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The Behavior of Restrained Reinforced Concrete Columns Under Sustained Load

by

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and

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STIFFNESS INFLUENCE COEFFICIENTS FOR NON-AXISYMMETRICAL LOADING ON CLOSED CYLINDRICAL SHELLS

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ABSTRACT

For cylindrical shells non-axisymmetrical loading is generally considered in the form of a periodic Fourier series. Since each term of the series must satisfy the boundary conditions, the solution is simplified by the use of stiffness coefficients. To permit a method of solution closely following the classical stiffness matrix method expressions for stiffness influence coefficients for all the harmonics of loading greater than zero are presented as general algebraic expressions in closed form. Two sets of expressions were obtained by introducing approximations into the classical theories. The range of applicability of these sets is given based on criteria involving the shell parameters and the harmonic number of the edge loading considered. A further simplification is possible for evaluating membrane stresses for very large values of the loading harmonic.

The values of coefficients and the stress resultants obtained for two examples are compared with those obtained using Flügge's equations for these shells. It is concluded that the use of these coefficients constitute a practical procedure for the solution of these types of problems.

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LIST OF SYMBOLS

$[C_1]$ = Coefficient matrix for displacements

$[C_2]$ = Coefficient matrix for forces

$[\overline{C}]$ = Coefficient matrix of constants

C_1 to C_8 = Arbitrary constants

$\overline{C}_1$ to $\overline{C}_8$ = Arbitrary constants

$[D]$ = Matrix of displacements

$\overline{D}$ = Extensional rigidity , $EP/(1-\upsilon^2)$

E = Young's Modulus

F_1 to F = Characteristic functions of the Basic Solution

$[K]$ = Stiffness matrix

$[K_1]$ = Stiffness matrix using Shell Theory sign convention

$[K_2]$ = Stiffness matrix using Stiffness matrix sign convention

$[K_3]$ = Stiffness matrix using modified stiffness matrix sign convention

$\overline{K}$ = Flexural rigidity, $EP^3/12(1 - \upsilon^2)$

M_x, M_ϕ = Bending moments

$M_{x\phi}, M_{\phi x}$ = Twisting moments

$\left.\begin{matrix} N_x, N_\phi \\ N_{x\phi}, N_{\phi x} \end{matrix}\right\}$ = Membrane stress resultants

Q_x, Q_ϕ = Radial shears

S_x, S_ϕ = Effective nomal shears

T_x, T_ϕ = Effective membrane shears

P = h/r

Q = ℓ/r

$[\overline{SM}]$ = Matrix obtained from $[SM1]$ and $[SM2]$

$[SM]$ = Shell stiffness matrix obtained from $[SM]$

$[SM1]$ = Stiffness matrix for Edge Effects Solution

$[SM2]$ = Stiffness matrix for Basic Solution

$[SM3]$ = Stiffness matrix for generalised plane stress solution

$[SM4]$ = Stiffness matrix for plate bending solution

a = $\frac{h}{\sqrt{3}\, r}$

a_1 to a_8 = Characteristic coefficients

c = Multiplier

c_1 = $(1+\upsilon)/(1-\upsilon)$

f = Any quantity either a displacement or a force

$\overline{f}$ = Maximum value of f in the expression $f \begin{matrix} \cos m\phi \\ \sin m\phi \end{matrix}$

f_1 to f_8 = Characteristic functions for displacements and stresses

g_1 = $e^{-x_1 y} \cos x_1 y$

g_2 = $e^{-x_1 y} \sin x_1 y$

h = Thickness

k = $h^2/12r^2$

k' = $h^2/12r^2(1-\upsilon^2)$

ℓ = Length

m = Harmonic number

r = Radius

s = $\bar{D}(1-\upsilon)/(3-\upsilon)$

s_1 to s_4 = Characteristic functions for edge effects solution

u,v,w = Non-dimensional displacements along the co-ordinate directions χ,ϕ,z respectively

x = A co-ordinate
Also a subscript to denote x direction

y = x/r

$\bar{y}$ = $Q - y$

α_j,β_j = Complex quantities relating u and v respectively to w

$\alpha_1 = -\alpha_6$ = $\alpha_1 + \alpha_2$

$\alpha_2 = -\alpha_5$ = $\alpha_1 + i\alpha_2$

$\alpha_3 = -\alpha_8$ = $\alpha_3 + i\alpha_4$

$\alpha_4 = -\alpha_7$ = $\alpha_3 - i\alpha_4$

$\beta_1 = \beta_6$ = $\beta_1 + i\beta_2$

$\beta_2 = \beta_5$ = $\beta_1 - i\beta_2$

$\beta_3 = \beta_8$ = $\beta_3 + i\beta_4$

$\beta_4 = \beta_7$ = $\beta_3 - i\beta_4$

γ = A constant

λ_1 to λ_8 = Roots of the characteristic equation

λ_1 = $-\chi_1 + i\mu_1$

λ_2 = $-\chi_1 - i\mu_1$

λ_3 = $-\chi_2 + i\mu_2$

λ_4 = $-\chi_2 - i\mu_2$

λ_5 = $\chi_1 + i\mu_1$

λ_6 = $\chi_1 - i\mu_1$

λ_7 = $\chi_2 + i\mu_2$

λ_8 = $\chi_2 - i\mu_2$

μ = $[3(1-\nu^2)\ r^2/h^2]^{1/4}$

ρ = A criterion for replacing the basic solution by the membrane solution

υ = Poisson's ratio

Φ = Potential function

ϕ = A co-ordinate and also a subscript to denote ϕ direction

=

χ_j, μ_j = Real and imaginary parts respectively in terms of which the root λ_j is expressed

Other Notations:

$(\ )'$ = $\frac{\partial}{\partial y}(\ )$

$(\ )^{\bullet}$ = $\frac{\partial}{\partial \phi}(\ )$

$(\ )^{bas}$ = $(\ )^{basic}$

$(\ )^{edg}$ = $(\ )^{edge}$

$(\ )^{t}$ = $(\ )^{top}$

$(\ )^{b}$ = $(\ )^{bottom}$

Note:- A displacement or a force term with a bar on the top indicates the maximum value

CHAPTER I

INTRODUCTION

1.1 PURPOSE OF THE STUDY

An unsymmetrical state of stress is obtained in shells of revolution such as water tanks, nuclear reactors, pressure vessels, silos and chimneys by one of these causes:

1. arbitrary loading varying along the directrix as in the case of wind and seismic actions,
2. the supporting reactions being 'unsymmetrical' or discontinuous as in the case of a shell supported on isolated columns,
3. the middle surface being incomplete.

The symmetrical state of stress results from particular cases of these conditions.

The classical method of analysis for arbitrary loading is to express the loading as a Fourier series, analyse for each harmonic separately and then superimpose the effects. The number of terms considered in this series must be sufficient to give the desired degree of convergence. An example of this procedure is the consideration of wind and seismic loading. With regard to wind action the normal component p_w of the equivalent static load is expressed as

$$p_w = g_w \ k(\phi)$$

where $k(\phi)$ is the shape coefficient and g_w is the basic dynamic pressure which depends on factors affecting wind pressure distribution. From studies on cylindrical models in wind tunnels several investigators have expressed the law of variation $k(\phi)$ for smooth and rough surfaces as a trigonometric series in the form,

$$k(\phi) = a_o + a_1\cos\phi + a_2\cos2\phi + a_3\cos3\phi + a_4\cos4\phi + \ldots$$

The usual procedure in design is to consider only the first harmonic $a_1\cos\phi$ ignoring the higher harmonics. The analysis is accomplished by the beam theory treating the structure as a cantilever beam or by the membrane theory of shells. This procedure is satisfactory only if the edges fully satisfy the required boundary conditions. According to Hruban (1) for certain stress resultants such as the bending moments, the higher harmonics, especially the second one, may represent very significant components.

An example of higher harmonic loading due to intermittent supporting reactions is the case of a cooling tower supported on eight columns. Here the reactions when expressed as a Fourier series will contain the harmonics 8, 16 , 24 etc.

With the advent of the digital computer, structural analysis may be conveniently done by matrix methods using influence coefficients. In the literature, the analysis for symmetrical loading is usually performed using influence coefficients which have been given in an explicit manner. For unsymmetrical bending there are no corresponding explicit expressions. Although one can theoretically write the exact expressions based on a rigorous theory, this procedure involves the solution of eight simultaneous equations, expressed in general terms of geometry of the shell surface and the physical constants. Moreover, the final expressions for influence coefficients, when written in an explicit manner, will be too unwieldy to be of practical use. For these reasons the expressions for influence coefficients have to be established based on approximations. Therefore the approximate methods of analysis proposed in the literature are reviewed in detail in the next section.

The purpose of this study is to investigate the possibility of establishing these influence coefficients in closed form for practical use.

1.2 A BRIEF HISTORICAL REVIEW OF SHELL THEORY

The general theory of thin shells was first initiated by Love (2) in 1890. Flügge (3) in 1932 presented the basic equations of the theory in the form mostly used today. Because of the complicated nature of the rigorous theories proposed by Love and Flügge, simplifications were proposed by Finsterwalder (4) in 1932, Donnell (5) in 1934 and Shorer (6) in 1935. It was not until after 1940 that Donnell's theory gained wide recognition. In establishing his equations Donnell rigorously discarded the lesser important terms. The Donnell theory was apparently derived independently by Von Kármán (7) in 1941, Jenkins (8) in 1947 and

Vlassov (9) in 1947.

The problem of unsymmetrical bending of closed cylinders was first investigated by Schwerin (10)in 1922 followed by Miesel (11)in 1930. Hoff (12) in 1954 gave an explicit solution for Donnell's equations. In 1957 Holand (13) proposed explicit solutions using a modified form of Flügge's equations. An approximate method of analysis with possible simplifications was given by Flügge (14) in 1960. Goldenveiser (15) in 1961 made a detailed study of the mathematical aspects of the shell theory and proposed simplified solutions.

1.3 REVIEW OF EXISTING METHODS OF ANALYSIS

A detailed examination of the more important works considering unsymmetrical bending of closed circular cylindrical shells follows. Goldenveiser and Flügge have presented discussions on the nature of the shell characteristic equations, while Hoff and Holand have proposed solutions to these equations.

GOLDENVEISER

Making the assumption that a<<1 where $a^2 = h^2/3r^2$, Goldenveiser obtains the following eighth order characteristic equation.

$$\lambda^8 - 4m^2\lambda^6 + [6m^4 - (8-2\upsilon^2)m^2 + \frac{(1-\upsilon^2)}{a^2}]\lambda^4 - 4m^2(m^2 - 1)^2 \lambda^2 + m^4 (m^2 - 1)^2 = o \qquad (1-1)$$

For values of $m \geqslant 1$, this equation reduces to the following forms.

For $m = 1$

$$\begin{aligned}(1 - \nu^2)\lambda^4 &= o\\ \lambda^4 + \frac{(1 - \nu^2)}{a^2} &= o\end{aligned}\qquad (1\text{-}2a,b)$$

For $1 < m < a^{-\frac{1}{2}}$

$$\begin{aligned}(1 - \nu^2)\lambda^4 + a^2m^8 &= o\\ \lambda^4 + \frac{(1 - \nu^2)}{a^2} &= o\end{aligned}\qquad (1\text{-}3a,b)$$

For $m \gg a^{-\frac{1}{2}}$

$$(\lambda^2 - m^2)^4 + \frac{(1 - \nu^2)}{a^2}\lambda^4 = o \qquad (1\text{-}4)$$

For $m \ggg a^{-\frac{1}{2}}$

$$(\lambda^2 - m^2)^4 = o \qquad (1\text{-}5)$$

Depending on the value of m each form of these eqations will have eight complex roots, λ_1 to λ_8, expressed in terms of real and imaginary parts, χ_1,χ_2 and μ_1,μ_2, respectively. EQUATIONS (1-2a) and (1-3a) are used to obtain the four roots expressed in terms of χ_2, μ_2 which are small in magnitude and EQUATIONS (1-2b) and (1-3b) are used to obtain the four roots expressed in terms of χ_1, μ_1 which are large in magnitude. EQUATION (1-4) as well as EQUATION (1-5) gives all the eight roots expressed in terms of χ_1, μ_1 and χ_2, μ_2, χ_1 and χ_2 being the same order of magnitude.

The stresses and displacements at any point in the shell are expressed in terms of a potential function Φ as

$$\begin{aligned}\Phi = \Big[&e^{-\chi_2 y}\left[C_1\cos\mu_2 y + C_2\sin\mu_2 y\right] + e^{+\chi_2 y}\left[C_3\cos\mu_2 y + C_4\sin\mu_2 y\right]\\ &+ e^{-\chi_1 y}\left[C_5\cos\mu_1 y + C_6\sin\mu_1 y\right] + e^{+\chi_1 y}\left[C_7\cos\mu_1 y + C_8\sin\mu_1 y\right]\Big]\cos m\phi\end{aligned} \qquad (1\text{-}6)$$

For $1 \leq m < a^{-\frac{1}{2}}$ the χ_1- solutions are termed the edge effects' because of their highly damped nature and the χ_2-solutions as the 'basic' because of their lightly damped nature. For $m \geqslant a^{-\frac{1}{2}}$ this classification does not apply since the rate of damping is similar in both the cases.

To consider the boundary conditions for any particular problem, it is necessary to evaluate the eight constants of integration C_1 to C_8. Goldenveiser proposes two methods for evaluating these constants, one applicable for $m < a^{-\frac{1}{2}}$ and the other for $m \geqslant a^{-\frac{1}{2}}$

For $m < a^{-\frac{1}{2}}$ the stress system corresponding to the basic state and the edge effects can be determined separately using EQUATIONS (1-2) or (1-3) and then superimposed. According to this any tangential displacement or force T can be expressed as,

$$T = T^{bas}(y,C_1,C_2,C_3,C_4) + T^{edg}(y,C_5,C_6,C_7,C_8) \qquad (1\text{-}7)$$

In a similar manner any non tangential displacement or force can be written as,

$$S = S^{bas}(y_1,C_1,C_2,C_3,C_4) + S^{edg}(y,C_5,C_6,C_7,C_8) \qquad (1\text{-}8)$$

In the basic state of stress, tangential forces are large in comparison with transverse forces and moments. In the edge effect, the strain in the middle surface of the shell is determined primarily by the normal deflection and rotation. It therefore follows that on the right hand side of EQATION (1-7) the second term is much smaller compared to the first term. Neglecting this term, EQUATION (1-7) can be written as,

$$T = T^{bas}(y, C_1, C_2, C_3, C_4) \qquad (1\text{-}9)$$

Using this equation the constants C_1 to C_4 can be determined from a set of four tangential boundary conditions. Recognising the highly damped nature of the χ_1 - solutions, EQUATION (1-8) can be written for the top and bottom edges separately as,

$$S^{top} = S^{bas}(y, C_1, C_2, C_3, C_4) + S^{edg}(y, C_5, C_6, o, o) \qquad (1\text{-}10a,b)$$

$$S^{bot} = S^{bas}(y, C_1, C_2, C_3, C_4) + S^{edg}(y, o, o, C_7, C_8)$$

As C_1 to C_4 have been already found, the remaining constants C_5 to C_8 can be determined using EQUATIONS (1-10a,b) from a set of four non tangential boundary conditions.

For $m > a^{-\frac{1}{2}}$ the problem is governed by the single eighth order EQUATION (1-4) or (1-5). Since both χ_1 and χ_2-solutions are highly damped the boundary conditions may be applied independently of each other at the two edges. The potential function Φ is expressed as

$$\Phi = \Big[e^{-\chi 1}\,[C_1 \cos\mu_1 y + C_2 \sin\mu_1 y] \; +$$

$$e^{-\chi 2}\,[C_5 \cos\mu_2 y + C_6 \sin\mu_2 y]\Big] \cos m\phi \qquad (1\text{-}11)$$

There are now only four constants, $C_1 C_2, C_5$ and C_6, to be determined from a set of four boundary conditions at the edge where the loads are applied.

For $m \gg a^{-\frac{1}{2}}$ that is for very large values of m the stress system can be subdivided into generalised plane state of stress and plate bending. The tangential displacements and forces are given by the equations of the generalised plane state of stress. The non tangential displacements and forces are given by the equation of plate bending. The equation of equilibrium for the shell element are given by these two groups of uncoupled equations each leading to a single characteristic equation of the following form.

$$(\lambda^2 - m^2)^2 = o \tag{1-12}$$

Thus for very large values of m the shell element behaves as if it were developed into a thin plate.

FLÜGGE

The eighth order characteristic equation given by Flügge is

$$\lambda^8 - 2\,(2m^2 - \nu)\lambda^6 + \left[\frac{1 - \nu^2}{k} + 6m^2(m^2 - 1)\right]\lambda^4$$
$$- 2m^2\left[2m^4 - (4 - \nu)m^2 + (2 - \nu)\right]\lambda^2 +$$
$$m^4(m^2 - 1)^2 = o \tag{1-13}$$

For small values of m this equation can be split into two independent fourth order equations as,

$$\lambda^4 - 2(2m^2 - \upsilon)\lambda^2 + \left[\frac{1 - \upsilon^2}{k} + 6m^2(m^2 - 1)\right] = o$$

$$\left[\frac{1 - \upsilon^2}{k} + 6m^2(m^2 - 1)\right]\lambda^4 - 2m^2[2m^4 - (4 - \upsilon)m^2 + (2 - \upsilon)]\lambda^2 + m^4(m^2 - 1) = o \qquad (1\text{-}14a,b)$$

EQUATION (1-14a) gives the larger roots expressed in terms of χ_1, μ_1 and EQUATION (1-14b) gives the smaller roots expressed in terms of χ_2, μ_2, the roots corresponding closely with those obtained from EQUATION (1-13). The large roots are of the order $k^{-1/4}$ and the small roots of the order $k^{1/4}$. From the table of characteristic coefficients for the displacements and the stress resultants (TABLE 1, p. 230, ref. 14) the states of stress corresponding to EQUATIONS (1-14a,b) can be determined separately. In this table, the quantities α_j are of the orders $k^{1/4}$ and $k^{-1/4}$ for the large and small roots respectively and β_j of the orders $k^{1/2}$ and 1. It is seen that for the state of stress and displacements given by EQUATION (1-14a) the membrane quantities are very small compared to the bending quantities and vice versa for the state given by EQUATION (1-14b). This means that the state of stress corresponding to EQUATION (1-13) can be found approximately by determining separately the two states of stress as mentioned above. Therefore EQUATION (1-14a) satisfy a pair of non tangential boundary conditions M_x or w' or S_x or w and EQUATION (1-14b) will satisfy a pair of tangential boundary conditions N_x or u and T_x or v. The boundary conditions corresponding to EQUATION (1-14b) are those considered in the membrane analysis. Therefore this part of the solution is only an improved form of the membrane solution and the membrane solution can be used as an approximation for the solution of EQUATION (1-14b).

For large values of m, the splitting of EQUATION (1-13) in the above manner is not valid and values of χ_1 approach χ_2 and values of μ_1 approach μ_2. Because of the highly damped nature of both χ_1 and χ_2 solutions the effect of far edge can be neglected, thus requiring the consideration of only four boundary conditions.

HOFF

Donnell's characteristic equation is expressed as

$$(\lambda^2 - m^2)^4 + \frac{1}{k'} \lambda^4 = o \tag{1-15}$$

Using this equation Hoff (12) obtained expressions for the middle surface displacements in terms of eight constants which are evaluated from the boundary conditions. This was accomplished by finding solutions for the root of the characteristic equation and the relations among the middle surface displacements.

HOLAND

In an attempt to obtain explicit solutions to the roots of Flügge's characteristic equation, Holand modified his equation in the following form,

$$\lambda^8 - 4m\sqrt{(m^2 - 1)}\lambda^6 + \left[\frac{1}{k'} + 6m^2(m^2 - 1)\right]\lambda^4 -$$

$$4m^2(m^2 - 1)\; m\sqrt{m^2 - 1}\lambda^2 + m^4(m^2 - 1)^2 = o \qquad (1\text{-}16)$$

The equation can also be written in the form

$$(\lambda^2 - m\sqrt{m^2 - 1})^4 + \frac{1}{k'}\lambda^4 = o \qquad (1\text{-}17)$$

The value of the roots as given in(SECTION A.1 of the Appendix) depend on the parameter g which is defined as $4\sqrt{k'}m\sqrt{m^2 - 1}$

For m = o or 1,

$$\chi_1 = \mu_1 = \frac{1}{\sqrt[4]{4k'}} \quad \text{and } \chi_2 = \mu_2 = o$$

The roots are therefore the same for both m = o and m = 1.

For small and large values of g, Holand gives approximate relations for the roots. He further suggests corrections to the roots obtained from EQUATION (1-17). But for all practical purposes where the shell is thin, the approximate solutions are very close to the exact ones.

Holand presents a discussion of the accuracy of the solution of the different characteristic equations, proposed by many investigators. If $(m^2 - 1)$ is replaced by m^2 in EQUATION (1-17), Donnell's equation is obtained. Neglecting terms which are of no importance to the design of thin shells, he obtains simple expressions for his characteristic coefficients.

Explicit expressions for influence coefficients neglecting the effect of far edge are also obtained by Holand . He considers a set of mixed boundary conditions expressed by the group of edge quantities M_x, N_x, and v and w. For m = 1, he obtains the solutions in two parts, a polynomial solution and a damped exponential solution. He concludes that the membrane theory can be used in place of the polynomial solution. Explicit expressions for both stresses and displacements are given for each part of the solution.

1.4 AUTHOR'S CLOSURE TO LITERATURE REVIEW

The differential equations obtained in the general theory of shells as encountered in the literature are quite complicated even when based on a consistent first approximation as in the sense conceived by Love (2). To obtain solutions it has been found necessary therefore to introduce simplifications. The review of literature points out to two types of simplifications. In the first type, approximate solutions result from a consideration of the states of stress. It is then necessary to proceed to a classification of the possible states of stress and for each state of stress, consider which simplifications or approximations are possible. This does of course result in a loss of the general character of the theory. The approximate methods of Goldenveiser and Flügge belong to this category. The second method is based on solutions to approximate eighth order characteristic eqations. Hoff's method based on Donnell's equation and Holand's simplification of Flügge's equation belong to the second category.

Only Goldenveiser (15) has done a detailed study of cylindrical shells with bending resistance under arbitrary loads. Within the mathematical framework he defines the region of applicability of his solutions with an estimate of errors in the roots. He has not made any quantitative study regarding the error in the final stress resultants.

Flügge (14) has suggested similar simplifications but has not presented any specific criteria to decide when the problems can be split.

By introducing various assumptions different investigators have obtained simplified characteristic equations. TABLE 1.1 gives a comparison of the different characteristic equations. Flügge's equation is the most accurate one based on a technical theory of shells in that he neglects only the fifth and higher powers of $\frac{h}{r}$ in the development of his theory. The other characteristic equations are therefore compared with that of Flügge.

The difference between the equations of Holand and Flügge appears in the coefficients of λ^6 and λ^2. For the usual type of shells k is of the order 10^{-4} to 10^{-7}. As such the coefficient of λ^4 is very large. The terms λ^6 and λ^2 are of little importance. Holands' equation is therefore the best approximation given so far to Flügge's exact equation.

Morley (16) makes assumptions which are tantamount to taking $\upsilon = 1$ in Flügge's equation , except in the term $(1 - \upsilon^2)/k$. This indicates that the roots are insensitive to variation in υ. Morley's computed values of λ are only slightly in error. Bijlaard's (17) equation is accurate for large values of m.

Donnell's equation is widely used because of its simplicity. In this equation only the higher powers of m are retained. Hoff (18), Morley (16) and Kempner (19) conclude that the accuracy of Donnell's equation improves as m increases. However, an investigation by Colbourne (22) shows that when m increases, the real parts of the roots, χ_1 and χ_2, approach the exact values but the imaginary parts, μ_1 and μ_2, do not agree with the exact values. Thus the effect of the error in the quantities, μ_1 and μ_2, on the final stress resultants, for high values of m, needs investigation and is discussed in CHAPTER V. Hoff points out that Donnell's equation should not be used when m is less than 4. Biezeno and Grammal (20) make a further approximation to Donnell's equation by neglecting $\frac{1 - \upsilon^2}{k}$ compared to $6m^4$. The solution is then independent of k.

It is seen that the values of the roots in the different characteristic equations depend on 'm' and $\frac{1 - \upsilon^2}{k}$. According to Hoff (16) when $\frac{1 - \upsilon^2}{k} < 2500$ the shell is relatively thick and the simplifying assumptions made in deriving the basic equations may invalidate the basic theory. When $\frac{1 - \upsilon^2}{k} > 2.5\text{x}10^7$ it represents the thinnest practical shell.

A different kind of inaccuracy results due to an inconsistency in the theory when one uses approximate equations like that of Donnell or Bijlaard. For m = 1 these equations do not give the four zero roots. These are essential for rigid body displacements required from equilibrium considerations. Any characteristic equations will have the necessary number of zero roots only if the elasticity relations are chosen in the proper way. By changing the relations even within the framework of the exact theory of shells one can lose the zero roots and obtain large errors in the final results.

On reviewing the literature it is seen that:

1. approximate solutions even where they exist have not been thoroughly investigated with regard to their effects on the final solutions and

2. no explicit expressions exist for stresses and displacements leading to the development of stiffness influence coefficients.

Therefore, the objectives of the present study can be stated as follows:

1. to study the effect of variation in the shell parameters and the harmonic number of the edge loading on the behaviour of the shell,

2. to arrive at values of stiffness influence coefficients for particular values of the shell parameters and harmonic number of the edge loading, using the exact theory,

3. to establish approximate closed form expressions for stiffness influence coefficients for all harmonics of loading and to establish the range of applicability of these expressions,

4. to solve certain specific problems using these expressions and to compare the results with those obtained by the exact theory.

TABLE 1.1

COMPARISON OF CHARACTERISTIC EQUATIONS FOR FOURIER SERIES REPRESENTATION CIRCUMFERENTIALLY

Flügge	$\lambda^8 + 2(2m^2-\upsilon)\lambda^6 + [\frac{1-\upsilon^2}{k} + 6m^2(m^2-1)\ \lambda^4$ $-2m^2[2m^4 - (4-\upsilon)m^2 + (2-\upsilon)]\lambda^2 + m^4(m^2-1)^2$	$= o$
Holand	$\lambda^8 - 4m\sqrt{m^2-1}\ \lambda^6 + [\frac{1-\upsilon^2}{k} + 6m^2(m^2-1)]\lambda^4$ $-4m^2(m^2-1)m\sqrt{m^2-1}\ \lambda^2 + m^4(m^2-1)^2$	$= o$
Morley	$\lambda^8 - 2\ (2m^2-1)\lambda^6 + [\frac{1-\upsilon^2}{k} + 6m^2(m^2-1)]\lambda^4$ $-2m^2(2m^4-3m^2+1)\lambda^2 + m^4(m^2-1)^2$	$= o$
Bijlaard	$\lambda^8 - 2\ (2m^2)\lambda^6 + [\frac{1-\upsilon^2}{k} + 6m^2(m^2-1) - \upsilon(1-\upsilon)m^2]\lambda^4$ $-2m^2\ [2m^4-(1/2)(7+\upsilon)m^2]\lambda^2 + m^4\ (m^4-2m^2)$	$= o$
Donnell	$\lambda^8 - 2(2m^2)\lambda^6 + [\frac{1-\upsilon^2}{k} + 6m^2(m^2)]\lambda^4$ $-\ 2m^2(2m^4)\lambda^2 + m^8$	$= o$
Donnell modified by Biezeno & Grammel	$\lambda^8 - 4m^2\lambda^6 + 6m^4\lambda^4 - 4m^6\lambda^2 + m^8$	$= o$

CHAPTER II

INFLUENCE COEFFICIENT PROCEDURE

2.1 APPLICATION OF MATRIX METHODS TO STRUCTURAL PROBLEMS

Structural problems which usually consist of an assemblage of many parts tend to be complex in nature. The true structure must generally be replaced by an idealised approximation or model suitable for mathematical analysis. Matrix methods using influence coefficients permit a more detailed idealisation of structures than was possible by the earlier classical procedures.

Matrix methods can be classified as force methods using flexibility influence coefficients and displacement methods using stiffness influence coefficients. Such influence coefficients relating forces and displacements reduce the given problem to a mathematical formulation in terms of a set of linear algebraic equations rather than differential equations. The solution of a large number of simultaneous equations is no more a problem with the advent of high speed electronic computers. Matrix methods have therefore come to be well recognised in structural analysis. Both force and displacement methods can be applied to a variety of complicated engineering problems.

The following basic principles are applied while setting up both types of influence coefficients.

1. Equilibrium of forces
2. Compatibility of deformations
3. Constitutive equations

2.2 FLEXIBILITY AND STIFFNESS INFLUENCE COEFFICIENTS

Assume an arbitrary linearly elastic structure supported against rigid body motion and subjected to forces $X_1, X_2, \ldots, X_n$ acting at nodes 1, 2, ..., n. Let the corresponding displacements be $d_1, d_2, \ldots, d_n$. The terms force and displacement are used here in a general sense. In matrix notation the relation between the forces and the displacements can be expressed as,

$$[d] = [C]\,[X] \tag{2-1}$$

where d is the column vector $\{d_1, d_2, \ldots, d_n\}$ and $[X]$ is another column vector $\{X_1, X_2, \ldots, X_n\}$

The square matrix [C] is known as the matrix of flexibility influence coefficients. The element c_{ij} is defined as the displacement at the node i due to unit force applied at the node j. EQUATION (2-1) represents load-deflection equations for the structure. The forces in terms of deflections can be solved from these equations.

For the same problem, the relations given by EQUATIONS (2-1) can also be restated as

$$[X] = [K]\,[d] \tag{2-2}$$

where [K] is known as the matrix of stiffness influence coefficients. The element k_{ij} is defined as the force at the node i due to unit displacement at the node j. EQUATIONS (2-2) represent the equilibrium conditions for the structure. The displacements in terms of forces can be solved from these equations.

Because of the reciprocal theorem both [C] and [K] are symmetric.

2-3 DIRECT STIFFNESS METHOD

While establishing the flexibility matrix the structure is assumed to be supported against rigid body motion, a condition not necessary for the stiffness matrix. For the latter case the structure can be free to move as a rigid body when the edge displacement is applied. The stiffness matrix [K] thus obtained is called the 'direct stiffness matrix.'

The symmetric direct stiffness matrix [K] is singular. The elements in any column represent a stress system in equilibrium. The singular nature of [K] can be removed by imposing boundary conditions on the problems. The advantage of the direct stiffness matrix method is that for a whole assembly of structural elements, the total structural stiffness matrix can be easily formed by superposition of the individual stiffness matrixes, irrespective of the boundary conditions. The boundary conditions are considered only in the actual solution of the system of equations. This permits the consideration of different boundary conditions while the total structural stiffness matrix remains unaltered.

2-4 SOLUTION PROCEDURE FOR DIRECT STIFFNESS METHOD

The total structural stiffness equations are represented by,

$$[X] = [K]\,[d] \qquad (2\text{-}3)$$

where $[X]$ is the total set of nodal forces and $[d]$, the corresponding total set of nodal displacements. Let $[d_a]$ represent the unknown nodal displacements and $[d_b]$ the known nodal displacements due to the boundary conditions. Correspondingly $[X]$ can be separated as the applied loads $[X_a]$ and the reactions $[X_b]$. Thus,

$$\left[\begin{array}{c}[X_a] \\ \hline [X_b]\end{array}\right] = \left[\begin{array}{c|c}K_{aa} & K_{ab} \\ \hline K_{ba} & K_{bb}\end{array}\right] \cdot \left[\begin{array}{c}[d_a] \\ \hline [d_b]\end{array}\right] , \qquad (2\text{-}4)$$

$$[X_a] = [K_{aa}] \quad [d_a] + [K_{ab}]\,[d_b] , \qquad (2\text{-}5)$$

$$[X_b] = [K_{ba}] \quad [d_a] + [K_{bb}]\,[d_b] . \qquad (2\text{-}6)$$

From EQUATION (2-5)

$$[d_a] = [K_{aa}]^{-1} \left[[X_a] - [K_{ab}]\,[d_b]\right] . \qquad (2\text{-}7)$$

If $[d_b]$ is a null matrix corresponding to a set of boundary displacements specified as zero , then

EQUATION (2-7) becomes,

$$[d_a] = [K_{aa}]^{-1}\,[X_a] . \qquad (2\text{-}8)$$

Values of the elements in $[X_b]$ can be found by substituting the values of the displacements $[d_a]$ obtained from the EQUATIONS (2-7) into EQUATION (2-6). If $[d_b]$ is a null matrix,

$$[X_b] = [K_{ba}]\,[K_{aa}]^{-1}\,[X_a]\,. \tag{2-9}$$

When all the nodal displacements are known, each structural element can be assumed to be subjected to an equivalent system of nodal forces designated by $[X_e]$, where

$$[X_e] = [K]\,[d]\,, \tag{2-10}$$

$[K]$ is the member stiffness matrix and $[d]$ the actual edge displacements for the members. The internal forces are related to $[X_e]$ by a relation directly obtained from the derivation of the member stiffness matrix.

