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STOCHASTIC FAULT TOLERANT CONTROL: ANALYSIS AND SYNTHESIS

by

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Abstract

Fault Tolerant Control Systems (FTCS) can maintain stability and restore as much performance as possible after system component/actuator/sensor faults occur in control systems. Therefore they are necessary for those safety-critical systems. Compared to passive FTCS, where there is no controller reconfiguration, active FTCS have a Fault Detection and Identification (FDI) scheme and a set of controllers for reconfiguration, and can provide superior fault tolerance capabilities.

In this thesis, the analysis and synthesis of stochastic active FTCS are studied. When the occurrence of parametric faults in the system is modeled by a homogeneous Markov chain, the open-loop system to study can be modeled as a linear system with Markovian jumping parameters. When applying this model to systems subject to parametric faults, FDI decision process should be modeled as another stochastic process, and the controller reconfiguration should be carried out based on FDI decisions.

This stochastic formulation of active FTCS concerns the imperfectness of FDI decision caused by disturbances and model uncertainties. Different stochastic processes can be used according to the characteristics of FDI decision processes. Two situations are considered in this thesis. The first and also the most widely used one is to model FDI decision process as another independent Markov chain, assuming that memoryless FDI algorithm is employed. The second one extends to a semi-Markov chain representation, where a random distributed fault detection delay is considered. For FTCS with model uncertainties,

state and output feedback controllers which only access FDI decisions are first designed for Mean Square Stability (MSS) using Linear Matrix Inequality (LMI) based techniques. Then, both H_2 and H_∞ performance objectives are considered to accommodate different performance requirements. In addition, iterative LMI algorithms are proposed to solve the nonlinear matrix inequalities resulted from controller synthesis.

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Contents

List of Tables

List of Figures

Acronyms

Acronyms	Definition
CCL	cone complementarity linearization
CUSUM	cumulative sum test
FMEA	Failure Mode and Effects Analysis
FDI	Fault Detection and Isolation/Identification
FTC	Fault Tolerant Control
FTCS	Fault Tolerant Control Systems
LMI	Linear Matrix Inequality
LSMJP	Linear Systems with Markovian Jumping Parameters
LQG	Linear Quadratic Gaussian
MJLS	Markovian Jump Linear Systems
MTBF	Mean Time Between Failures
MTTR	Mean Time To Repair
MSS	Mean Square Stable/Stability
PDF	probability density function
RV	random variable
SIQC	stochastic integral quadratic constraint
SLPMM	sequential linear programming matrix method
SPRT	Sequential Probability Ratio Test

List of Symbols

Symbol	Definition
$\mathbb{R}^n$	the set of n-dimensional real vectors
$\mathbb{R}^{m\times n}$	the set of $m \times n$ real matrices
$Tr(.)$	the trace of a square matrix (.)
$P > 0$ $(P \geq 0)$	the matrix P is positive definite (positive semi-definite)
A^T	the transpose of matrix A
$\mathscr{E}\{.\}$	mathematical expectation of $\{.\}$
$\Pr\{.\}$	the probability of $\{.\}$
$\mathrm{diag}\{.\}$	diagonal matrix
$\mathscr{A}$	the weak infinitesimal operator
$*$	symmetric block outside the main diagonal in a symmetric matrix
$\otimes$	the Kronecker product
$\oplus$	the Kronecker sum, $A \oplus B = A \otimes I + I \otimes B$
$o(\Delta t)$	an infinitesimal of higher order than Δt, i.e. $\lim_{\Delta t \to 0} \frac{o(\Delta t)}{\Delta t} = 0$
$\zeta(t)$, $r(t)$	Markov chain represents fault process
$\eta(t)$	Markov chain represents FDI decision process
$l(t)$	the semi-Markov chain represents FDI decision process
$\mathbf{1}_{\{.\}}$	the Dirac measure, equals one only if $\{.\}$ is true, otherwise zero.
$ker(.)$	orthonormal basis for the null space of (.)

Chapter 1

Introduction

1.1 Development of Fault Tolerant Control Systems

Nowadays, modern industrial systems become more and more sophisticated to accommodate the ever rising requirements on scale and performances, and they rely on complicated control systems to guarantee the smooth operation. Although modern technology has provided capability to manufacture components with enhanced reliability and longer lifespan, faults may still occur in systems' components due to wear and aging. Usually most of the control systems are designed based on the assumption that actuators, sensors and other components of the system function normally. When unexpected faults occur in a complex system, if not dealt with properly, they may result in degraded performance or even loss of system stability. For safety-critical systems, they may cause serious social and environmental diasters.

To overcome such limitations of conventional control strategies, Fault Tolerant Control (FTC) has been developed to accommodate component degradation and faults for an acceptable performance of the overall system. An informal definition of Fault Tolerant Control Systems (FTCS) can be found in [10]: "Fault-tolerant control systems employ redundancy in the plant and its automation system to make 'intelligent' software that monitors behavior of components and function blocks. Faults are isolated, and appropriate remedial actions taken to prevent that faults develop into critical failures. The overall FTC strategy is to keep plant availability and accept reduced performance when critical faults occur." This definition shows a fundamental philosophy of fault tolerant control, i.e. the overall reliability of a system is enhanced not by using the more reliable components, but by managing them in a way that the reliability of the whole system is greater than the reliability of its parts. The goal of such control systems is to maintain high priority design objectives, such as system stability, in case of a severe component fault, so that the "graceful" performance degradation is achieved rather than an abrupt system outage.

FTC is generally classified into passive and active approaches. In passive approaches, the robustness range of a controller is designed to be wide enough to accommodate specified faults within acceptable performance specifications. On the contrary, in active FTCS, detection and isolation of a fault lead to a change in the control operation to accommodate the fault. Actions include but are not limited to re-configuration, controller re-design, change of control path, change of path of measurements [9]. It should be noticed that the main difference lying between the two approaches is whether or not faults are explicitly detected and identified.

In the literature, special terminologies are used for passive fault tolerant control systems: **Integral Control** [82] deals with actuator faults and sensor faults in control loops, which can guarantee the system stability in case of loop failures (i.e several feedback loops become open) caused by the faults. **Reliable Stabilization** [112] aims at accommodating the possible controller faults. In this scheme, two or more controllers are used to stabilize the plant in parallel, when one or more of them work normally, the closed-loop system is stabilizable. **Simultaneous Stabilization** [95] is about designing a single controller to stabilize different plants. Such a scheme is suitable for dealing with system component faults.

In active FTCS, one major component is the reconfigurable controller, which can reconfigure or reconstruct the controller online according to the information of the post-fault system. "Reconfigure" means that only parameters of a controller are changed while "reconstruct" may result in a complete change of control system structure. The conventional classification of FTCS is shown in Figure 1.1, and a typical block diagram of active FTCS is shown in Figure 1.2.

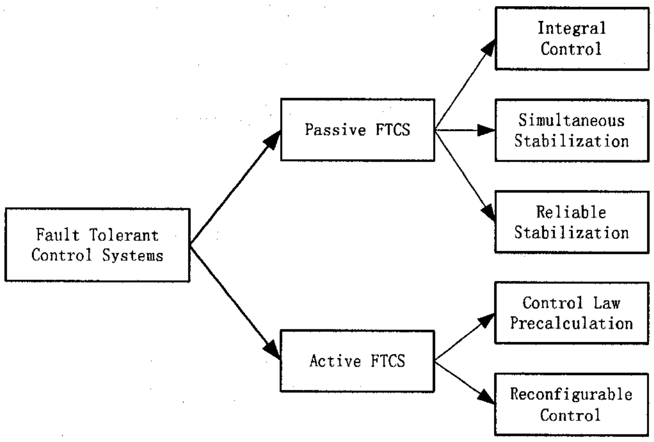

Figure 1.1. A simple classification of fault tolerant control systems

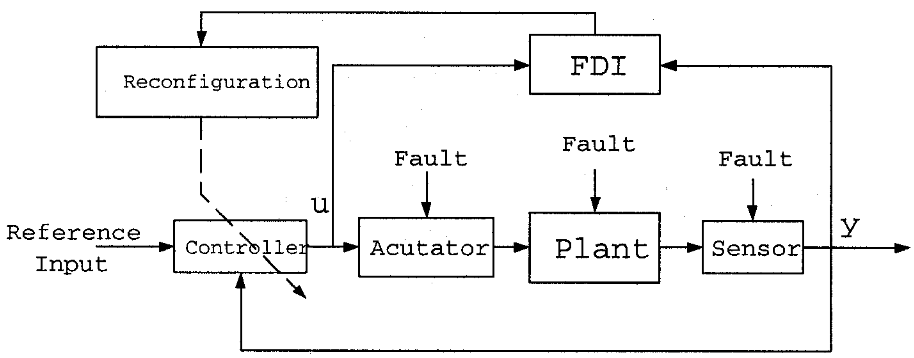

Figure 1.2. A scheme of active fault tolerant control systems

For practical applications, passive FTCS have obvious disadvantages. First, it has limited capability in fault tolerance since only known fault scenarios can be coped with. Because faults may have different sizes and effects, they may require different remedial actions. In this case, it is hard to design a passive fault tolerant controller to accommodate such faults. Secondly, it is possible that no feasible solutions exist for a given fault scenario. In fact, passive FTC algorithms are more suitable for baseline controllers design, whose function is to increase the robustness of the system, and to protect the system from sudden deterioration so that more time is gained for high level control reconfiguration mechanism to take remedial actions. In this research work, we are focused on active FTCS analysis and design. A brief review of FTC design methods is included in Section 1.3, while the more thorough introduction of FTCS can be found in [92], [109] and the references therein.

1.2 Fault Classification and Modeling

FTCS are designed and built to cope with faults occurred in control systems, therefore it is necessary to first define "faults" and associated terminologies.

IFAC SAFEPROCESS Technical Committee has given formal definitions to the following terminologies in the field of fault diagnosis and fault tolerant control [21], [57]:

Fault: an unpermitted deviation of at least one characteristic property or parameter of the system from the acceptable/usual/standard condition.

Failure: a permanent interruption of a system's ability to perform a required function under specified operation conditions.

Fault detection: determination of the faults in a system and the time of detection.

Fault Isolation: determination of the kind, location and time of detection of a fault. Follows fault detection.

Fault Identification: determination of the size and time-variant behavior of a fault. Follows fault isolation.

Fault Diagnosis: determination of the kind, size, location and time of detection of a fault. Includes fault detection, isolation and identification.

The conventional classification of faults in control systems is based on their locations, i.e. component faults, actuator faults and sensor faults (or instrument faults). Each category contains several types of faults [12].

Consider a continuous-time LTI system described by:

Consider a continuous-time LTI system described by:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + D_1 d(t) + f_1(t), \\ y(t) = Cx(t) + D_2 d(t) + f_2(t), \end{cases} \tag{1.1}$$

whe[illegible]te, control signal, system output and disturbance respectively; $f_1(t) \in \mathbb{R}^{r_1}$, $f_2(t) \in \mathbb{R}^{r_2}$ are fault signals imposed on the system state and the output respectively; all matrices have compatible dimensions. In the sequel, expressions of $f_1(t)$, $f_2(t)$ in different fault scenarios are given. For the sake of simplicity, it is assumed that a state feedback law $u(t) = K_c x(t)$ is used for the normal system.

System component fault:

$$f_1(t) = (A_f - A)x(t), \tag{1.2}$$

where A_f is the post-fault system matrix.

Actuator fault (for the i-th actuator):

$$f_1(t) = \begin{cases} (k_i - 1)b_i k_{ci} x(t),\ 0 < k_i < 1, & \text{lost of effectiveness,} \\ -b_i k_{ci} x(t), & \text{float,} \\ -b_i k_{ci}(x(t) - x(t_F)), & \text{lock-in-place,} \\ -b_i(K_{ci} x(t) - U_{iL} e_i),\ U_{iL} = U_{iMin} \text{ or } U_{iMax}, & \text{hard-over,} \end{cases} \tag{1.3}$$

where k_i is the effective coefficient of the faulty actuator, b_i is the i-th column of input matrix B, k_{ci} is the i-th row of K_c, t_F is the time instant of fault occurrence, and e_i is the i-th column of an identity matrix with appropriate dimensions.

Sensor fault (for the i-th sensor):

$$f_2(t) = \begin{cases} b_i e_i,\ b_i \neq 0, & \text{bias,} \\ b_i(t) e_i,\ b_i(t) \neq 0, & \text{drift,} \\ -C_i(x(t) - x(t_F)), & \text{sensor freezing,} \\ (k_i - 1)C_i x(t), & \text{calibration error,} \end{cases} \tag{1.4}$$

where $b_i/b_i(t)$ is the bias/drift term for the i-th faulty sensor, and k_i is the effective coefficient of the faulty sensor. The dimension and the i-th row of C_i are the same as those of C, while other entries of C_i are zeros.

From the above scenarios, a general representation of the fault signal $f(t)$ can be given as:

$$f(t) = Ex(t) + v(t). \tag{1.5}$$

For different types of faults, the specific expressions of E and $v(t)$ can be obtained from corresponding equations above.

The equations above show that component and actuator faults normally change the system matrix, same with sensor freezing and calibration error when the closed-loop system is considered. Therefore, these faults can result in the loss of system stability, this is why fault tolerant control design is so important for safety-critical applications. As for sensor bias/ drift, they only act like disturbance in the system and are relatively easier to deal with.

1.3 Methods of Fault Tolerant Control Design

The design process of an active FTCS is a systematic work from the engineering point of view. It ranges from FMEA (Failure Mode and Effects Analysis) to remedial actions selection, etc. The detailed discussion on these matters can be found in [7] [8] [9] [10] [57]. However, in this thesis, only the control related aspects of FTCS are considered, and other issues of the system analysis are out of the scope.

The premise for designing FTCS is that the system possesses hardware redundancy, such as redundant sensors and actuators. Otherwise, when some components break down, the corresponding measurement will be invalid and the necessary remedial action cannot exert to the process. The redundancy in the system also improves the FDI results, especially when system contains model uncertainty and disturbance, redundant measurements help eliminate faults' effects on diagnostic results. Meanwhile the existence of redundant components will simplify the design and improve the performance of FTCS. For example, if a system has more independent measurement, then the performance loss by using output feedback will be minimized as well. As for actuator redundancy, if post-fault system input matrix has full row rank, Pseudo-Inverse algorithm proposed in [46] can restore the closed-loop system matrix perfectly. Stability is the objective of highest priority for any control systems, and it is no exception for FTCS. Once the stability is retained, then fault tolerant controllers will try to restore other dynamic performances as much as possible depending on the degree of redundancy remainded in the post-fault system. Different algorithms use different control strategies to achieve preset objectives, and they are the kernel part of FTCS design. Common control strategies used in FTCS include Pseudo-Inverse Method (PIM), Eigenstructure Assignment (EA), Model Following, Multiple Model Adaptive Control, Four-Parameter Control and stochastic FTCS.

Pseudo-Inverse Method

Traditional Pseudo-Inverse Method (PIM) has been well used in flight control systems. If the input matrix has full row rank, the reconfigured system can fully restore the original system eigenvalues. But it is not the case without the rank condition, and in this case the post-fault system stability is not guaranteed. In [46], a modified PIM algorithm is proposed, which can guarantee the stability of the post-fault system. The design objective is to minimize the Frobenius norm of the difference matrix between the original and impaired system matrix, i.e.

$$\min_{K_f} J = \|(A+BK) - (A_f + B_f K_f)\|_F \ ,$$

where the matrices with subscript f stand for the corresponding post-fault matrices. This optimization objective is equivalent to minimize the distance between eigenvalues of pre-fault and post-fault closed-loop systems. However, this algorithm does not consider the modeling uncertainty, hence cannot provide robust stability conditions.

Eigenstructure Assignment

Since a system's dynamics is not solely determined by its eigenvalues but also by its eigenvectors, [62] uses eigenstructure assignment methods to keep the eigenstructure of the post-fault system as close as possible to that of the pre-fault system. If state feedback is used, all of the original eigenvalues can be restored exactly, while only part of them (most dominant ones) can be restored if using output feedback. Through projection, the corresponding eigenvectors are assigned close to those of the pre-fault system. However, since it is impossible to restore exactly the eigenvectors of the original system, such projection cannot guarantee the similarity of the dynamics between the pre-fault and post-fault systems. As same as PIM, the robust stability is not discussed. [68] adopts the similar idea, but with a constrained optimization to minimize the distance of the assigned eigenvectors with those of the pre-fault system. The constraints of this optimization reflect the requirement on system stability. Obviously, it leads to a better design.

Model Following Methods

For PIM, the main idea is to make the closed-loop system matrix of the reconfigured system close to that of the nominal one. An alternative design named as Model Following method shares the similar design philosophy, but with a different measure of the "closeness" of the two systems (post-fault and nominal) [47] [56] [64]. The objective of model following method is to make the state trajectories of the reconfigured system as close to those of nominal system as possible. For the implicit model following methods, a sufficient condition for the perfect model following is proposed for stability, i.e. the Erzberger's condition. However by using explicit model following, stability can be attained even when Erzberger's condition is not satisfied. One disadvantage is its complexity, since all the states of the reference model need to be generated for use. A block diagram of the explicit model following is given in Figure 1.3.

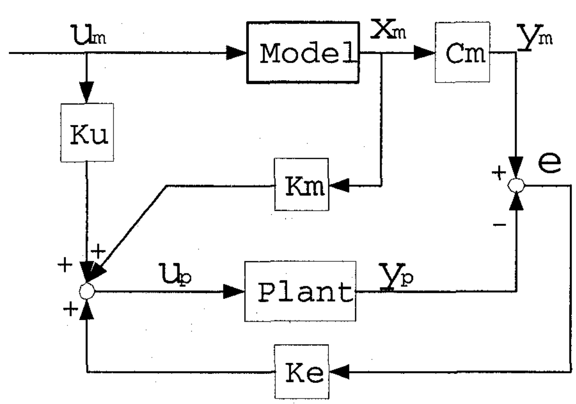

Figure 1.3. Block diagram of explicit model following

Adaptive Control

Due to the similar design strategies, adaptive control methods have been used in FTCS design. In [46], the application of multiple model adaptive control in reconfigurable flight control systems is discussed. In this scheme, a model set is built including one nominal model and other fault models. Kalman filter banks are applied to estimate the state and parameters. The control action is probability-weighted for each candidate model. For deterministic systems, the Kalman filter bank is replaced by a bank of observers. Their outputs are compared, and the one with the minimal value indicates the true plant model, then, the corresponding control law is applied [12]. Other applications of adaptive control can be found in [22], [106], [107], which use the plant-model matching to solve the actuator lock-in-place type of fault. Compared with conventional adaptive control, MMAC reacts faster to abrupt changes such as faults, and can greatly improve the transient response before convergence [84]. For faults with independent effects on system output, [63] gives a combined design of adaptive observer and controller to estimate the fault and stabilize the post-fault system.

Four-parameter Control

Generally speaking, the approaches discussed above only involve controller reconfiguration, i.e. no FDI scheme is considered. But there are also two main kinds of integrated fault tolerant control systems. Here, "integrated" means that fault detection scheme is designed with controller simultaneously or the characteristics of FDI are taken into consideration during the controller design. The so-called "four-parameter control" is one kind of passive FTC design, where both feedback controller design and fault detection design are carried out at the same time.

Four-parameter controller [58] took the name from the original setting, where controller/ observer to be designed can be split into four parts, i.e. four transfer functions. In this setting, FDI and controller are simultaneously designed using mixed optimization or multi-objective design methods. The following figures show its configuration [85]:
In Figure 1.4(a), w, u, z, y_c, are exogenous input, control signal, controlled variable and ideal output; f_a, f_s stand for actuator and sensor faults respectively, a is the diagnostic signal. This configuration, when considering model uncertainty, can be transformed into

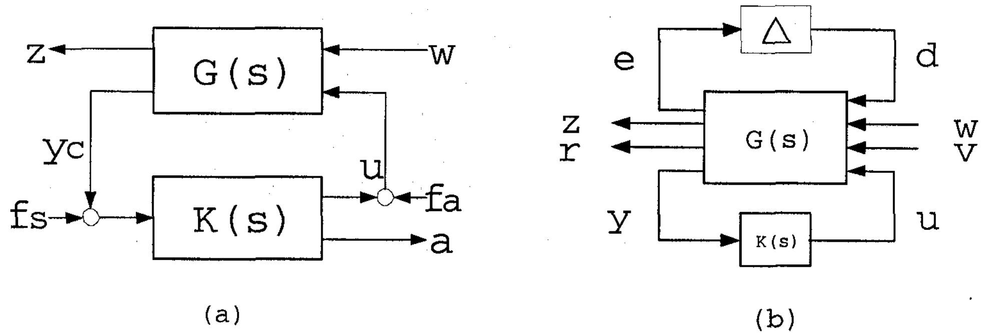

Figure 1.4. Four-parameter controller structure : (a) system setup (b) standard setup with model uncertainty

the standard formulation in robust control. In Figure 1.4(b), r is the diagnostic signal, fault models are included in the generalized system as frequency weight on the fault signals: $f = W_f(s)v$, where v is a signal that is anticipated to have a flat power spectrum. With this formulation, robust control theories can be applied for integrated system design to deal with disturbances and model uncertainties. For example, the following design objectives can be minimized for the system shown in Figure 1.4(b):

$||G_{zw}||_\infty$: disturbance rejection and system robustness,

$||G_{zv}||_\infty$: minimize effects of undetectable faults,

$||G_{rw}||_\infty$: reduce false alarms,

$||G_{rv}||_\infty$: alarm signal is a good estimate of potential faults.

Since such a design involves more than one objective, therefore multi-objective design or mixed-objective design must be used to handle the design problem.

This integrated FDI and controller design guarantees the performance and robustness. However, this scheme treats the faults as exogenous unknown input terms which are independent from system control or state, hence this design is quite restrictive since it is only applicable for sensor bias and drift among all kinds of faults discussed in Section 1.2. In other word, faults considered in this FTCS scheme do not challenge system stability. A similar system structure is used in [111]. For actuators belong to a specific set, stability can be guaranteed, and performance measured by H_∞ norm can be maintained.

Due to the increasing interests in multiple-objective control design, this integrated approach has attracted lots of attention recently. Some new results include: mixed H_2/H_∞ approach to simultaneous fault detection and feedback controller design [66]; and later that work has been extended to FDI design using the same methodology [67] to cope with additive fault signals with different characteristics, i.e. $\mathbb{L}_2$ bounded and white noise. Similar works include FDI design for uncertain systems under feedback control [53] etc.

The stochastic approach using Markovian Jump Linear System model

Because faults can change the system dynamics, a series of models can be used to represent different scenarios including the normal operation mode and multiple faults. A homogeneous Markov chain can then be used for representing such jumps among multiple models. In this formulation, the task of FDI is to retrieve the system modes from input-output data, and controller reconfiguration is carried out based on FDI decision.

This approach is the other integrated approach for FTCS. It provides an integrated framework to analyze and design for stochastic stability and performance of FTCS. The strength of this approach lies in its ability to accommodate random nature of fault occurrence and parametric faults. When FDI scheme is modeled by another stochastic process independent from the system fault process, imperfect FDI decisions such as detection delay, false alarms and missing detections can be quantized and taken into consideration when passive or active fault tolerant controller is designed.

Though this stochastic FTCS share the same representation with Markovian Jump Linear Systems (MJLS), the main difference lies in that the former have FDI scheme based on whose decision that controller reconfiguration is made, while in MJLS, the system fault mode is assumed to be instantly available for control.

As this approach "combines robustness design with reconfigurable control and FDI, it is a challenging and emerging theoretical field in fault tolerant control" [92]. The main work of this thesis focuses on the analysis and design within this stochastic framework, and the detailed introduction and discussion are given Chapter 3.

1.4 Motivation of the Research

FDI scheme and controller reconfiguration are two main ingredients of active FTCS. But in existing literatures, most of active FTCS designs neglect dynamics of the FDI mechanism, instead, they assume that FDI can provide immediate and correct post-fault system information for controller reconfiguration. However, when disturbances and model uncertainties are present, such a "perfect" FDI assumption does not hold in most practical applications. The decision and information provided by FDI may be incomplete or even incorrect, e.g. unisolatable faults, false alarms, missing detection and detection delay, etc. Without taking these characteristics into consideration, the fault tolerant controller designed cannot achieve satisfactory performance for overall FTCS. Hence an integrated FTCS is more desirable for achieving better fault tolerance capability and performance. An integrated design means that the characteristics of both FDI and control are considered in the closed-loop form, and trade-offs are taken into account in the design so that functions of FDI and control reconfiguration are well-balanced to guarantee the overall system performance. The integrated framework based on Linear Systems with Markovian Jumping Parameters (LSMJP) provides a good framework, where these issues can be studied.

The study of the stochastic active FTCS using MJLS as the open-loop system dates back to 1989, where besides the stochastic process representing fault process, there is another stochastic process standing for FDI decision. Since then, many researchers have worked in this area. However, most of the works in this area concentrate on the stability analysis or synthesis of controller accessing both system real fault mode and FDI decision, which is hard to satisfy in practical applications. The assumption that both system real fault mode and FDI mode are available for control obscures the role of FDI scheme, and makes the introduction of FDI scheme redundant and unnecessary. Therefore, a more reasonable assumption is that controller reconfiguration is solely based on FDI decisions. However, with this assumption, numerous available results on MJLS cannot be applied directly into the area of FTCS. To design a FDI-based controller, the main obstacle lies in that if assuming controller solely accesses FDI decision but no real fault mode, the number of controllers to be solved is much fewer than the total number of constraints. Hence it becomes a multi-objective design problem, which in general cannot be solved efficiently. To find effective methods to design FDI-based controller is the main motivation of the research.

In this stochastic active FTCS framework, FDI decisions are conventionally modeled by a piecewise Markov chain. However, it implies that the FDI algorithm is memoryless, which is not the case for most of advanced FDI schemes. Under some circumstances, such a modeling of FDI decision process should be extended to more complex situations, such as using a semi-Markov chain. For this reason, in this thesis, systems with random fault detection delay are studied. Furthermore, performances besides stability should be defined and introduced into the analysis and synthesis of controller for active stochastic FTCS.

1.5 Outline of the Thesis

Chapter 2 introduces some modeling methods of FDI decision process. These include modeling only fault detection delay when systems are subject to "strongly detectable faults" and modeling using stochastic processes. The Markov chain modeling is the focus and it will be shown in Chapter 6 that some semi-Markov chain modelings can be transformed into this case by augmentation of Markov chain.

Chapter 3 describes the stochastic fault tolerant control framework based on MJLS model. Since MJLS is adopted as the open-loop system model, the work on MJLS is reviewed first and the differences between MJLS and the stochastic FTCS are pointed out after the latter is introduced in details. For these stochastic systems, the stochastic stability should be chosen as the primary design objective. Different from deterministic systems, stochastic stability has many different definitions. These definitions are provided. Parallel to the deterministic cases, stochastic Lyapunov function will be the powerful tool in the stability analysis and stabilizing controller synthesis of the stochastic systems.

In Chapter 4, using the conventional stochastic FTCS model, where FDI scheme is modeled by a Markov chain, the stabilization results using the Mean Square Stability (MSS) criterion for controller with different forms are summarized and expressed in terms of Linear Matrix Inequalities (LMIs), which can be efficiently solved using many available convex optimization toolboxes. Compared to the available results that involve solving coupled Riccati equations, results given in this chapter are much simpler and easier to use. In this chapter, stabilizing controllers include the "full information" controller (controller accesses both real fault mode and mode of the FDI decision process) and the FDI-based controller are designed. Due to the complexity in the synthesis of the latter, a two-step procedure is given using Projection Lemma. For state feedback controller synthesis, techniques from mixed objective design can be employed to obtain an improved design approach.

In Chapter 5, besides the MSS, the analysis and synthesis for reconfigurable controllers satisfying H_2 performance are studied. Adopting the operator theory proposed by Costa and his colleagues, the conditions for MSS and the H_2 performance are expressed in terms of nonlinear matrix inequalities. iterative LMI algorithms are then presented, which adopt the techniques from multi-objective control design, to effectively solve the design problem. The synthesis for state feedback, output feedback in both continuous-time and discrete-time is discussed.

H_∞ performance with respect to $\mathbb{L}_2$ bounded disturbance is the main concern in Chapter 6. Different from settings in previous chapters, systems with fault detection delay are to analyze and design. The assumption made is that the fault detection delay is an exponentially distributed random variable. FDI decision process under the circumstances is actually a semi-Markov chain, which undoubtedly may be more realistic than the Markov chain modeling. In addition, two extensions are made so that the topic can be covered in complete. The result are first extended to the conventionally two-Markov-chain framework but with a more general performance than the H_∞ called stochastic integral quadratic constraint (SIQC). And the second extension discussed is on a non-exponential fault detection delay case, where the hypoexponential distribution is used to approximate an arbitrary distribution. This approach provides a feasible way to study FTCS employing complicated FDI algorithms such as Sequential Probability Ratio Test (SPRT) which does not satisfy the memoryless property.

Finally, the conclusions are drawn and the future work are discussed in Chapter 7.

1.6 Contribution

The contribution of this thesis includes three parts. First, available results on the "full information" controller synthesis are rewritten in terms of LMIs. Compared with the original conditions, results here are much simpler and more straightforward. In addition, three approaches are proposed for the more practical "FDI-based" stabilization controller design, including LMI based and iterative LMI based.

Next, besides Mean Square Stability, H_2 and H_∞ performances are introduced into this stochastic FTCS framework as design objectives, and controllers are designed using these performance objectives.

Last, analysis and synthesis results are extended to stochastic FTCS where FDI decision process is modeled by a semi-Markov chain. By using hypoexponentially distributed random variables to approximate the mode sojourn time of the semi-Markov chain, the system can be Markovianized, then previous results can be applied. This method provides an engineering approach to solve problems such as stochastic FTCS with semi-Markov chain modeling of FDI decision processes, and non-Markovian Jump Linear Systems.

Chapter 2

Modeling of the FDI Decision Process

2.1 Introduction

In the previous chapter, modeling of faults according to their occurrence locations was discussed. For active FTCS, the FDI scheme is an important component based on whose outcome the controller is reconfigured. Obviously, the performance of FDI will directly affect the stability and performance of FTCS. Therefore, modeling of FDI decision process is necessary in an integrated FTCS in order to perform the analysis and synthesis in a closed-loop configuration.

The role of FDI scheme is to detect and estimate faults after their occurrences. Since some performance criteria, such as missing detection rate, false alarm rate and detection delay may be contradictory in design, trade-offs are necessary. Therefore, performance of FDI is dependent on the nature of the faults, the chosen design objectives and the fault diagnosis algorithms.

When focused on the evolution of systems subject to random faults modeled by a Markov chain, modeling of FDI decision process using stochastic processes is natural. Such a problem will be discussed in this chapter, where faults are considered to be random and can be described by a Markov chain.

In this chapter, to simplify the analysis, a special category of faults named as "strongly detectable faults" is considered first, for which the FDI scheme can always make correct decisions after a fault detection delay. This delay can be an arbitrary distributed random variable depending on the specific faults present and FDI algorithms used. Exponential distribution is the simplest case that can model the memoryless FDI algorithm, while the hypoexponential distribution is more general and can be used to approximate other distributions.

2.2 FDI Decision Process: Detection Delays

Practical FDI subsystems may have imperfect decisions in terms of detection delays, false alarms and missing detections. A simple modeling of FDI scheme is just to consider fault detection delays, with the assumption that all faults present in the system can be eventually detected and identified. Actually, this modeling is quite simple yet can still capture the characteristics of many practical systems.

The assumption that all faults present in the system can be detected and identified can be satisfied if those faults belongs to the category named as "strongly detectable faults". This category of faults can be guaranteed to be detectable even in the presence of disturbance.

In the area of fault detection, the research on "strongly detectable faults" is the extension of the Unknown Input Observer (UIO) approach, which has the limitation when the perfect decoupling conditions cannot be satisfied due to the presence of model uncertainties and disturbances.

Hou and Patton found that when the residual becomes robust to disturbance, it becomes insensitive to faults as well [54]. To overcome this shortcoming, both robustness to disturbances and sensitivity to faults should be taken into consideration and properly compromised, where fault sensitivity can be measured by the H_- index, which is defined as the minimal eigenvalue of a transfer function.

However, obstacles still exist in this design. Since H_- is not a system norm, it prevents us from directly using robust control related theory to carry out the design. For this reason, many efforts are not very successful. Some trial and error is needed as for [54], and a heuristic iterative LMI algorithm was proposed in [113] without guarantee of convergence.

As a result, a more conservative H_∞/H_∞ design is used. Using the concept of "inverse system" and multi-objective optimization techniques, [108] proposed algorithms for both static and dynamic fault detection observers design using iterative LMI algorithm for systems with square transfer function. The main breakthrough was made in [73], where the analytical expression of H_- is given. With this result available, the multi-objective design can be applied for solving H_∞/H_- problem using available iterative LMI techniques.

Note that given the threshold J_{th}, the fault detection delay is defined as the minimum T_d satisfying

$$\int_0^{T_d} r(t)^T r(t) dt > \|G_{rd}(s)\|_\infty \|d\|_2$$

where $r(t)$ is the residual generated.

2.2.1 Exponentially Distributed Fault Detection Delay

It is proper to consider the randomness of the fault detection delay. From (1.5), the fault $f(t)$ is dependent on system state $x(t)$ and a fault related additive signal $v(t)$. Notice that the fault occurrence time instant t_F is random, which makes $x(t_F)$ and $v(t_F)$ random as well. Since the detection time T_d is dependent on $x(t_F)$ and $v(t_F)$, T_d is also a random variable.

The simplest fault detection logic is to compare the current residual with a threshold and make decision, which means only the current observation of the residual is used and discarded in the next test when a new observation is available. Under the circumstances, the FDI decision has the memoryless property, and the fault detection delay can be modeled as a memoryless random variable, e.g. one with the exponential distribution, whose probability density function (PDF) is shown in Figure 2.1.

In practice, exponential distribution arises in the modeling of the time between occurrence of events, the lifetime of devices and systems, and it can be obtained as a limiting form of the geometric random variable [72]. If MJLS model is used to represent open-loop FTCS, an advantage for using exponential distribution lies in its natural relationship with Markov chains. The time that a Markov chain occupies a specific state, also called the dwell time or sojourn time, is an exponential distributed random variable. Therefore,

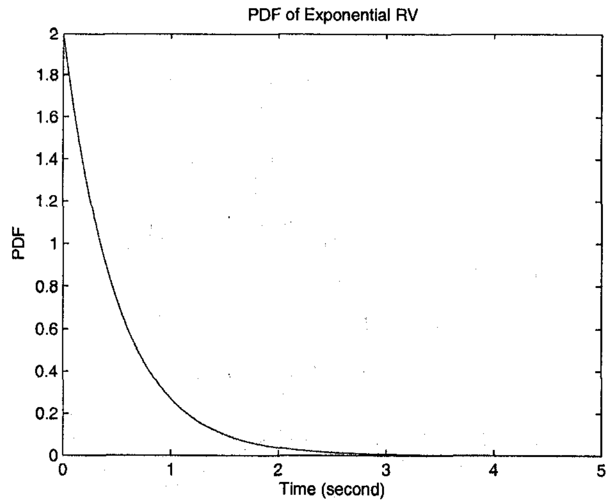

Figure 2.1. PDF of exponential random variable ($\lambda = 2$)

using the exponential distribution assumption for fault detection delays may lead to a joint Markov process in analysis of such a system.

