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A GENERALIZATION OF WATSON'S LEMMA

by

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A THESIS

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled "A Generalization of Watson's Lemma", submitted by Roderick Sue-Cheun Wong, in partial fulfilment of the requirements for the degree of Doctor of Philosophy.

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ABSTRACT

In this thesis we study the classical problem of determining the asymptotic behavior of a function $F(z)$ defined by

$$F(z) = \int_{\Gamma} g(w)\exp[-z\phi(w)]dw ,$$

where Γ is a continuous curve which may be finite or infinite in length. Although an alternative procedure is illustrated in The Method of Darboux (Chapter VII), it is generally true throughout the thesis that the use of the substitution $t = \phi(w)$ is envisaged, and $F(z)$ will have a canonical representation of the form

$$F(z) = \int_{\Gamma'} f(t)\exp(-zt)dt ,$$

where

$$f(t) = g(w)\,\frac{dw}{dt} ,$$

and Γ' is the image of Γ in the complex t-plane.

In the classical situations considered by Barnes [2] and Watson [21], the success of finding the asymptotic behavior of a Laplace integral $\int_L f(t)\exp(-zt)dt$ depends on $f(t)$ having at most a branch-point singularity at $t = o$ of the form $t^{\lambda-1}$. In this thesis, we study a situation in which $f(t)$ has a singularity at $t = o$ which is of the form $t^{\lambda-1}(\log t)^{\mu}$. Although some work along these lines has already been accomplished by Erdélyi [8], our proofs differ from those

used by Erdélyi and the results of the present thesis are more extensive than the results contained in the paper mentioned above.

As a justification for our study of logarithmic singularities, we find the complete asymptotic expansion of $\mu(z,\beta,\alpha)$, a result which, as far as we are aware, has not yet been obtained.

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CHAPTER I

Watson's Lemma

For many years, the study of the analytical properties of functions, $F(z)$, with integral representations of the form

$$(1.1)\qquad F(z) = \int_{\Gamma} f(t) \exp(-z\,\phi(t))dt, \quad z \text{ a complex variable,}$$

has been one of major mathematical interest. The path of integration, Γ, is normally taken to be a continuous curve in the complete complex t-plane, and Γ may or may not be a closed curve. In spite of the extremely restricted form which the integrand of (1.1) is assumed to have, the extent of the class of functions so defined is sufficiently large to embrace many of the higher transcendental functions of major importance in modern science. The following table illustrates some of the specific functions and classes of functions which have such an integral representation.

Obtaining information concerning the behavior of a function $F(z)$ within a neighborhood of some fixed point $z = z_0$ is of interest to, and indeed in one way or another seems to encompass, most parts of classical analysis. The limit process, the Landau order relations O and o, convergent expansions of various types, and some types of divergent expansions, all were designed to give information of the type described above. There is, however, considerable difference in the detailed information sought and obtained by using these different

TABLE 1

$\phi(t)$	$f(t)$	z	Γ	$F(z) = \int_\Gamma f(t)\exp(-z\phi(t))dt$	Identification
t	$f(t)$	z	straight line joining $t = o$ to $t = \infty$.	$\int_o^\infty f(t)\exp(-zt)dt$	Laplace integral or Laplace transform of $f(t)$.
$-\log t$	$f(t)$	$s - 1$	as above	$F(s-1) = \int_o^\infty t^{s-1}f(t)dt$	Mellin transform of $f(t)$.
$\log t$	$f(t)$	$(n+1)$, n an integer	suitable circle enclosing $t=o$	$2\pi i\, a_n = F(n + 1)$ $= \int_\Gamma t^{-n-1}f(t)dt$	a_n the Maclaurin or Laurent coefficients of $f(t)$, if they exist.
$-\frac{1}{2}(t-t^{-1})$	$t^{-\nu-1}$	z	suitable loop path of integration	$2\pi i J_\nu(z) = \int_{-\infty}^{(o+)} t^{-\nu-1}\exp[\frac{1}{2}z(t-t)^{-1})]dt$	Bessel function of order ν.
$-t$	$t^{a-1}(1-t)^{c-a-1}$	z	straight line joining $t = o$ to $t = 1$.	$\frac{\Gamma(a)\Gamma(c - a)}{\Gamma(c)}\Phi(a, c; z)$ $= \int_o^1 t^{a-1}(1-t)^{c-a-1}\exp(zt)dt$	Φ a confluent hypergeometric function.

procedures. Information of this type is called asymptotic information.

Although the situation in which the fixed point z_o is a finite point in the complex z-plane is of considerable interest, most asymptotic results are stated in a canonical form by taking z_o to be the point at infinity. A particular neighborhood system, $U(z, R, \alpha, \beta)$, defined as the point set for which

$$(1.2) \quad \alpha \leq \arg z \leq \beta, \quad |z| > R, \quad \alpha, \beta \text{ and } R > o \text{ real numbers}$$

plays an important role in deriving asymptotic information. Unless otherwise stated, α and β remain fixed during a discussion. Different members of the set of neighborhoods are obtained by choosing different values of R.

In particular, the Landau order relations are defined as follows:

(a) The notation $F(z) = O(G(z))$, as $z \to \infty$, implies the existence of a fixed constant K such that $|F(z)| \leq K\,|G(z)|$ for every z <u>in</u> some neighborhood $U(z, R, \alpha, \beta)$.

(b) The notation $F(z) = o(G(z))$, as $z \to \infty$, implies that for every choice of fixed $\varepsilon > o$, there exists a neighborhood $U(z, R(\varepsilon), \alpha, \beta)$ such that $|F(z)| \leq \varepsilon |G(z)|$ for every z in $U(z, R(\varepsilon), \alpha, \beta)$.

To illustrate the type of problems that will be considered in the present thesis, suppose that $F(z)$ is defined by a convergent Laplace integral

$$(1.3) \quad F(z) = \int_o^\infty f(t) \exp(-zt)dt, \quad |\arg z| \leq \frac{\pi}{2} - \Delta, \quad \Delta \text{ fixed and positive.}$$

Further, it is assumed that $f(t)$ has a convergent expansion of the form

$$(1.4) \qquad f(t) = \sum_{n=1}^{\infty} b_n \, t^{\frac{n}{r} - 1} , \qquad r \text{ fixed and positive,}$$

which can be integrated term-by-term to yield a convergent series representation of $F(z)$ of the form

$$(1.5) \qquad F(z) = \sum_{n=1}^{\infty} a_n \, z^{-\frac{n}{r}} ,$$

where $a_n = b_n \, \Gamma(n \, r^{-1})$, and each b_n is a fixed complex number.

The power series nature of (1.5) implies that for every fixed integer $N \geq o$,

$$(1.6) \quad F(z) = \sum_{n=1}^{N} a_n \, z^{-\frac{n}{r}} + o(z^{-\frac{N}{r}}), \quad \text{as} \quad z \to \infty \quad \text{in} \quad |\arg z| \leq \frac{\pi}{2} - \Delta .$$

Clearly, (1.6) gives very detailed information of the behavior of $F(z)$ in neighborhoods of the point at infinity. Poincaré made the observation that this information depends only on the result contained in (1.6), and is independent of the assumptions and the procedure by means of which (1.6) was obtained. Indeed, the validity of (1.6) does not imply the convergence of the series in (1.5). The introduction of powers of $z^{-\frac{1}{r}}$ is no more general than the situation in which $r = 1$, and only powers of z^{-1} are considered. One can be obtained from the other by a simple substitution of variables. With this in mind the Poincaré definition of an asymptotic expansion is formulated as follows:

Poincaré's Definition of an Asymptotic Expansion.

A formal power series, $\sum_{n=o}^{\infty} a_n \, z^{-n}$, convergent or divergent, is called an asymptotic expansion of a function $F(z)$ if, for every fixed integer $N \geq o$,

$$(1.7) \qquad F(z) = \sum_{n=o}^{N} a_n z^{-n} + O(z^{-(N+1)}), \quad \text{as} \quad z \to \infty.$$

The validity of (1.7) may be restricted to a sector $\alpha \leq \arg z \leq \infty$. The notation used to indicate the validity of (1.7) is

$$(1.8) \qquad F(z) \sim \sum_{n=o}^{\infty} a_n z^{-n}, \quad \text{as} \quad z \to \infty \quad \text{in} \quad \alpha \leq \arg z \leq \beta .$$

An important property of the Poincare asymptotic expansion is that the expansion, if it exists, is unique. The coefficients a_n are determined by the recurrence relations:

$$(1.9) \qquad a_o = \lim_{z \to \infty} F(z), \qquad a_m = \lim_{z \to \infty} z^m[F(z) - \sum_{n=o}^{m-1} a_n z^{-n}] .$$

These formulae, coupled with the fact that $\lim_{z \to \infty} z^m[\exp(-b|z|^p)] = 0$, for every non-negative integer m, and all fixed, positive, numbers b and p, imply that every function $g(z)$, satisfying $g(z) = O(\exp(-b|z|^p))$, as $z \to \infty$ in $\alpha \leq \arg z \leq \beta$, has the unique asymptotic expansion

$$(1.10) \qquad g(z) \sim 0, \quad \text{as} \quad z \to \infty \quad \text{in} \quad \alpha \leq \arg z \leq \beta.$$

In turn, it will be true, that two functions $F(z)$ and $G(z)$ such that

$$(1.11) \qquad F(z) = G(z) + g(z),$$

$g(z)$ as above, will have the same asymptotic expansion.

Functions satisfying the order relation $g(z) = O(\exp(-b|z|^p))$, as $z \to \infty$ in $\alpha \leq \arg z \leq \beta$ are said to be exponentially small, and it is usual to replace such functions by zero at any stage in a proof being used to establish the validity of Poincare asymptotic expansion.

Within the framework provided by Poincare, the determination of the asymptotic behavior of functions defined by (1.1) has been the object of intensive study. Among the many results now known, one due to G.N. Watson [21] is of major importance. In order to provide a basis for a generalization of the result due to Watson, now known as Watson's Lemma, his statement and proof of the result are reproduced below.

Watson's Lemma.

If (i) $f(t)$ is analytic when $|t| \leq a + \delta$, where $a > o$, $\delta > o$, except at a branch-point at the origin, and

(1.12) $$f(t) = \sum_{m=1}^{\infty} a_m t^{\frac{m}{r} - 1}$$

when $|t| \leq a$, r being positive; (ii) $|f(t)| < K e^{bt}$, where K and b are independent of t, when t is positive and $t \geq a$; (iii) $|\arg z| \leq \frac{\pi}{2} - \Delta$, where $\Delta > o$; and (iv) $|z|$ is sufficiently large: then there exists a complete asymptotic expansion given by the formula

(1.13) $$F(z) = \int_o^{\infty} f(t) \exp(-zt)dt \sim \sum_{m=1}^{\infty} a_m \Gamma(\tfrac{m}{r}) z^{-\frac{m}{r}} .$$

Proof: If M is any fixed integer, we have

(1.14) $$|f(t) - \sum_{m=1}^{M-1} a_m t^{\frac{m}{r} - 1}| < K_1 t^{\frac{M}{r} - 1} \exp(bt)$$

throughout the range of integration, where K_1 is some number independent of t.

Hence

$$(1.15) \qquad \int_0^\infty f(t)\exp(-zt)dt = \sum_{m=o}^{M-1} \int_0^\infty a_m t^{\frac{m}{r}-1} \exp(-zt)dt + R_M ,$$

where

$$(1.16) \qquad |R_M| < \int_0^\infty K_1 t^{\frac{M}{r}-1} \exp(bt) \; |\exp(-zt)|dt$$

$$< K_1 \Gamma(\tfrac{M}{r}) \{Re(z) - b\}^{-\frac{M}{r}} ,$$

provided that $Re z > b$, which is the case when $|z|$ is sufficiently large; and since $\{Re(z) - b\}^{-1} = 0(\frac{1}{z})$ for the range of values of z under consideration, we have

$$(1.17) \qquad \int_0^\infty f(t)\exp(-zt)dt = \sum_{m=1}^{M-1} a_m \Gamma(\tfrac{m}{r}) z^{-\frac{m}{r}} + 0(z^{-\frac{M}{r}}) ,$$

and so the integral possesses the complete asymptotic expansion, which is of Poincaré's type.

In attempting to generalize Watson's Lemma it is important to separate the necessary and sufficient conditions which he uses from those that are only sufficient. It is known [7] that a necessary and sufficient condition for a Laplace integral to converge is that fixed real numbers K and σ must exist for which

$$(1.18) \qquad \left|\int_0^t f(u)du\right| \leq K \exp(\sigma t), \quad t \geq 0 ,$$

and the integral will then converge when $Re z > \sigma > o$. If the integral in

$$(1.19) \qquad F(z) = \int_0^\infty g(t) \exp(-zt)dt$$

is convergent, then integration by parts gives

$$(1.20)\qquad F(z) = z \int_o^\infty f(t)\ \exp(-zt)dt, \quad f(t) = \int_o^t g(u)du\ ,$$

and therefore $|f(t)| \leq K \exp(\sigma t)$, $t \geq o$. Although this condition on $f(t)$ does not imply that the same condition is satisfied by $g(t)$ in (1.19), the integral representation of (1.20) can be considered as a canonical form for (1.19). For this reason, the condition $|f(t)| \leq K \exp(\sigma t)$, which is assumed by Watson, is actually implied, for the canonical form (1.20), by the convergence of the integral. In its canonical form, there is of course no hope that this condition can be either weakened or generalized.

Since the translation $t = t' + A$ places

$$(1.21)\qquad F(z, A) = \int_A^\infty g(t)\ \exp(-zt)dt$$

into the form

$$(1.22)\qquad F(z, A) = \exp(-zA) \int_o^\infty g(t + A)\ \exp(-zt)dt\ ,$$

it is clear that for every fixed $A > o$, the convergence of (1.21) is sufficient to ensure that

$$(1.23)\qquad F(z, A) = O(\exp(-zA)), \quad \text{as} \quad z \to \infty \quad \text{in} \quad |\arg z| \leq \frac{\pi}{2} - \Delta\ ,$$

and, therefore, $F(z, A)$ is, under these conditions, exponentially small. Since

$$(1.24)\qquad F(z) = \int_o^A g(t)\ \exp(-zt)dt + \int_A^\infty g(t)\ \exp(-zt)dt$$

$$= \int_o^A g(t)\ \exp(-zt)dt + O(\exp(-zA)), \quad \text{as} \quad z \to \infty \quad \text{in} \quad |\arg z| \leq \frac{\pi}{2} - \Delta,$$

one can and must obtain the desired asymptotic information about $F(z)$, up to the order of terms that are exponentially small, from $\int_{o}^{A} g(t)\exp(-zt)dt$. It is this property of Laplace integrals that enables one to see that the detailed asymptotic behavior of $F(z)$ must be determined by the behavior of $g(t)$ in an infinitesimal interval $o \leq t \leq A$. Watson has obtained this behavior when $g(t)$ has a singularity at $t = o$ of branch-point type. Although it is possible to obtain worthwhile generalizations of Watson's Lemma for branch-point singularities, the major areas in which new results now lie are in investigations in which the nature of the singularity differs from the one used in Watson's Lemma. In the present thesis, integrands with logarithmic singularities will receive detailed study.

An examination of (1.23) suggests that the essential feature of the result could be preserved without requiring A to be fixed. It seems possible that a requirement like $A \geq b|z|^{p-1}$, p fixed $(0 < p \leq 1)$, b fixed and positive, may imply that $F = O(\exp(-\varepsilon|z|^{p}))$, as $z \to \infty$ in $|\arg z| \leq \frac{\pi}{2} - \Delta$, for some fixed $\varepsilon > o$. Thus A might be a function of z with the property $\lim_{z \to \infty} A = 0$. The desired result is established in the following lemma.

<u>Lemma 1.1.</u>

Consider

$$(1.25) \qquad F(z, A) = \int_{A}^{\infty} f(t) \exp(-zt)dt, \qquad |\arg z| \leq \frac{\pi}{2} - \Delta ,$$

where $A = A(z)$ is positive for every finite z in $U(z, R, -\frac{\pi}{2} + \Delta, \frac{\pi}{2} - \Delta)$ for some fixed R, and $A(z)$ is bounded in U. If

(i) $\lim_{z \to \infty} |z| \; A(z) = \infty$

(ii) $|f(t)| \leq M \; t^{\lambda-1} \exp(\sigma t), \quad t \geq A,$

where M, σ and λ are arbitrary fixed real numbers, then

$$(1.26) \qquad |F(z, A)| = o(A^{\lambda} \exp(-zA)), \quad \text{as} \quad z \to \infty \quad \text{in} \quad U.$$

Proof: The assumptions imply

$$(1.27) \qquad |F(z, A)| \leq M \int_A^{\infty} t^{\lambda-1} \exp(\sigma t) \; |\exp(-zt)| dt.$$

By means of the substitution $t = A(1 + t')$ one obtains

$$(1.28) \quad |F(z,A)| \leq M \; A^{\lambda} |\exp(-(z-\sigma)A)| |\int_0^{\infty} (1 + t)^{\lambda-1} |\exp[-(z - \sigma)At]| dt.$$

If $\lambda \leq 1$, $(1 + t)^{\lambda-1} \leq 1$ in $t \geq o$, and if $\lambda \geq 1$, $(1 + t)^{\lambda-1} \leq \exp(\lambda t)$ in $t \geq o$. In either case, $(1 + t)^{\lambda-1} \leq \exp(|\lambda| t)$.

Hence

$$(1.29) \quad \int_0^{\infty} (1+t)^{\lambda-1} |\exp(-(z-\sigma)At)| dt \leq \int_0^{\infty} \exp[-(A|z|\sin\Delta - A\sigma - |\lambda|)t] dt$$

$$\leq \frac{1}{A|z| \; \sin \Delta - A\sigma - |\lambda|} \to o, \quad \text{as} \quad |z| \to \infty,$$

because $\lim_{z \to \infty} |z| A(z) = \infty$, A is bounded, and Δ, σ and $|\lambda|$ are fixed numbers. This, of course, implies the required result

$$(1.30) \qquad F(z,A) = o(A^{\lambda} \exp(-zA)), \quad \text{as} \quad z \to \infty \quad \text{in} \quad |\arg z| \leq \frac{\pi}{2} - \Delta.$$

A more general result than Lemma 1.1, with the exception that λ is more restricted, is contained in Lemma 3.1, Erdelyi and Wyman [12].

When $\lambda \geq o$, A bounded immediately gives $F(z,A) = c(\exp(-zA))$, and, as before, the condition $A \geq b|z|^{p-1}$ will imply $F(z,A)$ is exponentially small. On the other hand, $\lambda < o$ introduces a factor that is a fixed power of $|z|$, and hence, will not modify this particular result. This implies

$$(1.31) \qquad F(z,A) = o(\exp(-\varepsilon|z|^p)), \quad \text{as} \quad z \to \infty \quad \text{in} \quad |\arg z| \leq \frac{\pi}{2} - \Delta ,$$

as long as A is bounded and $A \geq b|z|^{p-1}$.

It will be of later interest that the result contained in (1.31) does not require the convergence of

$$(1.32) \qquad F(z) = F(z,o) = \int_o^\infty f(t) \exp(-zt)dt.$$

The particular instance in which convergence may fail, $\lambda < o$, will be of value when other paths of integration, which must of course avoid the point $t = o$, are considered.

Having described or established some of the general properties of Laplace integrals, procedures will now be discussed by means of which the conditions of Watson's Lemma can be relaxed. Suppose, by means of Watson's Lemma, it is established that

$$(1.33) \qquad F(z) = \int_o^\infty f(t) \exp(-zt)dt$$

has an asymptotic expansion of the form

$$(1.34) \qquad F(z) \sim \sum_{n=1}^{\infty} a_n z^{-\frac{n}{r}}, \quad z \to \infty \quad \text{in} \quad |\arg z| \leq \frac{\pi}{2} - \Delta.$$

Sometimes the expansion (1.34) might be valid even if $z \to \infty$ in a wider range of $\arg z$ than the one given above, and examples are known where z may be allowed to approach ∞ in $\alpha \leq \arg z \leq \beta$, where $\beta - \alpha$ might be greater than 2π. On the other hand, there may well be no relation between the asymptotic behavior of $F(z)$ in the region $|\arg z| \leq \frac{\pi}{2} - \Delta$ and in regions excluding $|\arg z| \leq \frac{\pi}{2} - \Delta$. For example,

$$(1.35) \qquad z/1 + z = \sum_{n=0}^{\infty} (-1)^n z^{-n}, \quad |z| > 1 ,$$

is a convergent expansion, which provides an asymptotic expansion as $z \to \infty$ with $\arg z$ unrestricted. The function $z(1 + e^{-z})/1 + z$ has the following asymptotic representation:

$$(1.36) \quad z(1+e^{-z})/(1+z) \sim \sum_{n=0}^{\infty} (-1)^n z^{-n}, \; z \to \infty \quad \text{in} \quad |\arg z| \leq \frac{\pi}{2} - \Delta \; ;$$

$$(1.37) \quad z(1+e^{-z})/(1+z) \sim e^{-z}[\sum_{n=0}^{\infty} (-1)^n z^{-n}], \quad z \to \infty \quad \text{in} \quad \frac{\pi}{2} + \Delta \leq |\arg z| \leq \frac{3\pi}{2} - \Delta.$$

The asymptotic behavior of $z(1 + e^{-z})/(1 + z)$ is quite different in the two sectors $|\arg z| \leq \frac{\pi}{2} - \Delta$ and $\frac{\pi}{2} + \Delta \leq |\arg z| \leq \frac{3\pi}{2} - \Delta$.

There exist quite powerful, but not necessarily infallible, methods by means of which the range of validity of an asymptotic expansion can be extended beyond that normally given by Watson's Lemma. Sometimes the integrand has sufficient regularity to allow a deformation of the path of integration to take place. Thus the same function $F(z)$ may have the integral representation

$$(1.38) \quad F(z) = \int_0^{\infty e^{i\gamma}} f(t) \exp(-zt)dt \qquad |\arg(ze^{i\gamma})| \leq \frac{\pi}{2} - \Delta .$$

$$= e^{i\gamma} \int_0^{\infty} f(te^{i\gamma}) \exp(-ze^{i\gamma}t)dt ,$$

where, by analytic continuation, the choice of the real number γ may be arbitrary within a certain interval. One might then use Watson's Lemma for (1.32) to retrieve the same asymptotic expansion

$$(1.39) \qquad F(z) \sim \sum_{n=1}^{\infty} a_n z^{-\frac{n}{r}}, \quad \text{as} \quad z \to \infty \quad \text{in} \quad |\arg(ze^{i\gamma})| \leq \frac{\pi}{2} - \Delta.$$

If, for example, the choice of γ is arbitrary in $|\gamma| < \frac{\pi}{2}$, then the range of validity of (1.39) becomes $|\arg z| \leq \pi - \Delta$, an extension of the range of validity of the expansion obtained, in the first instance, by the use of Watson's Lemma. An important situation in which this particular technique fails occurs when the path of integration is finite in length, say $f(t) = o$, $t \geq a > o$. Thus

$$(1.40) \qquad F(z) = \int_o^a f(t) \exp(-zt)dt.$$

It is interesting to note that while the integral in (1.40) may well converge for every finite z, and $\arg z$ unrestricted, the initial use of Watson's Lemma still imposes the condition $z \to \infty$ in $|\arg z| \leq \frac{\pi}{2} - \Delta$ for the validity of

$$(1.41) \qquad F(z) \sim \sum_{n=o}^{\infty} a_n z^{-\frac{n}{r}}.$$

The substitution $t = a - t'$ gives

$$(1.42) \qquad [\exp(az)]\, F(z) = \int_o^a f(a - t) \exp[-(-z)t]dt ,$$

and a second use of Watson's Lemma would give, under the condition of that Lemma, that

$$(1.43) \qquad [\exp(az)]\, F(z) \sim \sum_{n=1}^{\infty} b_n z^{-\frac{n}{r'}}, \quad z \to \infty \quad \text{in} \quad |\arg(-z)| \leq \frac{\pi}{2} - \Delta ,$$

the constants b_n and r' of (1.43) need not bear any relation to the constants a_n and r in (1.41). When the exponential factor, $\exp(az)$, is taken into account, the asymptotic behavior of $F(z)$ exhibited by (1.41) and (1.43) is quite different in the two different sectors $|\arg z| \leq \frac{\pi}{2} - \Delta$ and $|\arg z| \geq \frac{\pi}{2} + \Delta$.

Two important properties must be noted. The use of non-integral powers of z implies that the z-plane has been cut in a suitable manner to make z^α a single valued function, this cut normally being made from $z = o$ to $z = -\infty$, with the implication that $-\pi < \arg z \leq \pi$. The asymptotic behaviors exhibited in (1.41) and (1.43) do not provide a complete description of the asymptotic behavior of $F(z)$ as $z \to \infty$ in the complete-cut-plane, $-\pi < \arg z \leq \pi$. There exist gaps in the sectors which include $\arg z = \pm \frac{\pi}{2}$. To overcome this deficiency, the following technique is often useful. The straight line path of integration joining $t = o$ to $t = a$ is deformed in an infinitesimal manner as described below:

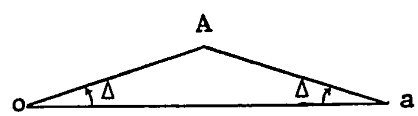


Figure 1.1.

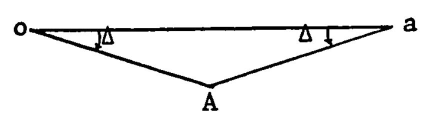


Figure 1.2.

One hopes that the integrand has sufficient regularity to allow a use of Cauchy's theorem to write

$$(1.44)\quad F(z) = \int_o^A f(t)\exp(-zt)dt + \int_A^a f(t)\exp(-zt)dt$$
$$= \int_o^A f(t)\exp(-zt)dt + \exp(-az)\int_o^{a-A} f(a-t)\exp(zt)dt.$$

The use of Watson's Lemma as illustrated for (1.38) would be expected to yield, for the situation in Figure 1.1,

$$F(z) \sim \sum_{n=1}^{\infty} a_n z^{-\frac{n}{r}} + \exp(-az) \sum_{n=1}^{\infty} b_n z^{-\frac{n}{r}}, \tag{1.45}$$

with the obvious meaning that, for every fixed integer $N \geq o$ and $M \geq o$,

$$F(z) = \sum_{n=1}^{N} a_n z^{-\frac{n}{r}} + O(z^{-\frac{(N+1)}{r}}) + \exp(-az)[\sum_{n=1}^{M} b_n z^{-\frac{n}{r'}} + O(z^{-\frac{(M+1)}{r'}})], \tag{1.46}$$

and a range of validity provided by the intersection of

$$|\arg(z \exp(i\Delta))| \leq \frac{\pi}{2} - \frac{\Delta}{2} \quad \text{and} \quad |\arg(-z \exp(-i\Delta))| \leq \frac{\pi}{2} - \frac{\Delta}{2}; \tag{1.47}$$

or

$$-\frac{\pi}{2} - \frac{1}{2}\Delta \leq \arg z \leq \frac{\pi}{2} - \frac{3}{2}\Delta \quad \text{and} \quad -\frac{3\pi}{2} + \frac{3}{2}\Delta \leq \arg z \leq -\frac{\pi}{2} + \frac{\Delta}{2}; \tag{1.48}$$

or

$$-\frac{\pi}{2} - \frac{\Delta}{2} \leq \arg z \leq -\frac{\pi}{2} + \frac{\Delta}{2}, \tag{1.49}$$

and one of the missing gaps has been covered. The validity in the remaining gap is obtained by the deformation indicated in Figure 1.2.

Although the proof used establishes the validity of (1.46) in a very narrow interval of $\arg z$, the actual validity is the complete-cut-complex z-plane, $|\arg z| \leq \pi - \Delta$. When $|\arg z| \leq \frac{\pi}{2} - \Delta$, all of the terms in (1.46) involving the exponential factor are $O(z^{-(N+1)/r})$ as $z \to \infty$ in $|\arg z| \leq \frac{\pi}{2} - \Delta$, and therefore

$$(1.50)\quad F(z) = \sum_{n=1}^{N} a_n z^{-\frac{n}{r}} + O(z^{-(N+1)/r}), \quad \text{as} \quad z \to \infty \quad \text{in} \quad |\arg z| \le \frac{\pi}{2} - \Delta,$$

or

$$(1.51)\qquad F(z) \sim \sum_{n=1}^{\infty} a_n z^{-\frac{n}{r}}, \quad \text{as} \quad z \to \infty \quad \text{in} \quad |\arg z| \le \frac{\pi}{2} - \Delta,$$

which is, of course, the correct result for the range of $\arg z$ given. Similarly, for $|\arg z| \ge \frac{\pi}{2} + \Delta$, all of the terms not possessing the exponential factor are $O((\exp(-az))z^{-(M+1)/r'})$, and therefore

$$(1.52)\quad F(z) \sim \exp(-az) \sum_{n=1}^{\infty} b_n z^{-\frac{n}{r'}}, \quad \text{as} \quad z \to \infty \quad \text{in} \quad |\arg z| \ge \frac{\pi}{2} + \Delta,$$

which is again the correct asymptotic expansion for the indicated range of $\arg z$. However, the proof of (1.45) was given for the missing gaps in $\arg z$, and hence (1.45) is valid as $z \to \infty$ in $|\arg z| \le \pi - \Delta$.

The use of (1.45) to describe the asymptotic behavior of $F(z)$ as $z \to \infty$ in $|\arg z| \le \pi - \Delta$ gives the first indication that strict interpretation of the Poincaré framework is not sufficiently general to describe adequately the asymptotic behavior of relatively simple Laplace integrals. Although the modifications required to accommodate (1.45) are simple to make, the existence of (1.45) does indicate that it may be desirable to provide a much more general framework in which asymptotic theory may be discussed. Such a framework will be described in the next chapter.

Before concluding the present chapter, it is desirable to call attention to some of the properties of the seemingly more general integral representation given in (1.1), (1.53) $$F(z) = \int_{\Gamma} f(t) \exp(-z\,\phi(t))dt.$$

If one desires to use the Watson's Lemma technique on (1.53), an obvious substitution to try is

(1.54) $$\phi(t) = \omega, \text{ with inverse } t = t(\omega),$$

resulting in

(1.55) $$F(z) = \int_{\Gamma'} f(t) \frac{dt}{dw} \exp(-zw)dw,$$

where Γ' is the image of Γ in the w-plane. Let us suppose that Γ begins at $t = A$ and ends at $t = B$. These points may be the same, making Γ a closed curve, or one, or both, may be the point at infinity. It is assumed that $\phi(t)$ has sufficient regularity to ensure that the image Γ' is a continuous curve in the w-plane, and $f(t)$ will allow an integration by parts to take place. It is not assumed that $\frac{dt}{dw}$ is continuous at every point of the path of integration.

Integrating by parts gives

(1.56) $$F(z) = \left(\int_{\phi(A)}^{w} f(t)\frac{dt}{dw}\,dw\right) \exp(-zw)\Big]_{w=\phi(A)}^{w=\phi(B)} + z\int_{\Gamma'} \Phi(w)\exp(-zw)dw.$$
$$= \left(\int_{A}^{B} f(t)dt\right) \exp(-z\,\phi(B)) + z\int_{\Gamma'} \Phi(w) \exp(-zw)dw,$$

where

(1.57) $$\Phi(w) = \int_{\phi(A)}^{w} f(t) \frac{dt}{dw}\,dw = \int_{A}^{t} f(t)dt.$$

Hence the determination of the asymptotic behavior of $F(z)$, given by (1.53), becomes equivalent to the determination of the asymptotic behavior of

$$G(z) = \int_{\Gamma'} \Phi(w) \exp(-zw)dw , \tag{1.58}$$

an integral that at least has the Laplace form of integrand. For definiteness in description, let us assume $\operatorname{Re} \phi(B) > \operatorname{Re} \phi(A)$. When $\Phi(w)$ has sufficient regularity to deform Γ' into the straight line joining $w = \phi(A)$ to $w = \phi(B)$, then the Watson's Lemma procedure might apply. Although far from obvious at this time, the possibility that $\operatorname{Re} \phi(B) = \operatorname{Re} \phi(A)$ does not actually cause a major difficulty. In the situation described above, the translation $w = t + \phi(A)$ gives

$$[\exp (z \phi(A))] G(z) = \int_0^{\phi(B)-\phi(A)} g(t) \exp(-zt)dt , \tag{1.59}$$

and the precise form of a Laplace integral has been obtained. The general properties of these integrals tells us that, up to the order of exponentially small terms, one need only discuss the asymptotic behavior of

$$G(z, E) = \int_0^E g(t) \exp(-zt)dt , \tag{1.60}$$

where $|E|$ can be chosen to be positive, and arbitrarily small. This very important fact makes the procedure outlined considerably easier than one might have expected from the original form of the integral. Once the existence of Laplace form of integral is established, it is not necessary to exhibit the inversion involved in (1.54)

$$\phi(t) = w, \qquad t = t(w) , \tag{1.61}$$

and the subsequent expression, (1.57), of $\int_A^t f(t)dt$ in terms of w, at every point of the path of integration. These need only be exhibited in an infinitesimal neighborhood. Since the Bürmann-Lagrange inversion formulae [4] were specifically designed to solve inversion problems of this type, it is fair to say that the technique described thus far provide a powerful tool which, under certain rather general conditions, will yield the asymptotic behavior of functions defined by generalized Laplace integrals of the type given in (1.53). In spite of this remark, the procedure does fail in certain important situations.

When an attempt is made to perform the inversion of $\phi(t) = w$ to $t = t(w)$, it is not unusual to find that $t = t(w)$ has one or more singularities. For the moment, it is sufficient to concentrate on the case when only one singularity is involved, and for simplicity, it will be assumed that a translation has been made so that the singularity is at the origin $w = o$. Further, it will be assumed that a cut along a straight line joining $w = o$ to $w = w_o$ (Re $w_o > o$), will make $t = t(w)$ single-valued along the path of integration Γ'. A possible situation is illustrated below.

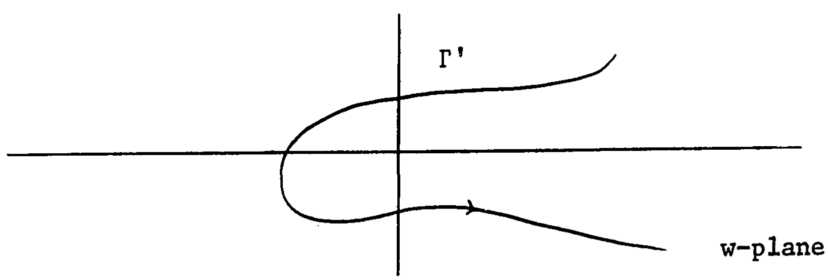


Figure 1.3.

