Local Well-Posedness and Prodi-Serrin type Conditions for models of Ferrohydrodynamics

by

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Abstract

Ferrofluids are liquids with a magnetic colloid suspended by Brownian motion which become magnetized in the presence of an external magnetic field. They have recently garnered a lot of interest due to their wide range of applications in industry and biomedicine. For example, ferrofluids are used in rotary seals, microchannel flows, nanotechnology, and even in experimental cancer treatments. There are two widely-used mathematical models which describe the motion of ferrofluids; the Rosensweig model derived by Ronald E. Rosensweig, and the Shliomis model derived by Mark I. Shliomis. In the mathematical literature, the ferrofluid models remain relatively unexplored. Only a handful of papers have been written on them, most of which are concerned with existence of weak solutions or strong solutions. Because the equations describing ferrofluids build upon the famous Navier-Stokes equations, we expect many properties which have been proven in the far more extensive literature for those equations to have an analogous version of themselves hold for the ferrofluid models. This thesis helps to narrow this gap in the literature by extending the analysis of classical solutions to both models. In particular, we first show local well-posedness of classical solutions on the whole three dimensional Euclidean space for a regularized version of each model– that a solution exists, is unique in this class of solutions, and varies continuously with the initial data in the appropriate topology. Then, we derive so-called Prodi-Serrin type conditions for the solutions, which are sufficient conditions to extend the solutions we constructed up to and beyond a time T.

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Chapter 1

Introduction

Ferrohydrodynamics is the study of fluid motion that is influenced by magnetic polarization [Ros85]. While many magnetic fluids arise in ferrohydrodynamics, of particular interest is the study of colloidal ferrofluids– liquids with nanoscale (3-15nm) ferro- or ferrimagnetic single-domain particles covered in a dispersant coating (colloid) suspended evenly throughout [Ros85, Cha02]. The colloid is usually manufactured by first producing magnetic powder, either by grinding or chemical precipitation, then mixing it with a dispersant coating and suspending it in the carrier fluid [1]. This process allows the fluid to become magnetized in the presence of an external magnetic field. Originally patented in the 1960s by NASA engineer Stephen Papell as a potential way to manipulate rocket fuel in zero gravity [Pap65], ferrofluids were later refined by Ronald E. Rosensweig and have since seen a wide variety of applications in engineering and biomedicine.

[1] For a more detailed account on the manufacturing of magnetic powders, see [Ros85]

In industry, ferrofluids have been studied for use in deformable mirrors [LBB+04, LCR+06, BBT07], plastic micropumps and valves [YCP+05, HBL04], microchannel flows [STK11, ZDV+13], nanotechnology [Zah01], bearings [MHN+03], and seals, loudspeakers, dampers, sensors and gauges [RMC95]. In biomedicine, ferrofluids have been used in targeted drug delivery [GR05] and in an experimental cancer treatment called magnetic hyperthermia [LR09, PCJD03], in which a magnetic fluid is directed to an infected area and then heated by oscillating magnetic fields to weaken cancer cells. Despite their wide range of applications, the mathematical literature contains only a handful of papers on the models which describe ferrofluids. To the best of my knowledge, the following works have been done so far (ordered chronologically by year):

Year	**Authors**	**Model**	**Topic**
2002	Venkatasubramanian Kaloni [VK02]	Rosensweig	Stability and Uniqueness
2008	Amirat, Hamdache, Murat [AHM08]	Rosensweig	Global Weak Solutions
2008	Amirat, Hamdache [AH08]	Shliomis	Global Weak Solutions
2009	Amirat, Hamdache [AH09]	Shliomis	Local Strong Solutions
2010	Amirat, Hamdache [AH10]	Rosensweig	Local Strong Solutions
2010	Tan, Wang [TW10]	Shliomis	Global Strong Solutions, Blow-up
2010	Wang, Tan [WT10]	Rosensweig	Global Weak Solutions, Asymptotics, Relation to Shliomis model
2015	Nochetto, Salgado, Thomas [NST15]	Rosensweig	Numerical Analysis

Table 1.1: Papers on Ferrohydrodynamics in the Mathematical Literature

The Rosensweig and Shliomis models listed in Table 1.1 are named for their creators, and are derived under different physical assumptions on the ferrofluid. Later on, we will learn about these differences, and how the models relate to each other.

The rest of this thesis is organized as follows: In Chapter 2, we provide a description of the magnetization, suspension, and magnetic stability of ferrofluids, and discuss the Rosensweig and Shliomis models and a magnetic regularization term that is sometimes added to the models. Next, in Chapter 3 we summarize the results from the literature listed in Table 1.1, grouped by the topic they cover. Then in Chapter 4, we provide some mathematical background that we will need to obtain our results, and prove some preliminary lemmas which will simplify our presentation in the following two chapters. In Chapter 5, we show local well-posedness of classical solutions for the regularized ferrofluid models in three-dimensional Euclidean space. That is, we show the existence, uniqueness, and continuous dependence on initial data of classical solutions to the equations. Then in Chapter 6, we derive Prodi-Serrin type conditions for both models. First we discuss the prototypal Prodi-Serrin conditions for solutions of the well-known Navier-Stokes equations. Then we derive archetypal Prodi-Serrin conditions for solutions of ferrofluid equations whose existence was shown in the previous step. These conditions guarantee regularity of our solutions in and beyond the time interval considered. Finally, in Chapter 7, we comment on some future work that could be done to extend our results.

Chapter 2

Overview of Ferrohydrodynamics

In this chapter, we describe in detail some physical properties of ferrofluids, the equations of ferrohydrodynamics, and the regularized equations of ferrohydrodynamics.

2.1 Properties of Ferrofluids

2.1.1 Magnetization

Each colloid particle in a ferrofluid has an embedded magnetic moment. In the absence of an external magnetic field, the particles are oriented randomly, so the fluid has no net magnetization [Ros85]. In the presence of an applied field, particles exhibit superparamagnetism – tendency of molecular moments to align with the applied field in the absence of long-range order, but with higher magnetization in low to moderate fields compared to paramagnetism [Ros85].

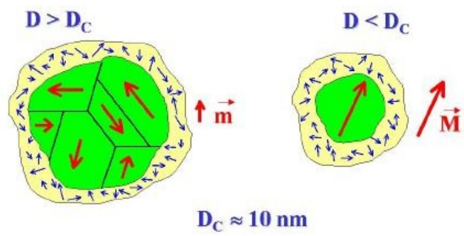

Figure 2.1: Multi-domain particle on left with single domain particle on right [Ger]

Since individual particles are small, the magnetization is treated as being collinear with the magnetic field (single-domain; see Figure 2.1). Conversely, if the particles are relatively large, the fluid needs to be treated as micropolar; i.e. particles are treated as rigid magnetic dipoles [Bus05]. For more information on micropolar fluids, one can check [Luk99]. Pictured below (Figure 2.2) is ferrofluid in the presence of a magnetic field applied normal to the surface. The spikes point in the direction of magnetic field lines.

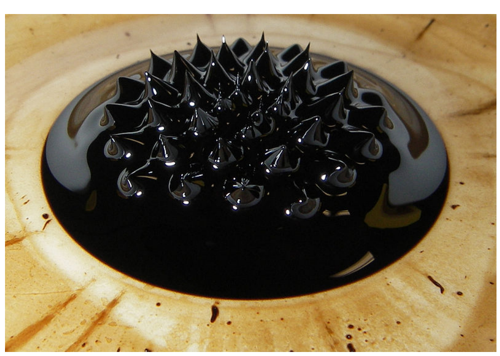

Figure 2.2: Ferrofluid in the presence of a magnetic field [Jur06]

2.1.2 Suspension

The colloid is suspended in the solvent by Brownian motion– the colloid's bombardment by fast-moving particles in the solvent [Ros85]. Brownian motion and the van der Waals attraction therefore present an enormous difficulty in the stability of ferrofluids: if the magnetic particles collide with each other, they will agglomerate and sink in the fluid. This is prevented by a dispersant coating: long chain molecules adsorbed onto the particles' surfaces [Ros85]. The coating causes steric repulsion between the adsorbed molecules via elastic "energy bumpers" (if two particles get too close, the energy cost for them to collide increases) [Ros85]. Molecules with similar properties to the solvent are chosen so as to disguise the colloid to look like part of the fluid. By these mechanisms, ferrofluids remain stable with respect to mutual agglomeration of the colloid particles, and the colloid remains suspended.

2.1.3 Magnetic Stability of Ferrofluids

The presence of a magnetic field can have two effects on a ferrofluid, depending on its alignment with the fluid's free surface: tangentially aligned fields stabilize the fluid surface, whereas fields applied normally to the free surface lead to normal surface instabilities (e.g. Figure 2.2) [RL09]. In applications such as rotary feedthroughs and pressure seals for blowers and compressors, and others mentioned in the introduction, a tangential magnetic field stabilizes the ferrofluid's free surface [RL09, Ros85]. On the other hand, the normal surface instabilities are of interest in studying pattern formation.

2.2 Models of Ferrohydrodynamics

In this section we introduce the Rosensweig and Shliomis models of ferrohydrodynamics. We begin by presenting the models and a table which explains each symbol. Next we provide a physical interpretation of individual terms in the equations. Finally, we discuss the addition of a regularization term to the magnetization equation of each model which is used in the literature and in this thesis to prove certain results.

