Analysis on Locally Compact Semitopological Semigroups

by

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Abstract

This thesis focuses on the measure algebra $M(S)$ of a locally compact semitopological semigroup S. In particular, we consider the analog of the group algebra $L_1(G)$ of a locally compact group G on S and the topological amenability of S. Among other results which shall be explained further in the introduction, the thesis answers the following open problems.

1. Baker 90' and Dzinotyiweyi 84' [6, 18]

   Let $L(S) = \{\mu \in M(S);\ s \mapsto \delta_s * |\mu| \text{ is weakly continuous}\}$. It is known that if $S = G$, then $L(S) = L_1(G)$. Is $L(S)$ a norm closed ideal of $M(S)$ that closed under absolute continuity in general? We shall answer this question in the positive in Section 3.3 and 3.4.

2. Day 82' [15]

   We say S is strong topological left amenable if there is a net of probability measure (μ_α) such that $\|\nu * \mu_\alpha - \mu_\alpha\| \to 0$ uniformly for all probability measures ν supported on a compact subset K of S. Does strong topological amenability implies non-trivial $L(S)$? The background for this question will be explained fully in Section 4.1, along with a counterexample that answers this problem in the negative.

3. Wong 79' [50]

We say S is topological left amenable if there is a net of probability measure (μ_α) such that $\|\nu * \mu_\alpha - \mu_\alpha\| \to 0$ for any probability measure ν on S. It was shown that when S is a discrete semigroup or a locally compact group, a locally compact Borel subsemigroup T is topological left amenable if and only if (1) S is topological left T-amenable, that is, there is a net of probability measures (μ_α) on S, such that $\|\nu * \mu_\alpha - \mu_\alpha\| \to 0$ for any probability measure ν on S that is supported on T, and (2) $\lim_\alpha \mu_\alpha(T) > 0$. Does similar result hold for locally compact semitopological semigroups? We shall prove this result in Section 4.2.

Preface

This dissertation is original, independent work by the author, Qianhong Huang. The author published the results of Chapter 4 and Example 3.5.7 in [27].

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Contents

Chapter 1

Introduction

Throughout the thesis, S is a locally compact semitopological semigroup, that is, a semigroup with a locally compact topology such that the multiplication is separately continuous. If the multiplication is jointly continuous, we say S is a locally compact topological semigroup. We write $M(S)$ as the Banach algebra of complex Radon measures on S with the total variation norm.

Abstract harmonic analysis is rooted in the study of topological groups. One of its primary goal is to extend Fourier analysis to non-commutative and locally compact groups. In the process of studying locally compact groups G, the Banach algebra $L_1(G)$ of complex-valued, integrable Borel functions with respect to Haar measure plays an important role. It allows us to use the functional analytic technique to study the underlying group structure. In the case of locally compact semitopological semigroup, there is no direct extension of $L_1(G)$ due to the lack of a Haar measure. However, it is well-known that $L_1(G)$ is isometrically isomorphic to the closed subalgebra of $M(S)$ consisting of measures that are absolutely continuous with respect to a Haar measure. These measures are precisely those complex Radon measures μ that are (left) translation continuous, i.e., the translation map $G \to M(G) : g \mapsto \delta_g * \mu$ is weakly continuous. Based on this characterization, Baker and Baker generalized $L_1(G)$ to locally compact topological semigroup in a paper series [3, 4, 5]. They defined the semigroup algebra $L(S)$ to be the set of complex Radon measures μ such that the translation maps $S \to M(S) : s \mapsto \delta_s * |\mu|$ and $S \to M(S) : s \mapsto |\mu| * \delta_s$ are weakly continuous. Note that if S is a locally compact group, $L(S)$ is exactly $L_1(S)$. Among other things, Baker and Baker proved that $L(S)$ is a norm closed two-sided ideal of $M(S)$ that is closed under absolute continuity. Subspaces of $M(S)$ that have this nature are sometimes called L-ideals follows the notation of [41]. In literature, the closure of the union of supports of all measures in $L(S)$ is called the foundation of S. When S is a locally compact topological semigroup, the foundation of S is a closed two-sided ideal of S that could be empty. We call S a foundation semigroup if its foundation is the whole of S. It is natural to expect $L(S)$ to reflect properties of the foundation of S, like $L_1(G)$ reflects that of G, when G is a locally compact group. Because $L(S)$ is closed under absolute continuity, a foundation of a semigroup is always a foundation semigroup. Thus we can always restrict our attention to the foundation semigroups.

Many authors have extensively studied $L(S)$ for topological foundation semigroups. For example, Dzinotyiweyi [17] studied $L(S)$ for topological semigroups that are not necessarily locally compact; for locally compact topological semigroups, Sleijpen [39] studied the algebra of right multipliers of $L(S)$; Bami [7] studied the relationship between the representations of S and the representations of $L(S)$ on reflexive Banach spaces; Amini and Medghalchi [1] studied Fourier algebras on locally compact topological semigroups with continuous involutions. However, it is not obvious to consider $L(S)$ for semitopological semigroups because most of the structure theorem of $L(S)$ depends heavily on the jointly continuous multiplication assumption of S. As mentioned in the survey papers of Dzinotyiweyi [18] and Baker [6], they expected the extensively studied theory of $L(S)$ for locally compact topological semigroup to hold for locally compact semitopological semigroup, but to prove that $L(S)$ is an L-ideal in the semitopological case remains a barrier. We shall give a proof of their conjecture in Chapter 2.

Unlike that of locally compact groups, the existence of $L(S)$ is not guaranteed for locally compact semigroups. There has been many attempts to find the existence a non-trivial $L(S)$, i.e, $L(S) \neq \{0\}$, for example, in the treatises of Sleijpen [37] and Dzinotyiweyi [17]. We shall give a characterization of $L(S)$ when S is a compact semitopological semigroup with a left invariant measure in Section 3.8. A compact semigroup with a dense subgroup is an example for such semigroups.

Another topic in this thesis is the topological left amenability of S, which will be introduced in Section 2.3. In short, topological left amenability can be characterized as the existence of a net of probability measures of left invariance under different topologies. Along with the studies of different types of topological amenability for locally compact semitopological semigroups, Day [15] inquired if a strong type of topological left amenability implies the existence of non-trivial $L(S)$, since the existence of non-trivial $L(S)$ implies the coincidence of the strong and the weak topological amenability. We shall discuss the background of this question in details in Section 4.1 along with a counterexample to the question.

In analysis, one often cares when we can preserve a property of a substructure and when we can extend the property from a substructure. For topological left amenability of a locally compact semitopological semigroup S, we consider the topological left amenability of its locally compact Borel subsemigroups T. Wilde [44] and Wong [50] proved respectively that when S is a discrete semigroup or a locally compact group, T is topological left amenable if and only if (1) S has a net of probability measure of left invariance under the translation of probability measures supported on T, and (2) the net of probability measure does not vanish on T. However, as Wong [50] pointed out, the proof for locally compact topological semigroup is left open. We shall give a proof of this result for locally compact semitopological semigroup in Section 4.2.

This thesis is organized as follows:

In Chapter 2, we provide preliminaries needed for this thesis, including the introduction of measure algebra of a locally compact semitopological semigroup and a representation of its dual. Then, we describe the origin of topological amenability and its characterizations in Section 2.3. Section 2.4 - 2.5 are materials closely related to the hereditary property of topological amenability. Section 2.6 lists some notations that we frequently use throughout this thesis.

In Chapter 3, we define $L(S)$ to be the set of all complex Radon measures μ, such that the map $s \mapsto \delta_s * |\mu|$ $(*)$ is weakly continuous. Such measures are called left translation continuous measures in this thesis. Recall in Baker and Baker's original definition for the topological case, they also require the right translation map $s \mapsto |\mu| * \delta_s$ to be weakly continuous. Since the thesis does not involve results that require this kind of symmetry, we only use the left translation to define the semigroup algebra $L(S)$. However, it should be clear that the results in this thesis also work for the two-sided definition. In Section 3.1 and 3.2, we look at the equivalent definitions of $L(S)$ and discuss if the total variation $|\cdot|$ is removable from the definition. In Section 3.3 and 3.4, we prove the conjecture that $L(S)$ is an L-ideal of $M(S)$. In Section 3.5, we consider the foundation of S, or in another word, the support of $L(S)$, where we mainly focus on the algebraic and topological properties of the foundation. In Section 3.6, we study when $L(S)$ has a right identity, and when it has a bounded left approximate identity. As a corollary, our result shows $L_1(G)$ has a right identity if and only if it is discrete. In Section 3.6, we prove that in the presence of a non-trivial $L(S)$, there are abundant measures such that the left translation map $(*)$ is norm continuous. Then in Section 3.8, we give a characterization of $L(S)$, when S is a compact semitopological semigroup with left invariant measures.

In Chapter 4, we answer the open problem raised by Day in [15] by providing a counterexample to show that the strong topological left amenability does not guarantee a non-trivial $L(S)$. In Section 4.2, we give a proof on the hereditary property of topological amenability, which answers an open problem raised by Wong [50].

In Chapter 5, we list some problems that are closely related to the thesis and are believed to be still open, along with some motivational remarks.

Chapter 2

Preliminaries

A semigroup S is a locally compact semitopological semigroup if (1) it is a locally compact space; (2) the multiplication on S is separately continuous. Throughout this thesis, S is a locally compact semitopological semigroup unless otherwise stated.

2.1 Measure Algebras

Let $C_0(S)$ be the set of all continuous functions on S that vanish at infinity. It is known that $C_0(S)$ is a Banach space under the supremum norm. Let $M(S)$ be the Banach space of complex Radon measures on S with total variation norm. By Riesz-Markov Theorem, we can identify $M(S)$ with $C_0(S)^*$. Let $\nu, \mu \in M(S)$. One can define convolution of the two measures through $\langle \nu * \mu, f \rangle = \iint f(st) d\nu(s) d\mu(t)$ for $f \in C_0(S)$. It is not hard to check that $M(S)$ with this convolution is a Banach algebra. Wong [48] showed that the order of this integration is interchangeable. In fact, for $f \in L_1(|\nu| * |\mu|)$,

$$\iint f(st) d\nu(s) d\mu(t) = \iint f(st) d\mu(t) d\nu(s) \qquad (\Delta)$$

Moreover, $\int f(st)d\nu(s) \in L_1(|\mu|)$ and $\int f(st)d\mu(s) \in L_1(|\nu|)$. The result is not trivially implied by Fubini's theorem since the map $f \circ \pi$ is not measurable on $S \times S$ in general. Here $\pi : S \times S \to S : (s,t) \mapsto st$. One may find the details for the argument is in [48, p. 608].

The Banach algebra $M(S)$ is often referred to as the measure algebra of S, which is the main topic of this thesis. Let $\mu \in M(S)$, the support of μ is the set $\text{supp}\mu := \{s \in S;\ |\mu|(U) > 0 \text{ for any open neighborhood } U \text{ of } S\}$. It is shown in [48, 4] that for $\nu, \mu \in M(S)$, $\text{supp}\,|\nu| * |\mu| = \overline{\text{supp}\nu \cdot \text{supp}\mu}$. It is also worth noting that a set might be $|\nu * \mu|$-measurable but not $|\nu| * |\mu|$-measurable (see e.g. [26, 19.25]).

The convolution on $M(S)$ introduces Arens product on $M(S)^{**}$. Let $m, n \in M(S)^{**}$, $f \in M(S)^*$, $\nu, \mu \in M(S)$. Then

$$\begin{aligned} m * n \in M(S)^{**} &\quad m * n(f) = m(n * f) \\ n * f \in M(S)^{*} &\quad n * f(\mu) = n(f * \mu) \\ f * \mu \in M(S)^{*} &\quad f * \mu(\nu) = f(\mu * \nu) \end{aligned}$$

Remark 2.1.1. 1. Symmetrically, we define $\mu * f \in M(S)^*$, such that $\mu * f(\nu) = f(\nu * \mu)$. We can see that $M(S)^*$ and $M(S)^{**}$ are both Banach $M(S)$-bimodule.

2. Let A be a Banach space. One can naturally embed A into A^{**}. In this thesis, we constantly treat A as a norm closed subspace of A^{**} whenever needed.

3. Let $n \in M(S)^{**}$. The map $M(S)^{**} \to M(S)^{**} : m \to m * n$ is weak* continuous. However $M(S)^{**} \to M(S)^{**} : m \to n * m$ is not weak* continuous in general. The map is weak* continuous if $n \in M(S)$. For details on Arens product, one may refer to [11, 2.6].

We can introduce a natural order on $M(S)$ by letting $\mu \geq 0$ if $\mu(E) \geq 0$ for all Borel subset E of S. Then $M(S)$ is a Banach lattice under this order. Let ν, μ. We say ν is absolutely continuous with respect to μ, denoted as $\nu \ll \mu$, if for any $\varepsilon > 0$, there is $\delta > 0$, such that whenever $|\mu|(E) < \delta$ for some Borel subset E, there is $|\nu|(E) < \varepsilon$. Note that if $\nu \ll \mu$, then $\nu = \lim_{n\to\infty} \nu \wedge n|\mu|$ (see e.g. [41]).

The following is a list of properties that characterize the weakly compact subsets of $M(S)$. We shall refer to this result frequently.

Theorem 2.1.2. [25, 16] Let X be a locally compact space. Let A be a subset of $M(X)$. Then the following are equivalent.

(i) A is relatively weakly compact.

(ii) For each pairwisely disjoint sequence (O_i) of open subsets, $\mu(O_i) \to 0$ uniformly for $\mu \in A$.

(iii) Let $\varepsilon > 0$. Then,

  a. there is a compact subset K of X, such that $|\mu|(X \backslash K) < \varepsilon$ for $\mu \in A$;

  b. for each compact subset K of X, there is an open subset $O \supseteq K$, such that $|\mu|(O \backslash K) < \epsilon$ for $\mu \in A$.