Although it may be possible to deform the contour into two straight lines, one on the top of the cut and the other on the bottom, it may be that the best that can be done is as illustrated below:

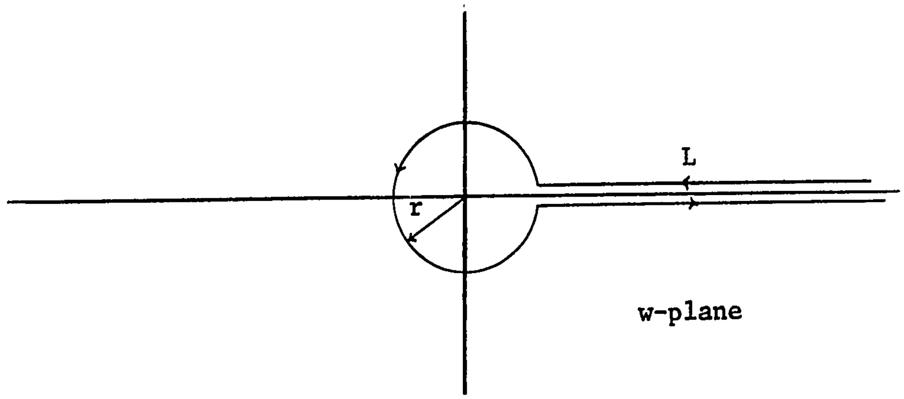


Figure 1.4.

The deformation of Γ' shown in Figure 1.4 is the well-known loop path of integration, except that it is not necessary to assume the path of integration is infinite in length. Although the radius of the circle r can be taken to be fixed and arbitrarily small, the choice $r = o$ may be forbidden because the integral involved

$$G(z) = \int_L \Phi(w) \exp(-zw)dw \tag{1.62}$$

may fail to converge when the singularity at $w = o$ is on the path of integration. In spite of this, the implication of the use of Lemma 1.1, would reduce the problem to the determination of the asymptotic behavior of

$$G(z, r) = \int_{|w|=r} \Phi(w) \exp(-zw)dw , \tag{1.63}$$

and the infinitesimal straight line for Laplace integrals has been replaced by an infinitesimal circle.

The first reasonably general theorem dealing with a problem of this type is, as far as we are aware, due to E.W. Barnes [2]. This result is of sufficient importance for the present thesis to reproduce the statement of the result, and the proof used by Barnes.

<u>Barnes' Lemma</u>.

Given the contour integral

$$I = \frac{i}{2\pi} \int_C (-t)^{\beta-1} f(t) \exp(-zt)dt , \tag{1.64}$$

where C is the Gamma function contour, Figure 1.5, which encloses the origin and embraces an axis P from the origin along which $\mathrm{Re}\,(zt)$ is positive, if

(A) the function $f(t)$ admits the convergent expansion

$$f(t) = \sum_{n=o}^{\infty} C_n(-t)^n , \quad \text{for } |t| < \ell ; \tag{1.65}$$

(B) the integral I is convergent;

and (C) the complex t-plane is dissected by lines passing away from the poles of $f(t)$ to infinity in a direction away from the origin, and the contour C does not contain or cut any of these lines; then

$$I \sim \sum_{n=o}^{\infty} \frac{C_n}{\Gamma(1 - \beta - n)\, z^{\beta+n}} , \quad \text{as } |z| \to \infty . \tag{1.66}$$

<u>Proof</u>: Divide up the contour C into two parts L and M. L lies wholly within the circle of convergence of $f(t)$ and, on L, $|z| \leq \ell'$, where $\ell' = \ell - \varepsilon$ and ε is a positive quantity as small as we please. M forms the remainder of the contour.

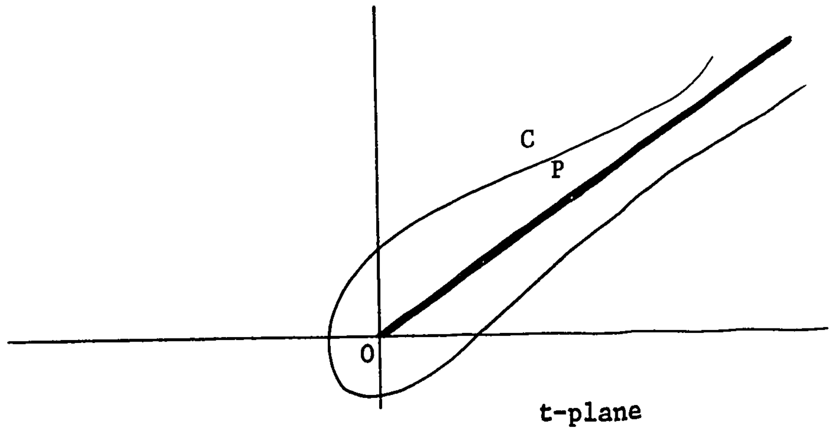


Figure 1.5.

We have

$$(1.67)\quad I - \sum_{n=o}^{N} \frac{C_n}{\Gamma(1-\beta-n)z^{n+\beta}} = \frac{i}{2\pi}\int_C (-t)^{\beta-1}\{f(t) - \sum_{n=o}^{N} C_n(-t)^n\}\exp(-zt)dt$$

$$= I_1 + I_2 ,$$

where I_1 is the integral taken along the contour L and I_2 the sum of the integrals along the two parts of the contour M.

In the integral I_1 put $zt = u$ and let L' be the transformed contour. The integral becomes

$$(1.68)\quad \frac{i}{2\pi\ z^{\beta+N+1}} \int_{L'} (-u)^{N+\beta}[(-\frac{z}{u})^{N+1} \sum_{n=N+1}^{\infty} C_n\ (-\frac{u}{z})^n]\ \exp(-u)du\ .$$

For any assigned finite value of N, however large,

$$(1.69)\quad |(-\frac{z}{u})^{N+1} \sum_{n=N+1}^{\infty} C_n(-\frac{u}{z})^n| = |C_{N+1} - C_{N+2}\frac{u}{z} + C_{N+3}(\frac{u}{z})^2 - \cdots|$$

$$\leq |C_{N+1}| + |C_{N+2}|\ell' + |C_{N+3}|\ell'^2 + \cdots .$$

This series is absolutely convergent and independent of z or u. We may therefore say that

(1.70) $$\left|\left(-\frac{z}{u}\right)^{N+1} \sum_{n=N+1}^{\infty} c_n \left(-\frac{u}{z}\right)^n\right| < R_N ,$$

where R_N is independent of z or u, and is finite when N is finite.

Hence

(1.71) $$|I_1| \leq \frac{1}{|z^{\beta+N+1}|} \left|\frac{i}{2\pi}\right| \int_{L'} |(-u)^{N+\beta}| \, R_N |\exp(-u)| \, |du|$$

$$= O(z^{-\beta-N-1}).$$

Consider in the next place the integral I_2 .

If the original contour cut none of the lines of dissection of the plane, it may be closed up as in the figure below:

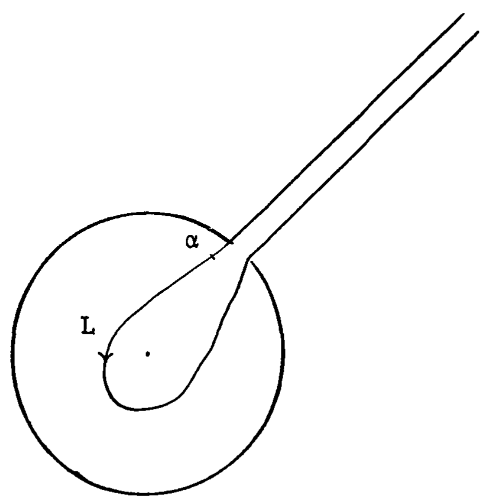


Figure 1.6.

For, as we pass over no poles of the subject of integration, by Cauchy's theorem we do not alter its value. The contour integral I_2 can therefore be replaced by

$$\text{(1.72)} \qquad \frac{\sin \pi \beta}{\pi} \int t^{\beta-1} [f(t) - \sum_{n=o}^{N} C_n(-t)^n] \exp(-zt)dt \ ,$$

a line integral taken from the point α along the axis P to infinity.

If we put $t = \alpha + u/z$, we get

$$\text{(1.73)} \quad I_2 = \frac{\sin \pi \beta}{\pi} \exp(-\alpha z) \int_o^{\infty} (\alpha + \frac{u}{z})^{\beta-1} [f(\alpha + \frac{u}{z}) - \sum_{n=o}^{N} C_n(-\alpha-u/z)^n] \exp(-u) du/z$$

and the integral is taken along a line for which $Re(u)$ is positive.

By our original hypothesis the integral is convergent. It is finite for any assigned finite value of N, and when $|z|$ is very large it tends to a finite limit. Hence $|I_2|$ tends exponentially to zero with $1/|z|$.

Therefore for all finite values of N however large we may take $|z|$ so large that

$$\text{(1.74)} \qquad |I - \sum_{n=o}^{N} C_n/z^{\beta+n}\, \Gamma(1 - \beta - n)| = O(z^{-\beta-N-1}), \quad \text{as} \quad |z| \to \infty \ .$$

This completes the proof of the lemma.

It is of historical interest to note that the essence of Watson's Lemma which first appeared in 1918 is actually contained in the result of Barnes published earlier in 1906. Indeed, by allowing P to be the real axis, and using the particular instance of the contour C shown below:

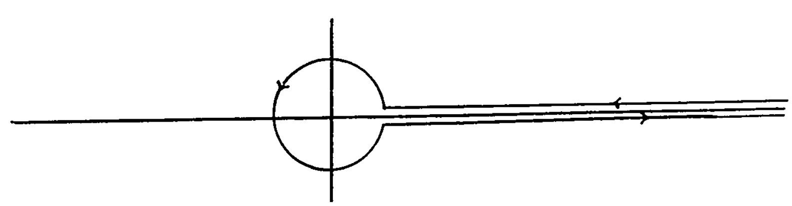


Figure 1.7.

one can, by shrinking the circular portion to zero, obtain

(1.75) $$I = \frac{\sin \pi\beta}{\pi} \int_0^\infty t^{\beta-1} f(t) \exp(-zt)dt, \quad \text{providing} \quad \text{Re}\beta > 0.$$

From the identity $\Gamma(1 - \beta - n)\, \Gamma(n + \beta) = (-1)^n\, \pi/\sin(\pi\beta)$, the result

(1.76) $$\int_0^\infty t^{\beta-1} f(t) \exp(-zt)dt \sim \sum_{n=0}^\infty (-1)^n\, c_n\, \Gamma(n+\beta) z^{-(n+\beta)}, \quad \text{as} \quad |z| \to \infty,$$

and $|\arg z| \leq \frac{\pi}{2} - \Delta$, follows. A minor difficulty appears when β is a positive integer.

The use of the loop path of integration allows Barnes to consider the possibility $\text{Re}\beta \leq 0$, a situation that makes the integral (1.76), used by Watson, to be divergent. At the same time, the expansion (1.65) seems more restrictive than the one used by Watson in (1.12). However, a detailed study of the proof used by Barnes shows that it is possible, with very minor modifications, to obtain a far more general result than the one proposed by Watson. It would seem that this latter result is misnamed in mathematical literature, and might well be called Barnes' Lemma.

Until fairly recently, it was not considered necessary to discuss problems of this type for more than one singularity. The reasons for this point of view are two-fold. Even though it was well known that the inversion of

(1.77) $$\phi(t) = w$$

might produce more than one singularity, it was envisaged that the singularities, say $w = a_0, a_1, \cdots$, were distinct and of finite affix.

If the deformation of Γ' could be made to take the form of a loop about one and only one singularity, then the existence of the others would be irrelevant as far as required asymptotic behavior was concerned. Even if this could not be done, the fact that one could order the singularities according to $\mathrm{Re}\, a_0 < \mathrm{Re}\, a_1 < \mathrm{Re}\, a_2 < \cdots$ would imply that one and only one of these singularities need be considered. Suppose, for example, the situation is as illustrated below, whereby a translation has been used to make $a_0 = 0$.

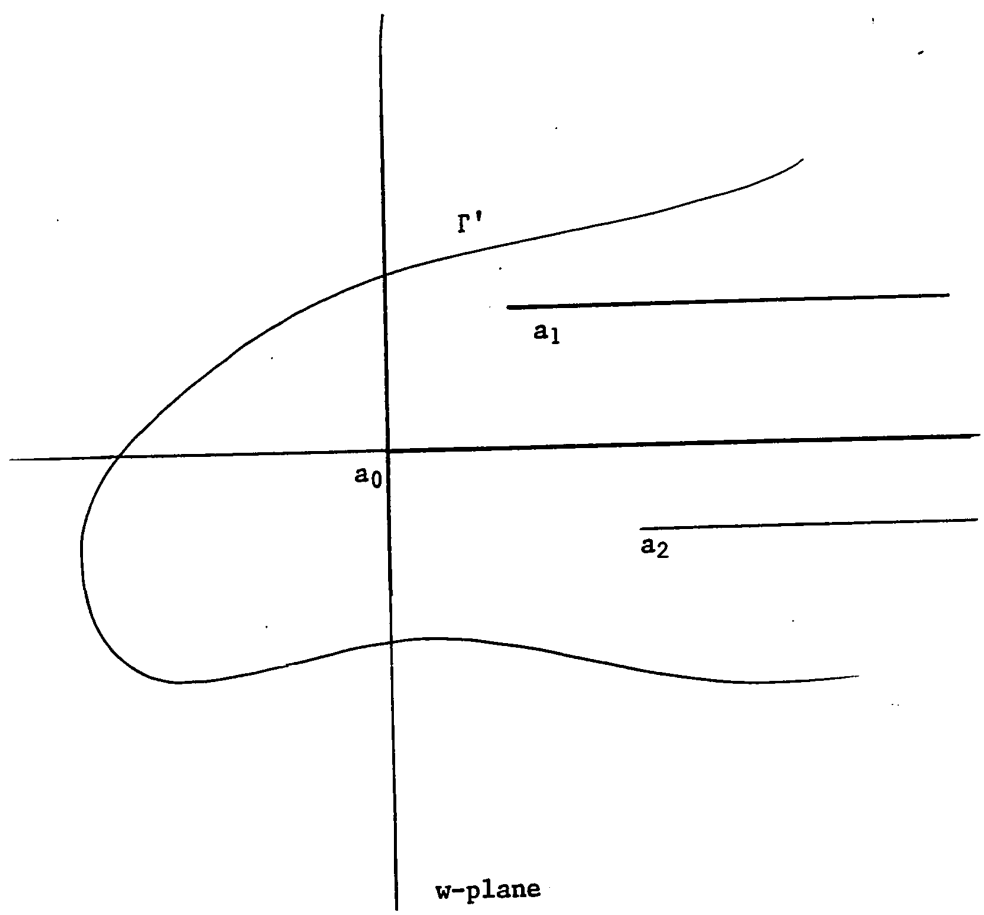


Figure 1.8.

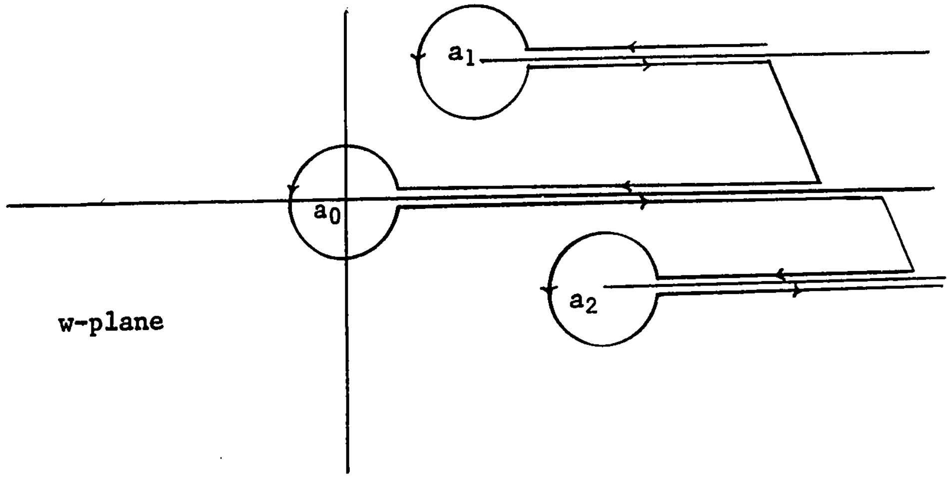


Figure 1.9.

One could of course envisage making an asymptotic evaluation involving each loop around a singularity separately and then adding the results together. In some instances the loops would only have the point at infinity as a point in common, and in the case of the finite paths between loops the contribution would be exponentially small. However, even this would not be necessary unless, say, $\text{Re } a_i = o$, for certain values of i. Normally the contribution of all parts of the path of integration contained entirely within $\text{Re}w > o$ are exponentially small, and, at least to this order, only one singularity need be taken into account.

More recently, however, the point of view has changed, and problems are being considered in which the singularities $\{a_0, a_1, \cdots\}$ are considered as variables, and interesting situations develop when two or more singularities begin to approach each other, and in the limit may coalesce. Further if the positions of the singularities are variable, it will not normally be possible to order them in a meaningful way like $\operatorname{Re} a_0 < \operatorname{Re} a_1 < \cdots$. Sometimes parameters become involved as in

$$(1.78)\quad F(z,\alpha_1,\alpha_2,\cdots,\alpha_m) = \int_{\Gamma} f(t,\alpha_1,\alpha_2,\cdots,\alpha_m) \exp(-z\ \phi(t,\alpha_1,\alpha_2,\cdots,\alpha_m))dt\ ,$$

and the inversion problem involves the solution of

$$(1.79)\qquad \phi(t,\alpha_1,\alpha_2,\cdots,\alpha_m) = w\ .$$

An extremely difficult problem is now envisaged if one seeks asymptotic expansions in z which are uniform in the α's. However, the present thesis will be devoted to the classical situation. Thus our discussion will center on a determination of the asymptotic behavior of

$$(1.80)\qquad F(z,r) = \int_{\Gamma_r} f(t) \exp(-zt)dt\ ,$$

where Γ_r is the infinitesimal interval $0 \le t \le r$, or the infinitesimal circle $|t| = r$. This implies of course that generalizations of Watson's Lemma are almost entirely obtained by considering several different forms of singularity which $f(t)$ is allowed to possess at $t = 0$.

CHAPTER II

Asymptotic Expansions

In Chapter I, it was indicated that it might be desirable to generalize the Poincaré framework in which asymptotic information can be discussed. Schmidt [18], Erdélyi [8], Erdélyi and Wyman [12], and others have developed or applied extremely general frameworks for this purpose. Although valuable, the definitions and concepts contained in the papers mentioned above are more general than is required to fulfil the purpose of the present thesis. These definitions and concepts will therefore be stated in a simpler form which is more appropriate for the present thesis.

Let $F(z)$ and $G(z)$ be two functions defined on a point set R in the complex z-plane, and let z_o be a limit point of R, possibly the point at infinity, (z_o itself may but need not belong to R). If z_o is a finite point, a neighborhood $U(z_o, \delta)$ of z_o will be the set of all points z such that $z \in R$ and $|z - z_o| < \delta$. If z_o is the point at infinity, a neighborhood $U(z_o, \delta)$ of z_o will be the set of all points z such that $z \in R$ and $|z| \geq \delta > o$. Although all functions of z must be defined for the points belonging to some neighborhood $U(z_o, \delta)$, they need not necessarily be defined for all points of R. Similarly the validity of statements or theorems is restricted to some neighborhood, and validity is not required for all points of R. Once the point set R is used to define the neighborhood system, it recedes into the background, and is seldom, if ever, used for any other purpose.

This neighborhood system introduces a topology, and the Landau order symbols mentioned in the previous chapter will now have the following meanings:

1. $F(z) = O(G(z))$, as $z \to z_o$, means that there exist a fixed constant K and a neighborhood $U(z_o, \delta)$ such that $|F| \leq K|G|$ for every $z \in U$.
2. $F(z) = o(G(z))$, as $z \to z_o$, means that for every choice of a fixed $\varepsilon > o$, there exists a neighborhood $U(z_o, \delta)$ such that $|F| \leq \varepsilon|G|$ for every $z \in U$.

Throughout this thesis, the symbol I will stand for one of two sets of integers. It may represent the finite set of integers $\{0,1,2,\cdots,M\}$, or it may represent the infinite set of integers $\{0,1,2,\cdots\}$. When convenient, summations of the form

(2.1) $$\sum_{n\varepsilon I} f_n(z) = \sum_{n=o}^{M} f_n(z) \ , \qquad I \text{ finite}$$

or

$$= \sum_{n=o}^{\infty} f_n(z) \ , \qquad I \text{ infinite}$$

will be written as

(2.2) $$\sum f_n(z).$$

When I is infinite the series may be formal and need not converge.

<u>Definition 2.1</u>. A sequence of functions $\{\phi_n(z)\}$, $n \varepsilon I$, is called an asymptotic sequence if

(2.3) $$\phi_{n+1} = o(\phi_n) , \quad \text{as} \quad z \to z_o ,$$

as long as n and n + 1 are fixed and both belong to I. Throughout the notation $\{\phi_n(z)\}$ will be used to denote an asymptotic sequence, and when ambiguity must be avoided the limit point z_o will in some way be specified.

Definition 2.2. The series $\sum_{n\varepsilon I} f_n(z) = \sum f_n(z)$ is called an asymptotic expansion of a function F(z) if, for every fixed integer $N \varepsilon I$,

(2.4) $$F(z) = \sum_{n=o}^{N} f_n(z) + o(\phi_N), \quad \text{as} \quad z \to z_o .$$

The notation

(2.5) $$F(z) \sim \sum f_n(z), \quad \{\phi_n\}, \quad \text{as} \quad z \to z_o ,$$

is used to denote that $\sum f_n(z)$ is an asymptotic expansion of F(z).

When $f_n(z) = a_n \phi_n(z)$, a_n a fixed complex number, for every $n \varepsilon I$, then the expansion is said to be of Poincaré's type. Furthermore, if the expansion is of this type, and $\phi_n(z) = (Z(z))^{\lambda_n}$, λ_n a fixed complex number, the expansion is said to be of power series form.

An asymptotic sequence $\{\phi_n(z)\}$ may be used to divide the class of complex functions defined for some neighborhood of a limit point z_o into equivalence classes. A concept of asymptotic equality is introduced in Definition 2.3.

Definition 2.3. Two functions F(z) and G(z) defined on some neighborhood $U(z_o, \delta)$ are said to be asymptotically equal, written

$$(2.6) \qquad F(z) \approx G(z), \quad \{\phi_n\}, \quad \text{as} \quad z \to z_o ,$$

if

$$(2.7) \qquad F(z) = G(z) + o(\phi_n) , \quad \text{as} \quad z \to z_o ,$$

for every fixed integer $n \in I$.

Two functions having the same asymptotic expansion are asymptotically equal, and the converse is true.

Even this degree of generality is not sufficient to describe the asymptotic behavior of many of the known functions of mathematics. The form

$$(2.8) \qquad F(z) \sim G_1(z) \left[\sum_{n \varepsilon I_1} f_n^{(1)}(z) ; \quad \{\phi_n^{(1)}\}\right] + G_2(z) \left[\sum_{n \varepsilon I_2} f_n^{(2)}(z) ; \quad \{\phi_n^{(2)}\}\right] + \cdots$$

as $z \to z_o$, with the meaning

$$(2.9) \qquad F(z) = G_1(z) \left[\sum_{n=o}^{N_1} f_n^{(1)}(z) + o(\phi_{N_1}^{(1)})\right] + G_2(z) \left[\sum_{n=o}^{N_2} f_n^{(2)}(z) + o(\phi_{N_2}^{(2)})\right] + \cdots ,$$

as $z \to z_o$, where $N_1, N_2, \cdots$ are arbitrary fixed integers chosen from $I_1, I_2, \cdots$, respectively, must often be used to give asymptotic information for many of the higher transcendental functions.

For the results to be obtained in this thesis, an asymptotic sequence $\{\phi_n(z)\}$, with

(2.10) $$\phi_n(z) = (\log z)^{\mu_n} \cdot z^{-\lambda_n},$$

where $\{\mu_n\}$ and $\{\lambda_n\}$, $n \in I$, are sequences of fixed complex numbers, plays an important role. The limit point z_o may be the point at infinity, $|z_o| = \infty$, or the origin, $z_o = 0$. In either case

(2.11) $$\phi_{n+1}/\phi_n = (\log z)^{\mu_{n+1} - \mu_n} \cdot z^{-(\lambda_{n+1} - \lambda_n)},$$

does not imply $\phi_{n+1} = o(\phi_n)$, as $|z| \to \infty$ or $|z| \to o$, unless further restrictions on $\{\mu_n\}$ or $\{\lambda_n\}$ are stated. In particular $\{\phi_n\}$ is an asymptotic sequence if $|z_o| = \infty$ and

(2.12) $\operatorname{Re} \lambda_{n+1} > \operatorname{Re} \lambda_n$, μ_n arbitrary, n and $(n+1)$ both in I;

or

(2.13) $\operatorname{Re} \lambda_{n+1} = \operatorname{Re} \lambda_n$, $\operatorname{Re} \mu_{n+1} < \operatorname{Re} \mu_n$, n and $(n+1)$ both in I.

Similarly, corresponding conditions can be stated whereby $\{\phi_n\}$ becomes an asymptotic sequence if the limit point is the origin.

Since

(2.14) $z^\alpha (\log z)^\beta \exp(-\varepsilon|z|^\delta) = o(\phi_n)$, as $z \to \infty$, every fixed n in I,

for any fixed complex numbers α and β, and any positive real numbers ε and δ, it will be true that terms which are exponentially small in $|z|$ can be replaced by zero in an asymptotic expansion. Similarly

(2.15) $(\log z)^\beta z^{-\alpha} = o((\log z)^{\mu_n} \cdot z^{-\lambda_n})$, as $|z| \to \infty$, every fixed n in I, (which will be true if $\text{Re } \alpha > \text{Re } \lambda_n$ for every fixed n in I), will imply

$$(2.16) \qquad (\log z)^{\beta} z^{-\alpha} \approx 0; \quad \{\phi_n\}, \quad \text{as } |z| \to \infty .$$

It is interesting to note that the condition $\text{Re } \lambda_{n+1} > \text{Re } \lambda_n$ does not imply $\overline{\lim_{n \to \infty}} \text{ Re } \lambda_n = \infty$. Hence (2.16) may, in some instances, allow terms to be dropped which are not exponentially small.

In the present thesis, when the limit point z_o is the point at infinity, R will be defined to be the sector

$$(2.17) \qquad S(\Delta): \ |\arg(z \exp(i\gamma))| \leq \frac{\pi}{2} - \Delta , \quad o < \Delta \leq \frac{\pi}{2} ,$$

where γ is some fixed real number. In some circumstances, (2.17) is equivalent to

$$(2.18) \qquad -\frac{\pi}{2} + \Delta - \gamma \leq \arg z \leq \frac{\pi}{2} - \Delta - \gamma ,$$

and in all cases R is equivalent to

$$(2.19) \qquad (4k - 1) \frac{\pi}{2} + \Delta - \gamma \leq \arg z \leq (4k + 1) \frac{\pi}{2} - \Delta - \gamma ,$$

for some fixed integer k.

As mentioned at the beginning of Chapter II, the definitions and concepts that have now been stated are just simpler forms of more general definitions and concepts given in [18], [8] and [12], and our statements are just paraphrases of the statements in their papers. All of the properties that have been proved in the above mentioned papers for these more general forms of asymptotic expansions will of course apply to the forms stated in the present chapter.

CHAPTER III

Branch-Point Singularities

The use of Watson's Lemma to determine, under certain conditions, an asymptotic expansion of a function F(z) defined by a Laplace integral,

$$(3.1) \qquad F(z) = \int_0^\infty f(t) \exp(-zt)dt ,$$

is one of the more important tools that have been designed to obtain the asymptotic behavior of functions of a complex variable. In spite of this, there do exist important generalizations of Watson's result which do not change the branch-point nature of the singularity that f(t) is assumed to have at $t = o$.

Among the generalizations which are more or less obvious, it might be noted that the requirement of a straight line path of integration can be relaxed in certain circumstances. Consider

$$(3.2) \qquad F(z) = \int_\Gamma f(t) \exp(-zt)dt ,$$

where Γ is a continuous curve beginning at $t = A$ and ending at $t = B$. It will be assumed that $t = A$ is a point of fixed finite affix, and that $t = B$ need not satisfy such a restriction.

If C is the closed contour shown in Figure 3.1, then

$$(3.3) \qquad \int_C f(t) \exp(-zt)dt = 2\pi i \, E ,$$

where E is the sum of the residues of f(t) at the poles of f(t) contained within C, providing $f(t)$ is continuous within and on C

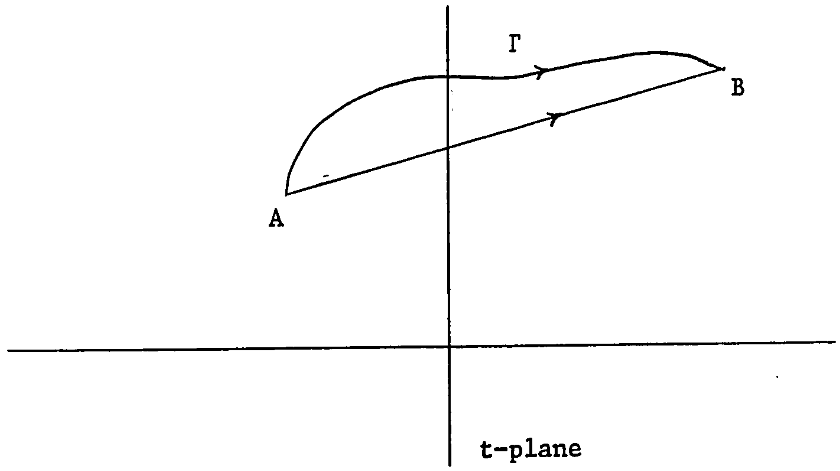


Figure 3.1.

and regular, except for a finite number of poles, within C. This would mean

$$(3.4)\qquad \int_{\Gamma} f(t)\exp(-zt)dt = \int_{A}^{B} f(t)\exp(-zt)dt - 2\pi i\, E\ ,$$

and analogous expressions can be obtained if the straight line AB happens to intersect Γ, or if Γ happens to be completely below AB. If the poles of $f(t)$ within C occur at $t = a_i$, $i = 1,2,\cdots,k$, and the poles are respectively of order m_i, then

$$(3.5)\qquad E = \sum_{i=1}^{k} \frac{\exp(za_i)}{(m_i - 1)!} P_{m_i-1}(z)\ ,$$

where

$$(3.6)\qquad P_{m_i-1}(z) = \frac{d^{m_i-1}}{dt^{m_i-1}} [t^{m_i} f(t + a_i)\exp(-zt)]\Big|_{t=o}\ ,$$

and $P_{m_i-1}(z)$ is a polynomial in z of degree m_i-1. This implies of course that the asymptotic behavior of E is completely known, and, therefore, the asymptotic behavior of $\int_A^B f(t)\exp(-zt)dt$ will, in turn, determine the asymptotic behavior of $\int_\Gamma f(t)\exp(-zt)dt$.

The substitution $t' = t - A$ gives

$$\int_A^B f(t)\exp(-zt)dt = \exp(-zA)\int_o^T f(t+A)\exp(-zt)dt, \; T = B - A, \tag{3.7}$$

and in (3.7) the path of integration is the straight line joining $t = o$ to $t = T$, where T may or may not be the point at infinity.

It is of interest to note that the exponential nature of the terms in E and $\int_A^B f(t)\exp(-zt)dt$ will imply, as $z \to \infty$ in certain sectors of the plane, that $\int_A^B f(t)\exp(-zt)dt$ is exponentially small with respect to one or more terms of E, and the converse will also be true, as $z \to \infty$ in other sectors. In either case, a rather simple expression for the asymptotic behavior of $\int_\Gamma f(t)\exp(-zt)dt$ might exist. If $z \to \infty$ in an unrestricted manner, the implied simplification, obtained by dropping exponentially small terms, will not exist.

Within the restrictions placed on $f(t)$ by the use of Cauchy's Residue Theorem, the straight line path of Laplace integrals may be considered as a canonical path of integration. For this reason, it now becomes possible to return to the more detailed discussion of the asymptotic behavior of a function $F(z)$ defined by

$$F(z) = \int_o^T f(t)\exp(-zt)dt \;, \tag{3.8}$$

where $T = |T| \exp(i\gamma) \neq 0$ is a fixed, finite or infinite point, and the path of integration is the straight line joining $t = o$ to $t = T$.

In order to generalize Watson's result, two asymptotic sequences $\{\phi_n(z)\}$ and $\{\psi_n(t)\}$ are defined in the following way. As before, I stands for the finite or infinite set of integers given in Chapter II. It will be assumed that $\{\lambda_n\}$, $n \in I$, is a sequence of fixed, complex numbers, for which

(3.9) $\operatorname{Re} \lambda_{n+1} > \operatorname{Re} \lambda_n$, for every n, providing n and $n+1$ are both members of I.

This condition ensures that

$$\{\phi_n(z) = z^{-\lambda_n}\} \tag{3.10}$$

is an asymptotic sequence as $z \to \infty$, and $\arg z$ is unrestricted. The same is true for the sequence

(3.11) $\psi_n(t) = t^{\lambda_n - 1}$, as $t \to o$, and $\arg t$ is unrestricted.

One must of course assume suitable cuts in the z and t-planes to make $z^{-\lambda_n}$ and $t^{\lambda_n - 1}$ single-valued functions of z and t respectively.

<u>Generalized Watson's Lemma</u>.

If:

(i) $F(z)$, as defined by (3.8), exists for some fixed $z = z_o$;

(ii) $f(t) \sim \sum_{n \in I} a_n t^{\lambda_n - 1}$; $\{t^{\lambda_n - 1}\}$;

as $t \to o$ along $\arg t = \gamma$, where each a_n is a fixed complex number and $\operatorname{Re} \lambda_o > 0$;

then

$$F(z) \sim \sum_{n\varepsilon I} a_n \Gamma(\lambda_n) z^{-\lambda_n} ; \quad \{z^{-\lambda_n}\} \tag{3.12}$$

as $z \to \infty$ in $|\arg(z \exp(i\gamma))| \leq \frac{\pi}{2} - \Delta$, where Δ is any fixed number in $0 < \Delta \leq \frac{\pi}{2}$. The result is uniform in the approach of $z \to \infty$ in the sector given above.

Proof: The general properties of Laplace integrals ensure that $F(z)$ exists as long as $\operatorname{Re}(z \exp(i\gamma)) > \operatorname{Re}(z_o \exp(i\gamma))$. Furthermore, for any arbitrary fixed choice of $t = c = |c| \exp(i\gamma)$, $0 < |c| < |T|$, it is true that

$$\int_c^T f(t) \exp(-zt)dt = O(\exp(-\varepsilon|c||z|)) , \tag{3.13}$$

uniformly in $\arg z$, as $z \to \infty$ in $|\arg(z \exp(i\gamma))| \leq \frac{\pi}{2} - \Delta$, for some fixed $\varepsilon > o$. Hence

$$\int_c^T f(t) \exp(-zt)dt \approx o; \quad \{z^{-\lambda_n}\} , \tag{3.14}$$

as $z \to \infty$ in $|\arg(z \exp(i\gamma))| \leq \frac{\pi}{2} - \Delta$. This, of course, implies

$$F(z) \approx \int_o^c f(t) \exp(-zt)dt; \quad \{z^{-\lambda_n}\} , \tag{3.15}$$

uniformly in $\arg z$, as $z \to \infty$ in $|\arg(z \exp(i\gamma))| \leq \frac{\pi}{2} - \Delta$.

