Extending WKB-Topological Recursion Connection

by

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Abstract:

It has been proven in other sources that spectral curves, (Σ, x, y), where Σ is a compact Riemann surface, and meromophic functions x and y satisfy a polynomial equation (and subject to certain admissibility conditions), can be used with the topological recursion to construct the WKB expansion for the quantization of said curve. In this paper we prove an extension of that connection for spectral curves, (Σ, u, y), where u is meromorphic only on an open region of Σ, and $x = e^u$ may or may not be meromorphic on Σ, so long as ydu is meromorphic on Σ; we will see that the admissibility condition still holds, and that there are added constraints. We provide a rigorous proof for dealing with spectral curves where u is meromorphic on Σ, but provide only a conceptual argument and affirmative examples for dealing with spectral curves where u is not meromorphic on Σ.

Preface:

This thesis is an original work by Anand Chotai. No part of this thesis has been previously published.

Acknowledgments:

Special thanks goes out to a number of people for helping me complete this thesis. To my supervisor, Vincent Bouchard, for being a patient and knowledgeable teacher. To my parents and sisters and friends for supporting me and being there when I needed them. Thanks to all the experts doing research in this field. And, thanks to the University of Alberta and my defense committee.

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1 Introduction

The topological recursion is a structure, that has emerged in the last few decades, with intimate connections to a wide range of fields in both mathematics and physics ([6][7][8][12][14]). We will introduce the details of the topological recursion in the following sections, and go into some of these connections; for now it is enough to understand that the topological recursion is a recursive structure for defining an infinite set of n-differentials, call them $W_{g,n}$'s, from the data of something called a 'spectral curve' which can be thought of as the data of an irreducible algebraic curve. The spectral curve can also be expressed as (Σ, x, y), where Σ is a compact Riemann surface, and x and y are meromorphic functions on Σ.

We will set up the definition of the spectral curve in Section 2.2, and of topological recursion in Sections 2.4 and 2.6. We will discuss the connections between the topological recursion and a few other relevant fields in Section 2.7.

The results of [2][3][4][9] establish, and prove in a broad class of cases, a particularly interesting connection between the topological recursion and the method of WKB in solving differential equations. The topic of this thesis is to extend the result of those papers to a broader class. The basic idea behind the connection between WKB and the topological recursion is that the $W_{g,n}$'s, defined through the recursion, can be used to construct a wavefunction, ψ, which is annihilated by a differential equation that is the 'quantization' of the spectral curve; this last process precisely mirrors WKB.

We will go over the details of WKB in Section 2.1, and quantization procedure in Section 2.3, in case the reader is not familiar. We will also introduce the connection between WKB and the topological recursion, in detail, in Section 2.5.

As was mentioned, the purpose of this thesis is to extend the pre-existing results to a wider class of spectral curves. Currently, in [2], the connection between WKB and topological recursion has been proven for spectral curves $S = (\Sigma, x, y)$, with x and y meromorphic on Σ (cases where the spectral curves are given by the zero locus of a polynomial in x and y), subject to an admissibility condition that we will define. The key here is that the differential, $y \cdot dx$, is meromorphic on Σ, since this object plays a key role in the topological recursion; as we will see, this condition needs to be maintained in the classes we will consider.

We would like to extend this to the following two main new classes:

Definition 1.1. The *first class*, for which we extend the WKB-TR connection, are spectral curves, (Σ, u, y), where Σ is a compact Riemann surface, y and $x = e^u$ are meromorphic functions on Σ, but u is a meromorphic function only on an open region of Σ. Then, $y \cdot du = y\frac{dx}{x}$ is meromorphic on Σ.

Here is an example of a spectral curve which falls into this class:

Example 1.2.

$$y^2 - e^u y + 1 = 0, \tag{1.0.1}$$

This is an example that we will solve explicitly in Section 5. We can parameterize the curve as: $(u, y) = (\log(z + \frac{1}{z}), z)$, where $z \in \Sigma = \mathbb{CP}^1$. Then, we can see that $u(z)$ is a meromorphic function only on the open subset $\mathbb{CP}^1 \backslash \{\pm i\}$. Here, $y \cdot du = y\frac{dx}{x} = \frac{z^2-1}{z^3+z}dz$ is meromorphic on the Riemann sphere.

As we will see, in Section 3, dealing with curves of this sort requires only a minor (but non-trivial) modification of the proof in [2]. We highlight that, while our approach for this class is proven rigorously, there are still significant challenges to overcome. In [2], and in this thesis, we can only prove the connection for a subclass of cases, cases which satisfy a particular 'admissibility' condition (which we will define later). For now it is enough to understand that extending the proof (being able to relax the admissibility condition) will require a modification of the method we use.[1] There are also added constraints, that we will address in Section 3.2.

Definition 1.3. The *second class* of cases we consider is spectral curves, (Σ, u, y), where Σ is a compact Riemann surface, y is meromorphic on Σ, u is meromorphic on an open region of Σ, but $x = e^u$ is not meromorphic on Σ. More specifically, we will restrict this class further to be such that $u = \log(f(z)) + p(z)$, where $f(z)$ and $p(z)$ are polynomials; then $y \cdot dx$ is still meromorphic on Σ.

An example of this class of cases is the following:

[1]It's not yet clear how large a modification is required.

Example 1.4.

$$ye^{-y} - x = 0, \quad x \in \mathbb{C}^* \tag{1.0.2}$$

Here, $x \in \mathbb{C}^*$, and so we can change variables to $x = e^u$, $u \in \mathbb{C}$. This is an example of a spectral curve which, we will see, arises out of the study of simple Hurwitz numbers. We can parameterize this curve as: $(u, y) = (\log(ze^{-z}), z)$. We see that y is meromorphic on the Riemann sphere, but $x = e^u$ is not, it has an essential singularity at ∞. Moreover, $y \cdot du = y\frac{dx}{x} = \frac{1-z}{z}dz$, which is meromorphic on the Riemann sphere.

To deal with this second class we consider a sequence of spectral curves, (Σ, u_r, y), with e^{u_r} meromorphic Σ, and such that $u = \lim_{r\to\infty} u_r$. Then, the claim is that we can apply the previously mentioned approach to this sequence of spectral curves, and take the limit $r \to \infty$ at the end to recover the quantum curve for this class. This approach for dealing with the second class of curves does not currently have a rigorous proof, but it does appear to work in the examples considered. We will discuss this in Section 3.3. For progress to be made in this field a rigorous proof of the material in Section 3.3 is required.

Finally, in Section 5, we will look at some examples that have been worked out using the methods of this thesis, these have all been verified computationally to low order.

2 Background

2.1 WKB Approximation

WKB[2] approximation is a method for solving finite, linear, differential equations, involving a very small parameter, ϵ:

$$\left[\epsilon^n \frac{d^n}{dx^n} + \epsilon^{n-1} a_{n-1}(x) \frac{d^{n-1}}{dx^{n-1}} + ... + a_0(x)\right] \psi(x) = 0 \tag{2.1.1}$$

The $a_k(x)$ here are rational functions in x, and they may also be polynomials in ϵ such that they can satisfy other orderings.

An approximate solution for (2.1.1) is found by assuming an asymptotic exponential solution for ψ, of the form:

$$\psi(x) \approx \exp\left(\frac{1}{\epsilon}\sum_{k=0}^{\infty} \epsilon^k S_k(x)\right) \tag{2.1.2}$$

The $S_k(x)$'s are functions of x that need to be solved for, by plugging (2.1.2) into (2.1.1). The result is a power series in ϵ, where each order can be solved in terms of the previous orders; we will see an example of this shortly.

WKB is an approach used to solve many problems in quantum mechanics; in that case we use the small parameter $\epsilon = \hbar$. Here $\hbar$ is Plank's constant, a parameter that appears ubiquitously in quantum mechanics: $\hbar \to 0$ represents the macroscopic (or non-quantum) limit, where the physical equations should reduce to those of classical mechanics.

[2] Gregor Wentzel [22, Hans Kramers [18], Leon Brillouin [10]

Example 2.1. As an example of the application of WKB, let's consider the differential equation, called the 'Airy differential equation':

$$\left[\hbar^2 \frac{d^2}{dx^2} - x\right]\psi = 0 \tag{2.1.3}$$

We will come back to this example multiple times in this paper, as an illustration of the WKB-topological recursion connection[3]. The solution of the Airy differential equation are the two Airy functions; the WKB expansion gives the asymptotic expansion of these functions.

Making the WKB assumption (2.1.2), and plugging into (2.1.3), gives the expression:

$$\left[\sum_{k=0}^{\infty} \hbar^{k+1} S_k''(x) + \left(\sum_{k=0}^{\infty} \hbar^k S_k'(x)\right)^2 - x\right] \exp\left(\frac{1}{\hbar}\sum_{k=0}^{\infty} \hbar^k S_k(x)\right) = 0 \tag{2.1.4}$$

Which we can write in orders of $\hbar$ to get:

$$\mathcal{O}(\hbar^0): \quad (S_0')^2 - x = 0 \tag{2.1.5}$$

$$\mathcal{O}(\hbar^1): \quad S_0'' + 2S_0'S_1' = 0 \tag{2.1.6}$$

$$\mathcal{O}(\hbar^m): \quad S_{m-1}'' + \sum_{k=0}^{m} S_k' S_{m-k}' = 0, \qquad m > 0 \tag{2.1.7}$$

The above expressions can be used to solve all S_k's, giving us the full WKB

[3] We will see, in Section 2.5, that it is related to something called the 'Airy curve', that we will define in Section 2.2

asymptotic solution.

The time-independent Schrödinger equation is a key equation in quantum mechanics:

$$\left(\hbar^2 \frac{d^2}{dx^2} - 2m(V(x) - E)\right)\psi(x) = 0 \tag{2.1.8}$$

The solution of this represents the spatial part of the quantum wavefunction, $\psi(x)$, of a particle moving in one-dimension in a potential energy field $V(x)$, and with energy E. This equation is obviously of the same form as (2.1.3), in an appropriate potential $V(x)$.

In some form or another, all modern physics can be expressed in terms of differential equations. The WKB-topological recursion connection is a connection between the fields of differential equations and geometric surfaces (namely, compact Riemann surfaces). From the above example alone we can see some of the importance of such a connection.

We note one possible alternative approach to dealing with the class of spectral curves considered in this thesis. Exponential operators are defined in terms of the series: $e^{\widehat{D}} = \sum_{n=0}^{\infty} \frac{\widehat{D}^n}{n!}$. The topic of this thesis will involve extending the WKB-topological recursion connection to the case of infinite order differential equations:

$$e^{\epsilon \frac{d}{dx}}\psi(x) = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!}\frac{d^n}{dx^n}\psi(x) \tag{2.1.9}$$

One way to view these kinds of equations is as follows: If we consider the

Maclaurin series of our function, $\psi(x) = \sum_{k=0}^{\infty} a_k x^k$. Then, for all $k \geq 0$:

$$e^{\epsilon \frac{d}{dx}}(a_k x^k) = a_k \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} \frac{d^n}{dx^n} x^k = a_k \sum_{n=0}^{k} \binom{k}{n} \epsilon^n x^{k-n} = a_k (x+\epsilon)^k \qquad (2.1.10)$$

Therefore, $e^{\epsilon \frac{d}{dx}} \psi(x) = \psi(x+\epsilon)$. Here we can see the connection between infinite order differential equations and difference equations.

It might be interesting, in future analysis, to consider how difference equations relate to the topological recursion, since one can apply WKB analysis to difference equations, [15]. That will not be the approach of this thesis though; here we will attempt a truncation method, where we will be truncating the infinite order equations at some r, dealing with finite differential equations and taking the limit $r \to \infty$ at the end; this is described in Section 3.3.

2.2 Spectral Curves and Their Geometry

The key ingredient in the topological recursion is the spectral curve, and we will introduce that structure in this section.

The spectral curve can be thought of in two ways, and to understand both we will introduce the concept of an algebraic curve, defined on the two dimensional complex plane:

$$P(x, y) = 0, \qquad (x, y) \in \mathbb{C} \times \mathbb{C}, \qquad (2.2.1)$$

where $P(x, y)$ is a polynomial; the algebraic curve itself is the zero locus of this polynomial.

For the spectral curve we will start with the following definition ([2][3][4][14]):

Definition 2.2. A *spectral curve*, written in this paper as $S = (\Sigma, x, y)$, is defined by a Torelli marked genus $\widehat{g}$ compact Riemann surface Σ, and meromorphic functions, x and y, defined on Σ.

'Torelli marked' means that the compact Riemann surface has a choice of symplectic basis of cycles. For $\widehat{g} = 0$ spectral curves (where Σ is the Riemann sphere) the basis is trivial. For higher genus curves 'symplectic basis of cycles' simply means that you have $2\widehat{g}$ non-contractible cycles satisfying the conditions, [14]:

$$A_i \cap B_j = \delta_{i,j}, \quad A_i \cap A_j = 0, \quad B_i \cap B_j = 0 \tag{2.2.2}$$

We note that this basis is not unique, and the $W_{g,n}$'s do depend upon the choice of basis.

Given that x and y are meromorphic they must satisfy a polynomial equation, as above:

$$P(x, y) = p_0(x)y^r + p_1(x)y^{r-1} + ... + p_{r-1}(x)y + p_r(x) = \sum_{i=0}^{r} p_{r-i}(x)y^i = 0 \tag{2.2.3}$$

And this is another way of viewing a spectral curve: as an irreducible algebraic curve given by the above equation.

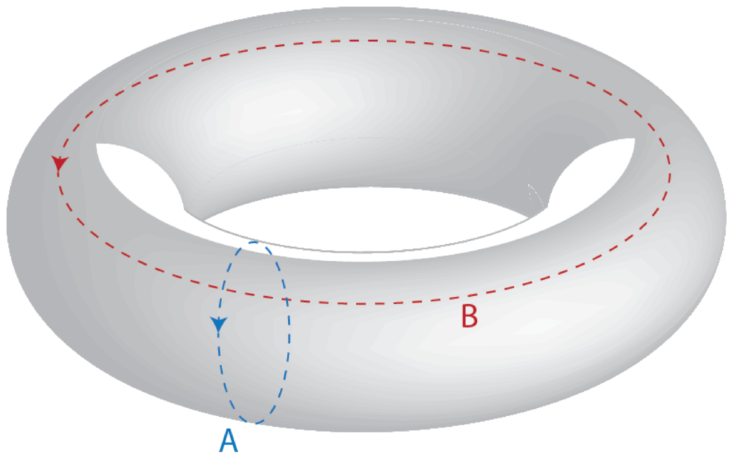

Figure 1: An example of symplectic basis of cycles for a torus ($\widehat{g} = 1$).

Definition 2.3. If $\widehat{g} = 0$ we say that the spectral curve is *rational*.

In this section we will be focusing only on rational spectral curves, where x and y can be parameterized by rational functions.

Definition 2.4. A spectral curve is called *regular* if the differential form dx has only a finite number of zeros, that are all simple, and the zeros of dy do not coincide with the zeros of dx.

Why this interest in the zeros of dx? One way of viewing the meromorphic function x is as a branched covering, $x : \Sigma \to \mathbb{CP}^1$; Figure 2. Let x be a degree d-branched covering of Σ: this means that, locally around any point $p \in \Sigma$, $x(z)$ can be seen as a map $z \mapsto z^m$, $|m| \leq d$. Here m is referred to as the 'multiplicity' of the point p, we'll write it m_p. Then, $m_p = ord_p(dx) + 1$ if p is not a pole of x, and $m_p = -ord_p(x)$ if p is a pole of x. This allows us to the define the following:

Definition 2.5. We can define the *ramification points* of Σ as the points $p \in \Sigma$ such that $m_p \geq 2$, which is equivalent to the set of points that are zeroes of dx, and poles of x with order greater than one. Seen another way, the ramification points are the points where sheets of the branched covering x collide.

Definition 2.6. The *branch points* are the points, $x(a_i)$, where a_i are the ramification points. They are the points whose pre-image, $x^{-1}(x(a_i))$ consists of a set of points which number $< d$.

The topological recursion involves calculating residues at the ramification points of our spectral curve, and summing over all these residues.

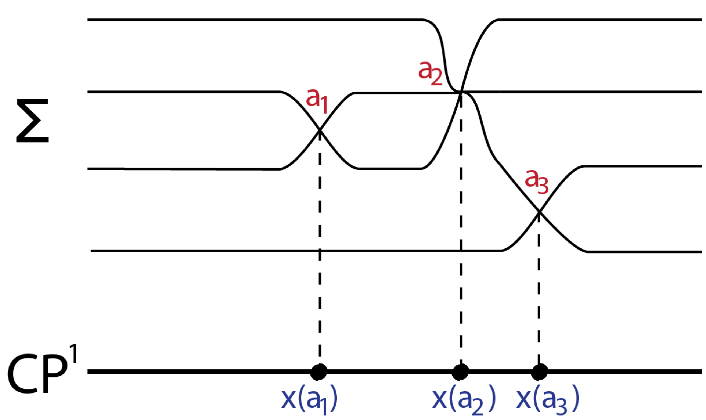

Figure 2: An example of a degree 4 branched covering, given by a function x. Ramification points a_1, a_2, a_3 and branch points $x(a_1), x(a_2), x(a_3)$, with $m_{a_1} = 2$, $m_{a_2} = 3$, and $m_{a_3} = 2$. All other points, $p \in \Sigma$, have $m_p = 1$.

On Σ, near the ramification point, for a regular spectral curve, there is then exactly one other point, called the 'conjugate point' such that $x(z) = x(\overline{z})$. The Eynard, Orantin topological recursion that we will introduce in

Section 2.4 deals only with regular spectral curves, but it has been extended, in [3] since then to include arbitrary ramification.

Definition 2.7. We can define the set R as being the ramification points of our spectral curve S, equivalently the zeros of dx and poles of x of order greater than one. $R = \{a_1, ..., a_n\}$, $n \in \mathbb{N}$.

Now, let's look at some examples of spectral curves.

Example 2.8. The first example is called the 'Airy curve'. This is an example we mentioned briefly in Section 2.1. The defining polynomial of the spectral curve is:

$$y^2 - x = 0, \qquad (x, y) \tag{2.2.4}$$

x and y can be parameterized by $x(z) = z^2$ and $y(z) = z$, both meromorphic on the Riemann sphere. Thus we can see x as a double branched covering, with $R = \{0, \infty\}$. The conjugate point, for any $z \in \mathbb{CP}^1 \backslash \{0, \infty\}$, is $\overline{z} = -z$.

Example 2.9. We can consider the following spectral curve, that will play key importance in this thesis; we can refer to this spectral curve as the 'r-spin Hurwitz' curve, since, as we wil see in Section 2.7, it comes out of the study of r-spin Hurwitz numbers.

$$ye^{-y^r} - x = 0, \qquad x \in \mathbb{C}^* \tag{2.2.5}$$

We can take $x = e^u$, and parameterize the curve as, $y = z$, $u = log(ze^{-z^r})$. Therefore, y is meromorphic on the Riemann sphere, but x is not (it has an essential singularity at ∞). R is then the set composed of the roots of: $1 - rz^{r-1} = 0$.

There are plenty of more examples of spectral curves in [2][3][4][12][14], and in Section 5.

2.3 Quantum Curves and Quantization

A key approach to solving problems connected to the Topological Recursion deals with what is called the 'quantization' of an algebraic curve. Quantization follows from the approach in going from classical to quantum physics, [9], whereby the classical coordinates are turned into quantum operators, that obey certain commutation relations.

Referring to the algebraic curve, defined in (2.2.1), we have that $(x, y) \to (\widehat{x}, \widehat{y})$, with $\widehat{x}$ and $\widehat{y}$ operators such that:

$$[\widehat{x}, \widehat{y}] = -\hbar \tag{2.3.1}$$

This turns the polynomial $P(x, y)$ into an operator, $\widehat{P}(\widehat{x}, \widehat{y})$, acting on the space of functions. We can expand the operator perturbatively, in powers of $\hbar$ as, [17]:

$$\widehat{P} = \widehat{P}_0 + \hbar\widehat{P}_1 + \hbar^2\widehat{P}_2 + ... \tag{2.3.2}$$

Where $\widehat{P}_0 = P$, and the $\mathcal{O}[\hbar]$ terms are referred to as 'quantum corrections'.

Because of the commutation relation (2.3.1), this expansion is not unique;

changing the ordering of x and y in $P(x, y)$ changes the powers of $\hbar$ in $\widehat{P}(\widehat{x}, \widehat{y})$, for example. We can choose a 'canonical' ordering, along with a choice of coordinates, mirroring the approach in quantizing classical mechanics. We define the operators:

$$\widehat{x} = x, \qquad \widehat{y} = \hbar \frac{d}{dx}, \tag{2.3.3}$$

where $\widehat{x}$ and $\widehat{y}$ are then analogous to the position and momentum operators, respectively, in quantum mechanics. There is then a 'natural' ordering, in which all instances of $\widehat{y}$ appear to the right of $\widehat{x}$.

When we look at the extension of spectral curves, given in Definitions 1.1 and 1.3, it will be natural to choose change of coordinates $x = e^u$, and the operators:

$$\widehat{u} = u, \quad \widehat{y} = \hbar \frac{d}{du}, \quad [\widehat{u}, \widehat{y}] = -\hbar \tag{2.3.4}$$

The idea here being that we are quantizing the symplectic space in the variables u and y, with the symplectic bilinear form $du \wedge dy$.

In the WKB-topological recursion connection, we find that a quantization of our spectral curve precisely annihilates a specific wavefunction, formed from the topological recursion. It is interesting to note that the quantization appeared, for a long time, to be picking out only a particular ordering (not necessarily the natural one), and it was not understood why. It was discovered in [2] that an assumption was being implicitly made, and that other orderings of the quantization do actually appear.[4]

[4] We will discuss this in more detail in Section 4.4, and we will see cases of this in the Examples. It is still not understood how to get all orderings of the quantum curve.