(iv) There is a positive measure λ in $M(S)$, such that A is uniformly absolutely continuous with respect to λ. That is, for each $\varepsilon > 0$, there is $\delta > 0$, such that if $\lambda(E) < \delta$ for some Borel subset E, then $|\mu|(E) < \varepsilon$ for $\mu \in A$.

2.2 Dual of Measure Algebra

In this section, we identify $M(S)^*$ as a subspace of $\prod\{L_\infty(|\mu|);\ \mu \in M(S)\}$. The details of the theory can be found in [47] and [42]. Let $\mu \in M(S)$, we equip $L_\infty(|\mu|)$ with the essential supremum norm $\|\cdot\|_{\mu,\infty}$ with respect to μ. Recall for $h \in L_\infty(|\mu|)$, $\|h\|_{\mu,\infty} = \inf_{\mu(N)=0} \sup_{s\notin N} |h(s)|$.

A function $f = (f_\mu)_{\mu\in M(S)} \in \prod\{L_\infty(|\mu|);\ \mu \in M(S)\}$ is a generalized function, if

1. $\sup\limits_{\mu\in M(S)} \|f_\mu\|_{\mu,\infty} < \infty$
2. If $\nu, \mu \in M(S)$ and $\nu \ll \mu$, then $f_\nu = f_\mu$ $|\nu|$-a.e.

Let $GL(S)$ be the set of all generalized functions. For $f \in GL(S)$, let $\|f\| = \sup_{\mu\in M(S)} \|f_\mu\|_{\mu,\infty}$ to be the norm of f. It is not hard to check that $GL(S)$ equipped with this norm is a Banach space.

In the previous section, we have shown that $M(S)^*$ is a Banach $M(S)$-bimodule. Let $f = (f_\mu)_{\mu\in M(S)} \in GL(S)$, define

$$f * \mu = (f_{|\mu|*|\nu|} * \mu)_{\nu\in M(S)} \qquad \mu * f = (\mu * f_{|\nu|*|\mu|})_{\nu\in M(S)}$$

where

$$f_{|\mu|*|\nu|} * \mu(t) = \int f_{|\mu|*|\nu|}(st)d\mu(s) \quad \mu * f_{|\mu|*|\nu|}(t) = \int f_{|\mu|*|\nu|}(ts)d\mu(s)$$

From last section, we know that $f_{|\mu|*|\nu|} * \mu$ and $\mu * f_{|\mu|*|\nu|}$ are $|\nu|$-measurable. By using the fact that $f_{|\mu|*|\nu|} \in L_\infty(|\mu| * |\nu|)$ and $\mathrm{supp}\,|\mu| * |\nu| = \overline{\mathrm{supp}\mu \cdot \mathrm{supp}\nu}$. It is not hard to check that both of them are in $L_\infty(|\nu|)$. Therefore, $GL(S)$ is also a Banach $M(S)$-bimodule.

Theorem 2.2.1. [47] Let $T : GL(S) \to M(S)^* : Tf(\mu) = \int f_\mu d\mu$. Then T is an isometric order preserving isomorphism that commutes with translations of $M(S)$, i.e., $T(f * \mu) = T(f) * \mu$, $T(\mu * f) = \mu * T(f)$.

Remark 2.2.2. 1. As the double dual of the commutative C^*-algebra $C_0(S)$, $M(S)^*$ itself is a commutative von Neumann algebra. Let $F, G \in M(S)^*$. Let $f = T^{-1}F$, $g = T^{-1}G$. The Arens product of F and G, induced by $C_0(S)$, is $F \cdot G = (f_\mu g_\mu)_{\mu \in M(S)}$. Then T is an isomorphism between unital Banach algebras if we equip $GL(S)$ with the coordinate-wise multiplication. The identity of $M(S)^*$ is $\mathbb{1}$, where $\mathbb{1}(\mu) = \mu(S)$ for $\mu \in M(S)$. The identity of $GL(S)$ is e, where $e_\mu = 1$ $|\mu|$-a.e.. It is clear that $T(e) = \mathbb{1}$.

2. Let $BM(S)$ be the set of bounded Borel measurable functions on S. One can view $BM(S)$ as a subspace of $GL(S)$. It is known that $BM(S)$ is a Banach space with the supremum norm $\|\cdot\|_{\sup}$. From the definition, $\|\cdot\|_{\sup}$ coincide with the norm on $GL(S)$ and equivalently the dual norm on $M(S)^*$, when consider $BM(S)$ as a subspace of $GL(S)$ and $M(S)^*$. Moreover, if $i : BM(S) \to M(S)^*$ or $GL(S)$ is the natural embedding, one has $i(l_s f) = i(f) * \delta_s$ for $s \in S$.

2.3 Amenability

The theory of amenability arose in finding finitely additive positive set functions on discrete groups that are invariant under group translations. In 1957, Day [12] extended the idea to discrete semigroups and named discrete semigroup with left invariant positive finitely additive set functions as left amenable semigroups. His formal definition involves the concept of left invariant means. Suppose S is a discrete semigroup. Let $l_\infty(S)$ be the Banach space of all bounded functions on S with supreme norm. An element $m \in l_\infty(S)^*$

is a mean if $m(\mathbf{1}) = 1 = \|m\|$, where $\mathbf{1}$ is the constant 1 function on S. Note that a mean is always positive. We say a mean $m \in l_\infty(S)^*$ is left invariant if $m(l_s f) = m(f)$ for all $f \in l_\infty(S)$, $s \in S$. Here $l_s f(t) = f(st)$ for $t \in S$. The discrete semigroup S is left amenable if there is a left invariant mean on $l_\infty(S)$.

Later, amenability was extended to locally compact groups. Let G be a locally compact group with a Haar measure λ. It is well-known that λ is unique up to scalar multiplication. Let $L_\infty(G)$ be the Banach space of essentially bounded functions on G, with respect to λ, with the essential supremum norm $\|\cdot\|_\infty$. Let $L_1(G)$ be the Banach space of integrable functions with respect to λ, with the L_1 norm $\|\cdot\|_1$. We say G is topological left amenable if there is $m \in L_\infty(G)^*$, such that $\|m\| = 1 = m(\mathbf{1})$ and $\nu * m = m$ for all positive ν in $L_1(G)$ with norm 1. The operation $*$ here is the Arens product on $L_\infty(G)^* = L_1(G)^{**}$ introduced by the convolution on $L_1(G)$. As suggested by Day [12], we may also consider to define amenability on $CB(G)$, the set of continuous bounded functions on G. In this case, we say G is left amenable if there is $m \in CB(G)^*$, such that $\|m\| = 1 = m(\mathbf{1})$ and $m(l_s f) = m(f)$ for $s \in S$ and $f \in CB(G)$. For locally compact groups, topological left amenability on $L_\infty(G)$ is equivalent to the left amenability on $CB(G)$. For details, one can see [23, Section 2.2].

2.3.1 Topological Left Invariant Means

There has been many efforts in literature to extend amenability to semi-topological semigroups, especially in the locally compact case. Researchers studied amenability on closed subspace of $CB(S)$ (e.g. [20], [30], [34], [53]), and on $M(S)^*$ (e.g. [15], [35], [45], [52]). This thesis mainly focus on the later. As we saw in Remark 2.2.2, $CB(S)$ can be considered as a closed subspace of

$M(S)^*$. Thus amenability defined on $M(S)^*$ can be viewed as a generalization of amenability defined on $CB(S)$.

Let $P(S)$ be the set of all probability measures on S. We say $m \in M(S)^{**}$ is a mean on $M(S)^*$ if $\|m\| = m(\mathbb{1}) = 1$. Recall $\mathbb{1}(\mu) = \mu(S)$ for $\mu \in M(S)$. Then the set $\mathfrak{M}(S)$ of all means on $M(S)^*$ is a weak* closed subsemigroup of $M(S)^{**}$. By Goldstine theorem, the embedding of $P(S)$ in $M(S)^{**}$ is weak* dense in $\mathfrak{M}(S)$. We say a mean $m \in M(S)^{**}$ is topological left invariant, if $\mu * m = m$ for all $\mu \in P(S)$. Let $n \in \mathfrak{M}(S)$. If m is a topological left invariant mean, then by Remark 2.1.1, $n * m = m$, while $m * n$ is again a topological left invariant mean. We let $\mathfrak{M}_l(S)$ denote the set of topological left invariant means on $M(S)^*$. It is easy to see that $\mathfrak{M}_l(S)$ is a weak* compact right ideal of $\mathfrak{M}(S)$ consisting of right zeros.

Let A be a Banach algebra, X is a Banach A-bimodule. A bounded linear map $D : A \to X$ is a derivation if $D(ab) = aD(b) + D(a)b$. A derivation $D : A \to X$ is inner if there is $x \in X$, such that $D(a) = ax - xa$.

B. E. Johnson proved in [29, Theorem 2.5] that locally compact group G is (topological) left amenable if and only if and only if for any Banach $L_1(G)$-bimodule X, any bounded derivation $D : L_1(G) \to X^*$ is inner. This is a groundbreaking result for abstract harmonic analysis and is the motivation to define amenability for Banach algebras. We say a Banach algebra A is left amenable if for any any Banach A-bimodule X, any bounded derivations $D : A \to X^*$ is inner.

However, this result does not hold for semigroups. The reason for the failure not only lies in the lack of proper definition for L_1 functions in general. For example, the discrete semigroup $\mathbb{N}$ with addition is a left amenable semigroup since $\mathbb{N}$ is abelian ([12]). However, $l_1(\mathbb{Z})$ is not left amenable as a Banach algebra.

A. T.-M. Lau [31] generalized the result of Johnson to make sure similar result holds for locally compact semitopological semigroups. Actually, Lau's result works not only for $M(S)$ but also for a class of Banach algebras called F-algebras or Lau-algebras in some literature. A Banach algebra A is an F-algebra if A is the predual of some W^*-algebra with an identity e and e is a character of A. An F-algebra A is left amenable if for any Banach A-bimodule X such that $a \cdot x = e(a)x$ for $a \in A$, $x \in X$, any bounded derivation $D : A \to X^*$ is inner. The group algebra $L_1(G)$ is an F-algebra since it is the predual of $L_\infty(G)$. By Remark 2.2.2, we see that $M(S)$ is also an F-algebra. Similar to topological left invariant means defined on $M(S)$, we can define topological left invariant means on any F-algebra. Lau [31] showed that for any F-algebra A, A has a topological left invariant mean if and only if A is left amenable as an F-algebra. We say this is a generalization of Johnson's result because of the following theorem.

Theorem 2.3.1. [32, Corollary 2.3] Let G be a locally compact group. Then the following are equivalent.

(i) G is (topological) left amenable.

(ii) $L_1(G)$ is left amenable as a Banach algebra.

(iii) $L_1(G)$ is left amenable as an F-algebra.

(iv) $M(G)$ is left amenable as an F-algebra.

The following gives a list of characterizations of topological left amenability on $M(S)^*$ that appeared in the literature.

Theorem 2.3.2. [31, 46, 47] Let S be a locally compact semitopological semigroup. Then the following are equivalent.

(a) $M(S)^*$ has a topological left invariant mean.

(b) There is $\phi \in GL(S)^*$, such that $\|\phi\| = \phi(e) = 1$ and $\phi(f * \mu) = \phi(f)$ for $f \in GL(S)$. Recall e is an identity of $GL(S)$.

(c) There is a net μ_α in $P(S)$ such that $\|\nu * \mu_\alpha - \mu_\alpha\| \to 0$ for each $\nu \in P(S)$. Such a net is called an (LSP) net (see [15]).

(d) For each $F \in M(S)^*$, $\overline{\{\mu * F;\ \mu \in P(S)\}}^{weak^*}$ contains a constant functional.

(e) $M(S)$ is left amenable as an F-algebra

(f) For each $\mu \in M(S)$, $|\mu(S)| = \inf\{\|\mu * \nu\|;\ \nu \in P(S)\}$.

Proof. The equivalence of (a) and (b) is proved in [47]; the equivalence of (a), (c), (d), (f) is proved in [46]; the equivalence of (a) and (e) was proved in [31]. □

Remark 2.3.3. 1. Actually, a mean on $M(S)^*$ is a topological left invariant mean if and only if it is the weak* limit of an (LSP) net. Suppose $m \in \mathfrak{M}(S)$ is the weak* limit of an (LSP) net (μ_α). Then $v * m - m = \text{weak-}\lim_\alpha (v * \mu_\alpha - \mu_\alpha) = 0$ for all $\nu \in P(S)$. Thus m is a topological left invariant mean on $M(S)^*$. On the other hand, suppose m is a topological left invariant mean on $M(S)^*$. By Goldstine theorem, there is a net (μ_α) in $P(S)$, such that weak*-$\lim_\alpha \mu_\alpha = m$. Then let $f \in M(S)^*$, $F(\nu * \mu_\alpha - \mu_\alpha) \to \nu * m(F) - m(F) = 0$. Thus $\nu * \mu_\alpha - \mu_\alpha \xrightarrow{\text{weakly}} 0$ for all $\nu \in P(S)$. Fix $\nu \in P(S)$, let $N_\nu = \overline{\text{co}}^{\|\cdot\|}\{\nu * \mu_\alpha - \mu_\alpha\}$. Then $0 \in N_\nu$. Hence there is a net $\sigma_\beta \in \text{co}\{\mu_\alpha\} \subset P(S)$, such that $\|\nu * \sigma_\beta - \sigma_\beta\| \to 0$ for each

$\nu \in P(S)$. Note that $\sigma_\beta \xrightarrow{\text{weak*}} m$. By applying the ideal of the proof of [12, Theorem 1], we know that there is a net ν_γ, such that weak*-$\lim \nu_\gamma = m$ and $\|\nu * \nu_\gamma - \nu_\gamma\| \to 0$ for all $\nu \in P(S)$.

