reliabilities of the selected measures of automatization were sufficiently high to be meaningful. For each problem type split-half correlations were calculated between the latencies on the set A and the latencies on the set B. These correlations are presented in Table 4.

It should be noted that each of these split-half correlations is based on only half the trials for each measure. The Spearman-Brown formula (Anastasi, 1976) can be used to assess the effect of doubling a test on its reliability coefficient. The use of the Spearman-Brown formula is based on the assumption that, other factors being equal, a longer test is more reliable. The reliability of the whole test should be, therefore, greater than the reliability of only half a test. As is evident from examination of numbers in the diagonals of Tables 5 and 6, all types of problems had acceptable reliabilities, ranging from the high .70s to the high .90s.

Relations Among Latency Measures

Correlations among latencies of all problem types are presented in Table 5 for Grade 4 and in Table 6 for Grade 6.

As clear from Tables 5 and 6, there were many relations among all problem types, with the exception of the type I for Grade 4. Such a finding is not surprising because the components of the multiplication algorithm are not independent and, therefore, many of the types shared common processes. For example, speed on the type V was probably related to speed of retrieval of simple multiplication facts (types IIa and IIb), as well as to efficiency in organizing and adding the numbers (types III and IV). Interestingly, speed of writing digits was correlated with all aspects of the multiplication algorithm in grade 6, but not in grade 4. This finding may indicate that writing and computational skill are relatively independent at the initial stages of learning. Later, however, they become related, probably in such a way that speed of writing starts to act as a limiting factor on the speed of cognitive process involved in performing the algorithm.

Relation to CCAT and Math Survey Test

Children's latencies on all algorithmic procedures were correlated with verbal (VCAT), nonverbal (NVCAT), and quantitative (QCAT) scales of Canadian Cognitive Ability Test (CCAT), and with Math Survey Test (MST). The results are presented in Table 7 for grade 4 and in Table 8 for grade 6.

The pattern of results was similar for both grades. There was no relation between the type I problems and any of the variables of interest for either Grade 4 or Grade 6. Neither problem type, on any grade level, correlated with the nonverbal scale of the CCAT. Interestingly, all types, with the exception of the type I, significantly correlated with the Math Survey Test. This finding suggests that multiplication skill may serve as a reliable predictor of general mathematical skill.

Summary

Latency was used as a primary measure of automatization of the multiplication algorithm. Latencies on all problem types decreased with age. All latency measures had acceptable reliabilities and, with exception of type I, they correlated highly with the scores of Math Survey Test.

III. TASK 2

Task 2 was designed to assess children's operational understanding of multiplication. The measure of understanding in this task was the ability to employ procedural heuristics that are based on conceptual knowledge about the selected principles of multiplication. A useful characteristic of this measure is its close relation to the algorithmic procedure, which makes it possible to determine whether conceptual knowledge is used in the process of executing the algorithm.

Children's ability to use heuristics was assessed with different types of problems, each representing an important aspect of knowledge about multiplication. For each problem type, use of heuristics was tested by comparing children's performance on two kinds of multiplication problems: standard problems, which could be solved only by executing the entire algorithm; and short-cut problems, which could be solved by employing heuristics if knowledge about certain important principles of multiplication were present and applied. The difference between solution strategies on these two kinds of problems served as an index of heuristic use.

A. Method

Subjects

The same children as in Task 1 participated.

Materials

Types of Problems

To determine whether children can invent and use knowledge-based heuristic procedures to increase the efficiency of problem solving, I asked them to verify the correctness of the products of 32 multi-digit multiplication problems. There were two kinds of problems. Standard problems were of the form:

125 x 12 = 1510

Verification of the correctness of this type of problem required execution of the entire algorithm and comparison of the obtained result with the provided answer. Sixteen standard problems were paired with sixteen short cut problems, which could be verified without actually performing the computations if certain aspects of knowledge about multiplication operation were present and used in the verification process. Short cut problems were further divided into four types, according to the areas of relevant knowledge of multiplication for the heuristics to be employed. The four types of short-cut problems are illustrated in Table 9.

When the product was smaller than the multiplicand, it could be identified easily and quickly as false upon recognition that the product of two whole numbers cannot be smaller in magnitude than either of these numbers. When the product was equal to the multiplicand, it could also be recognized as false without computation if knowledge that the product of two numbers can not be equal to any of these numbers (unless, of course, one of the numbers was zero or one) was present and applied to the verification process. Closure problems could be also verified as being false without computation if children understood that the product of two whole numbers must be a whole number. Understanding of the commutativity principle would allow the recognition that the products of two problems that had the same numbers but differed in the order of multiplication should be the same and that, therefore, the correctness of the second product can be determined without computation by comparing it with the product already verified for the first problem.

There were 32 problems: 16 short-cut problems (4 of each type), and 16 matching standard problems. They were constructed according to the principles described below.

Construction of the Problems

Multiplicand and multiplier. Children often are more likely to notice and understand certain numerical relations and principles when the numbers involved are small rather than large (Gelman, 1978). This tendency appears to be greater for younger than for older children (Bell, Costello, & Kucheman, 1983). To determine whether use of heuristics depends on the magnitude of the multiplied numbers, all types of short-cut and standard problems were constructed so that two out of four problems of each type involved multiplying 3-digit by 2-digit numbers ("3 x 2" denotes a 3-digit multiplicand and 2-digit multiplier), and the remaining two problems involved multiplying 2-digit by 1-digit numbers ("2 x 1" denotes 2-digit multiplicand and 1-digit multiplier).

Product. Half of all commutativity problems were provided with false products and half with true. The products of the closure, product-less-than-multiplicand, and product-equal-to-multiplicand short-cut problems were always false. Eight out of sixteen standard problems had true products, and the remaining eight problems had false. Therefore, some false short-cut problems were paired with the false standard problems, and some with the true ones. All false standard problems were constructed by changing the value of one digit in the correct products of these problems. However, the rightmost digit in all false products of both standard and short-cut problems was always correct to assure that accurate judgments were not based only on the last digit.

Presentation of the Problems

Four different sets of problems were used and each set was constructed according to the following constraints. The 32 problems were presented in a booklet, with eight problems per page (see Appendix 2). On each page problems were grouped into two four-problem sets. They were presented horizontally (e.g. 19 x 7 = 133). A horizontal (as opposed to vertical) presentation of the problems made it difficult to perform the algorithm mentally, and thus students were more likely to do paper-and-pencil calculations when they needed to compute the product. Therefore, children's overt behaviour was more likely reflect their actual solution strategies on the presented problems.

Arrangement of problems on a page

The problems were arranged in such a way that the example of each short-cut type together with its matching standard problem was presented only once on each page, and that no more than two short-cut problems of any type appeared in a row. The number of 3 x 2 and 2 x 1 problems was balanced so that there were four problems of each size on a page. The order of presentation of problems on a page was random but different for each page of the booklet to assure that the position of the problem on a page did not affect the results.

Procedure

Subjects were tested individually. Each child was asked to read aloud the instructions (see Appendix 2) that required verification of the correctness of the multiplication problems printed in the booklet. Although the children were asked to use paper and pencil whenever they needed to compute the answer, it also was emphasized that computing the result for each problem was not necessary if the correctness of the product could be determined without performing calculation. In order to make sure that children understood the instruction, the test was preceded with a practice set that contained sample problems of each type. Subjects' overt solution strategies on each problem, such as paper and pencil computation or obvious signs of mental calculation, were carefully watched and noted. Solution time for each problem was measured with a stopwatch. In order not to distract the child, a stopwatch was started for the first problem on a page and only the times of finishing each problem were recorded. The problem was considered to be completed as soon as the child marked it as being correct (1) or incorrect (0).

B. Results

The analysis of data was performed in five stages. First, latencies for standard problems were analyzed to determine the criteria for evaluation of children's strategies on the short-cut problems. Second, the use of heuristics was determined by comparing performance on each short-cut problem with the performance on the corresponding standard problems. Third, an analysis of variance on heuristics usage was conducted to examine the relation between the use of heuristics and age, sex, and problem type. Fourth, correlations were calculated to determine whether usage of of heuristics showed any commonality across problem types. Finally, correlations among heuristics of all types and the scores of CCAT and Math Survey Test were calculated.