2.4 Eynard, Orantin Topological Recursion

As was mentioned, the topological recursion is a device for generating, recursively, a series of meromorphic symmetric n-forms, living on:

$$S^n = \underbrace{S \times ... \times S}_{\text{n times}} \tag{2.4.1}$$

Where '$\times$' is the cartesian product, and S is the spectral curve (defined in Section 2.2). We will refer to these meromorphic differentials as $W_{g,n}$'s, with $g \geq 1$, $n \geq 0$, they are sometimes also referred to as 'Correlation Functions'. The Topological Recursion was originally derived from matrix model theory, but it was quickly found that it had deep underlying connections to many other fields. Some of these connections will be discussed in Section 2.7.

In this section we will introduce the original topological recursion, first presented formally by Eynard and Orantin in [14], in 2008. First, we need to set up some preliminary definitions, as they were first presented in [14]:

Definition 2.10. The *Canonical Bilinear Differential of the Second Kind* (sometimes referred to as the 'Bergmann Kernel'), written $W_{0,2}(z_1, z_2)$, is the unique bilinear differential living on S^2 satisfying the conditions that: A.) It is symmetric in its two variables z_1 and z_2. B.) Its only pole is a double pole at $z_1 = z_2$, and locally around this diagonal:

$$W_{0,2}(z_1, z_2) = \left(\frac{1}{(z_1 - z_2)^2} + regular \right) dz_1 dz_2 \tag{2.4.2}$$

C.) It satisfies the normalization condition:

$$\oint_{A_i} W_{0,2}(z_1, z_2) = 0, \quad \forall i = 1, ..., \widehat{g} \tag{2.4.3}$$

Where $\widehat{g}$ is the genus of the spectral curve, and A_i are the basis of cycles discussed in Section 2.2. The uniqueness of $W_{0,2}$ is guaranteed by conditions B and C above.

On genus zero spectral curves, which will be the focus of this paper, $W_{0,2}$ has the simple form:

$$W_{0,2}(z_1, z_2) = \frac{dz_1 dz_2}{(z_1 - z_2)^2} \tag{2.4.4}$$

On higher genus curves, $W_{0,2}$ can be a lot more complicated.

Definition 2.11. The *Recursion Kernel* is defined globally in z_0 on the spectral curve, near any branchpoint in z, as follows:

$$K(z_0, z) = \frac{\int_{z'=b}^{z} W_{0,2}(z_0, z')}{(y(z) - y(\overline{z}))dx(z)}, \tag{2.4.5}$$

where b is an arbitrary base point.

Given the above definitions we can now define the $W_{g,n}$'s on the spectral curve using the topological recursion, [14]:

Definition 2.12. Given a regular, rational, spectral curve S, we can define the following meromorphic differentials recursively, living on S^n. This recursion has the initial conditions: $W_{0,1}(z) = y(z)dx(z)$, and $W_{0,2}(z_1, z_2)$ as defined above. Then, for all other $2g-2+n>0$ we have:

$$
\begin{aligned}
W_{g,n+1}(z_0,\mathbf{J}) &= \sum_{a\in R}\operatorname*{Res}_{z=a}K(z_0,z) \\
&\times\left[W_{g-1,n+1}(z,\overline{z},\mathbf{J})+\sum_{h=0}^{g}\sum_{\mathbf{I}\subset\mathbf{J}}{}^{\prime}\, W_{h,1+|\mathbf{I}|}(z,\mathbf{I})W_{g-h,1+n-|\mathbf{I}|}(\overline{z},\mathbf{J}\backslash\mathbf{I})\right]
\end{aligned}
\tag{2.4.6}
$$

Where $\mathbf{J}=\{z_1,...,z_n\}$ and the $\sum'$ means that we exclude from the sum $(h,\mathbf{I})=\{(0,\emptyset),(g,\mathbf{J})\}$. This is a recursion in $2g-2+n$.

One very useful way of understanding the topological recursion is in terms of degeneration of compact Riemann surfaces, with marked points. There are two ways that you can you can pull apart a compact Riemann surface of genus g, that are demonstrated in Figure 3. One can view the degeneration as taking two cycles of our compact Riemann surface and pinching them to form a point. If the two pinched cycles are homologically unequal we get two new compact Riemann surfaces, of genus $g_1+g_2=g$. If they are homologically trivial we get a single new compact Riemann surface of genus $g-1$. This is exactly what (2.4.6) is doing: $W_{g,n+1}$ is constructed by summing over all the decompositions of the genus g compact Riemann surface with $n+1$ marked points (multiplied by an appropriate weight, this is discussed in more detail in [14]). These degenerations are themselves the previous $W_{g,n}$'s in the recursion. More on this graphical representation can be found in [3][4][14].

From the above constructed $W_{g,n}$'s we can define a number of other in-

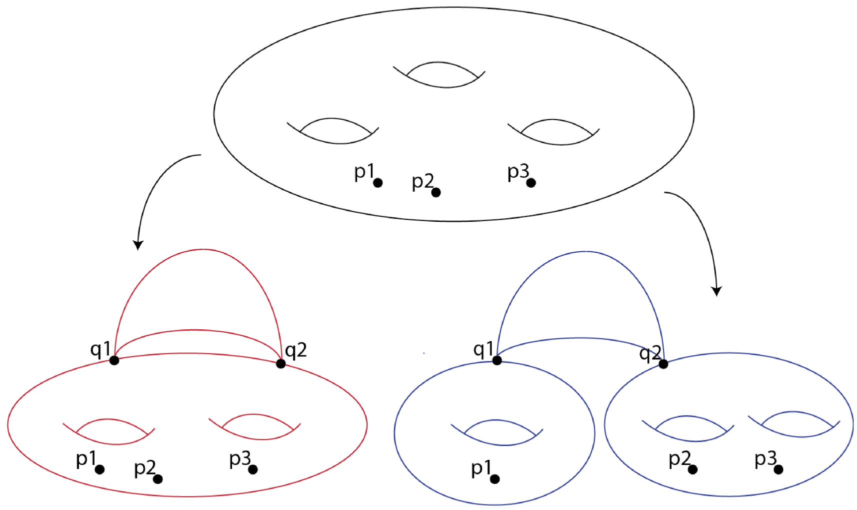

Figure 3: Deconstruction of a genus 3 torus with three marked points (p_1, p_2, p_3) into a genus 2 torus with three marked points, or into a genus 1 torus with one marked point and a genus 2 torus with two marked points. This is done through removing a sphere with two punctures.

teresting objects. An important construction is the F_g's, or 'free energies'.

Definition 2.13. The *free energies* are defined from the $W_{g,n}$'s as follows:

$$F_g := \frac{1}{2g-2} \sum_{a \in R} Res_{z=a} \left(\int_b^z y(t)dx(t) \right) W_{1,g}(z) \tag{2.4.7}$$

With arbitrary base point b.

2.5 WKB-Topological Recursion Connection

Now that we've introduced all the background information, as well as the topological recursion, we can introduce the connection that is the basis of this thesis; we have been calling it the WKB-topological recursion connection.

The basic idea is that, from the $W_{g,n}$'s generated by a spectral curve S, we can construct a wave-function Ψ, that mirrors (2.1.2) the assumption of WKB, and that is precisely annihilated by the quantization of S. Put more concretely, we can construct:

$$\Psi(z) = \exp\left(\frac{1}{\hbar}\sum_{g=0}^{\infty}\sum_{n=1}^{\infty}\frac{\hbar^{2g+n-1}}{n!}\int_b^z ... \int_b^z W_{g,n}(z_1, ..., z_n)\right), \qquad (2.5.1)$$

where b is an arbitrary base point. Then, the claim is that Ψ satisfies the following:

$$\widehat{P}\Psi = 0 \qquad (2.5.2)$$

Where $\widehat{P}$ is a quantization of our spectral curve, as discussed in (2.3.2). We could call (2.5.2) 'Schrödinger like' ([3]), in that it resembles the Schrödinger equation which operates in the same non-commutative algebra. The WKB-TR connection has been proven for all rational spectral curves of the form 2.2.1, subject to an admissibility condition (Definition 2.14), as we will see in Section 2.6; the purpose of this paper is to extend it to broader classes of spectral curves: the first class given by Definition 1.1, and the second class given by Definition 1.3.

We will see Section 2.7 where (2.5.1) originally came from. The connection was proven for a few cases individually[5]. A more general proof for all spectral

[5]The Airy curve case we saw in Section 2.1 and 2.2 is a fairly straight forward and useful example to work through. The proof involves many of the same steps as the general proof that we will do in Section 4. The first step is to deal with the residue in (2.4.6), using the fact that Σ is a compact Riemann surface we flip the contour integral to pick up all other poles not in R. Then, we integrate, sum over, and principally specialize, the

curves of the form (2.2.1), with $W_{0,1} = ydx$, was worked out in [2]. As was mentioned, this process mirrors the WKB approximation procedure, in the sense that the $W_{g,n}$'s are intimately connected to the S_k's defined in (2.1.2). The topological recursion itself mirrors the recursion arising out of the WKB process, (2.1.7).

Note too, from the form of (2.5.1) and (2.3), that $S_0' = y$; this too is something that we see in WKB, the $\hbar$ independent terms in the expansion simply give back the spectral curve under this substitution. In this way we can view the S_k's of (2.1.2), which are defined recursively in terms of $S_0' = y$ and x, as being functions living on our spectral curve.

2.6 Global Topological Recursion

In the years following the first presentation of the topological recursion modifications to the original Eynard, Orantin version were made. Specifically, in reference to the WKB-TR connection, what was needed was a way of constructing the wave-function Ψ, and proving that it is annihilated by the quantum curve, for some class of cases.

The first step was to extend the topological recursion to deal with spectral curves with arbitrary ramification, this was done in [4]. You can recall, from Section 2.4, that the Eynard, Orantin recursion is only applicable for spectral curves where the branched covering contains simple branch points.

$W_{g,n}$'s in a way that produces (2.5.1). The resultant wavefunction can then be seen to satisfy the equation (2.1.3). As we will see in Section 3.3 this flipping the residue trick brings up a new set of problems when we are dealing with log singularities.

In [4] they constructed formulation of the topological recursion with arbitrary ramification, is equivalent to the original recursion if we restrict to simple ramification points.

As we discussed in Section 2.4, we can understand the topological recursion as degenerations of Riemann surfaces, where we are removing a Riemann sphere with two marked points. The reformulation of the topological recursion for arbitrary ramification can be viewed graphically as forming the $W_{g,n}$'s by removing a Riemann sphere with k-marked points, and summing over all possible ways of doing this, for $k = 2, ..., n$ where n is the order of the branched covering given by x. More details on this graphical perspective can be found in [4]. In many ways, this graphical approach formed the basis for how this reformulation of the topological recursion was carried out.

The second major reformulation of the topological recursion came in [3]. The Eynard, Orantin topological recursion, of (2.4.6), involves a residue calculation where the integrand is defined locally around the ramification points. In [3] they extend the differential so that it is well-defined globally, on Σ, as a contour surrounding all the ramification points. This will simplify our approach to proving the WKB-TR connection.

Finally, we'll be examining closely the reformulation of the topological recursion carried out in [2], in which the formulation was changed again, and the WKB-topological recursion connection was proved for all spectral curves of the form (2.2.3), subject to an admissibility connection we will define shortly. This class of spectral curves for which the WKB conjecture has been proven are genus 0 curves, with x and y meromorphic, and which satisfy the following definition, [2]:

Definition 2.14. A spectral curve is considered *admissible* if: A.) Its Newton polygon has no interior points, and B.) The curve is smooth at the point $(x, y) = (0, 0)$, if that point is on the curve $\{P(x, y) = 0\} \subset \mathbb{C}^2$.

The definition of the Newton polygon is given explicitly in [2][12], and here in Definition 4.1. The method by which the WKB-topological recursion connection is proved will be gone over in more detail in Section 4.

2.7 Connections to Other Fields

One of the factors that make the topological recursion such an interesting area of study is its connections to many, seemingly disparate, fields of mathematics and physics. It is beyond the scope of this paper to go into all of these connections in depth, as each of the connected fields are broad subjects in their own right. For a more in depth discussion of some of these connections see [7][12][14][20][24]. Here, we will address only a few of the connections.

The topological recursion originally emerged out of the field of Matrix Models, and we will discuss this more in this section. Later we will examine some examples that will demonstrate the connection between the topological recursion and Hurwitz numbers, which we will also discuss in this section.

2.7.1 Matrix Models

Roughly speaking random matrix models is the study of random Hermitian matrices of size N, and their statistical properties, under certain conditions. More precisely, we are interested in the expectation value of the eigenvalues, specifically in the limit where N gets very large. The probability density function for the eigenvalues will converge to a density function, which has compact support, and is therefore represented by an algebraic curve. From the data of the matrix model one can construct a spectral curve, although the method by which one does this is complicated; many of the objects we will introduce in this section are invariants of that spectral curve, and lead towards the formulation of the topological recursion.

We define the partition function as:

$$Z := \int_{H_N} d\mu(M) = \int_{H_N} dM e^{-S_H} \tag{2.7.1}$$

Where we are integrating over the space of Hermitian matrices of size N, and $d\mu$ is a family of measures depending on N. Then, $S_H = N\mathrm{Tr}V(M)$ (where $V(M)$ is a particular polynomial) can be viewed as the action of a zero-dimensional quantum gauge theory.

The partition function has a large N expansion of the form:

$$\log(Z) = \sum_{g=0}^{\infty} N^{2-2g} F_g \tag{2.7.2}$$

These F_g's can in turn be calculated entirely from the data of the spectral curve, which comes out of the data of the matrix models. These are connected to the free energies that were discussed in Definition 2.13; calculating them is the key to calculating the partition function in the large N limit. The key is that these objects are invariants of the spectral curve, they depend only on it.

Given the gauge invariant quantum field theory of (2.7.1), we would obviously like to calculate expectation values of gauge invariant operators. For example the expectation value of some general operator such as: $\langle \mathrm{Tr} M^{n_1} ... \mathrm{Tr} M^{n_k} \rangle$, with $n_1, ..., n_k \in \{0\} \cup \mathbb{Z}^+$. Given that, one can Taylor expand the following expectation value as:

$$\langle \mathrm{Tr}\frac{1}{x_1 - M} ... \mathrm{Tr}\frac{1}{x_k - M} \rangle = \sum_{n_1,...,n_k} \frac{1}{x_1^{n_1} ... x_n^{n_k}} \langle \mathrm{Tr} M^{n_1} ... \mathrm{Tr} M^{n_k} \rangle \qquad (2.7.3)$$

Where x is just a variable. In this view, we are interested in is the following objects:

$$\widehat{W}_n(x_1, ..., x_n) = N^{n-2} \langle \mathrm{Tr}\frac{1}{x_1 - M} ... \mathrm{Tr}\frac{1}{x_n - M} \rangle_c \qquad (2.7.4)$$

Since Taylor expanding these gives all the expectation values of our matrix model theory. N^{n-2} is a normalization factor, and c here stands for the connected diagrams only of the path integral. These are n-point correlation functions in our quantum field theory; what we're interested in is the asymptotic expansion of these $\widehat{W}_n$, for very large N. These objects have a topological expansion in $1/N^2$ written in terms of functions, which when multiplied by the appropriate differential, are precisely the $W_{g,n}$'s the topological recursion produces (hence the reason we also refer to the $W_{g,n}$'s as correlation functions). These objects also only depend on the geometry of the spectral curve.

We can also construct what's called the 'wavefunction', by examining another kind of expectation value, [13]:

$$\begin{aligned}
\langle \det(x-M) \rangle =& \langle e^{\mathrm{Tr}(\log(x-M))} \rangle \qquad (2.7.5)\\
=& \langle e^{\int_\infty^x \mathrm{Tr}(\frac{dx'}{x'-M})} \rangle \\
=& \left\langle \sum_n \frac{1}{n!} \int_\infty^x \dots \int_\infty^x \mathrm{Tr}\left(\frac{dx'_1}{x'_1-M}\right) \dots \mathrm{Tr}\left(\frac{dx'_n}{x'_n-M}\right) \right\rangle \\
=& \sum_n \frac{1}{n!} \int_\infty^x \dots \int_\infty^x \widehat{W}_n(x'_1, \dots, x'_n) dx'_1 \dots dx'_n \\
=& e^{\sum_n \frac{1}{n!} \int_\infty^x \dots \int_\infty^x W_n(x'_1,\dots,x'_n) dx'_1 \dots dx'_n} \\
\psi(x) :=& x^N e^{\int_\infty^x (W_1(x') - \frac{N}{x'}) dx'} e^{\sum_{n\geq 2} \frac{1}{n!} \int_\infty^x \dots \int_\infty^x W_n(x'_1,\dots,x'_n) dx'_1 \dots dx'_n}
\end{aligned}$$

The first line use the determinant-trace relation: for some matrix L, $\det(L) = e^{\mathrm{Tr}(\log(L))}$. In the last line we define the wavefunction, which is precisely what we will be using in (4.4.12). The wavefunction, ψ constructed above can be shown to (under certain conditions, and making a number of assumptions) satisfy a specific differential equation. It is the form of ψ above, and it's connection to the correlation functions $W_{g,n}$'s, that leads us to the WKB-topological recursion connection; what we are doing in this thesis, and in [2] is to establish that connection directly from the topological recursion.

The evolution of this subject led to the dropping of matrix models altogether, and the focusing on the relationship between the topological recursion itself and the spectral curve. Recall that the probability density function for our random matrices gives us an spectral curve, which in turn gives us the $W_{g,n}$'s. These correlation functions are invariants of our spectral curve. Not all spectral curves come from the asymptotic behaviour of random matrices though; therefore we are justified in dropping the matrix dependence altogether and focusing just on the topological recursion itself.

We stress that the study of matrix models does not provide a proof of the WKB-TR connection. The above is established only in certain cases of spectral curves, under a large set of assumptions. The conjecture, of the WKB-TR connection, is that this connection between the topological recursion and WKB is that it is a general one.

For more details on this subject see [12][13][14].

2.7.2 Hurwitz Numbers

Simple Hurwitz numbers, $H_{g,n}(\mu)$, are defined as the number of homotopy classes of, genus g, branched coverings of the Riemann sphere $\mathbb{CP}^1$, which have a specific profile of ramification points. The profile of the ramification points is that the branched covering has a branch point at $\infty \in \mathbb{CP}^1$ with a ramification index $\mu = (\mu_1 \geq \mu_2 \geq ... \geq \mu_n)$, and a finite number of other simple branch points. We already discussed branched coverings in Section

2.2.

Simple Hurwitz numbers can also be viewed in a group theoretic sense. If we label the sheets, and view them as a set, then the simple branch points are transpositions, t_i; according to the Riemann Hurwitz formula, there are $2g-2+n-|\mu|$ such points; here $|\mu| = \sum_{i=1}^{n} \mu_i$, the degree of the branched covering. The special branch point at infinity is an element, σ, in the conjugacy class given by μ, of the symmetry group of the set of labeled sheets. We also have the condition that $t_1 \cdot t_2 \cdot ... \cdot t_{2g-2+n-|\mu|} \cdot \sigma = 1$, since our branched covering represents a compact smooth surface. Then, $H_{g,n}(\mu)$ counts the number of such cases that satisfy the above conditions. Calculating these numbers in this way is non-trivial.

As an example we can consider $H_{0,1}(\mu)$, the number of ramified coverings of the Riemann sphere, with one fully ramified point, and $\mu - 1$ simple ramification points. The result is very simple, it is Caley's formula, $H_{0,1}(\mu) = \mu^{\mu-2}$, but proving this is non-trivial. Now, we can define the following objects:

$$W_{g,n}(x_1, ..., x_n) = \sum_{\mu \in \mathbb{Z}_+^n} \frac{H_{g,n}(\mu)}{(2g-2+n-|\mu|)!} \sum_{\sigma \in S_n} \prod_{i=1}^{n} e^{\mu_i x_{\sigma(i)}} \tag{2.7.6}$$

These are, once again, related to the $W_{g,n}$'s that are calculated using the topological recursion; now we see them emerge as a type of discrete Laplace transform of the Hurwitz numbers. Then, we have:

$$W_{0,1}(x) = \sum_{k=1}^{\infty} \frac{k^{k-2}}{(k-1)!} e^{kx} =: y(x) \tag{2.7.7}$$

Applying the Lagrange Inversion formula to the middle infinite sum we get the equation:

$$e^x = y(x)e^{-y(x)} \tag{2.7.8}$$

The spectral curve of (2.7.8) is studied in [7] from the point of view of Hurwitz numbers, and we will study it in the examples in Section 5, from the point of view of the results of this thesis.

We can also define r-spin Hurwitz numbers, $H^{(r)}_{g,n}(\mu)$, as counting the same, homology equivalent, number of branch coverings of the Riemann sphere, with a special point at infinity with ramification profile μ, and all other branch points having ramification profiles as $r+1$ cycles. We saw the resultant spectral curve, for r-spin Hurwitz numbers, in Example 2.9. These results are studied specifically in [20], and we will see an example of this in Section 5.

For more details on Hurwitz numbers see [7][12][20][16].

3 Extension of WKB-TR Connection

3.1 Set Up

As we've seen in previous sections, the WKB-topological recursion connection has already been proven in a broad class of cases; in this section we will extend that to a broader class. We are extending the proof of the WKB-topological recursion connection to spectral curves that fit into the first class and second class of spectral curves, defined in Definitions 1.1 and 1.3 respectively. The class of spectral curves for which we are proving the connection is still subject to the admissibility condition of Definition 2.14.

3.2 First Class

As defined in Definition 1.1, the first class of spectral curves we prove an extension of the WKB-TR connection for spectral curves, (Σ, u, y), where Σ is a compact Riemann surface, y and $x = e^u$ are meromorphic functions on Σ, but u is a meromorphic function only on an open region of Σ.

A key feature of the topological recursion is that, for a spectral curve (Σ, x, y), it does not depend explicitly on the meromorphic function x; it depends instead on the differential dx; this has a direct impact when considering spectral curves in the first class. First, note that the first class of spectral curves is exactly equivalent to spectral curves of the form proven in [2], but with dx replaced by dx/x (the logarithmic differential).[6] Therefore, extend-

[6] In other words, spectral curves (Σ, x, y), with Σ a compact Riemann surface, x and y

ing the WKB-TR connection to the first class of curves involves the simple (but, non-trivial) modification of the proof of [2] such that $W_{0,1} = ydx/x$.