For every fixed integer N in I, and any fixed choice of $\varepsilon > o$, there exists a fixed number r such that

$$f(t) = \sum_{n=o}^{N} a_n t^{\lambda_n - 1} + R_N , \tag{3.16}$$

where

$$(3.17)\qquad |R_N| \le \varepsilon |t^{\lambda_N - 1}| , \quad \text{providing} \quad |t| \le r, \quad \arg t = \gamma .$$

Hence, for a fixed choice of c for which $|c| < r$,

$$(3.18)\qquad \int_o^c f(t) \exp(-zt)dt = \sum_{n=o}^{N} a_n \int_o^c t^{\lambda_n - 1} \exp(-zt)dt + S_N ,$$

with

$$(3.19)\qquad S_N = \int_o^c R_N \exp(-zt)dt .$$

From the properties of Laplace integrals it follows that

$$(3.20)\qquad \int_o^c t^{\lambda_n - 1} \exp(-zt)dt = \int_o^{\infty e^{i\gamma}} t^{\lambda_n - 1} \exp(-zt)dt + O(\exp(-\varepsilon|c||z|))$$

$$= \Gamma(\lambda_n)\, z^{-\lambda_n} + O(\exp(-\varepsilon|c||z|)) .$$

Further, (3.17) and (3.19) give

$$(3.21)\qquad |S_N| \le \varepsilon \int_o^c |t^{\lambda_N - 1} \exp(-zt)dt| \le \varepsilon \int_o^{\infty e^{i\gamma}} |t^{\lambda_N - 1} \exp(-zt)dt|$$

$$= o(z^{-\lambda_N}), \quad \text{as} \quad z \to \infty \quad \text{in} \quad |\arg(z \exp(i\gamma))| \le \frac{\pi}{2} - \Delta.$$

These results couple together to give

$$(3.22)\qquad F(z) \sim \sum_{n\varepsilon I} a_n\, \Gamma(\lambda_n)\, z^{-\lambda_n} ; \quad \{z^{-\lambda_n}\} ,$$

uniformly in $\arg z$, as $z \to \infty$ in $|\arg(z \exp(i\gamma))| \le \frac{\pi}{2} - \Delta$.

In the Barnes result, the function $F(z)$ is defined by

(3.23) $$F(z) = \frac{i}{2\pi} \int_T^{(o+)} f(t)\exp(-zt)dt \; ,$$

where the path of integration is shown below.

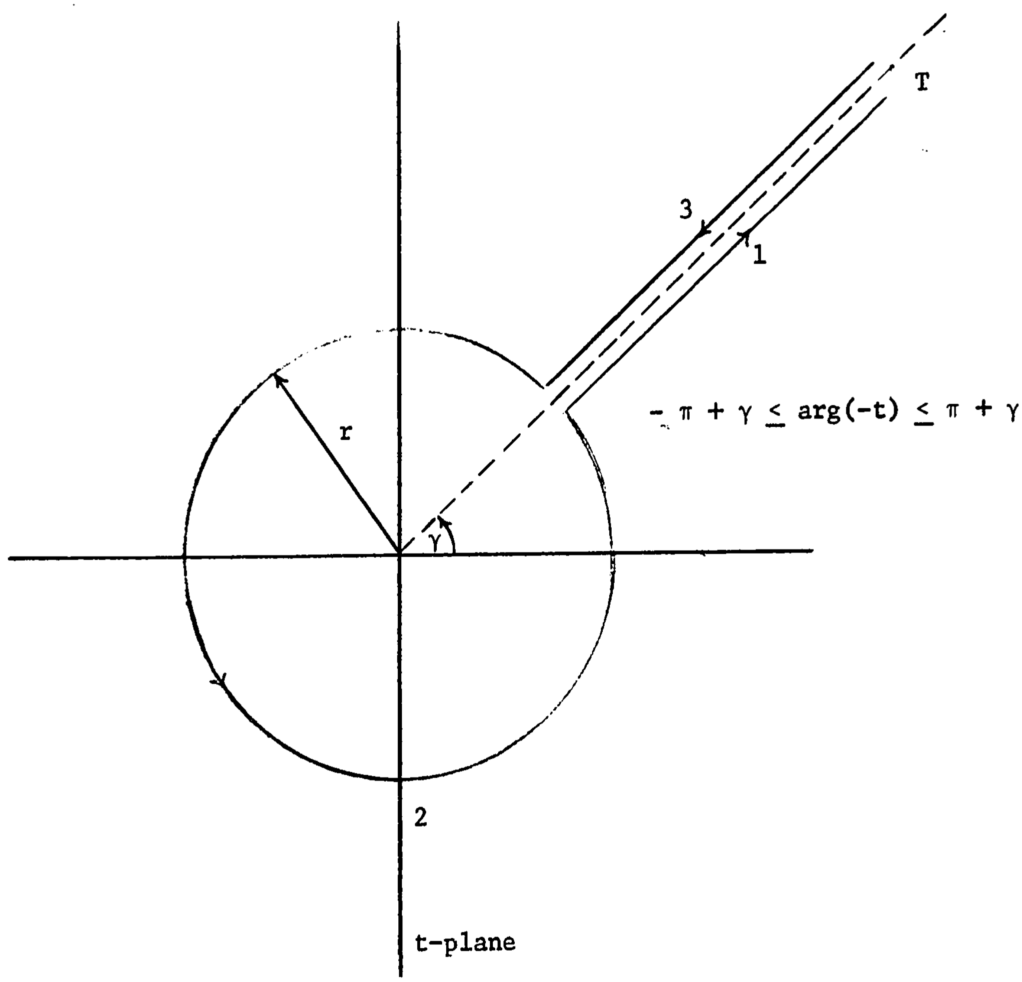


Figure 3.2.

The form of proof that will be used requires that the circle $|z| = r$ be deformed into a smaller circle, if necessary, and therefore regularity of $f(t)$ is required within the region shown below.

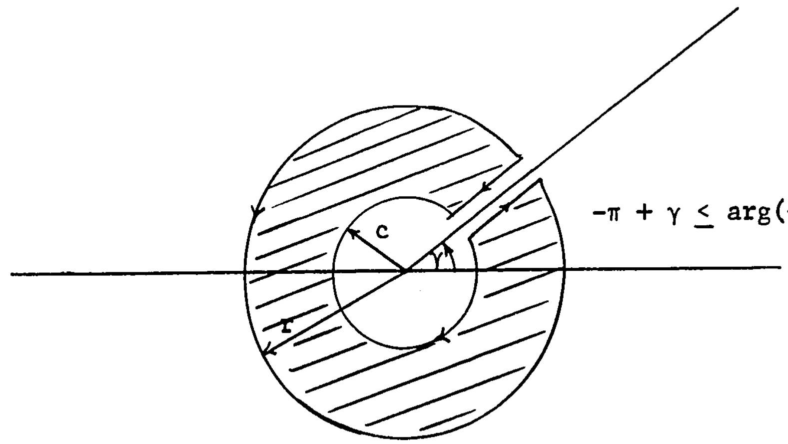


Figure 3.3.

Generalized Barnes' Lemma.

Consider $F(z) = \frac{i}{2\pi} \int_T^{(o+)} f(t) \exp(-zt)dt$, where the path of integration is illustrated in Figure 3.2. If:

(i) $F(z)$ exists for some fixed $z = z_o$;

(ii) $f(t)$ is regular within, and continuous on the boundary of, the shaded cut annulus of Figure 3.3, where the outer circle is of fixed radius, and the inner circle $|z| = c \neq o$ may be chosen to be fixed and arbitrarily small;

(iii) $$f(t) \sim \sum_{n\varepsilon I} a_n(-t)^{\lambda_n - 1} \; ; \quad \{t^{\lambda_n - 1}\} ,$$

as $t \to o$ in $-\pi + \gamma \leq \arg(-t) \leq \gamma + \pi$;

(iv) as before, $\text{Re}\,\lambda_{n+1} > \text{Re}\,\lambda_n$, for every $n \varepsilon I$, providing n and $n+1$ are members of I, and, in addition, for some fixed $n = k$ in I, $\text{Re}\,\lambda_k > o$;

then

$$(3.24) \qquad F(z) \sim \sum_{n\varepsilon I} a_n/\Gamma(1-\lambda_n)\, z^{\lambda_n} \; ; \quad \{z^{-\lambda_n}\} \, ,$$

uniformly in arg z, as $z \to \infty$ in $|\arg(z\exp(i\gamma))| \le \frac{\pi}{2} - \Delta$.

<u>Proof</u>: Let I_k be the set of integers $\{0,1,2,\cdots,k-1\}$, and define $f_k(t)$ by

$$(3.25) \qquad f_k(t) = f(t) - \sum_{n\varepsilon I_k} a_n(-t)^{\lambda_n - 1} \; .$$

Hence

$$(3.26) \quad \frac{i}{2\pi}\int_T^{(o+)} f_k(t)\exp(-zt)dt = F(z) - \sum_{n\varepsilon I_k} a_n \frac{i}{2\pi}\int_T^{(o+)} (-t)^{\lambda_n - 1}\exp(-zt)dt$$

$$= F(z) - \sum_{n\varepsilon I_k} a_n \frac{i}{2\pi}\int_{\infty e^{i\gamma}}^{(o+)} (-t)^{\lambda_n - 1}\exp(-zt)dt$$

$$+ O(\exp(-\varepsilon|z|)),$$

for some fixed $\varepsilon > o$, uniformly in arg z, as $z \to \infty$ in $|\arg(z\exp(i\gamma))| \le \frac{\pi}{2} - \Delta$. Anticipating the final result, the exponentially small term is dropped, and

$$(3.27) \quad F(z) \approx \sum_{n\varepsilon I_k} \frac{a_n}{z^{\lambda_n}\,\Gamma(1-\lambda_n)} + \frac{i}{2\pi}\int_T^{(o+)} f_k(t)\exp(-zt)dt; \quad \{\phi_n(z)\} \, ,$$

uniformly in arg z, as $z \to \infty$ in $|\arg(z\exp(i\gamma))| \le \frac{\pi}{2} - \Delta$, and $\{\phi_n\}$ can be any asymptotic sequence for which exponentially small terms can be replaced by zero.

The function $f_k(t)$ has the same properties of regularity possessed by $f(t)$, and in addition

$$(3.28)\qquad f_k(t) = O(t^{\lambda_k - 1}), \quad \text{as} \quad t \to o, \quad \text{Re}\,\lambda_k > o.$$

This implies that the circular part of the contour can be shrunk to zero and

$$(3.29)\quad \frac{i}{2\pi}\int_T^{(o+)} f_k(t)\exp(-zt)dt = \frac{i}{2\pi}\Big[\int_o^T f_k(te^{2\pi i})\exp(-zt)dt - \int_o^T f_k(t)\exp(-zt)dt\Big].$$

The conditions of the Generalized Watson's Lemma can be applied to both of these integrals to give

$$(3.30)\quad \frac{i}{2\pi}\int_T^{(o+)} f_k(t)\exp(-zt)dt \sim \frac{i}{2\pi}\Big[\sum_{n\varepsilon I-I_k} a_n\,\Gamma(\lambda_n)z^{-\lambda_n}e^{-i\pi\lambda_n} - \sum_{n\varepsilon I-I_k} a_n\,\Gamma(\lambda_n)z^{-\lambda_n}e^{i\pi\lambda_n}\Big]$$

$$\sim \sum_{n\varepsilon I-I_k} a_n\,\Gamma(\lambda_n)z^{-\lambda_n}\,\frac{\sin(\pi\lambda_n)}{\pi}$$

$$\sim \sum_{n\varepsilon I-I_k} a_n/z^{\lambda_n}\,\Gamma(1-\lambda_n); \quad \{z^{-\lambda_n}\},$$

uniformly in $\arg z$, as $z \to \infty$ in $|\arg(z\exp(i\gamma))| \leq \frac{\pi}{2} - \Delta$.

From (3.27) and (3.30), it then follows that

$$(3.31)\qquad F(z) = \frac{i}{2\pi}\int_T^{(o+)} f(t)\exp(-zt)dt$$

$$\sim \sum_{n\varepsilon I} a_n/z^{\lambda_n}\,\Gamma(1-\lambda_n), \quad \{z^{-\lambda_n}\},$$

uniformly in $\arg z$, as $z \to \infty$ in $|\arg(z \exp(i\gamma))| \leq \frac{\pi}{2} - \Delta$.

Generalizations of Watson's Lemma can be found in the book of Jeffrey's and Jeffrey's [13], Erdélyi [9], Wyman [22], and many others. Almost all of the ideas found in the work of these authors have been brought together to form the basis of the generalization of Watson's Lemma given in the present chapter. As mentioned previously, the work of Barnes seems to be relatively unknown, and consequently generalizations of his result, as far as we are aware, do not seem to have been attempted. In one essential feature the proof that was given for the generalized Barnes' Lemma followed a pattern given by Perron [17].

When

$$F(z) = \int_o^{\infty} f(t)\exp(-zt)dt \ , \tag{3.32}$$

the conditions under which the inverse

$$f(t) = \frac{1}{2\pi i} \int_{\sigma_o - i\infty}^{\sigma_o + i\infty} F(z)\exp(zt)dt \tag{3.33}$$

exists are well known. It is easily seen that the problem of determining the behavior of $f(t)$ as $t \to \infty$ is the same problem that is being considered in the present chapter. The determination of the behavior of $f(t)$ as $t \to o+$ presents, however, a difficult problem.

If $F(z)$ has a convergent expansion of the form

$$F(z) = \sum_{n=o}^{\infty} a_n \Gamma(\lambda_n) z^{-\lambda_n} \ , \tag{3.34}$$

which can be integrated term by term to give

$$(3.35) \qquad f(t) = \frac{1}{2\pi i} \int_{\sigma_o - i\infty}^{\sigma_o + i\infty} F(z) \exp(zt) dt$$

$$= \sum_{n=o}^{\infty} a_n \frac{\Gamma(\lambda_n)}{2\pi i} \int_{\sigma_o - i\infty}^{\sigma_o + i\infty} z^{-\lambda_n} \exp(zt) dt$$

$$= \sum_{n=o}^{\infty} a_n t^{\lambda_n - 1} ,$$

then the behavior as $t \to o$ would be established. Although we shall not pursue this problem further, it does seem worth while to investigate the theorems converse to those given in the present chapter, with the distinct possibility existing that all than can be said, concerning branch-point singularities of the type used in the present chapter, has been said, and no further generalizations are possible.

Watson's Lemma is so important in asymptotic analysis, and has been known for so many years, that there exist hundreds of important examples which can be used to illustrate the way the lemma can be used for specific functions. Since the function

$$(3.36) \qquad \mu(z, \beta, \alpha) = \int_{o}^{\infty} F(z, t) dt ,$$

where

$$(3.37) \qquad F(z,t) = z^{\alpha + t} t^{\beta} / \Gamma(\beta + 1) \Gamma(\alpha + t + 1), \quad \text{Re } \beta > -1 ,$$

plays an important role in the work of later chapters in this thesis, it will be used to illustrate the techniques developed in the present chapter. One can write

$$\mu = \frac{z^\alpha}{\Gamma(\beta + 1)} \int_o^\infty f(t) \exp[-t \log(z^{-1})]dt , \tag{3.38}$$

where

$$f(t) = t^\beta/\Gamma(\alpha + t + 1). \tag{3.39}$$

In some fixed neighborhood of the origin, $f(t)$ has a convergent expansion of the form

$$f(t) = \sum_{n=o}^{\infty} \frac{D^n[\Gamma^{-1}(\alpha + 1)]}{n!} t^{n+\beta} , \quad D = \frac{d}{d\alpha} . \tag{3.40}$$

Further, as $t \to \infty$, $t^\beta/\Gamma(\alpha + t + 1)$ is bounded. Hence,

$$\mu \sim \frac{z^\alpha}{\Gamma(\beta + 1)} [\sum_{n=o}^{\infty} \frac{D^n[\Gamma^{-1}(\alpha+1)]}{n!} \cdot \frac{\Gamma(n+\beta+1)}{[\log(z^{-1})]^{n+\beta+1}} ; \{(\log(z^{-1})^{-n-\beta-1}\}] \tag{3.41}$$

uniformly in $\arg z$, as $z \to o$ in $|\arg(\log(z^{-1}))| \le \frac{\pi}{2} - \Delta$. However, $\log(z^{-1}) = -\log|z| - i \arg z$, and therefore $\arg(\log(z^{-1})) \to o$ as $z \to o$ in an unrestricted manner. This implies

$$\mu \sim \frac{z^\alpha}{[\log(z^{-1})]^{\beta+1}} [\sum_{n=o}^{\infty} \frac{D^n[\Gamma^{-1}(\alpha+1)]}{n!} \frac{(\beta+1)_n}{[\log(z^{-1})]^n} ; \{(\log(z^{-1}))^{-n}\}], \tag{3.42}$$

uniformly in $\arg z$ as $z \to o$ in an unrestricted manner, providing $\text{Re } \beta > -1$. By using Barnes result, it is now possible to remove the restriction that $\text{Re } \beta > -1$. Consider

$$\psi(z,\beta,\alpha) = \Gamma(-\beta)z^\alpha \cdot \frac{i}{2\pi} \int_\infty^{(o+)} \frac{(-t)^\beta}{\Gamma(\alpha+t+1)} \exp[- t(\log(z^{-1}))]dt. \tag{3.43}$$

If $-1 < \text{Re } \beta < o$, one can easily show that $\psi(z,\beta,\alpha) = \mu(z,\beta,\alpha)$, and hence by analytic continuation, $\psi(z,\beta,\alpha)$ continued to represent $\mu(z,\beta,\alpha)$ when $\text{Re } \beta \le -1$. Using the generalized Barnes' Lemma immediately gives

$$(3.44) \quad \mu(z,\beta,\alpha) \sim \Gamma(-\beta)z^{\alpha}\left[\sum_{n=o}^{\infty} \frac{(-1)^n \, D^n[\Gamma^{-1}(\alpha+1)]}{n! \, \Gamma(-n-\beta)[\log(z^{-1})]^{n+\beta+1}}; \{(\log(z^{-1}))^{-n-\beta-1}\}\right]$$

as $z \to o$, $\mathrm{Re}\,\beta \leq -1$. As before,

$$(3.45) \quad \mu(z,\beta,\alpha) \sim \frac{z^{\alpha}}{(\log(z^{-1}))^{\beta+1}}\left[\sum_{n=o}^{\infty} \frac{(\beta+1)_n}{n!} \frac{D^n[\Gamma^{-1}(\alpha+1)]}{(\log(z^{-1}))^n} ; \{(\log(z^{-1}))^{-n}\}\right],$$

as $z \to o$, and α, β may now be considered to be unrestricted.

In the next chapter, a complete discussion of the behavior of $\mu(z,\beta,\alpha)$ as $|z| \to \infty$ will be given.

CHAPTER IV

The Asymptotic Behavior of $\mu(z, \beta, \alpha)$

As far as we are aware, the asymptotic behavior of the

$$(4.1) \qquad \mu(z, \beta, \alpha) = \int_0^\infty F(z, t)dt ,$$

where

$$(4.2) \qquad F(z, t) = z^{\alpha+t} \cdot t^{\beta}/\Gamma(\beta + 1) \cdot \Gamma(\alpha + t + 1) ,$$

has never been fully discussed. In addition to giving some of the important properties of this function - [11], a rather incomplete discussion of the asymptotic behavior is given. Unfortunately, in addition to being incomplete, the partial results given are incorrect.

In the present chapter the correct asymptotic behavior of μ as $|z| \to \infty$ will be obtained. It is of passing interest to note that the proof that will be used to obtain the behavior of $\mu(z, \beta, \alpha)$ as $|z| \to \infty$ can also be used, with a minor modification, to obtain the asymptotic behavior of μ as $z \to 0$, a result that was obtained in the previous chapter, and, in fact, is well known [11], p. 219. When this behavior becomes known, a significant generalization of Watson's lemma will be obtained.

Besides the theorem of Barnes which we stated in Chapter I, many other asymptotic results of major importance are obtained in the paper [2]. From a very broad point of view, the pattern of the present chapter will follow the pattern set by Barnes. However, the details of proof will differ so significantly from those used by Barnes that no detailed use will be made of his work.

The paper mentioned above obtains the complete asymptotic behavior of a function defined by

$$G_\beta(z, \alpha) = \sum_{n=o}^{\infty} \frac{z^n}{n!(n + \alpha)^\beta}, \tag{4.3}$$

a function that is not the same as $\mu(z, \beta, \alpha)$. However, their asymptotic behaviors are closely related.

In [11], p. 222, the Laplace transformation of $\mu(z, \beta, \alpha)$ is given by

$$\int_o^\infty \mu(z, \beta, \alpha) \exp(-sz)dz = s^{-\alpha-1}(\log s)^{-\beta-1}, \tag{4.4}$$

with $\operatorname{Re} \alpha > -1$ and $\operatorname{Re} s > 1$. The usual inversion formula gives

$$\mu(z, \beta, \alpha) = (2\pi i)^{-1} \int_{c-i\infty}^{c+i\infty} H \exp(sz)ds, \tag{4.5}$$

where $H = s^{-\alpha-1}(\log s)^{-\beta-1}$, and c is any arbitrary real number in $c > 1$. In (4.5), z is real and positive.

It is possible to use Cauchy's theorem to deform the path of integration in a variety of ways. The symbol L will be used as a generic symbol to denote paths of integration that begin and end at the point at infinity with the direction of approach to infinity being restricted to the second and third quadrants of the complex s-plane.

The contour L must loop aroung the origin in a counter-clockwise direction so that both $s = o$ and $s = 1$ are contained within the region bounded by L, and the contour must not cross any cuts placed in the s-plane in order to make $s^{-\alpha-1}(\log s)^{-\beta-1}$ a single-valued function of s. The main purpose of the deformation is to find an integral representation of $\mu(z, \beta, \alpha)$ in which the restriction $\text{Re}\ \alpha > -1$ can be removed, and the requirement that z be real and positive may be relaxed. An appeal to the principle of analytic continuation allows one to identify the function of (4.5) with the function defined when the path of integration becomes L.

Suppose that sz is replaced by s, and μ is defined by a suitable path of integration L, then

$$(4.6) \qquad \mu(z, \beta, \alpha) = (2\pi i)^{-1} z^{\alpha} \int_L s^{-\alpha-1}(\log \frac{s}{z})^{-\beta-1} \exp(s) ds .$$

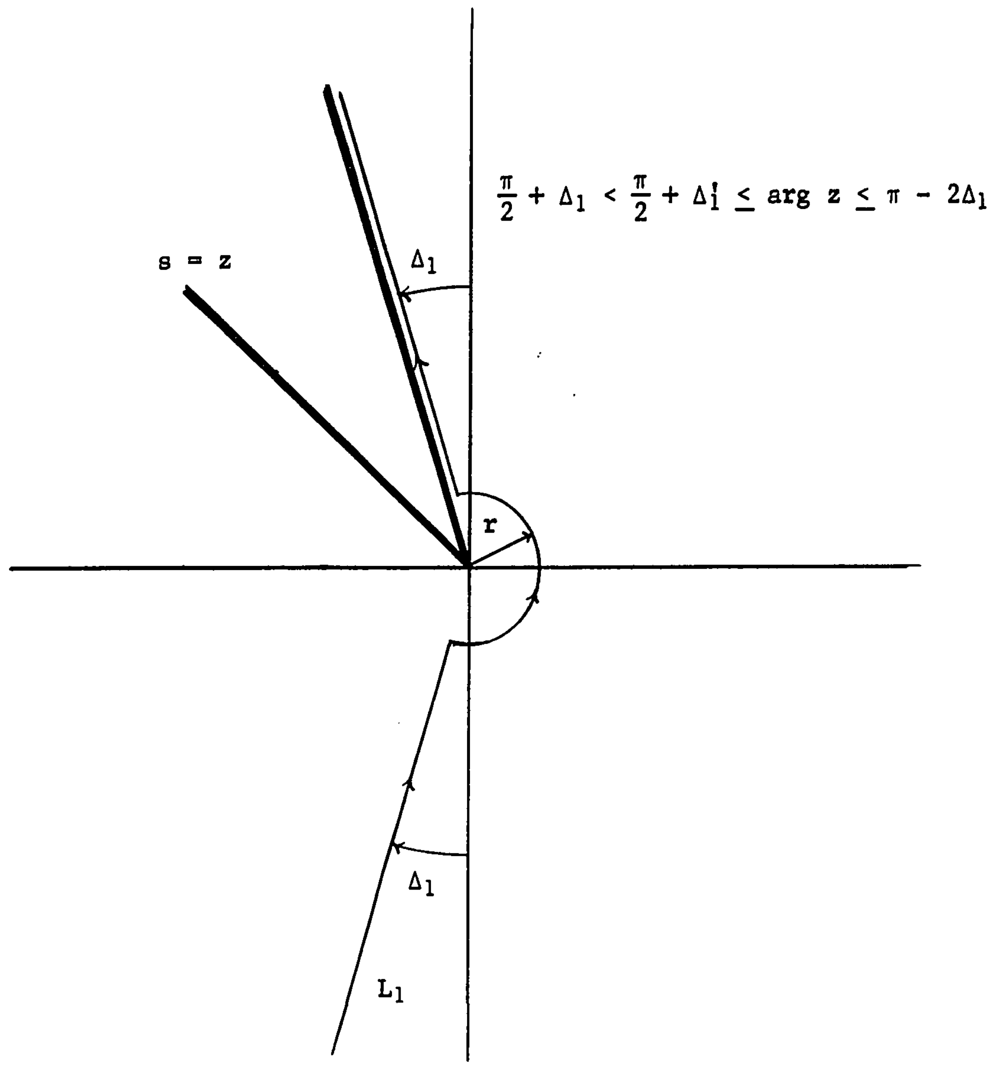
If z is now allowed to be complex, there are three possibilities that must be considered when choosing the contour L.

If Δ denotes an arbitrarily small fixed positive number, the paths of integration and cuts in the s-plane will be illustrated only for the case

$$(4.7) \qquad 0 \leq \arg z \leq \pi - 2\Delta$$

$$(4.8) \qquad -\frac{\pi}{2} - \Delta \leq \arg s \leq \frac{\pi}{2} + \Delta .$$

In this range $\arg(s/z) = \arg s - \arg z$, and $\log(s/z) = \log s - \log z$. Since $\overline{\mu(\overline{z}, \overline{\beta}, \overline{\alpha})} = \mu(z, \beta, \alpha)$, no other range of $\arg z$ need be considered. The choice of L and the cuts in the s-plane are illustrated below.



s-plane

Figure 4.1.

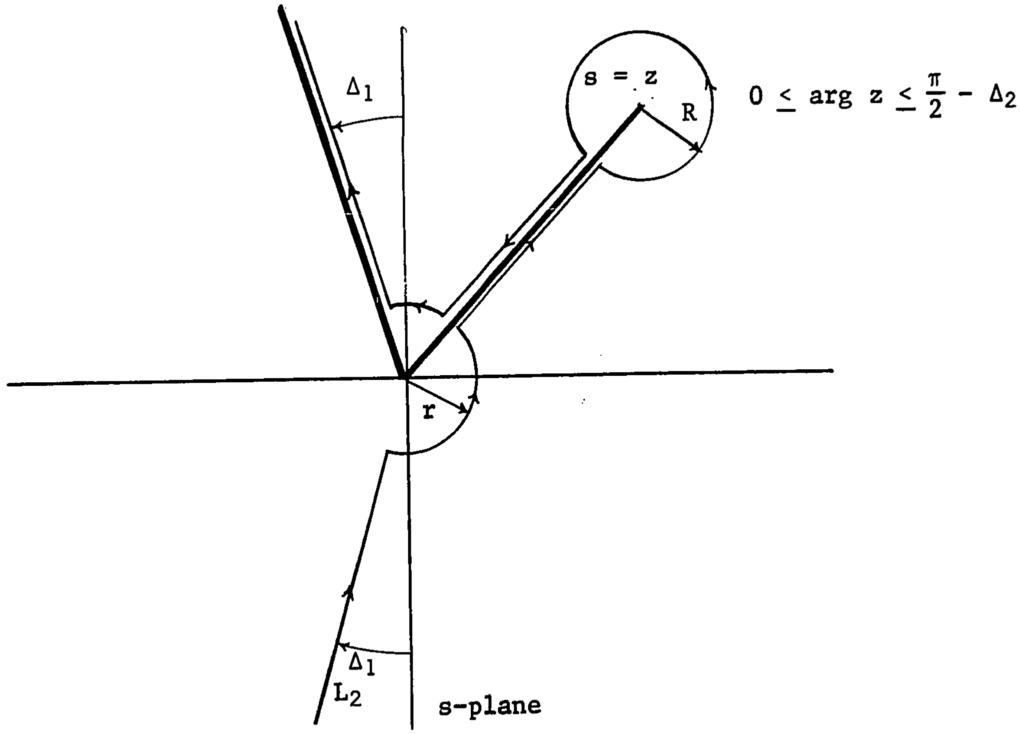


Figure 4.2.

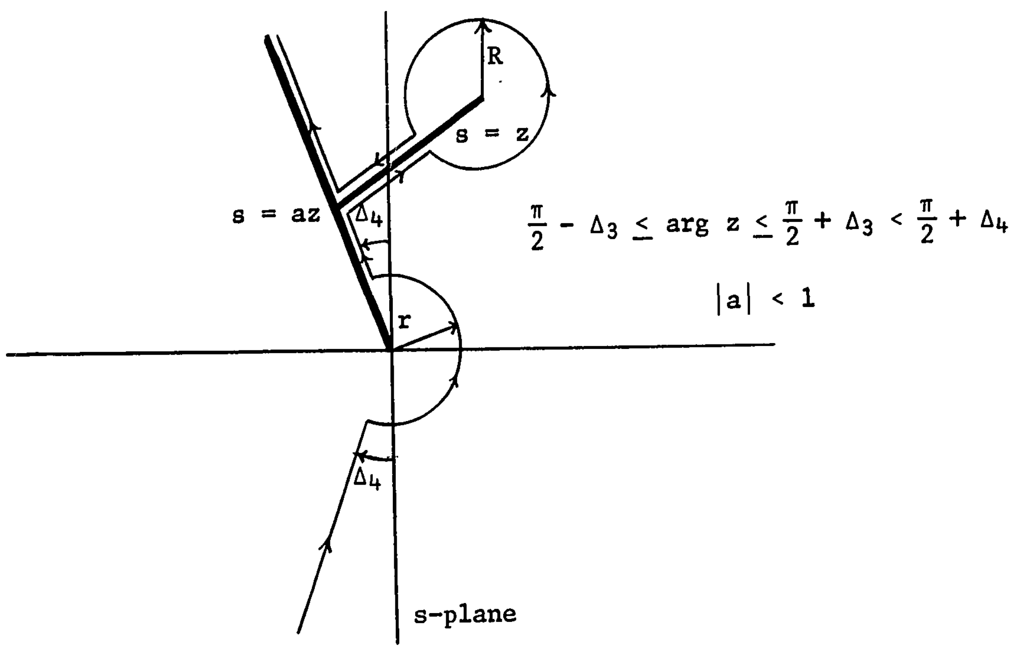


Figure 4.3.

Although three cases are listed, permitting the possibility that $\arg a = o$ allows the analysis to be accomplished in two steps. In every case it is necessary to discuss the asymptotic behavior of

$$(4.9) \qquad F(z,\beta,\alpha) = (2\pi i)^{-1} \int_{L_1} s^{-\alpha-1} (\log \frac{s}{z})^{-\beta-1} \exp(s)ds .$$

The path of integration, L_1, is divided into two parts $L_1 = A + B$, where A is that portion of L_1 contained within $|s| \leq |z|^\delta$ for a fixed δ in $0 < \delta < 1$. Clearly as $|z| \to \infty$, A will include all of the circular portion of L_1 and part of the straight line parts of L_1. Under these circumstances B will always consist of two disjoint straight line portions of L_1.

On B, $\log(\frac{s}{z})$ satisfies the inequalities

$$(4.10) \qquad |\arg(\frac{s}{z})| \leq |\log(\frac{s}{z})| \leq |\log|\frac{s}{z}|| + |\arg(\frac{s}{z})| .$$

Since $|\arg(\frac{s}{z})|$ is uniformly bounded away from zero, $|\log(\frac{s}{z})|$ is similarly uniformly bounded away from zero. Although $|\log(\frac{s}{z})|$ becomes unbounded on B, it is either bounded by $\log|z|$ or by $\log|s|$, depending on which is larger. An easy estimate gives, for $|s| \geq |z|^\delta$, that a fixed $\varepsilon > o$ must exist for which

$$(4.11) \qquad \int_B s^{-\alpha-1}(\log \frac{s}{z})^{-\beta-1} \exp(s)ds = O(\exp(-\varepsilon|z|^\delta)) ,$$

as $z \to \infty$, with α, β unrestricted. (Compare with the result (1.31)). The order relation holds uniformly.