Latencies for Standard Problems

A 2(Grade) x 2(Sex) x 2(Problem Size) x 2(Problem Validity—true or false) x 2(Practice— first or second half of the task) ANOVA was performed with repeated measures on the last three factors. Latency of standard problems served as a dependent variable in this analysis. Standard closure problems required multiplying more numbers than other problems and were eliminated from this analysis because their latencies were not homogenous with the latencies of the remaining standard problems.

Latencies decreased with grade, from 27.68 s for grade 4 to 16.20 s for grade 6, $F(1, 59) = 28.62$, $p < .001$, and increased with problem size, from 15.89 s for smaller problems to 27.82 s for larger problems, $F(1, 59) = 170.83$, $p < .001$. The significant interaction of grade and problem size, $F(1, 59) = 13.80$, $p < .001$, was qualified by two three-way interactions, Grade x Problem Size x Validity and Sex x Problem Size x Validity, which revealed that large false problems were solved more quickly than large true problems by children in grade 4 and by girls, $Fs(1, 236) > 15.72$, $p < .001$. The analysis of simple main effects of the marginally significant four-way interaction of Age x Sex x Problem Size x Validity showed that that large false problems were faster only for grade 4 girls, $F(1, 295) = 25.18$, $p < .001$.

The results of the analysis do not seem to confirm the finding that boys are superior to girls in mathematical performance. On the contrary, it appears that girls are more likely than boys to benefit in the situations in which solution strategies are important for performance. Verification of false multiplication problems might be faster than verification of true problems if a self-terminating strategy was employed that terminated computation as soon as the first incorrect digit in the product was found. It appears that only girls in grade 4 took advantage of this strategy for verification of large problems.

Scoring and Classification of Short-cut Strategies

Standard problems were selected for comparison with the short-cut problems using the most stringent criterion possible. For each child, his or her performance on the short-cut problem was compared with the performance on all six standard problems of the same size as the short-cut problem. This comparison is justified because there was no significant difference between the latencies of standard problems on the first and second part of the task, and because the overall difference between correct and incorrect problems was not significant for grade, problem size, or validity ($ts < 1$). Therefore, for each short-cut problem the comparison could be made using all six standard problems, independently of their accuracy and position in the task.

Because grade 4 girls were generally slower for large true than for large false problems, the comparison with all standard problems independently of their validity was an especially stringent criterion for the large true commutativity problems in this particular group of children. However, because the latencies of all these problems were shorter than the latencies of all six corresponding standard problems, no child was affected as a result of this restriction.

On standard problems children had no choice but to use the lengthy multiplication algorithm to verify each problem and so they were expected to use paper and pencil to record their computations. On the short-cut problems, if children used the same algorithm, then their performance on these problems would be similar to the performance on the standard problems. That is, written computations would be present and solution latencies would be comparable to the latencies of the corresponding standard problems. In contrast, if children used heuristics on the short-cut problems, then solution latencies should be shorter than those for the corresponding standard problems, and correct verifications should not be accompanied by writing.

According to this rationale, a child's strategy on each short-cut problem was classified as a heuristic if (a) the problem was verified correctly without written computation, (b) answers on all or almost all corresponding standard problems were accompanied by written computations, and (c) the short-cut problem was solved more quickly than all or most corresponding standard problems. Specific criteria for classification of children's strategies on the short-cut problems are presented in Table 10.

Strategies H1, H2, H1A, and H2A were classified generally as heuristics but the stringency of the criteria differed among these groups. For a given problem, a strategy was classified as H1 if a child correctly verified a short-cut problem without showing evidence of overt computational strategy, wrote on all six corresponding standard problems, and was faster on the short-cut problem than on any of the six corresponding standards. The strategy of a child who did not write on all standard problems could also be classified as H1, providing that the child was accurate on all six of them. This condition made it possible to classify the strategies of children who mentally solved one or more standard problems but were very accurate on them. In contrast, a strategy was classified as H2 if a child did not write on the correctly verified short-cut problem, but he also did not write on one or more standard problems, and was inaccurate on one or two of them. He or she also had to be faster on the short-cut problem than on at least five out of six standard problems.

Strategies H1A and H2A were in fact equivalent to the previously described strategies H1 and H2, except that writing was present for the part of the short-cut problem. This incomplete written strategy probably was related to the child's sudden realisation that computing the answer was not necessary, and that a shorter, more efficient way of determining the answer was possible.

Strategies G, CS1, CS2, and CS3 represent different patterns of performance that were characteristic for children who were not using heuristics on the short-cut problem. When a child correctly verified the short-cut problem without writing, but he rarely computed the answer on the standard problems and his performance on these problems was highly erratic, his strategy on this short-cut problem was classified as G. This pattern of performance was indicative of guessing. When a child correctly verified the short-cut problem without writing, and was accurate on most of the standard problems, but his latency on the short-cut problem was long compared to the latencies of the standards, his strategy was classified as CS1. This pattern of performance was indicative of some mental computational strategy. A child who wrote on the short-cut problem, or was inaccurate on this problem clearly showed the evidence of the lack of heuristic use. His strategy was then classified as CS2 or CS3. The frequency of each strategy is presented in Table 11.

My initial intention was to analyze each strategy usage. However, because the frequency of occurance of some of the strategies was too low for the statistical analysis to be meaningful (see Table 11), the initial fine-grained scheme had to be collapsed. For the purpose of further analysis strategies H1, H1A, H2, and H2A were classified as heuristics. Performance consistent with any pattern described for the strategies G, CS1, CS2, or CS3 was equivalent to the lack of heuristic use.

ANOVA on Heuristic Usage

A 2(Grade) x 2(Sex) x 4(Problem Type) x 2(Problem Size) ANOVA was performed with repeated measures on the last two factors. A number of problems on which heuristics were used (0–2) served as the dependent variable in this analysis. Percentages of short-cut problems on which heuristics were used are presented in Table 12 as a function of grade, problem type, and problem size.

Use of heuristics was highly related to the problem type, $\underline{F}(3, 177) = 19.78$, $\underline{p} < .001$. Newman-Keuls tests revealed that heuristics were used more frequently on the commutativity and product-equal-to-multiplicand problems (means 49.18% and 40.73% respectively) than on product-less-than-multiplicand and closure problems (means 23.75% and 19.75% respectively), $\underline{p} < .05$. It is plausible to suggest that this pattern of results may be directly related to children's familiarity with different types of problems. The commutativity principle was probably already familiar to children through the every-day school practice with analogous problems. The product-equal-to-multiplicand principle might be learned through the analogy to the often encountered and practiced problems that involve multiplying by one. The closure and product-less-than-multiplicand problems, in contrast, were more likely to be unfamiliar to children and, therefore, the underlying principles might be first discovered during testing.

One might expect that grade 6 children, who had been learning multiplication longer and had more experience in solving problems, should use heuristics more frequently than younger children. Surprisingly, there was no significant age difference in use of heuristics $\underline{F}(1, 59) < 2$, although means favored the older children in 7 of 8 cases (see Table 12). Sex differences were also not significant. Similarly, size of the problem did not have any influence on the frequency of heuristic use for any grade or problem type ($\underline{F} < 1$). This finding was contrary to the earlier prediction and seemed to indicate that even for young children use of heuristics is independent of the size of the problem, at least within the present range of problem sizes.

Relations Among Types of Heuristics

Presented in tables 13 and 14 are the correlations among heuristics of all the types: commutativity (HCOM), product less than multiplicand (HPLM), product equal to multiplicand (HPEM), and closure (HCLS). These correlations were calculated to find out whether use of heuristics on one type of problems was related to use of heuristics on the other types of problems. There was a considerable interrelation among the heuristics, with the exception of commutativity for Grade 4. This finding suggests that the use of heuristics shows certain commonality across different types of problems.

Relation to CCAT and Math Survey Test

Correlations among heuristics of all types and verbal, nonverbal, and quantitative scales of CCAT and Math Survey Test are presented in Table 15 for grade 4, and in Table 16 for grade 6. In addition to the already described types of heuristics the tables include the correlations with the total number of heuristics (HTOT) that were used on all problem types combined. As evident from Table 15, use of heuristics was not related to the scores of any scale of CCAT in grade 4. Correlations were uniformly low across all types of heuristics. Use of heuristics on the product-less-than-multiplicand and on all problem types combined was moderately related to Math Survey Test. The pattern of results was quite different for grade 6 (see Table 16). Use of heuristics of all types was highly related to all scales of CCAT and to Math Survey Test.

