University of Alberta

Operator ideals on ordered Banach spaces

by

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A thesis submitted to the Faculty of Graduate Studies and Research
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy
in
Mathematics

Department of Mathematical and Statistical Sciences

Spring, 2013
Edmonton, Alberta

To my family

Abstract

In this thesis we study operator ideals on ordered Banach spaces such as Banach lattices, C^*-algebras, and noncommutative function spaces.

The first part of this work is concerned with the domination problem: the relationship between order and algebraic ideals of operators. Fremlin, Dodds and Wickstead described all Banach lattices on which every operator dominated by a compact operator is always compact. First, we show that even if the dominated operator is not compact it still belongs to a relatively small class of operators, namely, the ideal of inessential operators. A similar question is studied for strictly singular operators. In particular, we show that the cube of every operator, dominated by a strictly singular operator, is inessential. Then we provide a complete solution of the domination problem for compact and weakly compact operators acting between C^*-algebras and noncommutative function spaces. Finally, we consider the domination problem for weakly compact operators acting on general noncommutative function spaces.

The second part is devoted to the operator ideal structure of the algebra of all linear bounded operators on a Banach space. First, we investigate the existence of non-trivial proper ideals on Lorentz sequence spaces and characterize some of them. Second, we look at the coincidence of some classical operator ideals, such as of compact, strictly singular, innesential, and Dunford-Pettis operators acting on noncommutative L_p-spaces. In particular, we obtain a characterization of strictly singular and inessential operators acting either between discrete noncommutative L_p-spaces or L_p-spaces, associated with a hyperfinite von Neumann algebras with finite trace.

Many of the results presented in this thesis were obtained jointly with other people. The thesis is based on papers [58, 70, 71, 93] by the author and his collaborators.

Acknowledgements

First and foremost, I would like to express my deep gratitude to my adviser, Vladimir G. Troitsky, who guided me through the PhD program, broadened my mathematical horizons, exposed me to the mathematical community, and bravely fought my written English. He has been supportive far beyond mathematics.

I would like to thank Timur Oikhberg, who has been more to me than just a collaborator, for introducing me to operator spaces and for the most stimulating correspondence on various mathematical topics.

I thank Anna Kaminska, Alexey Popov, Adi Tcaciuc for the results we proved together that I included in my thesis.

I also want to thank B. de Pagter for proofreading the thesis.

Contents

Chapter 1

Basic definitions and notations

1.1 Ordered Banach spaces

Let X be a Banach space and $C \subset X$ be a cone, that is, a set closed under addition and multiplication by a non-negative real number such that $C \cap (-C) = \{0\}$. This cone is *generating* in X if $X = C - C$. The set of all functionals $f \in X^*$, such that $f(x) \geqslant 0$ for every $x \in C$ forms a cone in X^*, that is referred as the *dual* cone of C. If the dual cone of C is generating then we call C *normal*.

We say that X is a real *ordered Banach space* (OBS) if it is equipped with a norm-closed cone $C \subset X$. The elements of C are called *positive* (denoted $x \geqslant 0$). One can define an order as follows: $x \leqslant y$ if and only if $y - x \in C$. An *order interval* $[x, y] \subset X$ is the set of all elements $z \in X$ such that $x \leqslant z \leqslant y$.

It was shown in [8] that a closed cone C is generating if and only if there exists $\delta_X > 0$, such that, for every $x \in X$, there exist a and b such that $x = a - b$ and $\max(\|a\|, \|b\|) \leqslant \delta_X \|a - b\|$ for every $a, b \in C$. And it is normal if and only if there exists $\gamma_X > 0$ such that $\|x\| \leqslant \gamma_X \max\{\|a\|, \|b\|\}$, whenever $a \leqslant x \leqslant b$.

An operator T acting between two OBS's is called positive if it maps positive elements to positive elements.

A complex OBS Y is the complexification of a real OBS Y_r that is $Y = Y_r + iY_r$. The positive elements of Y are exactly those belonging to the cone of Y_r. We say a cone is generating (normal) in Y if it is generating (normal) in Y_r.

Natural examples of OBS with proper normal generating cones and $\delta = \gamma = 1$ are Banach lattices, C^*-algebras, and noncommutative functions spaces. They will be discussed in the following sections of this chapter.

1.2 Banach lattices

We adhere to standard definitions and properties of Banach lattices that can be found in [1, 6, 63, 64, 67]

An OBS E is called a (real) *Banach lattice* with respect to its cone if for every two elements $x, y \in E$

- their supremum $x \vee y$ and infimum $x \wedge y$ are in E,
- $|x| \leqslant |y|$ implies $\|x\| \leqslant \|y\|$, where $|x| = x \vee (-x)$.

For a Banach lattice E its positive cone $E_+ = \{x \in E : x \geqslant 0\}$ is normal and generating.

A Banach lattice E is called *order continuous* if $\|x_\alpha\| \to 0$ for every decreasing net $(x_\alpha) \subset E$ such that $\wedge x_\alpha = 0$.

A linear subspace $I \subseteq E$ is an (order) *ideal* if $x \in I$ whenever $y \in E$ and $|x| < |y|$. A non-zero element $a \in E_+$ is an *atom* if $0 \leqslant b \leqslant a$ implies $b = \lambda a$, for some constant $\lambda \geqslant 0$. We say E is *atomic* if for every element $x \in E_+$, there exists an atom $a \leqslant x$ in E.

Any Banach space with a 1-unconditional basis can be realized as an order continuous atomic Banach lattice with the standard order defined coordinatwise. There are many examples of non-atomic order continuous Banach lattices, in particular, L_p $(1 \leqslant p < \infty)$, and various Lorentz and Orlicz function spaces on non-atomic measure space.

An element $e \in E$ is called an (order) unit if any ideal that contains e coincides with E. We say that E is an AM-space if $\|x+y\| = \max\{\|x\|, \|y\|\}$ for every disjoint $x, y \in E$. By Kakutani's theorem every AM- space with a unit is lattice isometric(there exists a surjective isometry that preserves lattice operations) to a $C(K)$ space for some compact Hausdorff topological space K.

1.3 C^* and von Neumann algebras

A C^*-*algebra* $\mathcal{A}$ is a Banach algebra with a $*$-map from $\mathcal{A}$ to $\mathcal{A}$ with the following properties:

(i) $(A+B)^* = A^* + B^*$,

(ii) $(cA)^* = \bar{c}A^*$,

(iii) $A^{**} = A$,

(iv) $(AB)^* = B^*A^*$,

(v) $\|A^*A\| = \|A\|^2$,

for every $A, B \in \mathcal{A}$ and $c \in \mathbb{C}$.

A C^*-algebra $\mathcal{A}$ is called unital if it has an algebraic unit $\mathbf{1}$. Every C^*-algebra $\mathcal{A}$ can be realized as a C^*-subalgebra of codimension one of a unital C^* algebra $\mathcal{A}_U$. This process is called the unitization. Thus, we can define the spectrum of $A \in \mathcal{A}$,

$$\sigma(A) = \{\lambda \in \mathcal{C} : \lambda\mathbf{1} - A \text{ is not invertible in } \mathcal{A}_U\}$$

We say that $A \in \mathcal{A}$ is positive if and only if it is self-adjoint $(A = A^*)$ and the $\sigma(A) \subset [0, \infty)$.

As usual we define $\mathcal{A}_+ = \{A \in \mathcal{A}, A \geqslant 0\}$. It is easy to see that $\mathcal{A}$ is a complex OBS with a normal generating cone $\mathcal{A}_+$ and $\delta_{\mathcal{A}} = \gamma_{\mathcal{A}} = 1$. Every element $A \in B_{\mathcal{A}}$ can be written as follows $A = A_1 - A_2 + i(A_3 - A_4)$, where $A_i \in B_{\mathcal{A}} \cap \mathcal{A}_+$. The following properties of positive elements in $\mathcal{A}$ will be required later:

(i) $A \geqslant B$ implies $C^*AC \geqslant C^*BC$, for any $C \in \mathcal{A}$.

(ii) $A \geqslant 0$ implies there exists a unique $B \in \mathcal{A}_+$, such that $B^2 = A$. B is denoted as $A^{\frac{1}{2}}$.

(iii) $A \geqslant B \geqslant 0$ implies $A^{\frac{1}{2}} \geqslant B^{\frac{1}{2}}$

(iv) $A \geqslant B$ implies $\|A\| \geqslant \|B\|$.

A *representation* π of $\mathcal{A}$ on a Hilbert space H is a $*$-homomorphism from $\mathcal{A}$ into $B(H)$, the space of all linear bounded operators on H.

It is called *faithful* if $\pi(A) \neq 0$ for every $A \in \mathcal{A}$ and *irreducible* if $\pi(\mathcal{A})$ has no proper invariant subspaces.

The Gelfand-Naimark theorem states that every C^*-algebra can be faithfully (isometrically) represented as a closed $*$-subalgebra of $B(H)$ for some Hilbert space H. Note that this representation preserves the order. That is, $A \geqslant 0$ if $(\pi(A)x, x) \geqslant 0$, for every $x \in H$, where π is a representation into $B(H)$.

Every element $A \in \mathcal{A} \subset B(H)$ can be written as A=$U|A|$, where $|A| = (A^*A)^{\frac{1}{2}}$ and $U \in B(H)$ is a partial isometry. This decomposition is called polar.

A unital C^*-subalgebra $\mathcal{A} \subseteq B(H)$ is a *von Neumann algebra* if, in addition, it is closed with respect to the weak operator topology. That is, $A \in \mathcal{A}$, whenever $((A_n - A)f, g) \to 0$, for every $f, g \in H$ and $(A_n) \in \mathcal{A}$. Equivalently, $\mathcal{A}$ is a von Neumann algebra if and only if $\mathcal{A}'' = (\mathcal{A}')' = \mathcal{A}$, where $\mathcal{A}' = \{C \in B(H),$ such that $AC = CA$ for every $A \in \mathcal{A}\}$ is the commutant of $\mathcal{A}$. Every von Neumann algebra $\mathcal{A}$ has a unique predual which is usually denoted as $\mathcal{A}_*$. From the definition it follows that $B(H)$ is a von Neumann algebra.

A *trace* τ on a von Neumann algebra $\mathcal{A}$ is an additive positively homogeneous function from $\mathcal{A}_+$ to $[0, \infty]$ such that $\tau(A^*A) = \tau(AA^*)$. It is *faithful* if $\tau(A) > 0$ for every positive $A \neq 0$, *normal* if $\tau(\sup A_i) = \sup \tau(A_i)$ for every bounded increasing net A_i, *finite* if $\tau(\mathbf{1}) < \infty$, and *semifinite* if for every nonzero $A \geqslant 0$, there exists a nonzero $0 \leqslant B \leqslant A$ such that $\tau(B) < \infty$. For additional properties of C^* and von Neumann algebras we refer the reader to [24, 12, 26, 27, 53, 95]

1.4 Noncommutative function spaces

Let H be a Hilbert space. A linear operator $A : D \subset H \to H$ is *densely defined* if its domain D is a linear subspace dense in H, and A is *closed* if its graph is closed. Suppose a von Neumann subalgebra $\mathcal{A} \subseteq B(H)$ is equipped with a normal faithful semi-finite trace τ. We say that a closed densely defined operator A is affiliated with $\mathcal{A}$ if A commutes with every unitary operator in $\mathcal{A}'$, that is, $U(D) \subseteq D$ and $UAU^* = A$ for every unitary operator $U \in \mathcal{A}'$. Let $\mathbf{P}(\mathcal{A}) \subset \mathcal{A}$ be the set of all projections in $\mathcal{A}$. We denote by $\tilde{\mathcal{A}}$ the set of closed, densely defined operators, affiliated with $\mathcal{A}$. An operator $A \in \tilde{A}$ is τ-measurable if for each $\epsilon > 0$ there exists a projection $P \in \mathcal{A}$ such that $P(H)$ lies in the domain of A and $\tau(1 - P) \leqslant \epsilon$. It is known that the set of τ-measurable operators in $\tilde{\mathcal{A}}$ is a $*$-algebra, equipped with the measure topology: the uniform topology given by the following system of neighbourhoods at 0: $\{U(\epsilon, \delta), \epsilon, \delta > 0\}$, where $U(\epsilon, \delta) = \{A \in \tilde{\mathcal{A}} :$ there exists $P \in \mathbf{P}(\mathcal{A}),\ \|PA\| < \epsilon,\ \tau(1 - P) < \delta\}$ [36]. We can define the *generalized singular value function*: for $x \in \mathcal{A}$ and $t \geq 0$, $\mu_x(t) = \inf\{\|xP\| : P \in \mathbf{P}(\mathcal{A}), \tau(1 - P) \leq t\}$ (see e.g. [77, 36] for other formulae for $\mu_x(\cdot)$). Define $L_1(\tau)$ to be the completion of the set $\{x \in \mathcal{A} : \tau(|x|) < \infty\}$ with respect to the norm: $\|x\| = \tau(|x|)$. One can show that $L_1(\tau) \subset \tilde{\mathcal{A}}$ [36, 80]. By [95, Theorem V.2.18] $L_1(\tau)$ can be identified with $\mathcal{A}_*$.

Now suppose $\mathcal{E}(\tau)$ is a linear subspace of $\tilde{A}$ equipped with a complete norm $\|\cdot\|_{\mathcal{E}}$. We say that $\mathcal{E}(\tau)$ is a *noncommutative function space* if:

(i) $L_1(\tau) \cap \mathcal{A} \subset \mathcal{E}(\tau) \subset L_1(\tau) + \mathcal{A}$.

(ii) For any $x \in \mathcal{E}(\tau)$ and $a, b \in \mathcal{A}$, we have $axb \in \mathcal{E}(\tau)$, and $\|axb\|_{\mathcal{E}} \leq$

$\|a\|\|x\|_{\mathcal{E}}\|b\|$.

We call $\mathcal{E}(\tau)$ *symmetric* if, whenever $x \in \mathcal{E}(\tau)$, $y \in \tilde{A}$, and $\mu_y \leq \mu_x$, then $y \in \mathcal{E}(\tau)$, with $\|y\|_{\mathcal{E}} \leq \|x\|_{\mathcal{E}}$. Following [33], we say that $\mathcal{E}(\tau)$ is *strongly symmetric* if, in addition, for any $x, y \in \mathcal{E}(\tau)$, with $y \prec\prec x$, we have $\|y\|_{\mathcal{E}} \leq \|x\|_{\mathcal{E}}$. Here, $\prec\prec$ refers to the *Hardy-Littlewood domination*: for any $\alpha > 0$, $\int_0^\alpha \mu_y(t)\,dt \leq \int_0^\alpha \mu_x(t)\,dt$. It is known that, as in the commutative case, $y \prec\prec x$ iff there exists an operator T, contractive both on $\mathcal{A}$ and $\mathcal{A}_\star = L_1(\tau)$, so that $y = Tx$ [28]. We say that $\mathcal{E}$ is *fully symmetric* if it is strongly symmetric and, for any $x \in \mathcal{E}(\tau)$ and $y \in \tilde{A}$, we have $y \in \mathcal{E}(\tau)$ whenever $y \prec\prec x$. By $\mathcal{E}^\times(\tau) = \{\tau\text{-measurable } A \in \tilde{\mathcal{A}} : \sup\{\tau(|BA|) : B \in \mathcal{E}(\tau), \|B\| \leqslant 1\} < \infty\}$ we denote the *Köthe dual* of $\mathcal{E}(\tau)$. If $\mathcal{E}(\tau)$ is strongly symmetric, then $\mathcal{E}^\times(\tau)$ is a fully symmetric noncommutative function space [33].

Any symmetric noncommutative function space $\mathcal{E}(\tau)$ has a generating and normal cone $\mathcal{E}(\tau) \cap \tilde{\mathcal{A}}_+$.

Many symmetric noncommutative function spaces arise from their commutative analogues. Indeed, suppose τ is a normal faithful semi-finite trace on a von Neumann algebra $\mathcal{A}$. Suppose $\mathcal{E}$ is a symmetric (commutative) function space (in the sense of e.g. [59]) on Ω, the range of τ. We can define the corresponding noncommutative function space $\mathcal{E}(\tau)$, consisting of those $x \in \tilde{A}$ for which the norm $\|x\|_{\mathcal{E}(\tau)} = \|\mu_x\|_{\mathcal{E}}$ is finite. By [55], this procedure yields a Banach space. It is well known (see e.g. [29, 33, 77]) that many properties of the function space $\mathcal{E}$ (for instance, being reflexive or order continuous) pass to the non-commutative space $\mathcal{E}(\tau)$.

In the discrete case ($\mathcal{E}$ is a symmetric sequence space on $\mathbb{N}$, and τ is the canonical trace on $B(H)$), the construction above produces a *noncommutative symmetric sequence space*, (often referred to as a *Schatten space*), denoted by $\mathfrak{C}_{\mathcal{E}}(H)$ (instead of $\mathcal{E}(\tau)$). When $H = \ell_2$ ($H = \ell_2^n$), we write $\mathfrak{C}_{\mathcal{E}}$ (resp. $\mathfrak{C}_{\mathcal{E}}^n$) instead of $\mathfrak{C}_{\mathcal{E}}(H)$. For properties of Schatten spaces, the reader is referred to e.g. [45, 92].

1.5 Operator ideals

Let X be a Banach space and B_X the unit ball of X. We say that J is an operator ideal if it is a two-sided algebraic ideal in the algebra of all linear bounded operators $L(X)$. Evidently, $\{0\}$ and $L(X)$ are operator ideals. We will be interested in non-trivial proper ideals in $L(X)$, that is, those that are different from the ones mentioned above.

An operator S, acting from X to a Banach space Y, is (*weakly*) *compact* if $S(B_X)$ is relatively (weakly) compact, *strictly singular* if it is not an isomorphism when restricted to any infinite-dimensional subspace of X, *finitely strictly singular* if for every $\epsilon > 0$ there exists $n \in \mathbb{N}$, such that in every n-dimensional subspace there exists a norm one vector with $\|Sx\| < \epsilon$, *strictly cosingular* if there is no an infinite-dimensional subspace $Z \subseteq Y$ and a corresponding quotient map Q_Z such that $Q_Z S$ is surjective, *Dunford-Pettis* if S maps every weakly null sequence to norm null, *inessential* if $I + US$ is Fredholm for every $U \in L(Y, X)$. We will make use of the following characterization of inessential operators.

Lemma 1.5.1. *[75] Suppose $T \in L(X, Y)$. Then T is inessential if and only if $I + AT$ has a finite-dimensional null space for every $A \in L(Y, X)$.*

We denote the classes of compact, weakly compact, finitely strictly singular, strictly singular, strictly cosingular, Dunford-Pettis, and inessential operators acting between X and Y as $\mathcal{K}(X,Y)$, $\mathcal{WK}(X,Y)$, $\mathcal{FSS}(X,Y)$, $\mathcal{SS}(X,Y)$, $\mathcal{SCS}(X,Y)$, $\mathcal{DP}(X,Y)$, $\mathcal{IN}(X,Y)$, respectively. The following chains of inclusions hold:

$$\mathcal{K}(X,Y) \subseteq \mathcal{FSS}(X,Y) \subseteq \mathcal{SS}(X,Y) \subseteq \mathcal{IN}(X,Y),$$
$$\mathcal{K}(X,Y) \subseteq \mathcal{SCS}(X,Y) \subseteq \mathcal{IN}(X,Y),$$

In the case when $X = Y$, it is known that these classes form norm-closed operator ideals in $L(X)$.

For two closed ideals J_1 and J_2 in $L(X)$, we will denote by $J_1 \wedge J_2$ the largest closed ideal J in $L(X)$ such that $J \subseteq J_1$ and $J \subseteq J_2$ (that is, $J_1 \wedge J_2 = J_1 \cap J_2$), and we will denote by $J_1 \vee J_2$ the smallest closed ideal J in $L(X)$ such that $J_1 \subseteq J$ and $J_2 \subseteq J$. We say that J_2 is a *successor* of J_1 if $J_1 \subsetneq J_2$. If, in addition, no closed ideal J in $L(X)$ satisfies $J_1 \subsetneq J \subsetneq J_2$, then we call J_2 an *immediate successor* of J_1.

Fore more information on operator ideals we refer the reader to [2, 76, 81].

Chapter 2

Domination problem

2.1 Introduction

Let X and Y are OBS and $T, S \in L(X, Y)$. We say that T is *dominated* by S if $0 \leqslant T \leqslant S$. Assume that S belongs to a certain ideal of operators, e.g., (weakly) compact, strictly singular, Dunford-Pettis and etc. Does this imply that T (or some power of T) is in the same ideal? We refer to this problem as the *domination problem.*

The domination problem has been extensively studied for Banach lattices, see [34, 40, 39, 54, 38, 42, 101, 100]. Let us mention several results on this subject. In the rest of this section X and Y are Banach lattices.

Theorem 2.1.1. *[34], [100, Theorem 1] The following statements are equivalent:*

(i) for any two operators $0 \leqslant T \leqslant S : X \to Y$*, if* S *is compact then* T *is compact;*

(ii) one of the following three (non-exclusive) conditions holds:

(a) both X^* *and* Y *are order continuous;*

(b) Y is atomic and order continuous;

(c) X^* is atomic and order continuous.

Theorem 2.1.2. *[5, Theorem 5.13] Let $0 \leqslant T \leqslant S : X \to X$. If S is compact, then T^3 is also compact.*

Theorem 2.1.3. *[5, Theorem 5.31] The following statements are equivalent:*

(i) either X^* or Y is order continuous;

(ii) every positive operator from X to Y dominated by a weakly compact operator is weakly compact.

Recently, similar results where obtained in the case when S is strictly singular.

Theorem 2.1.4. *[39, Theorem 1.1] Suppose S is a positive strictly singular operator and Y is order continuous. Then every operator dominated by S is strictly singular if either of the following conditions holds:*

(i) X is atomic and order continuous;

(ii) X and X^* are order continuous and X satisfies the subsequence splitting property.

Theorem 2.1.5. *[42, Corollary 2.4] Let $0 \leqslant T \leqslant S : X \to X$. If S is strictly singular, then T^4 is also strictly singular.*

The preceding results show that an operator acting between Banach lattices, dominated by a compact or strictly singular operator, does not necessarily belong to the same class. In the next section we will investigate whether it belongs to a 'slightly' larger class of operators, so called, the inessential operators. We will show that every operator, dominated by a compact operator, is inessential, and that the cube of any operator dominated by a strictly singular operator is inessential. There are many Banach spaces for which the class of inessential operators has been well studied. In particular, it is known that $\mathcal{SS}(X) = \mathcal{IN}(X)$ when X is subprojective (see [74]), or when $X = L_p(\mu)$ for some $(p \geqslant 1)$ and a finite measure μ (see [97]), or when $X = C(K)$ for some compact Hausdorff topological space K (see [68]), or when X is a Lorentz space with a certain weak additional condition on the generating function (see [88]).

In the last section of this chapter we will investigate the domination problem in the noncommutative setting, in particular, for C^*-algebras and noncommutative function spaces. Surprisingly, in this case the domination problem has drawn little attention so far, except in [30, 69] this problem was studied for completely positive maps.

We will obtain an analogue of Theorem 2.1.1 for C^*-algebras, and show that every operator dominated by a weakly compact operator is weakly compact. Finally, we will settle the domination problem for the noncomuttative sequence spaces and obtain some partial results in the general case.

2.2 Banach lattices

This section is based on [93]. Throughout this section X and Y are Banach lattices.

We say that an operator between two Banach spaces is c_0*-strictly singular* if it is not an isomorphism on every subspace isomorphic to c_0. Similarly we define ℓ_1-strictly singular operators. The following theorem is a generalization of [73, Corollary 4] that is discussed in [39, Remark 3.9].

Theorem 2.2.1. *Let X be an AM-space and Z an arbitrary Banach space. Then for every operator $S \in L(X, Z)$ the following are equivalent.*

(i) S is c_0-strictly singular;

(ii) S is weakly compact;

(iii) S is strictly singular.

Corollary 2.2.2. *Let X be an AM-space. Suppose $T \in L(X, Y)$ is dominated by a strictly singular operator S. Then $T \in \mathcal{SS}(X, Y)$.*

Proof. Theorem 2.2.1 implies that T is dominated by a weakly compact operator S. Therefore, T is weakly compact by Theorem 2.1.3 since X^* is order continuous. Hence the result follows by applying Theorem 2.2.1 again. ∎

The following fact is known [42]; however for the reader's convenience we provide the proof.

Proposition 2.2.3. *If $T \in L(X, Y)$ is dominated by a strictly singular operator S, then T is c_0-strictly singular.*

Proof. Assume that T is not c_0-strictly singular. Then [67, Theorem 3.4.11] implies that T is an isomorphism on a lattice copy of c_0, say Z. Consider the restrictions $A = T_{|Z}$ and $B = S_{|Z}$. Applying Corollary 2.2.2 to A and B we conclude that A is strictly singular. This is a contradiction. ∎

The following lemma is a trivial observation.

Lemma 2.2.4. *Let Z be a Banach space and $A, B \in L(Z)$. Then* $\dim \operatorname{Ker}(I - AB)$ *is finite if and only if* $\dim \operatorname{Ker}(I - BA)$ *is finite.*

Proof. Let $\operatorname{Ker}(I - BA)$ be finite dimensional. Clearly, $ABx = x$ for every $x \in \operatorname{Ker}(I - AB)$. Therefore $BA(Bx) = Bx$, thus $Bx \in \operatorname{Ker}(I - BA)$. Note that B is injective restricted to $\operatorname{Ker}(I - AB)$. This implies $\dim \operatorname{Ker}(I - AB) \leqslant \dim \operatorname{Ker}(I - BA)$. The other direction is obtained by switching A and B above. ∎

Lemma 2.2.5. *Let Y be an AM-space. Assume $T \in L(X, Y)$ is dominated by a strictly singular operator. Then $TU \in \mathcal{SS}(Y)$ and $UT \in \mathcal{IN}(X)$ for every $U \in L(Y, X)$.*

Proof. Proposition 2.2.3 guarantees that T and, consequently TU are c_0-strictly singular. Theorem 2.2.1 implies that $TU \in \mathcal{SS}(Y)$. Similarly for every $V \in L(X)$ we have $TVU \in \mathcal{SS}(Y)$. Therefore, dim $\operatorname{Ker}(I_Y - TVU) < \infty$ and, hence, dim $\operatorname{Ker}(I_X - VUT) < \infty$ by Lemma 2.2.4. As V was chosen arbitrarily, Lemma 1.5.1 implies $UT \in \mathcal{IN}(X)$. ∎

The next theorem is an immediate consequence of Corollary 2.2.2 and Lemma 2.2.5

Theorem 2.2.6. *Suppose that $T \in L(X, Y)$ factors through an AM-space such that at least one of the factors is dominated by a strictly singular operator. Then $T \in \mathcal{IN}(X, Y)$.*

It is a simple observation that an operator $\left(\begin{smallmatrix} S_1 & S_2 \\ S_3 & S_4 \end{smallmatrix}\right)$ acting on $X \oplus Y$ is strictly singular if and only if each S_i is strictly singular. Note also that $\left(\begin{smallmatrix} T_1 & T_2 \\ T_3 & T_4 \end{smallmatrix}\right) \leqslant \left(\begin{smallmatrix} S_1 & S_2 \\ S_3 & S_4 \end{smallmatrix}\right)$ if and only if $T_i \leqslant S_i$ as $i = 1, 2, 3, 4$.

Theorem 2.2.7. *Suppose Y is an AM-space. Consider $\tilde{T} \in L(X \oplus Y)$ where $\tilde{T} = \left(\begin{smallmatrix} 0 & T_2 \\ T_3 & T_4 \end{smallmatrix}\right)$. If $\tilde{T}$ is dominated by a strictly singular operator then it is inessential.*

Proof. Note that T_2 and T_4 are strictly singular by the preceding observation and Corollary 2.2.2. Therefore, $\left(\begin{smallmatrix} 0 & T_2 \\ 0 & T_4 \end{smallmatrix}\right)$ is strictly singular. It is left to show that $\left(\begin{smallmatrix} 0 & 0 \\ T_3 & 0 \end{smallmatrix}\right)$ is inessential. By Lemma 1.5.1 and Lemma 2.2.4 it suffices to show that for any $\left(\begin{smallmatrix} U_1 & U_2 \\ U_3 & U_4 \end{smallmatrix}\right) \in L(X \oplus Y)$ the kernel of

$$\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ T_3 & 0 \end{pmatrix} \begin{pmatrix} U_1 & U_2 \\ U_3 & U_4 \end{pmatrix}$$

is finite-dimensional in $L(X \oplus Y)$. Equivalently, the solution space of the following system of equations

$$x = 0, \qquad -T_3U_1x + (I - T_3U_2)y = 0$$

is finite-dimensional in $X \oplus Y$. Indeed, Lemma 2.2.5 implies that $T_3U_2 \in SS(Y)$ and, thus, $\ker(I - T_3U_2)$ is finite-dimensional. ■

We will use the following statement which follows from [67, Corollary 3.4.14] and [67, Theorem 3.4.17].

Theorem 2.2.8. *Let Y be an order continuous Banach lattice. If $T \in L(X, Y)$ is a positive isomorphism on a copy of ℓ_1, then T is an isomorphism on a lattice copy of ℓ_1 generated by a positive disjoint sequence.*

We say that an operator is *disjointly strictly singular* on X if it is not an isomorphism on every subspace spanned by an infinite disjoint sequence of vectors.

Theorem 2.2.9. *[38, Theorem 1.1] Let Y be an order continuous Banach lattice. If $T \in L(X,Y)$ is dominated by a disjointly strictly singular operator S then T is disjointly strictly singular.*

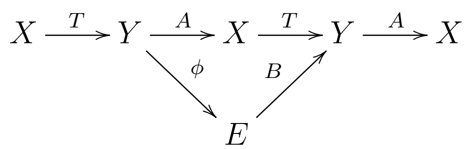
Theorem 2.2.10. *Let Y be an order continuous Banach lattice and $T \in L(X,Y)$ dominated by a Dunford-Pettis disjointly strictly singular operator S (in particular, S could be a compact operator). Then $T \in SS(X,Y)$.*

Proof. Suppose T is not strictly singular. Therefore, there exists an infinite dimensional subspace $M \subseteq X$ such that T is an isomorphism when restricted to M. Since Y is order continuous, T is Dunford-Pettis by [100, Theorem 2] and, thus, M contains a copy of ℓ_1 by [5, Theorem 5.80]. Then Theorem 2.2.8 yields that T must be an isomorphism on a lattice copy of ℓ_1 generated by a positive disjoint sequence, that contradicts Theorem 2.2.9. ■

For the rest of the section we use some ideas developed in [42]. The next theorem provides the affirmative answer to our conjecture for compact operators.

Theorem 2.2.11. *Let $T \in L(X,Y)$ be dominated by a Dunford-Pettis operator $S \in \mathcal{SS}(X,Y)$ (in particular, S could be a compact). Then $T \in \mathcal{IN}(X,Y)$.*

Proof. Assume $T \notin \mathcal{IN}(X,Y)$. Then, by Lemma 1.5.1, there exists $A \in L(Y,X)$ and an infinite dimensional subspace $M \subset X$ such that $ATx = x$ for every $x \in M$. Proposition 2.2.3 implies that T and, consequently, TA are c_0-strictly singular and, therefore, they are order weakly compact by [67, Corollary 3.4.5]. Then by [5, Theorem 5.58] we have the following factorization for TA:



where ϕ is a lattice homomorphism and E is an order continuous Banach lattice. Note that $\phi T \leqslant \phi S$ and ϕS is Dunford-Pettis and strictly singular. Now Theorem 2.2.10 yields that ϕT and, therefore, $ATAT$ are strictly singular. But $ATAT_{|M} = id_{|M}$ which is a contradiction. ■

Remark 2.2.12. We say that a seminormalized sequence $(x_n) \in X$ is **almost disjoint** if there exists a disjoint sequence $(y_n) \subset X$ such that $||x_n - y_n|| \to 0$. Let Z be a Banach space. Suppose that an infinite dimensional subspace $E \subseteq X$ contains an almost disjoint seminormalized sequence (x_n). If $T \in L(X, Z)$ is an isomorphism on E then T is an isomorphism on a subspace generated by a disjoint sequence.

Indeed, suppose $||y_n - x_n|| \to 0$, where (y_n) is a disjoint sequence. Then by passing to a subsequence we may assume that the sequences (x_n) and (Tx_n) are equivalent to (y_n) and (Ty_n) respectively, see [4, Theorem 1.3.9]. That is, there exist isomorphisms between the closed linear spans $U : [x_n] \to [y_n]$ and $V : [Tx_n] \to [Ty_n]$, such that $Ux_n = y_n$ and $VTx_n = Ty_n$ for all n. Clearly, $UT^{-1}V^{-1}T$ is the identity on $[y_n]$ and consequently T is an isomorphism on $[y_n]$.

Remark 2.2.13. Suppose that E is an order continuous Banach lattice and N is a separable closed subspace of E. We will use the following classical facts.

(i) N is contained in a closed ideal I with a weak unit ([64, Proposition 1.a.11, Vol 2]).

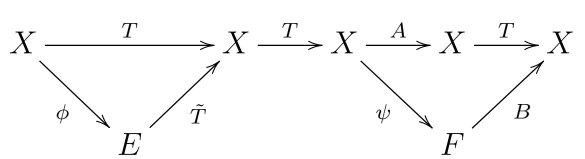
(ii) There is a positive projection P from E onto I ([64, Proposition 1.a.11, Vol 2]).

(iii) There is a positive one-to-one operator $j : I \to L_1(\mu)$ for some finite measure space (Ω, Σ, μ) ([64, Proposition 1.b.14]).

(iv) Either j is an isomorphism on N or N contains an almost disjoint seminormalized sequence ([64, Theorem 1.c.8 and its proof]).

