University of Alberta

THE LARGEST k-BALL IN AN n-BOX

by

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For my family:
Debra, Gordon, Erin, Jim and Christine

and

Angela

ABSTRACT

We develop a formula for the radius of the largest k-dimensional ball that can be contained inside an n-dimensional box, where $k = 1, \dots, n$, and develop an algorithm to give the equations for every k-dimensional flat that contains one of these balls of maximal radius.

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Chapter 1

Introduction

This thesis will deal with the problem of finding the radius and the location of the largest k-dimensional euclidian ball in an n-dimensional box where $1 \leq k \leq n$. A precise formula for the radius of such a ball was first found in [2] by Hazel Everett (Université du Québec à Montréal, Départemente d'Informatique), Ivan Stojmenovic (University of Ottawa, Department of Computer Science), Pavel Valtr (Charles University, Department of Applied Mathematics), and Sue Whitesides (McGill University, School of Computer Science). After showing that there exists a ball of maximal radius with its center at the origin, they approached the problem by calculating the distance from the origin to the intersection of an arbitrary k-dimensional linear subspace with each of the hyperplanes containing the faces of the box. This was done in terms of k vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ which formed an orthonormal basis for the k-dimensional linear subspace. They then gave conditions on these distances which determined if a k-dimensional linear subspace contained a k-dimensional ball of maximal radius in the box. However, the locations of these k-dimensional balls of maximal radius (or equivalently, the k-dimensional linear subspaces containing them) were not found, but were rather left as an unsolved problem. To date, no such solution appears in the literature. Here, we will again calculate the radius of the largest k-dimensional ball in an n-dimensional box, but we will use $n-k$ orthonormal vectors $\mathbf{m}_1, \ldots, \mathbf{m}_{n-k}$ which form an orthonormal basis of the $n-k$-dimensional subspace orthogonal to the k-dimensional subspaces mentioned above. We will obtain the results of [2], but in a far more geometrical way, and we will also derive the locations of the k-dimensional subspaces where these k-dimensional balls of maximal radius lie.

Since distance will play an important role in this thesis, we devote all of chapter 2 to this topic. Section 2.1 will define the Gram determinant, and list some of its properties. The section following this will show how to employ the Gram determinant to find the distance between affine subspaces of any dimension and show the equivalence of this method with one that was previously known (see [3], [1] and [4]).

In chapter 3, we address the problem of finding the largest k-dimensional ball in an n-dimensional box. Section 3.1 will introduce some terminology, as well as state a couple of useful facts. In section 3.2, we will find the radius and locations of the largest $n-1$ dimensional balls in an n-dimensional box, as well as stating the points where these balls of maximal radius contact the faces of the box. We generalize these results in section 3.3, where we find the radius and locations of the largest k-dimensional balls in an n-dimensional box for any $k = 1, \ldots, n-1$.

Since calculating specific solutions by hand will become increasingly unrealistic as $n - k$ becomes large, we include an algorithm in the Appendix that will quickly do the calculations.

Chapter 2

Gram Determinants and the Distance Between Flats

2.1 The Gram Determinant: Definition and Properties

We will begin with a brief study of the Gram determinant and its properties, which will be useful in the next chapter. We start with a couple of useful theorems from linear algebra which can be found in [5].

Theorem 2.1 (GRAM-SCHMIDT PROCESS). *Given a linearly independant set* $\{\boldsymbol{u}_1, \dots \boldsymbol{u}_k\} \subset \mathbb{R}^n$, *define*

$$\begin{aligned}
\boldsymbol{v}_1 &= \boldsymbol{u}_1, \\
\boldsymbol{v}_2 &= \boldsymbol{u}_2 - \frac{\langle \boldsymbol{u}_2, \boldsymbol{v}_1 \rangle}{\langle \boldsymbol{v}_1, \boldsymbol{v}_1 \rangle}\boldsymbol{v}_1, \\
\boldsymbol{v}_3 &= \boldsymbol{u}_3 - \frac{\langle \boldsymbol{u}_3, \boldsymbol{v}_1 \rangle}{\langle \boldsymbol{v}_1, \boldsymbol{v}_1 \rangle}\boldsymbol{v}_1 - \frac{\langle \boldsymbol{u}_3, \boldsymbol{v}_2 \rangle}{\langle \boldsymbol{v}_2, \boldsymbol{v}_2 \rangle}\boldsymbol{v}_2, \\
&\vdots \\
\boldsymbol{v}_k &= \boldsymbol{u}_k - \frac{\langle \boldsymbol{u}_k, \boldsymbol{v}_1 \rangle}{\langle \boldsymbol{v}_1, \boldsymbol{v}_1 \rangle}\boldsymbol{v}_1 - \frac{\langle \boldsymbol{u}_k, \boldsymbol{v}_2 \rangle}{\langle \boldsymbol{v}_2, \boldsymbol{v}_2 \rangle}\boldsymbol{v}_2 - \dots - \frac{\langle \boldsymbol{u}_k, \boldsymbol{v}_{k-1} \rangle}{\langle \boldsymbol{v}_{k-1}, \boldsymbol{v}_{k-1} \rangle}\boldsymbol{v}_{k-1}.
\end{aligned}$$

Then $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_k\}$ *is an orthogonal set and* $\mathrm{span}\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_p\} = \mathrm{span}\{\boldsymbol{u}_1, \dots, \boldsymbol{u}_p\}$ *for* $1 \le p \le k$.

Remark 2.1. We can reformulate Theorem 2.1 by saying that $\mathbf{v}_1 = \mathbf{u}_1$ and for every $k = 2, \dots, n$, $\mathbf{v}_k = \mathbf{u}_k - \mathrm{proj}_{V_{k-1}} \mathbf{u}_k$ where $V_{k-1} = \mathrm{span}\{\mathbf{v}_1, \dots, \mathbf{v}_{k-1}\}$.

Theorem 2.2 (QR FACTORIZATION). *If A is an $m \times n$ matrix with linearly independent columns $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n$, then A can be uniquely factored as $A = QR$, where Q is an $m \times n$ matrix whose columns, $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$, form an orthonormal basis for the column space of A, and R is an $n \times n$ upper triangular matrix with $r_{11} = \|\boldsymbol{v}_1\|$, $r_{jj} = \|\boldsymbol{v}_j - \operatorname{proj}_{V_{j-1}} \boldsymbol{v}_j\|$ for $j = 2, \ldots, n$, and $r_{ij} = \langle \boldsymbol{u}_i, \boldsymbol{v}_j \rangle$ for $i < j$, where $V_{j-1} = \operatorname{span}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{j-1}\}$.*

Remark 2.2. Setting $m = n$ in theorem 2.2 and keeping in mind that Q is an orthogonal matrix and R is an upper triangular matrix with positive entries on its main diagonal, we can write the absolute value of the determinant of A as:

$$\begin{aligned} |\det(A)| &= |\det(QR)| = |\det(Q)||\det(R)| = |\pm 1||\det(R)| = \det(R) \\ &= \|\mathbf{v}_1\|\|\mathbf{v}_2 - \operatorname{proj}_{V_1} \mathbf{v}_2\| \cdots \|\mathbf{v}_n - \operatorname{proj}_{V_{n-1}} \mathbf{v}_n\|. \end{aligned}$$

Definition 2.1 (GRAM MATRIX, GRAM DETERMINANT). Let $\mathbf{v}_1, \ldots, \mathbf{v}_k \in \mathbb{R}^n$ be the columns of an $n \times k$ matrix A. The matrix $A^\top A$ is known as the Gram matrix of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ and is denoted by $\mathrm{G}(\mathbf{v}_1, \ldots, \mathbf{v}_k)$. The determinant of $A^\top A$ is called the Gram determinant of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ and is denoted by $\mathrm{Gram}(\mathbf{v}_1, \ldots, \mathbf{v}_k)$. Thus,

$$\mathrm{Gram}(\mathbf{v}_1, \ldots, \mathbf{v}_k) = \det(A^\top A) = \begin{vmatrix} \langle \mathbf{v}_1, \mathbf{v}_1 \rangle & \ldots & \langle \mathbf{v}_1, \mathbf{v}_k \rangle \\ \vdots & \ddots & \vdots \\ \langle \mathbf{v}_k, \mathbf{v}_1 \rangle & \ldots & \langle \mathbf{v}_k, \mathbf{v}_k \rangle \end{vmatrix}.$$

Remark 2.3. If A is an $n \times k$ matrix with linearly independant columns, then we can apply the QR factorization:

$$\begin{aligned} A^\top A &= (QR)^\top QR \\ &= R^\top Q^\top QR \\ &= R^\top IR \\ &= R^\top R, \end{aligned}$$

so that

$$\begin{aligned}\det(A^\top A) &= \det(R^\top R)\\ &= \det(R^\top)\det(R)\\ &= (\det(R))^2\\ &= \|\mathbf{v}_1\|^2\|\mathbf{v}_2 - \operatorname{proj}_{V_1}\mathbf{v}_2\|^2 \dots \|\mathbf{v}_k - \operatorname{proj}_{V_{k-1}}\mathbf{v}_k\|^2.\end{aligned}$$

Theorem 2.3 (PROPERTIES OF THE GRAM DETERMINANT). *Let $v_1, \dots, v_k$ be vectors in $\mathbb{R}^n$. We have the following:*

1. $\mathrm{Gram}(v_{\sigma(1)}, \dots, v_{\sigma(k)}) = \mathrm{Gram}(v_1, \dots, v_k)$ *where σ is any permutation of $\{1, \dots, k\}$.*

2. $\mathrm{Gram}(v_1, \dots, \alpha v_i, \dots, v_k) = \alpha^2 \mathrm{Gram}(v_1, \dots, v_i, \dots, v_k)$ *for every $\alpha \in \mathbb{R}$ and for every $i = 1, \dots, k$.*

3. $\mathrm{Gram}(v_1, \dots, v_k) = 0$ *if and only if $\{v_1, \dots, v_k\}$ is a linearly dependant set.*

4. $0 < \mathrm{Gram}(v_1, \dots, v_k) \leq \prod_{i=1}^k \|v_i\|^2$ *for a linearly independant set $\{v_1, \dots, v_k\}$.*

Proof.

1. It will suffice to show

$$\mathrm{Gram}(\mathbf{v}_1, \dots, \mathbf{v}_i, \dots, \mathbf{v}_j, \dots, \mathbf{v}_k) = \mathrm{Gram}(\mathbf{v}_1, \dots, \mathbf{v}_j, \dots, \mathbf{v}_i, \dots, \mathbf{v}_k).$$

   By interchanging ith and jth vectors, we interchange the ith and jth rows and the ith and jth columns of the Gram matrix. Thus, the Gram determinant changes by a factor of $(-1)^2$.

2. Multiplying the ith vector by α results in multiplying the ith row and the ith column of the Gram matrix by α. Thus, the Gram determinant changes by a factor of α^2.

3. Let A be an $n \times k$ matrix with $\mathbf{v}_1, \dots, \mathbf{v}_k$ as it columns. If $\mathrm{Gram}(\mathbf{v}_1, \dots, \mathbf{v}_k) = 0$ then $\det(A^\top A) = 0$. Then there must be one column of $A^\top A$, say the ith

column, which is a linear combination of the other columns:

$$\begin{bmatrix} \langle \mathbf{v}_1, \mathbf{v}_i \rangle \\ \vdots \\ \langle \mathbf{v}_k, \mathbf{v}_i \rangle \end{bmatrix} = \sum_{j \neq i} c_j \begin{bmatrix} \langle \mathbf{v}_1, \mathbf{v}_j \rangle \\ \vdots \\ \langle \mathbf{v}_k, \mathbf{v}_j \rangle \end{bmatrix} = \begin{bmatrix} \langle \mathbf{v}_1, \sum_{j \neq i} c_j \mathbf{v}_j \rangle \\ \vdots \\ \langle \mathbf{v}_k, \sum_{j \neq i} c_j \mathbf{v}_j \rangle \end{bmatrix}. \tag{2.1}$$

Therefore, for $l = 1, \ldots, k$, $\langle \mathbf{v}_l, \mathbf{v}_i - \sum_{j \neq i} \mathbf{v}_j \rangle = 0$ so that $\mathbf{v}_l \perp \left(\mathbf{v}_i - \sum_{j \neq i} \mathbf{v}_j \right)$. Setting $\mathbf{v} = \mathbf{v}_i - \sum_{j \neq i} c_j \mathbf{v}_j$, we have $\mathbf{v} \in \operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$. But since $\mathbf{v}_l \perp \mathbf{v}$ for $l = 1, \ldots, k$, $\mathbf{v} = \mathbf{0}$ and $\mathbf{v}_i = \sum_{j \neq i} c_j \mathbf{v}_j$ and hence $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is linearly a dependant set. On the other hand, if the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is linearly dependant, then there is at least one vector, say $\mathbf{v}_i$ such that $\mathbf{v}_i = \sum_{j \neq i} \mathbf{v}_j$. It then follows from (2.1) that the ith column of $A^\top A$ is a linear combination of the other columns so that $\det(A^\top A) = \operatorname{Gram}(\mathbf{v}_1, \ldots, \mathbf{v}_k) = 0$.

4. $\operatorname{Gram}(\mathbf{v}_1, \ldots, \mathbf{v}_k) = \|\mathbf{v}_1\|^2 \|\mathbf{v}_2 - \operatorname{proj}_{V_1} \mathbf{v}_2\|^2 \ldots \|\mathbf{v}_k - \operatorname{proj}_{V_{k-1}} \mathbf{v}_k\|^2 > 0$ since $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is linearly independant. We will show that for every $\mathbf{x} \in \mathbb{R}^n$ and for every subspace L of $\mathbb{R}^n$ we have $\|\mathbf{x} - \operatorname{proj}_L \mathbf{x}\|^2 \leq \|\mathbf{x}\|^2$. Since $\mathbb{R}^n = L \oplus L^\perp$, we have that $\mathbf{x}$ can be uniquely written as $\mathbf{x} = \operatorname{proj}_L \mathbf{x} + \operatorname{proj}_{L^\perp} \mathbf{x}$. Since $L \perp L^\perp$, Pythagoras' theorem yields $\|\mathbf{x}\|^2 = \|\operatorname{proj}_L \mathbf{x}\|^2 + \|\operatorname{proj}_{L^\perp} \mathbf{x}\|^2$. But $\operatorname{proj}_{L^\perp} \mathbf{x} = \mathbf{x} - \operatorname{proj}_L \mathbf{x}$ gives $\|\mathbf{x}\|^2 = \|\operatorname{proj}_L \mathbf{x}\|^2 + \|\mathbf{x} - \operatorname{proj}_L \mathbf{x}\|^2$ so that $\|\mathbf{x} - \operatorname{proj}_L \mathbf{x}\|^2 \leq \|\mathbf{x}\|^2$. Thus, $\operatorname{Gram}(\mathbf{v}_1, \ldots, \mathbf{v}_k) \leq \prod_{i=1}^k \|\mathbf{v}_i\|^2$.

□

Remark 2.4. Note that $\|\mathbf{v}_i - \operatorname{proj}_{V_{i-1}} \mathbf{v}_i\|^2 = \|\mathbf{v}_i\|^2$ if and only if $\operatorname{proj}_{V_{i-1}} \mathbf{v}_i = 0$ if and only if $\mathbf{v}_i \perp V_{i-1}$. Thus, $\operatorname{Gram}(\mathbf{v}_1, \ldots, \mathbf{v}_k) = \prod_{i=1}^k \|\mathbf{v}_i\|^2$ if and only if $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is an orthogonal set.

Remark 2.5. Remark 2.3 states that for a linearly independent set $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$, we have that

$$\operatorname{Gram}(\mathbf{v}_1, \ldots, \mathbf{v}_k) = \|\mathbf{v}_1\|^2 \|\mathbf{v}_2 - \operatorname{proj}_{V_1} \mathbf{v}_2\|^2 \ldots \|\mathbf{v}_k - \operatorname{proj}_{V_{k-1}} \mathbf{v}_k\|^2. \tag{2.2}$$

However, if $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a linearly dependant set, then there is an i such that $\mathbf{v}_i \in V_{i-1}$. It follows that $\|\mathbf{v}_i - \operatorname{proj}_{V_{i-1}} \mathbf{v}_i\| = 0$ which of course implies $\|\mathbf{v}_1\|^2 \|\mathbf{v}_2 - \operatorname{proj}_{V_1} \mathbf{v}_2\|^2 \ldots \|\mathbf{v}_k - \operatorname{proj}_{V_{k-1}} \mathbf{v}_k\|^2 = 0$. Combining this with part 3 of theorem 2.3

verifies (2.2) for any set $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$.

2.2 Distance Formulae

In this section, we will be concerned with finding the distance between two flats, given that we know a basis for both of their underlying vector spaces as well as one point lying in each flat.

Theorem 2.4 (DISTANCE FROM A POINT TO A FLAT). *Let $\mathcal{L} = \boldsymbol{p} + L$ be a k-dimensional flat in $\mathbb{R}^n$, and let $\boldsymbol{x} \in \mathbb{R}^n$. Then the distance from $\boldsymbol{x}$ to $\mathcal{L}$ is given by*

$$\text{dist}(\boldsymbol{x}, \mathcal{L}) = \sqrt{\frac{\text{Gram}(\boldsymbol{x} - \boldsymbol{p}, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_k)}{\text{Gram}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k)}} \tag{2.3}$$

where $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\}$ is a basis for L.

Proof.

$$\begin{aligned}
\text{dist}(\mathbf{x}, \mathcal{L}) &= \text{dist}(\mathbf{x}, \mathbf{p} + L) = \text{dist}(\mathbf{x} - \mathbf{p}, L) \\
&= \|(\mathbf{x} - \mathbf{p}) - \text{proj}_L(\mathbf{x} - \mathbf{p})\| \\
&= \sqrt{\frac{\text{Gram}(\mathbf{v}_1, \ldots, \mathbf{v}_k)\|(\mathbf{x} - \mathbf{p}) - \text{proj}_L(\mathbf{x} - \mathbf{p})\|^2}{\text{Gram}(\mathbf{v}_1, \ldots, \mathbf{v}_k)}} \\
&= \sqrt{\frac{\text{Gram}(\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{x} - \mathbf{p})}{\text{Gram}(\mathbf{v}_1, \ldots, \mathbf{v}_k)}} \\
&= \sqrt{\frac{\text{Gram}(\mathbf{x} - \mathbf{p}, \mathbf{v}_1, \ldots, \mathbf{v}_k)}{\text{Gram}(\mathbf{v}_1, \ldots, \mathbf{v}_k)}}.
\end{aligned}$$

□

Theorem 2.5 (DISTANCE BETWEEN ARBITRARY FLATS). *Let $\mathcal{L}_1 = \boldsymbol{p}_1 + L_1, \mathcal{L}_2 = \boldsymbol{p}_2 + L_2$ be any two flats in $\mathbb{R}^n$. Then the distance from $\mathcal{L}_1$ to $\mathcal{L}_2$ is given by*

$$\text{dist}(\mathcal{L}_1, \mathcal{L}_2) = \sqrt{\frac{\text{Gram}(\boldsymbol{p}_1 - \boldsymbol{p}_2, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_k)}{\text{Gram}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k)}} \tag{2.4}$$

where $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\}$ is a basis for $L_1 + L_2$.

