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QUANTUM FIELD THEORY FOR ACCELERATED MIRRORS AND PARTICLE DETECTORS

BY

THOMAS G. STEELE

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ABSTRACT

The quantum field theory for accelerated systems is investigated in order to provide a physical interpretation for the particle content ambiguity which arises from alternate quantization methods. For uniform acceleration this ambiguity leads to thermal properties known as acceleration radiation. Uniformly accelerated dipole detectors are shown to have a non-thermal and non-isotropic response in agreement with previous work. Particle detectors following a Killing orbit are found to respond to Killing particle density, thus providing an interpretation for alternate particle content. The comparison of detectors following different trajectories leads to a description of acceleration radiation as an effective quantity. It is demonstrated that particle detectors co-moving with a single 1+1 dimensional mirror give a null response even in the presence of non-zero stress-energy. This agrees with equivalence arguments. A 1+1 dimensional uniformly accelerated box is shown to contain a negative energy density identical to the acceleration radiation result. This energy will be accounted for dynamically.

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Abbreviations, Units, Definitions and Notation

Abbreviations

h.c. - hermitian conjugate

c.c. - complex conjugate

Units

The units $\hbar=c=G=k_B=1$ will be used.

Definitions

The Minkowski metric is defined as

$$g_{\mu\nu}\equiv\eta_{\mu\nu} \qquad \eta_{00}=\eta^{00}=-1 \qquad \eta_{ij}=\eta^{ij}=\delta_{ij}$$

so that the line element is $ds^2=-dt^2+d\underline{x}^2$.

Notation

A general four vector with components v^μ and basis e^μ is represented by $v=v^\mu e_\mu$.

A vector in Euclidean space is represented by $\underline{v}$.

Partial derivatives are sometimes written as

$$\frac{\partial f}{\partial x^\mu}=f_{,\mu}$$

Covariant derivatives are

$$\nabla_v w=v^\nu w_{\mu;\nu}$$

The following will be considered standard notation.

$*$ ≡ complex conjugate

$\dagger$ ≡ hermitian conjugate

$H(t-t')$ ≡ Heavyside (step) function

β ≡ $1/T$, T=Temperature

Tr ≡ Trace

H ≡ Hamiltonian

sh ≡ sinh (hyperbolic sine)

ch ≡ cosh (hyperbolic cosine)

$\gamma \equiv (1-v^2)^{-\frac{1}{2}}$, v=velocity

$\hat{v}$ ≡ unit vector

u ≡ retarded time $u=t-x$

v ≡ advanced time $v=t+x$

Δ ≡ difference , $\Delta t = t - t'$

CHAPTER ONE

Quantum Field Theory of Accelerated Observers

1-1 Introduction

There were two original motivations for the study of quantum field theory for accelerated observers. These were Hawking's (1975) result that black holes are a source of thermal radiation, and Fulling's (1973) study of the non-uniqueness of canonical quantization in flat spacetime. It was then recognized that there were similarities between the causal structure of spacetime exterior to a black hole and that of Minkowski space as viewed by an accelerated observer.

Taking advantage of the non-uniqueness of quantization, Davies (1975), Israel (1976), and Unruh (1976), confirmed that quantization for a uniformly accelerated observer using natural Rindler coordinates yielded thermal properties. Specifically, the Minkowski vacuum was found to contain a thermal superposition of Rindler particles at a temperature equal to the acceleration over 2π ($T=a/2\pi$). This phenomenon is referred to as acceleration radiation. Israel's work considered a generic case involving an observer confined by null horizons, thereby containing a unified approach to black hole radiance and acceleration radiation.

Since in curved spacetime the quantization procedure is already ambiguous, it was originally intended that quantum field theory in Rindler space, where there is a well defined underlying field theory, would be an excellent test case for black hole radiance. However, it soon became apparent that the quantum field theory for accelerated observers was by itself an interesting subject.

It would be tempting to immediately dismiss the Rindler quantization procedure as non-physical because the basis for this method involves a non-inertial observer. It appears, however, that physical effects due to Rindler particles exist. At the same time as Unruh (1976) considered this alternate quantization, he also introduced model particle detectors. When these detectors were analysed for uniform acceleration he found that excitations were produced with a thermal spectrum at $T=a/2\pi$, implying that the detector was measuring Rindler particles. The technique of using detectors to study particle content has now become a standard procedure, following the conventions of Dewitt (1979), and Birrell and Davies (1982).

Certain detectors with a dipole interaction term have been studied by Hinton (1983), who finds that uniform acceleration results in a non-thermal and non-isotropic response. In an attempt to understand this discrepancy with other types of detectors we will examine the dipole detector using a different technique from that of Hinton. The general response of monopole detectors to a state $|N\rangle$

will also be studied in order to determine the difference in response for two distinct trajectories.

The magnitude of any physical effects arising from acceleration radiation is extremely small, requiring accelerations of the order 10^{20} m/s^2 for T=1K. However, such accelerations are commonly found in particle accelerators and high energy collisions. Bell and Leinaas (1983) have argued that the depolarization of cyclotron electrons could be viewed as a thermal effect although the problem has been handled theoretically by other techniques. Barshay and Troost (1978) have suggested that acceleration radiation may account for the observed thermal spectra of particles produced in certain collision experiments. These possibilities indicate that measurable effects may exist which can be qualitatively explained by acceleration radiation.

The physical effects discussed to this point have manifested themselves as transition probabilities in certain systems; a result which is intuitively reasonable since perturbation theory utilises proper time in calculating amplitudes. For uniformly accelerated systems this would single out Rindler proper time and particles as important, suggesting that they would be reflected in measurement. The general analysis of monopole detectors will confirm this belief for certain cases.

If the actual value of energy present in the

Minkowski vacuum is considered, then we know that the expectation value of the stress-energy tensor is zero for all observers. This implies that the thermal energy due to considering Rindler particles can at most be interpreted as an effective quantity which shows up in certain measurements.

It is well known that the presence of perfectly reflecting boundaries is capable of altering the field, producing a non-trivial stress-energy tensor. In the case of the well known Casimir effect, this modification results in a negative energy density causing an attractive force between the two parallel plates. This phenomenon suggests that thermal effects corresponding to acceleration radiation may be present in the stress-energy tensor produced by an accelerated mirror. The majority of these problems have been studied in 1+1 dimensions due to the difficulty of obtaining 3+1 solutions.

Candelas and Deutsch (1977) have found that a uniformly accelerated plane mirror in 3+1 dimensions will produce a stress-energy tensor with an asymptotic form corresponding to a thermal result which depends on the local acceleration. The difficulty with this result is that the energy balance cannot be verified dynamically by calculating the energy radiated while bringing an initially inertial mirror to uniform acceleration. For example, in the Casimir effect the energy present can be accounted for by the work done in bringing the mirrors from infinite to finite separation.

Fulling and Davies (1976) have studied general motion of a 1+1 dimensional boundary and conclude that an accelerated mirror radiates energy proportional to the time derivative of its acceleration. The difficulty with a one-mirror calculation is that once the energy is radiated it is no longer confined. It follows that the stress-energy tensor at a point which contains only uniform mirror motion in its causal past will be zero. For this reason we will consider the stress-energy tensor for a region bounded by two mirrors which define a uniformly accelerated box. To perform this calculation we will employ the Euclidean space techniques which Frolov and Serebriany (1979) utilised in evaluating stress-energy tensors for uniformly accelerated expanding spherical shells.

Throughout this study the focus of interest is on the observable effects of acceleration radiation. These effects, though negligible at conventional accelerations, may be important in high energy processes. By studying model particle detectors and accelerated mirrors we intend to determine the validity of acceleration radiation as a physical phenomenon. Before proceeding with this, however, we must first review the necessary quantum field theory as well as the formal aspects of quantization in Rindler space.

1-2 Topics in Quantum Field Theory

The quantum field theory we will consider will be for the neutral, massless scalar field ϕ. The wave equation for the field is

$$\Box\phi=\left(-\frac{\partial^2}{\partial t^2}+\nabla^2\right)\phi=0 \tag{1.1}$$

The field ϕ will be constructed as a decomposition of complete orthonormal modes as follows.

Let $\phi_1(x)$, $\phi_2(x)$ be solutions of equation 1.1. Then we define the inner product as

$$(\phi_1,\phi_2)\equiv-i\int d\Sigma^\mu\ \phi_1(x)(\overrightarrow{\partial}_\mu-\overleftarrow{\partial}_\mu)\phi_2^\star(x)$$

$$(\phi_1,\phi_2)\equiv-i\int d\Sigma^\mu\ \phi_1(x)\overleftrightarrow{\partial}_\mu\phi_2^\star(x) \tag{1.2}$$

where $d\Sigma^\mu$ defines a space-like surface, with a future directed normal. The inner product of 1.2 is usually written as

$$(\phi_1,\phi_2)=-i\int d^3\underline{x}\ \phi_1(x)\overleftrightarrow{\partial}_t\phi_2^\star(x) \tag{1.3a}$$

and has the property

$$(\phi_1,\phi_2)=(\phi_2,\phi_1)^\star=-(\phi_2^\star,\phi_1^\star) \tag{1.3b}$$

Now let the $u_k(x)$ be solutions to the wave equation and be orthonormal in the sense that

$$(u_k, u_{k'}) = \delta_{kk'} \qquad (u_k, u^{+}_{k'}) = 0 \qquad (u^{+}_k, u^{+}_{k'}) = -\delta_{kk'} \tag{1.4}$$

where $\delta_{kk'}$ may be either the discrete or continuous delta function depending on the situation. If the u_k are restricted to being complete then we may expand ϕ in terms of this set. That is

$$\phi(x) = a_k u_k(x) + a^{+}_k u^{+}_k(x)$$

$$a_k = (\phi, u_k) \qquad a^{+}_k = -(\phi, u^{+}_k)$$

so that

$$\phi(x) = -i\int d^3x' \Big\{\dot{\phi}(x')\,[\,-u^{+}_k(x')u_k(x) + u_k(x')u^{+}_k(x)\,]$$

$$+ \phi(x')\,[\dot{u}^{+}_k(x')u_k(x) - \dot{u}_k(x')u^{+}_k(x)]\Big\}$$

Therefore the complete set u_k has the additional properties

$$\sum_k -u^{+}_k(x')u_k(x) + u_k(x')u^{+}_k(x)\,\Big|_{t=t'} = 0 \tag{1.5}$$

$$\sum_k \dot{u}_k^{+}(x')u_k(x) - \dot{u}_k(x')u_k^{+}(x) \,\Big|_{t=t'} = i\delta(\underline{x}-\underline{x}') \tag{1.6}$$

As we shall see, the condition of completeness on the modes is crucial for both the Cauchy problem and quantization.

One more restriction on the modes is necessary. We must be able to distinguish between positive and negative frequency. This implies that a natural definition of "particle" exists. We will define positive frequency in the same way as it is used in quantum mechanics. If

$$\frac{\partial u_k}{\partial t} = -i\omega_k u_k \qquad , \quad \omega_k > 0 \tag{1.7}$$

then u_k is said to be a positive frequency mode. We see that this definition causes the positive and negative frequency modes to be orthogonal, which is consistent with 1.4. The problem that will later be encountered is that this definition is not unique, especially so in curved spacetime.

Quantization

Extending the quantum mechanical definition of commutators to fields gives the canonical quantization relations.

$$[\ \phi(x)\ ,\ \pi(x')\ ]_{t=t'} = i\delta(\underline{x}-\underline{x}') \tag{1.8}$$

$$\dot{\phi}(x) \equiv \pi(x) \tag{1.9}$$

$$[\ \phi(x)\ ,\ \phi(x')\ ]_{t=t'} = 0 \tag{1.10}$$

$$[\ \pi(x)\ ,\ \pi(x')\ ]_{t=t'} = 0 \tag{1.11}$$

Enforcing the condition 1.8 gives

$$[\ \phi(x)\ ,\ \pi(x')\ ]_{t=t'} = [\ a_k u_k(x) + a_k^\dagger u_k^\dagger(x)\ ,\ a_{k'}\dot{u}_{k'}(x) + \text{h.c.}]$$

$$= [\ a_k , a_{k'}\ ]\ u_k(x)\dot{u}_{k'}(x') + [\ a_k^\dagger , a_{k'}^\dagger\ ]\ u_k^\dagger(x)\dot{u}_{k'}^\dagger(x')$$

$$+ [\ a_k , a_{k'}^\dagger\ ]\ u_k(x)\dot{u}_{k'}^\dagger(x') - [\ a_{k'} , a_k^\dagger\ ]\ u_k^\dagger(x)\dot{u}_{k'}(x')\Big|_{t=t'}$$

$$= i\delta(\underline{x}-\underline{x}')$$

From the completeness relations in equations 1.5 and 1.6 we see that canonical quantization provides the following operator algebra.

$$[\, a_k, a_{k'} \,] = [\, a_k^+, a_{k'}^+ \,] = 0 \tag{1.12}$$

$$[\, a_k, a_{k'}^+ \,] = \delta_{kk'} \tag{1.13}$$

The condition 1.10 can be easily verified using 1.12 and 1.13. Condition 1.11 may be verified as follows.

$$[\, \pi(x), \pi(x') \,]_{t=t'} = [\, a_k \dot{u}_k(x) + \text{h.c.} \,,\ a_{k'} \dot{u}_{k'}(x') + \text{h.c.} \,]_{t=t'}$$

$$= \dot{u}_k^*(x)\dot{u}_k(x') - \dot{u}_k(x)\dot{u}_k^*(x') \Big|_{t=t'} \tag{1.14}$$

Using both the positive frequency restriction 1.7 and the wave equation 1.1 in expression 1.4 gives

$$[\, \pi(x), \pi(x') \,]_{t=t'} = -\omega_k^2 \, (u_k^*(x)u_k(x') - u_k(x)u_k^*(x') \Big|_{t=t'}$$

$$= \nabla^2 \, (\, u_k^*(x)u_k(x') - u_k(x)u_k^*(x') \Big|_{t=t'}$$

Therefore

$$[\, \pi(x), \pi(x') \,]_{t=t'} = 0$$

where we have used equation 1.5 in the final step.

To complete our quantization procedure there must be a set of states that a_k and a_k^+ operate on. These states are defined by the usual Fock representation where

$$a_k|o\rangle = 0 \qquad a_k^+ |o\rangle = |1_k\rangle \qquad a_k |1_k\rangle = |o\rangle \tag{1.15}$$

that is, the a_k operator annihilates a quantum identified by $|1_k\rangle$, and a_k^+ creates a particle $|1_k\rangle$. Many-particle states may be created from the vacuum state $|o\rangle$ using a_k and a_k^+.

$$|n_k\rangle = \frac{(a_k^+)^n}{\sqrt{n!}} |o\rangle \tag{1.16}$$

where the number operator is defined by

$$n_k |n_k\rangle = a_k^+ a_k |n_k\rangle \tag{1.17}$$

Green Functions

Expectation values for various combinations of field operators will be required in calculations throughout this study. These Green functions include the Pauli-Jordan, Wightman, Hadamard and Feynman functions, as well as their Euclidean counterparts which will be discussed in the next section.

The Pauli-Jordan Green function provides a covariant commutation relation

$$iG(x,x') = [\ \phi(x), \phi(x')\ ] \tag{1.18}$$

and is useful for solving Cauchy-type problems. The field at time t may be evaluated if it is known on a Cauchy surface at an earlier time t=0.

$$\phi(\underline{x},t) = \int d^3x'\ \phi(x')\ \overleftrightarrow{\partial}_{t'}\ G(x,x')\Big|_{t'=0} \tag{1.19}$$

The solution for ϕ in 1.19 is causal due to the property that G(x,x') has support only for time-like separation of x and x'. This may be verified as follows.

$$iG(x,x')\Big|_{t=t'=0} = [\ \phi(x), \phi(x')\ ]_{t=t'=0}$$

$$= [\ a_k u_k(x) + a_k^+ u_k^+(x')\ ,\ a_{k'} u_{k'}(x) + a_{k'}^+ u_{k'}^+(x')\ ]_{t=t'=0}$$

$$= u_k(x) u_k^+(x') - u_k^+(x) u_k(x')\ \Big|_{t=t'=0}$$

$$iG(x,x')\Big|_{t=t'=0} = 0 \qquad \text{(see equation 1.5)}$$

This justifies our previous assertion regarding the importance of completeness in defining a quantum field theory.

The Wightman functions are defined by

$$G^+(x,x') = \langle o| \phi(x)\phi(x') |o\rangle \tag{1.20}$$

$$G^-(x,x') = \langle o| \phi(x)\phi(x') |o\rangle \tag{1.21}$$

and will be required for analysis of particle detectors.

Hadamard's function may be defined in terms of the Wightman functions by

$$G^1(x,x') = G^+(x,x') + G^-(x,x')$$

$$G^1(x,x') = \langle o| \{ \phi(x),\phi(x') \} |o\rangle \tag{1.22}$$

The Feynman Green function is given by the time ordered product

$$iG_F(x,x') = \langle o| H(t-t') \phi(x)\phi(x') + H(t'-t) \phi(x')\phi(x) |o\rangle \tag{1.23}$$

Calculating $\Box G_F$

$$\Box G_F = \frac{1}{i}\left(- \frac{\partial^2}{\partial t^2} + \nabla^2 \right) \langle 0|\, H(t-t')\phi(x)\phi(x')$$

$$+ H(t'-t)\phi(x')\phi(x)\, |0\rangle$$

$$= \frac{1}{i}\, \delta(t-t')\, \frac{\partial}{\partial t}\, \langle 0|\, [\, \phi(x), \phi(x')\,]\, |0\rangle$$

Therefore

$$\Box G_F(x,x') = -\delta(x-x') \tag{1.24}$$

where we have used the canonical commutation relations 1.8 in the last step.

The Feynman Green function will be useful for the evaluation of stress-energy $T_{\mu\nu}$ in the presence of boundaries as discussed in chapter three. The reason for this is that G_F contains canonical quantization implicitly; so that we must merely solve 1.24 under certain boundary conditions.

Euclidean Green Functions

Euclidean Green functions may be obtained from the previous ones by letting $t=i\tau$. If we make this same substitution in the line element the metric becomes Euclidean. Substituting again into the wave equation results in a four dimensional "Laplacian".

The Euclidean Green function is useful for two reasons. First, it gives a method for avoiding poles that occur in the propagators by performing a Wick rotation from Euclidean space. Secondly, some boundary value problems become particularly simple in Euclidean space as will be seen in chapter three.

The Euclidean Green function G_E is defined as

$$G_E(\tau,\underline{x};\tau',\underline{x}') = \frac{1}{i}\, G_F(i\tau,\underline{x};i\tau',\underline{x}') \tag{1.25}$$

Then G_E will satisfy an equation similar to that of the Feynman Green function.

$$\Box G_E = \left(\frac{\partial^2}{\partial\tau^2} + \nabla^2\right) G_E(i\tau,\underline{x};i\tau',\underline{x}')$$

$$= \frac{1}{i}\left(\frac{\partial^2}{\partial\tau^2} + \nabla^2\right) G_F(i\tau,\underline{x};i\tau',\underline{x}')$$

$$= -\frac{1}{i}\,\delta(i\tau-i\tau')\,\delta(\underline{x}-\underline{x}')$$

$$\Box G_E(\tau,\underline{x};\tau',\underline{x}') = -\delta(\tau-\tau')\,\delta(\underline{x}-\underline{x}') \tag{1.26}$$

Thermal Green Functions

Thermal Green functions may be obtained by changing the vacuum expectation values to thermal ones as follows.

$$G^{+}_{\beta}(x,x') = \langle \phi(x)\phi(x') \rangle_{\beta}$$

$$G^{+}_{\beta}(x,x') = \mathrm{Tr}(\, e^{-\beta H}\, \phi(x)\phi(x')\,)\, /\, \mathrm{Tr}\, e^{-\beta H} \tag{1.27}$$

Using the equation of motion

$$\phi(t,\underline{x}) = e^{iHt}\, \phi(0,\underline{x})\, e^{-iHt}$$

and the cyclic property of traces, equation 1.27 implies that

$$G^{\pm}_{\beta}(t,\underline{x};t',\underline{x}') = G^{\mp}_{\beta}(t+i\beta,\underline{x};t',\underline{x}') \tag{1.28}$$

This periodicity is a fundamental property of thermal Green functions.

The only thermal Green function that we will use is the Hadamard one, which may be written as

$$G^{1}_{\beta}(t,\underline{x};t',\underline{x}') = \sum_{n=-\infty}^{\infty} G^{1}_{\beta}(t+in\beta,\underline{x};t',\underline{x}') \tag{1.29}$$

This function is necessary for our analysis of dipole particle detectors.

The Canonical Stress-Energy Tensor

The stress-energy tensor $T_{\mu\nu}$ may be generally defined by

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}}$$

where S is known as the action. In flat spacetime $T_{\mu\nu}$ may be defined in terms of the Lagrangian density.

$$T_{\mu\nu} = \frac{\delta L}{\delta \phi_{,\mu}} \phi_{,\nu} - g_{\mu\nu} L \qquad (1.30)$$

For the massless scalar field

$$L = \tfrac{1}{2} g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} \qquad (1.31)$$

so that 1.30 becomes

$$T_{\mu\nu} = \phi_{,\mu} \phi_{,\nu} - \tfrac{1}{2} g_{\mu\nu} g^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta} \qquad (1.32)$$

Conventionally $T_{\mu\nu}$ is interpreted as follows:

T^{oo} is the energy density.