2.2.1 Standard Models

Two mathematical models are given by,

- Rosensweig [RZ02]:

$$\rho(u_t + u \cdot \nabla u) - (\eta + \zeta) \triangle u + \nabla p = \mu_0 (M \cdot \nabla) H + 2\zeta \nabla \times \Omega \tag{2.1}$$

$$\rho\kappa(\Omega_t + u \cdot \nabla \Omega) - \eta' \triangle \Omega - (\eta' + \lambda')\nabla(\nabla \cdot \Omega) = \mu_0 M \times H + 2\zeta(\nabla \times u - 2\Omega) \tag{2.2}$$

$$M_t + u \cdot \nabla M = \Omega \times M - \frac{1}{\tau}(M - \chi_0 H) \tag{2.3}$$

$$\nabla \times H = 0,\ \nabla \cdot (H + M) = -\nabla \cdot H^{ext},\ \nabla \cdot u = 0, \tag{2.4}$$

and

- Shliomis [Shl02]:

$$\rho(u_t + u \cdot \nabla u) - \eta \triangle u + \nabla p = \mu_0 (M \cdot \nabla) H + \frac{\mu_0}{2} \nabla \times (M \times H) \tag{2.5}$$

$$M_t + u \cdot \nabla M = \frac{1}{2}(\nabla \times u) \times M - \frac{1}{\tau}(M - \chi_0 H) - \beta M \times (M \times H) \tag{2.6}$$

$$\nabla \times H = 0,\quad \nabla \cdot (H + M) = -\nabla \cdot H^{ext},\ \nabla \cdot u = 0, \tag{2.7}$$

where u, Ω, M, and H are functions of (x,t) for $x \in \mathbb{R}^3$ and $t \in (0,\infty)$. The meaning of each term can be found in Table 2.1 below.

$u(x,t)$	mass-average velocity
μ_0	permeability of free space; $4\pi \times 10^{-7} H/m$
$\Omega(x,t)$	spin velocity of suspension
η'	shear coefficient of spin viscosity
$M(x,t)$	magnetization vector of particles
λ'	shear coefficient of bulk viscosity
$H(x,t)$	magnetic field in suspension
τ	relaxation time constant
ρ	mass-density of ferrofluid
χ_0	magnetic susceptibility
η	coefficient of shear
κ	scalar moment
ζ	coefficient of vortex viscosity
p	dynamical pressure
ϕ	volume fraction of dispersed solid phase
β	$\frac{1}{6\eta\phi}$
σ	diffusion coefficient that carries spins

Table 2.1: Symbol Convention for Models of Ferrohydrodynamics

2.2.2 Physical Interpretation

Equations (2.1)-(2.4) and (2.5)-(2.7) can be thought of as an extension of the usual Navier-Stokes equations which describe fluid motion. Ferrohydrodynamics requires extra treatment for the magnetic forces at work. Let us examine (2.1)-(2.7) individually.

We begin with (2.1), which we will hereafter refer to as the momentum equation (we also refer to the equation for u in the Shliomis model as its momentum equation):

$$\underbrace{\rho(u_t + u \cdot \nabla u)}_{(a)} - \underbrace{(\eta + \zeta) \triangle u}_{(b)} + \underbrace{\nabla p}_{(c)} = \underbrace{\mu_0 (M \cdot \nabla) H}_{(d)} + \underbrace{2\zeta \nabla \times \Omega}_{(e)}.$$

First, (a) (mass-density multiplied by the fluid's material derivative) describes the transport of the fluid with respect to time and convection. Next, (b) is a diffusion term, where $\eta \triangle u$ is a shear stress term with coefficient of shear η that describes the viscosity of the fluid (see Figure 2.3, Figure 2.4 below).

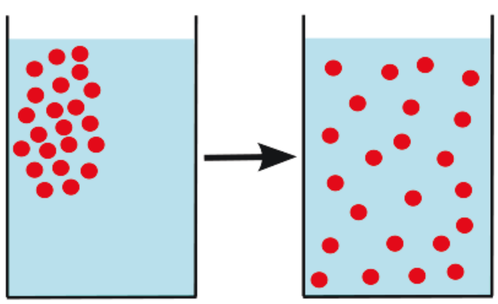

Figure 2.3: Diffusion process; particles tend to move from high to low concentration [Wik08]

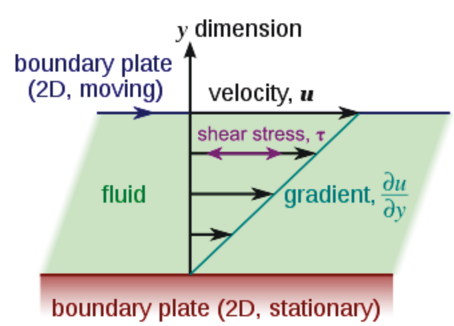

Figure 2.4: Shear stress on fluid between two boundary plates with top plate moving [Wik05]

$\zeta \triangle u$ represents the diffusion caused by vorticity, the local rotational behaviour of the fluid. The third term (c) corresponds to the pressure gradient, and accounts for the system's desire to move to a state of lower pressure. Next, (d) describes the magnetic force density acting on an incompressible ferrofluid [RCE+05]. Physically it corresponds to ferrofluids wanting to move in the direction of increasing magnetic field strength [RCE+05]. In particular, when the magnetic field is uniform, term (d) = 0 [RCE+05]. Finally, (e) represents the shear force from the particle's spin velocity. Together, the terms of this equation constitute a form of conservation of momentum.

Now we consider (2.2), which we will call the angular momentum equation:

$$\underbrace{\rho\kappa(\Omega_t + u\cdot\nabla\Omega)}_{(f)} - \underbrace{\eta' \triangle \Omega}_{(g)} - \underbrace{(\eta' + \lambda')\nabla(\nabla\cdot\Omega)}_{(h)} = \underbrace{\mu_0 M \times H}_{(i)} + \underbrace{2\zeta(\nabla\times u - 2\Omega)}_{(j)}.$$

The first term (f) represents the transport of the fluid's spin velocity with respect to time and convection. The second term (g) corresponds to diffusion, where η' is the shear coefficient of spin viscosity, and accounts for the fluid's viscosity. Next, (h) represents a shear-stress term that describes the fluid's viscosity with respect to spin and bulk deformation. (i) represents a magnetic torque density [RCE+05]. This arises in a rotating magnetic field, wherein the magnetization vector of the colloid particles are not aligned with the magnetic field in the suspension [RCE+05]. In particular, this causes the colloid and surrounding solvent to rotate [RCE+05]. Lastly, (j) describes the difference of external fluid angular momentum to internal angular momentum [Ros85].

We proceed to investigate (2.3),

$$\underbrace{M_t}_{(k)} + \underbrace{u \cdot \nabla M}_{(l)} = \underbrace{\Omega \times M}_{(m)} - \underbrace{\frac{1}{\tau}(M - \chi_0 H)}_{(n)}.$$

The above is called the magnetization equation for the Rosensweig model (similarly we will refer to the equation for M in the Shliomis model as its magnetization equation). The first two terms, (k), (l) again describe the material derivative, this time of the magnetization. The first term on the RHS, (m) represents the change of the fluid's magnetic vector [Ros85]. Finally, (n) will be described in parts. Firstly, $\frac{1}{\tau} = \frac{1}{\tau_N} + \frac{1}{\tau_B}$ is given by the sum of the inverses of the Néel time constant and the Brownian relaxation time. τ_N represents the time for a particle's magnetic moment to align with the magnetic field in the suspension, while τ_B describes the time for the whole particle to align with the magnetic field [RCE+05]. Meanwhile, inside the brackets χ_0 represents the particle's magnetic susceptibility, and in the low-field limit, $\chi_0 H$ is roughly the equilibrium magnetization. So the final term represents the deviation of the particles' magnetization vectors from equilibrium scaled by relaxation time for realignment with the magnetic field.

There are only three differing terms in the Shliomis model, namely

$\underbrace{\frac{\mu_0}{2}\nabla \times (M \times H)}_{(o)}$ from (2.5), $\underbrace{\frac{1}{2}(\nabla \times u) \times M}_{(p)}$ from (2.6), and $\underbrace{\beta M \times (M \times H)}_{(q)}$ from (2.6). The first, (o) is the curl of a magnetic torque density scaled by the permeability of free space, and accounts for the internal rotation of particles in (2.5). Together, terms (p) and (q) play the same role as (m) from the Rosensweig model, where $\Omega := \frac{1}{2}\nabla \times u + \frac{\mu_0}{4\zeta} M \times H$ (we will see this substitution gives the Shliomis model below). The last term, (q) in particular describes the magnetization approaching its equilibrium orientation without changing length [Shl02]. We remark that the trilinear term (q) will weaken certain bounds we derive later, on account of having two copies of M.

Finally, we describe (2.4), (2.7). The condition $\nabla \cdot u = 0$ means the fluid flow is incompressible (which is a standard assumption for most fluids) [Shl02]. Next, $\nabla \times H = 0$ means the fluid is non-conducting (which is usually assumed for ferrofluids) [Shl02]. Finally, $B = \mu_0(M + H)$ is the magnetic flux density of the fluid [VK02]. In particular, $\nabla \cdot (H + M) = -\nabla \cdot H^{ext}$ says that, for non-stationary magnetic fields ($\nabla \cdot H^{ext} \neq 0$), the "flow" of H combined with the "flow" of M exactly cancels the "flow" of the externally applied magnetic field.

The key difference between the Rosensweig and Shliomis models is how they describe internal rotations of the particles. Shliomis describes the rotation as a magnetic torque due to the ferrofluid magnetization's nonlinearity [RCE+05]. On the other hand, Rosensweig deals with the rotation by employing integral balance equations and thermodynamics. [Ros02]. It turns out that the Shliomis model can be derived from Rosensweig, stated by Wang and Tan [WT10]. Setting $\kappa = \eta' = \lambda' = 0$ (which eliminates all derivatives in the angular momentum equation (2.2) and therefore neglects the effects of terms (f), (g), and (h) – transport of spin velocity, angular diffusion, and spin shear/bulk viscosity) in (2.2) gives,

$$\Omega = \frac{1}{2}\nabla \times u + \frac{\mu_0}{4\zeta} M \times H. \tag{2.8}$$

Substituting (2.8) into the last term of (2.1), we get

$$2\zeta\nabla\times\Omega = 2\zeta\left(\nabla\times\left(\frac{1}{2}\nabla\times u + \frac{\mu_0}{4\zeta}M\times H\right)\right) = \zeta\triangle u + \frac{\mu_0}{2}\nabla\times M\times H. \quad (2.9)$$

where we have used $\nabla\cdot u = 0$ from (2.4).