The proof, with the above modification, is carried out in Section 4, and follows [2] almost exactly. We find that the admissibility condition of Definition 2.14 still holds in our case, and this severely restricts the curves that we would like to study. This admissibility condition comes about because a key part of producing the quantum curve involves taking a limit, which is only possible if this condition holds.

We find that the modification to [2], replacing dx with dx/x, leads to a nearly identical result with only minor modifications. Namely, the quantum curve that annihilates our wavefunction is given in the key result, Theorem 4.34, which is nearly identical to the result of [2]; one must be careful though as there are subtle differences. One key difference between our result and that of [2] is that we are able to explicitly use simple zeroes of x as our choice of integration divisor, to simplify the result of 4.34. On the other hand we are not able to use, as choice of integration divisor, points in R; in [2] they could sometimes use this choice of divisor so long as the $W_{g,n}$'s did not have a pole there; in our case the $W_{g,n}$'s will always have poles at points in R.

3.3 Second Class

The bigger challenge arises when we consider the second class of spectral curves, Definition 1.3. This second class was defined as the spectral curves,

meromorphic on Σ. Now, instead of taking $W_{0,1} = ydx$ as in [2], we take $W_{0,1} = ydx/x$.

(Σ, u, y), where Σ is a compact Riemann surface, y is meromorphic on Σ, $u = \log(f(z)) + p(z)$ (where $f(z)$ and $p(z)$ are polynomials) is meromorphic on an open region of Σ.

The spectral curves given by the above definition are not given by the zero locus of a polynomial (we can see that, for example, in equation (2.9)); but, $W_{0,1} = y \cdot du$ is still meromorphic on the Σ. Because this spectral curve is not given by a polynomial in x and y, we cannot proceed using the methods developed so far, in this thesis and in [2]. We therefore present a different method for dealing with these curves, involving a limit of spectral curves which converge to our desired one, and where we can evaluate each curve in the sequence.

If $T[S]$ is the topological recursion applied to some spectral curve, S as given in Definition 1.3, and we have that $\lim_{r\to\infty} S_r = S$, where all S_r are spectral curves falling into the class given by Definition 1.1, then the hypothesis of this section is that:

$$T[S] = T[\lim_{r\to\infty} S_r] = \lim_{r\to\infty} T[S_r] \tag{3.3.1}$$

Said more precisely, given some r-indexed sequence of spectral curves, S_r, each of which fall into the first class of spectral curves given in Definition 1.1, we can generate an infinite set of $W_{g,n}^{(r)}$ objects, which can be assembled into a wavefunction that is precisely annihilated by the quantum curve; the (unproven) hypothesis of this section is that the $W_{g,n}$'s generated by the limit of this sequence of spectral curves, S, is the limit of the set of $W_{g,n}^{(r)}$'s generated by the spectral curves in this sequence.

Diagrammatically:

$$\begin{array}{ccccc} S_r & \longrightarrow & \{W_{g,n}^{(r)}\}_{g,n=0}^{\infty} & \longrightarrow & (QuantumCurve)_r \\ \Big\downarrow{\scriptstyle r\to\infty} & & & & \Big\downarrow{\scriptstyle r\to\infty} \\ S & \longrightarrow & \{W_{g,n}\}_{g,n=0}^{\infty} & \dashrightarrow & (QuantumCurve) \end{array}$$

Where the dotted arrow represents that there is not currently a general way of constructing the quantum curve directly. The arrow from $(QuantumCurve)_r$ to $(QuantumCurve)$ is the basis of this section. This process, if true, would allow us to use the methods of Section 3.2, on spectral curves that fall into the, previously defined, second class. In this case, what we end up with is an infinite order differential equation; and the second question is whether complications arise in going from finite order to infinite order differential equations.

A key point here is that, as of this writing, the approach outlined in this section has not been proven; a concrete proof is needed to fully justify this approach. That being said, the approach described in this section, for dealing with the second class of spectral curves, does appear to work in all the examples considered; we will look at examples that can be evaluated in Section 5.

4 Proof of First Class

4.1 Defining Terms

A key feature of the topological recursion is that the meromorphic function x does not appear explicitly in the recursion, the differential dx appears only. This guides us in what approach we should use when extending the approach of [2] to curves that fall into the first class, of Definition 1.1. The idea is straightforward enough: We simply modify the proof of [2], replacing dx with the logarithmic differential, dx/x, and consider the implications of that change.

In the analysis that follows in this section we will follow the method of [2] very closely, we will state most of the definitions, theorems, etc, verbatim from that paper, with the alterations made clear; we will omit a lot of the more detailed analysis, referring the reader to other sources if they require more clarity.

If we rewrite (2.2.1) as follows:

$$P(x, y) = \sum_{(i,j)\in A} \alpha_{i,j} x^i y^j = 0 \tag{4.1.1}$$

Then, we can define the Newton Polygon as (Definition 2.3 in [2]):

Definition 4.1. The *Newton Polygon* Δ is the convex hull of the set A.

And we can define the objects, for $m = 0, ..., r$:

$$\alpha_m = \inf\{a|(a,m) \in \Delta\}, \qquad \beta_m = \sup\{a|(a,m) \in \Delta\} \tag{4.1.2}$$

Then we have that the number of *interior points* is given by:

$$\text{Number of interior points of } \Delta = \sum_{i=1}^{r-1}(\lceil\beta_i\rceil - \lfloor\alpha_i\rfloor - 1) \tag{4.1.3}$$

Which should clarify Definition 2.14, which will still hold in the cases we study here.

It is also useful to define the following:

$$P_n(x,y) := \sum_{k=1}^{n-1} p_{n-1-k}(x)y^k, \qquad n = 2, ..., r \tag{4.1.4}$$

Where the $p_n(x)$'s are coming from equation (2.2.3) for a degree r spectral curve.

If we write the divisor of a meromorphic function f as $div(f)$, and $div_0(f)$ as the divisor of zeroes, and $div_\infty(f)$ as the divisor of poles, then we have the following:

Lemma 4.2. For $m = 2, ..., r$,

$$div(P_m) \geq \alpha_{r-m+1}\mathrm{div}_0(x) - \beta_{r-m+1}\mathrm{div}_\infty(x) \tag{4.1.5}$$

This is Lemma 2.5 in [2]. The proof is in [1].

We still consider the set R to be as defined in Definition 2.7, as the zeroes of dx and poles of x of order 2 or greater.

We can define the following, useful:

Definition 4.3.

$$\tau(z) = \pi^{-1}(\pi(z)), \qquad \tau'(z) = \tau(z)\backslash\{z\} \tag{4.1.6}$$

$\tau(z)$ is a map that takes a point in Σ and maps it to the set of pre-images (see branched coverings in Section 2.2). $\pi : \Sigma \to \mathbb{CP}^1$ is the branched covering, that in our case is given by x.

We can also define the following, which are Definitions 2.11 and 2.12 in [2], but for genus 0 curves:

Definition 4.4.

$$w^{a-b}(z) := dz\left(\frac{1}{z-a} - \frac{1}{z-b}\right) \tag{4.1.7}$$

Where a and b are arbitrary points.

$$B(z_1, z_2) := \frac{dz_1 dz_2}{(z_1 - z_2)^2} \tag{4.1.8}$$

We also define the initial conditions for the recursion:

$$W_{0,1}(z) := y(z)\frac{dx(z)}{x(z)}, \qquad W_{0,2}(z_1, z_2) := B(z_1, z_2) \tag{4.1.9}$$

Noting the change made to $W_{0,1}$.

We can also define the following useful notation:

Definition 4.5. $A \subseteq_k B$ if $A \subseteq B$ and $|A| = k$.

Definition 4.6. $S(\mathbf{t})$ is the set of set partitions of ensemble $\mathbf{t}$

4.2 The Topological Recursion

We will now build towards a useful reformulation of the topological recursion, as stated in Definition 2.12. First, we define (Definition 3.4 in [2]):

Definition 4.7. Given the $W_{g,n+1}$'s, meromorphic differentials on Σ^n, with $g \geq 0$, $n \geq 0$ and $k \geq 1$. Also, $\mathbf{t} = \{t_1, \ldots, t_k\}$ and $\mathbf{z} = \{z_1, \ldots, z_n\}$. Define:

$$\mathcal{R}^{(k)} W_{g,n+1}(\mathbf{t};\mathbf{z}) := \sum_{\mu \in \mathcal{S}(\mathbf{t})} \sum_{\uplus_{i=1}^{\ell(\mu)} J_i = \mathbf{z}} \sideset{}{'}\sum_{\sum_{i=1}^{\ell(\mu)} g_i = g + \ell(\mu) - k} \left(\prod_{i=1}^{\ell(\mu)} W_{g_i, |\mu_i| + |J_i|}(\mu_i, J_i) \right) \tag{4.2.1}$$

The prime over the third sum means that we exclude all terms that have contributions from $W_{0,1}$. $\uplus$ means the disjoint union of sets. Also:

$$\mathcal{R}^{(0)} W_{g,n+1}(\mathbf{z}) := \delta_{g,0}\delta_{n,0}, \tag{4.2.2}$$

Now, we can give a rewritten version of the topological recursion (Definition 3.6 in [2]):

Definition 4.8. Take a spectral curve as defined previously, (Σ, x, y), and $\pi : \Sigma \to \mathbb{CP}^1$ degree r branched covering given by x, and the definitions 4.4. Also, let $\mathbf{z} = \{z_1, \ldots, z_n\} \in \Sigma^n$, $n \geq 0$, $g \geq 0$ and $2g - 2 + n \geq 0$. We can construct the unique symmetric meromorphic differentials $W_{g,n}$ on Σ^n with poles along R as follows:

$$W_{g,n+1}(z_0, \mathbf{z}) = \sum_{a \in R} \operatorname*{Res}_{z=a} \left(\sum_{k=1}^{r-1} \sum_{\beta(z) \subseteq_k \tau'(z)} \frac{(-1)^{k+1} \omega^{z-\alpha}(z_0)}{E^{(k)}(z; \beta(z))} \mathcal{R}^{(k+1)} W_{g,n+1}(z, \beta(z); \mathbf{z}) \right), \tag{4.2.3}$$

Where:

$$E^{(k)}(z; t_1, \ldots, t_k) = \prod_{i=1}^{k} (W_{0,1}(z) - W_{0,1}(t_i)). \tag{4.2.4}$$

α here is an arbitrary base point on Σ, that doesn't effect the result.

Given that statement of the topological recursion, we will now reformulate it into a form better suited for our purposes. This begins with the following definition.

Definition 4.9. Given the $W_{g,n+1}$'s, meromorphic differentials on Σ^n, with $g \geq 0$, $n \geq 0$ and $k \geq 1$. Also, $\mathbf{t} = \{t_1, \ldots, t_k\}$ and $\mathbf{z} = \{z_1, \ldots, z_n\}$. Define:

$$\mathcal{E}^{(k)} W_{g,n+1}(\mathbf{t}; \mathbf{z}) := \sum_{\mu \in \mathcal{S}(\mathbf{t})} \sum_{\uplus_{i=1}^{\ell(\mu)} J_i = \mathbf{z}} \sum_{\sum_{i=1}^{\ell(\mu)} g_i = g + \ell(\mu) - k} \left(\prod_{i=1}^{\ell(\mu)} W_{g_i, |\mu_i| + |J_i|}(\mu_i, J_i) \right) \tag{4.2.5}$$

Also:

$$\mathcal{E}^{(0)} W_{g,n+1}(\mathbf{z}) = \delta_{g,0} \delta_{n,0}, \tag{4.2.6}$$

The only difference between this and Definition 4.7 is that we have removed the prime over the third sum.

Then we have the following lemmas (Lemmas 3.12-3.14 in [2]):

Lemma 4.10. For all $g, n \geq 0$ and $k \geq 1$,

$$\begin{aligned}\mathcal{R}^{(k)}W_{g,n+1}(\mathbf{t};\mathbf{z}) &= \mathcal{R}^{(k-1)}W_{g-1,n+2}(\mathbf{t}\backslash\{t_k\};\mathbf{z},t_k)\\ &+\sum_{J_1\uplus J_2=\mathbf{z}}\sideset{}{'}\sum_{g_1+g_2=g}\left(\mathcal{R}^{(k-1)}W_{g_1,|J_1|+1}(\mathbf{t}\backslash\{t_k\};J_1)\right)W_{g_2,|J_2|+1}(t_k,J_2).\end{aligned} \tag{4.2.7}$$

The prime over the summation means that we exclude the case $(g_2, J_2) = (0, \emptyset)$.

Similarly, we have:

Lemma 4.11. For all $g, n, k \geq 0$,

$$\begin{aligned}\mathcal{E}^{(k)}W_{g,n+1}(\mathbf{t};\mathbf{z}) &= \mathcal{E}^{(k-1)}W_{g-1,n+2}(\mathbf{t}\backslash\{t_k\};\mathbf{z},t_k)\\ &+\sum_{J_1\uplus J_2=\mathbf{z}}\sum_{g_1+g_2=g}\left(\mathcal{E}^{(k-1)}W_{g_1,|J_1|+1}(\mathbf{t}\backslash\{t_k\};J_1)\right)W_{g_2,|J_2|+1}(t_k,J_2).\end{aligned} \tag{4.2.8}$$

Then we obtain the following relationship between $\mathcal{R}^{(k)}W_{g,n+1}(\mathbf{t};\mathbf{z})$ and $\mathcal{E}^{(k)}W_{g,n+1}(\mathbf{t};\mathbf{z})$:

Lemma 4.12. For all $g, n, k \geq 0$,

$$\begin{aligned}\mathcal{E}^{(k)} W_{g,n+1}(\mathbf{t};\mathbf{z}) =& \mathcal{R}^{(k)} W_{g,n+1}(\mathbf{t};\mathbf{z}) + \sum_{i=1}^{k} \sum_{\beta \subseteq_i \mathbf{t}} \mathcal{E}^{(i)} W_{0,1}(\beta) \mathcal{R}^{(k-i)} W_{g,n+1}(\mathbf{t} \setminus \beta; \mathbf{z}) \\ =& \sum_{i=0}^{k} \sum_{\beta \subseteq_i \mathbf{t}} \mathcal{E}^{(i)} W_{0,1}(\beta) \mathcal{R}^{(k-i)} W_{g,n+1}(\mathbf{t} \setminus \beta; \mathbf{z}), \qquad (4.2.9)\end{aligned}$$

The proof of these lemmas is exactly as in [2], as they do not depend on the form of $W_{0,1}$, and that is the only thing we have changed so far.

Definition 4.13. For $g, n, k \geq 0$:

$$Q^{(k)}_{g,n+1}(z;\mathbf{z}) = \sum_{\beta(z) \subseteq_k \tau(z)} \mathcal{E}^{(k)} W_{g,n+1}(\beta(z);\mathbf{z}). \qquad (4.2.10)$$

Also, note that, $\forall (g, n)$:

$$Q^{(k)}_{g,n+1}(z;\mathbf{z}) = 0, \qquad \text{for all } k > r. \qquad (4.2.11)$$

This is Definition 3.15 in [2].

Then,

$$Q^{(0)}_{g,n+1}(\mathbf{z}) = \delta_{g,0}\delta_{n,0}, \qquad (4.2.12)$$

Follows by definition.

For the $Q^{(k)}_{0,1}(z)$, by definition, we have:

$$Q^{(k)}_{0,1}(z) = \sum_{\beta(z) \subseteq_k \tau(z)} \prod_{i=1}^{k} W_{0,1}(\beta_i(z)), \qquad (4.2.13)$$

Which will be useful later. This implies that:

$$Q^{(k)}_{0,1}(z) = (-1)^k \frac{p_k(z)}{p_0(z)} \frac{dx(z)^k}{x(z)^k} \tag{4.2.14}$$

This refers to Example 3.19 in [2], but is modified by our change to $W_{0,1}$. The p_k here are the terms from (2.2.3). Note our change in notation though, where $p_i(z) := p_i(x(z))$, to simplify things.

This leads to the following, Lemma 3.21 in [2]:

Lemma 4.14. For $2g - 2 + n \geq 0$,

$$Q^{(1)}_{g,n+1}(z; \mathbf{z}) = 0. \tag{4.2.15}$$

And:

$$Q^{(1)}_{0,1}(z) = -\frac{p_1(z)}{p_0(z)} \frac{dx(z)}{x(z)}, \tag{4.2.16}$$

$$Q^{(1)}_{0,2}(z; z_1) = \pi^* B(x, x_1) = \frac{dx(z)dx(z_1)}{(x(z) - x(z_1))^2}, \tag{4.2.17}$$

Proof. The proof is trivial from above. ■

Now, we can state our reformulation of the topological recursion, this will be Theorem 3.22 in [2]:

Theorem 4.15. *The topological recursion as stated in 4.8 is equivalent to the following, for* $2g - 2 + n \geq 0$*:*

$$0 = \sum_{a \in R} \underset{z=a}{Res} \left(\omega^{z-\alpha}(z_0) Q_{g,n+1}(z; \mathbf{z}) \right), \tag{4.2.18}$$

where $Q_{g,n+1}(z;\mathbf{z})$ *is defined:*

$$Q_{g,n+1}(z;\mathbf{z}) := \frac{dx(z)}{x(z)\frac{\partial P}{\partial y}(z)}\left(p_0(z)\sum_{k=1}^{r}(-1)^k y(z)^{r-k}\frac{Q^{(k)}_{g,n+1}(z;\mathbf{z})x(z)^k}{dx(z)^k}\right), \tag{4.2.19}$$

with $P(x,y)=0$ *as defined in* (2.2.3).

Proof. The proof follows almost exactly as in [2], this is because up until now we have only changed $W_{0,1}$, and most of the proof does not depend on this change.

Starting with Definition 4.8:

$$W_{g,n+1}(z_0,\mathbf{z}) = \sum_{a\in R}\operatorname*{Res}_{z=a}\left(\sum_{k=1}^{r-1}\sum_{\beta(z)\subseteq_k\tau'(z)}\frac{(-1)^{k+1}\omega^{z-\alpha}(z_0)}{E^{(k)}(z;\beta(z))}\mathcal{R}^{(k+1)}W_{g,n+1}(z,\beta(z);\mathbf{z})\right). \tag{4.2.20}$$

We can put all the terms on a common denominator:

$$W_{g,n+1}(z_0,\mathbf{z}) = \sum_{a\in R}\operatorname*{Res}_{z=a}\left(\frac{p_0(z)\omega^{z-\alpha}(z_0)x(z)^{r-1}}{\frac{\partial P}{\partial y}(z)dx(z)^{r-1}}\right.$$
$$\left.\times\sum_{\alpha(z)\uplus\beta(z)=\tau'(z)}(-1)^{|\beta|+1}E^{(|\alpha|)}(z;\alpha(z))\mathcal{R}^{(|\beta|+1)}W_{g,n+1}(z,\beta(z);\mathbf{z})\right) \tag{4.2.21}$$

Where we are using the fact that $E^{(r-1)}(z,\tau'(z)) = \prod_{i=1}^{r-1}(W_{0,1}(z)-W_{0,1}(\tau'_i(z))) = \frac{dx(z)^{r-1}}{x(z)^{r-1}}\prod_{i=1}^{r-1}(y(z)-y(\tau'_i(z))) = \frac{dx(z)^{r-1}}{x(z)^{r-1}p_0(z)}\frac{\partial P}{\partial y}(z)$.

This we can replace, by the argument given in [2], with:

$$W_{g,n+1}(z_0,\mathbf{z}) = \sum_{a\in R} \operatorname*{Res}_{z=a} \left(\frac{p_0(z)\omega^{z-\alpha}(z_0)x(z)^{r-1}}{\frac{\partial P}{\partial y}(z)dx(z)^{r-1}} \right.$$
$$\left. \times \sum_{\substack{\alpha(z)\uplus\beta(z)=\tau(z)\\ |\beta(z)|\geq 2}} (-1)^{|\beta|} E^{(|\alpha|)}(z;\alpha(z))\mathcal{R}^{(|\beta|)} W_{g,n+1}(\beta(z);\mathbf{z}) \right). \quad (4.2.22)$$

Now, we bring the LHS to the RHS, so that the terms with $|\beta(z)| = 1$ are included in the sum:

$$\begin{aligned}
W_{g,n+1}(z_0,\mathbf{z}) &= -\operatorname*{Res}_{z=z_0} \omega^{z-\alpha}(z_0) W_{g,n+1}(z,\mathbf{z}) \\
&= \sum_{a\in R} \operatorname*{Res}_{z=a} \omega^{z-\alpha}(z_0) W_{g,n+1}(z,\mathbf{z}) \\
&\qquad + \frac{1}{2\pi i} \sum_{i=1}^{\hat{g}} \Big(\oint_{z\in A_i} B(z,z_0) \oint_{z\in B_i} W_{g,n+1}(z,\mathbf{z}) \\
&\qquad - \oint_{z\in B_i} B(z,z_0) \oint_{z\in A_i} W_{g,n+1}(z,\mathbf{z}) \Big) \\
&= \sum_{a\in R} \operatorname*{Res}_{z=a} \left(\omega^{z-\alpha}(z_0) W_{g,n+1}(z,\mathbf{z}) \right) \\
&= \sum_{a\in R} \operatorname*{Res}_{z=a} \left(\frac{p_0(z)\omega^{z-\alpha}(z_0)x(z)^{r-1}}{\frac{\partial P}{\partial y}(z)dx(z)^{r-1}} E^{(r-1)}(z;\tau'(z)) W_{g,n+1}(z,\mathbf{z}) \right) \\
&= \sum_{a\in R} \operatorname*{Res}_{z=a} \left(\frac{p_0(z)\omega^{z-\alpha}(z_0)x(z)^{r-1}}{\frac{\partial P}{\partial y}(z)dx(z)^{r-1}} \right. \\
&\qquad \left. \times \sum_{\beta(z)\subset_1\tau(z)} E^{(r-1)}(z;\tau(z)\setminus\beta(z)) W_{g,n+1}(\beta(z),\mathbf{z}) \right)
\end{aligned} \quad (4.2.23)$$

In the second line the Riemann's bilinear identity was used to pick up residues at the other poles. $B(z, z_0)$ and $W_{g,n}$ are normalized on A-cycles and thus the integrals vanish.