T^{io} is the $i^{\underline{th}}$ component of momentum density.

T^{ij} is the i,j component of stress.

T^{oi} is the $i^{\underline{th}}$ component of energy flux.

The stress-energy tensor 1.30 has the important property of being conserved, as we will now demonstrate.

Let $L = L(\phi, \phi_{,\mu})$ Then

$$T^{\mu\nu}{}_{,\nu} = \phi^{,\mu}{}_{\nu} \frac{\delta L}{\delta \phi_{,\nu}} + \phi^{,\mu} \frac{\partial}{\partial x^{\nu}} \frac{\delta L}{\delta \phi_{,\nu}}$$

$$- g^{\mu\nu}{}_{,\nu} L - g^{\mu\nu} L_{,\nu} \tag{1.33}$$

The condition of metric compatibility means that $g^{\mu\nu}{}_{,\nu} = 0$ so that 1.33 becomes

$$T^{\mu\nu}{}_{,\nu} = \phi^{,\mu}{}_{\nu} \frac{\delta L}{\delta \phi_{,\nu}} + \phi^{,\mu} \frac{\partial}{\partial x^{\nu}} \frac{\delta L}{\delta \phi_{,\nu}}$$

$$- g^{\mu\nu} \left(\frac{\delta L}{\delta \phi} \phi_{,\nu} + \frac{\delta L}{\delta \phi_{,\lambda}} \phi^{,\lambda}{}_{\nu} \right)$$

$$T^{\mu\nu}{}_{,\nu} = \phi^{,\mu} \left(\frac{\partial}{\partial x^{\nu}} \frac{\delta L}{\delta \phi_{,\nu}} - \frac{\delta L}{\delta \phi} \right) \tag{1.34}$$

If the field ϕ satisfies the wave equation then

$$\frac{\partial}{\partial x^{\nu}} \frac{\delta L}{\delta \phi_{,\nu}} - \frac{\delta L}{\delta \phi} = 0$$

so that (1.34) becomes

$$T^{\mu\nu}{}_{,\nu} = 0$$

which implies that $T^{\mu\nu}$ is conserved.

In 1+1 dimensions $T^{\mu\nu}$ has the additional property of being traceless;

$$T^{\mu}{}_{\mu} = \phi_{,\mu}\phi_{,\nu}\, g^{\mu\nu} - \tfrac{1}{2} g^{\mu\nu} g_{\mu\nu} g^{\alpha\beta}\, \phi_{,\alpha}\phi_{,\beta}$$

$$= \phi_{,\alpha}\phi_{,\beta}\, g^{\alpha\beta}\, (1 - \tfrac{1}{2}\delta^{\mu}{}_{\mu})$$

$$T^{\mu}{}_{\mu} = 0 \tag{1.35}$$

and thus $T_{\mu\nu}$ is traceless.

Regularization of Vacuum Stress-Energy

One of the principal problems of quantum field theory is the calculation and interpretation of divergent quantities such as the vacuum expectation value of $T_{\mu\nu}$. We will be using three different regularization techniques: normal ordering, point splitting, and Green functions.

Normal ordering is the conventional regularization technique used for free fields. To normal order we evaluate the object $\langle o|:T_{\mu\nu}:_N|o\rangle$ by placing the annihilation operators of the state $|N\rangle$ before the creation operators. Different choices of orderings will yield different results, so that the choice of N must be a physical one.

The formal phenomenon of acceleration radiation gives a thermal stress-energy tensor when the choice of N is the Rindler state, a topic discussed later in this chapter.

Point splitting is a technique requiring the displacement of one of the field operators in $T_{\mu\nu}$ as follows.

$$\phi(t,\underline{x}) \rightarrow \phi(t+\varepsilon,\underline{x})$$

where t and $\underline{x}$ are the traditional Minkowski coordinates. This displacement provides the convergence required for calculating the various integrals involved in the vacuum expectation value of $T_{\mu\nu}$. The result for $\langle T_{\mu\nu}\rangle$ is then expanded in powers of ε, and the limit as ε approaches zero is taken after discarding the divergent term symptomatic of quantum field theory.

The Green function technique is perhaps the most intuitive regularization procedure we will utilise. This method will be used for quantum fields in the presence of boundaries, so that the Green functions will be altered from their empty space versions due to the boundary conditions involved.

To calculate $T_{\mu\nu}$ we operate on the regularized

Feynman Green function with a differential operator as follows.

$$\langle T_{\mu\nu}\rangle = \lim_{x\to y} \left(\partial_{\mu_x}\partial_{\nu_y} - \tfrac{1}{2}g_{\mu\nu}g^{\alpha\beta}\partial_{\alpha_x}\partial_{\beta_y}\right) iG_{F_{reg}}(x,y) \tag{1.36}$$

$$G_{F_{reg}}(x,y) = G_F(x,y) - G_{F_0}(x,y) \tag{1.37}$$

where $G_{F_0}(x,y)$ is the Feynman Green function for unbounded Minkowski space, hereafter referred to as the empty space Green function. We may verify that the operator defined in 1.36 will lead to the proper expression for $T_{\mu\nu}$. This calculation is outlined in appendix one.

The Green function in 1.36 has been regularized by removing the usual empty space part. This makes the limiting procedure well defined since empty space Green functions diverge as x approaches y. The regularization process is intuitively reasonable since it represents the variation of $T_{\mu\nu}$ from the divergent empty space value. Note that in the case of no boundaries this gives the conventional result $T_{\mu\nu}=0$.

Non-Uniqueness of Canonical Quantization

The non-uniqueness of the canonical quantization procedure has been studied in detail by Fulling (1973). The thermal effects of quantum field theory for an accelerated observer arise from formal aspects of this ambiguity. The non-uniqueness of quantization means that there are many decompositions of the field, corresponding to a generalization of the positive frequency definition in 1.7.

Suppose that the spacetime under consideration has a time-like Killing vector ξ_μ. Then along the world line traced out by a particle of momentum p, $p\cdot\xi$ is a constant. This can be shown as follows.

$$\nabla_p(p\cdot\xi) = p^\mu(p^\nu\xi_\nu)_{;\mu} = p^\mu p^\nu{}_{;\mu}\xi_\nu + p^\mu p^\nu\xi_{\nu;\mu}$$

$$= \xi\cdot\nabla_p p + \tfrac{1}{2}p^\mu p^\nu\xi_{(\mu;\nu)} \tag{1.38}$$

But $\nabla_p p=0$ for a free particle and $\xi_{(\mu;\nu)}=0$ if ξ is a Killing vector. This means that 1.38 becomes

$$\nabla_p(p\cdot\xi) = 0 \tag{1.39}$$

so that $p\cdot\xi$ is a constant. Since ξ has been restricted to a time-like vector, $p\cdot\xi$ measures the particle energy which is thus a constant. This provides a method of distinguishing between positive and negative energy, leading to the generalization of 1.7.

If

$$\frac{\partial u_k}{\partial t} = -i\omega_k u_k \qquad , \quad \omega_k > 0 \tag{1.40}$$

then u_k is said to be a positive frequency mode.

The different alternatives for quantization may be related through Bogolubov transformations. Suppose that there are two decompositions of the field

$$\phi = a_k u_k + \text{h.c.} = \bar{a}_k \bar{u}_k + \text{h.c.} \tag{1.41}$$

where u_k and $\bar{u}_k$ are complete sets, so they may be expanded in terms of eacn other.

$$\bar{u}_j = \alpha_{ji} u_i + \beta_{ji} u_i^* \tag{1.42}$$

Taking inner products in 1.42 identifies α and β as

$$\alpha_{ij} = (\bar{u}_i, u_j) \qquad \beta_{ij} = -(\bar{u}_i, u_j^*) . \tag{1.43}$$

These coefficients define a Bogolubov transformation. The inverse of 1.42 is

$$u_i = \alpha^*_{ji} \bar{u}_j - \beta_{ji} \bar{u}_j^* \tag{1.44}$$

Using 1.41, 1.42, and 1.44 the relations between the operators a and $\bar{a}$ may be found to be as follows.

$$a_i = \alpha_{ji}\bar{a}_j + \beta^*_{ji}\bar{a}^+_j \qquad (1.45)$$

$$\bar{a}_j = \alpha^*_{ji}a_i - \beta^*_{ji}a^+_i \qquad (1.46)$$

Enforcing the condition that u and $\bar{u}$ (as given by 1.42 and 1.44) are still orthonormal gives the relationships between the Bogolubov coefficients

$$(\bar{u}_j,\bar{u}_k) = \delta_{jk} = (\alpha_{ji}u_i + \beta_{ji}u^*_i, \alpha_{kl}u_l + \beta_{kl}u^*_l)$$

$$= (\alpha_{ji}\alpha^*_{kl} - \beta_{ji}\beta^*_{kl})\,\delta_{il}$$

so that

$$\alpha_{ji}\alpha^*_{ki} - \beta_{ji}\beta^*_{ki} = \delta_{jk} \qquad (1.47)$$

Similarily, taking the other inner products yields

$$\alpha_{ji}\beta_{ki} = \beta_{ji}\alpha_{ki} \qquad (1.48)$$

$$\alpha_{lk}\alpha^*_{li} - \beta_{li}\beta^*_{lk} = \delta_{ik} \qquad (1.49)$$

$$\beta_{ji}\alpha^{\star}_{jk} = \alpha_{ji}\beta^{\star}_{jk} \tag{1.50}$$

The restrictions 1.47 to 1.50 on the Bogolubov transforms cause the operator algebra to remain invariant.

$$[\, a_i, a_j \,] = [\, \bar{a}_i, \bar{a}_j \,] = 0 \,.$$

$$[\, a_i, a_j^{+} \,] = [\, \bar{a}_i, \bar{a}_j^{+} \,] = \delta_{ij} \tag{1.51}$$

The various choices for quantization result in an ambiguity in particle content between different decompositions. As an example of this, consider the number of u particles in the vacuum corresponding to the $\bar{u}$ modes. Using 1.46 we find

$$\langle 0|\, \bar{a}_j^{+}\bar{a}_j \,|0\rangle = \langle 0|\, (\alpha_{ji} a_i^{+} - \beta_{ji} a_i)(\alpha^{\star}_{jk} a_k - \beta^{\star}_{jk} a_k^{+}) \,|0\rangle$$

$$= \langle 0|\, \beta_{ji}\beta^{\star}_{jk}\, a_i a_k^{+} \,|0\rangle$$

$$\langle 0|\, \bar{a}_j^{+} a_j \,|0\rangle = \beta_{ji}\beta^{\star}_{ji} \tag{1.52}$$

This implies that the particle content of the states $|0\rangle$ and $|\bar{0}\rangle$ will in general differ. The observable effects of this ambiguity are evident in the discussion of particle detectors in chapter two.

1-3 Relativistic Kinematics of Acceleration

Many of the calculations considered in subsequent chapters will involve uniformly accelerated observers. This section will review the kinematics of relativistically uniform acceleration following Misner, Thorne and Wheeler (1970).

The four-acceleration a^μ is defined by

$$\frac{du^\mu}{d\tau} = a^\mu \tag{1.53}$$

where u^μ is the four-velocity. Since

$$u \cdot u = -1 \quad , \quad \frac{d}{d\tau}(u \cdot u) = 0 = 2\, a \cdot u$$

then

$$u \cdot a = 0 \tag{1.54}$$

so that the acceleration is orthogonal to the four-velocity. This means that in the instantaneous rest frame, $u = e_0$, that $a^\mu = (0, a^i)$ and

$$\sum_i (a^i)^2 = \left(\frac{d^2 x}{dt^2}\right)^2$$

agreeing with the usual definition of acceleration. Since $a \cdot a$ is invariant then

$$a \cdot a = \left(\frac{d^2 x}{dt^2}\right)^2 \tag{1.55}$$

evaluated in the instantaneous rest frame. For uniform acceleration, $a \cdot a = 1/\alpha^2$ which is constant.

Consider solving for the Minkowski coordinates $(t, \underline{x})$ as a function of τ for uniform acceleration. Choosing acceleration in the x direction, and using 1.53 and 1.54 along with the condition that $a \cdot a$ is constant, results in the following relations.

$$\frac{du^0}{d\tau} = a^0 \qquad \frac{du^1}{d\tau} = a^1 \qquad u^0 a^0 = u^1 a^1 \tag{1.56}$$

$$a^{0^2} = -1/\alpha^2 + a^{1^2} = -1/\alpha^2 + a^{0^2} u^{0^2}/u^{1^2}$$

$$a^{0^2}(u^{1^2} - u^{0^2}) = -u^{1^2}/\alpha^2 \tag{1.57}$$

$$a^0 = u^1/\alpha \tag{1.58}$$

From 1.58 and 1.56 we obtain

$$a^1 = u^0/\alpha \tag{1.59}$$

Using 1.53, 1.58 and 1.59 we obtain the following differential equations for the four velocity.

$$\frac{d^2u^0}{d\tau^2} = \frac{1}{\alpha}\frac{du^0}{d\tau} = u^0/\alpha^2 \tag{1.60}$$

$$\frac{d^2u^1}{d\tau^2} = \frac{1}{\alpha}\frac{du^0}{d\tau} = u^1/\alpha^2 \tag{1.61}$$

Solving the differential equations 1.60 and 1.61 with the condition $u=(1,\underline{0})$ at $t=0$, gives (t,x) in terms of τ.

$$t = \alpha\,\mathrm{sh}(\tau/\alpha) \qquad x = \alpha\,\mathrm{ch}(\tau/\alpha) \tag{1.62}$$

Notice that

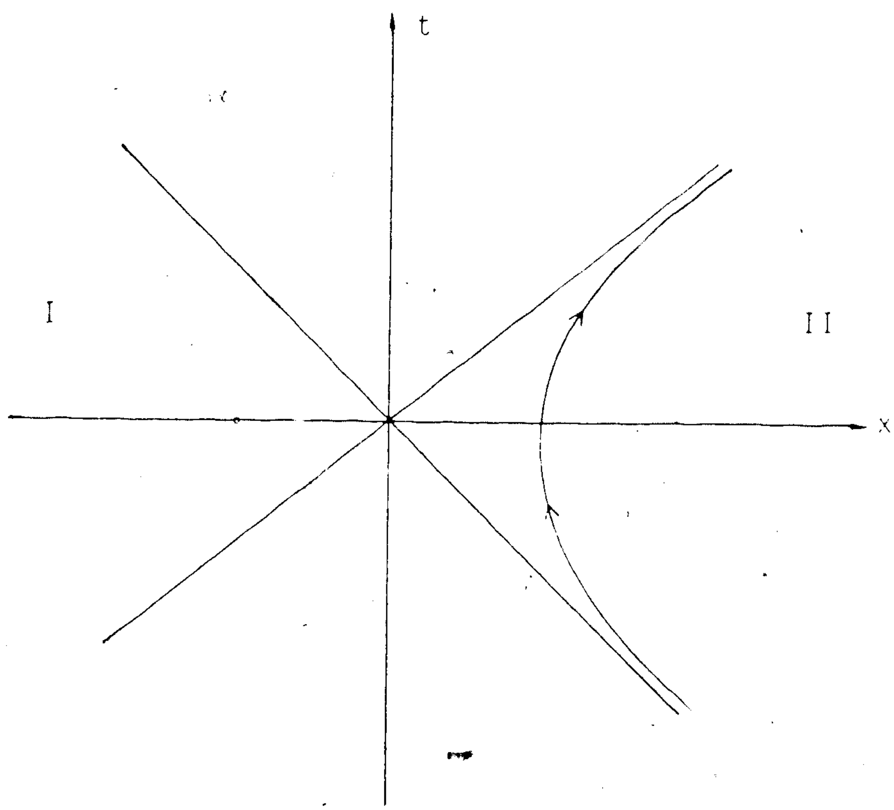
$$x^2 - t^2 = \alpha^2(\mathrm{ch}^2\tau/\alpha - \mathrm{sh}^2\tau/\alpha) = \alpha^2 = 1/a^2 \tag{1.63}$$

Thus the world-line of a uniformly accelerated object is a hyperbola in Minkowski coordinates (see Figure 1.1), causing the object to be causally disconnected from region II The causal structure of Rindler space is identical to that of a black hole represented by Kruskal coordinates. The presence of the null horizons bounding region I is responsible for the interesting features in the quantum field theory of both black holes and Rindler space (see Israel (1976)).

The natural coordinate system for a uniformly accelerated observer may now be obtained, resulting in

Figure 1.1

World Line for Uniform Acceleration in Minkowski Coordinates



the Rindler coordinates. To determine a coordinate basis with respect to the accelerated observer, we must be in the rest frame of the object and the basis must be non-rotating. The first condition is self-explanatory and may be satisfied by setting $e_0=u$. The second condition means that the basis system rotates only as much as required by the changing four velocity. Basis vectors orthogonal to both u and a are unaffected. The result of these requirements is the Fermi-Walker transport rule.

$$\frac{de_\alpha}{d\tau} = u(a\cdot e_\alpha) - a(u\cdot e_\alpha). \tag{1.64}$$

For uniform accelerations the choice of basis

$$e_0 = u \qquad e_1 = \hat{a} \qquad e_2 = \hat{y} \qquad e_3 = \hat{z} \tag{1.65}$$

will satisfy the restrictions of 1.64, along with the orthogonality conditions

$$e_\alpha\cdot e_\beta = 0 \quad , \alpha\neq\beta \tag{1.66}$$

This may be verified as follows;

$$\frac{de_0}{d\tau} = a = u(a\cdot u) - a(u\cdot u) = a$$

$$\frac{de_1}{d\tau} = \frac{1}{|a|}\frac{da}{d\tau} = \frac{1}{|a|}|a|\frac{d}{d\tau}(\gamma v,\gamma,0,0)$$

$$= |a|\ u \equiv u(a\cdot\hat{a}) - a(u\cdot\hat{a})$$

and

$$e_1\cdot e_0 = u\cdot a = 0$$

Thus the choice of basis 1.65 is appropriate. To define a coordinate system with respect to this basis we write a general vector as the sum of the object location and a space-like vector in the rest frame

$$z^\mu \equiv (\xi^k e_k)^\mu + x^\mu(\tau)$$

where $x^\mu(\tau)$ is the object location. This implies the following coordinate system as illustrated in Figure 1.2.

$$(t,x) = \xi(\text{sh}\ \tau/\alpha\ ,\ \text{ch}\ \tau/\alpha) + (\alpha\text{sh}\ \tau/\alpha\ ,\ \alpha\text{ch}\ \tau/\alpha)$$

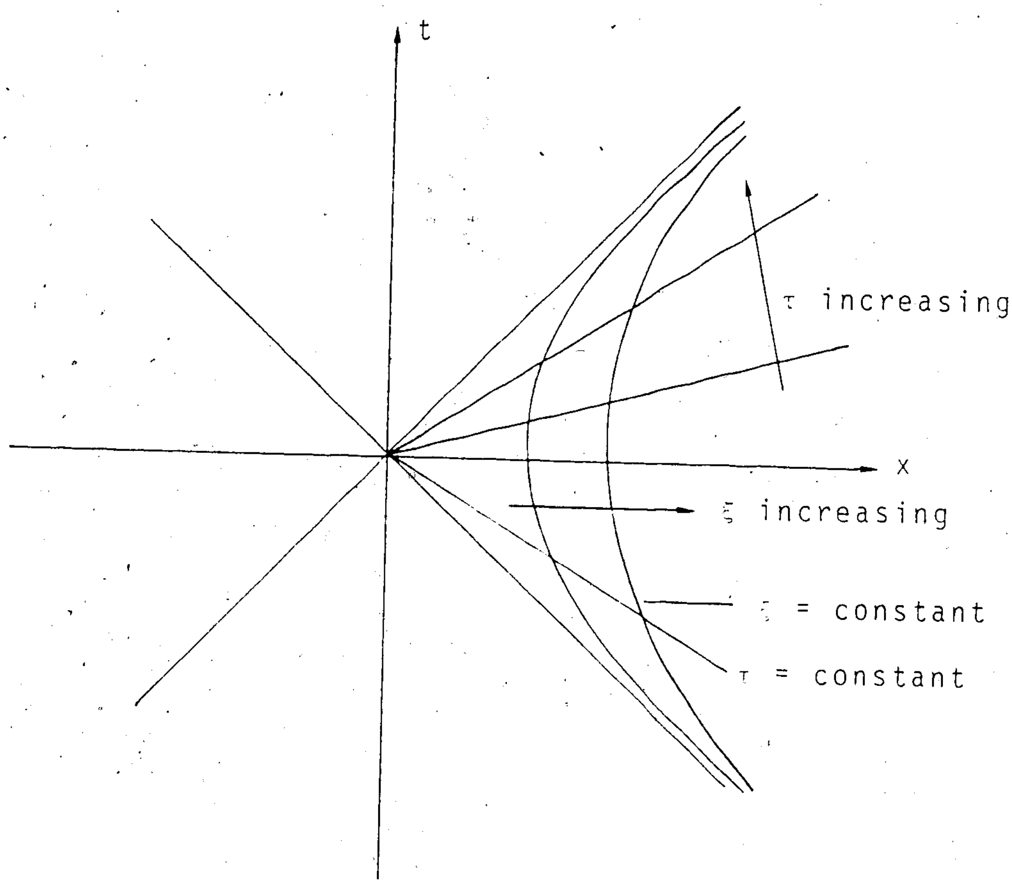
$$t = (\alpha+\xi)\ \text{sh}\ \tau/\alpha \qquad x = (\alpha+\xi)\ \text{ch}\ \tau/\alpha$$

(1.67)

Notice that

Figure 1.2

Rindler Coordinate System



$$x^2 - t^2 = (\alpha+\xi)^2 \tag{1.68}$$

so an observer with uniform acceleration remains at a fixed point ξ as would be expected.

