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Closures of $(\mathcal{U}+\mathcal{K})$-Orbits of Essentially Normal Models

by

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Abstract

Two operators A, B on a separable Hilbert space are $(\mathcal{U}+\mathcal{K})$-**equivalent** ($A \cong_{u+k} B$) if $A = R^{-1}BR$, where R invertible and $R = U + K, U$ unitary, K compact. The $(\mathcal{U}+\mathcal{K})$-**orbit** of A is defined as $(\mathcal{U}+\mathcal{K})(A) = \{B \in \mathcal{B}(\mathcal{H}) : A \cong_{u+k} B\}$. This orbit lies between the unitary and the similarity orbit. In addition, two $(\mathcal{U}+\mathcal{K})$-equivalent operators are compalent.

In this thesis, we construct essentially normal models — specific operators with various spectral pictures — and investigate the norm closures of the $(\mathcal{U}+\mathcal{K})$-orbits of these models. We start by considering a multiplication operator on a generalized Hardy space — this operator generalizes the forward unilateral shift. We then proceed to investigate operators with different indices, operators with disconnected spectra, operators with enlarged essential spectra, operators with isolated spectral points and operators whose spectra are connected but not simply connected.

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Introduction and Preliminaries

The concepts of $(\mathcal{U}+\mathcal{K})$-equivalence and $(\mathcal{U}+\mathcal{K})$-orbits were first introduced by Herrero in [**Her86**] in connection with the investigation of those quasidiagonal operators which are limits of block-diagonal nilpotents.

$(\mathcal{U}+\mathcal{K})$-orbits were subsequently studied by Guinand and Marcoux in [**GM93a**], where the closures of $(\mathcal{U}+\mathcal{K})$-orbits are described for normal and compact operators and for the unilateral shift. Weighted shifts are investigated in [**GM93b**]. A class of operators that includes the unilateral shift is dealt with in [**Mar92**]. A different generalization of the unilateral shift case due to Ji, Jiang and Wang is available in [**JJW96**]. Some more results pertaining to compact and essentially normal operators are due to Al-Musallam, [**AM90**].

In this thesis we shall study the closures of $(\mathcal{U}+\mathcal{K})$-orbits of operators (models) with various spectral pictures. In the introduction we shall start by recalling some well known facts about three well known equivalence relations for operators on a Hilbert space : similarity, unitary equivalence and compalence. We shall then proceed to introduce $(\mathcal{U}+\mathcal{K})$-orbits and review some of their basic properties. We shall also introduce the concept of essentially normal models.

In Chapter 1, various classes of essentially normal models will be constructed. The principal tools for this construction will be diagonal operators, modified Jordan blocks and a class of generalized Hardy spaces.

In Chapter 2 we shall describe the closure of the $(\mathcal{U}+\mathcal{K})$-orbit of the multiplication operator $M(\Omega,\mu)$, defined on a Hardy space $H^2(\Omega,\mu)$ of functions on a simply connected Cauchy domain. This operator is a generalization of the unilateral shift. This will be an important first step, since $M(\Omega,\mu)$ is not only a model for the class of operators sharing its spectral properties, but also a basic building block of the more involved models that will be investigated later. The technique used here will be an adaptation of the technique used in the description of the closure of the $(\mathcal{U}+\mathcal{K})$-orbit of the unilateral shift in [**GM93a**].

In Chapter 3 we shall develop some functional calculus techniques that will allow us to use conformal maps to generalize some results about the closures of $(\mathcal{U}+\mathcal{K})$-orbits. Most importantly, we will be able to generalize the main result of [**Mar92**].

In Chapter 4 we shall investigate the ampliations of the operator $M(\Omega,\mu)$. This will be accomplished by developing a tridiagonal decomposition technique. This technique will allow us to show that an operator is in the closure of the $(\mathcal{U}+\mathcal{K})$-orbit of another operator in some instances where the similarity of the two operators is easily observed. but the investigation of the $(\mathcal{U}+\mathcal{K})$-orbit requires a lengthier argument.

We shall then proceed to investigate models with disconnected spectrum in Chapter 5. It will turn out that the off-diagonal entries in the operator matrix of the Riesz decomposition of an essentially normal operator are compact. This will allow us to deal with this case using a corollary of Rosenblum's theorem.

In Chapter 6 we shall enlarge the essential spectrum of our model. The description of the closure of the $(\mathcal{U}+\mathcal{K})$-orbit of this model can then be accomplished by using some of the results developed in the previous chapters, together with the Weyl-von Neumann-Berg-Sikonia Theorem.

Models with isolated eigenvalues pose some specific problems. Fortunately. techniques that deal with this have already been developed in **[GM93a]** during the investigation of the closures of $(\mathcal{U}+\mathcal{K})$-orbits of a normal operator. In Chapter 7 we shall adapt their method to investigate a model whose spectrum includes isolated points.

Finally, in Chapter 8. we shall deal with a model with multiply connected spectrum. The crucial techniques here will be the functional calculus and block-tridiagonal decomposition techniques developed in Chapters 3 and 4.

In Chapter 9 we will state several open questions related to the results proved in this thesis.

Equivalence relations and orbits in $\mathcal{B}(\mathcal{H})$

In this paragraph we shall introduce some basic notation and we shall recall some of what is known about similarity, unitary equivalence and compalence.

Throughout this thesis, $\mathcal{H}$ will denote a complex separable Hilbert space. Let $\mathcal{B}(\mathcal{H})$ denote the Banach algebra of bounded linear operators on $\mathcal{H}$ equipped with the usual operator norm. Note that whenever we speak of closures of subsets of $\mathcal{B}(\mathcal{H})$ we have the norm topology in mind.

We shall use $\mathcal{K}(\mathcal{H})$ to denote the set of all compact operators on $\mathcal{H}$. Note that $\mathcal{K}(\mathcal{H})$ is a closed ideal in $\mathcal{H}$, and so we can construct the quotient Banach algebra $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$. We will denote this algebra by $\mathcal{A}(\mathcal{H})$ and call it the *Calkin Algebra*. The corresponding quotient map from $\mathcal{B}(\mathcal{H})$ to $\mathcal{A}(\mathcal{H})$ will be denoted by π.

Recall that the *spectrum* of an operator $T \in \mathcal{B}(\mathcal{H})$ is the set $\sigma(T) = \{\lambda \in \mathbb{C} : (T - \lambda I) \text{ is not invertible in } \mathcal{B}(\mathcal{H})\}$. ($I$ is the identity operator on $\mathcal{H}$.) We define the *essential spectrum* $\sigma_e(T)$ of T as the spectrum of $\pi(T)$ in the Calkin algebra, i.e.

$$\sigma_e(T) = \{\lambda \in \mathbb{C} : (\pi(T) - \lambda e) \text{ is not invertible in } \mathcal{A}(\mathcal{H})\}.$$

($e = \pi(I)$ is the unit in $\mathcal{A}(\mathcal{H})$.)

An operator T is called *Fredholm* if

(i) its range $T(\mathcal{H})$ is closed,
(ii) nul $(T) = \dim(\ker T)$ is finite,
(iii) nul (T^*) is finite.

An operator T is called *semi-Fredholm* if condition (i) holds and at least one of conditions (ii) and (iii) holds as well. Given a (semi-)Fredholm operator T we define its *Fredholm index* as

$$\text{ind}(T) = \text{nul } T - \text{nul } T^*.$$

For any operator T, we define its *Fredholm domain* as

$$\rho_F(T) = \{\lambda \in \mathbb{C} : (T - \lambda I) \text{ is Fredholm}\}$$

and its *semi-Fredholm domain* as

$$\rho_{sF}(T) = \{\lambda \in \mathbb{C} : (T - \lambda I) \text{ is semi-Fredholm}\}.$$

It is well known that an operator X is Fredholm if and only if the element $\pi(X)$ of the Calkin algebra is invertible. As a result,

$$\rho_F(X) = \mathbb{C} \setminus \sigma_e(X).$$

For every $n \in (\mathbb{Z} \cup \{-\infty, \infty\})$, the set $\{T \in \mathcal{B}(\mathcal{H}) | T \text{ is semi-Fredholm}, \text{ind}(T) = n\}$ is open. Therefore, ind(.) is a continuous function on the set of all semi-Fredholm operators.

If X is a semi-Fredholm operator and K is a compact operator, we have

$$\text{ind}(X + K) = \text{ind}(X).$$

This property turns the Fredholm index into a useful tool for investigating compalence and $(\mathcal{U} + \mathcal{K})$-equivalence.

For more details on the properties of (semi-)Fredholm operators, see [**CPY74**].

In this thesis, we shall use cl(Q) to denote the closure of Q when Q is a subset of a topological space. When Q is a subset of $\mathbb{C}$, Q^* will denote $\{\overline{z} : z \in Q\}$. The symbol $\overline{Q}$, where Q is a set, will be avoided here.

Two operators A, B in $\mathcal{B}(\mathcal{H})$ are said to be *similar* if there exists an invertible operator X such that $A = XBX^{-1}$. The *similarity orbit* of A is the set

$$\mathcal{S}(A) = \{XAX^{-1} : X \text{ is an invertible operator on } \mathcal{H}\}.$$

Note that whenever two operators A and B are similar, they share the same algebraic properties, such as spectrum, index and eigenvalues. In fact, if we disregard the inner product and consider A and B as operators on the *vector space* H, we can think of A and B as being the same operator on two isomorphic copies of H. (Note that we are talking about *vector space* isomorphism here.)

Two operators A, B in $\mathcal{B}(\mathcal{H})$ are said to be *unitarily equivalent* ($A \cong B$) if there exists a unitary operator U (i.e. U satisfies $UU^* = U^*U = I$, where I is the identity operator) such that $A = UBU^{-1}$. The *unitary orbit* of A is the set

$$\mathcal{U}(A) = \{UAU^{-1} : U \text{ is a unitary operator on } \mathcal{H}\}.$$

Unitarily equivalent operators are even closer to each other in terms of sharing properties than similar operators are. If A and B are unitarily equivalent we can think of B as A itself acting on an isomorphic copy of $\mathcal{H}$ (note that here we are talking about Hilbert space isomorphism).

Similarly, two elements a, b of the Calkin algebra $\mathcal{A}(\mathcal{H})$ are said to be *unitarily equivalent* if there exists a unitary $u \in \mathcal{A}(\mathcal{H})$ such that $a = ubu^*$. (By saying that u is unitary we mean that $uu^* = u^*u = e$, where $e = \pi(I)$ is the unit of the Calkin algebra.) The *unitary orbit* of a is the set

$$\mathcal{U}(a) = \{uau^{-1} : u \text{ is unitary in } \mathcal{A}(\mathcal{H})\}.$$

In what follows, we shall assume that all Hilbert spaces we are working with are infinite-dimensional, unless stated otherwise.

The fact that $\mathcal{U}(T)$ does not need to be closed motivates the definition of the following concept, closely related to the closure of the unitary orbit:

Two operators A and B are said to be *approximately unitarily equivalent* if for every $\epsilon > 0$ there exist a $K \in \mathcal{K}(\mathcal{H})$ with $\|K\| < \epsilon$ and a unitary operator U such that

$$A = U^*BU + K.$$

Approximate unitary equivalence is a stronger relation than the relation of compalence, defined below. For this reason, the approximate unitary equivalence of two operators A and B is often denoted by $A \sim_a B$.

For normal operators, we have the following description of approximate unitary equivalence and of closures of unitary orbits, due to Weyl, von Neumann, Berg and Sikonia, [**Ber71**], [**Sik71**], compare [**Hal70**], Problem 4.

0.1. Theorem. *Suppose that M and N are normal operators. Then M and N are approximately unitarily equivalent if and only if*

(i) $\sigma_e(M) = \sigma_e(N)$, *and*
(ii) nul $(M - \lambda I) =$ nul $(N - \lambda I)$ *for all* $\lambda \in \rho_F(M) = \rho_F(N)$.

0.2. Corollary. *Let N be normal. Then*

$$\begin{aligned} \mathrm{cl}(\mathcal{U}(N)) &= \{M \in \mathcal{B}(\mathcal{H}) : M \sim_a N\} \\ &= \{M \text{ normal on } \mathcal{H} : \sigma_e(M) = \sigma_e(N) \text{ and} \\ &\quad \text{nul } (M - \lambda I) = \text{nul } (N - \lambda I) \text{ for all } \lambda \in \rho_F(M)\} \end{aligned}$$

There also exists a description of the closure of the unitary orbit for an arbitrary operator, which is due to Hadwin, [**Had77**] . The *operator-valued spectrum* $\Sigma(T)$ of $T \in \mathcal{B}(\mathcal{H})$ is defined as the set of all those operators A acting on some finite-dimensional or infinite-dimensional (separable) Hilbert space such that T is the limit of a sequence $\{T_n\}_{n=1}^{\infty}$ in $\mathcal{B}(\mathcal{H})$ with T_n unitarily equivalent to $A \oplus T_n'$ (for suitable operators T_n').

With this definition, $A \sim_a B$ if and only if $\Sigma(A) = \Sigma(B)$. The equivalence $A \sim_a B \Leftrightarrow A \in \mathrm{cl}(\mathcal{U}(B))$ remains true for arbitrary operators in $\mathcal{B}(\mathcal{H})$ as well.

Note that there is also a description of $\mathrm{cl}(\mathcal{S}(T))$ for a dense class of operators due to Apostol, Fialkow, Herrero and Voiculescu. This description is in terms of spectral properties of operators. Since it is rather lengthy, we refer the reader to [**AFHV84**].

Two operators S and T are said to be *compalent* ($S \sim T$) if there exists a unitary operator U and a compact operator K so that $S = U^*TU + K$. In other words, we want S and T to be unitarily equivalent modulo the ideal of compact operators. Notice that this is an equivalence relation and that if S and T are compalent then their images $\pi(S)$ and $\pi(T)$ in the Calkin algebra are unitarily equivalent.

Recall that an operator A is said to be *essentially normal* if $A^*A - AA^*$ is compact, or, in other words, the element $\pi(A)$ of the Calkin algebra is normal. This generalizes the concept of a normal operator. Since normal operators are more tractable that other operators (more general functional calculus, spectral theorem), it should come as no surprise that essentially normal operators admit a richer theory than operators in general.

Recall also that for essentially normal operators the semi-Fredholm and Fredholm domains coincide, see [**Her90**], 4.1.1. The following theorem, due to Brown, Douglas and Fillmore, provides a spectral characterization of compalence for essentially normal operators:

0.3. Theorem. [BDF73] *Suppose that S and T are essentially normal. Then S and T are compalent if and only if*

(i) $\sigma_e(S) = \sigma_e(T)$
(ii) $\mathrm{ind}(S - \lambda I) = \mathrm{ind}(T - \lambda I)$ *for all* $\lambda \in \rho_F(T) = \rho_F(S)$

0.4. Corollary. *If A is essentially normal, then $\{B : A \sim B\}$ is closed and $\{B : A \sim B\} = \pi^{-1}(\mathcal{U}(\pi(A)))$.*

$(\mathcal{U}+\mathcal{K})$-orbit and $(\mathcal{U}+\mathcal{K})$-equivalence

In this paragraph we shall consider another orbit, which lies between the unitary and the similarity orbit of an operator and which is the principal object investigated in the present thesis.

For a Hilbert space $\mathcal{H}$, let

$$\mathcal{U}+\mathcal{K} = (\mathcal{U}+\mathcal{K})(\mathcal{H}) = \{R \in \mathcal{B}(\mathcal{H}) : \quad R \text{ is invertible in } \mathcal{B}(\mathcal{H}) \text{ and } R \text{ is of the form unitary plus compact}\}$$

Two operators $A, B \in \mathcal{B}(\mathcal{H})$ are $(\mathcal{U}+\mathcal{K})$-*equivalent* ($A \cong_{\mathcal{U}+\mathcal{K}} B$) if $A = R^{-1}BR$, for some $R \in (\mathcal{U}+\mathcal{K})(\mathcal{H})$. Note that this defines an equivalence relation on $\mathcal{B}(\mathcal{H})$. Given an operator T, we define its $(\mathcal{U}+\mathcal{K})$-*orbit* as

$$(\mathcal{U}+\mathcal{K})(A) = \{B \in \mathcal{B}(\mathcal{H}) : A \cong_{\mathcal{U}+\mathcal{K}} B\}.$$

Clearly, we have

$$\mathcal{U}(T) \subseteq (\mathcal{U}+\mathcal{K})(T) \subseteq \mathcal{S}(T).$$

Some basic properties of (necessary conditions for) $(\mathcal{U}+\mathcal{K})$-equivalence are as follows:

0.5. Proposition. *If $A \cong_{\mathcal{U}+\mathcal{K}} B$, we have*

(i) $\sigma(A) = \sigma(B)$,
(ii) $A \sim B$,
(iii) $\sigma_e(A) = \sigma_e(B)$,
(iv) $\mathrm{ind}(A - z) = \mathrm{ind}(B - z)$ *for* $z \in \rho_F(A) = \rho_F(B)$.

Proof. Statement (i) follows from the fact that A and B are similar. To see that A and B are compalent, write

$$A = (U+K)^{-1}B(U+K),$$

where U is unitary and K is compact. Now

$$\begin{aligned}(U+K)A &= B(U+K)\\ UA+KA &= BU+BK\\ UA &= BU+BK-KA\\ A &= U^*BU+(U^*BK-U^*KA).\end{aligned}$$

This shows that $A \sim B$, since the expression $U^*BK - U^*KA$ is compact. The statements (iii) and (iv) follow from the similarity of A and B (or from their compalence).

□

Since the $(\mathcal{U}+\mathcal{K})$-orbit of an operator need not be closed, it makes sense to also investigate the closures of $(\mathcal{U}+\mathcal{K})$-orbits. In fact one can find out more about these closures, which is consistent with what is known about other orbits — the results concerning the closures of orbits are usually more complete than those about the orbits themselves. We will use $A \to_{\mathcal{U}+\mathcal{K}} B$ to mean $B \in \mathrm{cl}(\mathcal{U}+\mathcal{K})(A)$. Note that this is a transitive relation.

$(\mathcal{U}+\mathcal{K})$-equivalence and compalence are closely related. In fact, descriptions of closures of $(\mathcal{U}+\mathcal{K})$-orbits usually depend on descriptions of compalence. We shall therefore study the closures of $(\mathcal{U}+\mathcal{K})$ orbits of essentially normal operators, for which a description of compalence is provided by the Brown-Douglas-Fillmore theorem.

Now let A be an essentially normal operator. We have

$$(\mathcal{U}+\mathcal{K})(A) \subseteq \pi^{-1}(\mathcal{U}(\pi(A))).$$

The set on the right is closed and therefore

$$\mathrm{cl}(\mathcal{U}+\mathcal{K})(A) \subseteq \pi^{-1}(\mathcal{U}(\pi(A)))$$

and

$$\mathrm{cl}(\mathcal{U}+\mathcal{K})(A) \subseteq \pi^{-1}(\mathcal{U}(\pi(A))) \cap \mathrm{cl}(\mathcal{S}(A)).$$

In fact, for all operators for which a description of the closure of the $(\mathcal{U}+\mathcal{K})$ orbit is known (including normal operators, compact operators, the unilateral shift, etc.,[**GM93a**]), we have

$$\mathrm{cl}(\mathcal{U}+\mathcal{K})(A) = \pi^{-1}(\mathcal{U}(\pi(A))) \cap \mathrm{cl}(\mathcal{S}(A)).$$

We do not know if this equality holds for all essentially normal operators.

Some basic properties of the relation $\to_{\mathcal{U}+\mathcal{K}}$ are as follows:

0.6. Proposition. *Let A be essentially normal. If $A \to_{\mathcal{U}+\mathcal{K}} B$, we have*

(i) $\sigma(A) \subseteq \sigma(B)$,
(ii) $A \sim B$ *(and hence B is essentially normal),*
(iii) $\sigma_e(A) = \sigma_e(B)$,
(iv) $\mathrm{ind}(A - z) = \mathrm{ind}(B - z)$ *for* $z \in \rho_F(A) = \rho_F(B)$,
(v) $\mathrm{nul}\ (A - z) \leq \mathrm{nul}\ (A - z)$ *for* $z \in \rho_F(A) = \rho_F(B)$,
(vi) $\mathrm{nul}\ (A - z)^* \leq \mathrm{nul}\ (A - z)^*$ *for* $z \in \rho_F(A) = \rho_F(B)$

Proof. We have $\sigma(A) = \sigma(B_0)$ for all B_0 in $(\mathcal{U} + \mathcal{K})(A)$. If $B \in \mathrm{cl}(\mathcal{U} + \mathcal{K})(A)$. we have $B = \lim_{n\to\infty} B_n$ for some $\{B_n\} \subseteq (\mathcal{U} + \mathcal{K})(A)$. Hence $\sigma(B) \supseteq \sigma(A)$, as the spectrum can only increase by passing to the limit. We have (i). Condition (ii) is the above observation $\mathrm{cl}(\mathcal{U} + \mathcal{K})(A) \subseteq \pi^{-1}(\mathcal{U}(\pi(A)))$. Conditions (iii) and (iv) follow from compalence. For conditions (v) and (vi) see Theorem 1.13 in [**Her90**].

□

To familiarize ourselves with some of these basic properties, let us now consider the $(\mathcal{U}+\mathcal{K})$-orbit of a specific operator:

0.7. Example. *Let S be the forward unilateral shift. Any operator $T \in \mathrm{cl}(\mathcal{U}+\mathcal{K})(S)$ must satisfy the following conditions:*

(i) *T is essentially normal,*
(ii) $\sigma(T) = \sigma(S) = \{z \in \mathbb{C} : |z| \leq 1\}$,
(iii) $\sigma_e(T) = \sigma_e(S) = \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$,
(iv) $\mathrm{ind}(T - z) = \mathrm{ind}(S - z) = -1$ *for* $|z| < 1$.

Proof. Since S is essentially normal, $\pi(S)$ is a normal element of the Calkin algebra. If now T_0 is in $(\mathcal{U}+\mathcal{K})(S)$, $\pi(T_0)$ is unitarily equivalent to $\pi(S)$ and therefore it is also normal. It follows that any T_0 in $(\mathcal{U} + \mathcal{K})(S)$ is essentially normal. The set of essentially normal operators is closed and hence all elements of $\mathrm{cl}(\mathcal{U} + \mathcal{K})(S)$ are essentially normal. We have shown (i).

Conditions (iii) and (iv), as well as the fact that $\sigma(T) \supseteq \sigma(S)$, follow from Proposition 0.6.

In the case of the unilateral shift, $\sigma(S)$ is simply connected. Newburgh's theorem (see [**Aup91**], Theorem 3.4.4) implies that $\sigma(T)$ cannot have a component that would be disjoint from $\sigma(S)$. This means that $\partial\sigma(T)$ has no isolated points, hence $\partial\sigma(T) \subseteq \sigma_e(T) = \mathbb{T}$ and we must have $\sigma(T) = \{z \in \mathbb{C} : |z| = 1\}$.

□

In [**GM93a**], Guinand and Marcoux showed that the conditions above are in fact sufficient:

0.8. Theorem.[**GM93a**] *Let S be the forward unilateral shift. An operator T is in* $\mathrm{cl}(\mathcal{U}+\mathcal{K})(S)$ *if and only if the conditions (i) to (iv) above are satisfied.*

Since many existing descriptions of closures of orbits of Hilbert space operators are in terms of spectral properties, one may now suspect that any operator T that shares the spectral properties of S will have $\mathrm{cl}(\mathcal{U}+\mathcal{K})(T) = \mathrm{cl}(\mathcal{U}+\mathcal{K})(S)$. This is indeed the case, as was shown in [**Mar92**].

There, an operator $T \in \mathcal{B}(\mathcal{H})$ is called *shift-like* if

(a) T is essentially normal,
(b) $\sigma(T) = \mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\}$,
(c) $\sigma_e(T) = \mathbb{T}$.
(d) $\mathrm{ind}(T-\lambda) = -1$ for all $\lambda \in \{z \in \mathbb{C} : |z| < 1\}$,
(e) nul $(T-\lambda) = 0$ for all $\lambda \in \{z \in \mathbb{C} : |z| < 1\}$.

With this definition we have:

0.9. Theorem.[**Mar92**] *Suppose that T is shift-like. Then $S \in \mathrm{cl}(\mathcal{U}+\mathcal{K})(T)$ and therefore* $\mathrm{cl}(\mathcal{U}+\mathcal{K})(T) = \mathrm{cl}(\mathcal{U}+\mathcal{K})(S)$.

Notice that describing the closure of the $(\mathcal{U}+\mathcal{K})$-orbit of one particular operator S proved to be the first step towards doing the same for the whole class of essentially normal operators with the same spectral properties.

In this thesis, we shall first construct essentially normal operators with various spectral pictures – models – and next we shall describe the closures of the $(\mathcal{U}+\mathcal{K})$-orbits of these models.

Notation. For two operators A, B, we shall write $A \cong_\epsilon B$ if there exists a unitary operator U such that $\|A - U^*BU\| < \epsilon$. Note that we do not require that $A - U^*BU$ be compact.

For $X \subseteq \mathbb{C}$, $\epsilon > 0$, we set $X_\epsilon = \{x \in \mathbb{C} : \mathrm{dist}(x, X) < \epsilon\}$.

For $T \in \mathcal{B}(\mathcal{H})$ and $\Delta \subseteq \sigma(T)$ closed and open, we will denote by $E(\Delta; T)$ the corresponding Riesz idempotent. (See [**DS57**] for the definition of the Riesz-Dunford functional calculus and related properties.) The range of $E(\Delta; T)$ will be denoted by $\mathcal{H}(\Delta; T)$. If $\Delta = \{\lambda\}$ is a single point and the dimension of $\mathcal{H}(\Delta; T)$ is finite, we shall call λ a *normal eigenvalue* of T. The set of all normal eigenvalues of T will be denoted by $\sigma_0(T)$.

For $T \in \mathcal{B}(\mathcal{H})$, $\sigma_{iso}(T)$ will denote the isolated points of $\sigma(T)$ and $\sigma_{acc}(T)$ will denote the accumulation points of $\sigma(T)$. By $\sigma_p(T)$ we shall denote the point spectrum (i.e. eigenvalues) of T.

For an operator A, $\rho^r_{sF}(A)$ denotes the *regular points* of the semi-Fredholm domain.

$$\rho^r_{sF}(A) = \{\lambda \in \rho_{sF}(A) \quad : \quad \text{nul } (A-\mu) \text{ and nul } (A-\mu)^* \\ \text{are continuous on some neighbourhood of } \lambda\},$$

and $\rho^s_{sF}(A) = \rho_{sF}(A) \setminus \rho^r_{sF}(A)$ denotes the *singular points* of the semi-Fredholm domain.

CHAPTER 1

Construction of the Models

A generalization of the unilateral shift

First of all we shall introduce a generalization of the Hardy space H^2, which will be the basic building block of our models. (See for instance [**Rud87**], Chapter 17, for the definition and basic properties of H^2.) Recall that a nonempty bounded open subset Ω of the complex plane $\mathbb{C}$ is a *Cauchy domain* if Ω has finitely many components, the closures of any two of which are disjoint, and the boundary $\partial\Omega$ of Ω is composed of a finite positive number of closed rectifiable Jordan curves, no two of which intersect. A Cauchy domain with an analytic boundary will be called an analytic Cauchy domain.

Let Ω be a simply connected analytic Cauchy domain. Then there exists a $\rho > 1$ and an invertible holomorphic function ϕ from $\{z : |z| < \rho\}$ to $\mathbb{C}$ such that $\phi|_{\mathbb{D}}$ is a conformal map of $\mathbb{D}$ onto Ω. Consequently, $|\phi'|$ is bounded and bounded away from zero on $\mathrm{cl}(\mathbb{D})$. (See [**Cur78**], Theorem 13.7.4.) Let us fix a ϕ with these properties.

We shall first of all define the set $H^2(\Omega)$ of holomorphic functions on Ω as follows:

$$H^2(\Omega) = \{f \circ \phi^{-1} : f \in H^2(\mathbb{D})\}.$$

If g is holomorphic on Ω, we shall denote

$$\hat{g}(z) = \lim_{r \to 1-} g(\phi(r.\phi^{-1}(z))),\ z \in \partial\Omega.$$

For $g \in H^2(\Omega)$, $\hat{g}(z)$ exists almost everywhere on $\partial\Omega$ with respect to the arc length measure λ and $\hat{g} \in L^2(\partial\Omega, \lambda)$. This is the case since $g \circ \phi \in H^2(\mathbb{D})$ and therefore

$$\lim_{r \to 1-} (g \circ \phi)(r.z_0),\ z_0 \in \mathbb{T}$$

exists almost everywhere on $\mathbb{T}$.

Next, let μ be a measure on $\partial\Omega$ equivalent to λ (i.e. μ and λ are assumed to be absolutely continuous with respect to each other). For $g, h \in H^2(\Omega)$, define

$$\langle g, h\rangle = \int_{\partial\Omega} g(z).\overline{h(z)}d\mu.$$

This is an inner product. $H^2(\Omega)$ with this inner product becomes a Hilbert space; we shall denote it $H^2(\Omega,\mu)$. This space inherits many of its properties from $H^2(\mathbb{D})$.

The proof of the following lemma is routine — the properties of a $g \in H^2(\Omega)$ discussed here follow from the corresponding properties of $g \circ \phi \in H^2(\mathbb{D})$.

1.1. Lemma. *Let Ω be a simply connected analytic Cauchy domain.*

(i) *$g \in H^2(\Omega)$ if and only if g is holomorphic on Ω, $\hat{g}(z)$ exists almost everywhere on $\partial\Omega$ and $\hat{g} \in L^2(\partial\Omega, \lambda)$;*
(ii) *Let $g \in H^2(\Omega)$. For $r \in [0,1)$, let*

$$g_r(z) = g(\phi(r.\phi^{-1}(z))),\ z \in \Omega$$

Then $g_r \in C(\mathrm{cl}(\Omega)) \cap H^2(\Omega)$ and $g_r \to g$ as $r \to 1-$ in any $H^2(\Omega,\mu)$. Note that g_r can also be viewed as a function which is holomorphic on an open set that includes $\mathrm{cl}(\Omega)$.

We can define $\widehat{H}^2(\Omega,\mu) = \{\hat{g} : g \in H^2(\Omega,\mu)\}$. Then there is a one-to-one correspondence $g \mapsto \hat{g}$ between these two sets. As is the custom for $H^2(\mathbb{D})$, we shall identify these two sets whenever it is convenient.

The next lemma shows that the construction of $H^2(\Omega,\mu)$ does not depend on the choice of ϕ.

