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Vibration Characteristics of a Ring Under General Boundary Conditions using Euler-Bernoulli and Timoshenko Theories

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Abstract— Ring vibration analysis is more challenging than straight beams analysis. In straight beams, the structural deformations depend on rotational and translational displacements; however, in rings, in addition to these, they also depend on the coupled tangential displacement caused by the curvature of structures. In this study, the free in-plane vibration problem for thin- and thick-walled rings are solved, and the explicit relationships between the radial, tangential, and rotational displacements are revealed. This paper introduces an analytical method for finding exact solutions for the natural frequencies and associated mode shapes of a ring under general boundary conditions. Both Euler-Bernoulli and Timoshenko theories are used in this work, and the corresponding results are compared.

Keywords- Ring Vibration, Mode Shape, Natural Frequency, Euler-Bernoulli Beam, Timoshenko Beam

I. INTRODUCTION

Rings and curved beams are widely used in many engineering applications. The vibration of rings has drawn the intensive interest of many researchers since the 19^{th} century. Hoppe [1] is known as the pioneer in dealing with thin-walled ring vibration. The early theoretical work is summarized in the classical theory of Love [2]. Many later research papers in the ring and curved-beam analysis [3-6] are based on his curve shell theory. The governing differential equations presented in Love's formulae were two coupled equations in the radial and the tangential directions. After some mathematical simplification, those two coupled differential equations can be reduced into a sixth-order ordinary differential equation with some constant coefficients in the tangential direction. Many researchers have followed this path and therefore, the bending theory of thin rings known as Euler-Bernoulli rings has been well established. However, the Love's theory does not work well for thick rings known as Timoshenko rings as it does not consider the rotatory inertia and shear deformation. [7-11].

Free vibration of both Euler-Bernoulli and Timoshenko rings without any constraints have been investigated by many researchers in the past by using different methods [12-16]. One suitable method that can also be used for both rings and even for the rings with constraints is known as the multiple-scales method. The same method has been used in the current work for the constrained ring.

For a ring with different elastic supports, the free vibration is inevitably more complicated. Represented works on free vibration of constrained rings include [8,17-24]. Similar to free-ring studies, here also different methods have used to study rings on elastic supports such as transfer matrix, Galerkin, perturbation, and other finite element methods.

Due to the large amount of work that has been developed in the above categories, the small amount of work previously carried out on the response of a ring under general boundary conditions, and from the few works in this area, almost all studied only the Euler-Bernoulli rings and took advantage of various numerical solutions. To the author's best knowledge, no work has been done to express the mode shape of a ring with general boundary conditions in a closed-form formulation. It is the aim of this paper to bring together the current work on such a case, to provide a single point of reference for solution methodology, and compare the results in both cases i.e., Euler-Bernoulli and Timoshenko theories.

This paper is organized as below: In section 2, the in-plane vibrational equations for both theories are derived, individually. Mode shapes and boundary conditions are discussed in sections 3 and 4, respectively. A case study is investigated in section 5, and conclusions are given in section 6.

II. DYNAMIC MODEL

Fig. 1 shows a ring that is mounted to its frame, and Fig. 2 represents that ring by its centerline, which is considered to remain undeformed. To avoid complexity in deriving the governing equations, the following assumptions are considered:

- The rigidity of the ring i.e., EI is assumed constant.
- The ring is being held by n bolted supports resulting in dividing the ring perimeter into n identical, equally spaced segments

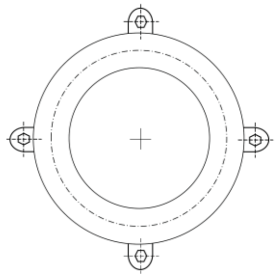

Figure 1. A ring and the graphical location of each support

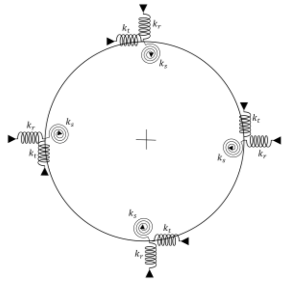

Figure 2. Representation of each support with three springs

To derive the governing equations, a typical element of the ring at an arbitrary position of θ is selected (Fig. 3). Fig. 4 represents this element. The forces and moments acting on this element can be expressed as functions of deformation components. The positive directions for normal force N, shear force Q, and the in-plane bending moment M are shown in Fig. 4.