So that, from (4.2.22), we get:

$$0 = \sum_{a \in R} \operatorname*{Res}_{z=a} \left(\frac{p_0(z)\omega^{z-\alpha}(z_0)x(z)^{r-1}}{\frac{\partial P}{\partial y}(z)dx(z)^{r-1}} \right.$$
$$\left. \times \sum_{\alpha(z) \uplus \beta(z) = \tau(z)} (-1)^{|\beta|} E^{(|\alpha|)}(z; \alpha(z)) \mathcal{R}^{(|\beta|)} W_{g,n+1}(\beta(z); \mathbf{z}) \right), \quad (4.2.24)$$

Where, now subsets $\beta(z) \subset_1 \tau(z)$ are included in the sum.

Now, recall:

$$E^{(|\alpha|)}(z; \alpha(z)) = \prod_{i=1}^{|\alpha|} (W_{0,1}(z) - W_{0,1}(\alpha_i(z)))$$
$$= \sum_{j=0}^{|\alpha|} (-1)^j \, W_{0,1}(z)^{|\alpha|-j} \sum_{\gamma(z) \subset_j \alpha(z)} \mathcal{E}^{(j)} W_{0,1}(\gamma(z)) \qquad (4.2.25)$$

Which follows just by definition.

We want to collect terms in the second sum of (4.2.24) by order in $W_{0,1}(z)$. We have the following: $j = |\gamma|$ and $k = |\beta| + |\gamma|$, we have $|\alpha| - j = r - |\beta| -$

$|\gamma| = r - k$. This gives us:

$$\begin{aligned}
&\sum_{\alpha(z)\uplus\beta(z)=\tau(z)} (-1)^{|\beta|} E^{(|\alpha|)}(z;\alpha(z))\mathcal{R}^{(|\beta|)} W_{g,n+1}(\beta(z);\mathbf{z}) \\
&= \sum_k (-1)^k\, W_{0,1}(z)^{r-k} \sum_{\gamma(z)\uplus\beta(z)\subset_k\tau(z)} \mathcal{E}^{(|\gamma|)} W_{0,1}(\gamma(z))\mathcal{R}^{(|\beta|)} W_{g,n+1}(\beta(z);\mathbf{z}) \\
&= \sum_k (-1)^k\, W_{0,1}(z)^{r-k} \sum_{\alpha\subset_k\tau(z)} \mathcal{E}^{(k)} W_{g,n+1}(\alpha(z);\mathbf{z}) \\
&= \sum_k (-1)^k\, W_{0,1}(z)^{r-k}\, Q^{(k)}_{g,n+1}(z;\mathbf{z}) \\
&= \frac{\frac{\partial P}{\partial y}(z)dx(z)^{r-1}}{x(z)^{r-1}p_0(z)}\, Q_{g,n+1}(z;\mathbf{z})
\end{aligned} \tag{4.2.26}$$

The second equality comes from Lemma 4.12. And that proves the Theorem.

■

4.3 Pole Analysis

Now we will try to get rid of the residue in Theorem 4.15. It is useful to define the following objects, Definition 4.1 in [2]:

Definition 4.16.

$$U^{(k)}_{g,n+1}(z;\mathbf{z}) = \sum_{\beta(z)\subseteq_k\tau'(z)} \mathcal{E}^{(k)} W_{g,n+1}(\beta(z);\mathbf{z}). \tag{4.3.1}$$

And, for $k = 0$, $g \geq 0$ and $n \geq 0$:

$$U^{(0)}_{g,n+1}(\mathbf{z}) = \delta_{g,0}\delta_{n,0}. \tag{4.3.2}$$

We have that for all admissible (g, n):

$$U^{(k)}_{g,n+1}(z;\mathbf{z}) = 0, \qquad \text{for all } k \geq r. \tag{4.3.3}$$

Again, just from definition, we have that:

$$\frac{\partial P}{\partial y}(z) = p_0(z) \sum_{k=0}^{r-1} (-1)^k y(z)^{r-1-k} \frac{U^{(k)}_{0,1}(z) x(z)^k}{dx(z)^k} \tag{4.3.4}$$

Or, rearranging:

$$p_0(z) U^{(m)}_{0,1}(z) = (-1)^m \frac{dx(z)^m}{x(z)^m} \sum_{k=0}^{m} p_{m-k}(z) y(z)^k \tag{4.3.5}$$

$$= (-1)^m \frac{dx(z)^m}{x(z)^m} (P_{m+1}(x(z), y(z)) + p_m(z)) \tag{4.3.6}$$

Using (4.1.4).

Then, we have:

Lemma 4.17. For all $g, n, k \geq 0$,

$$\begin{aligned} Q^{(k)}_{g,n+1}(z;\mathbf{z}) =& U^{(k)}_{g,n+1}(z;\mathbf{z}) + U^{(k-1)}_{g-1,n+2}(z;\mathbf{z}, z) \\ &+ \sum_{J_1 \uplus J_2 = \mathbf{z}} \sum_{g_1+g_2=g} U^{(k-1)}_{g_1,|J_1|+1}(z; J_1) W_{g_2,|J_2|+1}(z, J_2). \end{aligned} \tag{4.3.7}$$

Whose proof is exactly as in [2] (Lemma 4.5 in that paper).

The next Corollary is Corollary 4.6 in [2], only the fourth term has changed due to the changes we made previously:

Corollary 4.18. For all $g, n, k \geq 0$,

$$
\begin{aligned}
Q^{(k)}_{g,n+1}(z;\mathbf{z}) &= U^{(k)}_{g,n+1}(z;\mathbf{z}) + U^{(k-1)}_{g-1,n+2}(z;\mathbf{z},z) \\
&\quad - \sum_{J_1 \uplus J_2 = \mathbf{z}} \sum_{g_1+g_2=g} U^{(k-1)}_{g_1,|J_1|+1}(z;J_1) U^{(1)}_{g_2,|J_2|+1}(z,J_2) \\
&- \frac{p_1(z)}{p_0(z)} \frac{dx(z)}{x(z)} U^{(k-1)}_{g,n+1}(z;\mathbf{z}) + \sum_{i=1}^{n} \frac{dx(z)dx(z_i)}{(x(z)-x(z_i))^2} U^{(k-1)}_{g,n}(z;\mathbf{z} \setminus \{z_i\})
\end{aligned}
\tag{4.3.8}
$$

Proof. Follows from Lemma 4.17, Lemma 4.14 and:

$$
Q^{(1)}_{g,n+1}(z;\mathbf{z}) = W_{g,n+1}(z,\mathbf{z}) + U^{(1)}_{g,n+1}(z;\mathbf{z}). \tag{4.3.9}
$$

■

The following is Lemma 4.7 in [2], and the proof follows the same as in that paper with only minor modifications.

Lemma 4.19. For the topological recursion of Theorem 4.15, for $2g-2+n \geq 0$, the object,

$$
Q_{g,n+1}(z;\mathbf{z}) = \frac{dx(z)}{\frac{\partial P}{\partial y}(z)x(z)} \left(p_0(z) \sum_{k=1}^{r} (-1)^k y(z)^{r-k} \frac{Q^{(k)}_{g,n+1}(z;\mathbf{z})x(z)^k}{dx(z)^k} \right) \tag{4.3.10}
$$

only has poles at the coinciding points, ie. at $z \in \tau(z_i)$, for $i = 1, \ldots, n$.

Proof. First, we show that it does not have poles on R. We begin with the statement of Theorem 4.15:

$$0 = \sum_{i=1}^{m} \operatorname*{Res}_{z=a_i} \omega^{z-\alpha}(z_0) Q_{g,n+1}(z;\mathbf{z}). \tag{4.3.11}$$

Assuming $Q_{g,n+1}(z;\mathbf{z})$ has a pole of order $m+1 \geq 1$ at the ramification point $a \in R$, then in locally near a we can write:

$$Q_{g,n+1}(z;\mathbf{z}) \sim \frac{dz}{(z-a)^{m+1}} S_{g,n+1}(\mathbf{z}) \; (1 + \mathcal{O}(z-a)) \tag{4.3.12}$$

Assuming $S_{g,n+1}(\mathbf{z}) \neq 0$. Recall, $Q_{g,n+1}(z;\mathbf{z})$ is a meromorphic 1-form in z. We can also Taylor expand $\omega^{z-\alpha}(z_0)$, at a, in the same local neighborhood:

$$\omega^{z-\alpha}(z_0) \sim \sum_{k=0}^{\infty} (z-a)^k \xi_{a,k}(z_0;\alpha) \tag{4.3.13}$$

$\xi_{a,k}(z_0;\alpha)$ being a meromorphic 1-form of z_0 analytical everywhere but at a, where it has a pole of order $k+1$ (this comes from the definition of $\omega^{z-\alpha}(z_0)$, which we skipped over in this paper, but is given more explicitly in [2]).

Again, in the same local neighborhood, we have the Laurent expansion:

$$\xi_{a,k}(z_0) \sim \frac{dz_0}{(z_0-a)^{k+1}} \; (1 + \mathcal{O}(z_0-a)) \tag{4.3.14}$$

We can write:

$$\begin{aligned} 0 =& \sum_{b\in R} \operatorname*{Res}_{z=b} \omega^{z-\alpha}(z_0)\, Q_{g,n+1}(z;\mathbf{z}) \\ =& \operatorname*{Res}_{z=a} \omega^{z-\alpha}(z_0)\, Q_{g,n+1}(z;\mathbf{z}) + \sum_{b\neq a} \operatorname*{Res}_{z=b} \omega^{z-\alpha}(z_0)\, Q_{g,n+1}(z;\mathbf{z}), \end{aligned} \tag{4.3.15}$$

The second term doesn't have poles at $z_0 \to a$, only the term with the residue at $z \to a$ does. So:

$$\begin{aligned} \text{terms holomorphic at } z_0 \to a =& \operatorname*{Res}_{z=a}\, \omega^{z-\alpha}(z_0)\, Q_{g,n+1}(z;\mathbf{z}) \\ =& S_{g,n+1}(\mathbf{z})\ \xi_{a,m}(z_0)(1+\mathcal{O}(z_0-a)) \end{aligned} \tag{4.3.16}$$

A contradiction if the RHS is not analytical at $z_0 \to a$. Therefore, $Q_{g,n+1}(z;\mathbf{z})$ cannot have poles on R.

Where else can $Q_{g,n+1}(z;\mathbf{z})$ have poles? From (4.3.10), we can see that the only poles can come from the coinciding points, and the *punctures* (the points $p \in \Sigma$ that are poles of x and y). The $W_{g,n}$'s, $(g,n) \neq (0,1),(0,2)$, only have poles on R. $W_{0,2}$ only give poles at the coinciding points, and $W_{0,1}$ could only give poles at the punctures. The argument that $Q_{g,n+1}(z;\mathbf{z})$ doesn't have poles at the punctures follows exactly the same as the argument that it doesn't have poles on R.

Therefore $Q_{g,n+1}(z,\mathbf{z})$ can only have poles at coinciding points. ■

The purpose behind the previous Lemma is that we are interested in the following object:

$$\frac{p_0(z)Q^{(m)}_{g,n+1}(z;\mathbf{z})x(z)^m}{dx(z)^m}, \qquad m=1,\dots,r \tag{4.3.17}$$

Which will play a key role in what follows.

First, the following Lemmas (Lemma 4.8 and 4.9 in [2]):

Lemma 4.20. For $m=0,\dots,r$,

$$\frac{p_0(z)Q^{(m)}_{0,1}(z)x(z)^m}{dx(z)^m} = (-1)^m p_m(z). \tag{4.3.18}$$

Proof. This is (4.2.14). ■

Lemma 4.21. For $m=1,\dots,r$,

$$\begin{aligned}\frac{p_0(z)Q^{(m)}_{0,2}(z;z_1)x(z)^m}{x(z)^{\lfloor\alpha_{r-m+1}\rfloor}dx(z)^m} = d_{z_1}\Bigg(\frac{x(z)}{x(z)-x(z_1)}\Bigg(&\frac{U^{(m-1)}_{0,1}(z_1)x(z_1)^{m-1}}{dx(z_1)^{m-1}}\frac{p_0(z_1)}{x(z_1)^{\lfloor\alpha_{r-m+1}\rfloor}}\\ &+(-1)^{m-1}x(z)\left(\frac{p_{m-1}(z)}{x(z)^{\lfloor\alpha_{r-m+1}\rfloor}}-\frac{p_{m-1}(z_1)}{x(z_1)^{\lfloor\alpha_{r-m+1}\rfloor}}\right)\Bigg)\Bigg)\end{aligned} \tag{4.3.19}$$

α_m is defined in (4.1.2).

Note the differences here between this result and Lemma 4.9 of [2]. Here, as well as the $x(z_1)^{m-1}$ terms that arise out of our original change, we also have the appearance of $x(z)$ terms in the first and second line.

Proof. This proof follows almost exactly as in [2].

For $m=1$ is trivial, $U^{(0)}_{0,1}=1$ and $Q^{(1)}_{0,2}(z;z_1)=\frac{dx(z)dx(z_1)}{(x(z)-x(z_1))^2}$.

For $m = 2, ..., r$, we get:

$$\begin{aligned}\frac{p_0(z)Q_{0,2}^{(m)}(z)x(z)^m}{dx(z)^m} =& \sum_{k=0}^{r-1} \frac{B(\tau_k(z), z_1)x(z)}{dx(z)} \frac{U_{0,1}^{(m-1)}(\tau_k(z))p_0(z)x(z)^{m-1}}{dx(z)^{m-1}} \\ =& (-1)^{m-1} \left(\sum_{k=0}^{r-1} \frac{B(\tau_k(z), z_1)x(z)}{dx(z)} P_m(x(z), y(\tau_k(z))) \right. \\ & \left. + \sum_{k=0}^{r-1} \frac{B(\tau_k(z), z_1)x(z)}{dx(z)} p_{m-1}(z) \right) \end{aligned} \tag{4.3.20}$$

With the first equality coming by definition, and the second equality coming from (4.3.6).

We evaluate the second term by recognizing that:

$$\sum_{k=0}^{r-1} \frac{B(\tau_k(z), z_1)}{dx(z)} = \frac{dx(z_1)}{(x(z) - x(z_1))^2} \tag{4.3.21}$$

Therefore:

$$\begin{aligned}\sum_{k=0}^{r-1} \frac{B(\tau_k(z), z_1)x(z)}{dx(z)} p_{m-1}(z) =& p_{m-1}(z)x(z) \frac{dx(z_1)}{(x(z) - x(z_1))^2} \\ =& p_{m-1}(z)\, x(z)\, d_{z_1} \left(\frac{1}{x(z) - x(z_1)} \right) \end{aligned} \tag{4.3.22}$$

Then, the first term of (4.3.20) becomes:

$$\begin{aligned}\sum_{k=0}^{r-1}\frac{B(\tau_k(z),z_1)x(z)}{dx(z)}P_m(x(z),y(\tau_k(z)))\\ &=\sum_{k=0}^{r-1}\mathrm{Res}_{z'=\tau_k(z)}\frac{B(z',z_1)x(z)}{x(z')-x(z)}P_m(x(z'),y(z'))\\ &=\sum_{k=0}^{r-1}\mathrm{Res}_{z'=\tau_k(z)}\frac{B(z',z_1)x(z)}{x(z')-x(z)}\frac{x(z)^{\lfloor\alpha_{r-m+1}\rfloor}P_m(x(z'),y(z'))}{x(z')^{\lfloor\alpha_{r-m+1}\rfloor}}\end{aligned} \tag{4.3.23}$$

With α_m defined in (4.1.2).

Then, we have Lemma 4.2, and that the spectral curves are admissible, so have no interior points. This implies that:

$$\begin{aligned}\mathrm{div}\left(\frac{P_m}{x(z)^{\lfloor\alpha_{r-m+1}\rfloor}}\right)&\geq(\alpha_{r-m+1}-\lfloor\alpha_{r-m+1}\rfloor)\mathrm{div}_0(x)-(\beta_{r-m+1}-\lceil\beta_{r-m+1}\rceil+1)\mathrm{div}_\infty(x)\\ &\geq-\mathrm{div}_\infty(x)\end{aligned} \tag{4.3.24}$$

Therefore:

$$\frac{B(z',z_1)x(z)}{x(z')-x(z)}\frac{x(z)^{\lfloor\alpha_{r-m+1}\rfloor}P_m(x(z'),y(z'))}{x(z')^{\lfloor\alpha_{r-m+1}\rfloor}} \tag{4.3.25}$$

Only has poles in z' at $z'=\tau_k(z)$ and $z'=z_1$ from the $B(z',z_1)x(z)$. Therefore, we can flip the residue calculation giving:

$$\begin{aligned}\sum_{k=0}^{r-1} \frac{B(\tau_k(z), z_1)x(z)}{dx(z)} P_m(x(z), y(\tau_k(z))) \\ &= \mathrm{Res}_{z'=\tau_k(z)} \frac{B(z', z_1)x(z)}{x(z)-x(z')} \frac{x(z)^{\lfloor \alpha_{r-m+1} \rfloor} P_m(x(z'), y(z'))}{x(z')^{\lfloor \alpha_{r-m+1} \rfloor}} \\ &= x(z)^{\lfloor \alpha_{r-m+1} \rfloor} d_{z_1} \left(\frac{x(z)}{x(z)-x(z_1)} \frac{P_m(x(z_1), y(z_1))}{x(z_1)^{\lfloor \alpha_{r-m+1} \rfloor}} \right)\end{aligned} \tag{4.3.26}$$

Giving us the final result:

$$\begin{aligned}&\frac{p_0(z) Q_{0,2}^{(m)}(z) x(z)^m}{dx(z)^m} \\ &= (-1)^{m-1} d_{z_1} \left(\frac{x(z)\, x(z)^{\lfloor \alpha_{r-m+1} \rfloor}}{x(z)-x(z_1)} \frac{P_m(x(z_1), y(z_1))}{x(z_1)^{\lfloor \alpha_{r-m+1} \rfloor}} + \frac{x(z)\, p_{m-1}(z)}{x(z)-x(z_1)} \right) \\ &= x(z)^{\lfloor \alpha_{r-m+1} \rfloor} \left[d_{z_1} \left(\frac{x(z)}{x(z)-x(z_1)} \left(\frac{U_{0,1}^{(m-1)}(z_1) x(z_1)^{m-1}}{dx(z_1)^{m-1}} \frac{p_0(z_1)}{x(z_1)^{\lfloor \alpha_{r-m+1} \rfloor}} \right.\right.\right. \\ &\qquad \left.\left.\left. + (-1)^{m-1} x(z) \left(\frac{p_{m-1}(z)}{x(z)^{\lfloor \alpha_{r-m+1} \rfloor}} - \frac{p_{m-1}(z_1)}{x(z_1)^{\lfloor \alpha_{r-m+1} \rfloor}} \right) \right) \right) \right]\end{aligned} \tag{4.3.27}$$

■

Now, we can turn to the $(g, n) \neq (0,0), (0,1)$.

The following Lemma is Lemma 4.10 in [2]:

Lemma 4.22. For the following r-differential:

$$Q(z) = dx(z)^r \sum_{k=1}^{r} (-1)^k y(z)^{r-k} \frac{Q_k(x(z))}{dx(z)^k} \tag{4.3.28}$$

With $Q_k(x(z))$ being k-differentials, pulled-back from the base. Then these must have the form (denoting $\tau(z) = \{\tau_0(z), \ldots, \tau_{r-1}(z)\}$):

$$Q_k(x(z)) = -p_0(z)dx(z)^k \sum_{i=0}^{r-1} \left(\frac{1}{\frac{\partial P}{\partial y}(\tau_i(z))} \frac{Q(\tau_i(z))}{dx(z)^r} \frac{U_{0,1}^{(k-1)}(\tau_i(z))x(z)^{k-1}}{dx(z)^{k-1}} \right). \tag{4.3.29}$$

Proof. Again, this proof follows as in [2], with only minor changes.

Denoting $\tau(z) = \{\tau_0(z), \ldots, \tau_{r-1}(z)\}$, and such that $z = \tau_0(z)$.