2.2.2 Non-exponentially Distributed Detection Delay

The memoryless property of the exponentially distributed fault detection delay means that the time FDI needs to make a decision is independent of the time elapsed. Through its probability density function, it implies that most of the faults are more likely to be detected at the very beginning after faults occur. This seems to be contradicting with the common sense that at the early stage, information extracted from the residual is insufficient to make a good decision compared to that at a later time.

For more advanced FDI schemes, past observations are often utilized to provide as much information as possible. The FDI scheme will continuously update certain index as new samples are available until decision can be made, and FDI scheme will be reset thereafter. This category of algorithms for fault detection includes: Sequential Probability Ratio Test (SPRT) and cumulative sum test (CUSUM) [6]. For FDI schemes using these algorithms, the memoryless property does not hold. Therefore, exponential distribution may not be appropriate for modeling FDI delay in general, especially when more complex

FDI schemes are used.

In [114], for a fault tolerant navigation system employing SPRT to decrease the false alarm rate, it shows that a second order Erlang distribution can approximate the fault detection delay well.

A more general random distribution, namely, hypoexponential distribution, is a simple extension of the exponential distribution to model the FDI delay. A hypoexponentially distributed random variable is the sum of two or more independent exponential random variables.

The expression of the hypoexponential distribution is given as:

$$Y = \sum_{i=1}^{N} X_i \,, \tag{2.1}$$

where X_i, $i = 1, 2, \ldots, N$ are independent exponentially distributed random variables with the rate parameter λ_i. A special case is when $\forall \lambda_i = \lambda$, $i = 1, 2, \ldots, N$ and it is known as the "Erlang distribution".

The main reason to study hypoexponential distribution is due to its closeness to a so-called 'hyperexponential distribution", whose probability density function (PDF) is given as:

$$f_Y = \sum_{i=1}^{N} \rho_i \lambda_i e^{-\lambda_i x}, \quad \sum_{i=1}^{N} \rho_i = 1 \,.$$

An important characteristics of the "hyperexponential distribution" is that it can approximate arbitrary distributions on $[0, +\infty)$ with arbitrary precision [13]. Unfortunately, since ρ_i may be negative, it hinders the application of this distribution in modeling of FDI decision process. However, it will be shown later that the "hyperexponential distribution" can still approximate many random distributions well enough.

The PDF of the hypoexponential distribution is not trivial to compute. Consider two cases here. The first case is for the sum of $N = 2$ exponentially distributed RVs, X_i with PDF $f_{X_i}(x) = \lambda_i e^{-\lambda_i x}$, and $\lambda_1 \neq \lambda_2$. Then the PDF of $Y = X_1 + X_2$ can be obtained as

$$f_Y(y) = \frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1} e^{-\lambda_1 y} + \frac{\lambda_1 \lambda_2}{\lambda_1 - \lambda_2} e^{-\lambda_2 y}, \qquad y \geq 0. \tag{2.2}$$

Similarly, when $N=3$ and the rate parameters satisfy $\lambda_2=\lambda_3$ and $\lambda_1\neq\lambda_2$, the PDF of $Y=X_1+X_2+X_3$ becomes

$$f_Y(y)=\frac{\lambda_1\lambda_2^2}{(\lambda_1-\lambda_2)^2}e^{-\lambda_1 y}-\frac{\lambda_1\lambda_2^2}{(\lambda_1-\lambda_2)^2}e^{-\lambda_2 y}+\frac{\lambda_1\lambda_2^2}{\lambda_1-\lambda_2}ye^{-\lambda_2 y},\qquad y\geq 0. \qquad (2.3)$$

The "hypoexponential distribution" can be used to exactly represent the χ^2 distribution and the γ distribution. The following example shows how to approximate a non-exponential distributed detection delay. The objective is to approximate a log normal distribution, for which the logarithm of a variable has a normal distribution. This distribution is used instead of normal distribution due to the fact that fault detection delay should always be greater than zero.

Example 2.1 In this example, it is shown that the PDF of a random variable, which is the summation of exponentially distributed random variable, can be used to approximate the PDF of a log normal distribution.

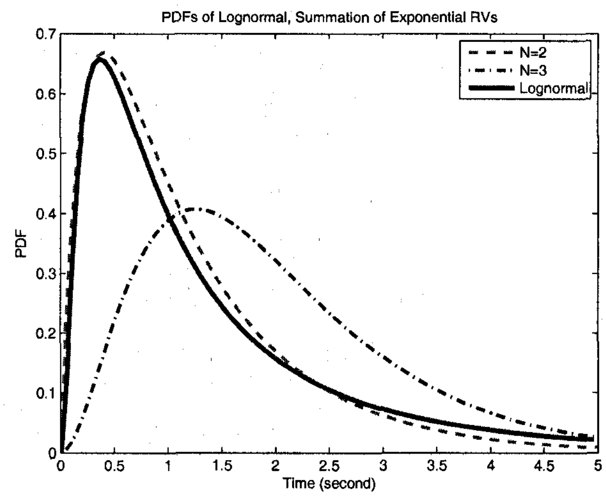

Figure 2.2. Approximation of non-exponential RV using summation of exponential RVs

In Figure 2.2, the solid curve shows the PDF of the log normal distribution, with $S=1$, $M=0$; the dashed curve represents the PDF of summation of two exponential random variables with $\lambda_1=1$, $\lambda_2=5$; the dash-dotted curve shows the PDF of summation of two exponential random variables with $\lambda_1=1$, $\lambda_2=2$, $\lambda_3=2$. Obviously, using summation of two exponentially distributed random variables, an approximation close to the log normal distribution is obtained.

2.3 Modeling of FDI Decision Process Using Stochastic Processes

2.3.1 FDI Modeling Using Stochastic Processes

The previous discussion only focused on modeling the random fault detection delay. Considering other characteristics, e.g. false alarms, missing detections, etc, the FDI decision process can be modeled as a separate stochastic process from the fault process.

Markov processes represent the simplest generalization of independent processes by allowing the outcome at any instant to depend only on the outcome that precedes it and none before that. In another word, for a Markov process $\zeta(t)$, the past has no influence on the future if the present is specified. This means that if $t_{n-1} < t_n$, then

$$Pr\{\zeta(t_n) \leq x_n | \zeta(t),\, t \leq t_{n-1}\} = Pr\{\zeta(t_n) \leq x_n | \zeta(t_{n-1})\}\ .$$

A Markov chain is a special kind of Markov process where the system can occupy a finite or countably infinite number of states, such that the future evolution of the process, once it is in a given state, depends only on the present state but not on how it arrives at that state [90]. A Markov chain is said to be homogeneous in time if the probability $Pr\{\zeta(t_m) = x_m | \zeta(t_n) = x_n\}$ depends only on the difference $m-n$.

The homogeneous Markov chain is one of the most common choices for modeling fault process, where countable finite states are used to indicate normal and faulty scenarios of the system. And the switching from one state to another may represent onset of a specific fault or resume of normal operation from a faulty scenario after reparations. The homogeneous assumption holds in general since for most of systems, the reliability related performances are time invariant. An important prerequisite for using Markov chain is that each transition should be an independent event. To use this representation of random faults, a priori information about the distribution of the mode sojourn time, and state transition probability must be obtained before analysis and design can be carried out within this framework. In discrete-time, it means that information in the form of an a priori probability distribution for the mode transition time is known. The a priori probability can be either obtained from historical data or from Monte Carlo simulations.

For systems subject to the fault process modeling by a Markov chain, FDI decision process is determined by the properties of faults and the algorithm used. Control engineers and researchers from signal processing field have different opinions on the modeling of FDI decision process. Engineers observed that in most high reliability fault-tolerant control systems, the occurrence of component faults is relatively rare compared to the frequent occurrence of redundancy management decision events. This property leads to a system with events of two time-scales: a slow time-scale for fault events and a fast time-scale for FDI events [115]. Therefore, common fault detection and identification algorithms can still be applied in this framework.

Researchers from signal processing field tend to construct specific algorithms to identify the fault mode from input-output data, and the transition rate matrix of the fault Markov chain. This was first investigated in early 1950's for online quality control problems. Nowadays many algorithms have been developed to solve the Markov chain mode estimation problem [6].

To be specific, consider the following Markovian Jump Linear System:

$$\begin{cases} \dot{x}(t) = A(\zeta(t),t)x(t) + B(\zeta(t),t)u(t) + D(\zeta(t))w(t), \\ y(t) = C(\zeta(t))x(t) + E(\zeta(t))w(t), \end{cases} \tag{2.4}$$

where $x(t) \in \mathbb{R}^n$ represents the system state vector, $u(t) \in \mathbb{R}^m$ the system input vector, $y(t) \in \mathbb{R}^s$ the system output, $w(t) \in \mathbb{R}^p$ is the unknown exogenous disturbance/noise. $\zeta(t)$ is assumed to be a measurable homogeneous continuous-time Markov chain with a finite state space $S_1 = \{1, 2, ..., q_1\}$.

The transition probability of the fault process $\zeta(t)$ is described as follows:

$$\zeta(t) : Pr\{\zeta(t+\Delta t) = j | \zeta(t) = i\} = \begin{cases} \alpha_{ij}\Delta t + o(\Delta t), & i \neq j, \\ 1 + \alpha_{ii}\Delta t + o(\Delta t), & i = j, \end{cases}, \ i, j \in S_1, \tag{2.5}$$

where $\alpha_{ii} = -\sum_{j\in S_1,\ j\neq i} \alpha_{ij}$, $\Delta t > 0$, $\lim_{\Delta t \to 0} \frac{o(\Delta t)}{\Delta t} = 0$, and α_{ij} is the transition rate of the fault process.

The objective of the FDI scheme can be achieved by estimating Markov state $\zeta(t)$ and/or continuous system states $x(t)$. Some recent related works include: stochastic sampling algorithm for maximum a posteriori (MAP) state estimate of Markov state [36], particle filter method [37], iterative algorithm for marginal MAP sequence estimate of the Markov chain (MMAP) [38], expectation maximization (EM) for MAP [74], interacting multiple model (IMM) method [65], and a research monograph on the Hidden Markov Model (HMM) approach [39].

However, it is worth mentioning that computing the conditional mean state estimates $\mathscr{E}\{x(k)|y(1),y(2),\ldots,y(k)\}$ and $\mathscr{E}\{\zeta(k)|y(1),y(2),\ldots,y(k)\}$ (consider discrete-time case here with $y(i)$ as system output) for MJLS is an NP-hard combinatorial optimization problem. The computation burden exponentially increases with the number of output samples $\{y(1),y(2),\ldots,y(k)\}$ k. Therefore, suboptimal methods are the only feasible approaches to handle this problem. Except computation burdens, some disadvantages are common with the current algorithms: local optimum and inconsistent state estimates. These factors make it extremely hard to analyze the exact distribution of FDI decision process. To the author's knowledge, there is no algorithm available for online use.

Therefore, in this thesis, it is assumed that those regular FDI algorithms can be applied here to provide estimation of system fault process. To simplify the analysis and synthesis, the Markov chain and the semi-Markov chain can be used to model the FDI decision process.

The Markov chain is most commonly used in the context of FDI decision process. The following formulation was firstly introduced by Srichander and Walker in 1993 [104]:

Assume that the FDI decision process $\eta(t)$ is modeled as separate Markov chain taking values on $S_2 = \{1,\ 2,\ldots,\ q_2\}$ with state transition rate depending on the current state of $\zeta(t)$:

$$\Pr\{\eta(t+\Delta t) = s|\eta(t) = j,\ \zeta(t) = i\} = \begin{cases} \beta^i_{js}\Delta t + o(\Delta t), & j \neq s, \\ 1 + \beta^i_{jj}\Delta t + o(\Delta t), & j = s. \end{cases}$$

As an extension of Markov chain modeling, the semi-Markov chain is already widely used in reliability analysis of systems. A semi-Markov process $Z(t)$, also denoted as a Markov renewal process, is associated with and can be constructed from a pair of processes (X,Y), where X is a Markov chain with state space S and state transition probability P, whereas Y is a process for which $Y(n)$ depends only on $r = X(n-1)$ and $s = X(n)$, and whose distribution function is F_{rs}. The semi-Markov process Z is then the process that chooses its state on S according to $X(n)$, and that chooses the transition time from $X(n-1)$ to $X(n)$ according to $Y(n)$. Since the properties of Y (such as mean transition time) may depend on which state X chooses next, the processes $Z(t)$ are in general not a Markov process. Yet, the associated process $\{X(n), Y(n)\}$ is a Markov process. Hence the name semi-Markov.

The difference between Markov chain and semi-Markov chain is that in the latter, the state occupancy time is not memoryless, i.e. exponentially distributed, but depends on the next state and how long it has been sojourned on the current state. According to the previous discussion, the semi-Markov chain undoubtedly provides more freedom and is more appropriate to describe the real nature of FDI scheme employing advanced algorithm such as SPRT. In the area of fault tolerant control systems, such a semi-Markov chain modeling was first proposed by Wereley and Walker in 1990 [115]. However, till now, though this modeling is widely employed in the reliability analysis of systems, it has not been used in analysis and synthesis in control systems due to its complexity. Analytical results, e.g. stability conditions are difficult to obtain for systems governed by a semi-Markov chain.

2.3.2 Determination of Transition Rates

As stated in the previous sections, to analyze and design systems with Markov chains, the prerequisite is that the transition rate matrices of Markov chains are available. In practice, such information can be obtained from Monte Carlo simulation and historical data.

For the homogeneous Markov chain $\zeta(t)$ representing fault process with the generator matrix $[\alpha_{ij}]$ (or denoted as transition rate matrix), the entries can be determined from historical data. The theoretical foundation is that a Markov process can be viewed as the composition of an embedded Markov chain with exponential sojourn times [90].

Consider the system with $q_1 - 1$ fault scenarios, e.g. the Markov chain $\zeta(t)$ has q_1 modes. Associate the system under normal operation with mode 1, and the other modes are used to denote faulty scenarios. Then the reliability index MTBF (Mean Time Between Failures) is just the reciprocal of $-\alpha_{11} = \sum_{i=2}^{q_1} \alpha_{1i}$, e.g. a fault is expected to happen after an exponentially distributed time with the rate parameter $-\alpha_{11}$. Other entries can be determined from the number of total faults recorded, e.g. n. If a specific fault associated with mode j happened m times, it follows that the transition rate $\alpha_{1j} = -\alpha_{11}\frac{m}{n}$. Similarly, for the fault associated with mode i, the average time staying in this fault (before it is repaired or transited to another type of fault) is $-\frac{1}{\alpha_{ii}}$, and other entries of the i-th row of the generator matrix can be determined following the same rules.

If the FDI decision process $\eta(t)$ is modeled as a Markov chain as well, then its transition rates can be determined using the same method as described above. If a semi-Markov chain is used instead, when the state transition is made from i to j, besides the state transition probability matrix, the distribution of sojourn time F_{ij} should also be determined. In this case, the distribution is not exponential any more. This can be determined by a priori knowledge of the random distribution type and using the historical data to estimate the parameter for that specific distribution.

The quality of transition rates determined by this method is dependent on the length of the historical data. They may not be very exact if the historical data is not sufficient. However, if only the upper and lower bounds of transition rates are known, or the transition rate lies in a convex set, there are certain techniques available to handle these kinds of transition rate uncertainties while maintaining robust stability and robust performance [29]. For more complex transition rate uncertainty description, iterative LMI algorithm can be used to reduce the number of inequalities involved [118].

Chapter 3

Stochastic FTCS: Framework and Analysis

3.1 Introduction

In this chapter, the integrated stochastic FTCS using MJLS model is introduced. The open-loop systems subject parametric faults are represented using Linear Systems with Markovian Jumping Parameters (LSMJP). However, unlike control of MJLS, where controller can access fault mode instantly, FDI mechanism is employed in active FTCS to fulfill the detection and identification of faults. The FDI decision process can be modeled by a stochastic process, conventionally by a separate Markov chain.

MJLS is introduced first and the literatures are reviewed, due to its relation with the stochastic FTCS. The latter is discussed thereafter and the differences from the former are analyzed, and literatures within this framework are reviewed then. Definitions on stochastic stability of different forms are given, and then stochastic Lyapunov functions are introduced, which are powerful in dealing with stability analysis and controller synthesis for stochastic systems.

3.2 Description of the Open-loop Stochastic FTCS

The occurrences of faults are random in nature. In other word, when a fault happens in practical systems can never be exactly predicted. Therefore, it is natural to use stochastic processes to describe such kind of events. When it comes to study of linear systems with parametric faults, Markovian Jump Linear Systems (MJLS), also called as Linear Systems with Markovian Jumping Parameters, can be used to represent such a kind of systems. The pre-fault and post-fault systems can be modeled by a linear system but with different parameters, while the switching, which stands for the occurrence of faults and reparation/replacement, is governed by a homogeneous Markov chain:

$$\begin{cases} \dot{x}(t) = A(\zeta(t),t)x(t) + B(\zeta(t),t)u(t) + D(\zeta(t))w(t), \\ y(t) = C(\zeta(t))x(t) + E(\zeta(t))w(t), \end{cases} \tag{3.1}$$

where $x(t) \in \mathbb{R}^n$ represents the system state vector, $u(t) \in \mathbb{R}^m$ is the system input vector, $y(t) \in \mathbb{R}^s$ is the system output and $w(t) \in \mathbb{R}^p$ is the unknown exogenous disturbance. $\zeta(t)$ is assumed to be a measurable homogeneous continuous-time Markov chain with a finite state space $S_1 = \{1, 2, ..., q_1\}$. In the background of FTCS, $\zeta(t)$ (also called as the form process) models the fault process, and $A(\zeta(t))$, $B(\zeta(t))$, $C(\zeta(t))$, $D(\zeta(t))$, $E(\zeta(t))$ are of appropriate dimensions. It represents a set of linear systems in accordance with different system conditions (so called "modes") modeled by $\zeta(t)$.

The transition probability of the fault process $\zeta(t)$, is described as follows:

$$\zeta(t) : Pr\{\zeta(t+\Delta t) = j | \zeta(t) = i\} = \begin{cases} \alpha_{ij}\Delta t + o(\Delta t), & i \neq j \\ 1 + \alpha_{ii}\Delta t + o(\Delta t), & i = j \end{cases}, \; i, j \in S_1. \tag{3.2}$$

Where $\alpha_{ii} = -\sum_{j \in S_1,\, j \neq i} \alpha_{ij}$, $\Delta t > 0$, $\lim\limits_{\Delta t \to 0} \frac{o(\Delta t)}{\Delta t} = 0$, and α_{ij} is the transition rate of the fault process.

The MJLS given in (3.1) is widely used to represent abrupt changes occurred in the system dynamics. These kind of changes may include system component fault occurrences and repairs, changes in subsystems interconnections, abrupt changes in operation points etc.

The study of MJLS can be dated back to early 1960's, and it attracted a lot of interest around 1990's. Before 1994, most of work was on the analysis of MJLS, especially on controllability, observability and stability conditions [59], [60], [42], some works tackled the optimal control problems such as LQG [117], [61]. However, since the synthesis of controller involves solving coupled Riccati equations, which is very complicated, less work touched the controller synthesis of MJLS, and researchers had to construct iterative methods to solve coupled Riccati equations with no guarantee of the global optimum.

This situation is changed after Linear Matrix Inequality (LMI) was introduced in systems and control [18]. In 1996, M.A. Rami and L.E. Ghaoui first successfully formulated controller synthesis for continuous-time MJLS in terms of a LMI, which is a convex optimization problem and the global optimum can be found. Hence it greatly improves the previous design [93], [49]. Since then, research on system analysis and controller synthesis for MJLS has been very active, especially on those Lyapunov function based design methods.

In the area of MJLS, a wide range of problems have been studied since the last two decades. Results on stability, optimal control and robust performance can be found in a flurry of published papers and research monographs. Important results are cited without any intention of being exhaustive here: state-feedback stabilization [49], dynamic output feedback control formulation [41], H_∞-based model reduction [119], necessary and sufficient conditions for robust stability of continuous-time systems [28], guaranteed cost control [14], H_∞ control design using game theory [89], H_2 control [26], [29], robust H_2 control [31], H_∞ filtering for continuous-time and discrete-time systems [101], [102], filtering [32], [44], H_∞ design for time-delay systems [48], "Separation Principle" in Kalman filter-based controller design [33], H_2 control with cluster observation [110], robust stabilization with uncertain transition rates [118], almost sure stability analysis [11], mean square stabilization with partial mode information [43], stability with respect to different kinds of disturbance [45], δ-moment stability conditions [40], singular systems stabilization [17]. And research monographes on MJLS include: summary of research on MJLS before 1990's by Mariton [80], the latest Analytical Point of View method by Costa [34], and the design for MJLS with time-delay by Boukas [16].

However, most of the works on MJLS assume that the system mode $\zeta(t)$ is immediately available for control, which implies that it is directly observable. But in most practical systems, such an assumption does not hold. In the context of active FTCS, the system mode indicates if a fault has occurred, and it is actually identified by the FDI scheme.

In the standard two-step procedure of fault detection, the first step involves generation of the residual signal and then it is evaluated to see if some fault happens. Due to the presence of disturbance and model uncertainties, a non-zero threshold is often used in the residual evaluation. Obviously, a time delay, called "detection delay", is caused during this procedure. On the other hand, for systems with model uncertainties or external disturbances, unless some conditions are satisfied, perfect detection and isolation of faults are extremely hard, and missing detection and false alarms are inevitable. Unfortunately, control of MJLS does not take these factors into consideration, and that is the reason to adapt this model by introducing an additional stochastic process standing for FDI decision into the formulation of stochastic FTCS.

3.3 Description of the Closed-loop Stochastic FTCS

As discussed before, the difference between MJLS and FTCS with stochastic framework lies in the control action. For MJLS, the control action has the form $u(\zeta(t))$. Since $\zeta(t)$ means the fault process, it implies in MJLS that the system fault mode is immediately available and can be used in feedback control.

In fault tolerant control systems, the form of control law varies depending on the type of the system and the assumption adopted. For passive FTCS, there is no FDI process, and the controller is the same for all circumstances, i.e. control action is independent of mode $\zeta(t)$. When it comes to active FTCS, the FDI subsystem, a main ingredient of active FTCS, fulfill the task to provide the detection and identification of the system fault mode changes, and output its decision as $\eta(t)$ to the controller reconfiguration scheme. $\eta(t)$ is generally a stochastic process since FDI subsystem always tries to follow the stochastic process $\zeta(t)$.

Therefore, in active FTCS, control action will access FDI decision $\eta(t)$.

To conclude, the control action in FTCS may have the following forms:

(1) Passive FTC: no controller reconfiguration, control action has the form $u(t)$.

(2) "Full information" active FTC: controller accesses both fault and FDI information, control action has the form $u(t,\zeta(t),\eta(t))$.

(3) "FDI-based" active FTC: controller access only FDI information, control action has the form $u(t,\eta(t))$.

In this thesis, the main focus is on FTCS analysis and controller design for the "FDI-based" active FTC, since it is reasonable in the context of active FTC. However, for completeness of the results, the results on stabilization of systems using "full information" active fault tolerant controller will also be covered in Chapter 4. And the passive fault tolerant controller design can be carried out using the same iterative LMI algorithm proposed in Chapter 5 and Chapter 6, although it is definitely not the focus.

Conventionally the FDI decision process $\eta(t)$ is assumed to be a separate and measurable piecewise homogeneous continuous-time Markov chains with finite state spaces $S_2 = \{1,2,...,q_2\}$, ($q_2 \leq q_1$ in general). It represents the decision process of the FDI, and its transition probabilities are given as:

$$\eta(t): Pr\{\eta(t+\Delta t) = j | \zeta(t) = k,\ \eta(t) = i\} = \begin{cases} \beta_{ij}^k \Delta t + o(\Delta t), & i \neq j, \\ 1 + \beta_{ii}^k \Delta t + o(\Delta t), & i = j, \end{cases},\ i,j \in S_2, \tag{3.3}$$

where $\beta_{ii}^k = -\sum_{j\in S_2,\ j\neq i} \beta_{ij}^k$, and β_{ij}^k is the transition rate of the FDI process conditioned on the current state of the fault process $\zeta(t)$.

This setting of FDI decision process together with the setting for open-loop FTCS (3.1) consists of the basic setup for stochastic FTCS to study throughout this chapter. Further extension to the semi-Markov chain modeling of FDI decision process can be handled based on this basic setup, which will be discussed in Chapter 6.

Remark 3.1 *In many works, researchers use separate Markov chains to describe faults of different types occurred in the system. i.e. system component faults, actuator faults and sensor faults are represented by separate independent homogeneous Markov chains, such as in [1] and [76]. For example, in [76], the FTCS is described as:*

$$\dot{x}(t) = A(\zeta(t))x(t) + B(\eta(t))u(x(t), \Psi(t), t)) ,$$

where $\zeta(t)$, $\eta(t)$, $\Psi(t)$ *are separate Markov chains.*

Although this formulation may look reasonable and give a more accurate description of the system, from a mathematical point of view, it is redundant and unnecessary. Since $\zeta(t)$ *and* $\eta(t)$ *here both belong to system fault, they can be augmented into a joint Markov chain as* $\{\zeta(t),\ \eta(t)\}$. *From stochastic process theory, the transition rates for the joint Markov chain can be explicitly expressed in terms of transition rates of* $\zeta(t)$ *and* $\eta(t)$. *This assertion applies to systems with an additional Markov chain standing for sensor faults. Therefore, all these situations can be reduced to a standard two-Markov-chain setting without loss of generality, where one Markov chain stands for system fault process and the other represents FDI decision process. This can simplify the notation of the system and save lots of efforts in derivations later.*

In 1989, Mariton first introduced this formulation into the study of stochastic FTCS, but with some modification [79]. In 1993, Srichander and Walker formalized the description of this framework [104] using MJLS with two Markov chains, and analyzed the stability of the FTCS. After that, a bunch of problems have been studied by many researchers using this framework. These works include: H_∞ control for the uncertain continuous-time systems [98], robust disturbance attenuation for discrete-time systems [99], uncertain continuous-time system stabilization [78], stability analysis for systems with noise [75], stochastic stability analysis with multiple failure processes [76], H_∞ control for sampled-data systems [55]. Other works include output feedback stabilization [1] and H_∞ stabilization using dynamic output feedback [2], bilinear system stability [3], performance analysis using SIQC index [23], [24], and there is also a research monograph within this framework [77].

However, most of the works on active FTC is based on the assumption that controller accesses both system fault mode $\zeta(t)$ and FDI decision $\eta(t)$. With this assumption, the number of controllers is the same as that of the constraints on stability or performance. Thus it generally can be converted into conditions in terms of LMIs, and results on MJLS control can be applied. However, this assumption, which can simplify the design of controller on one hand, obscures the role of FDI on the other hand. It also hinders the application of the theory into applications. Under these circumstances, it is necessary to build on controller which solely access FDI decision results in stead of system real fault mode.

3.4 Stochastic Stability

3.4.1 Definitions

The requirement of stability is always the objective of the highest priority in the control system design. Due to the presence of Markov chains $\zeta(t)$ and $\eta(t)$, the FTCS shown in (3.1) is a stochastic system, and the stability to be studied is also in the stochastic sense. Compared with the deterministic setting, stability in stochastic setting is more difficult in that there are several ways of defining stability.

Without loss of generality, consider the following stochastic system:

$$\dot{x}(t) = A(t, \Psi(t))x(t) , \tag{3.4}$$

where $\Psi(t)$ is a Markov process taking values on $S = \{1, 2, \ldots, q\}$, and without loss of generality it is assumed that $x = 0$ is an equilibrium point.

Definition 3.1 Almost Sure Stability: The solution $x = 0$ of system (3.4) is said to be almost surely stable in probability if for any $\Psi(0) \in S$, $\varepsilon > 0$ and $\rho > 0$, there exists a $\delta(\varepsilon, \rho) > 0$ such that if $||x(0)|| < \delta(\varepsilon, \rho)$

$$Pr\{\sup_{0 \leq t < \infty} ||x(t)|| \geq \varepsilon\} \leq \rho .$$

Definition 3.2 Almost Surely Asymptotical Stability: The solution $x = 0$ of system (3.4) is said to be almost surely asymptotically stable in probability if it is almost surely stable in probability and $x(t) \longrightarrow 0$ with probability one as $t \longrightarrow \infty$.

Definition 3.3 Mean Exponential Stability: The solution $x = 0$ of system (3.4) is said to be exponentially stable in the mean square if, for any $\Psi(0) \in S$ and some $\delta(\Psi(0)) > 0$, there exist two positive scalars $a > 0$, $b > 0$ such that when $\|x(0)\| < \delta(\Psi(0))$, the following inequality holds $\forall t \geq 0$ for all solutions of (3.4) with initial condition x_0:

$$\mathscr{E}\{\|x(t)\|^2\} \leq b\|x_0\|^2 e^{-at} .$$

There are also some definitions on stochastic stability in the study of MJLS:

Definition 3.4 Mean Square Stability (MSS): The solution $x = 0$ of system (3.4) is said to be asymptotically mean square stable, if for any $x(0)$, and any initial distribution of $\Psi(0) \in S$,

$$\lim_{t \longrightarrow +\infty} \mathscr{E}\{\|x(t)\|^2\} = 0 .$$

Definition 3.5 Stochastic Stability: The solution $x = 0$ of system (3.4) is said to be stochastic stable, if for any $x(0)$, and any initial distribution of $\Psi(0) \in S$,

$$\int_0^{+\infty} \mathscr{E}\{\|x(t)\|^2\} dt < +\infty .$$

For those different stability concepts, their relationships are revealed by the following theorem:

Theorem 3.1 *[42]: When $\Psi(t)$ is a finite state Markov chain, asymptotically mean square stable, exponentially mean square stable, stochastic stable are equivalent, and each implies almost surely asymptotically stable.*

As pointed out by Kozin [69], in practical applications, almost sure stability is more desirable because only a sample path of the system can be observed, while moment stability is a statistical performance. Therefore, moment stability criteria can be sometimes too conservative for practical use.

For the study of almost surely stability, the definition of δ-moment stability is proposed, and sufficient conditions are proposed in [40]:

Definition 3.6 Asymptotically δ-moment Stability: The solution $x = 0$ of system (3.4) is said to be asymptotically mean square stable, if for any $x(0)$, and any initial distribution of $\Psi(0) \in S$,

$$\lim_{t \to +\infty} \mathscr{E}\{\|x(t)\|^{\delta}\} = 0 .$$

The reason for studying δ-moment Stability is that for sufficiently small $\delta > 0$, the δ-moment Stability and almost sure stability are equivalent.

However, note that in the study of both MJLS and stochastic FTCS, MSS is still the dominant stability criterion to be used. With this stability criterion, a sufficient and necessary condition can be obtained using the stochastic Laypunov function method, which will be introduced in the next.

3.4.2 Stochastic Lyapunov Function

In deterministic systems, Lyapunov theorem is widely used for stability analysis and controller design. In stochastic systems analysis and synthesis, the stochastic Lyapunov function also plays a very important role [70].

Consider a function $V(x,\Psi,t)$ of the stochastic process $\{x,\Psi\}$. For a fixed $m \leq \infty$, assume the following conditions hold.

1. The function $V(x,\Psi(t),t)$ is positive definite and continuous in x and t in the open set $O_m = \{x(t) : V(x,\Psi,t) < m\}, \forall \Psi \in S, \forall t \geq t_0$, and $V(x,\Psi,t) = 0$ only if $x = 0$. (The function $V(x,\Psi,t)$ is said to be positive definite if $V(x,\Psi,t) \geq W(x)$, $\forall \Psi \in S$, $\forall t \geq t_0$, where $W(x)$ is positive definite in the sense of Lyapunov.)

2. The joint Markov process $\{x,\ \Psi\}$ is defined until $t = \tau$ where $\tau_m = \inf\{t : x(t) \notin O_m\}$ (or $\forall t < \infty$ if $x(t) \in O_m$ for $t < \infty$). If $x(t) \in O_m$ $\forall t < \infty$, then $\tau = \infty$.

3. The function $V(x,\Psi,t)$ is in the domain of $\mathscr{L}$ where $\mathscr{L}$ is the weak infinitesimal operator of the joint Markov process $\{x(\tau_t),\ \Psi(\tau_t)\}$, where $\tau_t = \min(t, \tau_m)$.

A function $V(x,\Psi,t)$ that satisfies the above conditions is said to be qualified as a stochastic Lyapunov function candidate for system (3.4).

Using a stochastic Lyapunov function, stochastic stability can be established.

First, the definition of weak infinitesimal operator is provided: for a joint Markov process $\{x,\Psi\}$ and a bounded function $V(x,\Psi,t)$, the weak infinitesimal operator $\mathscr{A}V(x,\Psi,t)$ defined on $V(x,\Psi,t)$ at the point of (x,Ψ,t) is given by

$$\begin{aligned} & \mathscr{A}V(x,\Psi,t) \\ = \; & \lim_{\Delta\to 0}\frac{1}{\Delta}\Big(\mathscr{E}\{V(x(t+\Delta),\Psi(t+\Delta),t+\Delta)|(x(t),\Psi(t),t)\} \\ & -V(x(t),\Psi(t),t)\Big) . \end{aligned}$$

With definition of the weak infinitesimal operator given above, the following theorems are the most important theoretical results on analysis and design of stochastic systems.

Theorem 3.2 *[70] Assume $V(x,\Psi,t)$ is a valid stochastic Lyapunov function, and*

$$\mathscr{A}V(x,\Psi,t) = -K(x,\Psi,t) < 0$$

in the open set O_m for system (3.4) when $\Psi \in S$, where $K(x,\Psi,t) > 0$ is continuous in x $\forall t \geq t_0$ and $K(x,\Psi,t) = 0$ only if $x = 0$. Then the solution $x(t) = 0$ of the stochastic system is almost surely asymptotically stable in probability.

Theorem 3.3 *[70] The solution $x(t) = 0$ of the linear stochastic system $\dot{x}(t) = A(t,\Psi)x(t)$ is Mean Square Stable (MSS) for $t \geq t_0$ if and only if there exists a stochastic Lyapunov function $V(x,\Psi,t)$ and the positive constants λ_i, $i = 1,2,3$, such that*

$$\lambda_1\|x(t)\|^2 \leq V(x,\Psi,t) \leq \lambda_2\|x(t)\|^2, \text{ and } \mathscr{A}V(x,\Psi,t) \leq -\lambda_3\|x(t)\|^2 .$$

3.4.3 Stability Criteria for MJLS and Stochastic FTCS

MJLS have been extensively studied for more than three decades. Besides stability condition using stochastic Lyapunov theorem, there are also some criteria for test the stability of MJLS. Since when control action is known, the closed-loop two-Markov-chain FTCS can be transformed into a MJLS after augmenting two Markov chains into a joint one, those criteria can be extended to the stochastic FTCS.