1.2. Lemma. *Let Ω be a simply connected analytic Cauchy domain. Then $\widehat{H}^2(\Omega,\mu)$ is the closure of the linear span of polynomials in $L^2(\partial\Omega,\mu)$.*

Proof. If $f \in H^2(\Omega,\mu)$, it can be approximated (in the L^2-norm) by a function f_r which is holomorphic on an open set that includes $\mathrm{cl}(\Omega)$. Next we can use Runge's theorem to approximate f_r by a polynomial (uniformly on $\mathrm{cl}(\Omega)$). This shows that $H^2(\Omega,\mu)$ is included in the closure of the linear span of polynomials in $L^2(\partial\Omega,\mu)$. The other inclusion is trivial.

□

Next, we define a multiplication operator $M(\Omega,\mu)$ on $H^2(\Omega,\mu)$ by

$$M(\Omega,\mu)(g)(z) = z.g(z),\ z \in \Omega.$$

Then $M(\Omega, \mu)$ is an essentially normal operator and the spectral properties of $M(\Omega. \mu)$ are as follows:

(i) $\sigma(M(\Omega, \mu)) = \mathrm{cl}(\Omega)$;
(ii) $\sigma_e(M(\Omega, \mu)) = \partial\Omega$;
(iii) $\mathrm{ind}(M(\Omega, \mu) - z) = -1$, $z \in \Omega$;
(iv) $\min\mathrm{ind}(M(\Omega, \mu) - z) = 0$, $z \in \Omega$

(See [**Her90**], sections 3.2 and 4.1.3)

Thus, for any choice of μ, $M(\Omega. \mu)$ can serve as an essentially normal model for the class of operators sharing these spectral properties.

Disconnected spectrum and increasing the value of the index

Next we shall use operators of this type to construct models with different spectral properties. First of all, let us consider an analytic Cauchy domain Ω consisting of n simply connected components, $\Omega = \bigcup_1^n \Omega_i$. Let μ_i be a measure on $\partial\Omega_i$ equivalent to the arc length measure. The essentially normal operator $M = \bigoplus_{i=1}^n M(\Omega_i, \mu_i)$ on the space $\bigoplus_{i=1}^n H^2(\Omega_i, \mu_i)$ now has the following spectral properties:

(i) $\sigma(M) = \mathrm{cl}(\Omega)$;
(ii) $\sigma_e(M) = \partial\Omega$;
(iii) $\mathrm{ind}(M - z) = -1$, $z \in \Omega$;
(iv) $\min\mathrm{ind}(M - z) = 0$, $z \in \Omega$.

M can now serve as a model for the class of operators sharing these spectral properties.

Note that any element $f = (f_1, f_2, \dots, f_n)$ of $\bigoplus_{i=1}^n H(\Omega_i, \mu_i)$ can also be viewed as a holomorphic function on $\Omega = \cup_1^n \Omega_i$ by setting $f(z) = f_i(z)$, where i is such that $z \in \Omega_i$. We shall adopt this point of view when it is convenient, mostly to simplify notation. With this in mind, note that the operator $M = \bigoplus_{i=1}^n M(\Omega_i, \mu_i)$ can also be defined by the formula $M(f)(z) = z.f(z)$, $z \in \Omega$.

We can also use direct sums to construct models with more general index properties. If $\Omega = \cup_1^n \Omega_i$, μ_i are as above and we have $j_i \in \mathbb{Z}, j_i \neq 0, i = 1, \dots n$, we can set $M_i = \bigoplus_{k=1}^{-j_i} M(\Omega_i, \mu_i)$ if $j_i < 0$ and $M_i = \bigoplus_{k=1}^{j_i} M^*(\Omega_i, \mu_i)$ if $j_i > 0$. (An operator of the type $M_i = \bigoplus_{k=1}^{-j_i} M(\Omega_i, \mu_i)$ is sometimes called the j_i*-fold ampliation* of $M(\Omega_i, \mu_i)$ and is usually denoted by $M(\Omega_i, \mu_i)^{(j_i)}$. Since we will occasionally need the superscript position for indices, we avoid this notation here.)

Setting $M = \bigoplus_{i=1}^n M_i$, we obtain an operator with the following spectral properties:

(i) $\sigma(M) = \mathrm{cl}(\Omega)$;
(ii) $\sigma_e(M) = \partial\Omega$;

(iii) $\operatorname{ind}(M - z) = j_i$, $z \in \Omega_i$;
(iv) $\min\operatorname{ind}(M - z) = 0$, $z \in \Omega$,

where $\min\operatorname{ind}(X)$ is defined as $\min(\operatorname{nul} X, \operatorname{nul} X^*)$.

Connected domain which is not simply connected

Next, let us consider a connected analytic Cauchy domain which is not simply connected. Assume that $\Omega = \Omega_1 \setminus \operatorname{cl}(\Omega_2)$, where Ω_1 is a simply connected analytic Cauchy domain, Ω_2 is an analytic Cauchy domain consisting of n simply connected components, $\Omega_2 = \cup_1^n \Omega_{2,i}$, $\operatorname{cl}(\Omega_2) \subseteq \Omega_1$. Once again, we want to construct an essentially normal operator M with the following spectral properties:

(i) $\sigma(M) = \operatorname{cl}(\Omega)$;
(ii) $\sigma_e(M) = \partial\Omega$;
(iii) $\operatorname{ind}(M - z) = -1$, $z \in \Omega$;
(iv) $\min\operatorname{ind}(M - z) = 0$, $z \in \Omega$.

This operator will then serve as a model for the class of operators sharing the same spectral properties.

One possible construction of the model would consist of constructing a Hardy space $H^2(\Omega)$ and using a multiplication operator on this space. Note that this would require a different, more general definition of the Hardy space than the one used at the beginning of this chapter.

While a model could be constructed in this way, we would run into difficulties if we attempted to describe the closure of its $(\mathcal{U}+\mathcal{K})$-orbit using the same techniques as in Chapter 2. In particular, the Lemmas 2.1 and 2.6 would no longer hold in this setting. We will therefore construct our model in a different manner. The operator $M(\Omega_1, \mu)$, where Ω_1 is as above, will be one of its building blocks. When we investigate the model we construct here in Chapter 8, we shall be able to make use of our investigation of the properties of $M(\Omega_1, \mu)$ in Chapter 2.

Let now μ be a measure on $\partial(\Omega_1)$ and for $i = 1, 2, \ldots, n$, let μ_i be a measure on $\partial(\Omega_{2,i}^*)$; all of these measures are assumed to be equivalent to the respective arc length measures. Let $A = M(\Omega_1, \mu)$ and let $B = \bigoplus_{i=1}^n M(\Omega_{2,i}^*, \mu_i)$. Both of these operators have already been considered as models and we are already familiar with their spectral properties. As a first step in constructing our model, let us consider the (essentially normal) operator $M_0 = A \oplus B^*$. The spectral properties of M_0 are as follows:

(i) $\sigma(M_0) = \operatorname{cl}(\Omega_1)$;
(ii) $\sigma_e(M_0) = \partial\Omega$;
(iii) $\operatorname{nul}(M_0 - z) = 0$, $z \in \Omega$;
(iv) $\operatorname{nul}(M_0^* - \overline{z}) = 1$, $z \in \Omega$;

(v) $\operatorname{ind}(M_0 - z) = -1,\ z \in \Omega;$
(vi) nul $(M_0 - z) = 1,\ z \in \Omega_2;$
(vii) nul $(M_0^* - \bar{z}) = 1,\ z \in \Omega_2;$
(viii) $\operatorname{ind}(M_0 - z) = 0,\ z \in \Omega_2.$

We see that M_0 has some of the properties we require of M : the properties (ii),(iii),(iv),(v) and (viii) are as required. We shall now construct M as compact perturbation of M_0. This will allow us to change the spectrum of our operator (to exclude Ω_2) without disturbing the already correct essential spectrum and index properties.

The following lemma shows how this can be accomplished. In fact, although the lemma is more general than necessary for the construction of the model, it will be useful when we investigate the closure of the $(\mathcal{U} + \mathcal{K})$ orbit of the model.

1.3. Lemma. *Let $\Omega, \Omega_1, \Omega_{2,1}, \Omega_{2,2}, \ldots, \Omega_{2,n}; \mu, \mu_1, \mu_2, \ldots, \mu_n$ be as above. Let 1_{Ω_1} be the constant function equal to 1 on Ω_1. Let A, B be as above. For $b \in \bigoplus_{i=1}^n H^2(\Omega_{2,i}^*, \mu_i)$, let C_b be an operator from $\bigoplus_{i=1}^n H^2(\Omega_{2,i}^*, \mu_i)$ into $H^2(\Omega_1, \mu)$ defined by $C_b g = (1_{\Omega_1} \otimes b^*)(g) = \langle g, b\rangle . 1_{\Omega_1}$, $g \in \bigoplus_{i=1}^n H^2(\Omega_{2,i}, \mu_i)$. Next, define an operator M_b on $H^2(\Omega_1, \mu) \oplus (\bigoplus_{i=1}^n H^2(\Omega_{2,i}^*, \mu_i))$ by*

$$M_b = \begin{pmatrix} A & C_b \\ 0 & B^* \end{pmatrix}.$$

(a) *If $b(z) \neq 0$ for $z \in \Omega_2$, we have*
  (i) $\sigma(M_b) = \operatorname{cl}(\Omega);$
  (ii) $\sigma_e(M_b) = \partial\Omega;$
  (iii) nul $(M_b - z) = 0,\ z \in \Omega;$
  (iv) nul $(M_b^* - \bar{z}) = 1,\ z \in \Omega;$
  (v) $\operatorname{ind}(M_b - z) = -1,\ z \in \Omega.$

(b) *If $b(z_0) = 0$ for some $z \in \Omega_2$, then z_0 is an eigenvalue of M_b.*

With this lemma in hand, we can finish the construction of the model by letting $b(z) = 1$ for $z \in \Omega_2$ and letting $M = M_b$.

Proof.

(a) Since the essential spectrum and index properties of M_b are already known (M_b being a compact perturbation of M_0), it suffices to show that nul $(M_b - z) = 0$ for $z \in \Omega_1 \setminus \operatorname{cl}(\Omega_2)$ and nul $(M_b - z) = 0$ for $z \in \Omega_2$.

Let $z \in \Omega_1 \setminus \operatorname{cl}(\Omega_2)$. We want to show that $M_b - z$ does not have any eigenvalues. Suppose that

$$\begin{pmatrix} A - z & C_b \\ 0 & B^* - z \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

for some $f \in H^2(\Omega_1, \mu)$, $g \in \bigoplus_{i=1}^n H^2(\Omega_{2,i}, \mu_i)$, i.e.

$$\begin{aligned} (A - z)f + C_b g &= 0 \\ (B^* - z)g &= 0. \end{aligned}$$

But $z \notin \sigma(B^*)$, so $g = 0$ and hence $(A - z)f = 0$. Since nul $(A - z) = 0$, we must have $f = 0$. We have shown that nul $(M_b - z) = 0$.

Suppose next that $z \in \Omega_2$ and again

$$\begin{pmatrix} A - z & C_b \\ 0 & B^* - z \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

for some $f \in H^2(\Omega_1, \mu)$, $g \in \bigoplus_{i=1}^n H^2(\Omega_{2,i}, \mu_i)$, i.e.

$$\begin{aligned} (A - z)f + C_b g &= 0 \\ (B^* - z)g &= 0. \end{aligned}$$

Assume that $g \neq 0$. Then $g \perp$ ran $(B - \overline{z})$. Since codim ran $(B - \overline{z}) = 1$, we see that ran $(B - \overline{z}) = \{g\}^\perp$. But $b(\overline{z}) \neq 0$, which means that $b \notin$ ran $(B - \overline{z})$ and so $\langle g, b \rangle \neq 0$. From above, we know that $(A - z)f = -C_b g = -\langle g, b \rangle . 1_{\Omega_1}$, hence $(A - z)f$ is a non-zero multiple of 1_{Ω_1}, i.e. a non-zero constant function on Ω_1. This is a contradiction, as $[(A - z)f](z) = 0$.

Hence we must have $g = 0$. This implies $(A - z)f = 0$ and so, as above, $f = 0$. We see that nul $(M - z) = 0$ in this case too.

(b) Suppose that $b(z_0) = 0$ for some $z_0 \in \Omega_2$. We can choose $g_0 \in \bigoplus_{i=1}^n H^2(\Omega_{2,i}, \mu_i)$ such that $(B^* - z_0)g_0 = 0$, $g_0 \neq 0$. We now have $g_0 \perp$ ran $(B - \overline{z_0})$ and $b \in$ ran $(B - \overline{z_0})$, hence $g_0 \perp b$. Consequently, $(M - z_0) \begin{pmatrix} 0 \\ g_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $z_0 \in \sigma(M)$.

□

Enlarging the essential spectrum

In the above models the essential spectrum was equal to the boundary of the spectrum. We shall now proceed to enlarge the essential spectrum. Let K be a compact subset of $\mathbb{C}$. Let $\{d_k\}_{k=1}^\infty$ be a sequence of complex numbers which is dense in K. In addition, assume that any isolated point z_0 of K appears in $\{d_k\}_{k=1}^\infty$ infinitely many times. Let D_K be a diagonal operator on $l^2(\mathbb{N})$ with diagonal entries $\{d_k\}_{k=1}^\infty$. Then we have

$$\sigma(D_K) = \sigma_e(D_K) = K.$$

We can again use direct sums to combine D_K with existing models to obtain models with more involved spectral properties. That is, given an existing model M as above, we define a new model $M_K = M \oplus D_K$.

Isolated eigenvalues

Finally, suppose that we have constructed an essentially normal model M_0 using the above techniques. We now want to attach isolated eigenvalues of finite multiplicity. We do not want to change the essential spectrum at this point. We will therefore assume that any accumulation point of the set of these isolated eigenvalues lies in the essential spectrum of the existing model M_0.

For $z, x \in \mathbb{C}$ we define

$$J(z, x, n) = \begin{pmatrix} z & x & & & \\ & z & x & & \\ & & \ddots & \ddots & \\ & & & z & x \\ & & & & z \end{pmatrix}.$$

The matrix is an n by n square matrix. Suppose we have a sequence $\{z_r\}_{r=1}^{\nu}$, here ν may be a finite number or $\nu = \infty$. We are assuming $\{z_r\}_{\text{acc}} \subseteq \sigma_e(M_0)$ and $\{z_r\} \cap \sigma(M_0) = \emptyset$. Next choose a sequence of complex numbers $\{x_r\}_{r=1}^{\nu}$ such that, if $\nu = \infty$, we have $\lim_{r\to\infty} x_r = 0$. Note that when ν is finite, we can do without the parameters x_r and work with the Jordan blocks $J(z_r, 1, n_r)$.

Let $\{n_r\}_{r=1}^{\nu}$ be a sequence of natural numbers. Consider the operator

$$M_1 = \bigoplus_{r=1}^{\nu} J(z_r, x_r, n_r).$$

Any $\{z_{r_0}\}$ is now an eigenvalue of M_1 of multiplicity

$$\sum_{r \in \{r: z_r = z_{r_0}\}} n_r.$$

Moreover, an easy calculation yields the fact that M_0 is essentially normal. An application of Theorem 5.42 in [**Her90**] verifies that the spectrum consists precisely of the eigenvalues $\{z_{r_0}\}$. (Both these statements rely on the fact that $\lim_{r\to\infty} x_r = 0$.)

The operator $M_0 \oplus M_1$ is now a new essentially normal model whose spectral picture differs from that of M_0 by the presence of the isolated eigenvalues of finite multiplicity described above.

CHAPTER 2

Generalized Shift

In this chapter we will be concerned with the closure of the $(\mathcal{U}+\mathcal{K})$-orbit of the essentially normal operator $M(\Omega, \mu)$, constructed in Chapter 1, with the following spectral properties: (Throughout this chapter Ω will be a simply connected Cauchy domain):

(i) $\sigma(M(\Omega, \mu)) = \mathrm{cl}(\Omega)$;
(ii) $\sigma_e(M(\Omega, \mu)) = \partial\Omega$;
(iii) $\mathrm{ind}(M(\Omega, \mu) - z) = -1$, $z \in \Omega$;
(iv) $\min\mathrm{ind}(M(\Omega, \mu) - z) = 0$, $z \in \Omega$.

We should notice the following important special case. If we set $\Omega = \mathbb{D}$ and if λ is the arc length measure, $M(\mathbb{D}, \lambda)$ will be unitarily equivalent to the unilateral shift S, the closure of the $(\mathcal{U}+\mathcal{K})$-orbit of which has already been described in [**GM93a**]. In this chapter, we shall generalize their result. In doing this, two approaches are possible:

(1) We can adapt the proof of [**GM93a**] by following its basic steps and making any changes that are needed to make the proof work for our operator $M(\Omega, \mu)$:
(2) We can take the description of $\mathrm{cl}((\mathcal{U}+\mathcal{K})(S))$ and use functional calculus to obtain a description of $\mathrm{cl}((\mathcal{U}+\mathcal{K})(M(\Omega, \mu)))$.

We shall pursue both of these directions. In fact, approach (2) will be easier and will ultimately yield a stronger result. The usefulness of method (1) will become apparent in the next chapters, where we will be able to use the auxiliary results which will be proven along the way towards a description of $\mathrm{cl}((\mathcal{U}+\mathcal{K})(M(\Omega, \mu)))$. We shall start with approach (1). Note that a similar result was proved in [**JJW96**], using a similar approach.

Suppose Ω and μ are fixed. First we shall construct an auxiliary function in $H^2(\Omega, \mu)$.

2.1. Lemma. *For $z_0 \in \Omega$ there is a function $\psi \in H^2(\Omega, \mu)$ such that ψ has a simple zero at z_0, $\psi(z) \neq 0$ for $z \neq z_0$, $|\psi(z)| = 1$ almost everywhere on $\partial\Omega$.*

Proof. Let ϕ be the function used in Chapter 1 to construct $H^2(\Omega, \mu)$. Let B be a Blaschke product with one simple zero at $\phi^{-1}(z_0)$. Set $\psi = B \circ \phi^{-1}$.Then ψ has the desired properties.

□

The next few auxiliary results are concerned with "pulling finite dimensional matrices out of $M(\Omega,\mu)$." More precisely, we want to know under what conditions are operators of the form

$$\begin{pmatrix} F & 0 \\ Q & M(\Omega,\mu) \end{pmatrix},$$

where F acts on a space of finite dimension, in (the closure of) the $(\mathcal{U}+\mathcal{K})$-orbit of $M(\Omega,\mu)$.

2.2. Lemma. *Let z_0 be in Ω. Then $M(\Omega,\mu)$ is unitarily equivalent to an operator of the form*

$$\begin{pmatrix} z_0 & 0 \\ Q & M(\Omega,\mu) \end{pmatrix},$$

for some $Q \in \mathcal{B}(\mathbb{C}, H^2(\Omega,\mu))$.

Proof. Let ψ be the function defined in the 2.1. Define an operator on $H^2(\Omega,\mu))$ by

$$Nf(z) = \psi(z).f(z),\ f \in H^2(\Omega,\mu),\ z \in \mathrm{cl}(\Omega).$$

Notice that N is an isometry, codim ran $N = 1$, $NM(\Omega,\mu) = M(\Omega,\mu)N$. Denote $H_1 =$ ran N, $H_0 = H_1^{\perp}$. Let

$$\begin{pmatrix} z_1 & P \\ Q' & R \end{pmatrix},$$

be the matrix of M with respect to the decomposition $H^2(\Omega,\mu) = H_0 \oplus H_1$. We want to show $z_1 = z_0$, $P = 0$, and R is unitarily equivalent to $M(\Omega,\mu)$. (No claim is being made about Q'.)

First, $R = P_{H_1} M(\Omega,\mu)|_{H_1} = NN^*M(\Omega,\mu)|_{H_1}$ and hence

$$\begin{aligned} N^*RN &= (N^*N)N^*M(\Omega,\mu)N \\ &= N^*NM(\Omega,\mu) \\ &= M(\Omega,\mu) \end{aligned}$$

This is what we need since N is unitary when considered as a map from $H^2(\Omega,\mu)$ to H_1.

Next, noting that H_1 is invariant for M, we see that $P = 0$.

Finally, we chose $e \in H_0$, $\|e\| = 1$, so $z_1 = \langle M(\Omega,\mu)e, e\rangle$. Consider these two linear functionals on $H^2(\Omega,\mu)$:

$$\begin{aligned} f &\mapsto f(z_0) \\ f &\mapsto \langle f, e\rangle \end{aligned}$$

Both of these have the same kernel H_1. Hence $\langle f, e\rangle = \alpha f(z_0)$ for some $\alpha \in \mathbb{C}$, $\alpha \neq 0$. Hence we have

$$\begin{aligned} z_1 &= \langle M(\Omega,\mu)e, e\rangle \\ &= \alpha(M(\Omega,\mu)e)(z_0) \\ &= \alpha . z_0 . e(z_0) \\ &= z_0\langle e, e\rangle \\ &= z_0 \end{aligned}$$

□

2.3. Corollary. *Let $\{z_1, z_2, \ldots, z_n\}$ be distinct elements of Ω. Then there exists an operator Q such that*

$$M(\Omega,\mu) \cong_{\mathcal{U}+\mathcal{K}} \begin{pmatrix} F_d & 0 \\ Q & M(\Omega,\mu) \end{pmatrix},$$

where F_d is the $n \times n$ diagonal matrix $F_d = \operatorname{diag}\{z_i\}_{i=1}^n$.

Proof. By applying Lemma 2.2 n times, we see that

$$M(\Omega,\mu) \cong \begin{pmatrix} E & 0 \\ Q' & M(\Omega,\mu) \end{pmatrix},$$

where E is a lower triangular matrix with $z_1, z_2, \ldots, z_n$ on the diagonal. Since E has no repeated eigenvalues, $F_d = RER^{-1}$ for some invertible matrix R. We can now apply the similarity transformation $\begin{pmatrix} R & 0 \\ 0 & I \end{pmatrix} \in \mathcal{U} + \mathcal{K}$ to $\begin{pmatrix} E & 0 \\ Q' & M(\Omega,\mu) \end{pmatrix}$ to obtain

$$M(\Omega,\mu) \cong_{\mathcal{U}+\mathcal{K}} \begin{pmatrix} F_d & 0 \\ Q & M(\Omega,\mu) \end{pmatrix},$$

where $Q = Q'R_{-1}$.

□

2.4. Lemma. *Let C be an operator of the form*

$$C = \begin{pmatrix} F_d & 0 \\ T & M(\Omega,\mu) \end{pmatrix},$$

where F_d is a diagonal matrix. Then the following statements are equivalent:

(i) $C \cong_{\mathcal{U}+\mathcal{K}} M(\Omega, \mu)$;
(ii) *C is similar to $M(\Omega, \mu)$;*
(iii) *the diagonal entries $\{z_1, z_2, \ldots, z_n\}$ of F_d are distinct, they lie in Ω, and C has no eigenvalues;*
(iv) *the diagonal entries $\{z_1, z_2, \ldots, z_n\}$ of F_d are distinct and lie in Ω, and, for $1 \leq i \leq n$, the i-th column t_i of T is not in*

$$\text{ran } (M(\Omega, \mu) - z_i I) = \{f \in H^2(\Omega, \mu) : f(z_i) = 0\}.$$

Proof. The implication (i) $\Rightarrow$ (ii) is trivial.

To obtain (ii) $\Rightarrow$ (iii), consider spectral properties of C and C^*. In particular, to see that the entries of F_d have to be distinct, note that if z appeared as a diagonal entry of F_d more than once, we would have nul $(C - z)^* > 1$, which contradicts (ii).

Consider the implication (iii) $\Rightarrow$ (iv). Without loss of generality, we can assume $z_1 = 0$ (if not, subtract $z_1 I$ from both C and $M(\Omega, \mu)$). We have

$$C\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \\ f \end{pmatrix} = \begin{pmatrix} 0 & & & & \\ & z_2 & & & \\ & & \ddots & & \\ & & & z_n & \\ t_1 & t_2 & \ldots & t_n & M(\Omega, \mu) \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \\ f \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ z_2 a_2 \\ \vdots \\ z_n a_n \\ t_1 a_1 + \cdots + t_n a_n + M(\Omega, \mu) f \end{pmatrix}$$

If now $t_1 \in \text{ran } M(\Omega, \mu)$, we can find f_0 such that $M(\Omega, \mu) f_0 = t_1$ and setting

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \\ f \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ -f_0 \end{pmatrix},$$

we see that $z_1 = 0$ is an eigenvalue of C. Similarly, we cannot have $t_i \in \text{ran }(M(\Omega,\mu) - z_i I)$, $i = 2, 3, \ldots, n$. Hence (iii) $\Rightarrow$ (iv).

To prove that (iv) $\Rightarrow$ (i), first note that given $z_1, z_2, \ldots, z_n$ satisfying the conditions in (iv), Lemma 2.2 says that there is an operator of the form

$$B = \begin{pmatrix} F_d & 0 \\ Q & M(\Omega,\mu) \end{pmatrix}.$$

such that $B \cong_{\mathcal{U}+\mathcal{K}} M(\Omega,\mu)$. We want to change the bottom left entry of this operator to T via a $(\mathcal{U}+\mathcal{K})$ similarity transformation. First of all, we can notice that by acting on B with similarities of the form $R = \begin{pmatrix} R_d & 0 \\ 0 & I \end{pmatrix}$, where R_d is an invertible diagonal matrix, the columns of Q can be scaled by arbitrary non-zero scalars. (Since F_d is diagonal, these similarities do not change F_d.) In what follows, we shall assume that this scaling has been performed so that $t_i(z_i) = q_i(z_i)$, where q_i is the i-th column of Q, $1 \le i \le n$. Notice that this is indeed possible since $t_i(z_i) \ne 0$ by (iv) and $q_i(z_i) \ne 0$ by the implication (i) $\Rightarrow$ (iv), which has already been proved.

Next consider the equation

$$\begin{pmatrix} I & 0 \\ D & I \end{pmatrix} \begin{pmatrix} F_d & 0 \\ Q & M(\Omega,\mu) \end{pmatrix} \begin{pmatrix} I & 0 \\ -D & I \end{pmatrix} = \begin{pmatrix} F_d & 0 \\ T & M(\Omega,\mu) \end{pmatrix}.$$

which is equivalent to $M(\Omega,\mu)D - DF_d = Q - T$. The i-th column of the left hand side is $(M(\Omega,\mu) - z_i I)d_i$, where d_i is the i-th column of D. Since the columns of Q have already been scaled so that $(q_i - t_i) \in \text{ran }(M(\Omega,\mu) - z_i I)$, it is now possible to choose the columns d_i so that the last equation can be solved and in this way the desired columns t_i will be obtained. In other words,

$$\begin{pmatrix} I & 0 \\ D & I \end{pmatrix} B \begin{pmatrix} I & 0 \\ -D & I \end{pmatrix} = C.$$

Finally, notice that both of the similarities that we have used, namely $R = \begin{pmatrix} R_d & 0 \\ 0 & I \end{pmatrix}$ and $\begin{pmatrix} I & 0 \\ D & I \end{pmatrix}$ are of the type $(\mathcal{U}+\mathcal{K})$, so we have shown that (iv) $\Rightarrow$ (i).

□

2.5. Corollary. *If C is an operator of the form*

$$C = \begin{pmatrix} F_d & 0 \\ T & M(\Omega,\mu) \end{pmatrix},$$

where F_d is a diagonal matrix with distinct diagonal entries $\{z_1, z_2, \dots, z_n\}$ lying in Ω then $C \in \mathrm{cl}((\mathcal{U}+\mathcal{K})(M(\Omega,\mu)))$.

Proof. An arbitrarily small perturbation of C will get the i-th column of T out of ran $(M(\Omega,\mu) - z_i I)$. Then by Lemma 2.4 this perturbed operator is in $(\mathcal{U}+\mathcal{K})(M(\Omega,\mu))$. Hence $C \in \mathrm{cl}((\mathcal{U}+\mathcal{K})(M(\Omega,\mu)))$.

□

2.6. Lemma. *Let $z_0 \in \Omega$. Then there exists an orthonormal basis $\{e_0, e_1, \dots\}$ of $H^2(\Omega,\mu)$ such that the matrix of the operator $M(\Omega,\mu)$ with respect to this basis is the Toeplitz matrix*

$$\begin{pmatrix} z_0 & 0 & & & \\ z_1 & z_0 & 0 & & \\ z_2 & z_1 & z_0 & 0 & \\ & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

Proof. Let ψ be the function constructed is Lemma 2.1. As in the proof of Lemma 2.2, we shall define an operator N on $H^2(\Omega,\mu)$ by

$$Nf(z) = \psi(z).f(z).\ f \in H^2(\Omega,\mu),\ z \in \mathrm{cl}(\Omega).$$

Next we choose $e_0 \in (\mathrm{ran}\ N)^\perp$ such that $\|e_0\| = 1$, and we set $e_n = N^n e_0$. Then $\{e_0, e_1, \dots\}$ is an orthonormal system (recall that N is an isometry). Moreover,

$$\{e_0\}^\perp = \mathrm{ran}\ N = \{f \in H^2(\Omega,\mu)) : f(z_0) = 0\},$$

and, by induction

$$\begin{aligned} \{e_0, e_1, \dots e_{n-1}\}^\perp &= \mathrm{ran}\ N^n \\ &= \{f \in H^2(\Omega,\mu)) : f \text{ has a zero of order at least } n \text{ at } z_0\}, \end{aligned}$$

and hence $\bigcap_{i=0}^{\infty}\{e_0, \dots, e_n\}^\perp = \{0\}$, i.e. $\{e_i\}_{i=0}^{\infty}$ is an orthonormal basis of $H^2(\Omega,\mu)$. Next, notice that span$\{e_0\}$ is the same as the subspace H_0 from the proof of Lemma 2.2. Hence the top left entry of the matrix of $M(\Omega,\mu)$ with respect to $\{e_0, e_1, \dots\}$ is equal to z_0. Finally, for $m, n \geq 0$,

$$\begin{aligned} \langle M(\Omega,\mu)e_{m+1}, e_{n+1}\rangle &= \langle M(\Omega,\mu)Ne_m, Ne_n\rangle \\ &= \langle NM(\Omega,\mu)e_m, Ne_n\rangle \end{aligned}$$

$$= \langle N^* N M(\Omega, \mu) e_m, e_n \rangle$$
$$= \langle M(\Omega, \mu) e_m, e_n \rangle$$

and, for $n \geq 1$,

$$\begin{aligned}\langle M(\Omega, \mu) e_n, e_0 \rangle &= \langle M(\Omega, \mu) N e_{n-1}, e_0 \rangle \\ &= \langle N M(\Omega, \mu) e_{n-1}, e_0 \rangle \\ &= 0.\end{aligned}$$

□

2.7. Theorem. *Let Ω be a simply connected analytic Cauchy domain. The closure of the $(\mathcal{U} + \mathcal{K})$ orbit of the operator $M(\Omega, \mu)$ is*

$$\begin{aligned}\mathrm{cl}((\mathcal{U} + \mathcal{K})(M(\Omega, \mu))) \quad &= \quad \{T \in \mathcal{B}(H^2(\Omega, \mu)) : \\ (i) \quad & \quad T \text{ is essentially normal}, \\ (ii) \quad & \quad \sigma(T) = \mathrm{cl}(\Omega), \\ (iii) \quad & \quad \sigma_e(T) = \partial\Omega, \\ (iv) \quad & \quad \mathrm{ind}(T - \lambda) = -1 \text{ for all } \lambda \in \Omega\}.\end{aligned}$$

Note that if we know that $T = M(\Omega, \mu) + K$, where K is compact, only the condition (ii) is not automatic.