Theorem 2.2.14. *Let $T \in L(X)$ be dominated by $S \in \mathcal{SS}(X)$. Then $T^3 \in \mathcal{IN}(X)$.*

Proof. Suppose $T^3 \notin \mathcal{IN}(X)$. Lemma 1.5.1 implies that there exists an operator $A \in L(X)$ and an infinite dimensional subspace $N_0 \subset X$ such that $AT^3x = x$ for every $x \in N_0$. Then, by Lemma 2.2.4, it follows that there exists an infinite dimensional subspace $M \subset X$ such that $TAT^2x = x$ for every $x \in M$. Without loss of generality we may assume M is separable. Note that T and, consequently, TA are c_0-strictly singular by Proposition 2.2.3 and, therefore, are order weakly compact by [67, Corollary 3.4.5]. Then [6, Theorem 5.58] implies that T and TA factor through order continuous Banach lattices E and F, and we have the following diagram:



where ϕ and ψ are lattice homomorphisms and $\tilde{T}$ is positive. It is clear that $\phi(M)$ is infinite dimensional (since T is an isomorphism on M) and separable. We apply Remark 2.2.13 to $N = \phi(M)$ and E. Then, either $\phi(M)$ contains an almost disjoint sequence, or $\phi(M)$ is isomorphic to a closed subspace of $L_1(\mu)$ and jP restricted to $\phi(M)$ is an isomorphism. Assume that the first case holds. Since $\psi T\tilde{T} \leqslant \psi S\tilde{T}$ and F is order continuous, it follows that $\psi T\tilde{T}$ is disjointly strictly singular by Theorem 2.2.9. Then, by Remark 2.2.12, $\psi T\tilde{T}$ can not be an isomorphism on $\phi(M)$. This contradicts to TAT^2 being the identity on M. Suppose the second case holds. Observe that $0 \leqslant jP\phi T \leqslant jP\phi S : X \to L_1(\mu)$. It was proved in [42, Proposition 3.2] that every operator into $L_1(\mu)$ which is dominated by a strictly singular operator is itself strictly singular. Since $jP\phi S$ is strictly singular, we conclude that so is $jP\phi T$ and, consequently, $jP\phi TAT^2$. On the other hand, since $jP\phi$ is an isomorphism on M and $jP\phi x = jP\phi TAT^2 x$ for every $x \in M$ it follows $jP\phi TAT^2$ is an isomorphism on M, which is a contradiction. ■

Theorem 2.2.15. *Let X and Y be an order continuous and $T \in L(X,Y)$ dominated by $S \in \mathcal{SS}(X,Y)$. Assume that T is ℓ_p-strictly singular for all $p > 1$. Then T is strictly singular.*

Proof. Suppose T is an isomorphism on an infinite dimensional subspace M. Without loss of generality, M is separable. By Theorem 2.2.9, T is disjointly strictly singular and, thus, M has no almost disjoint seminormalized infinite sequences by Remark 2.2.12. Therefore, Remark 2.2.13 implies that M is isomorphic to a subspace of $L_1(\mu)$, where μ is finite. By [47, Theorem IV.5.3]) ℓ_p is finitely representable in M for some $p \geqslant 1$, and thus, applying [47, Theorem IV.5.6], M contains a subspace isomorphic to ℓ_q, for some $1 \leqslant q \leqslant p$.

This is a contradiction: if $q > 1$, this contradicts the asumption of the problem, and $q = 1$ is imposible, because T, being disjointly strictly singular, is ℓ_1-strictly singular by Theorem 2.2.8. ■

As an application, we consider Tsirelson's space $\mathcal{T}$. By [19, Theorem I.8], $\mathcal{T}$ has a 1-unconditional basis and no super-reflexive subspaces, in particular, it contains no copy of ℓ_p for any $p > 1$. Since it has a 1-unconditional basis, it can be naturally given a Banach lattice structure. Moreover, it is order continuous by [5, Theorem 4.14]. Then [40, Proposition 0.5] implies that an operator A from $\mathcal{T}$ to an order continuous Banach lattice Y dominated by a strictly singular operator is strictly singular. Theorem 2.2.15 extends the preceding statement. That is, every operator to or from $\mathcal{T}$ dominated by a strictly singular operator is strictly singular, provided that the second lattice is order continuous. It is easy to check that the same statement holds for the original Tsirelson's space $\mathcal{T}^*$.

2.3 Domination problem in the noncommutative setting

This section is based on [70]. It is structured as follows. First, we prove some preliminary results about properties of positive operators, order intervals, and positive solids. In Subsection 2.3.1, we establish some basic facts about noncommutative function spaces. In Subsection 2.3.2, we investigate compact C^*-algebras, characterizing them in terms of compactness of order intervals. We also show that a C^*-algebra is compact iff it is hereditary in its enveloping algebra. Subsection 2.3.3 deals with the positive analogues of the Schur

Property. In Subsection 2.3.4, we study compactness of order intervals in preduals of von Neumann algebras.

Our main results are contained in the following sections. In Subsection 2.3.5, we investigate whether an operator to or from a noncommutative function space, dominated by a compact operator, must itself be compact. Subsection 2.3.6 is devoted to the same question for C^*-algebras. In Subsection 2.3.7, we consider domination by compact multiplication operators on C^*-algebras.

2.3.1 Compactness and positivity in Schatten spaces

To work with Schatten spaces, we need to introduce some notation. Denote the canonical basis in ℓ_2 by (e_k). Let P_n be the orthogonal projection onto $\text{span}[e_1, \ldots, e_n]$, and $P_n^\perp = \mathbf{1} - P_n$. For convenience, set $P_0 = 0$. If $\mathcal{E}$ is a non-commutative symmetric sequence space, let Q_n be the projection on $\mathcal{E}$, defined via $Q_n x = P_n x P_n$. Similarly, let $R_n x = P_n^\perp x P_n^\perp$. The usual approximation argument shows that $\lim_n \|(\mathbf{1} - Q_n)x\| = 0$.

Lemma 2.3.1. *Suppose $\mathcal{E}$ is a noncommutative symmetric sequence space on $B(\ell_2)$, Z is an ordered normed space, and $T : \mathcal{E} \to Z$ is a positive operator. Then, for any $x \in \mathcal{E}_+$, $\|T(x - R_n x - Q_n x)\|^2 \leq 4\mathbf{N}_Z \|T(Q_n x)\| \|T(R_n x)\|$, where $\mathbf{N}_Z$ is the normality constant of Z.*

Proof. For $t \in \mathbb{R}\backslash\{0\}$, consider $U(t) = tP_n + t^{-1}P_n^\perp$, and $V(t) = tP_n - t^{-1}P_n^\perp$. These operators are self-adjoint and invertible, hence $x(t) = U(t)xU(t)$ and $y(t) = V(t)xV(t)$ are positive elements of $\mathcal{E}$. An elementary calculation shows that $x(t) = t^2 Q_n x + t^{-2} R_n x + (x - Q_n x - R_n x)$, and $y(t) = t^2 Q_n x + t^{-2} R_n x - (x - Q_n x - R_n x)$. Let $a(t) = t^2 Q_n x + t^{-2} R_n x$, and $b = x - Q_n x - R_n x$.

By the above, $-a(t) \le b \le a(t)$. Therefore, for any t,

$$\mathbf{N}_Z^{-1}\|Tb\| \le \|Ta(t)\| \le t^2\|TQ_n x\| + t^{-2}\|TR_n x\|.$$

Taking $t = \|TR_n x\|^{1/4}/\|TQ_n x\|^{1/4}$, we obtain the desired inequality. ■

Corollary 2.3.2. *Suppose $\mathcal{E}$ is a noncommutative symmetric sequence space on $B(\ell_2)$, Z is a normal OBS, and $T : \mathcal{E} \to Z$ is a positive operator. Then*

$$\|T(I - Q_n)\| \le \|TR_n\| + 8{\mathbf{N}_Z}^{1/2}\|TR_n\|^{1/2}\|TQ_n\|^{1/2}.$$

Proof. Lemma 2.3.1 shows that, for $x \ge 0$,

$$\|T(I - R_n - Q_n)x\| \le 2\mathbf{N}_Z^{1/2}\|TR_n\|^{1/2}\|TQ_n\|^{1/2}\|x\|.$$

A polarization argument implies $\|T(I-R_n-Q_n)\| \le 8\mathbf{N}_Z^{1/2}\|TR_n\|^{1/2}\|TQ_n\|^{1/2}$. Finally, by the triangle inequality, $\|T(I-Q_n)\| \le \|TR_n\|+\|T(I-R_n-Q_n)\|$.
■

For future use, we need to quote a result from [22, Section 2].

Lemma 2.3.3. *Suppose τ is a normal faithful semi-finite trace on a von Neumann algebra $\mathcal{A}$, and a strongly symmetric noncommutative function space $\mathcal{E}$ is order continuous. Suppose, furthermore, that x is an element of $\mathcal{A}$, and a sequence of projections $p_n \in \mathcal{A}$ decreases to 0 in the strong operator topology. Then $\lim_n \|xp_n\| = \lim_n \|p_n x\| = \lim_n \|p_n x p_n\| = 0$.*

Specializing to Schatten spaces, we obtain:

Corollary 2.3.4. *Suppose $\mathcal{E}$ is an order continuous symmetric sequence space. Then, for every $x \in \mathfrak{C}_{\mathcal{E}}$, $\lim_n \|x - Q_n x\| = 0$.*

Proof. By [29, Section 3], $\mathfrak{C}_{\mathcal{E}}$ is order continuous iff $\mathcal{E}$ is order continuous. It suffices to show that, for $x \in \mathbf{B}(\mathfrak{C}_{\mathcal{E}})_+$, and $\varepsilon \in (0,1)$, $\|x - Q_n x\| < \varepsilon$ for n sufficiently large. By Lemma 2.3.3, $\|R_n x\| = \|P_n x P_n\| < \varepsilon^2/16$ for sufficiently large n. By Lemma 2.3.1 (applied when T is the identity map), $\|x - Q_n x - R_n x\| < \varepsilon/2$. We complete the proof by using the triangle inequality. ∎

Lemma 2.3.5. *Suppose $\mathcal{E}$ is an order continuous symmetric sequence space, not containing ℓ_1, and $S : \mathfrak{C}_{\mathcal{E}} \to Z$ is compact (Z is a Banach space). Then* $\lim_n \|S|_{R_n(\mathfrak{C}_{\mathcal{E}})}\| = 0$.

Proof. Suppose not. By Corollary 2.3.4, we have $\lim_n \|(I - Q_n)x\| = 0$. A standard approximation argument yields a sequence $0 = n_0 < n_2 < \ldots$ with the property that for each k there exists $x_k \in \mathfrak{C}_{\mathcal{E}}$, so that $\|x_k\| = 1$, and $(P_{n_k} - P_{n_{k-1}})x_k(P_{n_k} - P_{n_{k-1}}) = x_k$, and $\|Sx_k\| > c > 0$. By compactness, the sequence (Sx_k) must have a convergent subsequence (Sx_{k_i}). Then $\lim_N N^{-1}\|\sum_{i=1}^N Sx_{k_i}\| > 0$, while $\lim_N N^{-1}\|\sum_{i=1}^N x_{k_i}\| = 0$. Contradiction. ∎

Next we describe the Schatten spaces not containing ℓ_1.

Proposition 2.3.6. *Let $\mathcal{E}$ be a separable symmetric sequence space. For any infinite-dimensional Hilbert space H, the following are equivalent:*

(i) $\mathcal{E}$ contains a copy of ℓ_1.

(ii) $\mathcal{E}$ contains a lattice copy of ℓ_1 positively complemented.

(iii) $\mathfrak{C}_{\mathcal{E}}(H)$ contains a positively complemented copy of ℓ_1 spanned by a disjoint positive sequence.

(iv) $\mathfrak{C}_{\mathcal{E}}(H)$ *contains a copy of* ℓ_1.

Proof. The implications (2) $\Rightarrow$ (1) and (3) $\Rightarrow$ (4) are trivial. To show (2) $\Rightarrow$ (3), observe that $\mathfrak{C}_{\mathcal{E}}(H)$ contains $\mathcal{E}$ as a diagonal subspace, which is positively complemented. (4) $\Rightarrow$ (1) follows directly from [9, Corollary 3.2]. To prove (1) $\Rightarrow$ (2), apply a "gliding hump" argument to show that $\mathcal{E}$ contains disjoint vectors (x_i), equivalent to the canonical basis of ℓ_1. Then $X = \operatorname{span}[|x_i| : i \in \mathbb{N}]$ is a sublattice of $\mathcal{E}$, lattice isomorphic to ℓ_1. By [67, Theorem 2.3.11], X is positively complemented. ∎

For a subset $M \subset X_+$ (X is an OBS), define the *positive solid* of M:

$$\mathbf{PSol}(M) = \{x \in X_+, \text{ such that } 0 \leqslant x \leqslant y \text{ and } y \in M\}.$$

Lemma 2.3.7. *If* $\mathcal{E}$ *is an order continuous noncommutative symmetric sequence space, and* $M \subset \mathcal{E}$ *is relatively compact, then* $\mathbf{PSol}(M)$ *is relatively compact.*

For the proof, we need two technical results.

Lemma 2.3.8. *Suppose* $\mathcal{E}$ *and* M *are as in Lemma 2.3.7. Then there exists a projection* p *with separable range, so that* $M = pMp$.

Proof. The set M must contain a countable dense subset S. The elements of M are compact operators, hence, for any $x \in S$, there exists a projection p_x with separable range, so that $p_x x p_x = x$. Then $p = \vee_{x \in S} p_x$ has the desired properties. ∎

Lemma 2.3.9. *Suppose* $\mathcal{E}$ *is an order continuous noncommutative symmetric sequence space on* $B(\ell_2)$*, and* M *is relatively compact subset of* $\mathcal{E}$*. Then* $\lim_n \|R_n|_M\| = 0$.

Proof. For every $\varepsilon > 0$ there are $x_1, \ldots, x_k$ in M such that for every $x \in M$ there is an $1 \leqslant i \leqslant k$ such that $\|x - x_i\| < \varepsilon/2$. Pick $N \in \mathbb{N}$ such that $\|R_n x_i\| < \epsilon/2$ for every $n > N$ and $1 \leqslant i \leqslant k$. Hence, $\|R_n x\| \leqslant \|R_n x_i\| + \|R_n\|\|x - x_i\| < \epsilon$ for every $x \in M$ and $n > N$. ∎

Proof of Lemma 2.3.7. By Lemma 2.3.8, we can restrict ourselves to spaces on $B(\ell_2)$. As Q_n is a finite rank projection, it suffices to show that, for any $\varepsilon \in (0,1)$, there exists $n \in \mathbb{N}$ so that $\|(I - Q_n)x\| < \varepsilon$ for any $x \in \mathbf{PSol}(M)$. To this end, write $(I - Q_n)x = (x - Q_n x - R_n x) + R_n x$. Reasoning as in the proof of Lemma 2.3.1, we observe that

$$-\big(t^2 Q_n x + t^{-2} R_n x\big) \leq x - Q_n x - R_n x \leq t^2 Q_n x + t^{-2} R_n x$$

for any $t > 0$, hence $\|x - Q_n x - R_n x\| \leq t^2\|Q_n x\| + t^{-2}\|R_n x\|$. Taking $t = \|R_n x\|^{1/2}/\|Q_n x\|^{1/2}$, we obtain $\|x - Q_n x - R_n x\| \leq 2\|R_n x\|^{1/2}\|Q_n x\|^{1/2}$.

By scaling, we can assume that $\sup_{y \in M} \|y\| = 1$. By Lemma 2.3.9, there exists $n \in \mathbb{N}$ so that $\|R_n y\| < \varepsilon^2/16$ for any $y \in M$. For any $x \in \mathbf{PSol}(M)$, there exists $y \in M$ so that $0 \leq x \leq y$, hence $0 \leq R_n x \leq R_n y$. By the above, $\|x - Q_n x - R_n x\| \leq 2\|R_n y\|^{1/2} < \varepsilon/2$, hence

$$\|(I - Q_n)x\| = \|x - Q_n x - R_n x\| + \|R_n x\| \leq \frac{\varepsilon}{2} + \frac{\varepsilon^2}{16} < \varepsilon.$$

∎

Corollary 2.3.10. *Suppose $\mathcal{E}$ is a fully symmetric noncommutative sequence space. Then $\mathcal{E}$ is order continuous if and only if any order interval in $\mathcal{E}$ is compact.*

Lemma 2.3.11. *Suppose $\mathcal{E}$ is a fully symmetric noncommutative function or sequence space, which is not order continuous. Then there exists a positive complete isomorphism $j : \ell_\infty \to \mathcal{E}$.*

Proof. In the notation of [33, Section 6], there exists $x \in \mathcal{E}_+ \backslash \mathcal{E}^{an}$. Moreover, there exists a sequence of mutually orthogonal projections $e_i \in \mathcal{A}$ ($i \in \mathbb{N}$), so that $\inf_i \|e_i x e_i\| > 0$. The map $y \mapsto \sum_i e_i y e_i$ is contractive in $\mathcal{A}$, and in its predual, hence $\sum_i e_i y e_i \prec y$, for any $y \in \mathcal{A} + \mathcal{A}_\star$. Due to $\mathcal{A}$ being fully symmetric, $\sum_i e_i x e_i \in \mathcal{E}$, and $\| \sum_i e_i x e_i\| \leq \|x\|$. Therefore, the map

$$j : \ell_\infty \to \mathcal{E} : (\alpha_i) \mapsto (\sum_i \alpha_i e_i)(\sum_i e_i x e_i) = \sum_i \alpha_i e_i x e_i$$

has the desired properties. ■

Proof of Corollary 2.3.10. Note that an order interval $[0, x]$ is closed. If $\mathcal{E}$ is order continuous, an application of Lemma 2.3.7 to $M = \{x\}$ shows the compactness of $[0, x]$. If $\mathcal{E}$ is not order continuous, then, for x as in Lemma 2.3.11, $[0, x]$ is not (relatively) compact. ■

2.3.2 Compactness of order intervals in C^*-algebras

If Z is an OBS, and $x \in Z_+$, define the *order interval* $[0, x]$ as the set $\{y \in Z_+ : y \leq x\}$. In this subsection, we investigate the compactness of order intervals in C^*-algebras, and obtain a new description of compact C^*-algebras.

First we introduce some definitions. We say that an element a of a Banach algebra $\mathcal{A}$ is *multiplication compact* if the map $\mathcal{A} \to \mathcal{A} : b \mapsto aba$ is compact. Combining [109], [108], we see that, for an element a of a C^*-algebra $\mathcal{A}$, the following are equivalent:

(i) a is multiplication compact.

(ii) The map $\mathcal{A} \to \mathcal{A} : b \mapsto ab$ is weakly compact.

(iii) The map $\mathcal{A} \to \mathcal{A} : b \mapsto ba$ is weakly compact.

(iv) The map $\mathcal{A} \to \mathcal{A} : b \mapsto aba$ is weakly compact.

By [107], there exists a faithful representation $\pi : \mathcal{A} \to B(H)$ so that a is multiplication compact iff $\pi(a)$ is a compact operator on H. If, in addition, $\mathcal{A}$ is an irreducible C^*-subalgebra of $B(H)$, then $a \in \mathcal{A}$ is multiplication compact iff a is a compact operator [106].

Suppose $\mathcal{A}$ is a C^*-subalgebra of $B(H)$, where H is a Hilbert space. For $x \in B(H)$ we define an operator $M_x : \mathcal{A} \to B(H) : a \mapsto x^*ax$.

Lemma 2.3.12. *For an element a of a C^*-algebra $\mathcal{A}$, the following are equivalent.*

(i) a is multiplication compact.

(ii) The operator M_a is compact.

(iii) The operator M_a is weakly compact.

Proof. (2) $\Rightarrow$ (3) is trivial. To show (1) $\Rightarrow$ (2), recall that a is multiplication compact iff the map $\mathcal{A} \to \mathcal{A} : b \mapsto ab$ is weakly compact. Passing to the adjoint, we see that the last statement holds iff the map $\mathcal{A} \to \mathcal{A} : b \mapsto ba^*$ is weakly compact, or equivalently, iff a^* is multiplication compact. By [13], this implies the compactness of M_a.

To prove (3) $\Rightarrow$ (1), note that $M_a^{\star\star}$ takes $b \in \mathcal{A}^{\star\star}$ to a^*ba. We identify $M_a^{\star\star}$ with M_a, acting on $\mathcal{A}^{\star\star}$. Write $a = cu$, where $c = (aa^*)^{1/2}$, and u

(respectively, u^*) is a partial isometry from $(\ker a)^\perp = (\ker c)^\perp$ to $\overline{\operatorname{ran} a} = \overline{\operatorname{ran} c}$ (from $\overline{\operatorname{ran} a^*} = \overline{\operatorname{ran} c}$ to $(\ker a^*)^\perp = (\ker c)^\perp$). Then $M_a = M_u M_c$, and M_u is an isometry on $\operatorname{ran}(M_c) \subset \mathcal{A}^{\star\star}$. Writing $M_c = M_u^{-1} M_a$, we conclude that M_c is weakly compact. However, $M_c x = cxc$, hence, by the remarks preceding the lemma, c is multiplication compact. The operator $S : \mathcal{A}^{\star\star} \to \mathcal{A}^{\star\star} : b \mapsto aba$ can be written as $S = UM_cV$, where $Vb = ub$ and $Ub = bu$. Then S is weakly compact, and therefore, a is multiplication compact. ■

Multiplication compactness of elements of a C^*-algebra can be described in terms of compactness of order intervals.

Proposition 2.3.13. *For a positive element a of a C^*-algebra $\mathcal{A}$, the following are equivalent:*

(i) a is multiplication compact.

(ii) a^α is multiplication compact for any $\alpha > 0$.

(iii) The order interval $[0, a]$ is compact.

(iv) The order interval $[0, a]$ is weakly compact.

Proof. The implications (2) ⇒ (1) and (3) ⇒ (4) are immediate. To establish (1) ⇒ (2), pick a faithful representation π so that a is multiplication compact if and only if $\pi(a)$ is compact, and note that the compactness of $\pi(a)$ is equivalent to the compactness of $\pi(a)^\alpha = \pi(a^\alpha)$.

For (2) ⇒ (3), assume $\|a\| = 1$. By [24, Lemma I.5.2], for any $x \in [0, a]$ there exists $u \in \mathbf{B}(\mathcal{A})$, so that $x^{1/2} = ua^{1/4}$, hence $x = a^{1/4}u^*ua^{1/4}$. In particular, $[0,a] \subset M_{a^{1/4}}(\mathbf{B}(\mathcal{A}))$. If a is multiplication compact, then so is $a^{1/4}$. Therefore, $[0,a]$ is compact.

To prove (4) $\Rightarrow$ (1), suppose a is not multiplication compact. Then $a^{1/2}$ is not multiplication compact, hence $M_{a^{1/2}}(\mathbf{B}(\mathcal{A}))$ is not relatively compact. Note that any element $x \in \mathbf{B}(\mathcal{A})$ can be written as $x = x_1 - x_2 + i(x_3 - x_4)$, with $x_1, x_2, x_3, x_4 \in \mathbf{B}(\mathcal{A})_+$. Thus, $M_{a^{1/2}}(\mathbf{B}(\mathcal{A})_+)$ is not relatively weakly compact. However, $[0,a] \supset M_{a^{1/2}}(\mathbf{B}(\mathcal{A})_+)$. Indeed, if $0 \leq y \leq 1$, then $0 \leq a^{1/2} y a^{1/2} \leq a$. Therefore, $[0,a]$ is not relatively weakly compact. ∎

These results allow us to obtain new characterizations of compact C^*-algebras. Recall that a Banach algebra is called *compact* (or *dual*) if all of its elements are multiplication compact. By [9], compact C^*-algebras are precisely the algebras of the form $\mathcal{A} = (\sum_{i \in I} K(H_i))_{c_0}$, where each H_i is a complex Hilbert space, and $K(H)$ denotes the space of compact operators on H. Several alternative characterizations of compact C^*-algebras can be found in [26, 4.7.20].

Proposition 2.3.14. *For a C^*-algebra $\mathcal{A}$, the following four statements are equivalent.*

(i) *$\mathcal{A}$ is compact.*

(ii) *For any $c \in \mathcal{A}_+$, the order interval $[0,c]$ is compact.*

(iii) *For any $c \in \mathcal{A}_+$, the order interval $[0,c]$ is weakly compact.*

(iv) *For any relatively compact $M \subset \mathcal{A}_+$, $\mathbf{PSol}(M)$ is relatively compact.*

Proof. The implications (4) $\Rightarrow$ (2) $\Rightarrow$ (3) are immediate.

(3) $\Rightarrow$ (1): by Proposition 2.3.13, any positive $a \in \mathcal{A}$ is multiplication compact. By [13, Corollary 10.4], the map $\mathcal{A} \to \mathcal{A} : x \mapsto axb$ is compact for any $a, b \in \mathcal{A}_+$. As any $x \in \mathcal{A}$ is a linear combination of four positive elements, it is multiplication compact.

(1) $\Rightarrow$ (4): it suffices to show that, for any $\varepsilon > 0$, $\mathbf{PSol}(M)$ admits a finite ε-net. Assume, without loss of generality, that $M \subset \mathbf{B}(\mathcal{A})_+$. The map $\mathcal{A}_+ \to \mathcal{A}_+ : a \mapsto a^{1/4}$ is continuous, hence $M^{1/4} = \{a^{1/4} : a \in M\}$ is compact. Pick $(a_i)_{i=1}^n \subset M$ so that $(a_i^{1/4})_{i=1}^n$ is an $\varepsilon/4$-net in $M^{1/4}$. By Proposition 2.3.13, $a_i^{1/4}$ is multiplication compact for each i, hence $a_i^{1/4}\mathbf{B}(\mathcal{A})_+ a_i^{1/4}$ contains an $\varepsilon/4$-net $(b_{ij})_{j=1}^m$.

Now consider $x \in [0, a]$, for some $a \in M$. As noted in the proof of Proposition 2.3.13, there exists $u \in \mathbf{B}(\mathcal{A})$, so that $x = a^{1/4}u^*ua^{1/4}$. Pick i and j so that $\|a^{1/4} - a_i^{1/4}\| < \varepsilon/4$, and $\|a_i^{1/4}u^*ua_i^{1/4} - b_{ij}\| < \varepsilon/4$. Then

$$\|a^{1/4}u^*ua^{1/4} - b_{ij}\| \le \|(a_i^{1/4} - a^{1/4})u^*ua^{1/4}\|$$
$$+\|a_i^{1/4}u^*u(a_i^{1/4} - a^{1/4})\| + \|a_i^{1/4}u^*ua_i^{1/4} - b_{ij}\| < \varepsilon.$$

■

Recall that a C^*-subalgebra $\mathcal{A}$ of a C^*-algebra $\mathcal{B}$ is called *hereditary* if, for any $a \in \mathcal{A}_+$, we have $\{b \in \mathcal{B} : 0 \le b \le a\} \subset \mathcal{A}$.

Proposition 2.3.15. *A C^*-algebra $\mathcal{A}$ is a hereditary subalgebra of $\mathcal{A}^{\star\star}$ if and only if $\mathcal{A}$ is a compact C^*-algebra.*

Proof. If $\mathcal{A}$ is compact, then it is an ideal in $\mathcal{A}^{\star\star}$ [109]. It is well known (see e.g. [12, Proposition II.5.3.2]) that any ideal in a C^*-algebra is hereditary.

Now suppose $\mathcal{A}$ is a hereditary subalgebra of $\mathcal{A}^{\star\star}$. By [26, Exercise 4.7.20], it suffices to show that, for any $a \in \mathcal{A}_+$, any non-zero point of the spectrum of a is an isolated point. Suppose, for the sake of contradiction, that there exists $a \in \mathcal{A}_+$ whose spectrum contains a strictly positive non-isolated point α. In other words, for every $\delta > 0$, $((\alpha - \delta, \alpha) \cup (\alpha, \alpha + \delta)) \cap \sigma(a) \neq \emptyset$. Without loss of generality, we can assume $0 \leq a \leq 1$. Thus, we can find countably many mutually disjoint non-empty subsets S_i of $(\alpha/2, \infty) \cap \sigma(a)$. Denote the corresponding spectral projections by p_i (that is, $p_i = \chi_{S_i}(a)$). These projections belong to $\mathcal{A}^{\star\star}$. Furthermore, $p_i \leq (\inf S_i)^{-1} a$, hence, by the hereditary property, these projections belong to $\mathcal{A}$.

Now consider the linear map $T : \mathcal{A} \to \mathcal{A} : x \mapsto axa$. Then $T^{\star\star}$ is also implemented by $x \mapsto axa$. If $0 \leq x \leq \mathbf{1}$, then $axa \leq a^2$, hence $axa \in \mathcal{A}$. Therefore, $T^{\star\star}$ takes $\mathcal{A}^{\star\star}$ to $\mathcal{A}$. By Gantmacher's Theorem (see e.g. [6, Theorem 5.23]), T is weakly compact. However, T is an isomorphism on the copy of c_0, spanned by the projections p_i, leading to a contradiction. ∎

2.3.3 Positive Schur Property. Compactness of order intervals in Schatten spaces

An OBS X is said to have the *Positive Schur Property* (*PSP*) if every weakly null positive sequence is norm convergent to 0 and X has the *Super Positive Schur Property* (*SPSP*) if every positive weakly convergent sequence is norm convergent. Clearly, the Schur Property implies the SPSP, which, in turn, implies the PSP. Note that, if X has the SPSP, then, by the Eberlein-Smulian Theorem, any weakly compact subset of X_+ is compact.

The PSP and SPSP of Banach lattices have been investigated earlier. By [102], the Schur Property and the PSP coincide for atomic Banach lattices. In [56], it is shown that ℓ_1 is the only symmetric sequence space with the Schur

Property (by Remark 2.3.24 below, the symmetry assumption is essential). [57] gives a criterion for the PSP of Orlicz spaces.

Lemma 2.3.16. *Suppose $\mathcal{E}$ is a symmetric sequence space, and (A_n) is a positive bounded sequence in $\mathfrak{C}_{\mathcal{E}}$ without a convergent subsequence. Then there exist a subsequence (A_{n_k}) and $c > 0$ such that $\|R_k A_{n_k}\| > c$ for every k.*

Proof. Assume there is no such subsequence, that is

$$\lim_m \sup_n \|R_m A_n\| = 0.$$

Applying Lemma 2.3.1 when T is the identity operator, we obtain the inequality

$$\begin{aligned} \|A_n - Q_m A_n\| &\leqslant \|A_n - Q_m A_n - R_m A_n\| + \|R_m A_n\| \\ &\leqslant 2\|Q_m A_n\|^{\frac{1}{2}} \|R_m A_n\|^{\frac{1}{2}} + \|R_m A_n\|. \end{aligned}$$

Thus, $\lim_m \sup_n \|A_n - Q_m A_n\| = 0$. However, Q_m is a finite rank map, hence the set (A_n) is relatively compact, a contradiction. ■

Proposition 2.3.17. *Suppose $\mathcal{E}$ is a separable symmetric sequence space. Let (A_n) be a weakly null positive sequence in $\mathfrak{C}_{\mathcal{E}}(H)$, which contains no convergent subsequences. Then there exists $c > 0$ with the property that, for any $\varepsilon \in (0, 1)$, there exist sequences $1 = n_1 < n_2 < \ldots$ and $0 = m_0 < m_1 < \ldots$, so that $\inf_k \|A_{n_k}\| > c$, and*

$$\sum_k \|A_{n_k} - (P_{m_k} - P_{m_{k-1}}) A_{n_k} (P_{m_k} - P_{m_{k-1}})\| < \varepsilon.$$

Consequently, the sequence (A_{n_k}) is equivalent to a disjoint sequence of positive finite dimensional operators.

Proof. By the separability (equivalently, order continuity) of $\mathcal{E}$, there exists a projection $p \in B(H)$ with separable range, so that $pA_kp = A_k$ for any k. Thus, it suffices to prove our proposition in $\mathfrak{C}_{\mathcal{E}}$.

Furthermore, the order continuity of $\mathcal{E}$ implies that the finite rank operators are dense in $\mathfrak{C}_{\mathcal{E}}$. It is easy to see that, for any rank 1 operator u, $\lim_n \|u - Q_n u\| = 0$. Thus, $\lim_n \|x - Q_n x\| = 0$ for any $x \in \mathcal{E}$.

By scaling, we can assume $\sup_n \|A_n\| = 1$. Applying Lemma 2.3.16, and passing to a subsequence if necessary, we may assume that $\|R_n A_n\| > c$, for some positive number c. We construct the sequences (n_k) and (m_k) recursively. Set $n_1 = 1$ and $m_0 = 0$. As noted above, there exists $m_1 > m_0$ so that $\|A_{n_1} - P_{m_1} A_{n_1} P_{m_1}\| < \varepsilon/2$.

Suppose we have already selected $0 = m_0 < m_1 < \ldots < m_j$ and $1 = n_1 < n_2 < \ldots < n_j$ so that, for $1 \leq j \leq k$,

$$\|A_{n_k} - (P_{m_k} - P_{m_{k-1}}) A_{n_k} (P_{m_k} - P_{m_{k-1}})\| < 2^{-j}\varepsilon.$$

As Q_m is a finite rank operator for any m, and the sequence (A_n) is weakly null, $\lim_n \|Q_m A_n\| = 0$. Consequently, there exists $n_{k+1} > n_k$ so that $\|Q_{m_k} A_{n_{k+1}}\| < 2^{-2(k+1)-4}\varepsilon^2$. Then

$$\begin{aligned}\|A_{n_{k+1}} - R_{m_k} A_{n_{k+1}}\| &\leq \|A_{n_{k+1}} - R_{m_k} A_{n_{k+1}} - Q_{m_k} A_{n_{k+1}}\| + \|Q_{m_k} A_{n_{k+1}}\| \\ &\leq 2\|Q_{m_k} A_{n_{k+1}}\|^{1/2} \|R_{m_k} A_{n_{k+1}}\|^{1/2} + \|Q_{m_k} A_{n_{k+1}}\| < 2^{-(k+2)}\varepsilon.\end{aligned}$$

Now find m_{k+1} so that $\|R_{m_k} A_{n_{k+1}} - Q_{m_{k+1}} R_{m_k} A_{n_{k+1}}\| < 2^{-(k+2)}\varepsilon$. ■

Proposition 2.3.18. *For any Hilbert space H, $\mathfrak{C}_1(H)$ has the SPSP.*

Proof. It suffices to consider the case of infinite dimensional H. Suppose $A_0, A_1, A_2, \ldots$ are positive elements of $cs_1(H)$, and $A_n \to A_0$ weakly. Then there exist projections $p_0, p_1, p_2, \dots$ with separable range, so that $p_i A_i p_i = A_i$ for every i. Then $p = \vee_{i \geq 0} p_i$ has separable range, and $pA_i p = A_i$ for every i. Thus, we can assume that $H = \ell_2$.