In practice such a coordinate system 1.67 must remain small in the sense that

$$a\ell << 1 \tag{1.69}$$

where a is acceleration and ℓ is the size of the coordinate system. This can be seen as follows,

$$ds^2 = -dt^2 + dx^2 = -(d\xi\ \mathrm{sh}\tau/\alpha + (1+\xi/\alpha)\mathrm{ch}\tau/\alpha\ d\tau)^2$$

$$+ (d\xi\ \mathrm{ch}\tau/\alpha + (1+\xi/\alpha)d\tau\ \mathrm{sh}\tau/\alpha)^2$$

$$ds^2 = -(1+a\xi)^2\ d\tau^2 + d\xi^2 \tag{1.70}$$

but

$$1+a\xi > 0$$

therefore

$$-a\xi_{min} < 1$$

leading to the desired restriction 1.69.

The basis in equation 1.67 will be used in later sections, and a modification of the above coordinates will be useful for quantization in 1+1 dimensions.

$$t = \frac{e^{a\xi}}{a} \,\mathrm{sh}(a\tau) \qquad x = \frac{e^{a\xi}}{a} \,\mathrm{ch}(a\tau)$$

where the parameter a is a constant. (1.71)

Notice that

$$x^2 - t^2 = \frac{e^{2a\xi}}{a^2} \tag{1.72}$$

so for ξ constant we are on a world line of constant acceleration. The advantage to the Rindler coordinates given in 1.71 is that the line element is conformally Minkowskian.

$$ds^2 = -dt^2 + dx^2 = e^{2a\xi}(-d\tau^2 + d\xi^2) \tag{1.73}$$

In equations 1.70 and 1.73 the metric coefficients are independent of τ. This implies that the operator $\frac{\partial}{\partial\tau}$ defines a time-like Killing vector and is thus a basis for quantization.

1-4 Quantization in Rindler Space

Since the general quantization procedure is not unique, it is interesting to examine the consequences of quantum field theory for an accelerated observer. Uniform acceleration is essentially the first non-trivial case, since the usual quantization procedure is Poincaré invariant.

A uniformly accelerated observer has a natural coordinate system -- the Rindler coordinates. Thus we are essentially considering quantum field theory in Rindler space.

The interesting feature of this procedure is that the Minkowski vacuum is found to contain a thermal superposition of Rindler particles at $T=a/2\pi$. This is also reflected in the stress-energy tensor calculated by normal ordering with respect to Rindler annihilation and creation operators in the Minkowski vacuum.

This section will develop the standard quantization procedure for 1+1 dimensional Rindler space following the treatment of Birrell and Davies (1982).

Using the Rindler coordinates from equation 1.71

$$t = \frac{e^{a\xi}}{a}\,\mathrm{sh}(a\tau) \qquad x = \frac{e^{a\xi}}{a}\,\mathrm{ch}(a\tau)$$

the metric becomes

$$ds^2 = e^{2a\xi}(-d\tau^2 + d\xi^2)$$

Since the metric coefficients are independent of τ, $\frac{\partial}{\partial\tau}$ is a time-like Killing vector and thus can be used to define positive frequency (see section 1-2).

Under a conformal transformation, the wave equation for the massless scalar field is also conformal. This can be explicitly verified by considering the D'Alembertian

$$\Box = -\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2}$$

Calculating $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial x}$ gives

$$\frac{\partial}{\partial t} = \frac{\partial\xi}{\partial t}\frac{\partial}{\partial\xi} + \frac{\partial\tau}{\partial t}\frac{\partial}{\partial\tau} = -\frac{sha\tau}{e^{a\xi}}\frac{\partial}{\partial\xi} + \frac{cha\tau}{e^{a\xi}}\frac{\partial}{\partial\tau}$$

$$\frac{\partial}{\partial x} = \frac{cha\tau}{e^{a\xi}}\frac{\partial}{\partial\xi} - \frac{sha\tau}{e^{a\xi}}\frac{\partial}{\partial\tau}$$

Thus the D'Alembertian becomes

$$\Box = e^{-2a\xi}\left(-\frac{\partial^2}{\partial\tau^2} + \frac{\partial^2}{\partial\xi^2}\right) \tag{1.74}$$

The two causally distinct Rindler wedges, known as the left and right regions (L and R), are shown in Figure 1.3. As can be seen the L wedge has a sign problem in the τ coordinate. This can be handled in the mode analysis by changing the sign of τ to get the proper positive frequency behaviour.

We can now construct the field modes in Rindler coordinates.

$$
{}_R u_k(\tau,\xi) = \begin{cases} \dfrac{e^{-i\omega\tau+ik\xi}}{\sqrt{4\pi\omega}} & \text{in } R \\ 0 & \text{in } L \end{cases}
$$

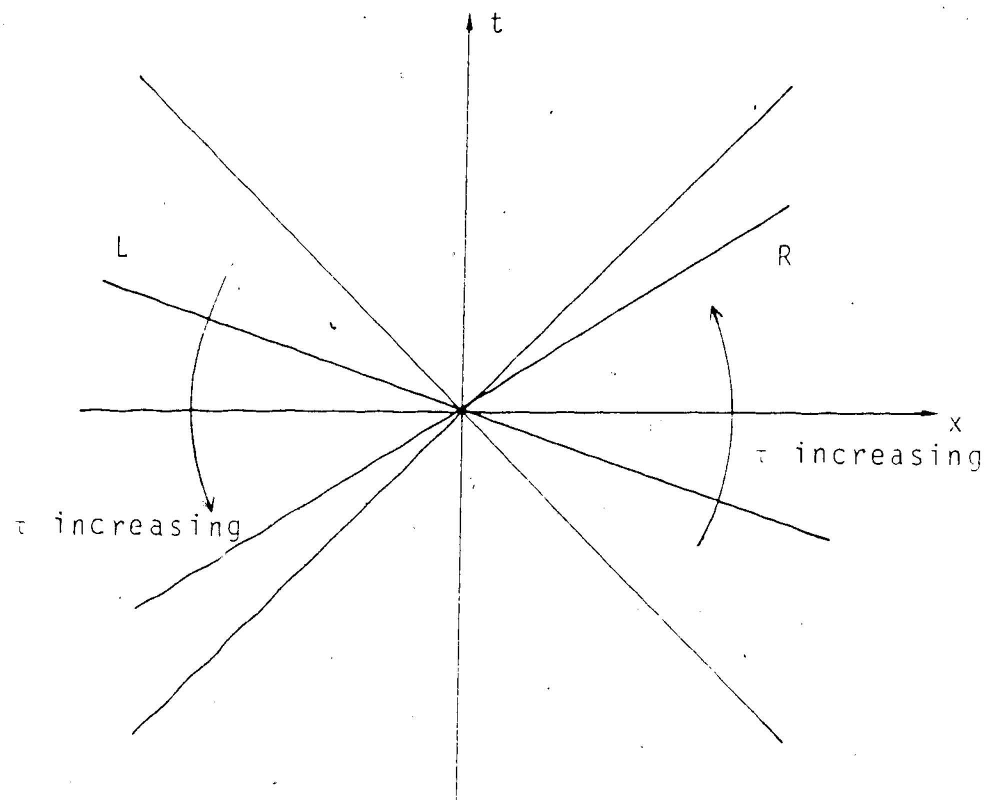
$$
{}_L u_k(\tau,\xi) = \begin{cases} \dfrac{e^{i\omega\tau+ik\xi}}{\sqrt{4\pi\omega}} & \text{in } L \\ 0 & \text{in } R \end{cases} \tag{1.75}
$$

Since neither set is complete over the Cauchy slice $t=0$ ($\tau=0$), both the L and R modes are required for the decomposition of ϕ.

$$
\phi(\tau,\xi) = \int dk \, [\, {}_R a_k \, {}_R u_k + h.c.] + [\, {}_L a_k \, {}_L u_k + h.c.] \tag{1.76}
$$

Figure 1.3

Causal Properties of the Rindler Wedges



Imposing canonical quantization gives

$$[\ \phi(x), \phi(x')\ ] = iG(x,x')$$

$$\frac{\partial G}{\partial t}\bigg|_{t=t'=0} = \frac{\partial G}{\partial \tau}\bigg|_{\tau=\tau'=0} = -\delta(\underline{x}-\underline{x}')$$

This forces the algebra of the annihilation and creation operators to be

$$[\ {}_{\iota}a_{k}, {}_{\sigma}a_{k'}\ ] = 0$$

$$[\ {}_{\iota}a_{k}, {}_{\sigma}\bar{a}_{k'}\ ] = \delta_{\iota\sigma}\delta_{kk'}$$

(1.77)

where ι and σ represent L or R indices. This indicates that the field can be decomposed either in Minkowski or Rindler coordinates.

In order to examine the particle content of the Rindler decomposition, we must evaluate the Bogolubov coefficients which relate Minkowski and Rindler creation and annihilation operators. This can be done in either of two ways. The inner products may be evaluated directly, as in equation 1.43, or complex analyticity arguments may be used. The second method which was introduced by Unruh (1976) is much more elegant in this case.

If

$$\phi(u) = \int_{-\infty}^{+\infty} e^{-i\omega u} \, \tilde{\phi}(\omega) \, d\omega \tag{1.78}$$

then $\phi(u)$ contains only positive frequency modes if it is analytic and bounded for all complex values of u whose imaginary parts are negative. This can easily be seen because the existence of $+i\omega$ terms in the exponent of 1.78 would cause the integral to be divergent for u with a negative imaginary part. By constructing combinations of u_R and u_L which obey the above analyticity properties, we can evaluate the relation between the two sets of creation and annihilation operators.

The linear combinations which satisfy the above argument are

$$_R u_k + e^{-\pi\omega/2a} \, _L u^{*}_{-k} \tag{1.79}$$

$$_R u^{*}_{-k} + e^{-\pi\omega/2a} \, _L u_k \tag{1.80}$$

which when written in Minkowski coordinates become

$$u^{i\omega/a}, \quad k>0 \qquad u = t - x$$

$$v^{-i\omega/a}, \quad k<0 \qquad v = t + x$$

and

$$\left.\begin{array}{ll} v^{i\omega/a}, & k>0 \\ u^{-i\omega/a}, & k<0 \end{array}\right\}$$

These expressions satisfy the analyticity requirements and thus contain the same information as the Minkowski decomposition. Imposing normalization on 1.79 and 1.80 leads to the field decomposition

$$\phi = \int dk \frac{1}{\sqrt{2sh\pi\omega/a}} \left[d_k^1 (e^{\pi\omega/2a} \; {}_Ru_k + e^{-\pi\omega/2a} \; {}_Lu^*_{-k}) \right.$$

$$\left. + d_k^2 (e^{-\pi\omega/2a} \; {}_Ru^*_{-k} + e^{\pi\omega/2a} \; {}_Lu_k) + h.c. \right]$$

(1.31)

where

$$d_k^1 |0\rangle = d_k^2 |0\rangle = 0$$

and

$$[d_k^i, d_{k'}^{j\dagger}] = \delta_{kk'} \delta^{ij}$$

If 1.81 is compared with 1.76 then we see that

$$R^a{}_K = \frac{1}{\sqrt{2sh\pi\omega/a}} \left(d^1_k \, e^{\pi\omega/2a} + d^{2+}_{-k} \, e^{-\pi\omega/2a} \right)$$

(1.82)

$$L^a{}_K = \frac{1}{\sqrt{2sh\pi\omega/a}} \left(d^2_k \, e^{\pi\omega/2a} + d^{1+}_k \, e^{-\pi\omega/2a} \right)$$

(1.83)

Since the L and R regions are causally disjoint the Rindler particle content of the Minkowski vacuum, as measured by an observer confined to the R region, is

$$\langle M | R^{a+}_k R^a_k | M \rangle = \langle M | d^2_{-k} d^{2+}_{-k} \frac{e^{-\pi\omega/a}}{2sh\pi\omega/a} | M \rangle$$

$$\langle M | R^{a+}_k R^a_k | M \rangle = \frac{1}{e^{2\pi\omega/a} - 1}$$

(1.84)

The expression for the number operator 1.84 is identical to a thermal superposition of particles at $T=a/2\pi$, implying that the Minkowski vacuum contains a thermal distribution of Rindler particles.

If the energy density of these particles is calculated by evaluating the Rindler ordered stress-energy tensor in the Minkowski vacuum, we find

$$\rho = \int_0^\infty \frac{d\omega}{2\pi} \frac{\omega}{e^{2\pi\omega/a} - 1}$$

$$\rho = \frac{1}{6}\left(\frac{a}{2\pi}\right)^2 = \frac{a^2}{24\pi} \qquad (1.85)$$

Taking into account the fact that

$$ds^2 = -e^{-2a\xi}\, d\xi^2$$

then 1.85 is modified by a factor of $e^{-2a\xi}$, so that

$$\rho = \frac{a^2}{24\pi} e^{-2a\xi}$$

Using 1.72 this becomes

$$\rho = \frac{a_{pr}^2}{24\pi} \qquad (1.86)$$

where a_{pr} is the proper acceleration at a point ξ.

If the total energy from 1.86 is calculated we obtain the following equations.

$$E = \int \rho \, d\ell = \int_0^\infty \frac{a^2}{24\pi} e^{-2a\xi} e^{a\xi} d\xi$$

$$E = \frac{a}{24\pi} \tag{1.87}$$

The results obtained for the energy density have been found by evaluating the Rindler ordered stress-energy in the Minkowski vacuum.

$$\langle M| :T_{\mu\nu}:_R |M\rangle$$

If we were to calculate a Minkowski ordered stress-energy for the Rindler vacuum we would find that

$$\langle R| :T_{\mu\nu}:_M |R\rangle = -\langle M| :T_{\mu\nu}:_R |M\rangle$$

which we will show in chapter two. This would cause the energy density 1.86 and the total energy 1.87 to change by a negative sign.

The results obtained in this section can be extended to the study of fermions and scalar fields in 3+1 dimensions. In 1+1 dimensions, Horibe (1979) has shown that the Rindler particle number in the Minkowski vacuum follows a Fermi-Dirac distribution at $T=a/2\pi$. The 3+1 results for the scalar field were considered in Unruh's original paper (1976), which differs from our previous analysis by having a more complicated set of normal modes.

CHAPTER TWO

Model Particle Detectors

2-1 Introduction

The ambiguity in the particle content of a quantum field lies in the definition of positive frequency, a concept which depends on the proper time chosen for quantization. A particle detector whose response depended on its trajectory and hence, proper time, could reveal whether or not this ambiguity is observable. More precisely, suppose the detector's proper time defines a time-like Killing vector giving a "natural" definition of both quantization and particles. If this natural particle content is reflected in the detector's response, then these particles are observable, giving the quantization procedure a physical interpretation.

The first model particle detectors were introduced by Unruh (1976) and DeWitt (1979), who represented a detector by a quantum system with Hamiltonian H_0 and eigenstates $|E\rangle$, such that

$$H_0|E\rangle = E|E\rangle \qquad (2.1)$$

with ground state

$$H_0|E_0\rangle = E_0|E_0\rangle$$

This system is then coupled to the quantum field of interest by an interaction Hamiltonian H_{int}. A particle is then said to be detected when the detector's quantum state jumps from the ground state to an excited one.

To study a specific model we assume that H_{int} is small and then use perturbation theory. For such small H_{int} the amplitude to go from the state $|E_0\rangle|0\rangle$ to $|E\rangle|\psi\rangle$ is given by

$$a_{E,\psi} = i\langle E|\langle\psi|\int_{-\infty}^{+\infty} d\tau'\ H_{int}(\tau')|0\rangle|E_0\rangle \qquad (2.2)$$

where τ is the detector proper time, and the states $|\psi\rangle$, $|0\rangle$ represent excited and vacuum states of the quantum field. The appearance of proper time in the amplitude causes the detector's response to be trajectory dependent.

The detector model studied by Unruh (1976) involved a Schroedinger particle ψ confined to a box, coupled to the field ϕ by the interaction $\varepsilon\phi\psi$. When this system is uniformly accelerated the excited states are occupied by ψ in a thermal distribution at $T=a/2\pi$. A relativistic model due to DeWitt (1979) has become the standard of detector analysis, the simplest such detector having a monopole interaction Hamiltonian.

2-2 Monopole Detectors

The interaction Hamiltonian for the monopole detector is

$$H_{int} = m(\tau)\phi(x(\tau)) \tag{2.3}$$

where $m(\tau)$ is the detector monopole moment, and $x(\tau)$ is the detector location. The equation of motion for $m(\tau)$ in the detector rest frame is

$$m(\tau) = e^{iH_0\tau} m(o) e^{-iH_0\tau} \tag{2.4}$$

Using this result the amplitude becomes

$$a_{E,\psi} = i\int_{-\infty}^{+\infty} d\tau \langle\psi|\phi(x(\tau'))|o\rangle \langle E|e^{iH_0\tau} m(o) e^{-iH_0\tau}|E_0\rangle \tag{2.5}$$

$$a_{E,\psi} = i\int_{-\infty}^{+\infty} d\tau\, e^{i\tau(E-E_0)} \langle\psi|\phi(x(\tau))|o\rangle\langle E|m(o)|E_0\rangle \tag{2.6}$$

Thus the total probability to jump to an excited state is

$$p = \sum_{\substack{\psi \\ E\neq E_0}} \int_{-\infty}^{+\infty} d\tau d\tau'\, e^{-i(E-E_0)(\tau-\tau')} |\langle E|m(o)|E_0\rangle|^2$$

$$\cdot \langle o|\phi(x(\tau))|\psi\rangle\langle\psi|\phi(x(\tau'))|o\rangle \tag{2.7}$$

The sum across the complete set of states $|\psi\rangle$ is the identity operator

$$\sum_{\psi} |\psi\rangle\langle\psi| = I \tag{2.8}$$

so that this becomes

$$p = |\langle E|m(o)|E_0\rangle|^2 \int_{-\infty}^{+\infty} d\tau d\tau' \, e^{-i(E-E_0)(\tau-\tau')}$$

$$\times \langle o|\phi(x(\tau))\phi(x(\tau'))|o\rangle \tag{2.9}$$

Recognizing the combination of field operators as the Wightman function $G^+(x(\tau),x(\tau'))$, results in

$$p = |\langle E|m(o)|E_0\rangle|^2 \int_{-\infty}^{+\infty} d\tau d\tau' \, e^{-i(E-E_0)(\tau-\tau')}$$

$$\times G^+(x(\tau),x(\tau')) \tag{2.10}$$

The expression

$$|\langle E|m(o)|E_0\rangle|^2 \tag{2.11}$$

is known as the detector sensitivity and is determined entirely by the details of the quantum system representing the detector. The part

$$F(E) = \int_{-\infty}^{+\infty} d\tau d\tau' \, e^{-iE(\tau-\tau')} G^+(x(\tau),x(\tau')) \tag{2.12}$$

is known as the response function and is trajectory dependent, which may reveal the apparent particle content for a given trajectory.

In cases where

$$G^+(x(\tau),x(\tau')) = g^+(\tau-\tau') \tag{2.13}$$

the whole process is invariant under τ translations, corresponding to a constant response per unit time.

$$F(E) = \int d(\tau-\tau')d(\tau+\tau')\ e^{-iE\Delta\tau}\ g^{+}(\Delta\tau) \tag{2.14a}$$

$$F(E)/\tau = \int d\Delta\tau\ e^{-iE\Delta\tau}\ g^{+}(\Delta\tau) \tag{2.14b}$$

If we now consider inertial detector motion as in Birrell and Davies (1982), the amplitude becomes

$$a_{E,\psi} = i\langle E|m(0)|E_0\rangle \int_{-\infty}^{+\infty} d\tau\ e^{i(E-E_0)\tau}\ \langle\psi|\phi(x(\tau))|0\rangle \tag{2.15}$$

But

$$\langle\psi|\phi(x(\tau))|0\rangle = \langle\psi|\int \frac{d^3k}{\sqrt{\omega}}\ (a_k^{+}\ e^{-i\underline{k}\cdot\underline{x}+i\omega t} + h.c.)|0\rangle$$

$$= \begin{cases} e^{-i\underline{k}'\cdot\underline{x}(\tau)+i\omega' t(\tau)} & ,\ \langle\psi| = \langle 1_{k'}| \\ 0 & \text{otherwise} \end{cases} \tag{2.16}$$

so that

$$a_{E,\psi} \sim \int_{-\infty}^{+\infty} d\tau\ e^{i(E-E_0)\tau}\ e^{-i\underline{k}'\cdot\underline{x}(\tau)+i\omega' t(\tau)} \tag{2.17}$$

For inertial motion $\underline{x}(\tau)=\gamma\underline{v}\tau$ and $t=\gamma\tau$ so that equation 2.17 becomes

$$a_{E,\psi} \propto \int_{-\infty}^{+\infty} d\tau \; e^{i(E-E_0)\tau} \; e^{i\tau(\omega'-\underline{k}'\cdot\underline{v})\gamma}$$

$$\propto \delta(E-E_0+(\omega'-\underline{k}'\cdot\underline{v})\gamma) \tag{2.18}$$

The argument of the delta function is never zero since $E-E_0>0$ and

$$\omega-\underline{k}\cdot\underline{v} \geq \omega-|\underline{k}||\underline{v}| = \omega(1-v) \geq 0 \tag{2.19}$$

meaning that $a_{E,\psi}=0$ for inertial motion. This result is due to Poincaré invariance; the Minkowski vacuum must be the empty state valid for all inertial observers.