Substituting (2.8) into the first term on the RHS of (2.2) gives,

$$\Omega\times M = \left(\frac{1}{2}\nabla\times u + \frac{\mu_0}{4\zeta}M\times H\right)\times H = \frac{1}{2}(\nabla\times u)\times M - \frac{\mu_0}{4\zeta}M\times(M\times H). \quad (2.10)$$

Putting (2.9) and (2.10) back into (2.1) and (2.3) gives the Shliomis model, (2.5)-(2.7).

2.2.3 Regularized Equations: Bloch-Torrey Magnetization

In situations where diffusion of the spin magnetic moment isn't negligible, one adds a $\sigma\Delta M$ term to the magnetization equations (2.3) and (2.6), where $\sigma > 0$ is a diffusion coefficent [AH08]. Equations (2.1)-(2.4) and (2.5)-(2.7) are replaced by

- Rosensweig [AHM08]:

$$\rho(u_t + u\cdot\nabla u) - (\eta+\zeta)\triangle u + \nabla p = \mu_0(M\cdot\nabla)H + 2\zeta\nabla\times\Omega, \quad (2.11)$$

$$\rho\kappa(\Omega_t + u\cdot\nabla\Omega) - \eta'\triangle\Omega - (\eta'+\lambda')\nabla(\nabla\cdot\Omega) = \mu_0 M\times H + 2\zeta(\nabla\times u - 2\Omega), \quad (2.12)$$

$$M_t + u\cdot\nabla M = \Omega\times M - \frac{1}{\tau}(M - \chi_0 H) + \sigma\Delta M, \quad (2.13)$$

$$\nabla\times H = 0,\ \nabla\cdot(H+M) = -\nabla\cdot H^{ext},\ \nabla\cdot u = 0; \quad (2.14)$$

- Shliomis [AH08]:

$$\rho(u_t + u \cdot \nabla u) - \eta \triangle u + \nabla p = \mu_0 (M \cdot \nabla) H + \frac{\mu_0}{2} \nabla \times (M \times H), \tag{2.15}$$

$$M_t + u \cdot \nabla M - \sigma \Delta M = \frac{1}{2}(\nabla \times u) \times M - \frac{1}{\tau}(M - \chi_0 H) - \beta M \times (M \times H), \tag{2.16}$$

$$\nabla \times H = 0, \ \nabla \cdot (H + M) = -\nabla \cdot H^{ext}, \ \nabla \cdot u = 0; \tag{2.17}$$

with initial data

$$u|_{t=0} = u_0, \quad \Omega|_{t=0} = \Omega_0, \quad M|_{t=0} = M_0, \quad H|_{t=0} = H_0, \tag{2.18}$$

and

$$u|_{t=0} = u_0, \quad M|_{t=0} = M_0, \quad H|_{t=0} = H_0, \tag{2.19}$$

respectively. These are called the Rosensweig model with Bloch-Torrey magnetization, and Shliomis model with Bloch-Torrey magnetization, respectively. They are also referred to as regularized ferrohydrodynamics equations, since the magnetic diffusion term provides extra regularity needed to prove the existence of weak solutions (see §3.2 on weak solutions in the next chapter). We will also require the addition of a Bloch-Torrey magnetization term to the models to prove our results. In addition to facilitating the proofs of stronger regularity results, adding a second-order derivative to the magnetization equation allows for additional control on the solution both analytically and numerically. In particular, one can impose additional control on the model through an extra boundary condition on the magnetization term. We postpone a discussion on boundary conditions for the models to the next chapter, where we will see a variety of approaches. On the other hand, our work will be done on $\mathbb{R}^3$, so no boundary conditions are necessary.

Chapter 3

Ferrohydrodynamics in the Mathematical Literature

In this chapter, we provide an overview of work that has been done in the literature on the ferrohydrodynamics equations. As noted in the introduction, most of the mathematical literature for ferrohydrodynamics is concerned with the existence of weak solutions or strong solutions. Therefore, we begin by summarizing papers which show existence of weak solutions to the ferrofluid models. We remark that these results require the use of a Bloch-Torrey magnetization term. Next, we look at papers which consider the existence and uniqueness of strong solutions. In contrast to the papers on weak solutions, the analysis in these papers can be done with or without a Bloch-Torrey magnetization term. Finally, we summarize some important points brought up in a paper on numerical methods by Nochetto et al. [NST15] about boundary conditions for the magnetization equation of the Rosensweig model.

3.1 Notations and Assumptions

Common to all papers is the use of certain functions spaces, and assumptions on the magnetostatic equation, domain, and boundary conditions. Firstly, most of the authors work on a domain $D_T := D \times (0,T)$ where D is a bounded smooth domain. Furthermore, for the papers about weak solutions, it is common for the authors to provide the following boundary conditions for u, Ω, and M:

$$u = 0, \quad \Omega = 0, \quad (\nabla \times M) \times n = 0, \quad M \cdot n = 0 \;\text{ on } \partial D \times (0,T), \tag{3.1}$$

(ignoring the condition on Ω for the Shliomis model) and initial conditions

$$u|_{t=0} = u_0, \quad \Omega|_{t=0} = 0, \quad M|_{t=0} = 0 \text{ in } D \tag{3.2}$$

(again ignoring the condition Ω for the Shliomis model). On the other hand, papers which consider strong solutions (which don't add a Bloch-Torrey magnetization term that requires additional boundary data for M) have the boundary conditions

$$u = 0, \quad \Omega = 0, \quad (H + M) \cdot n = 0 \;\text{ on } \partial D \times (0,T) \tag{3.3}$$

(ignoring the condition on Ω for the Shliomis model). Next, we introduce the function spaces referred to by the authors. They make use of the usual L^p and Sobolev spaces, and additionally define:

$$
\begin{aligned}
\mathcal{U} &:= \{v \in H_0^1(D) : \nabla \cdot v = 0 \text{ in } D\}, \\
\mathcal{U}_0 &:= \{v \in L^2(D) : \nabla \cdot v = 0 \text{ in } D, \ \ v \cdot n = 0 \text{ on } \partial D\}, \\
\mathcal{U}' &:= \text{ the dual space of } \mathcal{U}, \\
\mathcal{M} &:= \{q \in L^2(D) : \nabla \cdot q \in L^2(D), \ \nabla \times q \in L^2(D), q \cdot n = 0 \text{ on } \partial D\},
\end{aligned}
\tag{3.4}
$$

where $\mathcal{M}$ is a Hilbert space with inner product

$$
\langle q_1, q_2 \rangle_{\mathcal{M}} := (q_1, q_2) + (\nabla \cdot q_1, \nabla \cdot q_2) + (\nabla \times q_1, \nabla \times q_2).
$$

In general, we will denote (following the authors) the L^2 inner product by $(\cdot, \cdot)$, and denote the pairing between a Banach space and its dual by $\langle \cdot, \cdot \rangle_{\mathcal{X}' \times \mathcal{X}}$ (where the subscript is dropped when the meaning is clear). Also, denote by $\mathcal{C}([0, T]; \mathcal{X} \text{ weak})$ the set of functions from $[0, T]$ to the Hilbert space $\mathcal{X}$ which are continuous in the weak topology. In particular,

$$
v_n \rightharpoonup v \text{ in } \mathcal{C}([0, T]; \mathcal{X} \text{ weak}) \ \text{ if } \ \langle v_n(t), w \rangle \to \langle v(t), w \rangle
$$

uniformly in t for any $w \in \mathcal{X}'$. Finally, we will denote by $\mathcal{D}'(D_T)$ or $\mathcal{D}'(\mathbb{R}^3 \times (0, T))$ the space of distributions with respect to which an equation holds weakly on the domain D_T or $\mathbb{R}^3 \times (0, T)$, respectively.

3.2 Weak Solutions

The papers by Amirat, Hamdache, and Murat [AHM08], Amirat and Hamdache [AH08], and Wang and Tan [WT10] show the existence of globally-defined weak solutions of the Rosensweig model on D, Shliomis model on D, and the nonhomogeneous Rosensweig model on D, respectively. In all three papers, it is necessary for the authors to add a Bloch-Torrey magnetization term $\sigma \Delta M$ to the magnetization equation of the model they consider.

3.2.1 Amirat, Hamdache and Murat 2008 (Rosensweig Model)

In their paper "Global Weak Solutions to Equations of Motion for Magnetic Fluids" [AHM08], Amirat, Hamdache, and Murat show the existence of global weak solutions to the Rosensweig system with Bloch-Torrey magnetization (2.11)-(2.14) on the domain D_T with initial and boundary conditions given by (3.2), (3.1). Moreover, instead of (2.14), they consider

$$\nabla \times H = 0, \quad \nabla \cdot (H + 4\pi M) = F \ \text{ in } \ D_T, \tag{3.5}$$

with the boundary condition

$$H \cdot n = 0 \ \text{ on } \ \partial D \times (0, T), \tag{3.6}$$

where F satisfies $\int_D F dx = 0$ for all $t \in [0, T]$. Next, the initial data is assumed to satisfy, for some fixed $T > 0$:

$$u_0 \in \mathcal{U}, \quad \Omega_0 \in L^2(D), \quad M_0 \in L^2(D), \tag{3.7}$$

$$F \in H^1(0, T; L^2(D)), \quad \int_D F dx = 0 \text{ in } (0, T), \tag{3.8}$$

and $H_0 = \nabla \phi_0$, where ϕ_0 is the unique weak solution in $H^1(D)$ of

$$-\Delta \phi_0 = 4\pi(\nabla \cdot M_0) - F_0 \text{ in } D, \tag{3.9}$$

$$\frac{\partial \phi_0}{\partial n} = 0 \text{ on } \partial D, \quad \int_D \phi_0 dx = 0,$$

and $F_0 = F|_{t=0}$. With this setting in place, the authors define a global weak solution for their system:

Definition. (u, Ω, M, H) is a global weak solution of (2.11)-(2.13), (3.5), (3.2), (3.1), (3.6) provided the following hold:

(i)

$$u \in L^\infty(0, T; \mathcal{U}_0) \cap L^2(0, T; \mathcal{U}) \cap \mathcal{C}([0, T]; \mathcal{U}_0 \text{ weak}),$$
$$\Omega \in L^\infty(0, T; L^2(D)) \cap L^2(0, T; H_0^1(D)) \cap \mathcal{C}([0, T]; L^2(D) \text{ weak}),$$
$$M \in L^\infty(0, T; L^2(D)) \cap L^2(0, T; \mathcal{M}) \cap \mathcal{C}([0, T]; L^2(D) \text{ weak}),$$
$$H \in L^\infty(0, T; L^2(D)) \cap L^2(0, T; H^1(D));$$

(ii) the function H is such that $H = \nabla\phi$ where $\phi \in L^\infty(0,T;H^1(D)) \cap L^2(0,T;H^2(D))$ and solves

$$-\Delta\phi = 4\pi(\nabla \cdot M) - F \quad \text{in} \quad D_T,$$

$$\frac{\partial\phi}{\partial n} = 0 \quad \text{on} \quad \partial D \times (0,T), \qquad \int_D \phi dx = 0 \quad \text{in} \quad (0,T);$$

(iii) the equations (2.11)-(2.13), (3.5) hold weakly (when tested with $v \in \mathcal{U}$, $\omega \in H_0^1(D)$, and $q \in \mathcal{M}$ respectively).