By Lemma 2.3.16 there exist $c > 0$ and a subsequence such that $\|R_k A_{n_k}\| > c$. Since $R_m \geqslant R_k$ when $m \leqslant k$, we have $\mathrm{tr}(R_m A_{n_k}) > c$ for every k. On the other hand we can always pick m such that $\mathrm{tr}(R_m A) = \|R_m A\| < c$. This contradicts $A_n \to A$ weakly. ■

For Schatten spaces, Lemma 2.3.7 immediately implies:

Proposition 2.3.19. *Suppose $\mathcal{E}$ is a separable strongly symmetric noncommutative sequence space. Then any order interval in $\mathcal{E}$ is compact.*

Remark 2.3.20. (1) For $\mathcal{E} = \mathcal{S}_1$, this result has been known (see e.g. [95, Corollary III.5.11]). (2) As noted above, for symmetric sequence spaces order continuity is equivalent to separability.

Proposition 2.3.21. *Suppose $\mathcal{E}$ is a strongly symmetric sequence space, and H is an infinite dimensional Banach space. Then the following are equivalent:*

(i) $\mathcal{E} = \ell_1$.

(ii) $\mathcal{E}$ *has the Schur Property.*

(iii) $\mathcal{E}$ *has the PSP.*

(iv) $\mathcal{E}$ *has the SPSP.*

(v) $\mathfrak{C}_{\mathcal{E}}(H)$ *has the PSP.*

(vi) $\mathfrak{C}_{\mathcal{E}}(H)$ *has the SPSP.*

Proof. (1) ⇒ (2) is well known. The implications (2) ⇒ (4) ⇒ (3), (6) ⇒ (4), and (6) ⇒ (5) ⇒ (3) are obvious. (1) ⇒ (6) follows from Proposition 2.3.18.

(3) ⇒ (1). Assume that basis (e_n) of $\mathcal{E}$ is not equivalent to the canonical basis of ℓ_1. By symmetry, (e_n) contains no subsequence equivalent to the canonical basis of ℓ_1. By Rosenthal's dichotomy, the sequence (e_n) is weakly null, which contradicts the PSP. ■

We complete this section by (partially) describing Banach lattices possessing various versions of the Schur Property.

Proposition 2.3.22. *Any Banach lattice E with the SPSP is atomic.*

Recall that a Banach lattice is called *atomic* if it is the band generated by its atoms.

Proof. Clearly, a Banach lattice with the SPSP cannot contain a lattice copy of c_0. Theorems 2.4.12 and 2.5.6 of [67] show that E is a KB-space. In particular, E is order continuous. By [64, Proposition 1.a.9], without loss of generality, we may assume E is atomless and has a weak unit. Therefore, by [64, Theorem 1.b.4], there exists an atomless probability measure space (Ω, μ), so that $L_\infty(\mu) \subset E \subset L_1(\mu)$. Suppose, furthermore, that $e \in E_+ \backslash \{0\}$. Find $S \subset \Omega$ of finite measure, so that $e\chi_S > \alpha\chi_S$ for some positive number α. By the proof of [21, Proposition 2.1], there exists a weakly null sequence (f_n), so that $|f_n| = 1$ μ-a.e. on S, $f_n = 0$ on $\Omega \backslash S$, and $f_n \to 0$ in $\sigma(L_\infty(\mu), L_1(\mu))$. Letting $e_n = e + f_n$, we conclude that $e_n \geq 0$ for every n, and $e_n \to e$ weakly, but not in norm. ■

Proposition 2.3.23. *For any order continuous Banach lattice E the SPSP, the PSP, and the Schur Property are equivalent.*

Proof. Proposition 2.3.22 implies E is atomic. Therefore the result follows from the fact that the lattice operations are weakly sequentially continuous, see [67, Proposition 2.5.23]. ∎

Remark 2.3.24. An order continuous atomic Banach lattice with the Schur Property need not be isomorphic to ℓ_1, even as a Banach space. Indeed, suppose (E_n) is a sequence of finite dimensional lattices. Then $E = (\sum_{n=1}^{\infty} E_n)_{\ell_1}$ has the Schur Property. If, for instance, $E_n = \ell_2^n$, E is not isomorphic to ℓ_1. We do not know of any Banach lattice with the Schur Property which is not isomorphic to an ℓ_1 sum of finite dimensional spaces.

2.3.4 Compactness of order intervals in preduals of von Neumann algebras

Following [95, Definition III.5.9], we say that a von Neumann algebra $\mathcal{A}$ is *atomic* if every projection in $\mathcal{A}$ has a minimal Abelian subprojection. Note that $\mathcal{A}$ is atomic iff it is isomorphic to $(\sum_{i\in I} B(H_i))_{\ell_\infty(I)}$, for some index set I, and collection of Hilbert spaces $(H_i)_{i\in I}$. Indeed, any von Neumann algebra of the above form is atomic. To prove the converse, note that an atomic algebra must be of type I. Moreover, it can be written as $\mathcal{A} = (\sum_{j\in J} \mathcal{A}_j)_{\ell_\infty(J)}$, where $\mathcal{A}_j$ is an atomic algebra of type I_j. By [95, Theorem V.1.27] (see also [53, Theorem 6.6.5] and [12, III.1.5.3]), $\mathcal{A}_j$ is isomorphic to $\mathcal{C}_j \overline{\otimes} B(H_j)$, where $\mathcal{C}_j$ is the center of $\mathcal{A}_j$. Denote the set of all minimal projections in $\mathcal{C}_j$ by F_j. Then the elements of F_j are mutually orthogonal, and their join equals the identity of $\mathcal{C}_j$. Thus, $\mathcal{C}_j$ is isomorphic to $\ell_\infty(F_j)$. Alternatively, one could use

[12, III.1.5.18] and its proof to show that $\mathcal{C}_j$ is an ℓ_∞ space.

Theorem 2.3.25. *For a von Neumann algebra $\mathcal{A}$, the following are equivalent:*

(i) $\mathcal{A}$ is an atomic von Neumann algebra.

(ii) $\mathcal{A}_\star$ has the SPSP.

(iii) All order intervals in $\mathcal{A}_\star$ are compact.

Note that the predual of any von Neumann algebra has the PSP. Indeed, suppose (f_n) is a sequence of positive elements of $A_\star$, converging weakly to 0. Then $\|f_n\| = \langle f_n, \mathbf{1}\rangle$, hence $\lim_n \|f_n\| = \lim_n \langle f_n, \mathbf{1}\rangle = 0$.

The following auxiliary result may be known to experts. However, we have not been able to find it in the literature.

Lemma 2.3.26. *Any order interval in the predual of a von Neumann algebra is weakly compact.*

Proof. Suppose f is a positive element of $\mathcal{A}_\star$. Then $[0, f]$ is convex and closed. For any $g \in [0, f]$ and $a \in \mathcal{A}$, Cauchy-Schwarz Inequality [95, Proposition I.9.5] yields $|g(a)|^2 \leq g(\mathbf{1})g(a^*a) \leq f(\mathbf{1})f(a^*a)$. By [95, Theorem III.5.4], $[0, f]$ is relatively weakly compact. ∎

Proof of Theorem 2.3.25. If (1) holds, then $\mathcal{A} = (\sum_i B(H_i))_\infty$, hence $A_\star = (\sum_i \mathcal{S}_1(H_i))_1$. (2) and (3) follow from Propositions 2.3.21 and 2.3.19, respectively.

Now suppose $\mathcal{A}$ is not atomic. Write $\mathcal{A} = \mathcal{A}_I \oplus \mathcal{A}_{II} \oplus \mathcal{A}_{III}$, where $\mathcal{A}_I$, $\mathcal{A}_{II}$, and $\mathcal{A}_{III}$ are the summands of type I, II, and III, respectively. Then either $\mathcal{A} = \mathcal{A}_{II} \oplus \mathcal{A}_{III}$ is non-trivial, or $\mathcal{A}_I$ is not atomic.

(i) If $\mathcal{A}_I$ is not an atomic von Neumann algebra, write $\mathcal{A}_I = (\sum_{s \in S} \mathcal{A}_s)_{\ell_\infty(S)}$, with $\mathcal{A}_s = \mathcal{C}_s \overline{\otimes} B(H_s)$ ($\mathcal{C}_s$ is the center of $\mathcal{A}_s$). By [95, Theorem III.1.18], $\mathcal{C}_s$ is isomorphic to $L_\infty(\nu_s)$, for some locally finite measure ν_s. Consequently, $\mathcal{A}_\star$ contains $L_1(\nu_s) \otimes \mathcal{S}_1(H_s)$ as a positively and completely contractively complemented subspace. If $\mathcal{A}_I$ is not an atomic von Neumann algebra, then ν_s is not a purely atomic measure, for some s. By the above, $\mathcal{A}_\star$ contains $L_1(\nu_s) \otimes \mathcal{S}_1(H_s)$ as a positively and completely contractively complemented subspace. Furthermore, $L_1(\nu_s)$ is complemented in $L_1(\nu_s) \otimes \mathcal{S}_1(H_s)$ via a positive projection Q: just pick a rank one projection $e \in B(H_s)$, and set $Q(x) = (I_{L_1(\nu_s)} \otimes e)x(I_{L_1(\nu_s)} \otimes e)$. Finally, $L_1(\nu_s)$ contains a positively complemented copy of $L_1(0,1)$. Indeed, we can represent $L_1(\nu_s)$ a direct sum of spaces $L_1(\sigma_i)$, where σ_i is a finite measure. Since ν_s is not purely atomic, the same is true for $L_1(\sigma_i)$, for some i. By [95, Theorem III.1.22] (or [53, Theorem 9.4.1]), $L_1(\nu_s)$ contains a positively complemented copy of $L_1(0,1)$.

To finish the proof in this case, note that $L_1(0,1)$ fails the SPSP, and has non-compact order intervals. Indeed, let $f = \mathbf{1}$, and $f_n = \mathbf{1} + r_n$, where $r_1, r_2, \ldots$ are Rademacher functions. Then $f_n \to f$ weakly, but not in norm. This witnesses the failure of the SPSP. Moreover, $f_n/2 \in [0, \mathbf{1}]$, hence the order interval $[0, \mathbf{1}]$ is not compact.

(ii) Now suppose $\mathcal{A}_0 = \mathcal{A}_{II} \oplus \mathcal{A}_{III}$ is non-trivial. Let $\mathcal{B}$ be a MASA in $\mathcal{A}_0$ (hence a von Neumann subalgebra). As noted above, $\mathcal{B}$ is isomorphic (in the von Neumann algebra sense) to $L_\infty(\Omega, \nu)$, where ν is a locally finite measure. Moreover, $\mathcal{A}_0$ has no minimal projections, hence ν is non-atomic. Therefore, we can find $\Omega_0 \subset \Omega$, so that $L_\infty(\Omega_0, \nu)$ is isomorphic to $L_\infty(0,1)$. Then there exists a von Neumann algebra isomorphism $J : L_\infty(0,1) \to \mathcal{C} \subset \mathcal{B}$.

Define $\phi : \mathcal{C} \to \mathbb{C}$ by setting $\phi(x) = \int_0^1 J^{-1}(x)$. Clearly, ϕ is a norm one positive normal functional. By Hahn-Banach Theorem, ϕ has a norm 1 extension to a functional ψ on $\mathcal{A}_0$. By [95, Lemma III.3.2], ψ is positive. Moreover, $\mathcal{A}_{0\star}$ is L-embedded into its second dual, hence ψ is normal.

Show first that the order interval $[0, \psi]$ is not compact. To this end, let (r_n) be the sequence of Rademacher functions on $(0,1)$, and let $x_n = J(\mathbf{1} + r_n)/2$. For $a \in \mathcal{A}_0$, let $\psi_n(a) = \psi(x_n a x_n)$. Clearly, $0 \leq \psi_n \leq \psi$, for any n. However, for $n \neq m$, $\|\psi_n - \psi_m\| \geq 1/4$. Indeed, it is easy to observe that $\psi_n(x_n) = 8^{-1} \int_0^1 (\mathbf{1} + r_n)^3 = 1/2$, while for $n \neq m$, $\psi_m(x_n) = 8^{-1} \int_0^1 (\mathbf{1} + r_m)^2 (\mathbf{1} + r_n) = 1/4$.

By Lemma 2.3.26, $[0, \psi]$ is weakly compact, hence the sequence (ψ_n) has a weakly convergent subsequence. This witnesses the failure of the SPSP. ■

Complementing Theorem 2.3.25, we prove that $\mathcal{A}_\star$ contains an order copy of $L_1(0,1)$, complemented via a positive projection.

Proposition 2.3.27. *Suppose $\mathcal{A}$ is a von Neumann algebra, whose summands of types I, II, and III are denoted by $\mathcal{A}_I$, $\mathcal{A}_{II}$, and $\mathcal{A}_{III}$, respectively. Suppose at least one of the three conditions holds:* (*i*) *$\mathcal{A}_I$ is not atomic;* (*ii*) *$\mathcal{A}_{II}$ is non-trivial;* (*iii*) *$\mathcal{A}_{III}$ is non-trivial, and has separable predual. Then there exists an order isometry $j : L_1(0,1) \to \mathcal{A}_\star$, and a positive projection $P : \mathcal{A}_\star \to \mathrm{ran}\,(j)$.*

Proof. The case of $\mathcal{A}_I$ being non-atomic has been dealt with in the proof of Theorem 2.3.25.

Now suppose $\mathcal{A}_{II}$ is non-trivial. By [65], $\mathcal{A}_{II}$ contains an "appropriately embedded" copy of the hyperfinite II_1 factor $\mathcal{R}$, which (by [95, Theorem

V.2.36]) is the range of a weak* continuous projection. It therefore suffices to show that there exists an isometry $J : L_1(\mu) \to \mathcal{R}_\star$, so that the range of J is the range of a positive projection. Here, μ is the "canonical" measure on the Cantor set Δ, defined as follows: represent $\Delta = \{0,1\}^{\mathbb{N}}$, and write $\mu = \nu^{\mathbb{N}}$, where the measure ν on $\{0,1\}$ satisfies $\nu(0) = \nu(1) = 1/2$. For $\alpha = (i_1, \ldots, i_n) \in I = \{0,1\}^{<\mathbb{N}}$, define the function f_α by setting $f_\alpha(j_1, j_2, \ldots) = \prod_{k=1}^n \delta_{i_k, j_k}$ (here, $\delta_{i,j}$ stands for Kronecker's delta). Note that f_α and f_β have disjoint supports if α and β are different bit strings of the same length. Moreover, $f_\alpha = f_{(\alpha,0)} + f_{(\alpha,1)}$. Clearly, $L_1(\mu)$ is the closed linear span of the functions f_α.

We let $\Delta_n = \{0,1\}^n$, and denote by μ_n the product of n copies of ν. In this notation, $L_1(\mu_n)$ is isometric to $\ell_1^{2^n}$. We can also identify $L_1(\mu_n)$ with $\mathrm{span}[f_\alpha : |\alpha| = n]$. Let i_n be the formal identity $L_1(\mu_{n-1}) \to L_1(\mu_n)$ (taking f_α to itself, when $|\alpha| \leq n$).

For $n \in \mathbb{N}$, consider the map $j_n : M_{2^{n-1}} \to M_{2^n} : x \mapsto x \otimes M_2$. Denote by Tr_n the normalized trace on M_{2^n}, and by $M_{2^n}^\star$ the dual of M_{2^n} defined using Tr_n. Then $j_n : M_{2^{n-1}}^\star \to M_{2^n}^\star$ is an isometry. Furthermore, the diagonal embedding $u_n : L_1(\mu_n) \to M_{2^n}^\star$ is an isometry, and $u_n i_n = j_n u_{n-1}$. We can view both $M_{2^{n-1}}^\star$ and $L_1(\mu_n)$ as subspaces of $M_{2^n}^\star$, Furthermore, for any n there exist positive contractive unital projections $p_n : M_{2^n}^\star \to L_1(\mu_n)$ and $q_n : M_{2^n}^\star \to M_{2^{n-1}}^\star$ (the "diagonal" and "averaging" projections, respectively). We then have $p_n j_n = i_n p_{n-1}$.

It is well known (see e.g. [79, Theorem 3.4]) that $\mathcal{R}_\star$ can be viewed as $\overline{\cup_n M_{2^n}^\star}$. Moreover, for any n there exists a positive contractive unital projection $\tilde{q}_n : \mathcal{R}_\star \to M_{2^n}^\star$ (with $\tilde{q}_n|_{M_{2^N}^\star} = q_{n+1} \ldots q_N$). Now identify $L_1(\mu)$

with $\overline{\cup_n L_1(\mu_n)}$, and define the projection P by setting $P|_{M_{2n}^\star} = q_n$.

(iii) Suppose $\mathcal{A}$ is a type III von Neumann algebra with separable predual. By [89], it contains a weak*-closed subalgebra C, isomorphic of $L_\infty(0,1)$. Moreover, there exists a weak* continuous contractive conditional expectation σ from B to C (see e.g. [12, Section II.6.10] or [53, pp. 187-189] for properties of conditional expectations). Then $\rho_\star \circ \sigma_\star$ yields an order-preserving isometric embedding of $L_1(0,1)$ to $\mathcal{A}_\star$. ■

2.3.5 Compact operators on noncommutative function spaces

First we consider maps from ordered Banach spaces into Schatten spaces.

Proposition 2.3.28. *Suppose $\mathcal{E}$ is a separable symmetric sequence space, H is a Hilbert space, A is a generating OBS, and $0 \leqslant T \leqslant S : A \to \mathcal{S}_{\mathcal{E}}(H)$. If S is compact, then T is compact.*

Proof. It is enough to show $T(\mathbf{B}(A)_+)$ is relatively compact. Thus follows from Lemma 2.3.7, since $T(\mathbf{B}(A)_+) \subseteq \mathbf{PSol}(S(\mathbf{B}(A)_+))$. ■

For operators into Schatten spaces, we have:

Proposition 2.3.29. *Suppose $\mathcal{E}$ is a separable symmetric sequence space, and H is a Hilbert space.*

(1) *If $\mathcal{E}$ does not contain ℓ_1, and operators T and S from $\mathfrak{C}_{\mathcal{E}}(H)$ to a normal OBS Z satisfy $0 \leq T \leq S$, then the compactness of $S^\star$ implies the compactness of $T^\star$.*

(2) *Conversely, suppose $\mathcal{E}$ contains ℓ_1, and a Banach lattice Z is either not atomic, or not order continuous. Then there exist $0 \leq T \leq S : \mathfrak{C}_{\mathcal{E}}(H) \to Z$ so that S is compact, but T is not.*

Proof. (1) By [63, Theorem 1.c.9], $\mathcal{E}^\star$ is separable. Now apply Proposition 2.3.28.

(2) By [100], there exist $0 \leq \tilde{T} \leq \tilde{S} : \ell_1 \to Z$ so that $\tilde{S}$ is compact, but $\tilde{T}$ is not. By Proposition 2.3.6, there exists a lattice isomorphism $j : \ell_1 \to \mathfrak{C}_{\mathcal{E}}$, and a positive projection P from $\mathfrak{C}_{\mathcal{E}}$ onto $j(\ell_1)$. Then the operators $T = \tilde{T} j^{-1} P$ and $S = \tilde{S} j^{-1} P$ have the desired properties. ■

Finally we deal with operators on general noncommutative function spaces.

Proposition 2.3.30. *Suppose $\mathcal{E}$ is a strongly symmetric noncommutative function space, such that $\mathcal{E}^\times$ is not order continuous. Suppose, furthermore, that a symmetric noncommutative function space $\mathcal{F}$ contains non-compact order intervals. Then there exist $0 \leq T \leq S : \mathcal{E} \to \mathcal{F}$, so that S has rank 1, and T is not compact.*

Note that many spaces $\mathcal{F}$ contain non-compact order intervals. Suppose, for instance, that $\mathcal{F}$ arises from a von Neumann algebra $\mathcal{A}$ that is not atomic, and is equipped with a normal faithful semifinite trace τ. Using the type decomposition, we can find a projection $p \in \mathcal{A}$ with a finite trace. Then the interval $[0, p]$ is not compact. Indeed, [95, Proposition V.1.35] allows us to construct a family of projections (p_{ni}) ($n \in \mathbb{N}$, $1 \leq i \leq 2^n$), so that (i) $p = p_{11} + p_{12}$, and $p_{ni} = p_{n+1,2i-1} + p_{n+1,2i}$ for any n and i, and (ii) all projections p_{ni} are equivalent. Then the family $q_n = \sum_{i=1}^{2^{n-1}} p_{n,2i}$ is a sequence in $[0, p]$, with no convergent subsequences.

Note that, for fully symmetric noncommutative sequence spaces, order continuity is fully described by Corollary 2.3.10.

Lemma 2.3.31. *Suppose $\mathcal{E}$ is a strongly symmetric noncommutative function space, so that $\mathcal{E}^\times$ is not order continuous. Then there exists an isomorphism $j : \ell_1 \to \mathcal{E}$, so that both j and j^{-1} are positive, and $j(\ell_1)$ is the range of a positive projection.*

Proof. By [29], $\mathcal{E}^\times$ is fully symmetric. By Lemma 2.3.11, there exists $x \in \mathbf{B}(\mathcal{E}^\times)_+$, and a sequence of mutually orthogonal projections (e_i), so that $(\alpha_i) \mapsto \sum \alpha_i e_i x e_i$ determines a positive embedding of ℓ_∞ into $\mathcal{E}^\times$. For each i, find $y_i \in \mathcal{E}_+$ so that $e_i y_i e_i = y_i$, $\|y_i\| < 2\|e_i x e_i\|^{-1}$, and $\langle e_i x e_i, y_i \rangle = 1$. The map $j : \ell_1 \to \mathcal{E} : (\alpha_i) \mapsto \sum_i \alpha_i y_i$ determines a positive isomorphism. Furthermore, define $U : \mathcal{E} \to \ell_1 : y \mapsto (\langle e_i x e_i, y \rangle)_i$. Clearly, U is a bounded positive map, and $Uj = I_{\ell_1}$. Therefore, jU is a positive projection onto $j(\ell_1)$. ■

Proof of Proposition 2.3.30. In view of Lemma 2.3.31, it suffices to construct $0 \leq T \leq S : \ell_1 \to \mathcal{F}$, so that S has rank 1, and T is not compact. Pick $y \in \mathcal{F}$, so that $[0, y]$ is not compact. Then find a sequence $(y_i) \subset [0, y]$, without convergent subsequences. Denote the canonical basis of ℓ_1 by (δ_i). Let $\delta_i^\star$ be the biorthogonal functionals in ℓ_∞. Following [100], define S and T by setting $S\delta_i = y$, and $T\delta_i = y_i$. In other words, for $a = (\alpha_i) \in \ell_1$, $Sa = \langle \mathbf{1}, a \rangle y$, and $Ta = \sum_i \langle \delta_i^\star, a \rangle y_i$. It is easy to see that $\operatorname{rank} S = 1$, and $0 \leq T \leq S$. Moreover, $T(\mathbf{B}(\ell_1))$ contains the non-compact set $\{y_1, y_2, \ldots\}$, hence T is not compact. ■

2.3.6 Compact operators on C^*-algebras and their duals

In this section, we determine the C^*-algebras $\mathcal{A}$ with the property that every operator on $\mathcal{A}$, dominated by a compact operator, is itself compact. First we introduce some definitions. Let $\mathcal{A}$ be a C^*-algebra, and consider $f \in \mathcal{A}^\star$. Let $e \in \mathcal{A}^{\star\star}$ be its support projection. Following [50], we call f *atomic* if every non-zero projection $e_1 \leq e$ dominates a minimal projection (all projections are assumed to "live" in the enveloping algebra $\mathcal{A}^{\star\star}$). Equivalently, f is a sum of pure positive functionals. We say that $\mathcal{A}$ is *scattered* if every positive functional is atomic. By [49], [50], the following three statements are equivalent: (i) $\mathcal{A}$ is scattered; (ii) $\mathcal{A}^{\star\star} = (\sum_{i \in I} B(H_i))_\infty$; (iii) the spectrum of any self-adjoint element of $\mathcal{A}$ is countable. Consequently (see [26, Exercise 4.7.20]), any compact C^*-algebra is scattered. In [104], it is proven that a separable C^*-algebra has separable dual if and only if it is scattered.

The main result of this section is:

Theorem 2.3.32. *Suppose $\mathcal{A}$ and $\mathcal{B}$ are C^*-algebras, and E is a generating OBS.*

(i) Suppose $\mathcal{A}$ is a scattered. Then, for any $0 \leq T \leq S : E \to \mathcal{A}^\star$, the compactness of S implies the compactness of T.

(ii) Suppose $\mathcal{B}$ is a compact. Then, for any $0 \leq T \leq S : E \to \mathcal{B}$, the compactness of S implies the compactness of T.

(iii) Suppose $\mathcal{A}$ is not scattered, and $\mathcal{B}$ is not compact. Then there exist $0 \leq T \leq S : \mathcal{A} \to \mathcal{B}$, so that S has rank 1, while T is not compact.

From this, we immediately obtain:

Corollary 2.3.33. *Suppose $\mathcal{A}$ and $\mathcal{B}$ are C^*-algebras. Then the following are equivalent:*

(i) At least one of the two conditions holds: (i) $\mathcal{A}$ is scattered, (ii) $\mathcal{B}$ is compact.

(ii) If $0 \leq T \leq S : \mathcal{A} \to \mathcal{B}$, and S is compact, then T is compact.

It is easy to see that a von Neumann algebra is scattered if an only if it is finite dimensional if and only if it is compact. This leads to:

Corollary 2.3.34. *If von Neumann algebras $\mathcal{A}$ and $\mathcal{B}$ are infinite dimensional, then there exist $0 \leq T \leq S : \mathcal{A} \to \mathcal{B}$, so that S has rank 1, while T is not compact.*

We establish similar results about preduals of von Neumann algebras.

Lemma 2.3.35. *(1) Suppose $\mathcal{A}$ is an atomic von Neumann algebra, and E is a generating OBS. Then $0 \leq T \leq S : E \to \mathcal{A}_\star$, where S is a compact operator, implies T is compact.*

(2) Suppose $\mathcal{A}$ is a von Neumann algebra, and $\mathcal{A}_I, \mathcal{A}_{II}, \mathcal{A}_{III}$ are its summands of type I, II, and III, respectively. Suppose, furthermore, that one of the three statements is true: (i) $\mathcal{A}_I$ is not atomic, (ii) $\mathcal{A}_{II}$ is not empty, (iii) $\mathcal{A}_{III}$ is non-empty, and has separable predual. Then there exists $0 \leq T \leq S : \mathcal{A}_\star \to \mathcal{A}_\star$, so that S is compact, and T is not.

Proof. (1) The weak compactness of S implies, by Theorem 2.3.44, the weak compactness of T. By Theorem 2.3.25, $\mathcal{A}_\star$ has the SPSP, hence $T(\mathbf{B}(E)_+)$

is relatively compact. Thus, $T(\mathbf{B}(E))$ is relatively compact as well, hence T is compact.

(2) It suffices to show that there exists an order isomorphism $j : L_1(0,1) \to \mathcal{A}_\star$, so that there exists a positive projection P onto $\mathrm{ran}\,(j)$. Indeed, by [100], there exist operators $0 \leq T_0 \leq S_0 : L_1(0,1) \to L_1(0,1)$, so that S_0 is compact, and T_0 is not. Then $T = jT_0j^{-1}P$ and $S = jS_0j^{-1}P$ have the desired properties. The existence of j and P as above follows from the proof of Proposition 2.3.27. ∎

To establish Theorem 2.3.32, we need a series of lemmas.

Lemma 2.3.36. *Suppose $\mathcal{A}$ is a C^*-algebra for which $\mathcal{A}^\star$ has non-compact order intervals, and a Banach lattice E is not order continuous. Then there exist $0 \leq T \leq S : \mathcal{A} \to E$, so that S has rank 1, while T is not compact.*

Proof. By [67, Theorem 2.4.2], there exists $y \in E_+$, and normalized elements $y_1, y_2, \ldots \in [0, y]$ with disjoint supports. By our assumption there exist $\psi \in \mathcal{A}^\star_+$ and a sequence $(\phi_i) \subset [0, \psi]$ which does not have convergent subsequences. By Alaoglu's theorem we may assume $\phi_i \to \phi$ in weak* topology. Define two operators via

$$Sx = \psi(x)y \text{ and } Tx = \phi(x)y + \sum_{n=1}^{\infty} (\phi_n - \phi)(x)y_n.$$

Note that T is well defined: $(\phi_n - \phi)(x) \to 0$ for all x, hence

$$\|\sum_{n=m+1}^{k} (\phi_n - \phi)(x)y_n\| \leq \sup_{m>n} |(\phi_m - \phi)(x)| \|y\| \underset{n\to\infty}{\longrightarrow} 0.$$

Moreover, for any $x > 0$ and $N \in \mathbb{N}$ we have

$$\phi(x)y + \sum_{n=1}^{N} (\phi_n - \phi)(x)y_n = \phi(x)(y - \sum_{n=1}^{N} y_n) + \sum_{n=1}^{N} \phi_n(x)y_n \geqslant 0,$$

and

$$\psi(x)y - \phi(x)y - \sum_{n=1}^{N}(\phi_n - \phi)(x)y_n =$$
$$\psi(x)y - \sum_{n=1}^{n}\phi_n(x)y_n - \phi(x)\big(y - \sum_{n=1}^{N} y_n\big) \geq$$
$$\big(\psi(x) - \phi(x)\big)\big(y - \sum_{n=1}^{N} y_n\big).$$

By sending N to infinity, we obtain that $0 \leqslant Tx \leqslant Sx$ for every $x > 0$. Clearly, $\operatorname{rank} S = 1$. It remains to show that $T^\star$ is not compact. Note that there exist norm one $f_1, f_2, \ldots \in E^\star$ so that $f_n(y_m) = \delta_{nm}$. It is easy to see that $T^\star f = f(y)\phi + \sum_{n=1}^{\infty} f(y_n)(\phi_n - \phi)$, hence $T^\star f_m = (f_m(y) - 1)\phi + \phi_m$. The sequence $(T^\star f_m)$ has no convergent subsequences, since if it had, (ϕ_m) would have a convergent subsequence, too. This rules out the compactness of $T^\star$. ■

Corollary 2.3.37. *Suppose a C^*-algebra $\mathcal{B}$ is not compact, and $\mathcal{A}^\star$ has non-compact order intervals. Then there exist $0 \leq T \leq S : \mathcal{A} \to \mathcal{B}$, so that S has rank 1, while T is not compact.*

Proof. By Lemma 2.3.36, it suffices to show that $\mathcal{B}$ contains a Banach lattice which is not order continuous. By [26, Exercise 4.7.20], $\mathcal{B}$ contains a positive element b, whose spectrum contains a positive non-isolated point. Then the abelian C^*-algebra $\mathcal{B}_0$, generated by b, is not order continuous. Indeed, suppose $\alpha > 0$ is not an isolated point of $\sigma(a)$. Then there exist disjoint subintervals $I_i = (\beta_i, \gamma_i) \subset (\alpha/2, 3\alpha/2)$, so that $\delta_i = (\beta_i + \gamma_i)/2 \in \sigma(b)$ for every $i \in \mathbb{N}$. For each i, consider the function σ_i, so that $\sigma_i(\beta_i) = \sigma_i(\gamma_i) = 0$, $\sigma_i((\beta_i+\gamma_i)/2) = 1$, and σ_i is defined by linearity elsewhere. Then the elements $y_i = \sigma_i(b)$ belongs to $\mathcal{B}_0$, are disjoint and normalized, and $y_i \leq y = 2\alpha^{-1}b$. ■

Proof of Theorem 2.3.32. (1) If $\mathcal{A}$ is scattered, then $\mathcal{A}^{\star\star}$ is atomic. Now invoke Lemma 2.3.35(1).

(2) By assumption, $M = S(\mathbf{B}(E)_+)$ is relatively compact, and $T(\mathbf{B}(E)_+) \subset \mathbf{PSol}(M)$. By Proposition 2.3.14, $T(\mathbf{B}(E)_+)$ is relatively compact.

(3) Combine Theorem 2.3.25 with Corollary 2.3.37. ∎

2.3.7 Comparisons with multiplication operators

Suppose $\mathcal{A}$ is a C^*-subalgebra of $B(H)$, where H is a Hilbert space. For $x \in B(H)$ we define an operator $M_x : \mathcal{A} \to B(H) : a \mapsto x^*ax$. In this section, we study domination of, and by, multiplication operators, in relation to compactness. First, record some consequences of the results from Section 2.3.2.

Proposition 2.3.38. *Suppose x is an element of a C^*-algebra $\mathcal{A}$.*

(i) If M_x is weakly compact, and $0 \leq T \leq M_x : \mathcal{A} \to \mathcal{A}$, then T is compact.

(ii) If $0 \leq M_x \leq S : \mathcal{A} \to \mathcal{A}$, and S is weakly compact, then M_x is compact.