2-5 APPLICATION OF DIRECT STIFFNESS METHOD TO SHELL PROBLEMS

The procedure that is used in the classical shell analysis is to consider the general solution which consists of a homogeneous part and a particular part. The general solution must satisfy the specified boundary conditions. Using the stiffness matrix given by EQUATIONS (2-3) the homogeneous part of the solution can be written in terms of the unknown edge displacements. The particular solution or the membrane solution is superimposed on the homogeneous solution. The resulting general solution is made to satisfy the specified boundary conditions and the unknown displacements are evaluated.

As explained in the previous section, the internal forces corresponding to the homogeneous solution can be determined. To get the net values of the displacements and the stress resultants the particular solution is to be superimposed on the homogeneous solution. The classical shell analysis therefore does not give the net values of the edge displacements directly.

An alternative method which is analogous to the slope deflection procedure for beam elements permits a matrix formulation to the shell problem. In the local co-ordinate system in which the final displacements are directly obtained the total structural stiffness equations for a single shell with no surface loading is given by

$$[X'] = [K'] [d'] . \tag{2-11}$$

In a complete analysis the surface loading also has to be considered. Let $[X_p']$ and $[d'_p]$ be the edge forces and displacements corresponding to some particular solution. Applying a displacement system $-[d_p']$ the net force system corresponding to a set of displacements which are zero at the boundary can be obtained.

$$[X_o'] = - [K]' [d'_p] + [X_p'] \tag{2-12}$$

The system $[X_o]$ can be considered as the'initial nodal forces' corresponding to the surface loading on the single shell. These are the nodal forces corresponding to zero nodal displacements under the action of the applied loading. $[X'_o]$ can also be considered as a new particular solution with zero displacements at the edges.

The stiffness equations including consideration of surface loading can now be written for a single shell element in the local co-ordinate system as

$$[X'] = [K] [d'] + [X'_o] \quad . \tag{2-13}$$

For an assembly of shells the total structural stiffness matrix can be written from the element stiffness matrices, transformed to global co-ordinates as follows,

$$[X] = [K] [d] + [X_o] \quad . \tag{2-14}$$

In EQUATIONS (2-13) and (2-14) $[d']$ and $[d]$ represent the net nodal displacements.

In a similar manner the net equivalent nodal forces are given by .

$$[X_e'] = [K'] [d'] + [X'_o] \quad . \tag{2-15}$$

From the equivalent nodal forces, the internal forces can be obtained.

CHAPTER III

THEORY OF CLOSED CIRCULAR CYLINDERS SUBJECTED TO ARBITRARY LOADING ALONG THE EDGES

3.1 INTRODUCTION

In this chapter the exact general theory proposed by Flügge is outlined. For the special case of $m = 1$, the general solution given by Flügge cannot be directly applied because of the repeated roots of the shell characteristic equation. The solution is therefore modified for $m = 1$. As an exact analysis in general terms is tedious, approximate methods are considered. This requires a prior study of the different shell characteristic equations and the nature of damping of the solutions.

3.2 GENERAL THEORY

The exact theory has been proposed by Flügge and for a complete presentation of the development of the theory reference may be made to pp 208 to 239, CHAPTER 5, ref. 14.

The forces and displacements (FIGURE 3.1) are periodic functions of ϕ and hence are expressed as Fourier series. Any quantity f, either a force or a displacement is expressed in the general form as

$$f = c[\overline{C}_1 f_1 + \overline{C}_2 f_2 + \overline{C}_3 f_3 + \overline{C}_4 f_4 + \overline{C}_5 f_5 + \overline{C}_6 f_6 + \overline{C}_7 f_7 + \overline{C}_8 f_8] \begin{matrix}\cos\\ \sin\end{matrix} m\phi \quad (3\text{-}1)$$

The constants $\overline{C}_1$ to $\overline{C}_8$ are evaluated from the boundary conditions four at each edge. The quantities f_1 to f_8 are expressed as follows.

$$f_1 = a_1\Phi_1 - a_2\Phi_2 \qquad f_5 = -a_5\Phi_3 - a_6\Phi_6$$

$$f_2 = a_2\Phi_1 + a_1\Phi_2 \qquad f_6 = a_6\phi_5 - a_5\Phi_6$$

$$f_3 = a_3\Phi_3 - a_4\Phi_4 \qquad f_7 = -a_7\Phi_7 - a_8\Phi_8$$

$$f_4 = a_4\Phi_3 + a_3\Phi_4 \qquad f_8 = a_8\Phi 7 - a_7\Phi_8 \qquad (3\text{-}2)$$

The characteristic functions Φ_1 to Φ_8 are defined as

$$\Phi_1 = e^{-\chi_1 y} \cos\mu_1 y \qquad \Phi_5 = e^{+\chi_1 y} \cos\mu_1 y$$

$$\Phi_2 = e^{-\chi_1 y} \sin\mu_1 y \qquad \Phi_6 = e^{+\chi_1 y} \sin\mu_1 y$$

$$\Phi_3 = e^{-\chi_2 y} \cos\mu_2 y \qquad \Phi_7 = e^{+\chi_2 y} \cos\mu_2 y$$

$$\Phi_4 = e^{-\chi_2 y} \sin\mu_2 y \qquad \Phi_8 = e^{+\chi_2 y} \sin\mu_2 y \qquad (3\text{-}3)$$

By substituting the displacements expressed as periodic functions into the three differential equations (EQUATIONS 13a-c, p 219, ref. 14) Flügge obtaines an eighth order characteristic equation (EQUATION 1-12). The roots of this equation occur as pairs of complex conjugate quantities expressed in terms of χ_1, μ_1 and χ_2, μ_2. Values of a_1 and a_2 have been presented in a tabular form (TABLE 1, p. 230, ref. 14).Values of a_5 and a_6 can be obtained in turn from a_1 and a_2 as shown in TABLE 3.1. Values of a_3,a_4, a_7 and a_8 are obtained using these two tables by changing the subscripts of χ,μ, from 1 to 2 and those of $\overline{\alpha},\overline{\beta}$ from 1 and 2 to 3 and 4 respectively. Knowing the shell parameters and harmonic number m of the loading the roots can be obtained from the characteristic equation. Values of a_1 to a_8 can be determined from the above tables. Using these, values of f_1 to f_8 can be found at any point along the length of the shell by means of EQUATIONS (3-2) and (3-3). Substituting these into EQUATION (3-1), the value of any stress resultant or displacement f at any point along the length of the shell can be expressed in terms of the arbitrary constants $\overline{C}_1$ to $\overline{C}_8$.

Since in the derivation of EQUATION (3-1) the stress resultants and other displacements are expressed in terms of the displacement w it is pertinent to base further discussion on w. Using TABLE 3.1 the expression for w can be written as

$$\begin{aligned} w = \Big[& e^{-\chi_1 y}[\overline{C}_1 \cos\mu_1 y + \overline{C}_2 \sin\mu_1 y] + \\ & e^{-\chi_2 y}[\overline{C}_3 \cos\mu_2 y + \overline{C}_4 \sin\mu_2 y] + e^{+\chi_1 y}[\overline{C}_5 \cos\mu_1 y + \\ & \overline{C}_6 \sin\mu_1 y] + e^{+\chi_2 y}[\overline{C}_7 \cos\mu_2 y + \overline{C}_8 \sin\mu_2 y]\Big] \cos m\phi \end{aligned} \tag{3-4}$$

There are two cases in which the solutions can be simplified within the framework of the exact theory. These are the semi-infinite cylinder and the case for which m = o.

In a semi-infinite cylinder the solutions containing the terms $e^{+\chi_1 y}$ and $e^{+\chi_2 y}$ become unbounded as y increases. For a long shell if the applied loads are in self equilibrium, loads at one edge will not produce stresses and displacements at the other edge. We can therefore disregard the four solutions corresponding to $\overline{C}_5$ to $\overline{C}_8$. The expression for w can be written as

$$w = \left[e^{-\chi_1 y}[\overline{C}_1 \cos\mu_1 y + \overline{C}_2 \sin\mu_1 y] + e^{-\chi_2 y}[\overline{C}_3 \cos\mu_1 y + \overline{C}_4 \sin\mu_1 y]\right] \cos m\phi \qquad (3\text{-}5)$$

The case m = o gives rise to axisymmetric bending which has been fully covered in literature. This case is governed by a fourth order differential equation (EQUATION 73a, p. 273, ref. 14) giving rise to a solution in terms of four arbitrary constants of integration. The expression for w can be written as,

$$w = e^{-\overline{\mu}y}[C_1 \cos\overline{\mu}y + C_2 \sin\overline{\mu}y] + e^{+\overline{\mu}y}[C_3 \cos\overline{\mu}y + C_4 \sin\overline{\mu}y] \qquad (3\text{-}6)$$

As before, ignoring the interaction of the far edge for a long shell, in this expression the second term involving $e^{+\overline{\mu}y}$ can be disregarded, thus reducing the number of constants to two.

The case m = 1 requires special treatment in that the values of χ_2, μ_2 are zero. As the four roots expressed in terms of χ_2, μ_2 are repeated and zero the part of the general expression for w given by EQUATION (3-4) is not valid and must be replaced by a simple polynomial solution. The complete solution then takes the form,

$$\begin{aligned} w = \Big[& e^{-\chi_1 y}[\bar{C}_1 \cos\mu_1 y + \bar{C}_2 \sin\mu_1 y] + e^{+\chi_1 y}[\bar{C}_5 \cos\mu_1 y + \\ & \bar{C}_6 \sin\mu_1 y] + [\bar{C}_3 + \bar{C}_4 y + \bar{C}_7 y^2 + \bar{C}_8 y^3]\Big] \cos m\phi \end{aligned} \tag{3-7}$$

TABLE 3.2 gives the displacements and stress resultants corresponding to the polynomial part of the solution. Using this table, the membrane stresses can be written as,

$$\begin{aligned} \bar{N}_x &= \bar{D}\ (1-\nu^2)[2\bar{C}_1 + 6y\bar{C}_8] \\ \bar{N}_{x\phi} &= N_{\phi x} = 6\bar{D}\ (1-\nu^2)\ \bar{C}_8 \\ \bar{N}_\phi &= o \end{aligned} \tag{3-8}$$

This table gives also moments and transverse forces. These quantities are of the order k and therefore the resulting stresses are insignificantly small in comparison with those produced by membrane forces. Thus they can be ignored. Expressions identical to EQUATIONS (3-8) can also be obtained from the membrane theory for m = 1. Therefore for a thin shell the polynomial part of the general solution can be exactly replaced by the membrane solution.

It is seen that for the particular cases of m = o and a long shell the number of constants to be determined from the boundary conditions are reduced. In a general case one has to evaluate all the eight arbitrary constants. The evaluation of these constants in general terms using an exact theory as described above is very tedious and time consuming. However values of stress resultants can be tabulated for particular cases of ℓ/r, h/r, m and υ. A table similar in form to the one given in A.S.C.E. Manual 31 for open cylindrical shells for loads applied at the straight edges can be obtained. Such tables would have to be very exhaustive to consider sufficient values of these parameters to include all practical cases. As an example TABLE 3.3 has been presented giving values of the stress resultants for $m = 1$ and 2, for particular values of the shell parameters.

A solution in closed form in general terms may be possible if one uses approximate methods to the general shell problem. This requires a study of the different shell characteristic equations and the nature of the solutions and this is considered in the next section.

3.3 STUDY OF CHARACTERISTIC EQUATIONS

Flügge's exact characteristic equation and Holand's modified equation are used for an accurate analysis of the shell problem. Donnell's characteristic equation is widely used because of its relative simplifity. Two independent fourth order equations in place of a single eighth order equation have been proposed by Flügge for low values of m. (refer CHAPTER I).

The explicit solutions for the roots of Holand, Donnell and the split fourth order equations of Flügge have been given in APPENDIX A.

Flügge's eighth order characteristic equation is considered exact. The solutions of the other equations are therefore compared with those of Flügge. A value of 0.167 has been adopted for Poisson's ratio υ, this being the usual value for concrete. It has been found that variation in υ does not affect the roots appreciably. Therefore the qualitative nature of any further discussion in this thesis is not affected by variation in υ.

FIGURES 3.2 and 3.3 show the variation of the quantities χ_1, μ_1 and χ_2, μ_2 for various values of r/h computed using the exact theory. It is seen from the figures that values of χ_1 and χ_2 approach μ_1 and μ_2 respectively as r/h increases. This property is more pronounced for lower values of m.

TABLE (3-4) show a comparison of the roots obtained from the different characteristic equations. Holand's roots are found to agree with those of Flügge for all values of m. The agreement is quite good as k becomes very small. For values of m considered in this table χ_1, μ_1 obtained by Donnell's equation have a better agreement with those of Flügge than the ones given by Holand. For m = 1, Donnell's equation does not give the zero values of χ_2, μ_2 which are essential for the problem.

For lower values of m, values of χ_2, μ_2 obtained by Donnell's equation do not agree with the exact values. However the agreement improves as m increases, provided m is not very high. For lower values of m the roots obtained by the split equations of Flügge agree well with the exact values. The agreement becomes better as k becomes smaller. For m = 1, values of χ_2, μ_2 are zero indicating a perfect agreement with the exact values.

3.4 STUDY OF THE NATURE OF DAMPING OF THE SOLUTIONS

The nature of approximations to the shell problem depends on the nature of damping of the solutions. Therefore this aspect is investigated in detail in this section.

Making the substitutions $y=Q-\bar{y}$ in the last two terms of EQUATION (3-4) we get the expression for w as

$$\bar{w} = \Big[e^{-\chi_1 y}[\bar{C}_1\cos\mu_1 y + \bar{C}_2\sin\mu_1 y] + e^{-\chi_2 y}[\bar{C}_3\cos\mu_2 y + \bar{C}_4\sin\mu_2 y] + e^{-\chi_1\bar{y}}[\bar{C}_5'\cos\mu_1\bar{y} + \bar{C}_6'\sin\mu_1\bar{y}] + e^{-\chi_2\bar{y}}[\bar{C}_7'\cos\mu_2\bar{y} + \bar{C}_8'\sin\mu_2\bar{y}]\Big]\cos m\phi \quad (3\text{-}9)$$

where the constants $\bar{C}_5$ to $\bar{C}_8$ are modified to $\bar{C}_5'$ to $\bar{C}_8'$ to take into account this transformation.

EQUATION (3-9) now consists of four different components each representing damped oscillations like the one shown in FIGURE 3.4. As $\cos\mu_i y$ is periodic, the period of oscillation $\overline{L}_i$ is given by the expression,

$$\overline{L}_i = \frac{2\pi}{\mu_i} \quad \text{where } i = 1,2$$

As y is nondimensional, the absolute value of this period, L_i can be written as,

$$L_i = \frac{2\pi r}{\mu_i} \; .$$

L_i represents the length of the shell over which any edge disturbance again reaches a maximum value. The ratio of these maximum values represents the rate of damping which,referring to FIGURE 3.4 is given by the expression,

$$\frac{A_n}{A_{n+1}} = \frac{e^{-\chi_i y}}{e^{-\chi_i(y+\overline{L}_i)}} = e^{\chi_i \overline{L}_i} = e^{\frac{2\pi\chi_i}{\mu_i}} \; , \qquad (3\text{-}11)$$

For low values of m it is seen from FIGURES 3.2 and 3.4 that μ_i is approximately equal to χ_i. Therefore EQUATION (3-11) can be written as,

$$\frac{A_n}{A_{n+1}} = e^{2\pi} = 535.5 \quad .$$

Thus for low values of m, the rate of damping is independent of the dimensions of the shell.

For high values of m as μ_i is always less than χ_i the rate of damping is greater than $e^{2\pi}$. This means that for all values of m the rate of damping is very quick.

The purpose of the above study is to determine when a disturbance applied at the near edge may be neglected at the far edge. EQUATION (3-10) indicates that the period is a function of the shell dimensions and the harmonic number m. For example TABLE 3.5 gives values of the periods $\overline{L}_1$ and $\overline{L}_2$ for a moderately thick and a thin shell for two different harmonics. Consider first the low harmonic m = 2. For the thick shell for which h/r = 0.03, taking $\ell/r = 1$, the χ_1-solution is quickly damped, but the χ_2 solution is undamped. However when the length is very large say $\ell/r = 25$, the χ_2-solution is also damped. For thinner shells the situation is slightly different. For the thinner shell for which $\frac{h}{r} = 0.0014$, for all the lengths the χ_1-solutions are quickly damped, but the χ_2 - solutions are practically undamped. For the higher harmonic m = 10 the length over which damping occurs is slightly increased for the χ_1 solution, but this length is considerably reduced for the χ_2 solution.

From a study of the roots of the characteristic equations and the nature of damping of the solution the following conclusions can be drawn.

1. Holand's roots can be expected to give results which are pretty close to the exact values obtained from Flügge.

2. As the χ_1-solution is quickly damped for all values of ℓ/r, h/r and m, slight variation in the values of χ_1, μ_1 cannot affect the final solutions appreciably. Incidentally this means that for all values of m, except very high values Donnell's equation can be used to obtain the solution

corresponding to χ_1 , μ_1.

3. As the χ_2-solution is slowly damped for low values of m, slight variation in the values of χ_2, μ_2 can affect the final solutions appreciably. For low values of m the values of χ_2, μ_2 are to be evaluated with a higher degree of accuracy. This means that for low values of m, Donnell's equation cannot be used to obtain the solution corresponding to χ_2, μ_2.

4. As for low values of m, values of χ_1, μ_1 and χ_2, μ_2 computed by the split equation of Flügge are close to the exact ones, these equations can be expected to give comparable results.

TABLE 3.1

CHARACTERISTIC COEFFICIENTS

f	a_5	a_6
w	a_1	a_2
w'	$-a_1$	a_2
u	$-a_1$	a_2
v	a_1	$-a_2$
N_ϕ	a_1	$-a_2$
N_x	a_1	$-a_2$
$N_{\phi x}$	$-a_1$	a_2
$N_{x\phi}$	$-a_1$	a_2
T_x	$-a_1$	a_2
M_ϕ	a_1	$-a_2$
M_x	a_1	$-a_2$
$M_{\phi x}$	$-a_1$	a_2
$M_{x\phi}$	$-a_1$	a_2
Q_ϕ	$-a_1$	a_2
Q_x	$-a_1$	a_2
S_x	$-a_1$	a_2

TABLE 3.2

CHARACTERISTIC COEFFICIENTS FOR THE POLYNOMIAL PART OF THE SOLUTION

$\bar{f}$	c	f_3	f_4	f_7	f_8
$\bar{w}$	1	1	y	y^2	y^3
$\bar{w}'$	1	0	1	$2y$	$3y^2$
$\bar{u}$	1	0	-1	$-2y$	$-6(2+\nu)-3y^2$
$\bar{v}$	1	-1	$-y$	$2\nu-y^2$	$6\nu y-y^3$
$\bar{N}_\phi$	$\bar{D}$	0	0	0	0
$\bar{N}_x$	$\bar{D}$	0	0	$2[\nu^2-(1+k)]$	$[\nu^2-(1+k)]\ 6y$
$\bar{N}_{\phi x}$	$\bar{D}$	0	0	0	$6(1-\nu^2)+3k(2+\nu)$
$\bar{N}_{x\phi}$	$\bar{D}$	0	0	0	$6(1-\nu^2)+3k(\nu-\nu^2)$
$\bar{T}_x$	$\bar{D}$	0	0	0	$6(1-\nu^2)+9k(\nu-\nu^2)$
$\bar{M}_\phi$	$\bar{D}k$	0	0	2ν	$6\nu y$
$\bar{M}_x$	$\bar{D}k$	0	0	$2(2-\nu^2)$	$6(2-\nu^2)y$
$\bar{M}_{\phi x}$	$\bar{D}k$	0	0	0	$6(1-\nu)$
$\bar{M}_{x\phi}$	$\bar{D}k$	0	0	0	$-6\nu(1-\nu)$
$\bar{Q}_\phi$	$\bar{D}k$	0	0	-2ν	$-6\nu y$
$\bar{Q}_x$	$\bar{D}k$	0	0	0	$6[3-\nu-\nu^2]$
$\bar{S}_x$	$\bar{D}k$	0	0	0	$6(3-2\nu)$

$$\bar{f} = c\left[\bar{C}_3 f_3 + \bar{C}_4 f_4 + \bar{C}_7 f_7 + \bar{C}_8 f_8\right]$$

TABLE 3.3

STRESS RESULTANTS FOR UNIT VALUES OF DISPLACEMENTS FOR SPECIFIC CASES

	Edge Rotation				Normal Edge displacement				Longitudinal Edge Displacement				Tangential Edge Displacement			
	$M_x = w'_m r^2$(col.1) $10^{-4}\cos m\phi$ $Q_x = w'_m$ (col.2) $10^{-3}\cos m\phi$ $N_x = w'_m$ (col.3) $10^{-4}\cos m\phi$ $N_{x\phi} = w'_m$ (col.4) $10^{-3}\sin m\phi$				$M_x = w_m r$(col.5) $10^{-3}\cos m\phi$ $Q_x = w_m$ (col.6) $10^{-2}\cos m\phi$ $N_x = w_m$ (col.7) $10^{-3}\cos m\phi$ $N_{x\phi} = w_m$ (col.8) $10^{-2}\sin m\phi$				$M_x = u_m r$(col.9)$10^{-4}\cos m\phi$ $Q_x = u_m$ (col.10)$10^{-3}\cos m\phi$ $N_x = u_m$ (col.11)$10^{-1}\cos m\phi$ $N_{x\phi} = u_m$ (col.12)$10^{-2}\sin m\phi$				$M_x = v_m r$(col.13) $10^{-3}\cos m\phi$ $Q_x = v_m$ (col.14) $10^{-2}\cos m\phi$ $N_x = v_m$ (col.15) $10^{-2}\cos m\phi$ $N_{x\phi} = v_m$ (col.16) $10^{-2}\sin m\phi$			
y/Q	M_x	Q_x	N_x	$N_{x\phi}$	M_x	Q_x	N_x	$N_{x\phi}$	M_x	Q_x	N_x	$N_{x\phi}$	M_x	Q_x	N_x	$N_{x\phi}$
$h/r = 0.03$, $\ell/r = 2.0$, $m = 1$, $\upsilon = 0.167$																
0.0	-0.3462	0.2592	0.1234	-0.2361	-0.2592	0.3944	0.3001	-0.3714	0.1234	0.3008	-0.2066	-0.5335	-0.2361	0.3713	-0.5335	-0.9142
0.2	0.0163	-0.0104	0.3937	0.0335	0.0140	-0.0186	0.4813	0.0417	0.0215	0.0141	-0.1841	-0.5649	0.0134	-0.0177	-0.2914	-0.5251
0.4	0.0008	0.0039	0.2843	0.0227	-0.0080	0.0009	0.3744	0.0222	0.0250	0.0019	-0.1616	-0.5633	-0.0006	0.0007	-0.0757	-0.5435
0.6	0.0000	-0.0000	0.1928	0.0231	-0.0000	-0.0000	0.2831	0.0231	0.0212	-0.0008	-0.1391	-0.5634	-0.0002	-0.0002	0.1415	-0.5426
0.8	0.0000	-0.0000	0.1005	0.0231	-0.0000	-0.0000	0.1910	0.0230	0.0170	-0.0067	-0.1165	-0.5628	0.0005	0.0009	0.3586	-0.5437
1.0	0.0005	0.0004	0.0088	0.0227	0.0004	0.0002	0.0994	0.0228	0.0088	0.0994	-0.0940	-0.5734	-0.0227	-0.0228	0.5734	-0.5200
$h/r = 0.03$, $\ell/r = 2$, $m = 2$, $\upsilon = 0.167$																
0.0	-0.3369	0.2458	-0.3897	-0.4541	-0.2458	0.3828	-0.0387	-0.7315	-0.3897	0.0387	-0.3156	-0.8300	-0.4541	0.7315	-0.8300	-1.8193
0.2	0.0140	-0.0079	0.7305	0.0504	0.0127	-0.0164	-0.7745	0.0664	0.1190	0.0163	-0.2476	-0.8255	0.0277	-0.0331	-0.3443	-0.2882
0.4	-0.0008	0.0003	0.3955	0.0340	-0.0009	0.0007	-0.4503	0.0320	0.0795	-0.0056	-0.1820	-0.8195	0.0003	-0.0034	-0.0745	-0.3515
0.6	-0.0000	0.0001	0.1232	0.0342	-0.0000	0.0000	-0.1868	0.0332	0.0530	-0.0153	-0.1166	-0.8159	-0.0000	-0.0168	0.2049	-0.3473
0.8	-0.0000	0.0000	-0.1512	0.0343	-0.0000	0.0000	-0.0792	0.0333	0.0256	-0.0102	-0.0513	-0.8157	-0.0002	-0.0026	0.4837	-0.3494
1.0	-0.0024	0.0022	-0.4155	0.0300	0.0022	0.0020	-0.3359	0.0294	-0.4154	-0.3359	0.0121	-0.7505	-0.0300	-0.0294	0.7505	-0.2912

TABLE 3.4

VALUES OF ROOTS OBTAINED BY FLÜGGE, HOLAND DONNELL AND APPROXIMATE SOLUTIONS

k	m		χ_1	μ_1	χ_2	μ_2
0.8333×10^{-3}	1	Flügge	4.2419	4.0200	0.0	0.0
		Holand	4.1325	4.1325	0.0	0.0
		Donell	4.2568	4.0152	0.1243	0.1173
		Approx*	4.1325	4.1325	0.0	0.0
	5	Flügge	7.1238	2.8280	2.9407	1.2513
		Holand	7.0880	2.9164	2.9555	1.2161
		Donnell	7.1366	2.9082	3.0042	1.2242
		Approx.	5.8754	5.8754	2.0845	2.0845
	10	Flügge	12.0820	2.3441	7.8966	1.7330
		Holand	12.0500	2.4938	7.9178	1.6386
		Donnell	12.0750	2.4928	7.9429	1.6397
		Approx.	11.0930	11.0930	4.4848	4.4848
0.8333×10^{-5}	1	Flügge	13.1030	13.0330	0.0	0.0
		Holand	13.0680	13.0680	0.0	0.0
		Donnell	13.1060	13.0303	0.0384	0.0381
		Approx.	13.0680	13.0680	0.0	0.0
	5	Flügge	14.0770	12.1880	0.9924	0.8635
		Holand	14.0610	12.2060	0.9935	0.8623
		Donnell	14.0830	12.1900	1.0149	0.8784
		Approx.	13.1680	13.1680	0.9301	0.9301
	10	Flügge	17.2850	10.4890	4.2010	2.5632
		Holand	17.2720	10.5100	4.2041	2.5581
		Donnell	17.2930	10.5020	4.2246	2.5657
		Approx.	14.4840	14.4840	3.4347	3.4347
0.8333×10^{-7}	1	Flügge	41.3360	41.3140	0.0	0.0
		Holand	41.3250	41.3250	0.0	0.0
		Donnell	41.3370	41.3130	0.0121	0.0121
		Approx.	41.3250	41.3250	0.0	0.0
	5	Flügge	41.6280	41.0250	0.2984	0.2943
		Holand	41.6230	41.0300	0.2985	0.2942
		Donnell	41.6290	41.0240	o.3047	0.3000
		Approx.	41.3280	41.3280	0.2964	0.2964
	10	Flügge	42.5660	40.1530	1.2367	1.1671
		Holand	42.5610	40.1520	1.2368	1.1670
		Donnell	42.5680	40.1520	1.2431	1.1726
		Approx.	41.3770	41.3770	1.2023	1.2023

* Approx. Based on split equations of Flügge.