2. The condition (e) in the above theorem is known as topological right stationary property, which is important in the theory of amenability in general. Besides (LSP) nets, the nets that converge to left invariance uniformly are also studied in [15]. Such nets are called (LSU) nets, as in uniformly convergence to left invariance on compact sets. We say a locally compact semitopological semigroup is uniformly topological left amenable (U-TLA) if it has an (LSU) net. The following gives two equivalent definitions of (LSU) nets, whose equivalence was proved in [15].

A net (μ_α) in $P(S)$ is an (LSU) net if for each compact sets K in S, either,

(a) $\|\delta_s * \mu_\alpha - \mu_\alpha\| \to 0$ uniformly over $s \in K$; or

(b) $\|\nu * \mu_\alpha - \mu_\alpha\| \to 0$ uniformly for $\nu \in P(S)$ supported on K.

Theorem 2.3.4. [15] (i) If (μ_α) is an (LSP) net, then convoluted on the left by any probability measure or on the right by any net of probability measure still returns an (LSP) net.

(ii) If (μ_α) is an (LSU) net, then convoluted on the left by any point mass or on the right by any net of probability measure still returns an (LSU) net.

(iii) Suppose there is $\nu \in P(S)$ such that the map $s \mapsto \delta_s * \nu$ is norm continuous. If (μ_α) is an (LSP) net, then $(\nu * \mu_\alpha)$ is an (LSU) net.

It is interesting to note that the condition in (iii) is automatically satisfied for locally compact groups. Let G be a locally compact group. Let $M_a(G) =$

$\{\mu \in M(S);\ g \mapsto \delta_g * \mu$ is norm continuous.$\}$. Then $M_a(G)$ is a closed ideal in $M(G)$ and $M_a(G)$ is isometrically isomorphic with $L_1(G)$ ([26, Theorem 19.18]). Later we shall study such left translation continuous measures in Chapter 3.

2.4 Lumpy Subsets

In this section, we look at Borel subsets that approximately supports (LSP) and (LSU) nets and their translates. To be precise, we say a Borel subset T is,

(a) topological left lumpy (TLL), if for each $\epsilon > 0$ and $\nu \in P(S)$, there is $\mu \in P(S)$, such that $\nu * \mu(T) > 1 - \epsilon$;

(b) topological left thick (TLT), if for each $\epsilon > 0$ and compact subset $K \subset S$, there is $\mu \in P(S)$, such that $\nu * \mu(T) > 1 - \epsilon$ for all $\nu \in P(S)$ supports on K, i.e. supp$\nu \subseteq K$.

Day [15] extensively studies the equivalent definitions of TLL and TLT. Here we only include the extreme ones to use exchangeable with our definitions above. A Borel subset T is,

(a$'$) TLL, if for each $\epsilon > 0$, $\nu \in P(S)$, there is $s \in T$, such that $\nu * \delta_s(T) > 1 - \epsilon$.

(b$'$) TLT, if for each $\epsilon > 0$ and compact subset $K \subset S$,there is $\mu \in P(S)$ with compact support and $\mu(T) = 1$, such that $\nu * \mu(T) > 1 - \epsilon$ for all $\nu \in P(S)$ supports on K.

Theorem 2.4.1. [15, Section 5 and 7]

(i) Suppose S is topological left amenable, then a Borel subset is TLL if and only if there is a topological left invariant mean m on $M(S)^*$ such that $m(\chi_T) = 1$. Here χ_T is the characteristic functional of T on $M(S)$, that is, $\chi_T(\mu) = \mu(T)$ for $\mu \in M(S)$.

(ii) Suppose S has an (LSP) net, then a Borel subset is TLL if and only if there is an (LSP) net (μ_α) in P(S) such that $\mu_\alpha(T) = 1$.

(iii) Suppose S has an (LSU) net, then a Borel subset T is TLT if and only if there is an (LSU) net (μ_α) in P(S) such that $\mu_\alpha(T) = 1$.

(iv) If S has an (LSU) net, then a Borel subset is TLL if and only if it is TLT.

(v) If there is $\mu \in P(S)$, such that $s \mapsto \delta_s * \mu$ is norm continuous, then a Borel subset is TLT if and only if it is TLL.

Remark 2.4.2. 1. When S is discrete, it is easy to check that a subset T is TLL if and only if T is TLT if and only if for each finite subset $F \subseteq S$, there is $t \in T$, such that $Ft \subseteq T$. The last condition is actually the definition of left thick subsets, from which Day [14] and Wong [49] generalize the concept to define TLL and TLT subsets respectively. In general, if a Borel subset is TLL or TLT, then it is left thick. Left thick subsets for discrete semigroups were studied in [33] and is shown to be closely related to the hereditary property of left amenability. We shall see the hereditary property in details in the next section.

2. The set of TLL or TLT subsets is closed under union by definition. It is also clear that any Borel left ideal of S is a TLT subset and hence a TLL subset. When S is compact, we have the following closed relations between left thick subsets and left ideals.

Proposition 2.4.3. Let S be a compact semitopological semigroup, a closed subset $T \subseteq S$ is left thick if and only if T contains a left ideal.

Proof. By the above remark, we know that if T contains a left ideal, then T is a TLL or TLT subset, hence T is left thick.

Assume T is left thick, then by definition $\{s^{-1}T;\ s \in S\}$ has finite intersection property. Since T is closed, we know that $\{s^{-1}T^c;\ s \in S\}$ cannot form an open cover of S. Otherwise, by the compactness of S, there is a finite subset $\{s_i\}_{i=1}^n$ in S, such that $S = \cup_{i=1}^n s_i^{-1}T^c$. Hence $\cap_{i=1}^n s_i^{-1}T = \emptyset$, which is impossible. Hence there is $a \in S$, such that $Sa \subset T$. □

The relationship between left ideals and left thick subsets of discrete semigroups is studied in [44].

Theorem 2.4.4. [44, Lemma 5.1] Let T be a subset of a discrete semigroup S. Let βS be the Stone-Čech compactification of S. Then T is left thick in S if and only if the closure of T in βS contains a left ideal of βS.

Wong [51] later generalizes this result to locally compact semitopological semigroups.

Theorem 2.4.5. [51] Let S be a locally compact semitopological semigroup. Let T be a Borel subset of S. Recall $\mathfrak{M}(S)$ is the set of all means on $M(S)^*$. Let $\mathfrak{M}_T(S) = \{m \in \mathfrak{M}(S);\ m(\chi_T) = 1\}$. Then the following are equivalent.

(i) T is TLL.

(ii) $\mathfrak{M}_T(S)$ is a left ideal of $\mathfrak{M}(S)$.

(iii) $\mathfrak{M}_T(S)$ is left thick in $\mathfrak{M}(S)$.

We see that Proposition 2.4.3 gives an alternative proof for (iii) $\Longleftrightarrow$ (ii) in the above theorem.

2.5 Locally Compact Subsemigroups

In literature, we often care if a property can be passed down to a subset or if a property on a subset can be extended to the whole set. In our case, we look at locally compact subsemigroups to see when left amenability of the original semigroup can be passed down and when we can extend it from subsemigroups. The theory was first studied in [12, 33, 44] for discrete semigroups.

Theorem 2.5.1. (1) [12, Theorem 2] Let T be a subsemigroup of a discrete left amenable semigroup S. Suppose if there is a left invariant mean m on $l_\infty(S)$, such that $m(\xi_T) > 0$, then T is left amenable. Here ξ_T is the characteristic function of T.

(2) [33, Theorem 9] Let S be a discrete semigroup and T is a left thick subsemigroup of S. Then T is left amenable if and only if S is amenable.

Let S be a discrete semigroup. Let $T \subseteq S$ be a subsemigroup. We say S is left T-amenable if $l_\infty(S)$ has a left T-invariant mean m, i.e., $m(l_t f) = m(f)$ for $t \in T$, $f \in l_\infty(S)$. C. Wilde and K. Witz generalized (1) in the above theorem as follows.

Theorem 2.5.2. [44, Theorem 3.1] Let T be a subset of a discrete semigroup S. Then T is left amenable if and only if there is a left T-invariant mean m such that $m(\xi_T) > 0$.

In literature, Wong [50] is the first to consider hereditary property for topological left amenability on locally compact semitopological semigroups. He completely generalized (2) of Theorem 2.5.1 in a later paper.

Theorem 2.5.3. [51, Theorem 5.2] Let S be a locally compact semitopological semigroup. Let T be a locally compact Borel subsemigroup S that is TLL. Then T is topological left amenable if and only if S is topological left amenable.

However, as he points out in the remark on [50, p.309], the complete generalization of Theorem 2.5.2 is left open. We shall give a proof of the complete generalization in Chapter 4. The author has published the result in [27].

2.6 Some Notations

Throughout the thesis, we shall adopt the following notations depending on the context.

Suppose A, B are subsets of a semitopological semigroup S.

$M(S)$: the Banach algebra of complex Radon measures on S;

δ_s: the point evaluation measure at point $s \in S$;

suppμ: the support of a measure μ;

$\overline{A}$: the closure of A in S;

$AB = \{st;\ s \in A, t \in B\}$;

$A^{-1}B = \{t \in S;\ st \in B \text{ for some} s \in A\}$;

ξ_A: characteristic function of A on S;

χ_A: characteristic function of A on $M(S)$.

Suppose A is a subset of a locally convex space with topology τ.

$\overline{A}$: closure of A with respect to τ;

$\mathrm{co}(A)$: convex hull of A;

$\overline{\mathrm{co}}^{\tau}(A)$: closed convex hull of A

Suppose E is a Banach space.

E^* : the continuous dual of E.

$\langle x, f\rangle = \langle f, x\rangle = f(x)$, where $x \in E$, $f \in E^*$.

Chapter 3

Structure of L(S)

3.1 Introduction

In this section, we look at the left translation continuous measures. Throughout the section, S is a locally compact semitopological semigroup, G is a locally compact group unless otherwise specified.

A measure $\mu \in M(S)$ is left translation continuous if the left translation map $s \mapsto \delta_s * |\mu|$ of S into $M(S)$ is weakly continuous. Let $L(S)$ to be the set of left translation continuous measures. If S is a locally compact group, $L(S)$ is exactly the group algebra $L_1(S)$. When S is a locally compact topological semigroup, $L(S)$ has been studied extensively in the literature, especially in [3, 4, 5, 17, 37]. Their approaches depend heavily on the joint continuous multiplications on semigroups. We generalize their results on the structure of $L(S)$ to separately continuous setting.

We begin with the equivalent definitions of left translation continuous measures that we shall use interchangeably later. We discuss a particular case when the total variation in the definition can be removed. In section 3.3, we show that $L(S)$ is closed under absolute continuity, that is, if $\mu \in L(S)$

and $\nu \ll \mu$, then $\nu \in L(S)$. The result leads to the decomposition of $M(S)$ into $L(S)$ and its orthogonal complement. In section 3.4, we see that $L(S)$ is a closed two-sided ideal. These two properties of $L(S)$ are crucial in the study of group algebras. Baker mentioned in his survey paper [6] that unable to show these two properties is the main hurdle to argue $L(S)$ is a suitable generalization of group algebra for semitopological semigroups. In Section 3.5, we look at the support of $L(S)$, a subset of S that we call the foundation of S in literature. The structure of the foundation is believed to be closely related to the structure of $L(S)$. We study when $L(S)$ has an identity and bounded approximation identity in Section 3.6. In Section refnormcts, we shows the convolutions of any two measures in $L(S)$ is left norm translation continuous, i.e, the map $s \mapsto \delta_s * \nu * \mu$ is norm continuous, given $\nu, \mu \in L(S)$. Finally, in Section 3.8, we characterize $L(S)$ for a particular class of compact semitopological semigroups and list some examples.

3.2 Equivalent Definitions

Lemma 3.2.1. [22] Let X and Y be locally compact spaces. Suppose f is a complex valued bounded separately continuous function on $X \times Y$. Then for $\mu \in M(S)$, the map $y \mapsto \int f(x,y)d\mu(x)$ is continuous.

From this result, we have a direct corollary that is very important in this chapter.

Corollary 3.2.2. If S be a locally compact semitopological semigroup, the map $s \mapsto \delta_s * \mu$ is weak* continuous. In particular, if K is a compact subset of S, then the set $\{\delta_s * \mu;\ s \in K\}$ is weakly closed.

Lemma 3.2.3. [19, 4.22.1] Let X be a locally compact Hausdorff space. Then a bounded sequence μ_n in $M(X)$ is weakly convergent if and only if $\mu_n(F)$ is convergent to a finite limit for each closed set subset F of X.

The following theorem is adapted from [4]. This result offers four equivalent definitions for $L(S)$ that we use interchangeably later.

Theorem 3.2.4. Let $\mu \in M(S)$. The following statement are equivalent.

(i) If K is a compact subset of S, $\{\delta_s * \mu;\ s \in K\}$ is weakly compact.

(ii) If U is an open subset of S that is relatively compact, then $\{\delta_s * \mu;\ s \in U\}$ is relatively weakly compact.

(iii) The map $S \to M(S) : s \mapsto \delta_s * \mu$ is weakly continuous.

(iv) If F is a closed subset of S, the map $S \to \mathbb{C} : s \mapsto \delta_s * \mu(F)$ is continuous.

Proof. Clearly, (iii) $\Longrightarrow$ (i) and (iii) $\Longrightarrow$ (iv). Since S is locally compact, (i) is equivalent to (ii).

(iv) $\Longrightarrow$ (i) Assume the map $s \mapsto \delta_s * \mu(F)$ is continuous. Let $\delta_{s_n} * \mu$ be a sequence in $\{\delta_s * \mu;\ s \in K\}$. By Eberlien's Theorem, it sufficed to show $\delta_{s_n} * \mu$ is weakly convergent. Since s_n is a sequence in K, there exists $s \in K$, such that $s_n \to s$. By (iii), $\delta_{s_n} * \mu(F) \to \delta_s * \mu(F)$ for each closed subset F of S. Hence by Lemma 3.2.3, we show $\delta_{s_n} * \mu$ is weakly convergent.