On the part A of the path of integration,

$$(4.12)\quad s^{-\alpha-1}(\log(\tfrac{s}{z}))^{-\beta-1} = s^{-\alpha-1}(\log s - \log z)^{-\beta-1}$$

$$= (-\log z)^{-\beta-1}s^{-\alpha-1}(1 - \frac{\log s}{\log z})^{-\beta-1}$$

$$= (-\log z)^{-\beta-1}s^{-\alpha-1}\sum_{n=o}^{\infty}\frac{(\beta+1)_n}{n!}\frac{(\log s)^n}{(\log z)^n}$$

$$= (-\log z)^{-\beta-1}s^{-\alpha-1}[\sum_{n=o}^{N}\frac{(\beta+1)_n}{n!}\frac{(\log s)^n}{(\log z)^n} + O(\frac{(\log s)^{N+1}}{(\log z)^{N+1}})],$$

as $z \to \infty$, for every fixed integer $N \geq o$. Since $\int_{L_1} s^{-\alpha-1}(\log s)^n \exp(s)ds$ exists as an absolutely convergent integral for each fixed integer $n \geq o$, one must have

$$(4.13)\quad (2\pi i)^{-1}\int_A s^{-\alpha-1}(\log \tfrac{s}{z})^{-\beta-1}\exp(s)ds$$

$$= (-\log z)^{-\beta-1}[\sum_{n=o}^{N}\frac{(\beta+1)_n}{n!}\frac{(2\pi i)^{-1}}{(\log z)^n}\int_A s^{-\alpha-1}(\log s)^n\exp(s)ds$$

$$+ O((\log z)^{-N-1})],\quad \text{as} \quad z \to \infty.$$

However by the same argument used to obtain (4.11)

$$(4.14)\quad (2\pi i)^{-1}\int_A s^{-\alpha-1}(\log s)^n\exp(s)ds = (2\pi i)^{-1}\int_L s^{-\alpha-1}(\log s)^n\exp(s)ds$$

$$+ O(\exp(-\varepsilon|z|^{\delta})),\quad \text{as} \quad z \to \infty,$$

and, therefore from (2.14)

$$(4.15)\quad (2\pi i)^{-1}\int_A s^{-\alpha-1}(\log s)^n\exp(s)ds \approx (-1)^n D^n[\Gamma^{-1}(\alpha+1)];\quad \{\phi_n = (\log z)^{-n}\},$$

as $z \to \infty$, where $D = \frac{d}{d\alpha}$. These results coupled together give

$$(4.16)\quad F(z,\beta,\alpha) \sim (-\log z)^{-\beta-1}[\sum_{n=o}^{\infty}\frac{(\beta+1)_n D^n[\Gamma^{-1}(\alpha+1)]}{n!\quad(-\log z)^n};\quad \{(\log z)^{-n}\}]$$

as $z \to \infty$ in $0 \leq \arg z \leq \pi - 2\Delta_1$. Further, this implies

$$(4.17)\quad \mu(z,\beta,\alpha) \sim z^{\alpha}(-\log z)^{-\beta-1}\left[\sum_{n=o}^{\infty} \frac{(\beta+1)_n D^n[\Gamma^{-1}(\alpha+1)]}{n!\ (-\log z)^n}\ ;\ \{(\log z)^{-n}\}\right]$$

providing $z \to \infty$ in $\frac{\pi}{2} + \Delta_1' \leq \arg z \leq \pi - 2\Delta_1$, a result that automatically implies the formula holds if $-\pi + 2\Delta_1 \leq \arg z \leq -\frac{\pi}{2} - \Delta_1'$.

It is of passing interest to note that, with minor modification, the same proof can be used to prove (4.17) holds when $z \to o$ in an unrestricted manner, a result that was obtained in a different way in the previous chapter and also in [11], p. 219.

In Figure 4.3, γ will now be the portion of L_3 which is not part of L_1. It will therefore consist of the part encircling $s = z$, and the two straight lines necessary to join γ to L_1. If we replace s by $(1 - s)z$,

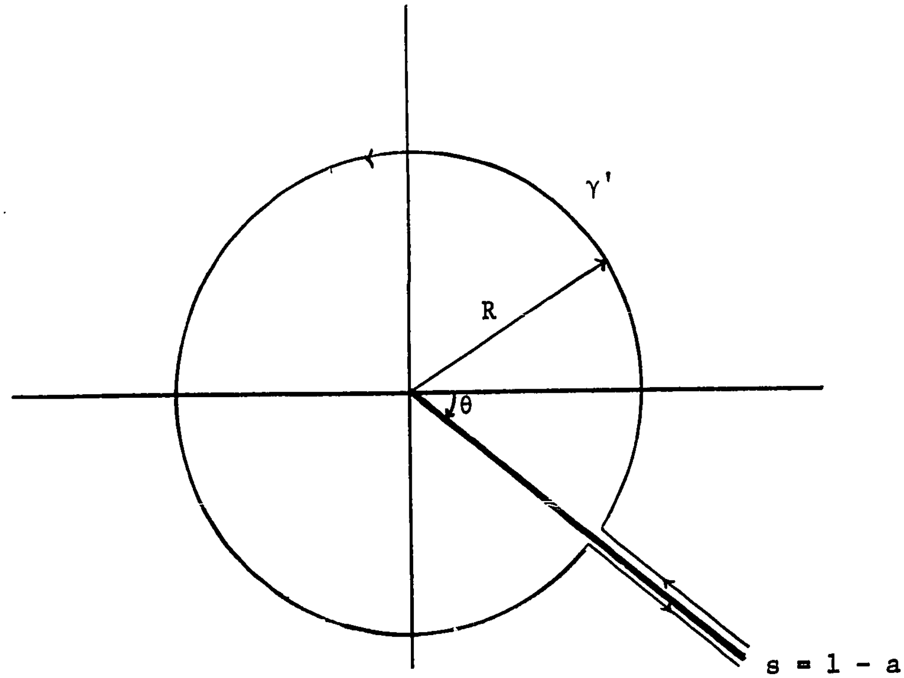
$$(4.18)\quad (2\pi i)^{-1}z^{\alpha} \int_{\gamma} s^{-\alpha-1}(\log(\tfrac{s}{z}))^{-\beta-1}\exp(s)ds = (2\pi i)^{-1}\exp(z) \int_{\gamma'} (1-s)^{-\alpha-1} \cdot (\log(1-s))^{-\beta-1}\exp(-sz)ds$$

with γ' as shown below. The radius of the circle can still be chosen arbitrarily in $0 < R < 1$.

Within and on γ', $(1-s)^{-\alpha-1}(\log(1-s))^{-\beta-1}$ has a convergent expansion of the form

$$(4.19)\qquad (1-s)^{-\alpha-1}(\log(1-s))^{-\beta-1} = (-s)^{\beta-1} \sum_{n=o}^{\infty} a_n(-s)^n, \qquad |s| < 1,$$

where



s-plane

Figure 4.4.

(4.20) $$a_n = \frac{(-1)^n}{n!} \frac{d^n}{ds^n} [(1-s)^{-\alpha-1} (\frac{\log(1-s)}{-s})^{-\beta-1}]_{s=o} .$$

Hence

(4.21) $$(1-s)^{-\alpha-1}(\log(1-s))^{-\beta-1} = (-s)^{-\beta-1} [\sum_{n=o}^{N} a_n(-s)^n + R_N] ,$$

where for any fixed integer $N \geq 0$, R_N is a regular function of s in $|s| < 1$, and

(4.22) $$|R_N| \leq K|s|^{N+1} , \quad |s| < 1,$$

for some fixed $K > o$. Hence

$$(4.23)\quad (2\pi i)^{-1} \int_{\gamma'} (1-s)^{-\alpha-1}(\log(1-s))^{-\beta-1}\exp(-zs)ds$$

$$= \sum_{n=o}^{N} a_n(2\pi i)^{-1} \int_{\gamma'} (-s)^{n-\beta-1}\exp(-zs)ds$$

$$+ (2\pi i)^{-1} \int_{\gamma'} (-s)^{-\beta-1} R_N \exp(-zs)ds.$$

If the integer N is chosen so that $N - \mathrm{Re}(\beta - 1) > o$, then the circular part of γ', for the remainder term only, can be shrunk to zero; leaving only straight line segments embracing the line joining $s = o$ to $s = 1 - a$. By using (4.22), one obtains

$$(4.24)\quad \left|\int_{\gamma'} (-s)^{-\beta-1} R_N \exp(-zs)ds\right| \le 2K \int_o^{1-a} |s^{N-\beta} \exp(-zs)ds|$$

$$\le \frac{2K}{|z|^{N+1-\beta}},$$

providing $|\arg(ze^{-i\theta})| < \frac{\pi}{2}$, or $-\frac{\pi}{2} + \theta < \arg z < \frac{\pi}{2} + \theta$, and this range will embrace the transitional region $\arg z = \frac{\pi}{2}$.

Again by an argument used before,

$$(4.25)\quad (2\pi i) \int_{\gamma'} (-s)^{n-\beta-1}\exp(-zs)ds = (2\pi i)^{-1} \int_L (-s)^{n-\beta-1}\exp(-zs)ds$$

$$+ O(\exp(-\varepsilon|z|)), \quad \text{as} \quad |z| \to \infty,$$

where L consists of γ' and the extension of the straight line portions to infinity in the direction $\arg s = -\theta$. This gives

$$(4.26)\quad (2\pi i)^{-1}\int_{\gamma'} (-s)^{n-\beta-1}\exp(-zs)ds \approx \frac{z^{\beta-n}}{\Gamma(\beta+1-n)} ; \quad \{\phi_n = z^{\beta-n}\}, \quad \text{as} \quad z \to \infty.$$

Coupling these results together gives

$$(4.27)\qquad (2\pi i)^{-1} \int_{\gamma'} (1-s)^{-\alpha-1}(\log(1-s))^{-\beta-1}\exp(-zs)ds$$
$$\sim z^{\beta} \left[\sum_{n=o}^{\infty} \frac{a_n}{\Gamma(\beta+1-n)z^n} ; \{z^{-n}\}\right].$$

Hence

$$(4.28)\qquad (2\pi i)^{-1} z^{\alpha} \int_{\gamma} s^{-\alpha-1}(\log(\tfrac{s}{z}))^{-\beta-1}\exp(s)ds$$
$$\sim z^{\beta} \exp(z) \left[\sum_{n=o}^{\infty} \frac{a_n}{\Gamma(\beta+1-n)z^n} ; \{z^{-n}\}\right],$$

as $z \to \infty$ in $0 \leq \arg z < \frac{\pi}{2} + \theta$. The exponential nature of this behavior allows one then to write

$$(4.29)\qquad \mu(z,\beta,\alpha) \sim z^{\beta} \exp(z) \left[\sum_{n=o}^{\infty} \frac{a_n}{\Gamma(\beta+1-n)z^n} ; \{z^{-n}\}\right]$$
$$+ z^{\alpha}(-\log z)^{-\beta-1}\left[\sum_{n=o}^{\infty} \frac{(\beta+1)_n \; D^n[\Gamma^{-1}(\alpha+1)]}{n! \; (-\log z)^n} ; \{(\log z)^{-n}\}\right]$$

as $z \to \infty$ in $|\arg z| \leq \pi - 2\Delta$, for any fixed Δ in $0 < \Delta \leq \frac{\pi}{2}$; a result that is uniform in the approach of $z \to \infty$. The meaning of this result is of course that for any fixed integer $M \geq o$ and $N \geq o$,

$$(4.30)\qquad \mu(z,\beta,\alpha) = z^{\beta} \exp(z)\left[\sum_{n=o}^{N} \frac{a_n}{\Gamma(\beta+1-n)z^n} + O(z^{-N-1})\right]$$
$$+ z^{\alpha}(-\log z)^{-\beta-1}\left[\sum_{n=o}^{M} \frac{(\beta+1)_n \; D^n[\Gamma^{-1}(\alpha+1)]}{n! \; (-\log z)^n} + O((\log z)^{-M-1})\right]$$

as $z \to \infty$. Clearly when $|\arg z| \leq \frac{\pi}{2} - \Delta$, the exponential dominates, and every term of the first series dominates every term of the second series. In $|\arg z| \geq \frac{\pi}{2} + \Delta$, the converse situation holds. When $\arg z \to \pm \frac{\pi}{2}$, no clear cut dominance exists except under special circumstances. If $\mathrm{Re}\,\alpha > \mathrm{Re}\,\beta$, then the first series can be dropped.

When $\beta = -m-1$, $m = 0,1,2,3,\cdots$, then all of the terms of the first series of (4.29) are zero, the second series has only a finite number of terms. In this case, there is no singularity at $s = 1$, and

$$(4.31)\qquad \mu(z,\beta,\alpha) = (2\pi i)^{-1} \int_{L_1} s^{-\alpha-1}(\log s)^m \exp(sz)ds$$

$$= (-1)^m \cdot D^m[(2\pi i)^{-1} \int_{L_1} s^{-\alpha-1} \exp(sz)ds]$$

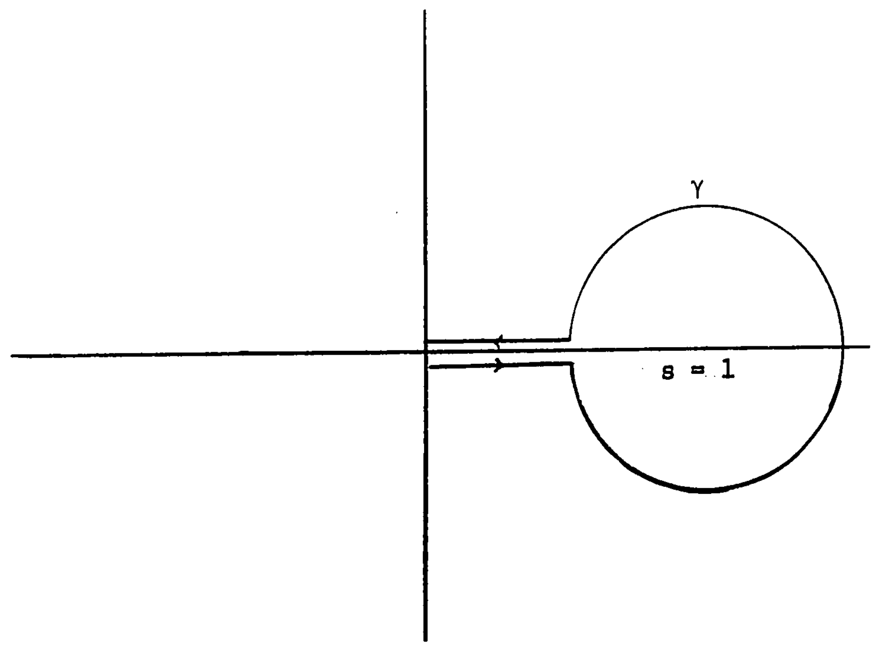
$$= (-1)^m D^m \{\frac{z^\alpha}{\Gamma(\alpha+1)}\}$$

$$= (-1)^m \cdot z^\alpha (\log z)^m \sum_{n=o}^{m} \binom{m}{n} \frac{D^n\{\Gamma^{-1}(\alpha+1)\}}{(\log z)^n},$$

and this exact well-known expansion adequately describes the asymptotic behavior of $\mu(z,\beta,\alpha)$ as $z \to \infty$ or $z \to o$ in an unrestricted manner.

The case $\beta = m$, $m = 0,1,2,\cdots$, is also a special case in that the singularity at $s = 1$ is a pole of order $m+1$, and is no longer a branch point. The path of integration to determine μ can be broken into two parts, a path of the form L_1, and a complete circle around $s = 1$, there exists a rather elegant evaluation of the integral that encircles $s = 1$, the value is of course the residue of the integral at this pole.

The circle around $s = 1$ is deformed into the path of integration γ shown below:



s-plane

Figure 4.5.

We consider

$$(4.32)\quad I = (2\pi i)^{-1} \int_{\gamma} s^{-\alpha-1}(\log s)^{-m-1} \exp(sz)ds, \quad m = 0,1,2,\cdots, \operatorname{Re} \alpha < o$$

Since the integral is now regular on the real axis $0 < s < 1$, the integral along the straight line portions vanishes. The substitution $s = e^{-w}$ gives

$$(4.33)\qquad I = (2\pi i)^{-1} \int_{L} (-w)^{-m-1} \exp(ze^{-w} + \alpha w)dw ,$$

where L begins at $w = \infty$, loops the origin in the counter-clockwise manner, and ends at $w = \infty$. Since

$$(4.34)\qquad \exp(ze^{-w}) \cdot \exp(\alpha w) = \sum_{n=o}^{\infty} \frac{z^n}{n!} \exp[-(n - \alpha)w],$$

$$(4.35)\qquad I = \frac{1}{m!} \sum_{n=o}^{\infty} \frac{z^n}{n!} (n - \alpha)^m$$

$$= \frac{1}{m!} \sum_{r=o}^{m} \binom{m}{r} (-\alpha)^{m-r} \sum_{n=o}^{\infty} \frac{z^n \cdot n^r}{n!}$$

$$= \frac{1}{m!} \sum_{n=o}^{m} \binom{m}{r} (-\alpha)^{m-r} \theta^r e^z , \quad \text{where} \quad \theta = z \frac{d}{dz} .$$

By the Leibnitz rule, one obtains

$$(4.36)\qquad I = z^{\alpha} \theta^m (z^{-\alpha} e^z)/m! ,$$

with this exact evaluation, the condition on α can now be removed. This result implies

$$(4.37)\qquad \mu(z,m,\alpha) \sim \frac{z^{\alpha}}{m!} \theta^m (z^{-\alpha} e^z)$$

$$+ \frac{(-1)^{m+1} z^{\alpha}}{(\log z)^{m+1}} [\sum_{n=o}^{\infty} \binom{-m-1}{n} D^n[\Gamma^{-1}(\alpha+1)](\log z)^{-n}; \quad \{(\log z)^{-n}\}]$$

where $m = 0,1,2,\cdots$.

This result can be used to correct a result given in [11], where the statement equivalent to

$$(4.38)\qquad \nu(z,\alpha) = \mu(z,0,\alpha) = e^z + 0(z^{\alpha-N}), \quad \text{as} \quad z \to \infty$$

in $|\arg z| \leq \pi$ is given, for every integer $N \geq o$. For $m = o$, (4.37) gives

$$(4.39)\qquad \nu(z,\alpha) = e^z - \frac{z^{\alpha}}{\log z} \{\frac{1}{\Gamma(\alpha+1)} + 0 (\frac{1}{\log z})\} ,$$

and the result given in (4.38) is incorrect because $z^{\alpha}/\log z \neq O(z^{\alpha-N})$, for any non-negative integer N. Although it is not difficult to pick up the error in [11], we omit it in view of the rather simple derivation of the correct result for $\nu(z,\alpha)$ which one could obtain by the method used in the present chapter.

In the earlier work of Barnes [2] and the later work of Watson [21] which we mentioned in Chapter I, the success of finding the asymptotic behavior of a Laplace integral $\int_L f(t)\exp(zt)dt$ depends on $f(t)$ having at most a branch-point singularity at $t = o$ of the form t^{α}. It is now clear from the asymptotic behavior of $\mu(z,\beta,\alpha)$ that this work can be extended to allow $f(t)$ to have combinations of branch-point singularities and logarithmic singularities of the form $t^{\alpha}(\log t)^{\beta}$. The extension of this work will be given in the next chapter.

CHAPTER V

Generalizations of Watson's Lemma

Although the discussion of the asymptotic behavior of a function $F(z)$ defined by

$$F(z) = \int_{\Gamma} f(t)\ \exp(-zt)dt \tag{5.1}$$

will be continued in this chapter, the function $f(t)$ will now be allowed to possess singularities which are of branch-point and logarithmic type.

When the identity

$$\int_{o}^{\infty e^{i\gamma}} t^{\lambda-1}\ \exp(-zt)dt = \Gamma(\lambda)z^{-\lambda}, \quad \text{Re}\ \lambda > o, \quad |\arg(ze^{i\gamma})| < \frac{\pi}{2} \tag{5.2}$$

is differentiated m times with respect to λ, the result is

$$\begin{aligned}\int_{o}^{\infty e^{i\gamma}} t^{\lambda-1}(\log\ t)^{m}\ \exp(-zt)dt &= \frac{d^{m}}{d\lambda^{m}}\ (\Gamma(\lambda)z^{-\lambda}) \\ &= \sum_{r=o}^{m} \binom{m}{r}\ \Gamma^{(r)}(\lambda)z^{-\lambda}(-\log\ z)^{m-r}\end{aligned} \tag{5.3}$$

or

$$\int_{o}^{\infty e^{i\gamma}} t^{\lambda-1}(-\log\ t)^{m}\exp(-zt)dt = (\log\ z)^{m} \cdot z^{-\lambda} \sum_{r=o}^{m} (-1)^{r}\binom{m}{r}\Gamma^{(r)}(\lambda)(\log\ z)^{-r}. \tag{5.4}$$

From (5.4), one might reasonably conjecture that

$$\int_{o}^{\infty e^{i\gamma}} t^{\lambda-1}(-\log\ t)^{\mu}\exp(-zt)dt = (\log\ z)^{\mu}\cdot z^{-\lambda}\cdot \sum_{r=o}^{\infty} (-1)^{r}\binom{\mu}{r}\Gamma^{(r)}(\lambda)(\log\ z)^{-r}, \tag{5.5}$$

a conjecture that is false. The series in (5.5) diverges for all finite values of $\log z$. However, the weaker result

(5.6) $$\int_{o}^{\infty e^{i\gamma}} t^{\lambda-1}(-\log t)^{\mu}\exp(-zt)dt \sim (\log z)^{\mu}\cdot z^{-\lambda}\cdot[\sum_{r=o}^{\infty}(-1)^{r}\binom{\mu}{r}\Gamma^{(r)}(\lambda)(\log z)^{-r};\ \{(\log z)^{-n}\}],$$

as $|z| \to \infty$, does hold under suitable conditions.

Since integrals of the type $\int_{A}^{B} t^{\lambda-1}(-\log t)^{\mu}\exp(-zt)dt$, and expressions of the form $\sum_{r=o}^{N}(-1)^{r}\binom{\mu}{r}\Gamma^{(r)}(\lambda)(\log z)^{-r}$, will constantly be used in the present chapter, the notations

(5.7) $$L(A,B,\lambda,\mu,z) = \int_{A}^{B} t^{\lambda-1}(-\log t)^{\mu}\exp(-zt)dt$$

(5.8) $$S_N(\lambda,\mu,\log z) = \sum_{r=o}^{N}(-1)^{r}\binom{\mu}{r}\Gamma^{(r)}(\lambda)(\log z)^{-r}$$

will be used to reduce the amount of repetition. In (5.7) the path of integration is the straight line joining $t = A$ to $t = B$.

For any fixed point $t = c = |c| \exp(i\gamma)$, the following result holds:

(5.9) $$L(c, \infty\exp(i\gamma), \lambda, \mu, z) = O(\exp(-\varepsilon|z|)), \text{ for some fixed } \varepsilon > o,$$

providing $\operatorname{Re}\lambda > o$, and $z \to \infty$ in any fixed interval contained within $|\arg(z\exp(i\gamma))| \leq \frac{\pi}{2} - \Delta < \frac{\pi}{2}$. If $\gamma = o$, and $o < c \leq 1$, then the condition $\operatorname{Re}\mu > -1$ must be added. This result, coupled with (5.4), gives

(5.10) $$L(o, c, \lambda, m, z) = (\log z)^{m} z^{-\lambda} S_m(\lambda, m, \log z) + O(\exp(-\varepsilon|z|)),$$

as $|z| \to \infty$, where in this instance m is a non-negative integer. One can interpret this result in two obvious but quite different ways. It is true that

$$(5.11)\qquad L(o, c, \lambda, m, z) \sim (\log z)^m z^{-\lambda} [S_m(\lambda, m, \log z); \{(\log z)^{-n}\}]$$

with the meaning that for every integer $o \leq N \leq m$,

$$(5.12)\qquad L(o, c, \lambda, m, z) = (\log z)^m z^{-\lambda} [S_N(\lambda, m, \log z) + o((\log z)^{-N})],$$

as $|z| \to \infty$.

It is also true that

$$(5.13)\qquad L(o, c, \lambda, m, z) \approx (-1)^m \frac{d^m}{d\lambda^m} (\Gamma(\lambda) z^{-\lambda}); \quad \{\phi_n\},$$

as $|z| \to \infty$, providing $\{\phi_n\}$ is an asymptotic sequence for which

$$(5.14)\qquad \exp(-\varepsilon|z|) = o(\phi_n) \quad \text{or} \quad \exp(-\varepsilon|z|) \approx o; \quad \{\phi_n\},$$

as $|z| \to \infty$.

Since every sequence

$$(5.15)\qquad \phi_n = P_s(\log z)/z^{\lambda_n},$$

where $\operatorname{Re} \lambda_{n+1} > \operatorname{Re} \lambda_n$ and $P_s(w)$ is a polynomial whose degree s is arbitrary, is an asymptotic sequence as $|z| \to \infty$, it becomes clear that the interpretation (5.13) will lead to asymptotic information which is more detailed in nature than that which would be obtained by using (5.11). The estimates of error involved are quite different in nature. Both points of view will be used to obtain generalizations of Watson's Lemma.

For

(5.16) $$F(z) = \int_o^a f(t) \exp(-zt)dt,$$

it will be assumed that:

(i) $t = a = |a| \exp(i\gamma)$, $|a| > o$, is a fixed point in the complex t-plane;

(ii) the path of integration is the straight line joining $t = o$ to $t = a$;

(iii) $S(\Delta)$ is defined as in (2.17) to be the point set in the z-plane for which $|\arg(z \exp(i\gamma))| \leq \frac{\pi}{2} - \Delta$, Δ fixed in $o < \Delta \leq \frac{\pi}{2}$;

(iv) $F(z)$ exists for some $z = z_o$.

Theorem 5.1. If:

(i) for each integer $N \varepsilon I$,

(5.17) $$f(t) = \sum_{n=o}^{N} a_n t^{\lambda_n - 1} P_n(\log t) + o(t^{\lambda_N - 1}(\log t)^{m(N)}),$$

as $t \to o$ along $\arg t = \gamma$;

(ii) $P_n(w)$ is a polynomial of degree $m = m(n)$;

(iii) $\{\lambda_n\}$ is a sequence of fixed complex numbers, with $\mathrm{Re}\,\lambda_{n+1} > \mathrm{Re}\,\lambda_n$, $\mathrm{Re}\,\lambda_o > o$, for all n such that n and $n + 1$ are both in I;

(iv) $\{a_n\}$ is a sequence of fixed complex numbers;

then as $z \to \infty$ in $S(\Delta)$

$$\text{(5.18)} \qquad F(z) \sim \sum_{n\varepsilon I} a_n P_n(D_n)[\Gamma(\lambda_n) z^{-\lambda_n}]; \quad \{z^{-\lambda_n}(\log z)^{m(n)}\},$$

where D_n is the operator $D_n = \frac{d}{d\lambda_n}$. The result is uniform in the approach of $z \to \infty$ in $S(\Delta)$.

Proof: Trivially $\{z^{-\lambda_n}(\log z)^{m(n)}\}$ is an asymptotic sequence as $|z| \to \infty$ in $S(\Delta)$. For any fixed choice of $t = c \neq o$ on the path of integration, (3.13) implies

$$\text{(5.19)} \qquad I(z, c) = \int_c^a f(t) \exp(-zt)dt$$

$$= O(\exp(-\delta|z|)) \approx 0; \quad \{z^{-\lambda_n}(\log z)^{m(n)}\},$$

uniformly, as $z \to \infty$ in $S(\Delta)$, where δ is some fixed positive number.

Writing

$$\text{(5.20)} \qquad f(t) = \sum_{n=o}^{N} a_n t^{\lambda_n - 1} P_n(\log t) + R_N$$

gives

$$\text{(5.21)} \qquad \int_o^c f(t)\exp(-zt)dt = \sum_{n=o}^{N} a_n \int_o^c t^{\lambda_n - 1} P_n(\log t) \exp(-zt)dt + r_N ,$$

where

$$\text{(5.22)} \qquad r_N = \int_o^c R_N \exp(-zt)dt.$$

Using (5.13) and (5.19) gives

$$\text{(5.23)} \qquad \int_o^a f(t) \exp(-zt)dt = \sum_{n=o}^{N} a_n P_n(D_n)[\Gamma(\lambda_n) z^{-\lambda_n}]$$

$$+ r_N + O(\exp(-\delta|z|))$$

for some fixed $\delta > o$, as $z \to \infty$ in $S(\Delta)$.

Since the choise of $t = c$ is arbitrary, it may be chosen sufficiently small so that

$$(5.24) \qquad r_N = \int_o^c R_N \exp(-zt)dt$$

satisfies

$$(5.25) \qquad |r_N| \leq \varepsilon \int_o^c |t^{\lambda_N-1}(\log t)^{m(N)}\exp(-zt)dt|.$$

Replacing zt by t gives

$$(5.26) \qquad |r_N| \leq \varepsilon |(\log z)^{m(N)} z^{-\lambda_N}| \int_o^{\infty\exp(i\arg(zc))} |t^{\lambda_N-1}\left(1 - \frac{\log t}{\log z}\right)^{m(N)} \exp(-t)dt|.$$

The existence of the integral, uniformly bounded in z as $z \to \infty$ in $S(\Delta)$, implies

$$(5.27) \qquad r_N = o(z^{-\lambda_N}(\log z)^{m(N)}), \text{ uniformly, as } z \to \infty \text{ in } S(\Delta).$$

These results give

$$(5.28) \qquad F(z) = \int_o^a f(t)\exp(-zt)dt$$

$$= \sum_{n=o}^{N} a_n P_n(D_n)[\Gamma(\lambda_n)z^{-\lambda_n}] + o(z^{-\lambda_N}(\log z)^{m(N)}),$$

uniformly, as $z \to \infty$ in $S(\Delta)$. Hence

$$(5.29) \qquad F(z) \sim \sum_{n\varepsilon I} a_n P_n(D_n)[\Gamma(\lambda_n)z^{-\lambda_n}]; \quad \{z^{-\lambda_n}(\log z)^{m(n)}\},$$

uniformly, as $z \to \infty$ in $S(\Delta)$.

When $m(n) = o$ for all $n \varepsilon I$, then $P_n(D_n)$ is a constant, and the result of Theorem 5.1 reduces to the result contained in the general form of Watson's Lemma given in Chapter III.

In particular, if

$$(5.30) \qquad f(t) \sim \sum a_n t^{\lambda_n - 1} (\log t), \quad \text{as} \quad t \to o,$$

theorem 5.1 gives

$$(5.31) \qquad \int_o^\infty f(t)\exp(-zt)dt \sim \sum a_n \frac{d}{d\lambda_n} [\Gamma(\lambda_n) z^{-\lambda_n}]$$

$$\sim \sum a_n \Gamma(\lambda_n)[\psi(\lambda_n) - \log z] z^{-\lambda_n}; \quad \{z^{-\lambda_n} \log z\},$$

where $\psi(\lambda) = \Gamma'(\lambda)/\Gamma(\lambda)$. This particular result is given by D.S. Jones in [14], p. 439.

It is interesting to note that Theorem 5.1 can be used to yield asymptotic results for Fourier integrals of the form

$$(5.32) \qquad I(z) = \int_a^b f(t)\exp(izt)dt, \quad z \text{ real and positive},$$

where the path of integration is the porcion of the real axis $a \leq t \leq b$. Erdélyi [10] and Jones and Kline [15] have considered the case where

$$(5.33) \qquad f(t) = (t - a)^{\lambda - 1}(b - t)^{\mu - 1} \phi(t) \log(t - a),$$

$\phi \varepsilon C^N[a,b]$, $o < \lambda \leq 1$, and $o < \mu \leq 1$. By repeated integration by parts, they obtain

$$(5.34)\quad I(z) = \sum_{n=o}^{N-1} a_n \Gamma(n+\lambda)\exp[i(az + \tfrac{\pi}{2}(n+\lambda))]\{\psi(n+\lambda) - \log z + i\,\tfrac{\pi}{2}\}z^{-(n+\lambda)}$$

$$+ \sum_{n=o}^{N-1} b_n \Gamma(n+\mu)\exp[i(bz + \tfrac{\pi}{2}(n-\mu))]z^{-(n+\mu)} + O(z^{-N}),$$

as $z \to +\infty$, where

$$(5.35)\qquad a_n = \frac{1}{n!}\frac{d^n}{da^n}\{(b - a)^{\mu-1}\ \phi(a)\}$$

and

$$(5.36)\qquad b_n = \frac{1}{n!}\frac{d^n}{db^n}\{(b - a)^{\lambda-1}\ \phi(b)\ \log(b - a)\}.$$

Although the method of repeated integration by parts gives a simple derivation of an asymptotic result when it can be applied, it is not normally a good procedure to use for proving general theorems. The method places unnecessary restrictions on the integrand. Although the function $\phi(t) = \sin(\sqrt{t - a})$ is not once differentiable at $t = a$, a complete asymptotic expansion of $I(z)$ can easily be obtained. Theorem 5.1 and a procedure outlined in Chapter I will be used to obtain a more general result than the one mentioned above.

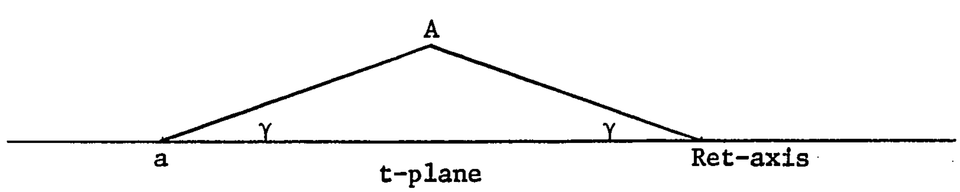


Figure 5.1.

Consider a function $I(z)$ defined by

$$(5.37) \qquad I(z) = \int_a^A f(t)\exp(izt)dt + \int_A^b f(t)\exp(izt)dt$$
$$= \int_a^A f(t)\exp(izt)dt - \int_b^A f(t)\exp(izt)dt.$$

In the first integral we replace t by $t + a$, and in the second we replace t by $b - t$. This will give

$$(5.38) \quad I(z) = \exp(iza)\int_o^{A-a} f(t+a)\exp(izt)dt + \exp(izb)\int_o^{b-A} f(b-t)\exp(-izt)dt.$$

If the conditions of Theorem 5.1 are satisfies by both integrals, then it is possible to write

$$(5.39) \quad I(z) \sim \exp(iza)[\sum_{n\in I_1} a_n^{(1)} P_n^{(1)}(D_n^{(1)})[\Gamma(\lambda_n)(-iz)^{-\lambda_n}]; \{z^{-\lambda_n}(\log z)^{m_1(n)}\}];$$
$$+ \exp(izb)[\sum_{n\in I_2} a_n^{(2)} P_n^{(2)}(D_n^{(2)})[\Gamma(\mu_n)(iz)^{-\mu_n}]; \{z^{-\mu_n}(\log z)^{m_2(n)}\}],$$

as $z \to \infty$ in the intersection of $|\arg(-iz \exp(i\gamma))| \leq \frac{\pi}{2} - \Delta_1$ and $|\arg(iz \exp(i\gamma_1))| \leq \frac{\pi}{2} - \Delta_2$, where

$$(5.40) \qquad \tan \gamma_1 = -\frac{|A - a| \sin \gamma}{b - a - |A - a| \cos \gamma}.$$

We are of course assuming, for each $N_1 \in I_1$

$$(5.41) \quad f(t) = \sum_{n=o}^{N_1} a_n^{(1)}(t-a)^{\lambda_n - 1} P_n^{(1)}(\log(t-a)) + o\left((t-a)^{\lambda_{N_1}-1}(\log(t-a))^{m_1(N_1)}\right)$$

as $t \to a$ along aA, and for each $N_2 \in I_2$,

$$(5.42) \quad f(t) = \sum_{n=o}^{N_2} a_n^{(2)}(b-t)^{\mu_n - 1} P_n^{(2)}(\log(b-t)) + o\left((b-t)^{\mu_{N_2}-1}(\log(b-t))^{m_2(N_2)}\right)$$

as $t \to b$ along Ab.

By taking γ sufficiently small, it can be proven that (5.39) holds as $z \to +\infty$, with z real and positive. Finally, if the conditiions of Cauchy's Theorem are satisfied, then (5.39) will hold when the path of integration is the straight line joining $t = a$ to $t = b$. From a purely pragmatic point of view, the result contained in (5.39) is far more general than the result obtained by integration by parts. In addition to a generalization of the integrand, Cauchy's Theorem does not require differentiability on the boundary. Since it does require regularity within the boundary, no mathematical proof has been given to show that one procedure is more general than the other.