This allows the authors to present their main theorem:

Theorem 3.2.1. (*Amirat Hamdache Murat Theorem 1* [AHM08]) Assume (3.7)-(3.8). Then problem (2.11)-(2.13), (3.5), (3.2), (3.1), (3.6) has a global weak solution (u, Ω, M, H) satisfying the energy inequality

$$E(t) + C_1 \int_0^t E^d(s)ds \le E_0 + C_2(\|F(s)\|_{L^2}^2 + \|\partial_t F(s)\|_{L^2}^2)ds \quad \text{a.e.} \quad t \in (0,T),$$

where

$$E(t) = \frac{\rho}{2}\|u\|_{L^2}^2 + \frac{\rho\kappa}{2}\|\Omega\|_{L^2}^2 + \frac{1}{2}\|M\|_{L^2}^2 + \frac{\mu_0}{8\pi}\|H\|_{L^2}^2,$$

$$E_0 = \frac{\rho}{2}\|u_0\|_{L^2}^2 + \frac{\rho\kappa}{2}\|\Omega_0\|_{L^2}^2 + \frac{1}{2}\|M_0\|_{L^2}^2 + \frac{\mu_0}{8\pi}\|H_0\|_{L^2}^2,$$

$$\begin{aligned} E^d(t) \ &= \eta\|\nabla u\|_{L^2}^2 + \eta'\|\nabla\Omega\|_{L^2}^2 + (\eta' + \lambda')\|\nabla \cdot \Omega\|_{L^2}^2 \\ &\quad + \zeta\|\nabla \times u - 2\Omega\|_{L^2}^2 + \frac{1}{\tau}\|M\|_{L^2}^2 + \sigma\|\nabla \times M\|_{L^2}^2 \\ &\quad \sigma(1 + 4\pi\mu_0)\|\nabla \cdot M\|_{L^2}^2 + \frac{1}{4\pi\tau}(\mu_0 + \chi_0 + 4\pi\mu_0\chi_0)\|H\|_{L^2}^2, \end{aligned}$$

and $\|\cdot\|_{L^2}$ denotes the L^2-norm on D, and C_1 and C_2 are positive constants depending only on $\|u_0\|_{L^2}, \|\Omega_0\|_{L^2}, \|M_0\|_{L^2}, \|F\|_{H^1(0,T;L^2(D))}, \sigma, D$, and the physical constants $\rho, \eta, \mu_0, \kappa, \eta', \lambda', \tau$ and χ_0; C_1 and C_2 are independent of ζ and T. Moreover, there exists $p \in W^{-1,\infty}(0,T; L^2(D))$ such that equations (2.11)-(2.13), (3.5), and $\nabla \cdot u = 0$ hold in $\mathcal{D}'(D_T)$.

3.2.2 Amirat and Hamdache 2008 (Shliomis Model)

The next paper of Amirat and Hamdache, "Global weak solutions to a ferrofluid flow model [AH08], proves the existence of global in time weak solutions with finite energy to the Shliomis system with Bloch-Torrey magnetization in the domain D_T. The authors considered system (2.15)-(2.17) on D_T with the boundary conditions (3.1), initial data (3.2), and the boundary condition for H:

$$H \cdot n = -H^{ext} \cdot n \quad \text{on} \quad \partial D \times (0,T),$$

where n is the unit outward normal to the boundary ∂D. The authors also let ϕ be the scalar function satisfying

$$H = \nabla\phi, \qquad \nabla \cdot (\nabla\phi + M) = F \text{ in } D_T, \qquad \frac{\partial\phi}{\partial n} = 0 \text{ on } \partial D \times (0,T), \qquad (3.10)$$

where $F = -\nabla \cdot H^{ext}$ as in the previous paper. The initial data are further assumed to satisfy,

$$u_0 \in \mathcal{U}_0, \quad M_0 \in \mathcal{M}, \quad \text{and} \quad F \in H^1(0,T; L^2(D)) \text{ with } \int_D F dx = 0 \quad \text{in} \quad (0,T),$$

where $\nabla \cdot (\nabla\phi + M) = F$. Finally, denote $H_0 = \nabla\phi_0$, where ϕ_0 is the unique weak solution in $H^1(D)$ of

$$\Delta\phi_0 = -\nabla \cdot M_0 + F_0 \text{ in } D, \quad \frac{\partial\phi_0}{\partial n} = 0 \text{ on } \partial D, \quad \int_D \phi_0 dx = 0. \tag{3.11}$$

Then the authors [AH08] provide their definition of weak solutions

Definition. (u, M, H) is a weak solution with finite energy to (2.15)-(2.17) if $u \in L^\infty(0,T;\mathcal{U}_0) \cap L^2(0,T;\mathcal{U}) \cap \mathcal{C}([0,T];\mathcal{U}_0 \text{ weak})$, $M \in L^\infty(0,T;L^2(D)) \cap L^2(0,T;\mathcal{M}) \cap \mathcal{C}([0,T];L^2(D) \text{ weak})$, $H \in L^\infty(0,T;L^2(D)) \cap L^2(0,T;\mathcal{M})$ and (u, M, H) satisfy the conditions:

(i) the function H is such that $H = \nabla\phi$ where $\phi \in L^\infty(0,T;H^1(D)) \cap L^2(0,T;H^2(D))$ and solves the problem

$$\Delta\phi = -\nabla\cdot M + F \text{ in } D_T,\ \partial\phi/\partial n = 0 \text{ on } \partial D\times(0,T),\ (\phi(t);1) = 0 \text{ in } (0,T);$$

(ii) for every $v \in \mathcal{U}$ and $q \in \mathcal{M}$, we have

$$\frac{d}{dt}(u,v) + \langle (u\cdot\nabla)u, v\rangle + \eta(\nabla u, \nabla v) = \langle S, v\rangle \text{ in } \mathcal{D}'(D_T),$$
$$u(0) = u_0,$$

$$\frac{d}{dt}(M,q) + \langle (u\cdot\nabla)M, q\rangle + \sigma(\nabla\times M, \nabla\times q) + \sigma(\nabla\cdot M, \nabla\cdot q)$$
$$= \langle T, q\rangle \text{ in } \mathcal{D}'(D_T),$$
$$M(0) = M_0,$$

where

$$S = \mu_0(M \cdot \nabla)H + \frac{\mu_0}{2}\nabla \times (M \times H),$$
$$T = \frac{1}{2}\nabla \times (u \times M) - \frac{1}{\tau}(M - \chi_0 H) - \beta M \times (M \times H);$$

(iii) the energy inequality

$$\mathcal{E}(t) + C_1 \int_0^t \mathcal{F}(s)ds \ \le \ \mathcal{E}_0 + C_2 \int_0^t (\|F(s)\|_{L^2}^2 + \|\partial_t F(s)\|_{L^2}^2)ds$$

holds for all $t \in [0, T]$, where C_1 and C_2 are positive constants which depend only on $\eta, \mu_0, \chi_0, \tau, \beta, \sigma$, and D. Here

$$\mathcal{E}_0 := \|u_0\|_{L^2}^2 + \|M_0\|_{L^2}^2 + \mu_0\|H_0\|_{L^2}^2,$$
$$\mathcal{E}(t) := \|u(t)\|_{L^2}^2 + \|M(t)\|_{L^2}^2 + \mu_0\|H(t)\|_{L^2}^2,$$

where $H_0 = \nabla\phi_0$ and ϕ_0 is defined by (3.11). The dissipated energy $\mathcal{F}$ is given by

$$\begin{aligned}\mathcal{F}(t) = 2(\eta\|\nabla u(t)\|_{L^2}^2 + \sigma\|\nabla \times M(t)\|_{L^2}^2 + \sigma(1+\mu_0)\|\nabla \cdot M(t)\|_{L^2}^2) \\ + 2\left(\frac{\mu_0}{\tau}(1+\chi_0)\|H(t)\|_{L^2}^2 + \frac{1}{\tau}\|M(t)\|_{L^2}^2 \right. \\ \left. + \frac{\chi_0}{\tau}\|H(t)\|_{L^2}^2 + \mu_0\beta\|M(t) \times H(t)\|_{L^2}^2\right).\end{aligned}$$

Finally, we can state the main theorem from their paper [AH08]:

Theorem 3.2.2. Let $u_0 \in \mathcal{U}_0, M_0 \in \mathcal{M}$ and $F \in H^1(0, T; L^2(D))$ with $\int_D F(t)dx = 0$ in $(0, T)$. Then there exists a weak solution with finite energy (u, M, H) to (2.15)-(2.17) in the sense of the above definition. Moreover, there exists $p \in W^{-1,\infty}(0, T; L^2(D))$ such that (2.15)-(2.17) hold in $\mathcal{D}'(D_T)$.