Assume that $\Psi(t)$ is a homogeneous Markov chain with transition rate matrix $Q=[q_{ij}]$:

Theorem 3.4 *The jump linear system $\dot{x}(t)=A(\Psi(t))x(t)$ is Mean Square Stable if and only if for any positive definite matrices $R_i>0,\ i\in S$, the coupled Lyapunov equation*

$$A_i^T P_i + P_i A_i + \sum_{j\in S} q_{ij}P_j = -R_i \tag{3.5}$$

has a positive definite solution $P_i>0,\ i\in S$, equivalently, if and only if F is Hurwitz stable.

$$\begin{aligned} F &= \begin{bmatrix} A_1^T\oplus A_1^T - q_{11}I & q_{12}I & \dots & q_{1s}I \\ q_{21}I & A_2^T\oplus A_2^T - q_{22}I & \dots & q_{2s}I \\ \vdots & \vdots & \ddots & \vdots \\ q_{s1}I & q_{s2}I & \dots & A_s^T\oplus A_s^T - q_{ss}I \end{bmatrix} \\ &= \operatorname{diag}\{A_1^T\oplus A_1^T,\dots,A_s^T\oplus A_s^T\} + Q\otimes I. \end{aligned} \tag{3.6}$$

Results above can be extended to the two-Markov-chain FTCS with conditions expressed in terms of matrix inequalities, so that they are easy to be solved using LMI techniques. The following corollary on stochastic stability can be obtained.

Corollary 3.1 *For the stochastic FTCS $\dot{x}(t)=\tilde{A}(\zeta(t),\eta(t))x(t)$, where $\zeta(t)$ and $\eta(t)$ take values on $S_1=\{1,2,\dots,q_1\}$, $S_2=\{1,2,\dots,q_2\}$ respectively, the stability criteria can be deduced as follows:*

The system is Mean Square Stable if and only if $\forall i\in S_1,\ j\in S_2$, there exist positive definite matrices $P_{ij}>0$ for the following matrix inequality:

$$P_{ij}\tilde{A}_{ij} + \tilde{A}_{ij}^T P_{ij} + \sum_{k\in S_1}\alpha_{ik}P_{kj} + \sum_{k\in S_2}\beta_{jk}^i P_{ik} < 0\,. \tag{3.7}$$

Or equivalently the following matrix M is Hurwitz stable.

$$M=\operatorname{diag}\{\tilde{A}_{11}^T\oplus\tilde{A}_{11}^T,\ \tilde{A}_{12}^T\oplus\tilde{A}_{12}^T,\ \cdots,\ \tilde{A}_{1q_2}^T\oplus\tilde{A}_{1q_2}^T,\ \tilde{A}_{21}^T\oplus\tilde{A}_{21}^T,\cdots,\ \tilde{A}_{q_1q_2}^T\oplus\tilde{A}_{q_1q_2}^T\}+\mu\otimes I,$$

where μ is the transition rate matrix for the augmented Markov chain $\{\zeta(t),\eta(t)\}$.

Remark 3.2 *In [104], [75], [76] and [77], time-dependant Lyapunov function $V(t,x,\Psi) = x^T P(t,\Psi)x$ is used. However, [59] showed that when system is Mean Square Stable, $P(t,\Psi)$ converges to a constant matrix $P(\Psi)$ when $t \longrightarrow \infty$. Therefore, using a time-independent Lyapunov matrix $P(\Psi)$ would not introduce any conservatism.*

Example 3.1 In this example, it is shown that when closed-loop FTCS is stable, positive definite Lyapunov matrices $P_{ij}(t)$ converge to constant matrices.

The simulation study is performed on a dual-motor ball beam system. The system has two driving motors, one at each end, so it has one degree of actuator redundancy. The function of the two motors is to move the beam at the two ends up and down to balance the ball at the desired position. With the actuator redundancy, the design of fault tolerant state-feedback control against the motor failure is possible. A linear model of the system is obtained. The system states $x = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$, where x_1 is the displacement of the ball from the center of the beam, and x_2 is the velocity. Two inputs $U = \begin{bmatrix} u_1 & u_2 \end{bmatrix}^T$, are the voltage signals to the driving motors. In this case study, two system modes are assumed, mode 1 represents the normal system operation, and mode 2 is the actuator failure case.

$$A_1 = A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0.8 \end{bmatrix}, \ B_1 = \begin{bmatrix} 0 & 1 \\ -0.25 & 0.25 \end{bmatrix}, \ B_2 = \begin{bmatrix} 0 & 0.2 \\ -0.25 & 0.05 \end{bmatrix},$$

$$[\alpha] = \begin{bmatrix} -0.5 & 0.5 \\ 2 & -2 \end{bmatrix}, \ [\beta^1] = \begin{bmatrix} -0.1 & 0.1 \\ 0.1 & -0.1 \end{bmatrix}, \ [\beta^2] = \begin{bmatrix} -0.1 & 0.1 \\ 0.1 & -0.1 \end{bmatrix},$$

$$K_1 = \begin{bmatrix} 1.2018 & 8.9006 \\ -7.4239 & -2.1148 \end{bmatrix}, \ K_2 = \begin{bmatrix} -0.20089 & 8.4035 \\ -7.7094 & -1.7643 \end{bmatrix}.$$

Using an ODE solver, the time-varying Lyapunov matrix $P(t, \zeta(t), \eta(t))$ can be solved. For this example, each entry of matrices is shown in Figure 3.1.

From the simulation results above, some important observations are obtained on the interactions between the FDI decisions and the controllers in the stochastic FTC framework:

The state transition rates for the fault Markov chain can be used to evaluate the system reliability. If the state i indicates the system normal condition, then $-\frac{1}{\alpha_{ii}}$ is the MTBF (Mean Time Between Failures). Larger MTBF means the system is more reliable. The

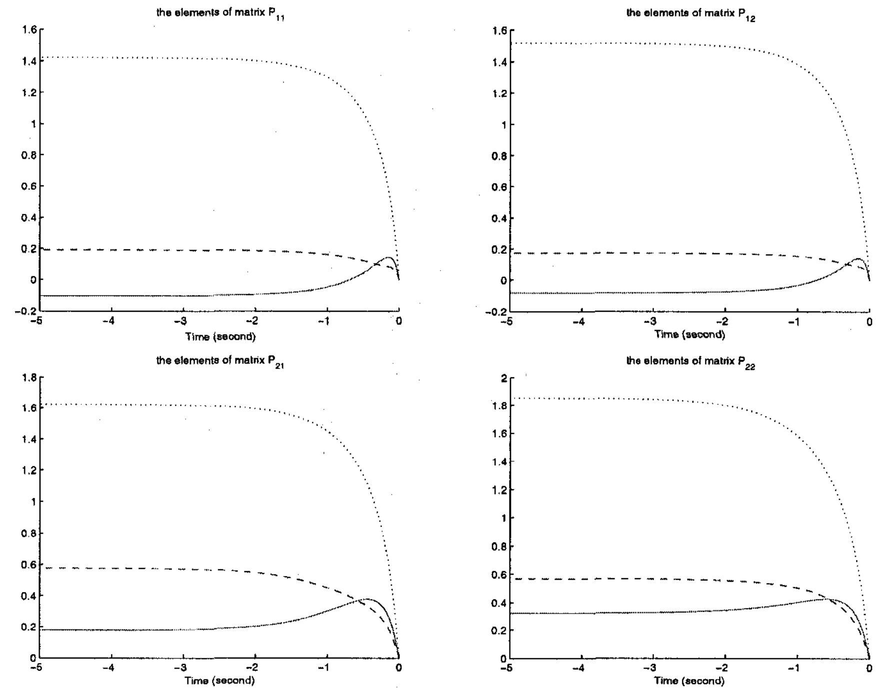

Figure 3.1. Entries of Lyapunov function obtained by ODE solver (Solid line for the off diagonal entries; dashed line for (1,1) entry; dotted line for (2,2) entry)

ratio $-\alpha_{ji}/\alpha_{jj}$ measures the probability of the system to restore to normal from fault, thus $\sum_{j\neq i}\alpha_{ji}/\alpha_{jj}^2$ is the MTTR (Mean Time To Repair).

The FDI Markov chain affects the overall system performance through its conditional transition rates. Hence, the performance of FDI should be interpreted in the sense of probabilities. It is the reason that the FDI process does not exactly follow the fault process in the single sample path simulation. One criterion can be given is the sojourn time of the FDI on different states, for example, the smaller $-\beta_{ii}^k$ $(i \neq k)$ means the longer sojourn time of the FDI on the wrong state i; similarly, the smaller $-\beta_{kk}^k$ means the longer time of the FDI on the correct state.

In FTCS, the reconfigurable controller is another important factor affecting the system performance. From the simulation, if the controller designed for the faulty case (mode 2 in the example) can stabilize the normal system, then the overall system will most likely be stabilized even with false alarms (which is similar to the reliable control problem). If the system cannot be stabilized when the FDI indicates an incorrect fault mode, then the sojourn time $-1/\beta_{ii}^{k}$ $(i \neq k)$ cannot be too long, otherwise the system will lose stability. When the $-\beta_{ii}^{k}$ is increased (the sojourn time $-1/\beta_{ii}^{k}$ is reduced) without changing anything else, the stochastic stability is achieved again.

In summary, the simulation study demonstrates a well known fact that the FDI and reconfigurable control should be designed together for better system performance.

3.5 Discrete-time Systems

For completeness of the work, the discrete-time FTCS is also be briefly handled. Consider a discrete-time open-loop FTCS as:

$$\mathscr{G}_d : \quad \begin{cases} x(k+1) = A(\zeta(k))x(k) + B(\zeta(k))u(k,\zeta(k),\eta(k)) + D(\zeta(k))w(k), \\ y(k) = C(\zeta(k))x(k) + E(\zeta(k))w(k), \end{cases} \tag{3.8}$$

where $\zeta(k)$ and $\eta(k)$ are two separate discrete-time homogeneous finite state Markov chains taking values on S_1, S_2, with their one-step transition rate matrices $[\alpha_{i_1,i_2}]$ and $[\beta^{i_1}_{j_1,j_2}]$ defined as follows:

$$\begin{aligned} Pr\{\zeta(k+1) = i_2 | \zeta(k) = i_1\} &= \alpha_{i_1,i_2} , \\ Pr\{\eta(k+1) = j_2 | \zeta(k) = i_1,\ \eta(k) = j_1\} &= \beta^{i_1}_{j_1,j_2} . \end{aligned} \tag{3.9}$$

Assume a control law $u(k,\zeta(k),\eta(k))$ is given, then the closed-loop system has the following description:

$$\tilde{x}(k+1) = \tilde{A}(\zeta(k),\ \eta(k))\tilde{x}(k) . \tag{3.10}$$

Using stochastic Lyapunov theorem, the MSS condition for the closed-loop system (3.10) is given in the following theorem:

Theorem 3.5 *The closed-loop system (3.10) is MSS if and only if* $\forall i \in S_1,\ j \in S_2$*, the following matrix inequality has a positive definite solution* $P_{ij} > 0$*:*

$$\tilde{A}_{ij}^T \left(\sum_{m \in S_1} \sum_{n \in S_2} \alpha_{im} \beta_{jn}^{i} P_{mn} \right) \tilde{A}_{ij} - P_{ij} < 0 . \tag{3.11}$$

Proof: The theorem can be proved by defining a stochastic Lyapunov function as

$$V(\tilde{x}(k), \zeta(k), \eta(k)) = \tilde{x}^T P(\zeta(k), \eta(k)) \tilde{x} .$$

For this Lyapunov function, the weak infinitesimal operator has the form:

$$\mathscr{A} V = \tilde{x}^T \left(\tilde{A}_{ij}^T \left(\sum_m \sum_n \alpha_{im} \beta_{jn}^i P_{mn} \right) \tilde{A}_{ij} - P_{ij} \right) \tilde{x} .$$

Therefore the condition shown in (3.11) can be obtained. □

Chapter 4

Stabilization of Stochastic FTCS

4.1 Introduction

In this chapter, for the stochastic FTCS discussed in Chapter 3, the results on stabilization will be presented, where different forms of controller are designed based on MSS criterion. Among these controllers, "full information" and "FDI-dependent" are the main focus. Both continuous-time and discrete-time systems are covered, and stabilization conditions are given for both state feedback and output feedback controllers, so that a complete treatment is made on this issue. Different design approaches have been employed while the conditions are expressed in terms of LMIs, which can be solved using many available convex optimization software efficiently. The more complicated algorithms using iterative LMI will be presented in Chapter 5 and Chapter 6.

4.2 Preliminaries

In this section, description of model uncertainties and some important lemmas used later in this thesis are presented.

4.2.1 Uncertainty Descriptions

For practical systems, exact mathematical models are extremely hard or even impossible to obtain. In this thesis, two types of the most commonly used uncertainties are considered. The first one is so-called polytopic type model uncertainty. It is assumed that system matrices lie within the uncertainty polytope Ω:

$$\Omega = \{(A_i, B_i, C_i, D_i) | (A_i, B_i, C_i, D_i) = \sum_{j=1}^{m} \tau_j (A_i^j, B_i^j, C_i^j, D_i^j); \tau_j \geq 0, \sum_{j=1}^{m} \tau_j = 1\}. \tag{4.1}$$

In this case, if the uncertainty is time-invariant or slowly time-varying, parameter-dependent Lyapunov function approach should be used to develop the stability conditions. That implies that for each vertex of the polytope, a separate Lyapunov function should be associated with the corresponding system. This approach is less conservative compared with quadratic stability, where a single Lyapunov function is used for all vertexes.

Another type of model uncertainty adopted in this thesis is the norm-bounded uncertainty, which can be used to describe those time-varying or time invariant uncertainties. The system matrices are assumed to have the form:

$$A_i = A_{0i} + A_{1i}\Delta_{1i}A_{2i}, \quad B_i = B_{0i} + B_{1i}\Delta_{2i}B_{2i}.$$

where $||\Delta_{1i}|| \leq 1$ and $||\Delta_{2i}|| \leq 1$.

The second source of model uncertainties comes from uncertainty on the transition rates, which can be on that of fault process $\zeta(t)$ or the FDI decision process $\eta(t)$. This type of uncertainty exists because those transition rates are determined by history data or Monte Carlo simulation, therefore the precision of the transition rates depends on the length of the data and the quality of the simulation. Generally the uncertainty on transition rates is assumed to be bounded by a upper value and a lower value, or lies in a polytope.

4.2.2 Preliminary Lemmas

Schur complement lemma is first given here, which is important and widely used in converting quadratic matrix inequality into a linear one.

Lemma 4.1 *Schur complement ([16]):*

Let the symmetric matrix M be partitioned as $M = \begin{bmatrix} X & Y \\ Y^T & Z \end{bmatrix}$ *with X, Z be symmetric matrices. Then following statements are equivalent:*

1. *M is nonnegative definite if and only if either*

$$\begin{cases} Z \geq 0 \\ Y = L_1 Z \\ X - L_1 Z L_1^T \geq 0 \end{cases} \quad or \quad \begin{cases} X \geq 0 \\ Y = X L_2 \\ Z - L_2^T X L_2 \geq 0 \end{cases}$$

   holds, where L_1, L_2 are some (nonunique) matrices of compatible dimensions.

2. *M is positive definite if and only if either*

$$\begin{cases} Z > 0 \\ X - YZ^{-1}Y^T > 0 \end{cases} \quad or \quad \begin{cases} X > 0 \\ Z - Y^T X^{-1} Y > 0 \,. \end{cases}$$

The next lemma is used to handle the norm-bounded uncertainties:

Lemma 4.2 *([5]): Let $E(t)$, $F(t)$ and $H(t)$ be real matrices of appropriate dimensions with $F(t)^T F(t) < I$, then for any scalar $\varepsilon > 0$, it is true that:*

$$H(t)F(t)E(t) + E^T(t)F^T(t)H^T(t) \leq \varepsilon H(t)H^T(t) + \frac{1}{\varepsilon} E^T(t)E(t) \,.$$

Projection lemma plays a very important role in the LMI based control system design, and it has been used in numerous areas. It is capable to obtain equivalent conditions for those approaches using transformation techniques or "change of variable" methods, but it is more straightforward.

Lemma 4.3 *Projection Lemma ([18]): Given Ψ, U, V, there exists X such that*

$$\Psi + U^T X^T V + V^T X U < 0$$

if and only if

$$N_U^T \Psi N_U < 0, \quad N_V^T \Psi N_V < 0$$

holds, where N_U and N_V are bases of null spaces of U and V respectively.

The following two lemmas were originally proposed to handle the analysis and design for systems with polytopic model uncertainties, but now they have been gradually used in multi-objective design and on transforming the nonlinear matrix inequality into linear ones.

Lemma 4.4 *Reciprocal Projection Lemma [4]: Let P be any given positive-definite matrix, the following statements are equivalent:*

(1) $\Psi + S + S^T < 0$,

(2) the LMI problem

$$\begin{bmatrix} \Psi + P - (W + W^T) & S^T + W^T \\ S + W & -P \end{bmatrix} < 0$$

is feasible with respect to W.

And in discrete-time, the following lemma, can be called as discrete-time Reciprocal Projection Lemma, plays the exactly same role as for Reciprocal Projection Lemma.

Lemma 4.5 *The following conditions are equivalent (*$f(P) > 0$ *is a matrix valued function of* P*):*

1. *There exists a symmetric matrix* $P > 0$ *such that*

$$A^T PA - f(P) < 0. \tag{4.2}$$

2. *There exist a symmetric matrix P and a matrix G such that*

$$\begin{bmatrix} f(P) & A^T G^T \\ GA & G + G^T - P \end{bmatrix} > 0. \tag{4.3}$$

Proof: The prototype of this lemma is shown in [87], and the proof is omitted here. □

4.3 System Formulations

Both continuous-time and discrete-time systems will be considered in this chapter. First, the description of the open-loop FTCS in continuous-time is given as follows:

$$\mathscr{G}_c : \quad \begin{cases} \dot{x}(t) = A(\zeta(t))x(t) + B(\zeta(t))u(t)\,, \\ y(t) = C(\zeta(t))x(t), \end{cases} \tag{4.4}$$

or a discrete-time system as:

$$\mathscr{G}_d : \quad \begin{cases} x(k+1) = A(\zeta(k))x(k) + B(\zeta(k))u(k)\,, \\ y(k) = C(\zeta(k))x(k)\,. \end{cases} \tag{4.5}$$

where the parameters for both Markov chains are the same as described in Chapter 3.

As mentioned in the previous chapter, for completeness of the results, three types of controllers will be designed in this chapter:

For open-loop systems given in (4.4) or (4.5), both passive FTC and active FTC can be studied, just depending on which type of controller is used in analysis and design. Depending on the information that the controller accesses, controller can have different forms:

- Controller access both fault and FDI information: $u(t, \zeta(t), \eta(t))$.
- Controller access only FDI information: $u(t, \eta(t))$.
- Passive fault tolerant controller: $u(t)$ without reconfiguration.

Moreover, for each category, both state-feedback and output-feedback controllers will be covered or briefly mentioned. It is well known that the former is simpler in design but the latter is inevitable when system states are not available for control.

It is obvious that the passive fault tolerant controller accesses least information while the first type of controller accesses most information from the system. Therefore, for controllers listed above in order, it can be expected that the system performance degrades gradually from the first to the third one, while the difficulty increases gradually.

All the stabilization controller design algorithms are based on the MSS conditions listed in Chapter 3, since when assuming the control law is known, the closed-loop system's stability can be determined using the stochastic Lyapunov theorem.

4.4 Full Information Controller Design

In the sequel of this chapter and also later chapters in this thesis, to simplify the notation, denote the arbitrary matrix $M(\zeta(t))$ or $M(\zeta(t),\eta(t))$ (or $M(\zeta(k))$ and $M(\zeta(k),\eta(k))$) associated with mode $\zeta(t)=i$, $\eta(t)=j$ (or $\zeta(k)=i$ and $\eta(k)=j$, by:

$$M_i = M(\zeta(t)=i) \quad \text{and} \quad M_{ij} = M(\zeta(t)=i,\eta(t)=j).$$

4.4.1 State Feedback Controller Design

For the continuous-time system (4.4), the state-feedback controller that stabilizes the system in MSS sense is given in the following theorem:

Theorem 4.1 *Under the state-feedback control law $u(t)=K_{ij}x(t)$, the FTCS (4.4) is MSS if and only if $\forall i \in S_1,\ j \in S_2$ there exist positive matrices $X_{ij}>0$ and $\bar{K}_{ij}$, as the solution for the following LMI:*

$$\begin{bmatrix} \left(\alpha_{ii}+\beta^i_{jj}\right)X_{ij}+A_iX_{ij}+X_{ij}A_i^T+B_i\bar{K}_{ij}+\bar{K}_{ij}^TB_i^T & H_1(i,j,X) \\ * & H_2(i,j,X) \end{bmatrix} < 0, \qquad (4.6)$$

where

$$H_1(i,j,X) = X_{ij}\left[\sqrt{\alpha_{i1}}I,\cdots,\sqrt{\alpha_{i,i-1}}I,\sqrt{\alpha_{i,i+1}}I,\cdots,\sqrt{\beta^i_{j1}}I,\cdots,\sqrt{\beta^i_{j,j-1}}I,\sqrt{\beta^i_{j,j+1}}I,\cdots\right],$$
$$H_2(i,j,X) = -\mathrm{diag}\{X_{1j},\cdots,X_{i-1,j},X_{i+1,j},\cdots,X_{i1},\cdots,X_{i,j-1},X_{i,j+1},\cdots\}.$$

The corresponding stabilizing state feedback gain is given as $K_{ij}=\bar{K}_{ij}X_{ij}^{-1}$.

Proof: For state-feedback controller $u(t)=K_{ij}x(t)$, the closed-loop system matrix is $\tilde{A}_{ij}=A_i+B_iK_{ij}$, therefore, from MSS conditions given in Chapter 3:

$$P_{ij}\tilde{A}_{ij}+\tilde{A}_{ij}^TP_{ij}+\sum_{k\in S_1}\alpha_{ik}P_{kj}+\sum_{k\in S_2}\beta^i_{jk}P_{ik}<0. \qquad (4.7)$$

Define $X_{ij}=P_{ij}^{-1}$, pre- and post-multiply the matrix inequality above by X_{ij} to obtain,

$$\left(A_i+B_iK_{ij}\right)X_{ij}+X_{ij}\left(A_i+B_iK_{ij}\right)^T+X_{ij}\left(\sum_{k\in S_1}\alpha_{ik}P_{kj}+\sum_{k\in S_2}\beta^i_{jk}P_{ik}\right)X_{ij}<0. \qquad (4.8)$$

Define $\bar{K}_{ij}=K_{ij}X_{ij}$, using Schur complement lemma, then the nonlinear matrix inequality above can be converted into a linear one as shown in (4.6). □

Theorem 4.2 *Under the state-feedback control law $u(k) = K_{ij}x(k)$, the FTCS (4.5) is MSS if and only if $\forall i \in S_1,\ j \in S_2$ there exist positive matrices $X_{ij} > 0$ and $\bar{K}_{ij}$, as the solution for the following LMI:*

$$\begin{bmatrix} -X_{ij} & (X_{ij}A_i^T + \bar{K}_{ij}^T B_i^T)H_3(i,j) \\ * & H_4(i,j,X) \end{bmatrix} < 0\,, \tag{4.9}$$

where

$$H_3(i,j) = \left[\sqrt{\alpha_{i1}\beta_{j1}^i}I,\ \sqrt{\alpha_{i1}\beta_{j2}^i}I,\ \ldots,\ \sqrt{\alpha_{i2}\beta_{j1}^i}I,\ \ldots,\ \sqrt{\alpha_{iq_1}\beta_{jq_2}^i}I\right],$$
$$H_4(i,j,X) = -\mathrm{diag}\{X_{11}, X_{12}\ldots, X_{1q_2}, X_{21}, \ldots, X_{q_1q_2}\}\,.$$

The corresponding stabilizing state feedback gain is given as $K_{ij} = \bar{K}_{ij}X_{ij}^{-1}$.

Proof: From Lyapunov stability theorem, the following sufficient and necessary condition for the MSS can be obtained:

$$(A_i + B_iK_{ij})^T\left(\sum_{m\in S_1}\sum_{n\in S_2}\alpha_{im}\beta_{jn}^i P_{mn}\right)(A_i + B_iK_{ij}) - P_{ij} < 0\,. \tag{4.10}$$

Apply Schur complement lemma to expand the inequality above to obtain:

$$\begin{bmatrix} -P_{ij} & (A_i + B_iK_{ij})^T H_3(i,j) \\ * & H_4(i,j,X) \end{bmatrix} < 0\,.$$

Define $X_{ij} = P_{ij}^{-1}$ as the same as before, pre- and post-multiply the inequality above by $\mathrm{diag}\{X_{ij}, I, I, \ldots, I\}$, then can get the LMI shown in (4.9). □

However, it should be noticed that in (4.9), if the system has an order n, and q_1, q_2 are numbers of Markov states for fault process and FDI process respectively, then the matrix inequality shown in (4.9) has a dimension as $n \times (1 + q_1 \times q_2)$, which can be a very large number. Therefore, more efficient conditions are prefered.

From the operator theory for MJLS by Costa et al [34], the condition on MSS is equivalent to that the spectral radiuses of several operators are less than 1. The details of the operator theory and its application will be stated in Chapter 6. Here, using results from the operator theory of MJLS, equivalent condition on MSS can be derived, but with lower dimension for matrix inequalities involved when compared with that in (4.9).

Lemma 4.6 *The closed-loop system $\dot{x}(k) = \tilde{A}_{ij}x(k)$ is MSS if and only if $\forall i \in S_1,\ j \in S_2$ there exists positive definite matrices $P_{ij} > 0$ such that*

$$-P_{mn} + \sum_{i\in S_1}\sum_{j\in S_2} \alpha_{im}\beta^i_{jn}\tilde{A}_{ij}P_{ij}\tilde{A}^T_{ij} < 0\,. \tag{4.11}$$

Theorem 4.3 *Under the state-feedback control law $u(k) = K_{ij}x(k)$, the FTCS (4.5) is MSS if and only if $\forall i \in S_1,\ j \in S_2$ there exist feasible solution $\bar{K}_{ij}$, Y_{ij} and positive definite matrices $P_{ij} > 0$ for the following LMIs:*

$$-P_{mn} + \sum_{i\in S_1}\sum_{j\in S_2} \left(\alpha_{im}\beta^i_{jn}(A_iP_{ij}A_i^T + A_i\bar{K}^T_{ij}B_i^T + B_i\bar{K}_{ij}A_i^T + B_iY^T_{ij}B_i^T)\right) < 0, \tag{4.12}$$

$$\begin{bmatrix} Y_{ij} & \bar{K}_{ij} \\ \bar{K}^T_{ij} & P_{ij} \end{bmatrix} > 0\,. \tag{4.13}$$

The corresponding state-feedback gain is given as $K_{ij} = \bar{K}_{ij}P^{-1}_{ij}$.

Proof:

Necessity: From the lemma above, it is known that the system is MSS, if and only if there exist $P_{ij} > 0$ and K_{ij} such that

$$-P_{mn} + \sum_{i\in S_1}\sum_{j\in S_2} \left(\alpha_{im}\beta^i_{jn}(A_i + B_iK_{ij})P_{ij}(A_i + B_iK_{ij})^T\right) < 0\,. \tag{4.14}$$

Expand the left-hand side of the inequality,

$$-P_{mn} + \sum_{i\in S_1}\sum_{j\in S_2} \left(\alpha_{im}\beta^i_{jn}(A_iP_{ij}A_i^T + A_iP_{ij}K^T_{ij}B_i^T + B_iK_{ij}P_{ij}A_i^T + B_iK_{ij}P_{ij}K^T_{ij}B_i^T)\right) < 0\,.$$

For the inequality above, there always exist sufficient small scalars $\varepsilon_{ij} > 0$ such that

$$\begin{aligned} &-P_{mn} + \sum_{i\in S_1}\sum_{j\in S_2} \left(\alpha_{im}\beta^i_{jn}(A_iP_{ij}A_i^T + A_iP_{ij}K^T_{ij}B_i^T + B_iK_{ij}P_{ij}A_i^T\right. \\ &\left. + B_iK_{ij}P_{ij}K^T_{ij}B_i^T + \varepsilon_{ij}B_iB_i^T)\right) < 0. \end{aligned}$$

Define $\bar{K}_{ij} = K_{ij}P_{ij}$, it implies that there exist Y_{ij} satisfy

$$Y_{ij} > K_{ij}P_{ij}K^T_{ij}$$

and

$$-P_{mn}+\sum_{i\in S_1}\sum_{j\in S_2}\left(\alpha_{im}\beta^i_{jn}(A_iP_{ij}A_i^T+A_i\bar{K}_{ij}^TB_i^T+B_i\bar{K}_{ij}A_i^T+B_iY_{ij}^TB_i^T)\right)<0\,.$$

This finishes the proof for necessity.

Sufficiency: For given $P_{ij}>0$, Y_{ij} and $\bar{K}_{ij}$, it is obvious that P_{ij} is the Lyapunov matrix and the controller $\bar{K}_{ij}$ stabilizes the closed-loop system $\tilde{A}_{ij}=A_i+B_i\bar{K}_{ij}P_{ij}^{-1}$, i.e. make (4.14) hold. □

4.4.2 Output Feedback Controller Design

When system states are not available for control, then output feedback control should be used. Here, the dynamic output feedback controller has the same order as the system is considered:

$$\begin{cases} \dot{x}_c(t)=\hat{A}_{ij}x_c(t)+\hat{B}_{ij}y(t), \\ u(t)=\hat{C}_{ij}x_c(t). \end{cases} \tag{4.15}$$

In this case, the closed-loop system matrix has the form:

$$\begin{bmatrix} \dot{x} \\ \dot{x}_c \end{bmatrix}=\begin{bmatrix} A_i & B_i\hat{C}_{ij} \\ \hat{B}_{ij}C_i & \hat{A}_{ij} \end{bmatrix}\begin{bmatrix} x \\ \hat{x}_c \end{bmatrix}. \tag{4.16}$$

The sufficient and necessary condition for the MSS using dynamic output feedback control is summarized in the following theorem:

Theorem 4.4 *The FTCS is MSS if and only if $\forall i\in S_1,\ j\in S_2$ there exists feasible solution F_{ij}, L_{ij}, X_{ij} and Y_{ij} for the following LMIs:*

$$\begin{bmatrix} A_iY_{ij}+Y_{ij}A_i^T+B_iF_{ij}+F_{ij}^TB_i^T+(\alpha_{ii}+\beta^i_{jj})Y_{ij} & H_1(i,j,Y) \\ * & H_2(i,j,Y) \end{bmatrix}<0\,, \tag{4.17}$$

$$X_{ij}A_i+A_i^TX_{ij}+L_{ij}C_i+C_i^TL_{ij}+\sum_{k\in S_1}\alpha_{ik}X_{kj}+\sum_{k\in S_2}\beta^i_{jk}X_{ik}<0, \tag{4.18}$$

$$\begin{bmatrix} Y_{ij} & I \\ I & X_{ij} \end{bmatrix}>0, \tag{4.19}$$

where

$$H_1(i,j,Y) = Y_{ij}\left[\sqrt{\alpha_{i1}}I,\cdots,\sqrt{\alpha_{i,i-1}}I,\sqrt{\alpha_{i,i+1}}I,\cdots,\sqrt{\beta^i_{j1}}I,\cdots,\sqrt{\beta^i_{j,j-1}}I,\sqrt{\beta^i_{j,j+1}}I,\cdots\right],$$
$$H_2(i,j,Y) = -\text{diag}\{Y_{1j},\cdots,Y_{i-1,j},Y_{i+1,j},\cdots,Y_{i1},\cdots,Y_{i,j-1},Y_{i,j+1},\cdots\}.$$

Furthermore, the parameters of the dynamic output feedback controller are given as follows:

$$\begin{aligned}\hat{A}_{ij} = \; & (X_{ij}-Y_{ij}^{-1})^{-1}(A_i^T + X_{ij}A_iY_{ij} + X_{ij}B_iF_{ij} + L_{ij}C_iY_{ij} \\ & + \textstyle\sum_{k\in S_1}\alpha_{ik}Y_{kj}^{-1}Y_{ij} + \sum_{k\in S_2}\beta^i_{jk}Y_{ik}^{-1}Y_{ij})Y_{ij}^{-1},\end{aligned} \tag{4.20}$$

$$\hat{B}_{ij} = \left(Y_{ij}^{-1} - X_{ij}\right)^{-1} L_{ij}, \tag{4.21}$$

$$\hat{C}_{ij} = F_{ij}Y_{ij}^{-1}. \tag{4.22}$$

Proof: Necessity: For the closed-loop system with $\zeta(t) = i, \eta(t) = j$, define the Lyapunov function as: $V(\tilde{x}, \zeta(t) = i, \eta(t) = j) = \tilde{x}^T P_{ij}\tilde{x}$, where

$$P_{ij} = \begin{bmatrix} P_{1ij} & P_{2ij} \\ P_{2ij}^T & P_{3ij} \end{bmatrix}.$$

Define

$$Y_{ij} = \left(P_{1ij} - P_{2ij}P_{3ij}^{-1}P_{2ij}^T\right)^{-1} > 0, \tag{4.23}$$

$$T_{1ij} = \begin{bmatrix} Y_{ij} & I \\ -P_{3ij}^{-1}P_{2ij}^TY_{ij} & 0 \end{bmatrix}. \tag{4.24}$$

Since there exists solution such that $P_{ij}\tilde{A}_{ij} + \tilde{A}_{ij}^TP_{ij} + \sum_{k\in S_1}\alpha_{ik}P_{kj} + \sum_{k\in S_2}\beta^i_{jk}P_{ik} < 0$, Apply similarity transformation with T_{1ij}, and define the left hand side of inequality as G_{ij}, $F_{ij} = -\hat{C}_{ij}P_{3ij}^{-1}P_{2ij}^TY_{ij}$ and $L_{ij} = P_{2ij}\hat{B}_{ij}$. Denote the (1,1) and (2,2) entries of G_{ij} as G_{11} and G_{22}, with

$$\begin{aligned} G_{11} = & A_iY_{ij} + Y_{ij}A_i^T + B_iF_{ij} + F_{ij}^TB_i^T \\ & + \sum_{k\in S_1}\alpha_{ik}Y_{ij}\left[Y_{kj}^{-1} + \left(P_{2kj} - P_{2ij}P_{3ij}^{-1}P_{3kj}\right)P_{3kj}^{-1}\left(P_{2kj} - P_{2ij}P_{3ij}^{-1}P_{3kj}\right)^T\right]Y_{ij} \\ & + \sum_{k\in S_2}\beta^i_{jk}Y_{ij}\left[Y_{ik}^{-1} + \left(P_{2ik} - P_{2ij}P_{3ij}^{-1}P_{3ik}\right)P_{3ik}^{-1}\left(P_{2ik} - P_{2ij}P_{3ij}^{-1}P_{3ik}\right)^T\right]Y_{ij}, \\ G_{22} = & P_{1ij}A_i + A_i^TP_{1ij} + L_{ij}C_i + C_i^TL_{ij}^T + \sum_{k\in S_1}\alpha_{ik}P_{1kj} + \sum_{k\in S_2}\beta^i_{jk}P_{1ik}. \end{aligned}$$

From the negative definiteness of the matrix G_{ij}, it follows that $G_{11} < 0$, $G_{22} < 0$.