When

$$f(t) = (t-a)^{\lambda-1}(b-t)^{\mu-1}\phi(t)\ \log(t-a), \tag{5.43}$$

and the conditions of our result are satisfied, then (5.39) reduces identically to the result obtained by integration by parts.

As a second example, consider

$$I(z) = \int_o^1 t^{\lambda-1}\log t\ \sin(\sqrt{t})\exp(izt)dt, \quad \text{Re } \lambda > o. \tag{5.44}$$

In a neighborhood of $t = o$, $f(t) = t^{\lambda-1} \log t \sin \sqrt{t}$ has the convergent expansion

$$t^{\lambda-1} \log t \sin \sqrt{t} = \sum_{n=o}^{\infty} \frac{(-1)^n}{(2n+1)!} t^{\lambda+n-1/2} \log t \tag{5.45}$$

and, in a neighborhood of $t = 1$,

$$t^{\lambda-1} \log t \sin \sqrt{t} = \sum_{n=o}^{\infty} b_n(1-t)^n, \quad b_n = \frac{(-1)^n}{n!} \frac{d^n}{dt^n} (t^{\lambda-1}\log t \sin\sqrt{t})\Big|_{t=1}, \tag{5.46}$$

is also convergent. Hence

$$(5.47)\quad a_n^{(1)} = \frac{(-1)^n}{(2n+1)!}, \quad \lambda_n = \lambda + n + \frac{1}{2}, \quad P_n^{(1)}(w) = w, \quad m_1(n) = 1,$$

$$a_n^{(2)} = b_n, \quad \mu_n = n + 1, \quad P_n^{(2)}(w) = 1, \quad m_2(m) = o,$$

and therefore

$$(5.48)\quad \int_o^1 t^{\lambda-1}\log t \sin\sqrt{t} \exp(izt)dt$$

$$\sim [\sum_{n=o}^{\infty} \frac{(-1)^n}{(2n+1)!} \Gamma(\lambda + n + \tfrac{1}{2})[\psi(\lambda + n + \tfrac{1}{2}) - \log z + i \tfrac{\pi}{2}](-iz)^{-(\lambda+n+\frac{1}{2})}$$

$$; \{z^{-(\lambda+n+\frac{1}{2})} \log z\}]$$

$$+ \exp(iz)[\sum_{n=o}^{\infty} b_n\, n!(iz)^{-(n+1)}; \{z^{-n}\}],$$

as $z \to \infty$, z real and positive.

The function $\sin \sqrt{t}$ is not even once differentiable at $t = o$, and yet the validity of the expansion given above is easily eestablished.

Erdélyi [8] has given an elegant proof of a result from which it is easily established that the function

$$(5.49)\qquad L(o,c,\lambda,\mu,z) = \int_o^c t^{\lambda-1}(-\log t)^{\mu} \exp(-zt)dt$$

has an asymptotic expansion

$$(5.50)\qquad L(o,c,\lambda,\mu,z) \sim z^{-\lambda}(\log z)^{\mu} [S_{\infty}(\lambda,\mu,\log z); \{(\log z)^{-n}\}],$$

as $z \to \infty$ through positive real values of z. Although elegant, the proof is not well adapted to extensions allowing $z \to \infty$ in the complex plane, and does not seem readily adapted to further generalizations. For this reason, a direct proof of the result given in (5.50) will be given. The ultimate result will be obtained by a sequence of lemmas.

Throughout, it will be assumed that $\mathrm{Re}\,\lambda > o$, $c = |c| \exp(i\gamma)$, where γ and $|c|$ are fixed, with $o < |c| < 1$. Although μ is fixed, it may be any complex number.

Lemma 5.1.

If $a = |z|^{-2+\delta}\exp(i\gamma)$, δ fixed, $o < \delta < 1$, then there will be a fixed $\rho > o$ such that

(5.51) $$L(o,a,\lambda,\mu,z) = O(z^{-\lambda-\rho}),$$

as $z \to \infty$ in $S(\Delta)$. The path of integration is the straight line joining $t = o$ to $t = a$.

Proof: Consider

(5.52) $$L(o,a,\lambda,\mu,z) = \int_o^a t^{\lambda-1}(-\log t)^{\mu} \exp(-zt)dt.$$

On the path of integration,

(5.53) $$|L(o,a,\lambda,\mu,z)| \leq \int_o^a t^{\lambda-1}(-\log t)^{\mu}dt|.$$

When $\mathrm{Re}\,\mu \geq o$, then for any $\eta > o$ $(-t^{\eta} \log t)^{\mu}$ is bounded on the path of integration. For the case $\mathrm{Re}\,\mu < o$, $(-\log t)^{\mu}$ is itself bounded on the path of integration. Hence, along the path of integration,

(5.54) $$|(-\log t)^{\mu}| \leq K \; | t^{-\eta\mu}| \; , \quad \text{for some fixed } K > o$$

where η is arbitrary when $\text{Re}\ \mu \geq o$, and $\eta = o$ when $\text{Re}\ \mu < o$. This coupled with (5.53) gives

$$(5.55)\qquad |L(o,a,\lambda,\mu,z)| \leq M\ |\ a^{\lambda-\eta\mu}|\ ,\quad \text{for some fixed } M > o.$$

Since $a = |z|^{-2+\delta} = |z|^{-1} \cdot |z|^{-(1-\delta)}$, it follows that

$$(5.56)\qquad |L(o,a,\lambda,\mu,z)| \leq M \cdot \frac{|z^{-\lambda}|}{|z^{(1-\delta)(\lambda-\eta\mu)-\eta\mu}|}\ .$$

The condition $\text{Re}\ \lambda > o$ ensures that $\text{Re}[\lambda(1-\delta)-(2-\delta)\eta\mu] > o$, under the conditions that μ is fixed, and either $\eta = o$ or η may be chosen arbitrarily small. Hence a fixed $\rho > o$ must exist such that

$$(5.57)\qquad L(o,a,\lambda,\mu,z) = O(z^{-\lambda-\rho}),\quad \text{as}\quad z \to \infty \quad \text{in}\quad S(\Delta).$$

For use at a latter stage, this particular result can be put into a more useful form. When $\text{Re}\ \mu \geq o$, (5.57) obviously implies

$$(5.58)\qquad L(o,a,\lambda,\mu,z) = O(z^{-\lambda-\rho}(\log z)^{\mu}),\quad \text{as}\quad z \to \infty \quad \text{in}\quad S(\Delta).$$

If $\text{Re}\ \mu \leq o$, write $\rho = \rho_1 + \rho_2$, where both ρ_1 and ρ_2 are positive. Since $(\log z)^{-\mu}/z^{\rho_2} \to o$ as $z \to \infty$, it will follow that

$$(5.59)\qquad L(o,a,\lambda,\mu,z) = O(z^{-\lambda-\rho_1}(\log z)^{\mu}),\quad \text{as}\quad z \to \infty \quad \text{in}\quad S(\Delta)$$

where again ρ_1 is some fixed positive number. Hence

$$(5.60)\qquad L(o,a,\lambda,\mu,z) = O(z^{-\lambda-\rho}(\log z)^{\mu}),\quad \text{as}\quad z \to \infty \quad \text{in}\quad S(\Delta),$$

for some fixed $\rho > o$, and μ unrestricted.

Lemma 5.2.

With a as in Lemma 5.1,

$$L(o,az,\lambda,n,1) = \int_o^{az} u^{\lambda-1}(-\log u)^n \exp(-u)du \tag{5.61}$$

$$= O(z^{-\rho}), \quad \text{as } z \to \infty \text{ in } S(\Delta),$$

for some fixed $\rho > o$.

The proof parallels that of Lemma 5.1.

If

$$b = |z|^{-\delta} \exp(i\gamma), \tag{5.62}$$

then by Lemma 1.1, it follows that

$$\int_b^c t^{\lambda-1}(-\log t)^\mu \exp(-zt)dt = O(\exp(-\varepsilon|z|^{1-\delta})), \tag{5.63}$$

for some fixed $\varepsilon > o$, as $z \to \infty$ in $S(\Delta)$. Further it is trivial to prove that

$$\int_{bz}^{\infty\exp[i \arg(ze^{i\gamma})]} u^{\lambda-1}(\log u)^n \exp(-u)du = O(\exp(-\varepsilon|z|^{1-\delta})), \tag{5.64}$$

for some fixed $\varepsilon > o$, as $z \to \infty$ in $S(\Delta)$.

Lemma 5.3.

With a and b defined as above, for any integer $n \geq o$,

$$L(za,ab,\lambda,n,1) = (-1)^n \int_{za}^{zb} u^{\lambda-1}(\log u)^n \exp(-u)du \tag{5.65}$$

$$\approx (-1)^n\, \Gamma^{(n)}(\lambda); \quad \{(\log z)^{-n}\},$$

as $z \to \infty$ in $S(\Delta)$.

Proof:

(5.66) $$L(za,zb,\lambda,n,1) = (-1)^n \int_0^{\infty \exp[i \arg(ze^{i\gamma})]} u^{\lambda-1}(\log u)^n \exp(-u)du$$

$$- L(o,az,\lambda,n,1) - L(bz,\infty\exp(i \arg(ze^{i\gamma})),\lambda,n,1)$$

$$= (-1)^n \int_0^{\infty} u^{\lambda-1}(\log u)^n \exp(-u)du + O(z^{-\rho}),$$

for some fixed ρ, as $z \to \infty$ in $S(\Delta)$.

Hence

(5.67) $$L(za,zb,\lambda,n,1) = (-1)^n \Gamma^{(n)}(\lambda) + O(z^{-\rho})$$

$$\approx (-1)^n \Gamma^{(n)}(\lambda) \; ; \; \{(\log z)^{-n}\},$$

as $z \to \infty$ in $S(\Delta)$.

Theorem 5.2.

If

(i) $L(o,c,\lambda,\mu,z) = \int_0^c t^{\lambda-1}(-\log t)^{\mu} \exp(-zt)dt;$

(ii) λ, μ fixed complex numbers, with $\mathrm{Re}\ \lambda > o$;

(iii) $c = |c| \exp(i\gamma)$, $|c| < 1$, and γ a fixed real number,

then

(5.68) $$L(o,c,\lambda,\mu,z) \sim z^{-\lambda}(\log z)^{\mu} \, [S_{\infty}(\lambda,\mu,\log z); \; \{(\log z)^{-n}\}],$$

as $z \to \infty$ in $S(\Delta)$.

Proof:

$$L(o,c,\lambda,\mu,z) = \int_a^b t^{\lambda-1}(-\log t)^{\mu} \exp(-zt)dt + L(o,a,\lambda,\mu,z) + L(b,c,\lambda,\mu,z), \tag{5.69}$$

where a and b have been chosen as points on the path of integration such that

$$\begin{aligned} a &= |a| \exp(i\gamma), \quad |a| = |z|^{-2+\delta}, \quad \delta \text{ fixed in } o < \delta < 1, \\ b &= |b| \exp(i\gamma), \quad |b| = |z|^{-\delta} . \end{aligned} \tag{5.70}$$

From (5.60) and 5.63), it follows that

$$L(o,c,\lambda,\mu,z) = \int_a^b t^{\lambda-1}(-\log t)^{\mu} \exp(-zt)dt + O(z^{-\lambda-\rho}(\log z)^{\mu}), \tag{5.71}$$

for some fixed ρ, as $z \to \infty$ in $S(\Delta)$.

Turning attention to

$$L(a,b,\lambda,\mu,z) = \int_a^b t^{\lambda-1}(-\log t)^{\mu} \exp(-zt)dt , \tag{5.72}$$

the substitution $u = zt$ gives

$$L(a,b,\lambda,\mu,z) = z^{-\lambda} \int_{az}^{bz} u^{\lambda-1}(\log z - \log u)^{\mu} \exp(-u)du \tag{5.73}$$

and

$$L(a,b,\lambda,\mu,z) = z^{-\lambda}(\log z)^{\mu} \int_{az}^{bz} u^{\lambda-1} \left(1 - \frac{\log u}{\log z}\right)^{\mu} \exp(-u)du. \tag{5.74}$$

Since $|\log u/\log z| \leq 1 - \delta_1$, for some fixed δ_1, $o < \delta_1 < 1$, as $z \to \infty$ in $S(\Delta)$, then for any fixed integer $N \geq o$, the finite binomial expansion, with remainder, gives for all points on the path of integration that

$$(5.75)\qquad \left(1 - \frac{\log u}{\log z}\right)^{\mu} = \sum_{n=o}^{N} (-1)^n \binom{\mu}{n} \frac{(\log u)^n}{(\log z)^n} + R_N$$

where

$$(5.76)\qquad |R_N| \le K\left|\frac{(\log u)^{N+1}}{(\log z)^{N+1}}\right| ,$$

for some fixed $K > o$. Hence

$$(5.77)\qquad L(a,b,\lambda,\mu,z) = z^{-\lambda}(\log z)^{\mu}\Big[\sum_{n=o}^{N} (-1)^n \binom{\mu}{n} (\log z)^{-n}\int_{az}^{bz} u^{\lambda-1}(\log u)^n \exp(-u)du + r_N\Big] ,$$

where

$$(5.78)\qquad r_N = \int_{az}^{bz} u^{\lambda-1} \exp(-u)R_N \, du .$$

Using (5.67) gives

$$(5.79)\qquad L(a,b,\lambda,\mu,z) = z^{-\lambda}(\log z)^{\mu}\Big[\sum_{n=o}^{N} (-1)^n \binom{\mu}{n}\Gamma^{(n)}(\lambda)(\log z)^{-n} + O(z^{-\rho}) + r_N\Big],$$

as $z \to \infty$ in $S(\Delta)$. Further

$$(5.80)\qquad |r_N| \le K|\log z|^{-(N+1)}\int_{az}^{bz} |u^{\lambda-1}(\log u)^{N+1} \exp(-u)du|$$
$$\le K|\log z|^{-(N+1)} \int_{o}^{\infty\exp[i \arg(ze^{i\gamma})]} |u^{\lambda-1}(\log u)^{N+1}\exp(-u)du| .$$

It is trivial to show that the integral in (5.80) exists and is bounded in $\arg z$. Hence

$$(5.81) \qquad L(a,b,\lambda,\mu,z) = z^{-\lambda}(\log z)^{\mu}[S_N(\lambda,\mu.\log z) + O((\log z)^{-N-1})],$$

as $z \to \infty$ in $S(\Delta)$. Further, the order relation does not depend on arg z. This proves the required result.

A combination of the results given thus far yields

$$(5.82) \quad L(o,\infty\exp(i\gamma),\lambda,\mu,z) \sim z^{-\lambda}(\log z)^{\mu}[S_\infty(\lambda,\mu,\log z): \{(\log z)^{-n}\}]$$

$$\sim z^{-\lambda}(\log z)^{\mu}[\sum_{n=o}^{\infty} (-1)^n \binom{\mu}{n} \Gamma^{(n)}(\lambda)(\log z)^{-n}; \{(\log z)^{-n}\}]$$

as $z \to \infty$ in $S(\Delta)$, providing $\operatorname{Re} \lambda > o$. If $\gamma = o$, the existence of L requires $\operatorname{Re} \mu > -1$. If

$$(5.83) \qquad L(o,\infty\exp(i\gamma),\lambda,\mu.z) = \int_o^{\infty e^{i\gamma}} t^{\lambda-1}(-\log t)^{\mu} \exp(-zt)dt$$

is examined, it is reasonable to expect that the result contained in (5.82) can be used to advantage to discuss the asymptotic behavior of integrals of the form

$$(5.84) \qquad F(z) = \int_o^a f(t) \exp(-zt)dt, \quad a = |a| \exp(i\gamma)$$

with $f(t)$ now allowed to have logarithmic type singularities, as well as singularities of branch-point type. In several respects, the asymptotic behavior is quite different than one might expect. Before proceeding, one further result is required.

<u>Lemma 5.4</u>.

There exists a fixed complex number $c = |c| \exp(i\gamma) \neq o$ such that

$$(5.85)\quad I = \int_{o}^{c} |t^{\lambda-1}(-\log t)^{\mu}\exp(-zt)dt| = O(z^{-\lambda}(\log z)^{\mu}), \quad \text{as} \quad z \to \infty \quad \text{in} \quad S(\Delta),$$

where λ, μ are fixed complex numbers, with $\mathrm{Re}\,\lambda > o$. The path of integration is the straight line $\arg t = \gamma$ joining $t = o$ to $t = c$.

Proof: Along the path of integration $t = \rho\exp(i\gamma)$, $o \leq \rho \leq |c|$. Hence

$$(5.86)\qquad |I| \leq K \int_{o}^{|c|} \rho^{\alpha-1}|(-\log \rho - i\gamma)^{\mu}| \exp(-|z|(\sin \Delta)\rho)d\rho,$$

where $\alpha = \mathrm{Re}\,\lambda$. As $|c| \to o$, $\log \rho \to \infty$. This implies, by taking $|c|$ sufficiently small, that

$$(5.87)\qquad |I| \leq K \int_{o}^{|c|} \rho^{\alpha-1}(-\log \rho)^{\beta} \exp(-|z|(\sin \Delta)\rho)d\rho$$

$$\leq K\, L(o,|c|,\alpha.\beta,|z|\sin \Delta),$$

where $\beta = \mathrm{Re}\,\mu$. The constant K is being used as a generic symbol whose value may change from time to time in the proof. Theorem 5.1 can be used to give

$$(5.88)\qquad I = O((|z|\sin \Delta)^{-\alpha}(\log(|z|\sin \Delta)^{\beta}), \quad \text{as} \quad |z| \to \infty$$

$$= O(|z|^{-\alpha}(\log|z|)^{\beta}), \quad \text{as} \quad |z| \to \infty$$

$$= O(z^{-\alpha}(\log z)^{\beta}), \quad \text{as} \quad z \to \infty \quad \text{in} \quad S(\Delta)$$

$$= O(z^{-\lambda}(\log z)^{\mu}), \quad \text{as} \quad z \to \infty \quad \text{in} \quad S(\Delta).$$

The result is uniform in $\arg z$ as $z \to \infty$ in $S(\Delta)$.

Theorem 5.3.

If:

(i) $F(z)$ as given in (5.84) exists for some fixed $z = z_o$;

(ii) $\{t^{\lambda_n - 1}(-\log t)^{\mu_n}\}$ is an asymptotic sequence as $t \to o$, where $\{\lambda_n\}$, $\{\mu_n\}$, $n \in I$, are both sequences of fixed complex numbers, with $\operatorname{Re} \lambda_o > o$;

(iii) $f(t) \sim \sum_{n\in I} a_n t^{\lambda_n - 1}(-\log t)^{\mu_n}$; $\{t^{\lambda_n - 1}(-\log t)^{\mu_n}\}$,

as $t \to o$ along $\arg t = \gamma$;

then

$$(5.89) \qquad F(z) \sim \sum_{n\in I} a_n L(o,a,\lambda_n,\mu_n,z); \quad \{z^{-\lambda_n}(\log z)^{\mu_n}\},$$

as $z \to \infty$ in $S(\Delta)$.

<u>Proof</u>: For any fixed choice of $c = |c|\exp(i\gamma) \neq o$, no matter how small,

$$(5.90) \qquad F(z) = \int_o^c f(t)\exp(-zt)dt + \int_c^a f(t)\exp(-zt)dt.$$

From (3.13),

$$(5.91) \qquad F(z) = \int_o^c f(t)\exp(-zt)dt + O(\exp(-\delta|z|)),$$

for some fixed $\delta > o$, as $z \to \infty$ in $S(\Delta)$.

For any fixed $N \in I$,

$$(5.92) \qquad f(t) = \sum_{n=o}^{N} a_n t^{\lambda_n - 1}(-\log t)^{\mu_n} + R_N$$

where for any given $\varepsilon > o$, there will exist a complex number $c \neq o$ such that

$$(5.93) \qquad |R_N| < \varepsilon|t^{\lambda_N - 1}(-\log t)^{\mu_N}|, \quad \arg t = \gamma, \quad |t| \leq |c|.$$

Hence

$$(5.94) \qquad F(z) = \sum_{n=o}^{N} a_n L(o,c,\lambda_n,\mu_n,z) + r_N + O(\exp(-\delta|z|)),$$

as $z \to \infty$ in $S(\Delta)$, and

$$(5.95) \qquad |r_N| \leq \varepsilon \int_o^c |t^{\lambda_N - 1}(-\log t)^{\mu_N} \exp(-zt)dt|$$

$$\leq K \cdot \varepsilon \cdot |z^{-\lambda_N}(\log z)^{\mu_N}|$$

by Lemma 5.4.

Since

$$(5.96) \qquad L(o,c,\lambda_n,\mu_n,z) = L(o.a,\lambda_n.\mu_n,z) - L(c,a,\lambda_n,\mu_n,z)$$

$$= L(o,a.\lambda_n,\mu_n.z) + O(\exp(-\delta|z|)),$$

for some fixed $\delta > o$, as $z \to \infty$ in $S(\Delta)$, it follows that

$$(5.97) \qquad F(z) = \sum_{n=o}^{N} a_n L(o,a,\lambda_n,\mu_n,z) + o(z^{-\lambda_N}(\log z)^{\mu_N}),$$

as $z \to \infty$ in $S(\Delta)$, and therefore

$$(5.98) \qquad F(z) \sim \sum_{n\varepsilon I} a_n L(o,a,\lambda_n,\mu_n,z); \quad \{z^{-\lambda_n}(\log z)^{\mu_n}\},$$

as $z \to \infty$ in $S(\Delta)$.

This result is a simplification of a result due to Erdélyi [8]. It should be pointed out that condition (ii) of Theorem 5.2 is stated in a manner different from that given in Erdélyi [8] in order to clarify a possible ambiguity in the statement of the Erdelyi result. If $\phi_n = t^{\lambda_n - 1}(-\log t)^{\mu_n}$, then

(5.99) $$\phi_{n+1}/\phi_n = t^{\lambda_{n+1}-\lambda_n}(-\log t)^{\mu_{n+1}-\mu_n}.$$

Hence $\phi_{n+1}/\phi_n \to o$, as $t \to o$, if

(5.100)
(a) $\text{Re } \lambda_{n+1} > \text{Re } \lambda_n$

(b) $\text{Re } \lambda_n = \text{Re } \lambda_{n+1}$ and $\text{Re } \mu_{n+1} < \text{Re } \mu_n$.

This means that $\{\phi_n\}$ will be an asymptotic sequence as $t \to o$ providing for every fixed $n \in I$ either (a) or (b) is true. It does not mean for example that as n varies either (a) is always true for every n, or that (b) is always true for every n. For example

$$\phi_n = t(-\log t), \quad \frac{t}{(-\log t)}, \quad t^2(-\log t), \quad \frac{t^2}{(-\log t)}, \ldots$$

is an asymptotic sequence as $t \to o$. If the substitution $t = \frac{1}{z}$ is made in ϕ_n, then

(5.101) $$\phi_n = \frac{1}{z^{\lambda_n - 1}} (\log z)^{\mu_n}$$

is an asymptotic sequence as $z \to \infty$, and of course $\{z^{-\lambda_n}(\log z)^{\mu_n}\}$ will also be an asymptotic sequence.

There is no real quarrel with the statement of the Erdélyi result which uses (5.100) instead of condition (ii) of Theorem 5.3. It is simply a matter of clarification of what (5.100) means that has suggested the alternative form of the condition.

The Erdélyi result may not be as useful in some circumstances as one might expect. To illustrate, consider Theorem 5.3 when $\text{Re } \lambda_{n+1} > \text{Re } \lambda_n$ whenever n and $n+1$ are both in I. It will also be assumed $a_o = a_1 = 1$ in (5.89).

From (5.96) and Theorem 5.1, it follows that

$$(5.102)\quad L(o,a,\lambda_o,\mu_o,z) \sim z^{-\lambda_o}(\log z)^{\mu_o}[\sum_{n=o}^{\infty}(-1)^n\binom{\mu_o}{n}\Gamma^{(n)}(\lambda_o)(\log z)^{-n};$$

$$;\ \{(\log z)^{-n}\}],$$

and

$$(5.103)\quad L(o,a,\lambda_1,\mu_1,z) \sim z^{-\lambda_1}(\log z)^{\mu_1}[\sum_{n=o}^{\infty}(-1)^n\binom{\mu_1}{n}\Gamma^{(n)}(\lambda_1)(\log z)^{-n};$$

$$;\ \{(\log z)^{-n}\}],$$

as $z \to \infty$ in $S(\Delta)$.

Hence for any fixed integers $N_o > o$ and $N_1 > o$,

$$(5.104)\quad F(z) = z^{-\lambda_o}(\log z)^{\mu_o}[\sum_{n=o}^{N_0}(-1)^n\binom{\mu_o}{n}\Gamma^{(n)}(\lambda_o)(\log z)^{-n} + o((\log z)^{-N_o})]$$

$$+ z^{-\lambda_1}(\log z)^{\mu_1}[\sum_{n=o}^{N_1}(-1)^n\binom{\mu_1}{n}\Gamma^{(n)}(\lambda_1)(\log z)^{-n} + o((\log z)^{-N_1})]$$

$$+ o(z^{-\lambda_1}(\log z)^{\mu_1}),\quad \text{as } z \to \infty \text{ in } S(\Delta).$$

If the term $z^{-\lambda_o}(\log z)^{\mu_o}$ is factored from the expressions in (5.104), one obtains

$$(5.105)\quad F(z) = z^{-\lambda_o}(\log z)^{\mu_o}[\sum_{n=o}^{N_0}(-1)^n\binom{\mu_o}{n}\Gamma^{(n)}(\lambda_o)(\log z)^{-n}$$

$$+ o((\log z)^{-N_o}) + O(z^{\lambda_o-\lambda_1}(\log z)^{\mu_1-\mu_o})],$$

as $z \to \infty$ in $S(\Delta)$. Since $\operatorname{Re}\lambda_1 > \operatorname{Re}\lambda_o$

$$(5.106) \qquad z^{\lambda_o-\lambda_1}(\log z)^{\mu_1-\mu_o} = o((\log z)^{-N_o}), \quad \text{as} \quad z \to \infty \quad \text{in} \quad S(\Delta).$$

Hence

$$(5.107) \quad F(z) = z^{-\lambda_o}(\log z)^{\mu_o}[\sum_{n=o}^{N_0}(-1)^n\binom{\mu_o}{n}\Gamma^{(n)}(\lambda_o)(\log z)^{-n} + o((\log z)^{-N_o})]$$

or

$$(5.108) \qquad F(z) \sim z^{-\lambda_o}(\log z)^{\mu_o}[S_\infty(\mu_o,\lambda_o,\log z); \quad \{(\log z)^{-n}\}]$$

as $z \to \infty$ in $S(\Delta)$.

If μ_o is not a non-negative integer. then $S_\infty(\mu_o,\lambda_o,\log z)$ has an infinite number of terms, each of which is larger than every term in $z^{-\lambda_1}(\log z)^{\mu_1}\ S_\infty(\mu_1,\lambda_1,\log z)$. From a pragmatic point of view the first term of (iii) gives the complete asymptotic expansion of $F(z)$. This is a somewhat surprising result because the situation when the μ_n are non-negative integers is quite different. In such a case, every term of (iii) gives a contribution to a much more accurate form of asymptotic expansion.

As an illustration of this phenomenon, consider

$$(5.109) \qquad F(z) = \int_o^1 (-\log t)^{\frac{1}{2}} \cos t \exp(-zt)dt.$$

This will have the asymptotic expansion

$$(5.110) \ F(z) \sim z^{-1}(\log z)^{\frac{1}{2}}[\sum_{n=o}^{\infty} (-1)^n\binom{\frac{1}{2}}{n}D_\lambda^n[\Gamma(\lambda)]_{\lambda=1}(\log z)^{-n}; \quad \{(\log z)^{-n}\}],$$

as $z \to \infty$ in $S(\Delta)$. It does seem strange that every function

$$(5.111) \qquad F(z) = \int_o^a (-\log t)^{\frac{1}{2}} f(t)\exp(-zt)dt,$$

will have exactly the same asymptotic expansion, for any fixed complex number a, as long as the integral exists, and

$$(5.112) \quad f(t) = 1 + o(t^{\lambda}), \quad \text{as} \quad t \to o \quad \text{along} \quad \arg t = \arg a, \quad \operatorname{Re} \lambda > o.$$

If the Erdélyi form of (5.89) is reduced to its natural pragmatic form of (5.108), then it is clear that the result is no longer a generalization of Watson's Lemma except in so far as the first term is concerned. It is therefore natural to ask whether such a generalization does exist in which the powers of $(-\log t)$ can take values which are not non-negative integers. It is possible, for example, to ask for conditions on $f(t)$ in

$$(5.113) \qquad F(z) = \int_o^a f(t) \exp(-zt)dt$$

which will ensure that

$$(5.114) \qquad F(z) \sim \sum_{n\varepsilon I} a_n z^{-\lambda_n}(\log z)^{\mu_n} \; ; \qquad \{z^{-\lambda_n}(\log z)^{\mu_n}\} .$$

where $\operatorname{Re} \lambda_{n+1} > \operatorname{Re} \lambda_n$, whenever n and $n+1$ are in I, and μ_n is an arbitrary complex number. This would then be a generalization of Watson's Lemma, with the result of this Lemma being obtained when $\mu_n = o$. In order to answer this question, we shall digress to discuss briefly the function $\mu(t,\beta,\alpha)$ of Chapter IV. This function has the asymptotic expansion

$$(5.115) \quad \mu(t,\beta,\alpha) \sim t^{\alpha}(-\log t)^{-\beta-1}[\sum_{n=o}^{\infty} (-1)^n \frac{(\beta+1)_n}{n!} \mu(1,-n-1,\alpha)(-\log t)^{-n}: \{(-\log t)^{-n}\}], \quad \text{as} \quad t \to o.$$

Further.

$$\mu(1,-n-1,\alpha) = (-1)^n \frac{d^n}{d\alpha^n} \left[\frac{t^\alpha}{\Gamma(\alpha+1)}\right]\Bigg|_{t=1} \quad . \tag{5.116}$$

This gives

$$\mu(t,\beta,\alpha) = \frac{t^\alpha(-\log t)^{-\beta-1}}{\Gamma(\alpha+1)} \left[1 + O\left(\frac{1}{\log t}\right)\right], \quad \text{as} \quad t \to o. \tag{5.117}$$

Hence

$$\mu(t,-\mu_n-1,\lambda_n-1) = \frac{t^{\lambda_n-1}(-\log t)^{\mu_n}}{\Gamma(\lambda_n)} \left[1 + O\left(\frac{1}{\log t}\right)\right], \quad \text{as} \quad t \to o. \tag{5.118}$$

The sequences $\{\mu(t,-\mu_n-1,\lambda_n-1)\}$ and $\{t^{\lambda_n-1}(-\log t)^{\mu_n}\}$ as $t \to o$ and $\{z^{-\lambda_n}(\log z)^{\mu_n}\}$ as $z \to \infty$ are all asymptotic sequences for the same conditions on $\{\lambda_n\}$ and $\{\mu_n\}$. It is assumed that such conditions are met, and all three are asymptotic sequences. In such a situation one can study functions $f(t)$ which have asymptotic expansions of the form

$$f(t) \sim \sum_{n\varepsilon I} a_n\, t^{\lambda_n-1}(-\log t)^{\mu_n}; \quad \{t^{\lambda_n-1}(-\log t)^{\mu_n}\}, \quad \text{as} \quad t \to o, \tag{5.119}$$

which has already been accomplished, or one can discuss the possibility that

$$f(t) \sim \sum_{n\varepsilon I} a_n\, \mu(t,-\mu_n-1,\lambda_n-1); \qquad \{\mu(t,-\mu_n-1,\lambda_n-1)\}, \tag{5.120}$$

as $t \to o$. If (5.120) exists then the constants a_n are given by

$$\begin{aligned} a_o &= \lim_{t \to o} f(t)/\mu(t,-\mu_o-1,\lambda_o-1) \\ &= \lim_{t \to o} f(t)\, \Gamma(\lambda_o)/t^{\lambda_o-1}(-\log t)^{\mu_o}\ , \end{aligned} \tag{5.121}$$

$$a_k = \lim_{t \to o} [f(t) - \sum_{n=o}^{k-1} a_n \, \mu(t,-\mu_n-1,\lambda_n-1)]/\mu(t,-\mu_k-1,\lambda_k-1)$$

$$= \lim_{t \to o} \Gamma(\lambda_k)[f(t) - \sum_{n=o}^{k-1} a_n \, \mu(t,-\mu_n-1,\lambda_n-1)]/t^{\lambda_k-1} (-\log t)^{\mu_k}.$$

Although these formulae give an explicit determination of these constants, the formulae are not useful in specific determinations. The problem of determining conditions on $f(t)$ for which (5.120) is valid is interesting, but is not a problem which will be considered in this thesis.

Theorem 5.4.

If

(i) $F(z) = \int_o^a f(t) \exp(-zt)dt, \qquad a = |a|\exp(i\gamma),$

exists for some fixed $z = z_o$;

(ii) $\{t^{\lambda_n-1} (-\log t)^{\mu_n}\}$ is an asymptotic sequence as $t \to o$, with $\text{Re}\, \lambda_n > o$ for all $n \in I$;

(iii) $f(t) \sim \sum_{n\in I} a_n \, \mu(t,-\mu_n-1,\lambda_n-1): \quad \{t^{\lambda_n-1} (-\log t)^{\mu_n}\}$,

as $t \to o$ along $\arg t = \gamma$:

then

$$F(z) \sim \sum_{n\in I} a_n \, z^{-\lambda_n}(\log z)^{\mu_n}; \qquad \{z^{-\lambda_n}(\log z)^{\mu_n}\}, \tag{5.122}$$

as $z \to \infty$ in $S(\Delta)$. The result holds uniformly in the approach of $z \to \infty$.