Proof.

$$\begin{aligned}
\mathrm{dist}(\mathcal{L}_1, \mathcal{L}_2) &= \mathrm{dist}(\mathbf{p}_1 + L_1, \mathbf{p}_2 + L_2) \\
&= \inf_{\mathbf{x}\in L_1, \mathbf{y}\in L_2} \mathrm{dist}(\mathbf{p}_1 + \mathbf{x}, \mathbf{p}_2 + \mathbf{y}) \\
&= \inf_{\mathbf{x}\in L_1, \mathbf{y}\in L_2} \mathrm{dist}(\mathbf{p}_1 - \mathbf{p}_2, \mathbf{y} - \mathbf{x}) \\
&= \inf_{\mathbf{x}\in L_1, \mathbf{y}\in L_2} \mathrm{dist}(\mathbf{p}_1 - \mathbf{p}_2, \mathbf{x} + \mathbf{y}) \\
&= \inf_{\mathbf{t}\in L_1 + L_2} \mathrm{dist}(\mathbf{p}_1 - \mathbf{p}_2, \mathbf{t}) \\
&= \mathrm{dist}(\mathbf{p}_1 - \mathbf{p}_2, L_1 + L_2).
\end{aligned}$$

But from the proof of theorem 2.4, we have

$$\mathrm{dist}(\mathbf{p}_1 - \mathbf{p}_2, L_1 + L_2) = \sqrt{\frac{\mathrm{Gram}(\mathbf{p}_1 - \mathbf{p}_2, \mathbf{v}_1, \ldots, \mathbf{v}_k)}{\mathrm{Gram}(\mathbf{v}_1, \ldots, \mathbf{v}_k)}}$$

where $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a basis for $L_1 + L_2$. □

The following alternate distance formula can be found in [3], [1] and [4]. Let A be an $m_1 \times n$ matrix and B be an $m_2 \times n$ matrix, both with orthonormal rows. Set $\mathcal{L}_1 = \{\mathbf{x} \in \mathbb{R}^n | A\mathbf{x} = \mathbf{a}\}$, and $\mathcal{L}_2 = \{\mathbf{x} \in \mathbb{R}^n | B\mathbf{x} = \mathbf{b}\}$.

Theorem 2.6. *Let p_1, p_2 be arbitrary elements of the two flats $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively and let C be any matrix with orthonormal rows such that* $\mathrm{Row}(C) = \mathrm{Row}(A) \cap \mathrm{Row}(B)$. *Then,*

$$\mathrm{dist}(\mathcal{L}_1, \mathcal{L}_2) = \|C(A^\top \boldsymbol{a} - B^\top \boldsymbol{b})\| = \|C(\boldsymbol{p}_1 - \boldsymbol{p}_2)\|. \tag{2.5}$$

□

We wish to show that (2.4) and (2.5) are equivalent. We have already seen that

$$\mathrm{dist}(\mathcal{L}_1, \mathcal{L}_2) = \mathrm{dist}(\mathbf{p}_1 - \mathbf{p}_2, L_1 + L_2).$$

It now follows that

$$\begin{aligned}
\operatorname{dist}(\mathcal{L}_1, \mathcal{L}_2) &= \|(\mathbf{p}_1 - \mathbf{p}_2) - \operatorname{proj}_{L_1+L_2}(\mathbf{p}_1 - \mathbf{p}_2)\| \\
&= \|(\mathbf{p}_1 - \mathbf{p}_2) - \operatorname{proj}_{\mathrm{Null}(A)+\mathrm{Null}(B)}(\mathbf{p}_1 - \mathbf{p}_2)\| \\
&= \|\operatorname{proj}_{\mathrm{Row}(A)\cap\mathrm{Row}(B)}(\mathbf{p}_1 - \mathbf{p}_2)\| \\
&= \|\operatorname{proj}_{\mathrm{Row}(C)}(\mathbf{p}_1 - \mathbf{p}_2)\| \\
&= \|C^\top C(\mathbf{p}_1 - \mathbf{p}_2)\| \\
&= \|C(\mathbf{p}_1 - \mathbf{p}_2)\|
\end{aligned}$$

since $C^\top$ has orthonormal columns. We find then that (2.4) and (2.5) are essentially equivalent, however, (2.4) does not require us to find an orthonormal basis.

Chapter 3

The Largest Ball in a Box Problem

We will now address the problem of finding the radius of the largest k-dimensional ball that can be contained in an n-dimensional box. Note that in this chapter, we will simply say n-box and k-ball, and drop the word *dimensional.*

3.1 Notation and Basic Ideas

We will begin with some notation that will be used throughout this chapter. First, let $\mathfrak{B} = \{x \in \mathbb{R}^n | -a_j \leq x^j \leq a_j, j = 1 \ldots n\}$ be the box under consideration. We have from the definition that $\mathfrak{B}$ is a convex and symmetric box of dimensions $2a_1 \times 2a_2 \times \ldots \times 2a_n$. There will be no loss of generality if we assume

$$0 < a_1 \leq a_2 \leq \ldots \leq a_n. \tag{3.1}$$

The $2n$ faces of $\mathfrak{B}$ lie in hyperplanes which we will denote by $\mathcal{H}_j^{\pm} = \{x \in \mathbb{R}^n | x^j = \pm a_j\}$. By the symmetry of $\mathfrak{B}$, we only need to consider $\mathcal{H}_j^+$, which we will simply denote as $\mathcal{H}_j$. We will refer to the face of $\mathfrak{B}$ lying in $\mathcal{H}_j$ as the jth face of $\mathfrak{B}$. $\mathcal{B}$ will denote any euclidian k-ball contained in $\mathfrak{B}$, and by $\mathcal{M}$, we mean the k-flat on which $\mathcal{B}$ lies. We define $\mathcal{M}_j = \mathcal{M} \cap \mathcal{H}_j$ to be the intersection of the k-flat with the hyperplane containing the jth face of $\mathfrak{B}$. Finally, by $r(k, n)$ we mean the largest possible radius $\mathcal{B}$ can have. So our goal then is to find equations for $r(k, n)$ and

$\mathcal{M}$ in terms of $a_1, \ldots, a_n$. To simplify notation we set $\delta_j = \sqrt{a_1^2 + \ldots + a_j^2}$, with $1 \leq j \leq n$, and $\delta_0 = 0$. We include here a couple of lemmas.

Lemma 3.1. *If $\mathcal{B} \subset \mathfrak{B}$ and has radius r, then there is a ball also having radius r with its center at the origin.*

Proof. Let $\mathcal{B}(b, r) \subset \mathfrak{B}$ be any ball with radius r centered at b. By the symmetry of $\mathfrak{B}$, the ball $\mathcal{B}(-b, r)$ is included in $\mathfrak{B}$. From the convexity of $\mathfrak{B}, \mathcal{B} = \left(\frac{1}{2}\mathcal{B}(b,r) + \frac{1}{2}\mathcal{B}(-b,r)\right) \subset \mathfrak{B}$. We will show $\mathcal{B} = \mathcal{B}(0, r)$. Let $z \in \mathcal{B}$. Then there exists an $x_1 \in \mathcal{B}(b, r)$, and an $x_2 \in \mathcal{B}(-b, r)$ such that $\|\mathbf{x}_1 - \mathbf{b}\| \leq r, \|\mathbf{x}_2 + \mathbf{b}\| \leq r$, with $z = \frac{x_1 + x_2}{2}$. Then,

$$\|\mathbf{z}\| = \|\tfrac{\mathbf{x}_1 + \mathbf{x}_2}{2}\| = \left(\tfrac{1}{2}\right) \|\mathbf{x}_1 - \mathbf{b} + \mathbf{x}_2 + \mathbf{b}\| \leq \left(\tfrac{1}{2}\right) (\|\mathbf{x}_1 - \mathbf{b}\| + \|\mathbf{x}_2 + \mathbf{b}\|) \leq r$$

and $z \in \mathcal{B}(0, r)$. On the other hand, suppose $z \in \mathcal{B}(0, r)$ Set $x_1 = z + b \in \mathcal{B}(b, r)$, $x_2 = z - b \in \mathcal{B}(-b, r)$. Then, $\frac{x_1 + x_2}{2} = \frac{z + b + z - b}{2} = z$ and we have that $z \in \mathcal{B}$. □

By lemma 3.1, we may assume that $\mathcal{M}$ is a linear subspace, which simplifies our work. In what follows, we will find the distance from the origin to $\mathcal{M}_j$ which will be denoted by d_j. This will be achieved by first finding d'_j, the distance from the origin to $p + \mathcal{M}_j^{\perp}$ where p is any point in $\mathcal{M}_j$. We can then find d_j using Pythagoras' theorem. We will show that the ball $\mathcal{B}$ of maximal radius $r(k, n)$ does not lie in a k-flat parallel to any of the faces of $\mathfrak{B}$, ensuring that $\mathcal{M}_j$ is non-empty for every $j = 1 \ldots, n$. The maximal radius, $r(k, n)$, that any k-ball can have on the given k-flat $\mathcal{M}$ will then be given by the minimum of the resulting d_j's. Finally, $\mathbf{e}_1, \ldots, \mathbf{e}_n$ is the standard basis of $\mathbb{R}^n$; the jth entry of $\mathbf{e}_j$ is 1, while all other entries are 0. We state here some inequalities involving the a_j's.

Lemma 3.2. *Let $k \leq n - 1$ and $j \leq n$. Then*

(3.2) $\quad a_j \leq d_j$ *for* $j = 1, \ldots, n$ *and* $\mathcal{M}_j \neq \emptyset$.

(3.3) $\quad a_j \leq \dfrac{\delta_{j-1}}{\sqrt{k-1}}$ *if and only if* $a_j \leq \dfrac{\delta_j}{\sqrt{k}}$ *for* $k \geq 2$ *and* $j \geq 2$.

(3.4) $\quad$ *If* $a_j \leq \dfrac{\delta_{j-1}}{\sqrt{k-1}}$ *then* $a_{j-1} \leq \dfrac{\delta_{j-2}}{\sqrt{k-2}}$ *for* $k \geq 3$ *and* $j \geq 3$.

Proof. (3.2) follows from the fact that $\mathcal{M}_j \subset \mathcal{H}_j$ and that the distance from the origin to $\mathcal{H}_j$ is clearly a_j. For (3.3), with $k, j \geq 2$ we have

$$a_j^2 \le \frac{\delta_j^2}{k} = \frac{\delta_{j-1}^2}{k} + \frac{a_j^2}{k}$$

which is equivalent to

$$\frac{(k-1)a_j^2}{k} \le \frac{\delta_{j-1}^2}{k}$$

or more simply

$$a_j^2 \le \frac{\delta_{j-1}^2}{k-1}.$$

For (3.4), with $k, j \ge 3$ we have using (3.1)

$$a_{j-1}^2 \le a_j^2 \le \frac{\delta_{j-1}^2}{k-1}.$$

It follows that

$$a_{j-1}^2 \le \frac{\delta_{j-2}^2}{k-1} + \frac{a_{j-1}^2}{k-1}$$

so that

$$\frac{(k-2)a_{j-1}^2}{k-1} \le \frac{\delta_{j-2}^2}{k-1}$$

from which it follows

$$a_{j-1}^2 \le \frac{\delta_{j-2}^2}{k-2}.$$

□

Before we begin our discussion on the largest $(n-1)$-ball in an n-box, we mention the trivial case of finding the radius of the largest n-ball in an n-box. In order for such a ball to be contained in a box of the same dimension, the diameter of the ball cannot exceed the smallest measurement of the box. In terms of the above definitions, we find that the diameter cannot exceed $2a_1$, or that the radius cannot exceed a_1. Thus, $r(n, n) = a_1$ for every n. Having dealt with this case, we now add the restriction that $n \ge 2$ and $1 \le k \le n-1$.

3.2 The $(n-1,n)$ Problem

Here, we will find the radius of the largest $(n-1)$-ball contained in an n-box, as well as the equations of the planes containing a ball of such radius. We consider an $(n-1)$-flat $\mathcal{M} = \{\mathbf{x} \in \mathbb{R}^n : \langle \mathbf{m}, \mathbf{x} \rangle = 0\}$ where $\mathbf{m} = (m^1, \ldots, m^n) \in \mathbb{R}^n$ and $\mathbf{m} \neq 0$. Without loss of generality, we can assume $\mathbf{m}$ is a unit vector. The simplest planes to consider are where $\mathbf{m} = \mathbf{e}_j$, $j = 1, \ldots, n$. Here, $\mathcal{M}$ is parallel to $\mathcal{H}_j$ and the maximal radius a ball can have in this case is $\min\{a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_n\}$. This minimum is a_2 if $j = 1$, and is a_1 otherwise. We now only consider $(n-1)$-flats $\mathcal{M}$ where $\mathbf{m} \neq \mathbf{e}_j$, $j = 1, \ldots, n$, or equivalently, any $(n-1)$-flat $\mathcal{M}$ such that for every $j = 1, \ldots, n$, $\mathcal{M}_j \neq \emptyset$.

3.2.1 A Formula for d_j

In this section, we will state a formula for d_j, $j = 1, \ldots, n$ when $k = n-1$. We will also give an equation that will prove useful when deciding if a given $(n-1)$-flat $\mathcal{M}$ contains a ball of maximal radius in $\mathfrak{B}$.

Lemma 3.3. *The distance, d_j, $j = 1, \ldots n$, from the origin to the $(n-2)$-flat $\mathcal{M}_j$ is given by*

$$d_j^2 = \frac{a_j^2}{1-(m^j)^2}. \tag{3.5}$$

Proof. Let $p \in \mathcal{M}_j$. Then $p \in \mathcal{M}$ from which it follows that $\langle \mathbf{p}, \mathbf{m} \rangle = 0$, and $p \in \mathcal{H}_j$ which gives us $\langle \mathbf{p}, \mathbf{e}_j \rangle = a_j$. Furthermore, $\mathbf{p}$ can be written as $a_j \mathbf{v}$ for some suitable

$\mathbf{v}$. We then have the distance, d'_j, from the origin to $p + \mathcal{M}_j^{\perp}$ as

$$
\begin{aligned}
(d'_j)^2 &= \frac{\mathrm{Gram}(\mathbf{p}, \mathbf{m}, \mathbf{e}_j)}{\mathrm{Gram}(\mathbf{m}, \mathbf{e}_j)} \\
&= \frac{\begin{vmatrix} a_j^2\|\mathbf{v}\|^2 & 0 & a_j \\ 0 & 1 & m^j \\ a_j & m^j & 1 \end{vmatrix}}{\begin{vmatrix} 1 & m^j \\ m^j & 1 \end{vmatrix}} \\
&= \frac{a_j^2\|\mathbf{v}\|^2(1-(m^j)^2) - a_j^2}{1-(m^j)^2}.
\end{aligned}
$$

Then,

$$
\begin{aligned}
d_j^2 &= \|\mathbf{p}\|^2 - (d'_j)^2 \\
&= a_j^2\|\mathbf{v}\|^2 - \frac{a_j^2\|\mathbf{v}\|^2(1-(m^j)^2) - a_j^2}{1-(m^j)^2} \\
&= \frac{a_j^2\|\mathbf{v}\|^2(1-(m^j)^2) - a_j^2\|\mathbf{v}\|^2(1-(m^j)^2) + a_j^2}{1-(m^j)^2} \\
&= \frac{a_j^2}{1-(m^j)^2}.
\end{aligned}
$$

□

Lemma 3.4. *For any $(n-1)$-flat $\mathcal{M}$ not parallel to $\mathcal{H}_j$, for $j = 1, \ldots, n$, we have the following equality:*

$$\sum_{j=1}^{n} \frac{a_j^2}{d_j^2} = n - 1. \tag{3.6}$$

Proof. $\sum_{j=1}^{n} \frac{a_j^2}{d_j^2} = \sum_{j=1}^{n}(1-(m^j)^2) = n - \sum_{j=1}^{n}(m^j)^2 = n-1.$ □

3.2.2 Finding $r(n-1, n)$ and $\mathcal{M}$

Theorem 3.1. *Let $\mathfrak{B}' = \{x \in \mathbb{R}^{n'} | -a_j \le x^j \le a_j, j = 1 \ldots n'\}$ with $0 < a_1 \le \ldots \le a_{n'}$ be a box of dimensions $2a_1 \times \ldots \times 2a_{n'}$. The following statements are equivalent:*

1. The $(n'-1)$-ball of maximal radius is tangent to all faces of $\mathfrak{B}'$,

2. $a_{n'} \leq \dfrac{\delta_{n'-1}}{\sqrt{n'-2}}$.