(i) $\Longrightarrow$ (iii) Let $s_\alpha \to s$ be a net in S. Let $K = \overline{\{x_\alpha\}}$. Then K is compact. By (i), $\{\delta_s * \mu;\ s \in K\}$ is weakly compact. By Corollary 3.2.2, $\{\delta_s * \mu;\ s \in K\}$ is weak* compact and Hausdorff, which implies weak topology and weak* topology coincide on the set. Therefore, $\delta_{s_\alpha} * \mu \xrightarrow{\text{weak}} \delta_s * \mu$. □

Corollary 3.2.5. If $\mu \in L(S)$, then the map $s \mapsto \delta_s * \mu$ is weakly continuous.

Proof. If $\mu \in L(S)$, then the set $\{\delta_s * |\mu|;\ s \in K\}$ is weakly compact. Hence by Theorem 2.1.2, there is a positive measure $\lambda \in M(S)$, such that $\delta_s * |\mu| \ll \lambda$, hence $\delta_s * \mu \ll \lambda$ for $s \in K$. That is, $\{\delta_s * \mu;\ s \in K\}$ is weakly compact. Hence, the map $s \mapsto \delta_s * \mu$ is weakly continuous. □

When S is a locally compact group, it is well-known that $s \mapsto \delta_s * \mu$ is weakly continuous if and only if $s \mapsto \delta_s * |\mu|$ is weakly continuous. Thus we can remove the $|\cdot|$ in the definition of $L(S)$ given that S is a locally compact group. However, it is generally not true for semigroups. Baker [5, P692] gives an example such that $s \mapsto \delta_s * \mu$ is weakly continuous but $\delta_s \mapsto \delta_s * |\mu|$ is not. The same paper also shows that if S is two-sided cancellative and with jointly continuous multiplication, then the weak continuity of the two maps are equivalent. We give another condition when the equivalence holds.

Proposition 3.2.6. If S has a dense subgroup, then

$$L(S) = \{\mu \in M(S);\ s \mapsto \delta_s * \mu \text{ is weakly continuous.}\}$$

Note that a semigroup with a dense subgroup might not be two-sided cancellative. It is known that a compact semitopological semigroup is two-sided cancellative if and only if it is a group (e.g., [8, 3.14]). It is also known that a compact semitopological semigroup with a dense subgroup is not necessarily a group (e.g., the weakly almost compactification of a group is such a semigroup). Thus our result complements that of [5].

Proof. Let G be a dense subgroup of S with identity e. Let U be an open subset of S. Let $x \in \overline{U}$, for each open neighborhood O_x of x, we have $O_x \cap U \neq \emptyset$.

Let $y \in O_x \cap U \neq 0$. In particular, $O_x \cap U$ is a neighborhood of y. Hence $O_x \cap U \cap G \neq \emptyset$, since G is dense in S. Thus $\overline{U} = \overline{U \cap G}$.

Now let $g \in G$, $\mu \in M(S)$. Since the multiplication on S is separately continuous, e is an identity of S. Hence $\{gF_i\}_{i=1}^n$ is a partition of a Borel subset E, if and only if $\{F_i\}_{i=1}^n$ is a partition of $g^{-1}E$. Then,

$$\begin{aligned}\delta_g * |\mu| (E) &= \sup \left\{ \sum |\mu(F_i)| ;\ F_i \text{ is a partition of } g^{-1}E \right\} \\ &= \sup \left\{ \sum |\delta_g * \mu(gF_i)| ;\ F_i \text{ is a partition of } g^{-1}E \right\} \\ &= |\delta_g * \mu| (E)\end{aligned}$$

Suppose U is relatively compact and the map $s \mapsto \delta_s * \mu$ is weakly continuous. Then $\{\delta_s * \mu;\ s \in U \cap G\}$ is relatively weakly compact. By Theorem 2.1.2(iv), the set $\{|\delta_s * \mu| = \delta_s * |\mu| ;\ s \in U \cap G\}$ is also relatively weakly compact. Recall the map $s \mapsto \delta_s * |\mu|$ is always weak* continuous. Then by the fact that weak* and weak topology coincide on relatively weak compact sets, $\left\{\delta_s * |\mu| ;\ s \in \overline{U}\right\} = \overline{\{\delta_s * |\mu| ;\ s \in U \cap G\}}^{\text{weak}}$ is weakly compact. Therefore $\mu \in L(S)$. □

3.3 Absolute Continuity

In order to prove $L(S)$ is closed under absolute continuity for a locally compact semitopological semigroup S. We need the following measure theoretic results. For a Borel subset A of S, let ξ_A be the characteristic function of A.

Lemma 3.3.1. [26, §11] Let $\mu \in M(S)^+$. Then for any open subset $U \subseteq S$,

$$\mu(U) = \sup \left\{ \int g d\mu;\ g \in C_c(S)^+,\ g \leq \xi_U \right\}$$

For any Borel subset $E \subseteq S$,

$$\mu(E) = \inf \left\{ \int g d\mu;\ \xi_E \leq f \text{ and } f \text{ is lower semicontinuous} \right\}$$

Lemma 3.3.2. Let $\mu \in M(S)^+$, let F be a closed subset of S, then

$$\mu(F) = \inf \left\{ \int f d\mu;\ f \in C(S)^+, f \geq \xi_F \right\}$$

Proof. By Lemma 3.3.1,

$$\begin{aligned}
\mu(F) &= \mu(S) - \mu(S \backslash F) \\
&= \mu(S) - \sup \left\{ \int f d\mu;\ f \in C_c(S)^+,\ f \leq \xi_{S \backslash F} \right\} \\
&= \inf \left\{ \int (1-f) d\mu;\ f \in C_c(S)^+,\ f \leq \xi_{S \backslash F} \right\} \\
&\geq \inf \left\{ \int f d\mu;\ f \in C(S)^+,\ f \geq \xi_F \right\} \\
&\geq \inf \left\{ \int f d\mu;\ f \geq \xi_F \text{ is lower semicontinuous} \right\} = \mu(F)
\end{aligned}$$

□

Lemma 3.3.3. Let $\mu \in M(S)^+$, let F be a closed subset of S, then $s \mapsto \delta_s * \mu(F)$ is upper semicontinuous.

Proof. By Lemma 3.3.2, $\delta_s * \mu(F) = \inf \left\{ \int f(st)\ d\mu(t);\ f \in C(S)^+,\ f \geq \xi_F \right\}$. Then by Lemma 3.2.2, the function $s \mapsto \delta_s * \mu(F)$ is the pointwise infimum of a net of continuous functions. Hence $s \mapsto \delta_s * \mu(F)$ is upper semicontinuous. □

Lemma 3.3.4. Let $\mu \in M(S)^+$, let F be a closed subset of S. Then for any convergent net $s_\alpha \to s_0$,

$$|\delta_{s_\alpha} * \mu(F) - \delta_{s_0} * \mu(F)| \to 0 \iff \mu(s_\alpha^{-1}F\Delta s_0^{-1}F) \to 0$$

Proof. "$\Leftarrow$" is clear. Assume $|\delta_{s_\alpha} * \mu(F) - \delta_{s_0} * \mu(F)| \to 0$. Let $\epsilon > 0$, there exists α_0, such that $\left|\mu(s_\alpha^{-1}F) - \mu(s_0^{-1}F)\right| < \epsilon$ whenever $\alpha > \alpha_0$. By Lemma 3.3.3, there exists $\alpha_1 > \alpha_0$ such that,

$$\mu(s_\alpha^{-1}F \backslash s_0^{-1}F) = \delta_{s_\alpha} * \mu\big|_{s\backslash s_0^{-1}F}(F) < \delta_{s_0} * \mu\big|_{s\backslash s_0^{-1}F}(F) + \epsilon = \epsilon$$

Therefore

$$\begin{aligned}\mu(s_\alpha^{-1}F\Delta s_0^{-1}F) &= \mu(s_\alpha^{-1}F\backslash s_0^{-1}F) + \mu(s_0^{-1}F\backslash s_\alpha^{-1}F)\\ &= 2\mu(s_\alpha^{-1}F\backslash s_0^{-1}F) + \mu(s_0^{-1}F) - \mu(s_\alpha^{-1}(F)) < 3\epsilon\end{aligned}$$

□

Now we are ready to prove the main result of this section.

Theorem 3.3.5. Let $\mu \in L(S)$, let $\nu \in M(S)$. If $\nu \ll \mu$, then $\nu \in L(S)$.

Proof. Since $\mu \in L(S)$ if and only if $|\mu| \in L(S)$. Thus without loss of generality, we assume μ is positive.

Let $\nu \ll \mu$. Then for each $\varepsilon > 0$, there exists $\delta > 0$, such that $|\nu|(E) < \varepsilon$ whenever $\mu(E) < \delta$ for some Borel subset E of S. Let $s_\alpha \to s_0$ be a net in S. Since, $\mu \in L(S)$, by Lemma 3.2, $|\delta_{s_\alpha} * \mu(F) - \delta_{s_0} * \mu(F)| \to 0$. Then by Lemma 3.3.4, $\mu(s_\alpha^{-1}F\Delta s_0^{-1}F) \to 0$. Hence there exists α_0 such that $\mu(s_\alpha^{-1}F\Delta s_0^{-1}F) < \delta$ and $|\nu|(s_\alpha^{-1}F\Delta s_0^{-1}F) < \varepsilon$. Therefore, $\delta_{s_\alpha} * |\nu| \to \delta_{s_0} * |\nu|$. Since s_α is an arbitrary convergent net in S, we have $\nu \in L(S)$. □

A direct implication from the absolute continuity of $L(S)$ is the restriction of a left translation continuous measure is still left translation continuous. Let $\mu \in M(S)$, let B be a Borel subset of S. We denote the measure μ restricted on B as $\mu|_B$, that is, $\mu|_B(E) = \mu(B \cap E)$ for Borel subsets $E \subseteq S$.

Corollary 3.3.6. Let B be a Borel subset of S. If $\mu \in L(S)$, then $\mu|_B \in L(S)$.

If we equip $M(S)$ with the natural order, that is, $\nu \leq \mu$ if $\nu(E) \leq \mu(E)$ holds for all Borel subset E in S, then the measure algebra $M(S)$ is a Banach lattice under this order.

Lemma 3.3.7. $L(S)$ is a sublattice of $M(S)$.

Proof. Let $\nu, \mu \in L(S)$, let $a, b \in \mathbb{C}$. Then from definition of $L(S)$, it is clear that $|a\mu| + |b\nu| \in L(S)$. Since $a\mu + b\nu \ll |a\mu| + |b\nu|$, by Theorem 3.3.5, $L(S)$ is a subspace of $M(S)$. Let $\omega = |\nu| + |\mu|$. By Radon-Nykodym Theorem, $\nu = f\omega$, $\mu = g\omega$ for some $f, g \in L_1(\omega)$. Note that $\nu \vee \mu = (f \vee g)\omega$, $\nu \wedge \mu = (f \wedge g)\omega$. By Theorem 3.3.5 again, $\nu \vee \mu,\ \nu \wedge \mu \in L(S)$. □

Since $L(S)$ is closed under absolute continuity, it is easy to see that $L(S)$ is also closed under $\leq$, that is, $L(S)$ is an order ideal in $M(S)$. Since $M(S)$ is order continuous, if we can show $L(S)$ is norm continuous, we may decompose $M(S)$ in terms of the direct sum of $L(S)$ and its orthogonal complement $L(S)^{\perp}$. For details on the theory of Banach lattices, please see [36].

Theorem 3.3.8. $L(S)$ is a norm closed order ideal of $M(S)$ and $M(S) = L(S) \oplus L(S)^{\perp}$.

Proof. From the previous remark, it suffices to show $L(S)$ is norm closed in

$M(S)$. Let μ_α be a net in $L(S)$ such that $\mu_\alpha \xrightarrow{\|\cdot\|} \mu$ for some $\mu \in M(S)$. For each $\varepsilon > 0$, there exists α_ε such that $\|\mu - \mu_{\alpha_\varepsilon}\| < \varepsilon$.

Let $F \in M(S)^*$. Suppose $s_\beta \to s_0$ is a net in S. Since $\mu_{\alpha_\varepsilon} \in L(S)$, there exists β_ε such that $\left|F(\delta_{s_\beta} * \mu_{\alpha_\varepsilon} - \delta_{s_0} * \mu_{\alpha_\varepsilon})\right| < \varepsilon$ whenever $\beta > \beta_\varepsilon$. Hence

$$\begin{aligned}\left|F(\delta_{s_\beta} * \mu - \delta_{s_0} * \mu)\right| &\leq \left|F(\delta_{s_\beta} * \mu - \delta_{s_\beta} * \mu_{\alpha_\varepsilon})\right| + \left|F(\delta_{s_\beta} * \mu_{\alpha_\varepsilon} - \delta_{s_0} * \mu_{\alpha_\varepsilon})\right| \\ &\quad + |F(\delta_{s_0} * \mu_{\alpha_\varepsilon} - \delta_{s_0} * \mu)| \\ &\leq 2\,\|\mu - \mu_{\alpha_\varepsilon}\| + \left|F(\delta_{s_\beta} * \mu_{\alpha_\varepsilon} - \delta_{s_0} * \mu_{\alpha_\varepsilon})\right| \\ &< 3\epsilon\end{aligned}$$

The last inequality hold when $\beta > \beta_\varepsilon$. Therefore $\mu \in L(S)$, hence $L(S)$ is a norm closed order ideal of $M(S)$. □

When S is a locally compact group, it is known that $L(S)^\perp$ contains all discrete measures unless S is discrete. For locally compact semitopological semigroup that is right cancellative, we have the following result.