Proof: For any fixed choice of $c = |c|\exp(i\gamma) \neq o$, it is true that

$$(5.123) \qquad F(z) = \int_o^c f(t)\exp(-zt)dt + \int_c^a f(t)\exp(-zt)dt$$

$$= \int_o^c f(t)\exp(-zt)dt + O(\exp(-\delta|z|)),$$

for some fixed $\delta > o$, as $z \to \infty$ in $S(\Delta)$. From (iii), it follows that for any fixed integer $N \varepsilon I$,

$$(5.124) \qquad f(t) = \sum_{n=o}^{N} a_n \, \mu(t,-\mu_n-1,\lambda_n-1) + R_N ,$$

where for every $\varepsilon > o$. there will exist a number $|c|$ such that

$$(5.125) \qquad |R_N| \leq \varepsilon |t^{\lambda_N-1}(-\log t)^{\mu_N}|, \text{ providing } |t| \leq |c|.$$

Since the choice of c in (5.123) is arbitrary, there is no loss of generality in identifying $|c|$ in (5.123) with that in (5.125), and assuming $|c|$ is small if so desired. These results give

$$(5.126) \quad F(z) = \sum_{n=o}^{N} a_n \int_o^c \exp(-zt)\mu(t,-\mu_n-1,\lambda_n-1)dt + r_N$$

$$+ O(\exp(-\delta|z|)). \text{ as } z \to \infty \text{ in } S(\Delta),$$

where

$$(5.127) \qquad |r_N| \leq \varepsilon \int_o^c |t^{\lambda_N-1}(-\log t)^{\mu_N} \exp(-zt)dt|,$$

which as before, (5.95), means

$$(5.128) \qquad r_N = o(z^{-\lambda_N}(\log z)^{\mu_N}), \text{ as } z \to \infty \text{ in } S(\Delta).$$

Further

$$(5.129)\quad \int_o^c \mu(t,-\mu_n-1,\lambda_n-1)\exp(-zt)dt = \int_o^{\infty e^{i\gamma}} \mu(t,-\mu_n-1,\lambda_n-1)\exp(-zt)dt + O(\exp(-\delta|z|)).$$

for some fixed $\delta > o$, as $z \to \infty$ in $S(\Delta)$. Hence

$$(5.130)\quad \int_o^{\infty e^{i\gamma}} \mu(t,-\mu_n-1,\lambda_n-1)\exp(-zt)dt = z^{-\lambda_n}(\log z)^{\mu_n}$$

coupled with the results above gives

$$(5.131)\quad F(z) = \sum_{n=o}^{N} a_n z^{-\lambda_n}(\log z)^{\mu_n} + o(z^{-\lambda_N}(\log z)^{\mu_N}),$$

as $z \to \infty$ in $S(\Delta)$, and the order relation is independent of z. This of course proves

$$(5.132)\quad F(z) \sim \sum_{n\varepsilon I} a_n z^{-\lambda_n}(\log z)^{\mu_n} \; ; \quad \{z^{-\lambda_n}(\log z)^{\mu_n}\} ,$$

uniformly, as $z \to \infty$ in $S(\Delta)$.

Since

$$(5.133)\quad \mu(t,-m,\alpha) = (-1)^{m-1} \frac{d^{m-1}}{d\alpha^{m-1}} \left(\frac{t^\alpha}{\Gamma(\alpha+1)}\right) .$$

$$(5.134)\quad \mu(t,-1,\lambda_n-1) = \frac{t^{\lambda_n-1}}{\Gamma(\lambda_n)} .$$

This result shows that $\mu_n = o$ will yield Watson's Lemma, and Theorem 5.3 is a true generalization of this latter result. In Theorem 5.4, μ_n is an arbitrary fixed complex number.

Returning to Theorem 5.3, the conditions under which

$$(5.135)\quad \text{Re } \lambda_n = \text{Re } \lambda_{n+1} , \quad \text{Re } \mu_{n+1} < \text{Re } \mu_n$$

whenever n and n+1 are members of I lead to a result that one might reasonably expect to hold. In this instance

$$(5.136)\quad F(z) = a_o z^{-\lambda_o}[\Gamma(\lambda_o)(\log z)^{\mu_o} - \binom{\mu_o}{1}\Gamma^{(1)}(\lambda_o)(\log z)^{\mu_o-1}$$

$$+ \cdots + (-1)^{r_o}\binom{\mu_o}{r_o}\Gamma^{(r_o)}(\lambda_o)(\log z)^{\mu_o-r_o} + o((\log z)^{\mu_o-r_o})]$$

$$+ a_1 z^{-\lambda_1}[\Gamma(\lambda_1)(\log z)^{\mu_1} - \binom{\mu_1}{1}\Gamma^{(1)}(\lambda_1)(\log z)^{\mu_1-1} +$$

$$+ \cdots + (-1)^{r_1}\binom{\mu_1}{r_1}\Gamma^{(r_1)}(\lambda_1)(\log z)^{\mu_1-r_1} + o((\log z)^{\mu_1-r_1})]$$

$$+ \cdots$$

$$+ a_M z^{-\lambda_M}[\Gamma(\lambda_M)(\log z)^{\mu_M} - \binom{\mu_M}{1}\Gamma^{(1)}(\lambda_M)(\log z)^{\mu_M-1}$$

$$+ \cdots + (-1)^{r_M}\binom{\mu_M}{r_M}\Gamma^{(r_M)}(\lambda_M)(\log z)^{\mu_M-r_M} + o((\log z)^{\mu_M-r_M})]$$

$$+ \cdots$$

Under the stated conditions, the ratio $z^{-\lambda_n}/z^{-\lambda_o}$ is bounded as $z \to \infty$. In (5.136), this will mean that none of the terms in (5.136) become ≈ 0 with respect to an asymptotic sequence composed of powers of $(\log z)^{-1}$. It is possible to regroup the terms of (5.136) so that $F(z)$ will exhibit an asymptotic expansion of the form

$$(5.137)\qquad F(z) \sim z^{-\lambda_o}[\sum b_n(z)(\log z)^{-\tau_n};\quad \{(\log z)^{-\tau_n}\}],$$

uniformly, as $z \to \infty$ in $S(\Delta)$, where the $b_n(z)$ are all bounded as $z \to \infty$ in $S(\Delta)$, and the sequence of fixed complex numbers $\{\tau_n\}$ satisfies $\tau_o = -\mu_o$, $\operatorname{Re}\tau_{n+1} > \operatorname{Re}\tau_n$. Although the explicit expression of $b_n(z)$ can be obtained, it is of such complexity that it is hardly worth while stating the formulae involved. In the situation just described, every term of the expansion of $f(t)$ contributes to the asymptotic expansion of $F(z)$. This is merely in keeping with what might be expected from an examination of the conditions and result given in Watson's Lemma.

As an illustration of this form of asymptotic expansion, the function $F(z)$ defined by

$$F(z) = \int_o^1 \frac{t^{\lambda-1}\exp(-zt)}{(1 - \log t)^{\mu}}\, dt \ , \qquad \text{Re } \lambda > o \tag{5.138}$$

will be considered. The general procedures of the present chapter may be used in two different ways. The function $(1 - \log t)^{-\mu}$ has the convergent expansion

$$(1 - \log t)^{-\mu} = \sum_{n=o}^{\infty} \binom{-\mu}{n} (-\log t)^{-\mu-n}, \tag{5.139}$$

providing $|t| < e^{-1}$. The conditions of Theorem 5.3 are then trivially satisfied and

$$F(z) \sim \sum_{n=o}^{\infty} \binom{-\mu}{n} L(o,1,\lambda,-\mu-n,z); \quad \{z^{-\lambda}(\log z)^{-\mu-n}\} \ , \tag{5.140}$$

uniformly, as $z \to \infty$ in $S(\Delta)$. As before

$$L(o,1,\lambda,-\mu-n,z) \sim z^{-\lambda}[\sum_{m=o}^{\infty} (-1)^m \binom{-\mu-n}{n} \Gamma^{(m)}(\lambda)(\log z)^{-\mu-n-m} : \{(\log z)^{-\mu-n-m}\}] \ , \tag{5.141}$$

uniformly, as $z \to \infty$ in $S(\Delta)$. Regrouping terms will then give

(5.142) $$F(z) \sim z^{-\lambda}(\log z)^{-\mu}[\sum_{n=o}^{\infty} a_n(\log z)^{-n} \;, \quad \{(\log z)^{-n}\}]$$

uniformly, as $z \to \infty$ in $S(\Delta)$, where

(5.143) $$a_n = \sum_{m=o}^{n} (-1)^m \binom{-\mu}{n-m}\binom{-\mu-n+m}{m}\Gamma^{(m)}(\lambda) \;.$$

There does however exist a much simpler asymptotic form of expansion of $F(z)$. In (5.138), replace t by et. Hence

(5.144) $$F(z) = e^{\lambda} \int_{o}^{e^{-1}} \frac{t^{\lambda-1} \exp[-(ze)t]}{(-\log t)^{\mu}} dt \;.$$

Directly, one therefore has

(5.145) $$F(z) \sim z^{-\lambda}(\log(ze))^{-\mu}[\sum_{n=o}^{\infty} (-1)^n\binom{-\mu}{n}\Gamma^{(n)}(\lambda)(\log(ze))^{-n}; \quad \{(\log(ze))^{-n}\}],$$

uniformly, as $z \to \infty$ in $S(\Delta)$, or

(5.146) $$F(z) \sim z^{-\lambda}(1 + \log z)^{-\mu}[\sum_{n=o}^{\infty} (-1)^n\binom{-\mu}{n}\Gamma^{(n)}(\lambda)(1 + \log z)^{-n}; \quad \{(\log z)^{-n}\}],$$

uniformly, as $z \to \infty$ in $S(\Delta)$. Because of the Poincaré nature of these expansions, (5.142) can be obtained from (5.146) by writing $(1 + \log z)^{-\mu-n} = (\log z)^{-\mu-n}[1 + \frac{1}{\log z}]^{-\mu-n}$, and then expanding $[1 + \frac{1}{\log z}]^{-\mu-n}$ in powers of $\frac{1}{\log z}$.

The examples given in this chapter are artificial in nature, and were chosen to illustrate rather simply some of the ways the theorems in this chapter may be applied. Later in the thesis, other non-artificial examples will be given to show these applications in a natural setting.

CHAPTER VI

Generalizations of Barnes' Lemma

In Chapter III, integrals of the form

$$F(z) = \frac{i}{2\pi} \int_T^{(o+)} f(t)\exp(-zt)dt \tag{6.1}$$

have been encountered, where the path of integration is the loop illustrated below.

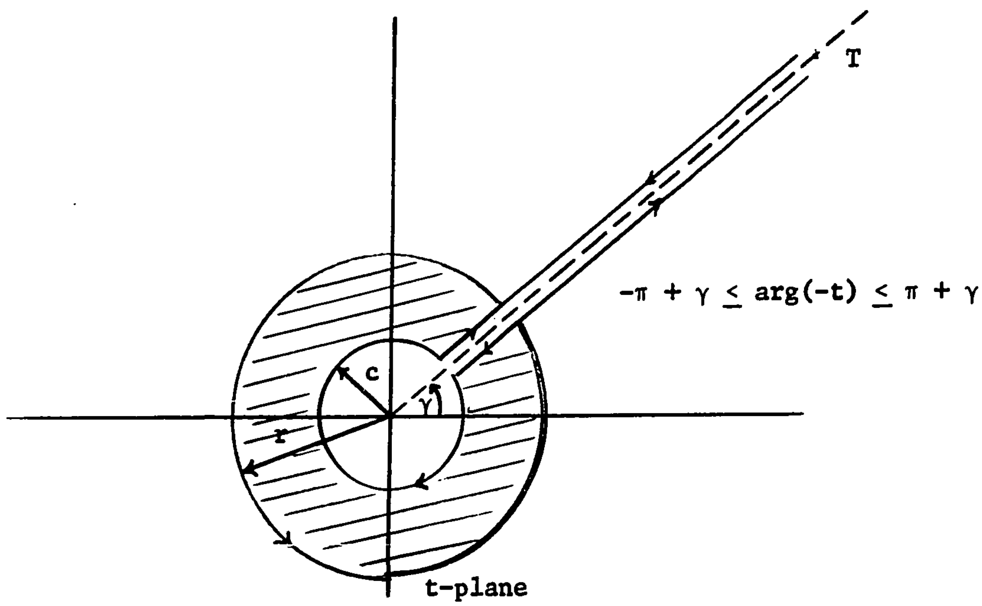


Figure 6.1.

The asymptotic behavior of F(z) was obtained under the restriction that f(t) had only branch-point singularities. In the present chapter, f(t) will be allowed to possess singularities of logarithmic and branch-point type, and results which correspond to theorems of Chapter V will be obtained.

For $F(z)$ defined by (6.1), it will be assumed that:

(i) $F(z)$ exists for some fixed $z = z_o$; and

(ii) $f(t)$ is regular within. and continuous on the boundary of, the shaded cut annulus of Figure 6.1, where the outer circle is of fixed radius, and the inner circle $|z| = c \neq o$ may be chosen to be fixed and arbitrarily small.

Differentiating the identity

(6.2) $$\frac{i}{2\pi} \int_{\infty e^{i\gamma}}^{(o+)} (-t)^{\lambda-1} \exp(-zt)dt = \frac{1}{\Gamma(1-)z^{\lambda}}, \quad |\arg(ze^{i\gamma})| < \frac{\pi}{2}$$

m times with respect to λ, yields

(6.3) $$\frac{i}{2\pi} \int_{\infty e^{i\gamma}}^{(o+)} (-t)^{\lambda-1}(\log(-t))^{m}\exp(-zt)dt = \frac{d^m}{d\lambda^m} [\frac{1}{\Gamma(1-\lambda)z^{\lambda}}] .$$

Setting

(6.4) $$M(T,\lambda,\mu,z) = \frac{i}{2\pi} \int_{T}^{(o+)} (-t)^{\lambda-1}(\log(-t))^{\mu} \exp(-zt)dt,$$

then (3.13) gives

(6.5) $$M(T,\lambda,m,z) \approx \frac{d^m}{d\lambda^m} [\frac{1}{\Gamma(1-\lambda)z^{\lambda}}] ; \quad \{\phi_n\} ,$$

uniformly in arg z, as $z \to \infty$ in $S(\Delta)$, where $\{\phi_n\}$ may be any asymptotic sequence as long as (5.14) holds.

<u>Theorem 6.1</u>.

Consider $F(z) = \frac{i}{2\pi} \int_{T}^{(o+)} f(t)\exp(-zt)dt$, and suppose that conditions (i) and (ii) hold. If, in addition,

(iii) $f(t) \sim \sum_{n\varepsilon I} a_n(-t)^{\lambda_n - 1} P_n(\log(-t)); \quad \{(-t)^{\lambda_n - 1}(\log(-t))^{m(n)}\}.$

as $t \to o$ in $-\pi + \gamma \leq \arg(-t) \leq \gamma + \pi$, where $P_n(w)$ is a polynomial of degree $m = m(n)$;

(iv) $\mathrm{Re}\, \lambda_{n+1} > \mathrm{Re}\, \lambda_n$, for every $n \varepsilon I$, providing n and $n+1$ both in I, and for some $n = k$ in I, $\mathrm{Re}\, \lambda_k > o$;

then

$$(6.6) \qquad F(z) \sim \sum_{n\varepsilon I} a_n P_n(D_n)[\Gamma^{-1}(1-\lambda_n) z^{-\lambda_n}]; \quad \{z^{-\lambda_n}(-\log z)^{m(n)}\},$$

uniformly in the approach of $z \to \infty$ in $S(\Delta)$, where $D_n = \frac{d}{d\lambda_n}$.

<u>Proof</u>: Let $I_k = \{0,1,2,\cdots,k-1\}$. and define

$$(6.7) \qquad f_k(t) = f(t) - \sum_{n\varepsilon I_k} a_n(-t)^{\lambda_n - 1} P_n(\log(-t)).$$

Then

$$(6.8) \quad \frac{i}{2\pi}\int_T^{(o+)} f_k(t)\exp(-zt)dt = F(z) - \sum_{n\varepsilon I_k} a_n \frac{i}{2\pi}\int_T^{(o+)} (-t)^{\lambda_n - 1} \cdot P_n(\log(-t))\exp(-zt)dt.$$

This result coupled with (6.5) implies

$$(6.9) \quad F(z) \approx \sum_{n\varepsilon I_k} a_n P_n(D_n)[\Gamma^{-1}(1-\lambda_n) z^{-\lambda_n}] + \frac{i}{2\pi}\int_T^{(o+)} f_k(t)\exp(-zt)dt; \quad \{z^{-\lambda_n}(-\log z)^{m(n)}\},$$

uniformly in $\arg z$, as $z \to \infty$ in $S(\Delta)$.

As in the proof of (3.19),

$$(6.10)\quad \frac{i}{2\pi}\int_T^{(o+)} f_k(t)\exp(-zt)dt = \frac{i}{2\pi}[\int_o^T f_k(te^{2\pi i})\exp(-zt)dt - \int_o^T f_k(t)\exp(-zt)dt]$$

$$= \frac{i}{2\pi}[I_+ - I_-].$$

With an argument similar to that used in Theorem 5.1, we obtain

$$(6.11)\quad I_\pm \sim -[\sum_{n\varepsilon I-I_k} a_n P_n(D_n)[\Gamma(\lambda_n)z^{-\lambda_n}e^{\pm i\pi\lambda_n}];\ \{z^{-\lambda_n}(-\log z)^{m(n)}\}],$$

where either all upper or all lower signs must be taken. The result in (6.11) is uniform in the approach of $z \to \infty$ in $S(\Delta)$. By adding the two expansions in (6.11), we have

$$(6.12)\quad \frac{i}{2\pi}\int_T^{(o+)} f_k(t)\exp(-zt)dt \sim \sum_{n\varepsilon I-I_k} a_n P_n(D_n)[\Gamma^{-1}(1-\lambda_n)z^{-\lambda_n}];$$

$$\{z^{-\lambda_n}(-\log z)^{m(n)}\},$$

uniformly in $\arg z$, as $z \to \infty$ in $S(\Delta)$. The required result (6.6) now follows from (6.9) and (6.12).

When $m(n) = o$ for all $n \varepsilon I$, and, therefore, without loss of generality $P_n(D_n) \equiv 1$, the result of Theorem 6.1 becomes as that contained in the Generalized Barnes' Lemma.

Consider the function $M(T,\lambda,\mu,z)$ given by (6.4). When μ is a non-negative integer, say $\mu = m$, one obtains from (6.5)

$$(6.13)\quad M(T,\lambda,\mu,z) \sim \frac{(-\log z)^m}{z^\lambda}[\sum_{n=o}^{m}\binom{m}{n}D^n[\Gamma^{-1}(1-\lambda)](-\log z)^{-n};\ \{(-\log z)^{-n}\}],$$

with the meaning that for every integer $o \leq N \leq m$,

$$(6.14)\quad M(T,\lambda,\mu,z) = \frac{(-\log z)^m}{z^\lambda}\left[\sum_{n=o}^{N}\binom{m}{n}D^n[\Gamma^{-1}(1-\lambda)](-\log z)^{-n} + o((\log z)^{-N})\right],$$

as $z \to \infty$ in $S(\Delta)$. To extend this result from integral to arbitrary complex values of μ, an analogue of Theorem 5.2 is given below.

Theorem 6.2.

If

(i) $M(T,\lambda,\mu,z) = \frac{i}{2\pi}\int_T^{(o+)} (-t)^{\lambda-1}(\log(-t))^\mu \exp(-zt)dt$;

(ii) λ, μ fixed complex numbers;

(iii) $T = |T| \exp(i\gamma)$, $|T| < 1$, and γ a fixed real number;

then

$$(6.15)\quad M(T,\lambda,\mu,z) \sim z^{-\lambda}(-\log z)^\mu\left[\sum_{n=o}^{\infty}\binom{\mu}{n}D^n[\Gamma^{-1}(1-\lambda)](-\log z)^{-n};\ \{(\log z)^{-n}\}\right]$$

as $z \to \infty$ in $S(\Delta)$.

Proof: Let L be a generic symbol to denote paths of integration that begin and end at the point at infinity with the direction of approach to infinity being restricted to the second and third quadrants of a complex plane. Further, L loops around the origin in a counter-clockwise direction so that the origin is contained within the region bounded by L.

We first consider $\gamma \neq \pi$, in which case we have, in view of (3.13),

$$(6.16)\quad M(T,\lambda,\mu,z) \approx \frac{i}{2\pi}\int_{\infty e^{i\gamma}}^{(o+)} (-t)^{\lambda-1}(\log(-t))^\mu \exp(-zt)dt:\ \{(\log z)^{-n}\},$$

as $z \to \infty$ in $S(\Delta/2)$. In (6.16), let $zt = (-s)$. By Cauchy's theorem, the integral becomes

$$(6.17)\qquad M(T,\lambda,\mu,z) \approx \frac{1}{2\pi i}\int_L \left(\frac{s}{z}\right)^{\lambda-1}\left(\log \frac{s}{z}\right)^{\mu}\exp(s)\, ds/z: \quad \{(\log z)^{-n}\},$$

as $z \to \infty$ in $S(\Delta/2)$. The path of integration and the cuts in the s-plane are illustrated below.

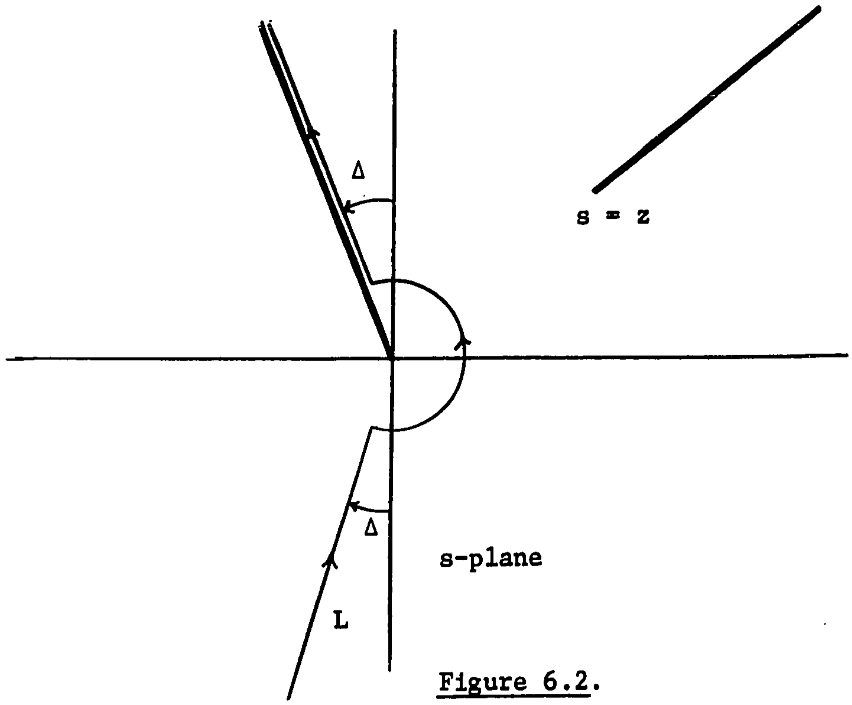


Figure 6.2.

For z restricted to $S(\Delta/2)$ and s confined to the path of integration L, $\arg(s/z) = \arg s - \arg z$, hence $(s/z)^{\lambda-1} = s^{\lambda-1} \cdot z^{1-\lambda}$. Therefore

$$(6.18)\qquad M(T,\lambda,\mu,z) \approx \frac{z^{-\lambda}}{2\pi i}\int_L s^{\lambda-1}\left(\log \frac{s}{z}\right)^{\mu} \exp(s)ds: \quad \{(\log z)^{-n}\},$$

as $z \to \infty$ in $S(\Delta/2)$. Since the last integral in (6.18) is of the form (4.9), we have from (4.16),

$$(6.19)\quad M(T,\lambda,\mu,z) \sim z^{-\lambda}(-\log z)^{\mu}[\sum_{n=o}^{\infty}\binom{\mu}{n}D^{n}[\Gamma^{-1}(1-\lambda)](-\log z)^{-n};\ \{(\log z)^{-n}\}],$$

as $z \to \infty$ in $S(\Delta/2)$, hence also as $z \to \infty$ in $S(\Delta)$.

We next consider $\gamma = \pi$. By the obvious substitution,

$$(6.20)\qquad M(T,\lambda,\mu,z) = \frac{1}{2\pi i}\int_{-Tz}^{(o+)} \left(\frac{s}{z}\right)^{\lambda-1}\left(\log \frac{s}{z}\right)^{\mu} \exp(s)ds/z.$$

Restricting z to be real and negative, we have

$$(6.21)\quad M(T,\lambda,\mu,z) = \frac{z^{-\lambda}}{2\pi i}(-\log z)^{\mu}\int_{-Tz}^{(o+)} s^{\lambda-1}\left(1-\frac{\log s}{\log z}\right)^{\mu} \exp(s)ds.$$

Furthermore, by analytic continuation, (6.21) must hold for all values of z in $S(\Delta)$. From here on the proof follows closely that of (4.16) and need not be given in full.

Returning to the consideration of the asymptotic behavior of

$$(6.22)\qquad F(z) = \frac{i}{2\pi}\int_{T}^{(o+)} f(t)\exp(-zt)dt,$$

$f(t)$ be allowed to have combinations of singularities of the form $(-t)^{\alpha}(\log(-t))^{\beta}$, with α and β both complex numbers. Before proceeding, a preliminary result is required, the proof of which follows from (5.85).

Lemma 6.1.

There exists a fixed number $r > o$ such that

$$(6.23)\ J = \int_{re^{i\gamma}}^{(o+)} |(-t)^{\lambda-1}(\log(-t))^{\mu}\exp(-zt)dt| = O(z^{-\lambda}(-\log z)^{\mu}),\ \text{as}\ z \to \infty\ \text{in}\ S(\Delta),$$

where λ, μ are fixed complex numbers, with $\mathrm{Re}\,\lambda > o$. The path of integration is the loop which starts at $t = re^{i\gamma}$ on the upper edge of the cut $\arg t = \gamma$, goes around the origin in the positive sense, and returns to $t = re^{i\gamma}$ on the lower edge of the cut.

Theorem 6.3.

Consider $F(z)$ as given in (6.22). If conditions (i) and (ii) hold and if condition (iii) and (iv) are replaced by

(iii)' $f(t) \sim \sum_{n\varepsilon I} a_n(-t)^{\lambda_n-1}(\log(-t))^{\mu_n}; \quad \{(-t)^{\lambda_n-1}(\log(-t))^{\mu_n}\}$,

as $t \to o$ in $-\pi + \gamma \leq \arg(-t) \leq \pi + \gamma$;

(iv)' $\{(-t)^{\lambda_n-1}(\log(-t))^{\mu_n}\}$ is an asymptotic sequence as $t \to o$,

where $\{\lambda_n\}$, $\{\mu_n\}$, $n \varepsilon I$, are both sequences of fixed complex numbers, with $\mathrm{Re}\,\lambda_k > o$ for some $k \varepsilon I$; then

$$(6.24) \qquad F(z) \sim \sum_{n\varepsilon I} a_n M(T,\lambda_n,\mu_n,z); \quad \{z^{-\lambda_n}(-\log z)^{\mu_n}\},$$

uniformly in $\arg z$, as $z \to \infty$ in $S(\Delta)$.

Proof: As before, we let $I_k = \{0,1,2,\cdots,k-1\}$ and set

$$(6.25) \qquad f_k(t) = f(t) - \sum_{n\varepsilon I_k} a_n(-t)^{\lambda_n-1}(\log(-t))^{\mu_n}.$$

Integrating on both sides of (6.25) gives

$$(6.26) \qquad F(z) = \sum_{n\varepsilon I_k} a_n M(T,\lambda_n,\mu_n,z) + \frac{i}{2\pi}\int_T^{(o+)} f_k(t)\exp(-zt)dt,$$

and, hence, only the last integral need be considered.

For any fixed integer $N \geq k$, $N \varepsilon I$, and for any fixed $\varepsilon > o$, we have from (iii)

$$f_k(t) = \sum_{n=k}^{N} a_n(-t)^{\lambda_n - 1}(\log(-t))^{\mu_n} + R_N \tag{6.27}$$

where

$$|R_N| \leq \varepsilon |(-t)^{\lambda_N - 1}(\log(-t))^{\mu_N}| , \quad \text{providing} \quad |t| \leq r. \tag{6.28}$$

With this fixed $r > o$,

$$\frac{i}{2\pi}\int_{re^{i\gamma}}^{(o+)} f_k(t)\exp(-zt)dt = \sum_{n=k}^{N} a_n \frac{i}{2\pi}\int_{re^{i\gamma}}^{(o+)} (-t)^{\lambda_n - 1}(\log(-t))^{\mu_n}\exp(-zt)dt + r_N \tag{6.29}$$

where

$$r_N = \frac{i}{2\pi}\int_{re^{i\gamma}}^{(o+)} R_N \exp(-zt)dt. \tag{6.30}$$

This result, as before, implies

$$\frac{i}{2\pi}\int_{T}^{(o+)} f_k(t)\exp(-zt)dt = \sum_{n=k}^{N} a_n M(T,\lambda_n,\mu_n,z) + r_N + O(\exp(-\delta|z|)) \tag{6.31}$$

for some fixed $\delta > o$.

From (6.28),

$$|r_N| \leq \frac{\varepsilon}{2\pi}\int_{re^{i\gamma}}^{(o+)} |(-t)^{\lambda_N - 1}(\log(-t))^{\mu_N} \exp(-zt)dt| . \tag{6.32}$$

Identifying r in (6.32) with that in (6.23), we have

$$r_N = O(z^{-\lambda_N}(-\log z)^{\mu_N}), \quad \text{as} \quad z \to \infty \quad \text{in} \quad S(\Delta), \tag{6.33}$$

by Lemma 6.1.

These results combined together give

$$(6.34)\qquad F(z) \sim \sum_{n\varepsilon I} a_n\, M(T,\lambda_n,\mu_n,z); \quad \{z^{-\lambda_n}(-\log z)^{\mu_n}\},$$

as $z \to \infty$ in $S(\Delta)$, and the result is uniform in the approach of $z \to \infty$ in $S(\Delta)$.

The remarks made on Theorem 5.3 apply to Theorem 6.3. That is to say, if μ_o is not a non-negative integer then the first term of (iii)' gives the complete asymptotic expansion of $F(z)$, and in case $\text{Re}\,\lambda_n = \text{Re}\,\lambda_{n+1}$ and $\text{Re}\,\mu_{n+1} < \text{Re}\,\mu_n$ then every term of the expansion (iii)' of $f(t)$ contributes to the asymptotic expansion of $F(z)$.

When the above remarks are taken into account, the result obtained in Theorem 6.3 is no longer a generalization of Barnes' Lemma. In Chapter V, a true generalization (Theorem 5.4) of Watson's Lemma was obtained, the proof of which is made very simply as a result of the identity (4.4)

$$(6.35)\qquad \int_o^\infty \mu(z,\beta,\alpha)\, \exp(-zt)dt = S^{-\alpha-1}(\log s)^{-\beta-1} .$$

An identity corresponding to (6.35) is not available for the integral

$$(6.36)\qquad \frac{i}{2\pi} \int_\infty^{(o+)} \mu(-t,\beta,\alpha)\, \exp(-zt)dt ,$$

instead, we have the rather complicated relation

$$(6.37)\quad \int_{\infty e^{i\gamma}}^{(o+)} \mu(-t,\beta,\alpha)\exp(-zt)dt = z^{-\alpha-1}[e^{\pi\alpha i}(\log z - i\pi)^{-\beta-1} - e^{-\pi\alpha i}(\log z + \pi i)^{-\beta-1}].$$

The following result is an analogue of Theorem 5.4. Since the procedures of the proof are very much the same as those outlined before, we state the theorem without proof.

<u>Theorem 6.4</u>.

Consider $F(z) = \frac{i}{2\pi}\int_T^{(o+)} f(t)\exp(-zt)dt$, and suppose that conditions (i), (ii) and (iv)' hold. If

$$(6.38)\quad f(t) \sim \sum_{n\varepsilon I} a_n\, \mu(-t,-\mu_n-1,\lambda_n-1); \quad \{(-t)^{\lambda_n-1}(\log(-t))^{\mu_n}\},$$

as $t \to o$ in $-\pi + \gamma \leq \arg(-t) \leq \pi + \gamma$, then

$$(6.39)\quad F(z) \sim \sum_{n\varepsilon I} a_n(2\pi i)^{-1} z^{-\lambda_n}[e^{\pi\lambda_n i}(\log z - i\pi)^{\mu_n} - e^{-\pi\lambda_n i}(\log z + i\pi)^{\mu_n}]; \quad \{z^{-\lambda_n}(-\log z)^{\mu_n}\},$$

as $z \to \infty$ in $S(\Delta)$. The result is valid uniformly in $\arg z$ as $z \to \infty$.

Although the expansion (6.39) seems complicated, the result obtained is a true generalization of Barnes' Lemma by setting $\mu_n = o$ for all n in I.

CHAPTER VII

The Method of Darboux

If a function $F(t)$ is regular at $t = o$, then it has a Maclaurin expansion of the form

$$F(t) = \sum_{n=o}^{\infty} f_n t^n , \tag{7.1}$$

which will converge within the so-called circle of convergence, say $|t| < R$. If, as Darbous [6] assumed, R is finite, then $F(t)$ must have at least one singularity on the circle $|t| = R$. When the Maclaurin coefficients f_n satisfy certain restrictions, Abel [1] was able to show the nature of the singularities which $F(t)$ must have on the circle of convergence. In a sense, Darboux investigated the converse of the Abel result. Darboux's major contribution was to show that, under specified conditions, the singularities of $F(t)$ on the circle of convergence acted as critical points, and the asymptotic behavior of f_n as $n \to \infty$ is obtained from infinitesimal contours surrounding the critical points. Since the pattern of proof initiated by Darboux will be followed in some detail, we shall not further describe the Darboux procedure.

We assume, as did Darboux, that $F(t)$ has a Maclaurin expansion with a finite radius of convergence, and that on the circle of convergence $F(t)$ has only a finite number of singularities. Anticipating the final result, it is possible to assume a canonical form in which $F(t)$ has one and only one singularity on the circle of convergence, with the more general result being obtained by adding the contribution of each singularity. Further, if the singularity occurs at $t = b$, then the substitution $t = bt'$ locates the singularity at $t' = 1$. Using the canonical form it is therefore assumed that $F(t)$ has a singularity at $t = 1$, and is regular within and on the contour shown below.

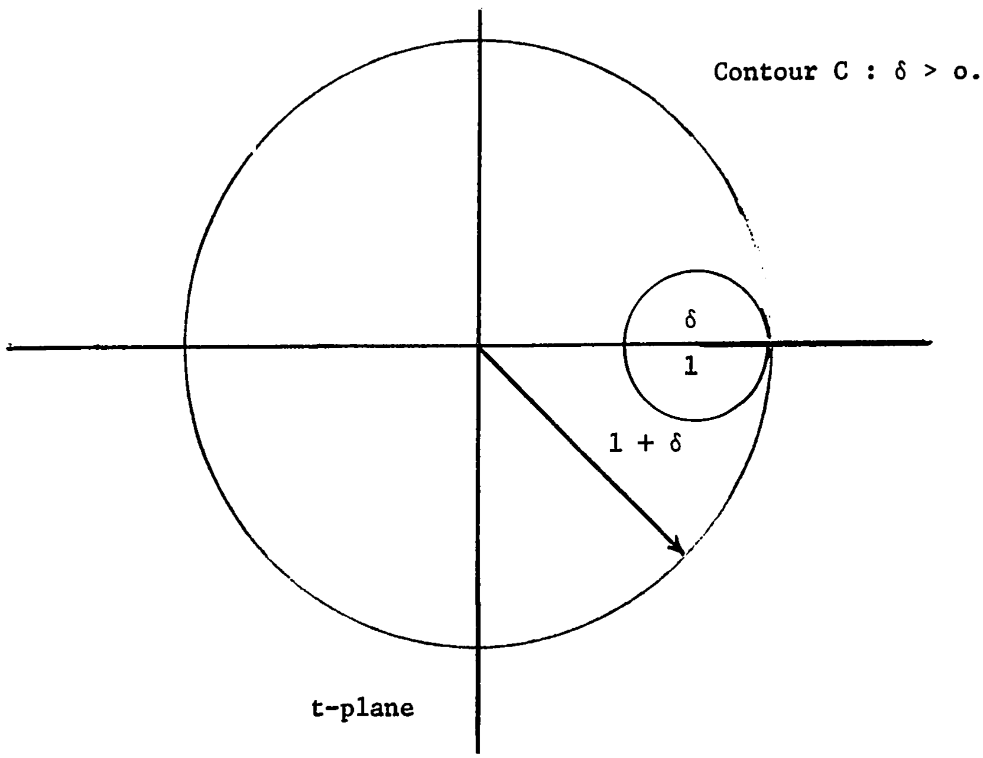


Figure 7.1.