It appears, therefore, that individual differences in the use of heuristics can be predicted from general cognitive and mathematical skills when multiplication is well established and mastered, as in Grade 6 students. At an earlier stage of learning, however, use of heuristics may be more related to some other variables, such as practice, or amount of exposure to various multiplication problems.

IV. TASK 3

Task 3 was designed to test verbal understanding of multiplication. In the present task children's understanding was assessed by asking them to explain explicitly the principles of multiplication that justified use of heuristics on the short-cut problems of all types. The quality of children's explanations was evaluated separately for each problem type and size.

A. Method

Subjects

The same children as in Tasks 1 and 2 participated.

Material and Procedure

Task 3 was administered immediately after Task 2 and was a structured interview with individual children. The interview was based on examples of the short-cut problems that had been verified by the child in the preceding task and some new problems of the same types. All children were presented with instructions that made them expect that some kind of explanation of their solutions on the preceding task was required. The children were questioned about their solution strategies on two out of four problems of each type, one 3 x 2, and one 2 x 1 problem. The set of problems for the interview was counterbalanced across children, so that all four problems of each type were used. For each short-cut type, the interview started with a 2 x 1 problem followed by a 3 x 2 problem for half of the children, and with a 3 x 2 problem followed by a 2 x 1 problem for the other half. The child was asked to explain how he knew that the answers to these problems were incorrect or correct ["How did you know that the answer to this problem was wrong (good)?" or "Why is this a wrong answer?"]. The purpose of these questions was to determine whether the child understood and could explain the relevant principles of multiplication that permitted use of efficient short-cut strategies. If child's responses on either one or both problems clearly reflected the reliance on the algorithmic procedure (e.g., "Because I computed it"), and not on the understanding of the principle, he was asked whether he knew some more efficient method of determining the correctness of the answer ("Can you think about any shorter way of finding out whether this answer is good or wrong, without computing it?"). This question was intended to help the child to realize that he might use some other knowledge of multiplication available to him, not only his knowledge about the algorithmic procedure.

The above questions testing understanding of the principle were followed with questions intended to determine whether children understood that two different problems of each type can be generalized as being the examples of the same, general principle of multiplication ("Do you think that these two problems are alike? How are they alike?"). Irrelevant or superficial generalizations (e.g., "They are both multiplication problems") were prompted with an additional question ("Can you think about some other way these two problems are alike?"). Children's ability to generalize was tested with new pairs of problems of each type (one 2 x 1, and one 3 x 2) that were not previously used for verification in the Task 2. Generalization of commutativity was tested separately for each problem size. All children's responses were recorded with a tape recorder and coded on a special answer sheet.

B. Results

The analysis of verbal data was performed in four stages. The first three were analogical to those performed on heuristics usage. First, the analysis of variance (ANOVA) on children's verbal explanations of principles was performed. Second, correlations were calculated to determine whether explanations were equally frequent across different principle types. Third, correlations among verbal data and the scores of CCAT and Math Survey Test were calculated. And finally, generalizations of principles were analyzed.

ANOVA on Explanations

In order to determine whether ability to explain tested principles of multiplication was related to age, sex, type of principle, and size of the problem, a 2(Age) x 2(Sex)x 4(Principle Type) x 2(Problem Size) ANOVA was performed with repeated measures on the last two factors. A number of correctly explained problems of each type (0 or 1) was a dependent variable in this analysis.

Percentages of children in each grade who correctly explained the principles on each problem type and size are presented in Table 17. Because for each child explanations were tested with only one problem of each type and size, the numbers in Table 17 also represent the percentages of correctly explained problems in each grade, and therefore Tables 12 and 17 are directly comparable.

Older children explained principles of multiplication more frequently than younger children, $F(1, 59) = 6.66$, $p <. 05$. Means were 57.26% for grade 4, and 78.90% for grade 6. Frequency of explanations was highly related to the type of principle, $F(1, 177) = 21.34$, $p < .001$. Commutativity was explained more frequently than any of the remaining principles, and the product-equal-to-multiplicand was explained more frequently than the product-less-than-multiplicand principle (Newman-Keuls, $p < .05$). Means for these four types of principles were 91.33% for commutativity, 71.14% for product equal to multiplicand, 61.67% for closure, and 48.18% for product less than multiplicand.

Grade and principle type interacted, $F(1, 177) = 6.29$, $p < .01$. Tests of simple main effects revealed that older children explained more frequently than younger children the product-less-than-multiplicand, $F(1, 236) = 4.46$, $p < .05$, product-equal-to-multiplicand, $F(1, 236) = 12.37$, $p < .01$, and closure principles, $F(1, 236) = 8.36$, $p < .01$. Commutativity, however, was explained equally frequently by both groups of children. Similar to use of heuristics, there was no sex differences in performance. Interestingly, all tested principles were explained more often on the small than on the large problems, $F(1, 59) = 7.81$, $p < .01$.

Relations Among Explanations

Correlations among explanations of all the principles are presented in Table 18 for grade 4 and in Table 19 for grade 6. In the tables explanation of the commutativity principle is denoted as ECOM, of product-less-than-multiplicand as EPLM, of product equal to multiplicand as EPEM, and of closure as ECLS.

In both grades explanations showed a considerable uniformity across different principle types. The only exception was explanation of commutativity in grade 4, which was not related to explanations of the remaining principles. This finding suggests that, at least for younger children, explanation of commutativty may capture somewhat different aspect of knowledge of multiplication than the explanation of the remaining principles.

Relation to CCAT and Math Survey Test

Presented in Table 20 and 21 are the correlations among explanations of each type and all types combined (ETOT) and the scores of CCAT and Math Survey Test. As evident from Table 20, explanation of principles was not related to any scale of CCAT in grade 4. Explanations of product less than multiplicand principle (EPLM) and all the principles combined (ETOT) were moderately related to Math Survey Test. In grade 6 the pattern of results showed two interesting trends. First, explanation of principles was not related to the nonverbal scale of CCAT. The correlations with this scale were uniformly low across all principle types. Second, there was no relation between explanation of product equal to multiplicand principle and any scale of CCAT and Math Survey Test. The correlations in all the remaining cells were significant, either moderate or high. The lack of correlation with product equal to multiplicand should be, however, interpreted with caution. Explanation of this principle was also not correlated with use of heuristics of the same type, which may suggest some reliability problem with this particular measure.

Generalizations

Only generalizations of the product-equal-to-multiplicand, product-less-than-multiplicand, and closure principles were analyzed. Commutativity was not included in this analysis because, for this particular principle, generalizations were tested separately for each problem size and, therefore, they were not comparable to the generalizations of other principles.

In comparison to explanations, generalizations were infrequent for all principles. The percentage of children who correctly generalized presented pairs of problems is presented in Table 22 as a function of grade and principle type.

In both age groups product-less-than-multiplicand was generalized less frequently than the product-equal-to-multiplicand and closure principles, X^2s(1) > 4, ps < .05. Older children generalized the product-less-than-multiplicand principle more frequently than younger children, $X^2(1) = 5.81$, $p < .05$. For the remaining two principles, the effect of age was not significant.

Examination of each child's performance on both explanation and generalization of each principle revealed that in most cases generalizations were only present when the problems of both sizes were correctly explained. The exceptions from this general tendency were present in only 2 cases (out of 93 possible) in grade 4 and in 4 cases (out of 96 possible) in grade 6. This finding suggests that ability to generalize represents a more advanced stage in understanding the principle than ability to explain it.

V. RELATIONS AMONG TASKS

A. Task 1 and Task 2

Relation Between Speed of Algorithm and Use of Heuristics

To determine whether individual differences in speed of execution of multiplication algorithm were related to individual differences in use of heuristics, correlations were calculated between the latencies of various multiplication algorithms and number of heuristics on each of the described types (product-less-than multiplicand, product-equal-to multiplicand, commutativity, and closure). These correlations are presented in Table 23 for grade 4 and in Table 24 for grade 6.

As evident from Table 23, there was no relation between use of heuristes of any type and speed of performance on the algorithmic task for grade 4. The correlations were uniformly very low across all algorithmic procedures.