On the other hand, due to the positive definiteness of Lyapunov matrix P_{ij}, after applying similarity transformation with T_{1ij}, it leads to:

$$T_{1ij}^T P_{ij} T_{1ij} = \begin{bmatrix} Y_{ij} & I \\ I & X_{ij} \end{bmatrix} > 0 \,.$$

Sufficiency:

Construct a specific Lyapunov matrix as $P_{ij} = \begin{bmatrix} X_{ij} & Y_{ij}^{-1} - X_{ij} \\ Y_{ij}^{-1} - X_{ij} & X_{ij} - Y_{ij}^{-1} \end{bmatrix} > 0$, and further define a similarity transformation matrix $T_{2ij} = \begin{bmatrix} Y_{ij} & I \\ Y_{ij} & 0 \end{bmatrix}$,

then one can obtain:

$$\begin{aligned}
\bar{G}_{ij} &= \begin{bmatrix} \bar{G}_{11} & \bar{G}_{12} \\ \bar{G}_{21} & \bar{G}_{22} \end{bmatrix} = T_{2ij}^T \left(P_{ij}\tilde{A}_{ij} + \tilde{A}^T P_{ij} + \Sigma_{k \in S_1} \alpha_{ik} P_{kj} + \Sigma_{k \in S_2} \beta_{jk}^i P_{ik} \right) T_{2ij} \,, \\
\bar{G}_{11} &= A_i Y_{ij} + Y_{ij} A_i^T + B_i F_{ij} + F_{ij}^T B_i^T + Y_{ij} \left(\Sigma_{k \in S_1} \alpha_{ik} Y_{kj} + \Sigma_{k \in S_2} \beta_{jk}^i Y_{ik} \right) Y_{ij} \,, \\
\bar{G}_{12} &= \bar{G}_{21} = 0 \,, \\
\bar{G}_{22} &= X_{ij} A_i + A_i^T X_{ij} + L_{ij} C_i + C_i^T L_{ij} + \Sigma_{k \in S_1} \alpha_{ik} X_{kj} + \Sigma_{k \in S_2} \beta_{jk}^i X_{ik} \,.
\end{aligned}$$

Therefore, it follows that $\bar{G}_{ij} < 0$ and hence the system is MSS. □

Remark 4.1 *The design here will generally lead to a full-order output feedback controller. As for reduced-order dynamic output feedback controller, the design would be much harder and has not been found yet using just LMI, even for ordinary MJLS. The design can be fulfilled by using iterative LMI algorithms, which will be discussed in Chapter 5 and Chapter 6.*

4.5 FDI-dependent Controller Design

In previous sections of this chapter, "full information" controller design for stochastic FTCS is studied, where no model uncertainties are considered, and all the conditions are expressed in terms of LMIs, which can be efficiently solved using various of available semi-definite optimization softwares.

However, in modeling practical systems, exact models are extremely hard and even impossible to obtain. In the context of active FTCS, model uncertainties are one of the sources that cause imperfect FDI decisions. Therefore, in the controller design, taking model uncertainties into consideration is meaningful and necessary. Stability should be maintained despite their presence, i.e. robust stability should be maintained.

Robust design is much harder than design for nominal systems. When using LMI techniques, the typical problem brought by model uncertainties is that nonlinear terms will appear in conditions derived, and generally it is difficult to convert them into LMIs.

In this section, for FTCS with norm-bounded model uncertainties, a design approach leading to the FDI-based controller for the uncertain systems will be proposed. This approach contains two steps. In the first step, the sufficient and necessary condition for the existence of K_{ij} is solved, and all the decision variables obtained in this step will be substituted back into the original nonlinear matrix inequality to solve for the controller K_j which only accesses the mode of the FDI decision process.

The system matrices are assumed to have the form:

$$A_i = A_{0i} + A_{1i}\Delta_{1i}A_{2i}, \quad B_i = B_{0i} + B_{1i}\Delta_{2i}B_{2i},$$

where $||\Delta_{1i}|| \leq 1$ and $||\Delta_{2i}|| \leq 1$.

4.5.1 State Feedback Controller Design

It is known that for the state feedback case, closed-loop being MSS is equivalent to the feasibility of the following matrix inequality:

$$P_{ij}\left(A_i + B_iK_j\right) + \left(A_i + B_iK_j\right)^T P_{ij} + \sum_{k\in S_1} \alpha_{ik}P_{kj} + \sum_{k\in S_2} \beta^i_{jk}P_{ik} < 0\,. \tag{4.25}$$

The first step is to remove the time-varying uncertainty matrix Δ_i from the inequality above. From Lemma 4.2, for positive scalars ε_{ij} and δ_{ij}, it is true that:

$$P_{ij}(A_{1i}\Delta_{1i}A_{2i}) + (A_{1i}\Delta_{1i}A_{2i})_i^T P_{ij} \leq \varepsilon_{ij}^{-1} A_{2i}^T A_{2i} + \varepsilon_{ij} P_{ij} A_{1i} A_{1i}^T P_{ij} \ ,$$
$$P_{ij}(B_{1i}\Delta_{2i}B_{2i})K_j + K_j^T (B_{1i}\Delta_{2i}B_{2i})^T P_{ij} \leq \delta_{ij}^{-1} K_j^T B_{2i}^T B_{2i} K_j + \delta_{ij} P_{ij} B_{1i} B_{1i}^T P_{ij} \ .$$

Substitute above inequalities into (4.25) to have the corresponding condition for MSS as follows:

$$\begin{aligned}&(A_{0i} + B_{0i}K_j)^T P_{ij} + P_{ij}(A_{0i} + B_{0i}K_j) + \textstyle\sum_{k\in S_1} \alpha_{ik} P_{kj} + \sum_{k\in S_2} \beta_{jk}^i P_{ik} \\ &+\varepsilon_{ij}^{-1} A_{2i}^T A_{2i} + \varepsilon_{ij} P_{ij} A_{1i} A_{1i}^T P_{ij} + \delta_{ij}^{-1} K_j^T B_{2i}^T B_{2i} K_j + \delta_{ij} P_{ij} B_{1i} B_{1i}^T P_{ij} < 0 \ .\end{aligned} \tag{4.26}$$

Since now K_j and P_{ij} have different subscripts, so the method used in the previous section would not work here. However, in this section, the methodology used is to obtain a local parametrization of K_{ij} first, then the controller K_j is the intersection of K_{ij} $\forall i \in S_1$. Therefore, first replace K_j in the inequality above with K_{ij}:

$$\begin{aligned}&(A_{0i} + B_{0i}K_{ij})^T P_{ij} + P_{ij}(A_{0i} + B_{0i}K_{ij}) + \textstyle\sum_{k\in S_1} \alpha_{ik} P_{kj} + \sum_{k\in S_2} \beta_{jk}^i P_{ik} \\ &+\varepsilon_{ij}^{-1} A_{2i}^T A_{2i} + \varepsilon_{ij} P_{ij} A_{1i} A_{1i}^T P_{ij} + \delta_{ij}^{-1} K_{ij}^T B_{2i}^T B_{2i} K_{ij} + \delta_{ij} P_{ij} B_{1i} B_{1i}^T P_{ij} < 0 \ .\end{aligned} \tag{4.27}$$

Theorem 4.5 *If the state feedback control law $x(t) = K_{ij}x(t)$ is used, nonlinear matrix inequality (4.27) has feasible solution if and only if $\forall i \in S_1,\ j \in S_2$ there exist positive definite matrices X_{ij} and positive scalars $\varepsilon_{ij}, \delta_{ij}$ for the following LMI:*

$$\begin{bmatrix} H_7(i,j) & W_{1i}^T X_{ij} A_{2i}^T & \varepsilon_{ij} W_{1i}^T A_{1i} & \delta_{ij} W_{1i}^T B_{1i} & W_{1i}^T H_1(i,j,X) \\ A_{2i} X_{ij} W_{1i} & -\varepsilon_{ij} I & 0 & 0 & 0 \\ \varepsilon_{ij} A_{1i}^T W_{1i} & 0 & -\varepsilon_{ij} I & 0 & 0 \\ \delta_{ij} B_{1i}^T W_{1i} & 0 & 0 & -\delta_{ij} I & 0 \\ H_1(i,j,X)^T W_{1i} & 0 & 0 & 0 & H_2(i,j,X) \end{bmatrix} < 0 \ , \tag{4.28}$$

where

$$\begin{bmatrix} W_{1i}^T & W_{2i}^T \end{bmatrix}^T = ker \begin{bmatrix} B_{0i}^T & B_{2i}^T \end{bmatrix} \ ,$$
$$H_7(i,j) = W_{1i}^T \left(X_{ij} A_{0i}^T + A_{0i} X_{ij} + (\alpha_{ii} + \beta_{jj}^i) X_{ij} \right) W_{1i} - \delta_{ij} W_{2i}^T W_{2i} \ .$$

Proof:

For matrix inequality (4.27) use Schur complement to expand those quadratic terms:

$$\begin{bmatrix} H_5(i,j) & \varepsilon_{ij}P_{ij}A_{1i} & \delta_{ij}P_{ij}B_{1i} & k_{ij}^T B_{2i}^T \\ \varepsilon_{ij}A_{1i}^T P_{ij} & -\varepsilon_{ij}I & 0 & 0 \\ \delta_{ij}B_{1i}^T P_{ij} & 0 & -\delta_{ij}I & 0 \\ B_{2i}K_{ij} & 0 & 0 & -\delta_{ij}I \end{bmatrix} < 0\,, \tag{4.29}$$

where

$$H_5(i,j) = P_{ij}A_{0i} + A_{0i}^T P_{ij} + P_{ij}B_{0i}K_{ij} + K_{ij}^T B_{0i}^T P_{ij} + \varepsilon_{ij}^{-1}A_{2i}^T A_{2i} + \sum_{k\in S_1}\alpha_{ik}P_{kj} + \sum_{k\in S_2}\beta_{jk}^i P_{ik}\,.$$

The inequality above can be rewritten in the same format as in Projection Lemma:

$$G(i,j) + U^T K_{ij}^T V_{ij} + V_{ij}^T K_{ij} U < 0\,,$$

where

$$G(i,j) = \begin{bmatrix} G_{11}(i,j) & \varepsilon_{ij}P_{ij}A_{1i} & \delta_{ij}P_{ij}B_{1i} & 0 \\ \varepsilon_{ij}A_{1i}^T P_{ij} & -\varepsilon_{ij}I & 0 & 0 \\ \delta_{ij}B_{1i}^T P_{ij} & 0 & -\delta_{ij}I & 0 \\ 0 & 0 & 0 & -\delta_{ij}I \end{bmatrix},$$

$$U = \begin{bmatrix} I & 0 & 0 & 0 \end{bmatrix},$$

$$V_{ij} = \begin{bmatrix} B_{0i}^T P_{ij} & 0 & 0 & B_{2i}^T \end{bmatrix},$$

$$G_{11}(i,j) = A_{0i}^T P_{ij} + P_{ij}A_{0i} + \varepsilon_{ij}^{-1}A_{2i}^T A_{2i} + \sum_{k\in S_1}\alpha_{ik}P_{kj} + \sum_{k\in S_2}\beta_{jk}^i P_{ik}\,.$$

Using Projection Lemma, it can be concluded that this nonlinear matrix inequality (4.27) has solution if and only if

$$N_U^T G(i,j) N_U < 0 \quad \text{and} \quad N_V^T G(i,j) N_V < 0$$

are satisfied. Note that

$$N_U = \begin{bmatrix} 0 & 0 & 0 \\ I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix},$$

$$N_U^T G(i,j) N_U = \begin{bmatrix} -\varepsilon_{ij} I & 0 & 0 \\ 0 & -\delta_{ij} I & 0 \\ 0 & 0 & -\delta_{ij} I \end{bmatrix} < 0$$

always holds, since ε_{ij}, δ_{ij} are all positive scalars. So the only constraint is

$$N_V^T G(i,j) N_V < 0 \,. \tag{4.30}$$

Define $V_{1i} = \begin{bmatrix} B_{0i}^T & 0 & 0 & B_{2i}^T \end{bmatrix}$, then from the definition of orthogonal complement, it can be obtained that

$$N_V = \mathrm{diag}\{P_{ij}^{-1}, I, I, I\} N_{V_{1i}} \,. \tag{4.31}$$

Substitute this expression into inequality (4.30), it becomes

$$N_{V_{1i}}^T \begin{bmatrix} H_6(i,j) & \varepsilon_{ij} A_{1i} & \delta_{ij} B_{1i} & 0 \\ \varepsilon_{ij} A_{1i}^T & -\varepsilon_{ij} I & 0 & 0 \\ \delta_{ij} B_{1i}^T & 0 & -\delta_{ij} I & 0 \\ 0 & 0 & 0 & -\delta_{ij} I \end{bmatrix} N_{V_{1i}} < 0 \,, \tag{4.32}$$

where $H_6(i,j) = A_{0i} P_{ij}^{-1} + P_{ij}^{-1} A_{0i}^T + P_{ij}^{-1} \left(\varepsilon_{ij}^{-1} A_{2i}^T A_{2i} + \sum_{k \in S_1} \alpha_{ik} P_{kj} + \sum_{k \in S_2} \beta_{jk}^i P_{ik} \right) P_{ij}^{-1}$.

From the definition of V_{1i}, it can be inferred that $N_{V_{1i}}$ has the following expression:

$$N_{V_{1i}} = \begin{bmatrix} W_{1i} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \\ W_{2i} & 0 & 0 \end{bmatrix}.$$

Substitute the above expression into inequality (4.32), then one can obtain

$$\begin{bmatrix} W_{1i}^T H_6(i,j)W_{1i} - \delta_{ij}W_{2i}^T W_{2i} & \varepsilon_{ij}W_{1i}^T A_{1i} & \delta_{ij}W_{1i}^T B_{1i} \\ \varepsilon_{ij}A_{1i}^T W_{1i} & -\varepsilon_{ij}I & 0 \\ \delta_{ij}B_{1i}^T W_{1i} & 0 & -\delta_{ij}I \end{bmatrix} < 0 . \qquad (4.33)$$

Inequality above implies that at the (1,1) entry, the term $W_{1i}^T H_{6ij} W_{1i} - \delta_{ij}W_{2i}^T W_{2i} < 0$. Define $X_{ij} = P_{ij}^{-1}$, this term can further be expanded using Schur complement:

$$\begin{bmatrix} H_7(i,j) & W_{1i}^T P_{ij}^{-1} A_{2i}^T & W_{1i}^T H_1(i,j,X) \\ A_{2i}P_{ij}^{-1}W_{1i} & -\varepsilon_{ij}I & 0 \\ H_1(i,j,X)^T W_{1i} & 0 & H_2(i,j,X) \end{bmatrix} < 0 .$$

Combined the two LMIs above, the LMIs shown in (4.28) can be obtained. □

With X_{ij}, ε_{ij} and δ_{ij} known, the following theorem shows that all stabilizing controller K_{ij} can be parameterized.

Theorem 4.6 *([100])*

Let matrices $B \in \mathbf{C}^{n\times m}$, $C \in \mathbf{C}^{k\times n}$ and $Q = Q^* \in \mathbf{C}^{n\times n}$ be given. Then the following statements are equivalent:

(1) There exists a matrix X satisfying

$$BXC + (BXC)^* + Q < 0 . \qquad (4.34)$$

(2) The following two conditions hold

$$B^{\perp}QB^{\perp *} < 0 \quad or \quad BB^* > 0 ,$$
$$C^{*\perp}QC^{*\perp *} < 0 \quad or \quad C*C > 0 .$$

Suppose the above statement hold. Let r_b and r_c be the rank of B and C, respectively, and (B_l, B_r) and (C_l, C_r) be any full rank factors of B and C, i.e. $B = B_l B_r$, $C = C_l C_r$. Then all matrices X in statement (1) are given by

$$X = B_r^+ K C_l^+ + Z - B_r^+ B_r Z C_l C_l^+ , \qquad (4.35)$$

where Z is an arbitrary matrix and

$$K \triangleq -R^{-1}B_l^*\Phi C_r^*(C_r\Phi C_r^*)^{-1} + S^{\frac{1}{2}}L(C_r\Phi C_r^*)^{-\frac{1}{2}} ,$$
$$S \triangleq R^{-1} - R^{-1}B_l^*[\Phi - \Phi C_r^*(C_r\Phi C_r^*)^{-1}C_r\Phi]B_lR^{-1} ,$$

where L is an arbitrary matrix such that $\|L\| < 1$ *and R is an arbitrary positive definite matrix such that*

$$\Phi \triangleq (B_lR^{-1}B_l^* - Q)^{-1} > 0 .$$

Obviously, the first part of the theorem is exactly the same as Projection Lemma, and the second part is on controller parametrization. When applying this theorem to solve the controller, a set of P_{ij}, ε_{ij} and δ_{ij} can be obtained as the solution of (4.28). Then based on this set of solution, the expression of B and C (here they also depend on i and j) can be calculated. This implies that a parametrization of controller K_{ij} can be found, with which the intersection of these controllers $\forall i \in S_1$ can be searched if exists. This constitutes the second step of the algorithm, and K_j can be found in this step if the solution from the first step is proper.

Remark 4.2 *In the second step, after substituting in the value of* P_{ij}, *it is trying to find FDI-dependent controller* K_j. *Applying the same method, it is also possible to find passive fault tolerant controller K, which is independent of both system real fault* $\zeta(t)$ *and FDI decision process* $\eta(t)$.

4.5.2 Output Feedback Controller Design

The dynamic output feedback controller which only uses system output and the FDI mode rather than the system real mode will be designed in the following part. System augmentation technique will be used to handle the synthesis problem. Unlike in the previous section, where a strictly proper and full order dynamic output feedback controller is designed, to be more general, there are no constraints applied on the dynamic output feedback controller designed here, i.e. the output feedback controller has the form (assuming that $\eta(t) = j$):

$$\begin{cases} \dot{x}_c(t) = \hat{A}_jx_c(t) + \hat{B}_jy(t), \\ u(t) = \hat{C}_jx_c(t) + \hat{D}_jy(t). \end{cases} \tag{4.36}$$

With this controller, the closed-loop matrix has the form:

$$\begin{bmatrix} \dot{x} \\ \dot{x}_c \end{bmatrix} = \begin{bmatrix} A_i + B_i\hat{D}_jC_i & B_i\hat{C}_j \\ \hat{B}_jC_i & \hat{A}_j \end{bmatrix} \begin{bmatrix} x \\ \hat{x}_c \end{bmatrix}. \tag{4.37}$$

As for the state feedback case, the design begins with separation of uncertain terms from other part of the expression. To accommodate the uncertainty terms, the closed-loop system state equation is first expanded as follows:

$$\begin{aligned} &\begin{bmatrix} A_i + B_i\hat{D}_jC_i & B_i\hat{C}_j \\ \hat{B}_jC_i & \hat{A}_j \end{bmatrix} \\ &= \begin{bmatrix} A_{0i} + B_{0i}\hat{D}_jC_i & B_{0i}\hat{C}_j \\ \hat{B}_jC_i & \hat{A}_j \end{bmatrix} + \begin{bmatrix} A_{1i} & B_{1i} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta_{1i} & 0 \\ 0 & \Delta_{2i} \end{bmatrix} \begin{bmatrix} A_{2i} & 0 \\ B_{2i}\hat{D}_jC_i & B_{2i}\hat{C}_j \end{bmatrix} \\ &= \bar{A}_{0i} + \bar{B}_{0i}K_j\bar{C}_i + \bar{A}_{1i}\Delta(\bar{A}_{2i} + \bar{B}_{2i}K_j\bar{C}_i) , \end{aligned} \tag{4.38}$$

where

$$K_j = \begin{bmatrix} \bar{A}_j & \bar{B}_j \\ \bar{C}_j & \bar{D}_j \end{bmatrix}, \bar{A}_{0i} = \begin{bmatrix} A_{0i} & 0 \\ 0 & 0 \end{bmatrix}, \bar{B}_{0i} = \begin{bmatrix} 0 & B_{0i} \\ I & 0 \end{bmatrix}, \bar{C}_i = \begin{bmatrix} 0 & I \\ C_i & 0 \end{bmatrix},$$

$$\bar{A}_{1i} = \begin{bmatrix} A_{1i} & B_{1i} \\ 0 & 0 \end{bmatrix}, \bar{A}_{2i} = \begin{bmatrix} A_{2i} & 0 \\ 0 & 0 \end{bmatrix}, \bar{B}_{2i} = \begin{bmatrix} 0 & 0 \\ 0 & B_{2i} \end{bmatrix}, \Delta_i = \begin{bmatrix} \Delta_{1i} & 0 \\ 0 & \Delta_{2i} \end{bmatrix}.$$

Then the condition on MSS is equivalent to the feasibility of the following matrix inequality:

$$\begin{aligned} &P_{ij}\bar{A}_{0i} + \bar{A}_{0i}^TP_{ij} + P_{ij}\bar{B}_{0i}K_j\bar{C}_i + \bar{C}_i^TK_j^T\bar{B}_{0i}^TP_{ij} + P_{ij}\bar{A}_{1i}\Delta_i(\bar{A}_{2i} + \bar{B}_{2i}K_j\bar{C}_i) \\ &+ (\bar{A}_{2i} + \bar{B}_{2i}K_j\bar{C}_i)^T\Delta_i^T\bar{A}_{1i}^TP_{ij} + \Sigma_{k\in S_1}\alpha_{ik}P_{kj} + \Sigma_{k\in S_2}\beta_{jk}^iP_{ik} < 0. \end{aligned}$$

Similar to the state feedback case, the sufficient condition for closed-loop system's MSS is that there exist K_j, $P_{ij} > 0$, scalar $\varepsilon_{ij} > 0$ satisfying the following matrix inequality:

$$\begin{aligned} &P_{ij}\bar{A}_{0i} + \bar{A}_{0i}^TP_{ij} + P_{ij}\bar{B}_{0i}K_j\bar{C}_i + \bar{C}_i^TK_j^T\bar{B}_{0i}^TP_{ij} + \varepsilon_{ij}P_{ij}\bar{A}_{1i}\bar{A}_{1i}^TP_{ij} \\ &+ \varepsilon_{ij}^{-1}(\bar{A}_{2i} + \bar{B}_{2i}K_j\bar{C}_i)^T(\bar{A}_{2i} + \bar{B}_{2i}K_j\bar{C}_i) + \Sigma_{k\in S_1}\alpha_{ik}P_{kj} + \Sigma_{k\in S_2}\beta_{jk}^iP_{ik} < 0. \end{aligned} \tag{4.39}$$

As same as the state-feedback control case, where a two-step procedure is proposed to solve FDI-dependent controller. The sufficient conditions for the existence of dynamic output feedback controller K_{ij} is given in the following theorem.

Theorem 4.7 *There exists dynamic output feedback controller of the form (4.36) that stabilizes the uncertain system (4.37) in the MSS sense if* $\forall i \in S_1,\ j \in S_2$ *there exist feasible solution* $P_{ij} > 0$, *scalar* $\varepsilon_{ij} > 0$ *for following matrix inequalities:*

$$\begin{bmatrix} H_8(i,j) & W_{3i}^T(P_{ij}-R_{ij})^T & W_{3i}^T\bar{A}_{2i}^T & W_{3i}^T\varepsilon_{ij}P_{ij}\bar{A}_{1i} \\ * & -I & 0 & 0 \\ * & * & -\varepsilon_{ij}I & 0 \\ * & * & * & -\varepsilon_{ij}I \end{bmatrix} < 0, \qquad (4.40)$$

$$\begin{bmatrix} H_9(i,j) & R_{ij}^TW_{1i}+\bar{A}_{2i}^TW_{2i} & \varepsilon_{ij}P_{ij}\bar{A}_{1i} & (P_{ij}-R_{ij})^T \\ * & -W_{1i}^TW_{1i}-\varepsilon_{ij}W_{2i}^TW_{2i} & 0 & 0 \\ * & * & -\varepsilon_{ij}I & 0 \\ * & * & * & -I \end{bmatrix} < 0, \qquad (4.41)$$

where

$$\begin{bmatrix} W_{1i} \\ W_{2i} \end{bmatrix} = ker(\begin{bmatrix} \bar{B}_{0i}^T & \bar{B}_{2i}^T \end{bmatrix}),$$

$$W_{3i} = ker(\bar{C}_i),$$

$$H_8(i,j) = W_{3i}^T\left(P_{ij}\bar{A}_{0i}+\bar{A}_{0i}^TP_{ij}+\Sigma_{k\in S_1}\alpha_{ik}P_{kj}+\Sigma_{k\in S_2}\beta_{jk}^iP_{ik}\right)W_{3i},$$

$$H_9(i,j) = P_{ij}\bar{A}_{0i}+\bar{A}_{0i}^TP_{ij}+\Sigma_{k\in S_1}\alpha_{ik}P_{kj}+\Sigma_{k\in S_2}\beta_{jk}^iP_{ik}.$$

Proof: Using Reciprocal Projection Lemma, it follows that: the feasibility of (4.39) is equivalent to the existence of P_{ij}, K_j, T_{ij}, and $Q_{ij} > 0$ satisfying:

$$\begin{bmatrix} \bar{H}_8(i,j) & \bar{C}_i^TK_j^T\bar{B}_{0i}^TP_{ij}+T_{ij}^T & (\bar{A}_{2i}+\bar{B}_{2i}K_j\bar{C}_i)^T & \varepsilon_{ij}P_{ij}\bar{A}_{1i} \\ * & -Q_{ij} & 0 & 0 \\ * & * & -\varepsilon_{ij}I & 0 \\ * & * & * & -\varepsilon_{ij}I \end{bmatrix} < 0, \qquad (4.42)$$

where

$$\bar{H}_8(i,j) = P_{ij}\bar{A}_{0i}+\bar{A}_{0i}^TP_{ij}+\sum_{k\in S_1}\alpha_{ik}P_{kj}+\sum_{k\in S_2}\beta_{jk}^iP_{ik}+Q_{ij}-(T_{ij}+T_{ij}^T).$$

Use Projection Lemma as in the state feedback case, rewrite the inequality above as:

$$H(i,j) + U_i^T X_j^T V_{ij} + V_{ij}^T X_j U_i < 0,$$

$$H(i,j) = \begin{bmatrix} \bar{H}_{8ij} & T_{ij}^T & \bar{A}_{2i}^T & \varepsilon_{ij} P_{ij} \bar{A}_{1i} \\ * & -Q_{ij} & 0 & 0 \\ * & * & -\varepsilon_{ij} I & 0 \\ * & * & * & -\varepsilon_{ij} I \end{bmatrix},$$

$$U_i = \begin{bmatrix} \bar{C}_i & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$V_{ij} = \begin{bmatrix} 0 & \bar{B}_{0i}^T P_{ij} & \bar{B}_{2i}^T & 0 & 0 \end{bmatrix},$$

$$X_j = K_j.$$

Then the original matrix feasibility is equivalent to that of:

$$N_{U_i}^T H N_{U_i} < 0, \quad N_{V_{ij}}^T H N_{V_{ij}} < 0. \tag{4.43}$$

Substitute all known expressions into (4.43) to obtain:

$$\begin{bmatrix} W_{3i}^T \bar{H}_{8ij} W_{3i} & W_{3i}^T T_{ij}^T & W_{3i}^T \bar{A}_{2i}^T & W_{3i}^T \varepsilon_{ij} P_{ij} \bar{A}_{1i} \\ * & -Q_{ij} & 0 & 0 \\ * & * & -\varepsilon_{ij} I & 0 \\ * & * & * & -\varepsilon_{ij} I \end{bmatrix} < 0, \tag{4.44}$$

$$\begin{bmatrix} -W_{1i}^T P_{ij}^{-1} Q_{ij} P_{ij}^{-1} W_{1i} - \varepsilon_{ij} W_{2i}^T W_{2i} & W_{1i}^T P_{ij}^{-1} T_{ij} + W_{2i}^T \bar{A}_{2i} & 0 \\ * & \bar{H}_{8ij} & \varepsilon_{ij} P_{ij} \bar{A}_{1i} \\ * & * & -\varepsilon_{ij} I \end{bmatrix} < 0. \tag{4.45}$$

Define $R_{ij} = P_{ij}^{-1} T_{ij}$ and $Q_{ij} = P_{ij} P_{ij}$, and use Schur complement, (4.44) is equivalent to (4.40). Similarly for (4.45), apply congruence transformation with

$$\begin{bmatrix} 0 & I & 0 \\ I & 0 & 0 \\ 0 & 0 & I \end{bmatrix},$$

and use the inequality

$$P_{ij}P_{ij} - (P_{ij}R_{ij} + R_{ij}^T P_{ij}) \leq (P_{ij} - R_{ij})^T (P_{ij} - R_{ij}),$$

it can be seen that (4.45) holds if (4.41) holds. This complete the proof. □

Just like in the state-feedback case, after calculating feasible solutions of (4.40) and (4.41), substitute these values into the original matrix inequality (4.39) to solve output feedback controller K_j.

Remark 4.3 *Unlike in the full information output feedback controller design, where the "change of variable" method is used and the constraint brought in is that only full-order strict proper dynamic output feedback controller can be designed using that algorithm. Here, the parameters of the controller are stacked up into a matrix, and no constraints are imposed on the order and structure of this matrix, therefore, both full-order and reduced-order controller can be calculated.*

Example 4.1 The purpose of this example is to provide a state-feedback design result where the algorithm proposed for FDI-based controller in this chapter is applied. Consider a second-order system, $S = \{1, 2\}$. (A_1, B_1, C_1) is assumed to be the normal system model and (A_2, B_2, C_2) is a faulty one:

$$A_{01} = A_{02} = \begin{bmatrix} 1 & 0 \\ 0 & 0.8 \end{bmatrix}, \; B_{01} = \begin{bmatrix} 0 & 1 \\ -0.25 & 0.25 \end{bmatrix},$$

$$B_{02} = \begin{bmatrix} 0 & 0.2 \\ -0.25 & 0.05 \end{bmatrix}, \; C_1 = C_2 = \begin{bmatrix} 1 & 2 \end{bmatrix}.$$

The weighting matrices for the disturbance imposed on the state equation, and the bounds for ΔA_i, ΔB_i, $i = 1, 2$, are given as:

$$D_1 = \begin{bmatrix} 0.05 \\ 0.05 \end{bmatrix}, D_2 = \begin{bmatrix} 0.05 \\ 0.1 \end{bmatrix}, A_{1i} = A_{2i} = B_{1i} = B_{2i} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \; i = 1, 2.$$

The transition rate matrix for fault and FDI Markov chain is chosen as: $\alpha = \begin{bmatrix} -0.5 & 0.5 \\ 2 & -2 \end{bmatrix}$, $[\beta^1] = [\beta^2] = \begin{bmatrix} -0.1 & 0.1 \\ 0.1 & -0.1 \end{bmatrix}$.

For this system, a state feedback control is designed by solving the LMIs developed in this section. The solutions are obtained as:

$$X_{11} = \begin{bmatrix} 1.3107 & -0.0364 \\ -0.0364 & 0.7634 \end{bmatrix}, X_{12} = \begin{bmatrix} 1.2347 & 0.0484 \\ 0.0484 & 0.6175 \end{bmatrix},$$

$$X_{21} = \begin{bmatrix} 0.7036 & -0.0880 \\ -0.0880 & 0.6203 \end{bmatrix}, X_{22} = \begin{bmatrix} 0.8821 & -0.1472 \\ -0.1472 & 0.7191 \end{bmatrix},$$

$$[\varepsilon_{ij}] = \begin{bmatrix} 1.2922 & 1.2922 \\ 1.2912 & 1.2912 \end{bmatrix}, [\delta_{ij}] = \begin{bmatrix} 1.365 & 1.3051 \\ 1.3519 & 1.4351 \end{bmatrix},$$

$$K_1 = \begin{bmatrix} 1.1208 & 8.9006 \\ -7.4239 & -2.1148 \end{bmatrix}, K_2 = \begin{bmatrix} -0.2009 & 8.4035 \\ -7.7094 & -1.7643 \end{bmatrix}.$$

By using the first set of controllers, a single sample path simulation is performed, and the results are shown in Figure 4.1. The disturbance is modeled as $w(t) = e^{-0.1t}\sin t$. In Figure 4.1(a) and (b), the mode of fault process $\zeta(t)$ and the FDI decision process $\eta(t)$ are present respectively. The controller used at the time constant t is dependent on the value of $\eta(t)$ at that moment. When zero and non-zero initial states are used in simulation, results show that the system is stable, and the disturbance $w(t)$ is attenuated.

4.5.3 State Feedback Controller Design: an Improved Approach

In the previous parts of this section, algorithms using Projection Lemma to solve for state-feedback controller and output-feedback controller are discussed, where in design of output feedback controller, approximation is used to simplify the design. However, the design involves two steps. Other design variables are solved in the first step while the controller is solved in the second. However, some trials and errors may be needed before getting the

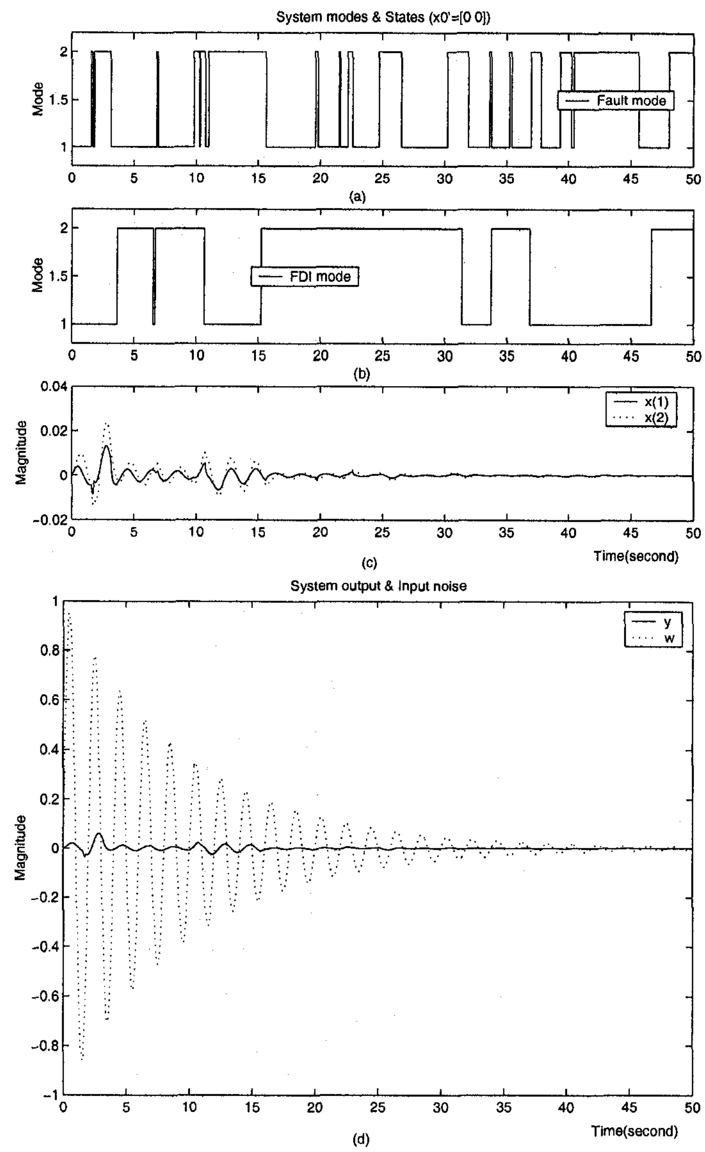

Figure 4.1. Single sample path simulation: (a) system modes; (b) FDI modes; (c) system state response; (d) the system output and disturbance

controller, which undoubtedly is not satisfactory. In this part, a much simpler method in designing FDI-based state-feedback controller will be given.