By Cauchy's theorem, the Maclaurin coefficients f_n of $F(t)$ are given by

$$(7.2) \qquad 2\pi i\, f_n = \int_C t^{-n-1} F(t)dt$$

$$= \int_{|t|=1+\delta} t^{-n-1} F(t)dt + \int_{|t-1|=\delta} t^{-n-1} F(t)dt.$$

Since $F(t)$ is regular on $|t| = 1 + \delta$, there must exist fixed numbers ε and M such that

(7.3) $$\left|\int_{|t|=1+\delta} t^{-n-1} F(t)dt\right| \leq M/(1+\delta)^n \leq M \exp(-\varepsilon n).$$

This will imply

(7.4) $$\int_{|t|=1+\delta} t^{-n-1} F(t)dt \approx 0; \quad \{\phi_m\}, \quad \text{as} \quad n \to \infty,$$

as long as $\{\phi_m(n)\}$ is any asymptotic sequence for which

(7.5) $$\exp(-\varepsilon n) \approx 0; \quad \{\phi_m(n)\}, \quad \text{as} \quad n \to \infty.$$

Again anticipating the final result, it will be assumed that $\{\phi_m(n)\}$ is such an asymptotic sequence. One can, under these circumstances, concentrate on the asymptotic behavior of

(7.6) $$I = \int_{|t-1|=\delta} t^{-n-1} F(t)dt.$$

The substitution $t = e^u$ gives

(7.7) $$I = \int_{\Gamma} F(e^u) \exp(-nu)du,$$

where Γ is the image of the circle $|t - 1| = \delta$ in the u-plane. Since the point $t = 1$ corresponds to $u = o$, $|t - 1| = |e^u - 1| = |u| + 0(u^2)$, as $u \to o$, one can by the regularity of the integrand assume Γ is the circle $|u| = \delta'$, where $\delta' > o$ is fixed, and may be chosen arbitrarily small. Hence

(7.8) $$I = -\int_{\delta'}^{(o+)} F(e^u) \exp(-nu)du ,$$

$$(7.9) \qquad f_n \approx \frac{i}{2\pi} \int_{\delta'}^{(o+)} F(e^u) \exp(-nu)du; \quad \{\phi_m(n)\}, \quad \text{as} \quad n \to \infty.$$

If the conditions of Theorem (6.3) hold, namely,

$$(7.10) \qquad F(e^u) \sim \sum_{m\varepsilon I} a_m (-u)^{\lambda_m - 1} (\log(-u))^{\mu_m}; \quad \{(-u)^{\lambda_m - 1} (\log(-u))^{\mu_m}\},$$

as $u \to o$ in $o \leq \arg u \leq 2\pi$, then

$$(7.11) \qquad f_n \sim \sum_{m\varepsilon I} a_m M(\delta', \lambda_m, \mu_m, n); \qquad \{n^{-\lambda_m}(-\log n)^{\mu_n}\},$$

as $n \to \infty$.

In (7.11),

$$(7.12) \quad M(\delta', \lambda_m, \mu_m, n) = \frac{i}{2\pi} \int_{\delta'}^{(o+)} (-u)^{\lambda_m - 1} (\log(-u))^{\mu_m} \exp(-nu)du,$$

$$\sim n^{-\lambda_m} (-\log n)^{\mu_m} \Big[\sum_{k=o}^{\infty} \binom{\mu_m}{k} D^k [\Gamma^{-1}(1-\lambda_m)] (-\log n)^{-k};$$

$$\{(\log n)^{-k}\}\Big],$$

as $n \to \infty$. Furthermore, the specialized results obtained in Chapter VI when special restrictions are placed on the μ_m can be translated into similar results for the asymptotic behavior of the Maclaurin coefficients f_n as $n \to \infty$. However, because of difficulties which will now be considered, the explicit results one can obtain are not as elegant as one might expect, and are not worth writing out in detail.

Returning to (7.10), the substitution $t = e^u$ gives

$$(7.13)\quad F(t) \sim \sum_{m\varepsilon I} a_m(-\log t)^{\lambda_m - 1}(\log(-\log t))^{\mu_m};\ \{(-\log t)^{\lambda_m - 1}(\log(-\log t))^{\mu_m}\},$$

as $t \to 1$ in $o \leq \arg(t-1) \leq 2\pi$. At $t = 1$, $(-\log t)^{\lambda_m - 1}$ has an algebraic branch point singularity of the type $(-(t-1))^{\lambda_m - 1}$, and $(\log(-\log t))^{\mu_m}$ has a logarithmic singularity of the type $[\log(-(t-1))]^{\mu_m}$. In terms of the function $F(t)$ an expansion of the form (7.13) is not a natural form, and an explicit determination of the a_m will be rarely available. A more natural form of expansion will be

$$(7.14)\quad F(t) \sim \sum_{m\varepsilon I} b_m(t-1)^{\lambda_m - 1}\{\log[-(t-1)]\}^{\mu_m};\ \{(t-1)^{\lambda_m - 1}(\log(-(t-1)))^{\mu_m}\},$$

as $t \to 1$ in $o \leq \arg(t-1) \leq 2\pi$. As long as one desires only one or two of the leading terms of the asymptotic expansion, there is no real difficulty, and one can determine a_o and a_1 with a reasonable amount of calculation. Because the form (7.13) is not natural, any specialized results which one might obtain, by the process mentioned above are not really worth the effort to obtain them. In order to overcome this difficulty, it is our intention to introduce an idea used first by Burkholder [3], and then later used by Perron [17], and Erdelyi and Wyman [12].

Returning to (7.2), the number δ will no longer be considered to be fixed, and will be chosen to be

$$(7.15)\qquad \delta = \frac{1}{\sqrt{n}}\ .$$

If on the circle $|t| = 1 + \delta$, $F(t)$ satisfies

$$(7.16)\qquad F(t) = O(n^s),\ \text{as}\ n \to \infty\ ,$$

for some fixed real number s, then

$$(7.17)\qquad \int_{|t|=1+\delta} t^{-n-1} F(t)dt = 0\left(\frac{n^s}{\left(1+\frac{1}{\sqrt{n}}\right)^n}\right), \quad \text{as } n \to \infty,$$

$$= 0(\exp(-\varepsilon\sqrt{n})), \quad \text{as } n \to \infty,$$

for some fixed $\varepsilon > o$. The result

$$(7.18)\qquad \int_{|t|=1+\delta} t^{-n-1} F(t)dt \approx o; \quad \{\phi_m(n)\}, \quad \text{as } n \to \infty$$

is recaptured except that the asymptotic sequence $\{\phi_m\}$ must now be chosen so that

$$(7.19)\qquad \exp(-\varepsilon\sqrt{n}) \approx o; \quad \{\phi_m(n)\}, \quad \text{as } n \to \infty .$$

Under these circumstances, the complete asymptotic behavior of the Maclaurin coefficients will again be determined by the asymptotic behavior of

$$(7.20)\qquad I = \int_{|t-1|=\delta} t^{-n-1} F(t)dt.$$

In (7.20), (t-1) is replaced by t to obtain

$$(7.21)\qquad I = \int_{|t|=\delta} f(t+1) \exp[-(n+1) \log(t+1)]dt$$

$$= \int_{|t|=\delta} \exp(-(n+1)t) F(t+1) \exp[-(n+1)\{\log(t+1) - t\}]dt.$$

At this stage, it will be assumed that $F(t+1)$ has the canonical form

(7.22) $$F(t+1) = (-t)^{\lambda-1}(\log(-t))^{\mu}G(t),$$

where λ and μ are fixed complex numbers, and $G(t)$ is regular at $t = o$. The factor $G(t)\exp[-(n+1)\{\log(t+1)-t\}]$ is written as

(7.23) $$G(t)\exp[-(n+1)\{\log(t+1)-t\}] = G(t)\exp[-\frac{1}{2}wt\{\frac{2(\log(t+1) - t)}{t^2}\}]$$

where

(7.24) $$w = (n+1)t.$$

The expression on the right side of (7.23) will have a convergent expansion of the form

(7.25) $$G(t)\ \exp[-\frac{1}{2}wt\{\frac{2(\log(t+1) - t)}{t^2}\}] = \sum_{m=o}^{\infty} P_m(w)t^m ,$$

as long as

(7.26) $$wt = 0(1), \quad \text{as} \quad n \to \infty.$$

The coefficients $P_m(w)$ are Appell polynomials whose degree does not exceed m. An explicit expression for $P_m(w)$ is given by

(7.27) $$P_m(w) = \frac{1}{m!}\ \frac{d^m}{dt^m}\ [G(t)\ \exp\{-\frac{1}{2}wt[\frac{2(\log(t+1) - t)}{t^2}]\}]_{t=o} .$$

Since $wt = (n+1)t^2$, $wt = 0(1)$, as $n \to \infty$, will certainly be satisfied within and on $|t| = \frac{K}{\sqrt{n}}$, where K is any fixed positive number, and may be taken as large as we please. For any fixed integer $N \geq o$, (7.25) can be written as

$$(7.28) \qquad G(t)\exp[-\frac{1}{2}\, wt\{\frac{2(\log(t+1) - t)}{t^2}\}]$$

$$= \sum_{m=o}^{N} P_m(w)t^m + R_N$$

where R_N is regular at $t = o$, and for which

$$(7.29) \qquad R_N = 0\left(n^{\frac{N+1}{2}} t^{N+1}\right), \quad \text{as} \quad n \to \infty,$$

providing $|t| \leq \frac{K}{\sqrt{n}}$. This result gives

$$(7.30) \qquad I = \sum_{m=o}^{N} (-1)^m \int_{|t|=\delta} (-t)^{\lambda+m-1}(\log(-t))^{\mu} P_m((n+1)t)\exp[-(n+1)t]dt$$

$$+ E_N ,$$

where

$$(7.31) \qquad E_N = \int_{|t|=\delta} (-t)^{\lambda-1}(\log(-t))^{\mu}\exp[-(n+1)t]\; R_N \; dt.$$

It is always possible to choose the integer N large enough so that $Re(\lambda+N+1) > o$, and therefore the regularity of the integrand will allow the replacement of the circular path of integration by two straight lines joining $t = o$ to $t = \delta$, one on the top side of the cut in the t-plane, and the other on the lower side of this cut.

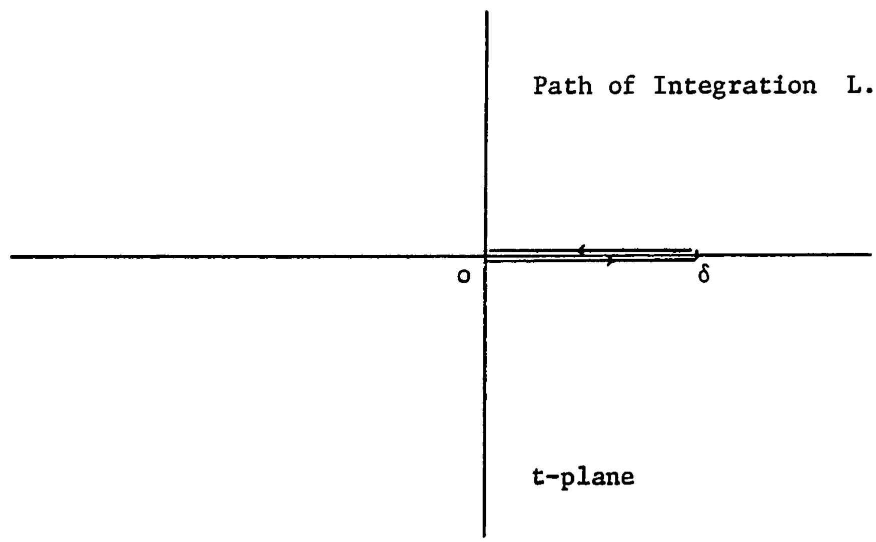


Figure 7.2.

Hence

(7.32) $$|E_N| = O\left(n^{\frac{N+1}{2}} \int_L |(-t)^{\lambda+N}(\log(-t))^{\mu}\exp(-(n+1)t)dt|\right), \quad \text{as} \quad n \to \infty,$$

and, as before,

(7.33) $$|E_N| = O\left(\frac{[\log n]^{\mu}}{n^{\lambda + \frac{N+1}{2}}}\right), \quad \text{as} \quad n \to \infty.$$

Since

(7.34) $$\phi_m = \frac{(\log n)^{\mu}}{n^{\lambda + \frac{m}{2}}}$$

is an asymptotic sequence,

(7.35) $$I \sim \sum_{m=o}^{\infty} (-1)^m I_m \; ; \quad \left\{\frac{(\log n)^{\mu}}{n^{\lambda + \frac{m}{2}}}\right\}, \quad \text{as} \quad n \to \infty.$$

Therefore the Maclaurin coefficients, f_n, of $F(t)$ have the asymptotic expansion

$$(7.36) \qquad f_n \sim \sum_{m=o}^{\infty} (-1)^m J_m \; ; \quad \{\frac{(\log n)^{\mu}}{n^{\lambda + \frac{m}{2}}}\} , \quad \text{as} \quad n \to \infty.$$

In (7.35), I_m is given by

$$(7.37) \qquad I_m = \int_{|t|=\delta} (-t)^{\lambda+m-1}(\log(-t))^{\mu} P_m((n+1)t)\exp(-(n+1)t)dt,$$

and J_m is given by

$$(7.38) \qquad J_m = \frac{i}{2\pi} \int_{|t|=\delta} (-t)^{\lambda+m-1}(\log(-t))^{\mu} P_m((n+1)t)\exp(-(n+1)t)dt.$$

In order to show that (7.36) is an asymptotic result worth having, the asymptotic behavior of J_m as $n \to \infty$ will be obtained. The Appell polynomials $P_m((n+1)t)$ may be written

$$(7.39) \quad P_m((n+1)t) = \sum_{s=o}^{m} p_s(n + 1)^s t^s, \quad \text{where } p_s \text{ is a fixed number.}$$

Hence

$$(7.40) \qquad J_m = \sum_{s=o}^{m} p_s(n + 1)^s M(\delta', \lambda+m+s, \mu, n+1),$$

where M is given in (7.12). Hence

$$(7.41) \quad J_m \sim \sum_{s=o}^{m} p_s \frac{(-\log(n+1))^{\mu}}{(n + 1)^{\lambda+m}} [\sum_{k=o}^{\infty} \binom{\mu}{k} D^k(\Gamma^{-1}(1-\lambda-m-s))(-\log(n+1))^{-k}$$

$$; \{(\log(n+1))^{-k}\}], \quad \text{as} \quad n \to \infty$$

$$\sim \frac{(-\log(n+1))^{\mu}}{(n+1)^{\lambda+m}} [\sum_{k=o}^{\infty} \binom{\mu}{k} (-\log(n+1))^{-k} \sum_{s=o}^{m} p_s D^k(\Gamma^{-1}(1-\lambda-m-s))$$

$$; \{(\log(n+1))^{-k}\}], \quad \text{as} \quad n \to \infty .$$

The order of the terms J_m indicates that the result in (7.36) can be improved to read

$$(7.42) \qquad f_n \sim \sum_{m=o}^{\infty} (-1)^m J_m \; ; \quad \{\frac{(\log n)^\mu}{n^{\lambda+m}}\} , \quad \text{as} \quad n \to \infty.$$

From (7.41),

$$(7.43) \qquad J_o \sim \frac{(-\log(n+1))^\mu}{(n+1)^\lambda} [\sum_{k=o}^{\infty} \binom{\mu}{k} (-\log(n+1))^{-k} \, p_o D^k [\Gamma^{-1}(1-\lambda)] ;$$

$$; \; \{(\log(n+1))^{-k}\}], \quad \text{as} \quad n \to \infty,$$

and

$$(7.44) \qquad J_1 \sim \frac{(-\log(n+1))^\mu}{(n+1)^{\lambda+1}} [\sum_{k=o}^{\infty} \binom{\mu}{k} (-\log(n+1))^{-k} \sum_{s=o}^{1} p_s D^k [\Gamma^{-1}(-\lambda-s)]$$

$$; \; \{(\log(n+1))^{-k}\}], \quad \text{as} \quad n \to \infty.$$

Hence, for any integer $N \geq o$

$$(7.45) \quad J_o - J_1 = G(o) \frac{(-\log(n+1))^\mu}{(n+1)^\lambda} [\sum_{k=o}^{N} \binom{\mu}{k} (-\log(n+1))^{-k} D^k [\Gamma^{-1}(1-\lambda)]$$

$$+ \, 0 \left[\frac{1}{(\log(n+1))^{N+1}} \right] + 0(\frac{1}{n})] ,$$

as $n \to \infty$. Clearly, none of the terms of J_1 can contribute to the asymptotic expansion unless the infinite asymptotic expansion for J_o terminates after a finite number of terms. The same will obviously be true for J_m, $m \geq 1$. Hence the general situation is

$$(7.46) \quad f_n \sim G(o) \frac{(-\log(n+1))^\mu}{(n+1)^\lambda} [\sum_{k=o}^{\infty} \binom{\mu}{k} (-\log(n+1))^{-k} D^k [\Gamma^{-1}(1-\lambda)] ; \; \{(\log(n+1))^{-k}\}]$$

as $n \to \infty$.

For the special case μ a non-negative integer, a more accurate asymptotic expansion does exist. This is true because the infinite series expansions for J_m all terminate. Returning to (7.38), the path of integration can be replaced by the infinite loop with the introduction of a term that is exponentially small. Hence

$$(7.47)\quad J_m \approx \frac{i}{2\pi}\int_{\infty}^{(o+)} (-t)^{\lambda+m-1}(\log(-t))^{\mu} P_m((n+1)t)\exp(-(n+1)t)dt$$

$$\approx \frac{i}{2\pi}\frac{d^{\mu}}{d\lambda^{\mu}}\int_{\infty}^{(o+)} (-t)^{\lambda+m-1} P_m((n+1)t)\exp(-(n+1)t)dt$$

$$\approx \frac{i}{2\pi}\frac{d^{\mu}}{d\lambda^{\mu}}\,[(n+1)^{-\lambda-m}\int_{\infty}^{(o+)} (-w)^{\lambda+m-1} P_m(w)\exp(-w)dw].$$

From (7.27), (7.47) can be written

$$(7.48)\quad J_m \approx \frac{i}{2\pi}\frac{d^{\mu}}{d\lambda^{\mu}}\,[(n+1)^{-\lambda-m}.\frac{1}{m!}\frac{d^m}{dt^m}\{G(t)\int_{\infty}^{(o+)} (-w)^{\lambda+m-1}\exp[-w(\frac{\log(t+1)}{t})]dw\}]$$

$$\approx \frac{1}{m!}\cdot\frac{d^{\mu}}{d\lambda^{\mu}}\,[\frac{d^m}{dt^m}\{G(t)[\frac{t}{\log(t+1)}]^{\lambda+m}\}_{t=o}(\frac{(n+1)^{-\lambda-m}}{\Gamma(1-\lambda-m)})]$$

the general form of (7.48) is

$$(7.49)\quad J_m \approx \frac{(-\log(n+1))^{\mu}}{(n+1)^{\lambda+m}}\,Q_{\mu}((\log(n+1))^{-1}) ,$$

where $Q_{\mu}(z)$ is a polynomial whose degree does not exceed μ. In (7.42), there is therefore no need to drop any of the terms, and the more accurate expansion is worth retaining.

The results thus far obtained, derived for a so-called canonical form, allow a direct derivation of a more general result. Returning to (7.21), it will now be assumed

$$(7.50)\qquad F(t+1) \sim \sum_{m\varepsilon I} (-t)^{\lambda_m - 1} (\log(-t))^{\mu_m} G_m(t); \quad \{(-t)^{\lambda_m - 1} (\log(-t))^{\mu_m}\},$$

as $t \to o$, where each $G_m(t)$ is regular at $t = o$. Hence for each fixed $N \varepsilon I$,

$$(7.51)\qquad F(t+1) = \sum_{m=o}^{N} (-t)^{\lambda_m - 1} (\log(-t))^{\mu_m} G_m(t) + R_N(t)$$

where

$$(7.52)\qquad R_N(t) = o((-t)^{\lambda_N - 1} (\log(-t))^{\mu_N}), \quad \text{as} \quad t \to o.$$

Hence, I of (7.21) becomes

$$(7.53)\qquad I = \int_{|t|=\delta} F(t+1)\exp[-(n+1)\log(t+1)]dt$$

$$= \sum_{m=o}^{N} \int_{|t|=\delta} (-t)^{\lambda_m - 1} (\log(-t))^{\mu_m} G_m(t)\exp[-(n+1)\log(t+1)]dt$$

$$+ \int_{|t|=\delta} R_N(t)\exp[-(n+1)\log(t+1)]dt.$$

As long as an integer N can be chosen so that

$$(7.54)\qquad \operatorname{Re} \lambda_N > o\ ,$$

and $R_N(t)$ is sufficiently regular in the cut neighborhood $|t| \leq \delta$ to replace the circular path of integration by the two straight line segments joining $t = o$ to $t = \delta$, above and below the cut, then

$$(7.55)\qquad \int_{|t|=\delta} R_N(t)\exp[-(n+1)\log(t+1)]dt = o\left(\frac{(\log(n+1))^{\mu_N}}{(n+1)^{\lambda_N}}\right), \quad \text{as} \quad n\to\infty.$$

Finally, each term of the asymptotic expansion is of the canonical form discussed in this chapter, and the asymptotic behavior of each term can be determined as $n \to \infty$.

The choice of the factor $(-t)^{\lambda_m}$ rather than t^{λ_m} in our general theorem was dictated by a desire to provide direct application of the Barnes' Lemma. Clearly if the natural factors in the expansion of $F(t+1)$ have the form $t^{\lambda_m}(\log t)^{\lambda_m}$, it would be possible to write $t = e^{ik\pi}(-t)$, $t^{\lambda_m} = e^{ik\pi\lambda_m}(-t)^{\lambda_m}$, and $\log t = \log(-t) + ik\pi$, for some integer k. This would imply an awkward re-expansion in order that a direct application of the results of this chapter be applicable to the factors $t^{\lambda_m}(\log t)^{\mu_m}$. Rather than follow this course, it is recommended that one recast the results of the chapter using

$$(7.57)\qquad \int_{\infty e^{i\gamma}}^{(o+)} t^{\lambda-1}\exp(-zt)dt = \frac{2\pi i}{(ze^{-i\pi})^{\lambda}\Gamma(1-\lambda)},$$

as the basic integral. For any positive integer μ, one obtains by differentiating μ times with respect to λ,

$$(7.58)\quad \int_{\infty e^{i\gamma}}^{(o+)} t^{\lambda-1}(\log t)^{\mu}\exp(-zt)dt = 2\pi i \sum_{r=o}^{\mu}\binom{\mu}{r}D^r[\Gamma^{-1}(1-\lambda)]D^{\mu-r}[(ze^{-i\pi})^{-\lambda}],$$

and the corresponding exact expression when μ is a non-negative integer is replaced by an asymptotic expansion for all other values of μ. Obviously everything can then be repeated to derive the analogous results when terms of the form $t^{\lambda-1}(\log t)^{\mu}$ are the natural factors to use in the expansion of $F(t+1)$.

As an example to illustrate the Method of Darboux applied to logarithmic type singularities, the function

$$F(t) = \left[\frac{\log(1-t)}{t}\right]^{\mu}, \quad F(o) = \exp(i\pi\mu), \tag{7.59}$$

will be used. Although this example does little else than repeat some of the theory, it is being used to compare the procedure with known methods of finding the asymptotic behavior of Stirling numbers of the first kind. It will also be used to illustrate that the final form of an asymptotic expansion so often depends on the procedure used to develop the form. Writing the Maclaurin expansion of $F(t)$ as

$$F(t) = \sum_{n=o}^{\infty} A_n^{(\mu)} t^n, \qquad A_o^{(\mu)} = \exp(i\pi\mu), \tag{7.60}$$

then

$$A_n^{(\mu)} = \frac{1}{2\pi i} \int_C \frac{F(t)}{t^{n+1}}\, dt ; \tag{7.61}$$

and as our general theorem shows

$$\begin{aligned} A_n^{(\mu)} &\approx \frac{1}{2\pi i} \int_{\gamma} \frac{F(t)}{t^{n+1}}\, dt, \quad \text{where } \gamma \text{ is the circle } |t-1| = \delta, \\ &\approx \frac{i}{2\pi} \int_{\delta}^{(o+)} \frac{F(t+1)}{(t+1)^{n+1}}\, dt, \\ &\approx \frac{i}{2\pi} \int_{\delta}^{(o+)} [\log(-t)]^{\mu} \exp[-(n+\mu+1)\log(t+1)]dt. \end{aligned} \tag{7.62}$$

In applying our previous results, $\lambda = 1$, $G(t) = \dfrac{1}{(t+1)^{\mu}}$, and the asymptotic expansion can be immediately obtained from (7.42). It is simpler however to consider $\lambda = 1$, $G(t) = 1$, and $z = n + \mu + 1$ as the asymptotic variable, and the results will apply even though μ may be a complex number.

Writing

$$(7.63)\quad A_n^{(\mu)} \approx \frac{i}{2\pi} \int_{\delta}^{(o+)} \exp[-(n+\mu+1)t][\log(-t)]^{\mu} \exp[-\frac{1}{2} wt \{\frac{2(\log(t+1) - t)}{t^2}\}]dt,$$

where $w = (n + \mu + 1)t$, we have seen that only the first term in the expansion of

$$(7.64)\quad \exp[-\frac{1}{2} wt \{\frac{2(\log(t+1) - t)}{t^2}\}] = \sum_{m=0}^{\infty} P_m(w)t^m$$

need be considered as long as μ is not a non-negative integer. In this case

$$(7.65)\quad A_n^{(\mu)} \approx \frac{i}{2\pi} \int_{\delta}^{(o+)} \exp(-(n+\mu+1)t)(\log(-t))^{\mu}dt = M(\delta,1,\mu,(n+\mu+1))$$

$$\sim \frac{e^{i\pi\mu}[\log(n+\mu+1)]^{\mu}}{(n+\mu+1)} [\sum_{m=0}^{\infty} \binom{\mu}{m} D^m[\Gamma(\lambda) \frac{\sin \pi\lambda}{\pi}]_{\lambda=1} (-\log(n+\mu+1))^{-m}$$

$$; \{(\log(n+\mu+1))^{-m}\}]$$

as $n \to \infty$. The first two non-zero terms of (7.65) are given by

$$(7.66)\quad A_n^{(\mu)} = \frac{e^{i\pi\mu}[\log(n+\mu+1)]^{\mu}}{(n+\mu+1)} [\frac{\mu}{\log(n+\mu+1)} - \frac{\mu(\mu-1)\Gamma'(1)}{(\log(n+\mu+1))^2} + O(\frac{1}{(\log(n+\mu+1))^3})]$$

$$= \frac{\mu e^{i\pi\mu}}{(n+\mu+1)} (\log(n+\mu+1))^{\mu-1}[1 - \frac{(\mu-1)\Gamma'(1)}{\log(n+\mu+1)} + O(\frac{1}{(\log(n+\mu+1))^2})]$$

as $n \to \infty$.

The Stirling numbers of the first kind S_n^m have the generating function

(7.67) $$[\log(1+t)]^m = \sum_{n=m}^{\infty} \frac{m!}{n!} S_n^m t^n$$

with the obvious relation to the $A_n^{(\mu)}$ given above by

(7.68) $$A_n^{(\mu)} = \frac{\mu!}{(n+\mu)!} S_{n+\mu}^{\mu} (-1)^{\mu+n},$$ when μ is a non-negative integer.

Jordan [16], page 161, gives the asymptotic result

(7.69) $$|S_{n+\mu}^{\mu}| \sim \frac{(n+\mu-1)!}{(\mu-1)!} [\log(n+\mu) + \gamma]^{\mu-1} ,$$

where $\gamma = -\Gamma'(1)$ is Euler's constant. The two results (7.66) and (7.69) agree to the order in (7.66). However, the procedures of the present chapter allow a much deeper result to be obtained when μ is a non-negative integer.

From (7.42), it follows that

(7.70) $$A_n^{(\mu)} \sim \sum_{m=o}^{\infty} (-1)^m J_m ; \qquad \left\{\frac{\{(\log(n+\mu+1)\}^{\mu}}{(n+\mu+1)^{1+m}}\right\} ,$$

where from (7.48)

(7.71) $$J_m \approx \frac{(-1)^m}{\pi \cdot m!} \frac{d^{\mu}}{d\lambda^{\mu}} \left[\frac{d^m}{dt^m} \{(\frac{t}{\log(t+1)})^{\lambda+m}\}_{t=o} (n+\mu+1)^{-\lambda-m} \Gamma(\lambda+m) \sin \pi\lambda\right]_{\lambda=1} .$$

Hence

(7.72) $$J_o \approx \frac{1}{\pi} \frac{d^{\mu}}{d\lambda^{\mu}} [(n+\mu+1)^{-\lambda} \Gamma(\lambda) \sin \pi\lambda]_{\lambda=1}$$

and

$$J_1 \approx -\frac{1}{\pi}\frac{d^\mu}{d\lambda^\mu}\left[\tfrac{1}{2}(\lambda+1)(n+\mu+1)^{-\lambda-1}\Gamma(\lambda+1)\sin\pi\lambda\right]_{\lambda=1} \tag{7.73}$$

$$= -\frac{1}{2\pi}\frac{d^\mu}{d\lambda^\mu}\left[(n+\mu+1)^{-\lambda-1}\Gamma(\lambda+2)\sin\pi\lambda\right]_{\lambda=1} .$$

If $n = 10^3$ and $\mu = 2$, then

$$J_o \sim \frac{1}{\pi}\frac{d^2}{d\lambda^2}\left[(1003)^{-\lambda}\Gamma(\lambda)\sin\pi\lambda\right]_{\lambda=1} \tag{7.74}$$

$$= 2(1003)^{-1}[\log 1003 + \gamma]$$

and

$$J_1 \sim -\frac{1}{2\pi}\frac{d^2}{d\lambda^2}\left[(1003)^{-\lambda-1}\Gamma(\lambda+2)\sin\pi\lambda\right]_{\lambda=1} \tag{7.75}$$

$$= -(1003)^{-2}[\log 1003 - 3 + 2\gamma].$$

By taking these first two terms of the asymptotic expansion (7.70) the result obtained in $A^{(2)}_{10^3} \sim 0.01493617441$ which is a reasonably accurate approximation to the exact value of $A^{(2)}_{10^3} = 0.01494504564$.

CHAPTER VIII

Examples

Although quite general theorems have been established by means of which Watson's Lemma has been generalized, no pretence is made that necessary and sufficient conditions for the validity of the results have been found. This point is stressed to emphasize the fact that it is the pattern, not the detail, of proof which is important. The pattern of proof need not be abandoned in a specific example just because one or more of the conditions of validity of a particular theorem does not happen to be true.

In the present chapter, the main ideas and concepts will be illustrated by specific examples.

Example 1. An Application to a Problem taken from Aerodynamics.

The function $F(z)$ defined by

$$(8.1)\quad F(z) = \int_0^\infty \cos(zx)dx \int_{-1}^{1} \left(1 - \frac{x}{\sqrt{x^2+y^2}}\right) \frac{dy}{\sqrt{1-y^2}}, \quad z \text{ real and positive,}$$

occurs in the theory of aerodynamics. F. Tricomi [20] showed that

$$(8.2)\qquad F(z) = \frac{2 \log z}{z^2}(1 + o(1)), \quad \text{as} \quad z \to +\infty .$$

To obtain this result, Tricomi first showed that for $z \geq o$,

$$(8.3)\qquad F(z) = 2 \int_0^1 K\left(\sqrt{1 - t^2}\right) t \exp(-zt)dt,$$

where $K(k)$ is the complete elliptic integral of the first kind, and is defined by

$$(8.4) \qquad K(k) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - k^2\sin^2\phi}}, \qquad |k| < 1,$$

$$= \frac{\pi}{2} F\left(\frac{1}{2}, \frac{1}{2}; 1; k^2\right),$$

where $F(a,b; c; z)$ is the hypergeometric function. Since

$$(8.5) \qquad K(\sqrt{1 - t^2}) = \frac{\pi}{2} F\left(\frac{1}{2}, \frac{1}{2}; 1; 1-t^2\right)$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n(\frac{1}{2})_n}{(n!)^2} [k_n - 2 \log t]t^{2n},$$

(Higher Transcendental Functions, volume I, page 74), where

$$(8.6) \qquad k_n = 2[\psi(n+1) - \psi(n + \tfrac{1}{2})],$$

it follows that

$$(8.7) \qquad K(\sqrt{1 - t^2}) = \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n(\frac{1}{2})_n}{(n!)^2} [h_n - \log t]t^{2n},$$

where

$$(8.8) \qquad h_n = \psi(n+1) - \psi(n + \tfrac{1}{2}).$$

By Theorem 5.1,

$$(8.9) \qquad F(z) \sim 2\left[\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n(\frac{1}{2})_n}{(n!)^2} \{(2n+1)!h_n - \Gamma'(2n+2) + (2n+1)! \log z\}z^{-(2n+2)} \right.$$

$$\left. ; \; \{(\log z) \cdot z^{-(2n+2)}\}\right],$$

as $z \to \infty$ in $|\arg z| < \frac{\pi}{2}$.

Computing the first two non-zero terms, one has

$$(8.10)\quad F(z) = \frac{2(h_o - \Gamma'(2) + \log z)}{z^2} + \frac{1}{2}\,\frac{(6h_1 - \Gamma'(4) + 6\log z)}{z^4} + O\left(\frac{\log z}{z^6}\right),$$

$$= \frac{2(\log 4 - 1 + \gamma + \log z)}{z^2} + \frac{1}{2}\,\frac{(6\log 4 - 17 + 6\gamma + 6\log z)}{z^4} + O\left(\frac{\log z}{z^6}\right),$$

as $z \to \infty$ in $|\arg z| < \frac{\pi}{2}$.

This is of course a more general result than that obtained by Tricomi.