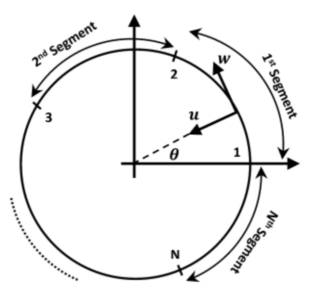

Figure 3. Ring centerline

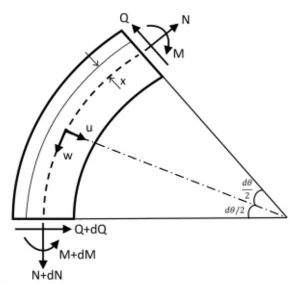

Figure 4. Element of a ring

Neglecting the small quantities of high order terms will result in the following equilibrium equations in tangential w, radial u, and rotational motion φ around the w axis.

$$\frac{\partial Q}{\partial \theta} + N = \rho A R\,\ddot{u} \tag{1}$$

$$\frac{\partial N}{\partial \theta} - Q = \rho A R\,\ddot{w} \tag{2}$$

$$\frac{\partial M}{\partial \theta} + R.Q = \rho I R\,\ddot{\varphi} \tag{3}$$

where R is the radius of ring centerline, ρ is the total mass density, A is the cross-sectional area, and I is the moment of inertia of the area A about w axis.

2.1 Equation derivation for Timoshenko ring

In Timoshenko beam theory for thick-walled rings, the effects of rotary inertia and shear deformation are both considered. In this section, the goal is to present a closed-form formulation for the system vibrational equation by solving (1-3). To do so, the following parameters need to be defined.

The rotation of a cross-section φ, the moment M and the normal force N are represented as below [3,24,26]:

$$\varphi = \frac{1}{R}\left(\frac{\partial u}{\partial \theta} + w - R\gamma\right) \tag{4}$$

$$M = \frac{EI}{R}\left(\frac{\partial \varphi}{\partial \theta}\right) = \frac{EI}{R^2}\left(\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial w}{\partial \theta} - R\frac{\partial \gamma}{\partial \theta}\right) \tag{5}$$

$$N = \frac{EA}{R}\left(\frac{\partial w}{\partial \theta} - u\right) - \frac{M}{R} = \frac{EA}{R}\left(\frac{\partial w}{\partial \theta} - u\right) - \frac{EI}{R^2}\left(\frac{\partial \varphi}{\partial \theta}\right) \tag{6}$$

where γ is the shear deformation.

Given that the centerline is assumed inextensible, the following relation is held [3]:

$$u = \frac{\partial w}{\partial \theta} \tag{7}$$

Equation (7) is called the inextensibility condition. Under this condition, the relationship between the shear force and the bending moment can be derived as.

$$Q = -k'AG\gamma = -\frac{k'AG}{R}\left(\frac{\partial u}{\partial \theta} + w - R\varphi\right) \tag{8}$$

By inserting (4-8) into (1-3) and doing some simplifications, (9) can be produced as follow: The detailed derivation can be found in many publications, such as [3,12,24]; thus, it is not given here for the sake of brevity.