For grade 6, use of heuristics was generally much more highly related to all aspects of algorithmic procedure, with exception of reading and writing digits (see Table 24). However, the rs were significant only for product-equal-to-multiplicand, and closure types, for which use of heuristics was highly related to speed of performance on all multiplication latencies. Interestingly, although the correlations for commutativity and product-less-than-multiplicand were not significant, speed of performance on all the algorithms was related to some general tendency to use heuristics, as evidenced by the significant correlations with total number of heuristics (HTOT). This pattern of results is quite interesting. First, it is clear that the hypothesis about the relation between speed of performing the algorithm and usage of heuristics was not confirmed at all for grade 4, and was confirmed, at least partially, for grade 6. Grade differences in the pattern of results may be explained by postulating that some critical level of speed of execution of the multiplication algorithm is necessary for the relation between the algorithmic skill and understanding of the principles to be established. Recall that, consistent with this explanation, grade 6 children were faster than grade 4 children on all aspects of the algorithmic procedure. Second, with respect to grade 6, speed of execution of the algorithm was related to the use of heuristics only on problems with product-equal-to-multiplicand and closure problems, but not on commutativity and product-less-than-multiplicand types. This particular pattern of results might be directly related to the degree of visual discrepancy between the "short-cut products" and the correct products, that is, the products one would get if the problems were solved correctly. The incorrectness of the closure products that had unnecessary decimals, and of the products that were equal to multiplicands, was much more apparent than the incorrectness of the other two types. It appears, therefore, that speed of performance of the algorithm facilitated recognition of only these most discrepant products.

B. Task 1 and Task 3

Relation Between Speed of Algorithm and Explanation of Principles

Presented in Tables 25 and 26 are the correlations between speed of execution of multiplication algorithm and explanations of principles of multiplication. Similar to correlations with use of heuristics, there was no relation between explanation of principles and any aspect of the algorithmic procedure for grade 4 (see Table 25). For grade 6 explanations of product-less-than-multiplicand (EPLM), commutativity (ECOM), closure (ECLS), and all the principles combined (ETOT) were related to speed of performance on all multiplication algorithms, with exception of reading and writing digits. As evident from Table 26 explanation of product-equal-to-multiplicand principle (EPEM) was not correlated with performance on the algorithm. This result should be, however, interpreted with caution. Recall that explanation of this principle was also not correlated with use of heuristics of the same type, which may suggest inadequate reliability of this measure. It is not clear why this particular measure might be unreliable, but the meaningfulness of the negligible correlation is questionable.

To summarize, individual differences in explanation of principles were related to speed of the algorithm only for grade 6. This pattern of results seems to suggest that some critical level of computational skill is necessary for the link with conceptual knowledge to be established.

C. Task 2 and Task 3

Relation Between Use and Explanation of Principles

Examination of each child's performance on both Tasks 2 and 3 revealed that, for each principle, four patterns of performance were possible. A child might neither use heuristics nor explain the principle ($\bar{H}\bar{E}$), or might use heuristics but be unable to explain the underlying principle ($H\bar{E}$), might not use heuristics for verification of the short-cut problems but correctly explain the principle when explicitly asked to do so ($\bar{H}E$), or might both use heuristics and explain the principle (HE). The number of children in each group is presented in Table 27 as a function of grade, problem type, and problem size.

As is evident from the table, on majority of problems children's performance was consistent across the tasks ($\bar{H}\bar{E}$ or HE). However, there was also a large number of children who demonstrated verbal knowledge about principles but failed to use heuristics for verification of the short-cut problems ($\bar{H}E$). This pattern of performance is indicative of a production deficiency, that is, an inability to spontaneously produce most appriopriate and efficient problem-solving strategies despite the demonstrated availability of relevant knowledge (Brown, 1975; Flavell, 1970).

It is plausible to suggest that the observed patterns of performance represent certain developmental stages in understanding principles of multiplication. First, no knowledge or understanding of principle is present ($\bar{H}\bar{E}$). Verbal responses of children in this stage, collected on the interview, were characterized by the certainty that computing the answer was necessary for verification of the presented short-cut problems. Several children in this group referred also to some indirect computational strategy, such as addition or division (e.g., "I could divide the result by the first number to see if I get the other number"), but not to the principle. This stage is followed by the "production deficiency" stage $(\overset{-}{H}\overset{+}{E})$ in which children already know about the principle in the sense that they are able to explain it when explicitly asked to do so, but they do not spontaneously use knowledge-related problem solving strategies. Children in this group were often unable to give any definite explanation for not using verbally demonstrated knowledge of strategies in the actual problem-solving situation (e.g., "I don't know why I didn't", "I didn't think about it", "I forgot"). Given the availability of relevant knowledge, the developmental course of acquisition of effective knowledge-related strategies may consist of gradual increase in the likelihood of its spontaneous occurance in appropriate problem-solving situations. The final stage in development of understanding of the principle is characterized by the ability to explain the principle and to spontaneously use effective problem-solving strategies $(\overset{+}{H}\overset{+}{E})$. Children in this group usually not only explained the principle instantly, but they often were surprised that such an easy and obvious statement was expected from them. The transition from nonproduction to production may proceed through the intermediate stage of "production inefficiency" (Flavell, 1970), during which the appropriate strategy appears to be within child's cognitive reach in the problem-solving task, but nonetheless it is not spontaneously used, or not used consistently. This stage, although behaviorally not distinguashable from production deficiency $(\overset{-}{H}\overset{+}{E})$, is characterized by the intuition of the possible utility of the strategy coupled with some uncertainty or lack of skill. Verbal comments of some children conveyed the awareness of the strategy during problem solving and, at the same time, the lack of confidence that the principle would work (e.g., "I wasn't sure so I wanted to check", "I wouldn't be absolutly sure if I didn't work it out"), or that it would work always (e.g., "I wanted to check whether it would work for for this problem as well").

An important step in development of understanding may be the ability to generalize different problems as being the examples of a common principle. Recall that explanations were more frequent for small problems but use of heuristics was independent of the size of the problems. It is plausible to assume, therefore, that verbal understanding develops first for smaller problems, and only later it becomes generalized to larger problems as well. At the stage of active employment of the strategy, when the principle is already well internalized, the magnitude of the problem may be not important.

If development of understanding does, in fact, follow the pattern in which verbal understanding always precedes operational use of the principle, one would expect that relatively few children would use heuristics before they are able to explain the underlying principle. Examination of Table 27 revealed that the $\overset{+-}{HE}$ pattern of performance was, indeed, least common for all principle types in both grades. Similarly, one would expect certain continuity and consistency in development of understanding, such as that principles that are more frequently explained should be also more frequently used. Recall that commutativity and product-equal-to-multiplicand principles were consistently most frequently explained and used.

VI. DISCUSSION

There were two main issues addressed in the present study: (a) the relation between automatization of multiplication skill and understanding of principles of multiplication and (b) differences among various aspects of understanding. Automatization of multiplication skill was assessed in terms of speed of performance of multi-digit multiplication algorithm. Understanding was assessed with two different but related measures: (a) use of heuristics that were based on conceptual knowledge about principles of multiplication, and (b) explicit verbal explanation of these principles. In the following sections most important findings are presented in terms of how they relate to each of these issues and to other relevant research on multiplication. Limitations of the measures and conclusions are discussed also.

A. Relation Between Speed and Understanding

Findings and Conclusions

The main hypothesis tested in the study was that automatization of multiplication algorithm was related to greater understanding of this operation as measured by the ability to use and explain principles of multiplication. The analysis of results revealed that this hypothesis was not confirmed for grade 4, in which speed of execution of the algorithm was not correlated with any aspect of understanding. For grade 6 speed of multiplication algorithm was related to use of heuristics on product-equal-to-multiplicand and closure problems, as well as to verbal explanations of all principles with exception of product-equal-to-multiplicand. Age differences in the pattern of results suggest that the relationship between skill and understanding is not steady or static but increases with learning and experience. These differences may also indicate that some critical, threshold level of speed of the algorithm is necessary for the multiplication skill and understanding to become related. It is tempting to suggest that for each principle this relation is first established with more advanced, verbal aspect of understanding and only later is generalized to active use of related strategies. This interpretation is potentially flawed by the fact that for product-equal-to-multiplicand principle speed of algorithm was correlated with use of heuristics but not with verbal understanding. Recall, however, that reliability of this particular measure was questionable and therefore this result should be treated as tentative.