Proof. By passing to the second adjoint if necessary, we can assume $\mathcal{A}$ is a von Neumann algebra. Note that $[0, x^*x] = M_x(\mathbf{B}(\mathcal{A})_+)$. Indeed, if $a \in \mathbf{B}(\mathcal{A})_+$, then $0 \leq a \leq \mathbf{1}$, hence $0 \leq M_x a \leq M_x \mathbf{1} = x^*x$, hence $M_x a \in [0, x^*x]$. Next show that any $b \in [0, x^*x]$ belongs to $M_x a \in [0, x^*x]$. By [27, p. 11], there exists $v \in \mathbf{B}(\mathcal{A})$ so that $b^{1/2} = vc$, where $c = (x^*x)^{1/2}$. Write $x = uc$, where u is a partial isometry from $(\ker x)^\perp$ onto $\overline{\operatorname{ran} x}$. Then $c = u^*x = x^*u$, and therefore, $b = M_x(uv^*vu^*)$.

Therefore, M_x is (weakly) compact if and only if the interval $[0, x^*x]$ is (weakly) compact. By Proposition 2.3.13, the compactness and weak compactness of $[0, x^*x]$ are equivalent. To establish (1), suppose $0 \leq T \leq M_x$, and M_x is weakly compact. Then $T(\mathbf{B}(\mathcal{A})_+)$ is relatively compact, as a subset of $[0, x^*x]$. Thus, T is compact. (2) is established similarly. ∎

If the "symbol" x of the operator M_x comes from the ambient $B(H)$, we obtain:

Proposition 2.3.39. *Suppose $\mathcal{A}$ is an irreducible C^*-subalgebra of $B(H)$, $x \in B(H)$, $M_x : \mathcal{A} \to B(H)$ is compact, and $0 \leq T \leq M_x$. Then T is compact.*

Proposition 2.3.40. *Suppose $\mathcal{A}$ is an irreducible C^*-subalgebra of $B(H)$, $S : \mathcal{A} \to B(H)$ is compact, $x \in B(H)$, and $0 \leq M_x \leq S$. Then M_x is compact.*

Remark 2.3.41. The irreducibility of $\mathcal{A}$ is essential here. Below we construct an abelian C^*-subalgebra $\mathcal{A} \subset B(H)$, and operators $x_1, x_2 \in B(H)$, so that $0 \leq M_{x_1} \leq M_{x_2}$, M_{x_2} is compact, while M_{x_1} is not (here, M_{x_1} and M_{x_2} are viewed as taking $\mathcal{A}$ to $B(H)$). By [100], there exist operators $0 \leq R_1 \leq R_2 : C[0,1] \to C[0,1]$ so that R_2 is compact, and R_1 is not. Let λ be the usual Lebesgue measure on $[0,1]$, and let $j : C[0,1] \to B(L_2(\lambda))$ be the diagonal embedding (taking a function f to the multiplication operator $\phi \mapsto \phi f$). By [72, Theorem 3.11], R_1 and R_2 are completely positive. Thus, by Stinespring Theorem, these operators can be represented as $R_i(f) = V_i^* \pi_i(f) V_i$ $(i = 1, 2)$, where $\pi_i : C[0,1] \to B(H_i)$ are representations, and $V_i \in B(L_2(\lambda), H_i)$. Let $H = L_2(\lambda) \oplus_2 H_1 \oplus_2 H_2$. Then $\pi = j \oplus \pi_1 \oplus \pi_2 : C[0,1] \to B(H)$ is an isometric representation. Let

$\mathcal{A} = \pi(C[0,1])$. Furthermore, consider operators x_1 and x_2 on H, defined via

$$x_1 = \begin{pmatrix} 0 & 0 & 0 \\ V_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } x_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ V_2 & 0 & 0 \end{pmatrix}.$$

Then, for any $f \in C[0,1]$, $jR_i(f) = x_i^* \pi(f) x_i$. Considering M_{x_1} and M_{x_2} as operators on $\mathcal{A}$, we see that $0 \leq M_{x_1} \leq M_{x_2}$, M_{x_2} is compact, and M_{x_1} is not.

The following lemma establishes a criterion for compactness of M_x. This result may be known to experts, but we could not find any references in the literature.

Lemma 2.3.42. *Suppose $\mathcal{A}$ is an irreducible C^*-subalgebra of $B(H)$, and $c \in B(H)$. Then $c^*\mathbf{B}(\mathcal{A})_+c$ is a relatively compact set if and only if c is a compact operator.*

Proof. By polar decomposition, it suffices to consider the case of $c \geq 0$. Indeed, write $c = du$, where $d = (cc^*)^{1/2}$, and u is a partial isometry from $(\ker c)^\perp = \overline{\operatorname{ran} c^*}$ to $(\ker c^*)^\perp = \overline{\operatorname{ran} c}$. Then $M_c = M_u M_d$, and $M_d = M_{u^*} M_c$ (here, we abuse the notation slightly, and allow M_u and M_{u^*} to act on $B(H)$). Therefore, the sets $c^*\mathbf{B}(\mathcal{A})_+c = M_c(\mathbf{B}(\mathcal{A})_+)$ and $d\mathbf{B}(\mathcal{A})_+d = M_d(\mathbf{B}(\mathcal{A})_+)$ are compact simultaneously.

If c is compact, then, by [107], $c\mathbf{B}(B(H))c$ is relatively compact. The set $c\mathbf{B}(\mathcal{A})_+c$ is also relatively compact, since it is contained in $c\mathbf{B}(B(H))c$.

Now suppose c is not compact. By scaling, we can assume that the spectral projection $p = \chi_{(1,\infty)}(c)$ has infinite rank. We shall show that, for every $n \in \mathbb{N}$, there exist $a_1, \ldots, a_n \in \mathbf{B}(\mathcal{A})_+$ so that $\|c(a_i - a_j)c\| > 1/3$ for $i \neq j$. Note first that there exist mutually orthogonal unit vectors $\xi_1, \ldots, \xi_n$ in

$\operatorname{ran} p$, so that $\langle \xi_i, \xi_j \rangle = \langle c\xi_i, c\xi_j \rangle = 0$ whenever $i \neq j$. Indeed, if $\sigma(c) \cap (1, \infty)$ is infinite, then there exist disjoint Borel sets $E_i \subset (1, \infty)$ $(1 \leq i \leq n)$, so that $\sigma(c) \cap E_i \neq \infty$. Then we can take $\xi_i \in \chi_{E_i}(c)$. On the other hand, if $\sigma(c) \cap (1, \infty)$ is finite, then for some $s \in \sigma(c) \cap (1, \infty)$, the projection $q = \chi_{\{s\}}(c)$ has infinite rank. Then we can take $\xi_1, \ldots, \xi_n \in \operatorname{ran} q$.

Let $\eta_i = c\xi_i / \|c\xi_i\|$ (by construction, these vectors are mutually orthogonal). As $\mathcal{A}$ is irreducible, its second commutant is $B(H)$. By Kaplansky Density Theorem (see e.g. [24, Theorem I.7.3]), $\mathbf{B}(\mathcal{A})_+$ is strongly dense in $\mathbf{B}(B(H))_+$. Thus, for every $1 \leq i \leq n$ there exist $a_i \in \mathbf{B}(\mathcal{A})_+$ so that $\|a_i \eta_k\| < 1/3$ for $i \neq k$, and $\|a_i \eta_i - \eta_i\| < 1/3$. Consider $b_i = c a_i c \in c(\mathbf{B}(\mathcal{A})_+)c$. For $i \neq j$,

$$\|b_i - b_j\| \geq \langle c(a_i - a_j)c\xi_i, \xi_i \rangle = \|c\xi_i\|^2 \langle (a_i - a_j)\eta_i, \eta_i \rangle > \frac{2}{3} - \frac{1}{3} = \frac{1}{3}.$$

As n is arbitrary, we conclude that $c(\mathbf{B}(\mathcal{A})_+)c$ is not relatively compact. ■

Proof of Proposition 2.3.39. Suppose $x \in B(H)$ is such that $M_x : \mathcal{A} \to B(H)$ is compact. By polar decomposition, we can assume that $x \geq 0$. Then $x\mathbf{B}(A)_+ x$ is relatively compact, and therefore, By Lemma 2.3.42, x is a compact operator. By Proposition 2.3.13, $[0, x^2]$ is compact. But $T(\mathbf{B}(\mathcal{A})_+) \subset [0, x^2]$, hence $T(\mathbf{B}(\mathcal{A})_+)$ is relatively compact. By polarization, $T(\mathbf{B}(\mathcal{A}))$ is compact. ■

To prove Proposition 2.3.40, we need a technical result.

Lemma 2.3.43. *Suppose $z \in B(H)$, and $x, y \in [0, \mathbf{1}_H]$. Then $zxz^* \geq zxyxz^*$.*

Proof. Note that $zxz^* - zxyxz^* = z(x - x^2)z^* + zx(\mathbf{1} - y)xz^*$, and both terms on the right are positive. ■

Proof of Proposition 2.3.40. As in the proof of Proposition 2.3.39, we can assume that $x \geq 0$, and that $p = \chi_{(1,\infty)}(x)$ is a projection of infinite rank. It suffices to show that there exist $a_0 \geq a_1 \geq \ldots \geq a_n$ in $\mathbf{B}(\mathcal{A})_+$, so that $\|x(a_{k-1} - a_k)x\| > 2/3$ for $1 \leq k \leq n$. Indeed, if S is compact, then there exist $u_1, \ldots, u_m \in B(H)$, so that for every $a \in \mathbf{B}(\mathcal{A})_+$ there exists $j \in \{1, \ldots, m\}$ so that $\|Sa - u_j\| < 1/3$. By the pigeon-hole principle, if $n > m$, there exist $i < j$ in $\{1, \ldots, n\}$ and k in $\{1, \ldots, m\}$, so that $\max\{\|Sa_i - u_k\|, \|Sa_j - u_k\|\} < 1/3$. However, $\|Sa_i - Sa_j\| \geq \|x(a_i - a_j)x\| > 2/3$, leading to a contradiction.

Imitating the proof of Proposition 2.3.39, we use the spectral decomposition of x to find mutually orthogonal unit vectors $\xi_1, \ldots, \xi_n$ in $\operatorname{ran} p$, so that (i) $x^k\xi_i$ is orthogonal to $x^\ell\xi_j$ for any $i \neq j$, and $k, \ell \in \{0, 1, \ldots\}$, and (ii) for any i, $1 = \|\xi_i\| \leq \|x\xi_i\| \leq \|x^2\xi_i\| \leq \ldots$. To construct $a_0, \ldots, a_n$, let $c = (2/3)^{1/(2n+1)}$, and let $\eta_i = x\xi_i/\|x\xi_i\|$. By Kaplansky Density Theorem, for $0 \leq k \leq n$ there exist $b_k \in \mathbf{B}(\mathcal{A})_+$, so that

$$b_k\eta_i = \begin{cases} c\eta_i & 1 \leq i \leq n - k \\ 0 & i > n - k \end{cases}$$

(we can take $b_n = 0$). Let $a_0 = b_0$, $a_1 = b_0b_1b_0$, $a_2 = b_0b_1b_2b_1b_0$, etc.. By Lemma 2.3.43, $a_0 \geq a_1 \geq \ldots \geq a_n$. Furthermore,

$$a_k\eta_i = \begin{cases} c^{2k-1}\eta_i & 1 \leq i \leq n - k \\ 0 & i > n - k \end{cases},$$

and therefore,

$$\begin{aligned} \|x(a_{k-1} - a_k)x\| & \geq \langle x(a_{k-1} - a_k)x\xi_{n-k+1}, \xi_{n-k+1}\rangle \\ & = \langle (a_{k-1} - a_k)\eta_{n-k+1}, \eta_{n-k+1}\rangle = c^{2k-1} > \tfrac{2}{3}. \end{aligned}$$

Therefore, the sequence $(a_k)_{k=0}^n$ has the desired properties. ■

2.3.8 Weakly compact operators

In this section, we show that, under certain conditions, weak compactness is inherited under domination. First consider operators on C^*-algebras and their duals.

Theorem 2.3.44. *Suppose E is an OBS, and $\mathcal{A}$ is a C^*-algebra, S is a weakly compact operator, and one of the following holds:*

(i) E is generating, and $0 \leq T \leq S : E \to \mathcal{A}^\star$.

(ii) E is normal, and $0 \leq T \leq S : \mathcal{A} \to E$.

Then T is weakly compact.

Note that, for commutative $\mathcal{A}$, this theorem follows from [99], and the order continuity of $\mathcal{A}^\star$.

Proof. (1) Suppose, for the sake of contradiction, that $T(\mathbf{B}(E)_+)$ is not weakly compact. By Pfitzner's Theorem [74], there exist a bounded sequence $(a_n) \subset \mathcal{A}$ of positive pairwise orthogonal elements, a sequence $(\phi_n) \subset \mathbf{B}(E)_+$, and $c > 0$, such that $T\phi_n(a_n) > c$. Therefore, $S\phi_n(a_n) > c$, which contradicts the weak compactness of $S(\mathbf{B}(E))$ (once again, by Pfitzner's Theorem).

(2) Apply part (1) to $0 \leq T^\star \leq S^\star$. ■

Remark 2.3.45. Theorem 2.3.44 fails for operators from duals of C^*-algebras to C^*-algebras, even in the commutative setting. Indeed, by [6, Theorem 5.31], there exist $0 \leq T \leq S : \ell_1 \to \ell_\infty$, so that S is weakly compact, whereas T is not.

For operators to or from general Banach lattices, we have:

Theorem 2.3.46. *Suppose either* (i) *A is a generating OBS, and B is order continuous Banach lattice, or* (ii) *A is a Banach lattice with order continuous dual, and B is an normal OBS. If* $0 \leqslant T \leqslant S : A \to B$*, and S is weakly compact, then T is weakly compact as well.*

Proof. The proof of (i) is contained in the first few lines of the proof of [6, Theorem 5.31]. (ii) follows by duality. ∎

Next we obtain a partial generalization of the above results for noncommutative function spaces. In the discrete case, we obtain a characterization of order continuous Banach lattices.

Proposition 2.3.47. *Suppose* $\mathcal{E}$ *is a symmetric sequence space, containing a copy of* ℓ_1*, H is an infinite dimensional Hilbert space, and X is a Banach lattice. Then the following are equivalent:*

(i) If $0 \leqslant T \leqslant S : \mathfrak{C}_{\mathcal{E}}(H) \to X$*, and S is weakly compact, then T is weakly compact.*

(ii) X is order continuous.

Proof. (2) ⇒ (1) follows from Theorem 2.3.46.

(1) ⇒ (2): By Proposition 2.3.6 $\mathfrak{C}_{\mathcal{E}}(H)$ contains a positive disjoint sequence, that spans a positively complemented copy of ℓ_1. Hence, the result follows from [6, Theorem 5.31]. ∎

Now consider domination by a weakly compact operator for noncommutative function spaces.

Recall that a noncommutative symmetric function space $\mathcal{E}$ is said to have the *Fatou Property* (sometimes referred to as the *Beppo Levi Property*) if for any norm-bounded increasing net $(x_i) \subset \mathcal{E}_+$, there exists $x \in \mathcal{E}$ so that $x_i \uparrow x$, and $\|x\| = \sup_i \|x_i\|$. In the commutative setting, any symmetric space with the Fatou Property is order complete.

We say that a noncommutative function space $\mathcal{E}$ is a *KB space* if any increasing norm bounded sequence in $\mathcal{E}$ is norm-convergent. Equivalently, $\mathcal{E}$ is order continuous, and has the Fatou Property (see [32]). Furthermore, the following are equivalent: (i) $\mathcal{E}$ is a KB space, (ii) $\mathcal{E}$ is weakly sequentially complete, and (iii) $\mathcal{E}$ contains no copy of c_0. It is clear from [29] that, if $\mathcal{E}$ is symmetric KB function space, then the same is true of $\mathcal{E}(\tau)$.

The following result is contained in [29, Section 5].

Proposition 2.3.48. *Suppose $\mathcal{E}$ is a noncommutative strongly symmetric function space. Then:*

(i) $\mathcal{E}^\times$ is strongly symmetric,

(ii) $\mathcal{E}^\times$ coincides with $\mathcal{E}^\star$ if and only if $\mathcal{E}$ is order continuous. In this case, for every $f \in \mathcal{E}^\star$ there exists a unique $y \in \mathcal{E}^\times$ so that $f(x) = \tau(xy)$, for any $x \in \mathcal{E}$.

(iii) $\mathcal{E}$ coincides with $\mathcal{E}^{\times\times}$ if and only if $\mathcal{E}$ has the Fatou Property.

Proposition 2.3.49. *Suppose $\mathcal{E} = \mathcal{E}(\tau)$ is a noncommutative strongly symmetric KB function space, X a generating OBS, and $0 \leq T \leq S : X \to \mathcal{E}$, with S weakly compact. Then T is weakly compact as well.*

Proof. By [29, Section 5], any positive element $\phi \in \mathcal{E}^{\star\star} = (\mathcal{E}^{\times})^{\star}$ can be written as $\phi(f) = \tau(af) + \psi(f)$, where $a \in \mathcal{E}$ is positive, and ψ is a positive singular functional. The canonical embedding of $\mathcal{E}$ into its double dual takes a to the linear functional $f \mapsto \tau(fa)$.

S is weakly compact, hence $S^{\star\star}(X) \subset \mathcal{E}$. A normal functional cannot dominate a singular one, hence $T^{\star\star}(\mathbf{B}(X^{\star\star})_+) \subset \mathcal{E}$. Since $X^{\star\star}$ is a generating OBS, then $T^{\star\star}(\mathbf{B}(X^{\star\star})) \subset \mathcal{E}$. Therefore, T is weakly compact. ∎

Alternatively, one can prove the above result using the characterization of $\sigma(\mathcal{F}^{\times}, \mathcal{F})$-compact sets given in [31, Proposition 6.2].

Remark 2.3.50. Note that the assumptions of Proposition 2.3.49 are stronger than those of its commutative counterpart – Theorem 2.3.46. For instance, the statement of Theorem 2.3.46(i) holds when the range space is order continuous. Propositions 2.3.49 is proved under the additional assumption of the Fatou property. One reason for this is that much more is known about order continuous Banach lattices (see e.g. [67, Section 2.4]). One useful characterization states that a Banach lattice $\mathcal{E}$ is order continuous iff it is an ideal in its second dual. No such description seems to be known in the non-commutative setting.

Chapter 3

Operator ideals

The purpose of the first section of this chapter is to uncover the structure of ideals on Lorentz sequence spaces. We show that (some of) these ideals can be arranged into the following diagram.

$$\begin{array}{ccccc} & \nearrow & \mathcal{SS} & \Searrow & \\ \{0\} \Rightarrow \mathcal{K} \subsetneq \overline{J^j} \to \overline{J^{\ell_p}} \wedge \mathcal{SS} & & & & \overline{J^{\ell_p}} \vee \mathcal{SS} \to \mathcal{SS}_{d_{w,p}} \Rightarrow L(d_{w,p}) \\ & \Searrow\,(\text{dotted}) & \overline{J^{\ell_p}} & \nearrow & \end{array}$$

On this diagram, a single arrow between ideals, $J_1 \longrightarrow J_2$, means that $J_1 \subseteq J_2$. A double arrow between ideals, $J_1 \Longrightarrow J_2$, means that J_2 is the only immediate successor of J_1 (in particular, $J_1 \neq J_2$), whereas a dotted double arrow between ideals, J_1 ⇢ J_2 , only shows that J_2 is an immediate successor for J_1 (in particular, J_1 may have other immediate successors).

While working with the diagram above, we obtain several important characterizations of some ideals in $L(d_{w,p})$. In particular, we show that $\mathcal{FSS}(d_{w,p}) = \mathcal{SS}(d_{w,p})$ (Theorem 3.1.19). We also characterize the ideal of weakly compact operators (Theorem 3.1.20) and Dunford-Pettis operators (Theorem 3.1.38) on $d_{w,p}$. We show in Theorem 3.1.27 that $\overline{J^j}$ is the only immediate successor of $\mathcal{K}$ under some assumption on the weights w. In the last section of the paper, we show that all strictly singular operators from ℓ_1 to $d_{w,1}$ can be approximated by operators factoring through the formal identity operator $j\colon \ell_1 \to d_{w,1}$ (see Section 3.1.3 for the definition). We also obtain a result on factoring positive operators from $\mathcal{SS}(d_{w,p})$ through the formal identity operator (Theorem 3.1.50).

In the last section we study the relationship between the ideals of compact, (finitely) strictly singular, inessential and Dunford-Pettis operators on noncommutative L_p-spaces. In particular, we obtain the characterization of strictly singular operators acting between noncommutative L_p, that generalizes the corresponding results of [97].

3.1 Operator ideals on Lorentz sequence spaces

This section is based on [58].

The spaces for which the structure of closed ideals in $L(X)$ is well-understood are very few. It was shown in [15] that the only non-trivial closed ideal in the algebra $L(\ell_2)$ is the ideal of compact operators. This result was generalized in [46] to the spaces ℓ_p ($1 \leqslant p < \infty$) and c_0. A space constructed recently in [11] is another space with this property. In [60] and [61], it was shown that the algebras $L\big((\oplus_{k=1}^{\infty}\ell_2^k)_{c_0}\big)$ and $L\big((\oplus_{k=1}^{\infty}\ell_2^k)_{\ell_1}\big)$ have exactly two non-trivial closed ideals. There are no other separable spaces for which the structure of closed ideals in $L(X)$ is completely known.

Partial results about the structure of closed ideals in $L(X)$ were obtained in [76, 5.3.9] for $X = L_p[0,1]$ ($1 < p < \infty$, $p \neq 2$) and in [90] and [91] for $L(\ell_p \oplus \ell_q)$ ($1 \leqslant p, q < \infty$). The purpose of this paper is to investigate the structure of ideals in $L(d_{w,p})$ where $d_{w,p}$ is a Lorentz sequence space (see the definition in Subsection 3.1).

It is well-known that if X is a Banach space then every non-zero ideal in the algebra $L(X)$ must contain the ideal $\mathcal{F}(X)$ of all finite-rank operators on X. It follows that, at least in the presence of the approximation property (in particular, if X has a Schauder basis), every non-zero closed ideal in $L(X)$ contains the closed ideal $\mathcal{K}(X)$ of all compact operators.

If X is a Banach space and $T \in L(X)$ then the ideal in $L(X)$ generated by T is denoted by J_T. It is easy to see that $J_T = \bigl\{\sum_{i=1}^{n} A_i T B_i \;:\; A_i, B_i \in L(X)\bigr\}$. It follows that if $S \in L(X)$ factors through T, i.e., $S = ATB$ for some $A, B \in L(X)$ then $J_S \subseteq J_T$.

Basic sequences

The main tool in this paper is the notion of a basic sequence. In this subsection, we will fix some terminology and remind some classical facts about basic sequences. For a thorough introduction to this topic, we refer the reader to [17] or [35].

If (x_n) is a sequence in a Banach space X then its closed span will be denoted by $[x_n]$. We say that a basic sequence (x_n) ***dominates*** a basic sequence (y_n) and write $(x_n) \succeq (y_n)$ if the convergence of a series $\sum_{n=1}^{\infty} a_n x_n$ implies the convergence of the series $\sum_{n=1}^{\infty} a_n y_n$. We say that (x_n) is ***equivalent*** to (y_n) and write $(x_n) \sim (y_n)$ if $(x_n) \succeq (y_n)$ and $(y_n) \succeq (x_n)$.

Remark 3.1.1. It follows from the Closed Graph Theorem that $(x_n) \succeq (y_n)$ if and only if the linear map from $\operatorname{span}\{x_n\}$ to $\operatorname{span}\{y_n\}$ defined by the formula $T\colon x_n \mapsto y_n$ is bounded.

If (x_n) is a basis in a Banach space X, $z = \sum_{i=1}^{\infty} z_i x_i \in X$, and $A \subseteq \mathbb{N}$ then the vector $\sum_{i\in A} z_i x_i$ will be denoted by $z|_A$ (provided the series converges; this is always the case when the basis is unconditional). We will refer to $z|_A$ as the ***restriction of** z **to** A*. The restrictions $z|_{[n,\infty)\cap\mathbb{N}}$ and $z|_{(n,\infty)\cap\mathbb{N}}$, where $n \in \mathbb{N}$, will be abbreviated as $z|_{[n,\infty)}$ and $z|_{(n,\infty)}$, respectively. We say that a vector v is a ***restriction*** of z if there exists $A \subseteq \mathbb{N}$ such that $v = z|_A$. The vector $z = \sum_{i=1}^{\infty} z_i x_i$ will also be denoted by $z = (z_i)$. If $z = \sum_{i=1}^{\infty} z_i x_i$ then the ***support*** of z is the set $\operatorname{supp} z = \{i \in \mathbb{N} \colon z_i \neq 0\}$.

Every 1-unconditional basis (x_n) in a Banach space X defines a Banach lattice order on X by $\sum_{i=1}^{\infty} a_i x_i \geqslant 0$ if and only if $a_i \geqslant 0$ for all $i \in \mathbb{N}$ (see, e.g., [64, page 2]). For $x \in X$, we have $|x| = x \vee (-x)$. A Banach lattice is said to have ***order continuous norm*** if the condition $x_\alpha \downarrow 0$ implies $\|x_\alpha\| \to 0$. For an introduction to Banach lattices and standard terminology, we refer the reader to [1, §1.2].

If (x_n) is a basic sequence in a Banach space X, then a sequence (y_n) in $\operatorname{span}\{x_n\}$ is a ***block sequence*** of (x_n) if there is a strictly increasing sequence (p_n) in $\mathbb{N}$ and a sequence of scalars (a_i) such that $y_n = \sum_{i=p_n+1}^{p_{n+1}} a_i x_i$ for all $n \in \mathbb{N}$.

The following two facts are classical and will sometimes be used without any references. The first fact is known as the Principle of Small Perturbations (see, e.g., [35, Theorem 4.23]).

Theorem 3.1.2. *Let X be a Banach space, (x_n) a basic sequence in X, and (x_n^*) the correspondent biorthogonal functionals defined on $[x_n]$. If (y_n) is a sequence such that $\sum_{n=1}^{\infty} \|x_n^*\| \cdot \|x_n - y_n\| < 1$ then (y_n) is a basic sequence equivalent to (x_n). Moreover, if $[x_n]$ is complemented in X then so is $[y_n]$. If $[x_n] = X$ then $[y_n] = X$.*

The next fact, which is often called the Bessaga-Pełczyński selection principle, is a result of combining the "gliding hump" argument (see, e.g., [17, Lemma 5.1]) with the Principle of Small Perturbations.

Theorem 3.1.3. *Let X be a Banach space with a seminormalized basis (x_n) and let (x_n^*) be the correspondent biorthogonal functionals. Let (y_n) be a seminormalized sequence in X such that $x_n^*(y_k) \overset{k\to\infty}{\longrightarrow} 0$ for all $n \in \mathbb{N}$. Then (y_n) has a subsequence (y_{n_k}) which is basic and equivalent to a block sequence (u_k) of (x_n). Moreover, $y_{n_k} - u_k \to 0$, and u_k is a restriction of y_{n_k}.*

Lorentz sequence spaces

Let $1 \leqslant p < \infty$ and $w = (w_n)$ be a sequence in $\mathbb{R}$ such that $w_1 = 1$, $w_n \downarrow 0$, and $\sum_{i=1}^{\infty} w_i = \infty$. The Lorentz sequence space $d_{w,p}$ is a Banach space of all vectors $x \in c_0$ such that $\|x\|_{d_{w,p}} < \infty$, where

$$\|(x_n)\|_{d_{w,p}} = \Big(\sum_{n=1}^{\infty} w_n x_n^{*p}\Big)^{1/p}$$

is the norm in $d_{w,p}$. Here (x_n^*) is the ***non-increasing rearrangement*** of the sequence $(|x_n|)$. An overview of properties of Lorentz sequence spaces can be found in [63, Section 4.e].

The vectors (e_n) in $d_{w,p}$ defined by $e_n(i) = \delta_{ni}$ $(n, i \in \mathbb{N})$ form a 1-symmetric basis in $d_{w,p}$. In particular, (e_n) is 1-unconditional, hence $d_{w,p}$ is a Banach lattice. We call (e_n) the unit vector basis of $d_{w,p}$. The unit vector basis of ℓ_p will be denoted by (f_n) throughout the paper.

Remark 3.1.4. It is proved in [7, Lemma 1] and [18, Lemma 15] that if (u_n) is a seminormalized block sequence of (e_n) in $d_{w,p}$, $u_n = \sum_{i=p_n+1}^{p_{n+1}} a_i e_i$, such that $a_i \to 0$, then there is a subsequence (u_{n_k}) such that $(u_{n_k}) \sim (f_n)$ and $[u_{n_k}]$ is complemented in $d_{w,p}$. Further, it was shown in [7, Corollary 3] that if (y_n) is a seminormalized block sequence of (e_n) then there is a seminormalized block sequence (u_n) of (y_n) such that $u_n = \sum_{i=p_n+1}^{p_{n+1}} a_i e_i$, with $a_i \to 0$. Therefore, every infinite dimensional subspace of $d_{w,p}$ contains a further subspace which is complemented in $d_{w,p}$ and isomorphic to ℓ_p ([18, Corollary 17]).

Remark 3.1.5. Remark 3.1.4 yields, in particular, that $d_{w,p}$ does not contain copies of c_0. Since the basis (e_n) of $d_{w,p}$ is unconditional, the space $d_{w,p}$ is weakly sequentially complete by [6, Theorem 4.60] (see also [63, Theorem 1.c.10]). Also, [6, Theorem 4.56] guarantees that $d_{w,p}$ has order continuous norm. In particular, if $x \in d_{w,p}$ then $\|x|_{[n,\infty)}\| \to 0$ as $n \to \infty$.

Remark 3.1.6. It was shown in [44] that if $p > 1$ then $d_{w,p}$ is reflexive. This can also be easily obtained from Remark 3.1.4 (cf. [63, Theorem 1.c.12]).

Remark 3.1.7. The unit vector basis (e_n) of $d_{w,p}$ is weakly null. Indeed, by Rosenthal's ℓ_1-theorem (see [87]; also [63, Theorem 2.e.5]), (e_n) is weakly Cauchy. Since it is symmetric, $(e_n) \sim (e_{2n} - e_{2n-1})$.

The next proposition will be used often in this section.

Proposition 3.1.8 ([7, Proposition 5 and Corollary 2])**.** *If (u_n) is a seminormalized block sequence of (e_n) then $(f_n) \succeq (u_n)$. If (u_n) does not contain subsequences equivalent to (f_n) then also $(u_n) \succeq (e_n)$.*

The following lemma is standard.

Lemma 3.1.9. *Let (x_n) be a block sequence of (e_n), $x_n = \sum_{i=p_n+1}^{p_{n+1}} a_i e_i$. If (y_n) is a basic sequence such that $y_n = \sum_{i=p_n+1}^{p_{n+1}} b_i e_i$, where $|b_i| \leqslant |a_i|$ for all $i \in \mathbb{N}$, then (x_n) is basic and $(x_n) \succeq (y_n)$.*

Proof. Let

$$\gamma_i = \begin{cases} \frac{b_i}{a_i}, & \text{if } a_i \neq 0, \\ 0, & \text{if } a_i = 0. \end{cases}$$

Define an operator $T \in L(d_{w,p})$ by $T\left(\sum_{i=1}^{\infty} c_i e_i\right) = \sum_{i=1}^{\infty} c_i \gamma_i e_i$. Then T is, clearly, linear and, since the basis (e_n) is 1-unconditional, T is bounded with $\|T\| \leqslant 1$. In particular, $T|_{[x_n]}$ is bounded. Also, $T(x_n) = y_n$ for all $n \in \mathbb{N}$, hence $(x_n) \succeq (y_n)$. ∎

3.1.1 Operators factorable through ℓ_p

Let X and Y be Banach spaces and $T \in L(X)$. We say that T ***factors through*** Y if there are two operators $A \in L(X,Y)$ and $B \in L(Y,X)$ such that $T = BA$.

The following two lemmas are standard. We present their proofs for the sake of completeness.

Lemma 3.1.10. *Let X and Y be Banach spaces and $T \in L(X,Y)$, $S \in L(Y,X)$ be such that $ST = \mathrm{id}_X$. Then T is an isomorphism and* Range T *is a complemented subspace of Y isomorphic to X.*

Proof. For all $x \in X$, we have $\|x\| = \|STx\| \leqslant \|S\|\|Tx\|$, so $\|Tx\| \geqslant \frac{1}{\|S\|}\|x\|$. This shows that T is an isomorphism. In particular, Range T is a closed subspace of Y isomorphic to X.

Put $P = TS \in L(Y)$. Then $P^2 = TSTS = T\mathrm{id}_X S = TS = P$, hence P is a projection. Clearly, Range $P \subseteq$ Range T. Also, $PT = TST = T$, so $\operatorname{Range} T \subseteq \operatorname{Range} P$. Therefore $\operatorname{Range} P = \operatorname{Range} T$, and $\operatorname{Range} T$ is complemented. ∎

Lemma 3.1.11. *Let X and Y be Banach spaces such that Y is isomorphic to $Y \oplus Y$. Then the set $J = \{T \in L(X) : T \text{ factors through } Y\}$ is an ideal in $L(X)$.*

Proof. It is clear that J is closed under multiplication by operators in $L(X)$. In particular, J is closed under scalar multiplication. Let $A, B \in J$. Write $A = A_1 A_2$ and $B = B_1 B_2$, where $A_1, B_1 \in L(Y, X)$ and $A_2, B_2 \in L(X, Y)$. Then $A + B = UV$ where $V : x \in X \mapsto (A_2 x, B_2 x) \in Y \oplus Y$ and $U : (x, y) \in Y \oplus Y \mapsto A_1 x + B_1 y \in Y$. Clearly, UV factors through $Y \oplus Y \simeq Y$. Hence $A + B \in J$. ∎

We will denote the set of all operators in $L(d_{w,p})$ which factor through a Banach space Y by J^Y.