TABLE 3.5

PERIODS OF DAMPING

P	m	χ_1	μ_1	L_1/r	χ_2	μ_2	L_2/r
0.03	2	7.810	7.292	0.87	0.235	0.223	28.00
0.03	10	13.867	5.134	1.22	6.293	2.383	2.64
0.0014	2	34.982	34.870	0.18	0.050	0.050	126.00
0.0014	10	36.409	33.557	0.19	1.477	1.136	5.50

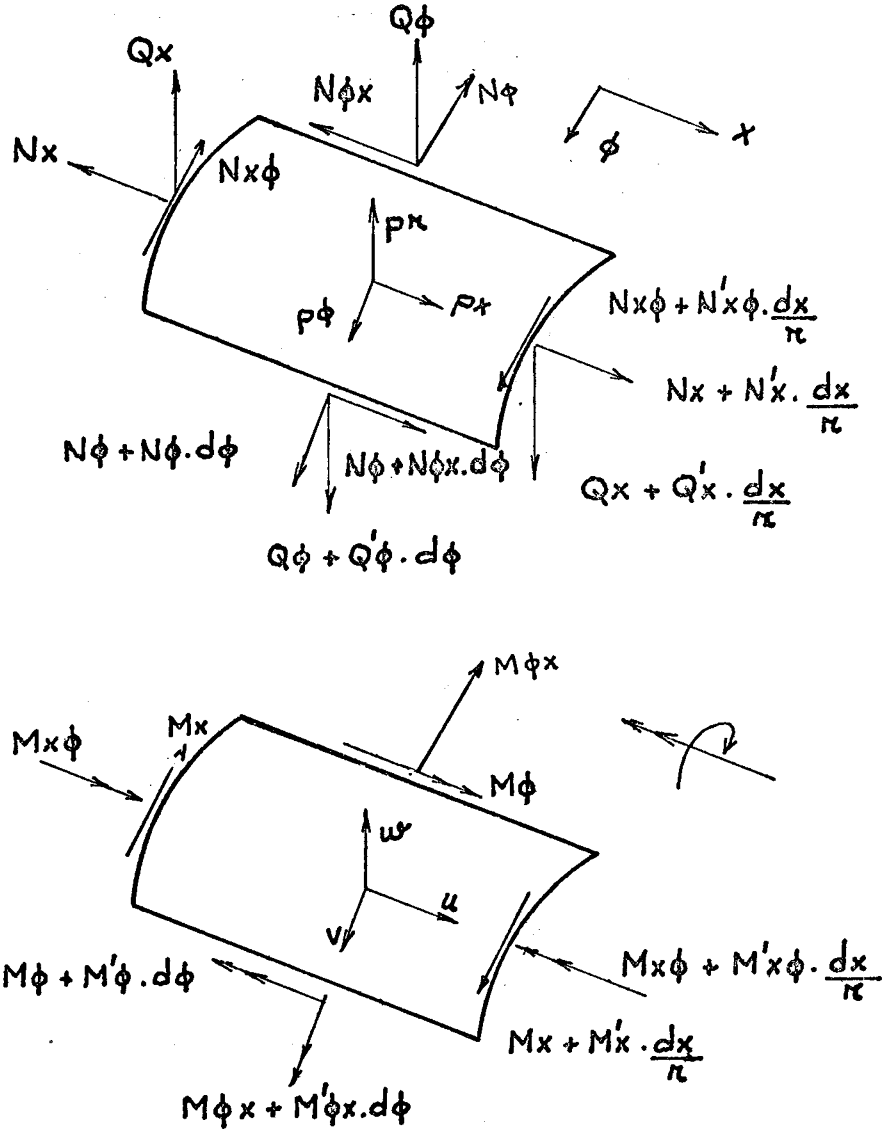
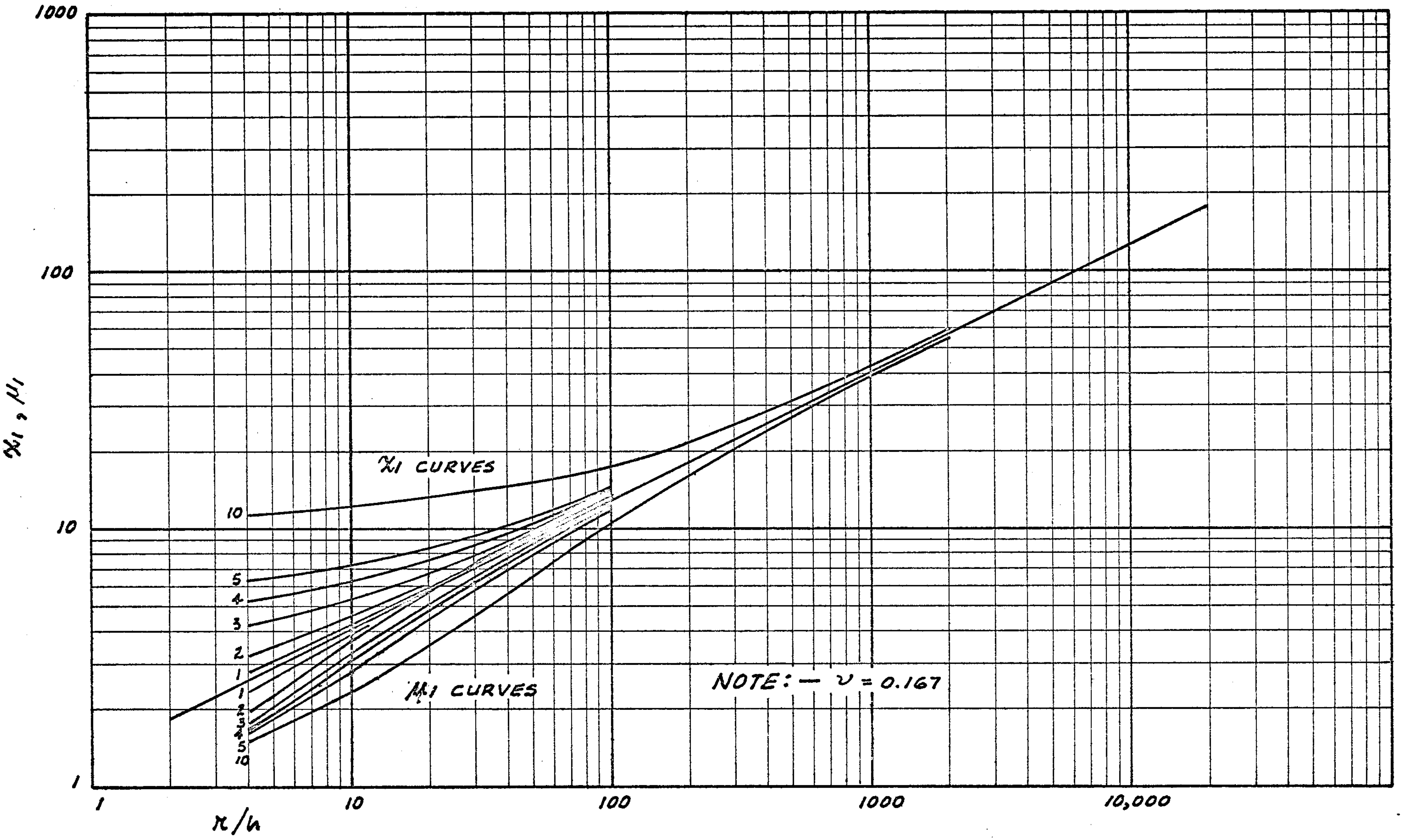


FIGURE 3.1 GENERAL SHELL THEORY SIGN CONVENTION



NOTE: — $\nu = 0.167$

FIGURE 3.2 VARIATION OF χ_1, μ_1 BY EXACT THEORY

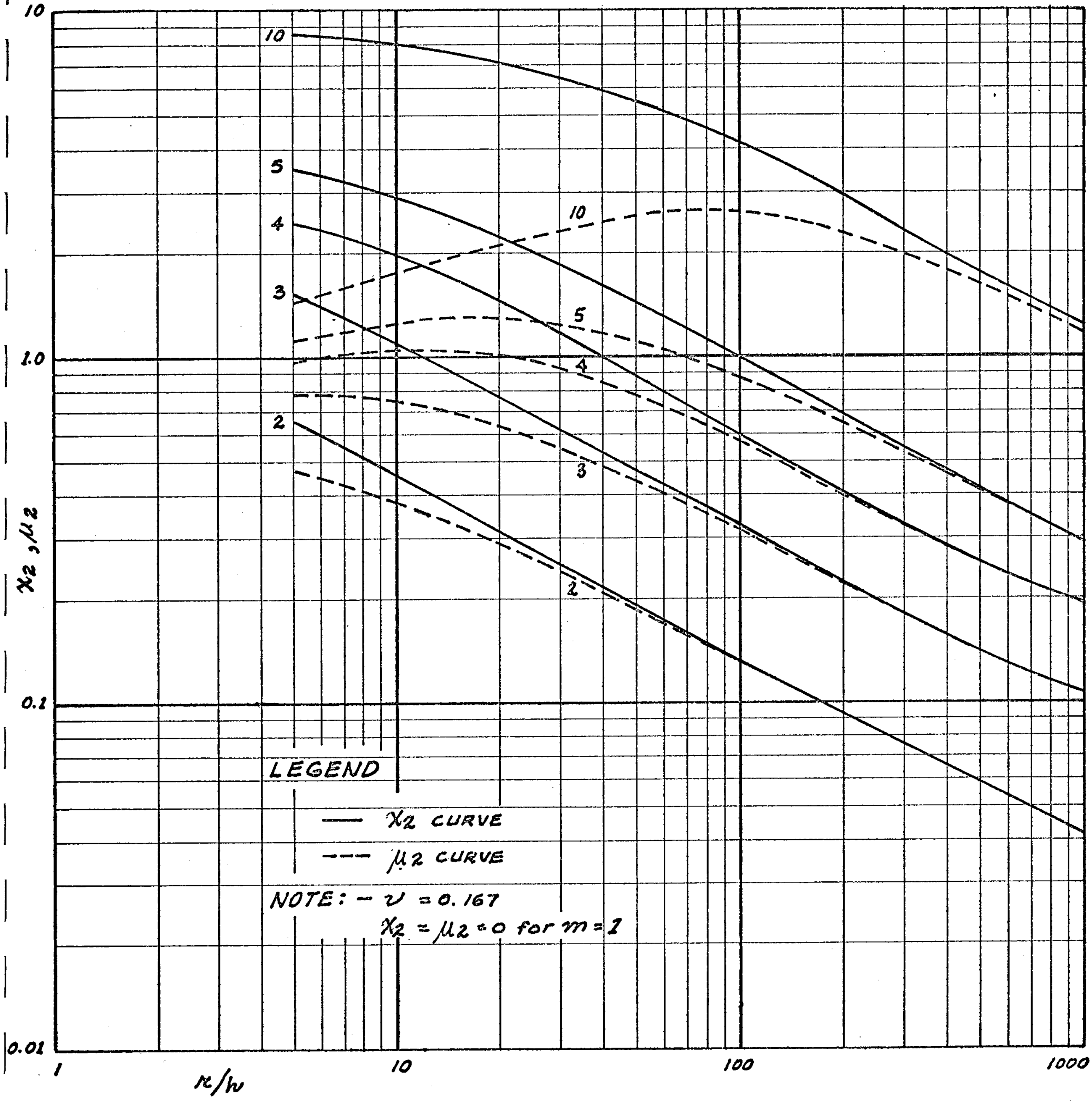


FIGURE 3.3 VARIATION OF χ_2, μ_2 BY EXACT THEORY

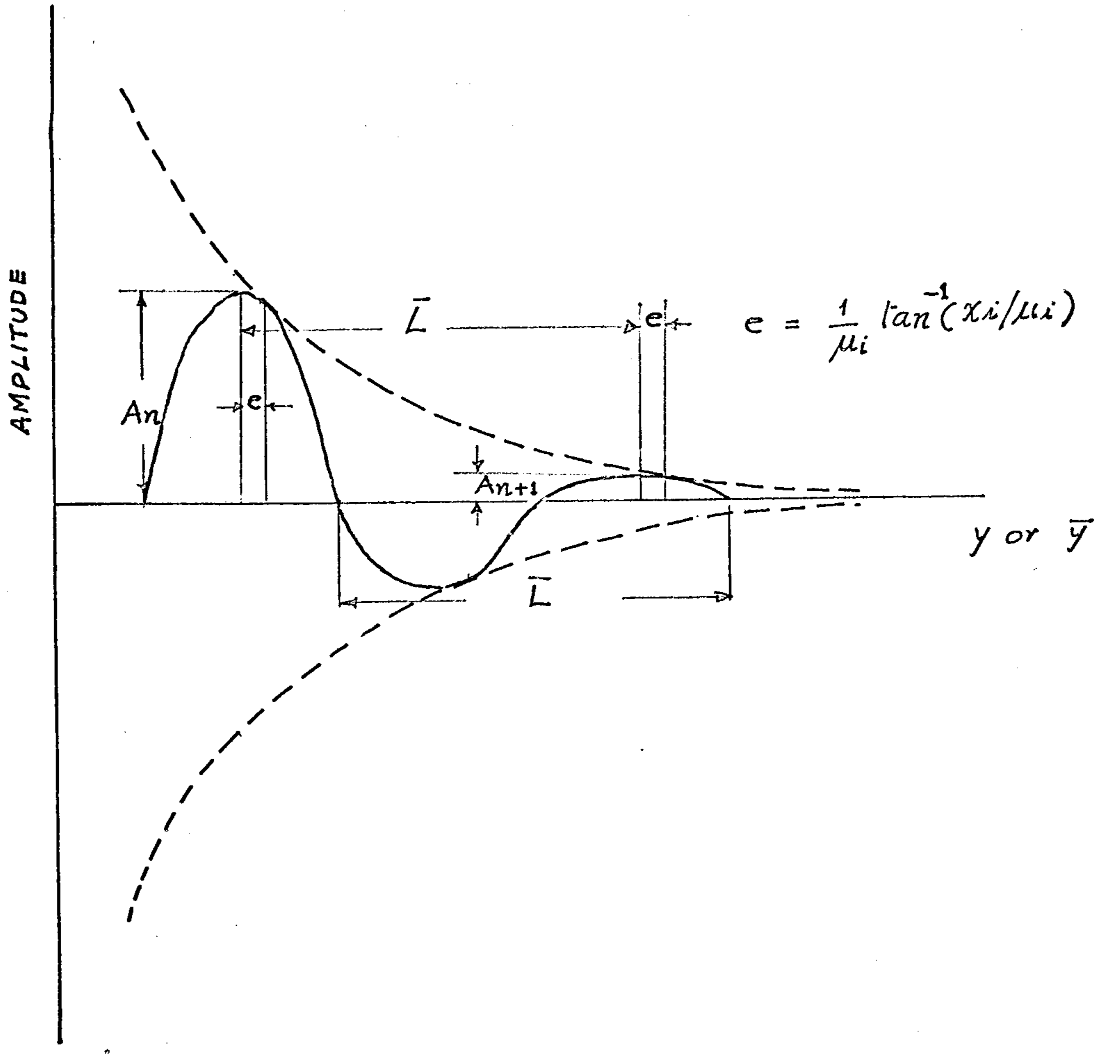


FIGURE 3.4 NATURE OF DAMPING

CHAPTER IV

INFLUENCE COEFFICIENTS

4.1 INTRODUCTION:

In this chapter, the method of finding the stiffness influence coefficients for the closed cylindrical shell using the exact theory is described. A specific numerical example has been given to illustrate the method. The resulting direct stiffness matrix contains 64 elements and can be made truely symmetric by adopting the stiffness matrix sign convention. However by adopting a slightly different sign convention this matrix can be reduced such that only 20 elements are different.

Establishing the influence coefficients in general terms is tedious. Therefore approximate methods are attempted and these are discussed in detail. It is seen that three types of approximations are possible depending on the shell geometry and the harmonic number of the applied edge displacement. A criterion for the limitation of replacing the basic solution by the membrane solution has been established.

4.2 METHOD OF FINDING INFLUENCE COEFFICIENTS BY THE EXACT THEORY

For any particular harmonic m the four displacements w', w, u and v at each of the two edges of a closed cylinder can be expressed in terms of the eight constants $\overline{C}_1$ to $\overline{C}_8$ as explained in CHAPTER III. Unit value of each displacement is applied individually at each edge while the remainder of the displacements are set as zero. The resulting equations obtained from EQUATION (3-1) can be written in matrix notation,with $[I]$ as the unit matrix, as

$$[C_1]\ [\overline{C}] = [I] \tag{4-1}$$

The coefficient matrix $[C_1]$ of size 8x8 found from the elements f_1 to f_8 is independent of the boundary conditions. Each column vector of the matrix $[\overline{C}]$ consists of the unknown constants $\overline{C}_1$ to $\overline{C}_8$ corresponding to each column vector of $[I]$. The elements in each of this column vector represents a set of displacement boundary conditions for each case of the applied unit edge displacement. In the analysis that follows, the displacements have been considered in the order w', w, u and v at each edge.

Solving EQUATION (4-1) we get

$$[\overline{C}] = [C_1]^{-1}[I] = [C_1]^{-1} \tag{4-2}$$

The direct stiffness matrix can now be determined. The stress resultants M_x, S_x, N_x and T_x considered in this order to correspond with the order of displacements can be expressed in terms of the eight constants $\overline{C}_1$ to $\overline{C}_8$ for each case of the applied edge displacements. The stiffness matrix can be written as,

$$[K] = [C_2]\ [C_1]^{-1} \tag{4-3}$$

$$[C_1] = \begin{array}{cc} \begin{array}{cccccccc} \bar{C}_1 & \bar{C}_2 & \bar{C}_3 & \bar{C}_4 & \bar{C}_5 & \bar{C}_6 & \bar{C}_7 & \bar{C}_8 \end{array} & \\ \left[\begin{array}{cccccccc} -\chi_1 & \mu_1 & -\chi_2 & \mu_2 & \chi_1 & \mu_1 & \chi_2 & \mu_2 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ \bar{\alpha}_1 & \bar{\alpha}_2 & \bar{\alpha}_3 & \bar{\alpha}_4 & -\bar{\alpha}_1 & \bar{\alpha}_2 & -\bar{\alpha}_3 & \bar{\alpha}_4 \\ \bar{\beta}_1 & \bar{\beta}_2 & \bar{\beta}_3 & \bar{\beta}_4 & \bar{\beta}_1 & -\bar{\beta}_2 & \bar{\beta}_3 & \bar{\beta}_4 \\ e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \\ \phi_{1s} & \phi_{2s} & \phi_{3s} & \phi_{4s} & \phi_{5s} & \phi_{6s} & \phi_{7s} & \phi_{8s} \\ e_9 & e_{10} & e_{11} & e_{12} & e_{13} & e_{14} & e_{15} & e_{16} \\ e_{17} & e_{18} & e_{19} & e_{20} & e_{21} & e_{22} & e_{23} & e_{24} \end{array}\right] & \begin{array}{l} w'^{t} \\ w^{t} \\ u^{t} \\ v^{t} \\ w'^{b} \\ w^{b} \\ u^{b} \\ v^{b} \end{array} \end{array}$$

$e_1 = (-\chi_1\Phi_{1s}-\mu_1\Phi_{2s})$ $e_9 = (\bar{\alpha}_1\Phi_{1s}-\bar{\alpha}_2\Phi_{2s})$ $e_{17} = (\bar{\beta}_1\Phi_{1s}-\bar{\beta}_2\Phi_{2s})$

$e_2 = (-\chi_1\Phi_{2s}+\mu_1\Phi_{1s})$ $e_{10} = (\bar{\alpha}_2\Phi_{1s}+\bar{\alpha}_1\Phi_{2s})$ $e_{18} = (\bar{\beta}_2\Phi_{1s}+\bar{\beta}_1\Phi_{1s})$

$e_3 = (-\chi_2\Phi_{3s}+\mu_2\Phi_{4s})$ $e_{11} = (\bar{\alpha}_3\Phi_{3s}-\bar{\alpha}_4\Phi_{4s})$ $e_{19} = (\bar{\beta}_3\Phi_{3s}-\bar{\beta}_4\Phi_{4s})$

$e_4 = (-\chi_2\Phi_{4s}+\mu_2\Phi_{3s})$ $e_{12} = (\bar{\alpha}_4\Phi_{3s}+\bar{\alpha}_3\Phi_{4s})$ $e_{20} = (\bar{\beta}_4\Phi_{3s}+\bar{\beta}_3\Phi_{4s})$

$e_5 = (\chi_1\Phi_{5s}-\mu_1\Phi_{6s})$ $e_{13} = (-\bar{\alpha}_1\Phi_{5s}-\bar{\alpha}_2\Phi_{6s})$ $e_{21} = (\bar{\beta}_1\Phi_{5s}+\bar{\beta}_2\Phi_{6s})$

$e_6 = (\chi_1\Phi_{6s}+\mu_1\Phi_{5s})$ $e_{14} = (\bar{\alpha}_2\Phi_{5s}-\bar{\alpha}_1\Phi_{6s})$ $e_{22} = (-\bar{\beta}_2\Phi_{5s}+\bar{\beta}_1\Phi_{6s})$

$e_7 = (\chi_2\Phi_{7s}-\mu_2\Phi_{8s})$ $e_{15} = (-\bar{\alpha}_3\Phi_{7s}-\bar{\alpha}_4\Phi_{8s})$ $e_{23} = (\bar{\beta}_3\Phi_{7s}+\bar{\beta}_4\Phi_{8s})$

$e_8 = (\chi_2\Phi_{8s}+\mu_2\Phi_{7s})$ $e_{16} = (\bar{\alpha}_4\Phi_{7s}-\bar{\alpha}_3\Phi_{8s})$ $e_{24} = (-\bar{\beta}_4\Phi_{7s}+\bar{\beta}_3\Phi_{8s})$

Φ_{is} - Value of Φ_i at $y = \ell/r$ (refer to p. 26)

(4-4)

$$[K_1]=\begin{array}{cccccccccc} & w'^t & w^t & u^t & v^t & w'^b & w^b & u^b & v^b & \\ & & & & & & & & & \\ \left[\begin{array}{c} \\ \\ \\ \\ \\ \\ \\ \\ \end{array}\right. & \begin{array}{c} -0.3369E-04 \\ 0.2458E-03 \\ -0.3898E-04 \\ -0.4541E-03 \\ 0.2385E-06 \\ 0.2195E-05 \\ -0.4155E-04 \\ 0.3000E-04 \end{array} & \begin{array}{c} -0.2458E-03 \\ 0.3828E-02 \\ -0.3867E-04 \\ -0.7315E-02 \\ 0.2195E-05 \\ 0.1984E-04 \\ -0.3359E-03 \\ 0.2937E-03 \end{array} & \begin{array}{c} -0.3898E-04 \\ 0.3867E-02 \\ -0.3156E-01 \\ -0.8300E-02 \\ -0.4155E-04 \\ -0.3359E-03 \\ 0.1214E-02 \\ -0.7505E-02 \end{array} & \begin{array}{c} -0.4541E-03 \\ 0.7315E-02 \\ -0.8300E-02 \\ -0.1819E-01 \\ -0.3000E-04 \\ -0.2937E-03 \\ 0.7505E-02 \\ -0.2912E-02 \end{array} & \begin{array}{c} -0.2385E-06 \\ 0.2195E-05 \\ 0.4154E-04 \\ 0.3000E-04 \\ 0.3369E-04 \\ 0.2458E-03 \\ 0.3898E-04 \\ -0.4541E-03 \end{array} & \begin{array}{c} 0.2195E-05 \\ -0.1984E-04 \\ -0.3359E-03 \\ -0.2936E-03 \\ -0.2458E-03 \\ -0.3828E-02 \\ -0.3867E-04 \\ 0.7315E-02 \end{array} & \begin{array}{c} 0.4155E-04 \\ -0.3359E-03 \\ -0.1214E-02 \\ -0.7505E-02 \\ 0.3898E-04 \\ 0.3867E-02 \\ 0.3156E-01 \\ -0.8300E-02 \end{array} & \begin{array}{c} -0.3000E-04 \\ 0.2937E-03 \\ 0.7505E-02 \\ 0.2912E-02 \\ -0.4541E-03 \\ -0.7315E-02 \\ -0.8300E-02 \\ 0.1819E-01 \end{array} & \left.\begin{array}{c} \\ \\ \\ \\ \\ \\ \\ \\ \end{array}\right] \begin{array}{c} M_x^t \\ S_x^t \\ N_x^t \\ T_x^t \\ M_x^b \\ S_x^b \\ N_x^b \\ T_x^b \end{array} \end{array}$$

EQUATIONS (4-5)

SIGN CONVENTION SHOWN IN FIGURE 4.1

From EQUATION (3-1) in a manner similar to that of $[C_1]$ for displacements, the coefficient matrix $[C_2]$ for the stress resultants is established, and as such is independent of the boundary conditions.

The crux of the problem lies in evaluating the constants given by each column vector of $[\overline{C}]$. This requires determination of the inverse of $[C_1]$. EQUATION (4-4) gives the matrix $[C_1]$ in general terms of shell geometry and loading. The inverse can be readily obtained if specific numerical values are used for these terms. In the course of the study the inverse of $[C_1]$ was in fact obtained in general terms, but the complexity of the resulting expressions was such that they were not suitable for practical applications.

An example illustrating the application of the exact theory as outlined above follows. EQUATION (4-5) gives the stiffness matrix $[K_1]$ for a particular case of Q=2, P=0.03, μ=0.167 and m=2. The matrix was evaluated using a digital computer. The sign convention adopted is as shown in FIGURE 4.1 which is based on the sign convention used in the general shell theory (FIGURE 3.1). Since the parameters Q and P are non dimensional, to obtain the absolute values of the elements in the stiffness matrix for absolute values of the edge displacements all the elements in the 1st and the 5th rows and all the elements in the 1st and the 5th columns are to be multiplied by r.

$$[K_2]= \begin{array}{cccccccc|c} w'^{t} & w^{t} & u^{t} & v^{t} & w'^{b} & w^{b} & u^{b} & v^{b} & \\ 0.3369E-04 & 0.2458E-03 & 0.3898E-04 & 0.4541E-03 & 0.2385E-06 & -0.2195E-05 & -0.4155E-04 & 0.3000E-04 & M_x^t \\ 0.2458E-03 & 0.3828E-02 & 0.3867E-02 & 0.7315E-02 & 0.2195E-05 & -0.1984E-04 & -0.3359E-03 & 0.2937E-03 & S_x^t \\ 0.3898E-04 & 0.3867E-04 & 0.3156E-01 & 0.8300E-02 & -0.4154E-04 & 0.3359E-03 & 0.1214E-02 & -0.7505E-02 & N_x^t \\ 0.4541E-03 & 0.7315E-02 & 0.8300E-02 & 0.1819E-01 & -0.3000E-04 & 0.2936E-03 & 0.7505E-02 & -0.2912E-02 & T_x^t \\ & & & & & & & & \\ 0.2385E-06 & 0.2195E-05 & -0.4155E-04 & -0.3000E-04 & 0.3369E-04 & -0.2458E-03 & 0.3898E-04 & -0.4541E-03 & M_x^b \\ -0.2195E-05 & -0.1984E-04 & 0.3359E-03 & 0.2937E-03 & -0.2458E-03 & 0.3828E-02 & -0.3867E-02 & 0.7315E-02 & S_x^b \\ -0.4155E-04 & -0.3359E-03 & 0.1214E-02 & 0.7505E-02 & 0.3898E-04 & -0.3867E-04 & 0.3156E-01 & -0.8300E-02 & N_x^b \\ 0.3000E-04 & 0.2937E-03 & -0.7505E-02 & -0.2912E-02 & -0.4541E-03 & 0.7315E-02 & -0.8300E-02 & 0.1819E-01 & T_x^b \end{array}$$

EQUATIONS (4-6)

SIGN CONVENTION SHOWN IN FIGURE 4.2

$$[K_3]=\begin{array}{cccccccc|c}
w'^t & w^t & u^t & v^t & w'^b & w^b & u^b & v^b & \\
\hline
0.3369E-04 & 0.2458E-03 & 0.3898E-04 & 0.4541E-03 & 0.2385E-06 & 0.2195E-05 & -0.4155E-04 & -0.3000E-04 & M_x^t \\
0.2458E-03 & 0.3828E-02 & 0.3867E-02 & 0.7315E-02 & 0.2195E-05 & 0.1984E-04 & -0.3359E-03 & -0.2937E-03 & S_x^t \\
0.3898E-04 & 0.3867E-04 & 0.3156E-01 & 0.8300E-02 & -0.4154E-04 & -0.3359E-03 & 0.1214E-02 & 0.7505E-02 & N_x^t \\
0.4541E-03 & 0.7315E-02 & 0.8300E-02 & 0.1819E-01 & -0.3000E-04 & -0.2936E-03 & 0.7505E-02 & 0.2912E-02 & T_x^t \\
 & & & & & & & & \\
0.2385E-06 & 0.2195E-05 & -0.4155E-04 & -0.3000E-04 & 0.3369E-04 & 0.2458E-03 & 0.3898E-04 & 0.4541E-03 & M_x^b \\
0.2195E-05 & 0.1984E-04 & -0.3359E-03 & -0.2937E-03 & 0.2458E-03 & 0.3828E-02 & 0.3867E-02 & 0.7315E-02 & S_x^b \\
-0.4155E-04 & -0.3359E-03 & 0.1214E-02 & 0.7505E-02 & 0.3898E-04 & 0.3867E-04 & 0.3156E-01 & 0.8300E-02 & N_x^b \\
-0.3000E-04 & -0.2937E-03 & 0.7505E-02 & 0.2912E-02 & 0.4541E-03 & 0.7315E-02 & 0.8300E-02 & 0.1819E-01 & T_x^b
\end{array}$$

EQUATIONS (4-9)

SIGN CONVENTION SHOWN IN FIGURE 4.3

It is noted that the stiffness matrix $[K_1]$ in EQUATION (4-5) is not symmetrical with regard to signs. This results because of the sign convention used in the general shell theory. A truly symmetrical stiffness matrix can be obtained by adopting the usual sign convention for the stiffness method. This sign convention is given in FIGURE 4.2 and the stiffness matrix $[K_2]$ using this sign convention is given in EQUATION (4-6). In accordance with this classical stiffness matrix formulation $[K_2]$ can be written as follows,

$$[K_2] = \begin{bmatrix} A & B^T \\ B & C \end{bmatrix} . \qquad (4\text{-}7)$$

In the matrix $[K_2]$ the numerical values of the elements of $[C]$ and $[B^T]$ are identical to those of $[A]$ and $[B]$ respectively. It would be convenient to introduce a sign convention which would also make the algebraic signs identical so that the matrix can be written as in EQUATION (4-8) .

$$[K_3] = \begin{bmatrix} A & B \\ B & A \end{bmatrix} . \qquad (4\text{-}8)$$

The sign convention for this matrix is given in FIGURE 4.3 and the matrix $[K_3]$ for the above problem written using this sign convention is given in EQUATION (4-9). It is seen that the matrix $[K_3]$ is not only symmetrical but also that it can be formed by two symmetrical submatrices $[A]$ and $[B]$. This means that out of the 64 elements of the matrix $[K_3]$ only 20 elements are different.

As explained earlier it is very difficult to obtain the stiffness influence coefficients in general terms because of the difficulty in determining the inverse of $[C_1]$. Since one of objectives of this study is to obtain general expressions for stiffness influence coefficients approximate solutions are considered.

4.3 APPROXIMATE SOLUTIONS

In accordance with Goldenveiser and Flügge as discussed in the literature review in CHAPTER 1, the approximate solutions to the problem of higher harmonic loading on the edges of closed cylinders are as outlined below:

1. The split type solution for low values of m, where the tangential and non tangential boundary conditions are considered separately.

2. The general solution for high values of m, where both the tangential and non tangential boundary conditions at one edge are considered together neglecting the effect of the far edge.

3. The solution for very high values of m, where the problem is split into two states of stress corresponding to generalised plane stress and plate bending.

There are thus 3 types of approximations possible depending on the loading harmonic, and stiffness influence coefficients based on these approximations are presented in this chapter.

4.4 BASED ON SPLIT TYPE SOLUTION FOR LOW VALUES OF m

The general solution is split into two parts, the edge effects and the basic state. The edge effects part is governed by the characteristic equation,

$$\lambda^4 - 2(2m^2 - \upsilon)\lambda^2 + \left[\frac{1 - \upsilon^2}{k} + 6m^2(m^2 - 1)\right] = o \quad . \qquad (4\text{-}10)$$

and the basic state part by the equation,

$$\left[\frac{1 - \upsilon^2}{k} + 6m^2(m^2 - 1)\right]\lambda^4 - 2m^2\left[2m^4 - (4 - \upsilon)m^2 + (2-\upsilon)\right]\lambda^2 + m^4(m^2 - 1)^2 = o \quad . \qquad (4\text{-}11)$$

These equations have been presented previously in CHAPTER 1 as EQUATIONS (1-13a,b). The equations are considered separately in the following sections.

4.4.1 EDGE EFFECTS SOLUTION

The evaluation of the roots for EQUATION (4-10) is given in SECTION A.3 of APPENDIX A. Making the assumption that,

$$\frac{1 - \upsilon^2}{k} + 6m^2(m^2 - 1) >> (2m^2 - \upsilon) \quad , \qquad (4\text{-}12)$$

we obtain ,

$$\chi_1 = \mu_1 \quad . \qquad (4\text{-}13)$$

Since k is very small and generally of the order 10^{-4} to 10^{-7} and as m is also low this assumption is justified.

The equality given by EQUATION (4-13) leads to further simplifications. Making also the assumption that

$$\frac{4\chi_1^2}{m^2} \gg 1 \tag{4-14}$$

analytical expressions for $\bar{\alpha}_1$, $\bar{\alpha}_2$, $\bar{\beta}_1$, $\bar{\beta}_2$ have been derived in SECTION A.5 of APPENDIX A. These terms relate the three displacements u, v and w and a complete explanation of these terms is presented on p. 228, Chapter 5, ref. 14. For the latter assumption to be valid χ_1 must be much greater than m, a condition that is satisfied for small values of k.

Values of $\bar{\alpha}_1$, $\bar{\alpha}_2$, $\bar{\beta}_1$, $\bar{\beta}_2$ have been evaluated for various values of m and k and presented in TABLE 4.1. The values labelled Flügge and Donnell in this table are obtained using exact solutions to their respective equations without any approximations whereas the simplified solutions are based on the split equations of Flügge with the above two assumptions.

From the table it is seen that the values of $\bar{\alpha}_1$, $\bar{\alpha}_2$, $\bar{\beta}_1$ and $\bar{\beta}_2$ found from the simplified solutions agree well with the exact values provided m and k are very small.

TABLE 4.1 also illustrates that for all values of m and k values of the parameters $\bar{\alpha}_j$ and $\bar{\beta}_j$ obtained from χ_1, μ_1 based on Donnell's theory agree well with the exact values of Flügge, indicating that the Donnell's theory can be used for this part of of the solution as concluded in SECTION 3.4.