Thus the results of the present study suggest that, at least for older children, automatization of the algorithm is related to greater availability and accessability of relevant conceptual knowledge in the problem solving. While not all the evidence is positive, the preponderence of it appears to be so.

The correlational design of the study does not permit the inference about the nature of the relation between automatization of the algorithm and conceptual understanding, but it is plausible to suggest that once this relationship is established, algorithmic skill and understanding may build upon each other. That is, practice in algorithmic procedure would make it more automatic, which would free processing resources for noticing mathematical regularities, checking for errors, and looking at the plausibility of the calculations one is performing. This more extensive processing might in turn further facilitate acquisition of multi-digit multiplication procedures.

A Comparison with Previous Research on Multiplication

The framework proposed in the present study differs in several respects from the previous accounts of children's multiplication. First, basic arithmetic facts and algorithms are often viewed as being merely rote skills, corresponding to no meaningful aspect of mathematics (Steffe & Blake, 1983). The present approach posits, instead, that skill at basic arithmetic may have deeper conceptual implications. The automatization argument advanced here suggests that, to the extent that arithmetic skill becomes automatic, it results in greater availability and accessibility of higher-level conceptual knowledge.

Second, it has been a prevalent view that acquisition and retrieval of basic arithmetic facts and algorithms is independent of existing conceptual knowledge. For example, with respect to single-digit multiplication it has been suggested that retrieval speed and the sequence of acquisition of multiplication facts are determined almost entirely by size of the problem and practice frequency (Campbell & Graham, 1985; Miller, et al., 1984; Stazyk, Ashcraft, & Hamann, 1982; Siegler, 1987). Smaller problems and the problems that have been extensively practiced are acquired earlier and recalled faster than the larger or relatively less frequent combinations. These models assume that, at any stage of learning, fact acquisition and recall do not interact with and are not influenced by the potentially relevant aspects of conceptual knowledge. Consequently, the recall of the pair of problems such as 5 x 7 = 35 and 7 x 5 = 35 are viewed as two psychologically unrelated events, unaffected by knowledge of commutativity. In contrast to association-learning models, the present approach suggests that, at least at later stage of learning, as existing number-combination knowledge is used and becomes more routine and automatized, it provides an opportunity to discover new relationships and principles which, in turn, can be used to facilitate acquisition and recall of new or harder problems. From this perspective, knowledge of the commutativity principle might obviate extensive practice of the 7 x 5 combination once related by this principle 5 x 7 problem has been memorized. Thus, discovery of relations and principles can give a child the sudden capacity to respond quickly to a whole range of related combinations. In effect, sudden qualitative changes can be produced that are not necessarily related to the amount of practice. Morever, it appears that, once discovered, knowledge of principles and relations might also influence the choice of backup strategies when the retrieval process goes wrong. For example, a child might intentionally use his or her knowledge about the relation between addition and multiplication for repeatedly executing addition procedure to solve a difficult multiplication problem. This suggestion is contrary to the Siegler's (1987) claim that the choice of strategies in multiplication is a self-regulatory, mindless process that does not depend on conceptual knowledge of this operation.

Finally, with respect to multi-digit multiplication, the study offers an useful framework for conceptualizing the relation between procedural and conceptual knowledge in computational performance. An interesting characteristic of the present approach is that it postulates active interplay and interdependence between conceptual knowledge and procedural skills in learning complex algorithmic procedures. The suggestion is that the relation between these two types of knowledge is more than a one-sided facilitation of procedural skills by conceptual understanding (Skemp, 1976). A distorted algorithm is more likely to be correctly "repaired" or "reinvented" when procedural skills are supported by conceptual understanding, but the inventions reflecting understanding can come about only when the procedures become well established and automatized so their results can be inspected and compared (Resnick, 1983). Thus, from this perspective automatization of the algorithmic procedure provides a necessary link between syntax of this agorithm and semantics of the corresponding conceptual knowledge.

Limitations of the Study

The present study also has certain limitations that need to be mentioned and evaluated. First, it should be noted that automatization of basic procedures and associated increase in attentional resources may be a necessary but not sufficient condition for corresponding increase in availability and accessability of conceptual knowledge. Motivational, emotional, or intellectual factors may limit the potential role of automatization in this respect. For example, a child might be not willing, not interested, or intellectually not capable of noticing and making use of mathematical regularities and principles despite the availability of attentional resources gained by means of extensive practice and automatization of basic algorithmic knowledge.

Second, the obvious limitation of the study is that all the conclusions and interpretations are based on the correlational data, showing that the same children who are fast and skillful on multiplication algorithm tend to have better understanding of principles of this operation. As such, the data do not necessarily indicate that increasing skill at algorithm will lead to any corresponding increase in conceptual sophistication. Within the obvious limits of such data, however, they do nothing to discourage the view that fluent performance of basic algorithmic procedures affects conceptual understanding, at least for older children. Going beyond such correlations to real understanding of how acquisition of procedural skill affects mathematical understanding would require detailed longitudinal studies of development of relation between these two aspects of mathematical knowledge.

B. Relation Between Measures of Understanding

Children's ability to employ heuristics was compared with their ability to explicitly explain the underlying principles of multiplication that justified use of heuristics. Both consistencies and differences in performance were revealed as a result of this comparison. Consistent across both measures, commutativity and product-equal-to-multiplicand principles were more frequently used and more frequently explained than the remaining principles. Superiority in understanding of these two principles has been explained in terms of different amount of practice and familiarity with the related problem types. The main difference in performance across the two tasks was that children were more advanced in explaining the principles that in spontaneously using them for verification of the short-cut problems. Consequently, in addition to children whose performance was consistent across both tasks ($\overset{-}{H}\overset{-}{E}$ or $\overset{+}{H}\overset{+}{E}$), there was also a large number of children who were able to explain the principles even though they did not use heuristics ($\overset{-}{H}\overset{+}{E}$). This pattern of performance has been characterized as production deficiency, that is, an inability to produce most appropriate strategies despite the presence of relevant knowledge (Flavell, 1970). It has been suggested that understanding of principles of multiplication follows a path of development in which verbal understanding (production deficiency) precedes spontaneous use of related efficient strategies (production).

Thus comparison of performance on the two tasks clearly showed that children do not possess a single understanding of principles of multiplication. Rather, their understanding varies as a function of the employed measure. The child may use a concept in some situations but fail to do that in others. This finding demonstrates that conceptual understanding is not all-or-nothing phenomenon. It has many aspects that cannot be fully assessed by examining performance on a single task. Only by investigating a concept both broadly and deeply, on a variety of tasks, corresponding to different aspects of understanding, can we discover what children know about the concept and how it is acquired.

VII. Tables

Table 1

Problem Types for Task 1

Problem Type	Example	Required Components of the Algorithm
I – Copying List of Digits	5	Reading and Writing Digits.
IIA – Single Digit Multiplication, Product Less than 10.	3 x 2	Reading and Writing Digits, Retrieval.
IIB – Single Digit Multiplication, Product Equal or Greater than 10.	9 x 8	Reading and Writing Digits, Retrieval.
III – 2 Digit x 1 Digit Multiplication, no Carries.	32 x 3	Reading and Writing Digits, Retrieval, Column Organization.
IV – 2 Digit x 2 Digit Multiplication, no Carries.	14 x 12	Reading and Writing Digits, Retrieval, Column Organization, Adding.
V – 2 Digit x 2 Digit Multiplication, with Carries	43 x 57	Reading and Writing Digits, Retrieval, Column Organization, Adding, Carrying.

Note. Multiplication problems were presented vertically. See Appendix A for the complete set of stimuli.

Table 2

Percentages of Errors on Task 1 as a Function of Age, Problem Type, and Administration

	Problem Type											
	I(15) Administration		IIa(10) Administration		IIb(10) Administration		III(6) Administration		IV(3) Administration		V(3) Administration	
Grade	1	2	1	2	1	2	1	2	1	2	1	2
4	0.2	0.4	1.2	1.2	4.1	5.9	4.2	4.2	21.9	12.5	33.3	34.47
6	0.0	0.0	2.2	1.2	3.1	4.4	1.6	3.6	9.4	17.7	29.1	25.0

Note. The number of problems for each type and administration is indicated in parentheses.