Theorem 3.1.12. *The sets J^{ℓ_p} and $\overline{J^{\ell_p}}$ are proper ideals in $L(d_{w,p})$.*

Proof. Since ℓ_p is isomorphic to $\ell_p \oplus \ell_p$, it follows from Lemma 3.1.11 that J^{ℓ_p} is an ideal in $L(d_{w,p})$. Let us show that $J^{\ell_p} \neq L(d_{w,p})$.

Assume that $J^{\ell_p} = L(d_{w,p})$, then the identity operator I on $d_{w,p}$ belongs to J. Write $I = ST$ where $T \in L(d_{w,p}, \ell_p)$ and $S \in L(\ell_p, d_{w,p})$. By Lemma 3.1.10, the range of T is complemented in ℓ_p and is isomorphic to $d_{w,p}$. This is a contradiction because all complemented infinite-dimensional subspaces of ℓ_p are isomorphic to ℓ_p (see, e.g., [63, Theorem 2.a.3]), while $d_{w,p}$ is not isomorphic to ℓ_p (see [14] for the case $p = 1$ and [44] for the case $1 < p < \infty$; see also [63, p. 176]).

Being the closure of a proper ideal, $\overline{J^{\ell_p}}$ is itself a proper ideal (see, e.g., [23, Corollary VII.2.4]). ∎

Proposition 3.1.13. *There exists a projection $P \in L(d_{w,p})$ such that* Range P *is isomorphic to ℓ_p. For every such P we have $J_P = J^{\ell_p}$.*

Proof. Such projections exist by Remark 3.1.4. Let $Y = \operatorname{Range} P$, $U \colon Y \to \ell_p$ be an isomorphism onto, and $i \colon Y \to d_{w,p}$ be the inclusion map. It is easy to see that $P = (iU^{-1})(UP)$, hence $P \in J^{\ell_p}$, so that $J_P \subseteq J^{\ell_p}$.

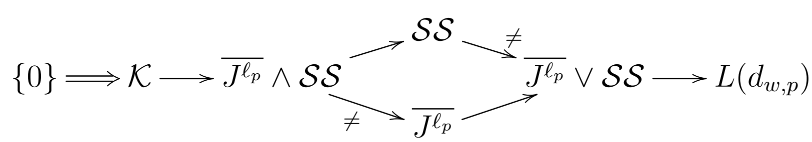
On the other hand, if $T \in J^{\ell_p}$ is arbitrary, $T = AB$ with $A \in L(\ell_p, d_{w,p})$, $B \in L(d_{w,p}, \ell_p)$, then one can write $T = (AUP)P(iU^{-1}B)$, so that $T \in J_P$. Thus $J^{\ell_p} \subseteq J_P$. ∎

Corollary 3.1.14. *The ideal $\overline{J^{\ell_p}}$ properly contains the ideal of compact operators $\mathcal{K}(d_{w,p})$.*

Proof. It was already mentioned in the introductory section that compact operators form the smallest closed ideal in $L(d_{w,p})$. Since a projection onto a subspace isomorphic to ℓ_p is not compact, it follows that $\mathcal{K}(d_{w,p}) \neq \overline{J^{\ell_p}}$. ∎

3.1.2 Strictly singular operators

In this section we will study properties of strictly singular operators in $L(d_{w,p})$. Since projections onto the subspaces of $d_{w,p}$ isomorphic to ℓ_p are clearly not strictly singular, it follows from Proposition 3.1.13 that $\mathcal{SS}(d_{w,p}) \neq J^{\ell_p}$. Moreover, $\mathcal{SS} \neq \overline{J^{\ell_p}} \vee \mathcal{SS}$ and $\overline{J^{\ell_p}} \wedge \mathcal{SS} \neq \overline{J^{\ell_p}}$. So, the ideals we discussed so far can be arranged as follows:



The following theorem shows that there can be no other closed ideals between $\mathcal{SS}$ and $\overline{J^{\ell_p}} \vee \mathcal{SS}$ on this diagram.

Theorem 3.1.15. *Let $T \in L(d_{w,p})$. If $T \notin \mathcal{SS}(d_{w,p})$ then $J^{\ell_p} \subseteq J_T$.*

Proof. Let $T \notin \mathcal{SS}(d_{w,p})$. Then there exists an infinite-dimensional subspace Y of $d_{w,p}$ such that $T|_Y$ is an isomorphism. By Remark 3.1.4, passing to a subspace, we may assume that Y is complemented in $d_{w,p}$ and isomorphic to ℓ_p. Let (x_n) be a basis of Y equivalent to the unit vector basis of ℓ_p. Define $z_n = Tx_n$, then (z_n) is also equivalent to the unit vector basis of ℓ_p. By Remark 3.1.4, (z_n) has a subsequence (z_{n_k}) such that $[z_{n_k}]$ is complemented in $d_{w,p}$ and isomorphic to ℓ_p.

Denote $W = [x_{n_k}]$. Then W and $T(W)$ are both complemented subspaces of $d_{w,p}$ isomorphic to ℓ_p. Let P and Q be projections onto W and $T(W)$, respectively. Put $S = \big(T|_W\big)^{-1}$, $S \in L\big(T(W), d_{w,p}\big)$. Then it is easy to see that $P = (SQ)TP$. Since SQ and P are in $L(d_{w,p})$, we have $J_P \subseteq J_T$. By Proposition 3.1.13, $J^{\ell_p} \subseteq J_T$. ■

Corollary 3.1.16. *$\overline{J^{\ell_p}} \bigvee \mathcal{SS}(d_{w,p})$ is the only immediate successor of $\mathcal{SS}(d_{w,p})$ and $\overline{J^{\ell_p}}$ is an immediate successor of $\overline{J^{\ell_p}} \wedge \mathcal{SS}(d_{w,p})$.*

Now we will investigate the ideal of finitely strictly singular operators on $d_{w,p}$. To prove the main statement (Theorem 3.1.19), we will need the following lemma due to Milman [68] (see also a thorough discussion in [90]). This lemma will be used more than once in this section.

Lemma 3.1.17 ([68])**.** *If F is a k-dimensional subspace of c_0 then there exists a vector $x \in F$ such that x attains its sup-norm at at least k coordinates (that is, x^* starts with a constant block of length k).*

We will also use the following simple lemma.

Lemma 3.1.18. *Let* $s_n = \sum_{i=1}^{n} w_i$ *(*$n \in \mathbb{N}$*) where* $w = (w_i)$ *is the sequence of weights for* $d_{w,p}$*. If* $x \in d_{w,p}$*,* $y = x^*$*, and* $N \in \mathbb{N}$ *then* $0 \leqslant y_N \leqslant \frac{\|x\|}{s_N^{1/p}}$*.*

Proof. $\|x\|^p = \|y\|^p = \sum_{i=1}^{\infty} y_i^p w_i \geqslant y_N^p \sum_{i=1}^{N} w_i = y_N^p s_N$. ■

Theorem 3.1.19. *Let* X *and* Y *be subspaces of* $d_{w,p}$*. Then* $\mathcal{FSS}(X,Y) = \mathcal{SS}(X,Y)$*. In particular,* $\mathcal{FSS}(\ell_p, d_{w,p}) = \mathcal{SS}(\ell_p, d_{w,p})$ *and* $\mathcal{FSS}(d_{w,p}) = \mathcal{SS}(d_{w,p})$*.*

Proof. Let $T \in L(X,Y)$. Suppose that T is not finitely strictly singular. We will show that it is not strictly singular. Since T is not finitely strictly singular, there exists a constant $c > 0$ and a sequence F_n of subspaces of X with $\dim F_n \geqslant n$ such that for each n and for all $x \in F_n$ we have $\|Tx\| \geqslant c\|x\|$.

Fix a sequence (ε_k) in $\mathbb{R}$ such that $1 > \varepsilon_k \downarrow 0$. We will inductively construct a sequence (x_k) in X and two strictly increasing sequences $(n_k), (m_k)$ in $\mathbb{N}$ such that:

(i) (x_k) and (Tx_k) are seminormalized; we will denote Tx_k by u_k;

(ii) for all $k \in \mathbb{N}$, $\operatorname{supp} x_k \subseteq [n_k, \infty)$ and $\operatorname{supp} u_k \subseteq [m_k, \infty)$;

(iii) if $k \geqslant 2$ then $\|x_{k-1}|_{[n_k,\infty)}\| < \varepsilon_k$, $\|u_{k-1}|_{[m_k,\infty)}\| < \varepsilon_k$, and all the coordinates of u_{k-1} where the sup-norm is attained are less than m_k;

(iv) for each $k \in \mathbb{N}$, the vector u_k^* begins with a constant block of length at least k.

That is, (x_n) and (u_n) are two almost disjoint sequences and u_n's have long "flat" sections.

Take x_1 to be any nonzero vector in F_1 and put $n_1 = m_1 = 1$. Suppose we have already constructed $x_1, \dots, x_{k-1}$, $n_1, \dots, n_{k-1}$, and $m_1, \dots, m_{k-1}$ such that the conditions (i)–(iv) are satisfied. Choose $n_k \in \mathbb{N}$ and $m_k \in \mathbb{N}$ such that $n_k > n_{k-1}$, $m_k > m_{k-1}$ and the condition (iii) is satisfied.

Consider the space

$$V = \{y = (y_i) \in F_{n_k+m_k+k} \, : \, y_i = 0 \text{ for } i < n_k\} \subseteq F_{n_k+m_k+k}.$$

It follows from $\dim F_{n_k+m_k+k} \geqslant n_k + m_k + k$ that $\dim V \geqslant m_k + k$. Since $V \subseteq F_{n_k+m_k+k}$, $\|Ty\| \geqslant c\|y\|$ for all $y \in V$. In particular, $\dim(TV) \geqslant m_k + k$. Define

$$Z = \{z = (z_i) \in TV \, : \, z_i = 0 \text{ for } i < m_k\}.$$

It follows that $\dim Z \geqslant k$.

Clearly, $\operatorname{supp} y \subseteq [n_k, \infty)$ for all $y \in V$ and $\operatorname{supp} z \subseteq [m_k, \infty)$ for all $z \in Z$. By Lemma 3.1.17, we can choose $u_k \in Z$ such that u_k is normalized and u_k^* starts with a constant block of length k. Put $x_k = (T|_V)^{-1}(u_k) \in Y$. Since $x_k \in V$ and $\|u_k\| = 1$, it follows that $\frac{1}{\|T\|} \leqslant \|x_k\| \leqslant \frac{1}{c}$, so the conditions (i)–(iv) are satisfied for (x_k).

For each $k \in \mathbb{N}$, let $x'_k = x_k|_{[n_k, n_{k+1})}$ and $u'_k = u_k|_{[m_k, m_{k+1})}$. Passing to tails of sequences, if necessary, we may assume that both (x'_k) and (u'_k) are seminormalized block sequences of (e_n).

Since the non-increasing rearrangement of each u'_k starts with a constant block of length k by (iii), the coefficients in u'_k converge to zero by Lemma 3.1.18. Therefore, passing to a subsequence, we may assume by Remark 3.1.4 that (u'_k) is equivalent to the unit vector basis (f_n) of ℓ_p. Using Theorem 3.1.2 and passing to a further subsequence, we may also assume that $(x_k) \sim (x'_k)$ and $(u_k) \sim (u'_k)$.

By Proposition 3.1.8, the sequence (x'_k) is dominated by (f_n). Notice that the condition $u_k = Tx_k$ implies $(x_k) \succeq (u_k)$. Therefore, we get the following chain of dominations and equivalences of basic sequences:

$$(f_n) \succeq (x'_k) \sim (x_k) \succeq (u_k) \sim (u'_k) \sim (f_n).$$

It follows that all the dominations in this chain are, actually, equivalences. In particular, $(x_k) \sim (u_k)$. Thus, T is an isomorphism on the space $[x_k]$, hence T is not strictly singular. ∎

Recall that an operator T on a Banach space X is weakly compact if the image of the unit ball of X under T is relatively weakly compact. Alternatively, T is weakly compact if and only if for every bounded sequence (x_n) in X there exists a subsequence (x_{n_k}) of (x_n) such that (Tx_{n_k}) is weakly convergent.

If $1 < p < \infty$ then $d_{w,p}$ is reflexive, and, hence, every operator in $L(d_{w,p})$ is weakly compact. In case $p = 1$ we have the following.

Theorem 3.1.20. *Let $T \in L(d_{w,1})$. Then T is weakly compact if and only if T is strictly singular.*

Proof. Suppose that T is strictly singular. We will show that T is weakly compact.

Let (x_n) be a bounded sequence in X. By Rosenthal's ℓ_1-theorem, there is a subsequence (x_{n_k}) of (x_n) such that (x_{n_k}) is either equivalent to the unit vector basis (f_n) of ℓ_1 or is weakly Cauchy. In the latter case, (Tx_{n_k}) is also weakly Cauchy. If $(x_{n_k}) \sim (f_n)$ then, since T is strictly singular, (Tx_{n_k}) cannot have subsequences equivalent to (f_n). Hence, using Rosenthal's theorem one more time and passing to a further subsequence, we may assume that, again, (Tx_{n_k}) is weakly Cauchy. Since $d_{w,1}$ is weakly sequentially complete, the sequence (Tx_{n_k}) is weakly convergent. It follows that T is weakly compact.

Conversely, let J be the closed ideal of weakly compact operators in $L(d_{w,1})$. By the first part of the proof, J is a successor of $\mathcal{SS}(d_{w,1})$. Suppose that $J \neq \mathcal{SS}(d_{w,1})$. By Theorem 3.1.15, $J^{\ell_1} \subseteq J$. This, however, is a contradiction since a projection onto a copy of ℓ_1 (which belongs to J^{ℓ_1} by Proposition 3.1.13) is not weakly compact. ∎

3.1.3 Operators factorable through the formal identity

The operator $j\colon \ell_p \to d_{w,p}$ defined by $j(e_n) = f_n$ is called ***the formal identity operator from*** ℓ_p ***to*** $d_{w,p}$. It follows immediately from the definition of the norm in $d_{w,p}$ that $\|j\| = 1$.

We will denote by the symbol J^j the set of all operators $T \in L(d_{w,p})$ which can be factored as $T = AjB$ where $A \in L(d_{w,p})$ and $B \in L(d_{w,p}, \ell_p)$.

Proposition 3.1.21. *J^j is an ideal in $L(d_{w,p})$.*

Proof. It is clear from the definition that the set J^j is closed under both right and left multiplication by operators from $L(d_{w,p})$. We have to show that if T_1 and T_2 are in J^j then $T_1 + T_2$ is in J^j, as well.

Write $T_1 = A_1jB_1$, $T_2 = A_2jB_2$ with $A_1, A_2 \in L(d_{w,p})$ and $B_1, B_2 \in L(d_{w,p}, \ell_p)$. Let $A \in L(d_{w,p} \oplus d_{w,p}, d_{w,p})$ and $B \in L(d_{w,p}, \ell_p \oplus \ell_p)$ be defined by

$$A(x_1, x_2) = A_1x_1 + A_2x_2 \quad \text{and} \quad Bx = (B_1x, B_2x).$$

Define also $U\colon \ell_p \to \ell_p \oplus \ell_p$ and $V\colon d_{w,p} \to d_{w,p} \oplus d_{w,p}$ by

$$U\big((x_n)\big) = \big((x_{2n-1}), (x_{2n})\big), \quad \text{and} \quad V\big((x_n)\big) = \big((x_{2n-1}), (x_{2n})\big).$$

Since the bases of ℓ_p and $d_{w,p}$ are both unconditional, U and V are bounded.

Now observe that for each $x = (x_n) \in d_{w,p}$ we can write

$$AVjU^{-1}Bx = AVjU^{-1}(B_1x, B_2x) = \\ A(jB_1x, jB_2x) = A_1jB_1x + A_2jB_2x = T_1x + T_2x.$$

This shows that $T_1 + T_2 = AVjU^{-1}B$ with $AV \in L(d_{w,p})$ and $U^{-1}B \in L(d_{w,p}, \ell_p)$, hence $T_1 + T_2 \in J^j$. ∎

As we already mentioned before, the space $d_{w,p}$ contains many complemented copies of ℓ_p. Consider the operator $jUP \in L(d_{w,p})$ where P is a projection onto any subspace Y isomorphic to ℓ_p and $U\colon Y \to \ell_p$ is an isomorphism onto. It turns out that the ideal generated by any such operator does not depend on the choice of Y and, in fact, coincides with J^j.

Proposition 3.1.22. *Let Y be a complemented subspace of $d_{w,p}$ isomorphic to ℓ_p, $P \in L(d_{w,p})$ be a projection with range Y, and $U\colon Y \to \ell_p$ be an isomorphism onto. If $T = jUP$ then $J_T = J^j$.*

Proof. Clearly, $J_T \subseteq J^j$. Let $S \in J^j$. Then $S = AjB$ where $A \in L(d_{w,p})$ and $B \in L(d_{w,p}, \ell_p)$. It follows that

$$S = AjB = Aj(UPU^{-1})B = AT(U^{-1}B) \in J_T.$$

∎

The next goal is to show that the ideal $\overline{J^j}$ "sits" between $\mathcal{K}(X)$ and $\mathcal{SS}(X) \wedge \overline{J^{\ell_p}}$.

Theorem 3.1.23. *The formal identity operator $j\colon \ell_p \to d_{w,p}$ is finitely strictly singular.*

Proof. Let $\varepsilon > 0$ be arbitrary. Take $n \in \mathbb{N}$ such that $\frac{1}{n}\sum_{i=1}^{n} w_i < \varepsilon$; such n exists by $w_n \to 0$. Since (w_n) is also a decreasing sequence, it follows that $w_i < \varepsilon$ for all $i \geqslant n$.

Let $Y \subseteq \ell_p$ be a subspace with $\dim Y \geqslant n$. By Lemma 3.1.17, there exists a vector $x \in Y$ such that $\|x\|_{\ell_p} = 1$ and x attains its sup-norm at at least n coordinates. Denote $\delta = \|x\|_{\sup} > 0$. Then $\|x\|_{\ell_p} \geqslant n^{1/p}\delta$, so $\delta \leqslant n^{-1/p}$.

Observe that the non-increasing rearrangement x^* of x satisfies the condition that $x_i^* = \delta$ for all $1 \leqslant 1 \leqslant n$. Therefore

$$\|jx\|_{d_{w,p}}^p = \sum_{i=1}^{\infty} x_i^{*p} w_i \leqslant \delta^p \sum_{i=1}^{n} w_i + \varepsilon \sum_{i=n+1}^{\infty} x_i^{*p} \leqslant \delta^p n\varepsilon + \varepsilon\|x\|_{\ell_p}^p \leqslant 2\varepsilon.$$

Hence $\|jx\|_{d_{w,p}} \leqslant (2\varepsilon)^{1/p}$. ∎

Corollary 3.1.24. *The following inclusions hold: $\mathcal{K}(d_{w,p}) \subsetneq \overline{J^j}$ and $J^j \subseteq \mathcal{SS}(d_{w,p}) \wedge J^{\ell_p}$.*

Proof. Let Y, P, and U be as in Proposition 3.1.22. Then $jUP \in J^j$. If $x_n = U^{-1}f_n \in d_{w,p}$ then (x_n) is seminormalized and $jUPx_n = e_n$. Hence the sequence $(jUPx_n)$ has no convergent subsequences, so that jUP is not compact.

The inclusion $J^j \subseteq \mathcal{SS}(d_{w,p}) \wedge J^{\ell_p}$ is obvious since j is strictly singular. ∎

Conjecture 3.1.25. The ideal $\overline{J^j}$ is the only immediate successor of $\mathcal{K}(d_{w,p})$.

In [7] and [18] (see also [63]), conditions on the weights $w = (w_n)$ are given under which $d_{w,p}$ has exactly two non-equivalent symmetric basic sequences. We will show that the conjecture holds true in this case.

Lemma 3.1.26. *If $T \in \mathcal{SS}(d_{w,p}) \backslash \mathcal{K}(d_{w,p})$ then there exists a seminormalized basic sequence (x_n) in $d_{w,p}$ such that $(f_n) \succeq (x_n)$ and (Tx_n) is weakly null and seminormalized.*

Proof. Let (z_n) be a bounded sequence in $d_{w,p}$ such that (Tz_n) has no convergent subsequences. Then (z_n) has no convergent subsequences either. Applying Rosenthal's ℓ_1-theorem and passing to a subsequence, we may assume that (z_n) is either equivalent to the unit vector basis of ℓ_1 or is weakly Cauchy.

Case: (z_n) is equivalent to the unit vector basis of ℓ_1. Since a reflexive space cannot contain a copy of ℓ_1, we conclude that $p = 1$, so $(z_n) \sim (f_n)$. Again, by Rosenthal's theorem, (Tz_n) has a subsequence which is either equivalent to (f_n) or is weakly Cauchy. If $(Tz_{n_k}) \sim (f_n)$ then T is an isomorphism on the space $[z_{n_k}]$, contrary to the assumption that $T \in \mathcal{SS}(d_{w,p})$. Therefore, (Tz_{n_k}) is weakly Cauchy. Put $x_k = z_{n_{2k}} - z_{n_{2k-1}}$. Then (x_k) is basic and (Tx_k) is weakly null. Passing to a further subsequence of (z_{n_k}) we may assume that (Tx_k) is seminormalized. Also, (x_k) is still equivalent to (f_n), hence is dominated by (f_n).

Case: (z_n) is weakly Cauchy. Clearly, (Tz_n) is also weakly Cauchy. Consider the sequence (u_n) in $d_{w,p}$ defined by $u_n = z_{2n} - z_{2n-1}$. Then both (u_n) and (Tu_n) are weakly null. Passing to a subsequence of (z_n), we may assume that (Tu_n) and, hence, (u_n) are seminormalized. Applying Theorem 3.1.3, we get a subsequence (u_{n_k}) of (u_n) which is basic and equivalent to a block sequence (v_n) of (e_n). Denote $x_k = u_{n_k}$. By Proposition 3.1.8, (f_n) dominates (v_n) and, hence, (x_k). ∎

Theorem 3.1.27. *If $d_{w,p}$ has exactly two non-equivalent symmetric basic sequences, then $\overline{J^j}$ is the only immediate successor of $\mathcal{K}(d_{w,p})$.*

Proof. Let T be a non-compact operator on $d_{w,p}$. It suffices to show that $J^j \subseteq J_T$. We may assume that T is strictly singular because, otherwise, we have $J^j \subseteq J^{\ell_p} \subseteq J_T$ by Theorem 3.1.15.

Let (x_n) be a sequence as in Lemma 3.1.26. Passing to a subsequence and using Theorem 3.1.3, we may assume that (Tx_n) is basic and equivalent to a block sequence (h_n) of (e_n) such that $Tx_n - h_n \to 0$. We claim that (h_n) has no subsequences equivalent to (f_n). Indeed, otherwise, for such a subsequence (h_{n_k}) of (h_n), we would have $(f_n) \sim (f_{n_k}) \succeq (x_{n_k}) \succeq (Tx_{n_k}) \sim (h_{n_k}) \sim (f_n)$, so $(x_{n_k}) \sim (Tx_{n_k})$, contrary to $T \in \mathcal{SS}(d_{w,p})$. By [18, Theorem 19], (h_n) has a subsequence which spans a complemented subspace in $d_{w,p}$ and is equivalent to (e_n). Therefore, by Theorem 3.1.2, we may assume (by passing to a further subsequence) that $(Tx_n) \sim (e_n)$ and $[Tx_n]$ is complemented in $d_{w,p}$.

We have proved that there exists a sequence (x_n) in $d_{w,p}$ such that $[Tx_n]$ is complemented in $d_{w,p}$ and

$$(f_n) \succeq (x_n) \succeq (Tx_n) \sim (e_n).$$

Let $A \in L(\ell_p, d_{w,p})$ and $B \in L([Tx_n], d_{w,p})$ be defined by $Af_n = x_n$ and $B(Tx_n) = e_n$. Let $Q \in L(d_{w,p})$ be a projection onto $[Tx_n]$. Then for all

$n \in \mathbb{N}$, we obtain: $BQTAf_n = BQTx_n = BTx_n = e_n$. It follows that $BQTA = j$, so that $J^j \subseteq J_T$. ■

In order to prove Conjecture 3.1.25 without additional conditions on w, it suffices to show that if $T \in \overline{J^j} \setminus \mathcal{K}(d_{w,p})$ then $J^j \subseteq \overline{J_T}$. We will prove a weaker statement: if $T \in J^j \setminus \mathcal{K}(d_{w,p})$ then $J^j \subseteq J_T$.

Recall (see [7, p.148]) that if $x = (a_n) \in d_{w,p}$ then a block sequence (y_n) of (e_n) is called a ***block of type I generated by*** x if it is of the form $y_n = \sum_{i=p_n+1}^{p_{n+1}} a_{i-p_n} e_i$ for all n. A set $A \subseteq d_{w,p}$ will be said to be ***almost lengthwise bounded*** if for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $\|x^*|_{[N,\infty)}\| < \varepsilon$ for all $x \in A$. We will usually use it in the case when $A = \{x_n\}$ for some sequence (x_n) in $d_{w,p}$. We need the following result, which is a slight extension of [7, Theorem 3]. We include the proof for completeness.

Theorem 3.1.28. *Let (x_n) be a seminormalized block sequence of (e_n) in $d_{w,p}$.*

(i) If (x_n) is not almost lengthwise bounded then there exists a subsequence (x_{n_k}) such that $(x_{n_k}) \sim (f_n)$.

(ii) If (x_n) is almost lengthwise bounded, then there exists a subsequence (x_{n_k}) equivalent to a block of type I generated by a vector $u = \sum_{i=1}^{\infty} b_i e_i \in d_{w,p}$ with $b_i \downarrow 0$. Moreover, if the sequence (x_n) is bounded in ℓ_p then u is in ℓ_p.[1]

Proof. (i) Without loss of generality, $\|x_n\| \leqslant 1$ for all $n \in \mathbb{N}$. By the assumption, there exists $\varepsilon > 0$ with the property that for each $k \in \mathbb{N}$, there is

[1] As a sequence space, ℓ_p is a subset of $d_{w,p}$. That is, we can identify ℓ_p with Range j. More precisely, we claim here that if $(j^{-1}x_n)$ is bounded in ℓ_p then u is in Range j. Being a block sequence of (e_n), (x_n) is contained in Range j.

$n_k \in \mathbb{N}$ such that $\|x^*_{n_k}|_{(k,\infty)}\| \geqslant \varepsilon$. Let u_k be a restriction of x_{n_k} such that $u^*_k = x^*_{n_k}|_{[1,k]}$ and $v_k = x_{n_k} - u_k$.

Clearly, each nonzero entry of u_k is greater than or equal to the greatest entry of v_k. By Lemma 3.1.18, the k-th coordinate of u^*_k is less than or equal to $\frac{1}{s_k^{1/p}}$ where $s_k = \sum_{i=1}^k w_i$. It follows that (v_k) is a block sequence of (e_n) such that $\varepsilon \leqslant \|v_k\| \leqslant 1$ and absolute values of the entries of v_k are all at most $\frac{1}{s_k^{1/p}}$. Since $\lim_k s_k = +\infty$ by the definition of $d_{w,p}$, passing to a subsequence and using Remark 3.1.4 we may assume that (v_k) is equivalent to (f_n). By Proposition 3.1.8, (f_n) dominates (x_{n_k}). Using also Lemma 3.1.9, we obtain the following diagram:

$$(f_n) \succeq (x_{n_k}) \succeq (v_k) \sim (f_n).$$

Hence (x_{n_k}) is equivalent to (f_n).

(ii) Suppose that $x_n = \sum_{i=p_n+1}^{p_{n+1}} a_i e_i$. Clearly, the sequence (a_i) is bounded. Without loss of generality, $a_{p_n+1} \geqslant \ldots \geqslant a_{p_{n+1}} \geqslant 0$ for each n. Put $y_n = x^*_n$. Using a standard diagonalization argument and passing to a subsequence, we may assume that (y_n) converges coordinate-wise; put $b_i = \lim\limits_{n\to\infty} y_{n,i}$. It is easy to see that $b_i \geqslant b_{i+1}$ for all i. Put $u = (b_i)$.

Case: the sequence $(p_{n+1} - p_n)$ *is bounded.* Passing to a subsequence, we may assume that $N := p_{n_k+1} - p_{n_k}$ is a constant. Note that $\operatorname{supp} u \subseteq [1, N]$ and $\operatorname{supp} y_{n_k} \subseteq [1, N]$ for all k. Put $u_k = \sum_{i=p_{n_k}+1}^{p_{n_k+1}} b_{i-p_{n_k}} e_i$, then $u = u^*_k$ and (u_k) as a block of type I generated by u. By compactness, $\|x_{n_k} - u_k\| = \|y_{n_k} - u\| \to 0$. Therefore, passing to a further subsequence, we have $(x_{n_k}) \sim (u_k)$. Being a vector with finite support, u belongs to ℓ_p.

Case: the sequence $(p_{n+1} - p_n)$ *is unbounded.* We will construct the required subsequence (x_{n_k}) and a sequence (N_k) inductively. Put $n_1 = N_1 =$

1 and let $k > 1$. Suppose that $n_1, \dots, n_{k-1}$ and $N_1, \dots, N_{k-1}$ have already been selected. Since (x_n) is almost lengthwise bounded, we can find $N_k > N_{k-1}$ such that $\|y_n|_{(N_k,\infty)}\| < \frac{1}{k}$ for all n. Put $v_k = u|_{[1,N_k]}$. Using coordinate-wise convergence, we can find $n_k > n_{k-1}$ such that $\|y_{n_k}|_{[1,N_k]} - v_k\|_{\ell_p} < \frac{1}{k}$ and $p_{n_k} + N_k \leqslant p_{n_k+1}$. Put $u_k = \sum_{i=p_{n_k}+1}^{p_{n_k}+N_k} b_{i-p_{n_k}} e_i$. Then $u_k^* = v_k$, so that

$$\|x_{n_k}|_{(p_{n_k}, p_{n_k}+N_k]} - u_k\|_{\ell_p} = \|y_{n_k}|_{[1,N_k]} - v_k\|_{\ell_p} < \tfrac{1}{k} \tag{3.1}$$

and

$$\|x_{n_k}|_{(p_{n_k}+N_k, p_{n_k+1}]}\| = \|y_{n_k}|_{(N_k,\infty)}\| < \tfrac{1}{k}.$$

It follows that $\|x_{n_k} - u_k\| \to 0$. Passing to a subsequence, we get $(x_{n_k}) \sim (u_k)$.

Next, we show that $u \in d_{w,p}$. Since $\|\cdot\| \leqslant \|\cdot\|_{\ell_p}$, it follows from (3.1) that

$$\|v_k\| = \|u_k\| \leqslant \|x_{n_k}|_{(p_{n_k}, p_{n_k}+N_k]}\| + \tfrac{1}{k} \leqslant \|x_{n_k}\| + \tfrac{1}{k}.$$

Since (x_n) is bounded, so is (v_k). Since $\operatorname{supp} v_k = N_k \to \infty$, we have $u \in d_{w,p}$. For the "moreover" part, we argue in a similar way. By (3.1), we have

$$\|v_k\|_{\ell_p} \leqslant \|u_k\|_{\ell_p} \leqslant \|x_{n_k}|_{(p_{n_k}, p_{n_k}+N_k]}\|_{\ell_p} + \tfrac{1}{k} \leqslant \|x_{n_k}\|_{\ell_p} + \tfrac{1}{k}.$$

Therefore, if (x_n) is bounded in ℓ_p then so is (v_k), hence $u \in \ell_p$. ■

Lemma 3.1.29. *Suppose that (u_n) is a block of type I in $d_{w,p}$ generated by some $u = \sum_{i=1}^{\infty} b_i e_i$. If $b_i \downarrow 0$ and $u \in \ell_p$ then (u_n) has a subsequence equivalent to (e_n)*

Proof. By Corollary 4 of [7], we may assume that the basic sequence (u_n) is symmetric. It suffices to show that $[u_n]$ is isomorphic to $d_{w,p}$ because all symmetric bases in $d_{w,p}$ are equivalent; see e.g., Theorem 4 of [7]. Without loss of generality, $\|u\| = 1$. Lemma 4 of [7] asserts that $[u_n]$ is isomorphic to $d_{w,p}$ iff $(s_n^{(u)}) \sim (s_n)$, where $s_n = \sum_{i=1}^n w_i$, $s_n^{(u)} = \sum_{i=1}^\infty b_i^p(s_{ni} - s_{n(i-1)})$, and $(\alpha_n) \sim (\beta_n)$ means that there exist positive constants A and B such that $A\alpha_n \leqslant \beta_n \leqslant B\alpha_n$ for all n. Let's verify that this condition is, indeed, satisfied. On one hand, taking only the first term in the definition of $s_n^{(u)}$, we get $s_n^{(u)} \geqslant b_1^p s_n$. On the other hand, it follows from $w_i \downarrow$ that $s_{ni} - s_{n(i-1)} \leqslant s_n$ for every i, hence $s_n^{(u)} \leqslant \sum_{i=1}^\infty b_i^p s_n = \|u\|_{\ell_p}^p s_n$. ∎

Lemma 3.1.30. *Let (x_n) be a block sequence of (f_n) in ℓ_p such that the sequences (x_n) and (jx_n) are seminormalized in ℓ_p and $d_{w,p}$, respectively. Then there exists a subsequence (x_{n_k}) such that $(jx_{n_k}) \sim (e_n)$.*

Proof. Clearly, $(x_n) \sim (f_n)$. It follows that $(jx_n) \not\sim (f_n)$ because, otherwise, j would be an isomorphism on $[x_n]$, which is impossible because j is strictly singular by Theorem 3.1.23. Applying Theorem 3.1.28 to (jx_n) and passing to a subsequence, we may assume that $(jx_n) \sim (u_n)$, where (u_n) is a block of type I generated by some $u = \sum_{i=1}^\infty b_i e_i$ such that $b_i \downarrow 0$ and $u \in \ell_p$. Applying Lemma 3.1.29 and passing to a subsequence, we get $(u_n) \sim (e_n)$. ∎

Theorem 3.1.31. *If $T \in J^j \setminus \mathcal{K}(d_{w,p})$ then $J^j \subseteq J_T$.*

Proof. Write $T = AjB$ where $B\colon d_{w,p} \to \ell_p$ and $A\colon d_{w,p} \to d_{w,p}$. Let (x_n) be as in Lemma 3.1.26. The sequence (Bx_n) is bounded, hence we may assume by passing to a subsequence that it converges coordinate-wise. Since (Tx_n) is weakly null and seminormalized, it has no convergent subsequences. It follows that, after passing to a subsequence of (x_n), we may assume that (Tz_n) is seminormalized, where $z_n = x_{2n} - x_{2n-1}$. In particular, (z_n), (Bz_n), and (jBz_n) are seminormalized. Also, (Bz_n) converges to zero coordinate-wise.