Proposition 3.3.9. Suppose $L(S)$ contains a discrete measure μ, then there exists $s \in S$ such that $|\mu|(\{s\}) \neq 0$. For each $t \in S$, there exists a neighborhood U_t of t, such that $U_t s = ts$.

Proof. It is clear that $\delta_s \ll \mu$, thus $\delta_s \in L(S)$ since $L(S)$ is closed under absolute continuity. Let $0 < \epsilon < 1/2$, then for every $t \in S$, there exists a neighborhood U_t of t such that $|\delta_z * \delta_s(ts) - \delta_t * \delta_s(ts)| < \epsilon$. Note that $\delta_z * \delta_s(ts) = \delta_{zs}(ts) = 1$ if $zs = ts$ and 0 otherwise. Therefore $U_t s = ts$. □

Corollary 3.3.10. If S right cancellative, then $L(S)$ contains a discrete measure if and only if S is discrete.

Proof. If S is discrete, the statement is trivial since $L(S) = l_1(S) = M(S)$. Assume $L(S)$ contains a discrete measure μ. By Proposition 3.3.9 and the fact that S is right cancellative, we know every point in S is open, which means S is discrete. □

3.4 Ideal Structure

In this section, we show that $L(S)$ is a two-sided ideal of $M(S)$. The results in this section can be found in [5] and [47]. Even though S is assumed to be topological in these papers, the ideas can be modified to suit the semitopological case. We include the proofs here for the sake of completeness.

Lemma 3.4.1. Let $\mu \in L(S)$, then $\delta_t * \mu$ for every $t \in S$, $\mu * \nu \in L(S)$ for every $\nu \in M(S)$.

Proof. Since $\delta_t * (\mu + \nu) = \delta_t * \mu + \delta_t * \nu$ for each $\nu, \mu \in M(S)$, it suffices to show the statement holds for positive measures. Suppose μ is positive.

Let K be a compact subset in S. Since multiplication in S is separately continuous, Kt is compact. Hence $\{\delta_s * \delta_t * \mu;\ s \in K\} = \{\delta_s * \mu;\ s \in Kt\}$ is weakly compact. Thus $\delta_t * \mu \in L(S)$.

For each $F \in M(S)^*$, we know $\nu * F \in M(S)^*$. Since $\mu \in L(S)$, the map $s \mapsto F(\delta_s * \mu * \nu) = \nu * F(\delta_s * \mu)$ is continuous. Hence $\mu * \nu \in L(S)$.

□

Lemma 3.4.2. [47, 4.1] Suppose $\mu \in L(S)$, then $\delta_t * |\mu| \ll |\nu| * |\mu|$ for each $t \in \text{supp}\nu$.

Proof. Suppose the statement does not hold, there exists $t \in \text{supp}\nu$ and a Borel subset E of S, such that $|\nu| * |\mu| = 0$ but $\delta_t * |\mu| = \delta > 0$. Since $s \mapsto \delta_s * |\mu|$ is weakly continuous, there exists a neighborhood U of t, such that $\delta_z * \mu(E) > \delta/2$ for each $z \in U$. Then

$$|\nu| * |\mu| (E) = \int \delta_z * \mu(E) d\nu(z) > \delta/2 \ |\nu| (U) > 0$$

which contradict the fact that $|\nu| * |\mu| (E) = 0$. □

Theorem 3.4.3. $L(S)$ is a two-sided ideal of $M(S)$.

Proof. By Lemma 3.4.1, it suffices to show for each $\mu \in L(S)$, $\nu * \mu \in L(S)$ for all $\nu \in M(S)$. Note that $\nu * \mu \ll |\nu| * |\mu|$, it suffices to show $|\nu| * |\mu| \in L(S)$, since $L(S)$ is closed under absolute continuity. If $L(S) = M(S)$, the proof is obvious. Suppose $L(S) \neq M(S)$. Since $L(S)$ is a norm closed, there exists some nonzero $F \in M(S)^*$ that vanishes only on $L(S)$. The restriction of F on $L_1(|\nu| * |\mu|)$ is $L_\infty(|\nu| * |\mu|)$. Thus there exists $f \in L_\infty(|\nu| * |\mu|)$ such that $F(\sigma) = \int f d\sigma$ for all $\sigma \ll |\nu| * |\mu|$. In particular,

$$F(\nu * \mu) = \int f d\nu * \mu = \iint f(ts) d\mu(s) d\nu(t) \tag{3.1}$$

$$= \int_{\text{supp}\nu} f(z) \ d\delta_t * \mu(z) d\nu(t) \overset{3.4.2}{=\!=} \int F(\delta_t * \mu) d\nu(t) \tag{3.2}$$

Since F vanishes on $L(S)$ and $\delta_t * \mu \in L(S)$, we have $F(\nu * \mu) = 0$. Therefore $\nu * \mu \in L(S)$. □

Actually, by using the proof of Theorem 3.4.3, we can prove a more general result.

Corollary 3.4.4. Let A be a closed subspace of $L(S)$. Then A is a two-sided ideal of $M(S)$ if and only if for $\mu \in A$, $\delta_s * \mu$, $\mu * \delta_s \in A$ for all $s \in S$.

3.5 Support of L(S) and Foundation Semigroup

Assume $L(S)$ is non-trivial. In this section, we look at the subset of S that "supports" left weakly translation continuous measures. Recall for $\mu \in M(S)$, $\text{supp}\mu = \{s \in S;\ |\mu|(U) > 0$ for any open neighborhood U of $S\}$. We define the foundation of S to be

$$\mathcal{F}(S) = \overline{\bigcup_{\mu \in L(S)} \text{supp}\mu}$$

From the definition, we can see that $t \in \mathcal{F}(S)$ if and only if there exists some $\mu \in L(S)$, such that $|\mu|(U) > 0$ for all open neighborhood U of t. This is because if $t \in \mathcal{F}(S)$ and U is a neighborhood of t, then there is $\mu \in L(S)$, such that $U \cap \text{supp}\mu \neq \emptyset$. Thus $\mu(U) > 0$, since U is an open neighborhood of any element in $U \cap \text{supp}\mu$.

Proposition 3.5.1. Let $t \in S$. Then $t \in \mathcal{F}(S)$, if and only if for any neighborhood U of t, there is $\nu \in L(S)$, such that $\text{supp}\nu \subseteq U$.

Proof. The only if case is trivial. Assume $t \in \mathcal{F}(S)$. Let U be a neighborhood of t. By the previous remark, there is $\mu \in L(S)$, such that $|\mu|(U) > 0$. Since S is locally compact, there is a compact subset K such that $t \in K \subseteq U$. Let $\nu = |\mu|\,|_K$. Then $\text{supp}\nu \subset U$. We have $\nu \in L(S)$ since $L(S)$ is closed under absolute continuity. □

The following result is due to Baker [4].

Proposition 3.5.2. $\mathcal{F}(S)$ is a two-sided ideal in S.

Proof. Let $t \in \mathcal{F}(S)$, $s \in S$. Let U be an open neighborhood of st. Then $s^{-1}U$ is an open neighborhood of t. Thus there exists $\mu \in L(S)$, such that $|\mu|(s^{-1}U) > 0$. Hence $\delta_s * \mu(U) > 0$. Since $\delta_s * \mu \in L(S)$, by the remark in the beginning of this section, we know that $st \in \mathcal{F}(S)$. □

We say S is a foundation semigroup if $\mathcal{F}(S) = S$. Then any locally compact group and discrete semigroups are foundation semigroups. To obtain structural information of the underlying semigroup S from that of $L(S)$, foundation semigroup seems to be a reasonable assumption. But how to find foundation semigroups? By Theorem 3.3.5, we know that the restriction of any left translation continuous measure on a closed subsemigroup is still a left translation continuous measure, thus as long as $L(S)$ is non-trivial, $\mathcal{F}(S)$ is a foundation semigroup. The following is a simple corollary of the above remark and Proposition 3.5.2.

Corollary 3.5.3. 1. If S is simple and has a non-trivial $L(S)$, then S is a foundation semigroup.

2. Suppose S has an identity e, then S is a foundation semigroup if and only if for each neighborhood O of e, there is $\mu \in L(S)$, such that $|\mu|(O) > 0$.

Later in Section 3.8, we shall see every compact right simple semigroup is a foundation semigroup.

Proposition 3.5.4. Let O be a Borel subset of S. If there is $\mu \in L(S)$ such that $|\mu|(O) > 0$, then for each $s \in S$, $(sO)O^{-1}$ is a neighborhood of s.

Proof. Without loss of generality, assume μ is positive. Let $\nu = \dfrac{\mu|_O}{\|\mu|_O\|}$. Then $v \in L(S)$, and $\|\nu\| = \nu(O) = 1$. Hence $\delta_s * \nu(sO) = |\nu|(s^{-1}(sO)) = 1$. Since the map $s \mapsto \delta_s * \nu$ is weakly continuous, there exists a neighborhood U of s, such that $\delta_z * \nu(sO) > 0$ for $z \in U$. Thus $z^{-1}(sO) \cap O \neq \emptyset$, whenever $z \in U$. Otherwise $\nu(z^{-1}(sO) \cup O) > 1$. Hence $z \in (sO)O^{-1}$ for $z \in U$. Therefore $(sO)O^{-1}$ is a neighborhood of s. □

With this result, we can have the following result which was shown in [37] for topological semigroups.

Corollary 3.5.5. Let S be a foundation semigroup. Let O be a subset of S with nonempty interior, let $s \in S$, then $(sO)O^{-1}$ is a neighborhood of s.

Proof. Since S is a foundation semigroup, if O is a subset of S with nonempty interior, there is $\mu \in L(S)$, such that $|\mu|(O) > 0$. Then by Proposition 3.5.4, $(sO)O^{-1}$ is a neighborhood of s. □

We say a point $x \in S$ has property α if, for any neighborhood O of x and any $s \in S$, the set $(sO)O^{-1}$ is a neighborhood of s. If $x \in \mathcal{F}(S)$, then automatically x has property α. The following result is an interesting application of property α adapted from [40, 4.2].

Proposition 3.5.6. Let $x \in S$. Suppose,

(1) x has property α;

(2) there is $\mu \in M(S)$, such that the map $s \mapsto \delta_s * \mu$ is norm continuous at x.

Then the map $s \mapsto \delta_{sx} * \mu$ is norm continuous on S.

Proof. Let $s_\alpha \to s_0$ be a net in S. From (1), there exists nets (x_α), (y_α) that converge to x, such that $s_\alpha x_\alpha = s_0 y_\alpha$. Then

$$\begin{aligned}\|\delta_{s_\alpha x} * \mu - \delta_{s_0 x} * \mu\| &= \|\delta_{s_\alpha x} * \mu - \delta_{s_\alpha x_\alpha} * \mu\| + \|\delta_{s_0 y_\alpha} * \mu - \delta_{s_0 x} * \mu\| \\ &\leq \|\delta_x * \mu - \delta_{x_\alpha} * \mu\| + \|\delta_{y_\alpha} * \mu - \delta_x * \mu\|\end{aligned}$$

By (2), the map $s \mapsto \delta_{sx} * \mu$ is norm continuous on S. □

One might wonder if the property α can be removed from the above result. The answer is negative as we can see from the following example.

Example 3.5.7. Let $S = [0, \infty)$ with $st = \max(s, t)$ and the usual topology. Then S is a commutative locally compact semitopological semigroup. Let $0 < a < b < c$, then $c[a, b)[a, b)^{-1} = c$. By Proposition 3.5.4, $|\mu|([a, b)) = 0$, for each $\mu \in L(S)$. However, since $[a, b)$ are arbitrary and the union of such sets can cover S. We know that $L(S) = \{0\}$. In particular, there is no left norm translation continuous measures on S.

On the other hand, let ν be a positive measure in $M(S)$ with norm 1. Let $p = \inf \operatorname{supp}\nu$. Since $\operatorname{supp}\nu$ is closed, $p \in \operatorname{supp}\nu$. If $p > 0$, we claim that the map $s \mapsto \delta_s * \mu$ is norm continuous on $[0, p)$. To see this, let $s < p$ and $E \subseteq S$ be a Borel subset.

If $s \in E$,	then	$\nu(s^{-1}E) = \nu([0, s) \cup E)) = \nu(E)$;
If $s \notin E$,	$s \leq \inf\ E$, then	$\nu(s^{-1}E) = \nu(E)$;
If $s \notin E$,	$\inf E \leq s \leq \sup\ E$, then	$\nu(s^{-1}E) = \nu((s, \infty) \cap E) = \nu(E)$;
If $s \notin E$,	$\sup E \leq s$, then	$\nu(s^{-1}E) = 0$.

Let $t_\alpha \to t_0 < p$. Without loss of generality, we may assume $t_\alpha < p$. If $E \cap \operatorname{supp}\nu = \emptyset$, then $\nu(s^{-1}E) = 0$ for all $s \in [0, p)$. If $E \cap \operatorname{supp}\nu \neq \emptyset$, then

$p \leq \sup E$, hence $\nu(s^{-1}E) = \nu(E)$ for all $s \in [0, p)$. Therefore in both cases, $\left|\nu(t_\alpha^{-1}E) - \nu(t_0^{-1}E)\right| = 0$. Since E is arbitrary, $s \mapsto \delta_s * \nu$ is continuous on $[0, p)$.

3.6 Identity and Bounded Approximate Identity of L(S)

It is well-known that for a locally compact group G, if $L_1(G)$ has an identity, then G has to be discrete. We generalize this result to foundation semigroups in this section as a consequence of the following theorem.

Theorem 3.6.1. Let S be a foundation semigroup. Then $L(S)$ has a positive right identity σ if and only if $\|\sigma\| = 1 = \sigma(x^{-1}x)$ for all $x \in S$.