Proof. The necessity of these conditions is easily verified (See Example 0.7). We now consider their sufficiency.

By the Brown-Douglas-Fillmore theorem [**BDF73**], if T satisfies the above conditions, then there exists a unitary U and a compact L so that, setting $K = ULU^*$, we have $T = U^* M(\Omega, \mu) U + L = U^*(M(\Omega, \mu) + K)U$. Thus it suffices to show that $M(\Omega, \mu) + K \in \mathrm{cl}((\mathcal{U} + \mathcal{K}) M(\Omega, \mu))$.

Choose an arbitrary $z_0 \in \Omega$ and let $\{e_0, e_1, \dots\}$ be the basis constructed in Lemma 2.6. Let P_n be the orthogonal projection onto $\mathrm{span}\{e_i\}_{i=0}^{n}$, $n = 0, 1, \dots$. Since K is compact, the sequence $\{M(\Omega, \mu) + P_n K P_n\}_{n=1}^{\infty}$ converges to $M(\Omega, \mu) + K$ in the norm. From what we know about the matrix of $M(\Omega,\mu)$, we see that these operators are of the form

$$M(\Omega,\mu)+P_nKP_n \cong \begin{pmatrix} F_n & 0 \\ Q_n & M(\Omega,\mu) \end{pmatrix}.$$

where the matrix on the right hand side is with respect to the decomposition $H^2(\Omega,\mu) = \text{ran } P_n \oplus (\text{ran } P_n)^{\perp}$.

By passing to a subsequence (if necessary), and by using the upper semicontinuity of the spectrum, we can make sure that $\sigma(F_n) \subseteq (\Omega)_{1/n}$. We can next perturb F_n to get a new operator G_n such that

- $\|G_n - F_n\| < \frac{1}{n}$,
- $\sigma(G_n) \subseteq \Omega$, and
- G_n has no multiple eigenvalues.

Clearly the sequence $T_n = \begin{pmatrix} G_n & 0 \\ Q_n & M(\Omega,\mu) \end{pmatrix}$ still converges to $M(\Omega,\mu)+K$.

Now if $G_d(n)$ is the diagonal matrix with the same eigenvalues as G_n then $G_n = R_n^{-1}G_d(n)R_n$ for some invertible matrices R_n (all eigenvalues here are of multiplicity one). Thus, by Corollary 2.5,

$$\begin{pmatrix} G_d(n) & 0 \\ Q_nR_n^{-1} & M(\Omega,\mu) \end{pmatrix} \in \text{cl}(\mathcal{U}+\mathcal{K})(M(\Omega,\mu)),$$

implying that

$$\begin{aligned} T_n &= \begin{pmatrix} G_n & 0 \\ Q_n & M(\Omega,\mu) \end{pmatrix} \\ &= \begin{pmatrix} R_n^{-1} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} G_d(n) & 0 \\ Q_nR_n^{-1} & M(\Omega,\mu) \end{pmatrix} \begin{pmatrix} R_n & 0 \\ 0 & I \end{pmatrix} \\ &\in \text{cl}((\mathcal{U}+\mathcal{K})(M(\Omega,\mu))) \end{aligned}$$

Since $T_n \in \text{cl}(\mathcal{U}+\mathcal{K})(M(\Omega,\mu))$ for all $n \geq 1$, $T = M(\Omega,\mu)+K = \lim T_n \in \text{cl}(\mathcal{U}+\mathcal{K})(M(\Omega,\mu))$.

□

CHAPTER 3

Functional Calculus

Suppose that A and B are essentially normal operators and ϕ is a function holomorphic on (a neighbourhood of) $\sigma(A)$. In this chapter, we shall see that under certain conditions $A \in \mathrm{cl}((\mathcal{U}+\mathcal{K})(B))$ implies that $\phi(A) \in \mathrm{cl}((\mathcal{U}+\mathcal{K})(\phi(B)))$. This will prove to be helpful in describing the closures of $(\mathcal{U}+\mathcal{K})$ orbits of a certain class of operators.

It is not without interest to note that that if A is one of the models we investigated in Chapter 2, then it can be written as $\phi(M(\mathbb{D},\mu))$ for a certain μ.

3.1. Proposition. *Let Ω be a simply connected analytic Cauchy domain. Let ϕ be an invertible holomorphic map from a neighbourhood of $\mathbb{D}$ to $\mathbb{C}$ such that $\phi|_{\mathbb{D}}$ is a conformal map of $\mathbb{D}$ onto Ω. (This is the map which was used in Chapter 1 to construct $H^2(\Omega)$.) Then for any measure μ on $\partial\Omega$, equivalent to the arc length measure, the model $M(\Omega,\mu)$ is unitarily equivalent to some $\phi(M(\mathbb{D},\nu))$, where ν is a suitable measure on $\mathbb{T}$, equivalent to the arc length measure. Conversely, for any measure ν on $\mathbb{T}$, equivalent to the arc length measure, there exists a measure μ on $\partial\Omega$, equivalent to the arc length measure, such that $M(\Omega,\mu)$ is unitarily equivalent to $\phi(M(\mathbb{D},\nu))$.*

Proof. Suppose first that ν is a measure equivalent to the arc length measure on $\mathbb{T}$. Notice that since ϕ is holomorphic on a neighbourhood of $\mathbb{D}$, it can be uniformly approximated by polynomials on $\sigma(M(\mathbb{D},\nu)) = \mathbb{D}$ (it is in fact the sum of a power series). It follows easily that

$$[(\phi M(\mathbb{D},\nu))(f)](z) = \phi(z)f(z), f \in H^2(\mathbb{D},\nu), z \in \mathbb{D}.$$

(This equation is trivial when ϕ is replaced by a polynomial, what we need follows then by passing to the limit.)

Next, we define a map U from $H^2(\Omega,\mu)$ to $H^2(\mathbb{D},\nu)$ by $Uf = f\circ\phi$. (μ is for the moment an arbitrary measure on $\partial\Omega$, equivalent to the arc length measure). The map U is invertible and bounded; it remains to be shown that U is an isometry. For $f \in H^2(\Omega,\mu)$, we have

$$\begin{aligned}||Uf||^2_{H^2(\mathbb{D},\nu)} &= \int_{\mathbb{T}} |f\circ\phi|^2 d\nu \\ &= \int_{\mathbb{T}} |f\circ\phi|^2 n \, d\lambda_{\mathbb{T}},\end{aligned}$$

where $\lambda_{\mathbb{T}}$ is the arc length measure on $\mathbb{T}$ and n is the density of ν with respect to $\lambda_{\mathbb{T}}$, and

$$\begin{aligned} ||f||^2_{H^2(\Omega,\mu)} &= \int_{\partial\Omega} |f|^2 d\mu \\ &= \int_{\partial\Omega} |f|^2 m \, d\lambda_{\partial\Omega} \\ &= \int_{\mathbb{T}} |f \circ \phi|^2 m \circ \phi . |\phi|' \, d\lambda_{\mathbb{T}}, \end{aligned}$$

where $\lambda_{\partial\Omega}$ is the arc length measure on $\partial\Omega$ and m is the density of μ with respect to $\lambda_{\partial\Omega}$. Hence if we choose μ so that $m \circ \phi.|\phi|' = n$, U will be unitary.

Similarly, if μ is given, a suitable ν can be chosen, so the Lemma is proved in both directions.

□

3.2. Lemma. *Suppose $A \in \mathrm{cl}(\mathcal{U} + \mathcal{K})(B)$ and let ϕ be a holomorphic function on (a neighbourhood of) $\sigma(A)$. Then $\phi(A) \in \mathrm{cl}(\mathcal{U} + \mathcal{K})(\phi(B))$.*

Proof. Suppose $R_n^{-1}BR_n \to A$, $R_n \in (\mathcal{U} + \mathcal{K})$. Then $p(R_n^{-1}BR_n) = R_n^{-1}p(B)R_n$ whenever p is a rational function with poles outside $\sigma(A)$ and hence, by passing to the limit, we see that $\phi(R_n^{-1}BR_n) = R_n^{-1}\phi(B)R_n$. We are using Runge's theorem here to approximate ϕ uniformly on $\sigma(A)$, see **[Rud87]**, Theorem 13.6.

Consequently, $R_n^{-1}\phi(B)R_n = \phi(R_n^{-1}BR_n) \to \phi(A)$ by the continuity of the functional calculus. This shows that $\phi(A) \in \mathrm{cl}(\mathcal{U} + \mathcal{K})(\phi(B))$.

□

3.3. Lemma. *Let $A \sim B$, A, B essentially normal. Let ϕ be a holomorphic function on (a neighbourhood of) $\sigma(A) \cup \sigma(B)$. Then $\phi(A)$ and $\phi(B)$ are essentially normal and $\phi(A) \sim \phi(B)$.*

Proof. Suppose that the sequence p_n of rational functions with poles outside $\sigma(A) \cup \sigma(B)$ converges uniformly to ϕ on $\sigma(A) \cup \sigma(B)$. Since A and B are essentially normal, so are $p_n(A)$ and $p_n(B)$. Therefore $\phi(A)$ and $\phi(B)$ are essentially normal.

Now $\pi(A)$ and $\pi(B)$ are unitarily equivalent in the Calkin algebra and hence so are $p_n(\pi(A))$ and $p_n(\pi(B))$,

$$p_n(\pi(A)) = u^* p_n(\pi(B))u,$$

for some unitary element u of the Calkin algebra. We have

$$\begin{aligned}\phi(\pi(A)) &= \lim_{n\to\infty} p_n(\pi(A)) \\ &= \lim_{n\to\infty} u^* p_n(\pi(B))u \\ &= \lim_{n\to\infty} u^* \phi(\pi(B))u\end{aligned}$$

Since the unitary orbit of a normal element of the Calkin algebra is closed (this follows from the Brown-Douglas-Fillmore Theorem [**BDF73**]), this means that $\phi(\pi(A)) \cong \phi(\pi(B))$ and hence

$$\pi(\phi(A)) = \phi(\pi(A)) \cong \phi(\pi(B)) = \pi(\phi(B)).$$

i.e. $\phi(A)$ and $\phi(B)$ are compalent.

□

3.4. Lemma. *Suppose that z_0 is an eigenvalue of A and ϕ is holomorphic on (a neighbourhood of) $\sigma(A)$. Then $\phi(z_0)$ is an eigenvalue of $\phi(A)$.*

Proof. Let $f(z) = \phi(z) - \phi(z_0)$. Then $f(z_0) = 0$ and hence $f(z) = (z - z_0)g(z)$ for some function z, holomorphic on $\sigma(A)$. Hence $\phi(A) - \phi(z_0)I = f(A) = g(A)(A - \lambda I)$, from which the result clearly follows.

□

By now we understand enough about the interaction of the functional calculus with $(\mathcal{U}+\mathcal{K})$-orbits and spectral properties to supply an alternative proof of Theorem 2.7. With the above lemmas at hand, Theorem 2.7 becomes a corollary of Theorem 4.7 in [**GM93a**], which describes the closure of the $(\mathcal{U} + \mathcal{K})$-orbit of the unilateral shift.

In fact, we can do more. The following result is deduced from the description of the closure of the $(\mathcal{U} + \mathcal{K})$-orbit of shift-like operators from [**Mar92**]. (Recall that shift-like operators are operators with the same spectral properties as the unilateral shift.) A similar result appears in [**JJW96**].

3.5. Theorem. *Suppose Ω is an simply connected analytic Cauchy Domain and A is an essentially normal operator on a separable Hilbert space H with the following spectral properties:*

(a) $\sigma(A) = \mathrm{cl}(\Omega)$;
(b) $\sigma_e(A) = \partial\Omega$;
(c) $\mathrm{ind}(A - z) = -1$, $z \in \Omega$;
(d) nul $(A - z) = 0$, $z \in \Omega$;

Then the closure of the $(\mathcal{U}+\mathcal{K})$ orbit of the operator A is

$$\begin{array}{rcl} \mathrm{cl}((\mathcal{U}+\mathcal{K})(A)) & = & \{T \in \mathcal{B}(H) : \\ & (i) & T \textit{ is essentially normal,} \\ & (ii) & \sigma(T) = \mathrm{cl}(\Omega), \\ & (iii) & \sigma_e(T) = \partial\Omega, \\ & (iv) & \mathrm{ind}(T-\lambda) = -1 \textit{ for all } \lambda \in \Omega\}. \end{array}$$

Proof. Let ϕ be an invertible holomorphic map from a neighbourhood of $\mathbb{D}$ to $\mathbb{C}$ such that $\phi|_{\mathbb{D}}$ is a conformal map of $\mathbb{D}$ onto Ω. Let A, T be as in the Theorem. First we will use [**Mar92**], Theorem 2.5, to show that $\phi^{-1}(T) \in \mathrm{cl}((\mathcal{U}+\mathcal{K})(\phi^{-1}(A)))$. Applying Lemma 3.3 to operators A, T and to the map ϕ^{-1}, we see that $\phi^{-1}(A), \phi^{-1}(T)$ are essentially normal the index properties of $\phi^{-1}(A), \phi^{-1}(T)$ are as needed. Similarly, Lemma 3.4 shows that nul $(\phi^{-1}(A) - z) = 0$ for $|z| < 1$. Hence by [**Mar92**], Theorem 2.5, $\phi^{-1}(T) \in \mathrm{cl}((\mathcal{U}+\mathcal{K})(\phi^{-1}(A)))$ and so by 3.2 $T \in \mathrm{cl}((\mathcal{U}+\mathcal{K})(A))$.

□

CHAPTER 4

Increasing the value of the index

In this chapter. we shall be mostly concerned with the following situation: Ω is a simply connected analytic Cauchy domain. μ is a measure on $\partial\Omega$. equivalent to the arc length measure. The essentially normal model operator $M = \bigoplus_{k=1}^{m} M(\Omega, \mu)$ has these spectral properties:

(i) $\sigma(M(\Omega,\mu)) = \mathrm{cl}(\Omega)$;
(ii) $\sigma_e(M(\Omega,\mu)) = \partial\Omega$;
(iii) $\mathrm{ind}(M(\Omega,\mu) - z) = -m$. $z \in \Omega$;
(iv) $\min\mathrm{ind}(M(\Omega,\mu) - z) = 0$, $z \in \Omega$;

We want to find a description of the closure of the $(\mathcal{U} + \mathcal{K})$-orbit of this operator. One of the techniques we will need can be introduced more easily by first concentrating on a different model. whose investigation will then be continued in Chapter 8. The lemma that we want to prove in this chapter is:

4.1. Lemma. *Assume that $\Omega = \Omega_1 \setminus \mathrm{cl}(\Omega_2)$. where Ω_1 is a simply connected analytic Cauchy domain. Ω_2 is an analytic Cauchy domain consisting of n simply connected components. $\Omega_2 = \cup_1^n \Omega_{2,i}$, $\mathrm{cl}(\Omega_2) \subseteq \Omega_1$. Let now μ be a measure on $\partial(\Omega_1)$ and for $i = 1, 2, \ldots, n$. let μ_i be a measure on $\partial(\Omega_{2,i}^*)$; all of these measures are assumed to be equivalent to the respective arc length measures. Let $A = M(\Omega_1, \mu)$ and let $B = \oplus_{i=1}^n M(\Omega_{2,i}^*, \mu_i)$. Let 1_{Ω_1} be the constant function equal to 1 on Ω_1 and let $1_{\Omega_2^*}$ be the constant function equal to 1 on Ω_2^*. Let C be an operator from $\bigoplus_{i=1}^n H^2(\Omega_{2,i}^*, \mu_i)$ into $H^2(\Omega_1, \mu)$ defined by $C = 1_{\Omega_1} \otimes 1_{\Omega_2^*}^*$. Define an operator M_1 on $H^2(\Omega_1,\mu) \oplus (\bigoplus_{i=1}^n H^2(\Omega_{2,i}^*, \mu_i))$ by*

$$M_1 = \begin{pmatrix} A & C \\ 0 & B^* \end{pmatrix}.$$

Next, let $D = 1_{\Omega_1} \otimes d^*$, where $d \in C(\mathrm{cl}(\Omega_2^*)) \cap H^2(\Omega_2^*)$ with $d(z) \neq 0$ for $z \in \mathrm{cl}(\Omega_2^*)$ and set

$$X = \begin{pmatrix} A & D \\ 0 & B^* \end{pmatrix}.$$

Then we have

$$X \in \mathrm{cl}(\mathcal{U}+\mathcal{K})(M_1).$$

Recall that the spectral properties of both M_1 and X were investigated in Lemma 1.3

While the similarity of X and M_1 will be verified very easily. showing that

$$X \in \mathrm{cl}(\mathcal{U}+\mathcal{K})(M_1)$$

will require a somewhat lengthy argument. We shall first develop a block tri-diagonal decomposition technique:

4.2. Lemma.

(a) *Suppose that $L, M, R_0, R_1, \ldots, R_n$ are operators on a Hilbert space $\mathcal{H}$ with the following properties:*
  (i) $LR_k - R_kM = 0,\ k = 0, 1, \ldots, n.$
  (ii) *$\mathcal{H}$ can be decomposed as $\mathcal{H} = \bigoplus_{i=0}^{\infty} \mathcal{H}_i$. where each subspace $\mathcal{H}_i$ is finite-dimensional and the operators $L, M, R_0, R_1, \ldots, R_n$ have a block tri-diagonal form with respect to this decomposition. i.e. $R_{ij}^{(k)} = L_{ij} = M_{ij} = 0$. if $|i-j| > 1$. $i \geq 0$. $j \geq 0$. where $R_{ij}^{(k)}$ is the (i, j)-entry of the operator matrix of R_k with respect to the above decomposition and L_{ij}, M_{ij} are the (i, j)-entries of L, M. respectively.*
  (iii) $||R_k - R_{k+1}|| < \epsilon$ *for* $k = 0, 1, \ldots n-1.$

Construct an operator R whose matrix is $\{R_{ij}\}_{i,j=0}^{\infty}$, where

$$\begin{aligned} R_{i,j} &= R_{i,j}^{(n-i-j)}, n-i-j \geq 0, \\ &= R_{i,j}^{(0)}, n-i-j < 0. \end{aligned}$$

Then $||LR - RM|| \leq 15\epsilon(||M|| + ||L||)$

(b) *Suppose that $R_0, R_1, \ldots, R_n, S_0, S_1, \ldots, S_n$ are operators on a Hilbert space $\mathcal{H}$ with the following properties:*
  (i) $R_0 = S_0 = I,\ S_k = R_k^{-1},\ k = 1, 2, \ldots, n.$
  (ii) *$\mathcal{H}$ can be decomposed as $\mathcal{H} = \bigoplus_{i=0}^{\infty} \mathcal{H}_i$, where each subspace $\mathcal{H}_i$ is finite-dimensional and the operators S_k and R_k, $k = 1, 2, \ldots, n$ have a block tri-diagonal form with respect to this decomposition.*
  (iii) $||R_k - R_{k+1}|| < \epsilon$ *and* $||S_k - S_{k+1}|| < \epsilon$ *for* $k = 0, 1, \ldots n-1$

Suppose that R is constructed as above and construct an operator S whose matrix is $\{S_{ij}\}_{i,j=0}^{\infty}$, where

$$\begin{aligned} S_{i,j} &= S_{i,j}^{(n-i-j)}, n-i-j \geq 0, \\ &= S_{i,j}^{(0)}, n-i-j < 0. \end{aligned}$$

Let $m = \max(||R_0||, ||R_1||, \dots, ||R_n||, ||S_0||, ||S_1||, \dots, ||S_n||)$. Then $||RS - I|| \leq 30\epsilon m$ and $||SR - I|| \leq 30\epsilon m$.

Note that the operators R and S constructed here also have block tri-diagonal matrices with respect to the decomposition $\mathcal{H} = \bigoplus_{i=0}^{\infty} \mathcal{H}_i$. This is what the matrix of R looks like when n is even:

$$\begin{pmatrix} R_{0,0}^{(n)} & R_{0,1}^{(n-1)} & & & & & & & \\ R_{1,0}^{(n-1)} & R_{1,1}^{(n-2)} & R_{1,2}^{(n-3)} & & & & & & \\ & R_{2,1}^{(n-3)} & R_{2,2}^{(n-4)} & R_{2,3}^{(n-5)} & & & & & \\ & & \ddots & \ddots & \ddots & & & & \\ & & & R_{\frac{n}{2}-1,\frac{n}{2}-2}^{(3)} & R_{\frac{n}{2}-1,\frac{n}{2}-1}^{(2)} & R_{\frac{n}{2}-1,\frac{n}{2}}^{(1)} & & & \\ & & & & R_{\frac{n}{2},\frac{n}{2}-1}^{(1)} & R_{\frac{n}{2},\frac{n}{2}}^{(0)} & R_{\frac{n}{2},\frac{n}{2}+1}^{(0)} & & \\ & & & & & R_{\frac{n}{2}+1,\frac{n}{2}}^{(0)} & R_{\frac{n}{2}+1,\frac{n}{2}+1}^{(0)} & R_{\frac{n}{2}+1,\frac{n}{2}+2}^{(0)} & \\ & & & & & & \ddots & \ddots & \ddots \end{pmatrix}$$

For an odd n, the picture is similar. The entries which are left blank equal zero.

Proof of lemma 4.2

(a) Denote $P = LR - RM$. We will investigate the entries P_{ij} of the matrix of P with respect to the decomposition $\mathcal{H} = \bigoplus_{i=0}^{\infty} \mathcal{H}_i$. As the the matrices of L, M and R are block tri-diagonal, we see that $P_{ij} = 0$ for $|i - j| > 2$.

To simplify notation, we shall set

$$\begin{aligned} R_k &= R_0 \text{ for } k < 0 \\ R_{ij}^{(k)} &= R_{ij}^{(0)} \text{ for } k < 0, i, j \geq 0. \end{aligned}$$

With this convention, for any $n \geq 0$, we have $R_{ij} = R_{ij}^{(n-i-j)}, i, j \geq 0$ and we still have $LR_k - R_k M = 0$ for $k \leq n$.

For $|i-j| \leq 2$, we have

$$\begin{aligned}
||P_{ij}|| &= ||(LR - RM)_{ij}|| \\
&= ||(LR - RM)_{ij} - (LR_{n-j-i} - R_{n-j-i}M)_{ij}|| \\
&= ||\sum_{l=0}^{\infty} L_{il}R_{lj} - L_{il}R_{lj}^{(n-i-j)} - R_{il}M_{lj} + R_{il}^{(n-i-j)}M_{lj}|| \\
&= ||\sum_{l=\max(0,i-1,j-1)}^{\min(i+1,j+1)} L_{il}R_{lj}^{(n-l-j)} - L_{il}R_{lj}^{(n-i-j)} \\
&\quad -R_{il}^{(n-l-i)}M_{lj} + R_{il}^{(n-i-j)}M_{lj}||
\end{aligned}$$

The restriction of the summation range is possible because of the block tri-diagonal form of L, M and R. Note that, as $|l-i| \leq 1$, $|l-j| \leq 1$, we have

$$\begin{aligned}
|(n-i-j)-(n-l-j)| &\leq 1 \\
|(n-i-j)-(n-l-i)| &\leq 1.
\end{aligned}$$

Hence

$$||P_{ij}|| \leq 3(||L||.\epsilon + ||M||.\epsilon) = 3.\epsilon(||M|| + ||L||).$$

Now

$$P = \sum_{r=-2}^{2} P_r,$$

where

$$\begin{aligned}
(P_r)_{ij} &= P_{ij} \text{ if } j = i + r \\
&= 0 \text{ otherwise.}
\end{aligned}$$

From the above estimate, we see that $||P_r|| \leq 3.\epsilon(||M|| + ||L||)$ and hence

$$||P|| \leq 15\epsilon(||M|| + ||L||).$$

(b) the proof is similar to that of (a). Since the whole situation is symmetric, it suffices to show that $||RS - I|| \leq 30\epsilon m$. Denote $Q = RS - I$. We will investigate the entries

Q_{ij} of the matrix of Q with respect to the decomposition $\mathcal{H} = \bigoplus_{i=0}^{\infty} \mathcal{H}_i$. We can again see that $Q_{ij} = 0$ for $|i-j| > 2$.

To simplify notation, we shall set

$$\begin{aligned} R_k &= R_0 \text{ for } k < 0 \\ R_{ij}^{(k)} &= R_{ij}^{(0)} \text{ for } k < 0, i, j \geq 0 \\ S_k &= S_0 \text{ for } k < 0 \\ S_{ij}^{(k)} &= S_{ij}^{(0)} \text{ for } k < 0, i, j \geq 0. \end{aligned}$$

With this convention, we have $R_{ij} = R_{ij}^{(n-i-j)}$ and $S_{ij} = S_{ij}^{(n-i-j)}$, $i, j \geq 0$ and we have $R_k S_k - I = 0$ for $k \leq n$.

For $|i-j| \leq 2$, we have

$$\begin{aligned} ||Q_{ij}|| &= ||(RS-I)_{ij}|| \\ &= ||(RS-I)_{ij} - (R_{n-j-i}S_{n-j-i} - I)_{ij}|| \\ &= ||\sum_{l=0}^{\infty} R_{il}S_{lj} - R_{il}^{(n-i-j)}S_{lj}^{(n-i-j)}|| \\ &= ||\sum_{l=\max(0,i-1,j-1)}^{\min(i+1,j+1)} R_{il}^{(n-l-i)}S_{lj}^{(n-l-j)} - R_{il}^{(n-i-j)}S_{lj}^{(n-i-j)}|| \\ &= ||\sum_{l=\max(0,i-1,j-1)}^{\min(i+1,j+1)} R_{il}^{(n-l-i)}(S_{lj}^{(n-l-j)} - S_{lj}^{(n-i-j)}) \\ &\quad + (R_{il}^{(n-l-i)} - R_{il}^{(n-i-j)})S_{lj}^{(n-i-j)}|| \\ &\leq 3(m\epsilon + \epsilon m) = 6\epsilon m \end{aligned}$$

Using the fact that $Q_{ij} = 0$ for $|i-j| > 2$, we can estimate

$$||Q|| \leq 30\epsilon m.$$

□

4.3. Corollary.

(a) *Suppose that $L, M, R_0, \ldots, R_n$ are operators satisfying (i), (iii) in Lemma 4.2 (a). Suppose that K is a compact operator and let $Q = R_n K$. Then there exists an operator R such that*
  (i) *$R - R_0$ has finite rank,*
  (ii) $||RK - Q|| < \epsilon(||K|| + 6\max\{||R_0||, ||R_1||, \ldots, ||R_n||\})$,
  (iii) $||LR - RM|| \leq 15\epsilon(||M|| + ||N||)$.

(b) *Suppose that we have in addition operators $S_0, S_1, \ldots, S_n$ satisfying (i), (iii) in Lemma 4.2 (b). Then we can also construct an operator S such that, in addition to the properties (i), (ii) and (iii) in part (a) of this lemma, we have*

$$\begin{aligned} ||RS - I|| &\leq 30\epsilon\max(||R_0||, ||R_1||, \ldots, ||R_n||, ||S_0||, ||S_1||, \ldots, ||S_n||) \\ ||SR - I|| &\leq 30\epsilon\max(||R_0||, ||R_1||, \ldots, ||R_n||, ||S_0||, ||S_1||, \ldots, ||S_n||). \end{aligned}$$

Proof.

(a) Since K is compact, we can fix a finite-dimensional $\mathcal{H}_0 \subset \mathcal{H}$ such that $||P_{\mathcal{H}_0}K - K|| < \epsilon$. Next, fix a basis $e_1, e_2, \ldots$ of $\mathcal{H}$. Set

$$\begin{aligned} \mathcal{K}_1 &= \text{span}\{\mathcal{H}_0, L\mathcal{H}_0, L^*\mathcal{H}_0, M\mathcal{H}_0, M^*\mathcal{H}_0, R_0\mathcal{H}_0, R_0^*\mathcal{H}_0, \ldots, e_1\}, \\ \mathcal{H}_1 &= \mathcal{K}_1 \ominus \mathcal{H}_0, \end{aligned}$$

and continue in this manner: With $\mathcal{K}_1, \mathcal{K}_2, \ldots, \mathcal{K}_i, \mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_i$ constructed, we set

$$\begin{aligned} \mathcal{K}_{i+1} &= \text{span}\{\mathcal{K}_i, L\mathcal{K}_i, L^*\mathcal{K}_i, M\mathcal{K}_i, M^*\mathcal{K}_i, R_0\mathcal{K}_i, R_0^*\mathcal{K}_i, \ldots, e_{i+1}\}, \\ \mathcal{H}_{i+1} &= \mathcal{K}_{i+1} \ominus \mathcal{H}_i, \end{aligned}$$

Then the conditions (i),(ii),(iii) of Lemma 4.2 (a) are satisfied. We can therefore construct R such that $||LR - RM|| \leq 15\epsilon(||M|| + ||L||)$, $R - R_0$ is finite dimensional and

$$\begin{aligned} ||RK - Q|| &= ||(R - R_n)K|| \\ &\leq ||(R - R_n)P_{\mathcal{H}_0}K|| + ||(R - R_n)(P_{\mathcal{H}_0}K - K)|| \\ &\leq \epsilon.||K|| + 6\max\{||R_0||, ||R_1||, \ldots, ||R_n||\}.\epsilon. \end{aligned}$$

To see why the last inequality holds, consider the (i, j) entry of the operator matrix of $(R - R_n)$:

$$||(R-R_n)_{ij}|| = ||R_{ij}^{(n-i-j)} - R_{ij}^{(n)}|| \leq 2.\max\{||R_0||, ||R_1||, \dots, ||R_n||\}.$$

and since $(R - R_n)$ is block tri-diagonal. we have

$$||R - R_n|| \leq 6.\max\{||R_0||, ||R_1||, \dots, ||R_n||\}.$$

(b) It suffices to alter the construction of the spaces $\mathcal{H}_k$ so that the resulting decomposition $\mathcal{H} = \bigoplus_{i=0}^{\infty} \mathcal{H}_i$ makes the matrices of $S_0, S_1, \dots, S_n$ also tridiagonal. Then we can finish the proof by applying Lemma 4.2 (b)

□

Proof of lemma 4.1 One can easily see that M_1 and X are similar. Indeed. since $d \in \mathcal{C}(\mathrm{cl}(\Omega_2^*))$ with $d(z) \neq 0$ for $z \in \mathrm{cl}(\Omega)$, the operator H on $H^2(\Omega_2^*, \mu)$ defined by

$$(Hf)(z) = d(z)f(z), z \in \Omega_2^*$$

is invertible. We also have $HB = BH$ and hence $H^*B^* = B^*H^*$. The operator

$$\begin{pmatrix} I & 0 \\ 0 & (H^{-1})^* \end{pmatrix} \begin{pmatrix} A & C \\ 0 & B^* \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & H^* \end{pmatrix} = \begin{pmatrix} A & CH^* \\ 0 & (H^{-1})^*B^*H^* \end{pmatrix} = \begin{pmatrix} A & CH^* \\ 0 & B^* \end{pmatrix}$$

is then similar to M_1. But for all $g \in H^2(\Omega_2^*, \mu)$

$$CH^*g = \langle H^*g, 1_{\Omega_2^*}\rangle 1_{\Omega_1} = \langle g, H1_{\Omega_2^*}\rangle 1_{\Omega_1} = \langle g, d\rangle 1_{\Omega_1} = Dg,$$

i.e. $CH^* = D$ and X is similar to M_1.