$$\begin{aligned}\frac{\partial^6 w(\theta,t)}{\partial \theta^6} + 2\frac{\partial^4 w(\theta,t)}{\partial \theta^4} + \frac{\partial^2 w(\theta,t)}{\partial \theta^2} &= \left(\frac{R^2\rho}{E} + \frac{R^2\rho}{k'G}\right)\frac{\partial^6 w(\theta,t)}{\partial t^2\,\partial \theta^4} - \\ \left(\frac{R^4\rho^2}{Ek'G}\right)\frac{\partial^6 w(\theta,t)}{\partial t^4\,\partial \theta^2} &+ \left(2\frac{R^2\rho}{E} - \frac{R^2\rho}{k'G} - \frac{R^4\rho A}{EI}\right)\frac{\partial^4 w(\theta,t)}{\partial t^2\,\partial \theta^2} + \\ \left(\frac{R^4\rho^2}{Ek'G}\right)\frac{\partial^4 w(\theta,t)}{\partial t^4} &+ \left(\frac{R^2\rho}{E} + \frac{R^4\rho A}{EI}\right)\frac{\partial^2 w(\theta,t)}{\partial t^2}\end{aligned} \tag{9}$$

Equation (9) can be solved with the variables separated by assuming the following.

$$w(\theta,t) = \sum_{j=1}^{\infty} C_j\,\Psi_j(\theta)\,\mathrm{q}_j(t) \simeq \Psi(\theta)\,q(t) \tag{10}$$

$$\mathrm{q}_j(t) = B_j\exp(i\omega_j t) \tag{11}$$

where $\Psi(\theta)$ is the mode shape and i is the imaginary unit number $i = \sqrt{-1}$. Substituting (10) into (9) gives

$$\frac{\partial^6 \Psi}{\partial \theta^6} + H_1\frac{\partial^4 \Psi}{\partial \theta^4} + H_2\frac{\partial^2 \Psi}{\partial \theta^2} + H_3\Psi = 0 \tag{12}$$

where:

$$H_1 = 2 + \left(\frac{R^2\rho}{E} + \frac{R^2\rho}{k'G}\right)\omega^2 \tag{13}$$

$$H_2 = 1 + \left(2\frac{R^2\rho}{E} - \frac{R^2\rho}{k'G} - \frac{R^4\rho A}{EI}\right)\omega^2 + \left(\frac{R^4\rho^2}{Ek'G}\right)\omega^4 \tag{14}$$

$$H_3 = \left(\frac{R^2\rho}{E} + \frac{R^4\rho A}{EI}\right)\omega^2 - \left(\frac{R^4\rho^2}{Ek'G}\right)\omega^4 \tag{15}$$

2.1 Equation derivation for Timoshenko ring

In Euler-Bernoulli beam theory for thin-walled rings, the effects of rotary inertia and shear deformation are both neglected. Thus, some modifications must be applied into the equations derived in the previous section. In this scenario, the rotation of a cross-section φ, the moment M and the normal force N can be represented as below [3,12,24,26]:

$$\varphi = \frac{1}{R}\left(\frac{\partial u}{\partial \theta} + w\right) \tag{16}$$

$$M = \frac{EI}{R}\left(\frac{\partial \varphi}{\partial \theta}\right) \tag{17}$$

$$N = \frac{EA}{R}\left(\frac{\partial w}{\partial \theta} - u\right) - \frac{M}{R} = \frac{EA}{R}\left(\frac{\partial w}{\partial \theta} - u\right) - \frac{EI}{R^2}\left(\frac{\partial \varphi}{\partial \theta}\right) \tag{18}$$

Furthermore, the relationship between the shear force and the bending moment can be derived as:

$$Q = -\frac{1}{R}\frac{\partial M}{\partial \theta} = -\frac{EI}{R}\left(\frac{\partial^2 \varphi}{\partial \theta^2}\right) \tag{19}$$

The terms containing ρI in (9) are due to rotary inertia and they must be removed for the Euler-Bernoulli beam. By defining $r_{gy} = \sqrt{I/A}$ as the radius of gyration of cross-sectional area A about the w axis, it can be shown that $\frac{R^2 A}{I} = \left(\frac{R}{r_{gy}}\right)^2$. Thus, the influence of rotary inertia is dependent on the thickness ratio $\frac{R}{r_{gy}}$. By doing those simplifications, the in-plane vibration equation of a Euler-Bernoulli ring can be represented in the following form.