Proof. Assume $L(S)$ has a positive right identity σ. Then $\mu(S) = \mu * \sigma(S) = \mu(S)\sigma(S)$ for all $\mu \in L(S)$. Thus $\|\sigma\| = \sigma(S) = 1$. Suppose there is $x \in X$, such that $\sigma(x^{-1}x) < 1$. By the outer regularity of $\delta_x * \sigma$, we may find an open neighborhood O of x, such that $\sigma(x^{-1}O) < 1$. Since $s \to \delta_s * \mu$ is weakly compact, there is an open neighborhood U of x, such that $U \subset O$ and $\sigma(s^{-1}O) < 1$ for $s \in U$. Since S is a foundation semigroup, by Proposition 3.5.1 there is a positive measure $\mu \in L(S)$, such that $\text{supp}\mu \subseteq U$. Then

$$\mu(O) = \mu * \sigma(O) = \int_{\text{supp}\mu} \delta_s * \sigma(O) d\mu(s) < \mu(O)$$

Therefore, $\sigma(x^{-1}x) = 1 = \|\sigma\|$ for $x \in S$.

On the other hand, assume $\|\sigma\| = 1 = \sigma(x^{-1}x)$ for all $x \in S$. Let $E \subseteq S$ be a Borel subset. Then $\mu * \sigma(E) = \int_{\text{supp}\mu \cap E} \delta_s * \sigma(E) d\mu(s) = \mu(E)$. □

The above theorem directly implies a well-known result: for a locally compact group G, the group algebra $L_1(G)$ has a (positive) right identity if and only if G is discrete. One might wonder if the similar result holds for locally compact semitopological semigroup. We shall give an affirmative answer in Corollary 3.6.5.

Corollary 3.6.2. If S is a foundation semigroup. If $L(S)$ has a positive right identity σ, then S has a right identity.

Proof. By Theorem 3.6.1, $\bigcap_{x \in S} x^{-1}x \neq \emptyset$, which is the set of right identities. Actually, supp$\sigma \subseteq \bigcap_{x \in S} x^{-1}x$. □

Corollary 3.6.3. If S is a foundation semigroup and has finitely many right identities, then $L(S)$ has a positive right identity if and only if S is discrete.

Proof. Suppose σ is a positive right identity of $L(S)$. By Theorem 3.6.1, the support of σ lies in the set of right identities. Thus if S has finitely many right identities, there is a right identity e, such that $\sigma(\{e\}) > 0$. Hence $\delta_e \in L(S)$. By Theorem 3.3.9, for each $t \in S$, there is a neighborhood U_t of t, such that $U_t = U_t e = te = t$. Therefore S is discrete. □

Note that in the previous results, we assume $L(S)$ has a positive right identity. The reason is we do not know if all right identities of $L(S)$ are positive unless S has a left identity.

Proposition 3.6.4. Let S be a foundation semigroup. If σ is a right identity of $L(S)$, then $\sigma(x^{-1}E) \geq 0$ for $x \in S$ and Borel subset $E \subseteq S$. In particular, if S in addition has a left identity, σ is positive. Actually in this case, S is a discrete semigroup with an identity.

Proof. The proof for this proposition is similar to that of Theorem 3.6.1.

Suppose $\sigma(x^{-1}E) < 0$ for some $x \in S$ and Borel subset E. Then there is an open subset U of x, such that $\sigma(s^{-1}E) < 0$ when $s \in U$. Since S is a foundation semigroup, there is a positive measure $\mu \in L(S)$ such that supp$\mu \subseteq U$. Hence $0 \leq \mu(E) = \mu * \sigma(E) < 0$. Therefore $\sigma(x^{-1}E) \geq 0$ for arbitrary $x \in S$ and Borel subset E.

Suppose S in addition has a left identity e, then for any Borel subset E of S, $\sigma(E) = \sigma(e^{-1}E) \geq 0$. Thus σ is positive. However, the existence of left identity implies that S has only one right identity, which is the identity of S at the same time. By Corollary 3.6.3, S is discrete. □

By using a previous result of this section, we have the following result which was first shown in [5] for locally compact topological semigroups.

Corollary 3.6.5. If S is a foundation semigroup with an identity, then $L(S)$ has an identity if and only if S is discrete.

Proof. Let e be the identity of S. If S is discrete, then $L(S) = M(S)$, hence δ_e is an identity in $L(S)$. Assume $L(S)$ has an identity σ. By Theorem 3.6.1 and Proposition 3.6.4, $\|\sigma\| = 1 = \sigma(e)$. Thus $\sigma = \delta_e$. Then by Proposition 3.3.9, there is a neighborhood U_s of each $s \in S$, such that $U_s = s$. Thus S is discrete. □

Approximate identity is a weaker sense of identity. A net (e_α) is a bounded left approximate identity of $L(S)$ if it is norm bounded and for each $\mu \in L(S)$, $e_\alpha * \mu \xrightarrow{\|\cdot\|} \mu$. It is known that $L_1(G)$ always has a bounded approximate identity. The existence of bounded (left) approximate identity is crucial in the study of group algebra $L_1(G)$. The result was generalized for topological semigroup in [17, 2.10]. Their proof works in the semitopological case. We include the proof here for completeness.

Proposition 3.6.6. If S be a foundation semigroup with left identity e, then $L(S)$ has a bounded left approximate identity.

Proof. Let U_α be a decreasing relatively compact neighborhood of e. Since $e \in \mathcal{F}(S)$, by 3.5.1, there is a positive net e_α in $L(S)$, such that $\mathrm{supp} e_\alpha \subseteq U_\alpha$ and $\|e_\alpha\| = 1$. It is clear that $e_\alpha \xrightarrow{\text{weak}^*} \delta_e$. Let $\mu \in L(S)$ and $F \in L(S)^*$. Then

$$F(e_\alpha * \mu) \overset{(3.2)}{=\!=} \langle e_\alpha, F(\delta_s * \mu)\rangle \to \langle \delta_e, F(\delta_s * \mu) = F(\mu)$$

Thus $e_\alpha * \mu \xrightarrow{\text{weak}} \mu$. By [9, P58], there is a net bounded $f_\alpha \in L(S)$, such that $f_\alpha * \mu \xrightarrow{\|\cdot\|} \mu$. □

3.7 Norm Continuity

As is well known, if S is a locally compact group, the map $s \mapsto \delta_s * \mu$ is weakly continuous if and only if it is norm continuous. However, this is in general not true for locally compact semigroups. Let

$$L^n(S) = \{\mu \in M(S);\ s \mapsto \delta_s * |\mu| \text{ is norm continuous.}\}$$

Sleijpen [38, 2.3] construct a locally compact semigroup with jointly continuous multiplication, such that $L^n(S) \subsetneq L(S)$ and $L^n(S)$ is not closed under absolute continuity.

We shall see later that if $\nu, \mu \in L(S)$, then $s \mapsto \delta_s * \nu * \mu$ is norm continuous. The same result for the topological case can be found in [37, 5.4]. However, their proof depends on joint continuity. We modified their proof to bypass the joint continuity requirement.

Let X and Y be Banach spaces. Let B_X be the unit ball of X. A linear operator $T : X \to Y$ is weakly compact if $T(B_X) = \{Tx;\ x \in B_X\}$ is weakly compact.

Theorem 3.7.1. [13, 4.4.5] If $T : M(S) \to M(S)$ is a weakly compact operator, then T maps relatively weak compact sets to relatively norm compact sets.

Lemma 3.7.2. Let $K \subset S$ be compact. Suppose $s \mapsto \delta_s * \mu$ is weakly continuous, then the map $\rho_\mu : M(K) \to M(S) : \nu \to \nu * \mu$ is weakly compact.

Proof. Let (O_i) be a sequence of open and pairwisely disjoint subsets. Since $\{\delta_s * \mu;\ s \in K\}$ is weakly compact, by Theorem 2.1.2, for each $\varepsilon > 0$, there is $i_0 \in \mathbb{N}$, such that whenever $i > i_0$, we have $|\delta_s * \mu(O_i)| < \varepsilon$ for $s \in K$. Take $\nu \in B_{M(K)}$. Then $|\nu * \mu(O_i)| = \left|\int_K \delta_s * \mu(O_i) d\nu(s)\right| < \epsilon$ if $i > i_0$, which implies $\left\{\nu * \mu;\ \nu \in B_{M(K)}\right\}$ is relatively weak compact. Therefore ρ_μ is weakly compact. □

Theorem 3.7.3. Let $\nu, \mu \in L(S)$, then the map $s \mapsto \delta_s * \nu * \mu$ is norm continuous.

Proof. Let $s_\alpha \to s_0$ be a net in S. Let $K = \overline{\{s_\alpha\}}$. Without loss of generality, we may assume K is compact. Let $\varepsilon > 0$. Since $\{\delta_s * \nu;\ s \in K\}$, there is a compact subset C of S, such that $|\delta_s * \nu|\,(S \backslash C) < \varepsilon$ for $s \in K$. Then

$$\|\delta_s * \nu * \mu - (\delta_s * \nu)|_C * \mu\| \le \|\delta_s * \nu * -(\delta_s * \nu)|_C\|\,\|\mu\| \le \varepsilon\,\|\mu\| \quad (*)$$

By Theorem 2.1.2, the set $\{(\delta_s * \nu)|_C;\ s \in K\}$ is weakly compact, since it is uniformly absolutely continuous with respect to some positive measure. Since $\rho_\mu : M(C) \to M(S)$ is weakly compact by Lemma 3.7.2, the set $\{(\delta_s * \nu)|_C * \mu;\ s \in K\}$ is relative norm compact by Theorem 3.7.1. Thus by passing $\{s_\alpha\}$ to a subnet, $\|(\delta_{s_\alpha} * \nu)|_C * \mu - (\delta_{s_0} * \nu)|_C * \mu\| \leq \varepsilon$ for $\alpha > \alpha_0$. Therefore by (*),

$$\|\delta_{s_\alpha} * \nu * \mu - \delta_{s_0} * \nu * \mu\| < (2\,\|\mu\| + 1)\varepsilon \quad (\alpha > \alpha_0)$$

Thus the map $s \mapsto \delta_s * \nu * \mu$ is norm continuous. □

Since $L(S)$ is closed under total variation, we show $|\nu| * |\mu| \in L^n(S)$, given $\nu, \mu \in L(S)$. However, it is not known if $\nu * \mu \in L^n(S)$. Below are some direct implications of the above theorem.

Corollary 3.7.4. If μ is a positive idempotent in $L(S)$, then $\mu \in L^n(S)$.

Corollary 3.7.5. If $L(S)$ has a bounded left approximation identity, then $L(S) = L^n(S)$. In particular, if S is a foundation semigroup with left identity, then $L(S) = L^n(S)$.

Proof. If $L(S)$ has a bounded left approximation identity, then by Cohen Factorization Theorem (e.g. see [11]), for any $\nu \in L(S)$, there is $\mu, \sigma \in L(S)$, such that $|\nu| = \mu * \sigma$. Then by Theorem 3.7.3, $\nu \in L^n(S)$. The second part of this result follows Corollary 3.6.6. □

3.8 Compact Semitopological Semigroups

In this section, we focus on compact semitopological semigroups S. Compact semitopological semigroup naturally arises from one point compactification and weakly almost periodic compactification of a group. As an application, our result gives a characterization of $L(S)$ on such semigroups. Before that, we introduce another decomposition of the measure algebra.

Let $M_{\text{inv}}(S) = \{\lambda \in M(S);\ \delta_s * \lambda = \lambda\}$. Measures in $M_{\text{inv}}(S)$ are called left invariant measures. Note that if λ is an invariant measure, so is $|\lambda|$. This is because $\delta * |\lambda| \geq |\delta_s \lambda| = |\lambda|$. Thus $\|\delta_s * |\lambda| - |\lambda|\| = (\delta_s * |\lambda| - |\lambda|)(S) = 0$. Let

$$M_{\text{ab}}(S) = \{\mu \in M(S);\ \mu \ll \lambda \text{ for some invariant measure } \lambda\}$$

$$M_{\text{s}}(S) = \{\mu \in M(S);\ \mu \perp \lambda \text{ for all } \lambda \in M_{\text{inv}}(S)\}$$

Proposition 3.8.1. The sets $M_{\text{inv}}(S)$ and $M_{\text{ab}}(S)$ are norm closed ideals of $M(S)$, while $M_{\text{s}}(S)$ is norm closed and invariant under translation of S. Moreover, $M(S) = M_{\text{ab}}(S) \oplus M_{\text{s}}(S)$.

Proof. As we mentioned in the earlier, $M_{\text{inv}}(S)$ is closed under total variation. It is also clear that $M_{\text{inv}}(S)$ is a norm closed subspace of $M(S)$. Let (μ_n) be a Cauchy sequence in $M_{\text{ab}}(S)$. Then there is $\mu \in M(S)$ such that $\mu_\alpha \xrightarrow{\|\cdot\|} \mu$, since $M(S)$ is norm closed. Meanwhile, there is a norm 1 sequence (λ_n) in $M_{\text{inv}}(S)$ such that $\mu_n \ll \lambda_n$. Since $M_{\text{inv}}(S)$ is norm closed, the series $\lambda = \sum_{n=1}^{\infty} \frac{1}{2^n} |\lambda_n|$ exists. Then $\mu_n \ll \lambda$ for all n, hence $\mu \ll \lambda$. Therefore $M_{\text{ab}}(S)$ is norm closed.

To see $M_{\text{inv}}(S)$ and $M_{\text{ab}}(S)$ are ideals, it suffices to see them closed under left and right translations under positive measures in $M(S)$. By the definition of $M_{\text{ab}}(S)$ and the fact that $M_{\text{inv}}(S)$ is closed under total variation. It suffices to show $M_{\text{inv}}(S)$ is an ideal. Let $\nu \in M(S)$ and $\lambda \in M_{\text{inv}}(S)$. It is easy to check that $\nu * \lambda = \lambda$ and $\lambda * \nu$ are left invariant measures. Therefore $M_{\text{inv}}(S)$ and $M_{\text{ab}}(S)$ are norm closed ideals of $M(S)$. As a particular consequence, it is easy to check that $M_s(S)$ is closed under both left and right translations of S.