Suppose that we now uniformly accelerate a detector so that it follows the world line given by equation 1.62

$$t = \alpha \,\mathrm{sh}\, \tau/\alpha \qquad x = \alpha \,\mathrm{ch}\, \tau/\alpha \qquad y,z = \text{constant} \tag{2.20}$$

Following the standard analysis due to DeWitt (1979), and Birrell and Davies (1982), the response of the detector may be calculated as follows.

The Wightman function for the massless scalar field is

$$G^{+}(x,x') = -\frac{1}{4\pi^2}\left[(t-t')^2-(\underline{x}-\underline{x}')^2\right]^{-1} \tag{2.21}$$

Evaluating this for the accelerated world line in equation 2.20 we find

$$G^{+}(x(\tau),x(\tau')) = -\frac{1}{4\pi^2}\left[(\mathrm{sh}\ \tau/\alpha - \mathrm{sh}\ \tau'/\alpha)^2 - (\mathrm{ch}\ \tau/\alpha - \mathrm{ch}\ \tau'/\alpha)^2\right]^{-1}$$

$$= -\frac{1}{16\pi^2}\left[\mathrm{sh}^2 \Delta\tau/2\alpha\ \mathrm{ch}^2 \frac{\tau+\tau'}{2\alpha} - \mathrm{sh}^2 \frac{\Delta\tau}{2\alpha}\ \mathrm{sh}^2 \frac{\tau+\tau'}{2\alpha}\right]^{-1}$$

$$G^{+}(x(\tau),x(\tau')) = -\frac{1}{16\pi^2}\left[\alpha^2\ \mathrm{sh}^2 \Delta\tau/2\alpha\right]^{-1} \tag{2.22}$$

The presence of τ invariance implies that there is a constant response per unit time.

$$F(E)/\tau = \int_{-\infty}^{+\infty} d\Delta\tau\ e^{-iE\Delta\tau}\left(-\frac{1}{16\pi^2\alpha^2 \mathrm{sh}^2 \Delta\tau/2\alpha}\right) \tag{2.23}$$

Using the expansion

$$\csc^2 x = \frac{1}{\pi^2}\sum_{n=-\infty}^{+\infty}\left(\frac{1}{x/\pi - n}\right)^2 \tag{2.24}$$

then

$$-\frac{1}{\mathrm{sh}^2 \Delta\tau/2\alpha} = \frac{1}{(i\ \sin\ i\Delta\tau/2\alpha)^2} = \frac{1}{\pi^2}\sum_{n=-\infty}^{+\infty}\frac{1}{\left(\frac{i\Delta\tau}{2\pi\alpha} - n\right)^2}\ .$$

$$-\frac{1}{\mathrm{sh}^2\ \Delta\tau/2\alpha} = -4\alpha^2\sum_{n=-\infty}^{+\infty}\frac{1}{(\Delta\tau + i2\pi n\alpha)^2} \tag{2.25}$$

The integral in equation 2.23 can be evaluated by completing the contour in the negative imaginary direction and ignoring the n=0 pole when calculating residues.

$$F(E)/\tau = \frac{1}{4\pi^2} \sum_{n=1}^{\infty} (2\pi i)(-iE)e^{iE(in2\pi\alpha)}$$

$$= \frac{E}{2\pi} \sum_{n=1}^{\infty} e^{-2\pi nE\alpha} = \frac{E}{2\pi} \frac{e^{-2\pi E\alpha}}{1-e^{-2\pi E\alpha}}$$

$$F(E)/\tau = \frac{E}{2\pi(e^{2\pi E\alpha}-1)} \qquad (2.26)$$

Using equation 2.26 and 2.10, the probability per unit time is

$$p/\tau = \frac{1}{2\pi} |\langle E|m(o)|E_0\rangle|^2 \frac{(E-E_0)}{e^{2\pi\alpha(E-E_0)}-1} \qquad (2.27)$$

This result is identical to the behaviour of the same detector at rest in a thermal bath at temperature $T=a/2\pi$.

Since the Minkowski vacuum contains a thermal superposition of Rindler particles at the same temperature, this implies that the detector is responding to the Rindler particles, giving the alternative quantization procedure a physical interpretation.

A uniformly accelerated detector will not yield a thermal result for an arbitrary H_{int}, the dipole detector being one such example.

2-3 Accelerated Dipole Detectors

Dipole detectors were introduced by Hinton (1983), who found a non-thermal and non-isotropic response in 3+1 dimensions and thermal response in 1+1 dimensions. Our analysis differs from the method of Hinton in that we work directly with the Green function as opposed to using a field decomposition in Rindler coordinates. Apart from a different choice of representing the detector dipole moment, the results we obtain are consistent with Hinton's.

The dipole interaction Hamiltonian is

$$H_{int} = m^{\mu}(\tau)\, \partial_{\mu} \phi\, x(\tau) \tag{2.28}$$

where $m^{\mu}(\tau)$ is the dipole moment and $x(\tau)$ is the detector location. Using the equation of motion for $m^{\mu}(\tau)$ leads to the transition probability (from equation 2.10)

$$m^{\mu}(\tau) = e^{iH_0 \tau}\, m^{\mu}(o)\, e^{-iH_0 \tau}$$

$$P = \langle E\, m^{\mu}(o)\, E_0 \rangle \langle E\, m^{\nu}(o)\, E_0 \rangle^{*}$$

$$\int d\tau\, d\tau'\, e^{-iE \Delta\tau} \ldots G^{+}(x,x') \Big|_{\substack{x = x(\tau) \\ x' = x(\tau')}} \tag{2.29}$$

Let

$$m(\tau) = m_0(\tau) u + m_1(\tau) n \tag{2.30}$$

where u is the detector velocity and n is the principal normal to the world line, satisfying $u \cdot n = 0$, $n \cdot n = +1$.

The vector n is a fixed vector which is Fermi-Walker transported. To illustrate this, consider the equation of motion for n (see 1.64).

$$\frac{dn}{d\tau} = u(a\cdot n) - a(u\cdot n) = u(a\cdot n) \qquad (2.31)$$

If we evaluate 2.31 in the rest frame $u=e_0$, then $n=(0,n^i)$ and

$$\frac{dn^i}{d\tau} = 0 \qquad (2.32)$$

so that in the rest frame, n is a fixed vector.

The conditions in 2.30 must remain valid under Fermi-Walker transport. This may be verified as follows.

$$\frac{d(u\cdot n)}{d\tau} = a\cdot n + u\cdot\frac{dn}{d\tau}$$

Using 2.31 the above expression becomes

$$\frac{d(u\cdot n)}{d\tau} = a\cdot n + u\cdot u(a\cdot n) = 0 \qquad (2.33)$$

It must also be verified that n remains normalized.

$$\frac{d(n\cdot n)}{d\tau} = 2n\cdot\frac{dn}{d\tau}$$

Using equation 2.31 for $\frac{dn}{d\tau}$ and the result 2.33

$$\frac{d(n\cdot n)}{d\tau} = 2(u\cdot n)(a\cdot n) = 0 \qquad (2.34)$$

Thus 2.33 and 2.34 indicate that the choice of the dipole moment in equation 2.30 is consistent with Fermi-Walker transport.

Using the coupling in 2.30, the probability for transition (equation 2.29) becomes

$$p = |\langle E|m_0(o)|E_0\rangle|^2 \int d\tau\, d\tau'\, e^{-i(E-E_0)\Delta\tau}\, \partial_\tau \partial_{\tau'} G^+$$

$$+ |\langle E|m_1(o)|E_0\rangle|^2 \int d\tau\, d\tau'\, e^{-i(E-E_0)\Delta\tau}$$

$$\cdot \Big\{ (n^\xi n^{\xi'} \partial_\xi \partial_{\xi'} + n^x n^{x'} \partial_x \partial_{x'} + n^y n^{y'} \partial_y \partial_{y'}) G^+$$

$$+ (n^\xi n^{x'} \partial_\xi \partial_{x'} + n^\xi n^{y'} \partial_\xi \partial_{y'} + \text{symmetric terms}) G^+ \Big\}$$

$$+ \langle E|m_0(o)|E_0\rangle \langle E|m_1(o)|E_0\rangle^*$$

$$\cdot \int d\tau\, d\tau'\, e^{-iE\Delta\tau}\, n^\xi \partial_\xi \partial_{\tau'} G^+ + \text{symmetric term}$$

$$+ \text{similar cross terms for } x, y \,. \qquad (2.35)$$

ξ and χ are defined by (equation 1.67)

$$t = (\chi+\xi)\,\mathrm{sh}\,\tau/\chi \qquad z = (\chi+\xi)\,\mathrm{ch}\,\tau/\chi$$

Evaluating the derivatives along the uniformly accelerated world line $\xi'=\xi=0$, $x=x'$, $y=y'$, gives

$$\partial_\tau \partial_{\tau'} G^+ = \frac{1}{16\pi^2\alpha^4 \mathrm{sh}^2 \Delta\tau/2\alpha} + \frac{3}{32\pi^2\alpha^4 \mathrm{sh}^4 \Delta\tau/2\alpha}$$

$$\partial_x \partial_{x'} G^+ = \partial_y \partial_{y'} G^+ = \frac{1}{32\pi^2\alpha^4 \mathrm{sh}^4 \Delta\tau/2\alpha}$$

$$\partial_\xi \partial_{\xi'} G^+ = -\frac{1}{16\pi^2\alpha^4 \mathrm{sh}^2 \Delta\tau/2\alpha} + \frac{1}{32\pi^2\alpha^4 \mathrm{sh}^4 \Delta\tau/2\alpha}$$

$$\partial_\xi \partial_{\tau'} G^+ = -\partial_{\xi'} \partial_\tau G^+ = \frac{\mathrm{ch}\,\Delta\tau/2\alpha}{16\pi^2\alpha^4 \mathrm{sh}^3 \Delta\tau/2\alpha} \tag{2.36}$$

All other derivatives are zero.

Notice that the τ translation symmetry exists, meaning a constant transition probability per unit time. Evaluating the integrals

$$\int_{-\infty}^{+\infty} \frac{e^{-iE\tau}\,d\tau}{\mathrm{sh}^2 \tau/2\alpha} = -\frac{8\pi E\alpha^2}{e^{2\pi E\alpha}-1} \tag{2.37}$$

$$\int_{-\infty}^{+\infty} \frac{e^{-iE\tau}\,d\tau}{\mathrm{sh}^4 \tau/2\alpha} = \frac{4\pi\alpha}{3}\,\frac{[(E\alpha)^2+4E\alpha]}{e^{2\pi E\alpha}-1} \tag{2.38}$$

$$\int_{-\infty}^{+\infty} \frac{e^{-iE\lambda} ch\lambda/2\alpha}{sh^3\lambda/2\alpha} d\lambda = \frac{i\alpha E 8\pi E\alpha^2}{e^{2\pi E\alpha}-1} \tag{2.39}$$

Finally this leads to the following result for equation 2.35 using 2.36 to 2.39.

$$p/\tau = \frac{E^3}{8\pi} \frac{|\langle E|m_0(o)|E_0\rangle|^2}{(e^{2\pi E\alpha}-1)}$$

$$+ |\langle E|m_1(o)|E_0\rangle|^2 \left(\frac{E^3 n^2}{24\pi(e^{2\pi E\alpha}-1)} \right.$$

$$\left. + \frac{Ea^2 n^2}{6\pi(E^{2\pi E\alpha}-1)} + \frac{2}{3\pi} \frac{n^{\xi 2} a^2 E}{(e^{2\pi E\alpha}-1)} \right)$$

$$+ (\langle E|m_0(o)|E_0\rangle\langle E|m_1(o)|E_0\rangle^* - c.c.)$$

$$\times - \frac{1}{16\pi^2\alpha^4} \frac{i\alpha E 8\pi\alpha^2 n^\xi}{(e^{2\pi E\alpha}-1)} \tag{2.40}$$

The combination $n^\xi a$ can be written as $n\cdot a$. Since $n^\xi a$ is time independent (see 2.32), $n\cdot a$ must also be constant. This may be verified as follows.

$$\frac{d(n\cdot a)}{d\tau} = \frac{dn}{d\tau} \cdot a + n \cdot \frac{da}{d\tau}$$

Using 2.32 this becomes

$$\frac{d(n\cdot a)}{d\tau} = (u\cdot a)(a\cdot n) + n \cdot \frac{da}{d\tau} = n \cdot \frac{da}{d\tau}$$

For uniform acceleration

$$\frac{da}{d\tau} = a^2 u \qquad \text{(see chapter one)}$$

$$\frac{d(n\cdot a)}{d\tau} \sim n\cdot u \equiv o \qquad \text{(equation 2.33)}$$

so that equation 2.40 can be written as

$$p/\tau = |\langle E|m_o(o)|E_o\rangle|^2 \frac{E^3}{8\pi(e^{+2\pi E\alpha}-1)}$$

$$+ \frac{|\langle E|m_1(o)|E_o\rangle|^2}{e^{2\pi E\alpha}-1} \left(\frac{E^3}{24\pi} + \frac{Ea^2}{6\pi} + \frac{2}{3\pi} E(n\cdot a)^2\right)$$

$$+ \mathrm{Im}[\langle E|m_o(o)|E_o\rangle\langle E|m_1(o)|E_o\rangle^*]$$

$$\times \frac{E^2 n\cdot a}{\pi(e^{2\pi E\alpha}-1)} \qquad (2.41)$$

This probability is both non-thermal and non-isotropic, where by non-thermal we mean that the response is not identical to that expected by the same detector at rest in a thermal bath. Both these effects occur in the space-like couplings so that for a purely time-like coupling ($m_1(\tau) \equiv 0$) the response will be thermal. The effect of the non-isotropic $n \cdot a$ term is to single out the direction of acceleration as preferred, a result which is reasonable.

For a general coupling the dominant contributions to equation 2.41 occur for $E/a \ll 1$ ($E \ll a$), which implies that the non-thermal contributions will tend to dominate the detector response. An explanation of this result is necessary; most of the confusion results from too much reasoning by analogy. Since all previous results are analogous to the behaviour of the same system at rest in a thermal bath, we would expect the same result in this case. This is definitely not true. The Rindler metric is identical to that of an infinite flat earth, so the correct analogy is that a uniformly accelerated detector responds as though it were at rest above an infinite flat earth. For the local monopole detector this is not a problem but when considering differences between two points, as with the dipole detector, discrepancies will occur. As a rough indication of what can happen, consider the Green function $G^1(x,x')$, as studied by Troost and Van Dam (1977) and in more detail by Dowker (1977).

$$G^1(x,x') = -\frac{1}{2\pi^2[(t-t')^2-(\underline{x}-\underline{x}')^2]} \tag{2.42}$$

For Rindler coordinates

$$t = (\alpha+\xi)\,\mathrm{sh}\,\tau/\alpha \qquad x = (\alpha+\xi)\,\mathrm{ch}\,\tau/\alpha$$

evaluated at $\xi=0$, 2.42 becomes (see 2.22 and 2.25)

$$G^1(\tau,\tau') = \frac{-1}{8\pi^2\,\mathrm{sh}^2\,\Delta\tau/2\alpha}$$

$$G^1(\tau,\tau') = -\frac{1}{2\pi^2}\sum_{n=-\infty}^{+\infty}\frac{1}{(\Delta\tau+i2\pi n\alpha)^2} \tag{2.43}$$

The corresponding thermal Green function is (1.29)

$$G^1_\beta(t,t') = -\frac{1}{2\pi^2}\sum_{n=-\infty}^{+\infty}\frac{1}{(\Delta t+in\beta)^2}$$

so that the Green function for the field evaluated on the world line is thermal.

The time-like couplings are confined strictly to the world line and will thus lead to a thermal response. The space-like couplings are sensitive to changes in the field between two world lines having different accelerations.

This implies that a space-like coupling is comparing points at varying temperatures, thus leading to a non-thermal response.

2-4 General Positive Frequency Response

The generalization of an accelerated trajectory is a Killing orbit which provides a natural method of quantization and particle definition. It is thus interesting to consider the behaviour of a detector with a world line of this type. We will show that a monopole detector following this generalized trajectory will respond to the Killing particle density. This is analagous to the accelerated detector's response to Rindler particles.

Let the trajectory proper time be τ and suppose that $\partial/\partial\tau$ represents a time-like Killing vector, then the field ϕ may be decomposed as

$$\phi = \int d^3k \left(e^{-i\omega\tau} u_{\underline{k}}(\underline{\xi}) b_{\underline{k}} + h.c.\right) \tag{2.44}$$

where $b_{\underline{k}}$ and $b^{+}_{\underline{k}}$ are the Killing annihilation and creation operators.

With a monopole interaction Hamiltonian, the response function is given by (see 2.12)

$$F(E) = \int_{-\infty}^{+\infty} d\tau d\tau' \, d^3k d^3k' \, e^{-iE(\tau-\tau')}$$

$$\times \left| \langle o | (e^{-i\omega\tau} u_{\underline{k}}(\underline{\xi}_o) b_{\underline{k}} + h.c.) \right.$$

$$\times \left. (e^{-i\omega'\tau'} u_{\underline{k}'}(\underline{\xi}_o) b_{\underline{k}'} + h.c.) | o \rangle \right|$$

$$F(E) = (2\pi)^2 \int d^3k\, d^3k'$$

$$\times \langle 0| \left(\delta(\omega+E) u_k(\underline{\xi}_0) b_k + \delta(\omega-E) u_k^*(\underline{\xi}_0) b_k^\dagger\right)$$

$$\times \left(\delta(\omega'-E) u_{k'}(\underline{\xi}_0) b_{k'} + \delta(\omega'+E) u_{k'}^*(\underline{\xi}_0) b_{k'}^\dagger\right)|0\rangle \tag{2.45}$$

where ξ_0 is the detector location.

Since $E, \omega > 0$, the only non-zero delta function contributions will come from the $b_k^\dagger$ part in the first term and the b_k piece in the second one. This results in the response

$$F(E) = (2\pi)^2 \int d^3k\, d^3k'\; \delta(\omega-E)\, \delta(\omega'-E)$$

$$\times \langle 0| b_k^\dagger b_{k'} |0\rangle u_k^*(\underline{\xi}_0) u_{k'}(\underline{\xi}_0) \tag{2.46}$$

This implies that the detector responds to the Killing particle density in the vacuum state $|0\rangle$. In the case of an accelerated detector the number of Rindler particles in the Minkowski state follows a thermal distribution which is then reflected in the response function. This also explains the behaviour of the dipole detector in a different way.

The response of the dipole detector with purely space-like couplings will be

$$F_{\mu\nu}(E) = \int d^3k d^3k' \, \delta(\omega-E)\delta(\omega'-E)$$

$$\times \, \partial_\mu \partial_\nu \, u_k^*(\xi) u_{k'}(\xi')\Big|_{\xi=\xi'=\xi_0} \langle 0|b_k^\dagger b_{k'}|0\rangle \; ;$$

$$\mu,\nu \neq 0$$

This means that the detector is measuring differences in the number density, a quantity which depends on the local acceleration and hence position, resulting in a non-thermal, non-isotropic response.

Equation 2.46 can be written in a different way by realizing that since the $b_k b_k^\dagger$ term will be destroyed by the τ,τ' integrations, the operators may be reordered as follows;

$$F(E) = \int d\tau d\tau' \, e^{-iE(\tau-\tau')} \langle 0| :\phi(x(\tau))\phi(x(\tau')):_k |0\rangle \tag{2.47}$$

where $: \, :_k$ represents normal ordering with respect to the Killing annihilation and creation operators.

Thus we have seen that a detector responds to the particles naturally defined by its world line. We now wish to consider the difference in response from two distinct trajectories in an attempt to determine if there is a state independent difference between them.

Suppose that the state of the field is $|\psi\rangle$; then consider two trajectories with positive frequency modes and operators represented by o,ō. From equation 2.47 we know that each detector will respond to the effective Green functions

$$\langle N|:\phi(x(\tau))\phi(x(\tau')):_o|N\rangle$$

$$\langle N|:\phi(x(\tau))\phi(x(\tau')):_{\bar{o}}|N\rangle$$

(2.43)

The difference between the two orderings in the above equation will be a c-number (as is always the case), so the task will be to determine the significance of this function.