Using Theorem 3.1.3 and passing to a further subsequence, we may assume that (Bz_n) is equivalent to a block sequence (u_n) of (f_n) and $Bz_n - u_n \to 0$. It follows from $(f_n) \succeq (x_n)$ that $(f_n) \succeq (z_n) \succeq (Bz_n) \sim (u_n) \sim (f_n)$. In particular, $(z_n) \sim (f_n)$.

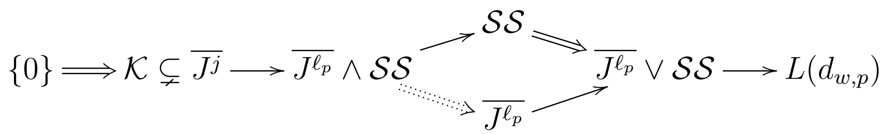
Since $Bz_n - u_n \to 0$ and (jBz_n) is seminormalized, we may assume that the sequence (ju_n) is seminormalized. By Lemma 3.1.30, passing to a further subsequence, we may assume that (ju_n) and, hence, (jBz_n) are equivalent to (e_n).

Passing to a subsequence and using Theorem 3.1.3, we may assume that (Tz_n) is equivalent to a block sequence (v_n) of (e_n) such that $Tz_n - v_n \to 0$. Since $T \in \mathcal{SS}(d_{w,p})$, no subsequence of (Tz_n) and, therefore, of (v_n) is equivalent to (f_n). By Proposition 3.1.8, $(v_n) \succeq (e_n)$. It follows from $(jBz_n) \sim (e_n)$ that $(e_n) \succeq (Tz_n)$, hence $(Tz_n) \sim (e_n) \sim (v_n)$.

Write $v_n = \sum_{i=p_n+1}^{p_{n+1}} a_n e_n$. By Remark 3.1.4, $a_n \not\to 0$. Hence, passing to a subsequence and using [18, Remark 9], we may assume that $[v_n]$ is complemented. By Theorem 3.1.3, we may assume that $[Tz_n]$ is complemented. Let $P \in L(d_{w,p})$ be a projection onto $[Tz_n]$ and $U \in L(\ell_p, d_{w,p})$ and $V \in L([Tz_n], d_{w,p})$ be defined by $Uf_n = z_n$ and $VTz_n = e_n$. Then we can write $j = VPTU$. Therefore $J^j \subseteq J_T$. ∎

3.1.4 $d_{w,p}$-strictly singular operators

The ideals in $L(d_{w,p})$ we have obtained so far can be arranged into the following diagram.



(see the Introduction for the notations). In this section, we will characterize the greatest ideal in the algebra $L(d_{w,p})$, that is, a proper ideal in $L(d_{w,p})$ that contains all other proper ideals in $L(d_{w,p})$.

If X and Y are two Banach spaces, then an operator $T \in L(X)$ is called ***Y-strictly singular*** if for any subspace Z of X isomorphic to Y, the restriction $T|_Z$ is not an isomorphism. The set of all Y-strictly singular operators in $L(d_{w,p})$ will be denoted by $\mathcal{SS}_Y$.

According to this notation, the symbol $\mathcal{SS}_{d_{w,p}}$ stands for the set of all $d_{w,p}$-strictly singular operators in $L(d_{w,p})$ (not to be confused with $\mathcal{SS}(d_{w,p})$).

Lemma 3.1.32. *Suppose that $T \in \mathcal{SS}_{d_{w,p}}$ and (x_n) is a basic sequence in $d_{w,p}$ equivalent to the unit vector basis (e_n). Then $Tx_n \to 0$.*

Proof. Suppose, by way of contradiction, that $Tx_n \not\to 0$. Then there is a subsequence (x_{n_k}) such that (Tx_{n_k}) is seminormalized. Since (x_n) is weakly null (Remark 3.1.7), we may assume by using Theorem 3.1.3 and passing to a further subsequence that (Tx_{n_k}) is a basic sequence equivalent to a block sequence (z_k) of (e_n).

By Proposition 3.1.8, either (z_k) has a subsequence equivalent to (f_n) or $(z_k) \succeq (e_n)$. Since (Tx_{n_k}) cannot have subsequences equivalent to (f_n) (this would contradict boundedness of T), the former is impossible. Therefore $(z_k) \succeq (e_n)$. We obtain the following diagram:

$$(e_n) \sim (x_{n_k}) \succeq (Tx_{n_k}) \sim (z_k) \succeq (e_n).$$

Therefore $T|_{[x_{n_k}]}$ is an isomorphism. This contradicts T being in $\mathcal{SS}_{d_{w,p}}$. ∎

Corollary 3.1.33. *Let $T \in \mathcal{SS}_{d_{w,p}}$. If $Y \subseteq d_{w,p}$ is a subspace isomorphic to $d_{w,p}$ then there is a subspace $Z \subseteq Y$ such that Z is isomorphic to $d_{w,p}$ and $T|_Z$ is compact.*

Proof. Let (x_n) be a basis of Y equivalent to (e_n). By Lemma 3.1.32, $Tx_n \to 0$. There is a subsequence (x_{n_k}) of (x_n) such that $\sum_{k=1}^{\infty} \frac{\|Tx_{n_k}\|}{\|x_{n_k}\|}$ is convergent. Let $Z = [x_{n_k}]$. It follows that Z is isomorphic to $d_{w,p}$ and $T|_Z$ is compact (see, e.g., [16, Lemma 5.4.10]). ∎

Theorem 3.1.34. *The set $\mathcal{SS}_{d_{w,p}}$ of all $d_{w,p}$-strictly singular operators in $L(d_{w,p})$ is the greatest proper ideal in the algebra $L(d_{w,p})$. In particular, $\mathcal{SS}_{d_{w,p}}$ is closed.*

Proof. First, let us show that $\mathcal{SS}_{d_{w,p}}$ is an ideal. Let $T \in \mathcal{SS}_{d_{w,p}}$. If $A \in L(d_{w,p})$ then, trivially, $AT \in \mathcal{SS}_{d_{w,p}}$. If $TA \notin \mathcal{SS}_{d_{w,p}}$ then there exists a subspace Y of $d_{w,p}$ such that Y and $TA(Y)$ are both isomorphic to $d_{w,p}$. Then $A|_Y$ is bounded below, hence $A(Y)$ is isomorphic to $d_{w,p}$. It follows that T is an isomorphism on a copy of $d_{w,p}$, contrary to $T \in \mathcal{SS}_{d_{w,p}}$. So, $\mathcal{SS}_{d_{w,p}}$ is closed under two-sided multiplication by bounded operators.

Let $T, S \in \mathcal{SS}_{d_{w,p}}$. We will show that $T + S \in \mathcal{SS}_{d_{w,p}}$. Let Y be a subspace of $d_{w,p}$ isomorphic to $d_{w,p}$. By Corolary 3.1.33, there exists a subspace

Z of Y such that Z is isomorphic to $d_{w,p}$ and $T|_Z$ is compact. Applying Corollary 3.1.33 again, we can find a subspace V of Z such that V is isomorphic to $d_{w,p}$ and $S|_V$ is compact. Therefore $(T+S)|_V$ is compact, so that $(T+S)|_Y$ is not an isomorphism. So, $\mathcal{SS}_{d_{w,p}}$ is an ideal.

Clearly, the identity operator I does not belong to $\mathcal{SS}_{d_{w,p}}$, so $\mathcal{SS}_{d_{w,p}}$ is proper. Let us show that $\mathcal{SS}_{d_{w,p}}$ is the greatest ideal in $L(d_{w,p})$.

Let $T \notin \mathcal{SS}_{d_{w,p}}$. Then there exists a subspace Y of $d_{w,p}$ such that Y and $T(Y)$ are isomorphic to $d_{w,p}$. By [18, Corollary 12], there exists a complemented (in $d_{w,p}$) subspace Z of $T(Y)$ such that Z is isomorphic to $d_{w,p}$. Let $P \in L(d_{w,p})$ be a projection onto Z. Put $H = T^{-1}(Z)$. It follows that H is isomorphic to $d_{w,p}$. Let $U \colon d_{w,p} \to H$ and $V \colon Z \to d_{w,p}$ be surjective isomorphisms. Then $S \in L(d_{w,p})$ defined by $S = (VP)TU$ is an invertible operator. Clearly $S \in J_T$, hence $J_T = L(X)$.

The fact that $\mathcal{SS}_{d_{w,p}}$ is closed follows from [23, Corollary VII.2.4]. ■

The next theorem provides a convenient characterization of $d_{w,p}$-strictly singular operators.

Lemma 3.1.35. *Let $T \in L(d_{w,p})$ be such that $Te_n \to 0$. Suppose that (x_n) is a bounded block sequence of (e_n) in $d_{w,p}$ such that (x_n) is almost lengthwise bounded. Then $Tx_n \to 0$.*

Proof. Write $x_n = \sum_{i=p_n+1}^{p_{n+1}} a_i e_i$. Since (x_n) is bounded, there is $C > 0$ such that $|a_i| \leqslant C$ for all i and $n \in \mathbb{N}$. Let $\varepsilon > 0$. Find $N \in \mathbb{N}$ such that $\|x_n^*|_{[N,\infty)}\| < \varepsilon$ for all $n \in \mathbb{N}$. Let u_n be a restriction of x_n such that $u_n^* = x_n^*|_{[1,N)}$ and $v_n = x_n - u_n$. It is clear that $\|v_n\| = \|x_n^*|_{[N,\infty)}\| < \varepsilon$. Also, $\|Tu_n\| \leqslant NC \cdot \max_{p_n+1 \leqslant i \leqslant p_{n+1}} \|Te_i\|$.

Pick $M \in \mathbb{N}$ such that $\|Te_k\| < \frac{\varepsilon}{N}$ for all $k \geqslant M$. Then

$$\|Tx_n\| \leqslant \|Tu_n\| + \|Tv_n\| \leqslant NC\frac{\varepsilon}{N} + \varepsilon\|T\| = \varepsilon\big(C + \|T\|\big)$$

for all n such that $p_n > M$. It follows that $Tx_n \to 0$. ■

Theorem 3.1.36. *An operator $T \in L(d_{w,p})$ is $d_{w,p}$-strictly singular if and only if $Te_n \to 0$.*

Proof. Suppose that $Te_n \to 0$ but $T \notin \mathcal{SS}_{d_{w,p}}$. Then there exists a subspace Y of $d_{w,p}$ such that Y is isomorphic to $d_{w,p}$ and $T|_Y$ is an isomorphism. Let (x_n) be a basis of Y equivalent to (e_n). By Remark 3.1.7, $x_n \xrightarrow{w} 0$. Using Theorem 3.1.3 and passing to a subsequence, we may assume that (x_n) is equivalent to a block sequence (z_n) of (e_n) such that $x_n - z_n \to 0$. Since (z_n) is equivalent to (e_n), it is almost lengthwise bounded by Theorem 3.1.28. By Lemma 3.1.35, $Tz_n \to 0$. Since $x_n - z_n \to 0$, we obtain $Tx_n \to 0$. This is a contradiction since (x_n) is seminormalized and $T|_{[x_n]}$ is an isomorphism.

The converse implication follows from Lemma 3.1.32. ■

Remark 3.1.37. In Theorem 3.1.34 we showed, in particular, that $\mathcal{SS}_{d_{w,p}}$ is closed under addition. Alternatively, we could have deduced this from Theorem 3.1.36.

Recall that an operator T on a Banach space X is called ***Dunford-Pettis*** if for any sequence (x_n) in X, $x_n \xrightarrow{w} 0$ implies $Tx_n \to 0$. If $1 < p < \infty$ then the class of Dunford-Pettis operators on $d_{w,p}$ coincides with $\mathcal{K}(d_{w,p})$ because $d_{w,p}$ is reflexive. For the case $p = 1$ we have the following result.

Theorem 3.1.38. *Let $T \in L(d_{w,1})$. Then T is $d_{w,1}$-strictly singular if and only if T is Dunford-Pettis.*

Proof. If T is Dunford-Pettis then then T is $d_{w,1}$-strictly singular by Theorem 3.1.36 because (e_n) is weakly null.

Conversely, suppose that T is $d_{w,1}$-strictly singular. Let (x_n) be a weakly null sequence. Suppose that (Tx_n) does not converge to zero. Then, passing to a subsequence, we may assume that (x_n) is a seminormalized weakly null basic sequence equivalent to a block sequence (u_n) of (e_n) such that $x_n - u_n \to 0$. Clearly, (u_n) is weakly null. In particular, (u_n) has no subsequences equivalent to (f_n). By Theorem 3.1.28, (u_n) is almost lengthwise bounded. Hence, by Lemma 3.1.35, $Tu_n \to 0$. It follows that $Tx_n \to 0$, contrary to the choice of (x_n). ■

3.1.5 Strictly singular operators between ℓ_p and $d_{w,p}$.

We do not know whether the ideals $\overline{J^j}$, $\mathcal{SS} \wedge \overline{J^{\ell_p}}$, and $\mathcal{SS}$ are distinct. In this section, we discuss some connections between these ideals.

Conjecture 3.1.39. $\overline{J^j} = \mathcal{SS} \wedge \overline{J_{\ell_p}}$. In particular, every strictly singular operator in $L(d_{w,p})$ which factors through ℓ_p can be approximated by operators that factor through j.

The following statement is a refinement of Lemma 3.1.9. Recall that $d_{w,p}$ is a Banach lattice with respect to the coordinate-wise order.

Lemma 3.1.40. *Suppose that (x_n) and (y_n) are seminormalized sequences in $d_{w,p}$ such that $|x_n| \geqslant |y_n|$ for all $n \in \mathbb{N}$ and $x_n \to 0$ coordinate-wise. Then there exists an increasing sequence (n_k) in $\mathbb{N}$ such that (x_{n_k}) and (y_{n_k}) are basic and $(x_{n_k}) \succeq (y_{n_k})$.*

Proof. Clearly, $y_n \to 0$ coordinate-wise. By Theorem 3.1.3, we can find a sequence (n_k) and two block sequences (u_k) and (v_k) of (e_n) such that (x_{n_k}) and (y_{n_k}) are basic, $(x_{n_k}) \sim (u_k)$, $(y_{n_k}) \sim (v_k)$, $x_{n_k} - u_k \to 0$, $y_{n_k} - v_k \to 0$, and for each $k \in \mathbb{N}$, the vector u_k (v_k, respectively) is a restriction of (x_{n_k}) (of (y_{n_k}), respectively).

For each $k \in \mathbb{N}$, define $h_k \in d_{w,p}$ by putting its i-th coordinate to be equal to $h_k(i) = \operatorname{sign}\bigl(v_k(i)\bigr) \cdot \bigl(|u_k(i)| \wedge |v_k(i)|\bigr)$. Then (h_k) is a block sequence of (e_n) such that $|h_k| \leqslant |u_k|$. A straightforward verification shows that $|h_k - v_k| \leqslant |u_k - x_{n_k}|$. It follows that $h_k - v_k \to 0$. By Theorem 3.1.2, passing to a subsequence, we may assume that (h_k) is basic and $(h_k) \sim (v_k)$. By Lemma 3.1.9, $(u_k) \succeq (h_k)$. Hence $(x_{n_k}) \succeq (y_{n_k})$. ∎

The next lemma is a version of Theorem 3.1.28 for the case (x_n) is an arbitrary bounded sequence.

Lemma 3.1.41. *If the bounded sequence (x_n) in $d_{w,p}$ is not almost length-wise bounded, then there is a subsequence (x_{n_k}) such that $(x_{n_{2k}} - x_{n_{2k-1}})$ is equivalent to the unit vector basis (f_n) of ℓ_p.*

Proof. We can assume without loss of generality that no subsequence of (x_n) is equivalent to the unit vector basis of ℓ_1. Indeed, if (x_{n_k}) is equivalent to the unit vector basis of ℓ_1 then $p = 1$. It follows that (x_{n_k}) is equivalent to (f_n) and hence $(x_{n_{2k}} - x_{n_{2k-1}})$ is equivalent to (f_n), as well.

Without loss of generality, $\sup_n \|x_n\| = 1$. Since (x_n) is not almost length-wise bounded, there exists $c > 0$ such that

$$\forall N \in \mathbb{N} \quad \exists n \in \mathbb{N} \quad \|x_n^*|_{[N,\infty)}\| > c. \tag{3.2}$$

Let $\frac{c}{4} > \varepsilon_k \downarrow 0$. We will inductively construct increasing sequences (n_k) and (N_k) in $\mathbb{N}$ and a sequence (y_k) in $d_{w,p}$ such that the following conditions are satisfied for each k:

(i) $\|x_{n_k}|_{[N_{k+1},\infty)}\| < \varepsilon_k$;

(ii) y_k is supported on $[N_k, N_{k+1})$;

(iii) y_k is a restriction of x_{n_k};

(iv) $\|y_k\| > \frac{c}{2}$;

(v) $\|y_k\|_\infty \leqslant s_{N_k}^{-1/p}$ where s_N is as in Lemma 3.1.18.

For $k = 1$, we put $N_1 = 1$, and define n_1 to be the first number n such that $\|x_n\| > c$; such an n exists by (3.2). Pick $N_2 \in \mathbb{N}$ such that $\|x_{n_1}|_{[N_2,\infty)}\| < \varepsilon_1$. Put $y_1 = x_{n_1}|_{[N_1,N_2)}$. It follows that $1 \geqslant \|y_1\| > c - \varepsilon_1 > \frac{c}{2}$, and the coordinates of y_1 are all at most 1 ($= s_1^{-1/p}$), hence all the conditions (i)–(v) are satisfied for $k = 1$.

Suppose that appropriate sequences $(n_i)_{i=1}^k$, $(N_i)_{i=1}^{k+1}$, and $(y_i)_{i=1}^k$ have been constructed. Use (3.2) to find n_{k+1} such that $\|x^*_{n_{k+1}}|_{[2N_{k+1},\infty)}\| > c$. Let z be the vector obtained from $x_{n_{k+1}}$ by replacing its N_{k+1} largest (in absolute value) entries with zeros. Then $\|z|_{[N_{k+1},\infty)}\| \geqslant \|z^*|_{[N_{k+1},\infty)}\| = \|x^*_{n_{k+1}}|_{[2N_{k+1},\infty)}\| > c$. By Lemma 3.1.18, $\|z\|_\infty \leqslant s_{N_{k+1}}^{-1/p}$. Choose N_{k+2} such that $\|x_{n_{k+1}}|_{[N_{k+2},\infty)}\| < \varepsilon_{k+1}$. It follows that $\|z|_{[N_{k+2},\infty)}\| < \varepsilon_{k+1}$. Put $y_{k+1} = z|_{[N_{k+1},N_{k+2})}$. Then $\|y_{k+1}\| \geqslant c - \varepsilon_{k+1} > \frac{c}{2}$, and the inductive construction is complete.

The sequence (y_k) constructed above is a seminormalized block sequence of (e_n) such that the coordinates of (y_k) converge to zero by condition (v).

Using Remark 3.1.4 and passing to a subsequence, we may assume that (y_k) is equivalent to the unit vector basis (f_n) of ℓ_p.

Since (x_n) contains no subsequences equivalent to the unit vector basis of ℓ_1, using the Rosenthal's ℓ_1-theorem and passing to a further subsequence, we may assume that (x_{n_k}) is weakly Cauchy. For all $m > k \in \mathbb{N}$, we have: $\|x_{n_k}|_{[N_m,\infty)}\| \leqslant \|x_{n_k}|_{[N_{k+1},\infty)}\| \leqslant \varepsilon_k$. Therefore $\|x_{n_m} - x_{n_k}\| \geqslant \|(x_{n_m} - x_{n_k})|_{[N_m,\infty)}\| \geqslant \|x_{n_m}|_{[N_m,\infty)}\| - \varepsilon_k \geqslant \|y_m\| - \varepsilon_k \geqslant \frac{c}{2} - \varepsilon_k > \frac{c}{4}$. It follows that the sequence (u_k) defined by $u_k = x_{n_{2k}} - x_{n_{2k-1}}$ is seminormalized and weakly null. Passing to a subsequence of (x_{n_k}), we may assume that (u_k) is equivalent to a block sequence of (e_n). By Proposition 3.1.8, $(f_n) \succeq (u_k)$.

Let $v_k = x_{n_{2k}} - \big(x_{n_{2k-1}}|_{[1,N_{2k})}\big)$. Then $\|u_k - v_k\| = \|x_{n_{2k-1}}|_{[N_{2k},\infty)}\| < \varepsilon_{2k-1} \to 0$. By Theorem 3.1.2, passing to a subsequence of (x_{n_k}), we may assume that (v_k) is basic and $(v_k) \sim (u_k)$. Also, (v_k) is weakly null. Note that $|y_{2k}| \leqslant |v_k|$ for all $k \in \mathbb{N}$, since $\operatorname{supp} y_{2k} \subseteq [N_{2k}, N_{2k+1})$, so that y_{2k} is a restriction of v_k. By Lemma 3.1.40, passing to a subsequence, we may assume that $(v_k) \succeq (y_{2k})$. Therefore we obtain the following diagram:

$$(f_k) \succeq (u_k) \sim (v_k) \succeq (y_{2k}) \sim (f_{2k}) \sim (f_n).$$

It follows that (u_k) is equivalent to (f_k). ■

Corollary 3.1.42. *If $T \in \mathcal{SS}(\ell_p, d_{w,p})$ then the sequence (Tf_n) is almost lengthwise bounded.*

Proof. Suppose that (Tf_n) is not almost lengthwise bounded. By Lemma 3.1.41, there is a subsequence (f_{n_k}) such that $(Tf_{n_{2k}} - Tf_{n_{2k-1}})$ is equivalent to (f_n). It follows that $T|_{[f_{n_{2k}} - f_{n_{2k-1}}]}$ is an isomorphism. ■

Remark 3.1.43. If we view T as an infinite matrix, the vectors (Tf_n) represent its columns.

Theorem 3.1.44. *If $T \in L(\ell_1, d_{w,1})$ is such that the sequence (Tf_n) is almost lengthwise bounded, then for any $\varepsilon > 0$ there exists $S \in L(\ell_1)$ such that $\|T - jS\| < \varepsilon$, where $j \in L(\ell_1, d_{w,1})$ is the formal identity operator.*

Proof. Let $\varepsilon > 0$ be fixed. Find $N \in \mathbb{N}$ such that $\|(Tf_n)^*|_{[N,\infty)}\| < \varepsilon$ for all n. Let $z_n \in d_{w,1}$ be the vector obtained from Tf_n by keeping its largest N coordinates and replacing the rest of the coordinates with zeros.

Define $S\colon \ell_1 \to d_{w,1}$ by $Sf_n = z_n$. Note that $\|T - S\| = \sup_n \|(T - S)f_n\| = \sup_n \|Tf_n - z_n\| \leqslant \varepsilon$; in particular, S is bounded. Let $F = \operatorname{span}\{e_1, \dots, e_N\}$. Since $\dim F < \infty$, there exists $C > 0$ such that

$$\frac{1}{C}\|x\|_{\ell_1} \leqslant \|x\|_{d_{w,1}} \leqslant C\|x\|_{\ell_1}$$

for all $x \in F$. Observe that for each $n \in \mathbb{N}$, the non-increasing rearrangement $(Sf_n)^*$ is in F. Therefore, for all $n \in \mathbb{N}$, we have

$$\|Sf_n\|_{\ell_1} = \|(Sf_n)^*\|_{\ell_1} \leqslant C\|(Sf_n)^*\|_{d_{w,1}} = C\|Sf_n\|_{d_{w,1}} \leqslant C\|S\|.$$

It follows that the operator $\widetilde{S}\colon \ell_1 \to \ell_1$ defined by $\widetilde{S}f_n = Sf_n$ belongs to $L(\ell_1)$. Obviously, $S = j\widetilde{S}$. So, $\|T - j\widetilde{S}\| < \varepsilon$. ∎

The next corollary follows immediately from Theorem 3.1.44 and Corollary 3.1.42. This corollary can be considered as a support for Conjecture 3.1.39.

Corollary 3.1.45. *$\mathcal{SS}(\ell_1, d_{w,1})$ is contained in the closure of $\{jS : S \in L(\ell_1)\}$.*

Question. Does Corollary 3.1.45 remain valid for $p > 1$?

The following fact is standard, we include its proof for convenience of the reader.

Proposition 3.1.46. *If X is a Banach space then $\mathcal{SS}(X, \ell_1) = \mathcal{K}(X, \ell_1)$.*

Proof. Let $T \notin \mathcal{K}(X, \ell_1)$. Pick a bounded sequence (x_n) in X such that (Tx_n) has no convergent subsequences. By Schur's theorem, (Tx_n) and, therefore, (x_n) have no weakly Cauchy subsequences. Applying Rosenthal's ℓ_1-theorem twice, we find a subsequence (x_{n_k}) such that (x_{n_k}) and (Tx_{n_k}) are both equivalent to the unit vector basis of ℓ_1. It follows that T is not strictly singular. ∎

Proposition 3.1.47. *For all $p \in [1, \infty)$, $\mathcal{SS}(d_{w,p}, \ell_p) = \mathcal{K}(d_{w,p}, \ell_p)$.*

Proof. By Proposition 3.1.46, we only have to consider the case $p > 1$. Let $T \notin \mathcal{K}(X, \ell_p)$. Pick a bounded sequence (x_n) in X such that (Tx_n) has no convergent subsequences. Since $d_{w,p}$ contains no copies of ℓ_1, by Rosenthal's ℓ_1-theorem we may assume that (x_n) is weakly Cauchy. Passing to a further subsequence, we may assume that the sequence (Ty_n), where $y_n = x_{2n} - x_{2n-1}$, is seminormalized. It follows that (y_n) is also seminormalized. Also, (y_n) and, therefore, (Ty_n) are weakly null. Passing to a subsequence of (x_n), we may assume that (y_n) and (Ty_n) are both basic, equivalent to block sequences of (e_n) and (f_n), respectively. By [7, Proposition 5] and [63, Proposition 2.a.1], $(f_n) \succeq (y_n)$ and $(f_n) \sim (Ty_n)$. So, we obtain the diagram

$$(f_n) \succeq (y_n) \succeq (Ty_n) \sim (f_n).$$

Hence $[y_n]$ is isomorphic to $[Ty_n]$, so that T is not strictly singular. ∎

The following lemma is standard.

Lemma 3.1.48. *Let X be a Banach space. Every seminormalized basic sequence in X is dominated by the unit vector basis of ℓ_1.*

Lemma 3.1.49. *Let (x_n) and (y_n) be two sequences in a Banach space X such that (x_n) is equivalent to the unit vector basis of ℓ_1 and (y_n) is convergent. Then the sequence (z_n) defined by $z_n = x_n + y_n$ has a subsequence equivalent to the unit vector basis of ℓ_1.*

Proof. Observe that (z_n) cannot have weakly Cauchy subsequences since (x_n) does not have such subsequences. Since (z_n) is bounded, the result follows from Rosenthal's ℓ_1-theorem. ∎

Recall that an operator A between two Banach lattices X and Y is called ***positive*** if $x \geqslant 0$ entails $Tx \geqslant 0$.

Conjecture 3.1.39 asserts, in particular, that if $T \in \mathcal{SS}(d_{w,p})$ and $T = AB$ for some $A\colon d_{w,p} \to \ell_p$ and $B\colon \ell_p \to d_{w,p}$ then $T \in \overline{J^j}$. In the next theorem, we prove this under the additional assumptions that $p = 1$ and both A and B are positive.

Theorem 3.1.50. *Let $T \in \mathcal{SS}(d_{w,1})$ be such that $T = AB$, where $A \in L(\ell_1, d_{w,1})$, $B \in L(d_{w,1}, \ell_1)$, and both A and B are positive. Then $T \in \overline{J^j}$.*

Proof. Define a sequence (A_N) of operators in $L(\ell_1, d_{w,1})$ by the following procedure. For each $n \in \mathbb{N}$, let $A_N f_n$ be obtained from Af_n by keeping the largest N coordinates and replacing the rest of the coordinates with zeros. Since $Af_n \geqslant 0$ for all $n \in \mathbb{N}$, this defines a positive operator $\ell_1 \to d_{w,1}$. Also, $\|A_N f_n\| \leqslant \|Af_n\| \leqslant \|A\|$ for all $n \in \mathbb{N}$, hence $\|A_N\| \leqslant \|A\|$.

Define $A'_N = A - A_N$. It is clear that $0 \leqslant A'_N f_n \leqslant Af_n$ for all $n \in \mathbb{N}$, hence $A'_N \geqslant 0$ and $\|A'_N\| \leqslant \|A\|$. We claim that $A'_N \to 0$ in the strong operator topology (SOT). Indeed, since $A'_N f_n$ is obtained from Af_n by removing the largest N coordinates, the elements of the matrix of A'_N are all smaller than $\frac{\|A\|}{s_N}$ by Lemma 3.1.18. In particular, if $0 \leqslant x \in \ell_1$, then $A'_N x \downarrow 0$; it follows that $\|A'_N x\| \to 0$ because $d_{w,1}$ has order continuous norm (see Remark 3.1.5). If $x \in \ell_1$ is arbitrary then $\|A'_N x\| \leqslant \big\|A'_N |x|\,\big\| \to 0$.

We will show that $\|A'_N B\| \to 0$ as $N \to \infty$, so that $\|AB - A_N B\| \to 0$ as $N \to \infty$. Since $(A_N f_n)_{n=1}^{\infty}$ is almost lengthwise bounded (in fact, the vectors in the sequence $(A_N f_n)_{n=1}^{\infty}$ all have at most N nonzero entries), the theorem will follow from Theorem 3.1.44.

Assume, by way of contradiction, that there are $c > 0$ and a sequence (N_k) in $\mathbb{N}$ such that $\|A'_{N_k} B\| > c$. Then there exists a normalized positive sequence (x_k) in $d_{w,p}$ such that $\|A'_{N_k} B x_k\| > c$. By Rosenthal's ℓ_1-theorem, we may assume that (x_k) is either weakly Cauchy or equivalent to (f_n).

Assume that (x_k) is weakly Cauchy. Then (Bx_k) is weakly Cauchy. Since (Bx_k) is a sequence in ℓ_1, it must converge to some $z \in \ell_1$ by the Schur property. Then $\|A'_{N_k} B x_k - A'_{N_k} z\| \leqslant \|A'_{N_k}\| \cdot \|Bx_k - z\| \leqslant \|A\| \cdot \|Bx_k - z\| \to 0$. Since $A'_{N_k} \to 0$ in SOT, it follows that $A'_{N_k} B x_k \to 0$, contrary to the assumption. Therefore (x_k) must be equivalent to (f_n).

Since the entries of the matrix of A'_N are all less than $\frac{\|A\|}{s_N}$, the coordinates of the vector $A'_{N_k} B x_k$ are all less than $\frac{\|A\|}{s_{N_k}} \|B\| \to 0$. Hence, passing to a subsequence, we may assume that $(A'_{N_k} B x_k)$ is equivalent to a block sequence (u_k) of (e_n) such that each u_k is a restriction of $A'_{N_k} B x_k$. In particular, the coordinates of (u_k) converge to zero. Passing to a further subsequence, we may assume by Remark 3.1.4 that $(A'_{N_k}Bx_k) \sim (f_n)$.

The sequence (Tx_k) cannot have subsequences equivalent to (f_n) since T is strictly singular. Therefore, by Rosenthal's ℓ_1-theorem, we may assume that (Tx_k) is weakly Cauchy. Since $d_{w,1}$ is weakly sequentially complete (Remark 3.1.5), the sequence (Tx_k) weakly converges to a vector $y \in d_{w,1}$. Since the positive cone in a Banach lattice is weakly closed, $y \geqslant 0$.

Note that $Tx_k \geqslant A'_{N_k}Bx_k \geqslant u_k \geqslant 0$ for every k. Since (u_k) is a seminormalized block sequence of (e_n), it follows that (Tx_k) is not norm convergent. Write $Tx_k = y + h_k$; then (h_k) converges to zero weakly but not in norm. Therefore, passing to a subsequence, we may assume that (h_k) is seminormalized and basic (but not, necessarily, positive).

Let $r_k = A'_{N_k}Bx_k - (A'_{N_k}Bx_k \wedge y) \geqslant 0$, $k \in \mathbb{N}$. Observe that $A'_{N_k}Bx_k \wedge y \in [0, y]$ for all k. Since $d_{w,1}$ has order continuous norm and the order in $d_{w,1}$ is defined by a 1-unconditional basis, order intervals in $d_{w,1}$ are compact (see, e.g., [103, Theorem 6.1]). Therefore, passing to a subsequence of (x_{n_k}), we may assume that $(A'_{N_k}Bx_k \wedge y)$ is convergent, hence, passing to a further subsequence, (r_k) is equivalent to (f_n) by Lemma 3.1.49 and Theorem 3.1.2.