For the split form of the EQUATION (4-10), the non tangential boundary conditions corresponding to w' and w are satisfactory for establishing the influence coefficients for the edge effects solution. Further if $\chi_1 = \mu_1$, the solution takes the form,

$$\overline{w} = \overline{C}_1(e^{-\chi_1 y} \cos \chi_1 y) + \overline{C}_2(e^{-\chi_1 y} \sin \chi_1 y)$$
$$+ \overline{C}_5(e^{+\chi_1 y} \cos \chi_1 y) + \overline{C}_6(e^{+\chi_1 y} \sin \chi_1 y) \, . \qquad (4\text{-}15)$$

As shown by the values given in TABLE 3.4, χ_1 is generally large. Therefore the last two terms become unbounded as y increases while the first two terms are highly damped. Because of the nature of the χ_1-solutions the last two terms can be neglected ignoring the effect of the far edge. The solution can now be written as,

$$\overline{w} = \overline{C}_1(e^{-\chi_1 y} \cos \chi_1 y) + \overline{C}_2(e^{-\chi_1 y} \sin \chi_1 y) \, . \qquad (4\text{-}16)$$

Designating $e^{-\chi_1 y} \cos \chi_1 y$ and $e^{-\chi_1 y} \sin \chi_1 y$ as g_1 and g_2 respectively, we can write $\overline{w}$ and $\overline{w}'$ as,

$$\overline{w} = \overline{C}_1 g_1 + \overline{C}_2 g_2 \, ,$$

$$\overline{w}' = \overline{C}_1(-\chi_1 g_1 - \chi_1 g_2) + \overline{C}_2(-\chi_1 g_2 + \chi_1 g_1) \, .$$

For the boundary conditions $\overline{w}' = 1$ and $\overline{w} = o$ at the top edge the following equations are obtained for the constants $\overline{C}_1$ and $\overline{C}_2$,

$$\begin{bmatrix} -x_1 & x_1 \\ 1 & o \end{bmatrix} \begin{bmatrix} \overline{C}_1 \\ \overline{C}_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

from which,

$$\overline{C}_1 = o \text{ and } \overline{C}_2 = \frac{1}{x_1} .$$

Using a relation similar to EQUATION (3-1) any quantity $\overline{f}$, either a stress resultant or a displacement, can be expressed as,

$$\overline{f} = c \left[(a_1\overline{C}_1 + a_2\overline{C}_2)g_1 + (a_1\overline{C}_2 - a_2\overline{C}_1)g_2\right] .$$

Substituting values of $\overline{C}_1$ and $\overline{C}_2$ we get,

$$\overline{f} = c \left[\frac{a_2}{x_1} g_1 + \frac{a_1}{x_1} g_2\right] . \tag{4-17}$$

Values of a_1 and a_2 for all the displacements and stress resultants are given in TABLE 4.3. This table is obtained in the following manner. Using the simplified expressions of Flügge for the stress resultants (see EQUATIONS (12a-f), p. 217, ref. 14), the table of characteristic coefficients (TABLE 1, p. 230, ref. 14) has been reduced to a much simpler form as given by TABLE 4.2. Making the substitution $\mu_1 = x_1$ in this table we obtain TABLE 4.3.

TABLE 4.4 which gives the characteristic coefficients for the boundary conditions $\overline{w}' = 1$ and $\overline{w} = o$ at the top edge is obtained by substituting the values of a_1 and a_2 from TABLE 4.3 in EQUATION (4-17). Using TABLE 4.4 explicit expressions for any displacement or stress resultant $\bar{f}$ for these boundary conditions can be written as,

$$\bar{f} = c(S_1 g_1 + S_2 g_2) \tag{4-18}$$

For the boundary conditions $\overline{w}' = o$ and $\overline{w} = 1$ at the top edge the following equations are obtained for the constants,

$$\begin{bmatrix} -X_1 & X_1 \\ 1 & o \end{bmatrix} \begin{bmatrix} \overline{C}_1 \\ \overline{C}_2 \end{bmatrix} = \begin{bmatrix} o \\ 1 \end{bmatrix}$$

from which

$$\overline{C}_1 = 1 \text{ and } \overline{C}_2 = 1.$$

As before any quantity $\bar{f}$ can be expressed as

$$\bar{f} = c[(a_1 C_1 + a_2 C_2) g_1 + (a_1 C_2 - a_2 C_1) g_2]$$

Substituting the values of $\overline{C}_1$ and $\overline{C}_2$ and a_1 and a_2 from TABLE 4.3,

$$\bar{f} = c (s_3 g_1 + s_4 g_2) \tag{4-19}$$

In a similar manner TABLE 4.5 giving the characteristic coefficients for the boundary conditions $\overline{w}' = o$ and $\overline{w} = 1$ at the top edge is obtained. Using this table explicit expressions for stress resultants and displacements for these boundary conditions can be written.

From the expressions given in TABLES 4.4 and 4.5, the displacements w', w, u and v and the stress resultants M_x, S_x, N_x and T_x at the top edge corresponding to y = o can be obtained. Both these displacements and stress resultants are quickly damped because of the nature of the x_1-solution. The values at y = o are obtained by making $g_1 = 1$ and $g_2 = o$ in the expressions given in TABLES 4.4 and 4.5.

It should be pointed out that as the other boundary conditions u = o and v = o have not been considered these quantities will appear as non zero displacements at the top edge. The effect of these displacements is discussed later in SECTION 4.4.3.

The stiffness matrix corresponding to the edge effect solution can be written from the values at y = o. As $m << x_1$ in the expressions for M_x and S_x the terms involving m^2 can be neglected compared to x_1^2. A symmetric matrix [SM1] is then obtained as shown below.

$$[SM1] = \bar{K} \begin{array}{cc} \begin{array}{cc} \bar{w}'=1 & \bar{w}=1 \end{array} & \\ \begin{bmatrix} -2x_1 & -2x_1^2 \\ -2x_1^2 & 8x_1^3 \end{bmatrix} & \begin{array}{c} \bar{M}_x \\ \bar{S}_x \end{array} \end{array} \qquad (4\text{-}20)$$

where the constant multipler $\overline{K}$ in this expression is the shell flexural stiffness.

4.4.2 BASIC SOLUTION

This solution is obtained from the characteristic EQUATION (4-11). The χ_2 solution of this equation is very slowly damping. It is found from investigations by the author that the final stress resultants are highly sensitive to the values of χ_2, μ_2, $\overline{\alpha}_3$, $\overline{\alpha}_4$, $\overline{\beta}_3$ and $\overline{\beta}_4$ and even small variations in these values will cause the matrix for the evaluation of the arbitrary constants from the boundary conditions to be highly ill conditioned. This precludes introducing approximations to simplify the expressions to determine these quantities. Their determination by the exact theory is tedious. However the values of χ_2, μ_2 are very small and almost zero. If χ_2, μ_2 are considered as zero the basic solution given by EQUATION (4-11) reduces to the following form

$$w = (D + Cy + By^2 + Ay^3) \cos m\phi \qquad (4\text{-}21)$$

This solution is identical to the one obtained by the membrane theory (SECTION A.8 of APPENDIX A). Hence the basic solution can be replaced by the membrane theory as a good approximation.

The limitation of the assumption that the basic solution can be replaced by the membrane solution, can be established in the following manner. The term $e^{\lambda y}$ in the basic solution corresponding to EQUATION (4-11), can be written as,

$$e^{\lambda y} = 1 + \frac{\lambda y}{1!} + \frac{\lambda^2 y^2}{2!} + \frac{\lambda^3 y^3}{3!} + \frac{\lambda^4 y^4}{4!} e^{\gamma \lambda y}$$

where γ is such that $o<\gamma<1$. Taking the polynomial solution given by EQUATION (4-21) implies the assumption

$$\frac{\lambda^4 y^4}{4!} \quad e^{\gamma \lambda y} << 1$$

As $\gamma\lambda y$ is very small $e^{\gamma\lambda y} \to 1$.

To obtain the largest error set $y = Q$.

$$\frac{\lambda^4 Q^4}{4!} << 1$$

It can be seen from EQUATION (4-11) that $\lambda \simeq h_2 k^{\frac{1}{4}} m^2$, where the modulus of h_2 is of the order unity. Substituting this value in the above,

$$\frac{k(m^2 - 1)^2 m^4 Q^4}{4!} << 1 \quad \text{or}$$

$$\frac{P^2 Q^4 (m^2 - 1)^2 m^4}{12(4)!} << 1 \qquad (4\text{-}22)$$

Let us designate the quantity on the left hand side of this inequality as ρ. ρ sets a limit on replacing the basic solution by the membrane solution. For $m = 1$, it is seen that $\rho = o$ indicating that the basic solution is exactly replaced by the membrane solution.

$$\begin{bmatrix} o & o & 1 & o \\ o & o & o & 1 \\ \frac{mQ^2}{6} & Q & 1 & o \\ \frac{m^2Q^3}{6}-2(1+\upsilon)Q & \frac{mQ^2}{2} & mQ & 1 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 1 \\ o \\ o \\ o \end{bmatrix} \qquad (4\text{-}23)$$

Influence coefficients are now determined using the membrane theory for edge loading as given in SECTION A.8 of APPENDIX A. The tangential boundary conditions u and v are considered for establishing the influence coefficients.

For the boundary conditions $\overline{u} = 1$ and $\overline{v} = o$ at the top edge and the corresponding displacements at the bottom edge set as zero, the EQUATIONS (4-23) are obtained for the arbitrary constants A,B,C and D. Values of these constants given in TABLE 4.6 are obtained from the solution of these equations.

Any displacement or stress resultant $\overline{f}$ can be expressed as,

$$\overline{f} = c(F_4 + F_3y + F_2y^2 + F_1y^3)\cdot \qquad (4\text{-}24a)$$

The expressions for F_1 to F_4 given in TABLE 4.7 can be obtained by substituting the values of the constants A,B,C and D into the approximate stress displacement relations given by EQUATIONS 12a-f, p. 217, ref. 14.

For the other pair of boundary conditions $\overline{u} = o$ and $\overline{v} = 1$ at the top edge and the corresponding displacements at the bottom edge set as zero, EQUATION (4-25) is obtained for the arbitrary constants. Values of the constants given in TABLE 4.6 were obtained from the solution of this equation.

$$\begin{bmatrix} o & o & 1 & o \\ o & o & o & 1 \\ \frac{mQ^2}{6} & Q & 1 & o \\ \frac{m^2Q^3}{6}-2(1+\upsilon)Q & \frac{mQ^2}{2} & mQ & 1 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} o \\ 1 \\ o \\ o \end{bmatrix} \qquad (4\text{-}25)$$

$$[SM2]=\begin{array}{cc} \begin{array}{cccc} u^{t}=1 & v^{t}=1 & u^{b}=1 & v^{b}=1 \end{array} & \\ \begin{bmatrix} \frac{b}{Q} & -\frac{n}{2} & \frac{n}{2}^{2}Q-\frac{b}{Q} & -\frac{n}{2} \\ -\frac{n}{2} & -\frac{1}{Q} & -\frac{n}{2} & \frac{1}{Q} \\ \frac{n}{2}^{2}Q-\frac{b}{Q} & \frac{n}{2} & \frac{b}{Q} & -\frac{n}{2} \\ -\frac{n}{2} & \frac{1}{Q} & -\frac{n}{2} & -\frac{1}{Q} \end{bmatrix} & \begin{array}{c} N_{x}^{t} \\ N_{x\phi}^{t} \\ N_{x}^{b} \\ N_{x\phi}^{b} \end{array} \end{array} \quad (4\text{-}26)$$

Note: $b = m^{2}Q^{2}/3 + 2(1+\upsilon)$

As before the corresponding displacements and stress resultants can be expressed as

$$\bar{f} = c (F_8 + F_7 y + F_6 y^2 + F_5 y^3) \qquad (4\text{-}24b)$$

where the values of F_1 to F_4 are given in TABLE 4.8.

In both cases, the membrane stress resultants are obtained by considering only the constants A and B. The constants C and D depend only on the inextensional deformation and give rise to the stress couples. For m = 1, the inextensional deformation is replaced by rigid body deformations and therefore C and D can be ignored.

From the expressions given in TABLES 4.7 and 4.8 the stress resultants M_x, S_x, N_x and T_x at the top and bottom edges corresponding to y = o and y = Q respectively can be obtained. The stiffness matrix corresponding to the basic part can now be written using these values. A symmetric matrix [SM2] is obtained as shown in EQUATION (4-26).

4.4.3 COMPLETE STIFFNESS MATRIX

To obtain a complete stiffness matrix for a shell element it is necessary to assign in turn to each boundary displacement a value of unity, while all other displacements are kept zero. Therefore the elements obtained for the submatrices [SM1] and [SM2] in the previous sections cannot be elements in the complete stiffness matrix since the interaction of the two systems of boundary conditions corresponding to the edge effects and the basic state has not been considered so far. This results in edge displacements u and v for the edge effects state and edge displacements w' and w for the basic state that are non-zero.

It is possible to obtain a complete stiffness matrix based on the split type solution in the following manner. Both the displacements and stress resultants at the top and bottom edges of the shell element can be computed corresponding to the edge effects state by EQUATIONS (4-18) and (4-19). The necessary coefficients are tabulated in TABLES 4.4 and 4.5. In a similar manner the displacements and stress resultants at the edges corresponding to the basic state can be obtained by EQUATIONS (4-24a) and (4-24b) and the required coefficients are tabulated in TABLES 4.7 and 4.8. Using the values of the stress resultants thus obtained a matrix [SM] can be written. This matrix will be 8×8 with the first four columns representing the stress resultants for the displacements applied at the top edge. The elements are tabulated in TABLE 4.9. The elements in the last four columns will be numerically equal to the corresponding elements in the first four columns with appropriate signs in accordance with Flügge's sign convention. It may be noted that the format of of [SM] , that is the labelling of the rows and columns resembles that of a stiffness matrix. However, although, the elements of [SM] are stress resultants at the edges, [SM] is not the desired stiffness matrix since the interaction of the displacements caused by the two systems has not been considered. To transform [SM] to the desired stiffness matrix it is necessary to correct the elements by eliminating the non-zero displacements in each system caused by the other system. This is done as follws.

The matrix formed from the elements corresponding to the boundary conditions leading to the stiffness matrix must be a unit matrix. In a manner similar to that adopted for writing a matrix [D] corresponding to the displacements can be written using EQUATIONS (4-18),(4-19),(4-24a) and (4-24b) and these elements are tabulated in TABLE 4.10. Any column in this matrix [D] represents the applied edge displacements and the resulting edge displacements considered in an order which is consistent with that of [SM] . To form a stiffness matrix, it is necessary that the matrix [D] be converted into a unit matrix. It is noted that all the terms on the major diagonal already have a value of unity.

The displacements in the first column of the matrix [D] are obtained from EQUATION (4-18) corresponding to the edge effects solution. Therefore the only non-zero elements other than on the major diagonal will be displacements corresponding to the basic state. These elements can be made zero by applying equal and opposite corrective displacements. Stress resultants corresponding to these displacements can be computed using the expressions for the basic state given in TABLES 4.7 and 4.8. The values are added to the corresponding elements in the first column of $[SM]$. This procedure is repeated for the second column. Correction for the third and fourth columns are due to displacements corresponding to the edge effects state. The corrections are obtained using TABLES 4.4 and 4.5 in a similar manner as for the first column. The corrections applied to $[SM]$ due to the membrane displacements in the first two columns of $[D]$ introduce non-zero bending displacements which will be smaller than the applied displacements. Similarly non-zero membrane displacements will be introduced when applying the corrections due to the bending displacements in the third and fourth columns of $[D]$. New corrective displacements can be applied to balance the non zero displacements resulting from the first cycle of corrections. This procedure can be repeated until the displacements introduced by applying the correction are to any desired degree of smallness. After a sufficient number of iterative cycles the elements of $[\overline{SM}]$ will approach those of the desired stiffness matrix $[SM]$.

Fortunately it is not necessary to use this iterative procedure to obtain a stiffness matrix corresponding to the split type solutions, since explicit solutions can be written for all elements of the stiffness matrix $[SM]$ in terms of the uncorrected elements of $[\overline{SM}]$ and $[D]$. These expressions are given in TABLE 4.11 and correspond to the sign convention given by Flügge.

The expressions given in TABLE 4.11 were obtained by applying the first cycle of corrections algebraically. The resulting elements for an extreme range of shell parameters were compared with the corresponding elements of the stiffness matrix obtained by Flügges's exact theory, described in SECTION 4.2. It was noted that in the two sub matrices corresponding to any edge the diagonal terms, terms corresponding to w' and u and the term T_x corresponding to w agree sufficiently close with the exact values to be used without any further correction. Since the final matrix is symmetrical all other elements can be written from symmetry. A numerical example for a specific shell geometry and loading harmonic is given as Example 5.1 in CHAPTER V.

4.5 BASED ON THE EIGHTH ORDER CHARACTERISTIC EQUATION FOR HIGH VALUES OF m

It is clear from the discussion in CHAPTER III that for high values of m,the general solution based on Donnell's theory will give sufficiently accurate results. Donnell's eighth order characteristic equation is given by EQUATION(4-27)

$$(\lambda^2 - m^2)^4 + \frac{1}{k'} \lambda^4 = o \qquad (4.27)$$

This equation has been already presented in CHAPTER I as EQUATION (1-15). For high values of m the solution may be simplified by ignoring the effect of the far edge. The solution for $\overline{w}$ can therefore be written in the form given by EQUATION(3-5). The constants $\overline{C}_1$ to $\overline{C}_4$ are evaluated from the boundary conditions at one edge.

Because the number of constants is reduced to four the general procedure explained in SECTION 4.2 for the exact theory of Flügge can be used to establish the influence coefficients. For the evaluation of the constants the four boundary conditions corresponding to w', w, u and v at the edge y = o are considered. A unit value of each of these displacements is applied keeping the remainder of the displacements as zero. Using EQUATION (3-1) any quantity $\overline{f}$ either a displacement or a stress resultant can be written as

$$\overline{f} = c[a_1\overline{C}_1 + a_2\overline{C}_2 + a_3\overline{C}_3 + a_4\overline{C}_4] \qquad (4\text{-}28)$$

$$[C_1] = \begin{bmatrix} -\chi_1 & \mu_1 & -\chi_2 & \mu_2 \\ 1 & 0 & 1 & 0 \\ \overline{\alpha}_1 & \overline{\alpha}_2 & \overline{\alpha}_3 & \overline{\alpha}_4 \\ \overline{\beta}_1 & \overline{\beta}_2 & \overline{\beta}_3 & \overline{\beta}_4 \end{bmatrix} \tag{4-30}$$

Using this equation, the matrix equation for the boundary conditions can be written in the following form which has been given earlier as EQUATION (4-1),

$$[C_1][\overline{C}] = [I] \cdot \tag{4-29}$$

The coefficient matrix $[C_1]$ is given by EQUATION (4-30). Each column vector of $[\overline{C}]$ consists of the constants $\overline{C}_1$ to $\overline{C}_4$ and each column vector of $[I]$ represents the displacement boundary condition for each case of the applied unit edge displacement. The matrix $[\overline{C}]$, obtained from the solution of EQUATIONS (4-29) is given by the EQUATIONS (4-31). The stress displacement relations on which Donnell's theory is based are identical to the corresponding simplified expressions of Flügge. Therefore TABLE 4.2 giving the values of the characteristic coefficients can be used for Donnell's theory also. Using this table and EQUATION (4-28) the coefficient matrix $[C_2]$ as given in EQUATION (4-32) for the stress resultants M_x, S_x, N_x and T_x corresponding to the displacements w', w, u and v is obtained.

The stiffness influence coefficients are obtained by the following equation.

$$[K] = [C_2][\overline{C}] \tag{4-33}$$

$$[\overline{C}] = \begin{bmatrix} -\varepsilon_1 & [-\chi_2\varepsilon_1-\mu_2\varepsilon_4-\mu_1\varepsilon_2] & [-\mu_2\overline{\beta}_2-\mu_1\overline{\beta}_4] & [\alpha_2\mu_2-\alpha_4\mu_1] \\ [-\varepsilon_2+\varepsilon_3] & [\chi_2\varepsilon_3-\chi_1\varepsilon_2+\mu_2\varepsilon_6] & [(\chi_1-\chi_2)\overline{\beta}_4+(\overline{\beta}_1-\overline{\beta}_3)\mu_2] & [(\chi_2-\chi_1)\overline{\alpha}_4-\mu_2(\overline{\alpha}_1-\overline{\alpha}_3) \\ \varepsilon_1 & [\chi_1\varepsilon_1-\mu_2\varepsilon_5+\mu_1\varepsilon_3] & [\mu_2\overline{\beta}_2+\mu_1\overline{\beta}_4] & -[\alpha_2\mu_2-\alpha_4\mu_1] \\ -(\varepsilon_4+\varepsilon_5) & [-\chi_2\varepsilon_5-\chi_1\varepsilon_4-\mu_1\varepsilon_6] & [(\chi_2-\chi_1)\overline{\beta}_2+(\overline{\beta}_3-\overline{\beta}_1)\mu_1] & [(\chi-\chi_2)\overline{\alpha}_a-\mu_1(\overline{\alpha}_3-\overline{\alpha}_1)] \end{bmatrix} \frac{1}{\theta}$$

Note:

$$\begin{aligned} \varepsilon_1 &= (\overline{\alpha}_2\overline{\beta}_1-\overline{\alpha}_4\overline{\beta}_2) \\ \varepsilon_2 &= (\overline{\alpha}_3\overline{\beta}_4-\overline{\alpha}_4\overline{\beta}_3) \\ \varepsilon_3 &= (\overline{\alpha}_1\overline{\beta}_4-\overline{\alpha}_4\overline{\beta}_1) \\ \varepsilon_4 &= (\overline{\alpha}_2\overline{\beta}_3-\overline{\alpha}_3\overline{\beta}_2) \\ \varepsilon_5 &= (\overline{\alpha}_1\overline{\beta}_2-\overline{\alpha}_2\overline{\beta}_1) \\ \varepsilon_6 &= (\overline{\alpha}_1\overline{\beta}_3-\overline{\alpha}_3\overline{\beta}_1) \\ \theta &= (\chi_1-\chi_2)\varepsilon_1-\mu_2(\varepsilon_4+\varepsilon_5)-\mu_1(\varepsilon_2-\varepsilon_3) \end{aligned} \qquad (4\text{-}31)$$

$$[C_2]= \overline{D} \begin{bmatrix} [(\chi_1{}^2-\mu_1{}^2)-\upsilon m^2]\ k & -2\chi_1\mu_1 k & [(\chi_2{}^2-\mu_2{}^2)-\upsilon m^2]\ k & -2\chi_2\mu_2\ k \\ \chi_1[(2-\upsilon)m^2-\chi_1{}^2+3\mu_1{}^2]\ k & -\mu_1[(2-\upsilon)m^2-3\chi_1{}^2+\mu_1{}^2]\ k & \chi_2[(2-\upsilon)m^2-\chi_2{}^2+3\mu_2{}^2]\ k & -\mu_2[(2-\upsilon)m^2-3\chi_2{}^2+\mu_2{}^2]\ k \\ \upsilon-(\chi_1\overline{\alpha}_1+\mu_1\overline{\alpha}_2)+\upsilon m\overline{\beta}_1 & [-(\chi_1\overline{\alpha}_2-\mu_1\overline{\alpha}_1)+\upsilon m\overline{\beta}_2] & [\upsilon-(\chi_2\overline{\alpha}_3+\mu_2\overline{\alpha}_4)+\upsilon m\overline{\beta}_3] & [-(\chi_2\overline{\alpha}_4-\mu_2\overline{\alpha}_3)+\upsilon m\overline{\beta}_4] \\ [-m\overline{\alpha}_1-(\chi_1\overline{\beta}_1+\mu_1\overline{\beta}_2)](1-\upsilon) & [-m\overline{\alpha}_2-(\chi_1\overline{\beta}_2-\mu_1\overline{\beta}_1)](1-\upsilon) & [-m\overline{\alpha}_3-(\chi_2\overline{\beta}_3+\mu_1\overline{\beta}_4)]\ (1-\upsilon) & [-m\overline{\alpha}_4-(\chi_2\overline{\beta}_4-\mu_2\overline{\beta}_3)]\ (1-\upsilon) \end{bmatrix}$$

(4-32)

The elements in the stiffness matrix [K] and the internal stress resultants in the shell can be found in the following manner. The elements in $[\overline{C}]$ and $[C_2]$ depend only on the quantities χ_i, μ_i, $\overline{\alpha}_j$, $\overline{\beta}_j$. Explicit expressions, for these derived by Hoff based on Donnell's theory, have been given in SECTIONS A.2 and A.6 of APPENDIX A. Substituting these values into EQUATIONS (4-31) and (4-32), $[\overline{C}]$ and $[C_2]$ can be obtained. The elements in [K] are found by premultiplying $[\overline{C}]$ by $[C_2]$. The internal stress resultants for any case of unit applied edge displacement can be found using EQUATION (4-28). The constants $\overline{C}_1$ to $\overline{C}_4$ in this equation are given by the particular column vector of $[\overline{C}]$ corresponding to the unit displacement considered. The quantities a_1 and a_2 have been given in TABLE 4.2. Values of a_3 and a_4 can be obtained from the same table by changing the subscripts of χ and μ from 1 to 2 and those of $\overline{\alpha}$ and $\overline{\beta}$ from 1 and 2 to 3 and 4 respectively.

It may be noted that in the above method Donnell's theory has been used without making any further approximations. However since the sign convention for the general shell theory has been used in evaluating the elements of the stiffness matrix, it will be symmetric except for the signs. A truly symmetric matrix can be obtained by adopting the stiffness matrix sign convention. (See FIGURE 4.2).

4.6 BASED ON SPLIT TYPE SOLUTION FOR VERY HIGH VALUES OF m

For very high values of m the shell behaves as if it were a thin plate. As a result the membrane stress resultants N_x and T_x and the corresponding displacements u and v may be determined by the equations of generalised plane stress and the bending stress resultants M_x and S_x and their corresponding displacements w' and w by the equations of plate bending. These equations can be obtained from the three simpler differential equations expressed in terms of displacements corresponding to the equations of equilibrium in the three co-ordinate directions as presented by Flügge (p. 219, ref. 14.). This is accomplished for plane stress by omitting terms involving w and the loading in the first two equilibrium equations and for plate bending by retaining only the terms involving the derivative of w in the third equilibrium equation.

The shell problem is thus split into two independent states and the analysis leading to the establishment of influence coefficients is done by considering these two states separately.

GENERALISED PLANE STRESS

The generalised plane stress equations are given as

$$(u'' + u^{\cdot\cdot}) \quad + \quad c_1(u'' + v'^{\cdot}) \quad = \quad o$$

$$(v'' + v^{\cdot\cdot}) \quad + \quad c_1(u'^{\cdot} + v'') \quad = \quad o \qquad (4\text{-}34)$$

where $c_1 = \frac{(1+\upsilon)}{(1-\upsilon)}$

Taking the solutions for u and v as $\overline{u}$ cos mϕ and $\overline{v}$ sin mϕ respectively, the above equations can be reduced to the following form.

$$\overline{u}'' (1+c_1) - m^2\overline{u} + mc_1\overline{v}' = o$$

$$\overline{v}'' - m^2(1 + c_1) \overline{v} - mc_1\overline{u}' = o \qquad (4\text{-}35a,b)$$

The solution for $\overline{u}$ and $\overline{v}$ can be written as $Ae^{\lambda y}$ and $Be^{\lambda y}$ respectively Substituting these in the above equation we get the following characteristic equation.

$$(\lambda^2 - m^2)^2 = o \qquad (4\text{-}36)$$

This equation has four roots which are all numerically equal to m, two of these being repeated. Therefore the solutions for $\overline{u}$ and $\overline{v}$ can be written as,

$$\overline{u} = e^{-my}[A_1 + A_2y] + e^{+my}[A_3 + A_4y]$$

$$\overline{v} = e^{-my}[B_1 + B_2y] + e^{+my}[B_3 + B_4y] \qquad (4\text{-}37)$$

Since m is very large the second term in these solutions can be ignored as being unbounded as y increases. Therefore,

$$\overline{u} = e^{-my}[A_1 + A_2y]$$

$$\overline{v} = e^{-my}[B_1 + B_2y] \qquad (4\text{-}38)$$

Substituting these into EQUATIONS (4-35a) ,

$$B_1 = A_1 - A_2 \frac{(2 + c_1)}{mc_1} \quad \text{and}$$

$$B_2 = A_2 \quad . \qquad (4\text{-}39)$$

There are thus two arbitrary constants of integration A_1 and A_2 which are to be evaluated from the boundary conditions. The stiffness influence coefficients for the generalised plane stress state are established by considering the boundary conditions corresponding to u and v as follows,

$$\begin{bmatrix} 1 & o \\ 1 & \frac{-mc_1}{2+c_1} \end{bmatrix} \begin{bmatrix} A_{1u} & A_{1v} \\ A_{2u} & A_{2v} \end{bmatrix} = \begin{bmatrix} 1 & o \\ o & 1 \end{bmatrix} . \qquad (4\text{-}40)$$

The column vector $\{A_{1u} \quad A_{2u}\}$ represents the constants A_1 and A_2 for the boundary conditions $\overline{u} = 1$ and $\overline{v} = o$. The vector $\{A_{1v} \quad A_{2v}\}$ represents the same constants for the boundary conditions $\overline{u} = o$ and $\overline{v} = 1$. EQUATIONS (4-40) are solved for the constants A_1 and A_2. Using EQUATIONS (4-38) and (4-39) the membrane displacements corresponding to the prescribed sets of boundary conditions can be evaluated.

The membrane stress resultants can now be evaluated from the corresponding displacements u and v obtained from EQUATIONS (4-38) for the boundary conditions specified as above and the stress displacement relations based on generalised plane stress theory. These relations can be obtained by omitting the terms corresponding to the bending displacement in the stress displacement equations as given by Flügge [EQUATIONS 12a-c, p. 217, ref. 14]. The expressions thus obtained are presented in TABLE 4-12.

From TABLE 4-12 the values of the stress resultants at $y = o$ can be found. The stiffness matrix (SM3] corresponding to the membrane displacements u and v can finally be written as,

$$[SM3] = ms \begin{array}{cc} \bar{u} = 1 & \bar{v} = 1 \\ \left[\begin{matrix} 2 & (1-\upsilon) \\ (1-\upsilon) & 2 \end{matrix}\right] \end{array} \begin{matrix} \bar{N}_x \\ \bar{N}_{x\phi} \end{matrix} \quad . \tag{4-41}$$

where $s = \dfrac{\bar{D}(1-\upsilon)}{r(3-\upsilon)}$

PLATE BENDING

The differential equation for plate bending is

$$w^{1v} + 2w^{''\cdot\cdot} + w^{::} = 0 \quad . \tag{4-42}$$

Taking w in the form $\bar{w}$ cos $m\phi$ this equation is reduced to the form,

$$\bar{w}^{1v} - 2m^2\bar{w} + m^4\bar{w} = o \quad . \tag{4-43}$$

Introducing $\overline{w} = C\, e^{\lambda y}$, we get from the above equation the following characteristic equation,

$$(\lambda^2 - m^2)^2 = o \; . \tag{4-44}$$

This equation is identically the same as EQUATION (4-36) and the solution for $\overline{w}$ is of the form,

$$\overline{w} = e^{-my}[C_1 + C_2 y] + e^{+my}[C_3 + C_4 y] \; .$$

Ignoring the unbounded solution,

$$\overline{w} = e^{-my}[C_1 + C_2 y] \; . \tag{4-45}$$

There are thus two arbitrary constants of integration C_1 and C_2 which are to be evaluated from the boundary conditions. The stiffness influence coefficients for the plate bending state are established by considering the boundary conditions corresponding to w' and w. The following equations are obtained for the constants ,

$$\begin{bmatrix} -m & 1 \\ 1 & o \end{bmatrix} \begin{bmatrix} C_{1w'} & C_{1w} \\ C_2 w' & C_{2w} \end{bmatrix} = \begin{bmatrix} 1 & o \\ o & 1 \end{bmatrix} \; . \tag{4-46}$$

The column vector $\{C_{1w'} \;\; C_{2w'}\}$ represents the constants C_1 and C_2 for the boundary conditions $\overline{w}' = 1$ and $\overline{w} = o$. The vector $\{C_{1w} \;\; C_{2w}\}$ represents the same constants for the boundary conditions $\overline{w}' = o$ and $\overline{w} = 1$. EQUATIONS (4-46) are solved for the constants C_1 and C_2. Using EQUATIONS (4-45) the bending dis-

$$[SM] = \begin{array}{cc} \begin{array}{cccc} \overline{w}'=1 & \overline{w}=1 & \overline{u}=1 & \overline{v}=1 \end{array} & \\ \left[\begin{array}{cccc} -2m\overline{K} & (1+\upsilon)m^2\overline{K} & 0 & 0 \\ (1+\upsilon)m^2\overline{K} & 2m^3\overline{K} & 0 & 0 \\ 0 & 0 & 2sm & (1-\upsilon)sm \\ 0 & 0 & (1-\upsilon)sm & 2sm \end{array}\right] & \begin{array}{c} \overline{M}_x \\ \overline{S}_x \\ \overline{N}_x \\ \overline{T}_x \end{array} \end{array}$$

(4-48)

placements corresponding to the prescribed set of boundary conditions can be evaluated.

The bending stress resultants can now be evaluated from the corresponding displacement w obtained from EQUATION (4-45) for the boundary conditions specified as above and the stress displacement relations based on plate bending theory. These relations can be obtained from the stress displacement equations as given by Flügge [EQUATIONS 12 d-f, p. 217, ref. 14]. The expressions thus obtained are presented in TABLE 4.13.