Using the notation

$$\phi(x) = a_i u_i + a_i^+ u_i^+ = \bar{a}_j \bar{u}_j + \bar{a}_j^+ \bar{u}_j^+$$

$$\phi(x') = a_i u_i' + a_i^+ u_i^{+\prime} = \bar{a}_j \bar{u}_j' + \text{h.c.}$$

(2.47)

then

$$:\phi(x)\phi(x'):_{\bar{o}} - :\phi(x)\phi(x'):_o$$

$$= \bar{u}_i \bar{u}_j'(\bar{a}_i \bar{a}_j - :\bar{a}_i \bar{a}_j:_o) + \bar{u}_i^+ \bar{u}_j^{+\prime}(\bar{a}_i^+ \bar{a}_j^+ - :\bar{a}_i^+ \bar{a}_j^+:_o)$$

$$+ \bar{u}_i^+ \bar{u}_j'(\bar{a}_i^+ \bar{a}_j - :\bar{a}_i^+ \bar{a}_j:_o) + \bar{u}_i \bar{u}_j^{+\prime}(\bar{a}_j^+ \bar{a}_i - :\bar{a}_i \bar{a}_j^+:_o)$$

(2.48)

The relation between the operators $a, \bar{a}$ in terms of the Bogolubov transforms (chapter one) is now used to obtain the following equations.

$$:\bar{a}_i \bar{a}_j:_o = :(\alpha^*_{ik} a_k - \beta^*_{ik} a^+_k)(\alpha^*_{jl} a_l - \beta^*_{jl} a^+_l):_o$$

$$:\bar{a}_i \bar{a}_j:_o = \bar{a}_i \bar{a}_j + \alpha^*_{ik} \beta^*_{jl} (a_k a^+_l - a^+_l a_k) = \bar{a}_i \bar{a}_j + \alpha^*_{ik} \beta^*_{jk} \qquad (2.49)$$

$$:\bar{a}^+_i \bar{a}^+_j:_o = :(\alpha_{ik} a^+_k - \beta_{ik} a_k)(\alpha_{jl} a^+_l - \beta_{jl} a_l):_o$$

$$:\bar{a}^+_i \bar{a}^+_j:_o = \bar{a}^+_i \bar{a}^+_j + \beta_{ik} \alpha_{jk} \qquad (2.50)$$

Similarly

$$:\bar{a}^+_i \bar{a}_j:_o = \bar{a}^+_i \bar{a}_j - \beta_{ik} \beta^*_{jk} \qquad (2.51)$$

$$:\bar{a}_i \bar{a}^+_j:_o = \bar{a}_i \bar{a}^+_j - \alpha^*_{ik} \alpha^*_{jk} \qquad (2.52)$$

Using 2.49 to 2.52 in equation 2.48 we obtain

$$:\phi(x)\phi(x'):_{\bar{0}} - :\phi(x)\phi(x'):_{0} =$$

$$- \bar{u}_i\bar{u}_j' \alpha^*_{ik}\beta^*_{jk} - \bar{u}^+_i\bar{u}^{+\prime}_j \beta_{ij}\alpha_{jk}$$

$$+ \bar{u}^+_i\bar{u}_j' \beta_{ik}\beta^*_{jk} + \bar{u}_i\bar{u}^{+\prime}_j(\bar{a}_j\bar{a}^+_i - \bar{a}_i\bar{a}^+_j + \alpha^*_{ik}\alpha_{jk})$$

$$= -\bar{u}_i\bar{u}_j' \alpha^*_{ik}\beta^*_{jk} - \bar{u}^+_i\bar{u}^{+\prime}_j \beta_{ik}\alpha_{jk}$$

$$+ \bar{u}^+_i\bar{u}_j' \beta_{ik}\beta^*_{jk} + \bar{u}_i\bar{u}^{+\prime}_j(-\delta_{ij} + \alpha^*_{ik}\alpha_{jk})$$

but (see equation 1.47)

$$\alpha_{ik}\alpha^*_{jk} - \beta_{ik}\beta^*_{jk} = \delta_{ij}$$

therefore

$$:\phi(x)\phi(x'):_{\bar{0}} - :\phi(x)\phi(x'):_{0} = -\bar{u}_i\bar{u}_j' \alpha^*_{ik}\beta^*_{jk}$$

$$+ \bar{u}^+_i\bar{u}_j' \beta_{ik}\beta^*_{jk} + \bar{u}_i\bar{u}^{+\prime}_j \beta^*_{ik}\beta_{jk} - \bar{u}^+_i\bar{u}^{+\prime}_j \beta_{ik}\alpha_{jk} \qquad (2.53)$$

Evaluating the following expectation value

$$\langle 0|:\phi(x)\phi(x'):|0\rangle = \langle 0|\bar{u}_i\bar{u}_j' \bar{a}_i\bar{a}_j$$

$$+ \bar{u}^+_i\bar{u}^{+\prime}_j \bar{a}^+_i\bar{a}^+_j + \bar{u}^+_i\bar{u}_j' \bar{a}^+_i\bar{a}_j + \bar{u}_i\bar{u}^{+\prime}_j \bar{a}^+_j\bar{a}_i|0\rangle$$

Once again, using

$$\bar{a}_i = \alpha^*_{ik} a_k - \beta^*_{ik} a^+_k$$

we find

$$\langle 0|\bar{a}_i \bar{a}_j|0\rangle = -\alpha^*_{ik}\beta^*_{jk} \qquad (2.54)$$

$$\langle 0|\bar{a}^+_i \bar{a}^+_j|0\rangle = -\beta_{ik}\alpha_{jk} \qquad (2.55)$$

$$\langle 0|\bar{a}^+_i \bar{a}_j|0\rangle = \beta_{ik}\beta^*_{jk} \qquad (2.56)$$

so that

$$\langle 0|:\rho(x)\rho(x'):_{\bar{0}}|0\rangle = -\bar{u}_i\bar{u}_j\,\alpha^*_{ik}\beta^*_{jk} - \bar{u}^+_i\bar{u}^+_j\,\beta_{ik}\alpha_{jk} + \bar{u}^+_i\bar{u}_j\,\beta_{ik}\beta^*_{jk} + \bar{u}_i\bar{u}^+_j\,\beta_{jk}\beta^*_{ik} \qquad (2.57)$$

This equation is identical to 2.53 implying that

$$:\rho(x)\rho(x'):_{\bar{0}} - :\rho(x)\rho(x'):_0 = \langle 0|:\rho(x)\rho(x'):_{\bar{0}}|0\rangle \qquad (2.58)$$

or

$$\langle N|:\rho(x)\rho(x'):_{\bar{0}}|N\rangle - \langle N|:\rho(x)\rho(x'):_0|N\rangle$$

$$= \langle 0|:\rho(x)\rho(x'):_{\bar{0}}|0\rangle \qquad (2.59)$$

Using the result 2.59, consider the difference between the response functions for two trajectories represented by τ_1, τ_2. Suppose that for both trajectories there is a constant response per unit time, then

$$F_1(E)/\tau_1 - F_2(E)/\tau_2 = \int d\Delta\tau_1\, e^{-iE\Delta\tau_1}\, G_1(\Delta\tau_1)$$

$$- \int d\Delta\tau_2\, e^{-iE\Delta\tau_2}\, G_2(\Delta\tau_2) \qquad (2.60)$$

where G_1, G_2 are the effective Green functions (see 2.47). Using the result 2.59, that

$$G_1(\Delta\tau_1) - G_2(\Delta\tau_1) = \langle 0_2 \,|\, \Delta(\tau_1)\Delta(\tau_1') \,|\, 0_1\, 0_2\rangle$$

then 2.60 becomes

$$F_1(E)/\tau_1 - F_2(E)/\tau_2 = \int d\Delta\tau_1\, e^{-iE\Delta\tau_1}$$

$$\left[G_2(\Delta\tau_1) + \langle 0_2 \,|\, \Delta(\tau_1)\Delta(\tau_1') \,|\, 0_1\, 0_2\rangle \right]$$

$$- \int d\Delta\tau_2\, e^{-iE\Delta\tau_2}\, G_2(\Delta\tau_2)$$

$$= \int d\Delta\tau_1\, e^{-iE\Delta\tau_1} \langle 0_2 \,|\, \Delta(\tau_1)\Delta(\tau_1') \,|\, 0_1\, 0_2\rangle \qquad (2.61)$$

The significance of this result is that the difference in response to the state $|N\rangle$ for two different trajectories, represented by $o, \bar{o}$, is governed by the state-independent object

$$\langle o| :\phi(x)\phi(x'):_{\bar{o}} |o\rangle ,$$

which was originally the response of the $\bar{o}$ detector to the state $|o\rangle$. This implies that there is a significance to this object. For example, in the accelerated-inertial case which we have been considering, the response function will always differ by a thermal contribution at $T=a/2\pi$ regardless of the state examined. This gives the acceleration radiation an objective meaning. It appears as if there is always a thermal stress-energy tensor present when comparing accelerated and inertial systems.

If the normal ordered stress-energy tensors appropriate to each observer are calculated, we obtain

$$\langle N| :T_{\mu\nu}:_{(\bar{o})} |N\rangle = \lim_{x\to x'} O_{\mu\nu} \langle N| :\phi(x)\phi(x') + x\leftrightarrow x': _{(\bar{o})} |N\rangle$$

where $O_{\mu\nu}$ is the appropriate differential operator. Then the result of equation 2.59 gives

$$\langle N| :T_{\mu\nu}:_{o} |N\rangle - \langle N| :T_{\mu\nu}:_{\bar{o}} |N\rangle = \langle o| :T_{\mu\nu}:_{\bar{o}} |o\rangle$$

(2.62)

where the term

is a generalization of acceleration radiation. Note that this result implies that

$$<\bar{0}|:T_{\mu\nu}:_0|\bar{0}> = -<0|:T_{\mu\nu}:_{\bar{0}}|0> \qquad (2.63)$$

as mentioned in chapter one.

In summary, response to state $|N>$ by two distinct detector trajectories differs by an amount attributable to the presence of a generalized acceleration radiation, thus providing it with a physical interpretation.

2-5 Energy Balance For Monopole Detectors

In previous sections we have been mainly concerned with detector behaviour and not with the state of the field. If equation 2.6 is considered, we see that the amplitude for the field to make a transition to state $|\psi\rangle$ is

$$a_{E,\psi} = i\int d\tau\ e^{i\tau(E-E_0)} \langle\psi|\ \phi(x(\tau))\ |0\rangle\langle E|m(0)|E_0\rangle$$

The linearity of $\phi(x)$ in annihilation and creation operators implies that the only possible candidate for $|\psi\rangle$ is the one-particle state. Thus for a non-zero amplitude, not only will the detector be excited but also the field will jump to a one-particle state. As viewed by an inertial observer, this process will appear as detector excitation with the emission of a scalar particle -- behaviour that seems paradoxical without an energy source. Although it is usually futile to question energy balance when employing perturbation theory, in this case we are trying to reconcile the behavioural difference between inertial and non-inertial detectors.

Consider the case of 1+1 uniform acceleration where the amplitude is

$$a_{E,\psi_0} \sim \int d\tau dk\ e^{i(E-E_0)\tau} \langle\psi|\ e^{-i\omega s+ik\xi}\ R_k + h.c.\ |0\rangle.$$

where, as in chapter 1, the Rindler coordinates s and ξ result in the line element evaluated at $\xi=0$.

$$-d\tau^2 = e^{-2a\xi}(-ds^2 + d\xi^2) = -ds^2$$

The objects R_k and R_k^+ are the Rindler annihilation and creation operators. But

$$R_k \sim \alpha a_k + \beta a_k^+$$

so the only choice for $|\psi\rangle$ is the state $|1_{k'}\rangle$.

The probability is the same for either positive or negative k because

$$P_{E,\psi} \sim \left| \int d\tau\, e^{i(E-E_0)\tau}\, e^{-i\omega\tau} \left[\langle 1_{k'}|R_{k'}|0\rangle + \langle 1_{k'}|R_{k'}^-|0\rangle \right] \right|^2$$

where

$$\langle 1_{k'}|R_{k'}|0\rangle + \langle 1_{k'}|R_{k'}|0\rangle$$

depends only on the magnitude of k' (see equation 1.32). Thus an inertial observer would on average see blue-shifted particles emitted to the right and red-shifted ones to the left. The net effect on the detector is a force opposing its motion.

This force, along with the one required due to increased detector mass, must be supplied by an external source for the detector to remain at constant acceleration. The work done by this source supplies the energy balance for the entire process.

CHAPTER THREE

Moving Boundary Effects

3-1. Introduction

The presence of perfectly reflecting boundaries can cause non-trivial vacuum stress-energy effects, a famous example being the Casimir effect. Can the motion of a boundary produce a non-trivial stress-energy tensor? In particular, will uniform acceleration produce a thermal stress-energy agreeing with that obtained in chapter one by considering Rindler particles? These questions will be investigated for spaces bounded by one and two mirrors, for 1+1 dimensions. This permits the use of conformal invariance of the wave equation to map the world lines of accelerated mirrors to inertial ones.

Some uniformly accelerated mirror problems have been solved in 3+1 dimensions by Candelas and Deutsch (1977), and by Frolov and Serebriany (1979). Candelas and Deutsch conclude that the asymptotic form of $T_{\mu\nu}$ at large distance from a uniformly accelerated plane mirror is identical to a negative energy thermal result related to the local acceleration. Frolov and Serebriany make use of Euclidean space to simplify problems involving uniformly accelerated expanding spheres, a technique we will employ in 1+1 dimensions.

To bridge the conceptual gap between particles and stress-energy calculations, the behaviour of particle detectors co-moving with the mirrors will be investigated. The results agree with the equivalence arguments intrinsic to relativity. For example, a co-moving observer would claim that in his rest frame the mirror is stationary so his detector should measure nothing. This result will be verified.

3-2 1+1 Non-Uniform Acceleration of One Mirror

In this problem first studied by Davies (1975), and DeWitt (1975), and in more detail by Fulling and Davies (1976), an initially stationary mirror follows a general trajectory. This section will review the analysis of Fulling and Davies leading to the calculation of $T_{\mu\nu}$ due to mirror motion. The problem is to solve:

$$\Box \phi = 0 \quad , \quad \phi(t,z(t)) = 0 \tag{3.1}$$

where $z(t)$ is the mirror position at time t, $z(t)=0$ for $t<0$.

The solution is

$$\phi = \int \frac{d\omega}{\sqrt{4\pi\omega}} \left(ia_\omega [e^{-i\omega v} - e^{-i\omega(2\tau_u - u)}] + \text{h.c.} \right)$$

$$[illegible] \equiv \tau_u - z(\tau_u) \tag{3.2}$$

which satisfies the conditions in 3.1 since each term depends only on u or v, so that

$$\Box \phi \sim \frac{\partial}{\partial u} \frac{\partial}{\partial v} \phi = 0 \tag{3.3}$$

The boundary condition can be verified using 3.2.

$$e^{-i\omega(\tau_u + z(\tau_u))} - e^{-i\omega(2\tau_u - \tau_u + z(\tau_u))}$$

$$= e^{-i\omega(\tau_u + z(\tau_u))} - e^{-i\omega(\tau_u + z(\tau_u))} \tag{3.4}$$

The parameter τ_u has the geometric interpretation shown in Figure 3.1. Then defining

$$p(u) \equiv 2\tau_u - u$$

we have

$$\phi = \int \frac{d\omega}{\sqrt{4\pi\omega}} \left\{ i a_\omega [e^{-i\omega v} - e^{-i\omega p(u)}] + h.c. \right\} \tag{3.5}$$

where a_ω and a_ω^+ are [illegible]sual Mi[illegible]kowski annihilation and creation operators s[illegible] mirror was initially at rest, with the field in th[illegible] vacuum state.

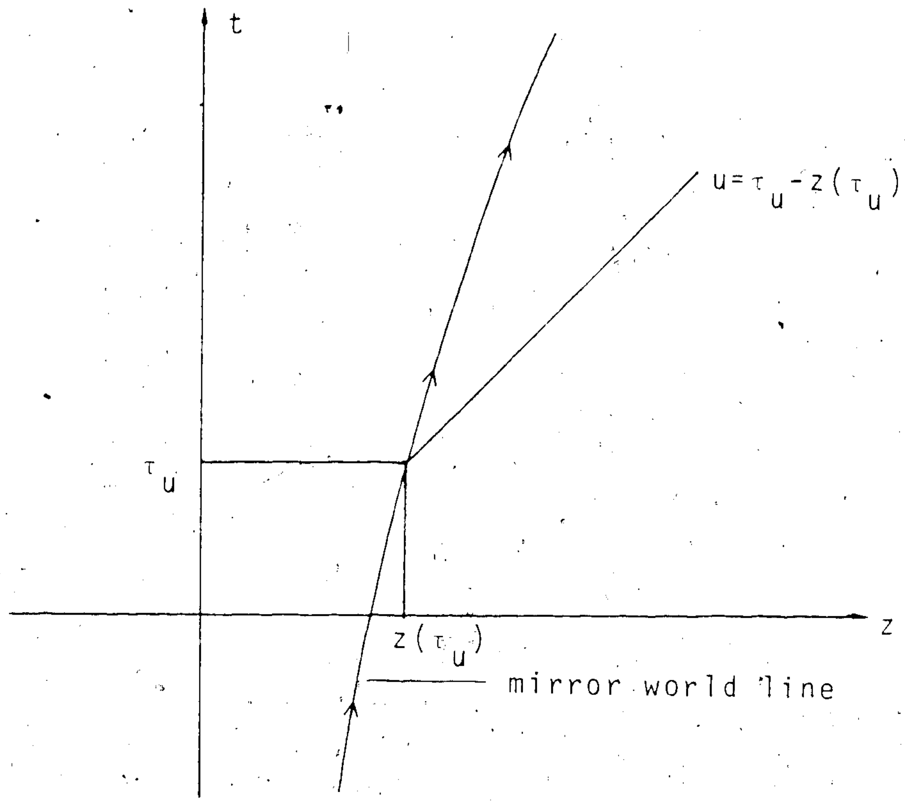
There is an i[illegible]cit initial condition contained in the above solution in that the $e^{-i\omega v}$ term represents unmodified left-moving waves. Any other choice would represent a particle flux incident from infinity, violating the condition that the field state is the Minkowski vacuum. For example, Rindler coordinates could be used for uniform acceleration in the above problem but this would alter the left-moving waves, violating the above condition and resulting in an incorrect value for $T_{\mu\nu}$.

The stress-energy tensor can now be calculated using the technique of point splitting (see chapter 1) to

Figure 3.1

Coordinate Description for Single Mirror

Non-Uniform Acceleration



regularize the tensor. This involves evaluating one of the fields at the point $(t+\varepsilon,x)$ and the other at (t,x) then expanding in powers of ε.

$$T_{oo} = T_{11} = \tfrac{1}{2}\left[(\partial_t\phi)^2 + (\partial_x\phi)^2\right]$$

$$T_{o1} = T_{1o} = \partial_t\phi\,\partial_x\phi \tag{3.6}$$

$$\partial_t\phi(t+\varepsilon,x) = \int d\omega\left(\frac{\omega}{4\pi}\right)^{\frac{1}{2}}$$

$$\times\left\{a_\omega\left[e^{-i\omega(v+\varepsilon)} - p'(u+\varepsilon)e^{-i\omega p(u+\varepsilon)}\right] + h.c.\right\}$$

$$\partial_x\phi(t+\varepsilon,x) = \int d\omega\left(\frac{\omega}{4\pi}\right)^{\frac{1}{2}}$$

$$\times\left\{a_\omega\left[e^{-i\omega(v+\varepsilon)} + p'(u+\varepsilon)e^{-i\omega p(u+\varepsilon)}\right] + h.c.\right\} \tag{3.7}$$

Now

$$T_{\mu\nu} = :T_{\mu\nu}: + \langle o|T_{\mu\nu}|o\rangle \tag{3.8}$$

where

$$\langle 0|T_{01}|0\rangle = \int d\omega \, \frac{\omega}{4\pi} \left\{ e^{i\omega(v+\varepsilon)} - p'(u+\varepsilon)\, e^{i\omega p(u+\varepsilon)} \right\}$$

$$\times \left\{ e^{-i\omega v} + p'(u)\, e^{-i\omega p(u)} \right\}$$

$$\langle 0|T_{01}|0\rangle = \int d\omega \, \frac{\omega}{4\pi} \left\{ e^{i\omega\varepsilon} - p'(u+\varepsilon)p'(u)\, e^{i\omega[p(u+\varepsilon)-p(u)]} \right\} \tag{3.9}$$

$$\langle 0|T_{00}|0\rangle = \int d\omega \, \frac{\omega}{8\pi} \left\{ \left\{ e^{i\omega(v+\varepsilon)} - p'(u+\varepsilon)\, e^{i\omega p(u+\varepsilon)} \right\} \right.$$

$$\times \left\{ e^{-i\omega v} - p'(u)e^{-i\omega p(u)} \right\} + \left\{ e^{i\omega(v+\varepsilon)} + p'(u+\varepsilon)e^{i\omega p(u+\varepsilon)} \right\}$$

$$\left. \times \left\{ e^{-i\omega v} + p'(u)\, e^{-i\omega p(u)} \right\} \right\}$$

$$\langle 0|T_{00}|0\rangle = \int d\omega \, \frac{\omega}{4\pi} \left\{ e^{i\omega\varepsilon} + p'(u+\varepsilon)p'(u)\, e^{i\omega[p(u+\varepsilon)-p(u)]} \right\} \tag{3.10}$$

The cross terms in the expressions leading to 3.9 and 3.10 have been ignored because the imaginary contributions to $T_{\mu\nu}$ vanish in the limit as ε approaches zero.