Now we will show that $X \in \mathrm{cl}(\mathcal{U} + \mathcal{K})(M_1)$. Let $m = \max(||d||_{\sup}, ||d^{-1}||_{\sup})$. Fix $\epsilon > 0$ so that $30\epsilon m < \frac{1}{2}$. Let γ be a holomorphic logarithm of d on Ω_2^* (see **[Rud87]**, Theorem 13.11), i.e. $e^{\gamma} = d$. Fix a large n so that $m.||e^{\gamma/n} - 1||_{\sup} < \epsilon$. Then we have

$$||e^{k.\gamma/n} - e^{(k-1).\gamma/n}||_{\sup} < \epsilon, k = -n, -n+1, \dots, n.$$

Let R_k be the multiplication by $e^{k.\gamma/n}$ on $H^2(\Omega_2^*)$, $k = 0, 1, \dots, n$ and let S_k be the multiplication by $e^{-k.\gamma/n}$.

We have $R_n = H$, $||R_k - R_{k+1}|| < \epsilon$, $S_n = H^{-1}$, $||S_k - S_{k+1}|| < \epsilon$ for $k = 0, 1, \dots n-1$ and $R_k H = H R_k$, $S_k = R_k^{-1}$, $k = 0, 1, \dots n$. If we let both L and M equal B, we see that the conditions (i), (iii) in Lemma 4.2 (a) as well as the conditions (i), (iii) in Lemma 4.2 (b) are satisfied. Moreover, we have $R_n C^* = HC^* = (CH^*)^* = D^*$. We can now apply Corollary 4.3 to see that there exist operators R, S such that

(i) $R - R_0 = R - I$ is finite-dimensional, and
(ii) we have

$$||CR^* - D|| = ||RC^* - D^*|| \;<\; \epsilon(||C|| + 6\max\{||R_0||, ||R_1||, \dots, ||R_n||\}) \;\leq\; \epsilon(||C|| + 6m),$$

(iii) $||RB - BR|| \leq 30\epsilon||H|| \leq 30\epsilon m$,
(iv) $||RS - I|| \leq 30\epsilon m < \frac{1}{2}$ and $||SR - I|| \leq 30\epsilon m < \frac{1}{2}$.

Note that (i) says that R is of the form unitary plus compact. The condition (iv) implies that SR and RS are both invertible and $||(SR)^{-1}|| < 2$, $||(RS)^{-1}|| < 2$. Therefore R is invertible and $||R^{-1}|| \leq 2||S|| \leq 6m$, and we have

$$\begin{aligned} ||(R^*)^{-1}B^*R^* - B^*|| &= ||RBR^{-1} - B|| \\ &= ||(RB - BR)R^{-1}|| \\ &\leq 180\epsilon m^2 \end{aligned}$$

Summing up, we get

$$\begin{aligned} & ||\begin{pmatrix} A & D \\ 0 & B^* \end{pmatrix} - \begin{pmatrix} I & 0 \\ 0 & (R^{-1})^* \end{pmatrix}\begin{pmatrix} A & C \\ 0 & B^* \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & R^* \end{pmatrix}|| \\ = \; & ||\begin{pmatrix} A - A & D - CR^* \\ 0 & B^* - (R^*)^{-1}B^*R^* \end{pmatrix}|| \\ \leq \; & \epsilon(||C|| + 6m) + 180\epsilon m^2 \end{aligned}$$

Since now $\begin{pmatrix} I & 0 \\ 0 & R^* \end{pmatrix} \in (\mathcal{U} + \mathcal{K})(H^2(\Omega_1, \mu) \oplus H^2(\Omega_2^*, \mu))$ and since ϵ can be chosen to be arbitrarily small, we see that

$$\begin{pmatrix} A & D \\ 0 & B^* \end{pmatrix} \in \mathrm{cl}(\mathcal{U}+\mathcal{K})(\begin{pmatrix} A & C \\ 0 & B^* \end{pmatrix})$$

□

We shall now return to the investigation of the closure of the $(\mathcal{U}+\mathcal{K})$ orbit of the operator $M = \oplus_{k=1}^{m} M(\Omega,\mu)$. As in Chapter 2, we want to know first under what conditions is an operator of the form

$$\begin{pmatrix} F & 0 \\ T & M \end{pmatrix},$$

where F is a finite-dimensional matrix, is in the (closure of the) $(\mathcal{U}+\mathcal{K})$-orbit of M.

4.4. Lemma. *Let C be an operator of the form*

$$\begin{pmatrix} F_d & & & \\ T_1 & M(\Omega,\mu) & & \\ \vdots & & \ddots & \\ T_m & & & M(\Omega,\mu) \end{pmatrix}.$$

Let t_{ij} denote the i-th column of T_j and suppose that F_d is a diagonal operator with distinct entries $\{z_1, z_2, \ldots, z_n\}$ in Ω. Suppose that

(i) *For each i, the i-th column $t_{i,1}$ of T_1* **is not** *in*

$$\mathrm{ran}\ (M(\Omega,\mu) - z_i I) = \{f \in H^2(\Omega,\mu) : f(z_i) = 0\}.$$

(ii) *For $j = 2, 3, \ldots, m$ and for each i, the i-th column $t_{i,j}$ of T_j* **is** *in*

$$\mathrm{ran}\ (M(\Omega,\mu) - z_i I) = \{f \in H^2(\Omega,\mu) : f(z_i) = 0\}.$$

Then $C \cong_{\mathcal{U}+\mathcal{K}} \bigoplus_{j=1}^{m} M(\Omega,\mu)$.

Proof. Using Lemma 2.4, we see that

$$C_0 = \begin{pmatrix} F_d & & & & \\ T_1 & M(\Omega,\mu) & & & \\ 0 & & M(\Omega,\mu) & & \\ \vdots & & & \ddots & \\ 0 & & & & M(\Omega,\mu) \end{pmatrix} \cong_{\mathcal{U}+\mathcal{K}} \bigoplus_{j=1}^{m} M(\Omega,\mu).$$

Suppose $D_j \in \mathcal{B}(\mathcal{H}_0, \mathcal{H}_j)$, $j = 2, 3, \ldots, m$, where $\mathcal{H}_0$ is the underlying space of F_d and H_j is the underlying space of the j-th copy of $M(\Omega, \mu)$. Consider the operator

$$C_1 = \begin{pmatrix} I & & & & \\ 0 & I & & & \\ D_2 & & I & & \\ \vdots & & & \ddots & \\ D_m & & & & I \end{pmatrix} C_0 \begin{pmatrix} I & & & & \\ 0 & I & & & \\ -D_2 & & I & & \\ \vdots & & & \ddots & \\ -D_m & & & & I \end{pmatrix}$$

$$= \begin{pmatrix} F_d & & & & \\ T_1 & M(\Omega, \mu) & & & \\ M(\Omega, \mu)D_2 - D_2F_d & & M(\Omega, \mu) & & \\ \vdots & & & \ddots & \\ M(\Omega, \mu)D_m - D_mF_d & & & & M(\Omega, \mu) \end{pmatrix}$$

Then $C_1 \cong_{\mathcal{U}+\mathcal{K}} C_0 \cong_{\mathcal{U}+\mathcal{K}} \bigoplus_{j=1}^m M(\Omega, \mu)$. We shall be able to choose the operators $D_2, D_3, \ldots, D_m$ so that $C_1 = C$. Indeed, the i-th column of of $M(\Omega, \mu)D_j - D_jF$ is

$$(M(\Omega, \mu) - z_iI)d_{ij},$$

where d_{ij} is the i-th column of D_j, $j = 2, 3, \ldots, m$. Since we have $t_{ij} \in \text{ran } (M(\Omega, \mu) - z_iI), i = 1, 2, \ldots, n, j = 2, 3, \ldots, m$, the columns of D_j can be chosen so that $T_j = M(\Omega, \mu)D_j - D_jF_d$ for $j = 2, 3, \ldots, m$ and so $C = C_1 \cong_{\mathcal{U}+\mathcal{K}} \oplus_{j=1}^m M(\Omega, \mu)$.

□

4.5. Lemma. *Let E be an operator of the form*

$$E = \begin{pmatrix} F_d & & & \\ S_1 & M(\Omega, \mu) & & \\ \vdots & & \ddots & \\ S_m & & & M(\Omega, \mu) \end{pmatrix},$$

where F_d is a diagonal operator with distinct entries $\{z_1, z_2, \ldots, z_n\}$ in Ω. Then $E \in \text{cl}(\mathcal{U} + \mathcal{K})[\bigoplus_{j=1}^m M(\Omega, \mu)]$.

Proof. Choose $\eta > 0$. Let T_1 be such that $||S_1 - T_1|| < \eta$ and the $i-th$ column of T_1 is not in ran $(M(\Omega, \mu) - z_iI)$. Let now j be an integer between 2 and m. Choose a function

$r_j \in H^2(\Omega, \mu)$ such that

$$r_j(z_i) = \frac{s_{ij}(z_i)}{t_{i,1}(z_i)}, i = 1, 2, \ldots, n.$$

where s_{ij} is the i-th column of S_j and set

$$t_{ij} = s_{ij} - r_j t_{i,1}, i = 1, 2, \ldots, n.$$

Let T_j be a matrix whose i-th column is t_{ij}. Notice that $t_{ij}(z_i) = 0$, i.e. $t_{ij} \in \operatorname{ran}\,(M(\Omega, \mu) - z_i I)$, and hence if

$$C = \begin{pmatrix} F_d & & & \\ T_1 & M(\Omega, \mu) & & \\ \vdots & & \ddots & \\ T_m & & & M(\Omega, \mu) \end{pmatrix},$$

we have $C \cong_{\mathcal{U}+\mathcal{K}} \bigoplus_{j=1}^m M(\Omega, \mu)$ (Lemma 4.4.). To finish the proof, it suffices to show that

$$D_1 = \begin{pmatrix} F_d & & & & \\ T_1 & M(\Omega, \mu) & & & \\ S_2 & & M(\Omega, \mu) & & \\ \vdots & & & \ddots & \\ S_m & & & & M(\Omega, \mu) \end{pmatrix}$$

is in $\mathrm{cl}(\mathcal{U}+\mathcal{K})(C) = \mathrm{cl}(\mathcal{U}+\mathcal{K})(\bigoplus_{j=1}^m M(\Omega, \mu))$. (Indeed, we have $||D_1 - E|| = ||S_1 - T_1|| < \eta$ and η can be chosen to be arbitrarily small.) Note that

$$\begin{pmatrix} I & & & & \\ 0 & I & & & \\ 0 & R_2 & I & & \\ \vdots & \vdots & & \ddots & \\ 0 & R_m & & & I \end{pmatrix} C \begin{pmatrix} I & & & & \\ 0 & I & & & \\ 0 & -R_2 & I & & \\ \vdots & \vdots & & \ddots & \\ 0 & -R_m & & & I \end{pmatrix}$$

$$= \begin{pmatrix} F_d & & & & \\ T_1 & M(\Omega,\mu) & & & \\ R_2T_1+T_2 & R_2M(\Omega,\mu)-M(\Omega,\mu)R_2 & M(\Omega,\mu) & & \\ \vdots & \vdots & & \ddots & \\ R_mT_1+T_m & R_mM(\Omega,\mu)-M(\Omega,\mu)R_m & & & M(\Omega,\mu) \end{pmatrix}$$

$$= \begin{pmatrix} F_d & & & & \\ T_1 & M(\Omega,\mu) & & & \\ S_2 & & M(\Omega,\mu) & & \\ \vdots & & & \ddots & \\ S_m & & & & M(\Omega,\mu) \end{pmatrix} = D_1,$$

where R_j is the multiplication operator

$$R_jf = r_jf, f \in H^2(\Omega,\mu).$$

Here we see that D_1 is similar to C. However, the operators R_j, $j = 2,3,\ldots,m$ need not be compact. Let $j \in \{2,3,\ldots,m\}$ be fixed. Fix also an $\epsilon > 0$. Let p be an integer such that $||R_j||/p < \epsilon$ and let $R_{jk} = \frac{k}{p}R_j, k = 0,1,\ldots,p$. Then we have

$$M(\Omega,\mu)R_{jk} - R_{jk}M(\Omega,\mu) = 0, k = 0,1,\ldots,p.$$

$$||R_{jk} - R_{j,k+1}|| < \epsilon, k = 0,1,\ldots,p-1.$$

Since T_1 is compact, Corollary 4.3 guarantees the existence of an operator $\tilde{R}_j$ such that

(i) $\tilde{R}_j = \tilde{R}_j - R_{j,0}$ is finite-dimensional,
(ii) $||R_jT_1 - \tilde{R}_jT_1|| < \epsilon(||T_1|| + 6||R_j||)$,
(iii) $||M(\Omega,\mu)\tilde{R}_j - \tilde{R}_jM(\Omega,\mu)|| \leq 30\epsilon||M(\Omega,\mu)||$.

Hence

$$||D_1 - \begin{pmatrix} I & & & & \\ 0 & I & & & \\ 0 & \tilde{R}_2 & I & & \\ \vdots & \vdots & & \ddots & \\ 0 & \tilde{R}_m & & & I \end{pmatrix} C \begin{pmatrix} I & & & & \\ 0 & I & & & \\ 0 & -\tilde{R}_2 & I & & \\ \vdots & \vdots & & \ddots & \\ 0 & -\tilde{R}_m & & & I \end{pmatrix} ||$$

$$= \left\| \begin{pmatrix} 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & \dots \\ R_2T_1 - \tilde{R}_2T_1 & M(\Omega,\mu)\tilde{R}_2 - \tilde{R}_2M(\Omega,\mu) & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \\ R_mT_1 - \tilde{R}_mT_1 & M(\Omega,\mu)\tilde{R}_m - \tilde{R}_mM(\Omega,\mu) & 0 & \dots \end{pmatrix} \right\|$$

$$\leq \quad (m-1)30\epsilon||M(\Omega,\mu)|| + \sum_{j=2}^{m} \epsilon(||T_1|| + 6||R_j||).$$

Hence $D_1 \in \mathrm{cl}(\mathcal{U}+\mathcal{K})(C)$ and consequently $E \in \mathrm{cl}(\mathcal{U}+\mathcal{K})(\bigoplus_{j=1}^m M(\Omega,\mu))$.

□

Note that a result similar to the following one is shown in [**JJW96**], using a different technique.

4.6. Theorem. *Let Ω be a simply connected analytic Cauchy domain. The closure of the $(\mathcal{U}+\mathcal{K})$ orbit of the operator $\bigoplus_{j=1}^m M(\Omega,\mu)$ is*

$$\begin{aligned} \mathrm{cl}((\mathcal{U}+\mathcal{K})(\bigoplus_{j=1}^{m} M(\Omega,\mu))) \quad = \quad & \{T \in \mathcal{B}(H^2(\Omega,\mu)) : \\ (i) \quad & T \text{ is essentially normal}, \\ (ii) \quad & \sigma(T) = \mathrm{cl}(\Omega), \\ (iii) \quad & \sigma_e(T) = \partial\Omega, \\ (iv) \quad & \mathrm{ind}(T-\lambda) = -m \text{ for all } \lambda \in \Omega\}. \end{aligned}$$

Again, if $T = \bigoplus_{j=1}^m M(\Omega,\mu) + K$, K compact, only condition (ii) is not automatic.

Proof. The necessity of these conditions is easily verified. We now consider their sufficiency.

By the Brown-Douglas-Fillmore theorem, if T satisfies the above conditions, then there exists a unitary U and a compact L so that, setting $K = ULU^*$, we have $T = U^* \bigoplus_{j=1}^m M(\Omega,\mu)U + L = U^*(\bigoplus_{j=1}^m M(\Omega,\mu) + K)U$. Thus it suffices to show that $\bigoplus_{j=1}^m M(\Omega,\mu) + K \in \mathrm{cl}((\mathcal{U}+\mathcal{K}) \bigoplus_{j=1}^m M(\Omega,\mu))$.

Choose an arbitrary $z_0 \in \Omega$ and use Lemma 2.6 to construct a basis of $H^2(\Omega,\mu)$ with respect to which the matrix of $M(\Omega,\mu)$ is a Toeplitz matrix. Denote the k-th element of this basis as e'_{jk}, where the index $j = 1,2,\ldots,m$ means that we regard this element as an element of the j-th copy of $H^2(\Omega,\mu)$ in the direct sum $\bigoplus_{j=1}^m H^2(\Omega,\mu)$.

We shall now construct a basis $\{e_k\}_{k=1}^\infty$ of $\bigoplus_{j=1}^m H^2(\Omega,\mu)$ by setting

$$e_{k+j} = e'_{j,l}, \text{ for } k = l.m, l = 0,1,\ldots, j = 1,2,\ldots,m.$$

Let P_n be the orthogonal projection onto $\text{span}\{e_i\}_{i=0}^n$, $n = 0,1,\ldots$. Since K is compact, the sequence $\{\bigoplus_{j=1}^m M(\Omega,\mu) + P_nKP_n\}_{n=1}^\infty$ converges to $\bigoplus_{j=1}^m M(\Omega,\mu) + K$ in the norm. We see that these operators are of the form

$$\bigoplus_{j=1}^m M(\Omega,\mu) + P_nKP_n \cong \begin{pmatrix} F_n & & & & \\ Q_{n1} & M(\Omega,\mu) & & & \\ Q_{n2} & & M(\Omega,\mu) & & \\ \vdots & & & \ddots & \\ Q_{nm} & & & & M(\Omega,\mu) \end{pmatrix},$$

where the matrix on the right hand side is with respect to the decomposition

$$\bigoplus_{j=1}^m H^2(\Omega,\mu) = [\text{ran } P_n] \oplus \bigoplus_{j=1}^m [(\text{ran } P_n)^\perp \cap H^2(\Omega,\mu)].$$

By passing to a subsequence (if necessary), and by using the upper semicontinuity of the spectrum, we may perturb F_n to get a new operator G_n such that

- $\|G_n - F_n\| < \frac{1}{n}$,
- $\sigma(G_n) \subseteq \Omega$, and
- G_n has no multiple eigenvalues.

Clearly the sequence

$$T_n = \begin{pmatrix} G_n & & & & \\ Q_{n1} & M(\Omega,\mu) & & & \\ Q_{n2} & & M(\Omega,\mu) & & \\ \vdots & & & \ddots & \\ Q_{nm} & & & & M(\Omega,\mu) \end{pmatrix}$$

still converges to $\bigoplus_{j=1}^{m} M(\Omega, \mu) + K$.

Now let $G_d(n)$ be the diagonal matrix with the same eigenvalues as G_n. This means that $G_n = R_n^{-1} G_d(n) R_n$ for some invertible matrix R_n (all eigenvalues here are of multiplicity one). Thus, by Lemma 4.5,

$$\begin{pmatrix} G_d(n) & & & & \\ Q_{n1}R_n^{-1} & M(\Omega,\mu) & & & \\ Q_{n2}R_n^{-1} & & M(\Omega,\mu) & & \\ \vdots & & & \ddots & \\ Q_{nm}R_n^{-1} & & & & M(\Omega,\mu) \end{pmatrix} \in \mathrm{cl}((\mathcal{U}+\mathcal{K}) \bigoplus_{j=1}^{m} M(\Omega,\mu)).$$

implying that

$$T_n = \begin{pmatrix} G_n & & & & \\ Q_{n1} & M(\Omega,\mu) & & & \\ Q_{n2} & & M(\Omega,\mu) & & \\ \vdots & & & \ddots & \\ Q_{nm} & & & & M(\Omega,\mu) \end{pmatrix}$$

$$= \begin{pmatrix} R_n^{-1} & & & \\ & I & & \\ & & \ddots & \\ & & & I \end{pmatrix} \begin{pmatrix} G_d(n) & & & & \\ Q_{n1}R_n^{-1} & M(\Omega,\mu) & & & \\ Q_{n2}R_n^{-1} & & M(\Omega,\mu) & & \\ \vdots & & & \ddots & \\ Q_{nm}R_n^{-1} & & & & M(\Omega,\mu) \end{pmatrix} \begin{pmatrix} R_n & & & \\ & I & & \\ & & \ddots & \\ & & & I \end{pmatrix}$$

$$\in \mathrm{cl}(\mathcal{U}+\mathcal{K})(\bigoplus_{j=1}^{m} M(\Omega,\mu)).$$

Since $T_n \in \mathrm{cl}((\mathcal{U}+\mathcal{K})(\bigoplus_{j=1}^{m} M(\Omega,\mu)))$ for all $n \geq 1$, $\bigoplus_{j=1}^{m} M(\Omega,\mu) + K = \lim T_n \in \mathrm{cl}((\mathcal{U}+\mathcal{K})(\bigoplus_{j=1}^{m} M(\Omega,\mu)))$.

□

CHAPTER 5

Disconnected spectrum

In this chapter we shall be concerned with the closures of the$(\mathcal{U}+\mathcal{K})$ orbits of models whose spectra are disconnected. We want to investigate the set $\mathrm{cl}((\mathcal{U}+\mathcal{K}))(\bigoplus_{i=1}^n A_1)$ if the spectra of A_i, $i = 1, 2, \ldots, n$ are pairwise disjoint and the sets $\mathrm{cl}((\mathcal{U}+\mathcal{K}))(A_i)$ are already known. The following two consequences of Rosenblum's theorem ([**Her90**], Corollary 3.20) are proved in [**GM93a**].

5.1. Lemma.[GM93a] *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two complex, separable Hilbert spaces and let $A \in \mathcal{B}(\mathcal{H}_1)$, $B \in \mathcal{B}(\mathcal{H}_2)$ and $Z \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$, Z compact. Assume that $\sigma(A) \cap \sigma(B) = \emptyset$. Then*

$$A \oplus B \cong_{\mathcal{U}+\mathcal{K}} \begin{pmatrix} A & Z \\ 0 & B \end{pmatrix}.$$

In fact, there is a compact $X \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ such that

$$A \oplus B = \begin{pmatrix} I & -X \\ 0 & I \end{pmatrix} \begin{pmatrix} A & Z \\ 0 & B \end{pmatrix} \begin{pmatrix} I & X \\ 0 & I \end{pmatrix}.$$

5.2. Proposition.[GM93a] *Suppose*

$$T = \begin{pmatrix} A_1 & Z_{1,2} & Z_{1,3} & \cdots & Z_{1,n} \\ & A_2 & Z_{2,3} & \cdots & Z_{1,n} \\ & & \ddots & & \vdots \\ & & & A_{n-1} & Z_{n-1,n} \\ & & & & A_n \end{pmatrix}$$

is an operator acting on the direct sum of Hilbert spaces $\bigoplus_{i=1}^n \mathcal{H}_i$. Suppose also that each $Z_{i,j}$, $1 \le i \le j \le n$, is a compact operator and suppose that the spectra of A_i, $i = 1, 2, \ldots, n$ are pairwise disjoint. Then $T \cong_{\mathcal{U}+\mathcal{K}} \bigoplus_{i=1}^n A_i$.

At this point we can also recall that a weaker conclusion, namely $A \oplus B \in \mathrm{cl}((\mathcal{U}+\mathcal{K}))(T)$ can be shown to hold under weaker assumptions. The following results have been shown in [**GM93a**] and [**AM90**]:

5.3. Lemma.[GM93a] *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces (finite-dimensional or infinite-dimensional) and let $T = \begin{pmatrix} A & Z \\ 0 & B \end{pmatrix}$ with respect to $\mathcal{H}_1 \oplus \mathcal{H}_2$. Suppose Z is compact. Then $A \oplus B \in \mathrm{cl}((\mathcal{U} + \mathcal{K}))(T)$.*

5.4. Proposition.[GM93a] *Suppose*

$$T = \begin{pmatrix} A_1 & Z_{1,2} & Z_{1,3} & \dots & Z_{1,n} \\ & A_2 & Z_{2,3} & \dots & Z_{1,n} \\ & & \ddots & & \vdots \\ & & & A_{n-1} & Z_{n-1,n} \\ & & & & A_n \end{pmatrix}$$

is an operator acting on the direct sum of Hilbert spaces $\bigoplus_{i=1}^n \mathcal{H}_i$. Suppose also that each $Z_{i,j}$, $1 \le i \le j \le n$, is a compact operator. Then $\bigoplus_{i=1}^n A_1 \in \mathrm{cl}((\mathcal{U} + \mathcal{K}))(T)$.

The following lemma will be crucial in our investigation. It tells us that that under certain conditions the Z entry in the operator $T = \begin{pmatrix} A & Z \\ 0 & B \end{pmatrix}$ has to be compact. which in turn will allow us to use Lemma 5.1 to show that $A \oplus B \cong_{\mathcal{U}+\mathcal{K}} \begin{pmatrix} A & Z \\ 0 & B \end{pmatrix}$.

5.5. Lemma. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces and let $A \in \mathcal{B}(\mathcal{H}_1)$. $B \in \mathcal{B}(\mathcal{H}_2)$ and $Z \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$. Assume that $\sigma_e(A) \cap \sigma_e(B) = \emptyset$. Assume moreover that*

$$T = \begin{pmatrix} A & Z \\ 0 & B \end{pmatrix}$$

is essentially normal. Then Z is compact and both A and B are essentially normal.

Proof. We know that

$$\begin{aligned} & \begin{pmatrix} A & Z \\ 0 & B \end{pmatrix} \begin{pmatrix} A & Z \\ 0 & B \end{pmatrix}^* - \begin{pmatrix} A & Z \\ 0 & B \end{pmatrix}^* \begin{pmatrix} A & Z \\ 0 & B \end{pmatrix} \\ = & \begin{pmatrix} A & Z \\ 0 & B \end{pmatrix} \begin{pmatrix} A^* & 0 \\ Z^* & B^* \end{pmatrix} - \begin{pmatrix} A^* & 0 \\ Z^* & B^* \end{pmatrix} \begin{pmatrix} A & Z \\ 0 & B \end{pmatrix} \\ = & \begin{pmatrix} AA^* - A^*A + ZZ^* & ZB^* - A^*Z \\ BZ^* - Z^*A & BB^* - B^*B - Z^*Z \end{pmatrix} \end{aligned}$$

is compact. i.e. the four operators $AA^* - A^*A + ZZ^*$, $ZB^* - A^*Z$, $BZ^* - Z^*A$ and $BB^* - B^*B - Z^*Z$ are compact. We can notice that once we know that Z is compact, this will imply that $AA^* - A^*A$ and $BB^* - B^*B$ are both compact, i.e. A and B are essentially normal. To finish the proof of the lemma, it suffices to show that Z is compact.

Since H_1 and H_2 are infinite-dimensional separable complex Hilbert spaces, they must be isomorphic and therefore we may assume that $H_1 = H_2$. This means that we regard A, B and Z as operators in $\mathcal{B}(H_1)$ such that $BZ^* - Z^*A$ is compact.

Let π denote the canonical quotient map from $\mathcal{H}_1$ to the Calkin algebra $\mathcal{C}(\mathcal{H}_1) = \mathcal{B}(\mathcal{H}_1)/\mathcal{K}(\mathcal{H}_1)$. Let a, b and z denote the images of A, B and Z under π. Since $BZ^* - Z^*A$ is compact, we have $bz^* - z^*a = 0$ in $\mathcal{C}(\mathcal{H}_1)$.

Recall that for two elements a, b of a Banach algebra X we define the Rosenblum operator $\tau_{b,a}$ on X by $\tau_{b,a}(x) = bx - xa$ (see **[Her90]**, section 3.1). When the spectra $\sigma(a)$, $\sigma(b)$ are disjoint, $\tau_{b,a}$ is invertible by **[Her90]**, Corollary 3.2.

Returning to our case, we see that $\sigma(a) = \sigma_e(A)$ and $\sigma(b) = \sigma_e(B)$ are indeed disjoint and $\tau_{b,a}$ is therefore invertible. We have $\tau_{b,a}(z^*) = bz^* - z^*a = 0$. Consequently, we have $z^* = 0$ in the Calkin algebra. In other words, Z^* is compact and so is Z. The rest of the conclusion of the lemma follows.

□

5.6. Corollary. *Suppose*

$$T = \begin{pmatrix} A_1 & Z_{1,2} & Z_{1,3} & \cdots & Z_{1,n} \\ & A_2 & Z_{2,3} & \cdots & Z_{1,n} \\ & & \ddots & & \vdots \\ & & & A_{n-1} & Z_{n-1,n} \\ & & & & A_n \end{pmatrix}$$

is an essentially normal operator acting on the direct sum of Hilbert spaces $\bigoplus_{i=1}^n \mathcal{H}_i$ such that the essential spectra of A_i, $i = 1, 2, \ldots, n$ are pairwise disjoint. Then each $Z_{i,j}$, $1 \le i \le j \le n$, is a compact operator and each A_i, $i = 1, 2, \ldots, n$ is essentially normal.