It follows from $y + h_k \geqslant A'_{N_k}Bx_k \geqslant 0$ that $|h_k| \geqslant r_k$ for all k. Passing to a subsequence, we may assume by Lemma 3.1.40 that $(h_k) \succeq (r_k) \sim (f_n)$. By Lemma 3.1.48, in fact $(h_k) \sim (f_n)$, and, hence, by Lemma 3.1.49, $(ABx_k) \sim (f_n)$. Since also $(x_k) \sim (f_n)$, this contradicts to $T = AB \in \mathcal{SS}(d_{w,1})$. ■

3.2 Strictly singular operators on noncommutative L_p

The results of this section are based on [71]. This section is structured as follows. In Subection 3.2.1, we consider operators on $L_p(\tau)$, where τ is a faithful normal trace on a finite hyperfinite algebra. Generalizing [97], we show that $T \in L(L_p(\tau))$ is strictly singular if and only if $L_p(\tau)$ contains a subspace E, isomorphic to either ℓ_2 or ℓ_p, so that $T|_E$ is an isomorphism, and both E and $T(E)$ are complemented (Theorem 4). We also show that, if either $2 \le u_2 \le u_1 < \infty$, or $1 < u_1 \le u_2 \le 2$, then $\mathcal{SS}(L_{u_1}(\tau_1), L_{u_2}(\tau_2))_+ = \mathcal{K}(L_{u_1}(\tau_1), L_{u_2}(\tau_2))_+$. (Proposition 4).

In Subection 3.2.2, we restrict our attention to the Schatten spaces $\mathfrak{C}_p$. We show that $\mathcal{SS}(\mathfrak{C}_p, \mathfrak{C}_q) = \mathcal{K}(\mathfrak{C}_p, \mathfrak{C}_q)$ if $\infty \geqslant p \geqslant 2 \geqslant q \geqslant 1$, and otherwise, $\mathcal{SS}(\mathfrak{C}_p, \mathfrak{C}_q) \supsetneq \mathcal{FSS}(\mathfrak{C}_p, \mathfrak{C}_q) \supsetneq K(\mathfrak{C}_p, \mathfrak{C}_q)$ (Theorem 3.2.25). Similar coincidence results are established for positive operators (Theorems 3.2.27 and 3.2.28). Although the dual of a strictly singular operator need not be strictly singular, we show that $T \in \mathcal{SS}(\mathfrak{C}_\infty)$ if and only if $T^* \in \mathcal{SS}(\mathfrak{C}_1)$ (Proposition 3.2.23). Eventually, we prove that $T \in B(\mathfrak{C}_1, Z)$ is Dunford-Pettis iff its restriction to every copy of ℓ_2 is compact (Proposition 3.2.31).

Finally, in Subsection 3.2.3, we investigate ideals of operators on C^*-algebras. Among other things, we prove that a von Neumann algebra $\mathcal{A}$ is of finite type I if and only if $\mathcal{FSS}(\mathcal{A}) = \mathcal{SS}(A) = \mathcal{IN}(\mathcal{A}) = \mathcal{WK}(\mathcal{A})$. Moreover, if $\mathcal{A}$ is not of finite type I, then all of this classes are different (Theorem 4). Incidentally, we establish some results for commutative function spaces.

Throughout this section, we shall use the term *ℓ_p-basis* as a shorthand for "a sequence equivalent to the canonical basis of ℓ_p."

3.2.1 Noncommutative L_p: continuous case

Characterization of strictly singular operators

The main result of this subsection is:

Theorem 3.2.1. *Suppose τ is a faithful normal finite trace on a hyperfinite von Neumann algebra $\mathcal{A}$, and $1 < p < \infty$. For $T \in L(L_p(\tau))$, the following statements are equivalent:*

(i) T is not strictly singular.

(ii) $L_p(\tau)$ contains a subspace E, isomorphic either to ℓ_p or ℓ_2, so that $T|_E$ is an isomorphism, and both E and $T(E)$ are complemented.

Throughout, we assume $p \neq 2$, and $\tau(\mathbf{1}) = 1$. The implication (2) $\Rightarrow$ (1) is clear. Proving (1) $\Rightarrow$ (2) is easy for $2 < p < \infty$, due to Kadec-Pelczynski dichotomy (see e.g. [85, Theorem 0.2]): any infinite dimensional subspace of $L_p(\tau)$ contains a further subspace E, isomorphic to either ℓ_p or ℓ_2, and complemented in $L_p(\tau)$. In fact, for $2 < p < \infty$ our conclusion remains true even for any normal faithful semifinite trace τ on a von Neumann algebra $\mathcal{A}$ (not necessarily hyperfinite). Below, we use some ideas from [97] to tackle the case of $1 < p < 2$.

Proposition 3.2.2. *Suppose $\mathcal{A}$, τ, and p are as in Theorem 4. Then any separable subspace of $L_p(\tau)$ $(1 \leqslant p < \infty)$ is contained in a subspace with an unconditional FDD. Consequently, if $\mathcal{A}$ is separably acting, then $L_p(\tau)$ has an unconditional FDD.*

Remark 3.2.3. By [105, Lemma 1.8], for a von Neumann algebra $\mathcal{A}$ with a normal faithful semifinite trace τ, the following are equivalent: (i) $\mathcal{A}$ is separably acting; (ii) $\mathcal{A}$ has a separable predual, and (iii) $L_2(\tau)$ is separable. Consequently, these statements are equivalent to $L_p(\tau)$ being separable, for any (equivalently, all) $p \in [1, \infty)$.

Remark 3.2.4. The hyperfiniteness of $\mathcal{A}$ is essential here. Indeed, by [52, Theorem 2.19], for $p \in (1, 80/79) \cup (80, \infty)$, there exists a von Neumann algebra $\mathcal{A}$ with separable predual, equipped with a finite faithful normal trace τ, so that $L_p(\tau)$ fails the Approximation Property.

Proof. If $\mathcal{A}$ is a hyperfinite von Neumann algebra, it contains a net $(\mathcal{A}_\alpha)$ of finite dimensional von Neumann subalgebras, ordered by inclusion, so that $\mathcal{A}$ is the weak*-closure of $\cup_\alpha \mathcal{A}_\alpha$. The conditional expectations $Q_\alpha : \mathcal{A} \to \mathcal{A}_\alpha$ are completely contractive, and satisfy $Q_\alpha Q_\beta = Q_\beta Q_\alpha = Q_\alpha$ whenever $\alpha \leq \beta$. By [79, Theorem 3.4], Q_α extends to a completely contractive map from $L_p(\tau)$ to $L_p(\tau_\alpha)$, where τ_α is the restriction of τ to N_α, and $L_p(\tau)$ is the norm closure of $\cup_\alpha L_p(\tau_\alpha)$.

Now suppose (x_k) is a dense subset of a subspace $X \subset L_p(\tau)$. Then there exists an increasing sequence (α_k) so that $\max_{j \leq k} \operatorname{dist}(x_j, \mathcal{A}_{\alpha_k}) < 4^{-k}$ for any k. Now define $\mathcal{A}'$ as the weak* closure of $\cup_k \mathcal{A}_{\alpha_k}$ in $\mathcal{A}$, and let τ' be the restriction of τ to $\mathcal{A}'$. As noted in the proof of [79, Theorem 3.4], $L_p(\tau') = \overline{\cup_k L_p(\tau_{\alpha_k})}$, and this space contains X. By [80, Section 7] (or [83]), the subspaces $L_p(\tau_{\alpha_k}) \cap \ker Q_{\alpha_{k-1}}$ form an unconditional FDD. ∎

We say that a Banach space has the *Unconditional Sequence Property* (*USP*) if every weakly null seminormalized sequence contains an unconditional subsequence. Theorem 3.2.2 combined with [63, Theorem 1.g.5], [63, Proposition 1.a.12], and the fact that every normalized block sequence of an unconditional basis is unconditional imply the following.

Corollary 3.2.5. *Suppose τ is a normal faithful semifinite trace on a hyperfinite von Neumann algebra $\mathcal{A}$. Then, for $1 < p < \infty$, $L_p(\tau)$ has the USP.*

The USP of commutative L_p spaces ($1 < p < \infty$) is well known, and follows from the unconditionality of the Haar basis. It was proved in [51] that $L_1(0, 1)$ fails the USP. In the case of noncommutative L_1 we have the following.

Proposition 3.2.6. *Let τ be a normal faithful semifinite trace on a von-Neumann algebra $\mathcal{A}$. Then $L_1(\tau)$ has USP if and only if A is atomic of type I.*

Proof. If A is not atomic of type I then $L_1(\tau)$ contains a complemented copy of $L_1(0, 1)$ by [70, Theorem 1.5.3], and, therefore, it fails the USP. Otherwise, $L_1(\tau)$ has the USP since it can be written as $\sum_i (\mathfrak{C}_1(H_i))_{\ell_1}$, where H_i is a Hilbert space. ∎

Question. Suppose τ is a normal faithful semifinite trace on a von Neumann algebra $\mathcal{A}$, and $1 < p < \infty$. Does $L_p(\tau)$ have the USP?

The following lemma can be deduced from Rosenthal's characterization of ℓ_1-bases. We present an easy proof for the sake of completeness.

Lemma 3.2.7. *A seminormalized unconditional sequence in a Banach space is either weakly null, or contains a subsequence equivalent to ℓ_1. Consequently, any bounded unconditional basic sequence in a reflexive space is weakly null.*

Proof. Suppose a normalized sequence (x_n), with an unconditional constant C, is not weakly null. Passing to a subsequence, we find a norm one $x^* \in X^*$ so that $|x^*(x_n)| > c > 0$ for every n. For any finite sequence (α_n) let $\omega_n = \mathrm{sgn}(\alpha_n)\frac{|x^*(x_n)|}{x^*(x_n)}$. Then

$$\sum_n |\alpha_n| \geq \| \sum_n \alpha_n x_n\| \geq C^{-1}\| \textstyle\sum_n \alpha_n \omega_n x_n\| \geqslant C^{-1}|x^*(\textstyle\sum_n \alpha_n\omega_n x_n)| \geq cC^{-1}\textstyle\sum_n |\alpha_n|.$$

Thus, (x_n) is equivalent to the ℓ_1-basis. ∎

Proposition 3.2.8. *If $1 < p < 2$, then any sequence in $L_p(\tau)$, equivalent to the ℓ_2-basis, has a subsequence whose linear span is complemented.*

Proof. Suppose (x_n) is a sequence, equivalent to the ℓ_2-basis. By Hahn-Banach Theorem, $L_q(\tau)$ (here, as before, $1/p + 1/q = 1$) contains a bounded biorthogonal sequence (y_n). By passing to a subsequence, we may assume $y_n \to y$ weakly. Note that, for any n, $y(x_n) = \lim_m y_m(x_n) = 0$, hence the sequence $z_n = y_n - y$ is weakly null, and biorthogonal to (x_n). By passing to a further subsequence, and using the noncommutative Kadec-Pelczynski dichotomy [85, Theorem 5.4], we assume that (z_n) is equivalent either to the ℓ_2-basis, or to the ℓ_q-basis, and complemented. The latter, however, is impossible. Indeed, then there exists a constant C so that, for every sequence (α_i), $C^{-1}(\sum_i |\alpha_i|^2)^{1/2} \leq \| \sum_i \alpha_i x_i\| \leq C(\sum_i |\alpha_i|^2)^{1/2}$, and $C^{-1}(\sum_i |\alpha_i|^q)^{1/q} \leq \| \sum_i \alpha_i z_i\| \leq C(\sum_i |\alpha_i|^q)^{1/q}$. In particular, for any m,

$$Cm^{1/q} \geqslant \| \textstyle\sum_{i=1}^m z_i\| = \sup_{\|x\|_p \leq 1} |(\sum_{i=1}^m z_i)(x)| \geq |(\sum_{i=1}^m z_i)(C^{-1}m^{-1/2}\sum_{i=1}^m x_i)| = C^{-1}m^{1/2},$$

which fails for sufficiently large values of m.

Thus, (z_n) is equivalent to the ℓ_2-basis, and there exists a projection P from $L_q(\tau)$ onto $Z = \text{span}[z_n : n \in \mathbb{N}]$. Note that the restriction of P^* onto $X = \text{span}[x_n : n \in \mathbb{N}]$ is an isomorphism. Indeed, for any sequence $(\alpha_n) \in \ell_2$, we have $\|\sum_n \alpha_n x_n\| \sim (\sum_n |\alpha_n|^2)^{1/2}$. Furthermore, let $z = (\sum_n |\alpha_n|^2)^{-1/2} \sum_n \overline{\alpha_n} z_n$. Then $Pz = z$, and $\|z\| \lesssim 1$, hence

$$\|P^*(\sum_n \alpha_n x_n)\| \gtrsim \|(P^*(\sum_n \alpha_n x_n))(z)\| = \|(\sum_n \alpha_n x_n)(Pz)\| \gtrsim (\sum_n |\alpha_n|^2)^{1/2}.$$

To complete the proof, note that $U^{-1}P^*$ is a bounded projection onto X where by U we denoted the restriction of P^* onto $X = \text{span}[x_n : n \in \mathbb{N}]$, viewed as an operator into $\text{ran}\, P^*$. ■

Suppose τ is a normal faithful semifinite trace on a von Neumann algebra $\mathcal{A}$. We say that $K \subset L_p(\tau)$ is *p-equiintegrable* if $\lim_\alpha \sup_{h \in K} \|e_\alpha h e_\alpha\|_p = 0$ for every net of projections (e_α), converging (weakly) to 0 (see e.g. [95, Section II.2] for a discussion on various modes of convergence). By [85, Section 4], the following are equivalent:

(i) K is p-equiintegrable.

(ii) $\lim_n \sup_{h \in K} \|e_n h e_n\|_p = 0$ for every sequence of projections (e_n), converging (weakly) to 0.

(iii) $\lim_\alpha \sup_{h \in K} \|x_\alpha h y_\alpha\|_p = 0$ if the nets of positive operators (x_α) and (y_α) converge to 0 weak*.

The following result seems to be folklore.

Lemma 3.2.9. *Suppose K is p-equiintegrable. Then for every $\varepsilon > 0$ there exists $\delta > 0$ so that $\sup_{f \in K} \max\{\|ef\|, \|fe\|\} < \varepsilon$ whenever e is a projection of trace not exceeding δ.*

Remark 3.2.10. If τ is finite, then a sequence of projections (e_n) converges weakly to 0 iff $\lim \tau(e_n) = 0$. In this setting, the above lemma shows that K is p-equiintegrable if and only if it is (in the terminology of [85]) K is p-biequiintegrable. If τ is not finite, p-equiintegrability need not imply p-biequiintegrability.

Proof. Note that, if (e_n) is a sequence of projections, and $\lim_n \tau(e_n) = 0$, then $e_n \to 0$ weak*. Indeed, otherwise, by passing to a subsequence, we can find $x \in L_1(\tau)$ and $c > 0$, so that $|\tau(xe_n)| > c$ for any n. By polarization, we can assume that $x \geq 0$. Denote by $\mu_x(t)$ the generalized singular value function of x. Then (see e.g. [29]) $\|x\|_1 = \int \mu_x(t)\,dt$, and $\tau(xe_n) \leq \int_0^{\tau(e_n)} \mu_x(t)\,dt$. The latter converges to 0, leading to a contradiction.

Find δ so that $\sup_{f \in K} \|rfr\| < \varepsilon/4$ whenever r is a projection with $\tau(r) < 2\delta$. If $\tau(e) < \delta$, denote by e' the range projection of fe. Clearly, $\tau(e') \leq \tau(e)$. Let $r = e \vee e'$. Then $\tau(r) \leq \tau(e) + \tau(e') < 2\delta$, hence $\|rfr\| < \varepsilon$ for $f \in K$. However, $fe = e'fe = e'(rfr)e$, hence $\|fe\| \leq \|rfr\| < \varepsilon$. An estimate for $\|ef\|$ is obtained similarly. ■

Proof of Theorem 4. It remains to establish (1) ⇒ (2) for $1 < p < 2$. Suppose $T \in B(L_p(\tau))$ is not strictly singular – that is, it fixes an infinite dimensional subspace X. To show that T is an isomorphism on $E \subset X$, so that E is isomorphic to either ℓ_p or ℓ_2, and both E and $T(E)$ are complemented, we consider two cases separately: (1) B_X is not p-equiintegrable (then $E \sim \ell_p$); (2) B_X is p-equiintegrable (then $E \sim \ell_2$).

Case 1. Let B_X be not p-equiintegrable. By [85, Theorem 5.1], X contains a complemented subspace Y, isomorphic to ℓ_p. Denote by (f_n) an ℓ_p-basis in Y.

(Tf_n) is an ℓ_p-basic sequence, hence, by [85, Theorem 5.1] again (or by [84]), there exists a normalized block sequence $h_k = \sum_{j \in I_k} \alpha_j T f_j$, whose linear span is complemented in $L_p(\tau)$. By [63, Proposition 2.a.1], the linear span of the vectors $g_k = T^{-1} h_k = \sum_{j \in I_k} \alpha_j f_j$ is complemented in $\mathrm{span}[f_n : n \in \mathbb{N}]$, hence also in $L_p(\tau)$.

Case 2. Suppose B_X is p-equiintegrable. By Corollary 3.2.5, X contains a normalized unconditional basic sequence (f_n). We shall use f_n's to produce the following sequence (g_n):

(i) $\|g_n\|_p \in [1/2, 2]$.

(ii) $\sup_n \|g_n\|_\infty < \infty$.

(iii) The sequence (g_n) is weakly null in both $L_p(\tau)$ and $L_2(\tau)$.

(iv) (g_n) is equivalent to an orthonormal basis in ℓ_2, in both $L_p(\tau)$ and $L_2(\tau)$.

(v) (Tg_n) is equivalent to the ℓ_2-basis.

Without loss of generality, we assume $\|f_n\|_p = 1$ for every n. Set $c = \inf_n \|Tf_n\|$, and fix $\varepsilon \in (0, \min\{1/40, c/(8\|T\|)\})$. The sequence (f_n) is p-equiintegrable, hence there exists $\delta > 0$ so that $\max\{\|ef_n\|_p, \|f_n e\|_p\} < \varepsilon$ whenever e is a projection of trace not exceeding δ. Let $M = \delta^{-p} + 1$. Write $f_n = u_n |f_n|$, where f_n is a partial isometry from $(\ker f_n)^\perp$ onto $\overline{\mathrm{ran}\, f_n}$. Let $\phi(t) = \begin{cases} t & t \le M \\ 0 & t > M \end{cases}$, and $\tilde{f}_n = u_n \phi(|f_n|)$. Note that $\|f_n - \tilde{f}_n\|_p < \varepsilon$.

Passing to a subsequence, we can assume $\tilde{f}_n \to f$ weakly. Then $\tilde{f}_n - f_n \to f$ weakly as well, hence $\|f\| \le \liminf \|\tilde{f}_n - f_n\| < \varepsilon$. Spectral calculus allows us to pick projections q_1 and q_2, so that $\tau(q_1^\perp), \tau(q_2^\perp) < \delta$, $f = q_1 f q_2 + q_1^\perp f q_2^\perp$, and $N = \|q_1 f q_2\| < \infty$. Let $g_n = q_1(\tilde{f}_n - f)q_2$. Then $\|g_n\|_\infty \leq M + N$. Furthermore,

$$f_n - g_n = q_1^\perp f_n + q_1 f_n q_2^\perp + q_1(f - \tilde{f}_n)q_2 + q_1 f q_2.$$

The fact that $\tau(q_1^\perp), \tau(q_2^\perp) < \delta$ leads to

$$\|f_n - g_n\|_p \leq \|q_1^\perp f_n\|_p + \|q_1 f_n q_2^\perp\|_p + \|q_1(f - \tilde{f}_n)q_2\|_p + \|q_1 f q_2\|_p < 4\varepsilon,$$

and therefore, $\|g_n\|_p \subset [1 - 4\varepsilon, 1 + 4\varepsilon] \subset [1/2, 2]$. We also have $\|g_n\|_\infty \leq M + N$. By Hölder's Inequality,

$$\|g_n\|_2 \leq \|g_n\|_p^{p/2} \|g_n\|_\infty^{1-p/2} \leq 2^{\frac{p}{2}} (M + N)^{1 - \frac{p}{2}}.$$

Note that $g_n \to 0$ weakly in $L_p(\tau)$. That is, for any $x^* \in L_q(\tau)$ $(1/p + 1/q = 1)$, $\lim_n x^*(g_n) = 0$. As $L_2(\tau) = \overline{L_q(\tau) \cap L_2(\tau)}^{\|\cdot\|_2}$, $g_n \to 0$ weakly in $L_2(\tau)$ as well. Therefore, by passing to a subsequence several times, and applying Proposition 3.2.2, we can assume that the sequence (g_n) is unconditional, both in $L_p(\tau)$ and in $L_2(\tau)$. Furthermore, the sequence $(Tg_n) \subset L_p(\tau)$ is weakly null, hence, by passing to a further subsequence, we can assume it is unconditional as well. On the other hand,

$$\|Tg_n\|_p \geq \|Tf_n\|_p - \|T\| \|f_n - g_n\|_p \geq c - 4\varepsilon \|T\| > \frac{c}{2}.$$

It remains to show that the sequence (Tg_n) is equivalent to the ℓ_2-basis. By unconditionality, there exists a constant C_1 so that, for any sequence (α_n), $\|\sum_n \alpha_n T g_n\| \geq C_1 \mathrm{Ave}_\pm \|\sum_n \pm \alpha_n T g_n\|$ (we are averaging over all possible signs). However, $L_p(\tau)$ has cotype 2, hence $\mathrm{Ave}_\pm \|\sum_n \pm \alpha_n T g_n\| \geq$

$C_2(\sum_n |\alpha_n|^2 \|Tg_n\|^2)^{1/2}$, for some C_2. Consequently, there exists $C_3 > 0$ so that $\|\sum_n \alpha_n Tg_n\| \geq C_3(\sum_n |\alpha_n|^2)^{1/2}$. On the other hand, $\|\sum_n \alpha_n Tg_n\| \leq \|T\| \|\sum_n \alpha_n g_n\| \leq C_4(\sum_n |\alpha_n|^2)^{1/2}$, for some constant C_4.

By Proposition 3.2.8, we can assume that T is an isomorphism on a complemented subspace Y, isomorphic to ℓ_2. Using Proposition 3.2.8 again, we can assume that $T(Y)$ is complemented as well.

■

Remark 3.2.11. The proof of Theorem 4 can be modified to yield: if $p_1, p_2 \in (1, \infty)$ are distinct, then, for T in $L(L_{p_1}(\tau_1), L_{p_2}(\tau_2))$, the following statements are equivalent: (i) T is not strictly singular; (ii) T is an isomorphism on E, where E is isomorphic to ℓ_2, and both E and $T(E)$ are complemented.

Remark 3.2.12. Note that we used hyperfiniteness only to claim the existence of unconditional basic sequence in every weakly null sequence. So in the statement of Theorem 4 we can replace hyperfiniteness with the USP. In general, we are not aware of any L_p space $(1 < p < \infty)$ without the USP.

Strict singularity and compactness of positive operators

From the previous section it can be noticed that a strictly singular operator on $L_p(\tau)$ $(p > 1)$ is the one that maps any ℓ_2-basis into ℓ_p-basis or vice versa. Therefore its second power is always a compact operator. The following results shows that the situation is even simpler in the case of positive operators.

Proposition 3.2.13. *Suppose τ_1 and τ_2 are normal faithful finite traces on hyperfinite von Neumann algebras $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively. Suppose, fur-*

thermore, that either $2 \le u_2 \le u_1 < \infty$*, or* $1 < u_1 \le u_2 \le 2$*. Then* $\mathcal{SS}(L_{u_1}(\tau_1), L_{u_2}(\tau_2))_+ = \mathcal{K}(L_{u_1}(\tau_1), L_{u_2}(\tau_2))_+$.

Particularizing to the case of $u_1 = u_2 = u$, we obtain:

Corollary 3.2.14. *Suppose* τ_1 *and* τ_2 *are as in Proposition 4, and* $1 < u < \infty$*. Then* $\mathcal{SS}(L_u(\tau_1), L_u(\tau_2))_+ = \mathcal{K}(L_u(\tau_1), L_u(\tau_2))_+$.

In the commutative case, similar results were obtained in [20, 37].

Below, we shall assume that all traces are normalized. Denote by $L_p(\tau)_{sa}$ the self-adjoint (real) part of $L_p(\tau)$. For the proof we need an auxiliary result.

Lemma 3.2.15. *Suppose* $2 < p < \infty$*, and* (x_k) *is an unconditional self-adjoint normalized sequence in* $L_p(\tau)$*, where* τ *is a finite normal faithful trace on a von Neumann algebra* $\mathcal{A}$*. Then either* (x_k) *is equivalent to the* ℓ_2*-basis, or there exist* $n_1 < n_2 < \ldots$*, and a sequence of mutually orthogonal projections* $p_k \in \mathcal{A}$*, so that* $\lim_k \|x_{n_k} - p_k x_{n_k} p_k\|_p = 0$.

Proof. The proof uses a variation on a well-known "Kadec-Pelczynski" method. Our exposition follows [94]. For $c > 0$, set

$$M_c = \big\{x \in L_p(\tau) : \tau(\chi_{(c\|x\|_p,\infty)}(|x|)) \ge c\big\}.$$

If there exists $c > 0$ so that $x_k \in M_c$ for every k, then, by the proof of [94, Theorem 2.4], (x_k) is equivalent to the ℓ_2-basis. Otherwise, by passing to a subsequence, we can assume that the projections

$$q_k = \chi_{\mathbb{R}\setminus(-4^{-k},4^{-k})}(x_k)$$

satisfy two conditions:

(i) $\tau(q_1) < 1/8$, and $\tau(q_k) < \tau(q_{k-1})/8$ for $k > 1$.

(ii) If q is a projection with $\tau(q) \leq 2\tau(q_k)$, then $\max_{i \leq k} \|qx_i\|_p < 4^{-(k+1)}$ (see [36, Theorem 4.2] to show that this can be satisfied).

Let $r_k = \vee_{j>k} q_j$, and $p_k = q_k \wedge r_k^\perp$. We claim that

$$\|x_k - p_k x_k p_k\|_p < 4^{-k}. \tag{3.3}$$

Indeed, write $p_k = q_k - q_k'$. Then

$$x_k - p_k x_k p_k = (x_k - q_k x_k q_k) + q_k' x_k (q_k - q_k') + q_k x_k q_k'.$$

Clearly, $\|x_k - q_k x_k q_k\|_p \leq \|x_k q_k^\perp\|_\infty < 8^{-k}$. Furthermore, $\tau(r_k) \leq \sum_{j>k} \tau(q_j) < 2\tau(q_{k+1})$, hence, by Kaplansky's Formula [53, Theorem 6.1.7],

$$\tau(q_k') = \tau(q_k) - \tau(q_k \wedge r_k^\perp) = \tau(q_k \vee r_k^\perp) - \tau(r_k^\perp) \leq 1 - \tau(r_k^\perp) = \tau(r_k) < 2\tau(q_{k+1}).$$

Thus, $\|q_k' x_k\|_p = \|x_k q_k'\|_p < 4^{-(k+1)}$. Together, these inequalities give us (3.3). ∎

Lemma 3.2.16. *Suppose τ is a faithful normal semifinite trace on a von Neumann algebra, and $1 \leq p < \infty$. Then every p-equiintegrable weakly null sequence $(f_n) \subset L_p(\tau)_+$ is norm null. In particular, no sequence in $L_p(\tau)_+$ is equivalent to a standard basis of ℓ_2.*

Proof. Consider a weakly null sequence $(f_j) \subset L_p(\tau)_+$. Then $\lim_n \tau(f_j) = 0$. The case of $p = 1$ is the easiest to handle: $\|f_j\|_1 = \tau(f_j) \to 0$.

Now let $1 < p < \infty$. Suppose, for the sake of contradiction, that (f_j) is not weakly null. Without loss of generality assume that (f_j) is normalized. Fix $0 < c < 1$. Since (f_n) is p-equiintegrable, by Lemma 3.2.9 there exists $C > 0$

so that, for any j, $\|f_j - w_j\|_p < c$, where $w_j = \boldsymbol{\varphi}_C(f_j)$ (the function $\boldsymbol{\varphi}_C(t)$ is defined as $\min\{t, C\}$). By Hölder Inequality, $\|w_j\|_p \leq \|w_j\|_1^{1/p} \|w_j\|_\infty^{1-1/p}$, hence

$$\tau(f_j) \geqslant \tau(w_j) = \|w_j\|_1 \geq \|w_j\|_p^p \|w_j\|_\infty^{1-p} > (1-c)^p C^{1-p}.$$

This contradicts $\lim_j \tau(f_j) = 0$. ■

Proof of Proposition 4. We have to show that any strictly singular positive $T \in B(L_{u_1}(\tau_1), L_{u_2}(\tau_2))$ is compact. First consider $2 \leq u_2 \leq u_1 < \infty$. Without loss of generality, we can assume $\|T\| \leq 1$. For $s \in \{1, 2\}$, let $v_s = u_s/(u_s - 1)$ (that is, $1/u_s + 1/v_s = 1$). Suppose, for the sake of contradiction, that T (or equivalently, T^*) is not compact. Note that T maps $L_{u_1}(\tau_1)_{sa}$ into $L_{u_2}(\tau_2)_{sa}$. Then there exists a weakly null normalized sequence (x_k) in $L_{u_1}(\tau_1)_{sa}$, so that $\|Tx_k\| > 5c > 0$ for any k. By passing to a subsequence twice, and invoking Corollary 3.2.5, we can assume that the sequences (x_k) and (Tx_k) are unconditional. Furthermore, by [85, Proposition 5.4], (x_k) ((Tx_k)) is equivalent either to the ℓ_2-basis, or to the ℓ_{u_1}-basis (respectively, either to the ℓ_2-basis, or to the ℓ_{u_2}-basis). As T is bounded, and strictly singular, only one possibility is open to us: (x_k) and (Tx_k) are equivalent to the ℓ_2-basis, and the ℓ_{u_2}-basis, respectively.

Applying Lemma 3.2.15 (and passing to a subsequence again if necessary), we conclude that there exist mutually orthogonal projections p_k so that $\|Tx_k - y_k\| < 100^{-k}c$. where $y_k = p_k(Tx_k)p_k$. Find a sequence of positive norm one elements $z_k \in L_{u_2}^*(\tau_2)$, so that $z_k = p_k z_k p_k$ for any k, and

$|\tau_2(z_k y_k)| > 5c/2$. Then, by passing to a subsequence, for any j,

$$\begin{aligned}\|T^* z_j\| &\geqslant |\tau_1\big((T^* z_j)\cdot x_j\big)| = |\tau_2(z_j \cdot T x_j)| \\ &\geqslant \Big||\tau_2(z_j y_j)| - |\tau_2(z_j \cdot (T x_j - y_j))|\Big| > 2c.\end{aligned}$$

Note that the sequence (z_j) is equivalent to the ℓ_{v_2}-basis ($1/u_i + 1/v_i = 1, i = 1, 2$), hence weakly null. The sequence $(T^* z_j)$ is weakly null as well. By passing to a subsequence if necessary, we can assume $(T^* z_j)$ is unconditional. However, this sequence has no subsequences equivalent to the ℓ_{v_1}-basis since $v_2 > v_1$. By [85, Theorem 5.3], $(T^* z_j)$ is v_1-equiintegrable. This contradicts Lemma 3.2.16.

Now suppose $1 < u_1 \leq u_2 \leq 2$. Let v_i ($i = 1, 2$) be such that $1/u_i + 1/v_i = 1$. Consider $T \in \mathcal{SS}(L_{u_1}(\tau_1), L_{u_2}(\tau_2))_+$. Note that $L_{u_2}(\tau_2)^* = L_{v_2}(\tau_2)$ is subprojective, hence, by [2, Theorem 7.54(ii)], T^* is strictly singular. By the above, T^* is compact, hence so is T. ■

Remark 3.2.17. Corollary 3.2.14 fails for $u = 1$, even in the commutative case: there exists a positive non-compact strictly singular operator on L_1. Indeed, let $(r_n)_1^\infty$ be a Rademacher system and e be the identity. Define $x_n = e + r_n$. Set $U : \ell_1 \to L_1$ as $U\delta_n = x_n$, where (δ_n) is the canonical basis for ℓ_1. It is easy to check that U is positive, and not compact. By Khintchine Inequality, $\mathrm{span}[e, r_1, r_2, \ldots]$ is isomorphic to ℓ_2, hence the same is true for $\mathrm{span}[x_n : n \in \mathbb{N}]$. Therefore, U is strictly singular. The required operator is the composition of a positive projection on a copy of ℓ_1 with U.

3.2.2 Noncommutative L_p: discrete case

In this section we study some operator ideals on the spaces $\mathfrak{C}_p$.

Strictly singular and weakly compact operators

We start by establishing:

Corollary 3.2.18. *Suppose the Banach space X satisfies one of two conditions:*

(i) $X = \mathfrak{C}_p$, *with* $1 \leq p < \infty$.

(ii) $X = L_p(\tau)$, *where* $1 < p < \infty$, *and* τ *is a normal faithful finite trace on a hyperfinite von Neumann algebra.*

Then $T \in L(X)$ *is strictly singular if and only it is inessential. For* $1 < p < \infty$, *these conditions are equivalent to* T *being strictly cosingular.*

Proof. By [2, Theorem 7.44], any strictly singular or strictly cosingular operator is inessential. Now suppose $T \in L(X)$ is not strictly singular. By Theorem 4 and [9, Theorem 1], there exists $E \subset X$ so that $T|_E$ is an isomorphism, and both E and $T(E)$ is complemented. A well-known description of inessential operators (see e.g. [2, Section 7.1]) shows that T is not inessential. Furthermore, such a T cannot be strictly cosingular. Finally, suppose $1 < p < \infty$, and T is strictly cosingular. By [2, Theorem 7.53], T^* is strictly singular, hence X^* contains a subspace E so that $T^*|_E$ is an isomorphism, and both E and $T^*(E)$ are complemented. Emulating the proof of [98, Theorem 2.2], we conclude that T is not strictly singular. ∎

Remark 3.2.19. Alternatively, one could show that, for operators on $\mathfrak{C}_p$, the ideals of strictly singular and inessential operators coincide by combining [2, Theorem 7.51] with the subprojectivity of $\mathfrak{C}_p$, established in [9].