3.2.3 Wang and Tan 2010 (Rosensweig Model)

In their paper "Global existence and asymptotic analysis of weak solutions to the equations of Ferrohydrodynamics" [WT10], Wang and Tan consider the nonhomogeneous Rosensweig model with Bloch-Torrey magnetization on D_T. In particular, their model has non-constant density. We copy it here for convenience ([WT10] equations (1.1)-(1.6)):

$$\rho_t + \nabla \cdot (\rho u) = 0, \tag{3.12}$$

$$\nabla \cdot u = 0, \tag{3.13}$$

$$\rho u_t + \rho(u \cdot \nabla)u - (\eta + \zeta)\Delta u + \nabla p = \mu_0 (M \cdot \nabla)H + 2\zeta \nabla \times \Omega, \tag{3.14}$$

$$k\rho\Omega_t + k\rho(u\cdot\nabla)\Omega - \eta'\Delta\Omega - (\eta' + \lambda')\nabla(\nabla\cdot\Omega) = \mu_0 M \times H + 2\zeta(\nabla\times u - 2\Omega), \tag{3.15}$$

$$M_t + (u \cdot \nabla)M - \sigma\Delta M = \Omega \times M - \frac{1}{\tau}(M - \chi_0 H), \tag{3.16}$$

$$\nabla \times H = 0, \quad \nabla \cdot (H + 4\pi M) = -\nabla \cdot H^{ext}. \tag{3.17}$$

They equip the model with the following initial data:

$$(\rho, \rho u, \rho\Omega, M)|_{t=0} = (\rho_0, q_0, \tilde{q}_0, M_0) \text{ in } D, \tag{3.18}$$

and boundary data

$$u = 0,\ \Omega = 0,\ (\nabla\times M)\times n = 0,\ M\cdot n = 0,\ H\cdot n = 0 \text{ on } \partial D\times(0,T). \tag{3.19}$$

For their main results, they assume (as Amirat et al. did) that $\nabla\cdot(H+4\pi M) = F$ and the following compatibility conditions on their initial data:

$$\begin{cases} \rho_0 \geq 0, \quad \rho_0 \in L^\gamma(D), \quad \gamma > \frac{3}{2}, \quad q_0, \tilde{q}_0 = 0 \text{ whenever } \rho_0 = 0, \\ \frac{|q_0|^2}{\rho_0} \in L^1(D), \quad \frac{|\tilde{q}_0|^2}{\rho_0} \in L^1(D), \quad M_0 \in L^2(D). \end{cases} \tag{3.20}$$

Then they give their definition of a weak solution:

Definition. For a given $T > 0$, the functions (ρ, u, Ω, M, H) are a finite-energy weak solution of (3.12)-(3.17), (3.18)-(3.19) if the following conditions are satisfied:

(i) $\rho \geq 0$, $\rho \in L^\infty(0,T; L^\gamma(D))$, $u \in L^2(0,T;\mathcal{U})$, $\Omega \in L^2(0,T; H^1_0(D))$, $M, H \in L^\infty(0,T; L^2(D)) \cap L^2(0,T;\mathcal{M})$;

(ii) Equation (3.12) is satisfied in $\mathcal{D}'(D_T)$. Moreover, provided ρ, u were prolonged to be zero on $\mathbb{R}^3 \setminus D$, (3.12) holds in $\mathcal{D}'(\mathbb{R}^3 \times (0,T))$, i.e.

$$\int_{D_T} \rho\phi_t + \rho u \cdot \nabla\phi dxdt = 0, \quad \text{for any} \quad \phi \in \mathcal{C}_0^\infty(\mathbb{R}^3 \times (0,T));$$

(iii) Equation (3.14) satisfy the weak formulation:

$$\int_{D_T} - \rho u \cdot \phi_t - \rho u \otimes u : \nabla\phi + (\eta + \zeta)\nabla u : \nabla\phi$$
$$- \mu_0(M \cdot \nabla H) \cdot \phi - 2\zeta\nabla \times \Omega \cdot \phi dxdt = 0$$

for any $\phi \in C_0^\infty(D_T)$ with $\nabla \cdot \phi = 0$.

(iv) Equations (3.15) hold in $\mathcal{D}'(D_T)$; i.e. for any $\phi \in C_0^\infty(D_T)$,

$$\int_{D_T} -k\rho\Omega \cdot \phi_t - k\rho\Omega \otimes u : \nabla\phi + \eta'\nabla\Omega : \nabla\phi + (\eta' + \lambda')(\nabla \cdot \Omega)(\nabla \cdot \phi)$$
$$-\mu_0(M \times H) \cdot \phi - 2\zeta(\nabla \times u - 2\Omega) \cdot \phi dxdt = 0;$$

(v) Equations (3.16) hold in the sense that for $\phi \in C^\infty(D_T)$ with $\phi \cdot n = 0$ on $\partial D \times (0, T)$,

$$\int_{D_T} -M \cdot \phi_t - M \otimes u : \nabla\phi + \sigma(\nabla \cdot M)(\nabla \cdot \phi) + \sigma(\nabla \times M)(\nabla \times \phi)$$
$$-(\Omega \times M) \cdot \phi + \frac{1}{\tau}(M - \chi_0 H) \cdot \phi dxdt = 0;$$

(vi) Equation (3.17) (with $F = -\nabla \cdot H^{ext}$) are satisfied pointwise. Moreover, we have $H = \nabla\phi$ where $\phi \in L^\infty(0, T; H^1(D)) \cap L^2(0, T; H^2(D))$ is the unique solution of the problem

$$-\Delta\phi = 4\pi\nabla \cdot M - F, \quad \frac{\partial\phi}{\partial n} = 0 \text{ on } \partial D \times (0, T), \quad \int_D \phi dx = 0. \quad (3.21)$$

The authors remark that (3.12)-(3.16) imply any finite energy weak solution belongs to the class,

$$\rho \in C([0, T]; L^\gamma(D) \text{ weak}), \quad \mathbb{P}(\rho u) \in C([0, T]; L^{\frac{2\gamma}{\gamma+1}}(D) \text{ weak}),$$
$$\rho\Omega \in C([0, T]; L^{\frac{2\gamma}{\gamma+1}}(D) \text{ weak}), \quad M \in C([0, T]; L^2(D) \text{ weak})$$

which they note implies the compatibility condition (3.20) is needed. Above, $\mathbb{P}$ is the projection of L^p onto the subspace of divergence-free functions $\{f \in L^p : \nabla \cdot f = 0\}$.

Next the authors assume the following conditions on F:

$$F \in H^1(0,\infty; L^2(D)), \qquad \int_D F(x,t)dx = 0 \quad \text{for all } \ t \in (0,\infty). \tag{3.22}$$

Therefore by classical regularity theory, $H \in C([0,T]; L^2(D)$ weak) so that the authors can impose the initial condition $H|_{t=0} = H_0 = \nabla\phi_0$, where $\phi_0 \in H^1(D)$ is the weak solution of

$$\begin{gathered} -\Delta\phi_0 = 4\pi\nabla \cdot M_0 - F_0, \\ \frac{\partial\phi_0}{\partial n} \ \text{ on } \partial D, \qquad \int_D \phi_0 dx = 0. \end{gathered}$$

This allows us to state their main result:

Theorem 3.2.3. (*Wang, Tan Theorem 2.1* [WT10]) Assume (3.20), (3.22), and $\gamma > \frac{3}{2}$. Given $T > 0$ arbitrarily, then there exists a global-in-time finite energy weak solution (ρ, u, Ω, M, H) of the system (3.12)-(3.17), (3.18)-(3.19). Moreover, for any $1 \leq p \leq \gamma$,

$$\int_D \rho^p(x,t)dx = \int_D \rho_0^p dx \quad \forall t \in [0,T],$$

and the energy inequality holds in the following sense:

$$\mathcal{E}(t) + C_1 \int_0^t \mathcal{F}(s)ds \leq \mathcal{E}_0 + C_2 \int_0^t (\|F(s)\|^2_{L^2(D)} + \|F_t(s)\|^2_{L^2(D)})ds$$

a.e. $t \in [0,T]$, where C_1, C_2 are positive constants depending only on D, and physical constants $\eta, \mu_0, \sigma, \tau, \chi_0$, but independent of $k, \eta', \lambda', \zeta, T$. The energy functionals $\mathcal{E}(t), \mathcal{F}(t)$ are defined by

$$
\begin{aligned}
\mathcal{E}(t) &= \int_D \left(\frac{1}{2}\rho|u|^2 + \frac{k}{2}\rho|\Omega|^2 + \frac{1}{2}|M|^2 + \frac{\mu_0}{8\pi}|H|^2\right) dx, \\
\mathcal{F}(t) &= \int_D \eta|\nabla u|^2 + \eta'|\nabla\Omega|^2 + (\eta' + \lambda')|\nabla \cdot \Omega|^2 + \zeta|\nabla \times u - 2\Omega|^2 dx \\
&\quad + \int_D \sigma|\nabla \times M|^2 + \sigma(1 + 4\pi\mu_0)|\nabla \cdot M|^2 dx + \int_D \frac{1}{\tau}|M|^2 dx \\
&\quad + \left(\frac{\mu_0\chi_0}{\tau} + \frac{\mu_0 + \chi_0}{4\pi\tau}\right)|H|^2 dx,
\end{aligned}
$$

and the initial energy by $\mathcal{E}_0 = \int_D (\frac{1}{2}\frac{|q_0|^2}{\rho_0} + \frac{k}{2}\frac{|\tilde{q}_0|^2}{\rho_0} + \frac{1}{2}|M_0|^2 + \frac{\mu_0}{8\pi}|H_0|^2)dx$.