Proof. (1) $\Rightarrow$ (2): Suppose there is an $(n'-1)$-ball of maximal radius tangent to all the faces of $\mathfrak{B}'$. Then $d_1 = \ldots = d_{n'} = t$, and (3.6) gives $t = \frac{\delta_{n'}}{\sqrt{n'-1}}$. (3.2) now ensures us that $a_{n'} \leq \frac{\delta_{n'}}{\sqrt{n'-1}}$, and (3.3) shows $a_{n'} \leq \frac{\delta_{n'-1}}{\sqrt{n'-2}}$.
(2) $\Rightarrow$ (1): Assume that $a_{n'} \leq \frac{\delta_{n'-1}}{\sqrt{n'-2}}$. Then (3.3) gives $a_{n'} \leq \frac{\delta_{n'}}{\sqrt{n'-1}}$, or equivalently, $(n'-1)a_{n'}^2 \leq \delta_{n'}^2$. Let $\mathcal{M}$ be the $(n'-1)$-flat such that

$$m^j = \frac{\sqrt{\delta_{n'}^2 - (n'-1)a_j^2}}{\delta_{n'}}, j = 1, \ldots, n'. \tag{3.7}$$

Defining the m^j's in this fashion, (3.5) gives

$$\begin{aligned} d_j^2 &= \frac{a_j^2}{1-(m^j)^2} \\ &= a_j^2 \left(1 - \frac{\delta_{n'}^2 - (n'-1)a_j^2}{\delta_{n'}^2}\right)^{-1} \\ &= a_j^2 \left(\frac{(n'-1)a_j^2}{\delta_{n'}^2}\right)^{-1} \\ &= \frac{\delta_{n'}^2}{n'-1} \end{aligned}$$

for every $j = 1, \ldots, n'$. The maximality follows from (3.6): If we want to increase the radius, we need to increase all the d_j's. But an increase in one, say d_{j_1}, requires a decrease in another, say d_{j_2}, where $j_1 \neq j_2$, which would only result in a ball of strictly smaller radius. It now remains only to show that the point of $\mathcal{M}_j$ closest to the origin lay in the jth face of $\mathfrak{B}'$. We will argue by contradiction. Suppose for some $j = 1, \ldots, n'$, the closest point of $\mathcal{M}_j$ to the origin, say p_j, is not contained in $\mathfrak{B}'$. Construct a line l from the origin to p_j. Since $0 \in \mathfrak{B}'$, and $l \subset \mathcal{M}$, there exists an $i = 1, \ldots, n'$, $i \neq j$, such that $l \cap \mathcal{M}_i = \{p_i\}$. It follows that $d_i \leq \|\mathbf{p}_i\| < \|\mathbf{p}_j\| = d_j$. Thus, we have that $d_i \neq d_j$, clearly contradicting their equality. Thus we have that the $(n'-1)$-flat $\mathcal{M}$ given by the equation

$$m^1 x^1 + \ldots + m^{n'} x^{n'} = 0$$

where the m^j's are defined as in (3.7), contains an $(n'-1)$-ball of maximal radius that is tangent to all faces of $\mathfrak{B}'$. ☐

Theorem 3.2. *Let s be the smallest integer satisfying*

$$a_{n-s} \leq \frac{\delta_{n-s-1}}{\sqrt{n-s-2}}. \tag{3.8}$$

Then the maximal radius of $\mathcal{B} \subset \mathfrak{B}$ occurs when $d_1 = \ldots = d_{n-s} = t$ with the radius being their common value. That is,

$$r(n-1, n) = t = \frac{\delta_{n-s}}{\sqrt{n-s-1}}. \tag{3.9}$$

Furthermore, $m^{n-s+1} = \ldots = m^n = 0$.

Proof. If $s = 0$ is the smallest integer satisfying (3.8) then $a_n \leq \frac{\delta_{n-1}}{\sqrt{n-2}}$ and from theorem 3.1 (with $n' = n$) we have

$$d_1 = \ldots = d_n = t = \frac{\delta_n}{\sqrt{n-1}}$$

which gives us (3.9).

Now suppose $s = 1$ is the smallest integer satisfying (3.8). Then $a_n > \frac{\delta_{n-1}}{\sqrt{n-2}}$ and $a_{n-1} \leq \frac{\delta_{n-2}}{\sqrt{n-3}}$. Intersecting $\mathfrak{B}$ with the hyperplane $x^n = 0$ yields an $(n-1)$-box of dimensions $2a_1 \times \ldots \times 2a_{n-1}$. Intersecting any $(n-1)$-ball contained in $\mathfrak{B}$ (not parallel to any face of $\mathfrak{B}$) with the same hyperplane gives an $(n-2)$ ball of the same radius contained in the $(n-1)$-box. This implies that $r(n-2, n-1) \geq r(n-1, n)$. We want to show that $r(n-2, n-1) = r(n-1, n)$. Applying theorem 3.1 with $n' = n-1$ we find $d_1 = \ldots = d_{n-1} = t$ where $t = \frac{\delta_{n-1}}{\sqrt{n-2}} = r(n-2, n-1)$ with the $(n-2)$-ball being tangent to all faces of the $(n-1)$-box. Using the m^j's from (3.7), and setting $m^n = 0$, we can extend the $(n-2)$-flat containing the $(n-2)$-ball to an $(n-1)$-flat $\mathcal{M}$ containing an $(n-1)$-ball of radius $\frac{\delta_{n-1}}{\sqrt{n-2}}$. The ball lies in $\mathfrak{B}$ since $a_n > \frac{\delta_{n-1}}{\sqrt{n-2}}$. It is maximal in $\mathfrak{B}$ because $r(n-1, n)$ cannot exceed $r(n-2, n-1)$. This $(n-1)$-flat $\mathcal{M}$ is unique (up to the symmetry of $\mathfrak{B}$) as guaranteed by (3.6).

Now suppose $s = 2$ is the smallest integer satisfying (3.8). Then $a_{n-1} > \frac{\delta_{n-2}}{\sqrt{n-3}}$ and $a_{n-2} \leq \frac{\delta_{n-3}}{\sqrt{n-4}}$. Intersecting $\mathfrak{B}$ with the hyperplanes $x^n = 0$ and $x^{n-1} = 0$ gives us an $(n-2)$-box of dimensions $2a_1 \times \ldots \times 2a_{n-2}$. Intersecting any $(n-1)$-ball contained in $\mathfrak{B}$ (again, not parallel to any face of $\mathfrak{B}$) with the same hyperplanes yields an $(n-3)$-ball of equal radius which is contained in the $(n-2)$-box. It follows that $r(n-3, n-2) \geq r(n-1, n)$. Applying theorem 3.1 with $n' = n-2$ we see $d_1 = \ldots = d_{n-2} = t$ with $t = \frac{\delta_{n-2}}{\sqrt{n-3}} = r(n-3, n-2)$ with the $(n-3)$-ball tangent to all faces of the $(n-2)$-box. Using the m^j's from (3.7), and setting $m^{n-1} = m^n = 0$, we extend the $(n-3)$-flat containing the $(n-3)$-ball to an $(n-1)$-flat $\mathcal{M}$ containing an $(n-1)$-ball of radius $\frac{\delta_{n-2}}{\sqrt{n-3}}$. This ball lies in $\mathfrak{B}$ since $a_n \geq a_{n-1} > \frac{\delta_{n-2}}{\sqrt{n-3}}$ and it is maximal as $r(n-1, n)$ cannot be greater than $r(n-3, n-2)$. (3.6) again guarantees that this $(n-1)$-flat $\mathcal{M}$ is unique up to the symmetries of $\mathfrak{B}$.

We continue in this fashion until an s is found so that (3.8) is satisfied, which must happen since it is clearly true for $s = n-2$.

We mentioned at the beginning of this section that if $\mathcal{M}$ was parallel to $\mathcal{H}_j$ for $j = 1, \ldots, n$, that the largest radius any ball B lying on $\mathcal{M}$ could have was a_2. We now need to show that (3.9) is always strictly greater than a_2.

$$
\begin{aligned}
[r(n-1, n)]^2 &= \frac{\delta_{n-s}^2}{n-s-1} = \frac{\sum_{i=1}^{n-s} a_i^2}{n-s-1} \\
&\geq \frac{a_1^2 + (n-s-1)a_2^2}{n-s-1} = \frac{a_1^2}{n-s-1} + a_2^2 \\
&> a_2^2
\end{aligned}
$$

so that $r(n-1, n) > a_2$. Thus (3.9) is that largest possible radius any $(n-1)$-ball B can have in $\mathfrak{B}$. □

Corollary 3.1. *The $(n-1)$-flat $\mathcal{M}$ containing the largest $(n-1)$-ball in $\mathfrak{B}$ has as its normal vector*

$$
m = \begin{bmatrix} \dfrac{\sqrt{\delta_{n-s}^2 - (n-s-1)a_1^2}}{\delta_{n-s}} \\ \vdots \\ \dfrac{\sqrt{\delta_{n-s}^2 - (n-s-1)a_{n-s}^2}}{\delta_{n-s}} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{3.10}
$$

where s is as in theorem 3.2

Proof. From theorem 3.2, $m^{n-s+1}, \ldots, m^n = 0$. From theorem 3.1, with $n' = n-s$, we have for $j = 1, \ldots, n-s$

$$(m^j) = \frac{\sqrt{\delta_{n-s}^2 - (n-s-1)a_j^2}}{\delta_{n-s}}.$$

□

3.2.3 Contact Points

In this section, we assume that $\mathcal{B}$ is an $(n-1)$-ball of maximal radius in the n-box $\mathfrak{B}$ with radius $r(n-1,n) = \frac{\delta_{n-s}}{\sqrt{n-s-1}}$ where s is as in theorem 3.2. Let $\mathcal{M}$ be the $(n-1)$-flat that contains $\mathcal{B}$ with normal vector $\mathbf{m}$ as defined in corollary 3.1. Theorem 3.2 gives $d_{n-s+1} = a_{n-s+1}$ and $a_{n-s+1} > \frac{\delta_{n-s}}{\sqrt{n-s-1}} = r(n-1,n)$ so that $\mathcal{B} \cap \mathcal{M}_{n-s+1} = \emptyset$. Also, since $d_i = a_i$ for $i = n-s+2, \ldots, n$, and since the a_i's form an increasing sequence, we have that $\mathcal{B} \cap \mathcal{M}_i = \emptyset$ for $i = n-s+2, \ldots, n$. Theorem 3.1 guarantees that $\mathcal{B}$ is tangent to the first $n-s$ faces of $\mathfrak{B}$ so that $\mathcal{B} \cap \mathcal{M}_i = \{c_i\}$ for $i = 1, \ldots, n-s$. We will write these c_i's as $(c_i^1, \ldots, c_i^n)$. We have immediately that $\|\mathbf{c}_i\| = r(n-1,n)$.

Theorem 3.3. *For $i = 1, \ldots, n-s$ and $j = 1, \ldots, n$,*

$$c_i^j = \begin{cases} -\dfrac{m^i m^j \delta_{n-s}^2}{(n-s-1)a_i} & \text{if } j = 1, \ldots, n-s \text{ and } j \neq i, \\ a_i & \text{if } j = i, \\ 0 & \text{if } j = n-s+1, \ldots, n. \end{cases} \tag{3.11}$$

Proof. We will examine c_{n-s}, the contact point $\mathcal{B}$ makes with the $(n-s)$ face of $\mathfrak{B}$. Since $\mathcal{M}_{n-s} = \mathcal{M} \cap \mathcal{H}_{n-s}$, Any point $x \in \mathcal{M}_{n-s}$ must satisfy

$$\sum_{j=1}^{n-s} x^j m^j = 0, \tag{3.12}$$

$$x^{n-s} = a_{n-s}. \tag{3.13}$$

It follows then that $c_{n-s}^{n-s} = a_{n-s}$ and that the points $p_l = (p_l^1, \ldots, p_l^n)$ for $l =$

$1, \ldots, n-1$ given by

$$p_l^j = \begin{cases} -\dfrac{a_{n-s} m^{n-s}}{m^j} & \text{if } j = l, \\ a_{n-s} & \text{if } j = n-s, \\ 0 & \text{otherwise} \end{cases}$$

when $l = 1, \ldots, n-s-1$ and by

$$p_l^j = \begin{cases} -\dfrac{a_{n-s} m^{n-s}}{m^1} & \text{if } j = 1, \\ a_{n-s} & \text{if } j = n-s, \\ 1 & \text{if } j = l+1, \\ 0 & \text{otherwise} \end{cases}$$

when $l = n-s, \ldots, n-1$ all belong to $\mathcal{M}_{n-s}$ since they satisfy conditions (3.12) and (3.13). We can then construct $n-2$ vectors $\mathbf{v}_l = \mathbf{p}_1\mathbf{p}_l$ where $l = 2, \ldots, n-1$. In terms of coordinates, we have

$$v_l^j = \begin{cases} \dfrac{a_{n-s} m^{n-s}}{m^1} & \text{if } j = 1, \\ -\dfrac{a_{n-s} m^{n-s}}{m^l} & \text{if } j = l, \\ 0 & \text{otherwise} \end{cases}$$

when $l = 2, \ldots, n-s-1$ and by

$$v_l^j = \begin{cases} 1 & \text{if } j = l+1, \\ 0 & \text{otherwise} \end{cases}$$

when $l = n-s, \ldots, n-1$. The $(n-2)$ $\mathbf{v}_l$'s form a linearly independent set and thus a basis for $\mathcal{M}_{n-s}$ since

$$\begin{aligned} \dim(\mathcal{M}_{n-s}) &= \dim(\mathcal{M}) + \dim(\mathcal{H}_{n-s}) - \dim(\mathcal{M} + \mathcal{H}_{n-s}) \\ &= (n-1) + (n-1) - n \\ &= n-2. \end{aligned}$$

Now, since c_{n-s} is the point of tangency of $\mathcal{B}$ with $\mathcal{M}_{n-s}$, we must have that

$\langle \mathbf{v}_l, \mathbf{c}_{n-s} \rangle = 0$ for $l = 2, \ldots, n-1$. First, for $l = n-s, \ldots, n-1$,

$$\langle \mathbf{v}_l, \mathbf{c}_{n-s} \rangle = c_{n-s}^{l+1} = 0$$

so that $c_{n-s}^{n-s+1} = \ldots = c_{n-s}^{n} = 0$. Then, for $l = 2, \ldots, n-s-1$,

$$\langle \mathbf{v}_l, \mathbf{c}_{n-s} \rangle = \frac{c_{n-s}^1 a_{n-s} m^{n-s}}{m^1} - \frac{c_{n-s}^l a_{n-s} m^{n-s}}{m^l} = 0$$

or equivalently

$$m^l = \frac{c_{n-s}^l m^1}{c_{n-s}^1}. \tag{3.14}$$

(3.12) can then be used to obtain

$$\sum_{l=1}^{n-s-1} m^l c_{n-s}^l = -a_{n-s} m^{n-s}$$

which, using (3.14) becomes

$$\begin{aligned} -a_{n-s} m^{n-s} &= \frac{m^1}{c_{n-s}^1} \sum_{l=1}^{n-s-1} (c_{n-s}^l)^2 \\ &= \frac{m^1}{c_{n-s}^1} \left(\frac{\delta_{n-s}^2}{n-s-1} - a_{n-s}^2 \right) \\ &= \frac{m^1}{c_{n-s}^1} \left(\frac{\delta_{n-s}^2 - (n-s-1) a_{n-s}^2}{n-s-1} \right) \\ &= \frac{m^1 (m^{n-s})^2 \delta_{n-s}^2}{(n-s-1) c_{n-s}^1}. \end{aligned}$$

Thus,

$$c_{n-s}^1 = -\frac{m^1 m^{n-s} \delta_{n-s}^2}{(n-s-1) a_{n-s}}.$$

Now we can use (3.14) to find the $c^2_{n-s}, \ldots, c^{n-s-1}_{n-s}$. For $l = 2, \ldots, n-s-1$,

$$\begin{aligned} c^l_{n-s} &= \frac{m^l c^1_{n-s}}{m^1} \\ &= -\frac{m^l m^1 m^{n-s} \delta^2_{n-s}}{m^1 (n-s-1) a_{n-s}} \\ &= -\frac{m^l m^{n-s} \delta^2_{n-s}}{(n-s-1) a_{n-s}}. \end{aligned}$$

A similar development follows for $c_1, \ldots, c_{n-s-1}$. □

3.2.4 The (2, 3) Problem

In this short section, we state the results of the previous section for the case $k = 2$ and $n = 3$. The contact points listed below are correct for the planes listed just before them. If one of the other planes are used, the contact points should be adjusted appropriately. We have that

$$r(2,3) = \begin{cases} \sqrt{\dfrac{a_1^2 + a_2^2 + a_3^2}{2}} & \text{if } a_3 \leq \sqrt{a_1^2 + a_2^2}, \\ \\ \sqrt{a_1^2 + a_2^2} & \text{if } a_3 > \sqrt{a_1^2 + a_2^2} \end{cases}$$

and $\mathcal{M} = \{\mathbf{x} \in \mathbb{R}^3 | \langle \mathbf{m}, \mathbf{x} \rangle = 0\}$ where

$$\mathbf{m} = \begin{bmatrix} \sqrt{\dfrac{-a_1^2 + a_2^2 + a_3^2}{a_1^2 + a_2^2 + a_3^2}} \\ \\ \sqrt{\dfrac{a_1^2 - a_2^2 + a_3^2}{a_1^2 + a_2^2 + a_3^2}} \\ \\ \sqrt{\dfrac{a_1^2 + a_2^2 - a_3^2}{a_1^2 + a_2^2 + a_3^2}} \end{bmatrix}$$

if $a_3 \leq \sqrt{a_1^2 + a_2^2}$. If however, $a_3 > \sqrt{a_1^2 + a_2^2}$, then we find

$$\mathbf{m} = \begin{bmatrix} \dfrac{a_2}{\sqrt{a_1^2 + a_2^2}} \\ \\ \dfrac{a_1}{\sqrt{a_1^2 + a_2^2}} \\ \\ 0 \end{bmatrix}.$$

The contact points are given as follows:

$$c_1 = \left(a_1, -\frac{\sqrt{-a_1^2 + a_2^2 + a_3^2}\sqrt{a_1^2 - a_2^2 + a_3^2}}{2a_1}, -\frac{\sqrt{-a_1^2 + a_2^2 + a_3^2}\sqrt{a_1^2 + a_2^2 - a_3^2}}{2a_1} \right)$$
$$c_2 = \left(-\frac{\sqrt{a_1^2 - a_2^2 + a_3^2}\sqrt{-a_1^2 + a_2^2 + a_3^2}}{2a_2}, a_2, -\frac{\sqrt{a_1^2 - a_2^2 + a_3^2}\sqrt{a_1^2 + a_2^2 - a_3^2}}{2a_2} \right)$$
$$c_3 = \left(-\frac{\sqrt{a_1^2 + a_2^2 - a_3^2}\sqrt{-a_1^2 + a_2^2 + a_3^2}}{2a_3}, -\frac{\sqrt{a_1^2 + a_2^2 - a_3^2}\sqrt{a_1^2 - a_2^2 + a_3^2}}{2a_3}, a_3 \right)$$

for $a_3 \leq \sqrt{a_1^2 + a_2^2}$, and they are given as

$$c_1 = (a_1, -a_2, 0)$$
$$c_2 = (-a_1, a_2, 0)$$

when $a_3 > \sqrt{a_1^2 + a_2^2}$. Figure 3.1 will help to illustrate the $(2,3)$ case.