Table 3

Mean Latencies and Standard Deviations for Each Problem Type and Grade for Task 1 (in seconds)

Type	Grade 4		Grade 6	
	Latencies	Standard deviations	Latencies	Standard deviations
I	11 96	1 58	10 35	2 24
IIA	15 21	6 89	12 16	3 77
IIB	40 12	38 59	27 20	23 03
III	22 29	11 39	17 19	5 35
IV	12 00	4 41	8 25	2 03
V	25 79	14 42	18 15	6 93

Table 4

Split half Correlations for Each Grade and Proble Type for Task 1

Grade	n	I	IIa	IIb	III	IV	V
4	31	602	913	958	898	822	823
6	32	885	601	935	673	808	763

Note All correlations are significant with ps < 001, two-tailed

Table 6

Correlations and Reliabilities Among Problem Types of Task 1 for Grade 6

	I	IIa	IIb	III	IV	V
I	939	554***	513**	635***	635***	419*
IIa		751	640***	853***	900***	868***
IIb			966	588***	615***	710***
III				805	827***	715***
IV					893	862***
V						866

Note All Correlations were calculated for n = 32
Reliabilities are provided in the diagonal
* p < 05, ** p < .01, *** p < 001, all two-tailed

Table 7

Correlations Among Algorithm Latencies of Task 1 and VCAT, QCAT, NVCAT, and Math Survey Test for Grade 4

	I	IIa	IIb	III	IV	V
VCAT n=28	-.208	-.479*	-.419*	-.478**	-.238	-.456*
QCAT n=28	.001	-.365*	-.193	-.427*	-.380*	-.344
NVCAT n=28	.249	-.010	-.142	-.127	.003	.034
M[illegible] n=29	-.001	-.403*	-.403*	-.526**	-.406*	.365*

* $p < .05$, ** $p < .01$, both two-tailed

Table 8

Correlations Among Algorithm Latencies of Task 1 and VCAT, QCAT, NVCAT, and Math Survey Test for Grade 6

	I	IIa	IIb	III	IV	V
VCAT n=32	.148	-.405*	-.390	-.297	-.300	-.506**
QCAT n=32	.150	-.360	-.364**	-.329	-.317	-.428*
NVCAT n=32	.095	-.169	-.280	-.147	-.122	-.191
MAT n=31	-.074	-.468	-.443*	-.412	-.465**	.513**

* $p < .05$, ** $p < .01$, both two-tailed.

Table 9

Types of Short-cut Problems

Problem Type	Example
Product less than multiplicand	136 x 19 = 124
Product equal to multiplicand	146 x 16 = 146
Closure	123 x 15 = 1845.5
Commutativity	215 x 13 = 2795 13 x 215 = 2795

Table 10

Criteria for Classification of Strategies on the Short-Cut Problems

Strategy	Writing on the short-cut problem	Accuracy on the short-cut problem[b]	Number of corresponding standard problems on which writing is present (max=6)	Number of correctly verified corresponding standard problems (max=6)	Number of corresponding standard problems with latencies greater than latency of the short-cut (max=6)
H1	No	1	a) 6	a) 0 - 6	a) 6
H1A	No[a]	1	b) 0 - 5	b) 6	b) 6
H2	No	1	5 or fewer	2 or fewer	5 or more
H2A	No[a]	1	5 or fewer	2 or fewer	5 or more
G	No	1	4 or fewer	3 or more	0 - 6
CS1	No	1	5 or fewer	2 or fewer	4 or fewer
CS2	No	0	0 - 6	0 - 6	0 - 6
CS3	Yes	a) 1 b) 0	0 - 6	0 - 6	0 - 6

[a] writing present but not completed. [b] 1 is correct, 0 is incorrect

Table 11

Frequency of Strategies as a Function of Grade

	Strategy							
Grade	H1	H1A	H2	H2A	G	CS1	CS2	CS3
4	101	11	33	1	11	26	308	5
6	123	36	28	3	37	31	244	10

Table 12

Total Percentages of Heuristic Use as a Function of Grade, Problem Type, and Problem Size

Grade	Product less than multiplicand		Product equal to multiplicand		Closure		Commutativity	
	2 x 1	3 x 2	2 x 1	3 x 2	2 x 1	3 x 2	2 x 1	3 x 2
4	16.13	24.19	37.10	27.42	16.12	12.90	53.22	43.54
6	21.87	32.81	51.56	46.87	21.87	28.12	45.31	54.68

Table 13

Correlations Among Types of Heuristics for Grade 4

	HPLM	HPEM	HCLS
HCOM	.259	.302	.076
HPLM		.763***	.734***
HPEM			.693***

Note. n = 31.
*** $p < .001$, two-tailed

Table 14

Correlations Among Types of Heuristics

for Grade 6

	HPLM	HPEM	HCLS
HCOM	.587***	.665***	.319
HPLM		.684***	.526***
HPEM			.506**

Note. n = 32.
** $p < .01$. *** $p < .001$, both two-tailed.

Table 15

Correlations Among Heuristics and VCAT, QCAT, NVCAT, and Math Survey Test for Grade 4

	HCOM	HPLM	HPEM	HCLS	HTOT
VCAT n=28	.022	.024	.139	.062	.086
QCAT n=28	.095	.025	.110	-.025	.078
NVCAT n=28	.082	.317	.336	.347	.353
MST n=29	.341	.245	.415*	.299	.426*

* $p < .05$, two-tailed.

Table 16

Correlations Among Heuristics and VCAT, QCAT, NVCAT, and Math Survey Test for Grade 6

	HCOM	HPLM	HPRM	HCLS	HTOT
VCAT n=32	375*	510**	543**	389*	560***
QCAT n=32	351	560***	454**	620***	594***
NVCAT n=32	200	493**	354*	543**	476**
MST n=31	454**	600***	639***	739***	735***

* p < 05, ** p < 01, *** p < 001, all two-tailed

Table 17

Percentage of Subjects who Correctly Explained Principles as a Function of Grade, Problem Type and Problem Size

Grade	Product less than multiplicand		Product equal to multiplicand		Closure		Commutativity	
	2 x 1	3 x 2	2 x 1	3 x 2	2 x 1	3 x 2	2 x 1	3 x 2
4	41.93	25.81	61.30	45.16	48.39	45.16	96.77	93.55
6	65.62	59.37	93.75	84.37	75.00	78.12	90.62	84.37

Table 18

Correlations Among Explanantions for

Grade 4

	EPLM	EPEM	ECLS
ECOM	045	170	019
EPLM		846***	595***
EPEM			618***

Note n = 31
*** p < 001, two-tailed

Table 19

Correlations Among Explanations for

Grade 6

	EPLM	EPEM	ECLS
ECOM	566***	408*	185
EPLM		554***	648***
EPEM			396*

Note n = 32
* p < 05, *** p < 001, both
two-tailed

Table 20

Correlations Among Explanations and VCAT, QCAT, NVCAT, and Math Survey Test for Grade 4

	ECOM	EPLM	EPHM	ECLS	ETOT
VCAT n=28	- 218	209	215	237	229
QCAT n=28	- 314	236	305	135	229
NVCAT n=28	- 338	287	200	258	249
MST n=29	137	420*	301	296	408*

* $p < .05$, two-tailed

Table 21

Correlations Among Explanations and VCAT, QCAT, NVCAT, and Math Survey Test for Grade 6

	ECOM	EPLM	EPEM	ECLS	ETOT
VCAT n=32	.404*	.406*	.043	.516**	.530**
QCAT n=32	.340*	.407*	.175	.566***	.510**
NVCAT n=32	.289	.205	.197	.153	.261
MST n=31	.517**	.514**	.296	.567***	.640***

* p < .05, ** p < .01, *** p < .001, all two-tailed.