Let now $\Omega = \cup_1^n \Omega_i$, where each Ω_i is a simply connected analytic Cauchy domain and let μ_i be measures on $\partial\Omega_i$ which are equivalent to the respective arc length measures. Choose $j_i \in \mathbb{Z}, j_i \neq 0, i = 1, \ldots n$ and set $M_i = \oplus_{k=1}^{-j_i} M(\Omega_i, \mu_i)$ if $j_i < 0$ and $M_i = \oplus_{k=1}^{j_i} M^*(\Omega_i, \mu_i)$ if $j_i > 0$. Finally, we set $M = \oplus_{i=1}^n M_i$. As we have noted in Chapter 1, M has the following spectral properties:

(i) $\sigma(M) = \mathrm{cl}(\Omega)$;

(ii) $\sigma_e(M) = \partial\Omega$;
(iii) $\operatorname{ind}(M - z) = i_j,\ z \in \Omega_i, i = 1, 2, \ldots, n$;
(iv) $\min \operatorname{ind}(M - z) = 0.\ z \in \Omega$.

In the preceding chapters we have investigated the closures of the $(\mathcal{U} + \mathcal{K})$ orbits of the "building blocks" M_i of M. Recalling what we know from Theorem 4.6 and using the fact that

$$\operatorname{cl}((\mathcal{U} + \mathcal{K}))(A^*) = \{X^* | X \in \operatorname{cl}((\mathcal{U} + \mathcal{K}))(A)\},$$

we see that

$$\begin{aligned} \operatorname{cl}((\mathcal{U} + \mathcal{K})(M_i)) \quad = \quad & \{T \in \mathcal{B}(H^2(\Omega_i, \mu_i)) : \\ \text{(i)} \quad & \text{T is essentially normal}, \\ \text{(ii)} \quad & \sigma(T) = \operatorname{cl}(\Omega_i). \\ \text{(iii)} \quad & \sigma_e(T) = \partial\Omega_i. \\ \text{(iv)} \quad & \operatorname{ind}(T - \lambda) = j_i \text{ for all } \lambda \in \Omega_i\}. \end{aligned}$$

The description of $\operatorname{cl}((\mathcal{U} + \mathcal{K})(M))$ is as follows:

5.7. Theorem. *The closure of the $(\mathcal{U} + \mathcal{K})$ orbit of the operator $M = \bigoplus_{i=1}^n M_i$ constructed above is*

$$\begin{aligned} \operatorname{cl}((\mathcal{U} + \mathcal{K})(M)) \quad = \quad & \{T \in \mathcal{B}(\bigoplus_{i=1}^{n} H^2(\Omega_i, \mu_i)) : \\ \textit{(i)} \quad & T \textit{ is essentially normal}. \\ \textit{(ii)} \quad & \sigma(T) = \operatorname{cl}(\Omega), \\ \textit{(iii)} \quad & \sigma_e(T) = \partial\Omega. \\ \textit{(iv)} \quad & \operatorname{ind}(T - \lambda) = j_i \textit{ for all } \lambda \in \Omega_i, i = 1, 2, \ldots, n\}. \end{aligned}$$

If T is of the form $M + K$, with K compact, the only condition which is not satisfied automatically is (ii).

Proof. The necessity of the above conditions is easily verified. Let us consider their sufficiency.

Let T be an operator that satisfies the above conditions. After using the Riesz decomposition $n-1$ times we can find spaces $\mathcal{H}_i$ such that $\mathcal{H} = \bigoplus_{i=1}^n \mathcal{H}_i$ and the matrix of T with respect to this decomposition is

$$T = \begin{pmatrix} A_1 & Z_{1,2} & Z_{1,3} & \dots & Z_{1,n} \\ & A_2 & Z_{2,3} & \dots & Z_{1,n} \\ & & \ddots & & \vdots \\ & & & A_{n-1} & Z_{n-1,n} \\ & & & & A_n \end{pmatrix},$$

where we also have, for $i = 1, 2, \dots, n$,

(i) $\sigma(A_i) = \mathrm{cl}(\Omega_i)$;
(ii) $\sigma_e(A_i) = \partial\Omega_i$;
(iii) $\mathrm{ind}(A_i - z) = j_i$, $z \in \Omega_i$.

In fact, we may assume that $\mathcal{H}_i = H^2(\Omega_i, \mu_i)$, $i = 1, 2, \dots, n$,. Indeed, if this is not the case, we can replace T by U^*TU, where U is a unitary operator that maps $H^2(\Omega_i, \mu_i)$ onto $\mathcal{H}_i$, $i = 1, 2, \dots, n$,. Note also that $U^*TU \in \mathrm{cl}((\mathcal{U}+\mathcal{K})(M))$ if and only if $T \in \mathrm{cl}((\mathcal{U}+\mathcal{K})(M))$.

Since T is essentially normal, Corollary 5.6 tells us that each $Z_{i,j}$, $1 \le i \le j \le n$, is a compact operator and each A_i, $i = 1, 2, \dots, n$ is essentially normal. We can now use Corollary 5.2 to see that $T \cong_{\mathcal{U}+\mathcal{K}} \bigoplus_{i=1}^n A_i$.

Recall the description of $\mathrm{cl}((\mathcal{U} + \mathcal{K})(M_i))$ that we have noted above. We see that $A_i \in \mathrm{cl}((\mathcal{U} + \mathcal{K})(M_i))$ and consequently

$$\bigoplus_{i=1}^n A_i \in \mathrm{cl}((\mathcal{U} + \mathcal{K})(\bigoplus_{i=1}^n M_i)).$$

Using the transitivity of the relation $\to_{\mathcal{U}+\mathcal{K}}$, we see finally that $T \in \mathrm{cl}((\mathcal{U} + \mathcal{K})(M))$.

□

CHAPTER 6

Enlarging the essential spectrum

Let us now consider a compact set $K \subseteq \mathbb{C}$. In Chapter 1 we have constructed a sequence $\{d_k\}_{k=1}^{\infty}$ and a diagonal operator D_K on $l^2(\mathbb{N})$ whose diagonal is is this sequence such that

$$\sigma(D_K) = \sigma_e(D_K) = K.$$

Since D_K is a normal operator. Corollary 0.2 provides a description of the closure of its unitary orbit:

$$\begin{aligned} \mathrm{cl}(\mathcal{U}(D_K)) = \{X \text{ normal} : \quad & \sigma_e(X) = \sigma_e(D_K) \text{ and} \\ & \text{nul } (X - zI) = \text{nul } (D_K - zI) \text{ for all } z \in \rho_{sF}(D_K)\}. \end{aligned}$$

The closure of the $(\mathcal{U} + \mathcal{K})$-orbit of the operator D_K has been described in [**GM93a**]:

$$\begin{aligned} \mathrm{cl}((\mathcal{U}+\mathcal{K})(D_K)) \quad = \quad & \{T \in \mathcal{B}(l^2(\mathbb{N})) : \\ \text{(i)} \quad & \text{T is essentially normal.} \\ \text{(ii)} \quad & \sigma(T) \supseteq \sigma(D_K) = K, \\ \text{(iii)} \quad & \text{every component of } \mathbb{C} \setminus \sigma(T) \\ & \text{is a component of } \mathbb{C} \setminus \sigma(D_K), \\ \text{(iv)} \quad & \sigma_e(T) = \sigma_e(D_K) = K, \\ \text{(v)} \quad & \text{if } z \text{ is an isolated point of } \sigma_e(T) = \sigma_e(D_K) = K, \\ & \text{we have } (T - z)|_{\mathcal{H}(z;T)} = 0, \text{ where } \mathcal{H}(z;T) \\ & \text{denotes the range of the Riesz idempotent} \\ & \text{corresponding to } \{z\}, \\ \text{(vi)} \quad & \text{ind}(T - z) = 0 \text{ for all } z \notin K\}. \end{aligned}$$

Next, we are interested in models which can be constructed from D_K and the models investigated in previous chapters using direct sum. Using the same techniques as in the proof of Theorem 5.7, we can easily show the following.

6.1. Theorem. *Let Ω_i be pairwise disjoint simply connected analytic Cauchy domains and let μ_i be measures on $\partial\Omega_i$ which are equivalent to the respective arc length measures. Let $j_i \in \mathbb{Z}, j_i \neq 0, i = 1, \dots n$ and set $M_i = \oplus_{k=1}^{-j_i} M(\Omega_i, \mu_i)$ if $j_i < 0$ and $M_i = \oplus_{k=1}^{j_i} M^*(\Omega_i, \mu_i)$ if $j_i > 0$.*

Next, let K be a compact subset of $\mathbb{C}$. Assume that $\mathrm{cl}(\Omega_i)$ lies in the unbounded component of $\mathbb{C} \setminus K$, for $i = 1, \dots n$.

Finally, we set $M = D_K \oplus \bigoplus_{i=1}^{n} M_i$, where D_K is a diagonal operator with $\sigma(D_K) = \sigma_e(D_K) = K$. The closure of the $(\mathcal{U} + \mathcal{K})$-orbit of the operator M is

$$
\begin{aligned}
\mathrm{cl}((\mathcal{U}+\mathcal{K})(D_K)) \quad = \quad & \{T \in \mathcal{B}(l^2(\mathbb{N}) \oplus \bigoplus_{i=1}^{n}(\bigoplus_{j=1}^{j_i} H^2(\Omega_i, \mu_i))) : \\
(i) \quad & T \text{ is essentially normal.} \\
(ii) \quad & \sigma(T) \supseteq \sigma(M), \\
(iii) \quad & \text{every component of } \mathbb{C} \setminus \sigma(T) \\
& \text{is a component of } \mathbb{C} \setminus \sigma(M). \\
(iv) \quad & \sigma_e(T) = \sigma_e(M). \\
(v) \quad & \text{if } z \text{ is an isolated point of } \sigma_e(T) = \sigma_e(M), \\
& \text{we have } (T - zI)|_{\mathcal{H}(z;T)} = 0, \text{ where } \mathcal{H}(z;T) \\
& \text{denotes the range of the Riesz idempotent} \\
& \text{corresponding to } \{z\}, \\
(vi) \quad & \mathrm{ind}(T - z) = \mathrm{ind}(M - z) \\
& \text{for all } z \notin \sigma_e(T) = \sigma_e(M)\}.
\end{aligned}
$$

Proof. The proof is analogous to that of Theorem 5.7.

□

We shall now consider the case where K is not necessarily disjoint from $\bigcup_{i=1}^{n} \mathrm{cl}(\Omega_i)$. We shall, however, restrict our investigation to a perfect K for the moment. We will return to the investigation of models with isolated spectral points in the next chapter.

6.2. Theorem. *Let Ω_i be pairwise disjoint simply connected analytic Cauchy domains and let μ_i be measures on $\partial\Omega_i$ which are equivalent to the respective arc length measures. Let $j_i \in \mathbb{Z}, j_i \neq 0, i = 1, \dots n$ and set $M_i = \oplus_{k=1}^{-j_i} M(\Omega_i, \mu_i)$ if $j_i < 0$ and $M_i = \oplus_{k=1}^{j_i} M^*(\Omega_i, \mu_i)$ if $j_i > 0$.*

Next, let K be a perfect compact subset of $\mathbb{C}$. Assume that K is disjoint from $\bigcup_{i=1}^n \Omega_i$. Assume that $\mathbb{C} \setminus K$ has only finitely many components.

Finally, we set $M = D_K \oplus \bigoplus_{i=1}^n M_i$, where D_K is a diagonal operator with $\sigma(D_K) = \sigma_e(D_K) = K$. The closure of the $(\mathcal{U} + \mathcal{K})$ orbit of the operator M is

$$\begin{aligned} \mathrm{cl}((\mathcal{U}+\mathcal{K})(M)) \quad = \quad & \{T \in \mathcal{B}(l^2(\mathbb{N}) \oplus \bigoplus_{i=1}^{n} (\bigoplus_{j=1}^{j_i} H^2(\Omega_i, \mu_i))) : \\ (i) \quad & T \text{ is essentially normal.} \\ (ii) \quad & \sigma(T) \supseteq \sigma(M). \\ (iii) \quad & \text{every component of } \mathbb{C} \setminus \sigma(T) \\ & \text{is a component of } \mathbb{C} \setminus \sigma(M). \\ (iv) \quad & \sigma_e(T) = \sigma_e(M). \\ (v) \quad & \mathrm{ind}(T - z) = \mathrm{ind}(M - z) \text{ for all } z \notin \sigma_e(T) = \sigma_e(M)\}. \end{aligned}$$

If $T = M + L$, L compact, the only condition which is not fulfilled automatically is condition (iii).

Proof. The necessity of these conditions is easily verified. We now consider their sufficiency.

Suppose that T satisfies the above conditions. Let $\eta > 0$. First we can use Proposition 4.4 of [**Apo76**] to find an operator T_0 such that $||T - T_0|| < 0$, $T - T_0$ is compact, T_0 satisfies (i), (iv) and (v), and we have

$$\sigma(T_0) = \sigma(M).$$

Note that T_0 is essentially normal. Clearly it suffices to show that $T_0 \in \mathrm{cl}(\mathcal{U} + \mathcal{K})(M)$.

By the Brown-Douglas-Fillmore Theorem there exists a unitary U and a compact L such that, setting $K_0 = ULU^*$, we have $T_0 = U^*MU + L = U^*(M + K_0)U$. Thus it suffices to show that $M + K_0 \in \mathrm{cl}((\mathcal{U} + \mathcal{K})M))$.

Fix $\epsilon > 0$. Let $\{e_0', e_1', \dots\}$ be the canonical basis of $l^2(\mathbb{N})$. For each $i = 1, \dots n$, use Lemma 2.6 to construct a basis of $H^2(\Omega_i, \mu_i)$ with respect to which the matrix of $M(\Omega_i, \mu_i)$

is a Toeplitz matrix. Denote the k-th element of this basis as e'_{ijk}. where the index $j = 1, 2, \ldots, j_i$ means that we regard this element as an element of the j-th copy of $H^2(\Omega_i, \mu_i)$ in the direct sum $l^2(\mathbb{N}) \oplus \bigoplus_{i=1}^{n}(\bigoplus_{j=1}^{j_i} H^2(\Omega_i, \mu_i))$.

We shall now construct a basis $\{e_k\}_{k=1}^{\infty}$ of $l^2(\mathbb{N}) \oplus \bigoplus_{i=1}^{n}(\bigoplus_{j=1}^{j_i} H^2(\Omega_i, \mu_i))$ as follows: Set $k_0 = 1 + \sum_{i=1}^{n} j_i$ and let now

$$
\begin{aligned}
e_k &= e'_l, \text{ for } k = l.k_0 + 1, l = 0, 1, \ldots, \\
e_{k+j} &= e'_{1,j,l}, \text{ for } k = l.k_0 + 1, l = 0, 1, \ldots, j = 1, 2, \ldots, j_1, \\
e_{k+j_1+j} &= e'_{2,j,l}, \text{ for } k = l.k_0 + 1, l = 0, 1, \ldots, j = 1, 2, \ldots, j_2, \\
&\vdots \\
e_{k+j_1+j_2+\cdots+j_{n-1}+j} &= e'_{n,j,l}, \text{ for } k = l.k_0 + 1, l = 0, 1, \ldots, j = 1, 2, \ldots, j_n.
\end{aligned}
$$

Let P_k be the orthogonal projection onto $\operatorname{span}\{e_i\}_{i=0}^{k}$, $k = 0, 1, \ldots$. Since K_0 is compact, the sequence $\{M + P_k K_0 P_k\}_{k=1}^{\infty}$ converges to $M + K_0$ in the norm.

Without loss of generality, we can assume that $j_i > 0$ for $i = 1, 2, \ldots, i_0$ and $j_i < 0$ for $i = i_0 + 1, i_0 + 2, \ldots, n$. Note that the operators $M + P_k K_0 P_k$ are of the form

$$
M + P_k K_0 P_k = \begin{pmatrix}
F_k & & & & & T_{i_0+1} & \cdots & T_n \\
& D_k & & & & & & \\
T_1 & & M_1 & & & & & \\
\vdots & & & \ddots & & & & \\
T_{i_0} & & & & M_{i_0} & & & \\
& & & & & M_{i_0+1} & & \\
& & & & & & \ddots & \\
& & & & & & & M_n
\end{pmatrix}.
$$

where the matrix on the right hand side is with respect to the decomposition

$$
[\operatorname{ran} P_k] \oplus [(\operatorname{ran} P_k)^{\perp} \cap l^2(\mathbb{N})] \oplus \bigoplus_{i=1}^{n} [(\operatorname{ran} P_k)^{\perp} \cap (\bigoplus_{j=1}^{j_i} H^2(\Omega_i, \mu_i))]
$$

and D_k, like D_K, is a diagonal operator such that

$$
\sigma(D_k) = \sigma_e(D_k) = K.
$$

By passing to a subsequence (if necessary), and by using the upper semicontinuity of the spectrum, we may perturb F_k to get a new operator G_k such that

- $\|G_k - F_k\| < \frac{1}{k}$,
- $\sigma(G_k) \subseteq (K \setminus \partial(\bigcup_{i=1}^{n} \Omega_i)) \cup \bigcup_{i=1}^{n} \Omega_i$, and
- G_k has no multiple eigenvalues.

Clearly the sequence

$$T_k = \begin{pmatrix} G_k & & & & & T_{i_0+1} & \cdots & T_n \\ & D_k & & & & & & \\ T_1 & & M_1 & & & & & \\ \vdots & & & \ddots & & & & \\ T_{i_0} & & & & M_{i_0} & & & \\ & & & & & M_{i_0+1} & & \\ & & & & & & \ddots & \\ & & & & & & & M_n \end{pmatrix},$$

where we have replaced F_k by G_k still converges to $M + K_0$.

Fix a k. Now the finite dimensional matrix G_k is similar to

$$R_k^{-1} G_k R_k = \begin{pmatrix} G_e & & \\ & G_- & \\ & & G_- \end{pmatrix}.$$

where G_e, G_+, G_- are diagonal matrices such that

$$\begin{aligned} \sigma(G_e) &\subseteq K \\ \sigma(G_+) &\subseteq \bigcup_{i=1}^{i_0} \Omega_i \\ \sigma(G_-) &\subseteq \bigcup_{i=i_0+1}^{n} \Omega_i \end{aligned}$$

We have

$$\tilde{T}_k = \begin{pmatrix} R_k^{-1} & \\ & I \end{pmatrix} T_k \begin{pmatrix} R_k & \\ & I \end{pmatrix}$$

$$
= \begin{pmatrix}
G_e & & & & & & & T^e_{i_0+1} & \cdots & T^e_n \\
& G_+ & & & & & & T^+_{i_0+1} & \cdots & T^+_n \\
& & G_- & & & & & T^-_{i_0+1} & \cdots & T^-_n \\
& & & D_k & & & & & & \\
T^e_1 & T^+_1 & T^-_1 & & M_1 & & & & & \\
\vdots & \vdots & \vdots & & & \ddots & & & & \\
T^e_{i_0} & T^+_{i_0} & T^-_{i_0} & & & & M_{i_0} & & & \\
& & & & & & & M_{i_0+1} & & \\
& & & & & & & & \ddots & \\
& & & & & & & & & M_n
\end{pmatrix}.
$$

where the matrices $\begin{pmatrix} R_k^{-1} & \\ & I \end{pmatrix}$ and $\begin{pmatrix} R_k & \\ & I \end{pmatrix}$ are written with respect to the decomposition ran $P_k \oplus (\text{ran } P_k)^{\perp}$.

By rearranging the decomposition of the underlying Hilbert space, we can write the matrix of $\tilde{T}_k$ as

$$
\begin{pmatrix}
G_e & & & & & & & T^e_{i_0+1} & \cdots & T^e_n \\
& D_k & & & & & & & & \\
& & G_+ & & & & & T^+_{i_0+1} & \cdots & T^+_n \\
T^e_1 & & T^+_1 & M_1 & & & T^-_1 & & & \\
\vdots & & \vdots & & \ddots & & \vdots & & & \\
T^e_{i_0} & & T^+_{i_0} & & & M_{i_0} & T^-_{i_0} & & & \\
& & & & & & G_- & T^-_{i_0+1} & \cdots & T^-_n \\
& & & & & & & M_{i_0+1} & & \\
& & & & & & & & \ddots & \\
& & & & & & & & & M_n
\end{pmatrix}
$$

We can now use Lemma 5.1 to see that $\tilde{T}_k$ is $(\mathcal{U} + \mathcal{K})$ equivalent to

$$
\begin{pmatrix}
G_e & & & & & & & 0 & \cdots & 0 \\
& D_k & & & & & & & & \\
& & G_+ & & & & & 0 & \cdots & 0 \\
T^e_1 & & T^+_1 & M_1 & & & 0 & & & \\
\vdots & & \vdots & & \ddots & & \vdots & & & \\
T^e_{i_0} & & T^+_{i_0} & & & M_{i_0} & 0 & & & \\
& & & & & & G_- & T^-_{i_0+1} & \cdots & T^-_n \\
& & & & & & & M_{i_0+1} & & \\
& & & & & & & & \ddots & \\
& & & & & & & & & M_n
\end{pmatrix}.
$$

Another application of Lemma 5.1 shows that this last operator is $(\mathcal{U}+\mathcal{K})$ equivalent to

$$T_k' = \begin{pmatrix} G_e & & & & & & & & & \\ & D_k & & & & & & & & \\ & & G_+ & & & & & & & \\ 0 & & T_1^+ & M_1 & & & & & & \\ \vdots & & \vdots & & \ddots & & & & & \\ 0 & & T_{i_0}^+ & & & M_{i_0} & & & & \\ & & & & & & G_- & T_{i_0+1}^- & \cdots & T_n^- \\ & & & & & & & M_{i_0+1} & & \\ & & & & & & & & \ddots & \\ & & & & & & & & & M_n \end{pmatrix}.$$

Summing up, we see that the operator T_k, which is one in a sequence of operators that converge to $M+K_0$, is $(\mathcal{U}+\mathcal{K})$ equivalent to

$$T_k' = \begin{pmatrix} G_e & \\ & D_k \end{pmatrix} \oplus \begin{pmatrix} G_+ & & & \\ T_1^+ & M_1 & & \\ \vdots & & \ddots & \\ T_{i_0}^+ & & & M_{i_0} \end{pmatrix} \oplus \begin{pmatrix} G_- & T_{i_0+1}^- & \cdots & T_n^- \\ & M_{i_0+1} & & \\ & & \ddots & \\ & & & M_n \end{pmatrix}.$$

We can now see that $T_k' \in \mathrm{cl}(\mathcal{U}+\mathcal{K})(M)$ by noticing that

$$\begin{pmatrix} G_e & \\ & D_k \end{pmatrix} \in \mathrm{cl}(\mathcal{U}+\mathcal{K})(D_K)$$

$$\begin{pmatrix} G_+ & & & \\ T_1^+ & M_1 & & \\ \vdots & & \ddots & \\ T_{i_0}^+ & & & M_{i_0} \end{pmatrix} \in \mathrm{cl}(\mathcal{U}+\mathcal{K})\Big(\bigoplus_{i=1}^{i_0} M_i\Big)$$

$$\begin{pmatrix} G_- & T_{i_0+1}^- & \cdots & T_n^- \\ & M_{i_0+1} & & \\ & & \ddots & \\ & & & M_n \end{pmatrix} \in \mathrm{cl}(\mathcal{U}+\mathcal{K})\Big(\bigoplus_{i=i_0+1}^{n} M_i\Big)$$

The last two statements follow from Lemma 5.7. The operator $\begin{pmatrix} G_e & \\ & D_k \end{pmatrix}$ is in fact in

$\mathrm{cl}(\mathcal{U}(D_K))$.

This finishes the proof, as $T'_k \in \mathrm{cl}(\mathcal{U} + \mathcal{K})(M)$ implies that $T_k \in \mathrm{cl}(\mathcal{U} + \mathcal{K})(M)$. Hence $M + K_0 \in \mathrm{cl}(\mathcal{U} + \mathcal{K})(M)$ and consequently T must be in $\mathrm{cl}(\mathcal{U} + \mathcal{K})(M)$.

□

CHAPTER 7

Isolated eigenvalues

We shall now investigate the closure of the $(\mathcal{U} + \mathcal{K})$ orbit of a model which differs from that investigated in Theorem 6.2 by the adding of isolated eigenvalues of both finite and infinite multiplicity.

With the results of Chapters 2, 4, 5 and 6 at our disposal, we are now able to adapt the proof of Theorem 2.14 in [**GM93a**] to show the following:

7.1. Theorem. *Let Ω_i be pairwise disjoint simply connected analytic Cauchy domains and let μ_i be measures on $\partial\Omega_i$ which are equivalent to the respective arc length measures. Let $j_i \in \mathbb{Z}, j_i \neq 0, i = 1, \dots n$ and set $M^{(i)} = \oplus_{k=1}^{-j_i} M(\Omega_i, \mu_i)$ if $j_i < 0$ and $M^{(i)} = \oplus_{k=1}^{j_i} M^*(\Omega_i, \mu_i)$ if $j_i > 0$.*

Next, let K be a compact subset of $\mathbb{C}$. Assume that K is disjoint from $\bigcup_{i=1}^{n} \Omega_i$.

Let $\{z_r\}_{r=1}^{\nu}$ be a sequence of complex numbers, here ν may be a finite number or $\nu = \infty$. We are assuming $\{z_r\}_{\mathrm{acc}} \subseteq K \cup \bigcup_{i=1}^{n} \partial\Omega_i$ and $\{z_r\} \cap (K \cup \bigcup_{i=1}^{n} \Omega_i) = \emptyset$. Choose a sequence of complex numbers $\{x_r\}_{r=1}^{\nu}$ such that, if $\nu = \infty$, we have $\lim_{r\to\infty} x_r = 0$. Let $\{n_r\}_{r=1}^{\nu}$ be a sequence of natural numbers.

Finally, we define an operator

$$M = D_K \oplus \bigoplus_{i=1}^{n} M^{(i)} \oplus \bigoplus_{r=1}^{\nu} J(z_r, x_r, n_r),$$

where the operators D_K and $J(z_r, x_r, n_r)$ are as defined in Chapter 1 (pages 16, 17), on the Hilbert space

$$\mathcal{H} = l^2(\mathbb{N}) \oplus \bigoplus_{i=1}^{n} (\bigoplus_{j=1}^{j_i} H^2(\Omega_i, \mu_i)) \oplus \bigoplus_{r=1}^{\nu} \mathbb{C}^{n_r}.$$

The closure of the $(\mathcal{U}+\mathcal{K})$ *orbit of the operator* M *is*

$$
\begin{aligned}
\mathrm{cl}((\mathcal{U}+\mathcal{K})(M)) \quad = \quad & \{T \in \mathcal{B}(\mathcal{H}) : \\
\textit{(i)} \quad & T \textit{ is essentially normal}, \\
\textit{(ii)} \quad & \sigma(T) \supseteq \sigma(M), \\
\textit{(iii)} \quad & \text{nul } (T-z)^l \geq \text{nul } (M-z)^l, l = 1, 2, \ldots, z \notin \sigma_e(M), \\
\textit{(iv)} \quad & \sigma_e(T) = \sigma_e(M), \\
\textit{(v)} \quad & \mathrm{ind}(T-z) = \mathrm{ind}(M-z) \textit{ for all } z \notin \sigma_e(T) = \sigma_e(M) \\
\textit{(vi)} \quad & \textit{if } z \in \sigma_{iso}(T) \cap \sigma_e(T), \textit{ then } (T-z)|_{\mathcal{H}(z:T)} = 0. \\
\textit{(vii)} \quad & \textit{if } z \in \sigma_0(T), \textit{ then } \text{rank } E(z;T) = \text{rank } E(z;M)\}.
\end{aligned}
$$

Note that if $T = M + L$, L compact, the conditions which are not fulfilled automatically are (ii), (iii) and (vii).

Proof. One can easily see that if $T \in \mathrm{cl}((\mathcal{U}+\mathcal{K})(M))$, the conditions (i) through (vii) must be satisfied.

Next, suppose that T satisfies the conditions (i) through (vii).

Step One: The isolated points of $\sigma(T)$. Denote the countable set $\sigma_{iso}(T)$ of isolated points of the spectrum of T by $\{\lambda_s\}_{s=1}^{\rho}$. Assume that these points are numbered in decreasing order of distance to $\sigma_{acc}(T)$, the set of all accumulation points of the spectrum.

Choose a small $\epsilon > 0$ subject to the condition $\partial((\sigma_{acc}(T))_\epsilon) \cap \sigma_{iso}(T) = \emptyset$ and set $\sigma_\epsilon = (\sigma_{acc}(T))_\epsilon$. Let s_0 be such that $\{\lambda_1, \lambda_2, \ldots, \lambda_{s_0}\} = \sigma(T) \setminus \sigma_\epsilon$. (If the set $\sigma_{iso}(T)$ is finite, we choose $\epsilon > 0$ small enough so that $\sigma_{iso}(T) \cap \sigma_\epsilon = \emptyset$.)

Then T has a block upper triangular matrix

$$
T = \begin{pmatrix}
T_{11} & \ldots & & & T_{10} \\
 & T_{22} & & T_{ij} & T_{20} \\
 & & \ddots & & \vdots \\
 & & & T_{s_0 s_0} & T_{s_0 0} \\
 & & & & T_0
\end{pmatrix}
$$

with respect to the decomposition

$$\mathcal{H} = \bigoplus_{s=1}^{s_0} \mathcal{H}(\lambda_s; T) \oplus [\mathcal{H} \ominus \bigoplus_{s=1}^{s_0} \mathcal{H}(\lambda_s; T)]$$

Corollary 5.6 shows that T_0 is essentially normal.

We can now rewrite the model M with respect to the decomposition

$$\mathcal{H} = \bigoplus_{s=1}^{s_0} \mathcal{H}(\lambda_s; M) \oplus [\mathcal{H} \ominus \bigoplus_{s=1}^{s_0} \mathcal{H}(\lambda_s; M)]$$

as

$$M = (\bigoplus_{s=1}^{s_0} M_s) \oplus M_0.$$

Here we have

$$M_0 = D_{K_0} \oplus (\bigoplus_{i=1}^{n} M^{(i)} \oplus \bigoplus_{r=r_0+1}^{\nu} J(z_r, x_r, n_r))),$$

where D_{K_0} is a diagonal operator with

$$\sigma(D_{K_0}) = \sigma_e(D_{K_0}) = K_0 = K \setminus \{\lambda_1, \lambda_2, \ldots, \lambda_{s_0}\}$$

and we are assuming, without loss of generality, that the blocks $J(z_r, x_r, n_r)$ are numbered in such a way that

$$\begin{aligned} \{z_r\}_{r=1}^{r_0} &\subseteq \{\lambda_s\}_{s=1}^{s_0} \\ \{z_r\}_{r=r_0+1}^{\nu} \cap \{\lambda_s\}_{s=1}^{s_0} &= \emptyset. \end{aligned}$$

Let us now consider the operators M_s and T_{ss}. We have two cases to distinguish.