The method used here is close to the common Lyapunov function approach. Common Lyapunov function is widely used in the design of complex systems, where a set of subsystems exists, e.g. for switching systems, or systems with multiple objectives, e.g. H_2/H_∞

control design [96]. Using common Lyapunov function may simplify the design, result in LMI rather than generally untractable multi-objective nonlinear matrix inequalities.

However, direct use of common Lyapunov function may be unrealistic for some situations. Just cite stochastic FTCS as an example, where the MSS condition for continuous-time systems are:

$$P_{ij}\left(A_i+B_iK_j\right)+\left(A_i+B_iK_j\right)^T P_{ij}+\sum_{k\in S_1}\alpha_{ik}P_{kj}+\sum_{k\in S_2}\beta^i_{jk}P_{ik}<0. \tag{4.46}$$

Setting either $P_{ij}=P_i$, $\forall j\in S_2$ or $P_{ij}=P_j$, $\forall i\in S_1$ will eliminate the term $\sum_{k\in S_2}\beta^i_{jk}P_{ik}$ or $\sum_{k\in S_1}\alpha_{ik}P_{kj}$ in the MSS condition shown in (4.46) respectively, which means information on FDI process or fault process is not utilized, and very conservative design results can be expected.

In the area of multi-objective design, it is well known the design generally cannot be directly solved using convex optimization unless some trade-offs are made. Therefore, the research on how to minimize the conservatism brought in, when cast the original conditions into LMI based ones, is quite active. A feasible approach is to introduce some instrumental variables, then set these variables common instead of Lyapunov function, thus can be expected to minimize the conservatism. The converted control problem is also denoted as "mixed control problem" to differentiate from "multi-objective control problem" [97].

Theorem 4.8 Under the state-feedback control law $u(t)=K_jx(t)$, the FTCS (4.4) is MSS if $\forall i\in S_1,\ j\in S_2$, there exist positive definite matrices $X_{ij}>0$, Q_j and $\bar{K}_j$ for the following LMI:

$$\begin{bmatrix} -(Q_j^T+Q_j) & Q_j^T(A_i^T+0.5(\alpha_{ii}+\beta^i_{jj})I)+\bar{K}_j^TB_i^T+X_{ij} & Q_j^TH_{1ij} & Q_j^T \\ * & -X_{ij} & 0 & 0 \\ * & * & H_{2ij} & 0 \\ * & * & * & -X_{ij} \end{bmatrix}<0, \tag{4.47}$$

where

$$H_{1ij}=\left[\sqrt{\alpha_{i1}}I,\cdots,\sqrt{\alpha_{i,i-1}}I,\sqrt{\alpha_{i,i+1}}I,\cdots,\sqrt{\beta^i_{j1}}I,\cdots,\sqrt{\beta^i_{j,j-1}}I,\sqrt{\beta^i_{j,j+1}}I,\cdots\right],$$
$$H_{2ij}=-\mathrm{diag}\{X_{1j},\cdots,X_{i-1,j},X_{i+1,j},\cdots,X_{i1},\cdots,X_{i,j-1},X_{i,j+1},\cdots\}.$$

Furthermore, the state-feedback control gain is given by $K_j = \bar{K}_j Q_j^{-1}$.

Proof: For matrix inequality (4.46), use Reciprocal Projection Lemma 4.4 to obtain:

$$\begin{bmatrix} \sum_{k\neq j}\alpha_{ik}P_{kj} + \sum_{k\neq j}\beta_{jk}^{i}P_{ik} + P_{ij} - (W_j^T + W_j) & (A_i + B_iK_j + 0.5(\alpha_{ii} + \beta_{jj}^{i})I)^T P_{ij} + W_j^T \\ * & -P_{ij} \end{bmatrix} < 0.$$

The (1,1)-entry implies that Q_j is nonsingular, then it is valid to define $X_{ij} = P_{ij}^{-1}$, $Q_j = W_j^{-1}$. Pre- and post-multiply the above matrix inequality by $T_{1ij} \triangleq \operatorname{diag}\{Q_j^T,\ X_{ij}\}$ and T_{1ij}^T,

$$\begin{bmatrix} Q_j^T(\sum_{k\neq j}\alpha_{ik}P_{kj} + \sum_{k\neq j}\beta_{jk}^{i}P_{ik} + P_{ij})Q_j - (Q_j^T + Q_j) & Q_j^T(A_i + B_iK_j + 0.5(\alpha_{ii} + \beta_{jj}^{i})I)^T + X_{ij} \\ * & -X_{ij} \end{bmatrix} < 0.$$

Use Schur complement lemma to expand the quadratic terms in the (1,1)-entry and define $\bar{K}_j = K_jQ_j$, then the results shown in (4.47) can be obtained. □

Similarly, for discrete-time systems, corresponding results for state-feedback controller design can be obtained.

Theorem 4.9 *Under the state-feedback control law $u(k) = K_jx(k)$, the FTCS (4.5) is MSS if $\forall i \in S_1,\ j \in S_2$, there exist positive definite matrices $R_{ij} > 0$, Q_j and $\bar{K}_j$ for the following LMI:*

$$\begin{bmatrix} R_{ij} & A_iG_j + B_i\bar{K}_j \\ G_j^TA_i^T + \bar{K}_j^TB_i^T & G_j^T + G_j - \sum_{m\in S_1}\sum_{n\in S_2}\alpha_{mi}\beta_{nj}^{m}R_{mn} \end{bmatrix} > 0. \tag{4.48}$$

Furthermore, the state feedback gain is given by $K_j = \bar{K}_jG_j^{-1}$.

Proof: Start from the other form of MSS condition for closed-loop FTCS:

$$-P_{mn} + \sum_{i\in S_1}\sum_{j\in S_2}\alpha_{im}\beta_{jn}^{i}\tilde{A}_{ij}P_{ij}\tilde{A}_{ij}^T < 0, \tag{4.49}$$

where $\tilde{A}_{ij} = A_i + B_iK_j$ is the closed-loop system matrix.

Since for discrete-times systems, $\forall i,m \in S_1,\ j,n \in S_2$, $\alpha_{im} \geq 0$ and $\beta^i_{jn} \geq 0$ hold, therefore positive definite matrices R_{ij} satisfying $P_{mn} = \sum_{i\in S_1}\sum_{j\in S_2}\alpha_{im}\beta^i_{jn}R_{ij}$ can be defined. Therefore the inequality (4.49) is satisfied if

$$-R_{ij} + \tilde{A}_{ij}P_{ij}\tilde{A}^T_{ij} < 0.$$

For this inequality, use Lemma 4.5 to obtain its equivalence as:

$$\begin{bmatrix} R_{ij} & \tilde{A}_{ij}G_{ij} \\ G^T_{ij}\tilde{A}^T & G^T_{ij} + G_{ij} - P_{ij} \end{bmatrix} > 0.$$

That is:

$$\begin{bmatrix} R_{ij} & \tilde{A}_{ij}G_{ij} \\ G^T_{ij}\tilde{A}^T & G^T_{ij} + G_{ij} - \sum_{m\in S_1}\sum_{n\in S_2}\alpha_{mi}\beta^m_{nj}R_{mn} \end{bmatrix} > 0.$$

To synthesize state-feedback controller for this case, one can set G_j in the inequality, and define $\bar{K}_j = K_jG_j$. i.e. The controller can be solved by the LMI shown in (4.48). □

4.6 Conclusion

In this chapter, miscellaneous stabilization results are summarized and presented. Both continuous-time and discrete-time system design are covered for completeness. Results on full information controller are given first, which are dominant in the available literatures in this two-Markov-chain FTCS framework. For more practical FDI-based controller design, a two-step approach using Projection Lemma is presented for both state-feedback and dynamic output feedback controllers, however, some trials and errors may be needed. To overcome this disadvantage, for state feedback controller synthesis, using mixed objective design techniques, the conditions can be expressed in terms of LMIs. This work can be regarded as the extension of the "cluster observation" of MJLS in [110].

Chapter 5

FTCS Design for H_2 Performance

5.1 Introduction

In Chapter 4, stabilization controller design for stochastic FTCS within the two-Markov-chain framework was discussed. The design objective is MSS and the conditions are expressed in terms of LMIs.

In this chapter, besides the MSS, input-output performance with respect to the additive disturbance $w(t)$ in terms of H_2 norm is studied. The H_2 norm of the FTCS is defined first parallel to the definition of MJLS. Similar problem for MJLS was solved by Costa and his colleagues using the "Analytical Point of View (APV)" approach, where operator theory is used for analysis and design. In this approach, definition of each operator generally only involves closed-loop system description, therefore there is no restriction on which form the controller should have.

In Chapter 4, the FDI-based controller is designed either using Projection Lemma or common Lyapunov function like approach. Unfortunately, both methods have limitations. The conditions derived by using Projection Lemma is not a sufficient condition, so some trials and errors may be needed before obtaining the controller. On the other hand, the common Lyapunov function like approach can only be used in state-feedback controller design with conservatism introduced inherited from common Lyapunov functions. Therefore, in this chapter and the next chapter, an iterative LMI algorithm is proposed to solve for the controller. Compared with existing ones in the previous chapters, the algorithm shown in this chapter has lots of advantages: It starts from a sufficient condition, and the iterative LMI is guaranteed for at least local convergence. And the controller can be designed with structure and order constraints.

In the remaining part of this chapter, the H_2 control synthesis for uncertain FTCS will be discussed, for both continuous-time and discrete-time systems using both state-feedback and output-feedback control. As in the previous chapter, the iterative LMI algorithm will be used to solve the nonconvex optimization problem.

5.2 Modeling and Problem Formulation

5.2.1 Notation

Some special notations are used in this chapter. Set $\mathscr{H}^{m,\,n}$ the linear space made up of all $q_1 \times q_2$-sequence of matrices $V = \left[V_{11}, V_{12}, \cdots, V_{1q_2}, V_{21}, \cdots, V_{q_1 q_2}\right]$, where each V_{ij}, $i = 1, 2, \cdots, q_1$, $j = 1, 2, \cdots, q_2$ is a $m \times n$ matrix. And denote $\mathscr{H}^{n+} = \{V \in \mathscr{H}^{n,n}; V_{ij} \geq 0\}$. For $H,\ L \in \mathscr{H}^{n+}$, the notation $H \geq L$ $(H > L)$ indicates that $H_{ij} \geq L_{ij}$ $(H_{ij} > L_{ij})$ for each $i = 1, 2, \cdots, q_1$, $j = 1, 2, \cdots, q_2$. In this chapter, stochastic signals are assumed to belong to the $(\Omega, \mathscr{F}, \mathbb{P})$ probability space. For square integrable signals, i.e. $w(t)$ (or $w(k)$) $\in L_2(\Omega, \mathscr{F}, \mathbb{P})$, define the signal norm as $\|w\|_2^2 = \mathscr{E}\{\int_0^\infty \|w(t)\|^2 dt\}$ for continuous-time signals or $\|w\|_2^2 = \mathscr{E}\{\sum_{k=0}^\infty \|w(k)\|^2\}$ for discrete-time signals.

5.2.2 Fault Tolerant Control Systems Modeling

The continuous-time system to be studied in this chapter is described by:

$$\mathscr{G}_c : \begin{cases} \dot{x}(t) = A(\zeta(t))x(t) + B(\zeta(t))u(t, \eta(t)) + D(\zeta(t))w(t), \\ y(t) = C(\zeta(t))x(t), \end{cases} \tag{5.1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $w(t) \in \mathbb{R}^l$, $y(t) \in \mathbb{R}^q$ are system state variable, control input, disturbance and output, respectively; all matrices have corresponding compatible dimensions. In this system, $\{\zeta(t), t \geq 0\}$ and $\{\eta(t), t \geq 0\}$ are Markov chains defined on

$S_1 = \{1, 2, \ldots, q_1\}$ and $S_2 = \{1, 2, \ldots, q_2\}$ with transition rate matrices $[\alpha_{ij}]$ and $[\beta^i_{ij}]$ respectively as defined in Chapter 3. In brief, the conventional two-Makov-chain framework with standard norm-bounded model uncertainties is considered in this chapter.

In the sequel, simplified notations are used with the mode $\zeta(t) = i$ and $\eta(t) = j$ as in previous chapters.

With the given state-feedback or output-feedback control law, the closed-loop system model can then be written as following forms:

- State-feedback case:

$$\begin{cases} \dot{x}(t) = \left(A_i + B_i K_j\right) x(t) + D_i w(t), \\ y(t) = C_i x(t). \end{cases} \tag{5.2}$$

- Output-feedback case:

$$\begin{cases} \begin{bmatrix} \dot{x}(t) \\ \dot{\hat{x}}(t) \end{bmatrix} = \begin{bmatrix} A_i + B_i \hat{D}_j C_i & B_i \hat{C}_j \\ \hat{B}_j C_i & \hat{A}_j \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix} + \begin{bmatrix} D_i \\ 0 \end{bmatrix} w(t), \\ y = \begin{bmatrix} C_i & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix}. \end{cases} \tag{5.3}$$

For both cases, unify the notation for the closed-loop system as:

$$\tilde{\mathscr{G}}_c : \begin{cases} \dot{\tilde{x}}(t) = \tilde{A}_{ij} \tilde{x}(t) + \tilde{D}_i w(t), \\ y(t) = \tilde{C}_i \tilde{x}(t), \end{cases} \tag{5.4}$$

where $\tilde{x}(t) \in \mathbb{R}^{\tilde{n}}$ is the state of the closed-loop system. And the expressions of $\tilde{x}(t)$, $\tilde{A}_{ij}$, $\tilde{D}_i$, $\tilde{C}_i$ can be determined from equations (5.2) and (5.3) respectively.

Remark 5.1 *The integrated FTC design can also be posed as a "dual control" problem, where dual goals, e.g. control of trajectory of $x(t)$ and estimation of $\zeta(t)$ are set for the control system [116]. This kind of problems is generally difficult and numerically intractable. For the FTCS model (5.1) adopted in this chapter, the characteristics of the FDI scheme is abstracted and described by a Markov chain. Then the emphasis is on designing the controller to accommodate for the imperfect FDI decisions.*

5.2.3 Definition of H_2 Norm for FTCS

With the given transition rates for Markov chains $\zeta(t)$ and $\eta(t)$, the augmented stochastic process $\{\zeta(t),\ \eta(t)\}$ taking values on $S_1 \times S_2$ is still a Markov chain with the following transition rate, where $\gamma_{(i_1,j_1),(i_2,j_2)}$ represents the transition rate from the state $\zeta = i_1,\ \eta = j_1$ to the state $\zeta = i_2,\ \eta = j_2$:

$$\gamma_{(i_1,j_1),(i_2,j_2)} = \begin{cases} \alpha_{i_1,i_1} + \beta^{i_1}_{j_1,j_1}, & i_1 = i_2,\ j_1 = j_2, \\ \beta^{i_1}_{j_1,j_2}, & i_1 = i_2,\ j_1 \neq j_2, \\ \alpha_{i_1,i_2}, & i_1 \neq i_2,\ j_1 = j_2, \\ 0, & i_1 \neq i_2,\ j_1 \neq j_2. \end{cases} \tag{5.5}$$

Therefore, the closed-loop FTCS described in Section 5.2.1 can be formulated as MJLS with clustered (partial) observations. For the joint Markov chain $\{\zeta(t), \eta(t)\}$, each state can be represented by a pair $(i,\ j), i \in S_1, j \in S_2$, in which i stands for the hidden part of Markov state, and j is the directly observable part.

By assuming the linear control law $u(t, \eta(t))$ is known, for the given open-loop system $\mathscr{G}_c$, the closed-loop system description can be written out as in (5.4), based on which the stability and the H_2 norm of the FTCS can be defined.

Definition 5.1 The H_2 norm of an MSS system $\tilde{\mathscr{G}}_c$ is given by

$$\|\tilde{\mathscr{G}}_c\|_2^2 = \sum_{i=1}^{q_1} \sum_{j=1}^{q_2} \sum_{m=1}^{l} \rho_{ij} \|y_{ijm}\|_2^2 , \tag{5.6}$$

where y_{ijm} is the output of the system with initial conditions $\zeta(0) = i,\ \eta(0) = j$, disturbed by $w(t) = e_m\delta(t)$. e_m is a l-dimensional unitary vector with its m-th entry as 1 and other entries as 0's. $\delta(t)$ is an impulse function and ρ_{ij} is the initial probability distribution for $\zeta(0) = i,\ \eta(0) = j$.

Considering the FTCS in (5.4) can be formulated as a MJLS with partial observations, the above definition of H_2 norm is analogous to that of MJLS in [29].

Furthermore, for matrices $S,\ P \in \mathscr{H}^{\tilde{n}+}$, the following definitions are given:

$$\mathscr{T}_{ij}(S) = \tilde{A}_{ij}S_{ij} + S_{ij}\tilde{A}_{ij}^T + \Sigma_{k=1}^{q_1}\alpha_{ki}S_{kj} + \Sigma_{k=1}^{q_2}\beta_{kj}^i S_{ik},$$
$$\mathscr{L}_{ij}(P) = \tilde{A}_{ij}^T P_{ij} + P_{ij}\tilde{A}_{ij} + \Sigma_{k=1}^{q_1}\alpha_{ik}P_{kj} + \Sigma_{k=1}^{q_2}\beta_{jk}^i P_{ik},$$
$$\mathscr{T}(S) = \left[\mathscr{T}_{11}(S),\ \mathscr{T}_{12}(S),\ \cdots,\ \mathscr{T}_{1q_2}(S),\ \mathscr{T}_{21}(S),\ \cdots,\ \mathscr{T}_{q_1q_2}(S)\right],$$
$$\mathscr{L}(P) = \left[\mathscr{L}_{11}(P),\ \mathscr{L}_{12}(P),\ \cdots,\ \mathscr{L}_{1q_2}(P),\ \mathscr{L}_{21}(P),\ \cdots,\ \mathscr{L}_{q_1q_2}(P)\right],$$
$$\mathscr{M} = \underbrace{[\tilde{C}_1^T\tilde{C}_1, \tilde{C}_1^T\tilde{C}_1, \cdots, \tilde{C}_1^T\tilde{C}_1,}_{q_2}\ \ \tilde{C}_2^T\tilde{C}_2, \cdots, \tilde{C}_{q_1}^T\tilde{C}_{q_1}] \in \mathscr{H}^{\tilde{n}+},$$
$$\rho\mathscr{N} = \left[\rho_{11}\tilde{D}_1\tilde{D}_1^T, \rho_{12}\tilde{D}_1\tilde{D}_1^T, \cdots, \rho_{1q_2}\tilde{D}_1\tilde{D}_1^T, \rho_{21}\tilde{D}_2\tilde{D}_2^T, \cdots, \rho_{q_1q_2}\tilde{D}_{q_1}^T\tilde{D}_{q_1}\right] \in \mathscr{H}^{\tilde{n}+}.$$

Assuming that the closed-loop system is MSS, use the available results on the H_2 norm to have

$$||\tilde{\mathscr{G}}_c||_2^2 = \Sigma_{i=1}^{q_1}\Sigma_{j=1}^{q_2}\rho_{ij}\text{Tr}(\tilde{D}_i^T P_{ij}\tilde{D}_i) = \Sigma_{i=1}^{q_1}\Sigma_{j=1}^{q_2}\text{Tr}(\tilde{C}_i S_{ij}\tilde{C}_i^T)\ , \tag{5.7}$$

where P and S are the solutions of $\mathscr{L}(P) + \mathscr{M} = 0$ and $\mathscr{T}(S) + \rho\mathscr{N} = 0$ respectively.

To avoid solving coupled equations, convert the result above into matrix inequalities:

Theorem 5.1 *The continuous-time FTCS (5.4) is MSS and the H_2 norm $||\tilde{\mathscr{G}}_c||_2$ can be determined, if $\forall i \in S_1, j \in S_2$, there exist Z_{ij} and positive symmetric matrices $P_{ij} > 0$ satisfying the following matrix inequality problem:*

$$||\tilde{\mathscr{G}}_c||_2^2 = \inf \Sigma_{i=1}^{q_1}\Sigma_{j=1}^{q_2}\rho_{ij}\text{Tr}(Z_{ij}) \tag{5.8}$$
$$s.t. \qquad \tilde{D}_i^T P_{ij}\tilde{D}_i < Z_{ij}, \tag{5.9}$$
$$\mathscr{L}_{ij}(P) + \tilde{C}_i^T\tilde{C}_i < 0. \tag{5.10}$$

Proof: Assume P_1 is the solution of $\mathscr{L}(P) + \mathscr{M} = 0$, i.e. $\mathscr{L}(P_1) + \mathscr{M} = 0$ and P_2 is a solution of $\mathscr{L}(P) + \mathscr{M} < 0$, i.e. $\mathscr{L}(P_2) + \mathscr{M} < 0$. Since $\mathscr{M} \geq 0$, it has $\mathscr{L}_{ij}(P_2) < 0, \forall\ i = 1, 2, \cdots, q_1; j = 1, 2, \cdots, q_2$. Define a stochastic Lyapunov function $v(\tilde{x}(t), \zeta(t), \eta(t)) = \tilde{x}^T(t)P_2(\zeta(t), \eta(t))\tilde{x}(t)$, then when $\zeta(t) = i,\ \eta(t) = j$, the weak infinitesimal operator defined on v is $\mathscr{A}v = \tilde{x}(t)^T\mathscr{L}_{ij}(P_2)\tilde{x}(t) < 0$. Hence the system is mean-square stable (MSS).

From Proposition 2 of [29], it follows that $P_2 > P_1$. Furthermore, for any given $\varepsilon > 0$ and the matrix P_1 satisfying $\mathscr{L}(P_1) + \mathscr{M} = 0$, there exists P_2 satisfying $\mathscr{L}(P) + \mathscr{M} < 0$ and $||P_{2_{ij}} - P_{1_{ij}}|| \leq \varepsilon$. Therefore, it leads to (5.8). □

5.3 Synthesis of Robust H_2 Controller: Continuous-time Case

With the stability and H_2 norm of the stochastic FTCS defined in Section 5.2, in this section, the synthesis of continuous-time robust H_2 control for the system in (5.1) is addressed. Due to the presence of model uncertainties, the design objective for synthesis of robust fault tolerant controllers is to make $\|\tilde{\mathscr{G}}_c\|_2 < \mu$, where μ is a given positive scalar.

5.3.1 Robust H_2 State-feedback Controller

First consider the state-feedback case, i.e. for $\zeta(t) = i$, $\eta(t) = j$, the controller has the form $u_j(t) = K_j x(t)$, and the closed-loop FTCS is given in (5.2) and (5.4). The following theorem provides sufficient conditions for solving the robust H_2 state-feedback control.

Theorem 5.2 *For the stochastic uncertain FTCS $\tilde{\mathscr{G}}_c$ in (5.2) and given a positive scalar μ, the system is MSS and satisfies the H_2 performance with $\|\tilde{\mathscr{G}}_c\|_2 < \mu$, if there exist $\bar{X}_{ij} > 0$, $\bar{W}_{1ij}$, $\bar{W}_{2ij}$, Q_{1ij}, Q_{2ij}, Z_{ij}, K_j and positive scalars $\bar{\varepsilon}_{ij}$, $\bar{\delta}_{ij}$, λ_{ij}, $i \in S_1$, $j \in S_2$, so that the following LMIs with equality constraints hold:*

$$\sum_{i=1}^{q_1} \sum_{j=1}^{q_2} \rho_{ij} \mathrm{Tr}(Z_{ij}) < \mu^2, \tag{5.11}$$

$$\mathbf{F}_{1ij} = \begin{bmatrix} -Q_{2ij} & \bar{W}_{2ij} - \frac{1}{2} Z_{ij} & 0 & \lambda_{ij} I - \bar{W}_{2ij} \\ * & -I & D_i^T & 0 \\ * & * & -\bar{X}_{ij} & 0 \\ * & * & * & -I \end{bmatrix} < 0, \tag{5.12}$$

$$\mathbf{F}_{2ij} = \begin{bmatrix} \bar{H}_{0ij} & \bar{X}_{ij}A_{2i}^T & \bar{X}_{ij}C_i^T & \bar{H}_{1ij} & \bar{X}_{ij}-\bar{W}_{1ij} & A_{0i}+B_{0i}K_j+\bar{W}_{1ij} & 0 \\ * & -\bar{\varepsilon}_{ij}I & 0 & 0 & 0 & 0 & 0 \\ * & * & -\lambda_{ij}I & 0 & 0 & 0 & 0 \\ * & * & * & \bar{H}_{2ij} & 0 & 0 & 0 \\ * & * & * & * & -I & 0 & 0 \\ * & * & * & * & * & -I & K_j^T B_{2i}^T \\ * & * & * & * & * & * & -\bar{\delta}_{ij}I \end{bmatrix} < 0, \tag{5.13}$$

where

$$\begin{aligned} \bar{H}_{0ij} &= (\alpha_{ii}+\beta_{jj}^i)\bar{X}_{ij} + \bar{\varepsilon}_{ij}A_{1i}A_{1i}^T + \bar{\delta}_{ij}B_{1i}B_{1i}^T - Q_{1ij}, \\ \bar{H}_{1ij} &= \bar{X}_{ij}\left[\sqrt{\alpha_{i1}}I, \cdots, \sqrt{\alpha_{i,i-1}}I, \sqrt{\alpha_{i,i+1}}I, \cdots, \sqrt{\beta_{j1}^i}I, \cdots, \sqrt{\beta_{j,j-1}^i}I, \sqrt{\beta_{j,j+1}^i}I, \cdots\right], \\ \bar{H}_{2ij} &= -diag\{\bar{X}_{1j}, \cdots, \bar{X}_{i-1,j}, \bar{X}_{i+1,j}, \cdots, \bar{X}_{i1}, \cdots, \bar{X}_{i,j-1}, \bar{X}_{i,j+1}, \cdots\}. \end{aligned}$$

And it should be satisfied that

$$Q_{1ij} = \bar{W}_{1ij}\bar{W}_{1ij}^T, \tag{5.14}$$

$$Q_{2ij} = \bar{W}_{2ij}\bar{W}_{2ij}^T. \tag{5.15}$$

Then the state feedback controller is obtained as $u_j(t) = K_j x(t)$.

Proof: This theorem can be shown by using Theorem 5.1. In the state-feedback case, the operator $\mathscr{L}_{ij}(P)$ in (5.10) can be expressed as

$$\mathscr{L}_{ij}(P) = P_{ij}\left(A_i + B_iK_j\right) + \left(A_i + B_iK_j\right)^T P_{ij} + \sum_{k=1}^{q_1} \alpha_{ik}P_{kj} + \sum_{k=1}^{q_2} \beta_{jk}^i P_{ik}.$$

Since A_i, B_i contain norm-bounded model uncertainties, use Lemma 4.2 to obtain the following inequalities, i.e. for positive scalars ε_{ij} and δ_{ij}, it is true that

$$P_{ij}(A_{1i}\Delta_{1i}A_{2i}) + (A_{1i}\Delta_{1i}A_{2i})_i^T P_{ij} \le \varepsilon_{ij}^{-1}A_{2i}^TA_{2i} + \varepsilon_{ij}P_{ij}A_{1i}A_{1i}^TP_{ij},$$

$$P_{ij}(B_{1i}\Delta_{2i}B_{2i})K_j + K_j^T(B_{1i}\Delta_{2i}B_{2i})^TP_{ij} \le \delta_{ij}^{-1}K_j^TB_{2i}^TB_{2i}K_j + \delta_{ij}P_{ij}B_{1i}B_{1i}^TP_{ij}.$$

By substituting these inequalities back into $\mathscr{L}_{ij}(P)$, it can be easily seen that (5.10) holds if there exist positive scalars ε_{ij} and δ_{ij}, $i \in S_1, j \in S_2$, such that

$$\begin{aligned} & P_{ij}\left(A_{0i}+B_{0i}K_j\right)+\left(A_{0i}+B_{0i}K_j\right)^T P_{ij}+\textstyle\sum_{k=1}^{q_1}\alpha_{ik}P_{kj}+\sum_{k=1}^{q_2}\beta_{jk}^{i}P_{ik}+\varepsilon_{ij}^{-1}A_{2i}^TA_{2i} \\ & +\varepsilon_{ij}P_{ij}A_{1i}A_{1i}^TP_{ij}+\delta_{ij}^{-1}K_j^TB_{2i}^TB_{2i}K_j+\delta_{ij}P_{ij}B_{1i}B_{1i}^TP_{ij}+C_i^TC_i<0. \end{aligned} \tag{5.16}$$

For nonlinear matrix inequality shown in (5.16), begin with removing the coupling between P_{ij} and the controllers K_j. For this purpose, the slack variables $\tilde{P}_{1ij} > 0$ and W_{1ij} with compatible dimensions are introduced. Use Lemma 4.4 to get

$$\begin{bmatrix} H_{0ij} & P_{ij}\left(A_{0i}+B_{0i}K_j\right)+W_{1ij} \\ * & -\tilde{P}_{1ij} \end{bmatrix} < 0, \tag{5.17}$$

where

$$\begin{aligned} H_{0ij} = & \sum_{k=1}^{q_1}\alpha_{ik}P_{kj}+\sum_{k=1}^{q_2}\beta_{jk}^{i}P_{ik}+\varepsilon_{ij}^{-1}A_{2i}^TA_{2i}+\delta_{ij}^{-1}K_j^TB_{2i}^TB_{2i}K_j+C_i^TC_i \\ & +P_{ij}\left(\varepsilon_{ij}A_{1i}A_{1i}^T+\delta_{ij}B_{1i}B_{1i}^T\right)P_{ij}+\tilde{P}_{1ij}-W_{1ij}-W_{1ij}^T. \end{aligned}$$

Since $\tilde{P}_{1ij}$ here can be any positive matrix, without loss of generality, set $\tilde{P}_{1ij} = \lambda_{ij}I - \delta_{ij}^{-1}K_j^TB_{2i}^TB_{2i}K_j$, where λ_{ij} is a positive scalar large enough to guarantee $\tilde{P}_{1ij} > 0$. Define $X_{ij} = P_{ij}^{-1}$, $\bar{W}_{1ij} = X_{ij}W_{1ij}$, pre- and post-multiply (5.17) with $diag\{X_{ij},\ I\}$. Use Schur complement to expand quadratic terms in the (1, 1) and (2, 2) entries to get the following matrix inequality:

$$\begin{bmatrix} \tilde{H}_{0ij} & X_{ij}A_{2i}^T & X_{ij}C_i^T & H_{1ij} & \lambda_{ij}X_{ij}-\bar{W}_{1ij} & A_{0i}+B_{0i}K_j+\bar{W}_{1ij} & 0 \\ * & -\varepsilon_{ij}I & 0 & 0 & 0 & 0 & 0 \\ * & * & -I & 0 & 0 & 0 & 0 \\ * & * & * & H_{2ij} & 0 & 0 & 0 \\ * & * & * & * & -\lambda_{ij}I & 0 & 0 \\ * & * & * & * & * & -\lambda_{ij}I & K_j^TB_{2i}^T \\ * & * & * & * & * & * & -\delta_{ij}I \end{bmatrix} < 0,$$

where

$$\begin{aligned} \tilde{H}_{0ij} &= (\alpha_{ii}+\beta_{jj}^{i})X_{ij}+\varepsilon_{ij}A_{1i}A_{1i}^T+\delta_{ij}B_{1i}B_{1i}^T-\lambda_{ij}^{-1}\bar{W}_{1ij}\bar{W}_{1ij}^T, \\ H_{1ij} &= X_{ij}\left[\sqrt{\alpha_{i1}}I,\cdots,\sqrt{\alpha_{i,i-1}}I,\sqrt{\alpha_{i,i+1}}I,\cdots,\sqrt{\beta_{j1}^{i}}I,\cdots,\sqrt{\beta_{j,j-1}^{i}}I,\sqrt{\beta_{j,j+1}^{i}}I,\cdots\right], \\ H_{2ij} &= -diag\{X_{1j},\cdots,X_{i-1,j},X_{i+1,j},\cdots,X_{i1},\cdots,X_{i,j-1},X_{i,j+1},\cdots\}. \end{aligned}$$

Notice that both λ_{ij} and λ_{ij}^{-1} appear in the inequality above, which makes it hard to choose λ_{ij} as a decision variable. To overcome this problem, again pre- and post-multiply the above matrix inequality by a diagonal matrix, $diag\{\lambda_{ij}^{\frac{1}{2}}I,\ \lambda_{ij}^{\frac{1}{2}}I,\ \lambda_{ij}^{\frac{1}{2}}I,\ \lambda_{ij}^{\frac{1}{2}}I,\ \lambda_{ij}^{-\frac{1}{2}}I,\ \lambda_{ij}^{-\frac{1}{2}}I,\ \lambda_{ij}^{\frac{1}{2}}I\}$, then define $\bar{\varepsilon}_{ij} = \lambda_{ij}\varepsilon_{ij},\ \bar{\delta}_{ij} = \lambda_{ij}\delta_{ij},\ \bar{X}_{ij} = \lambda_{ij}X_{ij},\ Q_{1ij} = \bar{W}_{1ij}\bar{W}_{1ij}^T$, to get the inequality (5.13).