3.3 The (k, n) Problem

In this section, we will generalize the results of section 3.2 to find the radius of the largest k-ball contained in an n-box. Let $\mathcal{M}$ be a k-flat which contains a k-ball $\mathcal{B} \subset \mathfrak{B}$. Let $\mathbf{m}_1, \ldots, \mathbf{m}_{n-k}$ be the $n-k$ normal vectors to $\mathcal{M}$ which satisfy

$$\langle \mathbf{m}_i, \mathbf{m}_j \rangle = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases} \tag{3.15}$$

We first consider the case of when a k-flat $\mathcal{M}$ (and thus the k-ball $\mathcal{B}$) is parallel to some of the $\mathcal{H}_j$'s. If $\mathcal{M}$ is parallel to $\mathcal{H}_j$ for some $j = 1, \ldots, n$, then $\mathcal{M} \subset \overrightarrow{\mathcal{H}}_j$,

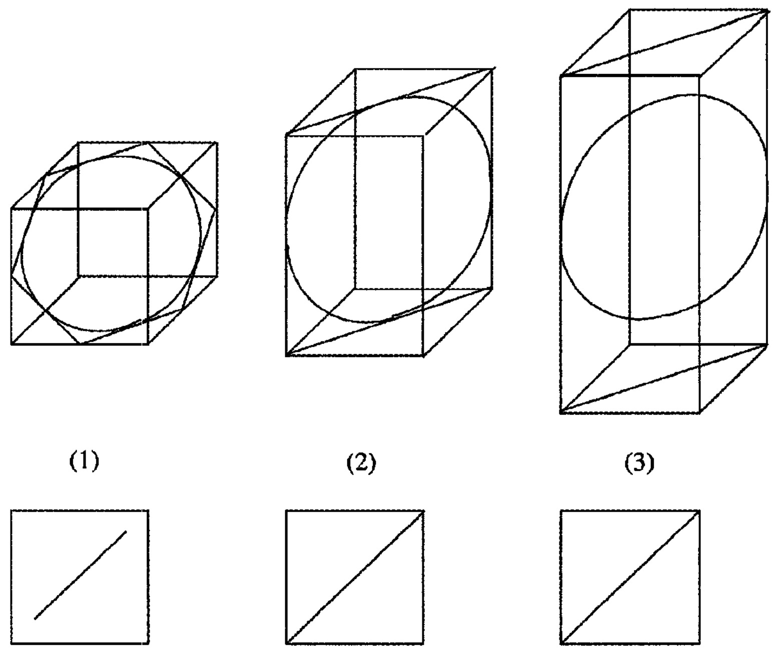

Figure 3.1:
The largest 2-ball in a 3-box for
$(1) : a_3 < \sqrt{a_1^2 + a_2^2}$, $(2) : a_3 = \sqrt{a_1^2 + a_2^2}$, $(3) : a_3 > \sqrt{a_1^2 + a_2^2}$
with cross sections in the xy-plane.

or equivalently, $\overrightarrow{\mathcal{H}}_j^{\perp} \subset \mathcal{M}^{\perp}$. Since $\overrightarrow{\mathcal{H}}_j^{\perp} = \text{span}\{\mathbf{e}_j\}$, it follows that $\mathbf{e}_j \in \mathcal{M}^{\perp}$, and since $\dim(\mathcal{M}^{\perp}) = n - k$, we have that $\mathcal{M}$ can be parallel to at most $n - k$ of the $\mathcal{H}_j$'s. For a given $f = 1, \ldots, n-k$, there are $\binom{n}{f}$ distinct k-flats $\mathcal{M}$ that are lying parallel to exactly f of the $\mathcal{H}_j$'s. We can view any k-ball $\mathcal{B}$ lying on such a k-flat $\mathcal{M}$ as being contained in an $(n - f)$-box of dimensions $2\hat{a}_1 \times \ldots \times 2\hat{a}_{n-f}$ where $0 < \hat{a}_1 \leq \ldots \leq \hat{a}_{n-f}$ are the a_j's associated with $\mathfrak{B}$ such that $\mathcal{M}$ is not parallel to $\mathcal{H}_j$. Thus, we have $\binom{n}{f}$ such boxes to consider. Our work is reduced when we realize that we only have to consider the largest of these $(n - f)$-boxes, since if any k-ball of radius r is contained in one of the smaller $(n - f)$-boxes, then a k-ball of the same radius can also be contained in the largest $(n - f)$-box since each of its measurements is no smaller then the corresponding measurements of the smaller $(n - f)$-boxes. Since the a_j's are increasing, we only need to consider the k-ball $\mathcal{B}$ lying on the k-flat $\mathcal{M}$ that is parallel to the first f of the $\mathcal{H}_j$'s. We can view this ball as being contained in an $(n-f)$-box of dimensions $2a_{f+1} \times \ldots \times 2a_n$. The important thing here to note is that $\mathcal{M}$ is not parallel to any of the faces of the $(n-f)$-box. Since we have not developed a formula for the maximal radius of a k-ball in an $(n-f)$-box yet, we will simply denote it as $\hat{r}(k, n-f)$. We will say more about this at the end of this section, but for now we will assume that $\mathcal{M}$ is not parallel to $\mathcal{H}_j$ for $j = 1, \ldots, n$, or equivalently, that none of the $\mathcal{M}_j$'s are empty.

3.3.1 A Formula for d_j

We will now develop a formula for d_j, $j = 1, \ldots, n$. Since $\mathcal{M}$ is not parallel to any $\mathcal{H}_j$, it follows that $\mathbf{e}_j \notin \text{span}\{\mathbf{m}_1, \ldots, \mathbf{m}_{n-k}\}$. Let $p \in \mathcal{M}_j$. Then $p \in \mathcal{M}$ so that $\langle \mathbf{p}, \mathbf{m}_i \rangle = 0$ for $i = 1, \ldots, n-k$, and $p \in \mathcal{H}_j$ so $\langle \mathbf{p}, \mathbf{e}_j \rangle = a_j$. Again, we can write $\mathbf{p}$ as $a_j \mathbf{v}$ for a suitable $\mathbf{v}$. The distance, d'_j, from the origin to the the $(n-k+1)$-flat $p + \mathcal{M}_j^{\perp}$ is given by

$$(d'_j)^2 = \frac{\text{Gram}(\mathbf{p}, \mathbf{m}_1, \ldots, \mathbf{m}_{n-k}, \mathbf{e}_j)}{\text{Gram}(\mathbf{m}_1, \ldots, \mathbf{m}_{n-k}, \mathbf{e}_j)}. \tag{3.16}$$

The following two lemmas give formulas for evaluating (3.16), the first deals with the denominator, the second with the numerator.

Lemma 3.5. $\text{Gram}(\boldsymbol{m}_1, \ldots, \boldsymbol{m}_n, \boldsymbol{e}_j) = 1 - \sum_{i=1}^{n} (m_i^j)^2.$

Proof. We will use induction on n. For $n = 1$,

$$\text{Gram}(\mathbf{m}_1, \mathbf{e}_j) = \begin{vmatrix} 1 & m_1^j \\ m_1^j & 1 \end{vmatrix} = 1 - (m_1^j)^2.$$

Now, assume $\text{Gram}(\mathbf{m}_1, \ldots, \mathbf{m}_n, \mathbf{e}_j) = 1 - \sum_{i=1}^{n} (m_i^j)^2$.

$$\text{Gram}(\mathbf{m}_1, \ldots, \mathbf{m}_{n+1}, \mathbf{e}_j) = \begin{vmatrix} 1 & 0 & \ldots & 0 & m_1^j \\ 0 & 1 & \ldots & 0 & m_2^j \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & m_{n+1}^j \\ m_1^j & m_2^j & \ldots & m_{n+1}^j & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & \dots & 0 & m_2^j \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 1 & m_{n+1}^j \\ m_2^j & \dots & m_{n+1}^j & 1 \end{vmatrix} + (-1)^{n+1} m_1^j \begin{vmatrix} 0 & \dots & 0 & m_1^j \\ 1 & \dots & 0 & m_2^j \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 1 & m_{n+1}^j \end{vmatrix}$$

$$= 1 - \sum_{i=2}^{n+1} (m_i^j)^2 + (-1)^{n+1}(-1)^n (m_1^j)^2$$

$$= 1 - \sum_{i=1}^{n+1} (m_i^j)^2.$$

□

Lemma 3.6. $\mathrm{Gram}(\boldsymbol{p}, \boldsymbol{m}_1, \dots, \boldsymbol{m}_n, \boldsymbol{e}_j) = a_j^2 \|\boldsymbol{v}\|^2 \left(1 - \sum_{i=1}^{n} (m_i^j)^2\right) - a_j^2.$

Proof.

$$\mathrm{Gram}(\mathbf{p}, \mathbf{m}_1, \dots, \mathbf{m}_n, \mathbf{e}_j) = \begin{vmatrix} a_j^2\|\mathbf{v}\|^2 & 0 & 0 & \dots & 0 & a_j \\ 0 & 1 & 0 & \dots & 0 & m_1^j \\ 0 & 0 & 1 & \dots & 0 & m_2^j \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & m_n^j \\ a_j & m_1^j & m_2^j & \dots & m_n^j & 1 \end{vmatrix}$$

$$= a_j^2 \|\mathbf{v}\|^2 \, \mathrm{Gram}(\mathbf{m}_1, \dots, \mathbf{m}_n, \mathbf{e}_j) + (-1)^{n+1} a_j \begin{vmatrix} 0 & 0 & \dots & 0 & a_j \\ 1 & 0 & \dots & 0 & m_1^j \\ 0 & 1 & \dots & 0 & m_2^j \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & m_n^j \end{vmatrix}$$

$$= a_j^2 \|\mathbf{v}\|^2 \left(1 - \sum_{i=1}^{n} (m_i^j)^2\right) + (-1)^{n+1}(-1)^n a_j^2$$

$$= a_j^2 \|\mathbf{v}\|^2 \left(1 - \sum_{i=1}^{n} (m_i^j)^2\right) - a_j^2.$$

□

We can now use the results of the previous two lemmas and (3.16) to find a formula for d_j.

Lemma 3.7. *For* $j = 1, \ldots, n$,

$$d_j^2 = \frac{a_j^2}{1 - \sum_{i=1}^{n-k} (m_i^j)^2}. \tag{3.17}$$

Proof.

$$\begin{aligned}
d_j^2 &= \|\mathbf{p}\|^2 - (d_j')^2 \\
&= a_j^2\|\mathbf{v}\|^2 - \frac{\mathrm{Gram}(\mathbf{p}, \mathbf{m}_1, \ldots, \mathbf{m}_{n-k}, \mathbf{e}_j)}{Gram(\mathbf{m}_1, \ldots, \mathbf{m}_{n-k}, \mathbf{e}_j)} \\
&= \frac{a_j^2\|\mathbf{v}\|^2 \left(1 - \sum_{i=1}^{n-k}(m_i^j)^2\right) - a_j^2\|\mathbf{v}\|^2 \left(1 - \sum_{i=1}^{n-k}(m_i^j)^2\right) + a_j^2}{1 - \sum_{i=1}^{n-k}(m_i^j)^2} \\
&= \frac{a_j^2}{1 - \sum_{i=1}^{n-k}(m_i^j)^2}.
\end{aligned}$$

□

Lemma 3.8. *For any k-flat* $\mathcal{M}$ *not parallel to* $\mathcal{H}_j$ *for* $j = 1, \ldots, n$, *we have*

$$\sum_{j=1}^{n} \frac{a_j^2}{d_j^2} = k. \tag{3.18}$$

Proof.

$$\begin{aligned}
\sum_{j=1}^{n} \frac{a_j^2}{d_j^2} &= \sum_{j=1}^{n} \frac{a_j^2\left(1 - \sum_{i=1}^{n-k}(m_i^j)^2\right)}{a_j^2} = \sum_{j=1}^{n} 1 - \sum_{j=1}^{n}\sum_{i=1}^{n-k}(m_i^j)^2 \\
&= n - \sum_{i=1}^{n-k}\sum_{j=1}^{n}(m_i^j)^2 = n - \sum_{i=1}^{n-k}(1) = n - (n-k) = k.
\end{aligned}$$

□

3.3.2 Choosing an Orthonormal Basis

Let $1 \leq k' < n'$. In the next section, we will be interested in finding a set of $n' - k'$ orthonormal vectors $\mathbf{m}_1, \ldots, \mathbf{m}_{n'-k'} \in \mathbb{R}^{n'}$ such that for each $j = 1, \ldots, n'$, we have

$$\sum_{i=1}^{n'-k'} (m_i^j)^2 = \frac{\delta_{n'}^2 - k' a_j^2}{\delta_{n'}^2} \tag{3.19}$$

where $0 < a_1 \leq a_2 \leq \ldots \leq a_{n'}$ and $a_{n'} \leq \frac{\delta_{n'-1}}{\sqrt{k'-1}}$. In this section, we find such a set of $n'-k'$ vectors. We will first consider an $n' \times (n'-1)$ matrix with orthonormal columns that will be useful in defining $\mathbf{m}_1, \ldots, \mathbf{m}_{n'-k'}$. In what follows, c_i denotes $\cos(\theta_i)$ and s_i denotes $\sin(\theta_i)$, for $i = 1, \ldots, n'-1$ and for each i, $0 \leq \theta_i < 2\pi$.

Lemma 3.9. *The $n' \times (n'-1)$ matrix*

$$\mathcal{R} = \begin{bmatrix} c_1 & 0 & \cdots & 0 & 0 \\ -s_1 s_2 & c_2 & \cdots & 0 & 0 \\ -s_1 c_2 s_3 & -s_2 s_3 & \cdots & 0 & 0 \\ -s_1 c_2 c_3 s_4 & -s_2 c_3 s_4 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -s_1 c_2 \cdots c_{n'-3} s_{n'-2} & -s_2 c_3 \cdots c_{n'-3} s_{n'-2} & \cdots & c_{n'-2} & 0 \\ -s_1 c_2 \cdots c_{n'-2} s_{n'-1} & -s_2 c_3 \cdots c_{n'-2} s_{n'-1} & \cdots & -s_{n'-2} s_{n'-1} & c_{n'-1} \\ -s_1 c_2 \cdots c_{n'-2} c_{n'-1} & -s_2 c_3 \cdots c_{n'-2} c_{n'-1} & \cdots & -s_{n'-2} c_{n'-1} & -s_{n'-1} \end{bmatrix} \tag{3.20}$$

has orthonormal columns.

Proof. Let $\mathbf{r}_i$ denote the ith column of $\mathcal{R}$. We will first show that the $\mathbf{r}_i$'s have unit length. Of course, this is clear for $\mathbf{r}_{n'-1}$, so we assume $1 \leq i \leq n'-2$.

$$\begin{aligned} \|\mathbf{r}_i\|^2 &= c_i^2 + s_i^2 \sum_{p=i+1}^{n'-1} \left(\prod_{l=i+1}^{p-1} c_l^2 \right) s_p^2 + s_i^2 \prod_{l=i+1}^{n'-1} c_l^2 \\ &= c_i^2 + s_i^2 \left(\sum_{p=i+1}^{n'-1} \left(s_p^2 \prod_{l=i+1}^{p-1} c_l^2 \right) + \prod_{l=i+1}^{n'-1} c_l^2 \right). \end{aligned}$$

Pulling the last term out of the sum and the second product, we have

$$\|\mathbf{r}_i\|^2 = c_i^2 + s_i^2 \left(\sum_{p=i+1}^{n'-2} \left(s_p^2 \prod_{l=i+1}^{p-1} c_l^2 \right) + s_{n'-1}^2 \prod_{l=i+1}^{n'-2} c_l^2 + c_{n'-1}^2 \prod_{l=i+1}^{n'-2} c_l^2 \right),$$

and since $c_i^2 + s_i^2 = 1$ for $i = 1, \ldots, n'-1$,

$$\|\mathbf{r}_i\|^2 = c_i^2 + s_i^2 \left(\sum_{p=i+1}^{n'-2} \left(s_p^2 \prod_{l=i+1}^{p-1} c_l^2 \right) + \prod_{l=i+1}^{n'-2} c_l^2 \right).$$

We continue in this fashion until we get

$$\|\mathbf{r}_i\|^2 = c_i^2 + s_i^2 \left(\sum_{p=i+1}^{i+2} \left(s_p^2 \prod_{l=i+1}^{p-1} c_l^2 \right) + \prod_{l=i+1}^{i+2} c_l^2 \right).$$

Now we use the convention that for any function f and any integers i, m with $m < i$, the empty product $\prod_{j=i}^{m} f(j) = 1$. With this in mind, we have

$$\begin{aligned}
\|\mathbf{r}_i\|^2 &= c_i^2 + s_i^2 \left(s_{i+1}^2 + s_{i+2}^2 c_{i+1}^2 + c_{i+1}^2 c_{i+2}^2 \right) \\
&= c_i^2 + s_i^2 \left(s_{i+1}^2 + c_{i+1}^2 \right) \\
&= c_i^2 + s_i^2 \\
&= 1,
\end{aligned}$$

and so each column of $\mathcal{R}$ has unit length. For orthogonality, we first assume $1 \leq i_1 < i_2 \leq n' - 2$. We have

$$\begin{aligned}
\langle \mathbf{r}_{i_1}, \mathbf{r}_{i_2} \rangle = s_{i_1} s_{i_2} \Bigg(& -c_{i_2} \prod_{l=i_1+1}^{i_2-1} c_l + \sum_{p=i_2+1}^{n'-1} \left(s_p^2 \prod_{l_1=i_1+1}^{p-1} c_{l_1} \prod_{l_2=i_2+1}^{p-1} c_{l_2} \right) + \\
& + \prod_{l_1=i_1+1}^{n'-1} c_{l_1} \prod_{l_2=i_2+1}^{n'-1} c_{l_2} \Bigg).
\end{aligned}$$

By pulling out the last term of the sum and each of the last two products, we get

$$\langle \mathbf{r}_{i_1}, \mathbf{r}_{i_2} \rangle = s_{i_1} s_{i_2} \left(-c_{i_2} \prod_{l=i_1+1}^{i_2-1} c_l + \sum_{p=i_2+1}^{n'-2} \left(s_p^2 \prod_{l_1=i_1+1}^{p-1} c_{l_1} \prod_{l_2=i_2+1}^{p-1} c_{l_2} \right) + \right.$$
$$\left. + s_{n'-1}^2 \prod_{l_1=i_1+1}^{n'-2} c_{l_1} \prod_{l_2=i_2+1}^{n'-2} c_{l_2} + c_{n'-1}^2 \prod_{l_1=i_1+1}^{n'-2} c_{l_1} \prod_{l_2=i_2+1}^{n'-2} c_{l_2} \right).$$