For any Y we have:

$$\begin{aligned} \frac{P(x(z), Y)}{Y - y(z)} =& p_0(z) \prod_{q \in \tau'(z)} (Y - y(q)) \\ =& p_0(z) \prod_{i=1}^{r-1} (Y - y(\tau_i(z))) \\ =& p_0(z) \sum_{k=0}^{r-1} (-1)^k Y^{r-1-k} \sum_{\beta \subset_k \tau'(z)} \prod_{q \in \beta} y(q) \\ =& p_0(z) \sum_{k=0}^{r-1} (-1)^k Y^{r-1-k} \frac{U_{0,1}^{(k)}(z)x(z)^k}{dx(z)^k} \end{aligned} \tag{4.3.30}$$

So, choosing $Y = y(z) = y(\tau_0(z))$ or $Y = y(\tau_i(z))$, $i \neq 0$, gives the result:

$$p_0(z) \sum_{k=0}^{r-1} (-1)^k y(z)^{r-k-1} \frac{U_{0,1}^{(k)}(\tau_i(z))}{dx(z)^k} = \frac{\partial P}{\partial y}(z)\delta_{i,0}. \tag{4.3.31}$$

Therefore:

$$
\begin{aligned}
-dx(z)^r \sum_{k=1}^{r}(-1)^k y(z)^{r-k} p_0(z) \sum_{i=0}^{r-1}&\left(\frac{1}{\frac{\partial P}{\partial y}(\tau_i(z))}\frac{Q(\tau_i(z))}{dx(z)^r}\frac{U_{0,1}^{(k-1)}(\tau_i(z))x(z)^{k-1}}{dx(z)^{k-1}}\right)\\
=&\sum_{i=0}^{r-1}\frac{Q(\tau_i(z))}{\frac{\partial P}{\partial y}(\tau_i(z))}p_0(z)\sum_{k=0}^{r-1}(-1)^k y(z)^{r-k-1}\frac{U_{0,1}^{(k)}(\tau_i(z))x(z)^k}{dx(z)^k}\\
=&\sum_{i=0}^{r-1}\frac{Q(\tau_i(z))}{\frac{\partial P}{\partial y}(\tau_i(z))}\frac{\partial P}{\partial y}(z)\delta_{i,0}\\
=&Q(z).
\end{aligned}
\tag{4.3.32}
$$

■

This gives us the following relation (exactly the same result as Corollary 4.11 in [2]):

Corollary 4.23. For $2g-2+n \geq 0$ and $m = 1, ..., r$,

$$Q_{g,n+1}^{(m)}(z;\mathbf{z}) = -\sum_{k=0}^{r-1}\left(Q_{g,n+1}(\tau_k(z);\mathbf{z})U_{0,1}^{(m-1)}(\tau_k(z))\right). \tag{4.3.33}$$

Proof. The proof is identical to that given in [2], with consideration given to the changes we have made so far. ■

This leads to the following, important, theorem (Theorem 4.12 in [2]):

Theorem 4.24. *For $2g-1+n \geq 0$, $m = 1, \ldots, r$:*

$$\frac{p_0(z)Q^{(m)}_{g,n+1}(z;\mathbf{z})x(z)^m}{x(z)^{\lfloor \alpha_{r-m+1} \rfloor}dx(z)^m} = \sum_{i=1}^{n} d_{z_i}\left(\frac{1}{x(z)-x(z_i)}\left(\frac{U^{(m-1)}_{g,n}(z_i;\mathbf{z}\setminus\{z_i\})x(z_i)^m}{dx(z_i)^{m-1}}\frac{p_0(z_i)}{x(z_i)^{\lfloor \alpha_{r-m+1} \rfloor}}\right)\right) \tag{4.3.34}$$

While, for $(g,n) = (0,1)$, $m = 1, \ldots, r$:

$$\frac{p_0(z)Q^{(m)}_{0,2}(z;z_1)x(z)^m}{x(z)^{\lfloor \alpha_{r-m+1} \rfloor}dx(z)^m} = d_{z_1}\left(\frac{x(z)}{x(z)-x(z_1)}\left(\frac{U^{(m-1)}_{0,1}(z_1)x(z_1)^{m-1}}{dx(z_1)^{m-1}}\frac{p_0(z_1)}{x(z_1)^{\lfloor \alpha_{r-m+1} \rfloor}} + (-1)^{m-1}x(z)\left(\frac{p_{m-1}(z)}{x(z)^{\lfloor \alpha_{r-m+1} \rfloor}} - \frac{p_{m-1}(z_1)}{x(z_1)^{\lfloor \alpha_{r-m+1} \rfloor}}\right)\right)\right) \tag{4.3.35}$$

And, with $(g,n) = (0,0)$, $m = 0, \ldots, r$:

$$\frac{p_0(z)Q^{(m)}_{0,1}(z)x(z)^m}{dx(z)^m} = (-1)^m p_m(z) \tag{4.3.36}$$

Proof. Again, of course, we follow the proof of [2] closely.

The $(g,n) = (0,0)$ and $(g,n) = (0,1)$ statements are just Lemmas 4.20 and 4.21.

Focusing on $2g-2+n \geq 0$. The case $m = 1$ gives just $0 = 0$, by definition.

Focusing on $m = 2, \dots, r$ then. By Corollary 4.23:

$$\begin{aligned}\frac{p_0(z)Q^{(m)}_{g,n+1}(z;\mathbf{z})x(z)^m}{dx(z)^m} &= -\sum_{k=0}^{r-1}\frac{Q_{g,n+1}(\tau_k(z);\mathbf{z})x(z)}{dx(z)}\frac{U^{(m-1)}_{0,1}(\tau_k(z))p_0(z)x(z)^{m-1}}{dx(z)^{m-1}}\\ &= (-1)^m\left(\sum_{k=0}^{r-1}\frac{Q_{g,n+1}(\tau_k(z);\mathbf{z})x(z)}{dx(z)}P_m(x(z),y(\tau_k(z)))\right.\\ &\qquad\left.+\sum_{k=0}^{r-1}\frac{Q_{g,n+1}(\tau_k(z);\mathbf{z})x(z)}{dx(z)}p_{m-1}(z)\right)\end{aligned} \tag{4.3.37}$$

Where the second line comes from (4.3.6).

The second term after the equality is zero, since:

$$\sum_{k=0}^{r-1}Q_{g,n+1}(\tau_k(z);\mathbf{z}) = -Q^{(1)}_{g,n+1}(z;\mathbf{z}) = 0 \tag{4.3.38}$$

Then, we have that:

$$\begin{aligned}&\sum_{k=0}^{r-1}\frac{Q_{g,n+1}(\tau_k(z);\mathbf{z})x(z)}{dx(z)}P_m(x(z),y(\tau_k(z)))\\ &\qquad= \sum_{k=0}^{r-1}\mathrm{Res}_{z'=\tau_k(z)}\frac{Q_{g,n+1}(z';\mathbf{z})x(z)}{x(z')-x(z)}P_m(x(z'),y(z'))\\ &\qquad= x(z)^{\lfloor\alpha_{r-m+1}\rfloor}\sum_{k=0}^{r-1}\mathrm{Res}_{z'=\tau_k(z)}\frac{Q_{g,n+1}(z';\mathbf{z})x(z)}{x(z')-x(z)}\frac{P_m(x(z'),y(z'))}{x(z')^{\lfloor\alpha_{r-m+1}\rfloor}}\end{aligned} \tag{4.3.39}$$

Then, by the argument used in the proof of 4.21 we can say that the only poles of the integrand are at $z' = \tau_k(z)$ and at the poles of $Q_{g,n+1}(z';z)$. We know from Lemma 4.19 that $Q_{g,n+1}(z';\mathbf{z})$ only has poles at $z' = \tau_k(z_j)$.

Therefore, (4.3.39) equals:

$$
\begin{aligned}
&- x(z)^{\lfloor \alpha_{r-m+1} \rfloor} \sum_{j=1}^{n} \sum_{k=0}^{r-1} \mathrm{Res}_{z'=\tau_k(z_j)} \frac{Q_{g,n+1}(z';\mathbf{z})x(z)}{x(z')-x(z)} \frac{P_m(x(z'),y(z'))}{x(z')^{\lfloor \alpha_{r-m+1} \rfloor}} \\
&= -x(z)^{\lfloor \alpha_{r-m+1} \rfloor} \sum_{j=1}^{n} \sum_{k=0}^{r-1} \mathrm{Res}_{z'=z_j} \frac{Q_{g,n+1}(\tau_k(z');\mathbf{z})x(z)}{x(z')-x(z)} \frac{P_m(x(z'),y(\tau_k(z')))}{x(z')^{\lfloor \alpha_{r-m+1} \rfloor}}
\end{aligned}
\tag{4.3.40}
$$

Then, from (4.3.37), we have:

$$
\begin{aligned}
&\frac{p_0(z)Q^{(m)}_{g,n+1}(z;\mathbf{z})x(z)^m}{dx(z)^m} \\
&\quad = x(z)^{\lfloor \alpha_{r-m+1} \rfloor}(-1)^{m-1} \sum_{j=1}^{n} \sum_{k=0}^{r-1} \operatorname*{Res}_{z'=z_j} \frac{Q_{g,n+1}(\tau_k(z');\mathbf{z})x(z)}{x(z')-x(z)} \frac{P_m(x(z'),y(\tau_k(z')))}{x(z')^{\lfloor \alpha_{r-m+1} \rfloor}} \\
&\quad = x(z)^{\lfloor \alpha_{r-m+1} \rfloor} \sum_{j=1}^{n} \sum_{k=0}^{r-1} \operatorname*{Res}_{z'=z_j} \frac{Q_{g,n+1}(\tau_k(z');\mathbf{z})x(z)}{x(z')-x(z)} \frac{U^{(m-1)}_{0,1}(\tau_k(z'))p_0(z')x^{m-1}(z')}{x(z')^{\lfloor \alpha_{r-m+1} \rfloor}dx(z')^{m-1}} \\
&\qquad + x(z)^{\lfloor \alpha_{r-m+1} \rfloor}(-1)^{m} \sum_{j=1}^{n} \sum_{k=0}^{r-1} \operatorname*{Res}_{z'=z_j} \frac{Q_{g,n+1}(\tau_k(z');\mathbf{z})x(z)}{x(z')-x(z)} \frac{p_{m-1}(z')}{x(z')^{\lfloor \alpha_{r-m+1} \rfloor}}
\end{aligned}
\tag{4.3.41}
$$

Using (4.3.6) again in the second equality. Again, the second term is zero by equation (4.3.38). So, we get:

$$\begin{aligned}
&\frac{p_0(z)Q^{(m)}_{g,n+1}(z;\mathbf{z})x(z)^m}{dx(z)^m}\\
&\quad= x(z)^{\lfloor\alpha_{r-m+1}\rfloor}\sum_{j=1}^{n}\sum_{k=0}^{r-1}\operatorname*{Res}_{z'=z_j}\frac{Q_{g,n+1}(\tau_k(z');\mathbf{z})x(z)}{x(z')-x(z)}\frac{U^{(m-1)}_{0,1}(\tau_k(z'))p_0(z')x^{m-1}(z')}{x(z')^{\lfloor\alpha_{r-m+1}\rfloor}dx(z')^{m-1}}\\
&\quad= x(z)^{\lfloor\alpha_{r-m+1}\rfloor}\sum_{j=1}^{n}\operatorname*{Res}_{z'=z_j}\frac{Q^{(m)}_{g,n+1}(z';\mathbf{z})p_0(z')\,x(z)x^{m-1}(z')}{(x(z')-x(z))x(z')^{\lfloor\alpha_{r-m+1}\rfloor}dx(z')^{m-1}}\\
&\quad= x(z)^{\lfloor\alpha_{r-m+1}\rfloor}\sum_{j=1}^{n}\operatorname*{Res}_{z'=z_j}\frac{B(z',z_j)U^{(m-1)}_{g,n}(z';\mathbf{z}\backslash\{z_j\})p_0(z')\,x(z)x^{m-1}(z')}{(x(z')-x(z))x(z')^{\lfloor\alpha_{r-m+1}\rfloor}dx(z')^{m-1}}\\
&\quad= x(z)^{\lfloor\alpha_{r-m+1}\rfloor}\sum_{j=1}^{n}d_{z_j}\left(\frac{U^{(m-1)}_{g,n}(z_j;\mathbf{z}\backslash\{z_j\})p_0(z_j)\,x(z)x^{m-1}(z_j)}{(x(z_j)-x(z))x(z_j)^{\lfloor\alpha_{r-m+1}\rfloor}dx(z_j)^{m-1}}\right)
\end{aligned} \tag{4.3.42}$$

■

The following is Lemma 4.13 in [2]:

Lemma 4.25.

$$
\begin{aligned}
&\frac{p_0(z)}{x(z)^{\lfloor \alpha_{r-m+1}\rfloor}} \frac{U^{(m)}_{g,n+1}(z;\mathbf{z})x(z)^m}{dx(z)^m} = -\frac{p_0(z)}{x(z)^{\lfloor \alpha_{r-m+1}\rfloor}} \frac{U^{(m-1)}_{g-1,n+2}(z;\mathbf{z},z)x(z)^{m-1}}{dx(z)^{m-1}} \frac{x(z)}{dx(z)} \\
&\quad + \frac{p_1(z)}{x(z)^{\lfloor \alpha_{r-m+1}\rfloor}} \frac{U^{(m-1)}_{g,n+1}(z;\mathbf{z})x(z)^{m-1}}{dx(z)^{m-1}} \\
&+ \frac{p_0(z)}{x(z)^{\lfloor \alpha_{r-m+1}\rfloor}} \sum_{J_1 \uplus J_2 = \mathbf{z}} \sum_{g_1+g_2=g} \frac{U^{(m-1)}_{g_1,|J_1|+1}(z;J_1)x(z)^{m-1}}{dx(z)^{m-1}} \frac{U^{(1)}_{g_2,|J_2|+1}(z;J_2)x(z)}{dx(z)} \\
&\quad - \sum_{i=1}^n \left[\frac{p_0(z)}{x(z)^{\lfloor \alpha_{r-m+1}\rfloor}} \frac{dx(z_i)}{(x(z)-x(z_i))^2} \frac{U^{(m-1)}_{g,n}(z;\mathbf{z}\setminus\{z_i\})x(z)^m}{dx(z)^{m-1}} \right. \\
&\quad \left. - d_{z_i}\left(\frac{p_0(z_i)}{x(z_i)^{\lfloor \alpha_{r-m+1}\rfloor}} \frac{x(z)}{x(z)-x(z_i)} \frac{U^{(m-1)}_{g,n}(z_i;\mathbf{z}\setminus\{z_i\})x(z_i)^{m-1}}{dx(z_i)^{m-1}} \right) \right] \\
&\quad + \delta_{g,0}\delta_{n,0}(-1)^m \frac{p_m(z)}{x(z)^{\lfloor \alpha_{r-m+1}\rfloor}} \\
&+ \delta_{g,0}\delta_{n,1}(-1)^{m-1}x(z)d_{z_1}\left(\frac{1}{x(z)-x(z_1)} \left(\frac{p_{m-1}(z)}{x(z)^{\lfloor \alpha_{r-m+1}\rfloor}} - \frac{p_{m-1}(z_1)}{x(z_1)^{\lfloor \alpha_{r-m+1}\rfloor}} \right)\right)
\end{aligned}
\tag{4.3.43}
$$

Proof. Theorem 4.24 and Corollary 4.18. ■

4.4 Quantum curves

Now, we can move on to the final step, by integrating (4.3.43).

The following is Definition 5.1 in [2].

Definition 4.26. For $D = \sum_i \alpha_i[p_i]$, a divisor on Σ, $p_i \in \Sigma$. Denoting its degree $\deg D = \sum_i \alpha_i$. The set of degree 0 divisors of Σ is called $\mathrm{Div}_0(\Sigma)$.

Focusing on $D \in \mathrm{Div}_0(\Sigma)$, then we can define *integration* of a meromorphic one-form $\nu(z)$ on Σ:

$$\int_D \nu(z) = \sum_i \alpha_i \int_b^{p_i} \nu(z), \tag{4.4.1}$$

$b \in \Sigma$ an arbitrary base point. The integration contours being unique homology chains (b, p_i) that do not intersect our basis of non-contractible cycles. Since we assumed that D is a degree 0 divisor, the integral does not depend on the choice of base point b.

We assume that the divisors are chosen so that the integrals converge.

Definition 4.27. Let $D_1, \ldots, D_n$ be n arbitrary degree 0 divisors on Σ. Define:

$$G^{(k)}_{g,n+1}(z; D_1, \ldots, D_n) := \int_{D_1} \cdots \int_{D_n} U^{(k)}_{g,n+1}(z; z_1, \ldots, z_n). \tag{4.4.2}$$

Then we can set all divisors equal (called *principle specialization*):

$$G^{(k)}_{g,n+1}(z; D) = \int_D \cdots \int_D U^{(k)}_{g,n+1}(z; z_1, \ldots, z_n), \tag{4.4.3}$$

Lemma 4.28. Denoting $\mathbf{D} = \{D_1, \ldots, D_n\}$, with $D_i = \sum_j \alpha_{i,j}[z_{i,j}]$, and

$D_{n+1} = \alpha z' + D'$ where D' is an arbitrary divisor of degree $-\alpha$:

$$\begin{aligned}
&\frac{p_0(z)}{x(z)^{\lfloor \alpha_{r-m+1} \rfloor}} \frac{G^{(k)}_{g,n+1}(z;\mathbf{D})x(z)^k}{dx(z)^k} \\
&\quad = -\frac{p_0(z)\,x(z)}{\alpha x(z)^{\lfloor \alpha_{r-m+1} \rfloor}} \frac{d}{dx(z')} \left(\frac{G^{(k-1)}_{g-1,n+2}(z;\mathbf{D},D_{n+1})x(z)^{k-1}}{dx(z)^{k-1}} \right)_{z'=z} \\
&\quad + \frac{p_1(z)}{x(z)^{\lfloor \alpha_{r-m+1} \rfloor}} \frac{G^{(k-1)}_{g,n+1}(z;\mathbf{D})x(z)^{k-1}}{dx(z)^{k-1}} + \delta_{g,0}\delta_{n,0}(-1)^k \frac{p_k(z)}{x(z)^{\lfloor \alpha_{r-m+1} \rfloor}} \\
&+ \frac{p_0(z)}{x(z)^{\lfloor \alpha_{r-m+1} \rfloor}} \sum_{J_1 \uplus J_2 = \mathbf{D}} \sum_{g_1+g_2=g} \frac{G^{(k-1)}_{g_1,|J_1|+1}(z;J_1)x(z)^{k-1}}{dx(z)^{k-1}} \frac{G^{(1)}_{g_2,|J_2|+1}(z;J_2)x(z)}{dx(z)} \\
&\quad - \sum_{i=1}^{n} \sum_j \alpha_{i,j} \left[\frac{p_0(z)}{x(z)^{\lfloor \alpha_{r-m+1} \rfloor}} \frac{x(z)}{x(z)-x(z_{i,j})} \frac{G^{(k-1)}_{g,n}(z;\mathbf{D}\setminus\{D_i\})x(z)^{k-1}}{dx(z)^{k-1}} \right. \\
&\qquad \left. - \frac{p_0(z_{i,j})}{x(z_{i,j})^{\lfloor \alpha_{r-m+1} \rfloor}} \frac{x(z)}{x(z)-x(z_{i,j})} \frac{G^{(k-1)}_{g,n}(z_{i,j};\mathbf{D}\setminus\{D_i\})x(z_{i,j})^{k-1}}{dx(z_{i,j})^{k-1}} \right] \\
&\quad + \delta_{g,0}\delta_{n,1} \sum_j \alpha_{1,j} \left(\frac{x(z)}{x(z)-x(z_{1,j})} \left(\frac{p_{k-1}(z)}{x(z)^{\lfloor \alpha_{r-m+1} \rfloor}} - \frac{p_{k-1}(z_{1,j}}{x(z_{1,j})^{\lfloor \alpha_{r-m+1} \rfloor}} \right) \right)
\end{aligned} \tag{4.4.4}$$

Proof. Integration of Lemma 4.25. ■

Now, we principal specialize giving us a modified version of Lemma 5.5 from [2]:

Lemma 4.29. Setting all divisors equal, containing the point z:

$$D_i = D = \alpha z + \sum_i \alpha_i z_i, \tag{4.4.5}$$

This gives from (4.4.4):

$$
\begin{aligned}
&\frac{p_0(z)}{x(z)^{\lfloor \alpha_{r-m+1}\rfloor}}\frac{G^{(k)}_{g,n+1}(z;D)x(z)^k}{dx(z)^k}\\
&\quad= -\frac{p_0(z)x(z)}{\alpha(n+1)x(z)^{\lfloor \alpha_{r-m+1}\rfloor}}\frac{d}{dx(z)}\left(\frac{G^{(k-1)}_{g-1,n+2}(z';D)x(z')^{k-1}}{dx(z')^{k-1}}\right)_{z=z'}\\
&\quad+\frac{p_1(z)}{x(z)^{\lfloor \alpha_{r-m+1}\rfloor}}\frac{G^{(k-1)}_{g,n+1}(z;D)x(z)^{k-1}}{dx(z)^{k-1}}+\delta_{g,0}\delta_{n,0}(-1)^k\frac{p_k(z)}{x(z)^{\lfloor \alpha_{r-m+1}\rfloor}}\\
&+\frac{p_0(z)}{x(z)^{\lfloor \alpha_{r-m+1}\rfloor}}\sum_{m=0}^{n}\sum_{g_1+g_2=g}\frac{n!}{m!(n-m)!}\frac{G^{(k-1)}_{g_1,m+1}(z;D)x(z)^{k-1}}{dx(z)^{k-1}}\frac{G^{(1)}_{g_2,n-m+1}(z;D)x(z)}{dx(z)}\\
&\quad-n\sum_j\alpha_j\left[\frac{p_0(z)}{x(z)^{\lfloor \alpha_{r-m+1}\rfloor}}\frac{x(z)}{x(z)-x(z_j)}\frac{G^{(k-1)}_{g,n}(z;D)x(z)^{k-1}}{dx(z)^{k-1}}\right.\\
&\quad\left.-\frac{p_0(z_j)}{x(z_j)^{\lfloor \alpha_{r-m+1}\rfloor}}\frac{x(z)}{x(z)-x(z_j)}\frac{G^{(k-1)}_{g,n}(z_j;D)x(z_j)^{k-1}}{dx(z_j)^{k-1}}\right]\\
&\quad-n\alpha\, x(z)\frac{d}{dx(z')}\left(\frac{p_0(z')}{x(z')^{\lfloor \alpha_{r-m+1}\rfloor}}\frac{G^{(k-1)}_{g,n}(z';D)x(z')^{k-1}}{dx(z')^{k-1}}\right)_{z'=z}\\
&\quad+\delta_{g,0}\delta_{n,1}\left[\sum_j\alpha_j\left(\frac{x(z)}{x(z)-x(z_j)}\left(\frac{p_{k-1}(z)}{x(z)^{\lfloor \alpha_{r-m+1}\rfloor}}-\frac{p_{k-1}(z_j)}{x(z_j)^{\lfloor \alpha_{r-m+1}\rfloor}}\right)\right)\right.\\
&\qquad\qquad\left.+\alpha\, x\frac{d}{dx}\left(\frac{p_{k-1}(z)}{x(z)^{\lfloor \alpha_{r-m+1}\rfloor}}\right)\right]
\end{aligned}
\tag{4.4.6}
$$

Proof. The specialization process is straightforward. The $1/(n+1)$ and n factors come from the nature of the specialization process, the derivatives actually being partial derivatives. For example, if we take some $F_{g,n}(t_1,...,t_n)$, function on the base, symmetric in all variables:

$$
\frac{d}{dt}\widehat{F}_{g,n}(t)=\sum_{i=1}^{n}\frac{\partial}{\partial t_i}F_{g,n}(t_1,...,t_n)|_{t_1=...t_n=t}=n\cdot\frac{\partial}{\partial t_1}F_{g,n}(t_1,...,t_n) \tag{4.4.7}
$$

Only the last derivative terms require attention. As $D_i \to D$, $1/(x(z) - x(z_{i,j})) \to 1/(x(z) - x(z_j))$, as long as $z_j \neq z$. When $z_{i,j} \to z$, the limit with the denominator $1/(x(z) - x(z_{i,j})$ tends to the derivative, giving the last lines. ■

Definition 4.30. For $m = 1, \ldots, r$, define:

$$\xi_m(z; D) = (-1)^m \sum_{g,n=0}^{\infty} \frac{\hbar^{2g+n}}{n!} \frac{G^{(m)}_{g,n+1}(z; D) x(z)^m}{dx(z)^m}. \tag{4.4.8}$$

Then $\xi_0(z; D) = 1$, and define $\xi_k(z; D) = 0$ for all $k < 0$. Also, $\xi_k(z; D) = 0$ for all $k \geq r$.