When the expectation value is taken in equation 3.8 the normal ordered term disappears leaving only the contributions from equations 3.9 and 3.10. The integrals from these equations may be evaluated by rotating the contour to the positive imaginary axis and including a negative sign.

$$\oint_C f(z)dz = 0 \to \int_0^\infty f(iy)\, i\,dy = \int_0^\infty f(x)\,dx$$

Therefore

$$\int_0^\infty \omega d\omega\, e^{i\omega\varepsilon} = \int_0^\infty i\omega\, i d\omega\, e^{-\omega\varepsilon} = -\int_0^\infty \omega d\omega\, e^{-\omega\varepsilon}$$

$$\int_0^\infty \frac{\omega d\omega}{4\pi} e^{i\omega\varepsilon} = \frac{-1}{4\pi\varepsilon^2} \tag{3.11a}$$

$$\int_0^\infty d\omega \frac{\omega}{4\pi} p'(u+\varepsilon)p'(u)\, e^{i\omega[p(u+\varepsilon)-p(u)]} = \frac{-1}{4\pi} \frac{p'(u+\varepsilon)p'(u)}{[p(u+\varepsilon)-p(u)]^2} \tag{3.11b}$$

Expanding in a power series for ε in 3.11b gives

$$\frac{1}{4\pi} \frac{p'(u+\varepsilon)p'(u)}{[p(u+\varepsilon)-p(u)]^2} = \frac{-1}{4\pi} p'(u) \frac{\partial}{\partial\varepsilon} \frac{1}{[p(u+\varepsilon)-p(u)]}$$

$$= \frac{-1}{4\pi} p'(u) \frac{\partial}{\partial\varepsilon} \left[\frac{1}{p'(u)\varepsilon + \frac{1}{2}\varepsilon^2 p''(u) + \frac{1}{6}\varepsilon^3 p'''(u) + \ldots} \right]$$

$$= \frac{-1}{4\pi} p'(u) \frac{\partial}{\partial\varepsilon} \frac{1}{p'(u)} \left[1 - \left(\frac{\varepsilon p''}{2p'} + \frac{\varepsilon^2 p'''}{6p'} \right) + \left(\frac{\varepsilon p''}{2p'} + \frac{\varepsilon^2 p'''}{6p'} \right)^2 \right]$$

$$= \frac{-1}{4\pi} \frac{\partial}{\partial\varepsilon} \left[\frac{1}{\varepsilon} - \frac{1}{2}\frac{p''}{p'} - \frac{\varepsilon p'''}{6p'} + \frac{\varepsilon}{4}\left(\frac{p''}{p'}\right)^2 + O(\varepsilon^2) \right]$$

$$\frac{p'(u+\varepsilon)p'(u)}{4\pi[p(u+\varepsilon)-p(u)]^2} = \frac{1}{4\pi\varepsilon^2} + \frac{1}{24\pi}\left[\frac{p'''}{p'} - \frac{3}{2}\left(\frac{p''}{p'}\right)^2 \right] + O(\varepsilon) \qquad (3.11c)$$

Using 3.9 to 3.11 we finally obtain $\langle T_{\mu\nu} \rangle$.

$$\langle T_{01} \rangle = \frac{1}{24\pi}\left[\frac{p'''}{p'} - \frac{3}{2}\left(\frac{p''}{p'}\right)^2 \right] + O(\varepsilon)$$

$$\langle T_{00} \rangle = -\langle T_{01} \rangle - \frac{1}{2\pi\varepsilon^2} \qquad (3.12)$$

If the point splitting technique is carried out for empty Minkowski space, a divergent term identical to that in 3.12 is obtained. As only differences from this divergent value are observable, it may be subtracted from 3.12, resulting in the regularized stress-energy tensor when the limit as ε approaches zero is taken.

$$\langle T_{01} \rangle = \langle T_{11} \rangle = -\langle T_{01} \rangle$$

$$\langle T_{01}\rangle = \frac{1}{24\pi}\left(\frac{p'''}{p'} - \frac{3}{2}\left(\frac{p''}{p'}\right)^2\right) = \frac{-1}{12\pi}\sqrt{p'(u)}\,\frac{d^2}{du^2}\,\frac{1}{\sqrt{p'(u)}} \qquad (3.13)$$

Using equations 3.2 and 3.5 the above expression may be evaluated in terms of kinematical properties of the world line.

$$p(u) = 2\;_u - u \qquad \frac{dp}{du} = 2\,\frac{d}{du}\;_u - 1$$

$$u = \;_u - z(\;_u) \qquad \frac{du}{d\;_u} = 1 - v \qquad (3.14)$$

Therefore

$$\frac{dp}{du} = \frac{1 + v}{1 - v} = \frac{\;+\;v}{\;-\;v} \qquad (3.15)$$

The expectation $\langle T_{01}\rangle$ is now calculated using 3.13 to 3.15.

$$\langle T_{01}\rangle = \frac{-1}{12\pi}\left(\frac{\;+\;v}{\;-\;v}\right)^{\frac{1}{2}}\frac{d^2}{du^2}\left(\frac{\;-\;v}{\;+\;v}\right)^{\frac{1}{2}}$$

$$= \frac{-1}{12\pi}\left(\frac{\;+\;v}{\;-\;v}\right)^{\frac{1}{2}}\frac{d}{du}\left(\frac{1}{\;-\;v}\,\frac{d}{d\;}\left(\frac{\;-\;v}{\;+\;v}\right)^{\frac{1}{2}}\right)$$

$$= \frac{1}{12\pi}\left(\frac{\;+\;v}{\;-\;v}\right)\;\frac{d}{du}\;\ldots$$

$$\langle T_{01}\rangle = \frac{1}{12\pi}\left(\frac{1+v}{1-v}\right)^{1/2}\frac{1}{1-v}\left.\frac{da}{d\tau}\right|_{mirror}$$

$$\langle T_{01}\rangle = \frac{1}{12\pi}\left(\frac{1+v}{1-v}\right)\left.\frac{da}{d\tau}\right|_{mirror}$$

Finally we have the result

$$\langle T_{01}\rangle = \langle T_{10}\rangle = -\langle T_{00}\rangle = -\langle T_{11}\rangle = \frac{1}{12\pi}\left(\frac{1+v}{1-v}\right)\left.\frac{da}{d\tau}\right|_{mirror} \quad (3.16)$$

where the derivative is calculated at the retarded time u.

Thus the mirror radiates negative energy to the right provided that the acceleration is increasing. The same technique is employed on the opposite side of the mirror, the difference being that advanced instead of retarded times are used. The result is identical to equation 3.16 except for a change in the sign of the energy and the blue-shift becomes a red-shift.

An inertial observer would see a net energy being radiated from the mirror.

$$\frac{dE}{d\tau} = \frac{-1}{12\pi}\,\dot{a}\left(\frac{1+v}{1-v} - \frac{1-v}{1+v}\right) = -\frac{\dot{a}\,v}{6\pi} \quad (3.17)$$

The energy balance is provided by the work done against the radiation reaction force.

$$F_\mu = T_{\mu\nu} n^\nu \Big|_+ - T_{\mu\nu} n^\nu \Big|_-$$

where n^ν is the normal to the mirror. In this case

$$n^\nu = \frac{a^\nu}{a} = (\gamma v, \gamma)$$

therefore

$$F_0 = \frac{\dot{a}}{12\pi} \left[\left(\frac{\gamma+\gamma v}{\gamma-\gamma v}\right)\gamma - \left(\frac{\gamma+\gamma v}{\gamma-\gamma v}\right)\gamma v \right]$$

$$- \frac{\dot{a}}{12\pi} \left[\left(\frac{\gamma-\gamma v}{\gamma+\gamma v}\right)\gamma v + \left(\frac{\gamma-\gamma v}{\gamma+\gamma v}\right)\gamma \right]$$

$$F_0 = \frac{\gamma v \dot{a}}{6\pi}$$

The work done against the radiation reaction force is

$$F^0 = - \frac{\gamma v \dot{a}}{6\pi} \tag{3.18}$$

which provides the energy balance for the radiating mirror.

If we now determine the total energy radiated by the mirror in moving from inertial motion to uniform acceleration, we find

$$E = -\int \frac{\dot{a}}{12\pi} \left(\frac{\gamma+\gamma v}{\gamma-\gamma v}\right) d\tau$$

With $\dot{a} \neq 0$ for only a short period of time, the velocity of the mirror will change very little so that

$$E \simeq -\int \frac{\dot{a}}{12\pi}\, dt = -\frac{a_{final}}{12\pi} \tag{3.19}$$

If we calculate the total energy expected from the Rindler particles as in chapter one, we find

$$E = \frac{a}{24\pi} \qquad \text{for} \quad \langle M|:T_{\mu\nu}:_R|M\rangle$$

and

$$E = -\frac{a}{24\pi} \qquad \text{for} \quad \langle R|:T_{\mu\nu}:_M|R\rangle$$

This means that the acceleration radiation cannot be thought of as a thermalized remnant of a dynamic process, as studied here. A more realistic model would be an accelerated box which can thus contain any energy radiated and possibly thermalize it through numerous reflections. This will be studied in the next section.

The response of a particle detector co-moving with the mirror will now be examined. Intuitively we would expect a null response since in the detector's rest frame the mirror is stationary. This is indeed the case, as we will now verify for a monopole detector.

For a monopole interaction Hamiltonian the amplitude is (see 2.6)

$$a_{E,\psi} = \int d\tau \, e^{i(E-E_0)\tau} \langle\psi|\phi(x(\tau))|0\rangle \tag{3.20}$$

Due to the conformal invariance of the field we can express the above equation as

$$a_{E,\psi} = \int d\tau d\omega \, e^{i(E-E_0)\tau} \langle\psi|(a_\omega e^{-i\omega\tau} + h.c.)\sin\omega\xi|0\rangle \tag{3.21}$$

where

$$p(u) = \tau - \xi \qquad v = \tau + \xi \tag{3.22}$$

This can be verified by substituting equation 3.22 into 3.21, resulting in modes identical to equation 3.5.

The line element for equation 3.22 becomes

$$-d\tau^2 + d\xi^2 = -p'(u)dudv = p'(u)(-dt^2 + dx^2)$$

$$= p'(u)ds^2$$

which is conformal to Minkowski space. Using this result with $\xi=\xi_d$=constant for a co-moving detector, 3.21 becomes

$$a_{E,\psi} \sim \int d\tau d\omega\, e^{i(E-E_0)\tau} \sin \omega\xi_d$$

$$\times \langle\psi|a_\omega e^{-i\omega\tau'} + \text{h.c.}|0\rangle$$

$$a_{E,\psi} \sim \int d\omega\, \sin\omega\xi_d \Big[\delta(E-E_0-\omega)\, \langle\psi|a_\omega|0\rangle$$

$$+\, \delta(E-E_0+\omega)\, \langle\psi|a^+_\omega|0\rangle\Big]$$

$$a_{E,\psi} \sim \int d\omega\, \sin\omega\xi_d\, \delta(E-E_0+\omega)\, \langle\psi|a^+_\omega|0\rangle$$

But $\omega>0$ and $E-E_0>0$ so the delta function will always be zero, that is $a_{E,\psi}=0$. This results in a zero response by a co-moving particle detector.

This phenomenon is different from those already examined. In this case $\langle T_{\mu\nu}\rangle$ reveals energy content but the co-moving detector measures nothing, as opposed to the previous cases where particle response existed in the absence of stress-energy. This analysis suggests that a particle detector is a probe sensitive to equivalence arguments whereas $T_{\mu\nu}$ is not. The reason for this is that any world line in 1+1 dimensions can be straightened by a conformal transformation, forcing co-moving detector response to be zero. The calculation of $T_{\mu\nu}$ however, is sensitive to the dependence of the conformal transformation on the Minkowski time parameter, which may lead to $T_{\mu\nu} \neq 0$.

3-3 1+1 Uniform Acceleration of Two Mirrors

This problem involves the region between two mirrors with different uniform accelerations and the condition that $\phi=0$ on the boundaries, as indicated in Figure 3.2. If the region between the hyperbolae is mapped into Euclidean space, then it becomes bounded by concentric circles as shown in Figure 3.3. The Euclidean sector illustrates that this system represents a uniformly accelerated rigid "box" because the proper distance between the two circles remains constant.

To evaluate $T_{\mu\nu}$, the Feynman's Green function will be obtained by solving

$$\Box G_F(x,x') = -\delta(x-x') \tag{3.23}$$

subject to $G_F(x,x')=0$ for x or x' on the boundary. The boundary condition on the Green function reflects the condition on the field. Regularization will be carried out by removing the usual Minkowski space contribution and then $T_{\mu\nu}$ can be obtained by operating on $G_{F_{reg}}$ with a suitable operator.

Before continuing with the two-mirror problem, the one-mirror case will be solved. This will set up the method for the more difficult two-mirror case and will verify the result obtained in the previous section that $T_{\mu\nu}=0$ for uniform acceleration.

Figure 3.2

World Line for Two Mirrors at Uniform Acceleration in Minkowski Space

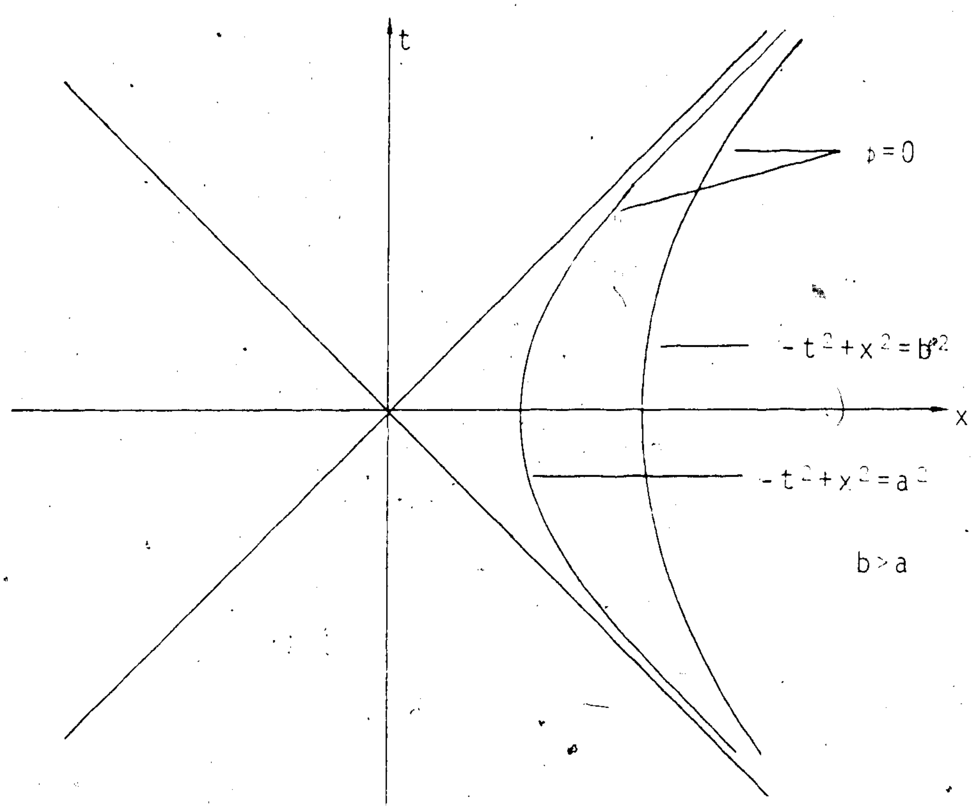
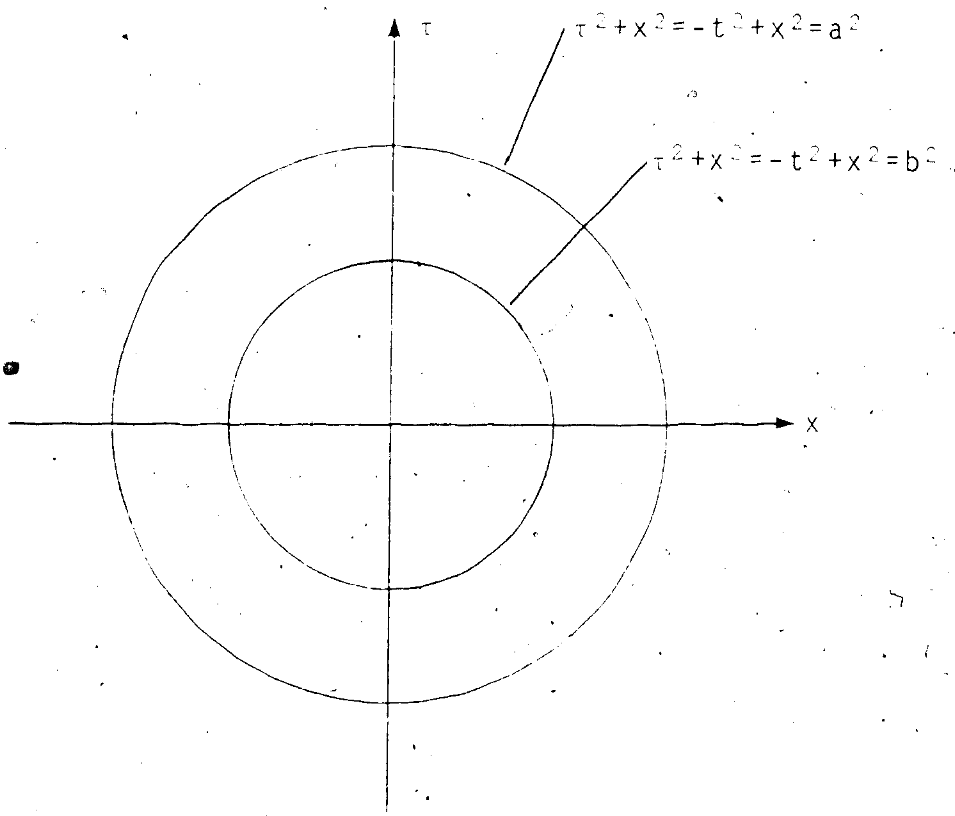


Figure 3.3

World Line for Two Mirrors at Uniform Acceleration in Euclidean Space



The one-mirror Green function may be obtained in the Euclidean sector by mapping the exterior of the circle onto a half plane as illustrated in Figure 3.4. This mapping is

$$w = \frac{1-z}{1+z} \tag{3.24}$$

where $z=x+iy$ is a point in Euclidean space and $w=u+iv$. The circle $|z|=1$ becomes the line $u=0$ in the w-plane. The Euclidean Green function is related to the Feynman Green function by

$$G_E(\tau,\underline{x};\tau,\underline{x}') = iG_F(it,\underline{x};it',\underline{x}') \tag{3.25}$$

Using the equation 3.24 and the method of images in the w-plane gives the Euclidean Green function.

$$G_E(w,w') = -\frac{1}{2\pi}\ln|w-w'| + \frac{1}{2\pi}\ln|w+\bar{w}'|$$

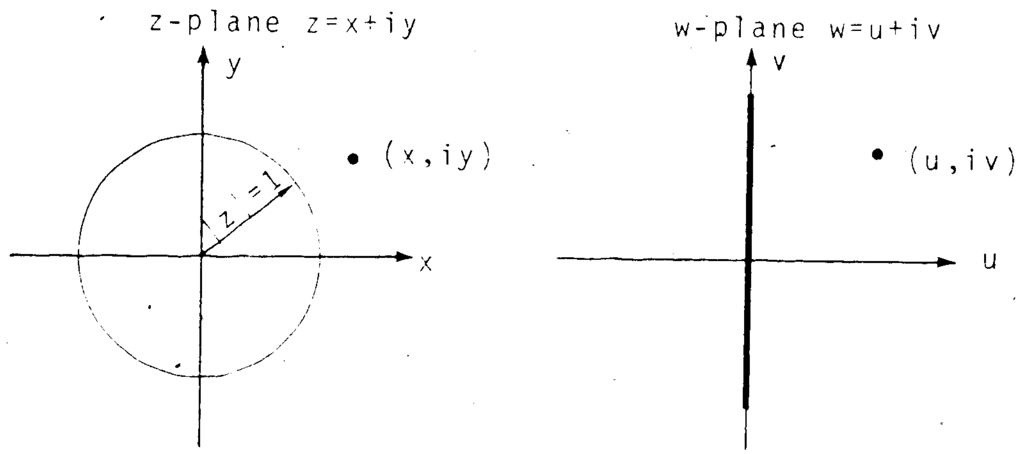
$$G_E(z,z') = -\frac{1}{2\pi}\ln\left|\frac{1-z}{1+z} - \frac{1-z'}{1+z'}\right| + \frac{1}{2\pi}\ln\left|\frac{1-\bar{z}}{1+\bar{z}} - \frac{1-\bar{z}'}{1+\bar{z}'}\right|$$

$$G_E(z,z') = -\frac{1}{2\pi}\ln\left|\frac{z-z'}{1-z\bar{z}'}\right|$$

Figure 3.4

Conformal Mapping for the Exterior of a Circle

$$w = \frac{1-z}{1+z}$$



$$G_E(\underline{x},\underline{y}) = -\frac{1}{4\pi}\ln|\underline{x}-\underline{y}|^2 + \frac{1}{4\pi}\ln|1+\underline{x}^2\underline{y}^2-2\underline{x}\cdot\underline{y}| \tag{3.26}$$

Scaling x by 1/a (x→x/a), gives the solution for an arbitrary radius since the mapping in Figure 3.4 (equation 3.24) presumes a unit circle. Thus 3.26 becomes

$$G_E(\underline{x},\underline{y}) = -\frac{1}{4\pi}\ln|\underline{x}-\underline{y}|^2 + \frac{1}{4\pi}\ln|a^4+\underline{x}^2\underline{y}^2-2a^2\underline{x}\cdot\underline{y}| + \text{constant} \tag{3.27}$$

The constant term in the above expression will be ignored since it does not contribute to stress-energy.