If $\lambda_s \in \sigma_0(T)$, we know from (viii) that rank $E(\lambda_s; T) =$ rank $E(\lambda_s; M)$. In other words, $\dim \mathcal{H}(\lambda_s; T) = \dim \mathcal{H}(\lambda_s; M)$. Without loss of generality, we may assume that $\mathcal{H}(\lambda_s; T) = \mathcal{H}(\lambda_s; M)$ (if necessary, replace T by a unitarily equivalent operator). We also have

$$\text{nul } (T - \lambda_s)^l \geq \text{nul } (M - \lambda_s)^l, l = 1, 2, \ldots,$$

and hence

$$\text{nul } (T_{ss} - \lambda_s)^l \geq \text{nul } (M_s - \lambda_s)^l, l = 1, 2, \ldots.$$

Therefore

$$T_{ss} \in \text{cl}(\mathcal{S}(M_s)) = \text{cl}(\mathcal{U} + \mathcal{K})(M_s)$$

(note that we are working on finite dimensional spaces right now).

If $\lambda_s \in \sigma_{iso}(T) \cap \sigma_e(T)$, we have rank $E(\lambda_s; T)$ = rank $E(\lambda_s; M) = \infty$. We can again assume $\mathcal{H}(\lambda_s; T) = \mathcal{H}(\lambda_s; M)$, and then we have, using (vi) and noting that M_s is a direct summand of a diagonal operator,

$$T_{ss} = \lambda_s I = M_s.$$

Summing up, we see that

$$\bigoplus_{s=1}^{s_0} T_{ss} \in \text{cl}(\mathcal{U} + \mathcal{K})(\bigoplus_{s=1}^{s_0} M_s).$$

Let us now consider the operators M_0 and T_0. Suppose, temporarily, that we can show that there exists a $V_0 \in (\mathcal{U} + \mathcal{K})(\mathcal{H} \ominus \bigoplus_{s=1}^{s_0} \mathcal{H}(\lambda_s; M))$ such that $||V_0^{-1} M_0 V_0 - T_0|| < 6\epsilon$. Then

$$M \rightarrow_{(\mathcal{U}+\mathcal{K})} \bigoplus_{s=1}^{s_0} T_s \oplus M_0 \cong M' = \begin{pmatrix} T_{11} & \dots & & & T_{10}V_0^{-1} \\ & T_{22} & & Tij & T_{20}V_0^{-1} \\ & & \ddots & & \vdots \\ & & & T_{s_0s_0} & T_{s_00}V_0^{-1} \\ & & & & M_0 \end{pmatrix}$$

by Corollary 5.2. Thus

$$\begin{aligned} M \quad &\rightarrow_{(\mathcal{U}+\mathcal{K})} \quad (\bigoplus_{s=1}^{s_0} I \oplus V_0^{-1}) M' (\bigoplus_{s=1}^{s_0} I \oplus V_0) \\ &= \quad M'' = \begin{pmatrix} T_{11} & \dots & & & T_{10} \\ & T_{22} & & T_{ij} & T_{20} \\ & & \ddots & & \vdots \\ & & & T_{s_0s_0} & T_{s_00} \\ & & & & V_0^{-1}M_0V_0 \end{pmatrix}. \end{aligned}$$

But then $||T - M''|| = ||T_0 - V_0^{-1} M_0 V_0|| < 6\epsilon$, implying that $T \in \text{cl}(\mathcal{U} + \mathcal{K})(M)$. We see that it is indeed sufficient to show that $\text{dist}(T_0, (\mathcal{U} + \mathcal{K})(M_0)) < 6\epsilon$.

Step Two: The points $\sigma_0(M) \setminus \sigma_{iso}(T)$. We are now working with the operators M_0 and T_0 which satisfy the same conditions (i) through (viii). In addition, we know that $\lambda \in \sigma(T_0)$ implies $\text{dist}(\lambda, \sigma_{acc}(T_0)) < \epsilon$. We shall now deal with the points $\beta \in \sigma_0(M)$ which lie in a hole of $\sigma(M)$, but are not isolated in $\sigma(T)$.

Let $\{\beta_s\}_{s=1}^{\zeta}$ $(0 \leq \zeta \leq \infty)$ denote the countable set $\sigma_0(M) \setminus \sigma_{iso}(T) = \sigma_0(M_0) \setminus \sigma_{iso}(T_0)$ in decreasing order of distance to $\sigma_{acc}(M_0) \subseteq \sigma_e(M_0)$.

The countability of $\sigma_0(M_0) \setminus \sigma_{iso}(T_0)$ allows us to choose $0 < \epsilon_1 < \epsilon$ such that $\partial((\sigma_{acc}(M_0))_{\epsilon_1}) \cap \{\beta_s\}_{s=1}^{\zeta} = \emptyset$. Choose p such that $\{\beta_s\}_{s=1}^{p}$ are those elements of $\{\beta_s\}_{s=1}^{\zeta}$ which do not lie in $(\sigma_{acc}(M_0))_{\epsilon_1}$. (As before, if $\zeta < \infty$, choose ϵ_1 so that $p = \zeta$.)

Conditions (iii) implies that nul $(T_0 - \beta_s)^l \geq$ nul $(M_0 - \beta_s)^l, l = 1, 2, \ldots, s = 1, 2, \ldots, p$. Therefore T_0 can be written as

$$\begin{pmatrix} T_{01} & \cdots & & & B_{10} \\ & T_{02} & & \beta_{ij} & B_{20} \\ & & \ddots & & \vdots \\ & & & T_{0p} & B_{p0} \\ & & & & T_1 \end{pmatrix},$$

where T_{0s} acts on a space of finite dimension rank $E(\beta_s; M)$ and

$$\text{nul } (T_{0s} - \beta_s)^l \geq \text{nul } (M_{0s} - \beta_s)^l, l = 1, 2, \ldots, s = 1, 2, \ldots, p,$$

where $M_{0s} = M|_{\mathcal{H}(\beta_s; M_0)}$. Consequently, we have $T_{0s} \in \text{cl}(\mathcal{U} + \mathcal{K})(M_{0s})$. (See **[Her90]**, Theorem 2.1.) An easy matrix calculation shows that T_1 is essentially normal. Moreover, the above matrix of T_0 may be assumed to be with respect to the decomposition

$$\bigoplus_{s=1}^{p} \mathcal{H}(\beta_s; M_0) \oplus (\bigoplus_{s=1}^{p} \mathcal{H}(\beta_s; M_0))^{\perp}.$$

We also have $\sigma(T_1) = \sigma(T_0)$, $\sigma_e(T_1) = \sigma_e(T_0)$ and nul $(T_1 - \beta) =$ nul $(T_0 - \beta)$ if $\beta \notin \sigma_e(T_1) \cup \{\beta_s\}_{s=1}^{p}$.

The corresponding decomposition of M_0 is

$$M_0 = \bigoplus_{s=1}^{p} M_{0s} \oplus M_1,$$

where

$$M_1 = M_0|_{(\bigoplus_{s=1}^{p} E(\beta_s; M_0))^{\perp}} = D_{K_0} \oplus \bigoplus_{i=1}^{n} M^{(i)} \oplus \bigoplus_{r=r_1+1}^{\nu} J(z_r, x_r, n_r).$$

We are assuming that that blocks $J(z_r, x_r, n_r)$ are numbered in such a way that $\bigoplus_{r=r_0}^{r_1} J(z_r, x_r, n_r) = \bigoplus_{s=1}^{p} M_{0s}$.

Note that $\beta_s \notin \sigma(M_1)$, $s = 1,2,\ldots,p$. As a result, if $\beta \in \sigma(M_1)$, we have either $\mathrm{dist}(\beta, \sigma_{acc}(M_1)) = \mathrm{dist}(\beta, \sigma_{acc}(M)) < \epsilon$ or $\beta \in \sigma_e(M_1) = \sigma_e(M_0)$ and β is in one of the holes of $\sigma(M_1)$ but is not isolated in $\sigma(T_1)$.

Step Three: The "Big Holes" of $\sigma(T_1)$. Note that $\sigma(T_1) = \sigma(T_0)$ looks like $\sigma(M_1)$ except that some of the (countably many) holes of $\sigma(M_1)$ are filled in. Let now $\{\tau_j\}_{j=1}^{\eta}$ ($1 \le \eta \le \infty$) denote the holes of $\sigma(M_1)$ which lie in $\sigma(T_1)$. Since $\sigma(M_1)$ is compact, all except finitely many of the holes τ_j must be very small in the sense that given an $\epsilon_2 > 0$, there exists an $N = N(\epsilon_2) > 0$ such that $\{\tau_j\}_{j=1}^{N} \subseteq (\sigma_{acc}(M_1))_{\epsilon_2}$.

Let us therefore choose $0 < \epsilon_2 < \epsilon/2$ and fix the appropriate $N = N(\epsilon_2)$. (Again, if $\eta < \infty$, we can choose ϵ_2 small enough so that $N = \eta$.) Denote $\kappa_j = \min \mathrm{ind}(T_1 - \lambda)$ for $\lambda \in \tau_j \cap \rho_{sF}^r(T_1)$, $1 \le j \le N$ and set $\kappa = \sum_{j=1}^{N} \kappa_j$.

We can now use Lemma 2.13 of [**GM93a**] κ times to find a compact operator K_2, $||K_2|| < \epsilon_2$ such that

$$T_1 - K_2 \cong \begin{pmatrix} N_1(\tau_1) & \cdots & & & & & W_{10} \\ & \ddots & & & W_{ij} & & W_{20} \\ & & N_{\kappa_1}(\tau_1) & & & & \\ & & & N_1(\tau_2) & & & \vdots \\ & & & & \ddots & & \\ & & & & & N_{\kappa_N}(\tau_N) & W_{\kappa 0} \\ & & & & & & T_2 \end{pmatrix}.$$

where

(i) each $N_i(\tau_j)$ is a compact perturbation of a normal operator.
(ii) $\sigma(N_i(\tau_j)) = \mathrm{cl}(\tau_j), \sigma_e(N_i(\tau_j)) = \partial(\mathrm{cl}(\tau_j))$,
(iii) $\sigma_p(N_i(\tau_j)) = (\sigma_p(N_i(\tau_j)^*))^* = \mathrm{cl}(\tau_j) \setminus \partial(\mathrm{cl}(\tau_j))$,
nul $(N_i(\tau_j) - \lambda) =$ nul $(N_i(\tau_j) - \lambda)^* = 1$ for all $\lambda \in \mathrm{cl}(\tau_j) \setminus \partial(\mathrm{cl}(\tau_j))$,
(iv) $\sigma_e(T_2) = \sigma_e(T_1)$,
(v) $\mathrm{ind}(T_2 - \lambda) = \mathrm{ind}(T_1 - \lambda)$ for $\lambda \notin \sigma_e(T_1)$,
(vi) nul $(T_2 - \lambda) =$ nul $(T_1 - \lambda) - \kappa_j$ if $\lambda \notin \tau_j$, $1 \le j \le N$,
nul $(T_2 - \lambda) =$ nul $(T_1 - \lambda)$ otherwise,
(vii) nul $(T_2 - \lambda)^* =$ nul $(T_1 - \lambda)^* - \kappa_j$ if $\lambda \notin \tau_j$, $1 \le j \le N$,
nul $(T_2 - \lambda)^* =$ nul $(T_1 - \lambda)^*$ otherwise.

An easy matrix calculation verifies that T_2 is essentially normal and all W_{ij} are compact.

We see now from conditions (iv) to (vii) that some of the singular points of $\sigma(T_1)$ have now become isolated eigenvalues of T_2:

$$\sigma_0(T_2) = \sigma_0(T_1) \cup (\rho_{sF}^s(T_1) \cap (\bigcup_{j=1}^{N} \tau_j)).$$

Moreover,

$$\sigma(T_2) = (\sigma(T_1) \setminus (\bigcup_{j=1}^{N} \tau_j) \cup (\rho_{sF}^s(T_1) \cap (\bigcup_{j=1}^{N} \tau_j)).$$

Step Four: The Singular Points of T_2. Consider the diagonal operator

$$D = D_{K_1} \oplus \bigoplus_{r=r_1+1}^{\nu} J(z_r, 0, n_r).$$

Note that for every $r_2 > r_1$, we have

$$\begin{aligned} & D_{K_1} \oplus \bigoplus_{r=r_1+1}^{r_2} J(z_r, 0, n_r) \oplus \bigoplus_{r=r_2+1}^{\nu} J(z_r, x_n, n_r) \\ \in\ & \mathrm{cl}(\mathcal{U} + \mathcal{K})(D_{K_1} \oplus \bigoplus_{r=r_1+1}^{\nu} J(z_r, x_n, n_r)) \end{aligned}$$

by Corollary 5.4, and therefore

$$\begin{aligned} D &= \lim_{r_2 \to \infty} D_{K_1} \oplus \bigoplus_{r=r_1+1}^{r_2} J(z_r, 0, n_r) \oplus \bigoplus_{r=r_2+1}^{\nu} J(z_r, x_n, n_r) \\ &\in \mathrm{cl}(\mathcal{U} + \mathcal{K})(D_{K_1} \oplus \bigoplus_{r=r_1+1}^{\nu} J(z_r, x_n, n_r)) \end{aligned}$$

Denote

$$M_2 = D_{K_0} \oplus \bigoplus_{i=1}^{n} M^{(i)} \oplus \bigoplus_{r=r_1+1}^{\nu} J(z_r, 0, n_r).$$

We shall now consider the operator T_2. We know that $\sigma_e(T_2) = \sigma_e(T_1) = \sigma_e(T_0) = \sigma_e(M_0) = \sigma_e(M_1) = \sigma_e(M_2)$ and, for $\lambda \notin \sigma_0(T_2)$, $\mathrm{ind}(T_2 - \lambda) = \mathrm{ind}(T_1 - \lambda) = \mathrm{ind}(M_2 -$

λ). Both T_2 and M_2 are essentially normal and hence, by the Brown-Douglas-Fillmore Theorem, there exists a compact operator L_1 and a unitary operator U such that

$$T_2 = UM_2U^* + L_1 = U(M_2 + L)U^*,$$

where $L = U^*L_1U$ is also compact.

Let $\{e_0', e_1', \dots\}$ be a basis of the underlying space of the operator D with respect to which D is diagonal. For each $i = 1, \dots n$, use Lemma 2.6 to construct a basis of $H^2(\Omega_i, \mu_i)$ with respect to which the matrix of $M(\Omega_i, \mu_i)$ is a Toeplitz matrix. Denote the k-th element of this basis as e_{ijk}', where the index $j = 1, 2, \dots, j_i$ means that we regard this element as an element of the j-th copy of $H^2(\Omega_i, \mu_i)$ in the direct sum $l^2(\mathbb{N}) \oplus \bigoplus_{i=1}^n (\bigoplus_{j=1}^{j_i} H^2(\Omega_i, \mu_i))$.

We shall now construct a basis $\{e_k\}_{k=1}^{\infty}$ of the underlying space of M_2 as follows: Set $k_0 = 1 + \sum_{i=1}^n j_i$ and let now

$$\begin{aligned}
e_k &= e_l', \text{ for } k = l.k_0 + 1, l = 0, 1, \dots, \\
e_{k+j} &= e_{1,j,l}', \text{ for } k = l.k_0 + 1, l = 0, 1, \dots, j = 1, 2, \dots, j_1, \\
e_{k+j_1+j} &= e_{2,j,l}', \text{ for } k = l.k_0 + 1, l = 0, 1, \dots, j = 1, 2, \dots, j_2, \\
&\vdots \\
e_{k+j_1+j_2+\dots+j_{n-1}+j} &= e_{n,j,l}', \text{ for } k = l.k_0 + 1, l = 0, 1, \dots, j = 1, 2, \dots, j_n.
\end{aligned}$$

Let P_k be the orthogonal projection onto $\operatorname{span}\{e_i\}_{i=0}^k$, $k = 0, 1, \dots$. Since L is compact, the sequence $F_i = P_iKP_i$ converges to L in the norm Now

$$M_2 + F_m \cong \begin{pmatrix}
\gamma_1 & \dots & & & & & \\
& \gamma_2 & \gamma_{ij} & & & & T^+ \\
& & \ddots & & & & \\
& & & \gamma_m & & & \\
& & & & \operatorname{diag}\{\gamma_i\}_{i>m} & & \\
& T^- & & & & M^- & \\
& & & & & & M^+
\end{pmatrix},$$

Here we have rewritten $\bigoplus_{i=1}^{n} M^{(i)}$ as $M^{-} \oplus M^{+}$, where

$$\begin{aligned} M^{-} &= \bigoplus\{M^{(i)} : j_i < 0\} \\ M^{-} &= \bigoplus\{M^{(i)} : j_i > 0\}. \end{aligned}$$

By choosing m large enough, we make sure that

$$\|(M_2 + F_m) - (M_2 + L)\| < \epsilon$$

and

$$\sigma(M_2 + F_m) \subseteq (\sigma(M_2 + L))_\epsilon = (\sigma(T_2))_\epsilon.$$

We have of course $\{\gamma_i\}_{i=1}^{\infty} \subseteq \sigma(M_2 + F_m) = (\sigma(T_2))_\epsilon$. Also, any accumulation points of $\{\gamma_i\}_{i=1}^{\infty}$ lie in $\sigma_e(T_2) = \sigma_e(M_2)$. We can now choose an $m_0 > m$ such that $\gamma_i \in (\sigma_e(M_2))_\epsilon$ for $i > m_0$. For each $i > m_0$, choose an $\gamma_i' \in \sigma_e(M_2)$ such that $|\gamma_i - \gamma_i'| < \epsilon$. Whenever $\gamma_i \in \sigma_e(M_2)$, let $\gamma_i' = \gamma_i$.

As for $i = 1, 2, \ldots, m_0$, we have either $\gamma_i \in \bigcup_{j=1}^{N} \tau_j \cup \bigcup_{i=1}^{n} \Omega_j$ or $\operatorname{dist}(\gamma_i, \sigma_{acc}(M_2)) < 2\epsilon$. If $\gamma_i \in \bigcup_{j=1}^{N} \tau_j \cup \bigcup_{i=1}^{n} \Omega_j$, let $\gamma_i' = \gamma_i$, otherwise choose $\gamma_i' \in \sigma_{acc}(M_2)$ such that $|\gamma_i - \gamma_i'| < 2\epsilon$.

Setting

$$T_3 = \begin{pmatrix} \gamma_1' & \cdots & & & & & \\ & \gamma_2' & \gamma_{ij} & & & & T^{+} \\ & & \ddots & & & & \\ & & & \gamma_m' & & & \\ & & & & \operatorname{diag}\{\gamma_i'\}_{i>m} & & \\ & T^{-} & & & & M^{-} & \\ & & & & & & M^{+} \end{pmatrix}.$$

we have

$$T_2 \cong M_2 + L \cong_\epsilon M_2 + F_m \cong_{2\epsilon} T_3.$$

Let $K_2' = 0 \oplus K_2$ with respect to the decomposition

$$[\bigoplus_{s=1}^{p} \mathcal{H}(\beta_s; M_0)] \oplus [\bigoplus_{s=1}^{p} \mathcal{H}(\beta_s; M_0)]^{\perp}.$$

Then $||K_2'|| = ||K_2|| < \epsilon_2$ and

$$T_0 - K_2' = \begin{pmatrix} T_{01} & \dots & & & B_{10} \\ & T_{02} & & \beta_{ij} & B_{20} \\ & & \ddots & & \vdots \\ & & & T_{0p} & Bp0 \\ & & & & T_1 - K_2 \end{pmatrix}$$

$$= \begin{pmatrix} T_{01} & \dots & & & C_{11} & \dots & C_{1\kappa} & C_{10} \\ & T_{02} & & \beta_{ij} & C_{21} & \dots & C_{2\kappa} & C_{20} \\ & & \ddots & & \vdots & \dots & \vdots & \vdots \\ & & & T_{0p} & C_{p1} & \dots & C_{p\kappa} & C_{p0} \\ & & & & N_1(\tau_1) & & & W_{10} \\ & & & & & \ddots & W_{ij} & \vdots \\ & & & & & & N_{\kappa_N}(\tau_N) & W_{\kappa 0} \\ & & & & & & & T_2 \end{pmatrix}.$$

Conjugating by $I \oplus U$, we get

$$T_0 \cong_\epsilon T_0 - K_2' \cong \begin{pmatrix} T_{01} & \dots & & & C_{11} & \dots & C_{1\kappa} & C_{10}U \\ & T_{02} & & \beta_{ij} & C_{21} & \dots & C_{2\kappa} & C_{20}U \\ & & \ddots & & \vdots & \dots & \vdots & \vdots \\ & & & T_{0p} & C_{p1} & \dots & C_{p\kappa} & C_{p0}U \\ & & & & N_1(\tau_1) & & & W_{10}U \\ & & & & & \ddots & W_{ij} & \vdots \\ & & & & & & N_{\kappa_N}(\tau_N) & W_{\kappa 0}U \\ & & & & & & & U^*T_2U \end{pmatrix}$$

$$\cong_{3\epsilon} \begin{pmatrix} T_{01} & \dots & & & C_{11} & \dots & C_{1\kappa} & C_{10}U \\ & T_{02} & & \beta_{ij} & C_{21} & \dots & C_{2\kappa} & C_{20}U \\ & & \ddots & & \vdots & \dots & \vdots & \vdots \\ & & & T_{0p} & C_{p1} & \dots & C_{p\kappa} & C_{p0}U \\ & & & & N_1(\tau_1) & & & W_{10}U \\ & & & & & \ddots & W_{ij} & \vdots \\ & & & & & & N_{\kappa_N}(\tau_N) & W_{\kappa 0}U \\ & & & & & & & T_3 \end{pmatrix}$$

Next we use the compactness of the operators $C_{i0}U$, $1 \le i \le p$, and $W_{i0}U$, $1 \le i \le \kappa$ to approximate each of them individually to within $\epsilon/(\kappa + p)$ by finite rank operators

$C'_{i0} = C_{i0}UP_r$ and $W'_{i0} = W_{i0}UP_r$ for some large $r > m_0$. r independent of i.

Then we have

$$T_0 \cong_{5\epsilon} \begin{pmatrix} T_{01} & \cdots & & C_{11} & \cdots & C_{1\kappa} & C'_{10} & 0 & 0 \\ & \ddots & \beta_{ij} & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots \\ & & T_{0p} & C_{p1} & \cdots & C_{p\kappa} & C'_{p0} & 0 & 0 \\ & & & N_1(\tau_1) & & & W'_{10} & 0 & 0 \\ & & & & \ddots & W_{ij} & \vdots & \vdots & \vdots \\ & & & & & N_{\kappa_N}(\tau_N) & W'_{\kappa 0} & 0 & 0 \\ & & & & & & \tilde{F} & & \tilde{T}^- \\ & & & & & & \tilde{T}^- & M^- & \\ & & & & & & & & M^+ \end{pmatrix} \oplus D_r.$$

where $D_r = (I - P_r)\text{diag}\{\gamma_i\}_{i>m}(I - P_r)$ is a direct summand of $\text{diag}\{\gamma_i\}_{i>m}$ and $\tilde{F}$ acts on a finite dimensional space.

Note that the eigenvalues of $\tilde{F}$ lie in

$$\bigcup_{j=1}^{N} \tau_j \cup \bigcup_{j=1}^{n} \Omega_j \cup (\sigma_{acc}(M_2))_{\epsilon_2}.$$

Denote

$$\tilde{M} = \begin{pmatrix} \tilde{F} & & \tilde{T}^- \\ \tilde{T}^- & M^- & \\ & & M^+ \end{pmatrix}.$$

One can verify that $\sigma(\tilde{M}) = \sigma(\tilde{F}) \cup \text{cl}(\Omega)$ and we can decompose $\tilde{M}$ as

$$\tilde{M} = \begin{pmatrix} M_\tau & T_a \\ & F_a \end{pmatrix},$$

where F_a acts on a finite-dimensional space and

$$\begin{aligned} \sigma(F_a) &= \sigma(F) \setminus \text{cl}(\bigcup_{j=1}^{N} \tau_j \cup \bigcup_{j=1}^{n} \Omega_j) \subseteq (\sigma_{acc}(M_2))_{\epsilon_2} \\ \sigma(M_\tau) &= \text{cl}(\Omega) \cup (\sigma(\tilde{F}) \cap \text{cl}(\bigcup_{j=1}^{N} \tau_j)) \end{aligned}$$

We have

$$T_0 \cong_{5\epsilon} \begin{pmatrix} T_{01} & \dots & & C_{11} & \dots & C_{1\kappa} & C''_{10} & C'''_{10} \\ & \ddots & \beta_{ij} & \vdots & \dots & \vdots & \vdots & \vdots \\ & & T_{0p} & C_{p1} & \dots & C_{p\kappa} & C''_{p0} & C'''_{p0} \\ & & & N_1(\tau_1) & & & W''_{10} & W'''_{10} \\ & & & & \ddots & W_{ij} & \vdots & \vdots \\ & & & & & N_{\kappa_N}(\tau_N) & W''_{\kappa 0} & W'''_{\kappa 0} \\ & & & & & & M_\tau & T_a \\ & & & & & & & F_a \end{pmatrix} \oplus D_\tau.$$

Step Five: Rebuilding T_0 from M_0. We shall now put together the facts assembled in steps two to four to show that T_0 is close to $(\mathcal{U}+\mathcal{K})(M_0)$.

We know that for $1 \le s \le p$, we have $T_{0s} \in \mathrm{cl}(\mathcal{U}+\mathcal{K})(M_{0s})$. Using this fact and Corollary 5.2, we see that

$$\begin{pmatrix} T_{01} & \dots & & \\ & T_{02} & & \beta_{ij} \\ & & \ddots & \\ & & & T_{0p} \end{pmatrix} \in \mathrm{cl}(\mathcal{U}+\mathcal{K})(\bigoplus_{s=1}^{p} M_{0s}) = \mathrm{cl}(\mathcal{U}+\mathcal{K})(\bigoplus_{r=r_0+1}^{r_1} J(z_r, x_r, n_r)).$$

We can now find R_1 such that

$$\left\| \begin{pmatrix} T_{01} & \dots & & \\ & T_{02} & & \beta_{ij} \\ & & \ddots & \\ & & & T_{0p} \end{pmatrix} - R_1^{-1}(\bigoplus_{r=r_0+1}^{r_1} J(z_r, x_r, n_r))R_1 \right\| < \epsilon.$$

Let next D_τ be a diagonal operator with $\sigma(D_\tau) = \sigma_e(D_\tau) = \bigcup_{j=1}^{N} \partial(\mathrm{cl}(\tau_j))$. Theorem 6.2 implies that there exists an $R_2 \in (\mathcal{U}+\mathcal{K})$ such that

$$\left\| \begin{pmatrix} N_1(\tau_1) & \dots & & W''_{10} \\ & \ddots & W_{ij} & \vdots \\ & & N_{\kappa_N}(\tau_N) & W''_{\kappa 0} \\ & & & M_\tau \end{pmatrix} - R_2^{-1}(D_\tau \oplus \bigoplus_{i=1}^{n} M^{(i)})R_2 \right\| < \epsilon.$$

one can easily verify that the spectral conditions needed here are satisfied.

Next, since $\sigma(F_a) \subseteq (\sigma_{acc}(M_2))_{\epsilon_2}$, we can find F_a' such that

(i) The eigenvalues of F_a' are simple.
(ii) $||F_a' - F_a|| < 2\epsilon_2 < \epsilon$.
(iii) $\sigma(F_a') \cap \{\beta_j\}_{j=1}^{p} = \emptyset$.
(iv) $\sigma(F_a') \cap \mathrm{cl}(\bigcup_{j=1}^{N} \tau_j) = \emptyset$.

We then know that F_a' is similar to a diagonal operator D_a, $F_a' = R_3^{-1} D_a R_3$.

Finally, to deal with D_r, let D_m be a diagonal operator such that

$$\sigma(D_m) = \sigma_e(D_m) \subseteq \bigcup_{i=1}^{n} \partial(\Omega_i)$$

and

$$\sigma_e(D \oplus D_m) = \sigma_e(D_r).$$

Then

$$\sigma(D \oplus D_m) = \sigma_e(D \oplus D_m) \cup \{z_r\}_{r_1+1}^{\nu}$$

and

$$\sigma(D_r) = \sigma_e(D \oplus D_m).$$

Since the Hausdorff distance $d_H(\sigma(D \oplus D_m), \sigma_e(D \oplus D_m)) < \epsilon$, it follows from **[Dav86]** that

$$\mathrm{dist}(D_r, \mathcal{U}(D \oplus D_m)) < \epsilon.$$

Choose R_4 unitary such that $||D_r - R_4^{-1}(D \oplus D_m)R_4|| < \epsilon$.

Summing up, and using Corollary 5.2, we have

$$T_0 \cong_{6\epsilon} \begin{pmatrix} R_1^{-1}(\bigoplus_{r=r_0+1}^{r_1} J(z_r, x_r, n_r))R_1 & \begin{matrix} C_{11} & \dots & C_{1\kappa} & C_{10}'' \\ \vdots & \dots & \vdots & \vdots \\ C_{p1} & \dots & C_{p\kappa} & C_{p0}'' \end{matrix} & \begin{matrix} C_{10}''' \\ \vdots \\ C_{p0}''' \end{matrix} \\ & R_2^{-1}(D_r \oplus \bigoplus_{i=1}^{n} M^{(i)})R_2 & \begin{matrix} W_{10}''' \\ \vdots \\ W_{\kappa 0}''' \end{matrix} \\ & & T_a \\ & & R_3^{-1} D_a R_3 \end{pmatrix}$$

$$\oplus R_4^{-1}(D \oplus D_m)R_4$$

$$\cong_{(\mathcal{U}+\mathcal{K})} \left(\bigoplus_{r=r_0+1}^{r_1} J(z_r, x_r, n_r) \right) \oplus \left(D_\tau \oplus \bigoplus_{i=1}^{n} M^{(i)} \right) \oplus D_a \oplus D \oplus D_m$$

$$= \left(\bigoplus_{r=r_0+1}^{r_1} J(z_r, x_r, n_r) \right) \oplus \left(D_\tau \oplus \bigoplus_{i=1}^{n} M^{(i)} \right) \oplus D_a \oplus D_{K_1} \oplus \bigoplus_{r=r_1+1}^{\nu} J(z_r, 0, n_r) \oplus D_m$$

$$\leftarrow_{(\mathcal{U}+\mathcal{K})} \left(\bigoplus_{r=r_0+1}^{r_1} J(z_r, x_r, n_r) \right) \oplus \left(D_\tau \oplus \bigoplus_{i=1}^{n} M^{(i)} \right) D_a \oplus D_{K_1} \oplus$$

$$\bigoplus_{r=r_1+1}^{\nu} J(z_r, x_r, n_r) \oplus D_m$$

$$\cong M_0 \oplus D_\tau \oplus D_a \oplus D_m$$

$$\cong_a M_0.$$

i.e. $\mathrm{dist}(T_0, (\mathcal{U}+\mathcal{K})(M_0)) < 6\epsilon$. As we saw in step one, this is what is need.