For the inequality (5.9), apply Lemma 4.4 again with slack variables $\tilde{P}_{2ij} > 0$ and W_{2ij} to obtain its equivalence as:

$$\begin{bmatrix} -Z_{ij} + D_i^T P_{ij} D_i + \tilde{P}_{2ij} - W_{2ij} - W_{2ij}^T & W_{2ij} \\ * & -\tilde{P}_{2ij} \end{bmatrix} < 0.$$

Without loss of generality, set $\tilde{P}_{2ij} = \lambda_{ij}I - D_i^T P_{ij} D_i$, and define $\bar{W}_{2ij} = W_{2ij} + \frac{1}{2}Z_{ij}$, then the inequality can be expressed in terms of new variables as:

$$\begin{bmatrix} \lambda_{ij}I - \bar{W}_{2ij} - \bar{W}_{2ij}^T & \bar{W}_{2ij} - \frac{1}{2}Z_{ij} \\ * & -\lambda_{ij}I + D_i^T P_{ij} D_i \end{bmatrix} < 0.$$

Use Schur complement to expand quadratic terms in (2, 2)-entry and obtain:

$$\begin{bmatrix} \lambda_{ij}I - \bar{W}_{2ij} - \bar{W}_{2ij}^T & \bar{W}_{2ij} - \frac{1}{2}Z_{ij} & 0 \\ * & -\lambda_{ij}I & D_i^T \\ * & * & -X_{ij} \end{bmatrix} < 0.$$

Since in (5.13), $\bar{X}_{ij}$ is the decision variable, so here pre- and post-multiply $diag\{\lambda_{ij}^{\frac{1}{2}}I,\ \lambda_{ij}^{-\frac{1}{2}}I,\ \lambda_{ij}^{\frac{1}{2}}I\}$, then expand quadratic terms in the (1, 1)-entry to obtain the inequality (5.12).

Due to the presence of model uncertainties, the design objective is chosen as $||\tilde{\mathscr{G}}_c||_2 < \mu$. Hence, it implies (5.11) by considering the expression of H_2 norm given in (5.8).

Finally, the inequalities (5.8)-(5.10) in Theorem 5.1 are converted to LMIs (5.11)-(5.13), and it follows that the MSS and the H_2 performance $||\tilde{\mathscr{G}}_c||_2 < \mu$ are satisfied upon their feasibilities. Hence the proof. □

Remark 5.2 *Although introducing slack variables W_{1ij}, W_{2ij} and λ_{ij} increases the number of decision variables, the advantages it brings are twofold. Firstly, there are more degrees* *of design freedom, which will reduce the conservatism introduced when converting the non-convex optimization into convex ones. Secondly, the coupling between P_{ij} and controller K_j is removed. This is especially useful in extending the design to output feedback control (to be discussed in Section 5.3.3). Furthermore, since the structural constraints of controllers (e.g., diagonal structure of gain for static decentralized controllers) can be reserved during the design procedure, and the reduced order controller and structural controllers can be found if they exist.*

Another advantage comes from the capability to handle polytopic model uncertainties, which is described in Chapter 4. Since X_{ij} does not couple with other decision variables, it is possible to solve X_{ij}^k for each A_i^k, B_i^k, C_i^k, D_i^k to avoid using a single X_{ij}. Thus the conservatism introduced in can can reduced. Similar approaches are used for discrete-time LTI systems in [88].

5.3.2 Algorithm for Solving LMIs with Equality Constraints

In Theorem 5.2, the nonlinear matrix inequalities have been converted into LMIs with equality constraints (5.14), (5.15). However, since the constraints are not in linear forms, this problem is still difficult to solve. A similar situation, where non-convex constraints appear, arises when dealing with *reduced-order output-feedback* (ROF) or *static output-feedback* (SOF) stabilization problem, and a number of numerical algorithms have been proposed [50]. The LMI-based algorithms include alternating projection, min-max algorithm, the XY-centering algorithm, and the cone complementarity linearization (CCL) algorithm. In [50], numerical simulations were performed to compare the performance and convergence of some of these algorithms, and the CCL algorithm has the best performance. In this section, we adopt the Cone Complementarity Linearization algorithm for solving the matrix inequality problem with equality constraints.

For $i \in S_1,\ j \in S_2$ and the sufficiently small positive scalars c_{1ij}, c_{2ij}, first of all define the following inequalities:

$$\mathbf{F}_{1ij} + diag\{c_{1ij}I,\ 0,\ 0,\ 0\} < 0, \tag{5.18}$$

$$\mathbf{F}_{2ij} + diag\{c_{2ij}I,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0\} < 0, \tag{5.19}$$

then the following sets can be defined:

$$\begin{aligned}&\Omega_1(\bar{X}_{ij}, \bar{\varepsilon}_{ij}, \bar{\delta}_{ij}, \lambda_{ij}, \bar{W}_{1ij}, \bar{W}_{2ij}, Q_{1ij}, Q_{2ij}, Z_{ij}, K_j)\\ &= \{(\bar{X}_{ij}, \bar{\varepsilon}_{ij}, \bar{\delta}_{ij}, \lambda_{ij}, \bar{W}_{1ij}, \bar{W}_{2ij}, Q_{1ij}, Q_{2ij}, Z_{ij}, K_j) \text{ satisfying } (5.11), (5.18) \text{ and } (5.19)\},\end{aligned} \tag{5.20}$$

$$\Omega_2(Q_{1ij}, \bar{W}_{1ij}) = \{(Q_{1ij}, \bar{W}_{1ij}) \text{ satisfying } \begin{bmatrix} Q_{1ij} & \bar{W}_{1ij} \\ \bar{W}_{1ij}^T & I \end{bmatrix} \geq 0\}, \tag{5.21}$$

$$\Omega_3(Q_{2ij}, \bar{W}_{2ij}) = \{(Q_{2ij}, \bar{W}_{2ij}) \text{ satisfying } \begin{bmatrix} Q_{2ij} & \bar{W}_{2ij} \\ \bar{W}_{2ij}^T & I \end{bmatrix} \geq 0\}. \tag{5.22}$$

The following iterative algorithm is used to solve the LMIs with equality constraints.

Algorithm 5.1:

(1) Preset c_{1ij}, c_{2ij} as sufficiently small positive scalars.

(2) Initialization: Set $k = 0$, determine $(\bar{X}_{ij}^0, \bar{\varepsilon}_{ij}^0, \bar{\delta}_{ij}^0, \lambda_{ij}^0, \bar{W}_{1ij}^0, \bar{W}_{2ij}^0, Q_{1ij}^0, Q_{2ij}^0, Z_{ij}^0, K_j^0) \in \Omega_1 \cap \Omega_2 \cap \Omega_3$.

(3) Determine $(\bar{X}_{ij}^{k+1}, \bar{\varepsilon}_{ij}^{k+1}, \bar{\delta}_{ij}^{k+1}, \lambda_{ij}^{k+1}, \bar{W}_{1ij}^{k+1}, \bar{W}_{2ij}^{k+1}, Q_{1ij}^{k+1}, Q_{2ij}^{k+1}, Z_{ij}^{k+1}, K_j^{k+1})$ as the unique solution of the optimization problem:

$$\begin{aligned}J^{k^*} &= \min \sum_{l=1}^{2} \sum_{i=1}^{q_1} \sum_{j=1}^{q_2} \mathrm{Tr}(Q_{lij} - \bar{W}_{lij}^k \bar{W}_{lij}^T - \bar{W}_{lij} \bar{W}_{lij}^{k^T})\\ &s.t. \quad (\bar{X}_{ij}, \bar{\varepsilon}_{ij}, \bar{\delta}_{ij}, \lambda_{ij}, \bar{W}_{1ij}, \bar{W}_{2ij}, Q_{1ij}, Q_{2ij}, Z_{ij}, K_j) \in \Omega_1 \cap \Omega_2 \cap \Omega_3.\end{aligned} \tag{5.23}$$

(4) If $Q_{lij}^{k+1} - \bar{W}_{lij}^{k+1} \bar{W}_{lij}^{k+1^T} < c_{lij}I$, for $l = 1, 2$, $i \in S_1, j \in S_2$, then stop and K_j^{k+1} is the feasible controller; otherwise, set $k = k + 1$, go to step (3).

The stop criterion in step (4) is to terminate the iteration when the solution ($\bar{X}_{ij}^{k+1}$, $\bar{\varepsilon}_{ij}^{k+1}$, $\bar{\delta}_{ij}^{k+1}$, λ_{ij}^{k+1}, $\bar{W}_{1ij}^{k+1}$, $\bar{W}_{2ij}^{k+1}$, Z_{ij}^{k+1}, K_j^{k+1}) from step (3) already satisfies (5.11)-(5.13) with equality constraints (5.14), (5.15) by choosing $Q_{1ij} = \bar{W}_{1ij}^{k+1}\bar{W}_{1ij}^{k+1^T}$ and $Q_{2ij} = \bar{W}_{2ij}^{k+1}\bar{W}_{2ij}^{k+1^T}$.

Definition 5.2 For the sets Ω_1, Ω_2, Ω_3 defined in (5.20)-(5.22), $l = 1,2$, $i \in S_1$, $j \in S_2$, and constant matrices Q_{lij}^k, $\bar{W}_{lij}^k$, define the constrained optimization problem at the k-th step as follows:

$$\begin{array}{l} \mathcal{J}^{k^*} = \min \mathcal{J}^k = \min \sum_{l=1}^{2} \sum_{i=1}^{q_1} \sum_{j=1}^{q_2} \mathrm{Tr}(Q_{lij}^k + Q_{lij} - \bar{W}_{lij}^k \bar{W}_{lij}^T - \bar{W}_{lij}\bar{W}_{lij}^{k^T}) \\ s.t. (\bar{X}_{ij}, \bar{\varepsilon}_{ij}, \bar{\delta}_{ij}, \lambda_{ij}, \bar{W}_{lij}, Q_{lij}) \in \Omega_1 \cap \Omega_2 \cap \Omega_3. \end{array} \tag{5.24}$$

In (5.24), at the k-th step, Q_{lij}^k is a constant matrix, so it does not affect the optimization result when dropped out of the objective function in (5.23) of Algorithm 5.1. Therefore, the two optimization problems in (5.23) and (5.24) have the same solution. The following theorem shows the convergence of Algorithm 5.1.

Theorem 5.3 *In Definition 5.2, $\mathcal{J}^{k^*} \geq 0$ and the sequence $\{\mathcal{J}^{1^*}, \mathcal{J}^{2^*}, \cdots\}$ is monotonically non-increasing and convergent. Furthermore, $\mathcal{J}^{k^*} = 0$ if and only if $Q_{lij} = \bar{W}_{lij}\bar{W}_{lij}^T$ holds at the optimum, $l = 1,2, i \in S_1, j \in S_2$.*

Proof: In (5.24), $Q_{lij}^k + Q_{lij} - \bar{W}_{lij}^k\bar{W}_{lij}^T - \bar{W}_{lij}\bar{W}_{lij}^{k^T}$ is a linearized form of the nonlinear function $Q_{lij} - \bar{W}_{lij}\bar{W}_{lij}^T$. From (5.21) and (5.22), it is obvious that $\mathrm{Tr}(Q_{lij}^k + Q_{lij} - \bar{W}_{lij}^k\bar{W}_{lij}^T - \bar{W}_{lij}\bar{W}_{lij}^{k^T}) \geq \mathrm{Tr}(\bar{W}_{lij}^k\bar{W}_{lij}^{k^T} + \bar{W}_{lij}\bar{W}_{lij}^T - \bar{W}_{lij}^k\bar{W}_{lij}^T - \bar{W}_{lij}\bar{W}_{lij}^{k^T}) = \mathrm{Tr}((\bar{W}_{lij}^k - \bar{W}_{lij})(\bar{W}_{lij}^k - \bar{W}_{lij})^T) \geq 0$. So $\mathcal{J}^{k^*} = \min \mathcal{J}^k \geq 0$.

From step (3) of Algorithm 5.1, it is clear that (Q_{lij}^k, $\bar{W}_{lij}^k$) is a feasible solution in the set $\Omega_1 \cap \Omega_2 \cap \Omega_3$ obtained from the $(k-1)$-th step, when substitute it into the objective function $\mathcal{J}^{k+1}$ in (5.24), the following equation can be obtained,

$$\mathcal{J}^{k+1}(Q_{lij}^k,\ \bar{W}_{lij}^k) = \sum_{l=1}^{2}\sum_{i=1}^{q_1}\sum_{j=1}^{q_2} \mathrm{Tr}(Q_{lij}^{k+1} + Q_{lij}^k - \bar{W}_{lij}^{k+1}\bar{W}_{lij}^{k^T} - \bar{W}_{lij}^k\bar{W}_{lij}^{k+1^T}) = \mathcal{J}^{k^*}.$$

Then due to the global optimization property of the LMI, the following is true:

$$\mathcal{J}^{(k+1)^*} \leq J_{lij}^{k+1}(Q_{lij}^k,\ \bar{W}_{lij}^k) = \mathcal{J}^{k^*},$$

which means that $\tilde{J}^{k^*}$ is monotonically non-increasing for all k, and with a lower bound 0. Therefore, the sequence is convergent. It is obvious that the optimum of the iteration process corresponds to $\tilde{J}^* = 0$ when $Q_{lij} = \bar{W}_{lij}\bar{W}_{lij}^T$. □

Remark 5.3 *The constraints $Q_{lij} = \bar{W}_{lij}\bar{W}_{lij}^T$, $i \in S_1$, $j \in S_2$, $l = 1,2$ are achieved at the boundary of the convex sets Ω_2 and Ω_3, while the LMI solver can only handle strict LMIs, e.g. finding solutions in the interior of the solution set. Therefore, in order to make sure that the inequalities (5.12) and (5.13) are satisfied under this circumstance, the set is enlarged so that the boundary is included inside. This is the purpose of introducing sufficiently small positive scalars c_{1ij} and c_{2ij} in (5.18) and (5.19).*

5.3.3 Robust H_2 Output Feedback Controller

Given the output feedback controller and the corresponding closed-loop system representation (5.3) and (5.4), the first step is to separate model uncertainties from the closed-loop system matrices, For this purpose, the following notations are used:

$$\hat{K}_j = \begin{bmatrix} \hat{A}_j & \hat{B}_j \\ \hat{C}_j & \hat{D}_j \end{bmatrix}, \bar{A}_{0i} = \begin{bmatrix} A_{0i} & 0 \\ 0 & 0 \end{bmatrix}, \bar{B}_{0i} = \begin{bmatrix} 0 & B_{0i} \\ I & 0 \end{bmatrix}, \bar{C}_i = \begin{bmatrix} 0 & I \\ C_i & 0 \end{bmatrix}, \bar{A}_{1i} = \begin{bmatrix} A_{1i} & B_{1i} \\ 0 & 0 \end{bmatrix},$$
$$\bar{A}_{2i} = \begin{bmatrix} A_{2i} & 0 \\ 0 & 0 \end{bmatrix}, \bar{B}_{2i} = \begin{bmatrix} 0 & 0 \\ 0 & B_{2i} \end{bmatrix}, \Delta = \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix}, \bar{D}_i = \begin{bmatrix} D_i \\ 0 \end{bmatrix}, \bar{C}_{1i} = \begin{bmatrix} C_i & 0 \end{bmatrix}.$$

Then the closed-loop system matrices can be written as:

$$\tilde{A}_{ij} = \bar{A}_{0i} + \bar{B}_{0i}\hat{K}_j\bar{C}_i + \bar{A}_{1i}\Delta(\bar{A}_{2i} + \bar{B}_{2i}\hat{K}_j\bar{C}_i), \quad \tilde{D}_i = \bar{D}_i, \quad \tilde{C}_i = \bar{C}_{1i}.$$

In this case, the expression of the operator $\mathscr{L}$ for the closed-loop system (5.3) is obtained:

$$\begin{aligned} \mathscr{L}_{ij}(P) = \; & P_{ij}\bar{A}_{0i} + \bar{A}_{0i}^T P_{ij} + P_{ij}\bar{B}_{0i}\hat{K}_j\bar{C}_i + \bar{C}_i^T\hat{K}_j^T\bar{B}_{0i}^T P_{ij} + P_{ij}\bar{A}_{1i}\Delta(\bar{A}_{2i} + \bar{B}_{2i}\hat{K}_j\bar{C}_i) \\ & + (\bar{A}_{2i} + \bar{B}_{2i}\hat{K}_j\bar{C}_i)^T\Delta^T\bar{A}_{1i}^T P_{ij} + \textstyle\sum_{k=1}^{q_1}\alpha_{ik}P_{kj} + \sum_{k=1}^{q_2}\beta_{jk}^i P_{ik}. \end{aligned}$$

From Lemma 4.2, for any positive scalars ε_{1ij} and ε_{2ij}, $i \in S_1, j \in S_2$, it is true:

$$\begin{aligned} & P_{ij}\bar{A}_{1i}\Delta(\bar{A}_{2i} + \bar{B}_{2i}\hat{K}_j\bar{C}_i) + (\bar{A}_{2i} + \bar{B}_{2i}\hat{K}_j\bar{C}_i)^T\Delta^T\bar{A}_{1i}^T P_{ij} \\ & \leq (\bar{A}_{2i} + \bar{B}_{2i}\hat{K}_j\bar{C}_i)^T\Gamma_{ij}^{-1}(\bar{A}_{2i} + \bar{B}_{2i}\hat{K}_j\bar{C}_i) + P_{ij}\bar{A}_{1i}\Gamma_{ij}\bar{A}_{1i}^T P_{ij}, \end{aligned}$$

where Γ_{ij} takes the form $\Gamma_{ij} = diag\{\varepsilon_{1ij}I,\ \varepsilon_{2ij}I\} > 0$ to accommodate the block diagonal norm-bounded uncertainty Δ.

With the augmented closed-loop system description, the remaining part of derivation is similar to that of the state-feedback case. For brevity, only main results are provided as shown in the following theorem:

Theorem 5.4 *For the stochastic FTCS in (5.3) and given a H_2 performance bound μ, the closed-loop system is MSS and satisfies the H_2 performance with $\|\mathscr{G}_c\|_2 < \mu$ if there exist $\bar{X}_{ij} > 0$, $\bar{W}_{1ij}$, $\bar{W}_{2ij}$, Q_{1ij}, Q_{2ij}, Z_{ij}, $\hat{K}_j$ and positive scalars λ_{ij}, $\bar{\varepsilon}_{1ij}$, $\bar{\varepsilon}_{2ij}$, $i \in S_1, j \in S_2$, so that the following LMIs with equality constraints hold:*

$$\sum_{i=1}^{q_1}\sum_{j=1}^{q_2} \rho_{ij}\mathrm{Tr}(Z_{ij}) < \mu^2, \tag{5.25}$$

$$\begin{bmatrix} -Q_{2ij} & \bar{W}_{2ij} - \frac{1}{2}Z_{ij} & 0 & \lambda_{ij}I - \bar{W}_{2ij} \\ * & -I & \bar{D}_i^T & 0 \\ * & * & -\bar{X}_{ij} & 0 \\ * & * & * & -I \end{bmatrix} < 0, \tag{5.26}$$

$$\begin{bmatrix} \hat{H}_{0ij} & \bar{X}_{ij}\bar{C}_{1i}^T & \bar{H}_{1ij} & \bar{X}_{ij} - \bar{W}_{1ij} & \bar{A}_{0i} + \bar{B}_{0i}\hat{K}_j\bar{C}_i + \bar{W}_{1ij} & 0 \\ * & -\lambda_{ij}I & 0 & 0 & 0 & 0 \\ * & * & \bar{H}_{2ij} & 0 & 0 & 0 \\ * & * & * & -I & 0 & 0 \\ * & * & * & * & -I & (\bar{A}_{2i} + \bar{B}_{2i}\hat{K}_j\bar{C}_i)^T \\ * & * & * & * & * & -\bar{\Gamma}_{ij} \end{bmatrix} < 0, \tag{5.27}$$

where $\hat{H}_{0ij} = (\alpha_{ii} + \beta_{jj}^i)\bar{X}_{ij} + \bar{A}_{1i}\bar{\Gamma}_{ij}\bar{A}_{1i}^T - Q_{1ij}$, $\bar{\Gamma}_{ij} = diag\{\bar{\varepsilon}_{1ij}I,\ \bar{\varepsilon}_{2ij}I\}$. The output feedback controller is obtained as $\hat{K}_j$. (Other variables are defined the same as in the state-feedback case.)

Proof: The proof is similar to that of Theorem 5.2, hence omitted. □

Similar as in the state feedback case, LMIs with equality constraints are obtained in this case, the Algorithm 5.1 can be applied with little modifications.

5.4 Synthesis of Robust H_2 Control: Discrete-time Case

In previous sections, the design of robust H_2 FTC with both state feedback and output feedback is discussed for the continuous-time systems. In this section, the same design problem is treated but for the discrete-time systems.

Consider a discrete-time open loop system as:

$$\mathscr{G}_d: \quad \begin{cases} x(k+1) = A(\zeta(k))x(k) + B(\zeta(k))u(\eta(k)) + D(\zeta(k))w(k), \\ y(k) = C(\zeta(k))x(k). \end{cases} \tag{5.28}$$

$\zeta(k)$ and $\eta(k)$ are discrete-time homogeneous Markov chains with transition rate matrices from $\zeta(k) = i_1$, $\eta(k) = j_1$ to $\zeta(k+1) = i_2$, $\eta(k+1) = j_2$ as $[\alpha_{i_1,i_2}]$ and $[\beta^{i_1}_{j_1,j_2}]$ respectively.

In this section, the state feedback control $u(\eta(k)) = K(\eta(k))x(k)$ is considered. The closed-loop system has the following state space representation:

$$\tilde{\mathscr{G}}_d: \quad \begin{cases} x(k) = \left(A(\zeta(k) + B(\zeta(k))K(\eta(k))\right)x(k) + D(\zeta(k))w(k), \\ y(k) = C(\zeta(k))x(k). \end{cases} \tag{5.29}$$

Consider joint process $\{\zeta(k),\ \eta(k)\}$, which is still a Markov chain, and the transition rate for this process from $\zeta(k) = i_1, \eta(k) = j_1$ to the next state $\zeta(k+1) = i_2, \eta(k+1) = j_2$ is $\alpha_{i_1,i_2}\beta^{i_1}_{j_1,j_2}$.

For $P \in \mathscr{H}^{n+}$ and $\zeta(k) = i, \eta(k) = j$, the following operator can be defined as:

$$\begin{aligned} \mathscr{R}_{ij}(P) &= \left(A_i + B_iK_j\right)^T \left(\textstyle\sum_{k=1}^{q_1}\sum_{l=1}^{q_2} \alpha_{ik}\beta^i_{jl}P_{kl}\right)\left(A_i + B_iK_j\right), \\ \mathscr{R}(P) &= [\mathscr{R}_{11}(P), \mathscr{R}_{12}(P), \cdots, \mathscr{R}_{1q_2}(P), \mathscr{R}_{21}(P), \cdots, \mathscr{R}_{q_1q_2}(P)]. \end{aligned} \tag{5.30}$$

In this case, the H_2 norm is given in the following definition.

Definition 5.3 The H_2 norm of a MSS system $\tilde{\mathscr{G}}_d$ is given by

$$||\tilde{\mathscr{G}}_d||_2^2 = \sum_{i=1}^{q_1}\sum_{j=1}^{q_2}\sum_{m=1}^{l} \rho_{ij}||y_{ijm}||_2^2, \tag{5.31}$$

where y_{ijm} is the output of the system with initial conditions $\zeta(0) = i$, $\eta(0) = j$ and is disturbed by $w(0) = e_m, w(k) = 0$ for $k = 1, 2, \cdots$. e_m is the unitary l-dimensional vector with its m-th entry as 1 and other entries as 0's. ρ_{ij} is the initial probability distribution for $\zeta(0) = i,\ \eta(0) = j$.

For such an H_2 norm, the following results are given:

Lemma 5.1 *([34]) The H_2 norm of the closed-loop system (5.29) is*

$$\begin{array}{ll} \|\mathscr{G}_d\|_2^2 = & \sum_{i=1}^{q_1}\sum_{j=1}^{q_2}\rho_{ij}\mathrm{Tr}\left\{D_i^T\left(\sum_{k=1}^{q_1}\sum_{l=1}^{q_2}\alpha_{ik}\beta_{jl}^{i}P_{kl}\right)D_i\right\} \\ s.t. & \mathscr{R}(P) - P + \mathscr{M} = 0, \end{array}$$

where $\mathscr{M}$ in this case is $\mathscr{M} = \left[C_1^T C_1, C_1^T C_1, \cdots, C_2^T C_2, \cdots, C_{q_1}^T C_{q_1}\right] \in \mathscr{H}^{\bar{n}+}$. □

As in the proof of Theorem 5.1, the condition in Lemma 5.1 can be expressed in the form of matrix inequalities. With the state-feedback control, the inequality $\mathscr{R}_{ij}(P) - P_{ij} + C_i^T C_i < 0$ is of the form:

$$(A_i + B_i K_j)^T\left(\sum_{k=1}^{q_1}\sum_{l=1}^{q_2}\alpha_{ik}\beta_{jl}^{i}P_{kl}\right)(A_i + B_i K_j) - P_{ij} + C_i^T C_i < 0. \tag{5.32}$$

For the inequality above, introducing slack variable G_{ij} with proper dimension using Lemma 4.5, then its feasibility is equivalent to that of the following inequality:

$$\begin{bmatrix} P_{ij} - C_i^T C_i & (A_i + B_i K_j)^T G_{ij}^T \\ G_{ij}(A_i + B_i K_j) & G_{ij} + G_{ij}^T - \left(\sum_{k=1}^{q_1}\sum_{l=1}^{q_2}\alpha_{ik}\beta_{jl}^{i}P_{kl}\right) \end{bmatrix} > 0. \tag{5.33}$$

Assuming that the system (5.28) contains the same form of norm-bounded model uncertainties except that uncertainties depend on $\zeta(k)$ now, the inequality above can also be expressed as follows:

$$\begin{aligned} &\begin{bmatrix} P_{ij} - C_i^T C_i & (A_{0i} + B_{0i} K_j)^T G_{ij}^T \\ G_{ij}(A_{0i} + B_{0i} K_j) & G_{ij} + G_{ij}^T - \left(\sum_{k=1}^{q_1}\sum_{l=1}^{q_2}\alpha_{ik}\beta_{jl}^{i}P_{kl}\right) \end{bmatrix} \\ &+ \begin{bmatrix} A_{2i}^T & K_j^T B_{2i}^T \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \Delta_{1i}^T & 0 \\ 0 & \Delta_{2i}^T \end{bmatrix}\begin{bmatrix} 0 & A_{1i}^T G_{ij}^T \\ 0 & B_{1i}^T G_{ij}^T \end{bmatrix} \\ &+ \begin{bmatrix} 0 & 0 \\ G_{ij} A_{1i} & G_{ij} B_{1i} \end{bmatrix}\begin{bmatrix} \Delta_{1i} & 0 \\ 0 & \Delta_{2i} \end{bmatrix}\begin{bmatrix} A_{2i} & 0 \\ B_{2i} K_j & 0 \end{bmatrix} > 0. \end{aligned} \tag{5.34}$$

Then the inequality (5.34) holds for all admissible model uncertainties, if one can find positive scalars $\varepsilon_{ij} > 0$ and $\delta_{ij} > 0$, $i \in S_1, j \in S_2$, such that the following inequality holds

$$\begin{bmatrix} H_{3ij} & (A_{0i}+B_{0i}K_j)^T G_{ij}^T \\ G_{ij}(A_{0i}+B_{0i}K_j) & H_{4ij} \end{bmatrix} < 0, \tag{5.35}$$

where

$$H_{3ij} = C_i^T C_i - P_{ij} + \varepsilon_{ij}^{-1} A_{2i}^T A_{2i} + \delta_{ij}^{-1} K_j^T B_{2i}^T B_{2i} K_j,$$
$$H_{4ij} = -G_{ij} - G_{ij}^T + \left(\sum_{k=1}^{q_1}\sum_{l=1}^{q_2} \alpha_{ik}\beta_{jl}^i P_{kl}\right) + G_{ij}(\varepsilon_{ij}A_{1i}A_{1i}^T + \delta_{ij}B_{1i}B_{1i}^T)G_{ij}^T.$$

Note that the (2, 2)-entry implies $\left(\sum_{k=1}^{q_1}\sum_{l=1}^{q_2} \alpha_{ik}\beta_{jl}^i P_{kl}\right) - G_{ij} - G_{ij}^T < 0$, hence $G_{ij} + G_{ij}^T > 0$, and G_{ij} is nonsingular. Therefore $\bar{G}_{ij} = G_{ij}^{-1}$ can be defined. Pre- and post-multiply (5.35) by $diag\{I, \bar{G}_{ij}\}$ and its transpose, then expand those quadratic terms in the (1, 1) and (2, 2)-entries using Schur complement to obtain:

$$F_{3ij} = \begin{bmatrix} C_i^T C_i - P_{ij} & A_{0i}+B_{0i}K_j & A_{2i}^T & K_j^T B_{2i}^T & 0 \\ * & -\bar{G}_{ij} - \bar{G}_{ij}^T + \varepsilon_{ij}A_{1i}A_{1i}^T + \delta_{ij}B_{1i}B_{1i}^T & 0 & 0 & H_{5ij} \\ * & * & -\varepsilon_{ij}I & 0 & 0 \\ * & * & * & -\delta_{ij}I & 0 \\ * & * & * & * & H_{6ij} \end{bmatrix} < 0, \tag{5.36}$$

where

$$H_{5ij} = \bar{G}_{ij}\left[\sqrt{\alpha_{i1}\beta_{j1}^i}I, \sqrt{\alpha_{i1}\beta_{j2}^i}I, \cdots, \sqrt{\alpha_{i2}\beta_{j1}^i}I, \cdots, \sqrt{\alpha_{i,q_1}\beta_{j,q_2}^i}I\right],$$
$$H_{6ij} = -diag\{X_{11}, X_{12}, \cdots, X_{21}, \cdots, X_{q_1q_2}\}.$$

The condition for finding a H_2 state-feedback controller can be summarized in the following theorem.

Theorem 5.5 *For the discrete-time FTCS (5.28) and a given positive scalar* μ*, the system is MSS and satisfies* $\|\mathscr{G}_d\|_2 < \mu$ *if there exist* K_j, $\bar{G}_{ij}$, $P_{ij} > 0$, $X_{ij} > 0$ *and positive scalars* ε_{ij}, δ_{ij}, $i \in S_1, j \in S_2$*, such that the following matrix inequalities are satisfied:*

$$\sum_{i=1}^{q_1}\sum_{j=1}^{q_2} \rho_{ij}\mathrm{Tr}\left\{D_i^T\left(\sum_{k=1}^{q_1}\sum_{l=1}^{q_2}\alpha_{ik}\beta_{jl}^i P_{kl}\right)D_i\right\} < \mu^2, \tag{5.37}$$

$$F_{3ij} < 0. \tag{5.38}$$

with equality constraint $P_{ij}X_{ij} = I$. *And the state feedback controller is obtained as* $u_j(k) = K_j x(k)$.

Note that in (5.36), both P_{ij}, X_{ij} appear and there are non-convex constraints $P_{ij}X_{ij} = I$. Similarly, the iterative LMI algorithm can be used to find feasible solution $P_{ij}, X_{ij}, \bar{G}_{ij}$, $Z_{ij}, \varepsilon_{ij}, \delta_{ij}$. Unlike in the continuous-time case, the equality constraints in this case are in the bilinear form. For this kind of problem, recently a new iterative algorithm was developed called SLPMM-*sequential linear programming matrix method* [71], which can be regarded as an improvement of CCL algorithm, and convergence of SLPMM algorithm has been shown in Theorem 3.9 of [71].

Similar as in the continuous-time case, for given constant c_{3ij}, first define

$$F_{3ij} + diag\{c_{3ij}I,\ 0,\ 0,\ 0,\ 0\} < 0, \tag{5.39}$$

then the following sets:

$$\Omega_4(P_{ij}, X_{ij}, \bar{G}_{ij}, \varepsilon_{ij}, \delta_{ij}, K_j) = \{(P_{ij}, X_{ij}, \bar{G}_{ij}, \varepsilon_{ij}, \delta_{ij}, K_j) \text{ satisfying } (5.37) and (5.39)\}, \tag{5.40}$$

$$\Omega_5(P_{ij}, X_{ij}) = \{(P_{ij}, X_{ij}) \text{ satisfying } \begin{bmatrix} P_{ij} & I \\ I & X_{ij} \end{bmatrix} \geq 0\}. \tag{5.41}$$

Algorithm 5.2:

(1) Preset c_{3ij} as sufficiently small positive scalars.

(2) Initialization: Set $k = 0$, determine $(P_{ij}^0, X_{ij}^0, \bar{G}_{ij}^0, \varepsilon_{ij}^0, \delta_{ij}^0, K_j^0) \in \Omega_4 \cap \Omega_5$.

(3) Determine $(\bar{P}_{ij}^{k+1}, \bar{X}_{ij}^{k+1}, \bar{G}_{ij}^{k+1}, \varepsilon_{ij}^{k+1}, \delta_{ij}^{k+1}, K_j^{k+1})$, as the unique solution of the optimization problem:

$$\begin{aligned} J^{k*} &= \min \sum_{i=1}^{q_1} \sum_{j=1}^{q_2} \mathrm{Tr}(P_{ij}^k X_{ij} + P_{ij} X_{ij}^k) \\ s.t. &\quad (P_{ij}, X_{ij}, \bar{G}_{ij}, \varepsilon_{ij}, \delta_{ij}, K_j) \in \Omega_4 \cap \Omega_5. \end{aligned} \tag{5.42}$$

(4) If $\bar{P}_{ij}^{k+1} - \bar{X}_{ij}^{k+1} < c_{3ij}I$, then stop and K_j^{k+1} is the feasible controller.

(5) Compute $\alpha^* \in [0,\ 1]$ by solving

$$\alpha^* = \min_{\alpha \in [0,\ 1]} \mathrm{Tr}\left(\left(P_{ij}^k + \alpha(\bar{P}_{ij}^k - P_{ij}^k)\right)\right)\left(X_{ij}^k + \alpha(\bar{X}_{ij}^k - X_{ij}^k)\right)\Big). \tag{5.43}$$

(6) Set $P_{ij}^{k+1} = (1-\alpha^*)P_{ij}^k + \alpha^* \bar{P}_{ij}^k$, $X_{ij}^{k+1} = (1-\alpha^*)X_{ij}^k + \alpha^* \bar{X}_{ij}^k$, $k = k+1$, go to step (3).

5.5 Design Examples and Simulation Results

In this section, design examples are presented to demonstrate the proposed results in this chapter. First of all, a comparison study is performed by considering the same system given in [110] but using the method proposed in this chapter. Then an aircraft model is used for which different fault tolerant controllers are synthesized.

Example 5.1 (Example 1, [110]) This example is to show how the performance can be improved by using the algorithm proposed, and the results obtained are presented in Table 1. The results of "complete observation" and "no observation" were given in [110].

$$A_1 = \begin{bmatrix} 2 & 2 \\ 3 & 1 \end{bmatrix},\ B_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, E_1 = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.4 \end{bmatrix},\ C_1 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix},\ D_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 1 & 0 \\ 0.5 & 1 \end{bmatrix},\ B_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, E_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0.8 \end{bmatrix},\ C_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},\ D_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

The transition probability matrix is $\begin{bmatrix} 0.9 & 0.1 \\ 0.8 & 0.2 \end{bmatrix}$, and initial distribution is $\begin{bmatrix} 0 & 1 \end{bmatrix}$.