From TABLE 4.13 the values of the stress resultants at y = o can be found. The stiffness matrix [SM4] corresponding to the bending displacements w' and w can finally be written as,

$$[SM4]=\begin{array}{cc} \begin{array}{cc} \bar{w}'=1 & \bar{w}=1 \end{array} & \\ \begin{bmatrix} -2 & (1+\upsilon)m \\ (1+\upsilon)m & 2m^2 \end{bmatrix} m\bar{K} & \begin{array}{c} \bar{M}_x \\ \bar{S}_x \end{array} \end{array} \qquad (4.47)$$

COMPLETE STIFFNESS MATRIX

Using EQUATIONS 4-41 and 4-27 the complete stiffness matrix considering all the four displacements w', w, u and v can be written as given in EQUATION 4.48. It is noted that the matrix thus obtained is truly symmetric.

TABLE 4.1

VALUES OF $\bar{\alpha}_1$, $\bar{\alpha}_2$, $\bar{\beta}_1$, $\bar{\beta}_2$

r/h	k	m		$\bar{\alpha}_1$	$\bar{\alpha}_2$	$\bar{\beta}_1$	$\bar{\beta}_2$
10	0.8333×10^{-3}	1	Flügge	0.01240	0.02765	0.00110	0.06334
			Donnell	0.01163	0.02797	0.00538	0.06335
			Simplified	0.01620	0.02445	0.00085	0.06339
		3	Flügge	0.01226	0.06355	0.01337	0.18318
			Donnell	0.00580	0.06130	0.01791	0.18182
			Simplified	0.03164	0.04297	0.07820	0.16035
		5	Flügge	0.03095	0.12528	0.03875	0.27316
			Donnell	0.02162	0.12285	0.04307	0.27316
			Simplified	0.05766	0.03435	0.12199	0.15090
		10	Flügge	0.04831	0.29466	0.07583	0.45888
			Donnell	0.03582	0.29160	0.07626	0.45762
			Simplified	0.04855	0.01888	0.08822	0.07451
100	0.8333×10^{-5}	1	Flügge	0.00615	0.00663	0.00001	0.00634
			Donnell	0.00626	0.00652	0.00000	0.00635
			Simplified	0.00613	0.00665	0.00005	0.00634
		3	Flügge	0.00512	0.00769	0.00017	0.01902
			Donnell	0.00524	0.00759	0.00023	0.01899
			Simplified	0.00487	0.00777	0.00104	0.01899
		5	Flügge	0.00325	0.00994	0.00095	0.03158
			Donnell	0.00338	0.00985	0.00104	0.03157
			Simplified	0.00216	0.00969	0.00458	0.03123
		10	Flügge	0.00259	0.02090	0.00620	0.06021
			Donnell	0.00237	0.02082	0.00634	0.06015
			Simplified	0.00931	0.01288	0.02478	0.05157
1000	0.8333×10^{-7}	1	Flügge	0.00201	0.00203	0.00000	0.00063
			Donnell	0.00202	0.00202	0.00000	0.00063
			Simplified	0.00201	0.00203	0.00000	0.00063
		3	Flügge	0.00198	0.00206	0.00000	0.00190
			Donnell	0.00198	0.00206	0.00000	0.00190
			Simplified	0.00197	0.00207	0.00001	0.00190
		5	Flügge	0.00194	0.00213	0.00000	0.00317
			Donnell	0.00192	0.00212	0.00001	0.00317
			Simplified	0.00190	0.00214	0.00004	0.00317
		10	Flügge	0.00161	0.00245	0.00008	0.00634
			Donnell	0.00162	0.00245	0.00008	0.00634
			Simplified	0.00152	0.00247	0.00037	0.00632

Note : 'Simplified' is based on split equations of Flügge.

TABLE 4.2

APPROXIMATE CHARACTERISTIC COEFFICIENTS

f	c	a_1	a_2	a_5	a_6	ϕ factor
w	1	1	0	a_1	$-a_2$	cos
w'	1	$-\chi_1$	μ_1	$-a_1$	a_2	cos
u	1	$\bar{\alpha}_1$	$\bar{\alpha}_2$	$-a_1$	a_2	cos
v	1	$\bar{\beta}_1$	$\bar{\beta}_2$	a_1	$-a_2$	sin
N_ϕ	$\bar{D}$	$1+m\bar{\beta}_1-\nu(\chi_1\bar{\alpha}_1+\mu_1\bar{\alpha}_2)$	$m\bar{\beta}_2-\nu(\chi_1\bar{\alpha}_2-\mu_1\bar{\alpha}_1)$	a_1	$-a_2$	cos
N_x	$\bar{D}$	$\nu-(\chi_1\bar{\alpha}_1+\mu_1\bar{\alpha}_2)+\nu m\bar{\beta}_1$	$-(\chi_1\bar{\alpha}_2-\mu_1\bar{\alpha}_1)+\nu m\bar{\beta}_2$	a_1	$-a_2$	cos
$N_{\phi x}$	$\bar{D}(1-\nu)/2$	$-m\bar{\alpha}_1-(\chi_1\bar{\beta}_1+\mu_1\bar{\beta}_2)$	$-m\bar{\alpha}_2-(\chi_1\bar{\beta}_2-\mu_1\bar{\beta}_1)$	$-a_1$	a_2	sin
T_x	$\bar{D}(1-\nu)/2$	$-m\bar{\alpha}_1-(\chi_1\bar{\beta}_1+\mu_1\bar{\beta}_2)$	$-m\bar{\alpha}_2-(\chi_1\bar{\beta}_2-\mu_1\bar{\beta}_1)$	$-a_1$	a_2	sin
M_ϕ	$\bar{K}$	$-m^2+\nu(\chi_1{}^2-\mu_1{}^2)$	$-2\nu\chi_1\mu_1$	a_1	$-a_2$	cos
M_x	$\bar{K}$	$(\chi_1{}^2-\mu_1{}^2)-\nu m^2$	$-\chi_1\mu_1$	a_1	$-a_2$	cos
$M_{\phi x}$	$\bar{K}(1-\nu)-2$	$2m\chi_1$	$2m\mu_1$	$-a_1$	a_2	sin
Q_ϕ	$\bar{K}$	$m(m^2-1)-m(\chi_1{}^2-\mu_1{}^2)$	$2m\chi_1\mu_1$	$-a_1$	a_2	cos
Q_x	$\bar{K}$	$2\chi_1(m^2-\chi_1{}^2+3\mu_1{}^2)$	$-2\mu_1(m^2-3\chi_1{}^2+\mu_1{}^2)$	$-a_1$	a_2	cos
S_x	$\bar{K}/2$	$2\chi_1[(2-\nu)m^2-\chi_1{}^2+3\mu_1{}^2]$	$-2\mu_1[(2-\nu)m^2-3\chi_1{}^2+\mu_1{}^2]$	$-a_1$	a_2	cos

Note: $N_{x\phi} = N_{\phi x}$, $M_{x\phi} = M_{\phi x}$

TABLE 4.3

VALUES OF a_1 AND a_2 WITH $\chi 1 = \mu 1$

f	c	a_1	a_2	ϕ factor
w	1	1	o	cos
w'	1	$-\chi_1$	χ_1	cos
u	1	$\bar{\alpha}_1$	$\bar{\alpha}_2$	cos
v	1	$\bar{\beta}_1$	$\bar{\beta}_2$	sin
N_ϕ	$\bar{D}$	$1+m\bar{\beta}_1-\upsilon\chi_1(\bar{\alpha}_1+\bar{\alpha}_2)$	$m\bar{\beta}_2-\upsilon\chi_1(\bar{\alpha}_2-\bar{\alpha}_1)$	cos
N_x	$\bar{D}$	$\upsilon-\chi_1(\bar{\alpha}_1+\bar{\alpha}_2)+\upsilon m\bar{\beta}_1$	$-\chi_1(\bar{\alpha}_2-\bar{\alpha}_1)+\upsilon m\bar{\beta}_2$	cos
$N_{\phi x}$	$\bar{D}(1-\upsilon)/2$	$-m\bar{\alpha}_1-\chi_1(\bar{\beta}_1+\bar{\beta}_2)$	$-m\bar{\alpha}_2-\chi_1(\bar{\beta}_2-\bar{\beta}_1)$	sin
T_x	$\bar{D}(1-\upsilon)/2$	$-m\bar{\alpha}_1-\chi_1(\bar{\beta}_1+\bar{\beta}_2)$	$-m\bar{\alpha}_2-\chi_1(\bar{\beta}_2-\bar{\beta}_1)$	sin
M_ϕ	$\bar{K}$	$-m^2$	$-2\upsilon\chi_1^2$	cos
M_x	$\bar{K}$	$-\upsilon m^2$	$-2\chi_1^2$	cos
$M_{\phi x}$	$\bar{K}(1-\upsilon)/2$	$2m\chi_1$	$-2m\chi_1$	sin
Q_ϕ	$\bar{K}$	$m(m^2-1)$	$2m\chi_1^2$	cos
Q_x	$\bar{K}/2$	$2\chi_1(m^2+2\chi_1^2)$	$-2\chi_1(m^2-2\chi_1^2)$	cos
S_x	$\bar{K}/2$	$2\chi_1[(2-\upsilon)m^2+2\chi_1^2]$	$-2\chi_1[(2-\upsilon)m^2-2\chi_1^2]$	cos

Note: $N_{x\phi} = N_{\phi x}$, $M_{x\phi} = M_{\phi x}$

TABLE 4.4

CHARACTERISTIC COEFFICIENTS

FOR w' = 1 AND w = o AT TOP

$\bar{f}$	c	s_1	s_2
$\bar{w}$	1	o	1
$\bar{w}'$	1	1	-1
$\bar{u}$	1	$\bar{\alpha}_2/\chi_1$	$\bar{\alpha}_1/\chi_1$
$\bar{v}$	1	$\bar{\beta}_2/\chi_1$	$\bar{\beta}_1/\chi_1$
$\bar{N}_\phi$	$\bar{D}$	$m\bar{\beta}_2/\chi_1-\upsilon(\bar{\alpha}_2-\bar{\alpha}_1)$	$(1+m\bar{\beta}_1)/\chi_1-\upsilon(\bar{\alpha}_1+\bar{\alpha}_2)$
$\bar{N}_x$	$\bar{D}$	$-(\bar{\alpha}_2-\bar{\alpha}_1)+\upsilon m\bar{\beta}_2/\chi_1$	$\upsilon(1+m\bar{\beta}_1)/\chi_1-(\bar{\alpha}_1+\bar{\alpha}_2)$
$\bar{N}_{\phi x}$	$\bar{D}(1-\upsilon)/2$	$-m\bar{\alpha}_2/\chi_1-(\bar{\beta}_2-\bar{\beta}_1)$	$-m\bar{\alpha}_1/\chi_1-(\bar{\beta}_1+\bar{\beta}_2)$
$\bar{T}_x$	$\bar{D}(1-\upsilon)/2$	$-m\bar{\alpha}_2/\chi_1-(\bar{\beta}_2-\bar{\beta}_1)$	$-m\bar{\alpha}_1/\chi_1-(\bar{\beta}_1+\bar{\beta}_2)$
$\bar{M}_\phi$	$\bar{K}$	$-2\upsilon\chi_1$	$-m^2/\chi_1$
$\bar{M}_x$	$\bar{K}$	$-2\chi_1$	$-\upsilon m^2/\chi_1$
$\bar{M}_{\phi x}$	$\bar{K}(1-\upsilon)/2$	-2m	2m
$\bar{Q}_\phi$	$\bar{K}$	$2m\chi_1$	$m(m^2-1)/\chi_1$
$\bar{Q}_x$	$\bar{K}/2$	$-2(m^2-2\chi_1{}^2)$	$2(m^2+2\chi_1{}^2)/\chi_1$
$\bar{S}_x$	$\bar{K}/2$	$-2[(2-\upsilon)m^2-2\chi_1{}^2]$	$2[(2-\upsilon)m^2+2\chi_1{}^2]/\chi_1$

Note: $\bar{N}_{x\phi} = \bar{N}_{\phi x}$, $\bar{M}_{x\phi} = \bar{M}_{\phi x}$

TABLE 4.5

CHARACTERISTIC COEFFICIENTS

FOR $\bar{w}' = o$ AND $\bar{w} = 1$ AT TOP

$\bar{f}$	c	s_3	s_4
$\bar{w}$	1	1	1
$\bar{w}'$	1	0	$-2\chi_1$
$\bar{u}$	1	$\bar{\alpha}_1+\bar{\alpha}_2$	$\bar{\alpha}_1-\bar{\alpha}_2$
$\bar{v}$	1	$\bar{\beta}_1+\bar{\beta}_2$	$\bar{\beta}_1-\bar{\beta}_2$
$\bar{N}_\phi$	$\bar{D}$	$1+m(\bar{\beta}_1+\bar{\beta}_2)-2\upsilon\chi_1\bar{\alpha}_2$	$1+m(\bar{\beta}_1-\bar{\beta}_2)-2v\chi_1\bar{\alpha}_1$
$\bar{N}_x$	$\bar{D}$	$\upsilon-2\chi_1\bar{\alpha}_2+\upsilon m(\bar{\beta}_1+\bar{\beta}_2)$	$\upsilon-2\chi_1\bar{\alpha}_1+\upsilon m(\bar{\beta}_1-\bar{\beta}_2)$
$\bar{N}_{\phi x}$	$\bar{D}(1-\upsilon)/2$	$-m(\bar{\alpha}_1+\bar{\alpha}_2)-2\chi_1\bar{\beta}_2$	$-m(\bar{\alpha}_1-\bar{\alpha}_2)-2\chi_1\bar{\beta}_1$
$\bar{T}_x$	$\bar{D}(1-\upsilon)/2$	$-m(\bar{\alpha}_1+\bar{\alpha}_2)-2\chi_1\bar{\beta}_2$	$-m(\bar{\alpha}_1-\bar{\alpha}_2)-2\chi_1\bar{\beta}_1$
$\bar{M}_\phi$	$\bar{K}$	$-m^2-2\upsilon\chi_1$	$-m^2+2\upsilon\chi_1$
$\bar{M}_{\phi x}$	$\bar{K}(1-\upsilon)/2$	0	$4mx_1$
$\bar{Q}_\phi$	$\bar{K}$	$m(m^2-1)+2m\chi_1{}^2$	$m(m^2-1)-2m\chi_1{}^2$
$\bar{Q}_x$	$\bar{K}/2$	$8\chi_1{}^3$	$4\chi_1 m^2$
$\bar{S}_x$	$\bar{K}/2$	$8\chi_1{}^3$	$4\chi_1(2-\upsilon)m^2$

Note: $\bar{N}_{x\phi} = \bar{N}_{\phi x}$, $\bar{M}_{x\phi} = \bar{M}_{\phi x}$

TABLE 4.6

VALUES OF THE CONSTANTS IN THE BASIC SOLUTION

constants	Case 1 $\bar{u}_m=1, \bar{v}=o$ at top $\bar{u}_m=o, \bar{v}=o$ at bottom	Case 2 $\bar{u}_m=0, \bar{v}_m=1$ at top $\bar{u}_m=0, \bar{v}_m=o$ at bottom
A	$\frac{m}{2G}$	$\frac{1}{QG}$
B	$\dfrac{-\left[\frac{m^2Q^2}{3} + 2(1+\upsilon)\right]}{QG}$	$\frac{-m}{2G}$
C	1	o
D	o	1

$$G = \left[\frac{m^2Q^2}{12} + 2(1+\upsilon)\right]$$

TABLE 4.7

CHARACTERISTIC COEFFICIENTS FOR $\bar{u}=1$ AND $\bar{v}=o$, AT TOP AND $\bar{u}=\bar{v}=0$ AT BOTTOM

$\bar{f}$	c	F_1	F_2	F_3	F_4
$\bar{w}$	1	$-m^4/12$	$m^2b/2Q$	$m^2[1+\upsilon/2-a]$	$\upsilon b/Q$
$\bar{w}'$	1	0	$-m^4/4$	m^2b/Q	$m^2[1+\upsilon/2-a]$
$\bar{u}$	1	0	$m^2/4$	$-b/Q$	a
$\bar{v}$	1	$m^3/12$	$-bm/2Q$	$am-m(1+\upsilon)$	0
$\bar{N}_\phi$	$\bar{D}$	0	0	0	0
$\bar{N}_x$	$\bar{D}$	0	0	$m^2/2$	$-b/Q$
$\left.\begin{array}{l}\bar{N}_{x\phi}\\ =\bar{N}_{\phi x}\end{array}\right\}$	$\bar{D}(1-\upsilon)/2$	0	0	0	$-m/2$

Note: - $a = m^2Q^2/12 + 2\,(1+\upsilon)$, $b = m^2Q^2/3 + 2\,(1+\upsilon)$

TABLE 4.8

CHARACTERISTIC COEFFICIENTS FOR $\bar{u}$=o AND $\bar{v}$=1 AT TOP AND $\bar{u}=\bar{v}=0$ AT BOTTOM

$\bar{f}$	c	F_5	F_6	F_7	F_8
$\bar{w}$	1	$-m^3/6Q$	$m^3/4$	$(2+\nu)m/Q$	$m\nu/2-am$
$\bar{w}'$	1	0	$-m^3/2Q$	$m^3/2$	$(2+\nu)m/Q$
$\bar{u}$	1	0	$m/2Q$	$-m/2$	0
$\bar{v}$	1	$m^3/6Q$	$-m^2/4$	$-2(1+\nu)/Q$	a
$\bar{N}_\phi$	$\bar{D}$	0	0	0	Q
$\bar{N}_x$	$\bar{D}$	0	0	m/Q	$-m/2$
$\left.\begin{array}{r}\bar{N}_{x\phi}\\ \approx \bar{N}_{\phi x}\end{array}\right\}$	$\bar{D}(1-\nu)/2$	0	0	0	$-2/Q$

Note : $a = m^2Q^2/12 + 2\ (1+\nu)$, $b = m^2Q^2/3 + 2\ (1+\nu)$

TABLE 4.9

ELEMENTS OF THE MATRIX $[\overline{SM}]$

SM(i,j)	Multiplier	Characteristic Values	$\overline{SM}$(i,j)	Multiplier	Characteristic Values
$\overline{SM}(1,1)$	$\overline{K}$	$2\chi_1$	$\overline{SM}(1,3)$	$\overline{K}/a$	0
$\overline{SM}(2,1)$	$\overline{K}$	$\upsilon m^2 + 2\chi_1{}^2 + 2\ (1-\upsilon)m^2$	$\overline{SM}(2,3)$	K/a	0
$\overline{SM}(3,1)$	$\overline{D}$	$-[\chi_1(\overline{\alpha}_1+\overline{\alpha}_2)-\upsilon m\beta_2]$	$\overline{SM}(3,3)$	P/a	$-b/Q$
$\overline{SM}(4,1)$	$\overline{D}(1-\upsilon)$	$-[(\overline{\beta}_1+\overline{\beta}_2) + m\ \overline{\alpha}_2/\chi_1]$	$\overline{SM}(4,3)$	P/a	$-m/2$
$\overline{SM}(1,2)$	$\overline{K}$	$-[\upsilon m^2 + 2\chi_1{}^2]$	$\overline{SM}(1,4)$	$\overline{K}/a$	0
$\overline{SM}(2,2)$	$\overline{K}$	$2\upsilon m^2\chi_1$	$\overline{SM}(2,4)$	$\overline{K}/a$	0
$\overline{SM}(3,2)$	$\overline{D}$	$[2\chi_1\alpha_2+\upsilon(1+m\beta_1-m\beta_2)]$	$\overline{SM}(3,4)$	P/a	$-m/2$
$\overline{SM}(4.2)$	$\overline{D}(1-\upsilon)$	$2[2\chi_1\beta_2-m(\overline{\alpha}_1-\overline{\alpha}_2)]$	$\overline{SM}(4,4)$	P/a	$-1/Q$
$\overline{SM}(5,1)$	$\overline{K}$	0	$\overline{SM}(5,3)$	$\overline{K}/a$	0
$\overline{SM}(6,1)$	$\overline{K}$	0	$\overline{SM}(6,3)$	$\overline{K}/a$	0
$\overline{SM}(7,1)$	$\overline{D}$	0	$\overline{SM}(7,3)$	P/a	$m^2Q/2 - b/Q$
$\overline{SM}(8,1)$	$\overline{D}(1-\upsilon)$	0	$\overline{SM}(8,3)$	P/a	$m/2$
$\overline{SM}(5,2)$	$\overline{K}$	0	$\overline{SM}(5,4)$	$\overline{K}/a$	0
$\overline{SM}(6,2)$	$\overline{K}$	0	$\overline{SM}(6,4)$	$\overline{K}/a$	0
$\overline{SM}(7,2)$	$\overline{D}$	0	$\overline{SM}(7,4)$	P/a	$m/2$
$\overline{SM}(8,2)$	$\overline{D}(1-\upsilon)$	0	$\overline{SM}(8,4)$	P/a	$1/Q$

Note: $a = m^2Q^2/12 + 2(1+\upsilon)$, $b = m^2Q^2/3 + 2\ (1+\upsilon)$

TABLE 4-10

ELEMENTS OF THE MATRIX [D]

D(i,j)	Values D(i,j)
D(1,1)	1
D(3,1)	$\bar{\alpha}_2/x_1$
D(4,1)	$\bar{\beta}_2/x_1$
D(2,2)	1
D(3,2)	$\bar{\alpha}_1-\bar{\alpha}_2$
D(4,2)	$\bar{\beta}_1-\bar{\beta}_2$
D(1,3)	$-[m^2(1+\upsilon)/2-a]/a$
D(2,3)	$\upsilon\ b/Q\ a$
D(3,3)	1
D(1,4)	$-(2+\upsilon)m/Q\ a$
D(2,4)	$[\upsilon m/2\ -\ a\ m]/\ a$
D(4,4)	1
D(5,3)	$[m^4Q^2/4\ m^2\ (b+1+\upsilon/2-b)]/b$
D(6,3)	$-m^4Q^3/12 + [b/2+(1+\upsilon/2-a)]\ m^2Q+\upsilon\ b/Q$
D(5,4)	$-(2+\upsilon)m/Q\ a$
D(6,4)	$[m^3Q^2/12+(2+\upsilon)m\ +\ m\upsilon/2\ -\ a\ m]/\ a$

Note: Only the left half of the matrix is given.

Values not shown in this half are zero.

TABLE 4.11

ELEMENTS IN THE FINAL MATRIX [SM]

SM(i,j)	Values of SM(i,j)
SM(1,1)	SM(1,1) - D(3,1).SM(1,3) - D(4,1)·SM(1,4)
SM(1,2)=SM(2,1)	SM(2,1) - D(3,1).SM(2,3) - D(4,1)·SM(2,4)
SM(1,3)=SM(3,1)	SM(1,3) - D(1,3).SM(1,1) - D(2,3)·SM(1,2)
SM(1,4)=SM(4,1)	SM(4,1) - D(3,1).SM(4,3) - D(4,1)·SM(4,4)
SM(2,2)	SM(2,2) - D(3,2).SM(2,3) - D(4,2)·SM(2,4)
SM(2,3)=SM(3,2)	SM(2,3) - D(1,3)·SM(2,1) - D(2,3)·SM(2,2)
SM(2,4)=SM(4,2)	SM(4,2) - D(3,2)·SM(4,3) - D(4,2).SM(4,4)
SM(3,3)	SM(3,3) - D(1,3)·SM(3,1) - D(2,3)·SM(3,2)
SM(3,4)=SM(4,3)	SM(3,4) - D(1,3)·SM(4,1) - D(2,3)·SM(4,2)
SM(4,4)	SM(4,4) - D(1,4)·SM(4,1) - D(2,4)·SM(4,2)
SM(5,1)	0
SM(5,2)=SM(6,1)	0
SM(5,3)=SM(3,5)	SM(5,3) + D(5,3).SM(1,1) - D(6,3)·SM(1,2)
SM(5,4)=SM(8,1)	SM(8,1) - D(3,1)·SM(8,3) - D(4,1)·SM(8,4)
SM(6,2)	0
SM(6,3)=SM(7,1)	SM(6,3) - D(5,3)·SM(2,1) + D(6,3)·SM(2,2)
SM(6,4)=SM(8,2)	SM(8,2) - D(3,2)·SM(8,3) - D(4,2)·SM(8,4)
SM(7,3)	SM(7,3) + D(5,3)·SM(3,1) - D(6,3)·SM(3,2)
SM(7,4)=SM(8,3)	SM(8,3) - D(5,3)·SM(4,1) + D(6,3)·SM(4,2)
SM(8,4)	SM(8,4) - D(5,4)·SM(4,1) + D(6,4)·SM(4,2)

TABLE 4.12

MEMBRANE STRESS RESULTANTS

FOR VERY HIGH VALUES OF m

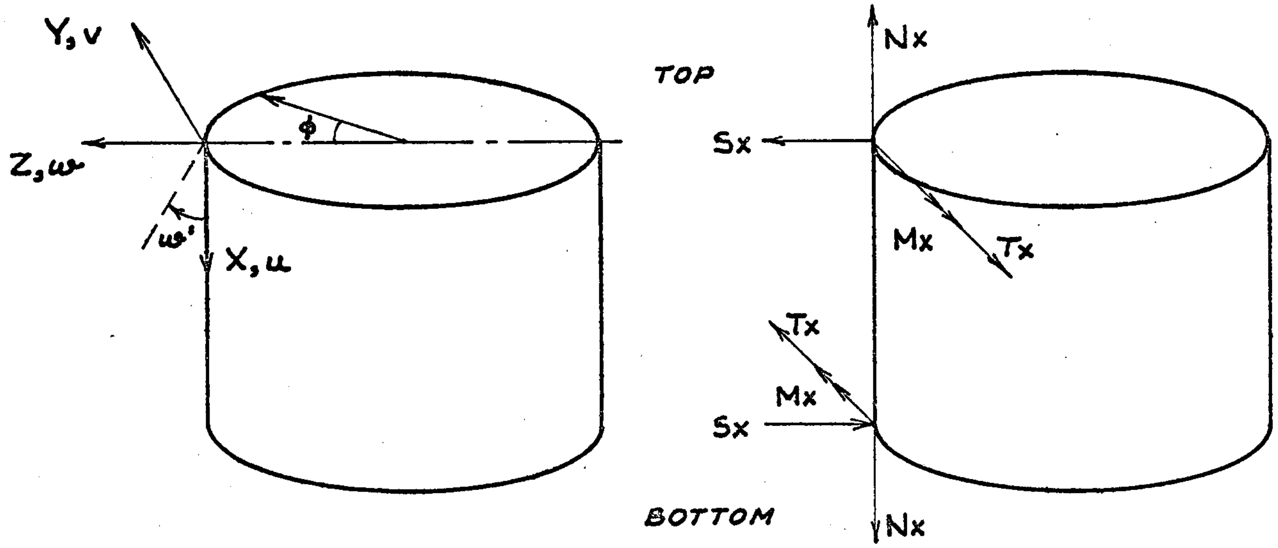
f	c	Case 1: $-\bar{u}=1$, $\bar{v}=0$	Case 2: $\bar{u}=0$, $\bar{v}=1$
$\bar{N}_x$	$\bar{D}$	$[(\bar{c}-m)-m\bar{c}(1-\upsilon)y]g$	$[-\bar{c}+\upsilon m+m\bar{c}(1-\upsilon)y]g$
N_ϕ	$\bar{D}$	$[\upsilon(\bar{c}-m)+m\bar{c}(1-\upsilon)y]g$	$[(m-\upsilon\bar{c})-m\bar{c}(1-\upsilon)y]g$
$\left.\begin{matrix}N_{x\phi}\\N_{\phi x}\end{matrix}\right\}$	$\bar{D}(1-\upsilon)/2$	$[(\bar{c}-m)-2m\bar{c}\ y]g$	$[-(\bar{c}+m)+2m\bar{c}\ y]g$

Note : $\bar{c} = m(1+\upsilon)/(3-\upsilon)$, $g = e^{-my}$

TABLE 4.13
BENDING STRESS RESULTANTS FOR VERY HIGH VALUES OF m

f	c	Case 1: - $\bar{w}'=1$ $\bar{w}=0$	Case 2: - $\bar{w}'=0$, $\bar{w}=1$
M_x	$\bar{K}$	$[-2m+m^2(1-\nu)y]\ g$	$[-(1+\nu)+m(1-\nu)y]\ m^2g$
$\bar{Q}_x$	$\bar{K}$	$[1-m(1-\nu)y]m^2g$	$2m^3g$
$\bar{M}_\phi$	$\bar{K}$	$[-2\nu m-m^2(1-\nu)y]g$	$[-(1+\nu)-m(1-\nu)y]m^2g$
$\bar{Q}_\phi$	$\bar{K}$	$2m^2g$	$[(1+\nu)+m(1-\nu)+m(1-\nu)(1-m)y]f$
$M_{\phi x}$	$\bar{K}$	$[-(1-\nu)+(1-\nu)my]mg$	$(1-\nu)m^4yg$

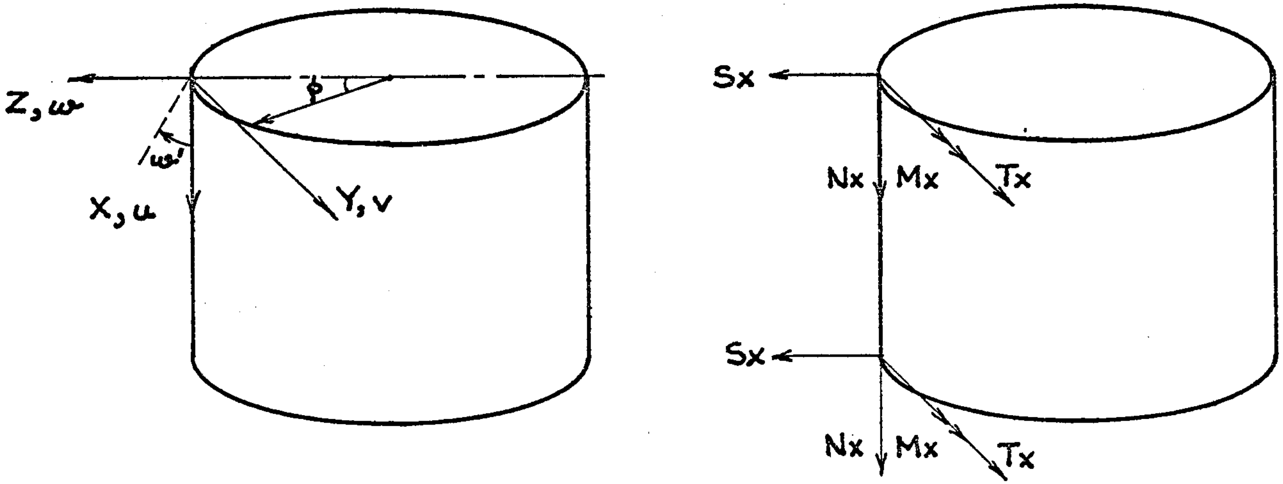
Note : - $g = c^{-my}$



COORDINATES AND DISPLACEMENTS

FORCES AND MOMENTS

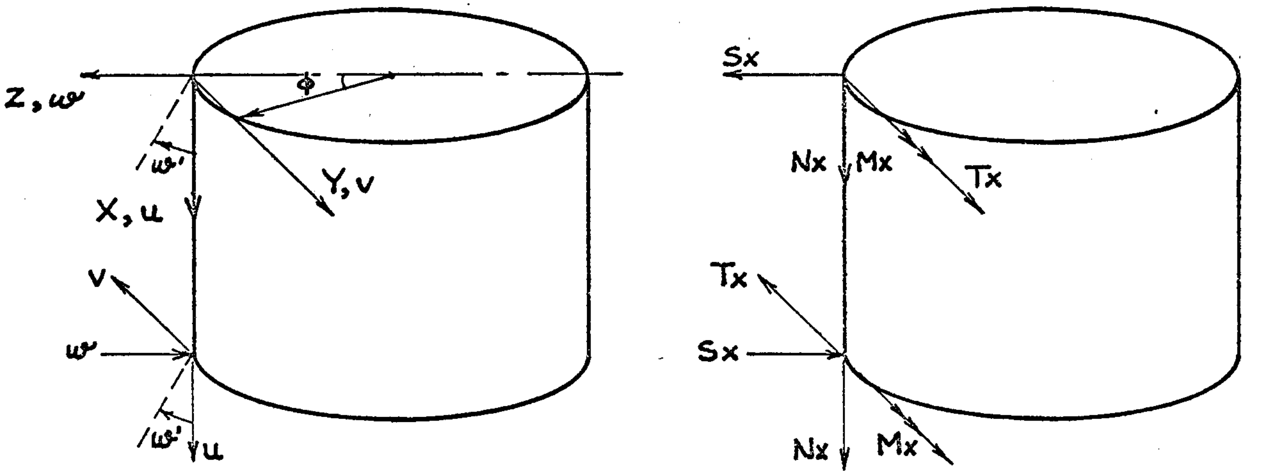
FIGURE 4.1 SHELL THEORY SIGN CONVENTION



COORDINATES AND DISPLACEMENTS

FORCES AND MOMENTS

FIGURE 4.2 STIFFNESS MATRIX SIGN CONVENTION



COORDINATES AND DISPLACEMENTS

FORCES AND MOMENTS

FIGURE 4.3 MODIFIED STIFFNESS MATRIX SIGN CONVENTION

CHAPTER V

CRITERIA FOR THE PROPOSED APPROXIMATE METHODS AND NUMERICAL EXAMPLES.