By returning to (8.3), the substitution $t = 1 - t'$ gives

$$(8.11)\qquad \exp(z)F(z) = 2\int_o^1 (1-t)K(\sqrt{(2-t)t})\,\exp(-(-zt))dt\ ,$$

where

$$(8.12)\qquad (1-t)K(\sqrt{(2-t)t}) = \frac{\pi}{2}(1-t)\,F\left(\frac{1}{2},\frac{1}{2};1;t(2-t)\right)$$

has a convergent Maclaurin expansion of the form

$$(8.13)\qquad (1-t)k(\sqrt{(2-t)t}) = \frac{\pi}{2}\sum_{n=o}^{\infty} a_n t^n\ ,$$

with

$$(8.14)\qquad a_o = 1,\quad a_1 = -\frac{1}{2}\ ,\ \cdots\ .$$

By the usual form of Watson's Lemma, one obtains

$$\text{(8.15)} \qquad F(z) \sim \pi \exp(-z) \; [\sum_{n=o}^{\infty} \frac{n!a_n}{(-z)^{n+1}} \; ; \quad \{\frac{1}{z^{n+1}}\}]$$

as $z \to \infty$ in $|\arg z| > \frac{\pi}{2}$.

In order to include the missing sectors in which $\arg z = \pm \frac{\pi}{2}$ lie, it is necessary to write

$$\text{(8.16)} \quad F(z) \sim 2[\sum_{n=o}^{\infty} \frac{(\frac{1}{2})_n(\frac{1}{2})_n}{(n!)^2} \{(2n+1)!h_n - \Gamma'(2n+2)+(2n+1)!\log z\}z^{-(2n+2)} \; ; \; \{(\log z)z^{-(2n+2)}\}]$$

$$+ \pi \exp(-z)[\sum_{n=o}^{\infty} \frac{n!a_n}{(-z)^{n+1}} \; ; \quad \{\frac{1}{z^{n+1}}\}] \; ,$$

and now $z \to \infty$ within $|\arg z| < \pi$.

Example 2. Parabolic Cylinder Functions $D_\gamma(z)$.

The function $D_\gamma(z)$ has the contour integral representation

$$\text{(8.17)} \quad D_\gamma(z) = -\frac{\Gamma(\nu+1)}{2\pi i} e^{-\frac{1}{2}z^2} \int_{\infty}^{(o+)} e^{-zt - \frac{1}{2}t^2} (-t)^{-\gamma-1}dt, \qquad |\arg(-t)| \le \pi,$$

where γ is a fixed arbitrary complex number. If we differentiate n-times with respect to γ, then

$$\text{(8.18)} \quad \frac{d^n}{d\gamma^n} D_\gamma(z) = e^{-\frac{1}{2}z^2} \sum_{m=o}^{n} \binom{n}{m} \Gamma^{(n-m)}(\nu+1) \frac{i}{2\pi} \int_{\infty}^{(o+)} e^{-zt - \frac{1}{2}t^2} (-\log(-t))^m(-t)^{-\nu-1}dt.$$

Since the sum on the right is finite, to obtain an asymptotic expansion for $\frac{d^n}{d\nu^n} D_\nu(z)$ we need only consider the integrals

$$(8.19) \qquad G_m(z,\nu) = \frac{i}{2\pi} \int_{\infty}^{(o+)} e^{-zt - \frac{1}{2}t^2} (-\log(-t))^m (-t)^{-\nu-1} dt.$$

Let us first expand the function $\exp(-t^2/2)$ into Maclaurin series, and hence

$$(8.20) \qquad \exp(-t^2/2) = \sum_{n=o}^{\infty} \frac{(-1)^n}{2^n \cdot n!} (-t)^{2n}.$$

Substituting (8.20) into (8.19) and carrying out the integration term by term, we have from Theorem 6.1

$$(8.21) \quad G_m(z,\nu) \sim \sum_{n=o}^{\infty} \frac{(-1)^n}{2^n \cdot n!} \frac{1}{z^{2n}} D^{(m)}[\frac{z^\nu}{\Gamma(1+\nu-2n)}]; \ \{(\log z)^m \cdot z^{\nu-2n}\} ,$$

as $z \to \infty$ in $|\arg z| \leq \frac{\pi}{2} - \Delta$. This result coupled with (8.18) yields

$$(8.22) \quad \frac{d^n}{d\nu^n} D_\nu(z) \sim e^{-\frac{1}{2}z^2} [\sum_{k=o}^{\infty} \frac{(-1)^k}{k!(2z^2)^k} D^{(n)}[c_k(\nu) z^\nu]; \ \{(\log z)^n \cdot z^{\nu-2k}\}]$$

as $z \to \infty$ in $|\arg z| \leq \frac{\pi}{2} - \Delta$, where $D^{(n)} = \frac{d^n}{d\nu^n}$ and

$$(8.23) \qquad C_n(\nu) = \nu(\nu-1) \cdots (\nu-(2n-1)), \quad c_o(\nu) = 1.$$

In particular, we have

$$(8.24) \qquad z^{-\nu} e^{-\frac{1}{2}z^2} D_\nu(z) \sim \sum_{n=o}^{\infty} \frac{(-1)^n \cdot C_n(\nu)}{n!(2z^2)^n}; \quad \{z^{-2n}\} ,$$

as $z \to \infty$ in $|\arg z| \leq \frac{\pi}{2} - \Delta$. The condition for validity, $|\arg z| \leq \frac{\pi}{2} - \Delta$, $\Delta > o$, can be weakened to $|\arg z| \leq \frac{3\pi}{4} - \Delta$ in the usual way. Thus (8.22) and (8.24) hold for $|\arg z| \leq \frac{3\pi}{4} - \Delta$, $\Delta > 0$ and $z \to \infty$.

Example 3.

In the subsequent example an integral of the form $\int_0^c \log(-\log t) \exp(-zt)dt$ will appear. Although the singularity of the integrand is not of the type considered in this thesis, we will use the more general integral

$$(8.25) \qquad F(z) = \int_0^c t^{\lambda-1}\log(-\log t)\exp(-zt)dt$$

to illustrate that the pattern of procedure established in this thesis will yield the asymptotic behavior of $F(z)$ as $z \to \infty$. Although it is our hope that useful theorems have been established, it should be noted again that the pattern of procedure is more important than the actual theorems which have been established.

In (8.25), it is assumed that $\text{Re}\,\lambda > 0$ and $0 < c < 1$. Furthermore, for the sake of simplicity we restrict ourselves to real z and let

$$(8.26) \qquad f(t) = \log(-\log t).$$

Thus

$$(8.27) \qquad f(uz^{-1}) = \log(\log z - \log u)$$

$$= \log \log z + \log \left(1 - \frac{\log u}{\log z}\right) .$$

For every fixed positive integer $N \geq o$, we have

$$(8.28) \qquad \log\left(1 - \frac{\log u}{\log z}\right) = -\sum_{n=1}^{N} \frac{1}{n} \frac{(\log u)^n}{(\log z)^n} + O\left(\frac{(\log u)^{N+1}}{(\log z)^{N+1}}\right)$$

as $z \to \infty$, providing

$$(8.29) \qquad \left|\frac{\log u}{\log z}\right| \leq 1 - \delta ,$$

or equivalently

$$(8.30) \qquad \frac{1}{z^{1-\delta}} \leq u \leq z^{1-\delta} , \quad \text{for some fixed } \delta > o.$$

Therefore the use of the approximation (8.28) must exclude $u = o$.

The substitution $u = tz$ gives

$$(8.31) \qquad \frac{1}{z^{2-\delta}} \leq t \leq \frac{1}{z^{\delta}} .$$

Let $a = z^{-2+\delta}$ and $b = z^{-\delta}$. It can then be shown that a fixed $\rho > o$ must exist such that

$$(8.32) \qquad \int_{o}^{a} t^{\lambda-1} f(t)\exp(-zt)dt = O(z^{-\lambda-\rho}), \quad \text{as} \quad z \to \infty.$$

Furthermore,

$$(8.33) \qquad \int_{b}^{c} t^{\lambda-1} f(t)\exp(-zt)dt = O(\exp(-z^{1-\delta})), \quad \text{as} \quad z \to \infty.$$

Therefore

$$(8.34) \qquad F(z) = \int_{a}^{b} t^{\lambda-1} f(t)\exp(-zt)dt + O(z^{-\lambda-\rho}), \quad \text{as} \quad z \to \infty.$$

To proceed further, we recall two facts from Chapter V. For any integer $n \geq o$, it is true that

$$(8.35) \qquad \int_{az}^{bz} u^{\lambda-1}(\log u)^n \exp(-u)du = \Gamma^{(n)}(\lambda) + O(z^{-\eta}), \quad \text{as} \quad z \to \infty,$$

for some fixed $\eta > o$. Moreover, the integral

$$(8.36) \qquad \int_{az}^{bz} |u^{\lambda-1}(\log u)^{N+1}| \exp(-u)du$$

exists and is bounded as $z \to \infty$.

Returning to (8.34), we now substitute, as before, $zt = u$. The results (8.28), (8.35) and (8.36) combined together now yield

$$(8.37) \qquad z^{\lambda}F(z) = (\log \log z)\{\Gamma(\lambda) + O(z^{-\eta})\}$$
$$+ [\sum_{n=1}^{N} (-\frac{1}{n})(\log z)^{-n}\Gamma^{(n)}(\lambda) + O((\log z)^{-N-1})]$$

as $z \to \infty$. This will imply

$$(8.38) \quad \{z^{\lambda}F(z) - \Gamma(\lambda)\log \log z\} \sim \sum_{n=1}^{\infty} (-\frac{1}{n})(\log z)^{-n}\Gamma^{(n)}(\lambda); \{(\log z)^{-n}\},$$

as $z \to \infty$.

<u>Example 4</u>. <u>An Application to Probability Theory</u>.

The integral $I(m)$ is defined by

$$(8.39) \qquad I(m) = \int_{-\infty}^{\infty} xe^{-x^2} \left(\frac{1 + \theta(x)}{2}\right)^m dx ,$$

where m is a positive integer, and $\theta(x)$ is the probability integral

(8.40) $$\theta(x) = \frac{2}{\sqrt{\pi}} \int_o^x e^{-u^2}\, du.$$

The substitution

(8.41) $$e^{-t} = \frac{1 + \theta(x)}{2}$$

places (8.39) in the form

(8.42) $$I(m) = \sqrt{\pi} \int_o^\infty x e^{-(m+1)t}\, dt ,$$

and therefore, as long as $x = x(t)$ satisfies certain conditions, it can be envisaged that the methods of the present thesis will yield the asymptotic behavior of $I(m)$ as $m \to \infty$. The function $\frac{1 + \theta(x)}{2}$ monotonically increases from o to 1 in the interval $-\infty \leq x \leq +\infty$. Similarly e^{-t} monotonically decreases from 1 to o in the interval $o \leq t \leq +\infty$. The inverse $x = x(t)$ is uniquely defined, and monotonicially decreases from $+\infty$ to $-\infty$ as t increases from o to ∞. In any closed finite interval $c \leq t \leq T$, $c > o$, $I(m)$ is exponentially small, and if terms of this order can be neglected then we need only discuss the behavior of the integral in an infinitesimal interval containing $t = o$, and an interval $t \geq T$, where T can be taken to be arbitrarily large.

The asymptotic behavior of $\theta(x)$ is known [5] as $x \to \infty$, being given by

(8.43) $$\theta(x) \sim 1 - \frac{e^{-x^2}}{\sqrt{\pi}\, x} \left(1 - \frac{1}{2x^2} + \frac{3}{4x^4} - + \cdots ;\ \{x^{2n}\}\right) ,$$

as $x \to \infty$. Further $\theta(-x) = -\theta(x)$ gives

(8.44) $$\theta(x) \sim -1 - \frac{e^{-x^2}}{\sqrt{\pi}\,x}\left(1 - \frac{1}{2x^2} + \frac{3}{4x^4} - + \cdots ;\ \{x^{2n}\}\right),$$

as $x \to -\infty$. Hence as $t \to \infty$, $x \to -\infty$,

(8.45) $$e^{-t} = \frac{1 + \theta(x)}{2} = -\frac{e^{-x^2}}{2\sqrt{\pi}x}\,(1 + o(1)),$$

and, trivially, a constant K will exist such that $|x| \le K\sqrt{t}$, for t sufficiently large. This is sufficient to show that $\int_T^\infty xe^{-(m+1)t}dt$ is exponentially small. Hence,

(8.46) $$I(m) \approx \int_o^c xe^{-(m+1)t}dt ,$$

where c is a fixed positive number whose value can be arbitrarily assigned to be as small as we wish, excluding the choice $c = o$. The asymptotic sequence involved in writing (8.46) must be such that terms which are exponentially small in m can be neglected. When $t \to o$ and $x \to +\infty$, then for any fixed integer $N \ge o$,

(8.47) $$e^{-t} = \frac{1+\theta(x)}{2} = 1 - \frac{e^{-x^2}}{\pi x}\left[\sum_{n=o}^{N} (-1)^n \frac{\Gamma(n+\frac{1}{2})}{x^{2n}} + 0\left(\frac{1}{x^{2N+2}}\right)\right],$$

as $x \to \infty$ and $t \to o$. Hence

(8.48) $$\frac{e^{-x^2}}{\pi x}\left[\sum_{n=o}^{N} (-1)^n \frac{\Gamma(n+\frac{1}{2})}{x^{2n}} + 0\left(\frac{1}{x^{2N+2}}\right)\right] = 1 - e^{-t} .$$

By a method of successive approximation, it is then easily established that

(8.49) $$x = \sqrt{-\log t} - \frac{\log(-\log t)}{4\sqrt{-\log t}} + 0\left(\frac{1}{\sqrt{-\log t}}\right), \quad \text{as } t \to o,$$

and additional terms may be calculated if desired. Hence

$$(8.50)\quad \int_0^c xe^{-(m+1)t}dt = \int_0^c (-\log t)^{1/2}e^{-(m+1)t}dt - \frac{1}{4}\int_0^c \frac{\log(-\log t)}{\sqrt{-\log t}} \cdot e^{-(m+1)t}dt + O\left(\frac{(\log(m+1))^{-(1/2)}}{(m+1)}\right),$$

as $m \to \infty$, the estimate of the remainder being obtained from Lemma 5.4. By Theorem 5.2,

$$(8.51)\quad \int_0^c (-\log t)^{\frac{1}{2}}e^{-(m+1)t}dt \sim \frac{(\log(m+1))^{\frac{1}{2}}}{(m+1)}\left[\sum_{k=o}^{\infty} \binom{1/2}{k}[\Gamma^{(k)}(\lambda)]_{\lambda=1} (\log(m+1))^{-k}; \quad \{(\log(m+1))^{-k}\}\right],$$

as $m \to \infty$, and by the method used in the previous example

$$(8.52)\quad \int_0^c \log(-\log t)(-\log t)^{-\frac{1}{2}}e^{-(m+1)t}dt$$

$$\sim \frac{\log\log(m+1)}{(m+1)\sqrt{\log(m+1)}}\left[\sum_{k=o}^{\infty} \binom{-(1/2)}{k}[\Gamma^{(k)}(\lambda)]_{\lambda=1}(\log(m+1))^{-k}; \quad \{(\log(m+1))^{-k}\}\right]$$

$$+ \frac{1}{(m+1)\sqrt{\log(m+1)}}\left[\sum_{k=1}^{\infty} c_k[\Gamma^{(k)}(\lambda)]_{\lambda=1}(\log(m+1))^{-k}; \quad \{(\log(m+1))^{-k}\}\right],$$

as $m \to \infty$. The c_k are all constants whose values are given by

$$(8.53)\quad c_k = \sum_{r=o}^{K} \frac{1}{r}\binom{-(1/2)}{r}.$$

Hence the first two terms of an asymptotic expansion of I(m) are

$$(8.54)\quad I(m) = \frac{\sqrt{\pi \log(m+1)}}{(m+1)}\left[1 - \frac{1}{4}\frac{\log\log(m+1)}{\log(m+1)} + O\left(\frac{1}{(\log(m+1))^{3/2}}\right)\right]$$

$$= \frac{\sqrt{\pi \log(m+1)}}{(m+1)} - \frac{\sqrt{\pi}}{4}\frac{\log\log(m+1)}{(m+1)(\log(m+1))^{1/2}} + O\left(\frac{1}{(m+1)\log(m+1)}\right) .$$

This compares with a result

$$(8.55)\qquad I(m) \sim \frac{\sqrt{\pi \log m}}{m}, \quad \text{as } m \to \infty ,$$

which was obtained by Tricomi [19].

If one takes m = 100, (8.54) and (8.55) give respectively

$$(8.56)\qquad I(100) = 0.034577083107$$

$$I(100) = 0.03803625747$$

and the additional term seems worth having.

Example 5. The Confluent Hypergeometric Function.

The function represented by

$$(8.57)\qquad \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\exp(-zt)dt$$

can be evaluated by means of the confluent hypergeometric function to be

$$(8.58)\quad \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\exp(-zt)dt = \frac{\Gamma(\beta)\Gamma(\alpha)}{\Gamma(\beta+\alpha)}\ {}_1F_1(\alpha;\beta+\alpha;-z) ,$$

Re $\alpha > o$, Re $\beta > o$. Differentiating m-times with respect to α and s-times with respect to β gives the function

$$(8.59)\quad F(z) = \int_o^1 t^{\alpha-1}(1-t)^{\beta-1}(\log t)^m(\log(1-t))^s \exp(-zt)dt$$

$$= \frac{\partial^{m+s}}{(\partial\alpha)^m(\partial\beta)^s}\left(\frac{\Gamma(\beta)\Gamma(\alpha)}{\Gamma(\beta+\alpha)}\ {}_1F_1(\alpha;\beta+\alpha;-z)\right).$$

By expanding $(1-t)^{\beta-1}(\log(1-t))^s$ into a Maclaurin expansion, one obtains

$$(8.60)\qquad (1-t)^{\beta-1}(\log(1-t))^s = \sum_{n=1}^{\infty} a_n(\beta,s)t^n, \qquad |t| < 1,$$

where

$$(8.61)\qquad a_n(\beta,s) = \frac{1}{n!}\frac{d^n}{dt^n}\left[(1-t)^{\beta-1}(\log(1-t))^s\right]_{t=o}.$$

For any fixed integer $N \geq 1$, the finite form of the Maclaurin expansion can be used to give

$$(8.62)\qquad (1-t)^{\beta-1}(\log(1-t))^s = \sum_{n=1}^{N} a_n(\beta,s)t^n + o(t^N)$$

as $t \to o$. This yields

$$(8.63)\quad t^{\alpha-1}(\log t)^m(1-t)^{\beta-1}(\log(1-t))^s = \sum_{n=1}^{N} a_n(\beta,s)t^{n+\alpha-1}(\log t)^m + o(t^{N+\alpha-1}(\log t)^m),$$

as $|t| \to o$. The other conditions of Theorem 5.1 are satisfied, and

$$(8.64)\quad F(z) \sim \sum a_n(\beta,s)D^m\{\Gamma(n+\alpha)z^{-(n+\alpha)}\};\quad \{\phi_n = (\log z)^m \cdot z^{-(n+\alpha)}\},$$

as $z \to \infty$ in $|\arg z| \le \frac{\pi}{2} - \Delta$. Since $a_n(\beta,s) = o$ when $1 \le n \le s-1$, the first two non-zero terms of (8.64) are

$$(8.65)\quad F(z) = (-1)^s D^m\{\Gamma(s+\alpha)z^{-s-\alpha}\} + (-1)^{s+1}\left(\beta - \frac{s+2}{2}\right)D^m\{\Gamma(s+\alpha+1)z^{-s-\alpha-1}\} + O((\log z)^m \cdot z^{-(s+\alpha+2)}),$$

as $z \to \infty$ in $|\arg z| \le \frac{\pi}{2} - \Delta$. Written in full, these terms give

$$(8.66)\quad F(z) = (-1)^s\left[\sum_{r=o}^{m} (-1)^{m-r}\binom{m}{r} \frac{\Gamma^{(r)}(\alpha+s)}{(\log z)^r}\right] \frac{(\log z)^m}{z^{\alpha+s}} + (-1)^{s+1}\left(\beta - \frac{s+2}{2}\right)\left[\sum_{r=o}^{m} (-1)^{m-r}\binom{m}{r}\frac{\Gamma^{(r)}(\alpha+s+1)}{(\log z)^r}\right] \frac{(\log z)^m}{z^{\alpha+s+1}} + O((\log z)^m\, z^{-(s+\alpha+2)}).$$

Hence

$$(8.67)\quad F(z) = (-1)^s \frac{(\log z)^m}{z^{\alpha+s}} \left\{ \sum_{r=o}^{m} (-1)^{m-r}\binom{m}{r} \frac{\Gamma^{(r)}(\alpha+s)}{(\log z)^r} - \frac{1}{z}\left(\beta - \frac{s+2}{2}\right) \times \sum_{r=o}^{m} (-1)^{m-r}\binom{m}{r} \frac{\Gamma^{(r)}(\alpha+s+1)}{(\log z)^r} + O(z^{-2})\right\}$$

as $z \to \infty$ in $|\arg z| \le \frac{\pi}{2} - \Delta$.

To obtain the behavior of $F(z)$ as $z \to \infty$ outside the region indicated above, we replace t by $1 - t$ and obtain

$$(8.68)\quad F(z) = \exp(-z) \int_o^1 t^{\beta-1}(1-t)^{\alpha-1}(\log t)^s(\log(1-t))^m \exp(-(-z)t)dt.$$

The integral on the right is of the form (8.59), and hence, Theorem 5.1 can again be used to give

$$(8.69)\quad F(z) \sim \exp(-z)[\Sigma a_n(\alpha,m)D^s\{\Gamma(n+\beta)(-z)^{-(n+\beta)}\};\{(\log(-z))^s z^{-(n+\beta)}\}]$$

as $z \to \infty$ in $\frac{\pi}{2} + \Delta \leq |\arg z| \leq \pi$, where the a_n are given by (8.5). In this sector, the first two nonzero terms of the expansion are

$$(8.70)\quad F(z) = \exp(-z)[(-1)^m D^s\{\Gamma(m+\beta)(-z)^{-m-\beta}\} + (-1)^{m+1}\left(\alpha - \frac{m+2}{2}\right)D^s\{\Gamma(m+\beta+1)(-z)^{-m-\beta-1}\} + O((\log(-z))^s \cdot z^{-(m+\beta+2)})].$$

Hence,

$$(8.71)\quad F(z) = (-1)^m \frac{(\log(-z))^s}{(-z)^{\beta+m}} \exp(-z)\left\{ \sum_{r=o}^{s} (-1)^{s-r}\binom{s}{r}\frac{\Gamma^{(r)}(m+\beta)}{(\log(-z))^r} + \right. \left. + \frac{1}{z}\left(\alpha - \frac{m+2}{2}\right) \sum_{r=o}^{s} (-1)^{s-r}\binom{s}{r} \frac{\Gamma^{(r)}(m+\beta+1)}{(\log(-z))^r} + O(z^{-2})\right\}$$

as $z \to \infty$ in $\frac{\pi}{2} + \Delta \leq |\arg z| \leq \pi$.

The total domain of validity of the two expansions (8.64) and (8.69) still does not cover the complete z-plane. There are gaps in the z-plane which include the positive and the negative imaginary axes. However, using a device outlined in Chapter I, one sees that the asymptotic behavior of $F(z)$ in these gaps is described by

$$(8.72)\quad F(z) \sim [\Sigma a_n(\beta,s)D^m\{\Gamma(n+\alpha)z^{-(n+\alpha)}\};\{(\log z)^m \cdot z^{-(n+\alpha)}\}] + \exp(-z)[\Sigma a_n(\alpha,m)D^s\{\Gamma(n+\beta)(-z)^{-(n+\beta)}\};\{(\log(-z))^s \cdot z^{-(n+\beta)}\}].$$

Because of the exponential factor $\exp(-z)$, the expansion (8.72) is valid for any values of $\arg z$. In particular,

$$(8.73)\quad F(z) = (-1)^s \frac{(\log z)^m}{z^{\alpha+s}}\Big\{ \sum_{r=o}^{m} (-1)^{m-r}\binom{m}{r}\frac{\Gamma^{(r)}(r+\alpha)}{(\log z)^r} -$$

$$- \frac{1}{z}\Big(\beta - \frac{s+2}{2}\Big) \sum_{r=o}^{m} (-1)^{m-r}\binom{m}{r}\frac{\Gamma^{(r)}(\alpha+s+1)}{(\log z)^r} + O(z^{-2})\Big\}$$

$$+ (-1)^m \frac{(\log(-z))^s}{(-z)^{\beta+m}} \exp(-z)\Big\{ \sum_{r=o}^{s} (-1)^{s-r}\binom{s}{r}\frac{\Gamma^{(r)}(m+\beta)}{(\log(-z))^r} +$$

$$+ \frac{1}{z}\Big(\alpha - \frac{m+2}{2}\Big) \sum_{r=o}^{s} (-1)^{s-r}\binom{s}{r}\frac{\Gamma^{(r)}(\beta+m+1)}{(\log(-z))^r} + O(z^{-2})\Big\}$$

as $z \to \infty$ in the complex z-plane.

CHAPTER IX

Conclusion

The present thesis has confined itself to the classical problem of determining the asymptotic behavior of a function $F(z)$ defined by

(9.1) $$F(z) = \int_{\Gamma} g(w) \exp[-z\phi(w)]dw ,$$

where Γ is a continuous curve which may be finite or infinite in length. Although an alternative procedure was illustrated, it was generally true throughout the thesis that the use of the substitution $t = \phi(w)$ was envisaged, and $F(z)$ would have a canonical representation of the form

(9.2) $$F(z) = \int_{\Gamma'} f(t)\exp(-zt)dt ,$$

where

(9.3) $$f(t) = g(w) \frac{dw}{dt} ,$$

and Γ' is the image of Γ in the complex t-plane. It was also envisaged that the so-called critical points would be found among the singularities of $f(t)$, or possibly at the end points of the path of integration, and that the asymptotic behavior would be determined by a finite number of these critical points, say $t = a_1, a_2, \cdots, a_k$, where all of the complex numbers a_i are fixed and finite. The first part of the general pattern therefore envisages the following situation:

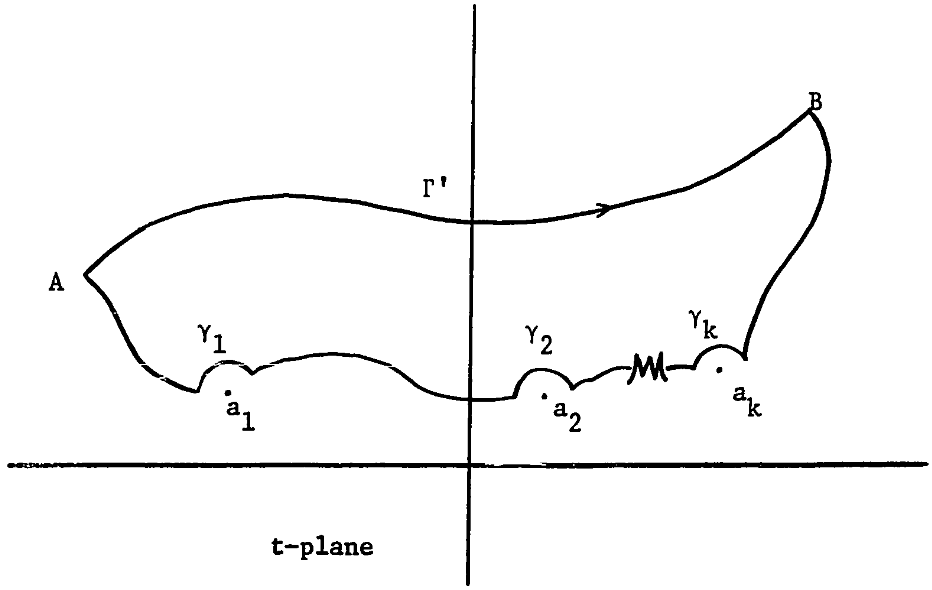


Figure 9.1.

Although A and B are shown as if they were both finite points and distinct, either or both may be the point at infinity, and the possibility of Γ' being a closed curve is not excluded. In order for the pattern to be successful it was assumed that Γ' could be deformed into a new contour Γ" in such a way that

(9.4) $$\int_{\Gamma'} f(t)\exp(-zt)dt = \int_{\Gamma''} f(t)\exp(-zt)dt$$

$+ 2\pi i$ (Sum of residues of $f(t)\exp(-zt)$ at a finite number of poles of $f(t)$ contained within the closed curve $\Gamma' + \Gamma''$).

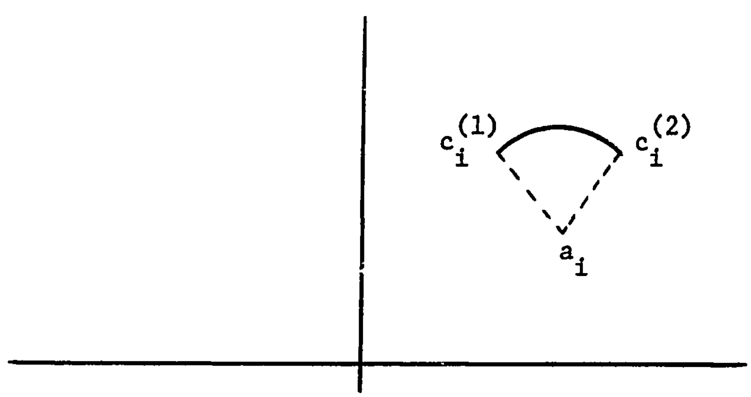
The pattern envisages that the standard of accuracy required in a specific asymptotic agency is determined by some agency external to the general theory of asymptotics, and that the standard of accuracy is obtained by the choice of a suitable asymptotic sequence $\{\phi_n(z)\}$. The success of the pattern of procedure then depends on establishing the result that

$$(9.5) \quad \int_{\Gamma''} f(t)\exp(-zt)dt \approx \sum_{i=1}^{k} \int_{\gamma_i} f(t)\exp(-zt)dt, \quad \{\phi_n(z)\} ,$$

as $|z| \to \infty$, where each γ_i is the portion of a circle with center at the critical points $t = a_i$, and whose radius is fixed and arbitrarily small. If (9.5) can be established, then further success depends on the study of the asymptotic behavior of the functions $F_i(z)$, defined by

$$(9.6) \qquad F_i(z) = \int_{\gamma_i} f(t)\exp(-zt)dt .$$

Although $f(t)$ may have a singularity at $t = a_i$ such that the contour γ_i cannot be replaced by the two straight lines shown below,



t-plane

Figure 9.2.

the pattern does require that it is possible to write $f(t)$ in the form

$$f(t) = g(t) + h(t) , \tag{9.7}$$

where the asymptotic behavior of $\int_{\gamma_i} g(t)\exp(-zt)dt$ is known, and

$$\int_{\gamma_i} h(t)\exp(-zt)dt = \int_{a_i}^{c_i^{(2)}} h(t)\exp(-zt)dt - \int_{a_i}^{c_i^{(1)}} h(t)\exp(-zt)dt \tag{9.8}$$

is valid. The reason for this requirement is that $|\exp(-zt)|$ is not normally exponentially small on the whole of the portion of a circle γ_i, and the estimates of remainders become difficult to make if a result like (9.8) is not true. This procedure allows us to consider the integral of Watson's Lemma

$$F(z) = \int_a^{a+c} f(t)\exp(-zt)dt , \tag{9.9}$$

where the path of integration is the straight line joining $t = a$ to $t = a + c$, as a canonical form for the more general problem being considered in the present thesis. From the development, $|c| \neq o$ may be taken to be as small as we please. If the translation $t = t' + a$ is effected, the canonical form is reduced to

$$F(z) = \int_o^c f(t)\exp(-zt)dt . \tag{9.10}$$

The pattern envisaged by Watson required an expansion of the form

(9.11) $$f(t) = \sum_{n=o}^{N} f_n(t) + R_N(t) ,$$

where

(9.12) $$\int_o^c R_N(t)\exp(-zt)dt = o(\phi_N(z)), \quad \text{as} \quad |z| \to \infty ,$$

and thus

(9.13) $$F(z) \sim \sum_{n\varepsilon I} F_n(z); \quad \{\phi_n(z)\} , \quad \text{as} \quad |z| \to \infty ,$$

where

(9.14) $$F_n(z) = \int_o^c f_n(t)\exp(-zt)dt .$$

In order to be a result worth having, the asymptotic behavior of $F_n(z)$ should be known. Indeed Watson's result included

(9.15) $$\int_o^c f_n(t)\exp(-zt)dt \approx \int_o^{\infty e^{i\arg c}} f_n(t)\exp(-zt)dt; \quad \{\phi_n(z)\} ,$$

as $|z| \to \infty$, and an explicit evaluation of $\int_o^{\infty\exp(i\arg c)} f_n(t)\exp(-zt)dt$ is known. The procedure used in the present thesis to discuss logarithmic singularities required a modification of this particular pattern. In the first instance, the exclusion of the point $t = o$ from the validity of (9.11) was required and the pattern required the establishment of

(9.14) $$F(z) \approx \int_\alpha^\beta f(t)\exp(-zt)dt,$$

where $o < |\alpha| < |\beta| < |c|$, $\arg \alpha = \arg \beta = \arg c$, and α and β

might now be required to be functions of z. The substitution $u = zt$ gives

(9.15) $$F(z) \approx z^{-1} \int_{\alpha z}^{\beta z} f(uz^{-1}) \exp(-u) du.$$

If, for the points on the path of integration, it is true that

(9.16) $$f(uz^{-1}) = \sum_{n=o}^{N} \psi_n(z)\rho_n(u) + o(\phi_N(z)\rho_N(u)), \quad \text{as} \quad |z| \to \infty,$$

where $\int_{\alpha z}^{\beta z} |\rho_N(u)\exp(-u)du|$ exists and is uniformly bounded in z as $|z| \to \infty$, then

(9.17) $$F(z) \sim z^{-1}[\sum_{n\epsilon I} \psi_n(z)h_n(z); \quad \{\phi_n(z)\}], \quad \text{as} \quad |z| \to \infty$$

where

(9.18) $$h_n(z) = \int_{\alpha z}^{\beta z} \rho_n(u)\exp(-u)du \; .$$

Again to be a result worth having, the asymptotic behavior of $\psi_n(z)h_n(z)$ must be known. Although this particular pattern was introduced to give a detailed study of the situation where $f(t)$ has a logarithmic singularity, it should be stressed that the pattern is capable of success for types of singularities other than the one mentioned above. It seems, therefore, worthwhile to do further investigation beyond the present thesis in the development of the pattern with respect to the other types of singularities.

In a paper [12], Erdélyi and Wyman have generalized the form of integrand to a study of

$$F(z) = \int_L h(z,t)dt \quad , \tag{9.19}$$

where the narrow form of integrand used in Watson's Lemma is replaced by an integral of far more general form. A combination of the ideas in that paper with the ideas of the present thesis should produce results of considerable significance.

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