In what follows we will write $\xi_m(x; D)$, and d/dx, with understanding that these are functions in $z \in \Sigma$.

Then, following the method we had discussed earlier, we will sum over g and n in (4.4.6):

$$\begin{aligned}
\frac{p_0(x)}{x^{\lfloor \alpha_{r-m+1} \rfloor}} \xi_k(x; D) &= \frac{p_k(x)}{x^{\lfloor \alpha_{r-m+1} \rfloor}} - \frac{p_1(x)}{x^{\lfloor \alpha_{r-m+1} \rfloor}} \xi_{k-1}(x; D) + \frac{p_0(x)}{x^{\lfloor \alpha_{r-m+1} \rfloor}} \xi_{k-1}(x; D) \xi_1(x; D) \\
&\quad + \hbar \sum_i \alpha_i \frac{x}{x - x_i} \left(\frac{p_0(x)}{x^{\lfloor \alpha_{r-m+1} \rfloor}} \xi_{k-1}(x; D) - \frac{p_0(x_i)}{x_i^{\lfloor \alpha_{r-m+1} \rfloor}} \xi_{k-1}(x_i; D) \right) \\
&+ \frac{\hbar}{\alpha} x \frac{d}{dx} \left(\frac{p_0(x')}{x'^{\lfloor \alpha_{r-m+1} \rfloor}} \xi_{k-1}(x'; D) \right)_{x'=x} + \hbar\alpha\, x' \frac{d}{dx'} \left(\frac{p_0(x')}{x'^{\lfloor \alpha_{r-m+1} \rfloor}} \xi_{k-1}(x'; D) \right)_{x'=x} \\
&- \hbar \sum_i \alpha_i \frac{x}{x - x_i} \left(\frac{p_{k-1}(x)}{x^{\lfloor \alpha_{r-m+1} \rfloor}} - \frac{p_{k-1}(x_i)}{x_i^{\lfloor \alpha_{r-m+1} \rfloor}} \right) - \hbar\alpha\, x \frac{d}{dx} \left(\frac{p_{k-1}(x)}{x^{\lfloor \alpha_{r-m+1} \rfloor}} \right)
\end{aligned} \tag{4.4.9}$$

And, we are nearing the end of our analysis. We note the changes between this result, and eq.(5.8) of [2]. Note the extra x terms in the second, third and fourth lines that aren't in the [2] version. These are key, since they will become part of the logarithmic derivative when we consider the transformation of variables: $x = e^u$, $x\frac{d}{dx} = \frac{d}{du}$.

Now, we assume that $\alpha = 1/\alpha$, or $\alpha = \pm 1$, which gives (Lemma 5.8 in [2]):

Lemma 4.31. If $D = \alpha[z] + \sum_i \alpha_i[z_i] \in \mathrm{Div}_0(\Sigma)$ with $\alpha = \pm 1$, we obtain the following differential recursion relation for the ξ_k's, $k \geq 1$:

$$\begin{aligned}\frac{p_0(x)}{x^{\lfloor \alpha_{r-k+1} \rfloor}}\xi_k(x;D) - \frac{p_k(x)}{x^{\lfloor \alpha_{r-k+1} \rfloor}} &= -\frac{p_1(x)}{x^{\lfloor \alpha_{r-k+1} \rfloor}}\,\xi_{k-1}(x;D) + \frac{p_0(x)}{x^{\lfloor \alpha_{r-k+1} \rfloor}}\xi_{k-1}(x;D)\xi_1(x;D) \\ &+\hbar \sum_i \alpha_i \frac{x}{x-x_i}\left(\frac{p_0(x)}{x^{\lfloor \alpha_{r-k+1} \rfloor}}\xi_{k-1}(x;D) - \frac{p_0(x_i)}{x_i^{\lfloor \alpha_{r-k+1} \rfloor}}\xi_{k-1}(x_i;D)\right) \\ &- \hbar \sum_i \alpha_i \frac{x}{x-x_i}\left(\frac{p_{k-1}(x)}{x^{\lfloor \alpha_{r-k+1} \rfloor}} - \frac{p_{k-1}(x_i)}{x_i^{\lfloor \alpha_{r-k+1} \rfloor}}\right) \\ &+ \hbar\alpha\, x\frac{d}{dx}\left(\frac{p_0(x)}{x^{\lfloor \alpha_{r-k+1} \rfloor}}\xi_{k-1}(x;D)\right) - \hbar\alpha\, x\frac{d}{dx}\left(\frac{p_{k-1}(x)}{x^{\lfloor \alpha_{r-k+1} \rfloor}}\right)\end{aligned} \tag{4.4.10}$$

Where again, the derivative in the last line comes about through the product rule.

Assume the z_i are not in R, so our integrals will converge. Let $\alpha_0 = \alpha = \pm 1$ and $z_0 = z$.

Definition 4.32. For $m = 0, \dots, r$, define:

$$\psi_m(x;D) = \psi(D)\frac{1}{x^{\lfloor \alpha_{r-m} \rfloor}}[p_0(x)\xi_m(x;D) - p_m(x)] \tag{4.4.11}$$

Where:

$$\psi(D) = e^{\frac{1}{\hbar}\int_D W_{0,1}} \; e^{\left(\sum_{(g,n)\neq(0,1)} \frac{\hbar^{2g+n-2}}{n!} \int_D \cdots \int_D \left(W_{g,n}(z_1,\ldots,z_n) - \delta_{g,0}\delta_{n,2}\frac{dx(z_1)dx(z_2)}{(x(z_1)-x(z_2))^2}\right)\right)} \tag{4.4.12}$$

Note, that the integral $\int_D W_{0,1}$ may need to be regularized. But this will play no role in the following because changing the regularization of $\int_D W_{0,1}$ only amounts to multiplying $\psi(D)$ by a constant.

Then, we have (the equivalent of Lemma 5.10 in [2]):

Lemma 4.33. For $m = r$:

$$\psi_r(x; D) = -\frac{p_r(x)}{x^{\lfloor \alpha_0 \rfloor}} \psi(D) \tag{4.4.13}$$

For $m = 1$:

$$\psi_1(x; D) = \alpha\hbar \frac{p_0(x)}{x^{\lfloor \alpha_{r-1} \rfloor}} x \frac{d}{dx} \psi(D) \tag{4.4.14}$$

Proof. This follows as in [2], with some minor modifications.

$\psi_r(x; D)$ follows since $\xi_r = 0$.

We have:

$$
\begin{aligned}
&p_0(x)\hbar\frac{d}{dx}ln\psi(D)\\
&\quad= p_0(x)\sum_{g,n}^{\infty}\frac{\hbar^{2g+n-1}}{n!}\frac{d}{dx}\int_D\cdots\int_D\left(W_{g,n}(z_1,\ldots,z_n)-\delta_{g,0}\delta_{n,2}\frac{dx(z_1)dx(z_2)}{(x(z_1)-x(z_2))^2}\right)\\
&\quad= \alpha\frac{p_0(x)}{dx(z)}\sum_{g,n}^{\infty}\frac{\hbar^{2g+n}}{n!}\int_D\cdots\int_D\left(W_{g,n+1}(z,z_1,\ldots,z_n)-\delta_{g,0}\delta_{n,1}\frac{dx(z)dx(z_1)}{(x(z)-x(z_1))^2}\right)
\end{aligned}
\tag{4.4.15}
$$

Where in the second line we are not integrating over z.

We also have the result, from Lemma 4.14:

$$
W_{g,n+1}(z,z_1,...,z_n)+U^{(1)}_{g,n+1}(z;z_1,...,z_n) = -\frac{p_1(x)}{p_0(x)}\frac{dx(z)}{x(z)}\delta_{g,0}\delta_{n,0}+\frac{dx(z)dx(z_1)}{(x(z)-x(z_1))^2}\delta_{g,0}\delta_{n,1} \tag{4.4.16}
$$

This, together with (4.4.15), gives:

$$
p_0(x)\hbar\frac{d}{dx}ln\psi(D) = \alpha\frac{p_0(x)}{x}\xi_1(z;D) - \alpha\frac{p_1(x)}{x} \tag{4.4.17}
$$

Therefore:

$$
\alpha p_0(x)\hbar x\frac{d}{dx}\psi(D) = \psi_1(x;D)x^{\lfloor \alpha_{r-1}\rfloor} \tag{4.4.18}
$$

∎

Then we get (Theorem 5.11 in [2]):

Theorem 4.34. *For $k = 2, ..., r$, we have:*

$$\hbar\alpha\, x\frac{d}{dx}(\psi_{k-1}(x;D)) = \frac{x^{\lfloor \alpha_{r-k} \rfloor}}{x^{\lfloor \alpha_{r-k+1} \rfloor}}\psi_k(x;D) - \frac{p_{k-1}(x)x^{\lfloor \alpha_{r-1} \rfloor}}{p_0(x)x^{\lfloor \alpha_{r-k+1} \rfloor}}\psi_1(x;D) - \hbar\sum_i \alpha_i \frac{x}{x-x_i}(\psi_{k-1}(x;D) - \psi_{k-1}(x_i;D)) \quad (4.4.19)$$

And:

$$\hbar\alpha x\frac{d}{dx}\psi_r(x;D) = \hbar\alpha x\left(\frac{p_r'(x)}{p_r(x)} - \frac{\lfloor \alpha_0 \rfloor}{x}\right)\psi_r(x;D) - \frac{p_r(x)x^{\lfloor \alpha_{r-1} \rfloor}}{p_0(x)x^{\lfloor \alpha_0 \rfloor}}\psi_1(x;D) \quad (4.4.20)$$

We can write the above in matrix form:

$$\hbar\alpha\, x\frac{d}{dx}\begin{pmatrix}\psi_1(x;D)\\ \vdots\\ \psi_{r-1}(x;D)\\ \psi_r(x;D)\end{pmatrix} =$$

$$\begin{pmatrix} -\frac{p_1(x)}{p_0(x)} & \frac{x^{\lfloor\alpha_{r-2}\rfloor}}{x^{\lfloor\alpha_{r-1}\rfloor}} & & \\ \vdots & & \ddots & \\ -\frac{p_{r-1}(x)x^{\lfloor\alpha_{r-1}\rfloor}}{p_0(x)x^{\lfloor\alpha_1\rfloor}} & & & \frac{x^{\lfloor\alpha_0\rfloor}}{x^{\lfloor\alpha_1\rfloor}} \\ -\frac{p_r(x)x^{\lfloor\alpha_{r-1}\rfloor}}{p_0(x)x^{\lfloor\alpha_0\rfloor}} & & & \hbar\alpha\left(x\frac{p_r'(x)}{p_r(x)} - \lfloor\alpha_0\rfloor\right) \end{pmatrix}\begin{pmatrix}\psi_1(x;D)\\ \vdots\\ \psi_{r-1}(x;D)\\ \psi_r(x;D)\end{pmatrix}$$

$$-\hbar\sum_i \frac{\alpha_i\, x}{x-x_i}\begin{pmatrix}\psi_1(x;D)\\ \vdots\\ \psi_{r-1}(x;D)\\ 0\end{pmatrix} + \hbar\sum_i \frac{\alpha_i\, x}{x-x_i}\begin{pmatrix}\psi_1(x_i;D)\\ \vdots\\ \psi_{r-1}(x_i;D)\\ 0\end{pmatrix} \qquad (4.4.21)$$

Proof. The proof of (4.4.20) is as follows. $\psi_r(x;D) = -\frac{p_r(x)}{x^{\lfloor\alpha_0\rfloor}}\psi(D)$. So, using

Lemma 4.33.:

$$\hbar\alpha x\frac{d}{dx}\psi_r(x;D) = -\hbar\alpha x\frac{d}{dx}\left(\frac{p_r(x)}{x^{\lfloor\alpha_0\rfloor}}\psi(D)\right) \tag{4.4.22}$$

$$= -\hbar\alpha x\left(\frac{p_r'(x)}{x^{\lfloor\alpha_0\rfloor}} - \lfloor\alpha_0\rfloor\frac{p_r(x)}{x^{\lfloor\alpha_0\rfloor+1}}\right)\psi(D) - \hbar\alpha x\frac{p_r(x)}{x^{\lfloor\alpha_0\rfloor}}\frac{d}{dx}\psi(D) \tag{4.4.23}$$

$$=\hbar\alpha x\left(\frac{p_r'(x)}{p_r(x)} - \frac{\lfloor\alpha_0\rfloor}{x}\right)\psi_r(x;D) - \frac{p_r(x)x^{\lfloor\alpha_{r-1}\rfloor}}{p_0(x)x^{\lfloor\alpha_0\rfloor}}\psi_1(x;D) \tag{4.4.24}$$

Then, for $k = 2, ..., r$ we take (4.4.10) multiplied by $\psi(D)$:

$$\begin{aligned}\frac{x^{\lfloor\alpha_{r-k}\rfloor}}{x^{\lfloor\alpha_{r-k+1}\rfloor}}\psi_k(x;D) = &\frac{x^{\lfloor\alpha_{r-1}\rfloor}}{x^{\lfloor\alpha_{r-k+1}\rfloor}}\psi_1(x;D)\xi_{k-1}(x;D)\\ &+\hbar\sum_i\alpha_i\frac{x}{x-x_i}(\psi_{k-1}(x;D)-\psi_{k-1}(x_i;D))\\ &+\hbar\alpha\psi(D)\,x\frac{d}{dx}\left(\frac{p_0(x)}{x^{\lfloor\alpha_{r-k+1}\rfloor}}\xi_{k-1}(x;D) - \frac{p_{k-1}(x)}{x^{\lfloor\alpha_{r-k+1}\rfloor}}\right)\end{aligned} \tag{4.4.25}$$

The last term can be re-written as:

$$\hbar\alpha\,x\frac{d}{dx}(\psi_{k-1}(x;D)) - \hbar\frac{\alpha}{x^{\lfloor\alpha_{r-k+1}\rfloor}}(p_0(x)\xi_{k-1}(x;D) - p_{k-1}(x))\,x\frac{d}{dx}\psi(D) \tag{4.4.26}$$

Then, using Lemma 4.33 again, this equals:

$$\hbar\alpha\,x\frac{d}{dx}(\psi_{k-1}(x;D)) - \frac{x^{\lfloor\alpha_{r-1}\rfloor}}{x^{\lfloor\alpha_{r-k+1}\rfloor}}\left(\xi_{k-1}(x;D) - \frac{p_{k-1}(x)}{p_0(x)}\right)\psi_1(x;D) \tag{4.4.27}$$

This gives (4.4.19).

■

Now, we do a change of coordinates to get the following Lemma (unique from [2]).

Lemma 4.35. Defining new coordinates $x = e^u$, we get the following result:

For $k = 2, ..., r$, we have:

$$\hbar\alpha \frac{d}{du}(\psi_{k-1}(e^u; D)) = \frac{e^{u(\lfloor\alpha_{r-k}\rfloor)}}{e^{u(\lfloor\alpha_{r-k+1}\rfloor)}}\psi_k(e^u; D) - \frac{p_{k-1}(e^u)e^{u(\lfloor\alpha_{r-1}\rfloor)}}{p_0(e^u)e^{u(\lfloor\alpha_{r-k+1}\rfloor)}}\psi_1(e^u; D) - \hbar\sum_i \alpha_i \frac{e^u}{e^u - e^{u_i}}(\psi_{k-1}(e^u; D) - \psi_{k-1}(e^{u_i}; D)) \quad (4.4.28)$$

And:

$$\hbar\alpha \frac{d}{du}\psi_r(e^u; D) = \hbar\alpha\left(\frac{p_r'(e^u)}{p_r(e^u)} - \lfloor\alpha_0\rfloor\right)\psi_r(e^u; D) - \frac{p_r(e^u)e^{u(\lfloor\alpha_{r-1}\rfloor)}}{p_0(e^u)e^{u(\lfloor\alpha_0\rfloor)}}\psi_1(e^u; D) \quad (4.4.29)$$

We can write the above in matrix form:

$$\hbar\alpha\,\frac{d}{du}\begin{pmatrix}\psi_1(e^u;D)\\ \vdots\\ \psi_{r-1}(e^u;D)\\ \psi_r(e^u;D)\end{pmatrix}=$$

$$\begin{pmatrix} -\frac{p_1(e^u)}{p_0(e^u)} & \frac{e^{u(\lfloor\alpha_{r-2}\rfloor)}}{e^{u(\lfloor\alpha_{r-1}\rfloor)}} & & \\ \vdots & & \ddots & \\ -\frac{p_{r-1}(e^u)e^{u(\lfloor\alpha_{r-1}\rfloor)}}{p_0(e^u)e^{u(\lfloor\alpha_1\rfloor)}} & & & \frac{e^{u(\lfloor\alpha_0\rfloor)}}{e^{u(\lfloor\alpha_1\rfloor)}} \\ -\frac{p_r(e^u)e^{u(\lfloor\alpha_{r-1}\rfloor)}}{p_0(e^u)e^{u(\lfloor\alpha_0\rfloor)}} & & & \frac{p_r'(e^u)}{p_r(e^u)}-\lfloor\alpha_0\rfloor \end{pmatrix}\begin{pmatrix}\psi_1(e^u;D)\\ \vdots\\ \psi_{r-1}(e^u;D)\\ \psi_r(e^u;D)\end{pmatrix}$$

$$-\hbar\sum_i\frac{\alpha_i\,e^u}{e^u-e^{u_i}}\begin{pmatrix}\psi_1(e^u;D)\\ \vdots\\ \psi_{r-1}(e^u;D)\\ 0\end{pmatrix}+\hbar\sum_i\frac{\alpha_i\,e^u}{e^u-e^{u_i}}\begin{pmatrix}\psi_1(e^{u_i};D)\\ \vdots\\ \psi_{r-1}(e^{u_i};D)\\ 0\end{pmatrix}\qquad(4.4.30)$$

Proof. This is a basic change of variables. Note that wherever there was a $\frac{d}{dx}$ in our result there was always an x joining it. This includes the result (4.4.29), where the $x\frac{d}{dx}p_r(x)$ becomes $\frac{d}{du}p_r(e^u)$. But, note the $1/x$ in (4.4.29) disappears, so the result is a bit different from [2]. ■

4.5 Special choices of integration divisors

Now, we will be looking at choices of integration divisors that simplify the results of Theorem 4.34. This analysis follows as in [2], but we will be switching between x and u coordinates as is convenient. We note here that, unlike in [2], we can consider as integration divisor zeroes of x. The reason for this will be explained in the subsection 4.5.2.

On the other hand though, we can only consider simple poles and simple zeroes of x. We can use, as our integration divisor, points in R so long as the $W_{g,n}$'s don't have poles there. In [2] this was the case if the recursion kernel had a zero there of degree ≥ 2, so that it could cancel out the degree 2 order pole coming from $B(z, \tau'(z))$. In our case this is never true though, since we have x/dx in the recursion kernel instead of $1/dx$; that always gives a simple zero or pole. So, we cannot consider integration divisors that are in R.

4.5.1 Simple Pole of x

The first choice we look at is:

$$D = [z] - [\beta], \tag{4.5.1}$$

So, $\alpha = 1$, and β is a simple pole of x (not in R).

Then, our result of Theorem 4.34 simplifies as:

$$\begin{aligned} \hbar\, x \frac{d}{dx}(\psi_{k-1}(x; D)) = \frac{x^{\lfloor \alpha_{r-k} \rfloor}}{x^{\lfloor \alpha_{r-k+1} \rfloor}} \psi_k(x; D) - \frac{p_{k-1}(x) x^{\lfloor \alpha_{r-1} \rfloor}}{p_0(x) x^{\lfloor \alpha_{r-k+1} \rfloor}} \psi_1(x; D) \\ + \hbar\, x \lim_{z_1 \to \beta} \frac{1}{x_1(z_1)} \psi_{k-1}(x_1(z_1); D) \end{aligned} \tag{4.5.2}$$

Then, we get the result (Lemma 5.13 in [2]):

Lemma 4.36. For β, simple pole of x (or a pole of x where the $W_{g,n}$'s are all holomorphic), and $k = 1, \ldots, r-1$,

$$\lim_{z_1 \to \beta} \frac{\psi_k(x(z_1); D)}{x(z_1)} = \psi(D) \lim_{z_1 \to \beta} \frac{P_{k+1}(x(z_1), y(z_1))}{x(z_1)^{\lfloor \alpha_{r-k} \rfloor + 1}} \tag{4.5.3}$$

Because of the admissibility condition in Definition 2.14, these limits are finite.