Again, since $c_i^2 + s_i^2 = 1$ for $i = 1, \ldots, n' - 1$,

$$\langle \mathbf{r}_{i_1}, \mathbf{r}_{i_2} \rangle = s_{i_1} s_{i_2} \left(-c_{i_2} \prod_{l=i_1+1}^{i_2-1} c_l + \sum_{p=i_2+1}^{n'-2} \left(s_p^2 \prod_{l_1=i_1+1}^{p-1} c_{l_1} \prod_{l_2=i_2+1}^{p-1} c_{l_2} \right) + \right.$$
$$\left. + \prod_{l_1=i_1+1}^{n'-2} c_{l_1} \prod_{l_2=i_2+1}^{n'-2} c_{l_2} \right).$$

We continue to pull the last term out of the sum and each of the last two products, until we arrive at

$$\langle \mathbf{r}_{i_1}, \mathbf{r}_{i_2} \rangle = s_{i_1} s_{i_2} \left(-c_{i_2} \prod_{l=i_1+1}^{i_2-1} c_l + \sum_{p=i_2+1}^{i_2+2} \left(s_p^2 \prod_{l_1=i_1+1}^{p-1} c_{l_1} \prod_{l_2=i_2+1}^{p-1} c_{l_2} \right) + \right.$$
$$\left. + \prod_{l_1=i_1+1}^{i_2+2} c_{l_1} \prod_{l_2=i_2+1}^{i_2+2} c_{l_2} \right).$$

Now, (recalling empty products are defined to be 1),

$$\sum_{p=i_2+1}^{i_2+2} \left(s_p^2 \prod_{l_1=i_1+1}^{p-1} c_{l_1} \prod_{l_2=i_2+1}^{p-1} c_{l_2} \right) = s_{i_2+1}^2 \prod_{l=i_1+1}^{i_2} c_l + s_{i_2+2}^2 \left(\prod_{l=i_1+1}^{i_2+1} c_l \right) c_{i_2+1},$$

and

$$\prod_{l_1=i_1+1}^{i_2+2} c_{l_1} \prod_{l_2=i_2+1}^{i_2+2} c_{l_2} = c_{i_2+2}^2 \left(\prod_{l=i_1+1}^{i_2+1} c_l \right) c_{i_2+1},$$

so that

$$
\begin{aligned}
\langle \mathbf{r}_{i_1}, \mathbf{r}_{i_2} \rangle &= s_{i_1} s_{i_2} \left(-c_{i_2} \prod_{l=i_1+1}^{i_2-1} c_l + s_{i_2+1}^2 \prod_{l=i_1+1}^{i_2} c_l + s_{i_2+2}^2 \left(\prod_{l=i_1+1}^{i_2+1} c_l \right) c_{i_2+1} + \right. \\
&\qquad \left. + c_{i_2+2}^2 \left(\prod_{l=i_1+1}^{i_2+1} c_l \right) c_{i_2+1} \right) \\
&= s_{i_1} s_{i_2} \left(-c_{i_2} \prod_{l=i_1+1}^{i_2-1} c_l + s_{i_2+1}^2 \prod_{l=i_1+1}^{i_2} c_l + c_{i_2+1} \prod_{l=i_1+1}^{i_2+1} c_l \right) \\
&= s_{i_1} s_{i_2} \left(-c_{i_2} \prod_{l=i_1+1}^{i_2-1} c_l + s_{i_2+1}^2 \prod_{l=i_1+1}^{i_2} c_l + c_{i_2+1}^2 \prod_{l=i_1+1}^{i_2} c_l \right) \\
&= s_{i_1} s_{i_2} \left(-c_{i_2} \prod_{l=i_1+1}^{i_2-1} c_l + \prod_{l=i_1+1}^{i_2} c_l \right) \\
&= s_{i_1} s_{i_2} \left(-c_{i_2} \prod_{l=i_1+1}^{i_2-1} c_l + c_{i_2} \prod_{l=i_1+1}^{i_2-1} c_l \right) \\
&= 0.
\end{aligned}
$$

Finally, for $1 \leq i \leq n' - 2$ we have (using empty products as necessary)

$$
\begin{aligned}
\langle \mathbf{r}_i, \mathbf{r}_{n'-1} \rangle &= -s_i \left(\prod_{l=i+1}^{n'-2} c_l \right) s_{n'-1} c_{n'-1} + s_i \left(\prod_{l=i+1}^{n'-1} c_l \right) c_{n'-1} s_{n'-1} \\
&= 0,
\end{aligned}
$$

and the $n' - 1$ columns of $\mathcal{R}$ form an orthonormal set. □

Our goal now is to decide which of the $n' - k'$ columns of $\mathcal{R}$ to use as the $\mathbf{m}_i$'s, and which values of $c_1, s_1 \ldots, c_{n'-1}, s_{n'-1}$ will satisfy (3.19) for $j = 1, \ldots, n'$. The next three lemmas will be useful in deciding which columns of $\mathcal{R}$ to use. They will be followed by a theorem that will give values for $c_1, s_1, \ldots, c_{n'-1}, s_{n'-1}$. First, for $i = 1, \ldots, n' - k' + 1$, consider the following $n' - k' + 1$ sequences:

$$
\left\{ (j - i + 1)\delta_{n'}^2 - k'\delta_j^2 \right\}_{j=i}^{k'+i-1}. \tag{3.21}
$$

Lemma 3.10. *For every $i = 1, \ldots, n' - k' + 1$, there is a $j = i, \ldots, k' + i - 1$ such* *that* $(j-i+1)\delta_{n'}^2 - k'\delta_j^2 \geq 0$. *Let* t_i *be the smallest such* j *for each* i. *Then the sequence* $\{t_i\}_{i=1}^{n'-k'+1}$ *is strictly increasing.*

Proof. To show the existence of such a j for each $i = 1, \dots, n' - k' + 1$, set $j = k' + i - 1$. We have that the $k' + i - 1$ term of each sequence is $k'\delta_{n'}^2 - k'\delta_{k'+i-1}^2$ which is nonnegative since $i \leq n' - k' + 1$ implies $k' + i - 1 \leq n'$. Now, for $i = 1, \dots, n' - k'$, let t_i be the smallest j such that $(j-i+1)\delta_{n'}^2 - k'\delta_j^2 \geq 0$. Then, for $j = i, \dots, t_i - 1$, $(j-i+1)\delta_{n'}^2 - k'\delta_j^2 < 0$. From this, it follows that $(j-i)\delta_{n'}^2 - k'\delta_j^2 < 0$ for $j = i+1, \dots, t_i$. Thus, the $j = t_i + 1$ term of the $i+1$ sequence $\{(j-i)\delta_{n'}^2 - k'\delta_j^2\}_{j=i+1}^{k'+i}$ is the first term that can be non-negative so that $t_{i+1} > t_i$. This shows that $\{t_i\}_{i=1}^{n'-k'+1}$ is a strictly increasing sequence. □

Lemma 3.11. *Let* $k' \geq 2$ *and* $i = 1, \dots, n' - k' + 1$. *If* $(j-i+1)\delta_{n'}^2 - k'\delta_j^2 > (j-i)\delta_{n'}^2 - k'\delta_{j-1}^2$ *for* $j \geq i+1$, *then* $(j-i)\delta_{n'}^2 - k'\delta_{j-1}^2 > (j-i-1)\delta_{n'}^2 - k'\delta_{j-2}^2$.

Proof. For $i = 1, \dots, n' - k' + 1$ and $j \geq i + 1$. We have

$$\begin{aligned}
(j-i+1)\delta_{n'}^2 - k'\delta_j^2 &> (j-i)\delta_{n'}^2 - k'\delta_{j-1}^2 \\
(j-i+1-(j-i))\delta_{n'}^2 &> k'(\delta_j^2 - \delta_{j-1}^2) \\
\delta_{n'}^2 &> k'a_j^2.
\end{aligned}$$

But since $a_j \geq a_{j-1}$,

$$\begin{aligned}
\delta_{n'}^2 &> k'a_{j-1}^2 \\
(j-i-(j-i-1))\delta_{n'}^2 &> k'(\delta_{j-1}^2 - \delta_{j-2}^2) \\
(j-i)\delta_{n'} - k'\delta_{j-1}^2 &> (j-i-1)\delta_{n'}^2 - k'\delta_{j-2}^2.
\end{aligned}$$

□

Lemma 3.12. *For* $i = 1, \dots, n' - k' + 1$, *the sequence* $\{(j-i+1)\delta_{n'}^2 - k'\delta_j^2\}_{j=i}^{t_i}$ *is strictly increasing and only the* $j = t_i$ *term is nonnegative.*

Proof. Let $i = 1, \dots, n' - k' + 1$. If $t_i = i$, the result is trivial as the ith sequence contains only one term, which is nonnegative. If $t_i > i$ then t_i is the smallest value of j such that $(j-i+1)\delta_{n'}^2 - k'\delta_j^2 \geq 0$. Setting $j = t_i - 1$, we then have $(j-i)\delta_{n'}^2 - k'\delta_{j-1}^2 <$

0 and so $(j-i+1)\delta_{n'}^2 - k'\delta_j^2 > (j-i)\delta_{n'}^2 - k'\delta_{j-1}^2$. Repeated applications of lemma 3.11 show for $j = i+1, \ldots, t_i$, $(j-i+1)\delta_{n'} - k'\delta_j^2 > (j-i)\delta_{n'} - k'\delta_{j-1}^2$ as required. □

Remark 3.1. For $k' = 1, \ldots, n'-1$ we have that $t_1 = 1$ since for $i = 1$, the sequence $\{j\delta_{n'}^2 - k'\delta_j^2\}_{j=1}^k$ has as its first term $\delta_{n'}^2 - k'\delta_1^2 = \delta_{n'}^2 - k'a_1^2$ which is clearly positive. Setting $i = n'-k'+1$, we find the resulting sequence $\{(j-n'+k')\delta_{n'}^2 - k'\delta_j^2\}_{j=n'-k'+1}^{n'-1}$ has as its last term $k'\delta_{n'}^2 - k'\delta_{n'}^2 = 0$. So $t_{n'-k'+1} \leq n'$. Thus, the first $n'-k'$ t_i's each correspond to a unique column of the matrix $\mathcal{R}$.

We can now use the t_i's and the matrix $\mathcal{R}$ to decide which vectors to use for the $\mathbf{m}_i$'s.

Theorem 3.4. *Let $\mathbf{m}_i$ be the t_ith column of $\mathcal{R}$ for $i = 1, \ldots, n'-k'$. Let*

$$c_j^2 = \begin{cases} \dfrac{(j-i+1)\delta_{n'}^2 - k'\delta_j^2}{(j-i+1)\delta_{n'}^2 - k'\delta_{j-1}^2} & \text{if } j = t_i, \\[2ex] \dfrac{(j-i)\delta_{n'}^2 - k'\delta_j^2}{(j-i-1)\delta_{n'}^2 - k'\delta_{j-1}^2} & \text{if } t_i < j < t_{i+1}, \\[2ex] 0 & \text{if } t_{n'-k'+1} < j < n', \end{cases} \tag{3.22}$$

$$s_j^2 = \begin{cases} \dfrac{k'a_j^2}{(j-i+1)\delta_{n'}^2 - k'\delta_{j-1}^2} & \text{if } j = t_i, \\[2ex] \dfrac{k'a_j^2 - \delta_{n'}^2}{(j-i-1)\delta_{n'}^2 - k'\delta_{j-1}^2} & \text{if } t_i < j < t_{i+1}, \\[2ex] 1 & \text{if } t_{n'-k'+1} < j < n', \end{cases} \tag{3.23}$$

where $i = 1, \ldots, n'-k'+1$ and $j = 1, \ldots, n'-1$. Then the $\mathbf{m}_i$'s are an orthonormal set and for $j = 1, \ldots, n'$,

$$\sum_{i=1}^{n'-k'} (m_i^j)^2 = \frac{\delta_{n'}^2 - k'a_j^2}{\delta_{n'}^2}. \tag{3.24}$$

Proof. First, we will show that our definitions for c_j^2 and s_j^2 make sense. First, for $j = t_i$ where $i = 1, \ldots, n'-k'+1$ we have

$$c_j^2 + s_j^2 = \frac{(j-i+1)\delta_{n'}^2 - k'\delta_j^2 + k'a_j^2}{(j-i+1)\delta_{n'}^2 - k'\delta_{j-1}^2} = \frac{(j-i+1)\delta_{n'}^2 - k'\delta_{j-1}^2}{(j-i+1)\delta_{n'}^2 - k'\delta_{j-1}^2} = 1,$$

and for $j \neq t_i$ where $i = 1, \ldots, n' - k'$ we have

$$c_j^2 + s_j^2 = \frac{(j-i)\delta_{n'}^2 - k'\delta_j^2 + k'a_j^2 - \delta_{n'}^2}{(j-i-1)\delta_{n'}^2 - k'\delta_{j-1}^2} = \frac{(j-i-1)\delta_{n'}^2 - k'\delta_{j-1}^2}{(j-i-1)\delta_{n'}^2 - k'\delta_{j-1}^2} = 1.$$

Of course, $c_j^2 + s_j^2 = 1$ for $j > t_{n'-k'+1}$. Now, for $j = t_i$ with $i = 1, \ldots, n' - k' + 1$, $(j-i+1)\delta_{n'}^2 - k'\delta_j^2 \geq 0$ since it is the first non-negative term of the ith sequence $\{(j-i+1)\delta_{n'}^2 - k'\delta_j^2\}_{j=i}^{k'+i-1}$. Now $(j-i+1)\delta_{n'}^2 - k'\delta_j^2 < (j-i+1)\delta_{n'}^2 - k'\delta_{j-1}^2$, so that $0 \leq c_j^2 < 1$ and so $0 < s_j^2 \leq 1$. For $t_i < j < t_{i+1}$ with $i = 1, \ldots, n' - k'$, $(j-i)\delta_{n'}^2 - k'\delta_j^2 < 0$ as it is the jth term of the sequence $\{(j-i)\delta_{n'}^2 - k'\delta_j^2\}_{j=i+1}^{k'+i}$ and since lemma 3.12 guarantees $(j-i-1)\delta_{n'}^2 - k'\delta_{j-1}^2 < (j-i)\delta_{n'}^2 - k'\delta_j^2$ we find $0 < c_j^2 < 1$ and thus $0 < s_j^2 < 1$. So our definitions of c_j^2 and s_j^2 are well defined. The fact that the $\mathbf{m}_i$'s form an orthonormal set follows from the result that the columns of $\mathcal{R}$ form an orthonormal set.

To prove (3.24), we will use induction on j. For $j = 1$, we have that $i = 1$ and $t_i = 1$ so that

$$\sum_{i=1}^{n'-k'} (m_i^1)^2 = c_1^2 = \frac{\delta_{n'}^2 - k'a_1^2}{\delta_{n'}^2}.$$

Our inductive step actually consists of six cases, which must be proved individually. For the first four cases, we assume $1 \leq j \leq n' - 2$.

Case I: $j = t_i$ and $t_i < j + 1 < t_{i+1}$ for some $i = 1, \ldots, n' - k'$.
If we assume that (3.24) is satisfied for $j = t_i$ for some $i = 1, \ldots, n' - k'$, then we have that

$$\begin{aligned}\sum_{i=1}^{n'-k'} (m_i^j)^2 &= \sum_{l=1}^{t_{i-1}} \left(s_{t_l}^2 c_{t_l+1}^2 \cdots c_{j-1}^2 s_j^2\right) + c_j^2 \\ &= \frac{\delta_{n'}^2 - k'a_j^2}{\delta_{n'}^2}.\end{aligned}$$

Then,

$$\begin{aligned}\sum_{i=1}^{n'-k'} (m_i^{j+1})^2 &= \sum_{l=1}^{t_{i-1}} \left(s_{t_l}^2 c_{t_l+1}^2 \cdots c_j^2 s_{j+1}^2\right) + s_j^2 s_{j+1}^2 \\ &= \left(\sum_{l=1}^{t_{i-1}} \left(s_{t_l}^2 c_{t_l+1}^2 \cdots c_{j-1}^2 s_j^2\right)\right) \frac{c_j^2 s_{j+1}^2}{s_j^2} + s_j^2 s_{j+1}^2,\end{aligned}$$

or more simply,

$$\sum_{i=1}^{n'-k'} (m_i^{j+1})^2 = s_{j+1}^2 \left(\left(\frac{\delta_{n'}^2 - k' a_j^2}{\delta_{n'}^2} - c_j^2 \right) \frac{c_j^2}{s_j^2} + s_j^2 \right). \tag{3.25}$$

Using (3.22) and (3.23), we find

$$\begin{aligned} c_j^2 &= \frac{(j-i+1)\delta_{n'}^2 - k'\delta_j^2}{(j-i+1)\delta_{n'}^2 - k'\delta_{j-1}^2}, \\ s_j^2 &= \frac{k' a_j^2}{(j-i+1)\delta_{n'}^2 - k'\delta_{j-1}^2}, \\ s_{j+1}^2 &= \frac{k' a_{j+1}^2 - \delta_{n'}^2}{(j-i)\delta_{n'}^2 - k'\delta_j^2}. \end{aligned}$$

We then find

$$\begin{aligned} \frac{\delta_{n'}^2 - k' a_j^2}{\delta_{n'}^2} - c_j^2 &= \frac{\delta_{n'}^2 - k' a_j^2}{\delta_{n'}^2} - \frac{(j-i+1)\delta_{n'}^2 - k'\delta_j^2}{(j-i+1)\delta_{n'}^2 - k'\delta_{j-1}^2} \\ &= 1 - \frac{k' a_j^2}{\delta_{n'}^2} - \frac{(j-i+1)\delta_{n'}^2 - k'\delta_{j-1}^2}{(j-i+1)\delta_{n'}^2 - k'\delta_{j-1}^2} + \frac{k' a_j^2}{(j-i+1)\delta_{n'}^2 - k'\delta_{j-1}^2} \\ &= \frac{k' a_j^2}{(j-i+1)\delta_{n'}^2 - k'\delta_{j-1}^2} - \frac{k' a_j^2}{\delta_{n'}^2}, \end{aligned}$$

and

$$\begin{aligned}\frac{c_j^2}{s_j^2} &= \frac{(j-i+1)\delta_{n'}^2 - k'\delta_j^2}{(j-i+1)\delta_{n'}^2 - k'\delta_{j-1}^2} \cdot \frac{(j-i+1)\delta_{n'}^2 - k'\delta_{j-1}^2}{k'a_j^2} \\ &= \frac{(j-i+1)\delta_{n'}^2 - k'\delta_j^2}{k'a_j^2} \\ &= \frac{(j-i+1)\delta_{n'}^2 - k'\delta_{j-1}^2}{k'a_j^2} - 1,\end{aligned}$$

so that

$$\left(\frac{\delta_{n'}^2 - k'a_j^2}{\delta_{n'}^2} - c_j^2\right)\frac{c_j^2}{s_j^2} = 1 - \frac{k'a_j^2}{(j-i+1)\delta_{n'}^2 - k'\delta_{j-1}^2} - \frac{(j-i+1)\delta_{n'}^2 - k'\delta_j^2}{\delta_{n'}^2}.$$

Adding s_j^2 yields

$$\begin{aligned}\left(\frac{\delta_{n'}^2 - k'a_j^2}{\delta_{n'}^2} - c_j^2\right)\frac{c_j^2}{s_j^2} + s_j^2 &= 1 - \frac{(j-i+1)\delta_{n'}^2 - k'\delta_j^2}{\delta_{n'}^2} \\ &= -\frac{(j-i)\delta_{n'}^2 - k'\delta_j^2}{\delta_{n'}^2},\end{aligned}$$

and we find, after multiplying by s_{j+1}^2, that (3.25) becomes

$$\begin{aligned}\sum_{i=1}^{n'-k'} (m_i^{j+1})^2 &= -\frac{k'a_{j+1}^2 - \delta_{n'}^2}{(j-i)\delta_{n'}^2 - k'\delta_j^2} \cdot \frac{(j-i)\delta_{n'}^2 - k'\delta_j^2}{\delta_{n'}^2} \\ &= \frac{\delta_{n'}^2 - k'a_{j+1}^2}{\delta_{n'}^2}.\end{aligned}$$

Thus (3.24) hold for $j = t_i$ and $t_i < j+1 < t_{i+1}$ for some $i = 1, \ldots, n' - k'$.