The no observation case here corresponds to passive FTC in this stochastic FTCS. In the simulation, set $c_{3ij} = e^{-10}$ and the bisection method is used to find the lowest upper bound of H_2 norm. When controller is known, the original nonlinear matrix inequalities

TABLE 5.1

$||\tilde{\mathscr{G}}_d||_2^2$ and the state feedback controller designed

	Complete observation	No observation	Algorithm 5.2
Upper bound	16.6301	37.3898	17.5
Attained value	16.6301	17.5202	17.4838
Controller	$K_1 = [-1.162\ -0.9849]$	$K_1 = [-1.4\ -0.9917]$	$K_1 = [-1.1654\ -0.9823]$
	$K_2 = [0\ 0]$		

used for design can be reformulated as a semi-definite programming problem, whose global optimum can be found. Therefore, the H_2 norm achieved with the given controller can be calculated. From the simulation result, compared to the results from [110], it shows that with the iterative algorithm, a sightly better H_2 norm is achieved but with the upper bound shrunken greatly. It implies that though the iterative algorithm is developed to synthesize FDI-based active fault tolerant controller, it can also improve the passive FTC performance.

Example 5.2 Consider a continuous-time aircraft example originally examined by Mudge and Patton in [83]. A fourth order model of the linearized lateral dynamics is given, and the system data are

$$A_{01} = A_{02} = \begin{bmatrix} -0.277 & 0 & -32.9 & 9.81 \\ -0.1033 & -8.525 & 3.75 & 0 \\ 0.3649 & 0 & -0.639 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \ B_{01} = \begin{bmatrix} -5.432 & 0 \\ 0 & -28.64 \\ -9.49 & 0 \\ 0 & 0 \end{bmatrix},$$

$$B_{02} = \begin{bmatrix} -5.432 & 0 \\ 0 & -14.32 \\ -9.49 & 0 \\ 0 & 0 \end{bmatrix}, \ C_1 = C_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$D_1^T = D_2^T = \begin{bmatrix} 0.5 & 0.3 & 0.2 & 0.1 \end{bmatrix}, A_{11} = A_{12} = B_{11} = B_{12} = \begin{bmatrix} 0.1 & 0.15 \\ 0 & 0.1 \\ 0.2 & 0 \\ 0.1 & 0.1 \end{bmatrix},$$

$$A_{21} = A_{22} = \begin{bmatrix} 0.1 & 0 & 0.2 & 0 \\ 0 & 0.2 & 0 & 0.1 \end{bmatrix}, B_{21} = B_{22} = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.3 \end{bmatrix},$$

$$[\alpha_{ij}] = \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix}, [\beta_{ij}^1] = \begin{bmatrix} -0.1 & 0.1 \\ 0.15 & -0.15 \end{bmatrix}, [\beta_{ij}^2] = \begin{bmatrix} -0.2 & 0.2 \\ 0.1 & -0.1 \end{bmatrix},$$

$$\rho = \begin{bmatrix} 1/3 & 1/3 & 1/6 & 1/6 \end{bmatrix}.$$

The fault scenario studied here is the lost effectiveness of the second actuator by 50% (i.e. the second column of B_{02} is half of that of B_{01}). And the upper bound of H_2 norm is set to $\mu^2 = 10$.

For this aircraft model, to fully test the performance of the proposed algorithm, different types of controller are designed based on the algorithm proposed in this chapter. They include: state feedback controller and reduced-order dynamic output feedback controllers (a second order controller is synthesized). In addition, for this two inputs two outputs (TITO) system, a decentralized static output feedback controller can be designed, i.e. $u_j(t) = \hat{D}_j y(t)$. All these controllers satisfy H_2 performance bound. However, static output feedback controller is the simplest in structure and has the lowest implementation cost. On the contrary, the dynamic output feedback controller is the most complex as to the implementation, but it may provide better performance than the other two types of controllers.

State feedback controller:

$$K_1 = \begin{bmatrix} -0.44859 & 0.3669 & 2.1065 & -0.54704 \\ 0.034464 & 1.6254 & 0.30266 & 0.56399 \end{bmatrix},$$

$$K_2 = \begin{bmatrix} -0.45025 & 0.38687 & 2.105 & -0.54321 \\ 0.029845 & 1.6206 & 0.28889 & 0.56427 \end{bmatrix}.$$

Dynamic output feedback controller:

$$\hat{K}_1 = \left[\begin{array}{cc|cc} -26.2105 & -2.30632 & 5.34131 & 3.45878 \\ 13.5667 & -15.6143 & 3.09282 & 2.35983 \\ \hline 0.574802 & 0.0101392 & 0.0964031 & -0.00543808 \\ -0.115127 & 0.15813 & 1.2241 & 0.628475 \end{array}\right],$$

$$\hat{K}_2 = \left[\begin{array}{cc|cc} -25.7435 & -2.50205 & 5.10348 & 3.59334 \\ 14.263 & -16.1354 & 2.31855 & 2.0014 \\ \hline 0.568548 & 0.0120306 & 0.0929235 & -0.0118122 \\ -0.0838294 & 0.156033 & 1.23362 & 0.63114 \end{array}\right].$$

Static output feedback controller:

$$\hat{D}_1 = \begin{bmatrix} 0.0088358 & 0 \\ 0 & 0.27012 \end{bmatrix}, \ \hat{D}_2 = \begin{bmatrix} 0.0092513 & 0 \\ 0 & 0.27231 \end{bmatrix}.$$

The advantages show that the proposed algorithm is very versatile, and it is capable of designing many different types of controller to satisfying the requirements on performance, simplicity and implementation cost, etc. Furthermore, with these sets of controllers, sample path simulations of the closed-loop system are performed and the results are shown in the following figures. Where the disturbance is set as white noise.

5.6 Conclusion

In this chapter, H_2 control within conventional two-Markov-chain framework is studied. The practical considerations such as model uncertainties and system disturbance are handled as well. Controller synthesis for both continuous-time and discrete-time systems are considered in this chapter. The design of reconfigurable controllers via iterative LMI algorithm can guarantee the H_2 performance of the closed-loop system, as well as the stochastic stability. Using the operator theory of MJLS, the design for this performance can be handled for stochastic FTCS. With the proposed method, state feedback, dynamic and static output feedback and even structural controllers can be synthesized in the same framework.

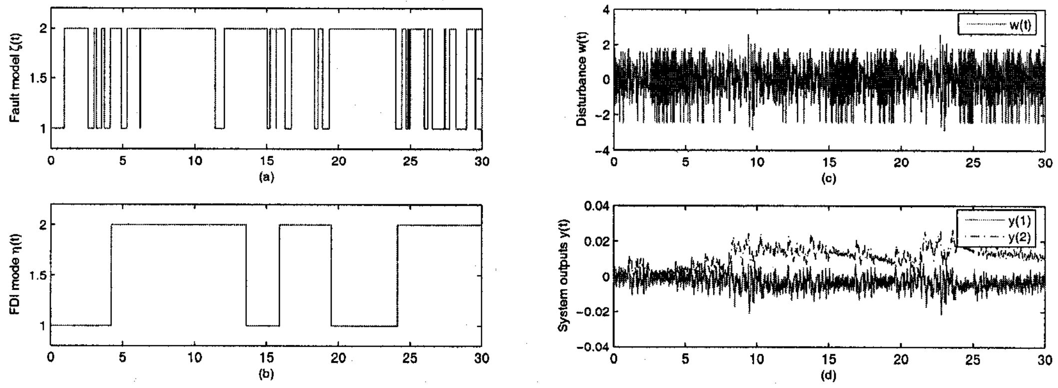

Figure 5.1. Single sample path simulation using state feedback controllers: (a) fault modes; (b) FDI modes; (c) system disturbance; (d) system outputs.

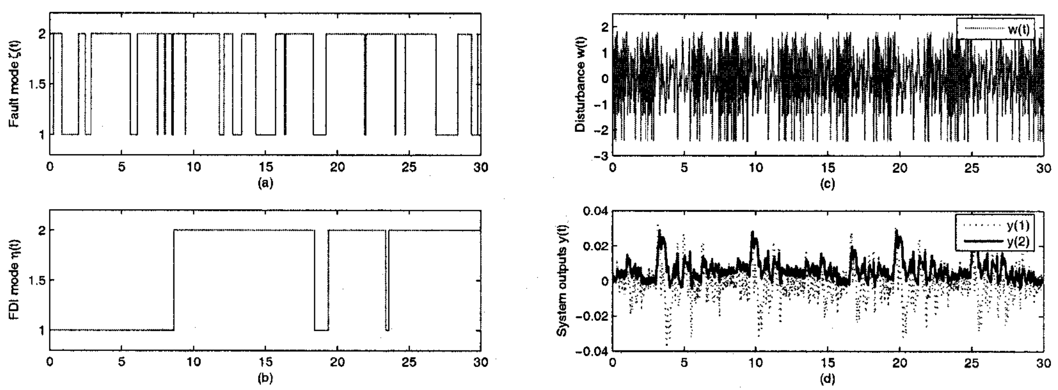

Figure 5.2. Single sample path simulation using the decentralized controllers: (a) fault modes; (b) FDI modes; (c) system disturbance; (d) system outputs.

Chapter 6

Analysis and Design for Systems with Random FDI Detection Delay

6.1 Introduction

In previous chapters, stabilization controller design and also H_2 control for FTCS have been investigated, all within the conventional two-Markov-chain FTCS framework. However, as described in Chapter 2, there are many cases where FDI scheme uses complicated algorithm and memoryless property of FDI decision does not hold. Under the circumstances, Markov chain cannot be used to represent FDI decision process in stochastic FTCS any more.

In this chapter, besides the MSS, input-output performance with respect to the additive disturbance $w(t) \in \mathbb{L}_2$ will be first considered, which will lead to the H_∞ design of stochastic FTCS. At the same time, the FDI modeling is extended from using a Markov chain to a semi-Markov chain, i.e. the random fault detection delay is considered. And the synthesis of output feedback controller will be solved using the iterative algorithm proposed in Chapter 5.

In addition, two further extensions will be discussed. The first extension is made by using a more general performance index, stochastic integral quadratic constraint, to replace the H_∞ norm, and then how to cope with non-exponential fault detection delay is briefly discussed.

Since discrete-time systems design has been covered in Chapter 5, only continuous-time output feedback control design will be discussed here to demonstrate the idea without occupying too much space. However, the design in this chapter can be extended to discrete-time case using the similar iterative LMI algorithm.

6.2 Problem Formulation

6.2.1 Notation

Some special notations are used in this chapter. Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a complete probability space carrying its natural filtration $\{\mathscr{F}_t, t \in \mathbb{R}^+\}$, as usual augmented by all null sets in the $\mathbb{P}$-completion of $\mathscr{F}$. Therefore, $L_2(\Omega, \mathscr{F}, \mathbb{P})$ is the space of square integrable stochastic process, where the 2-norm of the random variable $f(t), t \in \mathbb{R}^+$ is defined in Chapter 5.

6.2.2 Assumptions and Formulation

In this chapter, the open-loop system contains additive disturbance $w(t)$ besides parametric faults. It has the following state-space description:

$$\mathscr{G}_o : \begin{cases} \dot{x}(t) = (A(r(t)) + \Delta A(t, r(t)))x(t) + (B(r(t)) + \Delta B(t, r(t)))u(t, l(t)) + D(r(t))w(t), \\ y(t) = C(r(t))x(t) + E(r(t))w(t), \end{cases} \tag{6.1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $w(t) \in \mathbb{R}^p$, $y(t) \in \mathbb{R}^q$ are system state, control input, disturbance and output, respectively; it is assumed that $w(t) \in \mathbb{L}_2(\Omega, \mathscr{F}, \mathbb{P})$. All the matrices have corresponding compatible dimensions. $\{r(t), t \geq 0\}$ is a Markov chain representing the fault process of the system as usual, which takes value on a finite set $S = \{1, 2, \ldots, s\}$ and has the transition rate matrix $[\alpha_{ij}]$.

$A(r(t))$, $B(r(t))$, $C(r(t))$, $D(r(t))$, $E(r(t))$ are known constant matrices of appropriate dimensions for all $r(t) \in S$. $\Delta A(t, r(t))$ and $\Delta B(t, r(t))$ are unknown matrices representing time-varying parameter uncertainties in components and actuators. A simpler uncertainty description is used to lessen the notation complexity in this chapter:

$$\begin{bmatrix} \Delta A(t,r(t)) & \Delta B(t,r(t)) \end{bmatrix} = G(r(t))\Delta(t,r(t)) \begin{bmatrix} A_1(r(t)) & B_1(r(t)) \end{bmatrix}, \tag{6.2}$$

where $\|\Delta(r(t),t)\| \leq 1, \forall t \in \mathbb{R}^+$, $G(r(t))$, $A_1(r(t))$ and $B_1(r(t))$ are constant matrices for $r(t) \in S$ with proper dimensions.

The main difference is in the modeling of FDI decision process $\{l(t), t \geq 0,\ l(t) \in S\}$ in (6.1). In this chapter, the particular interest is on studying the random FDI delay and its effect on the controller synthesis. For simplicity but without loss of generality, the following assumptions for the characteristics of FDI are made:

(1) When system jumps from one mode to another, the FDI output can always follow and jump to the same mode after a time delay.

(2) The possibilities of multiple transitions of $r(t)$ between two consecutive transitions of $l(t)$ are negligible.

(3) The FDI delay is modeled by an independent exponentially distributed variable, whose mean value is given as $1/\beta_{ij}, j \neq i$, where i, j are the current modes of $l(t)$ and $r(t)$ respectively.

This formulation of FTC systems was firstly adopted by Mariton [79] when studying the stochastic stability. An illustration of the relation between $r(t)$ and $l(t)$ is given in Figure 1. $\Delta t_1 = t_2 - t_1$ is a delay occurring in the first transition of $l(t)$, an exponentially distributed random variable with mean value of $1/\beta_{12}$, and $\Delta t_2 = t_4 - t_3$ is another delay for the second transition with mean value of $1/\beta_{21}$.

Remark 6.1 *From the discussion on "strongly detectable faults" in Chapter 2, the first assumption that the FDI can identify the real system mode after a delay can be satisfied for faults of that category, therefore this assumption is valid.*

Remark 6.2 *Exponentially distributed random variables are used in this chapter for modeling the FDI delay. This kind of random variable has been widely adopted to model the*

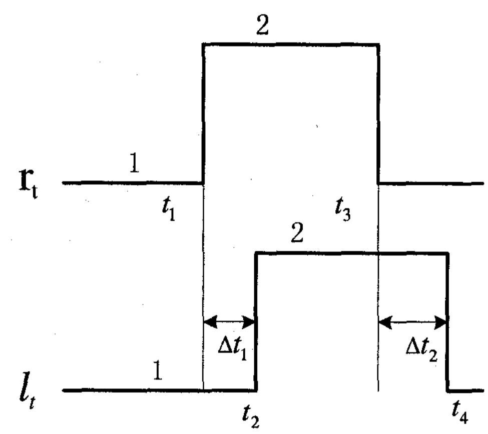

Figure 6.1. An illustration of the system fault process $r(t)$ and the FDI process $l(t)$

time between events (fault occurrence and FDI making up the decision in this context) that happen at a constant average rate [72]. Although the assumption of constant rate is rarely satisfied in real world scenarios, the exponential distribution can be used as a good approximation model for the time. Moreover, the memoryless property of the exponential distribution can lead to the joint Markov process in the analysis later.

It is worth noting that the FDI decision process $l(t)$ now is a semi-Markov chain since the sojourn time, or state occupancy time is not exponentially distributed any more. Although the stochastic process $l(t)$ is not a Markov Chain, the joint process $\{r(t), l(t)\}$ taking values on $S \times S$ is. The transition rate matrix of the augmented Markov Chain can be given as $\mu = [\mu_{ij,mn}] \in \mathbb{R}^{s^2 \times s^2}$ ($\mu_{ij,mn}$ is transition rate from the augmented state $(r(t) = i, l(t) = j)$ to the next state $(r(t+\Delta t) = m, l(t+\Delta t) = n)$):

$$\mu_{ij,mn} = \begin{cases} \alpha_{im}, & i = j,\ n = j, \\ \beta_{ji}, & i \neq j,\ m = n = i, \\ -\beta_{ji}, & i \neq j,\ m = i,\ n = j, \\ 0, & \text{otherwise.} \end{cases} \tag{6.3}$$

Remark 6.3 *Seen from (6.3), it is clear that the closed-loop system is a MJLS with partial Markov state observation. i.e. only $l(t)$ in the augmented Markov state $(r(t), l(t))$ is* *observable and accessible to the controllers. A similar situation was considered by [110] when studying the so called "cluster observation" problem for MJLS. However the common Lyapunov function approach, which is closed to that presented in Section 4.4.3, was used to tackle this difficult design problem, hence the result was relatively conservative.*

For the system model in (6.1), in this chapter, output feedback controllers of the following form will be designed,

$$\mathscr{G}_K: \quad \begin{cases} \dot{\hat{x}}(t) = \hat{A}(l(t))\hat{x}(t) + \hat{B}(l(t))y(t), \\ u(t,l(t)) = \hat{C}(l(t))\hat{x}(t) + \hat{D}(l(t))y(t), \end{cases} \tag{6.4}$$

where if controller has an order n_c, $\hat{A}(l(t)) \in \mathbb{R}^{n_c \times n_c}$, $\hat{B}(l(t)) \in \mathbb{R}^{n_c \times q}$, $\hat{C}(l(t)) \in \mathbb{R}^{m \times n_c}$ and $\hat{D}(l(t)) \in \mathbb{R}^{m \times q}$ are constant matrices to be designed for $l(t) \in S$.

In the sequel, for $r(t) =$ and $l(t) = j$, simplified notations are used as in Chapter 4 and Chapter 5.

With the controller given in (6.4), the closed-loop system model can then be written as following forms:

$$\mathscr{G}_{cl}: \begin{cases} \begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \begin{bmatrix} (A_i + \Delta A_i) + (B_i + \Delta B_i)\hat{D}_j C_i & (B_i + \Delta B_i)\hat{C}_j \\ \hat{B}_j C_i & \hat{A}_j \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} D_i + (B_i + \Delta B_i)\hat{D}_j E_i \\ \hat{B}_j E_i \end{bmatrix} w, \\ y = \begin{bmatrix} C_i & 0 \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} + E_i w. \end{cases} \tag{6.5}$$

Defining $\tilde{x}^T = \begin{bmatrix} x^T & \hat{x}^T \end{bmatrix}^T$, denote the closed-loop system in short as:

$$\mathscr{G}_{cl}: \quad \begin{cases} \dot{\tilde{x}} = \tilde{A}_{ij}\tilde{x} + \tilde{D}_{ij}w, \\ y = \tilde{C}_i\tilde{x} + E_i w, \end{cases} \tag{6.6}$$

where $\tilde{A}_{ij}$, $\tilde{D}_{ij}$, $\tilde{C}_i$ and $\tilde{E}_i$ have corresponding expressions in (6.5).

Definition 6.1 The closed-loop system (6.6) is said to have an H_∞ norm γ_0, denoted as $||\mathscr{G}_{cl}||_\infty = \gamma_0$, if for any arbitrary $w(t) \in L_2(\Omega, \mathscr{F}, \mathbb{P})$ and every $T_f \geq 0$, γ_0 is the smallest value that makes $\mathscr{E}\{\int_0^{T_f} y^T(t)y(t)dt\} < {\gamma_0}^2 \mathscr{E}\{\int_0^{T_f} w^T(t)w(t)dt\}$ hold.

With the definition of H_∞ for stochastic FTCS given, the design objectives of the proposed FTC can be defined as follows:

- **H_∞ controller synthesis problem:** for the FTCS subject to faults and disturbance $w(t) \in \mathbb{L}_2(\Omega, \mathscr{F}, \mathbb{P})$ shown in (6.1), design output feedback controllers of the form (6.4) so that for all admissible model uncertainties $\Delta A_i(t)$, $\Delta B_i(t)$: (1) the system is internally Mean Square Stable (MSS); and (2) the H_∞ norm of the closed-loop system $\|\mathscr{G}_{cl}\|_\infty \leq \gamma$.

6.3 Conditions on Nominal Stability and Performance

Now it is ready to develop conditions on MSS and H_∞ performance for the system $\mathscr{G}_{cl}$ given in (6.6). In this section, first restrict the focus on nominal systems without uncertainties, i.e. $\Delta A_i = 0$, $\Delta B_i = 0$. The results derived herein will then be extended to uncertain systems with uncertainties of the form (6.2).

The following theorem provides a more tractable condition that can lead to a valid design procedure for the output feedback controller.

Theorem 6.1 *The closed-loop system $\mathscr{G}_{cl}$ in the absence of disturbance $w(t)$ is MSS if and only if there exists $P_{ij} > 0$, such that*

$$N_{ij} = \tilde{A}_{ij}^T P_{ij} + P_{ij}\tilde{A}_{ij} + \mathbf{1}_{\{i=j\}}(\sum_{k\in S} \alpha_{ik}P_{kj}) + \mathbf{1}_{\{i\neq j\}}\beta_{ji}(P_{ii} - P_{ij}) < 0, \qquad i,j \in S, \quad (6.7)$$

where $\mathbf{1}_{\{.\}}$ stands for the Dirac measure, such that $\mathbf{1}_{\{x\}}$ equals one only if x is true, otherwise it equals zero.

Proof: Define the stochastic Lyapunov function of the joint stochastic process $\{\tilde{x}, r(t), l(t)\}$ as $V(\tilde{x}, r(t), l(t), t) = \tilde{x}^T P(r(t), l(t))\tilde{x}$. For such a Lyapunov function candidate, the weak infinitesimal operator $\mathscr{A}V$ can be defined as follows:

$$\begin{aligned} & \mathscr{A}V(\tilde{x}(t), r(t), l(t), t) \\ = \; & \lim_{\Delta t \to 0} \frac{1}{\Delta t}\Big(\mathscr{E}\{V(\tilde{x}(t+\Delta t), r(t+\Delta t), l(t+\Delta t), t+\Delta t | \tilde{x}(t), r(t), l(t), t)\} - V(\tilde{x}(t), r(t), l(t), t)\Big). \end{aligned} \quad (6.8)$$

The weak infinitesimal operator $\mathscr{A}V$ of the process $\{\tilde{x}, r(t), l(t), t\}$ at the point $\{\tilde{x}(t), r(t) = i,\ l(t) = j\}$ is calculated by

$$\mathscr{A}V(\tilde{x}(t), i, j, t) = V_t(\tilde{x}, i, j, t) + V_{\tilde{x}}(\tilde{x}, i, j, t)\dot{\tilde{x}}(i, j) + \sum_{m\in S}\sum_{n\in S} \mu_{ij,mn} V(\tilde{x}, m, n, t),$$

and it follows that,

$$\mathscr{A}V = \tilde{x}^T[\tilde{A}_{ij}^T P_{ij} + P_{ij}\tilde{A}_{ij} + \sum_{m\in S}\sum_{n\in S} \mu_{ij,mn} P_{mn}]\tilde{x} + w^T \tilde{D}_{ij}^T P_{ij}\tilde{x} + \tilde{x}^T P_{ij}\tilde{D}_{ij} w. \tag{6.9}$$

With the transition rate matrix $[\mu_{ij,mn}]$ given in (6.3), the expression of the weak infinitesimal operator above can be classified into two cases:

(1) If $i = j$, then $\mathscr{A}V$ can be calculated as:

$$\mathscr{A}V = \tilde{x}^T[\tilde{A}_{ij}^T P_{ij} + P_{ij}\tilde{A}_{ij} + \sum_{k\in S} \alpha_{ik} P_{kj}]\tilde{x} + w^T \tilde{D}_{ij}^T P_{ij}\tilde{x} + \tilde{x}^T P_{ij}\tilde{D}_{ij} w. \tag{6.10}$$

(2) If $i \neq j$, then $\mathscr{A}V$ can be calculated as:

$$\mathscr{A}V = \tilde{x}^T[\tilde{A}_{ij}^T P_{ij} + P_{ij}\tilde{A}_{ij} + \beta_{ji}(P_{ii} - P_{ij})]\tilde{x} + w^T \tilde{D}_{ij}^T P_{ij}\tilde{x} + \tilde{x}^T P_{ij}\tilde{D}_{ij} w. \tag{6.11}$$

By using the notation N_{ij} defined in (6.7), for both cases, the weak infinitesimal operator can be expressed in a unified form as:

$$\mathscr{A}V = \tilde{x}^T N_{ij}\tilde{x} + w^T \tilde{D}_{ij}^T P_{ij}\tilde{x} + \tilde{x}^T P_{ij}\tilde{D}_{ij} w. \tag{6.12}$$

By setting $w(t) = 0$ in the system $\mathscr{G}_{cl}$, it is known that the system is mean exponentially stable if and only if the weak infinitesimal operator $\mathscr{A}V < 0$. It is equivalent to that

$$N_{ij} < 0, \quad i,\ j \in S. \tag{6.13}$$

This completes the proof. □

In addition to the critical stability criterion, other important performance, such as disturbance/noise attenuation is also desirable. This type of performance is quantized as H_∞ norm of the system (i.e. the $L_2(\Omega, \mathscr{F}, \mathbb{P})$ gain of the system). By considering the H_∞ performance as defined in the Definition 5.1, the sufficient condition is given in the following theorem.

Theorem 6.2 *For a preset $\gamma > 0$, assume that the disturbance $w \in L_2(\Omega, \mathscr{F}, \mathbb{P})$, the system $\mathscr{G}_{cl}$ is internally MSS with $\|\mathscr{G}_{cl}\|_\infty \le \gamma$ achieved if there exists $P_{ij} > 0$ so that the following inequalities hold,*

$$\begin{bmatrix} N_{ij} + \tilde{C}_i^T \tilde{C}_i & P_{ij}\tilde{D}_{ij} + \tilde{C}_i^T E_i \\ \tilde{D}_{ij}^T P_{ij} + E_i^T \tilde{C}_i & -\gamma^2 I + E_i^T E_i \end{bmatrix} < 0, \tag{6.14}$$

where the expression of N_{ij} is given in (6.7).

Proof: For H_∞ performance, it has:

$$\begin{aligned} J &= \mathscr{E}\{\int_0^{T_f} y^T(t)y(t)dt\} - \gamma^2 \mathscr{E}\{\int_0^{T_f} w^T(t)w(t)dt\} \\ &= \mathscr{E}\{\int_0^{T_f} (y^T y - \gamma^2 w^T w)dt\} \\ &= \mathscr{E}\{\int_0^{T_f} (y^T y - \gamma^2 w^T w + \mathscr{A}V)dt\} - \mathscr{E}\{\int_0^{T_f} \mathscr{A}V dt\} \\ &= \mathscr{E}\{\int_0^{T_f} (y^T y - \gamma^2 w^T w + \mathscr{A}V)dt\} - \mathscr{E}\{V(T_f)\} + V(0). \end{aligned} \tag{6.15}$$

From Dynkin's formula, it has $\mathscr{E}\{\int_0^{T_f} \mathscr{A}V dt\} = \mathscr{E}\{V(T_f)\} - V(0)$. Since $w \in L_2(\Omega, \mathscr{F}, \mathbb{P})$ and system is internally MSS, so according to Theorem 5.2 of [44], $\tilde{x} \in L_2(\Omega, \mathscr{F}, \mathbb{P})$ and $\mathscr{E}\{V(T_f)\} \le \lambda_{\max}(P_{ij})\mathscr{E}\{x(T_f)^T x(T_f)\}$ is well defined for $\forall T_f \in \mathbb{R}^+$. If assume zero initial conditions, then the equation above becomes:

$$J \le \mathscr{E}\left\{ \sum_{k=0}^{h-1} \int_{t_k}^{t_{k+1}} \begin{bmatrix} \tilde{x} \\ w \end{bmatrix}^T \begin{bmatrix} N_{ij} + \tilde{C}_i^T \tilde{C}_i & P_{ij}\tilde{D}_{ij} + \tilde{C}_i^T E_i \\ \tilde{D}_{ij}^T P_{ij} + E_i^T \tilde{C}_i & -\gamma^2 I + E_i^T E_i \end{bmatrix} \begin{bmatrix} \tilde{x} \\ w \end{bmatrix} dt \right\} \tag{6.16}$$

where $t_0 = 0$, $t_h = T_f$, and $t_k, k = 1, 2, \cdots, h-1$ is the k-th transition time for joint process $\{r(t), l(t)\}$. The inequality $J < 0$ will hold if for $\forall i, j \in S$, the inequality (6.14) holds, which also implies MSS from (1,1)-entry of the inequality. This completes the proof. □

6.4 Controller Synthesis Using Iterative LMI Algorithm

In this section, the output feedback controller synthesis for MSS and H_∞ performance is discussed. For notational simplicity in the derivation, in Section 6.4.1 and 6.4.2, only the nominal system is considered, i.e., $\Delta A_i(t) = 0$, $\Delta B_i(t) = 0$. In Section 6.4.3, results for the uncertain system will be derived.

6.4.1 Output Feedback Based FTC Design

Within this stochastic FTCS framework, the design of output feedback controller was done by [1] based on the assumption that the controller can access the real system fault mode, i.e. controllers have the form $K(r(t),l(t))$. In this section, design is made for the FDI-based output feedback controllers.

For the open-loop system $\mathscr{G}_o$ in (6.1) with no model uncertainties, and the output feedback controller in (6.4), define the following matrices:

$$K_j = \begin{bmatrix} \hat{A}_j & \hat{B}_j \\ \hat{C}_j & \hat{D}_j \end{bmatrix}, \bar{A}_i = \begin{bmatrix} A_i & 0 \\ 0 & 0 \end{bmatrix}, \bar{B}_i = \begin{bmatrix} 0 & B_i \\ I & 0 \end{bmatrix}, \bar{C}_i = \begin{bmatrix} 0 & I \\ C_i & 0 \end{bmatrix},$$
$$\bar{D}_i = \begin{bmatrix} D_i \\ 0 \end{bmatrix}, \bar{E}_i = \begin{bmatrix} 0 \\ E_i \end{bmatrix}, F_i = \gamma^2 I - E_i^T E_i. \tag{6.17}$$

Then $\tilde{A}_{ij}$, $\tilde{D}_{ij}$ in the closed-loop system state space representation (6.6) can be expressed as affine functions of the controller K_j as:

$$\tilde{A}_{ij} = \bar{A}_i + \bar{B}_i K_j \bar{C}_i, \quad \tilde{D}_{ij} = \bar{D}_i + \bar{B}_i K_j \bar{E}_i. \tag{6.18}$$

Now, the controller synthesis problem is reduced to find feasible solutions K_j and P_{ij}, $i,j \in S$ for the inequality (6.14), which is rewritten here as:

$$\begin{bmatrix} N_{ij} + \tilde{C}_i^T \tilde{C}_i & P_{ij}\tilde{D}_{ij} + \tilde{C}_i^T E_i \\ \tilde{D}_{ij}^T P_{ij} + E_i^T \tilde{C}_i & -F_i \end{bmatrix} < 0. \tag{6.19}$$

The matrix inequality above is not linear in terms of the decision variables, in that the P_{ij} is coupled with $\tilde{A}_{ij}$ and $\tilde{D}_{ij}$ (see (6.7) for N_{ij}), therefore it cannot be solved directly using convex optimization techniques. In this subsection, first convert it into linear matrix inequalities with equality constraints, and then an iterative algorithm will be provided in the next subsection to solve for the controller K_j.

Using Schur complement, first rewrite the inequality (6.19) as:

$$N_{ij} + \tilde{C}_i^T \tilde{C}_i + \left(P_{ij}\tilde{D}_{ij} + \tilde{C}_i^T E_i\right) F_i^{-1} \left(\tilde{D}_{ij}^T P_{ij} + E_i^T \tilde{C}_i\right) < 0.$$

The next step is to remove the coupling between Lyapunov function P_{ij} and $\tilde{A}_{ij}$ or $\tilde{D}_{ij}$, and then cope with those quadratic terms. To this end, introduce slack variables $\bar{P}_{ij}$ and $\bar{W}_{ij}$, use Reciprocal Rrojection Lemma 4.4 and Schur complement to obtain:

$$\begin{bmatrix} H_{0ij} & P_{ij}\left(\tilde{A}_{ij}+\tilde{D}_{ij}F_i^{-1}E_i^T\tilde{C}_i\right)+\bar{W}_{ij} & P_{ij}\tilde{D}_{ij} & \tilde{C}_i^T & \tilde{C}_i^T E_i \\ * & -\bar{P}_{ij} & 0 & 0 & 0 \\ * & * & -F_i & 0 & 0 \\ * & * & * & -I & 0 \\ * & * & * & * & -F_i \end{bmatrix} < 0,$$

where $H_{0ij} = \mathbf{1}_{\{i=j\}}(\sum_{k\in S}\alpha_{ik}P_{kj}) + \mathbf{1}_{\{i\neq j\}}\beta_{ji}(P_{ii}-P_{ij}) + \bar{P}_{ij} - \bar{W}_{ij} - \bar{W}_{ij}^T$.

Since $\bar{P}_{ij}$ can be an arbitrary positive definite matrix, without loss of generality, set $\bar{P}_{ij} = I$, Further define $X_{ij} = P_{ij}^{-1}$, $W_{ij} = X_{ij}\bar{W}_{ij}$, $Q_{ij} = W_{ij}W_{ij}^T$, pre- and post-multiply the inequality above by $\text{diag}\{X_{ij},\ I,\ I,\ I,\ I\}$. For the resulted inequality, expand quadratic terms in the (1,1)-entry using Schur complement to obtain:

$$\begin{bmatrix} G_{11} & G_{12} & \tilde{D}_{ij} & X_{ij}\tilde{C}_i^T & X_{ij}\tilde{C}_i^T E_i & H_{1ij} & X_{ij}-W_{ij} \\ * & -I & 0 & 0 & 0 & 0 & 0 \\ * & * & -F_i & 0 & 0 & 0 & 0 \\ * & * & * & -I & 0 & 0 & 0 \\ * & * & * & * & -F_i & 0 & 0 \\ * & * & * & * & * & -H_{2ij} & 0 \\ * & * & * & * & * & * & -I \end{bmatrix} < 0, \tag{6.20}$$

where $\tilde{A}_{ij}$, $\tilde{D}_{ij}$ are defined in (6.18) and

$$\begin{aligned} G_{11} &= \left(\mathbf{1}_{\{i=j\}}\alpha_{ii} + \mathbf{1}_{\{i\neq j\}}\beta_{ji}\right)X_{ij} - Q_{ij}, \\ G_{12} &= \tilde{A}_{ij} + \tilde{D}_{ij}F_i^{-1}E_i^T\tilde{C}_i + W_{ij}, \\ H_{1ij} &= X_{ij}\left[\mathbf{1}_{\{i=j\}}\sqrt{\alpha_{i1}}I, \cdots, \mathbf{1}_{\{i=j\}}\sqrt{\alpha_{i,i-1}}I, \mathbf{1}_{\{i=j\}}\sqrt{\alpha_{i,i+1}}I, \cdots, \mathbf{1}_{\{i\neq j\}}\sqrt{\beta_{ji}}I\right], \\ H_{2ij} &= \text{diag}\{X_{1j}, \cdots, X_{i-1,j}, X_{i+1,j}, \cdots, X_{ii}\}. \end{aligned}$$

To this stage, the original nonlinear matrix inequalities have been successfully transformed to LMIs but with equality constraints ($Q_{ij} = W_{ij}W_{ij}^T,\ i,j \in S$). However, this problem is still non-convex. In the next subsection, an iterative optimization algorithm is used to solve this matrix inequality.