Proof. First, note:

$$\psi_k(x(z_1); D) = \psi(D) \frac{1}{x(z_1)^{\lfloor \alpha_{r-k} \rfloor}} (p_0(x(z_1)) \xi_k(x_1(z_1); D) - p_k(x(z_1))) \tag{4.5.4}$$

Then, as $z_1 \to \beta$ (remembering β is a pole of x where the correlation functions are holomorphic) the dominant term will be the one containing the most instances of $W_{0,1}(z_1) = y(z_1) \frac{dx_1(z_1)}{x_1(z_1)}$. In other words:

$$\begin{aligned} &\lim_{z_1 \to \beta} \frac{\psi_k(x(z_1); D)}{x(z_1)} \\ &= \psi(D) \lim_{z_1 \to \beta} \frac{1}{x_1(z_1)^{\lfloor \alpha_{r-k} \rfloor + 1}} \left((-1)^k \frac{U_{0,1}^{(k)}(z_1) p_0(x_1(z_1)) x_1(z_1)^k}{dx_1(z_1)^k} - p_k(x(z_1)) \right) \\ &= \psi(D) \lim_{z_1 \to \beta} \frac{P_{k+1}(x(z_1), y(z_1))}{x(z_1)^{\lfloor \alpha_{r-k} \rfloor + 1}} \end{aligned} \tag{4.5.5}$$

Note, this leading order behaviour of the limit argument is the same one used in [2]. The argument works in our case as well, because of the modified nature of $W_{0,1}$. β is a pole of x, so is a simple pole at $dx/x(\beta)$, and therefore $x/dx(\beta)$ is a simple zero. ξ_k has the form given in (4.4.8), and so all terms will be zero in the limit, except those that contribute a dx^k/x^k. The final limit is finite by equation (4.3.24).

■

To simplify our results further, we can define:

$$C_k := \lim_{z_1\to\beta} \frac{P_{k+1}(x(z_1), y(z_1))}{x(z_1)^{\lfloor\alpha_{r-k}\rfloor+1}}, \qquad k = 1, \ldots, r-1. \tag{4.5.6}$$

Allowing us to re-write the above as:

$$\lim_{z_1\to\beta} \frac{\psi_k(x(z_1); D)}{x(z_1)} = C_k\psi(D) = -\frac{x^{\lfloor\alpha_0\rfloor}}{p_r(x)} C_k\psi_r(x; D) \tag{4.5.7}$$

And, we can re-write the system of differential equations from the (4.5.2) as:

$$\begin{aligned}\hbar\, x \frac{d}{dx}(\psi_{k-1}(x; D)) = \frac{x^{\lfloor\alpha_{r-k}\rfloor}}{x^{\lfloor\alpha_{r-k+1}\rfloor}}\psi_k(x; D) - \frac{p_{k-1}(x)x^{\lfloor\alpha_{r-1}\rfloor}}{p_0(x)x^{\lfloor\alpha_{r-k+1}\rfloor}}\psi_1(x; D)\\ - \hbar\, x\, \frac{x^{\lfloor\alpha_0\rfloor}}{p_r(x)} C_{k-1}\psi_r(x; D)\end{aligned} \tag{4.5.8}$$

Then, we can define the following objects:

$$D_i := \hbar \frac{x^{\lfloor\alpha_i\rfloor}}{x^{\lfloor\alpha_{i-1}\rfloor}}\, x \frac{d}{dx}, \qquad i = 1, ..., r \tag{4.5.9}$$

This gives us (Lemma 5.14 from [2]):

Lemma 4.37.

$$\Bigg[D_1D_2...D_{r-1}\frac{p_0(x)}{x^{\lfloor\alpha_r\rfloor}}D_r + D_1D_2...D_{r-2}\frac{p_1(x)}{x^{\lfloor\alpha_{r-1}\rfloor}}D_{r-1} + ... + \frac{p_{r-1}(x)}{x^{\lfloor\alpha_1\rfloor}}D_1 + \frac{p_r(x)}{x^{\lfloor\alpha_0\rfloor}}$$
$$-\hbar C_1D_1D_2...D_{r-2}\,x\,\frac{x^{\lfloor\alpha_{r-1}\rfloor}}{x^{\lfloor\alpha_{r-2}\rfloor}} - \hbar C_2D_1D_2...D_{r-3}\,x\,\frac{x^{\lfloor\alpha_{r-2}\rfloor}}{x^{\lfloor\alpha_{r-3}\rfloor}} - ...$$
$$-\hbar C_{r-1}\,x\,\frac{x^{\lfloor\alpha_1\rfloor}}{x^{\lfloor\alpha_0\rfloor}}\Bigg]\psi(D) = 0 \quad (4.5.10)$$

This is a particular quantization of the spectral curve defined by $P(x,y) = 0$, as we will see below.

Proof. We can use, from Lemma 4.33:

$$\psi_r(x;D) = -\frac{p_r(x)}{x^{\lfloor\alpha_0\rfloor}}\psi(D), \qquad \psi_1(x;D) = \frac{p_0(x)}{x^{\lfloor\alpha_{r-1}\rfloor}}\hbar\,x\frac{d}{dx}\psi(D) \qquad (4.5.11)$$

To rewrite (4.5.8) as:

$$\psi_k(x;D) = D_{r-k+1}\psi_{k-1}(x;D) + \frac{p_{k-1}(x)}{x^{\lfloor\alpha_{r-k+1}\rfloor}}D_{r-k+1}\psi(D) - \hbar\,x\,\frac{x^{\lfloor\alpha_{r-k+1}\rfloor}}{x^{\lfloor\alpha_{r-k}\rfloor}}C_{k-1}\psi(D) \quad (4.5.12)$$

For $k = r$, the above becomes:

$$\psi_r(x;D) = D_1\psi_{r-1}(x;D) + \frac{p_{r-1}(x)}{x^{\lfloor\alpha_1\rfloor}}D_1\psi(D) - \hbar\,x\,\frac{x^{\lfloor\alpha_1\rfloor}}{x^{\lfloor\alpha_0\rfloor}}C_{r-1}\psi(D) \quad (4.5.13)$$

For $k = r-1$, gives:

$$\psi_{r-1}(x;D) = D_2\psi_{r-2}(x;D) + \frac{p_{r-2}(x)}{x^{\lfloor\alpha_2\rfloor}}D_2\psi(D) - \hbar\,x\,\frac{x^{\lfloor\alpha_2\rfloor}}{x^{\lfloor\alpha_1\rfloor}}C_{r-2}\psi(D) \quad (4.5.14)$$

Combining these gives:

$$\psi_r(x;D) = D_1 D_2 \psi_{r-2}(x;D) + D_1 \frac{p_{r-2}(x)}{x^{\lfloor \alpha_2 \rfloor}} D_2 \psi(D) + \frac{p_{r-1}(x)}{x^{\lfloor \alpha_1 \rfloor}} D_1 \psi(D) \\ - \hbar D_1\, x\, \frac{x^{\lfloor \alpha_2 \rfloor}}{x^{\lfloor \alpha_1 \rfloor}} C_{r-2}\psi(D) - \hbar\, x\, \frac{x^{\lfloor \alpha_1 \rfloor}}{x^{\lfloor \alpha_0 \rfloor}} C_{r-1}\psi(D) \quad (4.5.15)$$

Then, continuing this process gives the result. ■

If we have the very special case that $\lfloor \alpha_i \rfloor = 0$, for $i = 1, ..., r$ then this simplifies further. $D_i = \hbar\, x \frac{d}{dx} = \hbar \frac{d}{du}$, for $i = 0, ..., r$ and (writing the result in u coordinates):

$$\left[\hbar^r \frac{d^{r-1}}{du^{r-1}} p_0(e^u) \frac{d}{du} + \hbar^{r-1} \frac{d^{r-2}}{du^{r-2}} p_1(e^u) \frac{d}{du} + ... + \hbar p_{r-1}(e^u) \frac{d}{du} + p_r(e^u) \right. \\ \left. \hbar^{r-1} C_1 \frac{d^{r-2}}{du^{r-2}} e^u - \hbar^{r-2} C_2 \frac{d^{r-3}}{du^{r-3}} e^u - ... - \hbar C_{r-1} e^u \right] \psi(D) = 0 \quad (4.5.16)$$

And, if the $C_k = 0$, $k = 1, ..., r-1$, then this obviously simplifies further. In this case we have the quantization that was given in equation (2.3.4).

With ordering:

$$(\widehat{y}^{r-1} p_0(e^{\widehat{u}}) \widehat{y} + \widehat{y}^{r-2} p_1(e^{\widehat{u}}) \widehat{y} + ... + p_{r-1}(e^{\widehat{u}}) \widehat{y} + p_r(e^{\widehat{u}}))\psi = 0 \qquad (4.5.17)$$

As in [2], we can generalize this for different orderings of quantizations. If x has more poles, that satisfy our holomorphicity condition, β_i, $i = 1, ..., n$.

Then:

$$D = [z] - \sum_{i=1}^{n} \mu_i [\beta_i] \tag{4.5.18}$$

Where $\sum_{i=1}^{n} \mu_i = 1$. Then, the same steps as above, defining the objects:

$$C_k^{(i)} := \lim_{z_1 \to \beta_i} \frac{P_{k+1}(x(z_1), y(z_1))}{x(z_1)^{\lfloor \alpha_{r-k} \rfloor + 1}}, \quad k = 1, .., r-1, \;\; i = 1, .., n \tag{4.5.19}$$

Then, we get the result of Lemma 4.37, with C_k replaced with $\sum_{i=1}^{n} \mu_i C_k^{(i)}$.

4.5.2 Simple Zero of x

As a divergence on the analysis of [2], we can also examine what happens in the case that we choose the divisor:

$$D = [z] - [\gamma_0] \tag{4.5.20}$$

γ_0 is a simple zero of x, and therefore not in R.

Then, we have from Theorem 4.34, for $k = 2, ..., r$:

$$\hbar\alpha \, x \frac{d}{dx}(\psi_{k-1}(x; D)) = \frac{x^{\lfloor \alpha_{r-k} \rfloor}}{x^{\lfloor \alpha_{r-k+1} \rfloor}} \psi_k(x; D) - \frac{p_{k-1}(x) x^{\lfloor \alpha_{r-1} \rfloor}}{p_0(x) x^{\lfloor \alpha_{r-k+1} \rfloor}} \psi_1(x; D) + \hbar \psi_{k-1}(x; D) - \hbar \lim_{z_1 \to \gamma_0} \psi_{k-1}(x(z_1); D) \tag{4.5.21}$$

Then, we have the following Lemma:

Lemma 4.38. For γ_0, a simple zero of x, and $k = 1, \ldots, r-1$,

$$\lim_{z_1 \to \gamma_0} \psi_k(x(z_1); D) = \psi(D) \lim_{z_1 \to \gamma_0} \frac{P_{k+1}(x(z_1), y(z_1))}{x(z_1)^{\lfloor \alpha_{r-k} \rfloor}} \tag{4.5.22}$$

Because of the admissibility condition in Definition 2.14, these limits are finite.

Proof. First, note:

$$\psi_k(x(z_1); D) = \psi(D) \frac{1}{x(z_1)^{\lfloor \alpha_{r-k} \rfloor}} (p_0(x(z_1)) \xi_k(x_1(z_1); D) - p_k(x(z_1))) \tag{4.5.23}$$

Then:

$$\begin{aligned} &\lim_{z_1 \to \gamma_0} \psi_k(x(z_1); D) \\ &= \psi(D) \lim_{z_1 \to \gamma_0} \frac{1}{x_1(z_1)^{\lfloor \alpha_{r-k} \rfloor}} \left((-1)^k \frac{U_{0,1}^{(k)}(z_1) p_0(x_1(z_1)) x_1(z_1)^k}{dx_1(z_1)^k} - p_k(x(z_1)) \right) \\ &= \psi(D) \lim_{z_1 \to \gamma_0} \frac{P_{k+1}(x(z_1), y(z_1))}{x(z_1)^{\lfloor \alpha_{r-k} \rfloor}} \end{aligned} \tag{4.5.24}$$

By the same argument as Lemma as 4.36. $z_1 \to \gamma_0$ the dominant term will be the one containing the most instances of $W_{0,1}(z_1) = y(z_1) \frac{dx_1(z_1)}{x_1(z_1)}$. In other words: γ_0 is a simple zero of x, and therefore $x/dx(\gamma_0)$ is also a simple zero. ξ_k has the form given in (4.4.8), and so all terms will be zero in the limit (because γ_0 is not in R), except those that contribute a dx^k/x^k. Then, by equation (4.3.24), the limits are finite.

They could not use this argument in [2], since there they had a limit involving $1/dx$, and $W_{0,1}(z_1) = y(z_1)dx(z_1)$; at a zero of x the $W_{0,1}(z_1)$ terms will not dominate since they will all go to zero. ■

Then, we define:

$$E_k := \lim_{z_1 \to \gamma_0} \frac{P_{k+1}(x(z_1), y(z_1))}{x(z_1)^{\lfloor \alpha_{r-k} \rfloor}}, \qquad k = 1, \ldots, r-1. \tag{4.5.25}$$

Allowing us to re-write the above as:

$$\lim_{z_1 \to \gamma_0} \psi_k(x(z_1); D) = E_k \psi(D) = -\frac{x^{\lfloor \alpha_0 \rfloor}}{p_r(x)} E_k \psi_r(x; D) \tag{4.5.26}$$

And, we can simplify (4.5.21) as:

$$\begin{aligned}\hbar\alpha\, x\frac{d}{dx}(\psi_{k-1}(x; D)) = \frac{x^{\lfloor \alpha_{r-k} \rfloor}}{x^{\lfloor \alpha_{r-k+1} \rfloor}} \psi_k(x; D) - \frac{p_{k-1}(x) x^{\lfloor \alpha_{r-1} \rfloor}}{p_0(x) x^{\lfloor \alpha_{r-k+1} \rfloor}} \psi_1(x; D)\\ + \hbar\psi_{k-1}(x; D) + \hbar \frac{x^{\lfloor \alpha_0 \rfloor}}{p_r(x)} E_{k-1} \psi_r(x; D)\end{aligned} \tag{4.5.27}$$

Then, we have (4.5.9) again:

$$D_i := \hbar \frac{x^{\lfloor \alpha_i \rfloor}}{x^{\lfloor \alpha_{i-1} \rfloor}}\, x\frac{d}{dx}, \qquad i = 1, ..., r \tag{4.5.28}$$

And defining:

$$\begin{aligned}F_i :=& \left(D_i - \hbar \frac{x^{\lfloor \alpha_i \rfloor}}{x^{\lfloor \alpha_{i-1} \rfloor}} \right)\\ =& \hbar \frac{x^{\lfloor \alpha_i \rfloor}}{x^{\lfloor \alpha_{i-1} \rfloor}} \left(x\frac{d}{dx} - 1 \right)\end{aligned} \tag{4.5.29}$$

In this situation we have the following result:

Lemma 4.39.

$$\Big[F_1F_2...F_{r-1}\frac{p_0(x)}{x^{\lfloor\alpha_r\rfloor}}D_r + F_1F_2...F_{r-2}\frac{p_1(x)}{x^{\lfloor\alpha_{r-1}\rfloor}}D_{r-1} + ... + \frac{p_{r-1}(x)}{x^{\lfloor\alpha_1\rfloor}}D_1 + \frac{p_r(x)}{x^{\lfloor\alpha_0\rfloor}}$$
$$+\hbar E_1F_1F_2...F_{r-2}\frac{x^{\lfloor\alpha_{r-1}\rfloor}}{x^{\lfloor\alpha_{r-2}\rfloor}} + \hbar E_2F_1F_2...F_{r-3}\frac{x^{\lfloor\alpha_{r-2}\rfloor}}{x^{\lfloor\alpha_{r-3}\rfloor}} + ... + \hbar E_{r-1}\frac{x^{\lfloor\alpha_1\rfloor}}{x^{\lfloor\alpha_0\rfloor}}\Big]\psi(D) = 0 \tag{4.5.30}$$

Proof. Proof is same as what we did in 4.37

We can use, from Lemma 4.33:

$$\psi_r(x;D) = -\frac{p_r(x)}{x^{\lfloor\alpha_0\rfloor}}\psi(D), \qquad \psi_1(x;D) = \frac{p_0(x)}{x^{\lfloor\alpha_{r-1}\rfloor}}\hbar\, x\frac{d}{dx}\psi(D) \tag{4.5.31}$$

To rewrite (4.5.27) as:

$$\psi_k(x;D) = D_{r-k+1}\psi_{k-1}(x;D) + \frac{p_{k-1}(x)}{x^{\lfloor\alpha_{r-k+1}\rfloor}}D_{r-k+1}\psi(D)$$
$$- \hbar\frac{x^{\lfloor\alpha_{r-k+1}\rfloor}}{x^{\lfloor\alpha_{r-k}\rfloor}}\psi_{k-1}(x;D) + \hbar\frac{x^{\lfloor\alpha_{r-k+1}\rfloor}}{x^{\lfloor\alpha_{r-k}\rfloor}}E_{k-1}\psi(D) \tag{4.5.32}$$

For $k = r$, the above becomes:

$$\psi_r(x;D) = F_1\psi_{r-1}(x;D) + \frac{p_{r-1}(x)}{x^{\lfloor\alpha_1\rfloor}}D_1\psi(D) + \hbar\frac{x^{\lfloor\alpha_1\rfloor}}{x^{\lfloor\alpha_0\rfloor}}E_{r-1}\psi(D) \tag{4.5.33}$$

For $k = r-1$, gives:

$$\psi_{r-1}(x;D) = F_2\psi_{r-2}(x;D) + \frac{p_{r-2}(x)}{x^{\lfloor\alpha_2\rfloor}} D_2\psi(D) + \hbar\frac{x^{\lfloor\alpha_2\rfloor}}{x^{\lfloor\alpha_1\rfloor}} E_{r-2}\psi(D) \quad (4.5.34)$$

Combining these gives:

$$\psi_r(x;D) = F_1F_2\psi_{r-2}(x;D) + F_1\frac{p_{r-2}(x)}{x^{\lfloor\alpha_2\rfloor}} D_2\psi(D) + \frac{p_{r-1}(x)}{x^{\lfloor\alpha_1\rfloor}} D_1\psi(D) \\ + \hbar F_1\frac{x^{\lfloor\alpha_2\rfloor}}{x^{\lfloor\alpha_1\rfloor}} E_{r-2}\psi(D) + \hbar\frac{x^{\lfloor\alpha_1\rfloor}}{x^{\lfloor\alpha_0\rfloor}} E_{r-1}\psi(D) \quad (4.5.35)$$

This is continued all the way till $k = 2$, gives:

$$\psi_2(x;D) = F_{r-1}\frac{p_0(x)}{x^{\lfloor\alpha_r\rfloor}} D_r\psi(D) + \frac{p_1(x)}{x^{\lfloor\alpha_{r-1}\rfloor}} D_{r-1}\psi(D) + \hbar\frac{x^{\lfloor\alpha_{r-1}\rfloor}}{x^{\lfloor\alpha_{r-2}\rfloor}} E_1\psi(D) \quad (4.5.36)$$

Putting this all together gives the result. ■

In the special case that $\lfloor\alpha_i\rfloor = 0$, for $i = 1,...,r$, and $E_k = 0$ for $k = 1,...,r-1$, then this simplifies further. We get (writing the result in u coordinates):

$$\Bigg[\hbar^r\left(\frac{d}{du}-1\right)^{r-1} p_0(e^u)\frac{d}{du} + \hbar^{r-1}\left(\frac{d}{du}-1\right)^{r-2} p_1(e^u)\frac{d}{du} \\ +... + \hbar p_{r-1}(e^u)\frac{d}{du} + p_r(e^u)\Bigg]\psi(D) = 0 \quad (4.5.37)$$

Which, using the binomial theorem, can be written as:

$$\Bigg[\hbar^r \left(\sum_{k=0}^{r-1} \binom{r-1}{k} (-1)^{r-1-k} \frac{d^k}{du^k}\right) p_0(e^u)\frac{d}{du} + \hbar^{r-1} \left(\sum_{k=0}^{r-2} \binom{r-2}{k} (-1)^{r-2-k} \frac{d^k}{du^k}\right) p_1(e^u)\frac{d}{du} + ... + \hbar p_{r-1}(e^u)\frac{d}{du} + p_r(e^u)\Bigg] \psi(D) = 0 \quad (4.5.38)$$

This completes our modification of the result of [2]. As you can see, we followed that papers analysis very closely, and the result aligns very much with theirs, with some modifications. The reader can find examples of this result in Section 5.

5 Examples

To begin with we will look at a few examples of spectral curves that were also looked at in [2], only here we are using the results of Section 4.1; we can compare the similarities and differences between these results and those of [2]. Following that we will look at a couple of examples which use the results of Section 3.3.

All the examples we give in this section have been worked out computationally using Mathematica, for the first few $W_{g,n}$'s, to show that they give the matching quantum curve (to low order in $\hbar$). The approach here relies heavily on the work carried out in Section 4, and we will refer to the appropriate definitions and terms where needed.

5.1 $y^a - e^u y + 1 = 0$

Here $a \geq 2$. We consider that $x = e^u$ is meromorphic on the Riemann sphere, and u is meromorphic on $\mathbb{CP}^1 \backslash \{(-1)^{1/a}, \infty\}$. Its Newton polygon is the polygon with vertices at $(0,0)$, $(1,1)$ and $(0,a)$. The $\lfloor \alpha_i \rfloor = 0$, $i = 0, ..., a$. The α_i were defined in equation (4.1.2).

5.1.1 $a = 2$

The curve can be parameterized by $(u, y) = (\log(x), y) = (\log(z + \frac{1}{z}), z)$. Considering it as a curve in x, $R = \{\pm 1\}$. x has simple poles at $z = 0$, and $z = \infty$.

$p_0(x) = 1$, $p_1(x) = -x$, and $p_2(x) = 1$. So:

$$P_2(x, y) = p_0(x)y = y \tag{5.1.1}$$

If we consider the case where we evaluate using the pole at $z = \infty$ as our choice of integration divisor ($D = [z] - [\infty]$), then:

$$C_1 = \lim_{z_1 \to \infty} \frac{y(z_1)}{x_1(z_1)} = \lim_{z_1 \to \infty} \frac{z_1^2}{z_1^2 + 1} = 1 \tag{5.1.2}$$

The C_k were defined in equation (4.5.6).