Case II: $t_i < j$ and $j + 1 < t_{i+1}$ for some $i = 1, \ldots, n' - k'$.
Suppose (3.24) holds for such a value of j. That is,

$$\begin{aligned}\sum_{i=1}^{n'-k'} (m_i^j)^2 &= \sum_{l=1}^{t_i} s_{t_l}^2 c_{t_l+1}^2 \cdots c_{j-1}^2 s_j^2 \\ &= \frac{\delta_{n'}^2 - k'a_j^2}{\delta_{n'}^2}.\end{aligned}$$

Then

$$\sum_{i=1}^{n'-k'} (m_i^{j+1})^2 = \sum_{l=1}^{t_i} \left(s_{t_l}^2 c_{t_i+1}^2 \cdots c_j^2 s_{j+1}^2\right)$$
$$= \left(\sum_{l=1}^{t_i} \left(s_{t_l}^2 c_{t_i+1}^2 \cdots c_{j-1}^2 s_j^2\right)\right) \frac{c_j^2 s_{j+1}^2}{s_j^2},$$

and thus

$$\sum_{i=1}^{n'-k'} (m_i^{j+1})^2 = \left(\frac{\delta_{n'}^2 - k' a_j^2}{\delta_{n'}^2}\right) \frac{c_j^2 s_{j+1}^2}{s_j^2}. \tag{3.26}$$

Using equations (3.22) and (3.23) we find

$$c_j^2 = \frac{(j-i)\delta_{n'}^2 - k'\delta_j^2}{(j-i-1)\delta_{n'}^2 - k'\delta_{j-1}^2},$$
$$s_j^2 = \frac{k' a_j^2 - \delta_{n'}^2}{(j-i-1)\delta_{n'}^2 - k'\delta_{j-1}^2},$$
$$s_{j+1}^2 = \frac{k' a_{j+1}^2 - \delta_{n'}^2}{(j-i)\delta_{n'}^2 - k'\delta_j^2}.$$

Then we see

$$\frac{c_j^2 s_{j+1}^2}{s_j^2} = \frac{(j-i)\delta_{n'}^2 - k'\delta_j^2}{(j-i-1)\delta_{n'}^2 - k'\delta_{j-1}^2} \cdot \frac{k' a_{j+1}^2 - \delta_{n'}^2}{(j-i)\delta_{n'}^2 - k'\delta_j^2} \cdot \frac{(j-i-1)\delta_{n'}^2 - k'\delta_{j-1}^2}{k' a_j^2 - \delta_{n'}^2}$$
$$= \frac{k' a_{j+1}^2 - \delta_{n'}^2}{k' a_j^2 - \delta_{n'}^2},$$

and so (3.26) becomes

$$\sum_{i=1}^{n'-k'} (m_i^{j+1})^2 = \left(\frac{\delta_{n'}^2 - k' a_j^2}{\delta_{n'}^2}\right) \frac{k' a_{j+1}^2 - \delta_{n'}^2}{k' a_j^2 - \delta_{n'}^2}$$
$$= \frac{\delta_{n'}^2 - k' a_{j+1}^2}{\delta_{n'}^2}.$$

And thus (3.24) hold for all j such that $t_i < j$ and $j+1 < t_{i+1}$ for some $i = 1, \ldots, n'-k'$.

Case III: $t_i < j < t_{i+1}$ and $j+1 = t_{i+1}$ for some $i = 1, \ldots, n' - k' - 1$.
If we assume (3.24) holds for such a value of j, then we have

$$\begin{aligned}\sum_{i=1}^{n'-k'} (m_i^j)^2 &= \sum_{l=1}^{t_i} \left(s_{t_l}^2 c_{t_l+1}^2 \cdots c_{j-1}^2 s_j^2\right) \\ &= \frac{\delta_{n'}^2 - k' a_j^2}{\delta_{n'}^2}.\end{aligned}$$

Then,

$$\begin{aligned}\sum_{i=1}^{n'-k'} (m_i^{j+1})^2 &= \sum_{l=1}^{t_i} \left(s_{t_l}^2 c_{t_l+1}^2 \cdots c_j^2 s_{j+1}^2\right) + c_{j+1}^2 \\ &= \left(\sum_{l=1}^{t_i} \left(s_{t_l}^2 c_{t_l+1}^2 \cdots c_{j-1}^2 s_j^2\right)\right) \frac{c_j^2 s_{j+1}^2}{s_j^2} + c_{j+1}^2.\end{aligned}$$

So for $j+1 = t_{i+1}$ we have

$$\sum_{i=1}^{n'-k'} (m_i^{j+1})^2 = \left(\frac{\delta_{n'}^2 - k' a_j^2}{\delta_{n'}^2}\right) \frac{c_j^2 s_{j+1}^2}{s_j^2} + c_{j+1}^2. \tag{3.27}$$

Using (3.22) and (3.23), we find

$$\begin{aligned}c_j^2 &= \frac{(j-i)\delta_{n'}^2 - k'\delta_j^2}{(j-i-1)\delta_{n'}^2 - k'\delta_{j-1}^2}, \\ s_j^2 &= \frac{k' a_j^2 - \delta_{n'}^2}{(j-i-1)\delta_{n'}^2 - k'\delta_{j-1}^2}, \\ c_{j+1}^2 &= \frac{(j-i+1)\delta_{n'}^2 - k'\delta_{j+1}^2}{(j-i+1)\delta_{n'}^2 - k'\delta_j^2}, \\ s_{j+1}^2 &= \frac{k' a_{j+1}^2}{(j-i+1)\delta_{n'}^2 - k'\delta_j^2},\end{aligned}$$

so that

$$\begin{aligned}\frac{c_j^2 s_{j+1}^2}{s_j^2} &= \frac{(j-i)\delta_{n'}^2 - k'\delta_j^2}{(j-i-1)\delta_{n'}^2 - k'\delta_{j-1}^2} \cdot \frac{k' a_{j+1}^2}{(j-i+1)\delta_{n'}^2 - k'\delta_j^2} \cdot \frac{(j-i-1)\delta_{n'}^2 - k'\delta_{j-1}^2}{k' a_j^2 - \delta_{n'}^2} \\ &= \frac{\left((j-i)\delta_{n'}^2 - k'\delta_j^2\right)\left(k' a_{j+1}^2\right)}{\left((j-i+1)\delta_{n'}^2 - k'\delta_j^2\right)\left(k' a_j^2 - \delta_{n'}^2\right)},\end{aligned}$$

and (3.27) becomes

$$
\begin{aligned}
\sum_{i=1}^{n'-k'} (m_i^{j+1})^2 &= -\frac{k' a_{j+1}^2 \left((j-i)\delta_{n'}^2 - k'\delta_j^2\right)}{\delta_{n'}^2 \left((j-i+1)\delta_{n'}^2 - k'\delta_j^2\right)} + \frac{(j-i+1)\delta_{n'}^2 - k'\delta_{j+1}^2}{(j-i+1)\delta_{n'}^2 - k'\delta_j^2} \\
&= -\frac{k' a_{j+1}^2 \left((j-i+1)\delta_{n'}^2 - k'\delta_j^2 - \delta_{n'}^2\right)}{\delta_{n'}^2 \left((j-i+1)\delta_{n'}^2 - k'\delta_j^2\right)} + \frac{(j-i+1)\delta_{n'}^2 - k'\delta_{j+1}^2}{(j-i+1)\delta_{n'}^2 - k'\delta_j^2} \\
&= -\frac{k' a_{j+1}^2}{\delta_{n'}^2} + \frac{k' a_{j+1}^2 \delta_{n'}^2}{\delta_{n'}^2 \left((j-i+1)\delta_{n'}^2 - k'\delta_j^2\right)} + \frac{(j-i+1)\delta_{n'}^2 - k'\delta_{j+1}^2}{(j-i+1)\delta_{n'}^2 - k'\delta_j^2} \\
&= -\frac{k' a_{j+1}^2}{\delta_{n'}^2} + \frac{(j-i+1)\delta_{n'}^2 - k'\delta_{j+1}^2 + k' a_{j+1}^2}{(j-i+1)\delta_{n'}^2 - k'\delta_j^2} \\
&= -\frac{k' a_{j+1}^2}{\delta_{n'}^2} + 1 \\
&= \frac{\delta_{n'}^2 - k' a_{j+1}^2}{\delta_{n'}^2}.
\end{aligned}
$$

Thus, (3.24) is satisfied when $t_i < j < t_{i+1}$ and $j+1 = t_{i+1}$ for some $i = 1, \ldots, n'-k'-1$.

Case IV: $j = t_i$ and $j+1 = t_{i+1}$ for some $i = 1, \ldots, n'-k'-1$.
In this case, we assume that

$$
\begin{aligned}
\sum_{i=1}^{n'-k'} (m_i^j)^2 &= \sum_{l=1}^{t_{i-1}} \left(s_{t_l}^2 c_{t_l+1}^2 \cdots c_{j-1}^2 s_j^2\right) + c_j^2 \\
&= \frac{\delta_{n'}^2 - k' a_j^2}{\delta_{n'}^2}.
\end{aligned}
$$

Then,

$$
\begin{aligned}
\sum_{i=1}^{n'-k'} (m_i^{j+1})^2 &= \sum_{l=1}^{t_{i-1}} \left(s_{t_l}^2 c_{t_l+1}^2 \cdots c_j^2 s_{j+1}^2\right) + s_j^2 s_{j+1}^2 + c_{j+1}^2 \\
&= \left(\sum_{l=1}^{t_{i-1}} \left(s_{t_l}^2 c_{t_l+1}^2 \cdots c_{j-1}^2 s_j^2\right)\right) \frac{c_j^2 s_{j+1}^2}{s_j^2} + s_j^2 s_{j+1}^2 + c_{j+1}^2,
\end{aligned}
$$

so for $j = t_i$, $j+1 = t_{i+1}$ we have

$$\sum_{i=1}^{n'-k'} (m_i^{j+1})^2 = \left(\frac{\delta_{n'}^2 - k'a_j^2}{\delta_{n'}^2} - c_j^2\right) \frac{c_j^2 s_{j+1}^2}{s_j^2} + s_j^2 s_{j+1}^2 + c_{j+1}^2. \tag{3.28}$$

Using (3.22) and (3.23) we get

$$\begin{aligned}
c_j^2 &= \frac{(j-i+1)\delta_{n'}^2 - k'\delta_j^2}{(j-i+1)\delta_{n'}^2 - k'\delta_{j-1}^2}, \\
s_j^2 &= \frac{k'a_j^2}{(j-i+1)\delta_{n'}^2 - k'\delta_{j-1}^2}, \\
c_{j+1}^2 &= \frac{(j-i+1)\delta_{n'}^2 - k'\delta_{j+1}^2}{(j-i+1)\delta_{n'}^2 - k'\delta_j^2}, \\
s_{j+1}^2 &= \frac{k'a_{j+1}^2}{(j-i+1)\delta_{n'}^2 - k'\delta_j^2}.
\end{aligned}$$

We then find

$$\begin{aligned}
\frac{c_j^2 s_{j+1}^2}{s_j^2} &= \frac{(j-i+1)\delta_{n'}^2 - k'\delta_j^2}{(j-i+1)\delta_{n'}^2 - k'\delta_{j-1}^2} \cdot \frac{k'a_{j+1}^2}{(j-i+1)\delta_{n'}^2 - k'\delta_j^2} \cdot \frac{(j-i+1)\delta_{n'}^2 - k'\delta_{j-1}^2}{k'a_j^2} \\
&= \frac{a_{j+1}^2}{a_j^2}.
\end{aligned}$$

Also,

$$\begin{aligned}
\frac{\delta_{n'}^2 - k'a_j^2}{\delta_{n'}^2} - c_j^2 &= 1 - \frac{k'a_j^2}{\delta_{n'}^2} - \frac{(j-i+1)\delta_{n'}^2 - k'\delta_j^2}{(j-i+1)\delta_{n'}^2 - k'\delta_{j-1}^2} \\
&= \frac{-k'a_j^2}{\delta_{n'}^2} + \frac{(j-i+1)\delta_{n'}^2 - k'\delta_{j-1}^2}{(j-i+1)\delta_{n'}^2 - k'\delta_{j-1}^2} - \frac{(j-i+1)\delta_{n'}^2 - k'\delta_j^2}{(j-i+1)\delta_{n'}^2 - k'\delta_{j-1}^2} \\
&= -\frac{k'a_j^2}{\delta_{n'}^2} + \frac{k'a_j^2}{(j-i+1)\delta_{n'}^2 - k'\delta_{j-1}^2}.
\end{aligned}$$

Since

$$c_{j+1}^2 = \frac{(j-i+1)\delta_{n'}^2 - k'\delta_{j+1}^2}{(j-i+1)\delta_{n'}^2 - k'\delta_j^2} = 1 - \frac{k'a_{j+1}^2}{(j-i+1)\delta_{n'}^2 - k'\delta_j^2},$$

we can write $s_j^2 s_{j+1}^2 + c_{j+1}^2$ as

$$\begin{aligned}
&\frac{k' a_j^2}{(j-i+1)\delta_{n'}^2 - k'\delta_{j-1}^2} \cdot \frac{k' a_{j+1}^2}{(j-i+1)\delta_{n'}^2 + 1 - k'\delta_j^2} - \frac{k' a_{j+1}^2}{(j-i+1)\delta_{n'}^2 - k'\delta_j^2} \\
&= 1 + \frac{(k')^2 a_j^2 a_{j+1}^2 - k' a_{j+1}^2 \left((j-i+1)\delta_{n'}^2 - k'\delta_{j-1}^2\right)}{\left((j-i+1)\delta_{n'}^2 - k'\delta_{j-1}^2\right)\left((j-i+1)\delta_{n'}^2 - k'\delta_j^2\right)} \\
&= 1 + k' a_{j+1}^2 \frac{k' a_j^2 - (j-i+1)\delta_{n'}^2 + k'\delta_{j-1}^2}{\left((j-i+1)\delta_{n'}^2 - k'\delta_{j-1}^2\right)\left((j-i+1)\delta_{n'}^2 - k'\delta_j^2\right)} \\
&= 1 + k' a_{j+1}^2 \frac{k'\delta_j^2 - (j-i+1)\delta_{n'}^2}{\left((j-i+1)\delta_{n'}^2 - k'\delta_{j-1}^2\right)\left((j-i+1)\delta_{n'}^2 - k'\delta_j^2\right)} \\
&= 1 - \frac{k' a_{j+1}^2}{(j-i+1)\delta_{n'}^2 - k'\delta_{j-1}^2}.
\end{aligned}$$

Substituting into (3.28) and simplifying yields

$$\begin{aligned}
\sum_{i=1}^{n'-k'} (m_i^{j+1})^2 &= \left(\frac{-k' a_j^2}{\delta_{n'}^2} + \frac{k' a_j^2}{(j-i+1)\delta_{n'}^2 - k'\delta_{j-1}^2}\right)\frac{a_{j+1}^2}{a_j^2} + 1 - \frac{k' a_{j+1}^2}{(j-i+1)\delta_{n'}^2 - k'\delta_{j-1}^2} \\
&= \frac{-k' a_{j+1}^2}{\delta_{n'}^2} + \frac{k' a_{j+1}^2}{(j-i+1)\delta_{n'}^2 - k'\delta_{j-1}^2} + 1 - \frac{k' a_{j+1}^2}{(j-i+1)\delta_{n'}^2 - k'\delta_{j-1}^2} \\
&= 1 - \frac{k' a_{j+1}^2}{\delta_{n'}^2} \\
&= \frac{\delta_{n'}^2 - k' a_{j+1}^2}{\delta_{n'}^2}.
\end{aligned}$$

Thus, for $j = t_i$ and $j+1 = t_{i+1}$ for some $i = 1, \ldots, n'-k'-1$, we have that (3.24) holds.

So far, we have shown that (3.24) holds for all $j = 1, \ldots, t_{n'-k'+1} - 1$. We wish to show it holds for $j = t_{n'-k'+1}, \ldots, n'$ as well, so we must consider an additional two cases.

Case V: $t_{n'-k'} < n' - 1$.