$$\frac{\partial^6 w(\theta,t)}{\partial \theta^6} + 2\frac{\partial^4 w(\theta,t)}{\partial \theta^4} + \frac{\partial^2 w(\theta,t)}{\partial \theta^2} + \frac{R^4 \rho A}{E I}\frac{\partial^2}{\partial t^2}\left[\frac{\partial^2 w(\theta,t)}{\partial \theta^2} - w(\theta,t)\right] = 0 \tag{20}$$

Substituting (10) into (20) gives

$$\frac{\partial^6}{\partial \theta^6}\Psi(\theta)q(t) + 2\frac{\partial^4}{\partial \theta^4}\Psi(\theta)q(t) + \frac{\partial^2}{\partial \theta^2}\Psi(\theta)q(t) + \frac{R^4 \rho A}{E I}\left[\frac{\partial^2 \Psi(\theta)}{\partial \theta^2} - \Psi(\theta)\right]\ddot{q}(t) = 0 \tag{21}$$

Inserting (11) into above, produces

$$\left[\frac{\partial^6 \Psi(\theta)}{\partial \theta^6} + 2\frac{\partial^4 \Psi(\theta)}{\partial \theta^4} + \frac{\partial^2 \Psi(\theta)}{\partial \theta^2}\right]\exp(i\omega t) - \frac{R^4 \rho A \omega^2}{E I}\left[\frac{\partial^2 \Psi(\theta)}{\partial \theta^2} - \Psi(\theta)\right]\exp(i\omega t) = 0 \tag{22}$$

The following must be true if the above holds.

$$\frac{\partial^6 \Psi(\theta)}{\partial \theta^6} + a\,\frac{\partial^4 \Psi(\theta)}{\partial \theta^4} + b\,\frac{\partial^2 \Psi(\theta)}{\partial \theta^2} + c = 0 \tag{23}$$

where:

$$a = 2 \tag{24}$$

$$b = 1 - \Omega^2 \tag{25}$$

$$c = \Omega^2 \tag{26}$$

$$\Omega = R^2\sqrt{\rho A / E I}\,\omega \tag{27}$$

III. Solution Strategy

Assuming (12) for a Timoshenko ring or (23) for a Euler-Bernoulli ring has a solution of the following form

$$\Psi(\theta) = A\, e^{\Upsilon \theta} \tag{28}$$

The procedure for the Euler-Bernoulli ring is explained below. Same methodology can be followed for the Timoshenko ring that is not given here for the sake of brevity.

Substituting (27) into (23) gives the following algebraic equation.

$$\Upsilon^6 + a\,\Upsilon^4 + b\,\Upsilon^2 + c = 0 \tag{29}$$

Equation (28) can be simplified by letting $\lambda = \Upsilon^2$

$$\lambda^3 + a\,\lambda^2 + b\,\lambda + c = 0 \tag{30}$$

It is straightforward to solve (29). The roots of (29) can be expressed as $\lambda_r(r$=1-3). Once λ_r are obtained, $\Upsilon_s(s$=1-6) can be easily calculated by $\Upsilon_s = \pm\sqrt{\lambda_r}$. To solve (29), a discriminant parameter Δ is defined as below [26].

$$\Delta = S^3 + \bar{R}^2 \tag{31}$$

where:

$$\bar{S} = \frac{1}{9}(3b - a^2) \tag{32}$$

$$\bar{R} = \frac{1}{54}(9ab - 27c - 2a^3) \tag{33}$$

Equation (27) now can take the form of

$$\Psi(\theta) = A_1 e^{\sqrt{\lambda_1}\theta} + A_2 e^{-\sqrt{\lambda_1}\theta} + A_3 e^{\sqrt{\lambda_2}\theta} + A_4 e^{-\sqrt{\lambda_2}\theta} + A_5 e^{\sqrt{\lambda_3}\theta} + A_6 e^{-\sqrt{\lambda_3}\theta} \tag{34}$$