The authors prove this theorem via the Galerkin method. They also prove two additional theorems. The first is about the long-time behaviour of weak solutions:

Theorem 3.2.4. (*Wang, Tan Theorem 2.2* [WT10]) For any finite energy weak solution (ρ, u, Ω, M, H) of (3.12)-(3.17) (3.18)-(3.19) from (3.2.3), let $\{t_n\}_{n\in\mathbb{N}}$ be an arbitrary sequence of positive real numbers such that $\lim_{n\to\infty} t_n = \infty$. We define the time shifts

$$(\rho_n, u_n, \Omega_n, M_n, H_n)(x,t) = (\rho, u, \Omega, M, H)(x, t + t_n).$$

Then there exists a subsequence still denoted by $(\rho_n, u_n, \Omega_n, M_n, H_n)(x,t)$ and a stationary state $\rho_s(x)$, with $\int_D \rho_s(x)dx = \int_D \rho_0(x)dx$, such that as $n \to \infty$,

$$\rho_n \to \rho_s \text{ weakly star in } \ L^\infty(0,T; L^\gamma(D)),$$

and

$$(u_n, \Omega_n, M_n, H_n) \to (0,0,0,0) \ \text{ in } \ L^2(0,T; H^1(D)).$$

Lastly, the authors investigate the asymptotic limit of their weak solutions, and in particular whether letting some parameters go to zero will give the Shliomis model. They have the following theorem:

Theorem 3.2.5. (*Wang Tan Theorem 2.3* [WT10]) Setting $k = \eta' = \lambda'$, for every k, let $(\rho^k, u^k, \Omega^k, M^k, H^k)$ be weak solutions of (3.12)-(3.17) (3.18)-(3.19) from Theorem (3.2.3). Then we have, as $k \to 0$, extracting a subsequence if necessary, $(\rho^k, u^k, \Omega^k, M^k, H^k)$ tends to a limit (ρ, u, Ω, M, H), where (ρ, u, M, H) is a weak solution of the system (3.23)-(3.27) below with initial and boundary conditions (3.18), (3.19)

Theorems (3.2.3) and (3.2.5) generalize the work of Amirat et. al [AHM08, AH08] to systems with nonconstant density and show the relationship between the Rosensweig (3.12)-(3.17) and Shliomis (3.23)-(3.27) (presented below) systems with non-constant density, and boundary conditions (3.18), (3.19). The Shliomis system with non-constant density is given by

$$\rho_t + \nabla \cdot (\rho u) = 0, \tag{3.23}$$

$$\nabla \cdot u = 0, \tag{3.24}$$

$$\rho u_t + \rho(u \cdot \nabla)u - \eta \Delta u + \nabla p = \mu_0 (M \cdot \nabla) H + \frac{\mu_0}{2} \nabla \times (M \times H), \tag{3.25}$$

$$M_t + (u \cdot \nabla) u - \sigma \Delta M = \frac{1}{2} \nabla \times (u \times M) - \frac{1}{\tau}(M - \chi_0 H) - \beta M \times (M \times H), \tag{3.26}$$

$$\nabla \times H = 0, \quad \nabla \cdot (H + 4\pi M) = -\nabla \cdot H^{ext}. \tag{3.27}$$

3.3 Strong Solutions

The local existence and asymptotic properties of strong solutions to ferrofluid models are considered in papers by Amirat and Hamdache [AH09, AH10], Tan and Wang [TW10] and Venkatasubramanian and Kaloni [VK02]. In the first two papers, the authors show local existence of strong solutions to the Shliomis system and Rosensweig system without a Bloch-Torrey magnetization term. In the paper by Tan and Wang, a blow-up criterion is derived and a condition for a global strong solution found. Finally, the paper by Venkatasubramanian and Kaloni shows uniqueness and investigates asymptotic stability for the Rosensweig system.

3.3.1 Amirat and Hamdache 2009 (Shliomis Model)

In their paper "Strong solutions to the equations of a ferrofluid flow model" [AH09], Amirat and Hamdache consider the Shliomis system (2.5)-(2.7) on the domain D_T without a Bloch-Torrey regularization term. They prove local existence of strong solutions for the system. First, the authors give (2.5)-(2.7) the initial and boundary data (3.2), (3.3) Again the authors assume that H satisfies (instead of (2.7)),

$$\nabla \times H = 0, \quad \nabla \cdot (H + M) = F \ \text{ in } \ D_T,$$

where again F satisfies $\int_D F dx = 0$ for all $t \in [0, T]$.

Next they assume that

$$
\begin{aligned}
u_0 \in H^2(D) \cap \mathcal{U}, \quad M_0 \in W^{1,q}(D), \\
F \in W^{1,\infty}(0,T; L^q(D)) \text{ for } q > 3, \quad \text{and} \quad \int_D F dx = 0 \text{ in } (0,T).
\end{aligned} \tag{3.28}
$$

Their definition of strong solution was the following:

Definition. [AH09] Let $q > 3$ and $r = \min\{q, 6\}$. We say that (u, M, H) is a strong solution in D_T of (2.5)-(2.7) if the following are satisfied:

(i) $u \in C([0,T]; \mathcal{U} \cap H^2(D)) \cap W^{1,\infty}(0,T; L^2(D)) \cap L^2(0,T; W^{2,r}(D))$, $M, H \in L^\infty(0,T; W^{1,r}(D)) \cap W^{1,\infty}(0,T; L^r(D))$;

(ii) the function $H = \nabla\phi$ where $\phi \in L^\infty(0,T; W^{2,r}(D))$ and solves the problem

$$
\begin{aligned}
& -\Delta\phi = \nabla \cdot M - F \quad \text{in} \quad D_T, \\
& \frac{\partial\phi}{\partial n} = -M \cdot n \quad \text{on} \quad \partial D \times (0,T), \qquad \int_D \phi dx = 0 \quad \text{in} \quad (0,T);
\end{aligned}
$$

(iii) Equation (2.5) and $\nabla \cdot u = 0$ hold in the following sense: for every $v \in \mathcal{U}$,

$$
\begin{aligned}
& \rho \frac{d}{dt} \int_D u \cdot v dx + \rho \int_D (u \cdot \nabla) u \cdot v dx + \eta \int_D \nabla u \cdot \nabla v dx \\
& = \mu_0 \int_D (M \cdot \nabla) H \cdot v dx + \frac{\mu_0}{2} \int_D (\nabla \times (M \times H)) \cdot v dx \quad \text{in} \quad \mathcal{D}'((0,T)), \\
& u|_{t=0} = u_0;
\end{aligned}
$$

(iv) there exists $p \in L^2(0,T; W^{1,r}(D))$ such that equations (2.5)-(2.7) hold a.e. in D_T and the initial condition on M holds in the sense of traces.

Then, their main result was the following:

Theorem 3.3.1. (*Amirat, Hamdache Theorem 1* [AH09]) Under the assumptions (3.28), there is a time $T^* > 0$ such that (2.5)-(2.7) admits a unique strong solution (u, M, H) in D_{T^*} in the sense of the above definition.

3.3.2 Amirat and Hamdache 2010 (Rosensweig Model)

The most recent paper "Unique solvability of equations of motion for ferrofluids" [AH10], of Amirat and Hamdache shows local existence of the unique strong solution to the Rosensweig model (2.1)-(2.4) with no Bloch-Torrey magnetization term on D_T with initial and boundary conditions (3.2), (3.3). As before, the authors suppose that instead of (2.4), the magnetic field satisfies

$$\nabla \times H = 0, \quad \nabla \cdot (H + M) = F \text{ in } D_T,$$

where F is such that $\int_D F dx = 0$ for all $t \in [0, T]$. Then, they make the following assumptions on the initial data and F:

$$\begin{aligned} &u_0 \in H^2(D) \cap \mathcal{U}, \quad \Omega_0 \in H^2(D) \cap H_0^1(D), \\ &M_0 \in W^{1,q}(D), \quad F \in W^{1,\infty}(0, T; L^q(D)), \text{ for } q > 3. \end{aligned} \tag{3.29}$$

This allows the authors to give their definition of strong solution,

Definition. (*Amirat, Hamdache Definition 1* [AH10]) Let $q > 3$ and $r = \min\{q, 6\}$. We say that (u, Ω, M, H) is a strong solution in D_T of (2.1)-(2.3) with the above conditions if the following conditions are satisfied

(i)

$$u \in L^\infty(0,T;H^2(D)\cap\mathcal{U})\cap W^{1,\infty}(0,T;L^2(D))\cap L^2(0,T;W^{2,r}(D)),$$

$$\Omega \in L^\infty(0,T;H^2(D)\cap H_0^1(D))\cap W^{1,\infty}(0,T;L^2(D))\cap L^2(0,T;H^3(D)),$$

$$M,H \in L^\infty(0,T;W^{1,q}(D))\cap W^{1,\infty}(0,T;L^q(D));$$

(ii) the function H is such that $H=\nabla\phi$ where $\phi\in L^\infty(0,T;W^{2,q}(D))$ and solves the problem

$$-\Delta\phi = \nabla\cdot M - F \text{ in } D_T,$$

$$\frac{\partial\phi}{\partial n} = -M\cdot n \text{ on } \partial D\times(0,T), \qquad \int_D \phi dx = 0 \text{ in } (0,T);$$

(iii) (2.1) holds weakly; i.e. for every $v\in\mathcal{U}$,

$$\begin{aligned}\rho\frac{d}{dt}\int_D u\cdot\nabla v dx + \rho\int_D (u\cdot\nabla)u\cdot v dx + (\eta+\zeta)\int_D \nabla u\cdot\nabla v dx \\ = \mu_0\int_D (M\cdot\nabla)H\cdot v dx + 2\zeta\int_D (\nabla\times\Omega)\cdot v dx \text{ in } \mathcal{D}'(D_T),\end{aligned}$$

and $u|_{t=0}=u_0$;

(iv) (2.2) and (2.3) hold a.e. in D_T and the initial conditions on Ω and M hold in the sense of traces.

Next, the authors state their main result:

Theorem 3.3.2. (*Amirat, Hamdache Theorem 1* [AH10]) Under assumptions (3.29), there is a time $T^*>0$ such that the above problem admits a unique strong solution (u,Ω,M,H) in D_{T^*}, in the sense of Definition (3.3.2). Moreover, there exists $p \in L^2(0,T^*;W^{1,r}(D))$, $r = \min\{q,6\}$, such that (2.1), $\nabla \cdot u = 0$ hold a.e. in D_{T^*}.