Remark 3.2.20. Note that the ideal of cosingular operators acting on $\mathfrak{C}_1$ sits properly between the ideals of compact and strictly singular operators. Indeed, the strictly cosingular operators are contained by the ideal of inessential operators. Thus, Corollary 3.2.18 yelds $\mathcal{SCS}(\mathfrak{C}_1) \subseteq \mathcal{SS}(\mathfrak{C}_1)$. By [4, theorem 2.3.1] there exists a surjective operator $T : X \to Y$, where X and Y are complemented subspaces of $\mathfrak{C}_1$ isomorphic to ℓ_1 and ℓ_2, respectively. Clearly, T is a strictly singular operator. At the same, being surjective, it is not strictly cosingular. This implies that $S = TP \in \mathcal{SS}(\mathfrak{C}_1) \setminus \mathcal{SCS}(\mathfrak{C}_1)$ where P is a projection from $\mathfrak{C}_1$ onto X. Also there is a strictly cosingular non-compact operator on $\mathfrak{C}_1$, because such is the canonical embedding of ℓ_1 into ℓ_2.

From the preceding corollary and the fact that $T \in \mathcal{IN}(X)$ if and only if $T^* \in \mathcal{IN}(X^*)$ for every reflexive X, we obtain:

Corollary 3.2.21. *Suppose* $1 < p < \infty$, *and* X *is either* $\mathfrak{C}_p$, *or* $X = L_p(\tau)$, *where* τ *is a normal faithful finite trace on a hyperfinite von Neumann algebra. Then* $T \in B(X)$ *is strictly singular if and only if* T^* *is strictly singular.*

The following two proposition complement Corollary 3.2.21.

Proposition 3.2.22. $T^* \in \mathcal{SS}(B(H))$ *implies* $T \in \mathcal{SS}(\mathfrak{C}_1)$.

Proof. It follows immediately from [98, Theorem 2.2] and [43]. ∎

Proposition 3.2.23. $T \in \mathcal{SS}(\mathfrak{C}_\infty)$ *if and only if* $T^* \in \mathcal{SS}(\mathfrak{C}_1)$.

Proof. By [98, Theorem 2.2], the strict singularity of T^* implies the strict singularity of T. To prove the converse, suppose, for the sake of contradiction, that T is strictly singular, but T^* is not. Then there exists an infinite dimensional $X \subset \mathfrak{C}_1$ so that $\|T^*x\| \geqslant c\|x\|$ for every $x \in X$ (here $c > 0$). By

[43], X contains either ℓ_1, or ℓ_2. By Remark 3.2.24, T is weakly compact, hence so is T^*. Thus, by passing to a subspace if necessary, we can assume $X \approx \ell_2$. Then $T^*(X)$ is also isomorphic to ℓ_2. Consequently, there exists $c_0 > 0$ so that, for every $z \in X \cup T^*X$, we have $c_0\|z\|_1 \leq \|z\|_\infty \leq \|z\|_1$, see [43, Proposition 2 and the proof of Proposition 1].

Now consider the space $Y = (J(T^*X))^\star \subset \mathfrak{C}_\infty$ (here and below, $\star$ stands for taking the adjoint, and J is the formal identity from $\mathfrak{C}_1$ to $\mathfrak{C}_\infty$). We claim that T is an isometry on Y. Indeed, pick $y \in Y$, with $\|y\|_\infty = 1$. Then $\|J^{-1}y\|_1 \leq c_0^{-1}$, and consequently, $x = cc_0(T^*)^{-1}J^{-1}y^\star$ satisfies $\|x\|_1 \leq 1$. Then

$$\|Ty\|_\infty \geq \mathrm{Tr}((Ty)x) = \mathrm{Tr}(y(T^*x)) = cc_0\mathrm{Tr}(y(J^{-1}y^\star)) = cc_0\|y\|_2^2 \geq cc_0^3.$$

■

Remark 3.2.24.) Observe first that $\mathcal{SS}(\mathfrak{C}_\infty, X) \subseteq \mathcal{WK}(\mathfrak{C}_\infty, X)$ for any Banach space X. Indeed, by [74], every non-weakly compact operator from a C^*-algebra preserves a copy of c_0.

Theorem 3.2.25. *The following holds:*

(i) $\mathcal{SS}(\mathfrak{C}_p, \mathfrak{C}_q) = \mathcal{K}(\mathfrak{C}_p, \mathfrak{C}_q)$, *if* $\infty \geqslant p \geqslant 2 \geqslant q \geqslant 1$,

(ii) $\mathcal{SS}(\mathfrak{C}_p, \mathfrak{C}_q) \supsetneq \mathcal{FSS}(\mathfrak{C}_p, \mathfrak{C}_q) \supsetneq K(\mathfrak{C}_p, \mathfrak{C}_q)$ *otherwise.*

First, we establish a technical result.

Lemma 3.2.26. *Suppose* $T \in B(X, Y)$ *is non-compact, and* X *does not contain a copy of* ℓ_1. *Then there exists a weakly null sequence* $(x_n) \subset B_X$, *so that* $\inf \|Tx_n\| > 0$.

Proof. By the noncompactness of X, there exists a sequence $(z_n) \subset B_X$, so that $\inf_{n \neq m} \|Tz_n - Tz_m\| > 0$. By Rosenthal's characterization of ℓ_1 (see e.g. [4, Theorem 10.2.1]), we can assume, by passing to a subsequence if necessary, that (z_n) is weakly Cauchy. Then the sequence $x_n = (z_{2n} - z_{2n+1})/2$ has the desired properties. ■

Proof of Theorem 3.2.25. (1) Let p and q be as in the statement of the theorem. Suppose $T : \mathfrak{C}_p \to \mathfrak{C}_q$ is not a compact operator. By Lemma 3.2.26, there is a weakly null sequence (x_n) such that Tx_n is bounded away from 0. First, consider $p \neq \infty$. Clearly, (x_n) contains a basic subsequence, thus, from [9, Theorem 3.1] by passing to a subsequence, (x_n) can be considered equivalent to either ℓ_2 or ℓ_p-basis. Similar (Tx_n) is equivalent to either ℓ_2 or ℓ_q . We recall Pitt's theorem and the fact that $q < 2$ to deduce that T is an isomorphism on a copy of ℓ_2. Hence the result follows. We note that the proof of [9, Theorem 3.1] works for $p = \infty$, and, thus, every basic sequence in $\mathfrak{C}_\infty$ contains either an ℓ_2 or c_0-bases. The rest of the argument is similar to the one above.

To show that finitely strictly singular operators do not coincide with strictly singular operators, we note that every $\mathfrak{C}_p$ contains complimented copies of $(\oplus \ell_2^n)_p$, ℓ_2, and ℓ_p [9]. Therefore we can proceed as in [81, Example 1]. If $p \geqslant q > 2$, we build an operator from $(\oplus \ell_2^{n_i})_2 \subset \mathfrak{C}_p$ to $\ell_q \subset \mathfrak{C}_q$ which is strictly singular, but not finitely strictly singular. Similar if $2 > p \geqslant q$, then we construct such an operator from $(\oplus \ell_2^{n_i})_p \subset \mathfrak{C}_p$ to $\ell_2 \subset \mathfrak{C}_q$.

To distinguish between the ideals of finitely strictly singular and compact operators, we note that the embedding of ℓ_u into ℓ_v is a non-compact finitely strictly singular operator when $u < v$, see [90, Proposition 3.3]. ■

More can be said about positive operators.

These results allow us to prove a "noncommutative Pitt's Theorem" for positive operators.

Theorem 3.2.27. *For* $1 \leq q < p < \infty$, $B(\mathfrak{C}_p, \mathfrak{C}_q)_+ = \mathcal{K}(\mathfrak{C}_p, \mathfrak{C}_q)_+$.

Proof. Suppose, for the sake of contradiction, that there exists a non-compact $T \in B(\mathfrak{C}_p, \mathfrak{C}_q)_+$. By Lemma 2.3.5 $\inf_n \|TR_n\| > 0$. Then there exists a sequence (n_k), and normalized positive sequence (x_k) in C_p, so that $x_k = (P_{n_k} - P_{n_{k-1}})x_k(P_{n_k} - P_{n_{k-1}})$, and $\|Tx_k\| > c > 0$ for every k. By polarization, we can assume that $x_k \geq 0$ for every k. The sequence (x_k) is equivalent to the standard basis of ℓ_p, hence weakly null. Therefore, the sequence (Tx_k) is weakly null as well. Proposition 2.3.17 implies the existence of $k_1 < k_2 < \dots$ so that the sequence (Tx_{k_j}) is equivalent to a standard basis of ℓ_q. Thus, T maps an ℓ_p-basis to an ℓ_q-basis, which contradicts the boundedness of T. ∎

Theorem 3.2.28. *For* $1 \leq p < \infty$, *and a positive* $T \in L(\mathfrak{C}_p)_+$, *the following are equivalent:*

(i) T *is compact.*

(ii) T *is strictly singular.*

(iii) There is no a subspace $E \subset \mathfrak{C}_p$, *isomorphic to* ℓ_p, *so that* $T|_E$ *is an isomorphism, and both* E *and* $T(E)$ *are complemented.*

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are trivial. To establish (3) $\Rightarrow$ (1), it suffices to show that any $T \in B(\mathfrak{C}_p)_+ \backslash \mathcal{K}(\mathfrak{C}_p)_+$ fixes a copy of ℓ_p.

Suppose first $1 < p < \infty$. Proceeding as in the proof of Theorem 3.2.27, we show that T maps an ℓ_p-basis to an ℓ_p-basis, hence T is not strictly singular.

Let $p = 1$. Then there exists a positive seminormalized sequence (x_n) such that Tx_n does not contain any convergent subsequences. Since $\mathfrak{C}_1$ is sequentially weakly complete, see [3], by passing to a subsequence, we may assume that (x_n) is ether isomorphic to the ℓ_1-basis or it is weakly convergent. The later yields (x_n) is norm convergent by [70, Theorem 1.4.3], which contradicts the way we have chosen (x_n). Similar we obtain that $T(x_n)$ is equivalent to the ℓ_1-basis. Therefore T is an isomorphism on a copy of ℓ_1. ■

Due to [70, Theorem 1.4.3] the following holds.

Proposition 3.2.29. $\mathcal{WK}(X, \mathfrak{C}_1)_+ = \mathcal{K}(X, \mathfrak{C}_1)_+$, *where* X *is an ordered Banach space with a proper generating cone.*

Dunford-Pettis operators

Suppose X, Y are Banach spaces. We say that $T \in B(X, Y)$ is Z*-compact* if $T|_{Z'}$ is compact whenever $Z' \subset X$ is isomorphic to Z. And it is Dunford-Pettis if it maps relatively weakly compact sets to relatively compact.

Remark 3.2.30. In [86], H. Rosenthal proved that an operator $T \in B(L_1, Z)$ is Dunford-Pettis if and only if it is ℓ_2-strictly singular. As ℓ_2-compactness implies ℓ_2-strict singularity, we conclude that $T \in B(L_1, Z)$ is ℓ_2-strictly singular iff it is ℓ_2-compact.

The following proposition is of the same spirit.

Proposition 3.2.31. *Suppose Z is a Banach space. Then an operator $T \in B(\mathfrak{C}_1, Z)$ is Dunford-Pettis if and only if it is ℓ_2-compact.*

Proof. Since ℓ_2 is a reflexive space any Dunford-Pettis operator is ℓ_2-compact. Suppose T is ℓ_2-compact. Then [10, Theorem 2.2 (i)] implies that for any $n \in \mathbb{N}$ and $\varepsilon > 0$ there exists $N = N(n, \varepsilon)$ so that

$$\|T|_{\mathrm{span}[E_{ij}: i \le n, j > N]}\|, \|T|_{\mathrm{span}[E_{ij}: i > N, j \le n]}\| < \varepsilon.$$

Now select a sequence $1 = u_1 < v_1 < u_2 < \ldots$, so that, for any k, $\|T|_{X_k}\|, \|T|_{Y_k}\| < 4^{-k}$. For convenience, let $u_0 = v_0 = 0$. Here,

$$X_k = \mathrm{span}[E_{ij} : (i,j) \in A_k], \; Y_k = \mathrm{span}[E_{ij} : (i,j) \in B_k],$$

$$A_k = \{(i,j) : i \le u_k, j > v_k\} = [1, u_k] \times (v_k, \infty),$$
$$B_k = \{(i,j) : j \le v_k, i > u_{k+1}\} = (u_{k+1}, \infty) \times [1, v_k].$$

Note that the spaces X_k and Y_k are isomorphic to ℓ_2, hence $T|_{X_k}$ and T_{Y_k} are compact. Therefore, $T|_{\mathrm{span}[E_{ij}:(i,j) \in \cup_k A_k \times B_k]}$ is compact.

Moreover, $C = \mathbb{N} \times \mathbb{N} \backslash (\cup_k A_k \times B_k)$ is the disjoint union of the sets $C_k = [u_{k-1}, u_{k+1}] \times [v_{k-1}, v_k]$. Then $\mathrm{span}[E_{ij} : (i,j) \in C] \subset \mathfrak{C}_1$ is isomorphic to $X_0 = (\oplus_k \mathfrak{C}_1^{a_k, b_k})_{\ell_1}$, where $a_k = u_{k+1} - u_{k-1} + 1$, and $b_k = v_k - v_{k-1} + 1$. As X_0 is an ℓ_1 sum of finite dimensional spaces, it has the Schur property. Consequently, any operator on X_0 is Dunford-Pettis.

Let P be the coordinate projection from $\mathfrak{C}_1$ onto $\mathrm{span}[E_{ij} : (i,j) \in C]$, see [9, Proposition 3]. Note that TP factors through X_0, while $T(1-P)$ factors through $T|_{\mathrm{span}[E_{ij}:(i,j) \in \cup_k A_k \times B_k]}$. Thus, both TP and $T(1-P)$ are Dunford-Pettis. The same property is inherited by $T = TP + T(1-P)$. ∎

3.2.3 Operator ideals on C^*-algebras and function spaces

In this section, we investigate the coincidence of operator ideals, when the domain and/or range space is either a function space, or a C^*-algebra.

We start with a technical lemma. But first, recall that an operator T from the Banach spaces X to Y is called (p,q)-summing if there is a $K \geqslant 0$ such that

$$\Big(\sum_{k=1}^{n} \|Tx_k\|^p\Big)^{\frac{1}{p}} \leqslant K \sup\{(\sum_{k=1}^{n} |x^*(x_k)|^q)^{\frac{1}{q}}, x^* \in B_{X^*}\},$$

for any $(x_i)_{i=1}^n \subset X$. If $p = q$, then T is p-summing.

Proposition 3.2.32. *If $1 \leq q \leq p < \infty$, and $1/q - 1/p < 1/2$, then any (p,q)-summing operator is finitely strictly singular. Moreover, any p-summing operator is weakly compact, and Dunford-Pettis.*

Proof. The "moreover" statement about p-summing operators is [25, Theorem 2.17]. To prove the first part, note (reasoning as in the proof of [25, Theorem 10.5]) that it suffices to consider the case of $q = 2$. Suppose $T \in \Pi_{pq}(X,Y)$, and a $2n$-dimensional $E \subset X$ is such that $\|Tx\| \geq c\|x\|$ for any $x \in E$. We show that $c \leq 2n^{-1/p}\pi_{p2}(T)$. Indeed, by Dvoretzky-Rogers Lemma (see e.g. [25, Lemma 1.3]), one can find $x_1, \ldots, x_n \in E$, so that $\min_j \|x_j\| \geq 1/2$, yet $\|\sum \alpha_j x_j\|^2 \leq \sum_j |\alpha_j|^2$ for any sequence of scalars $(\alpha_j)_{j=1}^n$. Equivalently, $\sup_{f \in X^*, \|f\| \leq 1} \sum_j |\langle f, x_j \rangle|^2 \leq 1$. Thus,

$$\frac{c}{2} n^{1/p} \leq \Big(\sum_j \|x_j\|^p\Big)^{1/p} \leq \pi_{p2}(T),$$

which yields the desired estimate for c. ■

As the ideals of finitely strictly singular, weakly compact, and Dunford-Pettis operators are norm closed, we conclude:

Corollary 3.2.33. *Suppose* $T, T_n, \in B(X,Y)$ *are such that* $\lim_n \|T_n - T\| = 0$*, and* T_n *is* (p_n, q_n)*-summing, with* $1 \le q_n \le p_n < \infty$*, and* $1/q_n - 1/p_n < 1/2$*. Then* T *is finitely strictly singular. If, in addition, each* T_n *is* p_n *summing, with* $1 \le p_n < \infty$*, then* T *is weakly compact and Dunford-Pettis.*

Let $1 \leqslant p \leqslant \infty$. A Banach space X is called a $\mathcal{L}_{p,\lambda}$*-space* if every finite dimensional subspace $Y \subset X$ is contained in a subspace Z such that there exists an isomorphism $U : Z \to \ell_p^{\dim Z}$ with $\|U\|\|U^{-1}\| \leqslant \lambda$. X is an $\mathcal{L}_p$*-space* if it is a $\mathcal{L}_{p,\lambda}$*-space* for some $\lambda \ge 1$. The obvious examples of such spaces are $L_p(\mu)$ and $C(K)$ spaces, see [25, Chapter 3] for details.

Proposition 3.2.34. (1) *If* X *is a* $\mathcal{L}_\infty$ *space, and* Y *has non-trivial cotype, then* $B(X,Y) = \mathcal{FSS}(X,Y) = \mathcal{WK}(X,Y)$*.*

(2) *If* X *is a* $\mathcal{L}_1$ *space, and* Y *is a* $\mathcal{L}_p$ *space with* $1 < p < \infty$*, then* $B(X,Y) = \mathcal{FSS}(X,Y)$*.*

Proof. (1) Suppose Y has cotype $q \in [2, \infty)$. By [25, Theorem 11.14], $B(X,Y) = \Pi_p(X,Y)$ for any $p > q$ (if $q = 2$, we can take $p = 2$). To complete the proof, invoke Corollary 3.2.33 Similarly, (2) follows from [96, Theorem 11.11], stating that $\pi_{q1}(X,Y) = B(X,Y)$, with $1/q = 1 - |1/p - 1/2|$. ■

For more pairs (X,Y) where $B(X,Y) = \Pi_p(X,Y)$, see [78, Section 6].

In [68] it was proved that $\mathcal{SS}(C(K),Y) = \mathcal{WK}(C(K),Y)$ and $\mathcal{SS}(L_1(\mu)) = \mathcal{WK}(L_1(\mu))$. Bellow we show that this ideals coincide with the ideal of finitely strictly singular operators.

Theorem 3.2.35. *$\mathcal{FSS}(C(K), Y) = \mathcal{WK}(C(K), Y)$ for any Banach space Y and a compact Hausdorff topological space K.*

Proof. By [25, Theorem 15.2], any $T \in \mathcal{WK}(C(K), Y)$ is a norm limit of a sequence of operators (T_n), which factor through ℓ_2. However, the T_n's are 2-summing. By Corollary 3.2.33, $T \in \mathcal{FSS}(C(K), Y)$. ■

Corollary 3.2.36. *Let $T : C(K) \to X$ and $S : Y \to L_1$, where K is compact and Hausdorff. Then T and S are weakly compact if and only if there ultrapowers are weakly compact.*

Proof. The weak compactness of ultrapowers, obviously, implies weak compactness of the operators itself. For S, the converse statement follows from [41, Proposition 5.5]. Assume T is compact then T is finitely strictly singular by Theorem 3.2.35. Therefore [66, Lemma 4] implies any ultrapower of T is strictly singular and therefore T is weakly compact, since the ultrapower of $C(K)$ is $C(M)$ for some compact Hausdorff M, [48, Theorem 3.3]. ■

Theorem 3.2.37. *$\mathcal{FSS}(L_1(\mu)) = \mathcal{WK}(L_1(\mu))$, where μ is σ-additive.*

Proof. Let $T \in \mathcal{WK}(L_1(\mu))$. Then Corollary 3.2.36 implies an ultrapower of T is weakly compact. Since the ultrapower of $L_1(\mu)$ is L_1-space [48, Theorem 3.3] the ultrapower of T is strictly singular by [68, Theorem 4].

Hence, $T \in \mathcal{FSS}(L_1(\mu))$ by [66, Lemma 4]. ■

Proposition 3.2.38. *Suppose that Banach spaces X and Y satisfy $B(Y, X) = \mathcal{SS}(Y, X)$, and let $Z = X \oplus Y$. Then, for any $T \in B(X, Y)$, the operator $S = \begin{pmatrix} 0 & 0 \\ T & 0 \end{pmatrix} \in B(Z)$ is inessential.*

Proof. By [75], S is inessential if and only if, for any $A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \in B(Z)$, $I - AS$ has finite dimensional kernel. But $\ker(I - AS)$ consists of all vectors $x \oplus y$ ($x \in X$, $y \in Y$) satisfying $x \in \ker(I - A_2T)$, and $y = A_4Tx$. However, $A_2 \in B(Y, X)$ is strictly singular, hence $I - A_2T$ is Fredholm, hence its kernel is finite dimensional. Thus, $\ker(I - AS)$ is finite dimensional. ∎

Corollary 3.2.39. *Suppose Y is a separable Banach space, and let $Z = Y \oplus \ell_\infty$. Then $\mathcal{IN}(Z) \neq \mathcal{SS}(Z)$. Moreover, for $Y = c_0$, the ideal $\mathcal{IN}(Z)$ properly contains $\mathcal{WK}(Z)$.*

Proof. Since ℓ_∞ is universal for every separable Banach space, there exists an isomorphism $T : Y \to \ell_\infty$. By [4, Theorem 5.5.5], any operator from ℓ_∞ to Y is strictly singular. By Proposition 3.2.38, the operator $\begin{pmatrix} 0 & 0 \\ T & 0 \end{pmatrix}$ is inessential. On the other hand, it is clearly not strictly singular. The last statement follows from the fact that, for $Y = c_0$, $\mathcal{SS}(Z) = \mathcal{WK}(Z)$. ∎

Theorem 3.2.40. *A von Neumann algebra $\mathcal{A}$ is of finite type I if and only if $\mathcal{FSS}(\mathcal{A}) = \mathcal{SS}(\mathcal{A}) = \mathcal{IN}(\mathcal{A}) = \mathcal{WK}(\mathcal{A})$. Moreover, if $\mathcal{A}$ is not of finite type I, then all of this classes are different.*

Proof. Recall that $\mathcal{A}$ is finite type I if it is a direct sum of finitely many algebras of type I_n, where n is a positive integer. By [53, Theorem 6.6.5], any type I_n algebra is isomorphic to $M_n \otimes C$, where C is a commutative von Neumann algebra. Therefore it is isomorphic to $L_\infty(\mu)$ which, together with Theorem 3.2.35, imply $\mathcal{FSS}(\mathcal{A}) = \mathcal{SS}(\mathcal{A}) = \mathcal{IN}(\mathcal{A}) = \mathcal{WK}(\mathcal{A})$.

If $\mathcal{A}$ is not of finite type I, then (see e.g. [82]) there exists a complete isometry $J : B(\ell_2) \to \mathcal{A}$ (in fact, J and J^{-1} are completely positive). By Stinespring-Wittstock-Arveson-Paulsen Theorem, there exists a complete contraction $S : \mathcal{A} \to B(\ell_2)$, so that $S = J^{-1}$ on $J(B(\ell_2)))$. Denote by E_{ij} the matrix units in $B(\ell_2)$, and consider the map T, taking E_{1j} to E_{kj} (k is the unique integer satisfying $2^{k-1} \leq j < 2^k$), and E_{ij} to 0 for $i > 1$. Clearly, T can be viewed as a "formal identity" from ℓ_2 to $(\oplus_k \ell_2^{2^{k-1}})_{c_0}$, thus it is not finitely strictly singular. Hence, $JTS \in \mathcal{SS}(\mathcal{A}) \setminus \mathcal{FSS}(\mathcal{A})$.

Moreover, $B(\ell_2)$ contains a subspace Z, isometric to $\ell_2 \oplus_\infty \ell_\infty$, and complemented via a projection P. By Corollary 3.2.39, there exists $T \in \mathcal{IN}(Z) \backslash \mathcal{SS}(Z)$. Then $JTPS \in \mathcal{IN}(A) \backslash \mathcal{SS}(A)$.

Finally, note that there is a projection on a copy of ℓ_2, which is a weakly compact but, evidently, not an inessential operator. ■

Any commutative von Neumann algebra $\mathcal{A}$ is of finite type I, hence $\mathcal{FSS}(\mathcal{A}) = \mathcal{SS}(\mathcal{A}) = \mathcal{IN}(\mathcal{A}) = \mathcal{WK}(\mathcal{A})$. Corollary 3.2.39 (together with Theorem 3.2.35) shows that, for a commutative C^*-algebra $\mathcal{A} = c_0 \oplus_\infty \ell_\infty$, $\mathcal{FSS}(\mathcal{A}) = \mathcal{SS}(\mathcal{A}) = \mathcal{WK}(\mathcal{A}) \subsetneq \mathcal{IN}(\mathcal{A})$. However, in many cases, $\mathcal{IN}(\mathcal{A}) \subset \mathcal{WK}(\mathcal{A})$.

Proposition 3.2.41. *Let $\mathcal{A}$ be either a separable C^*-algebra or a von Neumann algebra. Then $\mathcal{IN}(A) \subset \mathcal{WK}(\mathcal{A})$.*

Proof. It suffices to show that, for any $T \notin \mathcal{WK}(\mathcal{A})$, there exists an infinite dimensional subspace M such that $T(M)$ is complemented. Indeed, then $T^{-1}PT|_M = I_M$, where P is a projection on $T(M)$. This witnesses $T \notin \mathcal{IN}(\mathcal{A})$.

As T^* is not weakly compact, [74, Theorem 1] yields $\varepsilon > 0$ and a disjoint normalized sequence of self-adjoint elements $x_n \in A$ such that $\sup_{f \in B_{X^*}} |T^* f(x_n)| > \varepsilon$ for every n. In particular, $T^*|_{\mathrm{span}[x_n : n \in \mathbb{N}]}$ is not weakly compact. The space

c_0 has Property (V) (see e.g. [4, Theorem 5.5.3]), hence there exists a subspace E of $\text{span}[x_n : n \in \mathbb{N}]$, isomorphic to c_0, so that $T|_E$ is an isomorphism. Using a gliding hump argument, we can assume that $E = \text{span}[y_m : m \in \mathbb{N}]$, where (y_m) is a normalized block basis of (x_n) (hence the operators y_m are also disjoint, in the sense that $y_m^* y_k = y_m y_k^* = 0$ if $k \neq m$).

If $\mathcal{A}$ is separable, then $M = E$ works, since c_0 is separably injective. If $\mathcal{A}$ is a von Neumann algebra, consider the space $F \subset \mathcal{A}$ of operators $\sum_m \omega_m y_m$, with $\sup_m |\omega_m| < \infty$. Then F is isometric to ℓ_∞, and $T|_F : F \to \mathcal{A}$ is not weakly compact. By [4, Theorem 5.5.5], F contains a subspace $M \approx \ell_\infty$, so that $T|_M$ is an isomorphism. By the injectivity of ℓ_∞, $T(M)$ is complemented. ■

Chapter 4

Summary

This thesis is devoted to operator ideals on various ordered Banach spaces.

In **Chapter 2** we considered the following question: what is the relationship between order and algebraic ideals in $L(X)$, where X is an ordered Banach space? In other words, assume that two positive operators T and S act on X, and S is greater then T (i.e. $S - T$ is positive). If S belongs to a certain operator ideal, does T (or its power) belong to the same ideal? This question has been extensively studied for various classes of operators, acting between Banach lattices [5, 34, 38, 39, 40, 42, 54, 99, 100].

In **Section 2.2** (which is based on [93]) we looked at the classical domination problem for compact and strictly singular operators on Banach lattices and established the connection with the inessential operators. Dodds and Fremlin [34] noticed that compactness of S does not necessary imply compactness of T. We considered the question whether T will belong to a 'slightly' larger class of operators, namely, to the ideal of inessential operators. It seemed natural to consider this ideal since it contains the ideal of strictly singular operators which, in turn, contains the ideal of compact operators. At the same time it is not too large since all operators in this ideal share the same spectral properties as compact operators. We showed that if S is compact, then T is inessential, and, moreover, if X is order continuous then T is strictly singular.

In the case $X = Y$, Flores, Hernandez and Tradacete [42] discovered that if S is strictly singular then T^4 is strictly singular. They asked whether the fourth power is optimal. We proved that T^3 must be inessential. This suggests that the fourth power might not be optimal since inessential and strictly singular operators coincide on many Banach lattices.

The results of **Section 2.3** are based on joint work with T. Oikhberg [70]. We were among the first who considered the domination problem for operators acting between either C^*-algebras or noncommutative function spaces. Among the most interesting results of this section are the following statements:

Theorem. Suppose $\mathcal{A}$ and $\mathcal{B}$ are C^*-algebras. Then the following are equivalent:

(i) At least one of the two conditions holds: $\mathcal{A}$ is scattered or $\mathcal{B}$ is compact.

(ii) If $0 \leq T \leq S : \mathcal{A} \to \mathcal{B}$, and S is compact, then T is compact.

Theorem. Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras and $0 \leqslant T \leqslant S : \mathcal{A} \to B$. If S is weakly compact operator then T is weakly compact.

While working on the domination problem, we established a few structural results on C^*-algebras and noncommutative function spaces of their own interest. For instance, we characterized the C^*-algebras with compact order intervals and discovered a new characterization of compact C^*-algebras.

In **Chapter 3** we studied how the geometry of a Banach space X affects the ideal structure of the operator algebra $L(X)$. How many proper operator ideals are there? Which classical operator ideals coincide? Can we characterize all operators belonging to a certain ideal? Mostly, we were interested in the following classes of operators: (weakly) compact, strictly singular, finitely strictly singular, inessential, Dunford-Pettis, and p-summable operators. When X is an ordered Banach space, for example a Banach lattice, a C^*-algebra, or a non-commutative function/sequence space, we also considered the above questions for positive operators.

Section 3.1 is based on my joint work with A. Kaminska, A. Popov, A. Tcaciuc, and V. Troitsky [58].

The problem of classifying closed ideals of operators on a given Banch space is considered of great difficulty. There have been very few advances since the celebrated result of Gohberg, Markus, Feldman [46], who proved that there is a unique non-trivial ideal in the algebra of operators on ℓ_p $(1 \leqslant p < \infty)$ and on c_0. The area has recently been revived by the series of papers of Laustsen et al. that classified all ideals on $(\oplus \ell_2^n)_0$ [60], and $(\oplus \ell_2^n)_1$ [61], and the construction of Haydon and Argyros [11] of a special HI-space with the ideal structure exactly as on ℓ_p. Sari, Schlumprecht, Tomczak-Jaegermann, Troitsky also studied operator ideals on $\ell_p \oplus \ell_q$ [90].

In this section we presented our progress on ideals on Lorentz sequence spaces. Even though Lorentz and ℓ_p-spaces have similar Banach space geometries, their operator ideal structures turned out to be quite different. We identified several proper non-trivial ideals, showed that some classical ideals coincide, and also proved an interesting result about the factorization of operators through ℓ_1. I also note that, later, Lin, Sari and Zheng [62] produced several similar results for Orlicz sequence spaces.

Section 3.2 is based on a joint work with T. Oikhberg [71]. We extended the results of Weis [97], Caselles and Gonzalez [20] and Flores [37] by characterizing the ideals of strictly singular operators on certain noncommutative L_p-spaces:

Theorem. Suppose τ is a faithful normal finite trace on a hyperfinite von Neumann algebra $\mathcal{A}$, and $1 < p < \infty$. For $T \in L(L_p(\tau))$, the following statements are equivalent:

(i) T is strictly singular.

(ii) $L_p(\tau)$ does not contains a subspace E, isomorphic either to ℓ_p or ℓ_2, such that $T|_E$ is an isomorphism, and both E and $T(E)$ are complemented.

(iii) T is inessential.

Theorem. Suppose τ_1 and τ_2 are normal faithful finite traces on hyperfinite von Neumann algebras $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively. Suppose, furthermore, that either $2 \leq u_2 \leq u_1 < \infty$, or $1 < u_1 \leq u_2 \leq 2$. Then all positive strictly singular operators between $L_{u_1}(\tau_1)$ and $L_{u_2}(\tau_2)$ are compact.

To establish these results we had to identify when noncommutative L_p-spaces have an unconditional subsequence property (USP), that is from every weakly null seminormalized sequence one can extarct It is well known that commutative L_p $(p > 1)$ has an unconditional basis and, therefore, the unconditional subsequence property (USP), that is, from every basic sequence we can extract an unconditional subsequence. Only recently Johnson, Maurey and Schechtman [51] proved that $L_1[0,1]$ fails the USP. We proved that a noncommutative L_p-space ($p > 1$) associated with a hyperfinite von Neumann algebra has the USP and that the noncommutative L_1 has the USP if and only if the associated von Neumann algebra is atomic.

Then we presented similar results for discrete noncommutative L_p spaces (p-Schatten classes). There we also proved various statements on when the ideals of finitely strictly singular, Dunford-Petis, and weakly compact operators are the same.

In the last part of this section we studied the structure of operator ideals on some commutative function spaces and C^*-algebras. In particular, we complemented the results of Milman [68] by showing that the ideals of weakly compact and finitely strictly singular operators acting either from the space of continuous functions into any Banach space or on the space of integrable functions coincide. For von Neumann algebras we showed the following.

Theorem. Let $\mathcal{A}$ be either a separable C^*-algebra or a von Neumann algebra. Then $\mathcal{IN}(A) \subset \mathcal{WK}(\mathcal{A})$.

Theorem. A von Neumann algebra $\mathcal{A}$ is of finite type I if and only if $\mathcal{FSS}(\mathcal{A}) = \mathcal{SS}(\mathcal{A}) = \mathcal{IN}(\mathcal{A}) = \mathcal{WK}(\mathcal{A})$. Moreover, if $\mathcal{A}$ is not of finite type I, then all of this classes are different.

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