λ_r $(r = 1,2,3)$ takes different forms based on the discriminant Δ. This is discussed below:

- $\Delta > 0$: There are one real root and two complex roots, which are complex conjugates.
- $\Delta = 0$: All three roots are real; however, two of them are identical. In other words, there are a set of repeated roots in the system. In this case, the corresponding terms of (33) associated with the repeated roots will take the forms of $\left(A_{rep} + A_{rep+1}\theta\right)e^{\sqrt{\lambda_{rep}}\theta}$.
- $\Delta < 0$: All roots are real and distinct. For this case, the discriminant is expressed below:

$$\Delta = -\frac{c^3}{27} + \frac{71\,c^2}{108} - \frac{2\,c}{27} \tag{35}$$

Depending on the three cases above, the mode shape will take different forms, which are discussed in the next section.

IV. Compatibility and Equilibrium Conditions

Assuming the n bolts are equally spaced as shown in Fig. 2. The angle between any two adjacent supports ζ is as below:

$$\zeta = \frac{2\pi}{n} \tag{36}$$

The radial displacement u and rotational motion φ can be determined once tangential displacement w is found. They can be expressed as below with the mode shape.

$$u(\theta,t) = \sum_{j=1}^{\infty} D_j\, U_j(\theta)\, \mathrm{q}_j(t) \simeq U(\theta)\, q(t) \tag{37}$$

$$\varphi(\theta,t) = \sum_{j=1}^{\infty} E_j\, \Phi_j(\theta)\, \mathrm{q}_j(t) \simeq \Phi(\theta)\, q(t) \tag{38}$$

Then the boundary conditions for a bolt support are represented as below:

$$\Psi_i(\theta_i) = \Psi_{i+1}(\theta_i) \tag{39}$$

$$\mathrm{U}_i(\theta_i) = \mathrm{U}_{i+1}(\theta_i) \tag{40}$$

$$\Phi_i(\theta_i) = \Phi_{i+1}(\theta_i) \tag{41}$$

$$\bar{N}_i(\theta_i) = \bar{N}_{i+1}(\theta_i) - k_t \Psi_i(\theta_i) \tag{42}$$

$$\bar{Q}_i(\theta_i) = \bar{Q}_{i+1}(\theta_i) - k_r \mathrm{U}_i(\theta_i) \tag{43}$$

$$\bar{M}_i(\theta_i) = \bar{M}_{i+1}(\theta_i) - k_s \Phi_i(\theta_i) \tag{44}$$

Equations (36-43) simply state the continuity of the tangential, the radial, and the rotational displacements across the support. Φ, $\bar{N}$, $\bar{Q}$, and $\bar{M}$ are the rotation angle, the normal force, the shear force, and the bending moment of the cross-section at the support. They are calculated according to (4-6,8,16-19) as [12]:

$$\bar{N} = \frac{EA}{R}\left(\frac{\partial \Psi}{\partial \theta} - U\right) - \frac{EI}{R^2}\left(\frac{\partial \Phi}{\partial \theta}\right) \tag{45}$$

$$\bar{M} = \frac{EI}{R}\left(\frac{\mathrm{d}\Phi}{\mathrm{d}\theta}\right) \tag{46}$$

$$\Phi_{EB} = \frac{1}{R}\left(\frac{\partial U}{\partial \theta} + \Psi\right) \tag{47–a}$$

$$\Phi_T = \frac{1}{R}\left(\frac{\mathrm{d}^2\Psi}{\mathrm{d}\theta^2} + \frac{EI/R^2}{\rho A R^2 \omega^2 - EA}\Psi\right) \tag{47–b}$$

$$\bar{Q}_{EB} = -\frac{EI}{R^2}\left(\frac{\mathrm{d}^2\Phi}{\mathrm{d}\theta^2}\right) \tag{48–a}$$

$$\bar{Q}_T = -\frac{k' AG}{R}\left(\frac{\partial U}{\partial \theta} + \Psi - R\Phi_T\right) \tag{48–b}$$

where the indices "EB" and "T" represent Euler-Bernoulli and Timoshenko, respectively. Same methodology can be used for both theories, however, the correct versions of (47,48) need to be used for each scenario.