□

CHAPTER 8

Connected domain which is not simply connected

We shall now continue the investigation of the following model: we assume that $\Omega = \Omega_1 \setminus \mathrm{cl}(\Omega_2)$, where Ω_1 is a simply connected analytic Cauchy domain, Ω_2 is an analytic Cauchy domain consisting of n simply connected components, $\Omega_2 = \cup_1^n \Omega_{2,i}$, $\mathrm{cl}(\Omega_2) \subseteq \Omega_1$. Now let μ be a measure on $\partial(\Omega_1)$ and for $i = 1, 2, \ldots, n$, let μ_i be a measure on $\partial(\Omega_{2,i}^*)$; all these measures are assumed to be equivalent to the respective arc length measures. Let $A = M(\Omega_1, \mu)$ and let $B = \oplus_{i=1}^n M(\Omega_{2,i}^*, \mu_i)$. Recall that 1_{Ω_1} and $1_{\Omega_2^*}$ are constant functions equal to 1 on their respective domains Ω_1 and Ω_2^*.

Let $C = 1_{\Omega_1} \otimes 1_{\Omega_2^*}^*$. Define an operator M on $H^2(\Omega_1, \mu) \oplus (\bigoplus_{i=1}^n H^2(\Omega_{2,i}^*, \mu_i))$ by

$$M = \begin{pmatrix} A & C \\ 0 & B^* \end{pmatrix}.$$

As we have noted in Chapter 1, the spectral properties of M are :

(i) $\sigma(M) = \mathrm{cl}(\Omega)$;
(ii) $\sigma_e(M) = \partial\Omega$;
(iii) $\mathrm{ind}(M - z) = -1$, $z \in \Omega$;
(iv) $\min\mathrm{ind}(M - z) = 0$, $z \in \Omega$;

We can now resume investigating which operators of the form $\begin{pmatrix} A & D \\ 0 & B^* \end{pmatrix}$ are in $\mathrm{cl}(\mathcal{U} + \mathcal{K})(M)$. The following is an easy corollary of Lemma 4.1.

8.1. Lemma. *Let $M = \begin{pmatrix} A & C \\ 0 & B^* \end{pmatrix}$ be as constructed above. Let $D = 1_{\Omega_1} \otimes d^*$, where $d \in H^2(\Omega_2^*)$ and set*

$$X = \begin{pmatrix} A & D \\ 0 & B^* \end{pmatrix}.$$

Then we have (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii), where

(i) $d(z) \neq 0$ *for* $z \in \Omega_2^*$,
(ii) $\sigma(X) \cap \Omega_2 = \emptyset$.

(iii) $X \in \mathrm{cl}(\mathcal{U}+\mathcal{K})(M)$.

Proof. The equivalence $(i) \Leftrightarrow (ii)$ follows from Lemma 1.3.

Assume that d satisfies (i). Lemma 1.1 allows us to find a $d' \in \mathcal{C}(\mathrm{cl}(\Omega_2^*))$ such that $d'(z) \neq 0$ for $z \in \mathrm{cl}(\Omega_2^*) \cap H^2(\Omega_2^*)$ and $\|d-d'\|_{H^2(\Omega_2^*)}$ can be made arbitrarily small. (This construction is to be done on each component of Ω_2^* separately.)

By Lemma 4.1, we have

$$\begin{pmatrix} A & 1_{\Omega_1} \otimes d'^* \\ 0 & B^* \end{pmatrix} \in \mathrm{cl}(\mathcal{U}+\mathcal{K})(M),$$

and hence $X \in \mathrm{cl}(\mathcal{U}+\mathcal{K})(M)$.

□

8.2. Lemma. *Let $k \in \mathbb{N}$. Denote by $\mathcal{F}_k$ the subspace of $H^2(\Omega_1)$ spanned by $1, z, z^2, \ldots, z^k$. Let X be an operator of the form*

$$X = \begin{pmatrix} A & F \\ 0 & B^* \end{pmatrix},$$

where F is a finite rank operator with $\mathrm{ran}\, F \subseteq \mathcal{F}_k$. *Then X is $(\mathcal{U}+\mathcal{K})$-equivalent to an operator of the form*

$$\begin{pmatrix} A & D \\ 0 & B^* \end{pmatrix},$$

where $D = 1_{\Omega_1} \otimes d^*$, for some $d \in H^2(\Omega_2^*)$. This $(\mathcal{U}+\mathcal{K})$-equivalence is of the form

$$X = \begin{pmatrix} I & Z \\ 0 & I \end{pmatrix} \begin{pmatrix} A & D \\ 0 & B^* \end{pmatrix} \begin{pmatrix} I & -Z \\ 0 & I \end{pmatrix},$$

where Z is a finite-rank operator.

Moreover, d can be calculated as $d = \mathcal{G}(F)$, where $\mathcal{G}$ is the linear map from

$$\{Y \in \mathcal{B}(H^2(\Omega_2^*), H^2(\Omega_1)) : \mathrm{ran}\, Y \subseteq \mathcal{F}_k \text{ for some } k\}$$

into $H^\infty(\Omega_2^*)$ such that

$$\mathcal{G}(p \otimes g^*) = \theta(p) g,$$

where p is a polynomial on Ω_1. $g \in H^2(\Omega_2^*)$ and θ is a bounded map from $H^2(\Omega_1)$ into $H^\infty(\Omega_2^*)$ defined by

$$\theta(f)(z) = \overline{f(\overline{z})}, z \in \Omega_2^*.$$

Proof. Observe that

$$\begin{pmatrix} I & Z \\ 0 & I \end{pmatrix} \begin{pmatrix} A & F \\ 0 & B^* \end{pmatrix} \begin{pmatrix} I & -Z \\ 0 & I \end{pmatrix} = \begin{pmatrix} A & F + ZB^* - AZ \\ 0 & B^* \end{pmatrix}$$

We will find a compact Z such that $G = F + ZB^* - AZ$ satisfies ran $G \subseteq F_{k-1}$ (for $k > 1$). The lemma will then follow by induction. (Note that the expression $ZB^* - AZ$ is linear in the variable Z.)

Let g_0 be such that $F - z^n \otimes g_0^* \in \mathcal{F}_{k-1}$ and let $Z = z^{n-1} \otimes g_0^*$. Then

$$\begin{aligned} (F + ZB^* - AZ)g &= Fg + \langle B^* g, g_0 \rangle z^{n-1} - \langle g, g_0 \rangle A z^{n-1} \\ &= (Fg - \langle g, g_0 \rangle z^n) + \langle g, z.g_0 \rangle z^{n-1} \\ &\in F_{k-1}. \end{aligned}$$

The fact that $d = \mathcal{G}(F)$ can be verified by an easy calculation. The boundedness of θ follows from the Cauchy theorem.

□

8.3. Lemma. *Suppose $f \in H^2(\Omega_1)$, $g \in H^2(\Omega_2^*)$. Define $g_0 \in H^2(\Omega_2^*)$ by $g_0 = \theta(f)g$, where θ is the map introduced in Lemma 8.2. Then for every $\epsilon > 0$ there exists a finite-rank operator $X : H^2(\Omega_2^*) \to H^2(\Omega_1)$ for which*

$$\|f \otimes g^* - XB^* + AX - 1_{\Omega_1} \otimes g_0^*\| < \epsilon.$$

In particular, we have

$$\begin{aligned} \begin{pmatrix} A & f \otimes g^* \\ 0 & B^* \end{pmatrix} &\in \mathrm{cl}(\mathcal{U} + \mathcal{K})\left(\begin{pmatrix} A & 1_{\Omega_1} \otimes g_0^* \\ 0 & B^* \end{pmatrix}\right), \\ \begin{pmatrix} A & 1_{\Omega_1} \otimes g_0^* \\ 0 & B^* \end{pmatrix} &\in \mathrm{cl}(\mathcal{U} + \mathcal{K})\left(\begin{pmatrix} A & f \otimes g^* \\ 0 & B^* \end{pmatrix}\right). \end{aligned}$$

Proof. Let $\{f_k\}_{k=1}^{\infty}$ be a sequence of polynomials such that $f_k \to f$ in $H^2(\Omega_1)$. Then by Lemma 8.2, there exist a finite-rank $Z_k : H^2(\Omega_2^*) \to H^2(\Omega_1)$ and $g_k \in H^2(\Omega_2^*)$ such that

$$f_n \otimes g^* - Z_k B^* + AZ_k = 1_{\Omega_1} \otimes g_k^*.$$

for $k = 1, 2, \ldots$.

Recall that $g_k = \theta(f_k).g$. We have $f_k \to f$ in $H^2(\Omega_1)$, hence $\theta(f_k) \to \theta(f)$ in $H^\infty(\Omega_2^*)$ and $g_k = \theta(f_k).g \to g_0 = \theta(f).g$ in $H^2(\Omega_2^*)$.

With $\epsilon > 0$ given, choose k such that $||g_k - g_0|| < \epsilon/2$, $||f_k - f|| < \epsilon/2||g||$ and set $X = Z_k$. Then we have

$$\begin{aligned} & ||f \otimes g^* - XB^* + AX - 1_{\Omega_1} \otimes g_0^*|| \\ = \; & ||(f_k \otimes g^* - Z_k B^* + AZ_k - 1_{\Omega_1} \otimes g_0^*) + (f - f_k) \otimes g^* + 1_{\Omega_1} \otimes (g - g_0)^*|| \\ = \; & ||0 + (f - f_k) \otimes g^* + 1_{\Omega_1} \otimes (g - g_0)^*|| < \epsilon \end{aligned}$$

The second statement follows easily using $(\mathcal{U} + \mathcal{K})$ similarities of the form $\begin{pmatrix} I & X \\ 0 & I \end{pmatrix}$.

□

8.4. Corollary. *Let $W \in \mathcal{B}(H^2(\Omega_2^*), H^2(\Omega_1))$ be finite-rank, say $W = \sum_{i=1}^k f_i \otimes g_i^*$. Set $g_0 = \sum_{i=1}^k \theta(f_i).g_i$. Then for every $\epsilon > 0$ there exists a finite-dimensional operator $X : H^2(\Omega_2^*) \to H^2(\Omega_1)$ for which*

$$||W - XB^* + AX - 1_{\Omega_1} \otimes g_0^*|| < \epsilon.$$

Proof. This follows from the linearity of the expression $XB^* - AX$ in X.

□

8.5. Lemma. *Let $W \in \mathcal{B}(H^2(\Omega_2^*), H^2(\Omega_1))$ be compact. Then there exists a $g \in$*

$H^2(\Omega_2^*)$ *such that*

$$\begin{pmatrix} A & W \\ 0 & B^* \end{pmatrix} \in \text{cl}(\mathcal{U}+\mathcal{K})(\begin{pmatrix} A & 1_{\Omega_1}\otimes g^* \\ 0 & B^* \end{pmatrix})$$
$$\begin{pmatrix} A & 1_{\Omega_1}\otimes g^* \\ 0 & B^* \end{pmatrix} \in \text{cl}(\mathcal{U}+\mathcal{K})(\begin{pmatrix} A & W \\ 0 & B^* \end{pmatrix})$$

Proof. Choose $\epsilon > 0$. Choose $z_0 \in \Omega_1$ arbitrarily. Let $\psi \in H^2(\Omega_1)$ be the function constructed in Lemma 2.1. Let $\{e_k\}_{k=1}^{\infty}$. $e_n = \psi^n.e_0$ be the basis of $H^2(\Omega_1)$ constructed in Lemma 2.6. Denote $r = \max\{|\psi(z)| : z \in \text{cl}(\Omega_2)\}$ and note that. by the maximum modulus principle, we have $r < 1$.

Write W as
$$W = \sum_{i=0}^{\infty} e_i \otimes h_i^*$$
and let
$$W_k = \sum_{i=0}^{k} e_i \otimes h_i^*, k = 0, 1, \ldots .$$
Then by Corollary 8.4, if we define $g_k \in H^2(\Omega_2^*)$ by $g_k = \sum_{i=0}^{k} \theta(e_i).h_i$, one can find a finite dimensional $X_k : H^2(\Omega_2^*) \to H^2(\Omega_1)$ such that

$$||W_k - X_k B^* + AX_k - a \otimes g_k^*|| < \epsilon.$$

Observe that
$$||\theta(e_i)||_{H^\infty(\Omega_2^*)} \leq M||\psi^i|_{\Omega_2}||_\infty = Mr^i, \quad M = \max_{\text{cl}(\Omega_2^*)}\{e_0\}$$
and hence
$$||\sum_{i=k}^{\infty} \theta(e_i).h_i|| \leq \sum_{i=k}^{\infty} Mr^i ||W|| = M||W||\frac{r^k}{1-r}.$$
Therefore the sequence g_k has a limit in $H^2(\Omega_2^*)$. We shall call it g.

We can now choose k_0 such that $||W - W_{k_0}|| < \epsilon$ and $M\frac{r^{k_0+1}}{1-r} < 1$. Then we have $||h_i|| < ||W - W_{k_0}|| < \epsilon, i > k_0$ and hence

$$||g - g_{k_0}|| \leq \sum_{i=k_0+1}^{\infty} Mr^i\epsilon = \epsilon M\frac{r^{k_0+1}}{1-r} < \epsilon.$$

We have

$$\begin{aligned} & \|W - X_{k_0}B^* + AX_{n_0} - 1_{\Omega_1} \otimes g^*\| \\ \leq\ & \|W_{k_0} - X_{k_0}B^* + AX_{n_0} - 1_{\Omega_1} \otimes g^*_{k_0}\| + \|W - W_{k_0}\| + \|g - g_{k_0}\| \\ \leq\ & \epsilon + \epsilon + \epsilon = 3\epsilon. \end{aligned}$$

Both statements of the lemma now follow easily using $(\mathcal{U} + \mathcal{K})$ similarities of the form $\begin{pmatrix} I & X_{k_0} \\ 0 & I \end{pmatrix}$.

□

Lemmas 8.5 and 8.1 together give

8.6. Corollary. *Let* $X = \begin{pmatrix} A & W \\ 0 & B^* \end{pmatrix}$, *$W$ compact. Suppose that* $\sigma(X) = \sigma(M) = \mathrm{cl}(\Omega_1) \setminus \Omega_2$. *Then* $X \in \mathrm{cl}(\mathcal{U} + \mathcal{K})(M)$.

We shall now restrict the class of models we are investigating. This will also restrict the class of spectral pictures. We will subsequently use functional calculus (the techniques developed in Chapter 3) to get back to the original class of spectral pictures.

8.7. Theorem. *Assume that in the model* $M = \begin{pmatrix} A & C \\ 0 & B^* \end{pmatrix}$ *constructed above we have* $\Omega_1 = \mathbb{D}$ *and μ is the arc length measure. In other words, we have* $A \cong S$, *where S is the forward unilateral shift. Let X be an essentially normal operator such that*

(i) $\sigma(X) = \sigma(M) = \mathrm{cl}(\Omega_1) \setminus \Omega_2$,
(ii) $\sigma_e(X) = \sigma_e(M) = \partial\Omega_1 \cup \partial\Omega_2$,
(iii) $\mathrm{ind}(X - \lambda) = \mathrm{ind}(M - \lambda) = -1$ *for* $\lambda \in \Omega_1 \setminus \mathrm{cl}(\Omega_2)$.

Then $X \in \mathrm{cl}(\mathcal{U} + \mathcal{K})(M)$.

Proof. As in the proof of Theorems 2.7, 4.6 and 6.2, we can again use the Brown-Douglas-Fillmore Theorem to see that we may assume without loss of generality that $X = M + K$, where K is compact.

Fix $\epsilon > 0$. Let $\{1, z, z^2, \dots\}$ be the canonical basis of $H^2(\Omega_1, \mu) = H^2(\mathbb{D})$. For each $i = 1, \dots n$, use Lemma 2.6 to construct a basis of $H^2(\Omega_{2,i}, \mu_i)$ with respect to which the matrix of $M(\Omega_{2,i}, \mu_i)$ is a Toeplitz matrix. Denote the k-th element of this basis as e'_{ik}.

We shall now construct a basis $\{e_k\}_{k=1}^{\infty}$ of $H^2(\Omega_1, \mu) \oplus (\bigoplus_{i=1}^{n} H^2(\Omega_{2,i}^*, \mu_i))$ as follows:

$$\begin{aligned} e_k &= z^l, \text{ for } k = l(n+1)+1, l = 0, 1, \ldots, \\ e_k &= e'_{j,l}, \text{ for } k = l(n+1)+j, l = 0, 1, \ldots, j = 1, 2, \ldots, n. \end{aligned}$$

Let P_k be the orthogonal projection onto $\text{span}\{e_i\}_{i=0}^{k}$, $k = 0, 1, \ldots$. Since K is compact, the sequence $\{M + P_k K P_k\}_{k=1}^{\infty}$ converges to $M + K$ in the norm. Find k_0 such that $||P_{k_0} K P_{k_0} - K|| < \epsilon$ and $\sigma(M + P_{k_0} K P_{k_0} - K) \subseteq (\sigma(X))_\epsilon$. Denote $K_0 = P_{k_0} K P_{k_0} - K$ and notice that $X_1 = M + K_0$ is of the form

$$\begin{pmatrix} A & C_1 & 0 \\ & F_1 & C_3 \\ & & B^* \end{pmatrix}.$$

Notice that the entries of the matrix of C_1 are zeros except for the bottom row, i.e. ran $C_1 \subseteq \text{span}\{1_{\Omega_1}\}$, if we consider C_1 as an operator from $\mathbb{C}^{2k_0}$ into $H^2(\mathbb{D})$. We want to show that X_1 is close to $(\mathcal{U} + \mathcal{K})(M)$.

Let F_2 be a perturbation of F_1 such that

- $||F_2 - F_1|| < \epsilon$,
- $\sigma(F_2) \subseteq \Omega_1$,
- the eigenvalues of F_2 are simple.

Let

$$X_2 = \begin{pmatrix} A & C_1 & 0 \\ & F_2 & C_3 \\ & & B^* \end{pmatrix}.$$

Then we have $||X_2 - X|| < 2\epsilon$ and $\sigma(X_2) \subseteq \text{cl}(\Omega_1)$.

The fact that F_2 has simple eigenvalues allows us now to use Lemma 2.4 to see that an arbitrarily small perturbation of C_1 to C_1' will cause $\begin{pmatrix} A & C_1' \\ & F_2 \end{pmatrix}$ to have no eigenvalues and to be $(\mathcal{U}+\mathcal{K})$ equivalent to A. Moreover, one can do this so that ran $(C_1 - C_1') \subseteq \text{span}\{1_{\Omega_1}\}$. Choose such a perturbation small enough so that in addition to this we have $||C_1 - C_1'||$ and the spectrum of

$$X_3 = \begin{pmatrix} A & C_1' & 0 \\ & F_2 & C_3 \\ & & B^* \end{pmatrix}$$

lies in $(\sigma(X))_\epsilon \cap \text{cl}(\Omega_1)$.

We now have the following situation:

(i) $||X_3 - X|| < 3\epsilon$.
(ii) $\sigma(X_3) \subseteq \mathrm{cl}(\Omega_1)$.
(iii) $\begin{pmatrix} A & C_1' \\ & F_2 \end{pmatrix}$ has no eigenvalues and is $(\mathcal{U}+\mathcal{K})$ equivalent to A.
(iv) X_3 may have eigenvalues in Ω_2. These are not more than ϵ away from $\partial\Omega_2$. There may be infinitely many of these. If this is the case, any cluster point of the set of eigenvalues will be in $\partial\Omega_2$ (because $\sigma_e(X_3) = \partial(\Omega_1) \cup \partial(\Omega_2)$).

In the next step, we want to find X_4 close to X_3 with the same properties except that there will be only finitely many eigenvalues in Ω_2.

Notice that condition (iii) implies that X_3 is $(\mathcal{U}+\mathcal{K})$ equivalent to an operator of the form $Y = \begin{pmatrix} A & L \\ & B^* \end{pmatrix}$, L compact, which of course has the same spectral properties as X_3. We want to show that there is is an X_4 of the form

$$X_4 = \begin{pmatrix} A & C_1' & 0 \\ & F_2 & C_3' \\ & & B^* \end{pmatrix}$$

such that $||X_3 - X_4|| < \epsilon$, and X_4 has the desired spectral properties. This will easily follow once we prove:

Claim: Let $Y = \begin{pmatrix} A & F \\ & B^* \end{pmatrix}$, F finite-dimensional. Suppose that Y has the spectral properties described above for X_3. Let $\eta > 0$. Then there exists a a F' such that $Y' = \begin{pmatrix} A & F' \\ & B^* \end{pmatrix}$ has only finitely many eigenvalues in Ω_2 and $||F - F'|| < \eta$, ran $F' \subseteq$ ran F.

Proof of the Claim: Suppose $F = \sum_{i=1}^{no} f_i \otimes g_i^*$, $g_i \in H^2(\Omega_2^*)$, $f_i \in H^2(\Omega_1)$. Using Runge's theorem and the definition of $H^2(\Omega_2^*)$, we can find polynomials g_i' such that for $F' = \sum_{i=1}^{no} f_i' \otimes g_i^*$ we have $||F - F'|| < \epsilon$ and moreover $||F - F'||$ is small enough so that $\Omega_2 \setminus \sigma(Y') \neq \emptyset$. From Lemmas 1.3 and 8.3, we know that the eigenvalues of Y' inside Ω_2 correspond to the zeros of $k = \sum_{i=1}^{no} \theta(f_i).g_i'$. This is (can be extended to) a holomorphic function on Ω_1. If k had infinitely many zeros in Ω_2, it would be a constant equal to zero, causing $\Omega_2 \setminus \sigma(Y') = \emptyset$, contradiction. This proves the claim, we can now resume the proof of Proposition 8.7.

We now have

$$X_4 = \begin{pmatrix} A & C_1' & 0 \\ & F_2 & C_3' \\ & & B^* \end{pmatrix}$$

with respect to $\mathcal{H} = H^2(\Omega_1) \oplus \mathbb{C}^{k_0} \oplus H^2(\Omega_2^*)$. We know that ran $C_1' \subseteq \mathrm{span}\{1_{\Omega_1}\}$, $||X_4 - X|| < 4\epsilon$, $\sigma(X_4) \subseteq \mathrm{cl}(\Omega_2)$, $\begin{pmatrix} A & C_1' \\ & F_2 \end{pmatrix}$ has no eigenvalues and is $(\mathcal{U}+\mathcal{K})$ equivalent to A.

X_4 may have eigenvalues in Ω_2. These are not further than ϵ away from $\partial\Omega_2$ and there are only finitely many of them. Suppose these eigenvalues are $\lambda_1, \lambda_2, \dots, \lambda_m$.

Denote $H_0 = \text{span}\{H(\lambda_1; X_4), H(\lambda_2; X_4), \dots, H(\lambda_m; X_4)\}$. If we knew that $H_0 \perp H^2(\Omega_1)$ and $H_0 \perp H^2(\Omega_2^*)$, i.e. H_0 is a subspace of the underlying space of F_2, we could move the eigenvalues λ_i away from Ω_2 by perturbing F_2. As we shall see next, it is true that $H_0 \perp H^2(\Omega_1)$ and $H_0 \perp H^2(\Omega_2^*)$ can be achieved by altering the decomposition of $\mathcal{H}$.

For each $i = 1, 2, \dots, m$, let n_i be such that $H(\lambda_i; X_4) \subseteq \ker(X_4 - \lambda_i)^{n_i}$. Note that for any λ and k, X_4^k is of the form

$$\begin{pmatrix} A^k & C_1'' & C_2'' \\ & F_2^k & C_3'' \\ & & B^{*k} \end{pmatrix}.$$

One can verify by induction that

$$\text{ran } \begin{pmatrix} C_1'' & C_2'' \end{pmatrix} \subseteq \text{span}\{1, (x-\lambda), \dots, (x-\lambda)^{k-1}\}.$$

If now $\begin{pmatrix} f \\ h \\ g \end{pmatrix}$ is in $\ker(X_4 - \lambda_i)^k$, we have

$$X_4 \begin{pmatrix} f \\ h \\ g \end{pmatrix} = \begin{pmatrix} A^k & C_1'' & C_2'' \\ & F_2^k & C_3'' \\ & & B^{*k} \end{pmatrix} \begin{pmatrix} f \\ h \\ g \end{pmatrix} = \begin{pmatrix} A^k f + C_1'' h + C_2'' g \\ F_2^k h + C_3'' g \\ B^{*k} g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Now $A^k f$ is linearly independent of ran $\begin{pmatrix} C_1'' & C_2'' \end{pmatrix}$, forcing $A^k f = 0$ and hence $f = 0$. This implies that $H_0 \perp H^2(\Omega_1)$.

Notice also that the $H^2(\Omega_2^*)$ component of any vector in $\ker(X_4 - \lambda_i)^k$ (denoted here by g) is in $\ker(B^* - \lambda_i)^k$.

Use Lemma 2.2 n_1 times to find vectors $\{f_1, f_2, \dots, f_{n_1}\}$ such that the matrix of B^* is

$$\begin{pmatrix} \lambda_1 & \dots & & & \\ & \lambda_1 & \dots & & \\ & & \ddots & & \\ & & & \lambda_1 & \dots \\ & & & & B^{*\prime} \end{pmatrix}$$

with respect to the decomposition $\text{span}\{f_1, f_2, \dots, f_{n_1}\} \oplus \text{span}\{f_1, f_2, \dots, f_{n_1}\}^\perp$, where $B^{*\prime}$ is a unitarily equivalent copy of B^*. Note that $\ker(B^* - \lambda_1)^{n_1} = \text{span}\{f_1, f_2, \dots, f_{n_1}\}$. We can now continue in this manner until we can write B^* as

$$\begin{pmatrix} \lambda_1 & \cdots & & & & & & & \\ & \lambda_1 & \cdots & & & & & & \\ & & \ddots & & & & & & \\ & & & \lambda_1 & \cdots & & & & \\ & & & & \lambda_2 & \cdots & & & \\ & & & & & \ddots & & & \\ & & & & & & \lambda_m & \cdots & \\ & & & & & & & & B^{*\prime\prime} \end{pmatrix}$$

with respect to the decomposition $\operatorname{span}\{f_1, f_2, \ldots, f_{n_0}\} \oplus \operatorname{span}\{f_1, f_2, \ldots, f_{n_0}\}^{\perp}$, where $n_0 = \sum_{i=1}^{m} n_i$ and $B^{*\prime\prime}$ is another unitarily equivalent copy of B^*. Note that $\ker(B^* - \lambda_i)^{n_i} \subseteq \operatorname{span}\{f_1, f_2, \ldots, f_{n_0}\}$ for $i = 1, 2, \ldots, m$ and consequently we have

$$H_0 \perp (H^2(\Omega_2^*) \ominus \operatorname{span}\{f_1, f_2, \ldots, f_{n_0}\}).$$

We can now rewrite X_4 as

$$X_4 = \begin{pmatrix} A & C_1' & & \\ & F_2 & C_{31}' & C_{32}' \\ & & B_1 & B_2 \\ & & & B^{*\prime\prime} \end{pmatrix} = \begin{pmatrix} A & C_1' & 0 \\ & F_3 & B_2' \\ & & B^{*\prime\prime} \end{pmatrix}$$

with respect to the decomposition

$$\begin{aligned} \mathcal{H} &= H^2(\Omega_1) \oplus \mathbb{C}^{k_0} \oplus \operatorname{span}\{f_1, f_2, \ldots, f_{n_0}\} \oplus H^2(\Omega_2^*)' \\ &\cong \mathcal{H} = H^2(\Omega_1) \oplus \mathbb{C}^{k_0 + n_0} \oplus H^2(\Omega_2^*)' \end{aligned}$$

where $H^2(\Omega_2^*)'$ is an isometrically isomorphic copy of $H^2(\Omega_2^*)$. We have now $H_1 \subseteq \mathbb{C}^{k_0} \oplus \operatorname{span}\{f_1, f_2, \ldots, f_{n_0}\} \cong \mathbb{C}^{k_0+n_0}$. We shall denote $B^{*\prime\prime}$ as B^* from now on.

For $i = 1, 2, \ldots, m$, let I_i be the identity operator on

$$\operatorname{span}_{j=1}^{i} H(\lambda_j; X_4) \ominus \operatorname{span}_{j=1}^{i-1} H(\lambda_j; X_4)$$

and let $\lambda_i' \in \Omega_1 \setminus cl(\Omega_2)$ be chosen so that $|\lambda_i - \lambda_i'| < \epsilon$. Set

$$X_5 = X_4 + \sum_{i=1}^{m}(\lambda_i' - \lambda_i)I_i.$$

Then X_5 is of the form

$$X_5 = \begin{pmatrix} A & C_1' & 0 \\ & F_4 & B_2' \\ & & B^{*''} \end{pmatrix}.$$

we have $\sigma(X_5) = \text{cl}(\Omega_1) \setminus \Omega_2$, $||X_5 - X|| < 5\epsilon$ and λ_i'. $i = 1, 2, \ldots, m$. are the only points with nul $(X_5 - \lambda_i) > 0$.