As for the decomposition, since norm closed order ideals in $M(S)$ are protective bands complemented by their disjoint complements. As the disjoint complement $M_{ab}(S)$ in $M(S)$, we have $M_s(S)$ is norm closed and $M(S) = M_{ab}(S) \oplus M_s(S)$. □

Of course the above decomposition is trivial if $M_{inv}(S) = \{0\}$. When $M_{inv}(S)$ is non-trivial, we can identity $L(S)$ with $M_{ab}(S)$ as we show in the next result.

Theorem 3.8.2. If $M_{inv}(S) \neq \{0\}$, then $L(S) = M_{ab}(S)$.

Proof. The direction $M_{ab}(S) \subseteq L(S)$ is obvious since every left invariant measure is automatically left translation continuous.

Let $\mu \in L(S)$. By Proposition 3.8.1, $\mu = \mu_1 + \mu_2$ where $\mu_1 \in M_{ab}(S)$ and $\mu_2 \in M_s(S)$. If $\mu_2 = 0$, the proof is trivial. Suppose $\mu_2 \neq 0$, since $\mu_2 \perp \mu_1$, we have $\mu_2 \ll \mu$. Hence $\{\delta_s * |\mu_2|;\ s \in S\}$ is weakly compact. Further, $K = \overline{\text{co}}^w \{\delta_s * |\mu_2|;\ s \in S\}$ is weakly compact by Krein-Šmulian theorem. At the same time, the set is contained in $M_s(S)$, since $M_s(S)$ is norm closed and closed under translation of S.

On the other hand, if $M_{inv}(S) \neq 0$, $C(S)$ is left amenable. By Lemma 3.2.1, it is easy to check the map $S \times M(S) \to M(S) : (s, \mu) \to \delta_s * \mu$ is separately continuous when $M(S) = C_0(S)^*$ is equipped with weak* topology. Thus by [34], there is a fixed point λ of S in K. By the fact that $K \subseteq M_s(S)$, we have $\lambda = 0$. Note that $|\mu(S)| = \delta_s * |\mu_2|(S)$. Thus $0 \in K$ implies $|\mu_2|(S) = 0$. Therefore $\mu_2 = 0$ and $\mu \in L(S)$. □

Proposition 3.8.3. If S is a compact semigroup with a dense subgroup, then

$L(S) = M_{ab}(S)$.

Proof. [8, 4.2.15] showed that if S be a semitopological semigroup with a dense subgroup, then $WAP(S)$ is left amenable. When S is compact, $WAP(S) = C(S)$. Hence $M_{\text{inv}}(S)$ is non-trivial by Riesz representation theorem. Therefore by Theorem 3.8.2, $L(S) = M_{ab}(S)$. □

The previous result is based on the assumption that non-trivial left invariant (Borel regular) measure exists. Invariant means on compact semitopological semigroup has been thoroughly studied in the literature. The following gives some characterizations of the existence of left invariant measures on S.

Theorem 3.8.4. Let S be a compact semitopological semigroup. Then the following are equivalent.

(i) S has a left invariant measure.

(ii) $C(S)$ is left amenable.

(iii) S contains a compact left invariant subset F, i.e, $sF = F$ for all $s \in S$.

(iv) S has a unique minimal right ideal.

(v) S is left reversible, that is, the intersection of any two closed right ideal is non-empty.

Proof. (i) and (ii) are equivalent by Riesz representation theorem. The equivalence of (i) and (iii) was shown in [2, 2.2]. The equivalence of (i) and (iv) was shown in [8, 3.14]. The equivalence of (iv) and (v) is trivial. □

As an application of the above results, the next result gives an algebraic description of $\mathcal{F}(S)$ for compact semitopological semigroup S.

Theorem 3.8.5. If $M_{\text{inv}}(S)$ is non-trivial, then $\mathcal{F}(S) = K(S)$, the minimal ideal of S. Moreover, $\mathcal{F}(S)$ is a closed right simple subsemigroup with left identity that is a union of disjoint closed groups.

Proof. From Theorem 3.8.4, we know S has a unique minimal right ideal. Then by [8, Theorem 2.7 and 2.11], $K(S)$ is the minimal right ideal and a union of disjoint closed subgroups, each of which is a minimal left ideal of S. Then the Haar measure on each subgroup canonically extends to invariant measures on S. Hence by Theorem 3.8.2, $K(S) \subseteq \cup\{\text{supp}\lambda;\ \lambda \in M_{\text{inv}}(S)\} = \mathcal{F}(S)$.

On the other hand, let λ be a positive non-zero left invariant measure on S, then $\lambda(S) \geq \lambda(aS) = \delta_a * \lambda(aS) \geq \lambda(S)$ for all $a \in S$. Hence $\text{supp}\lambda \subseteq K(S)$. As λ is arbitrary, $\mathcal{F}(S) \subseteq K(S)$.

Since $\mathcal{F}(S)$ is a minimal right ideal, it has a left identity. Let $a \in \mathcal{F}(S)$, then $a\mathcal{F}(S) \subseteq \mathcal{F}(S) = aS$. Since $\mathcal{F}(S)$ is the disjoint union of minimal left ideals, a might falls in one of them. Since each minimal left ideal is a group, suppose the one that contains a has an identity e. Then $\mathcal{F}(S) = aS = aeS \subset a\mathcal{F}(S)$. Therefore $\mathcal{F}(S) = a\mathcal{F}(S)$ for all $a \in \mathcal{F}(S)$ and thus $\mathcal{F}(S)$ is right simple. □

Corollary 3.8.6. Let S be a compact semitopological semigroup with non-trivial left invariant measures. Then S is a foundation semigroup if and only if it is right simple. In this case, S is topological.

Proof. This result is a direct consequence of Theorem 3.8.5. By [8, 4.6], such S is topological. □

One might wonder if non-trivial $L(S)$ implies the existence of left invariant measures. We do not know the answer, but we were able to loosen the existence of left invariant measures to the existence of left invariant mean on $LUC(S)$

in the presence of non-trivial left translation continuous measure. When S is a locally compact group, the equivalence of left invariant mean on $C(S)$ and $UC(S)$ is a convenient result for the study of amenability on locally compact groups.

Theorem 3.8.7. If $L(S)$ is non-trivial, $UC(S)$ has a left invariant mean if and only if $C(S)$ has a left invariant mean.

Proof. By Theorem 3.7.3, since $L(S)$ is non-trivial, there is non-zero $\mu \in L^n(S)$. Let $F \in M(S)^*$. Define $f(t) = F(\delta_t * |\mu|)$ on S. Suppose $s_\alpha \to s$ is a net in S.

$$\|r_{s_\alpha} f - r_s f\| = \sup_{t \in S} |F(\delta_t * \delta_{s_\alpha} * \mu - \delta_t * \delta_s * \mu)| \leq \|\delta_{s_\alpha} * \mu - \delta_s * \mu\| \to 0$$

Thus $f \in RUC(S)$. Since S is compact, $UC(S) = RUC(S)$.

Let ϕ be the invariant mean on $UC(S)$. Thus there is a net of finite means $\nu_\alpha = \sum_{i=0}^{n_\alpha} \lambda_{\alpha_i} \delta_{s_{\alpha_i}}$ such that $\nu_\alpha \xrightarrow{weak^*} \phi$. Then let $t \in S$,

$$\phi(f) = \lim_\alpha \sum_{i=0}^{n_\alpha} \lambda_{\alpha_i} \langle F, \delta_{s_{\alpha_i} * \mu} \rangle = \langle F, \lim_\alpha \sum_{i=0}^{n_\alpha} \lambda_{\alpha_i} \delta_{s_{\alpha_i} * \mu} \rangle$$
$$\phi(l_t f) = \lim_\alpha \sum_{i=0}^{n_\alpha} \lambda_{\alpha_i} \langle F, \delta_{t s_{\alpha_i} * \mu} \rangle = \langle F, \delta_t * \lim_\alpha \sum_{i=0}^{n_\alpha} \lambda_{\alpha_i} \delta_{s_{\alpha_i} * \mu} \rangle$$

Since $K = \overline{\text{co}}\{\delta_s * |\mu|;\ s \in S\}$ is weakly compact, by passing to a subnet, there is $\sigma = \lim_\alpha \sum_{i=0}^{n_\alpha} \lambda_{\alpha_i} \delta_{s_{\alpha_i} * \mu} \in K$. Thus $\langle F, \sigma \rangle = \langle F, \delta_t * \sigma \rangle$, because $\phi(f) = \phi(l_t f)$ for all $t \in S$. Since F is arbitrary in $M(S)^*$, we have $\delta_t * \sigma = \sigma$. Therefore σ is a left invariant measure and $C(S)$ has a left invariant mean. $\square$

Remark 3.8.8. For a compact semitopological semigroup S, $UC(S)$ has a right invariant mean if and only if $C(S)$ has a right invariant mean (see [20, Corollary 4.11] and [8, Theorem 3.14]). However, the author does not know if the result is true in general for left invariant means.

Example 3.8.9. (a) Let G be a locally compact group. Let $S = G \cup \{\infty\}$ be the one-point compactification of G. Suppose $\lambda \in M_{\text{inv}}(S)$ is a positive left invariant measure on S. Then $\lambda(G) = \delta_g * \lambda(G) = \delta_\infty * \lambda(G) = 0$. Thus λ has to be a multiple of δ_∞. It is easy to check that $\lambda = c\delta_\infty$ is indeed a left invariant measure. Thus $L(S) = \mathbb{C}\delta_\infty$.

(b) Let G be a locally compact group. Let $S = G^w$ be the weakly almost periodic compactification of G. Let $R : G \to G^w$ be the canonical embedding of G to a dense subset of G^w. R is injective since $C_0(G) \subseteq WAP(G)$. Then $R^* : C(G^w) \to WAP(G) : F \mapsto R^*F(x) = F(Rx)$ is isometric and bijective. Hence $R^{**} : WAP(G)^* \to M(G^w) : \phi \mapsto R^{**}\phi(F) = \langle \phi, R^*F \rangle$ is bijective and is invariant under left translations of s. Then S has a left invariant measure λ if and only if $(R^{**})^{-1}(|\lambda| \,/\, \|\lambda\|)$ is a left invariant mean on $WAP(G)$. Note that a left invariant mean on $WAP(G)$ is actually an invariant mean and there is a unique invariant mean on $WAP(G)$ (see e.g. [10]). Therefore, $M_{\text{inv}}(S)$ is spanned by a unique probability invariant measure λ. Thus $\mathcal{F}(S) = K(S) = \text{supp}\lambda$ is a compact group by [8, 3.14] and $L(S) = \{\mu \in M(S);\ \mu \ll \lambda\}$. Note that $K(S)$ as an ideal of S is disjoint with $R(G)$.

(c) Let $G = (\mathbb{C}, +) \rtimes (\mathbb{T}, \cdot)$, the 2-motion group.

The open sets $U_\delta = \{(z, w) \in G;\ |z| < \delta, |\arg w| < \delta\}$ forms a neighborhood basis of the identity $(0, 1)$ in G. Since G is minimal weakly almost periodic, $WAP(G) = C_0(G) \oplus AP(G)$. It can be shown that $AP(G) \simeq C(\mathbb{T})$. To be specific, if $f \in AP(G)$, there exists $g \in C(\mathbb{T})$, such that $f = g \circ q$, where q is the natural projection from G to $\mathbb{T}$. Thus $wG \simeq G \oplus \mathbb{T}$ with semigroup operation "·" defined as

1) If $x, y \in G$, $x \cdot y = xy$ in G.

2) If $x, y \in \mathbb{T}$, $x \cdot y = xy$ in $\mathbb{T}$.

3) If $x \in G$, $y \in \mathbb{T}$, $x \cdot y = q(x)y$ in $\mathbb{T}$, $y \cdot x = yq(x)$ in $\mathbb{T}$.

Therefore $\mathbb{T}$ is an ideal in wG. Thus the canonical extension of normalized Haar measure on $\mathbb{T}$ to wG is the unique probability measure that spans $M_{\text{inv}}(wG)$.

Chapter 4

Topological Left Amenability

4.1 (LSU) nets and L(S)

Previously in Chapter 1, we have seen the following results.

1. (Theorem 2.3.4) If there is $\nu \in P(S)$, such that the map $s \mapsto \delta_s * \nu$ is norm continuous, then S has an (LSU) net if and only if it has an (LSP) net.

2. (Theorem 2.4.1) If S has an (LSU) net or there is $\mu \in P(S)$, such that $s \mapsto \delta_s * \mu$ is norm continuous, then a Borel subset T is TLT if and only if it is TLL.

Because of these results, Day[15, p.84] inquired if the existence of an (LSU) net implies that of $\mu \in P(S)$, such that $s \mapsto \delta_s * \nu$ is norm continuous. From Chapter 2, we know that such a measure exists if and only if $L(S)$ is none trivial. If $L(S)$ is non-trivial, there is $\mu \in L(S) \cap P(S)$, since $L(S)$ is a two-sided ideal of the Banach algebra $M(S)$ that is closed under absolutely continuity (Theorem 3.3.5 and Theorem 3.4.3). Then by Theorem 3.7.3, $\mu * \mu \in P(S)$ is such a measure. We answer Day's question in the negative by giving a counterexample.

Example 4.1.1. Consider $S = [0, \infty)$ with max operation and subspace topology deduced from the natural topology of $\mathbb{R}$. Then S is a commutative locally compact semitopological semigroup with identity. Let $x_n \to \infty$ be a sequence in S. Then for each compact subset K, there is $n_K \in \mathbb{N}$, such that $sx_n = x_n$ for $n > n_K$, $s \in K$. Hence $\|\delta_s * \delta_{x_n} - \delta_{x_n}\| \to 0$ uniformly on compact subsets and δ_{x_n} is a (LSU) net.