For an inextensible ring, $u = \frac{\partial w}{\partial \theta}$ (45) can then be shortened into

$$\bar{N} = -\frac{EI}{R^2}\left(\frac{\partial \varphi}{\partial \theta}\right) \tag{49}$$

By substituting (34,45-49) into (39-44), the following equations can be obtained.

$$\sum_{k=1}^{6} \left(A_{i,k} - A_{i+1,k}\right) e^{\Upsilon_k \beta_s} = 0 \tag{50}$$

$$\sum_{k=1}^{6} \left(A_{i,k} - A_{i+1,k}\right) \Upsilon_k \, e^{\Upsilon_k \beta_s} = 0 \tag{51}$$

$$\sum_{k=1}^{6} \left(A_{i,k} - A_{i+1,k}\right) \left(1 + {\Upsilon_k}^2\right) e^{\Upsilon_k \beta_s} = 0 \tag{52}$$

$$\sum_{k=1}^{6} \left[\left(-\frac{EI}{R^3}\left({\Upsilon_k}^3 + \Upsilon_k\right) + k_t\right) A_{i,k} + \frac{EI}{R^3}\left({\Upsilon_k}^3 + \Upsilon_k\right) A_{i+1,k}\right] e^{\Upsilon_k \beta_s} = 0 \tag{53}$$

$$\sum_{k=1}^{6} \left[\left(-\frac{EI}{R^3}\left({\Upsilon_k}^3 + \Upsilon_k\right) + k_r\right) A_{i,k} + \frac{EI}{R^3}\left({\Upsilon_k}^3 + \Upsilon_k\right) A_{i+1,k}\right] \Upsilon_k \, e^{\Upsilon_k \beta_s} = 0 \tag{54}$$

$$\sum_{k=1}^{6} \left[\left(\frac{EI}{R^2}\Upsilon_k + \frac{k_s}{R}\right) A_{i,k} - \left(\frac{EI}{R^2}\Upsilon_k\right) A_{i+1,k}\right] \left(1 + {\Upsilon_k}^2\right) e^{\Upsilon_k \beta_s} = 0 \tag{55}$$

where "β_s" is the position of the selected node (or the support).

For each support, six equations, (50-55) can be formed. The whole ring then has 6n equations, which can be recast into compact matrix form of:

$$[G(\omega)]\left\{A_{1,1}\, A_{1,2}\, A_{1,3}\, A_{1,4}\, A_{1,5}\, A_{1,6} \cdots A_{n,1}\ A_{n,2}\ A_{n,3}\ A_{n,4}\ A_{n,5}\ A_{n,6}\right\}^T = \{0\} \tag{56}$$

where $G(\omega)$ is a matrix of $6n \times 6n$. For the particular case of four supports n=4, $G(\omega)$ are given in Appendix A.

A non-trivial solution of (56) exists if the determinant of $G(\omega)$ is zero. By solving

$$|G(\omega)| = 0, \tag{57}$$

the natural frequencies ω will be obtained. Then the mode shape can be obtained through (57).

V. Case Study

A symmetrically supported ring with a rectangular cross-section is studied in this section. The ring is supported by four equally spaced bolts (n=4). The parameters of the ring and supports are given in Table 1 below.

Table 1 Parameters of a ring.

Definition	Parameter	Value	Unit
Radius of ring	R	0.484	m
Density	ρ	7858	kg/m^3
Cross-sectional area	A	1.96×10^{-3}	m^2
Young's modulus	E	207.17	GPa
Moment of inertia	I	3.068×10^{-7}	kg-m^2
Tangential spring constant	k_t	7.263×10^7	N/m
Radial spring constant	k_r	7.263×10^7	N/m
Rotational spring constant	k_s	4.315×10^8	N-m/rad

Following the methodology explained in section 4, (56) can be derived for both Euler-Bernoulli and Timoshenko theories. For instance, the detailed entries of the matrix in Euler-Bernoulli case are presented in Appendix A. Equation (57) generally must be solved numerically for either case. In this paper, the Maple package is used. The first six natural frequencies are found by using the modified Newton-Raphson and Bisection methods as given in Table 2.