3.3.3 Tan and Wang 2010 (Shliomis Model)

In their paper "Global analysis for strong solutions to the equations of a ferrofluid flow model" [TW10], Tan and Wang consider the Shliomis model (2.5)-(2.7) without a Bloch-Torrey magnetization term on $\mathbb{R}^3$. They prove local existence of a unique strong solution, find a finite-time blow-up criterion for strong solutions, and prove global existence of the strong solutions under certain smallness assumptions on the initial data and H^{ext}. Since the domain is $\mathbb{R}^3$, the authors don't need to prescribe boundary conditions. They impose the usual initial conditions (3.2) and assume

$$u_0 \in H^2 \ \text{ with } \nabla \cdot u_0 = 0, \quad M_0 \in H^2. \tag{3.30}$$

Additionally they assume the external magnetic field H^{ext} satisfies,

$$H^{ext} \in L^2 \cap L^\infty(0,\infty;H^2), \qquad H^{ext}_t \in L^2 \cap L^\infty(0,\infty;H^1). \tag{3.31}$$

Next they give their definition of strong solution (which differs from the definition given by Amirat and Hamdache in [AH09]):

Definition. We call (u,M,H) a strong solution of (2.5)-(2.7) on $(0,T)$ if (u,M,H) satisfies equations (2.5), (2.6) a.e. in $(0,T) \times \mathbb{R}^3$ for some pressure function p, with regularity

$$u \in C([0,T); H^2) \cap L^2(0,T; \dot{H}^3), \qquad u_t \in C([0,T); L^2) \cap L^2(0,T; H^1),$$
$$M, H \in C([0,T); H^2), \quad M_t, H_t \in C([0,T); H^1).$$

and $H = \nabla\phi$ where ϕ is the unique solution of

$$-\Delta\phi = \nabla \cdot M + \nabla \cdot H^{ext} \text{ in } \mathbb{R}^3, \qquad \phi_0(x) \to 0 \text{ as } |x| \to \infty.$$

Then the authors define a blow-up time for a solution of (2.5)-(2.7):

Definition. (*Tan and Wang Definition 1.2* [TW10]) We shall call the finite number T^* a finite blow-up time of the solution (u, M, H) provided that

$$J(t) < \infty \text{ for } 0 \leq t < T^* \quad \text{and} \quad \lim_{t \to T^*} J(t) = \infty,$$

where the functional $J(t)$ is defined by

$$J(t) := \sup_{0 \leq s \leq t} \{\|u(s)\|_{H^2} + \|u_t(s)\|_{L^2} + \|M(s)\|_{H^2} + \|H(s)\|_{H^2} + \|M_t(s)\|_{H^1}$$
$$+ \|H_t(s)\|_{H^1}\} + \int_0^t (\|u(s)\|^2_{\dot{H}^3} + \|u_t(s)\|^2_{H^1}) ds, \qquad t \geq 0.$$

Next the authors state their main results. The first is about local existence of a unique strong solution to the problem:

Theorem 3.3.3. (*Tan, Wang Theorem 1.1* [TW10]) Under the assumptions (3.30)-(3.31), there is a time T^* such that problem (2.5)-(2.7) admits a unique strong solution (u, M, H) on $(0, T^*)$.

The second is their finite-time blow-up criterion:

Theorem 3.3.4. (*Tan, Wang Theorem 1.2* [TW10] Assume (3.30)-(3.31), and let (u, M, H) be the strong solution of (2.5)-(2.7) on $(0, T^*)$. There exists $K > 0$ such that if T^* is the finite blow-up time of (u, M, H), then

$$\int_0^{T^*} (\|\nabla u(s)\|_{L^2} + \|M(s)\|_{H^2})^K dt = \infty.$$

Third, they find smallness criteria for the initial data so that the solution is global:

Theorem 3.3.5. (*Tan, Wang Theorem 1.3* [TW10]) There exists a small constant $\varepsilon_0 > 0$ depending only on the physical constants such that if the initial data u_0, M_0, H_0 in (3.30) and H^{ext} in (3.31) satisfy

$$\Psi(0) + N^{ext} \leq \varepsilon_0,$$

then there exists a unique global strong solution (u, M, H) of (2.5)-(2.7) on $(0, \infty)$ satisfying the regularity (3.3.3) with $T = \infty$, and

$$\Psi(t) \leq C_1\varepsilon_0 \qquad \forall t \geq 0,$$

where C_1 is a positive constant depending only on some physical constants.

In the above,

$$\Phi(t) := \{\|\nabla u(t)\|_{L^2}^2 + \|u_t(t)\|_{L^2}^2 + \|M(t)\|_{H^2}^2\}, \ \Psi(t) = \Phi(t) + \|u(t)\|_{L^2}^2, \ t \geq 0,$$

$$N^{ext} = \sup_{0 \leq t < \infty} \{\|H^{ext}(t)\|_{H^2}^2 + \|H_t^{ext}\|_{H^1}^2\} + \int_0^{\infty} (\|H^{ext}(t)\|_{H^2}^2 + \|H_t^{ext}(t)\|_{H^1}^2) dt.$$

Finally, the authors investigate how fast their solution decays:

Theorem 3.3.6. (*Tan, Wang Theorem 1.4* [TW10]) Suppose (u, M, H) is the global unique strong solution of (2.5)-(2.7) on $(0, \infty)$ obtained in Theorem (3.3.5) and assume that H^{ext} satisfies

$$\|H^{ext}(t)\|_{L^2} \leq C(1+t)^{-\alpha}, \qquad \|\nabla \cdot H^{ext}(t)\|_{L^2} \leq C(1+t)^{-\beta},$$

with $\alpha \geq 0$, $\alpha + \beta > 1$. Then we have

$$\|M(t)\|_{L^2}^2 + \|H(t)\|_{L^2} \leq C_0(1+t)^{-\alpha},$$

and if in addition $u_0 \in L^p$ with $p \in [1, 2]$, then

$$\|u(t)\|_{L^2} \leq C_0(1+t)^{\alpha_0},$$

where $\alpha_0 = \min\{\frac{3}{2}(\frac{1}{p} - \frac{1}{2}), 2\alpha\}$.

3.3.4 Venkatasubramanian and Kaloni 2002 (Rosensweig Model)

In their paper "Stability and uniqueness of magnetic fluid motions" [VK02], the authors investigate the Rosensweig model (2.1)-(2.4) on a bounded (possibly time-dependent) domain $D = D(t)$. They impose the additional condition $B = \mu_0(M + H)$, which along with $\nabla \times H = 0$ and $\nabla \cdot B = 0$ (i.e. $\nabla \cdot H^{ext} = 0$) constitutes the magnetostatic limit of Maxwell's equations. Also, instead of the hydrodynamic pressure p, the authors consider a modified pressure which we will ignore for the subsequent discussion (for details see [VK02]). The authors impose the following boundary conditions on the model:

$$
\begin{aligned}
u = u_0 &\quad \text{on solid boundaries,}\\
\Omega = \Omega_0 &\quad \text{on solid boundaries,}\\
\hat{n} \cdot (B^0 - B^i) = 0 &\quad \hat{n} \text{ is the unit normal and the superscripts `i' and `o'}\\
\hat{n} \times (H^0 - H^i) = 0 &\quad \text{denote the inner and outer sides of the boundary.}
\end{aligned}
$$

Next, the authors prove results for "mechanically isolated systems" (systems where the power supply is negligible):

Theorem 3.3.7. (*Venkatasubramanian, Kanoli Theorem 3.1* [VK02]) The motion of a magnetic fluid initially at rest and which is mechanically isolated for all $t \geq 0$ is given by

$$
\begin{aligned}
\frac{\rho}{2}\frac{d}{dt}\|u\|^2_{L^2(D)} + \frac{\rho\kappa}{2}\frac{d}{dt}\|\Omega\|^2_{L^2(D)} + \frac{\mu_0}{2}\frac{d}{dt}\|H\|_{L^2(\mathbb{R}^3)} + \eta'\|\nabla \times \Omega\|^2_{L^2(D)}\\
+(\lambda' + \eta')\|\nabla \cdot \Omega\|^2_{L^2(D)} + (\eta + \zeta)\|\nabla \times u\|^2_{L^2(D)} + \zeta\|\nabla \times u - 2\Omega\|^2_{L^2(D)}\\
+\frac{\mu_0(1+\chi)}{\tau}\|H\|^2_{L^2(\mathbb{R}^3)} = 0.
\end{aligned}
$$

For their next result, they define a measure K of the kinetic energy of translation

$$
K(t) := \frac{1}{2}\rho\|u\|^2_{L^2(D)} + \frac{1}{2}\rho\kappa\|\Omega\|^2_{L^2(D)} + \frac{\mu_0}{2}\|H\|^2_{L^2(\mathbb{R}^3)}.
$$

Then they prove the following corollary to their Theorem,

Corollary 3.3.1. (*Venkatasubramanian Kaloni Corollary 3.2* [VK02]) A magnetic fluid flow which is mechanically isolated for all $t \geq 0$ satisfies

$$\dot{K}(t) \leq 0 \quad \forall t \in [0, \infty).$$

Next, they improve this result by arriving at the following stronger inequality:

Theorem 3.3.8. (*Venkatasubramanian, Kaloni Theorem 3.3* [VK02]) Let the flow of a magnetic fluid be mechanically isolated for all $t \geq 0$. Then there exists $\lambda \in \mathbb{R}^+$ such that

$$\dot{K}(t) + \lambda K(t) \leq 0,$$

where λ is the minimum of some constants. (See [VK02] for details)

This gives the asymptotic stability of mechanically isolated equilibrium flows of the Rosensweig model. Next, the authors consider the difference of two flows and check if the energy functional of the difference approaches zero as $t \to \infty$. This would prove stability of the flow. They show that under certain conditions on the Reynolds numbers, the energy of the difference of flows decays exponentially in time, and if the flows additionally satisfy the same boundary conditions, they must be identical (i.e. uniqueness). For more details see [VK02].