6.4.2 Using Iterative Algorithm to Solve LMIs with Equality Constraints

In this section, the iterative algorithm adapted from SLPMM [71] will be presented. For notational simplicity, the constants c_{ij} introduced in Chapter 5 will be left out here though in simulation, they are still used. First of all, define the following sets:

$$\Omega_6(X_{ij}, W_{ij}, Q_{ij}, K_j) = \{(X_{ij}, W_{ij}, Q_{ij}, K_j) \text{ satisfying } (6.20)\}, \tag{6.21}$$

$$\Omega_7(Q_{ij}, W_{ij}) = \{(Q_{ij}, W_{ij}) \text{ satisfying } \begin{bmatrix} Q_{ij} & W_{ij} \\ W_{ij}^T & I \end{bmatrix} \geq 0\}. \tag{6.22}$$

Then the following iterative algorithm can be used to solve the LMIs with equality constraints $Q_{ij} = W_{ij}W_{ij}^T,\ i, j \in S$.

Algorithm 6.1:

(1) Initialization: Set $k = 0$, determine $(X_{ij}^0, W_{ij}^0, Q_{ij}^0, K_j^0) \in \Omega_6 \cap \Omega_7$.

(2) Determine $(\tilde{X}_{ij}^k, \tilde{W}_{ij}^k, \tilde{Q}_{ij}^k, \tilde{K}_j^k)$, as the unique solution of the optimization problem:

$$\begin{aligned} &\min \sum_{i\in S}\sum_{j\in S} \mathrm{Tr}(Q_{ij} - W_{ij}^k W_{ij}^T - W_{ij} W_{ij}^{k^T}), \\ &s.t. \quad (X_{ij}, W_{ij}, Q_{ij}, K_j) \in \Omega_6 \cap \Omega_7. \end{aligned} \tag{6.23}$$

(3) If $\mathrm{Tr}(Q_{ij}^k + \tilde{Q}_{ij}^k - W_{ij}^k \tilde{W}_{ij}^{k^T} - \tilde{W}_{ij}^k W_{ij}^{k^T}) = 2 \cdot \mathrm{Tr}(Q_{ij}^k - W_{ij}^k W_{ij}^{k^T})$, for $i, j \in S$, then stop, and $\tilde{K}_j^k$ is the solution for controller.

(4) Calculate $\theta^* \in [0,\ 1]$ by solving

$$\theta^* = \min_{\theta \in [0,\ 1]} \sum_{i\in S}\sum_{j\in S} \mathrm{Tr}\Big((1-\theta)Q_{ij}^k + \theta\tilde{Q}_{ij}^k - ((1-\theta)W_{ij}^k + \theta\tilde{W}_{ij}^k)((1-\theta)W_{ij}^k + \theta\tilde{W}_{ij}^k)^T\Big).$$

(5) Set $W_{ij}^{k+1} = (1-\theta^*)W_{ij}^k + \theta^*\tilde{W}_{ij}^k$, $Q_{ij}^{k+1} = (1-\theta^*)Q_{ij}^k + \theta^*\tilde{Q}_{ij}^k$, $k = k+1$, go to Step (2).

In the above algorithm, each step only involves solving LMI. Step (1) is a feasibility problem and step (2) is an "eigenvalue" problem (minimizing a linear function with LMI constraints). Hence for each iteration, the global optimum can be found via convex optimization, and this is vital for the convergence of the algorithm. The stop criterion in step (3) is to terminate the iteration when no improvement shows in terms of the equality approximation error $J_{ij}^k = \mathrm{Tr}(Q_{ij}^k - W_{ij}^k W_{ij}^{k^T})$.

Remark 6.4 *In the above algorithm, the optimization objective is set as the linearized form of the equality approximation error, i.e.*

$$\tilde{J}^k = \sum_{i\in S}\sum_{j\in S} \mathrm{Tr}(Q_{ij} + Q_{ij}^k - W_{ij}^k W_{ij}^T - W_{ij} W_{ij}^{k^T}),$$

where Q_{ij}^k can be dropped out of the expression without affecting the optimization result since it is a constant matrix at the k-th step. With this linearized objective function, the SLPMM algorithm adopted here can guarantee that the equality approximation error $\tilde{J}^k$ is always strictly decreasing, and therefore the algorithm is convergent ([71], Theorem 3.9).

6.4.3 Synthesis for Uncertain Systems

In FTC systems, modeling errors and unknown disturbances are the major causes of an imperfect FDI result, which in turn affects the control performance. In this section, the output feedback based FTC design problem is considered for the same stochastic system treated in the previous section, but with modeling uncertainties $\Delta A_i(t)$, $\Delta B_i(t)$ of the form (6.2).

In addition to those definitions in (6.17), further define the following matrices:

$$\bar{G}_i = \begin{bmatrix} G_i \\ 0 \end{bmatrix}, \bar{A}_{1i} = \begin{bmatrix} A_{1i} & 0 \end{bmatrix}, \bar{B}_{1i} = \begin{bmatrix} 0 & B_{1i} \end{bmatrix}.$$

Therefore, in this case, $\tilde{A}_{ij}$ and $\tilde{D}_{ij}$ can be written as follows:

$$\begin{aligned}\tilde{A}_{ij} &= \bar{A}_i + \bar{B}_i K_j \bar{C}_i + \bar{G}_i \Delta_i \left(\bar{A}_{1i} + \bar{B}_{1i} K_j \bar{C}_i\right), \\ \tilde{D}_{ij} &= \bar{D}_i + \bar{B}_i K_j \bar{E}_i + \bar{G}_i \Delta_i \bar{B}_{1i} K_j \bar{E}_i .\end{aligned} \tag{6.24}$$

Now replace the expression of $\tilde{A}_{ij}$, $\tilde{D}_{ij}$ in the inequality (6.20) with (6.24), and apply Lemma 4.2 to this inequality, and can have the following result:

Theorem 6.3 *For uncertain FTC system (6.1) with arbitrary admissible model uncertainties (6.2), there exist output feedback controllers making closed-loop system MSS with* $\|\mathscr{G}_{cl}\|_\infty \leq \gamma$, *if there exist positive scalars* ε_{ij}, δ_{ij} *and matrices* $X_{ij} > 0$, W_{ij}, Q_{ij}, K_j *such that the following matrix inequalities hold:*

$$\begin{bmatrix} H_{3ij} & H_{4ij} & \bar{D}_i + \bar{B}_i K_j \bar{E}_i & X_{ij}\tilde{C}_i^T & X_{ij}\tilde{C}_i^T E_i & H_{1ij} & X_{ij} - W_{ij} & 0 & 0 \\ * & -I & 0 & 0 & 0 & 0 & 0 & H_{5ij}^T & 0 \\ * & * & -F_i & 0 & 0 & 0 & 0 & 0 & \bar{E}_i^T K_j^T \bar{B}_{1i}^T \\ * & * & * & -I & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & -F_i & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -H_{2ij} & 0 & 0 & 0 \\ * & * & * & * & * & * & -I & 0 & 0 \\ * & * & * & * & * & * & * & -\varepsilon_{ij} I & 0 \\ * & * & * & * & * & * & * & * & -\delta_{ij} I \end{bmatrix} < 0, \tag{6.25}$$

where

$$\begin{aligned} H_{3ij} &= \left(\mathbf{1}_{\{i=j\}}\alpha_{ii} + \mathbf{1}_{\{i\neq j\}}\beta_{ji}\right) X_{ij} - Q_{ij} + \left(\varepsilon_{ij} + \delta_{ij}\right) \bar{G}_i \bar{G}_i^T, \\ H_{4ij} &= \left(\bar{A}_i + \bar{D}_i F_i^{-1} E_i^T \tilde{C}_i\right) + \bar{B}_i K_j \left(\bar{C}_i + \bar{E}_i F_i^{-1} E_i^T \tilde{C}_i\right) + W_{ij}, \\ H_{5ij} &= \bar{A}_{1i} + \bar{B}_{1i} K_j \left(\bar{C}_i + \bar{E}_i F_i^{-1} E_i^T \tilde{C}_i\right). \end{aligned}$$

and the definitions of Q_{ij}, H_{1ij}, H_{2ij}, X_{ij}, W_{ij} *can be found in Section 6.4.2.*

Proof: For inequality (6.20), apply Lemma 4.2 for uncertainties in (1,2), (1,3) entries and their symmetric entries, then use Schur complement to expand those quadratic terms in (2,2) and (3,3) entries to obtain the above inequality. □

The algorithm given in Section 6.4.2 can still be applied to find K_j, but with minor changes by redefining the set $\Omega_1(X_{ij}, W_{ij}, Q_{ij}, K_j, \varepsilon_{ij}, \delta ij) = \{(X_{ij}, W_{ij}, Q_{ij}, K_j, \varepsilon_{ij}, \delta ij)$ satisfying $(6.25)\}$.

6.4.4 Numerical Example

Example 6.1 In this example, simulations are performed to demonstrate the proposed methods. Consider a continuous-time aircraft example originally examined by [83], the state space model of the linearized lateral dynamics is given, where the normal case and the actuator fault are considered,

$$A_1 = A_2 = \begin{bmatrix} -0.277 & 0 & -32.9 & 9.81 \\ -0.1033 & -8.525 & 3.75 & 0 \\ 0.3649 & 0 & -0.639 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} -5.432 & 0 \\ 0 & -28.64 \\ -9.49 & 0 \\ 0 & 0 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} -5.432 & 0 \\ 0 & -14.32 \\ -9.49 & 0 \\ 0 & 0 \end{bmatrix}, \quad C_1 = C_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$D_1^T = D_2^T = \begin{bmatrix} 0.5 & 0.3 & 0.2 & 0.1 \end{bmatrix}, \quad E_1 = E_2 = \begin{bmatrix} 0.5 \\ 0.25 \end{bmatrix}.$$

And the transition rate matrices are given as: $[\alpha_{ij}] = \begin{bmatrix} -0.5 & 0.5 \\ 1 & -1 \end{bmatrix}$, $\beta_{12} = 1$, $\beta_{21} = 1.5$.

The fault scenario studied here is the lost effectiveness of the second actuator by 50% (i.e. the second column of B_2 is half of that of the B_1). The upper bound of H_∞ norm to achieve is preset to $\gamma = 1$.

Generally speaking, with given performance requirement, a lower order controller is more desirable, which can reduce the complexity and the implementation cost. For this reason, firstly begin the design by choosing the controller order $n_c = 0$, and find the output feedback controllers as follows by using the proposed method:

$$K_1 = \begin{bmatrix} -0.10928 & 0.2687 \\ 0.014011 & 0.28306 \end{bmatrix}, \quad K_2 = \begin{bmatrix} -0.11072 & 0.27327 \\ 0.014239 & 0.27943 \end{bmatrix}.$$

Actually, the algorithm proposed in this chapter can even design decentralized output feedback controllers to further reduce the implementation cost.

It is assumed in the system there exist model uncertainties given as:

$$G_1 = G_2 = \begin{bmatrix} 0.1 & 0.15 \\ 0 & 0.1 \\ 0.2 & 0 \\ 0.1 & 0.1 \end{bmatrix}, A_{11} = A_{12} = \begin{bmatrix} 0.1 & 0 & 0.2 & 0 \\ 0 & 0.2 & 0 & 0.1 \end{bmatrix}, B_{11} = B_{12} = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.3 \end{bmatrix}.$$

Then for this uncertain system, the robust static decentralized controllers are computed as:

$$K_1 = \begin{bmatrix} 0.052222 & 0 \\ 0 & 0.13356 \end{bmatrix}, \quad K_2 = \begin{bmatrix} 0.052283 & 0 \\ 0 & 0.12579 \end{bmatrix}.$$

By using the first set of controllers, a single sample path simulation is performed, and the closed-loop system responses are obtained. The disturbance is modeled as truncated random noise $w = N(s,t)\mathbf{1}_{\{0 \le t \le 20\}}$, where $N(s,t)$ is a Gaussian stochastic process.

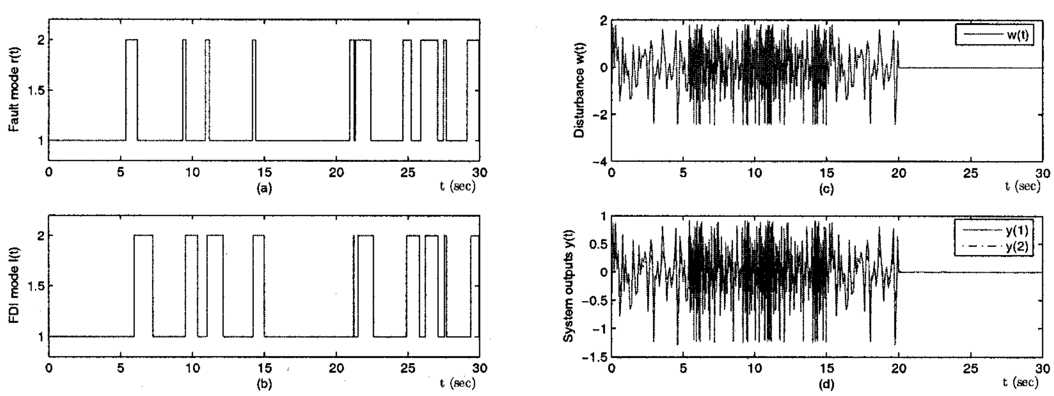

Figure 6.2. Single sample path simulation: (a) fault modes; (b) FDI modes; (c) system disturbance; (d) the system outputs

Figure 6.2(a)(b) shows the single sample path of the process $r(t)$ and $l(t)$. They differ by the exponentially distributed random delay for each jump. From Figure 6.2(c) and 6.2(d), the system is stabilized and the disturbance attenuation is achieved.

With the decentralized controllers, another sample path is simulated with the disturbance signal given as $w(t) = e^{-20\,|a(s)|t}$, where $a(s) \sim N(0,1)$ is a Gaussian random variable. The corresponding sample path is shown in Figure 6.3(a)(b), the system response is shown in Figure 6.3(d).

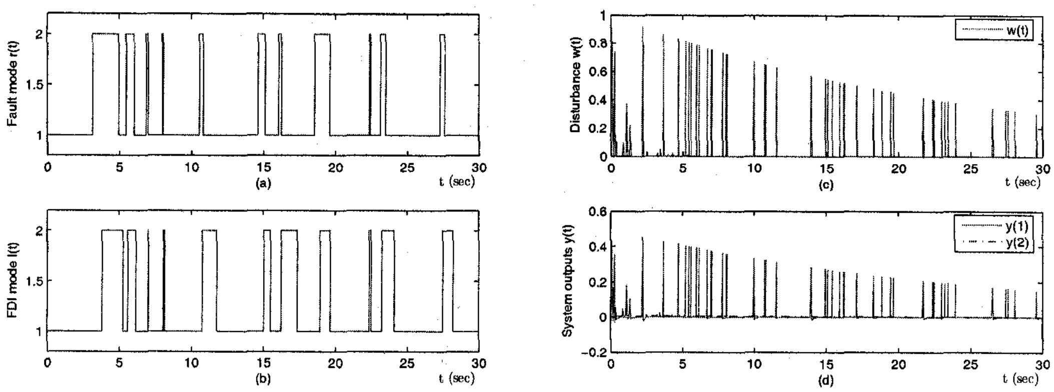

Figure 6.3. Single sample path simulation using the decentralized controllers: (a) fault modes; (b) FDI modes; (c) system disturbance; (d) the system outputs

6.5 Extension I: SIQC Performance

In this section, stochastic integral quadratic constraint performance is studied, which is more general than the H_∞ performance. In this section, the framework used is still the conventional two-Markov chain framework, and the systems have the regular norm-bounded model uncertainty description as shown Section 4.2.1. From discussion in previous sections of this chapter, it can be seen that the design results here can be applied to FTCS with setting the same as in previous sections of this chapter, i.e. with more complex FDI modeling.

Definition 6.2 The signals $w(t)$ and $y(t)$ are called to satisfy the stochastic integral quadratic constraint (SIQC) defined by $\Pi = \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{21} & \Pi_{22} \end{bmatrix}$ if

$$\mathscr{E}\{J\} = \mathscr{E}\left\{ \int_0^{T_f} \begin{bmatrix} w \\ y \end{bmatrix}^T \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{21} & \Pi_{22} \end{bmatrix} \begin{bmatrix} w \\ y \end{bmatrix} dt \right\} > 0. \tag{6.26}$$

Π_{11} is symmetric and positive definite, Π_{22} is symmetric, and $\Pi_{12} = \Pi_{21}^T$. T_f is the final time. J is called the SIQC index number defined over finite horizon.

Obviously, the H_∞ is the special case with $\Pi = \begin{bmatrix} \gamma_0^2 I & 0 \\ 0 & -I \end{bmatrix}$. With this performance index in addition to MSS, the corresponding condition for uncertain FTCS state-feedback controller is given as follows, for simplicity, only the results are listed out:

$$\begin{bmatrix} \bar{G}_{ij} & \bar{X}_{ij}A_{2i}^T & \bar{X}_{ij}C_i^T & \bar{H}_{3ij} & A_{0i}+B_{0i}K_j-D_i\Pi_{11}^{-1}\Pi_{12}C_i+\bar{W}_{ij} & 0 & \\ * & -\bar{\varepsilon}_{ij}I & 0 & 0 & 0 & 0 & 0 \\ * & * & \bar{H}_{4ij} & 0 & 0 & 0 & 0 \\ * & * & * & \bar{H}_{2ij} & 0 & 0 & 0 \\ * & * & * & * & -I & 0 & 0 \\ * & * & * & * & * & -I & K_j^T B_{2i}^T \\ * & * & * & * & * & * & -\bar{\delta}_{ij}I \end{bmatrix} < 0, \tag{6.27}$$

where

$$\begin{aligned} \bar{G}_{ij} &= (\alpha_{ii}+\beta_{jj}^i)\bar{X}_{ij} + D_i\Pi_{11}^{-1}D_i^T\lambda_{ij} + \bar{\varepsilon}_{ij}A_{1i}A_{1i}^T + \bar{\delta}_{ij}B_{1i}B_{1i}^T - Q_{ij}, \\ \bar{H}_{2ij} &= -\mathrm{diag}\{\bar{X}_{1j},\cdots,\bar{X}_{i-1,j},\bar{X}_{i+1,j},\cdots,\bar{X}_{i1},\cdots,\bar{X}_{i,j-1},\bar{X}_{i,j+1},\cdots\}, \\ \bar{H}_{3ij} &= A_{0i}+B_{0i}K_j-D_i\Pi_{11}^{-1}\Pi_{12}C_i+\bar{W}_{ij}, \\ \bar{H}_{4ij} &= -(\Pi_{21}\Pi_{11}^{-1}\Pi_{12}-\Pi_{22})^{-1}\lambda_{ij}. \end{aligned}$$

The same iterative algorithm can be applied to solve the linear matrix inequality with equality constrains as before.

Example 6.2 To be complementary to the previous simulation, state-feedback simulation of SIQC algorithm is performed in this example. The system models are given as:

$$A_{01} = A_{02} = \begin{bmatrix} -0.277 & 0 & -32.9 & 9.81 \\ -0.1033 & -8.525 & 3.75 & 0 \\ 0.3649 & 0 & -0.639 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad B_{01} = \begin{bmatrix} -5.432 & 0 \\ 0 & -28.64 \\ -9.49 & 0 \\ 0 & 0 \end{bmatrix},$$

$$B_{02} = \begin{bmatrix} -5.432 & 0 \\ 0 & -14.32 \\ -9.49 & 0 \\ 0 & 0 \end{bmatrix}, \ C_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}, \ C_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix},$$

$$D_1 = \begin{bmatrix} 0 & 0.1 & 0 & 0 \end{bmatrix}^T, \ D_2 = \begin{bmatrix} 0 & 0.11 & 0.1 & 0 \end{bmatrix}^T.$$

Furthermore, the model uncertainties of the general form as shown in Chapter 4 are also assumed in this example with

$$A_{11} = A_{12} = 0.5 \times I_{4\times4}, \ A_{21} = A_{22} = 0.1 \times I_{4\times4},$$

$$B_{11} = B_{12} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}^T, \ B_{21} = B_{22} = 0.1 \times I_{2\times2}.$$

And the transition matrices are given as: $[\alpha_{ij}] = \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix}$, $[\beta_{ij}^1] = \begin{bmatrix} -0.1 & 0.1 \\ 0.15 & -0.15 \end{bmatrix}$, $[\beta_{ij}^2] = \begin{bmatrix} -0.2 & 0.2 \\ 0.1 & -0.1 \end{bmatrix}$.

The fault scenario studied here is the lost effectiveness of the second actuator by 50%. The SIQC weighting matrix is given as $\Pi = \begin{bmatrix} 0.8 & 0.5 \\ 0.5 & -1 \end{bmatrix}$.

By implementing the iterative algorithm proposed in section IV, the following controllers are obtained:

$$K_1 = \begin{bmatrix} -0.1320 & 0.0714 & 1.5013 & -0.4742 \\ -0.1226 & 0.9766 & 0.0832 & 0.2090 \end{bmatrix},$$

$$K_2 = \begin{bmatrix} -0.1359 & 0.0655 & 1.5038 & -0.4753 \\ -0.1189 & 0.8693 & 0.0903 & 0.2040 \end{bmatrix}.$$

The decision variable $[\lambda_{ij}] = \begin{bmatrix} 887.6028 & 835.0881 \\ 213.7141 & 224.3147 \end{bmatrix}$, other simulation results are omitted here due to the limit of space. In the simulation, $c_{ij} = 0.01$ is set. The iterative algorithm terminates after 3 steps, and the optimization objective function values and the error of the

TABLE 6.1

Optimal objective function values and equality approximation errors

k	$\sum_i \sum_j J_{ij}^{k*}$	$\tilde{J}_{11}^k$	$\tilde{J}_{12}^k$	$\tilde{J}_{21}^k$	$\tilde{J}_{22}^k$
0	—	5063.0689	5059.8999	5293.8660	5302.7193
1	2.1031×10^4	9.8914	9.7459	14.2492	14.5193
2	60.0858	6.5100×10^{-12}	7.7300×10^{-12}	1.9947	2.0621
3	5.399	2.927×10^{-12}	3.098×10^{-12}	6.651×10^{-12}	9.436×10^{-12}

equality constraint are shown in Table 5.1. It is clear that the algorithm converges quickly. Furthermore, with this set of controllers, sample path simulation of the closed-loop system is performed and the results are shown in Figure 6.4, where the disturbance $w(t)$ is chosen as a random noise with mean 0 and variance 1. In the simulated sample path, the SIQC index of 36.0793 is achieved.

To further study the statistical characteristics of the stochastic FTCS, Monte Carlo simulation method is applied for the above system subject to a deterministic disturbance signal $w(t) = e^{-0.1t} \sin t$. 1000 sample paths are simulated, and the mean values of the system states $\mathscr{E}\{x(t)\}$ are shown in Figure 6.5 while the histogram of SIQC index value J is given in Figure 6.6. The histogram is suitable for showing the distribution of quantized performance index, so that different controllers' performance can be compared.

6.6 Extension II: Non-exponentially Distributed Fault Detection Delay

Till now, the analysis and design for stochastic FTCS with exponentially distributed random fault detection delay have been studied. The assumption on exponential distribution is vital since otherwise difficult to calculate the weak infinitesimal operator.

Using the method of decomposing non-exponential distributed fault detection delay into the sum of N exponentially distributed random variables is shown in Figure 6.7. Where

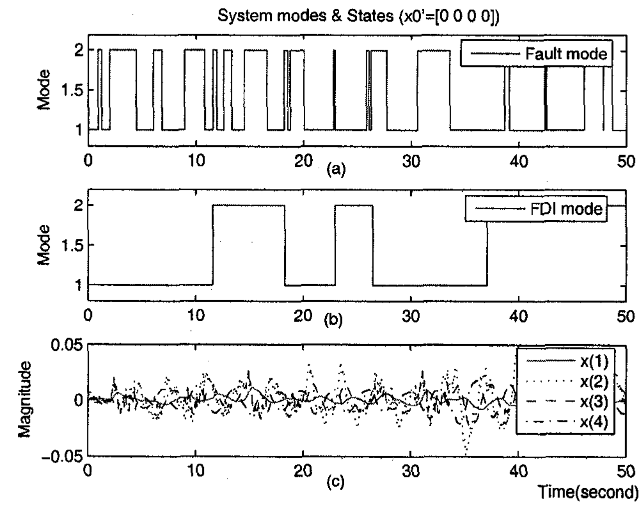

Figure 6.4. Single sample path simulation: (a) system modes; (b) FDI modes; (c) system states

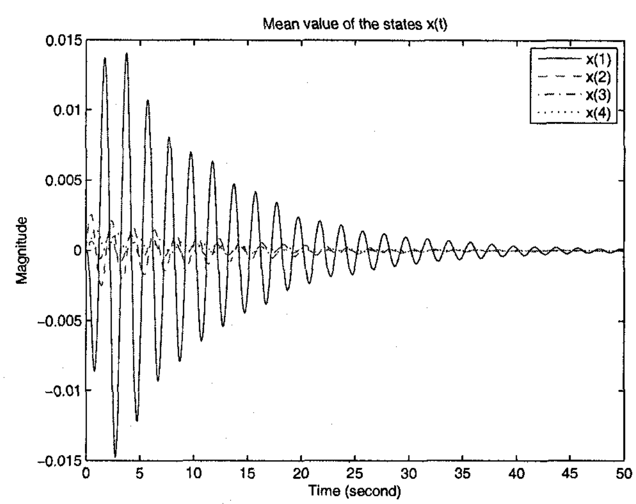

Figure 6.5. Mean value of the states $x(t)$ over 1000 sample paths

$r(t)$ is the fault process, $l^N(t)$ is the FDI process, and $l^i(t),\ i = 1,2,\ldots,N-1$ are fictitious random processes such that the transition delay between the adjacent processes is exponentially distributed. For example, Δt_{11} is an exponential RV with parameter β_{12}^1, Δt_{12} with β_{12}^2, and $\Delta t_1 = \sum_{i=1}^{N} \Delta t_{1i}$.

The formulation of the problem is given as follows:

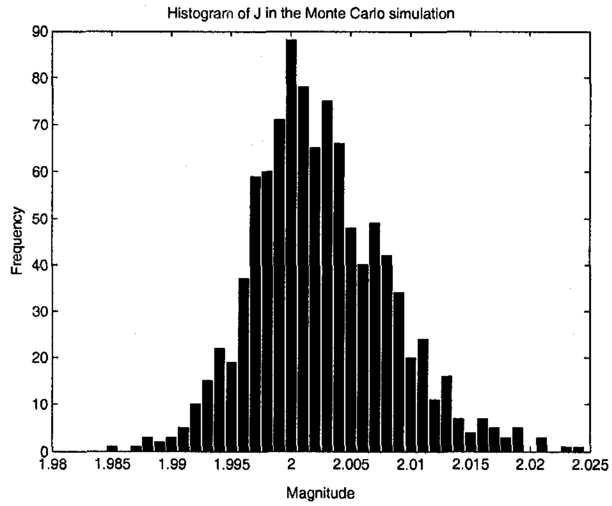

Figure 6.6. Histogram of SIQC index value J ($\bar{J} = \frac{1}{1000}\sum_{i=1}^{1000} J_i = 2.003$)

(1) The random distributed fault detection delay can be approximated by using sum of N exponential random variables.

(2) For the random process $l^i(t)$, when the previous random process jumps from state j to k, after an exponentially distributed delay with parameter β^i_{jk}.

(3) After $r(t)$ jumps from state j to k, it will not jump until $l^N(t)$ jumps from state j to k.

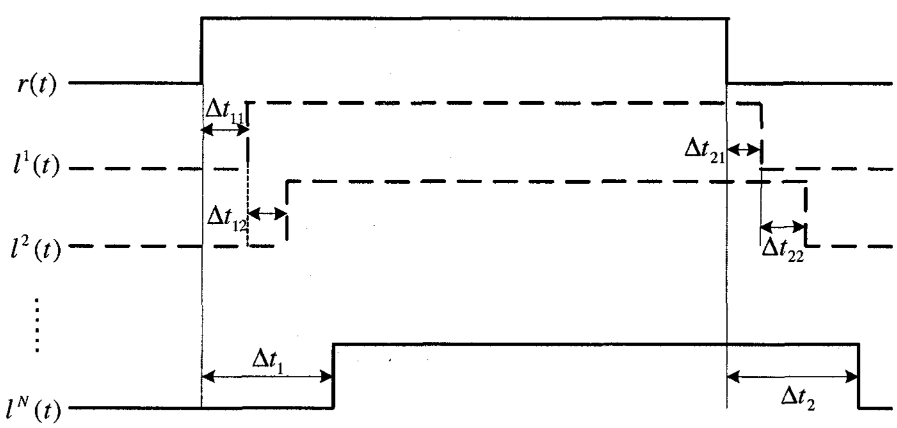

Figure 6.7. Non-exponential fault detection delay case

In this setting, the augmented process $\{r(t), l^1(t), l^2(t), \ldots l^N(t)\}$ determined the system mode, while the analysis and design method given in previous sections can be applied here. In the total s^{N+1} possible states, only $s \times (N+1)$ modes are valid, thus greatly reduces the number of matrix inequality to solve. And when using more fictitious random processes to approximate the random fault detection delay, the number of constraints only increases linearly.

Same as in the previous sections of this chapter, the augmented process $\{r(t), l^1(t), l^2(t), \ldots l^N(t)\}$ is a Markov chain. Assuming the current state is represented by $\{i_1, j_1^1, j_1^2, \ldots, j_1^N$, and the next state is $\{i_2, j_2^1, j_2^2, \ldots, j_2^N\}$, then the elements of transition rate matrix $[\mu] \in \mathbb{R}^{s^{N+1} \times s^{N+1}}$ can be determined similarly as in (6.3). Therefore, for FTCS with arbitrary FDI fault detection delay distribution, it is possible to approximate that distribution using hypoexponential distribution, and use the Markov state augmentation technique to handle the design.

6.7 Conclusion

In this chapter, the design of stochastic FTC in the presence of random FDI delays is discussed. The FDI delay is represented by an exponentially distributed random variable. It is assumed that the FDI can identify the true system fault mode after this delay. The main difficulty in the design lies in the fact that the mode of controller is solely dependent on the mode of the FDI process, whose transitions depend on those of the system fault mode as well as the FDI delay. The sufficient conditions for the desirable Mean Square Stability and H_∞ performance are obtained for the system with modeling uncertainties and external disturbances. By transforming the given nonlinear matrix inequalities into LMIs with equality constraints, an iterative LMI algorithm is used to solve the synthesis problem. Output feedback controllers can be solved conveniently by using the algorithm with available semi-definite programming software packages. Illustrative examples are given and simulation results are shown to demonstrate the effectiveness of the proposed design algorithm. It is shown in the last that the design can be extended to the cases using the more general

SIQC performance index and with non-exponential fault detection delay.

Chapter 7

Conclusions and Future Work

7.1 Conclusions

In the previous chapters, stabilization of the integrated FTCS is systematically reviewed and summarized. Further, results on performance besides MSS, such as H_2 and H_∞ are discussed. To design controller that only accesses FDI decision instead of system real fault mode, some techniques drawn from multi-objective control field of research have been used to separate the Lyapunov function from the controller, so that the conditions expressed in terms of nonlinear matrix inequality can be converted into LMI with equality constraints. Then an iterative LMI algorithm is used to solve that problem.

The effectiveness of this design has been confirmed from numerical simulations, where stochastic stability and performances are verified using single sample path and Monte-Carlo simulations, and the efficiency of the iterative algorithm is also shown through the equality approximation error of each iteration.

While the analysis and design are mainly carried out in the conventional two-Markov-chain framework, FTCS with FDI detection delay case is discussed as well. And it shows that similar design algorithm can be applied for this case.

7.2 Future Work

Although basic problems have been solved in this area, there are still some challenging problems need to study:

1 In Chapter 4, as to "full information controller" design, the syntheses of stabilizing controllers for continuous-time full-order dynamic output feedback controller are proposed in terms of LMIs. However, there are no corresponding results for discrete-time systems using LMI yet. Obviously, this is a meaningful problem since LMI is more efficient than iterative LMI algorithms.

2 In Chapter 2, the identification of FDI detection delay and the approximation using summation of independent exponential distribution is briefly discussed through an example. However, for more complicated situations, the PDF of the hypoexponential distribution should be developed first, so that it is possible using least square technique to approximate that specific distribution. Then the results in Chapter 6 should be extended to handle FTCS with non-exponentially distributed fault detection delay.

   The second problem on this issue is to determine what should be the criteria for the distribution approximation. Two issues should be noted here. The first is that the larger the number of independent exponential distributions used, the more matrix inequalities are involved in stability/performance conditions. The second is that iterative LMI algorithm may converge locally, therefore, the smaller the approximation error does not imply better performance.

   The third problem is how to handle the remained distribution estimation error. Should it be treated as transition rate uncertainty or else? How to guarantee that the system is robust stable with distribution estimation error?

3 In Chapter 2, for systems subject to "strongly detectable faults", the FDI scheme can always give the right decision and the only imperfectness of FDI is the fault detection delay. However, for those faults do not belong to that category, their detection cannot be guaranteed. Then a natural question is how to design baseline controller so that if

those faults happen, the baseline controller can still stabilize the system or maintain the minimum performance requirement.

The difficulties lie in that this type of parametric faults must be treated as model uncertainties. However, the difficulties for this problem are due to the uncertainties therein may not lie in a convex set, therefore, current available methods cannot be applied to solve robust controller design for stabilizing systems with this kind of model uncertainties.

4 In the design of controller throughout this thesis, we did not impose any constraints on the range of control signals. However, practical systems under normal operation always have a range for manipulated variables and process variables. When those variables stay in the given range, the operation is more efficient. However, most of the design does not take this factor into consideration. Even in the area of MJLS, this issue is rarely considered, and only [30] discussed state feedback control with constraints: $\|F_i x(k) + G_i u(k)\| \leq \rho_i$ with probability 1. As for FTCS, the similar and more complicated situations should be discussed, so that the results can be directly put into application.

5 Open-loop FTCS discussed in this thesis is MJLS, i.e. the pre-fault and post-fault systems are all linear systems. However, some nonlinear components may exist in the systems, such as saturation of inputs. Such a problem is studied in [77], but the controller is assumed to access both real fault mode and FDI mode. The synthesis of FDI-based controller should be studied.

6 Further study on nonlinearity of the system subject to faults can be done for bilinear systems. This kind of system should also be a meaningful extension of the current work. Some work on MJLS control has tackled this issue. However, control synthesis for FTCS has not been discussed.

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