Regularizing the Green function by removing the empty space part and rotating back to Minkowski space gives the regularized Feynman Green function.

$$G_{F_{reg}}(x,y) = -\frac{i}{4\pi}\ln|a^4+x^2y^2-2a^2x\cdot y| \tag{3.28}$$

This expression still contains a divergence as x^2 and y^2 approach a^2, that is, as the mirror is approached. This divergence is symptomatic of all problems involving boundaries. Its origin in this case is that the image charge is being moved until it reaches the boundary, without the physical charge being present (removed in regularization) to cancel the divergence.

The stress-energy tensor may be evaluated by operating on the regularized Green function in equation 3.28, and taking limits as follows (see chapter one).

$$T_{\mu\nu} = \lim_{y\to x} i\left(\partial^x_\mu \partial^y_\nu - \tfrac{1}{2}\eta_{\mu\nu}\eta^{\alpha\beta}\partial^x_\alpha \partial^y_\beta\right) G_{F_{reg}}(x,y) \qquad (3.29)$$

Calculating the derivatives appearing in 3.29

$$i\partial^x_\mu \partial^y_\nu G_{F_{reg}}(x,y) = \frac{1}{2\pi}\,\frac{(2y_\nu x_\mu - a^2\eta_{\mu\nu})}{(a^4+x^2y^2-2a^2x\cdot y)}$$

$$-\ \frac{1}{\pi}\,\frac{(y^2x_\mu - a^2y_\mu)(x^2y_\nu - a^2x_\nu)}{(a^4+x^2y^2-2a^2x\cdot y)^2}$$

Taking the limit as y approaches x

$$\lim_{y\to x} i\partial^x_\mu \partial^y_\nu G_{F_{reg}} =$$

$$\frac{1}{2\pi}\,\frac{(2x_\mu x_\nu - a^2\eta_{\mu\nu})}{(x^2-a^2)^2} - \frac{1}{\pi}\,x_\mu x_\nu\,\frac{(x^2-a^2)^2}{(x^2-a^2)^4}$$

$$\lim_{y\to x} i\partial^x_\mu \partial^y_\nu G_{F_{reg}} = -\,\frac{a^2\eta_{\mu\nu}}{(x^2-a^2)^2} \qquad (3.30)$$

Using this result and equation 3.29 to evaluate $T_{\mu\nu}$ gives

$$T_{\mu\nu} = -\frac{a^2}{2\pi(x^2-a^2)^2}\eta_{\mu\nu} + \frac{a^2}{4\pi(x^2-a^2)^2}\eta_{\mu\nu}\eta^{\alpha\beta}\eta_{\alpha\beta}$$

$$T_{\mu\nu} = -\frac{a^2}{2\pi(x^2-a^2)^2}(\eta_{\mu\nu}-\eta_{\mu\nu}) = 0 \tag{3.31}$$

As expected, $T_{\mu\nu}=0$ in agreement with the calculations of the previous section.

The obtained Feynman Green function will now be used in the two mirror problem by constructing an infinite number of image and physical charges using the functional form of equation 3.27. The process begins with the direct solution

$$G_E(\underline{x},\underline{y}) = -\frac{1}{4\pi}\ln|\underline{x}-\underline{y}|^2$$

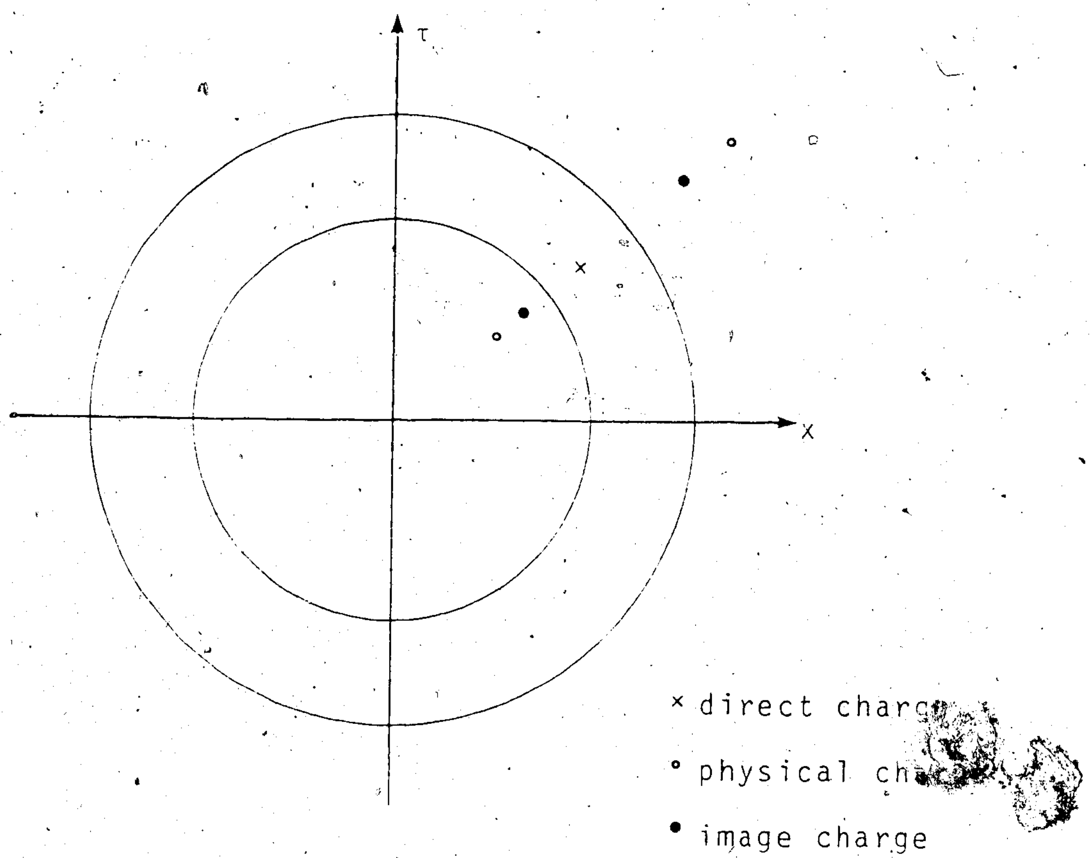
then image charges are placed both outside and inside the ring-shaped region as indicated in Figure 3.5.

$$G_E(\underline{x},\underline{y}) = -\frac{1}{4\pi}\ln|\underline{x}-\underline{y}|^2 + \frac{1}{4\pi}\ln|\underline{x}^2\underline{y}^2-2b^2\underline{x}\cdot\underline{y}+b^4|$$

$$+\frac{1}{4\pi}\ln|\underline{x}^2\underline{y}^2-2a^2\underline{x}\cdot\underline{y}+a^4|$$

Figure 3.5

Method of Images Between Two Concentric Circles



The boundary condition is violated on the inside mirror by the second term and on the outside one by the third term. These contributions may be cancelled by placing physical charges in the image region and noticing that to within a constant, scaling $\underline{y}$ by a^2/b^2 (or b^2/a^2) gives a term proportional to the image contribution.

$$G_E(\underline{x},\underline{y}) = -\frac{1}{4\pi}\ln|\underline{x}-\underline{y}|^2 - \frac{1}{4\pi}\ln\left|\underline{x}-\frac{a^2}{b^2}\underline{y}\right|^2 - \frac{1}{4\pi}\ln\left|\underline{x}-\frac{b^2}{a^2}\underline{y}\right|^2$$

$$+ \frac{1}{4\pi}\ln|x^2y^2-2a^2\underline{x}\cdot\underline{y}+a^4| + \frac{1}{4\pi}\ln|x^2y^2-2b^2\underline{x}\cdot\underline{y}+b^4|$$

Repeating this process leads to the Euclidean Green function

$$G_E(\underline{x},\underline{y}) = -\frac{1}{4\pi}\sum_{n=-\infty}^{\infty}\ln\left|\underline{x}-\left(\frac{a}{b}\right)^{2n}\underline{y}\right|^2$$

$$+ \frac{1}{4\pi}\ln\left|x^2y^2-2b^2\left(\frac{a}{b}\right)^{2n}\underline{x}\cdot\underline{y}+\left(\frac{a}{b}\right)^{4n}b^4\right| \qquad (3.32)$$

where once again the constant term has been ignored.

Three of the terms in the sum are divergent in the region of interest. They are; the n=0 term in the first sum which leads to the empty space solution, and the n=0,1 terms in the second sum which are the divergences encountered in the one-mirror problem.

Removing the empty space term and rotating to Minkowski coordinates results in the regularized Feynman Green function

$$iG_{F_{reg}}(x,y) = -\frac{1}{4\pi}\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \ln\left|x-\left(\frac{a}{b}\right)^{2n} y\right|^2$$

$$+\frac{1}{4\pi}\sum_{n=-\infty}^{\infty} \ln\left|x^2y^2-2b^2\left(\frac{a}{b}\right)^{2n} x\cdot y+\left(\frac{a}{b}\right)^{4n} b^4\right| \qquad (3.33)$$

For the calculation of $T_{\mu\nu}$ we note that the second sum in equation 3.33 will not contribute since its functional form is identical to that of 3.28 which resulted in $T_{\mu\nu}=0$. We now evaluate $T_{\mu\nu}$ for the first sum.

$$\partial_{\mu_x}\partial_{\nu_y} G_{F_{reg}}(x,y) = -\frac{1}{2\pi}\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{\left(\frac{a}{b}\right)^{2n}\eta_{\mu\nu}}{\left(x^2-2x\cdot y\left(\frac{a}{b}\right)^{2n}+\left(\frac{a}{b}\right)^{4n} y^2\right)}$$

$$+\frac{1}{\pi}\left(\frac{a}{b}\right)^{2n} \frac{\left(x_\mu - y_\mu\left(\frac{a}{b}\right)^{2n}\right)\cdot\left(y_\nu\left(\frac{a}{b}\right)^{2n}-x_\nu\right)}{\left(x^2-2x\cdot y\left(\frac{a}{b}\right)^{2n}+\left(\frac{a}{b}\right)^{4n} y^2\right)^2}$$

$$\lim_{y\to x} \partial_{\mu_x}\partial_{\nu_y} iG_{F_{reg}}(x,y) = -\frac{1}{2\pi}\sum_{\substack{n=\infty \\ n\neq 0}}^{\infty} \frac{\eta_{\mu\nu}}{x^2}\frac{\left(\frac{a}{b}\right)^{2n}}{\left(1-\left(\frac{a}{b}\right)^{2n}\right)^2}$$

$$+ \frac{1}{\pi}\left(\frac{a}{b}\right)^{2n} \frac{x_\mu x_\nu \left(1-\left(\frac{a}{b}\right)^{2n}\right)\left(\left(\frac{a}{b}\right)^{2n}-1\right)}{\left(1-\left(\frac{a}{b}\right)^{2n}\right)^4 x^4}$$

$$\lim_{y\to x} i\partial_{\mu_x}\partial_{\nu_y} G_{F_{reg}}(x,y) = \sum_{n\neq 0} \frac{-x_\mu x_\nu}{x^4} \frac{\left(\frac{a}{b}\right)^{2n}}{\left(1-\left(\frac{a}{b}\right)^{2n}\right)^2}$$

$$- \frac{\eta_{\mu\nu}}{2\pi x^2} \frac{\left(\frac{a}{b}\right)^{2n}}{\left(1-\left(\frac{a}{b}\right)^{2n}\right)^2}$$

Therefore

$$\langle T_{\mu\nu}\rangle = \frac{-(x_\mu x_\nu - \frac{1}{2}\eta_{\mu\nu} x^2)}{\pi x^4} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{\left(\frac{a}{b}\right)^{2n}}{\left(1-\left(\frac{a}{b}\right)^{2n}\right)^2} \qquad (3.34)$$

In the limit as a/b approaches zero (a is held constant while b becomes large), a condition which describes a box of infinite length, the summation in the above equation will be zero, causing $T_{\mu\nu}$ to agree with the one mirror results. The sum in 3.34 is now evaluated.

$$\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{r^{2n}}{(r^{2n}-1)^2} = \sum_{n=1}^{\infty} \frac{r^{2n}}{(r^{2n}-1)^2} + \frac{r^{-2n}}{(r^{-2n}-1)^2} \cdot$$

$$\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{r^{2n}}{(r^{2n}-1)^2} = 2\sum_{n=1}^{\infty} \frac{r^{2n}}{(r^{2n}-1)^2} \quad , \; r = \frac{a}{b} < 1$$

From the tabulations of Hansen (1975), we get

$$\sum_{n=1}^{\infty} \frac{r^{2n}}{(r^{2n}-1)^2} = \frac{1}{24} + \frac{1}{6\pi^2} K(k)\left\{(2-k^2)K(k)-3E(k)\right\} \tag{3.35}$$

where $r<1$, and

$$r = \exp[-\pi K'(k)/K(k)] \tag{3.36}$$

$$K(k) = \int_0^{\frac{\pi}{2}} (1-k^2\sin^2 t)^{-\frac{1}{2}}\, dt \tag{3.37}$$

$$K'(k) = K(k') \quad , \; k' = (1-k^2)^{\frac{1}{2}} \tag{3.38}$$

$$E(k) = \int_0^{\frac{\pi}{2}} (1-k^2\sin^2 t)^{\frac{1}{2}}\, dt \qquad (3.39)$$

Substituting the result of 3.35 into 3.34, $T_{\mu\nu}$ becomes

$$T_{\mu\nu} = -\frac{2}{24\pi x^4}(x_\mu x_\nu - \tfrac{1}{2}\eta_{\mu\nu}x^2) - \frac{2}{6\pi^3 x^4}(x_\mu x_\nu - \tfrac{1}{2}\eta_{\mu\nu}x^2)$$

$$\times\ K(k)\left[(2-k^2)K(\kappa)-3E(\kappa)\right] \qquad (3.40)$$

If the condition $a\ell \ll 1$ (ℓ is the length of the box and a is acceleration) is enforced so that the co-moving observer's coordinate system is well behaved throughout the box (see chapter one), then

$$a_{front} \equiv a_f = 1/b \qquad a_{back} \equiv a_b = 1/a \quad , \ell = b-a$$

$$a_f = \frac{1}{b} = \frac{1}{a+\ell} = \frac{1}{a(1+\ell/a)} \simeq \frac{1}{a}(1-\ell/a)$$

so that

$$a_f \simeq a_b - a_b \ell \qquad (3.41)$$

Using the above expression and equation 3.36, we find

$$\ln(1-a_b\ell) \simeq -\pi K'(k)/K(k)$$

$$a_b\ell \simeq \pi K'(k)/K(k)$$

Since $a_b\ell$ is small, then $K'(k)$ is small and $K(k)$ is large. To first approximation, let $k=1$ so that $K'(k)=\pi/2$ and $K(k) \to \infty$. Then ignoring $K(k)E(k)$ as insignificant compared with the $K(k)K(k)$ term, equation 3.40 for $T_{\mu\nu}$ becomes

$$T_{\mu\nu} \simeq -\frac{2}{24\pi x^4}(x_\mu x_\nu - \tfrac{1}{2}\eta_{\mu\nu}x^2)$$

$$-\frac{2}{\pi x^4}(x_\mu x_\nu - \tfrac{1}{2}\eta_{\mu\nu}x^2)\frac{1}{6\pi^2}\left(\frac{\pi^2}{2a_b\ell}\right)^2$$

Therefore

$$T_{\mu\nu} \simeq -\frac{2}{24\pi x^4}(x_\mu x_\nu - \tfrac{1}{2}\eta_{\mu\nu}x^2)$$

$$-\frac{2}{x^4}(x_\mu x_\nu - \tfrac{1}{2}\eta_{\mu\nu}x^2)\frac{\pi}{24a_b^2\ell^2} \qquad (3.42)$$

To determine $T_{\mu\nu}$ in the rest frame of the box, the tensorial part of 3.42 must be evaluated at $t=0$ (see

With this choice the stress-energy tensor in the instantaneous rest frame is

$$T_{00}\Big|_{t=0} \simeq -\frac{1}{24\pi \underline{x}^2} - \frac{\pi}{24 a_b^2 \ell^2 \underline{x}^2} \tag{3.43}$$

$$T_{01}\Big|_{t=0} = 0$$

The first term in expression 3.43 is identical to a thermal energy density at $T=a/2\pi$, since $1/\underline{x}^2$ is the proper acceleration as we move through the box. The second term is merely the Casimir effect since to first order in $(a\ell)$, $\underline{x}^2\ a_b^2=1$, giving the usual result.

Calculating the total energy in the box at t=0 yields

$$E \simeq \int_{1/a_b}^{1/a_f} dx \left(-\frac{1}{24} - \frac{\pi}{24 a_b^2 \ell^2}\right) \frac{1}{x^2}$$

$$= -\frac{(a_b - a_f)}{24\pi} - \frac{\pi(a_b - a_f)}{24 a_b^2 \ell^2}$$

Using equation 3.41 this becomes

$$E = -\frac{1}{24\pi} a_b^2 \ell - \frac{\pi}{24\ell^2} \frac{a_b^2 \ell}{a_b^2}$$

$$E \simeq -\frac{a_b^2 \ell}{24\pi} - \frac{\pi}{24\ell} \qquad (3.44)$$

which indicates that the uniformly accelerated box responds as if the field were in the Rindler state (thermal energy density at $T=a/2\pi$), with the Casimir effect as a correction for the finiteness of the box. This implies that there is an observable stress-energy effect of acceleration radiation.

3-4 Energy Balance for a Uniformly Accelerated Box

The result 3.44 raises the question of energy balance. For example with the Casimir effect, we can think of the walls being an infinite distance apart, then holding one wall fixed we bring the second one to a separation L. There is an attractive force between the walls equal to

$$F = \frac{\pi}{24\ell^2}$$

Thus the work done in moving the wall with no acceleration is given by

$$W = \int_{\infty}^{L} d\ell \, \frac{\pi}{24\ell^2} = -\frac{\pi}{24L}$$

so the work done in bringing the mirrors from infinite separation to separation L accounts for the energy present.

Can a procedure similar to this be carried out for an accelerated box? If the box is initially at rest and is brought to a constant acceleration, what will be the final energy contained? As a first approximation to this problem, let the two mirrors radiate energy independently, then consider the rate of energy flow from the walls as measured by a co-moving observer at the back wall. Using 3.16 the rate of energy change is

$$\frac{dE}{d\tau} = -\frac{\dot{a}_b}{12\pi} + \frac{\dot{a}_f}{12\pi}(1+a \ldots \tag{3.45}$$

where we have modified the energy flow from the front wall to include the "gravitational" blue shift. If we integrate 3.45 for mirror motion that is initially inertial and finally uniformly accelerated, we find the total energy

$$E = \int d\tau \; - \frac{\dot{a}_b}{12\pi} + \frac{\dot{a}_f}{12\pi}(1+a_f \ell)$$

$$E = -\frac{a_b}{12\pi} + \frac{a_f}{12\pi} + \frac{\ell a_f^2}{24\pi}$$

(3.46a)

where a_b, a_f are the final accelerations of the front and back walls. If the condition $a\ell \ll 1$ is imposed, then 3.46a becomes (using 3.41)

$$E \simeq -\frac{a_b}{12\pi} + \frac{(a_b - a_b^2 \ell)}{24\pi} + \frac{a_b^2}{24\pi}(1-a_b\ell)^2$$

$$E \simeq -\frac{a_b^2 \ell}{24\pi}$$

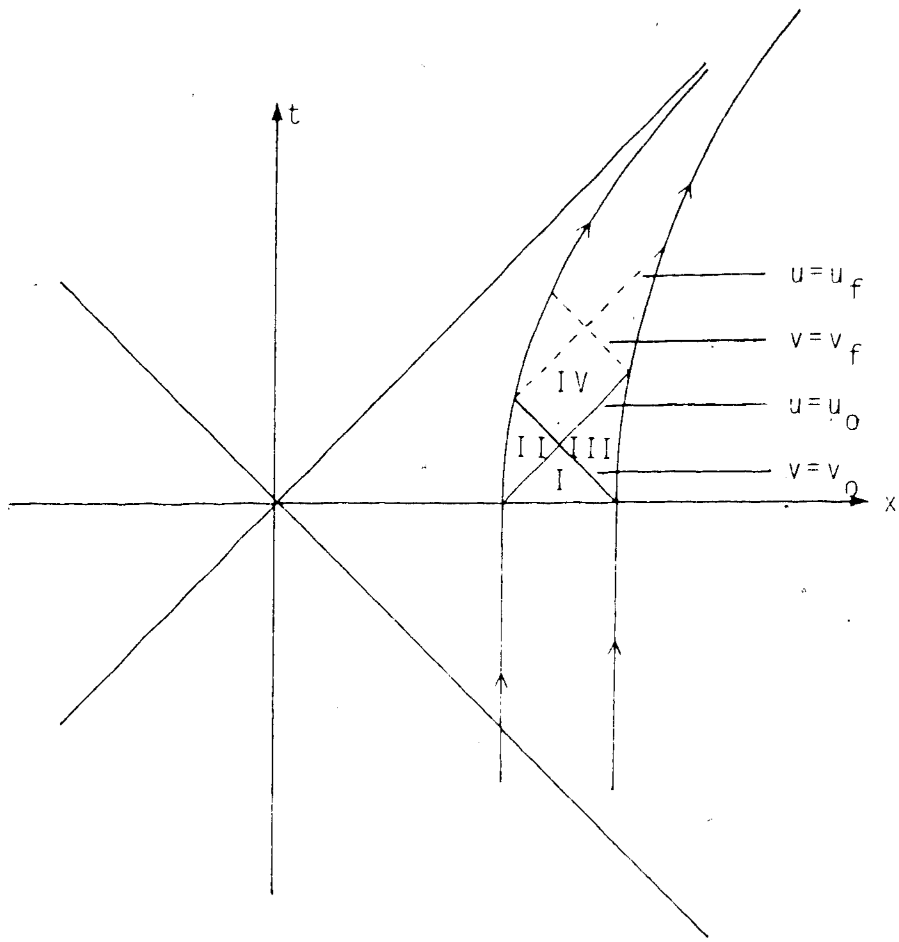
(3.46b)

If we include the energy from the Casimir effect then the above result will agree with that obtained in 3.44. The above argument can be verified for an explicit calculation.