So, from (4.5.16), we have:

$$\left(\hbar^2 \frac{d^2}{du^2} - \hbar e^u \frac{d}{du} + 1 - \hbar e^u\right) \psi(D) = 0 \tag{5.1.3}$$

Which is equivalent to:

$$\left(\hbar^2 \frac{d^2}{du^2} - \hbar \frac{d}{du} e^u + 1\right) \psi(D) = 0 \tag{5.1.4}$$

This is the following quantization:

$$(u, y) \mapsto (\widehat{u}, \widehat{y}) = \left(u, \hbar \frac{d}{du}\right) \tag{5.1.5}$$

With choice of ordering:

$$(\widehat{y}^2 - \widehat{y} e^{\widehat{u}} + 1)\psi = 0 \tag{5.1.6}$$

We could also consider the case of choosing $z = 0$ as our integration divisor ($D = [z] - [0]$). Then we get:

$$C_1 = \lim_{z_1 \to 0} \frac{z_1^2}{z_1^2 + 1} = 0 \tag{5.1.7}$$

Our result, from (4.5.16), then is:

$$\left(\hbar^2 \frac{d^2}{du^2} - \hbar e^u \frac{d}{du} + 1\right) \psi(D) = 0 \tag{5.1.8}$$

Which is a choice of ordering:

$$(\widehat{y}^2 - e^{\widehat{u}} \widehat{y} + 1)\psi = 0 \tag{5.1.9}$$

5.1.2 $a > 2$

Now, we have a parameterization $(u, y) = (\log(x), y) = (\log(\frac{z^a+1}{z}), z)$. Then, R equals the z such that $z^a = \frac{1}{a-1}$ and infinity. x has a simple pole at $z = 0$ and a pole at infinity. $z = \infty$ is in R; in [2] they can still use this as an integration divisor since their $W_{g,n}$'s do not diverge there; ours do (an algorithmic computation confirms this), and so we cannot use $z = \infty$.

Choosing $D = [z] - [0]$, we get:

$$C_k = \lim_{z_1 \to 0} \frac{y(z_1)^k}{x(z_1)} = \lim_{z_1 \to 0} \frac{z_1^{k+1}}{z_1^a + 1} = 0, \quad k = 1, ..., a - 1 \tag{5.1.10}$$

And, so we get the quantum curve:

$$\left(\hbar^a \frac{d^a}{du^a} - \hbar e^u \frac{d}{du} + 1\right)\psi = 0 \tag{5.1.11}$$

In this case, we only got one ordering of the quantization of our spectral curve.

5.2 $e^u y^a + y + 1 = 0$

With $a \geq 2$. Again, we consider that $x = e^u$ is meromorphic on the Riemann sphere, and u is meromorphic on $\mathbb{CP}^1 \backslash \{-1, \infty\}$. In this case we will see that we will get three different orderings, depending on our choice of integration divisor.

The Newton polygon here is the polygon with vertices at $(0, 0)$, $(0, 1)$, and $(1, a)$. Then $\lfloor \alpha_i \rfloor = 0$ and $\lfloor \alpha_a \rfloor = 1$.

5.2.1 $a = 2$

We have a parameterization $(u, y) = (\log(x), y) = (\log(\frac{-z-1}{z^2}), z)$. In this case $R = \{-2, 0\}$. We have, as choices for integration divisor, $z = -1$ and $z = \infty$, being the zeros of x, not in R.

If we take $D = [z] - [-1]$, we have that:

$$\lim_{z_1 \to -1} \psi_1(x(z_1); D) = \psi(D) \lim_{z_1 \to -1} p_0(x(z_1) y(z_1) = 0 \tag{5.2.1}$$

Then, from Theorem 4.34:

$$\hbar\frac{d}{du}(\psi_1(e^u; D)) = \psi_2(e^u; D) - e^{-u}\psi_1(e^u; D) + \hbar\psi_1(x; D) \tag{5.2.2}$$

We have that:

$$\psi_2(e^u; D) = -\psi(D), \qquad \psi_1(e^u; D) = e^u\hbar\frac{d}{du}\psi(D) \tag{5.2.3}$$

Therefore:

$$\left(\hbar^2\frac{d}{du}e^u\frac{d}{du} + \hbar\frac{d}{du} - \hbar^2 e^u\frac{d}{du} + 1\right)\psi = 0 \tag{5.2.4}$$

Which, is equivalent to:

$$\left(\hbar^2 e^u\frac{d^2}{du^2} + \hbar\frac{d}{du} + 1\right)\psi = 0 \tag{5.2.5}$$

Or, we can take $D = [z] - [\infty]$:

$$\lim_{z_1\to\infty}\psi_1(x(z_1); D) = \psi(D)\lim_{z_1\to\infty}p_0(x(z_1)y(z_1) = -\psi(D) \tag{5.2.6}$$

Then, from Theorem 4.34:

$$\hbar\frac{d}{du}(\psi_1(e^u; D)) = \psi_2(e^u; D) - e^{-u}\psi_1(e^u; D) + \hbar(\psi_1(x; D) + \psi(D)) \tag{5.2.7}$$

And, since (5.2.3), we get:

$$\left(\hbar^2 \frac{d}{du} e^u \frac{d}{du} + \hbar \frac{d}{du} - \hbar^2 e^u \frac{d}{du} + 1 - \hbar\right)\psi = 0 \tag{5.2.8}$$

Which, is equivalent to:

$$\left(\hbar^2 e^u \frac{d^2}{du^2} + \hbar \frac{d}{du} + 1 - \hbar\right)\psi = 0 \tag{5.2.9}$$

If we then do the transformation $\tilde{\psi} = e^{-u}\psi$, so that $\tilde{\psi}' = e^{-u}(\psi' - \psi)$, and $\tilde{\psi}'' = e^{-u}(\psi'' - 2\psi' + \psi)$. So, we get:

$$\left(\hbar^2 \frac{d^2}{du^2} e^u + \hbar \frac{d}{du} + 1\right)\tilde{\psi} = 0 \tag{5.2.10}$$

And, these are two of the three different orderings of the quantization that we can get using the approach of this thesis.

5.2.2 $a > 2$

We note, in [2] they study the spectral curve $xy^a + y + 1$ (which is very similar to ours, only theirs was a spectral curve in x and y, ours is a spectral curve in u and y, with $x = e^u$). They cannot analyze in any great detail the case $a > 2$, because of the challenges in investigating divisors that are zeroes of x, in that paper. Here, we can investigate this case, since we have a way for dealing with divisors that are simple zeroes of x.

We have the parameterization $(u, y) = (\log(x), y) = (\log(\frac{-z-1}{z^a}), z)$. Ramification points at $z = \frac{a}{1-a}$ and infinity, and $z = 0$.

We can consider the divisor $D = [z] - [-1]$:

$$E_k = \lim_{z_1 \to -1} p_0(z_1) y(z_1)^k = \lim_{z_1 \to -1} \frac{-z_1^k(z_1+1)}{z_1^a} = 0, \quad k = 1, ..., a-1 \tag{5.2.11}$$

where the E_k were defined in equation (4.5.25).

Then, from (4.5.38):

$$\left[\hbar^a \left(\sum_{k=0}^{a-1} \binom{a-1}{k} (-1)^{a-1-k} \frac{d^k}{du^k}\right) e^u \frac{d}{du} + \hbar \frac{d}{du} + 1\right] \psi(D) = 0 \tag{5.2.12}$$

Which, is equivalent to:

$$\left(\hbar^a e^u \frac{d^a}{du^a} + \hbar \frac{d}{du} + 1\right) \psi = 0 \tag{5.2.13}$$

Then, we consider $D = [z] - [\infty]$:

$$E_k = \lim_{z_1 \to \infty} p_0(z_1) y(z_1)^k = \lim_{z_1 \to \infty} \frac{-z_1^{k+1} - z_1^k}{z_1^a} = 0, \quad k = 1, ..., a-2 \tag{5.2.14}$$

$$E_{a-1} = \lim_{z_1 \to \infty} \frac{-z_1^a - z_1^{a-1}}{z_1^a} = -1 \tag{5.2.15}$$

Then, we get:

$$\left[\hbar^a \left(\sum_{k=0}^{a-1} \binom{a-1}{k} (-1)^{a-1-k} \frac{d^k}{du^k}\right) e^u \frac{d}{du} + \hbar \frac{d}{du} + 1 - \hbar\right] \psi(D) = 0 \tag{5.2.16}$$

If we then do the transformation $\tilde{\psi} = e^{-u}\psi$, so that $\tilde{\psi}' = e^{-u}(\psi' - \psi)$, and $\tilde{\psi}'' = e^{-u}(\psi'' - 2\psi' + \psi)$. So, we get:

$$\left(\hbar^a \frac{d^a}{du^a} e^u + \hbar \frac{d}{du} + 1\right) \tilde{\psi} = 0 \tag{5.2.17}$$

This gives us three possible orderings of the quantization of our spectral curve. The question then arises, how do we get the other orderings? This is a question to which we do not yet have an answer.

5.3 $e^u y^2 - e^u y + 1 = 0$

Once again, we consider that $x = e^u$ is meromorphic on the Riemann sphere, and u is meromorphic on $\mathbb{CP}^1 \backslash \{0, \infty\}$. By the Newton polygon we have $\lfloor \alpha_i \rfloor = 0$ for $i = 0, 1$, and $\lfloor \alpha_2 \rfloor = 1$. We can parameterize this as $(u, y) = (\log(x), y) = (\log(\frac{z^2}{z-1}), \frac{1}{z})$. So, $R = \{0, 2\}$, x has simple poles at $z = 1$ and $z = \infty$, and a zero at $z = 0$ but this is in R.

If we choose $D = [z] - [1]$:

$$C_1 = \lim_{z_1 \to 1} \frac{p_0(z_1) y(z_1)}{x(z_1)} = \lim_{z_1 \to 1} \left(\frac{1}{z_1}\right) = 1 \tag{5.3.1}$$

So, from Lemma 4.37:

$$\left[\hbar^2 \frac{d}{du} e^u \frac{d}{du} - \hbar e^u \frac{d}{du} + 1 - \hbar e^u\right] \psi(D) = 0 \tag{5.3.2}$$

Which, is the same as:

$$\left[\hbar^2 \frac{d}{du} e^u \frac{d}{du} - \hbar \frac{d}{du} e^u + 1\right] \psi = 0 \tag{5.3.3}$$

Or, we can choose $D = [z] - [\infty]$:

$$C_1 = \lim_{z_1 \to \infty} \frac{p_0(z_1) y(z_1)}{x(z_1)} = \lim_{z_1 \to \infty} \left(\frac{1}{z_1}\right) = 0 \tag{5.3.4}$$

So, we get the quantum curve:

$$\left[\hbar^2 \frac{d}{du} e^u \frac{d}{du} - \hbar e^u \frac{d}{du} + 1\right] \psi = 0 \tag{5.3.5}$$

And so, given the quantization $(u, y) \mapsto (\widehat{u}, \widehat{y}) = \left(u, \hbar \frac{d}{du}\right)$, we get the orderings:

$$\begin{aligned} (\widehat{y} e^{\widehat{u}} \widehat{y} - \widehat{y} e^{\widehat{u}} + 1)\psi =& 0, \\ (\widehat{y} e^{\widehat{u}} \widehat{y} - e^{\widehat{u}} \widehat{y} + 1)\psi =& 0 \end{aligned} \tag{5.3.6}$$

5.4 $ye^{-y^q} - x = 0$

This is a curve such that $x = e^u$ is not meromorphic on the Riemann sphere, with $q \in \mathbb{Z}^+$. This is an important curve, coming out of the connection to Hurwitz numbers; we saw an outline of this in Section 2.7. The case where $q = 1$ arises out of simple Hurwitz numbers, the relevant application of which was studied in [7]. The case $q > 1$ comes from what are called r-spin Hurwitz numbers, these were studied in [20].

The standard parameterization of this curve is $(u, y) = (\log(x), y) = (\log\left(ze^{-z^q}\right), z)$. This spectral curve falls into the second class of spectral curves we considered an extension for, and therefore we use the method of Section 3.3.

5.4.1 Applying Truncation Method

Therefore, using the method of Section 3.3, we consider the truncated curve:

$$\sum_{n=0}^{r} \frac{(-1)^n y^{qn+1}}{n!} - x = \frac{(-1)^r}{r!} y^{qr+1} + \frac{(-1)^{r-1}}{(r-1)!} y^{q(r-1)+1} + ... + (-1)y^{q+1} + y - x = 0 \tag{5.4.1}$$

This is a curve of order $qr + 1$, where x and y are meromorphic functions on the Riemann sphere. Then, we will take $r \to \infty$ at the end to get back to our original curve. The Newton polygon is the polygon with vertices $(1, 0)$, $(0, 1)$ and $(0, qr + 1)$. Then $\lfloor \alpha_i \rfloor = 0$, for $i = 1, ..., qr + 1$, and $\lfloor \alpha_0 \rfloor = 1$.

So, $p_{qn}(x) = \frac{(-1)^{r-n}}{(r-n)!}$, $n = 0, 1, 2, ..., r$, and $p_{qr+1}(x) = -x$, and zero for all others. We can parameterize this curve as: $y = z$, $x = \sum_{n=0}^{r} \frac{(-1)^n z^{qn+1}}{n!}$. Then:

$$\frac{dx(z)}{x(z)} = \frac{\sum_{n=0}^{r} \frac{(-1)^n (qn+1) z^{qn}}{n!}}{\sum_{n=0}^{r} \frac{(-1)^n z^{qn+1}}{n!}} = \frac{1 + \mathcal{O}(z)}{\mathcal{O}(z)} \tag{5.4.2}$$

R equals the roots of this, and infinity. We can see that $z = 0$ is a simple zero of x not in R. So, we will consider the result of Lemma 4.39. In our case, $E_k = 0$, for $k = 1, ..., r - 1$.

Therefore, we have, (4.5.38):

$$\left[\hbar^{qr} p_0 \left(\sum_{k=0}^{qr} \binom{qr}{k} (-1)^k \frac{d^{qr-k}}{du^{qr-k}}\right) \hbar \frac{d}{du} + \hbar^{qr-1} p_1 \left(\sum_{k=0}^{qr-1} \binom{qr-1}{k} (-1)^k \frac{d^{qr-1-k}}{du^{qr-1-k}}\right) \hbar \frac{d}{du} + ... + p_{qr} \hbar \frac{d}{du} - e^u\right] \psi(D) = 0 \quad (5.4.3)$$

Or, simplified as:

$$\left[\sum_{n=0}^{r} \hbar^{qn} \sum_{k=0}^{qn} \frac{(-1)^{n-k}(qn)!}{k!n!(qn-k)!} \frac{d^{qn-k}}{du^{qn-k}} \cdot \hbar \frac{d}{du} - e^u\right] \psi = 0 \quad (5.4.4)$$

Which, taking $r \to \infty$, gives:

$$\left[\exp\left(-\hbar^q \left(\frac{d}{du} - 1\right)^q\right) \hbar \frac{d}{du} - e^u\right] \psi = 0$$

$$\implies \left[\hbar \frac{d}{du} - \exp\left(\hbar^q \left(\frac{d}{du} - 1\right)^q\right) e^u\right] \psi = 0$$

Or, using:

$$e^{c\hbar^q \frac{d^m}{du^m}} e^{bu} \phi = e^{bu} \sum_{n=0}^{\infty} \frac{c^n \hbar^{qn}}{n!} \sum_{k=0}^{mn} \binom{mn}{k} b^k \frac{d^{mn-k}}{du^{mn-k}} \phi = e^{bu} e^{c\hbar^q \left(\frac{d}{du} + b\right)^m} \phi$$

a, b, c are constants.

We get the result:

$$\left[\hbar \frac{d}{du} - e^{u} e^{\hbar^q \frac{d^q}{du^q}}\right] \psi = 0 \tag{5.4.5}$$

Which can be seen clearly as the quantization of our original spectral curve, with the operators $\widehat{u} = u$ and $\widehat{y} = \hbar \frac{d}{du}$, and choice of ordering:

$$\left[\widehat{y} - e^{\widehat{u}} e^{\widehat{y}^q}\right] \psi = 0 \tag{5.4.6}$$

5.4.2 Quantum Curve From Hurwitz Numbers

Now, we can compare our result to the quantum curve arrived at in [7][20], which is arrived at using methods coming directly out of Hurwitz numbers. This result is achieved completely independently from the topological recursion; but we should arrive at the same final quantum curve, since (as was discussed in Section 2.7) the topological recursion structure can be seen as being intimately connected to Hurwitz numbers. In [20] they achieve:

$$\left[\hbar \frac{d}{du} - e^{\frac{3}{2}u} \exp\left(\frac{\hbar^q \sum_{i=0}^{q} e^{-u} \frac{d^i}{du^i} e^{u} \frac{d^{q-i}}{du^{q-i}}}{q+1}\right) e^{\frac{-1}{2}u}\right] \tilde{\psi} = 0 \tag{5.4.7}$$

$$\implies \left[\hbar \frac{d}{du} - e^{\frac{3}{2}u} \prod_{i=0}^{q} \prod_{j=0}^{i} \exp\left(\frac{\hbar^q}{q+1} \binom{i}{j} \frac{d^{q-j}}{du^{q-j}}\right) e^{\frac{-1}{2}u}\right] \tilde{\psi} = 0$$

$$\implies \left[\hbar\frac{d}{du} - e^u \prod_{i=0}^{q} \exp\left(\frac{\hbar^q}{q-i+1}\binom{q}{i}\left[\frac{d}{du} - \frac{1}{2}\right]^i\right)\right]\tilde{\psi} = 0 \qquad (5.4.8)$$

We can compare the results of (5.4.5) and (5.4.8), for different choice of q, we see that the operator is not the same, except for $q = 1$. But, we can do a transformation that shows that our result actually is the same as that achieved in [20].

Assume an operator, $\widehat{A}$, such that $\widehat{A}\psi = \tilde{\psi}$, where ψ is our constructed wavefunction, and $\tilde{\psi}$ is the function defined in [20]. Then, if we assume that $\widehat{A} = \exp\left(\sum_{k=1}^{p} a_k \frac{d^k}{du^k}\right)$, where $q < p \in \mathbb{Z}^+$ and a_k are constants. We get (rewriting our result, (5.4.5), and acting on it with $\widehat{A}$):

$$\begin{aligned}
&\widehat{A}\left[\hbar\frac{d}{du} - e^u \prod_{i=0}^{q} \exp\left(\hbar^q(-1)^{q-i}\binom{q}{i}\left[\frac{d}{du} + 1\right]^u\right)\right]\psi \\
&= \left[\hbar\frac{d}{du}\widehat{A} - e^u \prod_{i=0}^{q} \exp\left(\hbar^q(-1)^{q-i}\binom{q}{i}\left[\frac{d}{du} + 1\right]^u\right)\exp\left(\sum_{k=1}^{p} a_k\left[\frac{d}{du} + 1\right]^k\right)\right]\psi \\
&= \left[\hbar\frac{d}{du} - e^u \prod_{i=0}^{q} \exp\left(\hbar^q(-1)^{q-i}\binom{q}{i}\left[\frac{d}{du} + 1\right]^u\right)\exp\left(\sum_{k=1}^{p}\sum_{i=0}^{k-1} a_k\binom{k}{i}\frac{d^i}{du^i}\right)\right]\tilde{\psi} = 0
\end{aligned} \qquad (5.4.9)$$

Therefore, comparing (5.4.9) and (5.4.8), we want the two operators to be the same. For this to be true we must have that:

$$\exp\left(\sum_{k=1}^{p}\sum_{i=0}^{k-1} a_k \binom{k}{i} \frac{d^i}{du^i}\right) = \begin{cases} \prod_{k=0,2,...}^{q-2} \exp\left(\hbar^q \binom{q}{k} \frac{2^{k-q}}{(q+1-k)} \frac{d^k}{du^k}\right), & \text{if q is even} \\ \prod_{k=1,3,...}^{q-2} \exp\left(\hbar^q \binom{q}{k} \frac{2^{k-q}}{(q+1-k)} \frac{d^k}{du^k}\right), & \text{if q is odd} \end{cases} \tag{5.4.10}$$

Matching the coefficients for the derivatives gives $p-1$ equations to solve for a_k, $k = 1, .., p-1$. It is easy to show that there exists a unique solution for all a_k, for any given q. What this shows is that the function arrived at in [20] is the same as our wavefunction multiplied by $\widehat{A}$, as given by above; it would be interesting to study why this is the case, and what this can tell us about the connection between the topological recursion and Hurwitz numbers.

5.5 $y^2e^{y^2+y} - x = 0$

Standard parameterization: $y = z$, $x = y^2e^{y^2+y}$. Truncated as:

$$\sum_{n=0}^{r}\sum_{k=0}^{n} \frac{y^{n+k+2}}{k(n-k)!} - x = 0 \tag{5.5.1}$$

This is a spectral curve of order $2r+2$. Can parameterize as $y = z$, $x = \sum_{n=0}^{r}\sum_{k=0}^{n} \frac{z^{n+k+2}}{k(n-k)!}$. $p_{2r+2-m} = \frac{1}{r!}$

6 Conclusion

To summarize, we found an extension to the results of [2], proving the WKB-topological recursion connection for a a broader class of spectral curve - those falling into the classes of Definitions 1.1 and 1.3. We found that the admissibility condition, Definition 2.14, still holds as in [2]. Also, the extension for the second class, of Definition 1.3, does not have a formal proof as of the writing of this thesis, but does appear to work in the examples considered. We found that, unlike in [2], we can only consider the simple poles and simple zeroes of x as integration divisors; but, also unlike that paper, we can consider the simple zeroes of x in any case considered.

At the conclusion of this thesis, the following open questions remain:

1. We require a rigorous proof of the method for dealing with spectral curves in the second class.

2. For what spectral curves does the limiting procedure, described in Section 3.3, succeed or fail in producing the quantum curve? Presumably by answering question one, this question would be answered. For example, the spectral curve:

$$\sum_{k=0}^{p} a_k e^{n_k y} - e^u = 0, \tag{6.0.1}$$

where the a_k's and n_k's are constants, and $p \in \mathbb{Z}^+$. This curve clearly does not fall into the second class, and yet the methods of this thesis do appear to produce the correct quantum curve (although the result has not been checked computationally).

3. The admissibility condition, of Definition 2.14, needs to be overcome. This is a challenge, not only for this thesis, but also in [2].

4. How does one arrive at all possible orderings of the quantum curve. This is also a challenge in [2]. By answering question 3 this problem may become more clear.

This concludes the analysis of this thesis.

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