Table 22

Percentages of Subjects who Generalized Principles as a Function of Grade and Problem Type

Grade	Product less than multiplicand	Product equal to multiplicand	Closure
4	16.12	6.45	29.03
6	43.75	12 50	43.75

Table 23

Correlations Between Speed of Execution of Multiplication Algorithms and Use of Heuristics for Grade 4

	HCOM	HPLM	HPEM	HCLS	HTOT
I	-.043	.062	.012	.222	.109
IIa	.068	-.088	-.142	-.071	-.077
IIb	.082	-.196	-.215	-.150	-.153
III	.090	-.012	-.069	-.023	-.006
IV	.047	.089	-.004	.073	.059
V	.093	-.062	-.128	-.057	-.051

Note. n = 31

Table 24

Correlations Between Speed of Execution of Multiplication Algorithms and Use of Heuristics for Grade 6

	HCOM	HPLM	HPEM	HCLS	HTOT
I	206	077	081	-.200	006
IIa	-262	-202	-375*	-426*	-.388*
IIb	-.146	-246	-.449**	-.351*	-.373*
III	-243	-.269	-.413**	-533**	-.441*
IV	-161	-142	-321	-.477**	-331
V	-.187	-.289	-.440**	-.384*	-.399*

Note n = 32.
* $p < 05$. ** $p < 01$, both two-tailed.

Table 25

Correlations Between Speed of Execution of Multiplication Algorithms and Explanations for Grade 4

	ECOM	EPLM	EPEM	ECLS	ETOT
I	.088	-.098	.060	.083	.031
IIa	.032	-.245	-.237	-.159	-.236
IIb	.085	-.341	-.315	-.336	-.337
III	-.004	-.142	-.164	-.186	-.185
IV	.011	-.159	-.190	.018	-.124
V	.074	-.230	-.267	-.195	-.247

Note. n = 31

* $p < .05$, two-tailed.

Table 26

Correlations Between Speed of Execution of Multiplication Algorithms and Explanatons for Grade 6

	ECOM	EPLM	EPEM	ECLS	ETOT
I	-.008	-.122	-.021	-.155	-.150
IIa	-.398*	-.475**	-.164	-.562***	-.587***
IIb	-.457**	-.509**	-.014	-.501**	-.558***
III	-.278	-.411*	-.012	-.508**	-.453**
IV	-.481**	-.473**	-.230	-.501**	-.608***
V	-.523**	-.482**	-.218	-.532**	-.629***

Note $n = 32$.
* $p < .05$, ** $p < .01$, *** $p < .001$,
all two-tailed

Table 27

Patterns of Performance on Task 2 and 3 and Number of Subjects in Each Group as a Function of Grade and Problem Type

Grade	n	Pattern	Product less than multiplicand		Product equal to multiplicand		Closure		Commutativity	
			Small	Large	Small	Large	Small	Large	Small	Large
4	31	$\bar{H}\bar{E}$	16	17	11	14	14	17	0	2
		$\bar{H}E$	7	3	7	7	8	9	9	10
		$H\bar{E}$	2	4	1	3	2	0	1	0
		HE	6	7	12	7	7	5	21	19
6	32	$\bar{H}\bar{E}$	8	9	1	5	8	6	3	4
		$\bar{H}E$	6	5	10	10	14	12	9	7
		$H\bar{E}$	3	3	1	1	0	3	0	1
		HE	15	15	20	16	10	11	20	20

VIII. References

Aiken, L.R. (1971). Intellective variables and mathematics achievement: Directions for research. Journal of School Psychology, 9, 201 - 212.

Anastasi, A. (1976). Psychological testing. New York: Macmillan.

Baroody, A.J., & Mason, C.A. (1984). The case of Brian: An additional explanation for production deficiences. In J. Moser (Ed.), Proceedings of the Sixth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Madison: Wisconsin Center for Educational Research.

Bell, A.W., Costello, J., & Kuchemann, D. (1983). A review in research in mathematical education. Windsor: Nfer-Nelson.

Brown, A.L. (1975). The development of memory: Knowing, knowing about knowing, and knowing how to know. In H.W. Reese (Ed.), Advances in child development and behavior: Vol. 10. New York: Academic Press.

Brown, J.S., & Van Lehn, K. (1980). Repair theory: A generative theory of bugs in procedural skills. Cognitive Science, 4, 379-426.

Brown, J.S., & Van Lehn, K. (1982). Toward a generative theory of bugs in procedural skills. In T. Carpenter, J.M. Moser, & T. Romberg (Eds.), Addition and subtraction: A cognitive perspective (pp. 117 - 135). Hillsdale, New York: Lawrence Erlbaum Associates.

Campbell, J.I., & Graham, D.J. (1985). Mental multiplication skill: Structure, process, and acquisition. Canadian Journal of Psychology, 39, 338 - 366.

Case, R. (1984). The process of stage transition: A neo-Piagetian view. In R.J. Sternberg (Ed.), Mechanisms of cognitive development (pp. 19 - 44). New York: Freeman.

Cohn, R. (1968). Developmental dyscalculia. Pediatric Clinics of North America, 15, 3,

Flavell, J.H. (1970). Developmental studies of mediated memory. In H.W. Reese & L.P. Lipsitt (Eds.), Advances in child development and behavior: Vol. 5. New York: Academic Press.

Gelman, R. & Meck, E. (1983). Preschoolers counting: Principles before skill. Cognition, 13, 343 - 359.

Ginsburg, H.P. (1977). Children's arithmetic: The learning process. New York: D.Van Nostrand Company.

Greeno, J.G., Riley, M.S., & Gelman, R. (1984). Conceptual competence and children's counting. Cognitive Psychology, 16, 94 - 143.

Kahneman, D. (1973). Attention and effort. Englewood Cliffs, New York: Prentice Hall.

Kaye, D.B. (1986). The development of mathematical cognition. Cognitive Development, 1, 157 - 172.

LaBerge, D., & Samuels, S.J. (1974). Toward a theory of automatic processing in reading. Cognitive Psychology, 6, 293 - 323.

McLeod, T., & Armstrong, S.W. (1982). Learning disabilities in mathematics: skill deficits and remedial approaches at the intermediate and secondary level. Learning Disability Quarterly, 5, 305 - 316.

Miller, K., Perlmutter, M., & Keating, D. (1984). Cognitive arithmetic: Comparison of operations. Journal of Experimental Psychology: Learning, Memory, and Cognition, 10, 46 - 60.

Maccoby, E.E., & Jacklin, C.N. (1974). The psychology of sex differences. Stanford, California: Stanford University Press.

Parkman, J.M. (1972). Temporal aspects of simple multiplication and comparison. Journal of Experimental Psychology, 95, 437 - 444.

Perfetti, C.A., & Lesgold, A.M. (1977). Discourse comprehension and source of individual differences. In P. Carpenter & M. Just (Eds.), Cognitive processes in comprehension. Hillsdale, New York: Erlbaum.

Posner, M.J., & Snyder, C.R.R. (1975). Attention and cognitive control. In R.L. Solso (Ed.), Information processing and cognition: The Loyola Symposium. Hillsdale, New York: Erlbaum.

Resnick, L.B. (1980). The role of invention in the development of mathematical competence. In R. Kluwe, & H. Spada (Eds.), Developmental models of thinking (pp. 213 - 243). New York: Academic Press.

Resnick, L. B. (1983). A developmental theory of number understanding. In H.P Ginsburg (Ed.), The development of mathematical thinking, (pp. 110 - 149). New York: Academic Press.

Ross, R. (1964). A description of twenty arithmetic underachievers. The Arithmetic Teacher, 11, 235 - 241.

Shiffrin, R.M., & Schneider, W. (1977). Controlled and automatic human information processing: Perceptual learning, automatic attending, and a general theory. Psychological Review, 84, 127 - 190.

Salatas, H., & Flavell, J.H. (1976). Behavioral and metamnemonic indicators of strategic behaviors under remember instructions in first grade. Child Development, 47, 81 - 89.

Siegler, R.S. (1987). Strategy choice procedures and the development of multiplication skills. Manuscript submitted for publication.

Skemp, R.R. (1976). Relational understanding and instrumental understanding. Math Teaching, 77, 1 - 7.

Stazyk, E.H., Ashcraft, M.H., & Hamann, M.S. (1982). A network approach to simple multiplication. Journal of Experimental Psychology: Learning, Memory, and Cognition, 8, 320 - 335.

Steffe, L.P., & Blake, R.N. (1983). Seeking meaning in mathematical instruction: A response to Gagne. Journal in Research in Mathematics Education, 14, 210 - 213.

Sternberg, R.J. (1984). Mechanisms of cognitive development: A componential approach. In R.J. Sternberg (Ed.), Mechanisms of cognitive development (pp. 163 - 186). New York: Freeman.