Since $t_{n'-k'} < n'-1$, it follows that $t_{n'-k'} + 1 \leq t_{n'-k'+1} \leq n'$. If $t_{n'-k'+1} = n'$, then setting $j = n' - 1$, we have by assumption

$$\sum_{i=1}^{n'-k'} (m_i^{n'-1})^2 = \sum_{l=1}^{t_{n'-k'}} \left(s_{t_l}^2 c_{t_l+1}^2 \cdots c_{n'-2}^2 s_{n'-1}^2\right) = \frac{\delta_{n'}^2 - k' a_{n'-1}^2}{\delta_{n'}^2}.$$

Then,

$$\sum_{i=1}^{n'-k'} (m_i^{n'})^2 = \sum_{l=1}^{t_{n'-k'}} \left(s_{t_l}^2 c_{t_l+1}^2 \cdots c_{n'-2}^2 c_{n'-1}^2\right) = \left(\sum_{l=1}^{t_{n'-k'}} \left(s_{t_l}^2 c_{t_l+1}^2 \cdots c_{n'-2}^2 s_{n'-1}^2\right)\right) \frac{c_{n'-1}^2}{s_{n'-1}^2},$$

or more simply,

$$\sum_{i=1}^{n'-k'} (m_i^{n'})^2 = \left(\frac{\delta_{n'}^2 - k' a_{n'-1}^2}{\delta_{n'}^2}\right) \frac{c_{n'-1}^2}{s_{n'-1}^2}. \tag{3.29}$$

From (3.22) and (3.23) we have

$$c_{n'-1}^2 = \frac{(k'-1)\delta_{n'}^2 - k'\delta_{n'-1}^2}{(k'-2)\delta_{n'}^2 - k'\delta_{n'-2}^2},$$
$$s_{n'-1}^2 = \frac{k' a_{n'-1}^2 - \delta_{n'}^2}{(k'-2)\delta_{n'}^2 - k'\delta_{n'-2}^2}.$$

Then we find

$$\frac{c_{n'-1}^2}{s_{n'-1}^2} = \frac{(k'-1)\delta_{n'}^2 - k'\delta_{n'-1}^2}{k' a_{n'-1}^2 - \delta_{n'}^2} = \frac{k'(\delta_{n'}^2 - \delta_{n'-1}^2) - \delta_{n'}^2}{k' a_{n-'1}^2 - \delta_{n'}^2} = \frac{k' a_{n'}^2 - \delta_{n'}^2}{k' a_{n'-1}^2 - \delta_{n'}^2}.$$

(3.29) then becomes

$$\sum_{i=1}^{n'-k'} (m_i^{n'})^2 = \left(\frac{\delta_{n'}^2 - k'a_{n'-1}^2}{\delta_{n'}^2}\right) \frac{k'a_{n'}^2 - \delta_{n'}^2}{k'a_{n'-1}^2 - \delta_{n'}^2}$$
$$= \frac{\delta_{n'}^2 - k'a_{n'}^2}{\delta_{n'}^2},$$

and we see that (3.24) is satisfied for $j = n'$ when $t_{n'-k'} < t_{n'-k'+1} - 1$ and $t_{n'-k'+1} = n'$. If however, $t_{n'-k'+1} < n'$, setting $j - 1 = t_{n'-k'+1} - 1$, we have that (3.24) is satisfied for $j = t_{n'-k'+1}$ by Case III. Now, for $i = n' - k' + 1$, the sequence $\{(j - n' + k')\delta_{n'}^2 - k'\delta_j^2\}_{j=t_{n'-k'+1}}^{n'}$ is increasing since

$$((j - n' + k')\delta_{n'} - k'\delta_j^2) + \delta_{n'} - k'a_{j+1}^2 = (j + 1 - n' + k')\delta_{n'} - k'\delta_{j+1}^2$$

and $\delta_{n'} - k'a_{j+1}^2 \geq 0$. But since the first term of $\{(j - n' + k')\delta_{n'}^2 - k'\delta_j^2\}_{j=t_{n'-k'+1}}^{n'}$ is not negative, and the last term is 0 by remark 3.1, it follows that

$$\{(j - n' + k')\delta_{n'}^2 - k'\delta_j^2\}_{j=t_{n'-k'+1}}^{n'} = \{0\}_{j=t_{n'-k'+1}}^{n'},$$

from which we find that $\delta_{n'} - k'a_j^2 = 0$ for $j = t_{n'-k'+1} + 1, \ldots, n'$. Thus,

$$\sum_{i=1}^{n'-k'} (m_i^j) = 0$$

for $j = t_{n'-k'+1} + 1, \ldots, n'$. Since $c_{t_{n'-k'+1}}^2 = 0$, we have that $m_i^j = 0$ for every $i = 1, \ldots, n' - k'$, where $j = t_{n'-k'+1} + 1, \ldots, n'$, so (3.24) is satisfied for $j = t_{n'-k'+1}, \ldots, n'$ when $t_{n'-k'} < n' - 1$

Case VI: $t_{n'-k'} = n' - 1$.
That $t_{n'-k'} = n' - 1$ implies $t_{n'-k'+1} = n'$. Thus, $t_{n'-k'} = t_{n'-k'+1} - 1$. Setting $j = n' - 1$, we have by assumption

$$\sum_{i=1}^{n'-k'} (m_i^{n'-1})^2 = \sum_{l=1}^{t_{n'-k'-1}} \left(s_{t_l}^2 c_{t_l+1}^2 \cdots c_{n'-2}^2 s_{n'-1}^2\right) + c_{n'-1}^2$$
$$= \frac{\delta_{n'}^2 - k'a_{n'-1}^2}{\delta_{n'}^2},$$

from which it follows that

$$\begin{aligned}\sum_{i=1}^{n'-k'} (m_i^{n'})^2 &= \sum_{l=1}^{t_{n'-k'-1}} \left(s_{t_l}^2 c_{t_l+1}^2 \cdots c_{n'-2}^2 c_{n'-1}^2\right) + s_{n'-1}^2 \\ &= \sum_{l=1}^{t_{n'-k'-1}} \left(s_{t_l}^2 c_{t_l+1}^2 \cdots c_{n'-2}^2 s_{n'-1}^2\right) \frac{c_{n'-1}^2}{s_{n'-1}^2} + s_{n'-1}^2,\end{aligned}$$

giving us

$$\sum_{i=1}^{n'-k'} (m_i^{n'})^2 = \left(\frac{\delta_{n'}^2 - k'a_{n'-1}^2}{\delta_{n'}^2} - c_{n'-1}^2\right) \frac{c_{n'-1}^2}{s_{n'-1}^2} + s_{n'-1}^2. \tag{3.30}$$

We again use (3.22) and (3.23) to find

$$\begin{aligned}c_{n'-1}^2 &= \frac{k'\delta_{n'}^2 - k'\delta_{n'-1}^2}{k'\delta_{n'}^2 - k'\delta_{n'-2}^2}, \\ s_{n'-1}^2 &= \frac{k'a_{n'-1}^2}{k'\delta_{n'}^2 - k'\delta_{n'-2}^2}.\end{aligned}$$

Now, we calculate

$$\begin{aligned}\frac{c_{n'-1}^2}{s_{n'-1}^2} &= \frac{k'\delta_{n'}^2 - k'\delta_{n'-1}^2}{k'a_{n'-1}^2} \\ &= \frac{a_{n'}^2}{a_{n'-1}^2}\end{aligned}$$

and

$$\frac{\delta_{n'}^2 - k'a_{n'-1}^2}{\delta_{n'}^2} - c_{n'-1}^2 = 1 - \frac{k'a_{n'-1}^2}{\delta_{n'}^2} - \frac{k'a_{n'}^2}{k'\delta_{n'}^2 - k'\delta_{n'-2}^2},$$

whence

$$\left(\frac{\delta_{n'}^2 - k'a_{n'-1}^2}{\delta_{n'}^2} - c_{n'-1}^2\right) \frac{c_{n'-1}^2}{s_{n'-1}^2} = \frac{a_{n'}^2}{a_{n'-1}^2} - \frac{k'a_{n'}^2}{\delta_{n'}^2} - \frac{a_{n'}^2}{a_{n'-1}^2} \cdot \frac{k'a_{n'}^2}{k'\delta_{n'}^2 - k'\delta_{n'-2}^2},$$

and adding $s_{n'-1}^2$ we see (3.30) becomes

$$\begin{aligned}
\sum_{i=1}^{n'-k'} (m_i^{n'})^2 &= \frac{a_{n'}^2}{a_{n'-1}^2} - \frac{k'a_{n'}^2}{\delta_{n'}^2} - \frac{a_{n'}^2}{a_{n'-1}^2} \cdot \frac{k'a_{n'}^2}{k'\delta_{n'}^2 - k'\delta_{n'-2}^2} + \frac{k'a_{n'-1}^2}{k'\delta_{n'}^2 - k'\delta_{n'-2}^2} \\
&= -\frac{k'a_{n'}^2}{\delta_{n'}^2} + \frac{a_{n'}^2}{a_{n'-1}^2}\left(1 - \frac{k'a_{n'}^2}{k'\delta_{n'}^2 - k'\delta_{n'-2}^2}\right) + \frac{k'a_{n'-1}^2}{k'\delta_{n'}^2 - k'\delta_{n'-2}^2} \\
&= -\frac{k'a_{n'}^2}{\delta_{n'}^2} + \frac{a_{n'}^2}{a_{n'-1}^2}\left(\frac{k'\delta_{n'}^2 - k'\delta_{n'-2}^2 - k'a_{n'}^2}{k'\delta_{n'}^2 - k'\delta_{n'-2}^2}\right) + \frac{k'a_{n'-1}^2}{k'\delta_{n'}^2 - k'\delta_{n'-2}^2} \\
&= -\frac{k'a_{n'}^2}{\delta_{n'}^2} + \frac{a_{n'}^2}{a_{n'-1}^2}\left(\frac{k'\delta_{n'-1}^2 - k'\delta_{n'-2}^2}{k'\delta_{n'}^2 - k'\delta_{n'-2}^2}\right) + \frac{k'a_{n'-1}^2}{k'\delta_{n'}^2 - k'\delta_{n'-2}^2} \\
&= -\frac{k'a_{n'}^2}{\delta_{n'}^2} + \frac{a_{n'}^2}{a_{n'-1}^2}\left(\frac{k'a_{n'-1}^2}{k'\delta_{n'}^2 - k'\delta_{n'-2}^2}\right) + \frac{k'a_{n'-1}^2}{k'\delta_{n'}^2 - k'\delta_{n'-2}^2} \\
&= -\frac{k'a_{n'}^2}{\delta_{n'}^2} + \frac{k'a_{n'}^2 + k'a_{n'-1}^2}{k'\delta_{n'}^2 - k'\delta_{n'-2}^2} \\
&= -\frac{k'a_{n'}^2}{\delta_{n'}^2} + 1 \\
&= \frac{\delta_{n'}^2 - k'a_{n'}^2}{\delta_{n'}^2},
\end{aligned}$$

and (3.24) is satisfied for $j = n'$ and $t_{n'-k'} = n' - 1$. Finally, we have that (3.24) is satisfied for all $j = 1, \ldots, n'$ and the theorem is proven. □

3.3.3 Finding $r(k, n)$ and $\mathcal{M}$

Theorem 3.5. *Let $\mathfrak{B}'$ be a box of dimensions $2a_1 \times \ldots \times 2a_{n'}$ where $0 < a_1 \leq \ldots \leq a_n'$. The following statements are equivalent:*

1. *There is a k'-ball of maximal radius tangent to all faces of $\mathfrak{B}'$,*
2. $a_{n'} \leq \dfrac{\delta_{n'-1}}{\sqrt{k'-1}}$.

Proof. (1) ⇒ (2) : Suppose there is an k'-ball of maximal radius tangent to all the faces of $\mathfrak{B}'$. Then $d_1, \ldots, d_{n'} = t$, and (3.18) gives $t = \frac{\delta_{n'}}{\sqrt{k'}}$. (3.2) now ensures us that $a_{n'} \leq \frac{\delta_{n'}}{\sqrt{k'}}$, and (3.3) shows $a_{n'} \leq \frac{\delta_{n'-1}}{\sqrt{k'-1}}$.
(2) ⇒ (1) : Assume $a_{n'} \leq \frac{\delta_{n'-1}}{\sqrt{k'-1}}$. Then (3.3) gives $a_{n'} \leq \frac{\delta_{n'}}{\sqrt{k'}}$, or equivalently, $k'a_{n'}^2 \leq \delta_{n'}^2$. Let $\mathbf{m}_1, \ldots, \mathbf{m}_{n'-k'}$ be as in theorem 3.4 and let $\mathcal{M}$ be a k'-flat such

that $\mathcal{M} \perp \mathbf{m}_i$ for $i = 1, \ldots, n' - k'$. Then,

$$\sum_{i=1}^{n'-k'} (m_i^j)^2 = \frac{\delta_{n'}^2 - k' a_j^2}{\delta_{n'}^2}, \; j = 1, \ldots, n'. \tag{3.31}$$

Therefore, (3.17) gives

$$\begin{aligned} d_j^2 &= \frac{a_j^2}{1 - \sum_{i=1}^{n'-k'} (m_i^j)^2} \\ &= a_j^2 \left(1 - \frac{\delta_{n'}^2 - k' a_j^2}{\delta_{n'}^2}\right)^{-1} \\ &= a_j^2 \left(\frac{k' a_j^2}{\delta_{n'}^2}\right)^{-1} \\ &= \frac{\delta_{n'}^2}{k'} \end{aligned}$$

for every $j = 1, \ldots, n'$. The maximality follows from (3.18): If we want to increase the radius, we need to increase all the d_j's. But an increase in one, say d_{j_1}, requires a decrease in another, say d_{j_2}, where $j_i \neq j_2$ which would only result in a ball of strictly smaller radius. It now remains only to show that the point of $\mathcal{M}_j$ closest to the origin lay in the jth face of $\mathfrak{B}'$. We will argue by contradiction. Suppose for some $j = 1, \ldots, n'$, the closest point of $\mathcal{M}_j$ to the origin, say p_j, is not contained in $\mathfrak{B}'$. Construct a line l from the origin to p_j. Since $0 \in \mathfrak{B}'$, and $l \subset \mathcal{M}$, there exists an $i = 1, \ldots, n'$, $i \neq j$, such that $l \cap \mathcal{M}_i = \{p_i\}$. It follows that $d_i \leq \|\mathbf{p}_i\| < \|\mathbf{p}_j\| = d_j$. Thus, we have that $d_i \neq d_j$, clearly contradicting their equality. Thus we have that the k'-flat $\mathcal{M}$ given by the equations

$$\langle \mathbf{m}_i, \mathbf{x} \rangle = 0 \text{ for } i = 1, \ldots, n' - k'$$

with the m_i^j's defined as in (3.31), contains an k'-ball of maximal radius that is tangent to all faces of $\mathfrak{B}'$. □

Theorem 3.6. *Let s be the smallest integer satisfying*

$$a_{n-s} \leq \frac{\delta_{n-s-1}}{\sqrt{k - s - 1}}. \tag{3.32}$$

Then the maximal radius of $\mathcal{B} \subset \mathfrak{B}$ occurs when $d_1 = \ldots = d_{n-s} = t$ with the radius *being their common value. That is,*

$$r(k,n) = t = \frac{\delta_{n-s}}{\sqrt{k-s}}. \tag{3.33}$$

Furthermore, $m_i^{n-s+1} = \ldots = m_i^n = 0$ *for* $i = 1, \ldots n-k$.

Proof. If $s = 0$ is the smallest integer satisfying (3.32) then $a_n \leq \frac{\delta_{n-1}}{\sqrt{k-1}}$ and we have (from theorem 3.5 with $n' = n$, $k' = k$)

$$d_1 = \ldots = d_n = t = \frac{\delta_n}{\sqrt{k}}$$

which gives us (3.33).

Now suppose $s = 1$ is the smallest integer satisfying (3.32). Then $a_n > \frac{\delta_{n-1}}{\sqrt{k-1}}$ and $a_{n-1} \leq \frac{\delta_{n-2}}{\sqrt{k-2}}$. Intersecting $\mathfrak{B}$ with the hyperplane $x^n = 0$ yields an $(n-1)$-box of dimensions $2a_1 \times \ldots \times 2a_{n-1}$. Intersecting any k-ball contained in $\mathfrak{B}$ that doesn't lie parallel to any face of $\mathfrak{B}$ with the same hyperplane gives a $(k-1)$ ball of the same radius that is contained in the $(n-1)$-box. From this, we have $r(k-1, n-1) \geq r(k,n)$. Applying theorem 3.5 with $n' = n-1$ and $k' = k-1$, we find $d_1 = \ldots = d_{n-1} = t$ where $t = \frac{\delta_{n-1}}{\sqrt{k-1}} = r(k-1, n-1)$ where the $(k-1)$-ball is tangent to all faces of the $(n-1)$-box. Using the m_i^j's from (3.31), and setting $m_i^n = 0$ for $i = 1, \ldots, n-k$, we extend the $(k-1)$-flat containing the $(k-1)$-ball to a k-flat $\mathcal{M}$ containing a k-ball of radius $\frac{\delta_{n-1}}{\sqrt{k-1}}$. The ball lies in $\mathfrak{B}$ since $a_n > \frac{\delta_{n-1}}{\sqrt{k-1}}$. It is maximal in $\mathfrak{B}$ because $r(k,n)$ cannot exceed $r(k-1, n-1)$. The k-flat $\mathcal{M}$ is unique (up to the symmetry of $\mathfrak{B}$) as guaranteed by (3.6).

Now suppose $s = 2$ is the smallest integer satisfying (3.32). Then $a_{n-1} > \frac{\delta_{n-2}}{\sqrt{k-2}}$ and $a_{n-2} \leq \frac{\delta_{n-3}}{\sqrt{k-3}}$. Intersecting $\mathfrak{B}$ with the two hyperplanes $x^n = 0$ and $x^{n-1} = 0$ gives us an $(n-2)$-box of dimensions $2a_1 \times \ldots \times 2a_{n-2}$. Intersecting any k-ball contained in $\mathfrak{B}$ (again, not parallel to any face of $\mathfrak{B}$) with the same two hyperplanes yields a $(k-2)$-ball of equal radius which is contained in the $(n-2)$-box. It follows that $r(k-2, n-2) \geq r(k,n)$. Applying theorem 3.5 with $n' = n-2$ and $k' = k-2$, we have $d_1 = \ldots = d_{n-2} = t$ with $t = \frac{\delta_{n-2}}{\sqrt{k-2}} = r(k-2, n-2)$ and that the $(k-2)$-ball is tangent to all faces of the $(n-2)$-box. Using the m_i^j's from (3.31), and setting $m_i^{n-1} = m_i^n = 0$ for $i = 1, \ldots, n-k$, we extend the $(k-2)$-flat containing the $(k-2)$-ball to a k-flat $\mathcal{M}$ containing a k-ball of radius $\frac{\delta_{n-2}}{\sqrt{k-2}}$. This ball lies in $\mathfrak{B}$ since $a_n \geq a_{n-1} > \frac{\delta_{n-2}}{\sqrt{k-2}}$ and it is maximal as $r(k,n)$ is bounded from above by

$r(k-2, n-2)$. $\mathcal{M}$ is unique up to the symmetries of $\mathfrak{B}$ by (3.6).