Suppose $L(S)$ is non-trivial, let $\nu \in P(S) \cap L(S)$. Let $x \in \text{supp}\nu$. Consider $E = [x+a, x+b]$ with $a, b > 0$. Then $(x+a)^{-1}E = [0, x+b]$, while $(x+a-\frac{1}{n})^{-1}E = E$ for any $n \in \mathbb{N}$. Hence $\delta_{(x+a)} * \nu(E) - \delta_{(x+a-\frac{1}{n})} * \nu(E) = \nu[0, x+a) > 0$ independent of the choice of $n \in \mathbb{N}$. Therefore, $s \to \delta_s * \nu$ is not continuous at $s = x+a$, which is a contradiction.

Remark 4.1.2. When consider the one-point compactification $\bar{S}$ of $S = [0, \infty)$, it is not hard to see δ_∞ is the only normalized (left) invariant measure on $\bar{S}$. Thus by Theorem 3.8.2, we know $L(\bar{S}) = \mathbb{C}\delta_\infty$. The above example shows that the existence of nonzero left translation continuous measure on the $\bar{S}$ can't even pass down to its open dense subsemigroups S. The reason is that $L(\bar{S})$ vanishes on S. Thus as we mentioned earlier, it is more reasonable to expect $L(S)$ to reflect properties of $F(S)$, rather than that of the whole semigroup.

Proposition 4.1.3. Suppose $L(S)$ is non-trivial. Then,

(i) S is TLA if and only if there is an (LSU) net (μ_α) in $L(S) \cap P(S)$. Actually, every topological left invariant mean m on $M(S)^*$ is a weak* limit of such a net.

(ii) $M(S)^*$ has a topological left invariant mean if and only if there is $m \in L(S)^{**}$, such that $\|m\| = m(\mathbf{1}) = 1$ and $v * m = m$ for all $\nu \in L(S)$.

Remark 4.1.4. Let G be a locally compact group. It is well-known that $M(G)^*$ has a topological left invariant mean if and only if $L_\infty(G)$ has a topological left invariant mean (eg. [31, Corollary 4.3]). In the previous proposition, (ii) is a generalization of this result.

Proof. (i) If $L(S)$ is non-trivial, there is $\nu \in P(S)$ such that $s \mapsto \delta_s * \nu$ is norm continuous as we argued at the beginning of the section. Then by Theorem 2.3.4 (iii), $(\nu * \mu_\alpha)$ is an (LSU) net in $P(S)$. Since we have shown in the previous chapter that $L(S)$ is an ideal of $M(S)$, the net is actually in $L(S) \cap P(S)$.

(ii) Let $\phi \in \mathfrak{M}(S)$. Since $L(S)$ is a left ideal of $M(S)$, then ϕ is topological left invariant if and only if it is invariant under left translation of $L(S) \cap P(S)$. Consequently, S is TLA if and only if there is a net (σ_α) in $P(S)$ such that $\|\sigma * \sigma_\alpha - \sigma_\alpha\| \to 0$ for $\sigma \in L(S) \cap P(S)$. Let $\mu_\alpha = \nu * \sigma_\alpha$. Then μ_α is a net in $P(S) \cap L(S)$ such that $\|\sigma * \mu_\alpha - \mu_\alpha\| \to 0$. Let $m = \text{weak*-}\lim \mu_\alpha \in L(S)^{**}$. It is easy to check that $\|m\| = m(\mathbf{1}) = 1$ and $v * m = m$ for all $\nu \in L(S)$. Conversely, assume there is such an m in $L(S)^{**}$. Then by Goldstine theorem, there is $\mu_\alpha \in L(S) \cap P(S)$, such that $\mu_\alpha \xrightarrow{\text{weak*}} m$. Consequently, $\sigma * \mu_\alpha - \mu_\alpha \xrightarrow{\text{weak}} 0$ for $\sigma \in L(S) \cap P(S)$. Hence $0 \in \overline{\{\sigma * \mu - \mu;\ \mu \in P(S) \cap L(S)\}}^{\text{weak}}$. Since the set $\{\sigma * \mu - \mu;\ \mu \in P(S) \cap L(S)\}$ is convex, there is $f_\alpha \in L(S) \cap P(S)$, such that $\|\sigma * f_\alpha - f_\alpha\| \to 0$.

□

4.2 Hereditary Property

Throughout this section we let T be a locally compact Borel subsemigroup of S, for example, take T as an open or closed subsemigroup of S. As we mentioned in Section 2.5, the generalization of Theorem 2.5.2 is still open for locally compact semitopological semigroups. We shall give an affirmative answer to this question in this section.

Define $i : M(T) \to M(S) : \mu \mapsto i\mu$, where $i\mu(E) = \mu(E \cap T)$ for Borel subset $E \subseteq S$. Define $p : M(S) \mapsto M(T) : \nu \mapsto p\nu$, where $p\nu(F) = \nu(F)$ for Borel subset $F \subset T$. Then $p \circ i$ is the identity map on $M(T)$. Thus in this section, we let ν, μ to denote measures in $M(S)$, while $\nu_T = p\nu$ and $\mu_T = p\mu$ to denote their corresponding restriction on T. We know that $M(T) = \{\mu_T;\ \mu \in M(S)\}$ by previous argument. For each Borel measurable function h on T, we define

$$\bar{h}(s) = \begin{cases} \text{h(s)} & \text{if } s \in T \\ 0 & \text{otherwise} \end{cases}$$

Let $\mu \in M(S)$. If $h \in L_1(|\mu|)$, then clearly $\int h d\mu_T = \int_T \bar{h} d\mu$.

Before we prove the main theorem, we need the following result, which is a generalization of Wong [50, Lemma 3.1].

Lemma 4.2.1. Let $\nu, \mu \in M(S)$. Then

$$\|(\nu * \mu)_T - \nu_T * \mu_T\| \leq \int |\mu| (s^{-1}T \cap T^c) d|\nu|(s) + \int_T |\nu| (Tt^{-1} \cap T^c) d|\mu|(t)$$

Proof. Let h be a bounded Borel measurable function on T. Note that $\overline{r_t h}(s) = h(st) = r_t\overline{h}(s)$ for all $s, t \in T$. Since complex Radon measures are bounded, h is integrable against any positive measure in $M(T)$. Thus by (Δ) on p.1,

$\int h(st)d\nu_T(s) = \int_T r_t\bar{h}(s)d\nu(s)$ is $|\mu_T|$-integrable for all $\mu \in M(S)$. Hence

$$\begin{aligned}\int h\ d\nu_T * \mu_T &\overset{(\Delta)}{=\!=} \iint h(st)d\nu_T(s)d\mu_T(t)\\ &= \int \int_T r_t\bar{h}(s)d\nu(s)d\mu_T(t)\\ &= \int_T \overline{\int_T r_t\bar{h}(s)d\nu(s)}d\mu(t)\\ &= \int_T \int_T \bar{h}(st)d\nu(s)d\mu(t)\end{aligned}$$

On the other hand,

$$\int hd(\nu * \mu)_T = \int_x \bar{h}(x)d\nu * \mu(x) = \iint_{st\in T} \bar{h}(st)d\nu(s)d\mu(t)$$

Thus,

$$\begin{aligned}&\left|\int h\ d\nu_T * \mu_T - \int hd(\nu * \mu)_T\right|\\ &\leq \left|\int \int_{s^{-1}T\cap T^c} h(st)d\mu(t)d\nu(s)\right| + \left|\int_T \int_{Tt^{-1}\cap T^c} h(st)d\nu(s)d\mu(t)\right|\\ &\leq \|h\| \left(\int |\mu| (s^{-1}T \cap T)d|\nu|(s) + \int_T |\nu| (Tt^{-1} \cap T)d|\mu|(t)\right)\end{aligned}$$

By letting h runs out of $C_0(S)$, we have the desired result. □

Corollary 4.2.2. Let T be a Borel locally compact subsemigroup of S. Let $\nu, \mu \in M(S)$ such that suppν and suppμ are contained in T. Then $(\nu * \mu)_T = \nu_T * \mu_T$.

Let $m \in \mathfrak{M}(S)$, we say m is topological left T-invariant if $\nu * m = m$ for $\nu \in P(S)$ with supp$\nu \subseteq T$. We say S is topological left T-amenable if there is a topological left T-invariant mean on $M(S)^*$. Equivalently, there is exists a net

σ_α in P(S), such that $\|\nu * \sigma_\alpha - \sigma_\alpha\| \to 0$ for each $\nu \in P(S)$ with supp$\nu \subseteq T$. The argument is similar to that of the proof of Proposition 4.1.3. Like earlier, we say T is uniformly topological left T-invariant, if for each compact subset $K \subseteq T$, there is a net μ_α in $M(S)$ such that $\|\nu * \mu_\alpha - \mu_\alpha\| \to 0$ uniformly for $\nu \in P(S)$ with supp$\nu \subseteq K$.

Theorem 4.2.3. Let S be a locally compact semitopological semigroup. Let T be a Borel locally compact subsemigroup of S. Then the following are equivalent.

(i) T is TLA [resp. U-TLA].

(ii) There is a topological left T-invariant mean [resp. uniformly topological left T-invariant mean] m on $M(S)^*$ such that $m(\chi_T) = 1$.

(iii) There is a topological left T-invariant mean [resp. uniformly topological left T-invariant mean] m on $M(S)^*$ such that $m(\chi_T) > 0$.

Proof. The equivalence of (i) and (ii) was shown in [50, Theorem 4.1]. (ii) implies (iii) is trivial. Thus it suffices to prove (iii) implies (i).

Assume m is a topological left T-invariant mean on $M(S)^*$ such that $m(\chi_T) = c > 0$. Then there is a net (μ_α) in $P(S)$ that weak* converges to m and $\|\nu * \mu_\alpha - \mu_\alpha\| \to 0$ for $\nu \in P(S)$. Without loss of generality, we may assume $\mu_\alpha(T) > \frac{c}{2} > 0$. Let $\phi_\alpha = (\mu_\alpha)_T/\mu_\alpha(T)$. Then for each $\nu \in P(S)$ with supp$\nu \subseteq T$,

$$\begin{aligned}\|\nu_T * \phi_\alpha - \phi_\alpha\| &= \frac{1}{\mu_\alpha(T)} \|\nu_T * (\mu_\alpha)_T - (\mu_\alpha)_T\| \\ &\leq \frac{1}{\mu_\alpha(T)} \|\nu_T * (\mu_\alpha)_T - (\nu * \mu_\alpha)_T\| + \|(\nu * \mu_\alpha)_T - (\mu_\alpha)_T\| \\ &\leq \frac{2}{c} \int \mu_\alpha(s^{-1}T \backslash T) d\nu(s) + \|\nu * \mu_\alpha - \mu_\alpha\|\end{aligned}$$

Since T is a subsemigroup, for $s \in T$, $T \subset s^{-1}T$. Therefore,

$$\begin{aligned}\int \mu_\alpha(s^{-1}T \cap T^c) d\nu(s) &= \int \mu_\alpha(s^{-1}T) - \mu_\alpha(T) d\nu(s) \\ &= \nu * \mu_\alpha(T) - \mu_\alpha(T) \leq \|\nu * \mu_\alpha - \mu_\alpha\|\end{aligned}$$

Therefore $\|\nu_T * \phi_\alpha - \phi_\alpha\| \to 0$. That is, T is TLA. The proof of uniform case is exactly the same. $\square$

Chapter 5

Some Open Problems

In the chapter, we continue to let S be a locally compact semitopological semigroup. The following lists some questions that are closely related to the thesis and believed to be open.

1. We have seen that if either S has an (LSU) net or has non-trivial $L(S)$, then a Borel subset is TLL if and only if it is TLT. In particular, when S is a discrete semigroup, or a locally compact group, or a compact semitopological semigroup with a left invariant Radon measure, the two definitions coincide. Are the two definitions equivalent in general?

2. We have characterized $L(S)$ for compact semitopological semigroups with left invariant Radon measures. In this case $\mathcal{F}(S)$ is the minimal right ideal of S and is a topological semigroup. Till now, all examples of foundation semigroups are topological semigroups. Is there a semitopological foundation semigroup?

3. We have shown in Corollary 3.5.5 that if S is a foundation semigroup, then (α) for any subset O of S with nonempty interior, $(sO)O^{-1}$ is a neighborhood of s for any $s \in S$. Does (α) imply the existence of a

non-trivial $L(S)$?

4. A semigroup S is called a Clifford semigroup if it is the union of algebraic subgroups of S. It has been shown in [37, Theorem 11.5] that every Clifford semigroup with an identity and a "stip" structure is a foundation semigroup. However, not every Clifford semigroup with an identity is a foundation semigroup, for example, the weakly almost compactification of the 2- motion group in Example 3.8.9 (c). Is there always a non-trivial $L(S)$ for a Clifford semigroup?

5. In literature, one say a Banach algebra $(A, *)$ is right (left) weakly completely continuous, if the map $\rho_a : b \to ba$ ($\lambda_a : b \to ab$) from A to A is weakly compact for any $a \in A$. A Banach algebra A is weakly completely continuous (w.c.c) if it is both right and left weakly completely continuous. The study for w.c.c. Banach algebras arise from the study of the group algebra $L_1(G)$, where G is a compact group. It has been shown that a Banach algebra A is right (left) w.c.c. if and only if A is a left (right) ideal of A^{**} (see [21]). It is also known that $L_1(G)$ is w.c.c. if and only if G is compact (see [43] and [24]). When S is a compact semitopological semigroup, it is obvious that $L(S)$ is a right w.c.c.. Does the converse hold?

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