5.1 INTRODUCTION

In this chapter criteria for the range of applicability of the approximate methods for establishing the stiffness influence coefficients described in CHAPTER IV are presented. The criteria are based not only on the values of the loading harmonic m, but also on the shell geometric parameters k and Q. For ease of reference the different approximate procedures are classified as follows,

Method 1 : solution based on splitting of the differential equation for low values of m (SECTION 4.4).

Method 2 : solution based on simplified Donnell's theory for high values of m (SECTION 4.5).

Method 3 : solution based on equations of generalised plane stress and plate bending for very high values of m. (SECTION 4.6).

Examples are included to illustrate the regions of applicability of the above methods. Example 1 considers the analysis of a steel bin for wind effects which involves low values of m and Example 2 considers the analysis of a water tank on isolated supports which involves high and very high values of m. The range of applicability of Method 3 is discussed in this latter example.

5.2 RANGE OF APPLICABILITY OF METHODS 1 AND 2

The criteria defining the range of applicability of Methods 1 and 2 were established by comparing the final stress resultants obtained by these methods with those obtained using the exact theory proposed by Flügge. For this purpose a closed cylindrical shell free at the top and fixed at the bottom was considered. The shell was analysed for wind effects in which the loading was taken in the form of a Fourier series as given by the following equation,

$$q = q_0 + q_1\cos\phi + q_2\cos2\phi + q_3\cos3\phi + \ldots$$

For each harmonic beyond the first term of this series the analysis was obtained by Flügge's exact theory, Method 1 with one cycle correction and Method 2 neglecting the far edge effect.

The effects of variations in k and Q on the final stress resultants M_x, S_x, N_x and T_x at the support are presented graphically. The comparison of the results obtained by the exact theory and Method 1 are given in FIGURES 5.1 to 5.3 and values obtained by the exact theory and method 2 are given in FIGURES 5.4 to 5.6.

The figures indicate the following. For a given shell, as m increases Method 1 becomes less accurate and Method 2 becomes more accurate. However when either the thickness or the length of the shell is reduced (k or Q is reduced) Method 2 becomes less accurate. The reasons are as follows. From SECTION 4.4 it is clear that the splitting of the original eighth order characteristic equation for low values of m is more justified when k is very small. Also in Method 1 the basic solution is replaced by the membrane solution. It can be seen from the inequality given by EQUATION (4-22) that when values of either k or Q become smaller, the value of m for which this inequality is satisfied becomes greater. Therefore for low values of k or Q the assumptions on which Method 1 is based become more justified and the accuracy increases. However Method 2 does not give good results for these conditions as the assumptions on which this method is based do not permit an accurate evaluation of χ_2 and μ_2. The value of χ_2 is very small and cause slowly damped solutions. For any given value of m, when k is large, χ_2 also becomes large and is evaluated more accurately. In addition, when Q is large, the damping of the χ_2-solution is faster for any given value of m and k and the assumption of neglecting the far edge becomes more justified. Therefore for high values of k and Q the accuracy of Method 2 increases.

Based on the above discussion regarding the final stress resultants for any loading harmonic, it is seen that the criteria in terms of m defining the ranges of validity of Methods 1 and 2 are functions of not only m but also k and Q. These are therefore established on the basis of the maximum tolerance that can be permitted in the final stress resultants for any values of k and Q.

From FIGURES 5.1 to 5.3, the values given by Method 1 become much greater than the exact values for higher values of m and the maximum percentage difference is observed in T_x. Therefore the maximum value of m for which Method 1 is valid is found by specifying a maximum tolerance for the difference in T_x. The tolerance arbitrarily specified was 30 percent for the worst term of T_x. It should be noted that this difference of 30 percent specified for T_x corresponds to the difference for only a particular loading harmonic and not the difference for the complete analysis for wind load which is expressed as a trigonometric polynomial. The following equation has been proposed by Soare (21) for wind load on a smooth surface.

$$w = g_w [-0.804 + 0.140\cos\phi + 1.380\cos 2\phi + 0.490\cos 3\phi - 0.318\cos 4\phi] \tag{5-1}$$

Where g_w is the basic dynamic pressure. In this equation the coefficient is maximum for the term for which m = 2 and the coefficients for higher loading harmonics become increasingly smaller. Because of this in a complete analysis for wind load the net difference is expected to be much smaller than the maximum error in any given term. This is illustrated later in EXAMPLE 1, SECTION 5.3 where the maximum difference of 30 percent occurs for the term T_x for which m = 3, but the difference from summing the terms up to and including this term is only 18 percent. The difference for other stress resultants will of course be smaller.

Thus the maximum tolerance of 30 percent in a given term of T_x is therefore reasonable and can be accepted. On this basis, considering the stress resultant T_x the maximum value of m for which Method 1 is valid is found from FIGURES 5.1 to 5.3. With these maximum values FIGURE 5.7 is obtained which shows the relation among the maximum values of m, k and Q.

The criterion for Method 2 is established as follows. Referring to FIGURES 5.4 to 5.6 it can be seen that the stress resultants given by this method are always less than the exact values for low values of m, the maximum difference being in N_x. As the values were not conservative the minimum value of m which gave conservative values of the worst term for the stress resultant N_x was noted from FIGURES 5.4 to 5.6 for each value of k and Q. FIGURE 5.8 thus obtained shows the relation among the minimum value of m, k and Q. A similar figure could be obtained by permitting non-conservative value of N_x, say, corresponding to a difference of 10 percent in the worst term N_x. This would have the effect of changing the criterion increasing the range of applicability of Method 2.

The criteria for the ranges of validity of Methods 1 and 2 are thus given by FIGURES 5.7 and 5.8. For any given values of k and Q the maximum and minimum values of m for which, respectively, Methods 1 and 2 are valid can be found using these figures. The use of these figures is illustrated in EXAMPLE 1 given in SECTION 5.3.

It is noted from FIGURES 5.4 to 5.6 that Method 2 gives accurate values as m becomes large. Therefore for high values of m Method 2 can be expected to give results which are close to those that can be obtained by the exact theory.

For most shells of dimensions encountered in practice all of the terms in the series for wind load can be evaluated within the tolerance established by the above criteria by either Method 1 or by Methods 1 and 2. For certain cases there may be a particular loading harmonic that does not fall within the range of applicability of these methods as established by the above proposed criteria. In these cases Method 1 should be used. This means that the maximum difference for this term only will exceed the maximum difference specified for a term on which the criterion was based, but the result will always be on the conservative side. The final values of the stress resultants will be sufficiently accurate for design purposes.

5.3 NUMERICAL EXAMPLES

EXAMPLE 1.

This example illustrates the use of Method 1 for the analysis of a vertical cylindrical steel bin for wind effects. The structure is assumed to be fully fixed at the bottom and free at the top. Therefore the boundary conditions are $w' = w = u = v = o$ at the bottom and $M_x = S_x = N_x = T_x = o$ at the top. The design data are

Radius $r = 25$ ft

Thickness $h = {}^{1}/2$ in

Length $\ell = 125$ ft

Poisson's ratio $\upsilon = 0.3$

The wind load is taken in the form of a Fourier series along the directrix as given by EQUATION (5-1). The particular solution for the surface loading due to wind may be obtained by considering sufficient terms of the load expressed as a Fourier series along the generator of the shell. However in this example as is usual in the analysis of shell structures the membrane solution has been taken as a close approximation for the particular solution. The homogeneous solution which accounts for the edge effects is determined from the stiffness matrix established by Method 1.

A complete analysis for the stresses in the structure requires that each term of the Fourier series for the loading along the directrix be evaluated separately. The first step is to determine the number of terms in this series for which Method 1 is applicable. This is done as follows,

$$P = \frac{h}{r} = 0.0016667$$

$$Q = \frac{\ell}{r} = 5$$

$$k = \frac{P^2}{12} = 0.23148 \times 10^{-6}$$

$$[\overline{SM}]=\left[\begin{array}{cccc|c}
-0.2669E-07 & -0.8411E-06 & 0.0 & 0.0 & \\
0.8337E-06 & 0.5297E-04 & 0.0 & 0.0 & \\
0.1335E-06 & -0.4196E-05 & -0.1096E-02 & -0.1524E-03 & \\
0.1690E-05 & -0.1059E-03 & -0.1524E-03 & -0.3049E-04 & \\
\hline
0.0 & 0.0 & 0.0 & 0.0 & \\
0.0 & 0.0 & 0.0 & 0.0 & \\
0.0 & 0.0 & 0.4289E-03 & 0.1524E-03 & \\
0.0 & 0.0 & -0.1524E-03 & -0.3049E-04 &
\end{array}\right] \tag{5-2}$$

$$[D]=\left[\begin{array}{cccc:c}
0.100E01 & 0.0 & 0.3579E\ 01 & -0.8415E-01 & \\
0.0 & 0.1000E\ 01 & 0.1972E\ 00 & -0.1973E\ 01 & \\
-0.1532E-03 & 0.9528E-02 & 0.1000E\ 01 & 0.0 & \\
-0.7368E-04 & 0.2309E-02 & 0.0 & 0.1000E\ 01 & \\
\hdashline
0.0 & 0.0 & -0.4207E\ 00 & -0.8415E-01 & \\
0.0 & 0.0 & -0.7721E-01 & -0.2744E-01 & \\
0.0 & 0.0 & 0.0 & 0.0 & \\
0.0 & 0.0 & 0.0 & 0.0 &
\end{array}\right]$$

(5-3)

Referring to FIGURE 5.7, for these values of k and Q,

$$m_{max} = 3$$

This means that Method 1 is valid for evaluating all stress resultants for which m is equal to 3 or less within the criterion used to obtain FIGURE 5.7. This can also be demonstrated by checking the value of ρ given by EQUATION (4-21) and reproduced here for reference,

$$\rho = \frac{m^4(m^2-1)\ P^2Q^4}{12(4!)} << 1$$

Since,with the values of P and Q in this example,ρ is equal to 0.0039 which is much smaller than 1,the assumptions on which Method 1 is based,are valid and Method 1 can be used.

To illustrate the use of the stiffness matrix, detailed calculations are shown for the term m = 2 only. The terms for m = 1 and 3 are evaluated in a similar manner.

The stiffness matrix for m = 2 is established using the procedure explained in SECTIONS 4.4. From EQUATIONS (A-3) and (A-5) values of χ_1 and $\overline{\alpha}_1$, $\overline{\alpha}_2$, $\overline{\beta}_1$ and $\overline{\beta}_2$ are evaluated. Substituting these values in the expressions given in TABLES 4.9 and 4.10 the matrix $[\overline{SM}]$ and the matrix of unbalanced displacements$\lfloor D \rceil$

$$
[SM] = \left[\begin{array}{cccc|c}
0.2670E-07 & 0.8387E-06 & -0.7132E-07 & 0.1665E-05 & \\
0.8387E-06 & 0.5297E-04 & -0.7433E-05 & 0.1043E-03 & \\
-0.7132E-07 & -0.7433E-05 & 0.1097E-02 & 0.1378E-03 & \\
0.1665E-05 & 0.1043E-03 & 0.1378E-03 & 0.2362E-03 & \\
\hline
0.0 & 0.0 & -0.7657E-07 & 0.2559E-07 & \\
0.0 & 0.0 & -0.4443E-05 & 0.1523E-05 & \\
-0.7657E-07 & -0.4444E-05 & 0.4294E-03 & 0.1437E-03 & \\
-0.2559E-07 & -0.1523E-05 & 0.1437E-03 & 0.2749E-04 &
\end{array}\right]
$$

(5-4)

$$
\left[K_3\right] = \left[\begin{array}{cccc|c}
0.2664E-07 & 0.8378E-06 & -0.7124E-07 & 0.1649E-05 & \\
0.8378E-06 & 0.5287E-04 & -0.7424E-05 & 0.1042E-03 & \\
-0.7125E-07 & -0.7424E-05 & 0.1097E-02 & 0.1380E-03 & \\
0.1649E-05 & 0.1042E-03 & 0.1380E-03 & 0.2360E-03 & \\
\hdashline
0.1348E-10 & 0.7858E-09 & -0.7649E-07 & -0.2386E-07 & \\
0.7858E-09 & 0.4573E-07 & -0.4440E-05 & -0.1431E-05 & \\
-0.7649E-07 & -0.4440E-05 & 0.4300E-03 & 0.1439E-03 & \\
-0.2386E-07 & -0.1431E-05 & 0.1439E-03 & 0.2466E-04 &
\end{array}\right]
\tag{5-5}
$$

are evaluated. These values are given in EQUATION› (5-2) and (5-3). The elements in these matrices are in accordance with the sign convention used in the general shell theory and as given by FIGURE 4.1. Only half the portion of the matrices $[\overline{SM}]$ and [D] are given as the correction to be applied to $[\overline{SM}]$ involve the elements in these halves only. The correction is applied as described in SECTION (4.4.3). The corrected matrix is written using the sign convention shown in EQUATION (5-4). Again only half of the final stiffness matrix [SM] is shown in EQUATION (5-4) as the other half can be written using EQUATION (4-8). It is seen that [SM] agrees very well with the stiffness matrix $[K_3]$ given in EQUATION (5-5), obtained by the exact theory.

Once the stiffness matrix [SM] is established for a particular harmonic m there are two procedures as described in CHAPTER 2 to obtain a final solution. The first procedure is to use the classical procedure by writing the homogeneous solution and the particular solution and superimposing these. The unknown displacements in the homogeneous solution are evaluated by making the general solution satisfy the specified boundary conditions. In this problem there are eight boundary conditions, four expressed in terms of force at the top edge and the remaining four expressed in terms of displacements at the bottom edge. The four homogeneous displacements at the bottom edge must be equal and opposite to the displacements given by the particular solution to satisfy the displacement boundary conditions at the bottom edge. Therefore these four quantities can be evaluated directly. There are now only four unknown homogeneous displacements at the top and these are evaluated by inverting a 4x4 matrix written by using the upper half portion of the stiffness matrix [SM] and the four force boundary conditions at the top.

If one were to use the alternative stiffness matrix procedure explained in CHAPTER 11 the initial forces $[X'_o]$ as given by EQUATION (2-12) must be found. In this equation the stiffness matrix [SM] must be substituted for [K]. The quantities $[d'_p]$ and $[X'_p]$ are the edge displacements and forces obtained from the particular solution or the membrane solution. Using EQUATION (2-13) the unknown displacements given by the vector $[d']$ is found. The elements in the vector $[X']$ are the applied forces, four of which are known to be zero at the top. In the vector $[d']$ the four elements at the bottom are known to be zero and thus there are only four unknowns to be found by inverting a 4x4 matrix as before. The advantage in using this method is that it permits the evaluation of the net displacements at the edges directly.

However in this example the complete analysis was done using the former classical approach. TABLE 5.1 gives the values of the stress resultants at the bottom of the shell both by Method 1 and the exact theory. The analysis for the fourth harmonic was also done by Method 1 even though this harmonic was outside the range of validity established for Method 1. The results given are for unit value of the basic dynamic pressure g_w (see EQUATION (5-1)). It is seen from this table that even though the error in T_x is 30 percent corresponding to the value of m = 3, in the complete analysis the error in T_x is only 18 percent and further this is on the conservative side. The errors in the other stress resultants are less. This means that Method 1 can be used for harmonics beyond the maximum value of m obtained using the criterion for the range of validity for this method. However the magnitude of error in the final stress resultants depends on the relative magnitudes of the Fourier coefficients in the series expressing the load.

EXAMPLE 2

To illustrate the application of Methods 2 and 3 a circular tank supported on 6 columns is analysed. It is assumed that the tank has been first analysed by the ordinary theory of symmetrically loaded tanks on the assumption of a continuous support along the circumference. The scope of the calculation shown below is to find the additional stresses caused by the discontinuity due to the supports being isolated.

The design data are:

Length ℓ = 39 ft

Thickness h = 5.9 ins

Radius r = 32.8 ft

Column width = $\frac{\pi r}{24}$ = 4.3 ft

Column distance = $\frac{\pi r}{3} = 34.3$ ft

Poisson's ratio υ = 0.25

N_x from the symmetrical load analysis is distributed uniformly along the bottom edge. The total load is 33.6 πr kips/ft. A self equilibrating system of N_x distributed as shown in FIGURE 5.9 is applied to the bottom edge to eliminate the value of N_x between columns. The load on the column is assumed to be constant over the column width. This edge load N_x is assumed as the Fourier series

$$N_{xo} = - \frac{268.8}{\pi} \sum \frac{1}{n} \sin \frac{n\pi}{8} \cos m\phi$$

where m = 6n.

The other boundary conditions are:

$$v_o = w_o = w'_o = o.$$

The influence from the upper edge is neglected.

The stiffness matrix can be evaluated for each harmonic using either Method 2 or 3 using the procedure explained in SECTIONS (4-5) and (4-6). For illustration, EQUATION (5-6a) and (5-6b) give the stiffness matrices obtained by Method 2 and Method 3 respectively for m = 6. EQUATION (5-7a) and (5-7b) give the corresponding quantities for m = 120. It is seen that for the very high value of m (m = 120) the bending terms corresponding to the applied bending displacements and the membrane terms corresponding to the applied membrane displacements obtained by

$$[K] = \begin{bmatrix} 0.5713E-05 & 0.5143E-04 & 0.7657E-04 & 0.2423E-01 \\ 0.5743E-04 & 0.1215E-02 & 0.7308E-01 & 0.1315E-01 \\ 0.7657E-04 & 0.7308E-03 & 0.4182E-01 & 0.1315E-01 \\ 0.2423E-03 & 0.6516E-02 & 0.1315E-01 & 0.4185E-01 \end{bmatrix}$$

(5-6a)

$$[K] = \begin{bmatrix} 0.8600E-05 & 0.1350E-04 & 0.0 & 0.0 \\ 0.1350E-04 & 0.1296E-03 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.5236E-01 & 0.1964E-01 \\ 0.0 & 0.0 & 0.1963E-01 & 0.5236E-01 \end{bmatrix}$$

(5-6b)

$$[K] = \begin{bmatrix} 0.7200E-04 & 0.5400E-02 & 0.6818E-05 & 0.1818E-04 \\ 0.5400E-02 & 0.1037E-01 & 0.1091E-02 & 0.7909E-02 \\ 0.6818E-05 & 0.1091E-02 & 0.1047E\ 01 & 0.3927E-00 \\ -0.8817E-05 & 0.7909E-02 & 0.3927E\ 00 & 0.1047E\ 01 \end{bmatrix} \tag{5-7a}$$

$$[K] = \begin{bmatrix} 0.7200E-04 & 0.5400E-02 & 0.0 & 0.0 \\ 0.5400E-02 & 0.1037E-01 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.1047E\ 01 & 0.3927E-00 \\ 0.0 & 0.0 & 0.3927E\ 00 & 0.1047E\ 01 \end{bmatrix} \tag{5-7b}$$

Method 3 agree very well with those by Method 2. The agreement is not good for the lower value of m = 6.

The shell is fully analysed considering the first 20 terms to study the nature of convergence of the final stress resultants. Because of the uncoupled nature of the stiffness matrix in Method 3 only the boundary conditions corresponding to N_{xo} and v_o are to be considered when using this method for this problem and as such this method gives values of only N_x, $N_{x\phi}$ and N_ϕ.

TABLE 5.2 gives the values of the stress resultants at the support after considering 20 terms. It is seen that the values of the membrane stress resultants at the support obtained by Method 3 agree exactly with the results of Method 2.

FIGURES 5.10 and 5.11 illustrate the convergence of the stress resultants M_x and N_x at the support and at a distance of $\ell/8$ from the support. The values were obtained by Method 2. The convergence of M_x is very rapid and the value of M_x can be obtained by considering the first 3 or 4 terms, but the convergence of N_x is slow and that of $N_{x\phi}$ very slow so that more than 20 terms are required. At $\ell/8$ from the support however the convergence of both N_x and $N_{x\phi}$ also are very fast and only 4 to 5 terms are required.

FIGURE 5.12 shows the variation of the stress resultants along the length of the shell, these values being obtained by considering 20 terms in the series. It is seen that M_x and N_x

become damped at a faster rate than $N_{x\phi}$. The maximum value of $N_{x\phi}$ occurs at a distance of $\ell/6$from the support at which location fortunately as indicated by FIGURE 5.11 the convergence of $N_{x\phi}$ is also rapid. Therefore this maximum value can be obtained by considering only a few terms in the Fourier series.

Even though Donnell's theory does not give correct values of $\mu_1$and μ_2 (refer to page 14) for very high values of m, Method 2 which is based on Donnell's theory was found to give final results within an accuracy of 1 to 2 percent when compared with the exact results obtained by Flügge's theory . This indicates that Donnell's theory can be used in place of the exact theory for problems of this nature. From Figure 5.12 it is seen that the membrane stress resultants obtained by Method 3 agree well with the results obtained by Method 2. Therefore similar problems can be completely analysed by Method 3 for the membrane stress resultants. The advantage of this method is that only a 2 × 2 matrix must be considered. The bending stress resultants are obtained by Method 2 considering the first 3 or 4 terms in the series depending on the desired convergence.

TABLE 5.1

COMPARISON OF STRESS RESULTANTS BY EXACT THEORY AND METHOD 1.

	Exact Theory				Method 1			
m	M_x	S_x	N_x	T_x	M_x	S_x	N_x	T_x
0	-0.24	-0.66	0	0	-0.25	-0.66	0	0
1	0.23	0.54	44	-12	0.23	0.54	44	-12
2	7.60	18.00	1720	-307	7.60	18.00	1700	-350
3	5.60	13.30	1300	-97	5.50	13.00	1253	-126
4	-4.50	-10.00	-1047	0	-5.00	-12.00	-1107	0
Total	8.60	21.70	1917	-416	7.98	19.00	1890	-494
Difference expressed in percentage					-7	-12	-1	+18

Note: Values of M_x, S_x and N_x at $\phi = 0$ and $N_{x\phi}$ at $\phi = 45^0$

TABLE 5.2

VALUES OF STRESS RESULTANTS AT THE SUPPORT

	M_x at $\phi=0$ kips.ft/ft	N_x at $\phi=0$ kips /ft	$N_{x\phi}$ at $\phi=15°$ kips/ft	N_ϕ at $\phi=0°$ kips/ft
Method 2	-3.00	-118.2	-5.6	-29.95
Method 3	0	-118.2	-5.6	-29.95