Weinstein, M.L. (1978). Dyscalculia: A Psychological and Neurological Approach to Learning Disabilities in Mathematics in School Children. Doctoral dissertation, University of Pennsylvania.

Zbrodoff, N.J., & Logan, G.D. (1986). On the autonomy of mental processes: A case study of arithmetic. Journal of Experimental Psychology: General, 115, 118-130.

IX. Appendix 1: Task 1

TASK 1

NAME:

GRADE:

DATE:

I would like to know how quickly and accurately you can write and multiply numbers. I am going to ask you to write some numbers and to solve several multiplication problems. Some of these problems will be easy and some may be hard. Please, work as quickly as you can but try not to make any errors.

This is a task to see how quickly you can write numbers. There are several numbers below. On the right side of each number there is a line. I want you to copy each number onto this line. Please, do that as quickly as you can but do not skip any number.

5 ________

0 ________

1 ________

9 ________

There are more numbers on the next two pages. Please, copy them as quickly as you can without skipping any. When you have finished the first page, stop. Do not go to the next page, until you are asked to do so.

IA

4 ________

3 ________

9 ________

2 ________

5 ________

1 ________

7 ________

0 ________

8 ________

6 ________

4 ________

2 ________

8 ________

1 ________

6 ________

STOP

IB

3 ______

1 ______

7 ______

0 ______

5 ______

9 ______

4 ______

2 ______

6 ______

8 ______

5 ______

1 ______

9 ______

3 ______

0 ______

STOP

This is a task to see how quickly and accurately you can multiply numbers. There are several multiplication problems below. Please, solve them as quickly as you can but try not to make any errors:

2 x 3	7 x 5
34 x 2	31 x 3
20 x 14	34 x 22
94 x 37	58 x [illegible]4

There are more multiplication problems for you to solve on the next pages. Solve them as quickly and accurately as you can. When you have finished each page, stop. Please, do not go to a next page until you are asked to do so.

IIaA

```
  2        3
x 4      x 2
---      ---

  2        3
x 2      x 3
---      ---

  4        8
x 2      x 1
---      ---

  5        2
x 0      x 3
---      ---

  0        3
x 9      x 1
---      ---
```

STOP

IIaB

$$\begin{array}{r} 7 \\ \times\ 0 \\ \hline \end{array}$$

$$\begin{array}{r} 2 \\ \times\ 3 \\ \hline \end{array}$$

$$\begin{array}{r} 2 \\ \times\ 2 \\ \hline \end{array}$$

$$\begin{array}{r} 1 \\ \times\ 5 \\ \hline \end{array}$$

$$\begin{array}{r} 2 \\ \times\ 4 \\ \hline \end{array}$$

$$\begin{array}{r} 3 \\ \times\ 3 \\ \hline \end{array}$$

$$\begin{array}{r} 9 \\ \times\ 1 \\ \hline \end{array}$$

$$\begin{array}{r} 3 \\ \times\ 2 \\ \hline \end{array}$$

$$\begin{array}{r} 4 \\ \times\ 2 \\ \hline \end{array}$$

$$\begin{array}{r} 0 \\ \times\ 4 \\ \hline \end{array}$$

STOP

IIbA

$$\begin{array}{r} 4 \\ \times\ 7 \\ \hline \end{array} \qquad \begin{array}{r} 8 \\ \times\ 8 \\ \hline \end{array}$$

$$\begin{array}{r} 9 \\ \times\ 5 \\ \hline \end{array} \qquad \begin{array}{r} 5 \\ \times\ 3 \\ \hline \end{array}$$

$$\begin{array}{r} 6 \\ \times\ 8 \\ \hline \end{array} \qquad \begin{array}{r} 7 \\ \times\ 5 \\ \hline \end{array}$$

$$\begin{array}{r} 5 \\ \times\ 5 \\ \hline \end{array} \qquad \begin{array}{r} 3 \\ \times\ 9 \\ \hline \end{array}$$

$$\begin{array}{r} 3 \\ \times\ 4 \\ \hline \end{array} \qquad \begin{array}{r} 6 \\ \times\ 3 \\ \hline \end{array}$$

STOP

IIbB

7 x 6 = ___	4 x 4 = ___
5 x 8 = ___	9 x 7 = ___
9 x 9 = ___	4 x 8 = ___
5 x 6 = ___	3 x 7 = ___
8 x 2 = ___	6 x 9 = ___

STOP

IIIA

```
   10        40
x   3     x   2
-----     -----

   21        32
x   4     x   2
-----     -----

   43        22
x   2     x   3
-----     -----
```

STOP

IIIB

[illegible] x 4 ___	23 x 3 ___
32 x 3 ___	34 x 2 ___
14 x 2 ___	21 x 3 ___

STOP

IVA1

$$\begin{array}{r} 12 \\ \times\ 24 \\ \hline \end{array}$$

STOP

IVA2

$$\begin{array}{r} 23 \\ \times\ 32 \\ \hline \end{array}$$

STOP

IVA3

```
   14
x  12
-----
```

STOP

IVB1

$$
\begin{array}{r} 31 \\ \times\ 13 \\ \hline \end{array}
$$

STOP

IVB2

```
  43
x 12
----
```

STOP

IVB3

```
  14
x 21
----
```

STOP

VA1

$$\begin{array}{r} 29 \\ \times\ 83 \\ \hline \end{array}$$

STOP

VA2

$$\begin{array}{r} 52 \\ \times\ 75 \\ \hline \end{array}$$

STOP

VA3

$$\begin{array}{r} 63 \\ \times\ 78 \\ \hline \end{array}$$

STOP

VB1

```
  67
x 28
____
```

STOP

VB2

```
  47
x 53
```

STOP

VB3

X. Appendix 2: Tasks 2 and 3

TASK 2

NAME:

GRADE:

DATE:

I want you to imagine that you are an arithmetic teacher. You asked your class to solve a number of multiplication problems as a homework exercise. Now you want to find out whether each student solved the problems correctly or not. Because you are going to ask the students to correct their own errors, you do not have to find the correct answer for every problem. As soon as you are sure that the answer is wrong, mark it with "0". If the answer is right, mark it with "1". There are several problems below. Mark them as quickly as you can but try not to miss any wrong answer. If you need to multiply the numbers yourself, you can write on the side of the page. If you do not have to multiply the numbers yourself, just mark "0" or "1".

74 x 5 = 370

37 x 3 = 32

123 x 12 = 1476

12 x 123 = 1456

24 x 2 = 48.8

On the next four pages there is a complete work of one student. Mark it as quickly as you can but try not to miss any errors. Remember that you do not have to find the answer for every problem, but if you need to multiply the numbers yourself, do that on the side of the page. When you have finished each page, stop. Please, do not go to a next page until you are asked to do so.

A

A1) 84 x 7 = 568

A2) 215 x 13 = 2795

A3) 13 x 215 = 2795

A4) 44 x 7 = 308

A5) 87 x 4 = 78

A6) 253 x 12.3 = 3131.9

A7) 241 x 14 = 3374.4

A8) 32 x 6 = 32

STOP

B

B1) 143 x 12 = 126

B2) 49 x 6.3 = 308.7

B3) 214 x 16 = 214

B4) 231 x 15 = 3485

B5) 54 x 3 = 162.2

B6) 64 x 4 = 236

B7) 4 x 64 = 256

B8) 127 x 14 = 1778

STOP

C

C1) 135 x 17 = 135

C2) 148 x 14 = 2072

C3) 134 x 12 = 1628

C4) 12 x 134 = 1608

C5) 36.1 x 5 = 170.5.

C6) 67 x 5 = 55

C7) 98 x 6 = 588.8

C8) 69 x 6 = 414

STOP

D

D1) 134.5 x 13 = 1748.5

D2) 216 x 11 = 206

D3) 72 x 5 = 360

D4) 5 x 72 = 360

D5) 213 x 14 = 2882

D6) 123 x 11 = 1353.3

D7) 84 x 6 = 84

D8) 79 x 4 = 326

STOP

345 x 12 = 320

29 x 3 = 27

208 x 23 = 4784

23 x 208 = 4784

421 x 21 = 421

96 x 6 = 96

194 x 23 = 4462.2

54 x 6 = 324.4

48 x 9 = 432

9 x 48 = 432