Table 2 System natural frequencies.

Theory	ω_1 (Hertz)	ω_2 (Hertz)	ω_3 (Hertz)	ω_4 (Hertz)	ω_5 (Hertz)	ω_6 (Hertz)
Euler-Bernoulli	53.46	81.60	242.88	331.12	487.42	722.54
Timoshenko	52.19	79.78	195.43	356.08	512.52	798.87

Once natural frequencies of the system are obtained, the corresponding mode shape functions for each of those natural frequencies can be plotted by using (34). Figs, (5,6) illustrate the mode shape in a radial direction about the ring centerline for the first and second natural frequency, respectively.

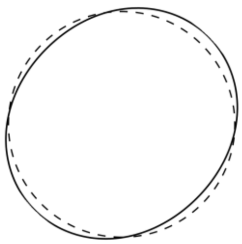

Figure 5. Radial mode shape corresponding to ω_1

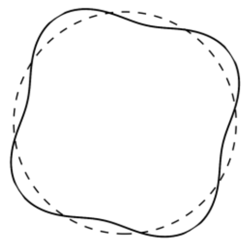

Figure 6. Radial mode shape corresponding to ω_2

In the figures, the dashed lines represent the undeformed shape**s**, and the solid lines are the mode shapes with deformation. It is noted that only the radial displacement is visible.

VI. CONCLUSIONS

This paper represents an analytical method to derive natural frequencies and mode shapes of a thin- or thick-walled ring under general boundary conditions. Results categorized in Table 2 shows that for the first three natural frequencies, the Timoshenko theory predicts lower numbers in comparison with the Euler-Bernoulli theory, unlike for higher frequencies where the Euler-Bernoulli theory is more conservative. This is because for low frequencies, the effects of shear deformation and rotary inertia are significant and that will result in lower natural frequencies of the vibrating system when the Timoshenko theory is in use.

Appendix (A)

For a ring hold at four equally spaced points, Eq. (48) will take the form of (A1) where $[G(\omega)]$ is represented as a matrix with $B_{s,i}$ (i=1,2 and s=1-4) components, as follows:

$$[G(\omega)]\,.\,\{A_{ij}\} = \begin{bmatrix} B_{1,1} & B_{1,2} & 0 & 0 \\ 0 & B_{2,1} & B_{2,2} & 0 \\ 0 & 0 & B_{3,1} & B_{3,2} \\ B_{4,2} & 0 & 0 & B_{4,1} \end{bmatrix}_{24\times24} . \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \end{Bmatrix}_{24\times1} = 0 \qquad \text{(A1)}$$

where:

$$D_p = \{A_{p,1}\ \ A_{p,2}\ \ A_{p,3}\ \ A_{p,4}\ \ A_{p,5}\ \ A_{p,6}\}^T \qquad (p = 1-4) \qquad \text{(A2)}$$