We continue in this fashion until an s is found so that (3.32) is satisfied, which must happen since it is clearly true for $s = k-1$.

It has been our assumption in this proof that the k-flat $\mathcal{M}$ was not parallel to $\mathcal{H}_j$ for $j = 1, \ldots, n$. At the beginning of this section, we commented that $\mathcal{M}$ could actually be parallel to at most $n-k$ faces of $\mathfrak{B}$. We said we could view the k-ball lying on such a k-flat $\mathcal{M}$ as a k-ball in the $(n-f)$-box of dimensions $2a_{f+1}, \ldots, 2a_n$, and that the maximal radius such a k-ball could have in this case was $\hat{r}(k, n-f)$ where $f = 1, \ldots, n-k$ was the number of faces of $\mathcal{B}$ that the k-flat $\mathcal{M}$ is parallel to. We also remarked that any k-ball lying on $\mathcal{M}$ was not parallel to any of the faces of the $(n-f)$-box. We will now show that $\hat{r}(k, n-f) < r(k, n)$ where $r(k, n)$ is defined by (3.33) and s is defined by (3.32). If $f = n-k$, then $\hat{r}(k, n-f) = \hat{r}(k, k) = a_{n-k+1}$. But, since $s < k$,

$$
\begin{aligned}
\frac{\delta_{n-s}^2}{k-s} &\geq \frac{\delta_{n-k}^2 + (k-s)a_{n-k+1}^2}{k-s} \\
&= \frac{\delta_{n-k}^2}{k-s} + a_{n-k+1}^2 \\
&> a_{n-k+1}^2
\end{aligned}
$$

and so the k-ball cannot be maximal in this case. If $f = 1, \ldots, n-k-1$, then in light of (3.33), we have that

$$
\hat{r}(k, n-f) = \frac{\sqrt{\sum_{i=f+1}^{n-\hat{s}} a_i^2}}{\sqrt{k-\hat{s}}},
$$

where $\hat{s}$ is the smallest integer satisfying $a_{n-\hat{s}} \leq \frac{\sqrt{\sum_{i=f+1}^{n-\hat{s}-1} a_i^2}}{\sqrt{k-\hat{s}-1}}$. Since $\frac{\sqrt{\sum_{i=f+1}^{n-\hat{s}-1} a_i^2}}{\sqrt{k-\hat{s}-1}} < \frac{\delta_{n-\hat{s}-1}}{\sqrt{k-\hat{s}-1}}$, we have $a_{n-\hat{s}} < \frac{\delta_{n-\hat{s}-1}}{\sqrt{k-\hat{s}-1}}$ from which we can conclude that $s \leq \hat{s}$. For $j = 1, \ldots, \hat{s}-s$, we have $a_{n-\hat{s}+j} > \frac{\sum_{i=f+1}^{n-\hat{s}+j-1} a_i^2}{k-\hat{s}+j-1}$, and thus

$$
\frac{(k-\hat{s}+j-1)a_{n-\hat{s}+j}^2 - \sum_{i=f+1}^{n-\hat{s}+j-1} a_i^2}{(k-\hat{s}+j-1)(k-\hat{s}+j)} = \frac{a_{n-\hat{s}+j}^2}{k-\hat{s}+j} - \frac{\sum_{i=f+1}^{n-\hat{s}+j-1} a_i^2}{(k-\hat{s}+j-1)(k-\hat{s}+j)} > 0.
$$

Then,

$$\begin{aligned}
\frac{\sum_{i=f+1}^{n-\hat{s}+j-1} a_i^2}{k-\hat{s}+j-1} &< \frac{\sum_{i=f+1}^{n-\hat{s}+j-1} a_i^2}{k-\hat{s}+j-1} + \frac{a_{n-\hat{s}+j}^2}{k-\hat{s}+j} - \frac{\sum_{i=f+1}^{n-\hat{s}+j-1} a_i^2}{(k-\hat{s}+j-1)(k-\hat{s}+j)} \\
&= \frac{\sum_{i=f+1}^{n-\hat{s}+j-1} a_i^2}{k-\hat{s}+j} + \frac{a_{n-\hat{s}+j}^2}{k-\hat{s}+j} \\
&= \frac{\sum_{i=f+1}^{n-\hat{s}+j} a_i^2}{k-\hat{s}+j}.
\end{aligned}$$

It then follows that

$$[r(k,n-f)]^2 = \frac{\sum_{i=1}^{n-\hat{s}} a_i^2}{k-\hat{s}} < \frac{\sum_{i=1}^{n-\hat{s}+1} a_i^2}{k-\hat{s}+1} < \ldots < \frac{\sum_{i=f+1}^{n-s} a_i^2}{k-s} < \frac{\delta_{n-s}^2}{k-s} = [r(k,n)]^2.$$

and so the k-ball cannot be maximal if $f = 1, \ldots, n-k-1$, or in other words, if any k-ball $\mathcal{B}$ lies in a k-flat $\mathcal{M}$ that is parallel to any of the $\mathcal{H}_j$'s then $\mathcal{B}$ cannot be of maximal radius in $\mathfrak{B}$. □

Corollary 3.2. *The $n-k$ normal vectors to $\mathcal{M}$ satisfy the following conditions:*

$$(3.34) \qquad \langle \boldsymbol{m}_i, \boldsymbol{m}_j \rangle = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

$$(3.35) \qquad \sum_{i=1}^{n-k} (m_i^j)^2 = \begin{cases} \dfrac{\delta_{n-s}^2 - (k-s)a_j^2}{\delta_{n-s}^2} & \text{if } j = 1, \ldots, n-s, \\ 0 & \text{if } j = n-s+1, \ldots, n, \end{cases}$$

for $i = 1, \ldots, n-k$ where s is as in theorem 3.6.

Proof. (3.34) is just (3.15). From theorem 3.6, $m_i^{n-s+1}, \ldots, m_i^n = 0$ for $i = 1, \ldots, n-k$. From theorem 3.5, with $n' = n-s$, $k' = k-s$, we have

$$\sum_{i=1}^{n-k} (m_i^j)^2 = \frac{\delta_{n-s}^2 - (k-s)a_j^2}{\delta_{n-s}^2}.$$

□

3.3.4 The $(k, 4)$ Problem

Finally, we state all results for $(k, 4)$ problem. As mentioned in the introduction, writing out explicit solutions becomes difficult when $n-k$ becomes large. Because of this, the need for a computer algorithm arises, and the reader will find one supplied in the Appendix. We will examine three cases, one each for $k = 1$, $k = 2$, and $k = 3$. In each case, we state first all possible forms of the radius $r(k, 4)$, followed by the resulting $4 - k$ equations for the location of these balls.

Case I: $k = 1$

This is the simplest case, as we do not need to check any conditions on the a_j's or the δ_j's. For any box $\mathfrak{B}$, we have

$$r(1, 4) = \delta_4,$$

and

$$\mathbf{m}_1 = \begin{bmatrix} \dfrac{\sqrt{\delta_4^2 - a_1^2}}{\delta_4} \\ -\dfrac{a_1 a_2}{\delta_4\sqrt{\delta_4^2 - a_1^2}} \\ -\dfrac{a_1 a_3}{\delta_4\sqrt{\delta_4^2 - a_1^2}} \\ -\dfrac{a_1 a_4}{\delta_4\sqrt{\delta_4^2 - a_1^2}} \end{bmatrix}, \mathbf{m}_2 = \begin{bmatrix} 0 \\ \sqrt{\dfrac{\delta_4^2 - \delta_2^2}{\delta_4^2 - a_1^2}} \\ -\dfrac{a_2 a_3}{\sqrt{(\delta_4^2 - a_1^2)(\delta_4^2 - \delta_2^2)}} \\ -\dfrac{a_2 a_4}{\sqrt{(\delta_4^2 - a_1^2)(\delta_4^2 - \delta_2^2)}} \end{bmatrix}, \mathbf{m}_3 = \begin{bmatrix} 0 \\ 0 \\ \dfrac{a_4}{\sqrt{\delta_4^2 - \delta_2^2}} \\ -\dfrac{a_3}{\sqrt{\delta_4^2 - \delta_2^2}} \end{bmatrix}.$$

Case II: $k = 2$

This would seem like the most complicated case, since s can be either 0 or 1, and we can pick either the first and second or the first and third columns of $\mathcal{R}$ depending on whether $\delta_n^2 \geq 2\delta_2^2$ or $\delta_n^2 < 2\delta_2^2$ respectively. However, it is not possible for $\delta_n^2 < 2\delta_2^2$, so we only need to consider using the first and second columns of $\mathcal{R}$ to locate the ball of maximal radius. We have the radius as

$$r(2, 4) = \begin{cases} \dfrac{\delta_4}{\sqrt{2}} & \text{if } a_4 \leq \delta_3, \\ \delta_3 & \text{if } a_4 > \delta_3, \end{cases}$$

and our normal vectors are given by

$$\mathbf{m}_1 = \begin{bmatrix} \frac{\sqrt{\delta_4^2 - 2a_1^2}}{\delta_4} \\ -\frac{2a_1a_2}{\delta_4\sqrt{\delta_4^2 - 2a_1^2}} \\ -\frac{a_1}{\delta_2\delta_4}\sqrt{\frac{(\delta_4^2 - 2\delta_2^2)(\delta_4^2 - 2a_3^2)}{\delta_4^2 - 2a_1^2}} \\ -\frac{a_1}{\delta_2\delta_4}\sqrt{\frac{(\delta_4^2 - 2\delta_2^2)(\delta_4^2 - 2a_4^2)}{\delta_4^2 - 2a_1^2}} \end{bmatrix}, \mathbf{m}_2 = \begin{bmatrix} 0 \\ \sqrt{\frac{\delta_4^2 - 2\delta_2^2}{\delta_4^2 - 2a_1^2}} \\ -\frac{a_2}{\delta_2}\sqrt{\frac{\delta_4^2 - 2a_3^2}{\delta_4^2 - 2a_1^2}} \\ -\frac{a_2}{\delta_2}\sqrt{\frac{\delta_4^2 - 2a_4^2}{\delta_4^2 - 2a_1^2}} \end{bmatrix}$$

when $a_4 \leq \delta_3$, and they are given by

$$\mathbf{m}_1 = \begin{bmatrix} \frac{\sqrt{\delta_3^2 - a_1^2}}{\delta_3} \\ -\frac{a_1a_2}{\delta_3\sqrt{\delta_3^2 - a_1^2}} \\ -\frac{a_1a_3}{\delta_3\sqrt{a_2^2 + a_3^2}} \\ 0 \end{bmatrix}, \mathbf{m}_2 = \begin{bmatrix} 0 \\ \sqrt{\frac{\delta_3^2 - \delta_2^2}{\delta_3^2 - a_1^2}} \\ -\frac{a_2}{\sqrt{a_2^2 + a_3^2}} \\ 0 \end{bmatrix}$$

when $a_4 > \delta_3$.

Case III: $k = 3$

In this last case, we have to consider the cases when $s = 0, 1$, or 2. We find that the radius is

$$r(3,4) = \begin{cases} \frac{\delta_4}{\sqrt{3}} & \text{if } a_4 \leq \frac{\delta_3}{\sqrt{2}}, \\ \frac{\delta_3}{\sqrt{2}} & \text{if } a_4 > \frac{\delta_3}{\sqrt{2}}, \text{ and } a_3 \leq \delta_2, \\ \delta_2 & \text{if } a_3 > \delta_2, \end{cases}$$

and $\mathbf{m}_1$ is one of

$$\begin{bmatrix} \frac{\sqrt{\delta_4^2 - 3a_1^2}}{\delta_4} \\ -\frac{\sqrt{\delta_4^2 - 3a_2^2}}{\delta_4} \\ -\frac{\sqrt{\delta_4^2 - 3a_3^2}}{\delta_4} \\ -\frac{\sqrt{\delta_4^2 - 3a_4^2}}{\delta_4} \end{bmatrix}, \begin{bmatrix} \frac{\sqrt{\delta_3^2 - 2a_1^2}}{\delta_3} \\ -\frac{\sqrt{\delta_3^2 - 2a_2^2}}{\delta_3} \\ -\frac{\sqrt{\delta_3^2 - 2a_3^2}}{\delta_3} \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{\sqrt{\delta_2^2 - a_1^2}}{\delta_2} \\ -\frac{\sqrt{\delta_2^2 - a_2^2}}{\delta_2} \\ 0 \\ 0 \end{bmatrix}$$

when $a_4 \leq \frac{\delta_3}{\sqrt{2}}$, $a_4 > \frac{\delta_3}{\sqrt{2}}$ and $a_3 \leq \delta_2$, or $a_3 > \delta_2$ respectively. This concludes all possible cases of the $(k, 4)$ problem.

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Appendix A

An Algorithm to Find $\mathbf{m}_1, \ldots, \mathbf{m}_{n-k}$

The following Maple code calculates the $n-k$ normal vectors to the k-flat $\mathcal{M}$ which contains a k-ball of maximal radius $r(k,n)$ in a box $\mathfrak{B}$ of dimensions $2a_1 \times \ldots \times 2a_n$ with $0 < a_1 \leq \ldots \leq a_n$. Simply adjust the values of k and n with $k < n$ at the beginning of the program, as well as the values for $a_1, \ldots, a_n$. The vector $\mathbf{t} = (t^1, \ldots, t^{n-1})$ is such that $t^j = 1$ if the jth column of $\mathcal{R}$ is used, and $t^j = 0$ if it is not, and the jth entries of **cosvector** and **sinvector** give the value of $\cos^2(\theta_j)$ and $\sin^2(\theta_j)$ respectively for $j = 1, \ldots, n-s-1$. The italicized text before each block of code are just comments, and do not need to be copied into a maple file.

```
restart:
```

Input k and n values here

```
k:=7:
n:=8:
```

Create an array called avector for the measurements of the box in each dimension, then input a value for each of the measurements $a_1, \ldots, a_n$ in ascending order and find the resulting value for sval, which is just the value of s in chapter 3.

```
avector:=array(1..n):
```

```
for count1 from 1 to n do
   avector[count1]:=1:
od:
sval:=0:
conditioncheck:=-1:
for counter from 0 to k-2 while conditioncheck<=0 do
   conditioncheck:=sum(avector[c]^2,c=1..n-sval-1)/(k-sval-1)-avector[n-sval]^2:
   if(conditioncheck<=0) then sval:=sval+1:
   fi:
od:
```

Create arrays containing:
1) tvector - an array which records the values of the t_i's
(For example, $t = [1, 0, 0, 1, 1]$ implies $t_1 = 1$, $t_2 = 4$, $t_3 = 5$)
2) solvector - jth entry contains $\frac{\delta^2_{n-sval} - a^2_j}{\delta^2_{n-sval}}$
3) cosvector - contains the values of $c_1, \ldots, c_{n-s-1}$
4) sinvector - contains the values of $s_1, \ldots, s_{n-s-1}$
5) mmatrix - matrix whose columns are the $n - k$ normal vectors to the k flat containing the k-ball of maximal radius in the n-box with dimensions given in avector

```
tvector:=array(1..n-sval):
solvector:=array(1..n-sval):
cosvector:=array(1..n-sval-1):
sinvector:=array(1..n-sval-1):
mmatrix:=array(1..n-k):
```

Initializations:
1) tvector[1] set to 1 since $t_1 = 1$, all else set to 0
2) solvector
3) cosvector[1] set to solvector[1], all else unassigned
4) sinvector[1] set to 1-cosvector[1], all else unassigned
5) mmatrix set to zero matrix
6) tcounter set to 1, to keep track of how many of the t_i's have been assigned.

```
tvector[1]:=1:
for count2 from 2 to n-sval-1 do
   tvector[count2]:=0
od:
for count3 from 1 to n-sval do
   solvector[count3]:=1-((k-sval)*avector[count3]^2)/sum(avector[b]^2,b=1..n-sval):
od:
cosvector[1]:=solvector[1]:
sinvector[1]:=1-cosvector[1]:
for count4 from 1 to n-k do
   mmatrix[count4]:=array(1..n):
   for count5 from 1 to n do
```

```
      mmatrix[count4][count5]:=0:
   od:
od:
tcounter:=1:
```

Searches for t_i's to use, stops when tcounter $= n - k$ and assigns values to cosvector and sinvector along the way. Breakpoint used to record where the last t_i was found

```
breakpoint:=1:
for count6 from 2 to n-sval-1 while tcounter<n-k do
   check:=sum(tvector[j]*sinvector[j]*product(cosvector[h],h=j+1..count6-1),j=1..count6-1):
   if (check<solvector[count6])
      then
         cosvector[count6]:=(solvector[count6]-check)/(1-check):
         sinvector[count6]:=1-cosvector[count6]:
         tcounter:=tcounter+1:
         tvector[count6]:=1:
      else
         sinvector[count6]:=solvector[count6]/check:
         cosvector[count6]:=1-sinvector[count6]:
   fi:
   breakpoint:=count6:
od:
```

Computes any remaining entries in cosvector and sinvector from breakpoint onwards

```
for i from breakpoint+1 to n-sval-1 do
   sinval:=sum(tvector[j]*sinvector[j]*product(cosvector[h],h=j+1..i-1),j=1..i-1):
   sinvector[i]:=solvector[i]/sinval:
   cosvector[i]:=1-sinvector[i]:
od:
```

Assigns entries to mmatrix

```
position:=1:
for count7 from 1 to n-sval-1  do
   if (tvector[count7]=1)
      then
         mmatrix[position][count7]:=sqrt(cosvector[count7]):
         for count8 from count7+1 to n-sval-1 do
            mmatrix[position][count8]:=
                  -sqrt(sinvector[count7]*product(cosvector[h],h=count7+1..count8-1)
                  *sinvector[count8]):
         od:
         mmatrix[position][n-sval]:=
                  -sqrt(sinvector[count7]*product(cosvector[h],h=count7+1..n-sval-1)):
```

```
        position:=position+1:
  fi:
od:
```

Outputs the $n-k$ normal vectors

```
print ("THE", n-k ,"NORMAL VECTORS");
for count9 from 1 to n-k do
   print (count9,mmatrix[count9]);
od;
```