Consider a box which changes extremely quickly from inertial motion to uniform acceleration at t=0, as indicated in Figure 3.6. Invoking causality, $T_{\mu\nu}$ in region I should only register the Casimir effect while regions II and III will contain particle flux from one of

Figure 3.6

World Line for a Box Changing Discontinuously from Inertial Motion to Uniform Acceleration



the mirrors. In region IV there will be particle flux present from both mirrors.

Recalling the discussion in section 3-2 concerning the importance of initial conditions on the field modes, and using the above causality arguments, the field must be decomposed as follows;

$$\phi_I = \sum_n \frac{1}{\sqrt{n\pi}} \left(a_n e^{-i\omega_n t} + h.c.\right) \sin \omega_n (x - 1/a_b)$$

$$\phi_I = \sum_n \frac{1}{\sqrt{4n\pi}} \left\{ ia_n \left(e^{-i\omega_n (v - 1/a_b)} - e^{-i\omega_n (u + 1/a_b)} \right) + h.c \right\}$$

(3.47)

where

$$\omega_n \left(\frac{1}{a_f} - \frac{1}{a_b} \right) \equiv \omega_n \ell = n\pi$$

$$\phi_{II} = \sum_n \frac{1}{\sqrt{4n\pi}} \left\{ ia_n \left(e^{-i\omega_n (v - 1/a_b)} - e^{-i\omega_n p(u)} \right) + h.c. \right\}$$

(3.48)

$$\phi_{III} = \sum_n \frac{1}{\sqrt{4n\pi}} \left\{ ia_n \left(e^{-i\omega_n q(v)} - e^{-i\omega_n (u + 1/a_b)} \right) + h.c. \right\}$$

(3.49)

$$\phi_{IV} = \sum_n \frac{1}{\sqrt{4n\pi}} \left\{ i a_n \left(e^{-i\omega_n q(v)} - e^{-i\omega_n p(u)} \right) + h.c. \right\}$$

(3.50)

The natural choice for $q(v)$ and $p(u)$ in 3.50 will lead to the Rindler coordinates since the boundaries are stationary as seen by the accelerated observer. From equation 1.72

$$t = \frac{1}{a_b} \exp(a_b \xi) \, \mathrm{sh}\, a_b \tau \qquad x = \frac{1}{a_b} \exp(a_b \xi) \, \mathrm{ch}\, a_b \tau$$

so that

$$\tau - \xi = -\frac{1}{a_b} \ln(-a_b u) \equiv p(u)$$

(3.51)

$$\tau + \xi = \frac{1}{a_b} \ln(a_b v) \equiv q(v)$$

(3.52)

The mirror in region II is bounded by the lines $v=v_0$, u_0 and in region III by $u=u_0$, v_f (see Figure 3.6). The values for v_f and u_f may be calculated as follows.

$$(t+x)(t-x) = uv = -1/a^2$$

Therefore

$$u_o v_f = -1/a_f^2 \qquad v_o u_f = -1/a_b^2$$

but

$$v_o = 1/a_f \qquad u_o = -1/a_b$$

(3.53a)

so

$$u_f = -a_f/a_b^2 \qquad v_f = a_b/a_f^2$$

(3.53b)

If we now expand p(u) and q(v) from 3.51 and 3.52 we obtain

$$p(u) = -\frac{1}{a_b}\ln(-a_b u) = -\frac{1}{a_b}\left\{(-a_b u - 1) + O[(a_b u+1)^2]\right\}$$

$$p(u) = u + 1/a_b + O[(a_b u+1)^2]$$

(3.54)

$$q(v) = \frac{1}{a_b}\ln(a_b v) = v - 1/a_b + O[(a_b v-1)^2]$$

(3.55)

For the regions II and III, the values of u and v have the ranges indicated in 3.53. For these values the higher order terms in 3.54 and 3.55 are negligible. This may be verified as follows;

$$a_b u_o + 1 = a_b(-1/a_b) + 1 = 0$$

$$a_b u_f + 1 = a_b(-a_f/a_b^2) + 1 = -a_f/a_b + 1 \simeq a_b \ell << 1$$

$$a_b v_o - 1 = a_b/a_f - 1 \simeq 1 + a_b \ell - 1 = a_b \ell << 1$$

$$a_b v_f - 1 = a_b^2/a_f^2 - 1 \simeq 1 + 2a_b \ell - 1 = 2a_b \ell << 1$$

where the result of 3.41 has been used to obtain the approximation for a_f/a_b.

The above results imply that to a good approximation

$$p(u) \simeq u + 1/a_b$$

in region III, and

$$q(v) \simeq v - 1/a_b$$

in region II. This means that we may satisfy the required form of the field expansions in 3.47 to 3.50 by the decomposition

$$\phi = \sum_n \frac{1}{\sqrt{4n\pi}} \left\{ i a_n \left[e^{-i\omega_n q(v)} - e^{-i\omega_n p(u)} \right] + h.c. \right\} \tag{3.56}$$

where

$$p(u) = -\frac{1}{a_b} \ln(-a_b u)\; H(u+1/a_b) + (u+1/a_b)\; H(-u-1/a_b) \tag{3.57}$$

and

$$q(v) = \frac{1}{a_f} \ln(a_f v)\; H(v-1/a_f) + (v-1/a_f)\; H(-v+1/a_f) \tag{3.58}$$

The expectation of $T_{\mu\nu}$ may now be evaluated using the method of point splitting. Following steps identical to those leading to 3.9 and 3.10 in section 3-2, we find

$$\left.\begin{matrix} \langle T_{01} \rangle \\ \langle T_{00} \rangle \end{matrix}\right\} = \frac{1}{\ell^2} \sum_n n\pi \Big\{ q'(v) q'(v+\varepsilon) e^{in(q(v+\varepsilon)-q(v))\pi/\ell}$$

$$\mp p'(u) p'(u+\varepsilon) e^{in(p(u+\varepsilon)-p(u))\pi/\ell} \Big\} \tag{3.59}$$

The summations in 3.59 have been evaluated by Fulling and Davies (1976) with the result

$$\langle T_{01} \rangle = f(u) - g(v) \tag{3.60}$$

$$\langle T_{00} \rangle = -f(u) - g(v) \tag{3.61}$$

where

$$f(u) = \frac{1}{24\pi}\left(\frac{p'''}{p'} - \frac{3}{2}\left(\frac{p''}{p'}\right)^2 + \frac{1}{2}\left(\frac{p'}{\ell}\right)^2\right) \tag{3.62}$$

$$g(v) = \frac{1}{24\pi}\left(\frac{q'''}{q'} - \frac{3}{2}\left(\frac{q''}{q'}\right)^2 + \frac{1}{2}\left(\frac{q'}{\ell}\right)^2\right) \tag{3.63}$$

The above results differ from those obtained in the one-mirror problem by the inclusion of the final term in 3.62 and 3.63 which is a Casimir-like contribution. If we evaluate the derivatives in 3.62 and 3.63 using 3.57 and 3.58, we find

$$p'(u) = \delta(u+1/a_b)\left(-\frac{1}{a_b}\ln(-a_b u) - u - 1/a_b\right) + H(-u-1/a_b) - \frac{1}{a_b u} H(u+1/a_b) \tag{3.64}$$

$$p''(u) = \frac{1}{a_b u^2} H(u+1/a_b) + \delta(u+1/a_b)\left(-1 - \frac{1}{a_b u}\right) \tag{3.65}$$

$$p'''(u) = -\frac{2}{a_b u^3} H(u+1/a_b) + \delta(u+1/a_b)\,\frac{1}{a_b u^2} \tag{3.66}$$

$$q'(v) = \frac{1}{a_f v} H(v-1/a_f) + H(-v+1/a_f)$$

$$+ \delta(v-1/a_f)\left(\frac{1}{a_f}\ln(a_f v) - v + \frac{1}{a_f}\right) \tag{3.67}$$

$$q''(v) = -\frac{1}{a_f v} H(v-1/a_f) + \delta(v-1/a_f)\left(-1 + \frac{1}{a_f v}\right) \tag{3.68}$$

$$q'''(v) = \frac{2}{a_f v} H(v-1/a_f) - \delta(v-1/a_f)\,\frac{1}{a_f v} \tag{3.69}$$

Notice that in the first and second derivatives the delta function terms do not contribute to first order in $(a\ell)$.

Finally, we may write $T_{\mu\nu}$ for the region IV as

$$\langle T_{00}\rangle = -\frac{1}{24\pi}\left(\frac{1}{2u^2} + \frac{1}{2v^2} + \frac{\pi^2}{\ell^2 a_b^2}\left(\frac{1}{2u^2} + \frac{1}{2v^2}\right)\right)$$

$$+ \frac{1}{24\pi}\,\delta(u+1/a_b)\,\frac{1}{u} + \frac{1}{24\pi}\,\delta(v-1/a_f)\,\frac{1}{v}. \tag{3.70}$$

$$\langle T_{01}\rangle = \frac{-1}{24\pi}\left(-\frac{1}{2u^2}+\frac{1}{2v^2}+\frac{\pi^2}{\ell^2 a_b^2}\left(-\frac{1}{2u^2}+\frac{1}{2v^2}\right)\right)$$

$$-\frac{1}{24\pi}\delta(u+1/a_b)\,\frac{1}{u}+\frac{1}{24\pi}\delta(v-1/a_f)\,\frac{1}{v} \qquad (3.71)$$

Now we calculate the following quantity in order to simplify $T_{\mu\nu}$.

$$\pm\frac{1}{2u^2}+\frac{1}{2v^2} = \frac{1}{2u^2v^2}(\pm v^2+u^2)$$

$$= \frac{1}{2(-t^2+x^2)^2}\left\{\pm(t^2+x^2+2xt) + t^2+x^2-2xt\right\}$$

$$= \begin{cases} \dfrac{t^2+x^2}{(-t^2+x^2)^2} \\ \dfrac{-2xt}{(-t^2+x^2)^2} \end{cases} = \frac{x_\mu x_\nu - \frac{1}{2}\eta_{\mu\nu}x^2}{x^4} \qquad (3.72)$$

Using the above results $T_{\mu\nu}$ becomes

$$\langle T_{\mu\nu}\rangle = -\frac{2}{24\pi x^4}(x_\mu x_\nu-\tfrac{1}{2}\eta_{\mu\nu}x^2) - \frac{\pi(x_\mu x_\nu-\frac{1}{2}\eta_{\mu\nu}x^2)}{24\ell^2 a_b^2 x^4}$$

$$+\frac{1}{24\pi}\delta(v-1/a_f)\,\frac{1}{v} + (\partial_\mu u\,\partial_\nu u)\,\frac{1}{24\pi}\delta(u+1/a_b)\,\frac{1}{u} \qquad (3.73)$$

Apart from the delta function terms, this result agrees with that previously obtained in equation 3.42 for $(a\ell)<<1$.

If we now calculate the energy radiated from the mirrors due to the delta function contributions in 3.73, we find

$$E_{rad} = \int dt \; - \frac{1}{24\pi} \delta(t) \; (a_b - a_f)$$

$$E_{rad} = - \frac{1}{24\pi} (a_b - a_f)$$

Finally this becomes

$$E_{rad} \simeq - \frac{1}{24\pi} (a_b - a_b + a_b^2 \ell) = - \frac{a_b^2 \ell}{24\pi} \qquad (3.74)$$

where we have used expression 3.41.

This result agrees with the total energy present in the instantaneous rest frame of a uniformly accelerated box (see equation 3.44), thus solving the problem of energy balance. This implies that the acceleration radiation can be thought of as the total energy radiated by the mirrors in moving from inertial motion to uniform acceleration.

The restriction placed on this system $(a\ell<<1)$, permits a co-moving observer to have a well defined coordinate system throughout the box. The fact that the box will register a non-zero stress-energy identical to that obtained by Minkowski ordering in the Rindler vacuum, is perhaps, an intuitive result. Since the box is getting shorter as viewed by an inertial observer, the state of the field should match this behaviour. The Rindler vacuum restricted to a finite length is the natural candidate for such a state.

SUMMARY

The goal of this study has been to provide a physical interpretation for the formal ambiguity in particle content which arises from different choices of quantization. In particular we have been interested in quantum field theory for accelerated systems, which could reflect the thermal properties of the Rindler quantization procedure.

In chapter two, the response of a uniformly accelerated dipole detector was analysed by a different method from that used by Hinton in his 1983 study. We find and qualitatively justify, a non-thermal and non-isotropic response thus agreeing with the previous work.

We then determined that a particle detector following a trajectory defining a time-like Killing vector, would respond to the Killing particle density in the Minkowski vacuum. The variation in response to a state $|N\rangle$ between detectors following different trajectories was then analysed. As a result of this calculation, the generalized acceleration radiation was interpreted as the effective quantity present when comparing systems following different trajectories.

In chapter three we found that particle detectors comoving with a single 1+1 dimensional mirror have a null response even for trajectories resulting in non-trivial $T_{\mu\nu}$. This example suggests that the particle content of

a quantum field is truly an observer dependent concept, and will thus reflect equivalence principle arguments.

In the same chapter we also found that a 1+1 dimensional uniformly accelerated box contained a negative energy density which was identical to a thermal result at $T=a/2\pi$. The presence of this energy was dynamically accounted for using the results of the one-mirror problem and by considering a box which moved discontinuously from inertial motion to uniform acceleration.

Our results point to future study of $T_{\mu\nu}$ for completely general motion of two mirrors. It would then be possible to examine both the uniformly accelerated box and the one-mirror problem as limiting cases of completely general motion.

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APPENDIX ONE

Verification of the Definition for $T_{\mu\nu}$ from the Feynman Green Function:

We will demonstrate that the operator defined in equation 1.36 is suitable for calculating $T_{\mu\nu}$. The Feynman Green function is

$$iG_F(x;x') = \langle 0|H(t-t')\phi(x)\phi(x') + x \leftrightarrow x'|0\rangle$$

Now calculating the derivatives of G_F

$$\lim_{x' \to x} \partial_{j'}\partial_i \, iG_F = \lim_{\underline{x}' \to \underline{x}} \tfrac{1}{2}\langle 0|\partial_i\phi\partial_{j'}\phi' + \partial_{j'}\phi'\partial_i\phi|0\rangle \tag{A.1}$$

$$\lim_{x' \to x} \partial_t\partial_{j'} iG_F = \lim_{\underline{x}' \to \underline{x}} \tfrac{1}{2}\langle 0|\partial_t\phi\partial_{j'}\phi' + \partial_{j'}\phi'\partial_t\phi|0\rangle$$

$$+ \lim_{x' \to x} \delta(t-t')\langle 0|[\phi, \partial_{j'}\phi']|0\rangle$$

$$= \lim_{\underline{x}' \to \underline{x}} \langle 0|\tfrac{1}{2}\{\partial_t\phi, \partial_{j'}\phi'\}|0\rangle$$

$$+ \lim_{x' \to x} \delta(t-t')\partial_{j'}\langle 0|[\phi,\phi']|0\rangle\Big|_{t=t'}.$$

But from 1.10 we have

$$[\,\phi(x),\phi(x')\,]_{t=t'} = 0$$

therefore

$$\lim_{x'\to x} i\partial_t\partial_{j'}G_F = \lim_{\underline{x}'\to\underline{x}} \langle 0|\tfrac{1}{2}\{\partial_t\phi,\partial_{j'}\phi'\}|0\rangle$$

(A.2)

Similarily

$$\lim_{x'\to x} i\partial_t\partial_{t'}G_F = \lim_{\underline{x}'\to\underline{x}} \tfrac{1}{2}\langle 0|\{\partial_t\phi,\partial_{t'}\phi'\}|0\rangle$$

$$+ \lim_{x'\to x} \langle 0|-\delta(t-t')\,[\,\text{[illegible]},\phi'\,]\,|0\rangle$$

$$+ \lim_{x'\to x} \langle 0|-\dot{\delta}(t-t')\,[\,\phi,\phi'\,]\,|0\rangle$$

$$= \lim_{\underline{x}'\to\underline{x}} \tfrac{1}{2}\langle 0|\{\partial_t\phi,\partial_{t'}\phi'\}|0\rangle$$

$$+ \lim_{x'\to x}\Big\{\langle 0|-\delta(t-t')\,[\,\text{[illegible]},\phi'\,]\,|0\rangle$$

$$+\ \delta(t-t')\,\langle 0|\,[\,\text{[illegible]},\phi'\,]\,|0\rangle\Big\}$$

Finally we obtain

$$\lim_{x'\to x} \partial_t \partial_{t'} iG_F = \lim_{\underline{x}'\to \underline{x}} \tfrac{1}{2}\langle o|\ \{\partial_t \phi, \partial_{t'} \phi\}\ |o\rangle \tag{A.3}$$

Since

$$T_{\mu\nu} = \phi_{,\mu}\phi_{,\nu} - \tfrac{1}{2} g_{\mu\nu} g^{\alpha\beta} \phi_{,\alpha}\phi_{,\beta}$$

then the results of A.1 to A.3 imply that the definition 1.36 will yield the correct form for $T_{\mu\nu}$.

APPENDIX ONE

Verification of the Definition for $T_{\mu\nu}$ from the Feynman Green Function:

We will demonstrate that the operator defined in equation 1.36 is suitable for calculating $T_{\mu\nu}$. The Feynman Green function is

$$iG_F(x;x') = \langle 0|H(t-t')\phi(x)\phi(x') + x \leftrightarrow x'|0\rangle$$

Now calculating the derivatives of G_F

$$\lim_{x'\to x} \partial_{j'}\partial_i \, iG_F = \lim_{\underline{x}'\to\underline{x}} \tfrac{1}{2}\langle 0|\partial_i\phi\,\partial_{j'}\phi' + \partial_{j'}\phi'\,\partial_i\phi|0\rangle \tag{A.1}$$

$$\lim_{x'\to x} \partial_t\partial_{j'} \, iG_F = \lim_{\underline{x}'\to\underline{x}} \tfrac{1}{2}\langle 0|\partial_t\phi\,\partial_{j'}\phi' + \partial_{j'}\phi'\,\partial_t\phi|0\rangle$$

$$+ \lim_{x'\to x} \delta(t-t')\langle 0|[\phi,\partial_{j'}\phi']|0\rangle$$

$$= \lim_{\underline{x}'\to\underline{x}} \langle 0|\tfrac{1}{2}\{\partial_t\phi,\partial_{j'}\phi'\}|0\rangle$$

$$+ \lim_{x'\to x} \delta(t-t')\,\partial_{j'}\langle 0|[\phi,\phi']|0\rangle\Big|_{t=t'} .$$

But from 1.10 we have

$$[\ \phi(x), \phi(x')\ ]_{t=t'} = 0$$

therefore

$$\lim_{x' \to x} i\partial_t \partial_{j'} G_F = \lim_{\underline{x}' \to \underline{x}} \ldots = 0 \qquad \text{(A.2)}$$

Similarily

$$\lim_{x' \to x} i\partial_t \partial_{t'} G_F = \lim_{\underline{x}' \to \underline{x}} \ldots$$

$$+ \lim_{x' \to x} \ldots (t-t') \ [\ \ldots\ ] \ldots$$

$$+ \lim_{x' \to x} \ldots (t-t') \ [\ \ldots\ ] \ldots$$

$$= \lim_{\underline{x}' \to \underline{x}} \ldots$$

$$+ \lim_{x' \to x} \ldots (t-t') \ [\ \ldots\ ] \ldots$$

$$+ \ldots (t-t') \ldots [\ \ldots\ ] \ldots$$

Finally we obtain

$$\lim_{x'\to x} \partial_t \partial_{t'} iG_F = \lim_{\underline{x}'\to\underline{x}} \tfrac{1}{2}\langle o| \{\partial_t \phi, \partial_{t'} \phi\} |o\rangle \tag{A.3}$$

Since

$$T_{\mu\nu} = \phi_{,\mu}\phi_{,\nu} - \tfrac{1}{2} g_{\mu\nu} g^{\alpha\beta} \phi_{,\alpha}\phi_{,\beta}$$

then the results of A.1 to A.3 imply that the definition 1.36 will yield the correct form for $T_{\mu\nu}$.

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