Next, we decompose the finite dimensional operator F_4 as

$$F_4 = \begin{pmatrix} F_5 & D_1 \\ & F_6 \end{pmatrix}.$$

where $\sigma(F_5) \subseteq \text{cl}(\Omega_2)$ and $\sigma(F_6) \cap \text{cl}(\Omega_2) = \emptyset$. We have

$$X_5 = \begin{pmatrix} A & D_2 & D_3 & \\ & F_5 & D_1 & D_4 \\ & & F_6 & D_5 \\ & & & B^* \end{pmatrix}$$

Since $\sigma(F_6) \cap \sigma(B^*) = \emptyset$, Lemma 5.1 allows us to find a Z such that

$$\begin{pmatrix} I & -Z \\ & I \end{pmatrix} \begin{pmatrix} F_6 & D_5 \\ & B^* \end{pmatrix} \begin{pmatrix} I & Z \\ & I \end{pmatrix} = \begin{pmatrix} F_6 & 0 \\ & B^* \end{pmatrix}.$$

The operator X_5 is now $(\mathcal{U} + \mathcal{K})$ equivalent to

$$X_6 = \begin{pmatrix} I & -Z \\ & I \end{pmatrix} X_5 \begin{pmatrix} I & Z \\ & I \end{pmatrix} = \begin{pmatrix} A & D_2 & D_3' & D_6' \\ & F_5 & D_1' & D_4' \\ & & F_6 & 0 \\ & & & B^* \end{pmatrix}.$$

Let $M_1 = || \begin{pmatrix} I & -Z \\ & I \end{pmatrix} ||.|| \begin{pmatrix} I & Z \\ & I \end{pmatrix} ||$. Recall that the only eigenvectors of X_5 were of the form $\begin{pmatrix} 0 \\ h_i' \\ g_i' \\ 0 \end{pmatrix}$ and hence the eigenvectors of X_6 will in fact be the same:

$$\begin{pmatrix} I & & & \\ & I & & \\ & & I & -Z \\ & & & I \end{pmatrix} \begin{pmatrix} 0 \\ h_i' \\ g_i' \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ h_i' \\ g_i' \\ 0 \end{pmatrix}.$$

Next, consider Lemma 2.4 and perturb D_3', D_1', F_6 so that

$$\begin{pmatrix} A & D_2 & E_1 \\ & F_5 & E_2 \\ & & F_7 \end{pmatrix} \cong_{\mathcal{U}+\mathcal{K}} A,$$

$$\left\| \begin{pmatrix} D_3' \\ D_1' \\ F_6 \end{pmatrix} - \begin{pmatrix} E_1 \\ E_2 \\ F_7 \end{pmatrix} \right\| < \epsilon/M_1.$$

and F_7 has simple eigenvalues in $\Omega_1 \setminus \mathrm{cl}(\Omega_2)$.

Then

$$X_7 = \begin{pmatrix} A & D_2 & E_1 & D_6' \\ & F_5 & E_2 & D_4' \\ & & F_7 & 0 \\ & & & B^* \end{pmatrix}$$

is $(\mathcal{U} + \mathcal{K})$ equivalent to an operator of the form

$$\begin{pmatrix} A & K_7 \\ & B^* \end{pmatrix}.$$

where K_7 is compact.

Let us check if X_7 has any eigenvalues in Ω_2. Suppose $\lambda \in \Omega_2$ and

$$(X_7 - \lambda) \begin{pmatrix} f \\ g \\ h \\ k \end{pmatrix} = \begin{pmatrix} A - \lambda & D_2 & E_1 & D_6' \\ & F_5 - \lambda & E_2 & D_4' \\ & & F_7 - \lambda & 0 \\ & & & B^* - \lambda \end{pmatrix} \begin{pmatrix} f \\ g \\ h \\ k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Then we have $(F_7 - \lambda)h = 0$ and hence $h = 0$. But then

$$(X_7 - \lambda)\begin{pmatrix} f \\ g \\ 0 \\ k \end{pmatrix} = (X_6 - \lambda)\begin{pmatrix} f \\ g \\ 0 \\ k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

which is a contradiction — X_6 does not have any such eigenvalues.

So X_7 has no eigenvalues in Ω_2 and by Corollary 8.6. we have $X_7 \in \mathrm{cl}(\mathcal{U} + \mathcal{K})(M)$. Hence

$$\mathrm{dist}(X_6, (\mathcal{U} + \mathcal{K})(M)) < \epsilon/M_1,$$

which implies

$$\mathrm{dist}(X_5, (\mathcal{U} + \mathcal{K})(M)) < \epsilon.$$

and hence

$$\mathrm{dist}(X, (\mathcal{U} + \mathcal{K})(M)) < 6\epsilon.$$

This last statement holds for any $\epsilon > 0$, so finally $X \in \mathrm{cl}(\mathcal{U} + \mathcal{K})(M)$.

□

8.8. Corollary. *Let $\Omega = \Omega_1 \setminus \mathrm{cl}(\Omega_2)$, where Ω_1 is a simply connected analytic Cauchy domain, Ω_2 is an analytic Cauchy domain consisting of n simply connected components, $\Omega_2 = \cup_1^n \Omega_{2,i}$, $\mathrm{cl}(\Omega_2) \subseteq \Omega_1$. Let ϕ be an invertible holomorphic map from a neighbourhood of $\mathbb{D}$ to $\mathbb{C}$ such that $\phi|_{\mathbb{D}}$ is a conformal map of $\mathbb{D}$ onto Ω_1. (This is the map which was used in Chapter 1 to construct $H^2(\Omega_1)$.) Then we have $\phi^{-1}(\Omega) = \mathbb{D} \setminus \mathrm{cl}(\Omega'_2)$, where Ω'_2 is an analytic Cauchy domain consisting of n simply connected components, $\Omega'_2 = \cup_1^n \Omega'_{2,i}$, $\mathrm{cl}(\Omega'_2) \subseteq \mathbb{D}$. Let now μ be a measure on $\partial(\mathbb{D})$ and for $i = 1, 2, \ldots, n$, let μ_i be a measure on $\partial(\Omega'^{*}_{2,i})$; all these measures are assumed to be equivalent to the respective arc length measures. Let now $M' = \begin{pmatrix} M(\mathbb{D}, \mu) & C \\ 0 & M(\Omega'^{*}_{2,i}) \end{pmatrix}$ be the model constructed at the beginning of this chapter. Then $M = \phi(M')$ has the following spectral properties.*

(i) $\sigma(M) = \mathrm{cl}(\Omega)$;
(ii) $\sigma_e(M) = \partial\Omega$;
(iii) nul $(M - z) = 0$, $z \in \Omega$;
(iv) nul $(M^* - \overline{z}) = 1$, $z \in \Omega$;
(v) $\mathrm{ind}(M - z) = -1$, $z \in \Omega$.

Let X be an essentially normal operator such that

(i) $\sigma(X) = \sigma(M) = \mathrm{cl}(\Omega_1) \setminus \Omega_2$,

(ii) $\sigma_e(X) = \sigma_e(M) = \partial\Omega_1 \cup \partial\Omega_2$.
(iii) $\text{ind}(X - \lambda) = \text{ind}(M - \lambda) = -1$ *for* $\lambda \in \Omega_1 \setminus \text{cl}(\Omega_2)$.

Then $X \in \text{cl}(\mathcal{U} + \mathcal{K})(M)$.

Proof. The proof is very similar to that of Theorem 3.5. Lemmas 3.3 and 3.4 allow us to verify that M has the spectral properties described in the Corollary. Note that M' is the type of operator for which Proposition 8.7 provides conditions that are sufficient for an operator to lie in $\text{cl}(\mathcal{U} + \mathcal{K})(M')$. Using Lemmas 3.3 and 3.4 again, we see that $\phi^{-1}(X)$ satisfies the conditions of Proposition 8.7 and hence $\phi^{-1}(X) \in \text{cl}(\mathcal{U} + \mathcal{K})(M')$. Now $X \in \text{cl}(\mathcal{U} + \mathcal{K})(\phi(M')) = \text{cl}(\mathcal{U} + \mathcal{K})(M)$ by Lemma 3.2.

□

8.9. Corollary. *Let* $M = \phi(M')$ *be as in 8.8. Then* $X \in \text{cl}(\mathcal{U} + \mathcal{K})(M)$ *if and only if*

(i) $\sigma(X) = \sigma(M) = \text{cl}(\Omega_1) \setminus \Omega_2$ *or* $\sigma(X) = \text{cl}(\Omega_1)$.
(ii) $\sigma_e(X) = \sigma_e(M) = \partial\Omega_1 \cup \partial\Omega_2$.
(iii) $\text{ind}(X - \lambda) = \text{ind}(M - \lambda) = -1$ *for* $\lambda \in \Omega_1 \setminus \text{cl}(\Omega_2)$.
(iv) $\text{ind}(X - \lambda) = \text{ind}(M - \lambda) = 0$ *for* $\lambda \in \Omega_2$.

Note that if we know that $X = M + K$, where K is compact, the only condition which is not satisfied automatically is condition (i).

Proof. The necessity of these conditions is easily verified.

Suppose that X satisfies the conditions of the theorem. We can use Proposition 4.4 of [**Apo76**] to find an operator X_0 such that $||X - X_0||$ is arbitrarily small, and X_0 satisfies the conditions of Corollary 8.8. Then $X_0 \in \text{cl}(\mathcal{U} + \mathcal{K})(M)$ and consequently $X \in \text{cl}(\mathcal{U} + \mathcal{K})(M)$.

□

We can now use Corollaries 5.2 and 5.6 to put this result together with Theorem 7.1 as follows:

8.10. Proposition. *Let* M_0 *be a model of the type constructed in Theorem 7.1. For* $i = 1, 2, \ldots, n$, *let* M_i *be a model of the type constructed in Corollary 8.9. Assume that*

X_4 may have eigenvalues in Ω_2. These are not further than ϵ away from $\partial\Omega_2$ and there are only finitely many of them. Suppose these eigenvalues are $\lambda_1, \lambda_2, \ldots, \lambda_m$.

Denote $H_0 = \text{span}\{H(\lambda_1; X_4), H(\lambda_2; X_4), \ldots, H(\lambda_m; X_4)\}$. If we knew that $H_0 \perp H^2(\Omega_1)$ and $H_0 \perp H^2(\Omega_2^*)$, i.e. H_0 is a subspace of the underlying space of F_2, we could move the eigenvalues λ_i away from Ω_2 by perturbing F_2. As we shall see next, it is true that $H_0 \perp H^2(\Omega_1)$ and $H_0 \perp H^2(\Omega_2^*)$ can be achieved by altering the decomposition of $\mathcal{H}$.

For each $i = 1, 2, \ldots, m$, let n_i be such that $H(\lambda_i; X_4) \subseteq \ker(X_4 - \lambda_i)^{n_i}$. Note that for any λ and k, X_4^k is of the form

$$\begin{pmatrix} A^k & C_1'' & C_2'' \\ & F_2^k & C_3'' \\ & & B^{*k} \end{pmatrix}.$$

One can verify by induction that

$$\text{ran}\ \begin{pmatrix} C_1'' & C_2'' \end{pmatrix} \subseteq \text{span}\{1, (x-\lambda), \ldots, (x-\lambda)^{k-1}\}.$$

If now $\begin{pmatrix} f \\ h \\ g \end{pmatrix}$ is in $\ker(X_4 - \lambda_i)^k$, we have

$$X_4 \begin{pmatrix} f \\ h \\ g \end{pmatrix} = \begin{pmatrix} A^k & C_1'' & C_2'' \\ & F_2^k & C_3'' \\ & & B^{*k} \end{pmatrix} \begin{pmatrix} f \\ h \\ g \end{pmatrix} = \begin{pmatrix} A^k f + C_1'' h + C_2'' g \\ F_2^k h + C_3'' g \\ B^{*k} g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Now $A^k f$ is linearly independent of ran $\begin{pmatrix} C_1'' & C_2'' \end{pmatrix}$, forcing $A^k f = 0$ and hence $f = 0$. This implies that $H_0 \perp H^2(\Omega_1)$.

Notice also that the $H^2(\Omega_2^*)$ component of any vector in $\ker(X_4 - \lambda_i)^k$ (denoted here by g) is in $\ker(B^* - \lambda_i)^k$.

Use Lemma 2.2 n_1 times to find vectors $\{f_1, f_2, \ldots, f_{n_1}\}$ such that the matrix of B^* is

$$\begin{pmatrix} \lambda_1 & \ldots & & & \\ & \lambda_1 & \ldots & & \\ & & \ddots & & \\ & & & \lambda_1 & \ldots \\ & & & & B^{*\prime} \end{pmatrix}$$

with respect to the decomposition $\text{span}\{f_1, f_2, \ldots, f_{n_1}\} \oplus \text{span}\{f_1, f_2, \ldots, f_{n_1}\}^\perp$, where $B^{*\prime}$ is a unitarily equivalent copy of B^*. Note that $\ker(B^* - \lambda_1)^{n_1} = \text{span}\{f_1, f_2, \ldots, f_{n_1}\}$. We can now continue in this manner until we can write B^* as

$$\begin{pmatrix} \lambda_1 & \cdots & & & & & & & \\ & \lambda_1 & \cdots & & & & & & \\ & & \ddots & & & & & & \\ & & & \lambda_1 & \cdots & & & & \\ & & & & \lambda_2 & \cdots & & & \\ & & & & & \ddots & & & \\ & & & & & & \lambda_m & \cdots & \\ & & & & & & & & B^{*\prime\prime} \end{pmatrix}$$

with respect to the decomposition $\text{span}\{f_1, f_2, \ldots, f_{n_0}\} \oplus \text{span}\{f_1, f_2, \ldots, f_{n_0}\}^\perp$, where $n_0 = \sum_{i=1}^m n_i$ and $B^{*\prime\prime}$ is another unitarily equivalent copy of B^*. Note that $\ker(B^* - \lambda_i)^{n_i} \subseteq \text{span}\{f_1, f_2, \ldots, f_{n_0}\}$ for $i = 1, 2, \ldots, m$ and consequently we have

$$H_0 \perp (H^2(\Omega_2^*) \ominus \text{span}\{f_1, f_2, \ldots, f_{n_0}\}).$$

We can now rewrite X_4 as

$$X_4 = \begin{pmatrix} A & C_1' & & \\ & F_2 & C_{31}' & C_{32}' \\ & & B_1 & B_2 \\ & & & B^{*\prime\prime} \end{pmatrix} = \begin{pmatrix} A & C_1' & 0 \\ & F_3 & B_2' \\ & & B^{*\prime\prime} \end{pmatrix}$$

with respect to the decomposition

$$\begin{aligned} \mathcal{H} &= H^2(\Omega_1) \oplus \mathbb{C}^{k_0} \oplus \text{span}\{f_1, f_2, \ldots, f_{n_0}\} \oplus H^2(\Omega_2^*)' \\ &\cong \mathcal{H} = H^2(\Omega_1) \oplus \mathbb{C}^{k_0+n_0} \oplus H^2(\Omega_2^*)' \end{aligned}$$

where $H^2(\Omega_2^*)'$ is an isometrically isomorphic copy of $H^2(\Omega_2^*)$. We have now $H_1 \subseteq \mathbb{C}^{k_0} \oplus \text{span}\{f_1, f_2, \ldots, f_{n_0}\} \cong \mathbb{C}^{k_0+n_0}$. We shall denote $B^{*\prime\prime}$ as B^* from now on.

For $i = 1, 2, \ldots, m$, let I_i be the identity operator on

$$\text{span}_{j=1}^{i} H(\lambda_j; X_4) \ominus \text{span}_{j=1}^{i-1} H(\lambda_j; X_4)$$

and let $\lambda_i' \in \Omega_1 \setminus cl(\Omega_2)$ be chosen so that $|\lambda_i - \lambda_i'| < \epsilon$. Set

$$X_5 = X_4 + \sum_{i=1}^{m} (\lambda_i' - \lambda_i) I_i.$$

Then X_5 is of the form

$$X_5 = \begin{pmatrix} A & C_1' & 0 \\ & F_4 & B_2' \\ & & B^{*''} \end{pmatrix}.$$

we have $\sigma(X_5) = \text{cl}(\Omega_1) \setminus \Omega_2$, $||X_5 - X|| < 5\epsilon$ and λ_i', $i = 1, 2, \ldots, m$, are the only points with nul $(X_5 - \lambda_i) > 0$.

Next, we decompose the finite dimensional operator F_4 as

$$F_4 = \begin{pmatrix} F_5 & D_1 \\ & F_6 \end{pmatrix}.$$

where $\sigma(F_5) \subseteq \text{cl}(\Omega_2)$ and $\sigma(F_6) \cap \text{cl}(\Omega_2) = \emptyset$. We have

$$X_5 = \begin{pmatrix} A & D_2 & D_3 & \\ & F_5 & D_1 & D_4 \\ & & F_6 & D_5 \\ & & & B^* \end{pmatrix}$$

Since $\sigma(F_6) \cap \sigma(B^*) = \emptyset$, Lemma 5.1 allows us to find a Z such that

$$\begin{pmatrix} I & -Z \\ & I \end{pmatrix} \begin{pmatrix} F_6 & D_5 \\ & B^* \end{pmatrix} \begin{pmatrix} I & Z \\ & I \end{pmatrix} = \begin{pmatrix} F_6 & 0 \\ & B^* \end{pmatrix}.$$

The operator X_5 is now $(\mathcal{U} + \mathcal{K})$ equivalent to

$$X_6 = \begin{pmatrix} I & -Z \\ & I \end{pmatrix} X_5 \begin{pmatrix} I & Z \\ & I \end{pmatrix} = \begin{pmatrix} A & D_2 & D_3' & D_6' \\ & F_5 & D_1' & D_4' \\ & & F_6 & 0 \\ & & & B^* \end{pmatrix}.$$

Let $M_1 = \left|\left| \begin{pmatrix} I & -Z \\ & I \end{pmatrix} \right|\right| . \left|\left| \begin{pmatrix} I & Z \\ & I \end{pmatrix} \right|\right|$. Recall that the only eigenvectors of X_5 were of the form $\begin{pmatrix} 0 \\ h_i' \\ g_i' \\ 0 \end{pmatrix}$ and hence the eigenvectors of X_6 will in fact be the same:

$$\begin{pmatrix} I & & & \\ & I & & \\ & & I & -Z \\ & & & I \end{pmatrix} \begin{pmatrix} 0 \\ h_i' \\ g_i' \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ h_i' \\ g_i' \\ 0 \end{pmatrix}.$$

Next, consider Lemma 2.4 and perturb D_3', D_1', F_6 so that

$$\begin{pmatrix} A & D_2 & E_1 \\ & F_5 & E_2 \\ & & F_7 \end{pmatrix} \cong_{\mathcal{U}+\mathcal{K}} A,$$

$$\| \begin{pmatrix} D_3' \\ D_1' \\ F_6 \end{pmatrix} - \begin{pmatrix} E_1 \\ E_2 \\ F_7 \end{pmatrix} \| < \epsilon/M_1.$$

and F_7 has simple eigenvalues in $\Omega_1 \setminus \mathrm{cl}(\Omega_2)$.

Then

$$X_7 = \begin{pmatrix} A & D_2 & E_1 & D_6' \\ & F_5 & E_2 & D_4' \\ & & F_7 & 0 \\ & & & B^* \end{pmatrix}$$

is $(\mathcal{U} + \mathcal{K})$ equivalent to an operator of the form

$$\begin{pmatrix} A & K_7 \\ & B^* \end{pmatrix}.$$

where K_7 is compact.

Let us check if X_7 has any eigenvalues in Ω_2. Suppose $\lambda \in \Omega_2$ and

$$(X_7 - \lambda) \begin{pmatrix} f \\ g \\ h \\ k \end{pmatrix} = \begin{pmatrix} A-\lambda & D_2 & E_1 & D_6' \\ & F_5 - \lambda & E_2 & D_4' \\ & & F_7 - \lambda & 0 \\ & & & B^* - \lambda \end{pmatrix} \begin{pmatrix} f \\ g \\ h \\ k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Then we have $(F_7 - \lambda)h = 0$ and hence $h = 0$. But then

$$(X_7 - \lambda)\begin{pmatrix} f \\ g \\ 0 \\ k \end{pmatrix} = (X_6 - \lambda)\begin{pmatrix} f \\ g \\ 0 \\ k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

which is a contradiction — X_6 does not have any such eigenvalues.

So X_7 has no eigenvalues in Ω_2 and by Corollary 8.6. we have $X_7 \in \mathrm{cl}(\mathcal{U} + \mathcal{K})(M)$. Hence

$$\mathrm{dist}(X_6, (\mathcal{U} + \mathcal{K})(M)) < \epsilon/M_1,$$

which implies

$$\mathrm{dist}(X_5, (\mathcal{U} + \mathcal{K})(M)) < \epsilon,$$

and hence

$$\mathrm{dist}(X, (\mathcal{U} + \mathcal{K})(M)) < 6\epsilon.$$

This last statement holds for any $\epsilon > 0$, so finally $X \in \mathrm{cl}(\mathcal{U} + \mathcal{K})(M)$.

□

8.8. Corollary. *Let $\Omega = \Omega_1 \setminus \mathrm{cl}(\Omega_2)$, where Ω_1 is a simply connected analytic Cauchy domain, Ω_2 is an analytic Cauchy domain consisting of n simply connected components, $\Omega_2 = \cup_1^n \Omega_{2,i}$, $\mathrm{cl}(\Omega_2) \subseteq \Omega_1$. Let ϕ be an invertible holomorphic map from a neighbourhood of $\mathbb{D}$ to $\mathbb{C}$ such that $\phi|_{\mathbb{D}}$ is a conformal map of $\mathbb{D}$ onto Ω_1. (This is the map which was used in Chapter 1 to construct $H^2(\Omega_1)$.) Then we have $\phi^{-1}(\Omega) = \mathbb{D} \setminus \mathrm{cl}(\Omega_2')$, where Ω_2' is an analytic Cauchy domain consisting of n simply connected components, $\Omega_2' = \cup_1^n \Omega_{2,i}'$, $\mathrm{cl}(\Omega_2') \subseteq \mathbb{D}$. Let now μ be a measure on $\partial(\mathbb{D})$ and for $i = 1, 2, \ldots, n$, let μ_i be a measure on $\partial(\Omega_{2,i}'^{\,*})$; all these measures are assumed to be equivalent to the respective arc length measures. Let now $M' = \begin{pmatrix} M(\mathbb{D}, \mu) & C \\ 0 & M(\Omega_{2,i}'^{\,*}) \end{pmatrix}$ be the model constructed at the beginning of this chapter. Then $M = \phi(M')$ has the following spectral properties.*

(i) $\sigma(M) = \mathrm{cl}(\Omega)$;
(ii) $\sigma_e(M) = \partial\Omega$;
(iii) nul $(M - z) = 0$, $z \in \Omega$;
(iv) nul $(M^* - \bar{z}) = 1$, $z \in \Omega$;
(v) $\mathrm{ind}(M - z) = -1$, $z \in \Omega$.

Let X be an essentially normal operator such that

(i) $\sigma(X) = \sigma(M) = \mathrm{cl}(\Omega_1) \setminus \Omega_2$,

(ii) $\sigma_e(X) = \sigma_e(M) = \partial\Omega_1 \cup \partial\Omega_2$.
(iii) $\operatorname{ind}(X - \lambda) = \operatorname{ind}(M - \lambda) = -1$ *for* $\lambda \in \Omega_1 \setminus \operatorname{cl}(\Omega_2)$.

Then $X \in \operatorname{cl}(\mathcal{U} + \mathcal{K})(M)$.

Proof. The proof is very similar to that of Theorem 3.5. Lemmas 3.3 and 3.4 allow us to verify that M has the spectral properties described in the Corollary. Note that M' is the type of operator for which Proposition 8.7 provides conditions that are sufficient for an operator to lie in $\operatorname{cl}(\mathcal{U} + \mathcal{K})(M')$. Using Lemmas 3.3 and 3.4 again, we see that $\phi^{-1}(X)$ satisfies the conditions of Proposition 8.7 and hence $\phi^{-1}(X) \in \operatorname{cl}(\mathcal{U} + \mathcal{K})(M')$. Now $X \in \operatorname{cl}(\mathcal{U} + \mathcal{K})(\phi(M')) = \operatorname{cl}(\mathcal{U} + \mathcal{K})(M)$ by Lemma 3.2.

□

8.9. Corollary. *Let* $M = \phi(M')$ *be as in 8.8. Then* $X \in \operatorname{cl}(\mathcal{U} + \mathcal{K})(M)$ *if and only if*

(i) $\sigma(X) = \sigma(M) = \operatorname{cl}(\Omega_1) \setminus \Omega_2$ *or* $\sigma(X) = \operatorname{cl}(\Omega_1)$,
(ii) $\sigma_e(X) = \sigma_e(M) = \partial\Omega_1 \cup \partial\Omega_2$,
(iii) $\operatorname{ind}(X - \lambda) = \operatorname{ind}(M - \lambda) = -1$ *for* $\lambda \in \Omega_1 \setminus \operatorname{cl}(\Omega_2)$,
(iv) $\operatorname{ind}(X - \lambda) = \operatorname{ind}(M - \lambda) = 0$ *for* $\lambda \in \Omega_2$.

Note that if we know that $X = M + K$, where K is compact, the only condition which is not satisfied automatically is condition (i).

Proof. The necessity of these conditions is easily verified.

Suppose that X satisfies the conditions of the theorem. We can use Proposition 4.4 of [**Apo76**] to find an operator X_0 such that $||X - X_0||$ is arbitrarily small, and X_0 satisfies the conditions of Corollary 8.8. Then $X_0 \in \operatorname{cl}(\mathcal{U} + \mathcal{K})(M)$ and consequently $X \in \operatorname{cl}(\mathcal{U} + \mathcal{K})(M)$.

□

We can now use Corollaries 5.2 and 5.6 to put this result together with Theorem 7.1 as follows:

8.10. Proposition. *Let* M_0 *be a model of the type constructed in Theorem 7.1. For* $i = 1, 2, \ldots, n$, *let* M_i *be a model of the type constructed in Corollary 8.9. Assume that*

$\sigma(M_i) \subseteq \Sigma_i$, $i = 0, 1, \ldots, n$, where Σ_i are pairwise disjoint simply connected open sets of $\mathbb{C}$. Set $M = \bigoplus_{i=0}^{n} M_i$. Then the closure of the $(\mathcal{U}+\mathcal{K})$-orbit of the operator M is

$$\mathrm{cl}((\mathcal{U}+\mathcal{K})(M)) = \{T \in \mathcal{B}(\mathcal{H}) :$$

$$\text{(i) } T \text{ is essentially normal.}$$

$$\text{(ii) } \sigma(T) \supseteq \sigma(M).$$

$$\text{(iii) } \mathrm{nul}\ (T-z)^l \geq \mathrm{nul}\ (M-z)^l, l = 1, 2, \ldots, z \notin \sigma_e(M).$$

$$\text{(iv) } \sigma_e(T) = \sigma_e(M).$$

$$\text{(v) } \mathrm{ind}(T-z) = \mathrm{ind}(M-z) \text{ for all } z \notin \sigma_e(T) = \sigma_e(M)$$

$$\text{(vi) if } z \in \sigma_{iso}(T) \cap \sigma_e(T), \text{ then } (T-z)|_{\mathcal{H}(z:T)} = 0.$$

$$\text{(vii) if } z \in \sigma_0(T), \text{ then } \mathrm{rank}\ E(z:T) = \mathrm{rank}\ E(z:M)\}.$$

Proof. The proof is analogous to that of Theorem 5.7.

□

CHAPTER 9

Further Comments

In this chapter we shall state several open questions in the present area of research.

Notice that if M is one of the models investigated in Chapters 2 and 4, we have

$$\min \operatorname{ind}(M - z) = 0, \quad z \in \rho_F(M).$$

One would want to know what the closure of the $(\mathcal{U} + \mathcal{K})$-orbit is in the following case:

9.1. Question. *Let Ω be a simply connected Cauchy domain. Suppose that μ and μ^* are measures on $\partial\Omega$ and $\partial\Omega^*$ equivalent to the arc-length measure. Let $i, j \geq 0$ be integers. Set*

$$M = \bigoplus_{k=1}^{i} M(\Omega^*, \mu^*)^* \oplus \bigoplus_{k=1}^{j} M(\Omega, \mu).$$

Note that we have

$$\begin{aligned} \text{nul } (M - z) &= i, \\ \text{nul } (M - z)^* &= j, \end{aligned}$$

for $z \in \Omega$.

What is $\mathrm{cl}(\mathcal{U} + \mathcal{K})(M)$ *?*

Note, however, that the models we dealt with in Chapters 2 and 4 are maximal in the following sense: The model $M = \bigoplus_{k=1}^{i} M(\Omega^*, \mu^*)^* \oplus \bigoplus_{k=1}^{j} M(\Omega, \mu)$ is in $\mathrm{cl}(\mathcal{U} + \mathcal{K})(M_0)$, where $M_0 = \bigoplus_{k=1}^{i-j} M(\Omega^*, \mu^*)^*)$ if $i > j$ or $M_0 = \bigoplus_{k=1}^{j-i} M(\Omega, \mu)$ if $i < j$. (See Theorem 4.6.) Since the relation $\rightarrow_{\mathcal{U}+\mathcal{K}}$ is transitive, this implies that $\mathrm{cl}(\mathcal{U}+\mathcal{K})(M) \subseteq \mathrm{cl}(\mathcal{U}+\mathcal{K})(M_0)$.

Next, note that in Theorem 7.1, we required that K (the set by which the essential spectrum is being enlarged) and the sequence $\{z_r\}$ (the isolated eigenvalues of the model) are both disjoint from $\bigcup_{i=1}^{n} \Omega_i$. One may now ask:

9.2. Question. *Consider the model M investigated in Theorem 7.1. If we remove the condition that K be disjoint from $\bigcup_{i=1}^{n} \Omega_i$ and if we replace the condition $\{z_r\} \cap (K \cup \bigcup_{i=1}^{n} \Omega_i) = \emptyset$ by $\{z_r\} \cap K = \emptyset$. what is* $\mathrm{cl}(\mathcal{U} + \mathcal{K})(M)$ *?*

As for the model investigated in Chapter 8. the most urgent concern would be the following:

9.3. Question. *Suppose M is the model investigated in Corollary 8.9. Let $i > 1$. What is* $\mathrm{cl}(\mathcal{U} + \mathcal{K})(\bigoplus_{k=1}^{i} M)$ *?*

Finally, whenever the closure of the $(\mathcal{U} + \mathcal{K})$-orbit is described for a model with a certain spectral picture. one may wish to go beyond the model and investigate the whole class of operators sharing the same spectral picture.

9.4. Question. *Suppose M is one of the models investigated here. Call an operator X* M-like *if it has the same spectral picture (including nullity) as M. Is it true that for any M-like operator, we have* $M \in \mathrm{cl}(\mathcal{U} + \mathcal{K})(X)$ *?*

If the answer to this question is affirmative, transitivity of the relation $\to_{\mathcal{U}+\mathcal{K}}$ implies that $\mathrm{cl}(\mathcal{U} + \mathcal{K})(X) = \mathrm{cl}(\mathcal{U} + \mathcal{K})(M)$ whenever X is an M-like operator. (Compare [**Mar92**].)

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