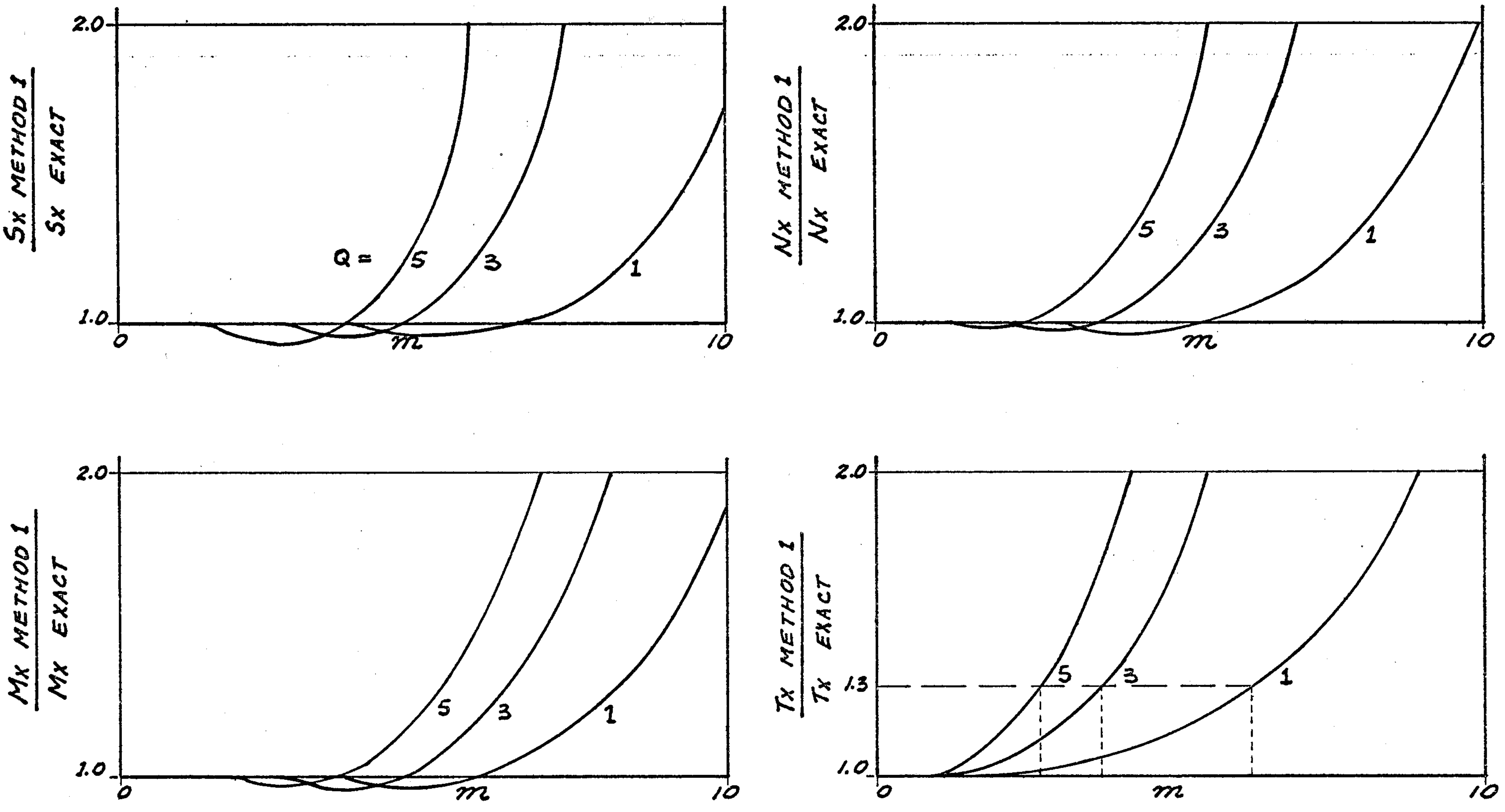


FIGURE 5.1 COMPARISON OF RESULTS FOR $k = 0.23148 \times 10^{-6}$ BY EXACT THEORY AND METHOD 1

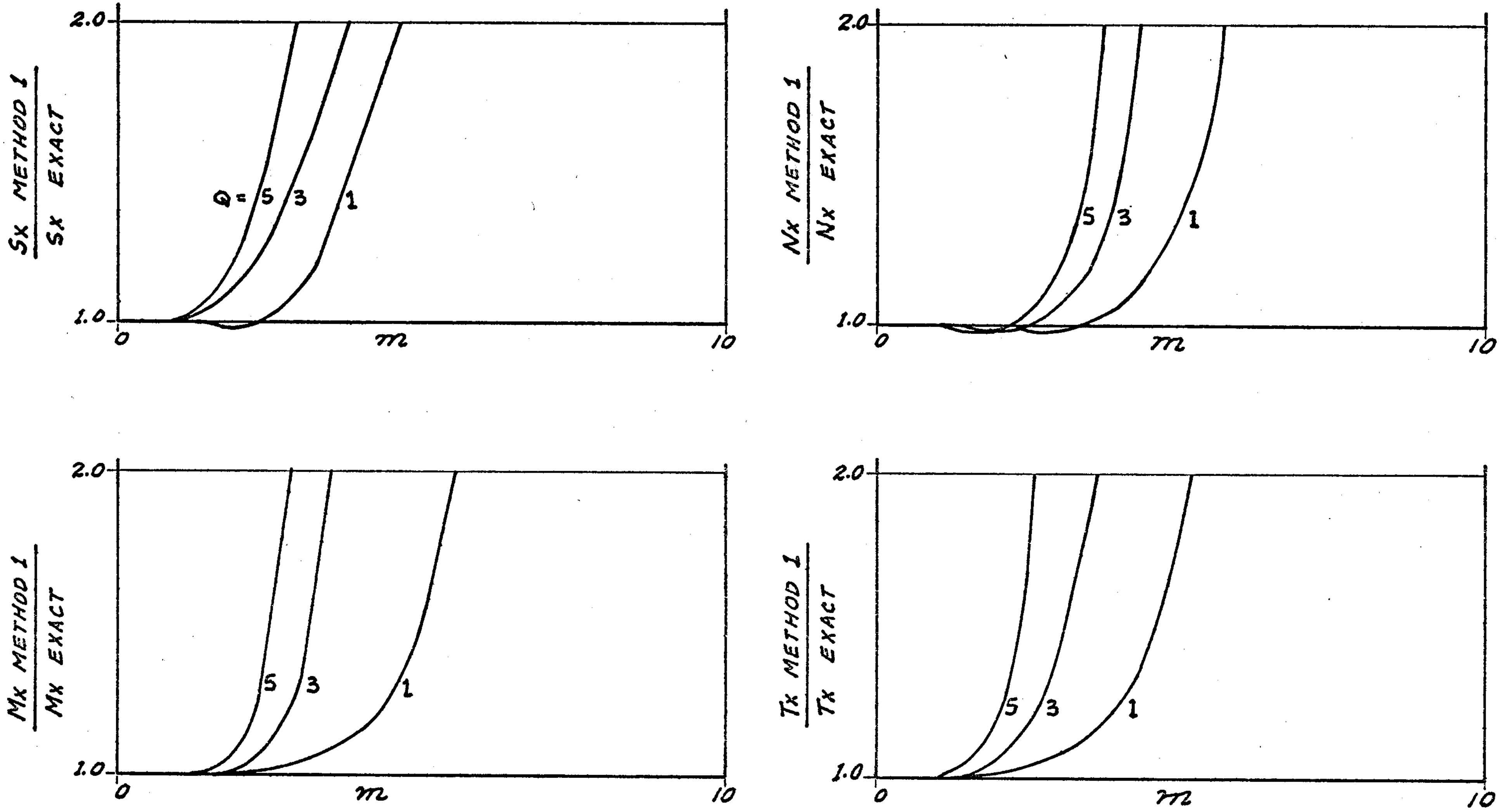


FIGURE 5.2 COMPARISON OF RESULTS FOR $k = 0.7500 \times 10^{-5}$ BY EXACT THEORY AND METHOD 1

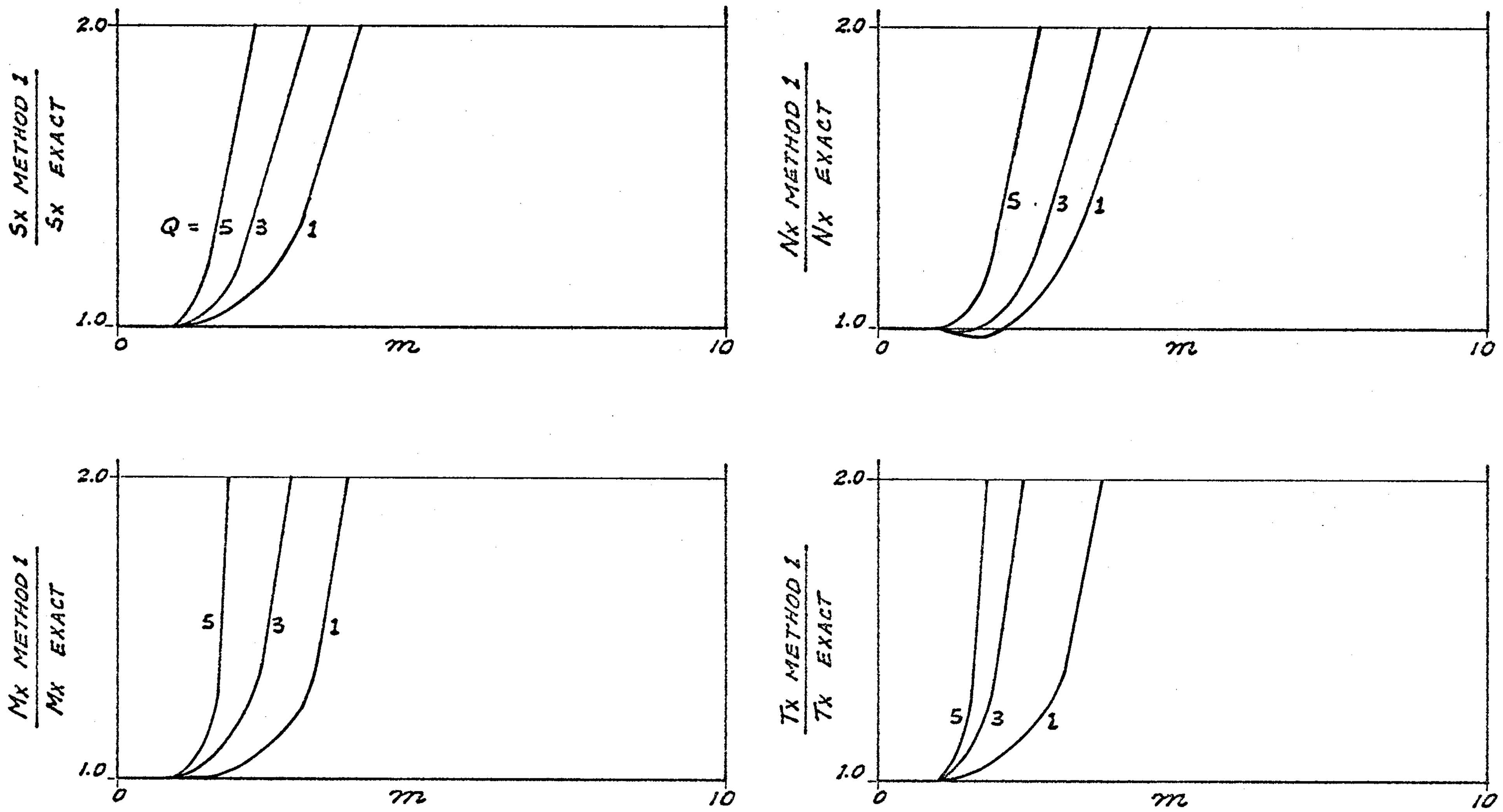


FIGURE 5.3 COMPARISON OF RESULTS FOR $k = 0.7500 \times 10^{-4}$ BY EXACT THEORY AND METHOD 1

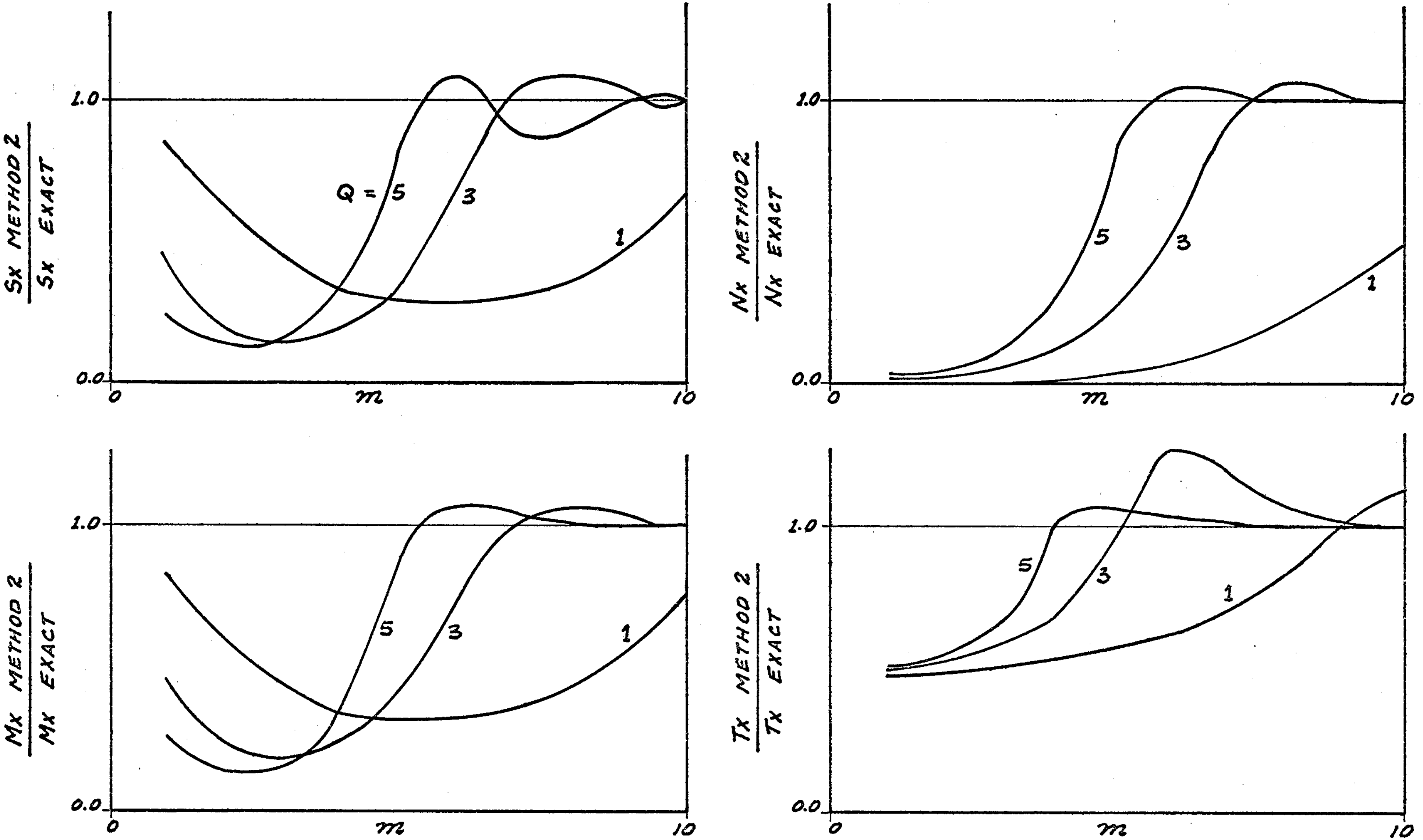


FIGURE 5.4 COMPARISON OF RESULTS FOR $k = 0.23148 \times 10^{-6}$ BY EXACT THEORY AND METHOD 2

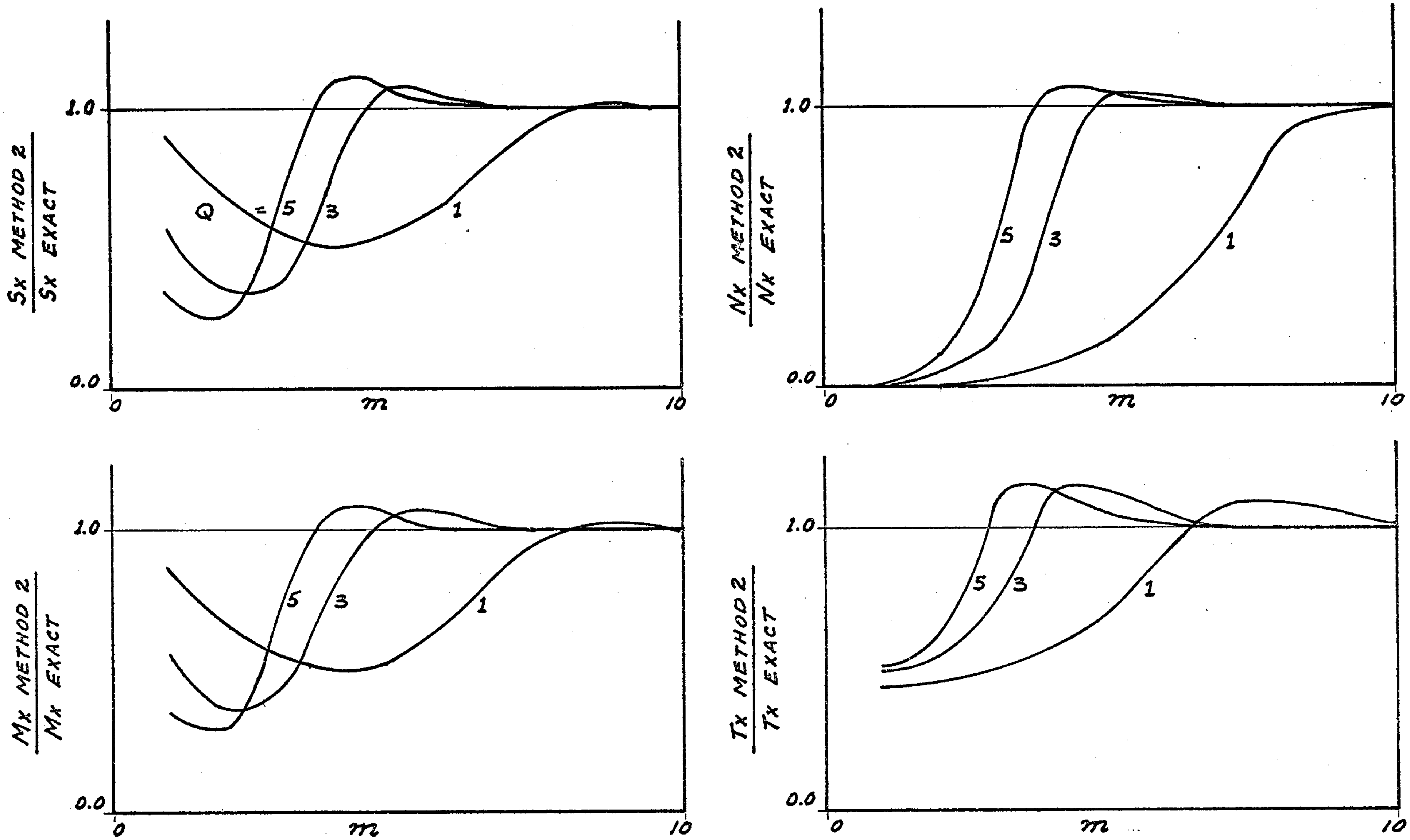


FIGURE 5.5 COMPARISON OF RESULTS FOR $k = 0.7500 \times 10^{-5}$ BY EXACT THEORY AND METHOD 2

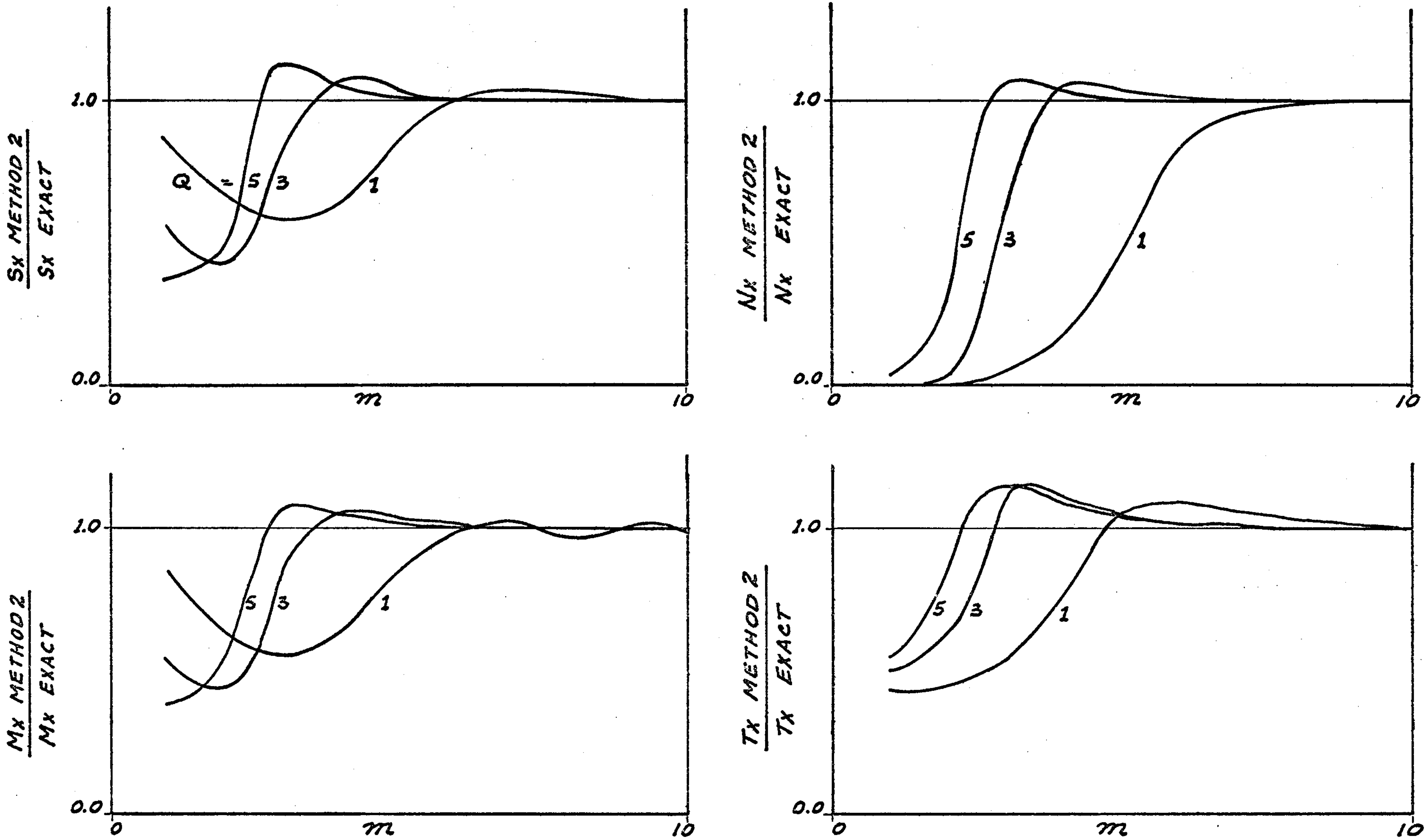


FIGURE 5.6 COMPARISON OF RESULTS FOR $k = 0.7500 \times 10^{-4}$ BY EXACT THEORY AND METHOD 2

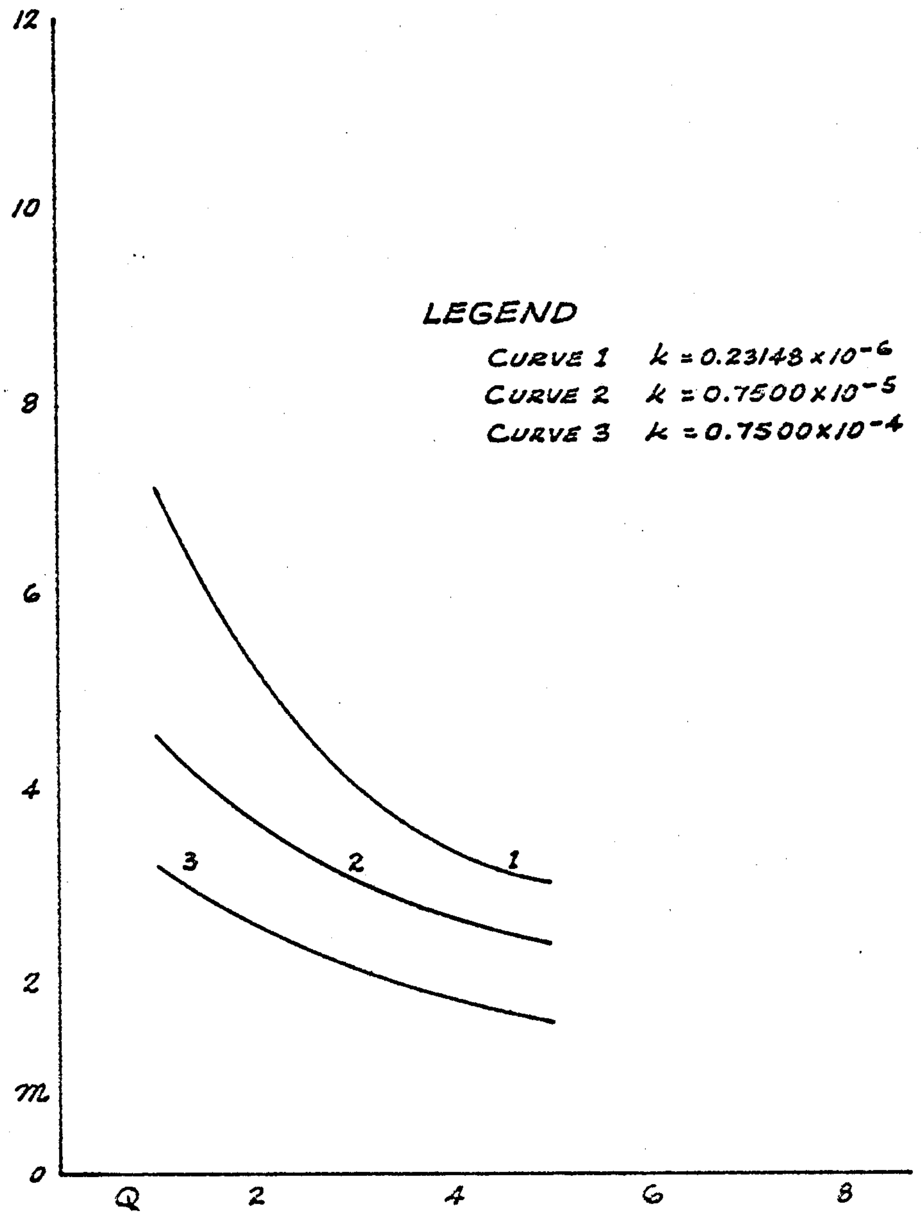


FIGURE 5.7 CRITERION FOR METHOD 1

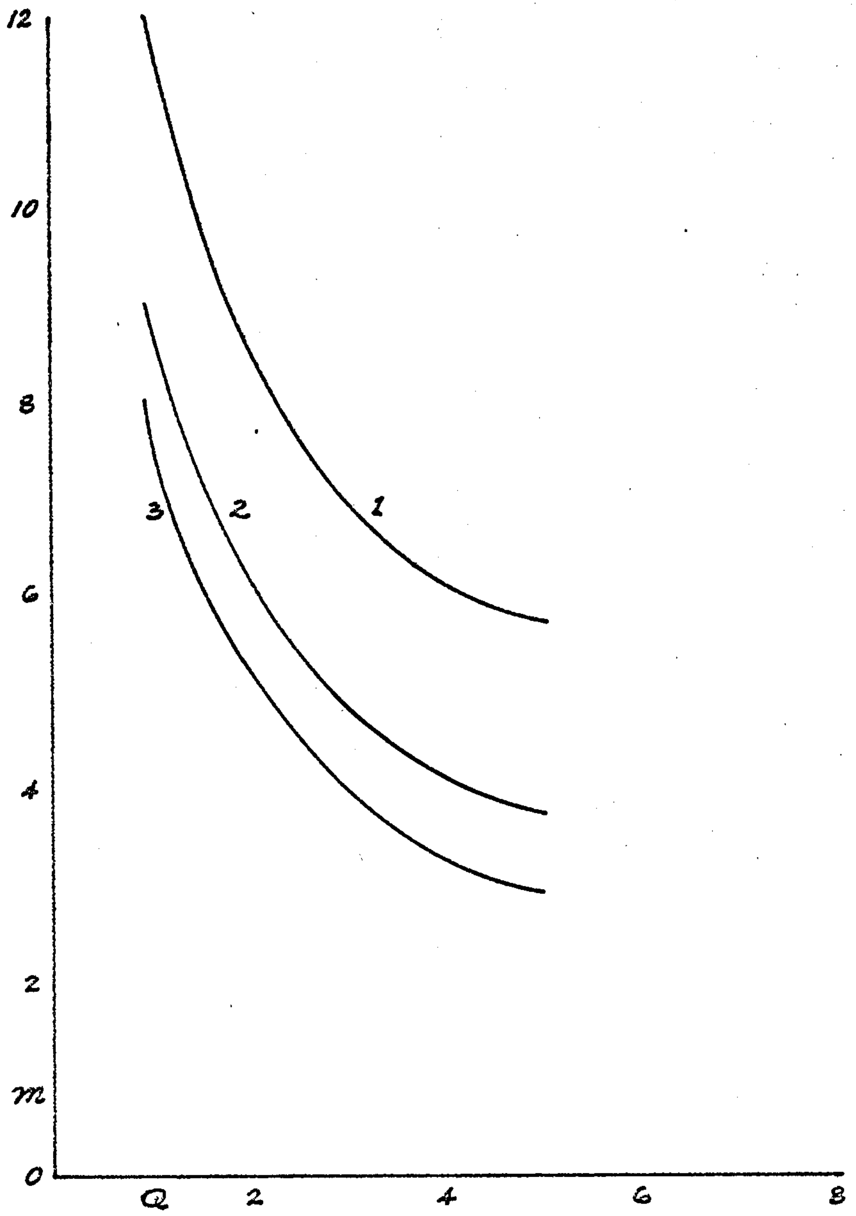


FIGURE 5.8 CRITERION FOR METHOD 2

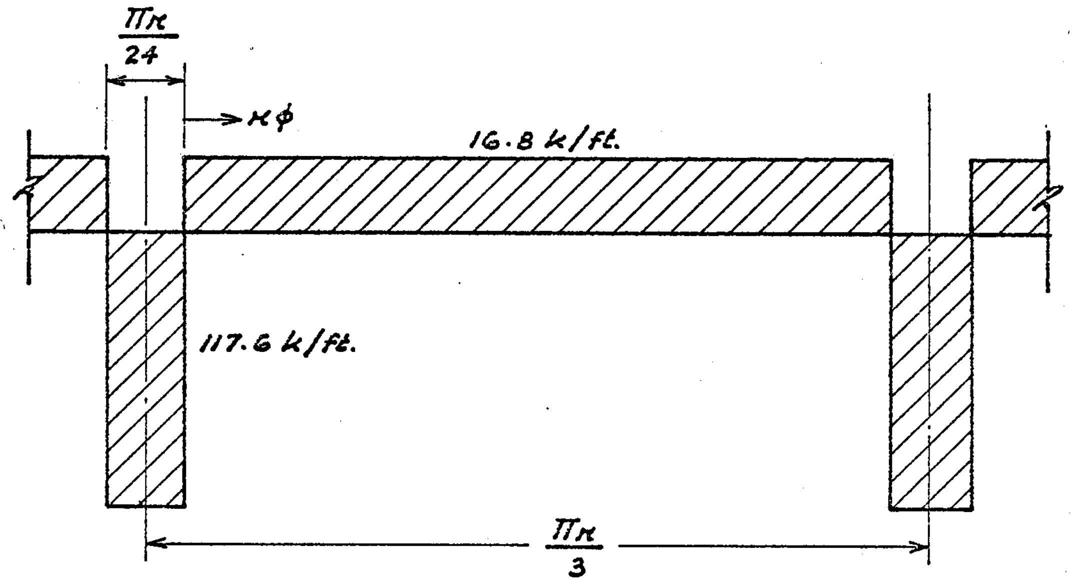


FIGURE 5.9 EDGE LOAD AT SUPPORT IN EXAMPLE 2

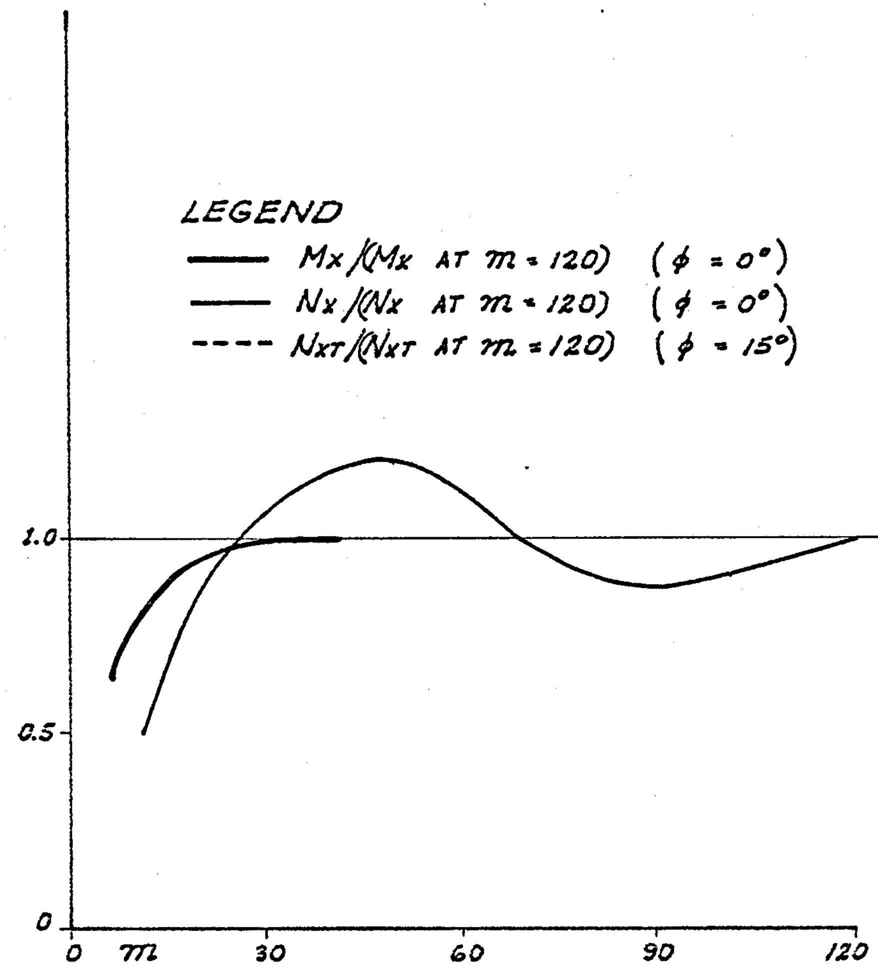


FIGURE 5.10 CONVERGENCE AT $y = 0$ FOR EXAMPLE 2

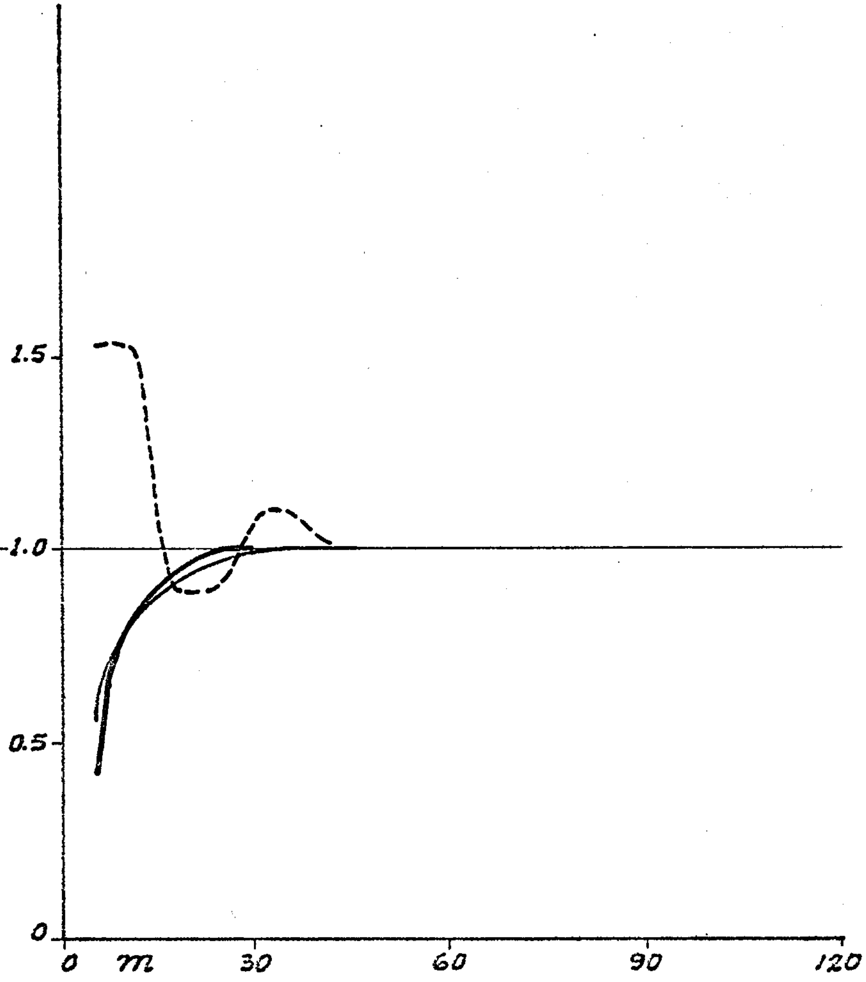


FIGURE 5.11 CONVERGENCE AT $y = \ell/8$ FOR EXAMPLE 2

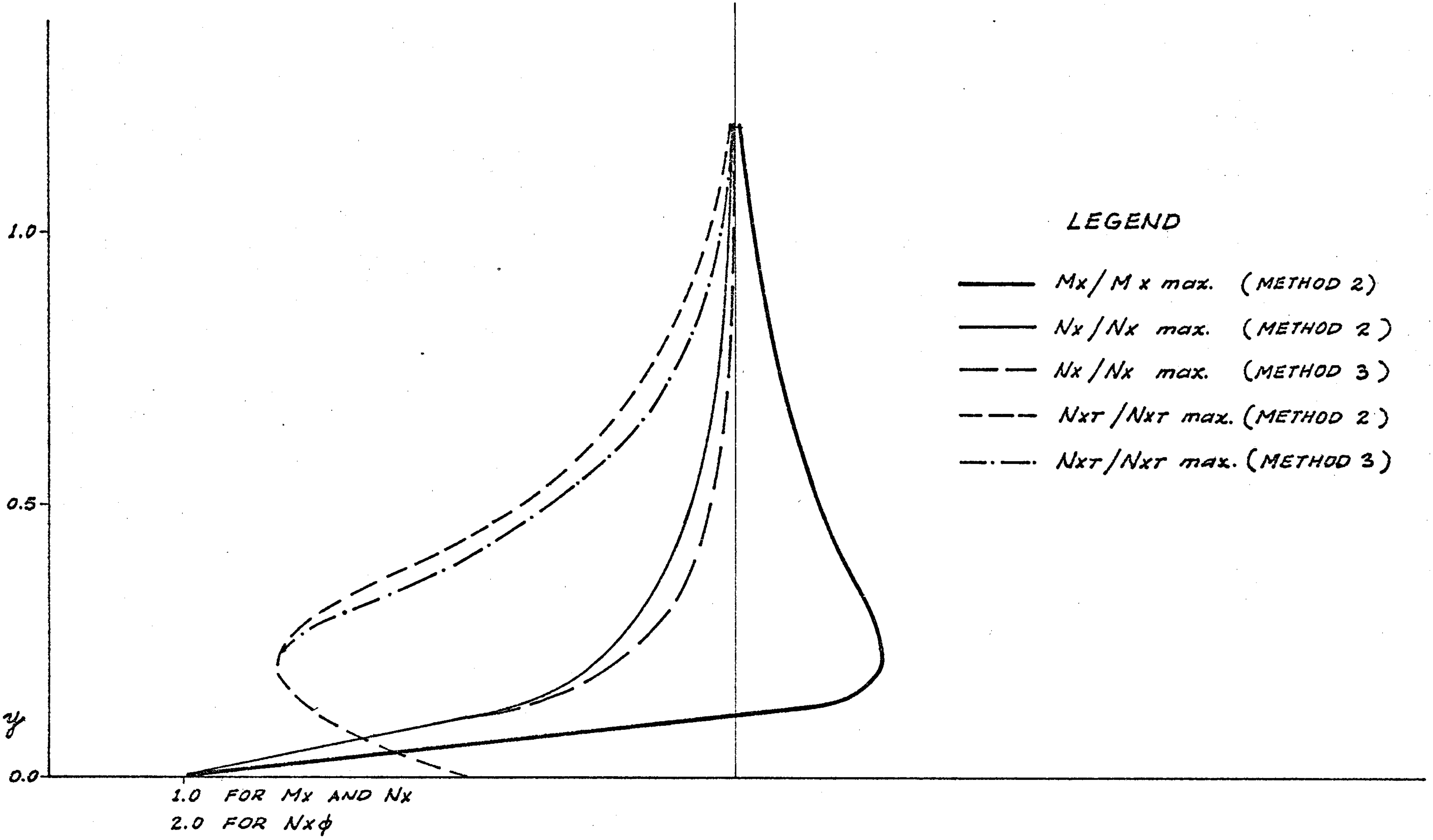


FIGURE 5.12 STRESS RESULTANTS ALONG THE DIRECTRIX IN EXAMPLE 2

CHAPTER VI

SUMMARY AND CONCLUSIONS

6.1 SUMMARY

The objectives of this study are:

1. to obtain usable closed form expressions for stiffness influence coefficients for closed circular cylinders for all harmonics of edge loading greater than zero and,

2. to establish the ranges of validity of these expressions.