$$B_{s,k} = \begin{bmatrix} B_{s,k_{[1,1]}} & B_{s,k_{[1,2]}} & B_{s,k_{[1,3]}} & B_{s,k_{[1,4]}} & B_{s,k_{[1,5]}} & B_{s,k_{[1,6]}} \\ B_{s,k_{[2,1]}} & B_{s,k_{[2,2]}} & B_{s,k_{[2,3]}} & B_{s,k_{[2,4]}} & B_{s,k_{[2,5]}} & B_{s,k_{[2,6]}} \\ B_{s,k_{[3,1]}} & B_{s,k_{[3,2]}} & B_{s,k_{[3,3]}} & B_{s,k_{[3,4]}} & B_{s,k_{[3,5]}} & B_{s,k_{[3,6]}} \\ B_{s,k_{[4,1]}} & B_{s,k_{[4,2]}} & B_{s,k_{[4,3]}} & B_{s,k_{[4,4]}} & B_{s,k_{[4,5]}} & B_{s,k_{[4,6]}} \\ B_{s,k_{[5,1]}} & B_{s,k_{[5,2]}} & B_{s,k_{[5,3]}} & B_{s,k_{[5,4]}} & B_{s,k_{[5,5]}} & B_{s,k_{[5,6]}} \\ B_{s,k_{[6,1]}} & B_{s,k_{[6,2]}} & B_{s,k_{[6,3]}} & B_{s,k_{[6,4]}} & B_{s,k_{[6,5]}} & B_{s,k_{[6,6]}} \end{bmatrix} \quad (k = 1,2) \qquad \text{(A3)}$$

$$B_{s,1_{[1,n]}} = e^{\Upsilon_n \beta_s} \qquad (n = 1-6) \qquad \text{(A4)}$$

$$B_{s,1_{[2,n]}} = \Upsilon_n e^{\Upsilon_n \beta_s} \qquad (n = 1-6) \qquad \text{(A5)}$$

$$B_{s,1_{[3,n]}} = \left(1 + \Upsilon_n^{\,2}\right) e^{\Upsilon_n \beta_s} \qquad (n = 1-6) \qquad \text{(A6)}$$

$$B_{s,1_{[4,n]}} = \left(-\frac{EI}{R^3}\left(\Upsilon_n^{\,3} + \Upsilon_n\right) + k_t\right) e^{\Upsilon_n \beta_s} \qquad (n = 1-6) \qquad \text{(A7)}$$

$$B_{s,1_{[5,n]}} = \left(-\frac{EI}{R^3}\left(\Upsilon_n^{\,3} + \Upsilon_n\right) + k_r\right) \Upsilon_n e^{\Upsilon_n \beta_s} \qquad (n = 1-6) \qquad \text{(A8)}$$

$$B_{s,1_{[6,n]}} = \left(\frac{EI}{R^2}\Upsilon_n + \frac{k_s}{R}\right)\left(1 + \Upsilon_n^{\,2}\right) e^{\Upsilon_n \beta_s} \qquad (n = 1-6) \qquad \text{(A9)}$$

$$B_{s,2_{[1,n]}} = -e^{\Upsilon_n \beta_s} \qquad (n = 1-6) \qquad \text{(A10)}$$

$$B_{s,2_{[2,n]}} = -\Upsilon_n e^{\Upsilon_n \beta_s} \qquad (n = 1-6) \qquad \text{(A11)}$$

$$B_{s,2_{[3,n]}} = -\left(1 + \Upsilon_n^{\,2}\right) e^{\Upsilon_n \beta_s} \qquad (n = 1-6) \qquad \text{(A12)}$$

$$B_{s,2_{[4,n]}} = \left(\frac{EI}{R^3}\left(\Upsilon_n^{\,3} + \Upsilon_n\right) + k_t\right) e^{\Upsilon_n \beta_s} \qquad (n = 1-6) \qquad \text{(A13)}$$

$$B_{s,2_{[5,n]}} = \left(\frac{EI}{R^3}\left(\Upsilon_n^{\,3} + \Upsilon_n\right) + k_r\right) \Upsilon_n e^{\Upsilon_n \beta_s} \qquad (n = 1-6) \qquad \text{(A14)}$$

$$B_{s,2_{[6,n]}} = -\left(\frac{EI}{R^2}\Upsilon_n + \frac{k_s}{R}\right)\left(1 + \Upsilon_n^{\,2}\right) e^{\Upsilon_n \beta_s} \qquad (n = 1-6) \qquad \text{(A15)}$$

In all above equations, the index 's' represented the position of the nodes. Since those matrices are derived for a ring with four supports, thus, 's' will be varied from 1 to 4.

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