Explicit expressions for influence coefficients based on the exact theory are very unwieldy and their evaluation is tedious. Therefore approximate solutions discussed in literature were considered. Influence coefficients were developed based on the following three approximate methods.

Method 1 is based on the split equations of Flugge for low values of m and the solutions obtained are termed as the 'edge effects' and the'basic state.' The latter is replaced by the membrane solution within certain limitations. The 'edge effects' solution considers M_x and S_x and the boundary conditions corresponding to w' and w whereas the membrane solution considers N_x and T_x and the boundary conditions corresponding to u and v. The stiffness matrices obtained are combined to get the matrix $[\overline{SM}]$. A method of successive correction has been proposed to consider the interaction of the above two stress systems and obtain the final stiffness matrix [SM]. Only one cycle of correction is required to obtain [SM] from $[\overline{SM}]$.

Method 2 is based on Donnell's theory neglecting the effects of the far edge for high values of m.

Method 3 is based on the split type stress systems, the generalised plane stress and the plate bending for very high values of m.

In these three methods explicit expressions were developed for all elements in the stiffness matrices. Criteria have been established for the ranges of validity for these methods. These are based on establishing a limiting value of m corresponding to a maximum specified tolerance of the most critical term of the most sensitive stress resultant for that method. Examples illustrating the application of these stiffness matrices to obtain final stress resultants are given.

6.2 CONCLUSIONS

1. The solution of closed cylindrical shells subjected to non-axisymmetric loading may be conveniently analysed by the method of stiffness influence coefficients. Within the limitations presented it is possible to develop simplified expressions for the stiffness influence coefficients in closed form suitable

for practical use.

2. Expressions for the roots of the split equations proposed by Flügge for the characteristic equation for cylindrical shells can be simplified by equating the real and imaginary parts (i.e. $\chi_1=\mu_1$, $\chi_2=\mu_2$). The roots so obtained are in close agreement with the exact values for low values of m.

3. Values of the roots expressed in terms of χ_1, μ_1 evaluated by Donnell's theory agree well with the exact ones for low and high of m. This means that Donnell's theory also can be used to obtain the edge effects solution in Method 1.

4. For low values of m the roots based on the parameters χ_2, μ_2 are sufficiently small to be considered as zero. This results in permitting the replacement of the basic solution by the membrane solution in Method 1. The maximum value of m is dependent on the shell geometry and increases as the shell becomes thinner and shorter.

5. For high values of m,the stress resultants can be evaluated using Donnell's theory and neglecting the effect of the far edge. (Method 2). The minimum value of m also depends on the shell geometry and decreases as the shell becomes thicker and longer.

6. Based on a maximum specified error in the final stress resultant criteria can be developed to establish values of m

for any given shell geometry to define the range of validity of Methods 1 and 2.

7. For most shells of dimensions encountered in practice within reasonable limits of the maximum specified error the criteria will uniquely specify the method for evaluating the stiffness influence coefficients. However if extremely small values are specified for the maximum error it is possible to obtain a few values of m which are between the limits of validity established for Methods 1 and 2. For these values of m Method 1 should be used, since this gives conservative values of the stress resultants for these harmonics.

8. In the analysis of cylindrical shells on isolated supports involving very high values of m,the membrane stress resultants converge very slowly and the bending stress resultants converge very quickly. The solution for the membrane stress resultants which involves a large number of terms of the series can be simplified by using the split solutions of Method 3.

LIST OF REFERENCES

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1. Hruban, K,"*Statics - General Report*," Int. Assoc. for shell structures, Bull No. 29, Mar, 1967 pp. 41-43.

2. Love, A.E.H., "*The Small Free Vibrations and Deformations of a Thin Elastic Shell*," Phil. of the Royal Society, London, England, Series A, 179, 1889, p. 491.

3. Flügge, W., "*Die Stabilität der Kreiszylinderschale*," Ing. Arch., 3, Dec, 1932, p. 463.

4. Finsterwalder, V., "*Die Theorie der zylindrischen Schalengewolbe System Zeiss - Dywidag und ihre Anwendung auf die Grossmarkthalle in Budapest*," Int. Assoc. Bridge Struct. Eng., Publ., 1, 1932 pp. 127-150.

5. Donnell, L.H., "A *New Theory for the Buckling of Thin Cylinders under Axial Compression and Bending*," Trans., Am. Soc. Mech. Eng., 56, 1934, pp. 795-806.

6. Shorer, H., "*Line Load Action in Thin Cylindrical Shells*," Proc., Am. Soc. Civ. Eng., 61, 1935, pp. 281-316.

7. Von Kármán, T., and Tsien, H.S., "*The Buckling of Thin Cylindrical Shells under Axial Compression*," Jour. Ac. Naut. Sc, 8, 1941, p. 303.

8. Jenkins, R.S., "*Theory and Design of Cylindrical Shell Structures*," O, Arup and Partners, London, 1947.

9. Vlassov, V.Z., "*Some New Problems on Shells and Thin Structures*," Izv. Akademii Nauk, No. 1, 1947, pp. 27-53.

10. Schwerin, E., "*Über die Spannungen in freitragenden gefüllten Rohren*," Z, angew. Math. Mech., 2, 1922, pp. 340-353.

11. Miesel, K., "*Über die Festigkeit von Kreiszylinderschalen mit nicht-achsensymmetrischer Belastung,*" Ing. Arch., 1, 1930, pp. 22-71.

12. Hoff, N.J., "*Boundary Value Problems of the Thin Walled Circular Cylinder,*" J. App. Mech., 21, Dec. 1954, pp. 343-350.

13. Holand, I., "*Design of Circular Cylindrical Shells,*" Norges Tekniske Vitenskapsakademi, Series 2, No. 3, Oslo University Press, Oslo, 1957.

14. Flügge, W., "*Stresses in Shells,*" Springer-Verlag, Berlin, Germany, 1960, pp. 208-239.

15. Goldenveiser, A.L., "*Theory of Thin Elastic Shells,*" Pergamon Press, New York, 1961, pp. 229-290.

16. Morley, L.S.D., "*An Improvement on Donnell's Approximation for Thin Walled Circular Cylinders,*" Quart. J. Mech., 12, Part 1, 1959, pp. 89-99.

17. Bijlaard, P.P., "*Stresses from Local Loadings in Cylindrical Pressure Vessels,*" Trans., Am. Soc. Mech., Eng., 77, 1955, p. 805.

18. Hoff, N.J., "*The Accuracy of Donnell's Equations,*" J. App. Mech., 22, 1955, pp. 329-334.

19. Kempner, J., "*Remarks on Donnell's Equations,*" Trans., Am. Soc. Mech. Eng., 77, 1955, p. 117.

20. Biezeno, C.B., and Grammel, R., "*Engineering Dynamics,*" Vol. 11, p. 319, Blackie, London, 1956.

21. Soare, M., "*Application of Finite Difference Equations to Shell Analysis,*" Pergamon Press, New York, 1967, p. 135.

22. Colbourne, J.R., "*Approximate Roots of Flügge's Characteristic Equation for the Closed Cylinderical Shell,*" Trans., Am. Soc. Mech., Eng., , 1969, pp. 352-355.

APPENDIX A

A.1 HOLAND'S CHARACTERISTIC EQUATION

For a complete presentation of the theory refer refer to Holand (13).

The eighth order characteristic equation according to Holand is

$$\lambda^8 - 4m\sqrt{m^2-1}\,\lambda^6 + [\frac{1}{k'} + 6m^2(m^2-1)]\,\lambda^4$$

$$-4m^2(m^2-1)]\,m\sqrt{m^2-1}\,\lambda^2 + m^4\,(m^2-1)^2 = o \;. \qquad A.1$$

where $k' = \frac{h^2}{12(1-\upsilon^2)r^2}$

The roots for this equation are expressed as,

$$\lambda_{1,2,5,6} = \pm\, {}^{1}/_{2}\sqrt[4]{\frac{1}{4k'}}\,[1 + \sqrt{\sqrt{1+\varepsilon^2} + \varepsilon} \pm i\;(1 + \sqrt{\sqrt{1+\varepsilon^2} - \varepsilon}\,)] \quad ,$$

$$\lambda_{3,4,7,8} = \pm\, {}^{1}/_{2}{}^{4}\,\frac{1}{4k'}\,[1 - \sqrt{\sqrt{1+\varepsilon^2} + \varepsilon} \pm i\;(1 - \sqrt{\sqrt{1+\varepsilon^2} - \varepsilon}\,)]$$

where, $\varepsilon = 4\;k'\;m\sqrt{m^2 - 1}$. A.2

A.2 DONNELL'S CHARACTERISTIC EQUATION

Donnell's equation is given in the form,

$$(\lambda^2 - m^2) + \frac{1}{k'}\lambda^4 = o \qquad A.3$$

This equation can be obtained from Holand's equation by making $m^2 - 1 = m^2$. The roots are expressed as,

$$\lambda_{1,2,5,6} = \pm\, {}^{1}/_{2}\sqrt[4]{\frac{1}{4k'}}\;[1+\sqrt{\sqrt{1+\varepsilon_o^{\,2}} + \varepsilon_o} \pm i(1 + \sqrt{\sqrt{1+\varepsilon_o^2} - \varepsilon_o})] \;,$$

$$\lambda_{3,4,7,8} = {}^{1}/{}_{2}\sqrt[4]{\frac{1}{4k'}}\,[1 - \sqrt{\sqrt{1 + \varepsilon_o^{2}} + \varepsilon_o} \pm i(1 - \sqrt{\sqrt{1 + \varepsilon_o^{2}} - \varepsilon_o})] \quad (A.4)$$

where $\varepsilon_o = 4 \ k' \ m^2$.

A.3 SPLIT EQUATIONS OF FLÜGGE

The eighth order characteristic equation derived by Flügge has been split into two fourth order characteristic equations in the following form.

$$\lambda^4 - 2(2m^2 - \upsilon)\lambda^2 + [\frac{1 - \upsilon^2}{k} + 6m^2(m^2 - 1)] = o \ .$$

$$[\frac{1 - \upsilon^2}{k} + 6m^2(m^2 - 1)]\lambda^4 - 2m^2\ [2m^4 - (4 - \upsilon)m^2 + (2 - \upsilon)]\lambda^2 = o \cdot (A\text{-}5a,b)$$

The large roots $\lambda_{1,2,5,6}$ are given by the first equation and the small roots $\lambda_{3,4,7,8}$ by the second equation.

EQUATION (A-5a) is of the form,

$\lambda^4 + b\lambda^2 + c = o$, where ,

$b = -2(2m^2 - \upsilon)$ and $c = \frac{1 - \upsilon^2}{k} + 6m^2(m^2 - 1)$.

$$\lambda^2 = [-\frac{b}{2} \pm [{}^{1}/{}_{2}]\sqrt{b^2 - 4c}] \ .$$

As $4c > b^2$, $(b^2 - 4c)$ is negative ,

$\lambda^2 = [q_1 \pm iq_2]$, where ,

$q_1 = -\frac{b}{2}$ and $q_2 = [{}^{1}/{}_{2}]\sqrt{(4c - b^2)}$.

$\lambda_{1,2} = \pm (\chi_1 - i\mu_1)$.

$\lambda_{5,6} = \pm (\chi_1 + i\mu_1)$,

where, $\chi_1 = \frac{1}{2} [\sqrt{q_1{}^2 + q_2{}^2} + q_1]^{1/2}$,

$$\mu_1 = \frac{1}{2} [\sqrt{q_1{}^2 + q_2{}^2} - q_1]^{1/2} .$$

$c >> \frac{b}{2}$ as k is small and m is low. Therefore ignoring q_1 compared to $\sqrt{q_1{}^2 + q_2{}^2}$,

$$\chi_1 = [{}^1/2][\underset{k}{\underline{1 - \upsilon}}{}^2 + 6m^2(m^2 - 1)]^{1/4} ,$$

$$\mu_1 = [{}^1/2][\underset{k}{\underline{1 - \upsilon}}{}^2 + 6m^2(m^2 - 1)]^{1/4} . \quad \text{(A-6a,b)}$$

It is seen that,

$$\chi_1 = \mu_1 . \quad \text{(A-7)}$$

In a similar manner we get ,

$$\chi_2 = [{}^1/2]\left[\frac{m^4(m^2 - 1)^2}{(\underset{k}{\underline{1 - \upsilon}}{}^2) + 6m^2(m^2 - 1)}\right]^{1/4} ,$$

$$\mu_2 = [{}^1/2]\left[\frac{m^4(m^2 - 1)^2}{(\underset{k}{\underline{1 - \upsilon}}{}^2) + 6m^2(m^2 - 1)}\right]^{1/4} . \quad \text{(A-8a,b)}$$

It is seen that,

$$\chi_2 = \mu_2 \ .$$

A.4 EVALUATION OF α_j AND β_j USING THE EXACT THEORY

The derivation of analytical expressions for $\overline{\alpha}_j$ and $\overline{\beta}_j$ using EQUATIONS(25-a,b), p. 227. ref.14 is very tedious. For particular cases they can be determined numerically. The procedure is outlined below.

The above equations can be reduced to the form,

$$(\lambda^2 + s_1)A + s_2\lambda B + (s_3\lambda - k\lambda^3)C = o \ ,$$

$$s_2\lambda A + (s_4\lambda^2 + m^2)B + (m + s_5\lambda^2)\ C = o \ , \qquad \text{(A-9a,b)}$$

where the following substitution have been made.

$$s_1 = -\ \left(\frac{1 - \upsilon}{2}\right)m^2\ (1 + k)$$

$$s_2 = \left(\frac{1 + \upsilon}{2}\right)m$$

$$s_3 = \upsilon - k\left(\frac{1 - \upsilon}{2}\right)m^2$$

$$s_4 = -\ \left(\frac{1 - \upsilon}{2}\right)\ (1 + 3k)$$

$$s_5 = -\ \left(\frac{3 - \upsilon}{2}\right)\ km$$

Taking any root λ_j in the form a - ib, and substituting in these equations, we get ,

$$(x_1 + ix_2)A + (x_3 + ix_4)B + (x_5 + ix_6)C = o \quad ,$$

$$(x_3 + ix_4)A + (x_7 + ix_8)B + (x_9 + ix_{10})\, C = o \qquad \text{(A-10a,b)}$$

where,

$$x_1 = a^2 - b^2 + s_1$$

$$x_2 = 2a_1b_1$$

$$x_3 = s_3a_1$$

$$x_4 = s_2b_1$$

$$x_5 = s_3a_1 - k[a_1(a_1{}^2 - b_1{}^2) - 2a_1b_1{}^2]$$

$$x_6 = s_3b_1 - k[2a^2b_1 + (a_1{}^2 - b_1{}^2)b_1]$$

$$x_7 = s_4(a_1{}^2 - b_1{}^2) + m^2$$

$$x_8 = 2s_4a_1b_1$$

$$x_9 = m + s_5(a_1{}^2 - b_1{}^2)$$

$$x_{10} = 2s_5a_1b_1$$

EQUATIONS (A-10a,b) can further be reduced to the form ,

$$A + (z_1 + iz_2)B + (z_3 + iz_4)C = o \quad ,$$

$$A + (z_5 + iz_6)B + (z_7 + iz_8)C = o \quad , \qquad \text{(A-11a,b)}$$

where,

$$z_1 = \frac{(x_1x_3 + x_2x_4)}{(x_1^2 + x_2^2)} \qquad z_5 = \frac{(x_3x_7 - x_4x_8)}{(x_3^2 + x_4^2)}$$

$$z_2 = \frac{-(x_2x_3 - x_1x_4)}{(x_1^2 + x_2^2)} \qquad z_6 = \frac{-(x_4x_7 - x_3x_8)}{(x_3^2 + x_4^2)}$$

$$z_3 = \frac{(x_1x_5 + x_2x_6)}{(x_1^2 + x_2^2)} \qquad z_7 = \frac{(x_3x_9 + x_4x_{10})}{(x_3^2 + x_4^2)}$$

$$z_4 = \frac{-(x_2x_5 - x_1x_6)}{(x_1^2 + x_2^2)} \qquad z_8 = \frac{-(x_4x_9 - x_3x_{10})}{(x_3^2 + x_4^2)} \quad .$$

Solving EQUATIONS (A-11a,b) we can express A and B in terms as,

$A = \alpha_j C$ and

$B = \beta_j C$.

where α_j and β_j are as follows,

$$\alpha_j = - \frac{(z_1 + iz_2)(c_1 + ic_2) + (z_3 + iz_4)(c_3 + ic_4)}{(c_3 + ic_4)}$$

$$\beta_j = \frac{(c_1 + ic_2)}{(c_3 + ic_4)} \quad .$$

The following substitution have been made in the above,

$$c_1 = z_7 - z_3$$

$$c_2 = z_8 - z_4$$

$$c_3 = z_1 - z_5$$

$$c_4 = z_2 - z_6$$

$$c_5 = -c_1z_1 + c_2z_2 - c_3z_3 + c_4z_4$$

$$c_6 = -c_1z_2 - c_2z_1 - c_3z_4 - c_4z_3$$

α_j and β_j are complex numbers. The real and imaginary parts of these can be written as follows,

$$\mathrm{Re}[\alpha_j] = \frac{(c_3c_5 + c_6c_4)}{(c_3^2 + c_4^2)}$$

$$\mathrm{Im}[\alpha_j] = \frac{-(c_4c_5 - c_3c_6)}{(c_3^2 + c_4^2)}$$

$$\mathrm{Re}[\beta_j] = \frac{(c_1c_3 + c_2c_4)}{(c_3^2 + c_4^2)}$$

$$\mathrm{Im}[\beta_j] = \frac{-(c_1c_4 - c_2c_3)}{(c_3^2 + c_4^2)} .$$

From these quantities values of $\overline{\alpha}_1$, $\overline{\alpha}_2$, $\overline{\alpha}_3$, $\overline{\alpha}_4$ and $\overline{\beta}_1$, $\overline{\beta}_2$, $\overline{\beta}_3$, $\overline{\beta}_4$ in terms of which α_j and β_j are expressed for all the eight roots can be found (see p. 228, ref. 14).

A.5 EVALUATION OF $\overline{\alpha}_j$, $\overline{\beta}_j$ FOR THE ROOTS EXPRESSED IN TERMS OF χ_1, μ_1 FOR THE SPLIT EQUATIONS OF FLUGGE

The following assumptions have been made.

1. $\chi_1 = \mu_1$ (see SECTION A.3)
2. $k << 1$ and
3. $1 << \frac{4\chi_1^2}{m^2}$ (see SECTION 4.4.1)

Analytical expressions for the root $\chi_1 - i\mu_1$, derived using the above assumptions and the procedure outlined in SECTION A.4 have been given below.

$$\overline{\alpha}_1 = \left[- \nu + \frac{(m^2 + 4k\chi_1^4)}{2\chi_1^2 m^2}\right] \frac{m}{2\chi_1}$$

$$\overline{\alpha}_2 = -\left[\nu + \frac{(m^2 + 4k\chi_1^4)}{2\chi_1^2 m^2}\right] \frac{m}{2\chi_1}$$

$$\overline{\beta}_1 = \left[(2 + \nu) + \frac{4k\chi_1^4}{m^2}\right] \frac{m^3}{2\chi_1^4}$$

$$\overline{\beta}_2 = - \left[(2 + \nu) \frac{\chi_1^2}{m^2} - 4k\chi_1^2 \right] \frac{m^3}{2\chi_1^4}$$

A.6 EVALUATION OF $\overline{\alpha}_j$, $\overline{\beta}_j$ USING DONNELL's THEORY

For a complete description of the method of derivation the reader is referred to Hoff (12). Only the final expressions are given here.

$$\overline{\alpha}_1 = \frac{1}{2k'^2}\left[\upsilon\mu_1 - \frac{m^2\mu_1}{\chi_1^2 + \mu_1^2}\right]$$

$$\overline{\alpha}_2 = -\frac{1}{2k'^2}\left[\upsilon\chi_1 + \frac{m^2\chi_1}{\chi_1^2 + \mu_1^2}\right]$$

$$\overline{\beta}_1 = \frac{-\chi_1\mu_1 m^3}{k'^2(\chi_1^2 + \mu_1^2)^2}$$

$$\overline{\beta}_2 = \frac{m}{2k'^2}\left[2 + \upsilon - \frac{m^2(\chi_1^2 - \mu_1^2)}{(\chi_1^2 + \mu_1^2)^2}\right]$$

$$\overline{\alpha}_3 = \frac{1}{2k'^2}\left[\frac{\upsilon\mu_2 - m^2\mu_2}{\chi_1^2 + \mu_2^2}\right]$$

$$\overline{\alpha}_4 = -\frac{1}{2k'^2}\left[\upsilon\chi_2 + \frac{m^2\chi_2}{\chi_2^2 + \mu_2^2}\right]$$

$$\overline{\beta}_3 = -\frac{\chi_2\mu_2 m^3}{k'^2(\chi_2^2 + \mu_2^2)}$$

$$\overline{\beta}_4 = \frac{m}{2k'^2}\left[2 + \upsilon - \frac{m^2(\chi_2^2 - \mu_2^2)}{(\chi_2^2 + \mu_2^2)}\right]$$

A.7 MEMBRANE SOLUTION FOR SURFACE LOADING

For a complete presentation of the theory proposed by Flügge refer to pp 107 to 132, chapter 3, ref 14. In this reference Flügge has given EQUATIONS (4) for the stress resultants and EQUATIONS (22) for the displacements in the dimensional form using the sign convention shown in FIGURE 3.1. The coordinates are measured from the centre of the shell as shown in FIGURE A.1. Taking the surface load in the form $p_m = q_m \cos m\phi$, $p_\phi = p_x = 0$ and making the substitution $y = \frac{x}{r}$, $p = \frac{h}{r}$, and $Q = \frac{\ell}{r}$ the above equations have been reduced to the following form in terms of the non dimensional parameters y and .

$$\overline{N}_\phi = q_m$$

$$\overline{N}_{x\phi} = m\, q_m y$$

$$\overline{N}_x = \frac{m^2}{8}(Q^2-4y^2) q_m$$

$$EP\, \overline{u} = \left[\frac{m^2(3Q^2-4y^2)}{24} - \upsilon y\right] q_m$$

$$EP\, \overline{v} = -(Q^2-4y^2)\left[\frac{(5Q^2-4y^2)m^2}{384} + \frac{2+\upsilon}{8}\right] m\, q_m$$

$$EP\, \overline{w} = (Q^2-4y^2)\left[\frac{(5Q^2-4y^2)m^2}{384} + {}^1/4\; m^2 + 1\right] q_m$$

$$EP\, \overline{w}' = -8y\left[\frac{(5Q^2-4y^2)}{384} m^2 + {}^1/4 + \frac{(Q^2-4y^2)}{384} m^2\right] m^2 q_m$$

To get the absolute values of the stress resultants $\overline{N}_\phi$, $\overline{N}_{x\phi}$ and $\overline{N}_x$ and the displacements $\overline{u}$, $\overline{v}$ and $\overline{w}$, these expressions are to be multiplied by the radius r.

A.8 FOURIER SOLUTION FOR EDGE LOADING BY MEMBRANE THEORY

For a detailed description of the method refer to section 3.2.2.3, chapter 3, ref 14. In this reference Flügge has given the homogeneous solutions for the displacements given by EQUATIONS (25) and (26) in the dimensional form. Making the same substitutions as explained in SECTION A.7 these equations can be reduced to the following non dimensional form,

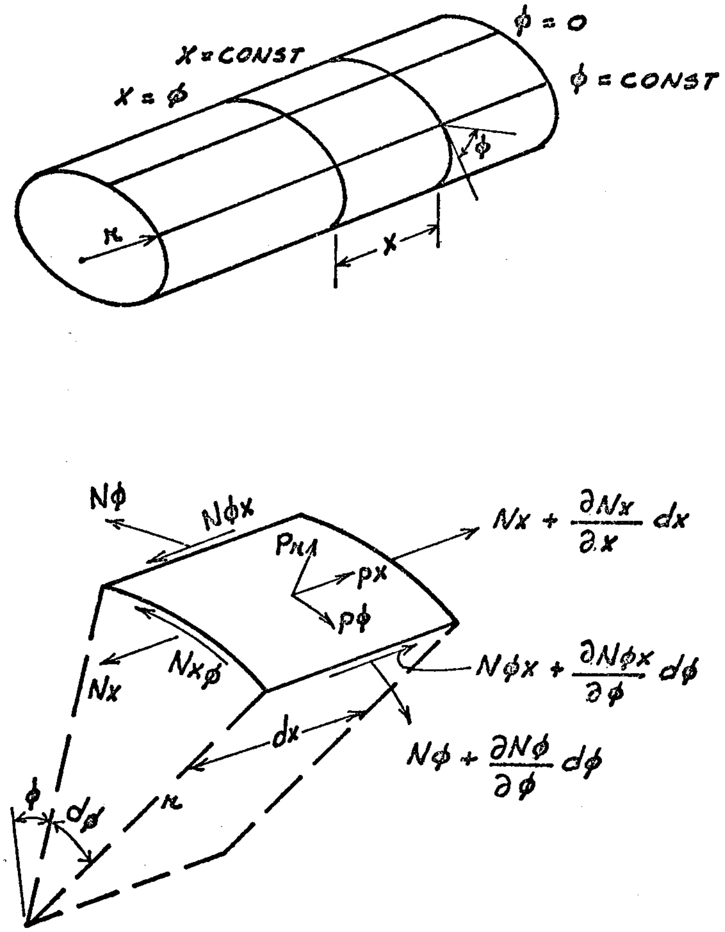
$$EP\,\overline{u} = \frac{A}{2}y^2 + By + c$$

$$EP\,\overline{v} = A\left[\frac{m^2y^3}{6} - 2(1+\upsilon)my\right] + \frac{Bmy^2}{2} + Cmy + D$$

$$EP\,\overline{w} = -A\left[\frac{m^2y^3}{6} - (2+\upsilon)my\right] - B\left[\frac{m^2y^2}{2} + \upsilon\right] - Cm^2y - mD$$

$$EP\,\overline{w}' = -A\left[\frac{m^2y^2}{2} - (2+\upsilon)m\right] - Bm^2y - Cm^2$$

To get the absolute values of the displacements $\overline{u}$, $\overline{v}$ and $\overline{w}$ these expressions are to be multiplied by r.



A-1. SHELL ELEMENT FOR MEMBRANE ANALYSIS

STRUCTURAL ENGINEERING REPORTS

1. THE BEHAVIOR OF SHEAR PLATES IN SLOPING GRAIN SURFACES - No. 1, by J. Longworth, September, 1963.
2. TESTS OF ECCENTRICALLY LOADED CONCRETE COLUMNS BENT IN DOUBLE CURVATURE, by S.L. Barter, May, 1964.
3. THE BEHAVIOR OF SHEAR PLATES IN SLOPING GRAIN SURFACES - No. 2, by J. Longworth, June, 1964.
4. METHOD OF ANALYSIS OF SHALLOW TRANSLATIONAL SHELLS, by I. Rajendram and S. H. Simmonds, February, 1965.
5. TORSIONAL STRENGTH AND BEHAVIOR OF CONCRETE BEAMS IN COMBINED LOADING, by G.S. Pandit and J. Warwaruk, July 1965.
6. ULTIMATE STRENGTH AND BEHAVIOR OF PLATES, by A. Gyi and S. H. Simmonds, November, 1965.
7. THE BEHAVIOR OF RESTRAINED REINFORCED CONCRETE COLUMNS UNDER SUSTAINED LOAD, by R.F. Manuel and J. G. MacGregor, January, 1966.
8. PRETENSIONED BEAMS WITH CONFINED COMPRESSED CONCRETE, by R. L. Ward and J. Warwaruk, April, 1966.
9. AN ANALYTICAL STUDY OF INCLINED CRACKING IN REINFORCED CONCRETE BEAMS, by J.R.V. Walters and J. G. MacGregor, July, 1966.
10. ROTATION CAPACITY OF WIDE-FLANGE BEAMS UNDER MOMENT GRADIENT, by A. F. Lukey and P. F. Adams, May, 1967.
11. TORSIONAL STRENGTH OF RECTANGULAR REINFORCED CONCRETE BEAMS SUBJECTED TO COMBINED LOADING, A. E. McMullen and J. Warwaruk, July, 1967.
12. THE INFLUENCE OF COLUMN SHAPE ON THE BEHAVIOR OF FLAT PLATE SLABS, by R. J. Kavanagh and S. H. Simmonds, November, 1967.
13. EXPERIMENTS ON WIDE-FLANGE BEAMS UNDER MOMENT GRADIENT, by R. J. Smith and P. F. Adams, May, 1968.
14. APPROXIMATE ANALYSIS OF FRAME-SHEAR WALL STRUCTURES, by S. N. Guhamajumdar, R. P. Nikhed, J. G. MacGregor, and P. F. Adams, May, 1968.
15. ELASTIC TORSIONAL ANALYSIS OF MULTI-STORY STRUCTURES, by J. H. Wynhoven and P. F. Adams, October, 1968.
16. ANALYSIS OF REINFORCED CONCRETE SHEAR WALL-FRAME STRUCTURES, by W. J. Clark and J. G. MacGregor, November, 1968.
17. MOMENT CURVATURE RELATIONSHIPS OF PRESTRESSED CONCRETE BEAMS WITH CONFINED COMPRESSED CONCRETE, by M. K. Wilkinson and J. Warwaruk, January, 1969.
18. FATIGUE STRENGTH OF DEFORMED REINFORCING BARS, by I. C. Jhamb and J. G. MacGregor, May, 1969.

19. SHALLOW SHELLS SUPPORTED BY ELASTIC EDGE BEAMS, by D. H. Quapp and S. H. Simmonds, May, 1969.

20. COMPOSITE BEAMS IN NEGATIVE BENDING, by J.H. Davison and J. Longworth, May, 1969.

21. TEST OF FLAT PLATE SUPPORTED ON COLUMNS ELONGATED IN PLAN, by A.E. Smith and S.H. Simmonds, May, 1969.

22. CONTINUOUS PRESTRESSED CONCRETE BEAMS WITH CONFINEMENT, by I.H.G. Duncan and J. Warwaruk, September, 1969.

23. REINFORCED CONCRETE CELLULAR ORTHOTROPIC SLABS, by R.G. Quinton and S.H. Simmonds, October, 1969.

24. PRESTRESSED CONCRETE BEAMS WITH WEB REINFORCEMENT UNDER COMBINED LOADING, by P. Mukherjee and J. Warwaruk, May, 1970.

25. STUDIES OF REINFORCED CONCRETE SHEAR WALL-FRAME STRUCTURES, by R.P. Nikhed, J.G. MacGregor and P.F. Adams, June, 1970.

26. BUCKLING STRENGTHS OF HOT ROLLED HAT SHAPED SECTIONS, by D.A. Heaton and P.F. Adams, July, 1970.

27. EXPERIMENTAL AND ANALYTICAL INVESTIGATION OF THE BEHAVIOR OF COUPLED SHEAR WALL-FRAME STRUCTURES, S.N.G. Majumdar and P.F. Adams, August, 1970.

28. COMPARATIVE STUDY OF SLAB-BEAM SYSTEMS, by J. Misic and S.H. Simmonds, September, 1970.

29. ELASTIC-PLASTIC ANALYSIS OF THREE DIMENSIONAL STRUCTURES, by J.H. Wynhoven and P.F. Adams, September, 1970.

30. FLEXURAL AND LATERAL-TORSIONAL BUCKLING STRENGTHS AND DOUBLE ANGLE STRUTS, by N.J. Nuttall and P.F. Adams, September, 1970.

31. STIFFNESS INFLUENCE COEFFICIENTS FOR NON-AXISYMMETRICAL LOADING ON CLOSED CYLINDRICAL SHELLS by S.H. Iyer and S.H. Simmonds, October, 1970.