3.4 Numerical Analysis

3.4.1 Nochetto, Salgado, and Tomas 2015 (Rosensweig Model)

In their very recent paper "The equations of ferrohydrodynamics: modelling and numerical methods" [NST15], the authors devise a numerical scheme for the Rosensweig model on the domain D_T and investigate potential applications. They also provide a discussion of various physical considerations for the model.

First of all, the authors note that the magnetic diffusion $\sigma\Delta M$ in (2.13), (2.16) introduced by Amirat, Hamdache and Murat in [AHM08], [AH08] to prove the existence of global weak solutions, and which we will use in our proof of local well-posedness and derivation of Prodi-Serrin conditions may not be physical (see reference in [NST15]). In practice, σ is small enough to make the diffusion term negligible. However, introducing this term allows for additional (numerical) control via boundary conditions for the vector Laplacian. Next, the authors mention a disparity between a common physical assumption for the model, and the assumption made in [AH10], [AHM08] which prove existence results. The authors define H_a to be an applied magnetizing field, which induces the magnetization field M and a demagnetizing field H_d. Then $H = H_a + H_d$ is the effective magnetizing field in the model. The authors mention that physically it is common to assume that the magnetic susceptibility $\chi_0 \ll 1$ and consequently set $H = H_a$. Indeed since $H = H_d + H_a = \nabla\psi + \nabla\phi$, and the estimate

$$\|\nabla\psi\|_{L^2} \leq \|M\|_{L^2} \tag{3.32}$$

holds (see [NST15] §2.2, §5.1), the authors conclude $H \approx H_a$ for $\|\cdot\|_{L^2}$-small M. Near equlibrium, $M \approx \chi_0 H$ (see [NST15] §2), so that if $\chi_0 \ll 1$, $\|M\|_{L^2}$ is small, and $H \approx H_a$. Conversely, in [AH10] and [AHM08], the authors define the effective magnetizing field H in such a way that $H = H_d$, which is equivalent to considering the unforced case (see [NST15] Remark 2.2).

Now we will discuss initial and boundary conditions for the model. As in the other papers, the authors impose the initial conditions (3.2), and boundary (no-slip and no-spin) conditions

$$u|_{\partial D\times(0,T)} = 0, \quad \Omega|_{\partial D\times(0,T)} = 0. \tag{3.33}$$

for the momentum and angular momentum equations. On the other hand, the authors discuss a variety of possible boundary conditions for M. They note that if $\sigma = 0$, no boundary conditions are needed for M (in agreement with [AH10]). On the other hand, if $\sigma > 0$, the authors suggest four different types of boundary conditions:

- Magnetic boundary conditions (considered in [AHM08] with $g = r = 0$):

  $$M \cdot n = g, \quad (\nabla\times M)\times n = r \text{ on } \partial D\times(0,T),$$

  where g and r are the boundary data.

- Electric boundary conditions:

$$\nabla \cdot M = q, \quad M \times n = y \;\text{ on } \partial D \times (0,T).$$

- Robin-like boundary conditions:

$$\begin{aligned} \nabla \times (M \times n) + \gamma_1(M - (M \cdot n)n - y) &= r \text{ on } \partial D \times (0,T), \\ \nabla \cdot M + \gamma_2(M \cdot n - g) &= q \text{ on } \partial D \times (0,T). \end{aligned}$$

- Natural boundary conditions (considered in [WT10]):

$$\nabla \times (M \times n) = 0, \quad \nabla \cdot M = 0 \;\text{ on } \partial D \times (0,T).$$

In the current paper, the authors consider the following three cases of boundary conditions:

(i) $\sigma = 0$ with no boundary conditions for M;

(ii) $\sigma > 0$ with natural boundary conditions (same as in [WT10]);

(iii) $\sigma > 0$ with a variant of the Robin-like boundary conditions.

See [NST15] for their analysis. Next the authors define terms for a formal energy estimate.

They denote (switched to the notation of this thesis where applicable)

$$
\begin{aligned}
\mathcal{E}(u,\Omega,M,H;s) &:= \frac{1}{2}(\|u(s)\|_{L^2}^2 + \rho\kappa\|\Omega(s)\|_{L^2}^2 + \mu_0\|M(s)\|_{L^2}^2 + \mu_0\|H(s)\|_{L^2}^2),\\
\mathcal{D}(u,\Omega,M,H;s) &:= \eta\|\nabla u(s)\|_{L^2}^2 + \eta'\|\nabla\Omega(s)\|_{L^2}^2 + \sigma\mu_0\|\nabla\times M(s)\|_{L^2}^2\\
&\quad + \sigma\mu_0\|\nabla\cdot M(s)\|_{L^2}^2 + \sigma\mu_0\|\nabla\cdot H(s)\|_{L^2}^2 + (\eta'+\lambda')\|\nabla\cdot\Omega(s)\|_{L^2}^2\\
&\quad + \zeta\|(\nabla\times u - 2\Omega)(s)\|_{L^2}^2 + \frac{\mu_0}{\tau}\|M(s)\|_{L^2}^2\\
&\quad + \frac{\mu_0}{2\tau}\left(\frac{1}{2}+3\chi_0\right)\|H(s)\|_{L^2}^2,\\
\mathcal{F}(H_a;s) &:= \tau\mu_0\|\partial_t H_a(s)\|_{L^2}^2 + \frac{\mu_0}{2\tau}(1+\chi_0)\|H_a(s)\|_{L^2}^2,
\end{aligned}
$$

and then prove the following:

Proposition 3.4.1. (*Nochetto et al. Proposition 3.1* [NST15]) A solution (u,p,Ω,M,H) of (2.11)-(2.14), (3.2), (3.33) satisfying

$$-\Delta\phi = \nabla\cdot M \ \text{ in } \ D, \quad \frac{\partial\phi}{\partial n} = (H_a - M)\cdot n \ \text{ on } \ \partial D$$

has the energy estimate

$$\mathcal{E}(u,\Omega,M,H;T) + \int_0^T \mathcal{D}(u,\Omega,M,H;s)ds \le \int_0^T \mathcal{F}(H_a;s)ds + \mathcal{E}(u,\Omega,M,H;0),$$

so that the system is energy-stable. The authors remark that the system is also energy-stable under their modified Robin-like boundary conditions. Most of the rest of the paper is devoted to deriving a numerical scheme for the Rosensweig model. Because numerics are far away from the topic of this thesis, we refer the reader to the paper [NST15] for details on the scheme, its stability, etc.

However we remark that at the end of their paper, the authors perform a variety of numerical experiments for different physical situations (we list them here for interested readers):

- ◦ Spinning magnet (see §7.1 of [NST15])
- ◦ Ferrofluid pumping (see §7.2 of [NST15])
- ◦ Ferromagnetic stirring of a passive scalar (see §7.3 of [NST15])

Chapter 4

Background and Preliminary Lemmas

In this chapter, we begin by listing some mathematical results we will need for the proofs in subsequent chapters. Then we prove some lemmas which will clarify the presentation of our main results. We omit very standard definitions and results from the literature of applied mathematics (e.g. Sobolev spaces).

4.1 Mathematical Background

The purpose of this section is to introduce key definitions and results from the literature that we will need in subsequent chapters. In particular, we present some background results needed for the proof of local well-posedness of classical solutions and for our derivation of Prodi-Serrin type conditions.

In our proof of local well-posedness, we will use the technique of mollification, which requires the definition of a standard family of mollifiers:

Definition. (*Vicol and Bedrossian Definition D.6* [VB15]) Let $\psi : \mathbb{R}^3 \to \mathbb{R}$ be a smooth, non-increasing radial function such that

$$\int_{\mathbb{R}^3} \psi(x)dx = 1,$$

and for any $n \geq 0$,

$$\int_{\mathbb{R}^3} |x|^n \psi(x)dx < \infty$$

(ψ has finite moments). For any $\varepsilon > 0$, define

$$\psi_\varepsilon(x) := \frac{1}{\varepsilon^3}\psi\left(\frac{x}{\varepsilon}\right). \tag{4.1}$$

Then $\{\psi_\varepsilon\}_{\varepsilon>0}$ is a standard family of mollifiers. Moreover, it is common to define the corresponding mollification operator

$$\mathcal{J}_\varepsilon f(x) := (\psi_\varepsilon * f)(x), \tag{4.2}$$

for any $f \in L^1_{\text{loc}}(\mathbb{R}^3)$.

We will also need the following properties of mollifiers, copied from Majda and Bertozzi's book for functions in $\mathbb{R}^3$:

Lemma 4.1.1. Properties of Mollifiers (*Majda, Bertozzi Lemma 3.5* [MB02]). Let $\mathcal{J}_\varepsilon$ be defined by (4.2). Then $\mathcal{J}_\varepsilon$ is a C^∞ function and

(i) for all $v \in C^0(\mathbb{R}^3)$, $\mathcal{J}_\varepsilon v \to v$ uniformly on any compact subset Ω in $\mathbb{R}^3$,

and

$$\|\mathcal{J}_\varepsilon v\|_{L^\infty} \leq \|v\|_{L^\infty}; \tag{4.3}$$

(ii) Mollifiers commute with distribution derivatives,

$$D^\alpha \mathcal{J}_\varepsilon v = \mathcal{J}_\varepsilon D^\alpha v, \qquad \forall |\alpha| \leq k, \qquad v \in H^k(\mathbb{R}^3); \tag{4.4}$$

(iii) For all $u \in L^p(\mathbb{R}^3),\ v \in L^q(\mathbb{R}^3),\ \frac{1}{p} + \frac{1}{q} = 1$,

$$\int_{\mathbb{R}^3} (\mathcal{J}_\varepsilon u) v dx = \int_{\mathbb{R}^3} u(\mathcal{J}_\varepsilon v) dx; \tag{4.5}$$

(iv) For all $v \in H^k(\mathbb{R}^3)$, $\mathcal{J}_\varepsilon v$ converges to v in H^k and the rate of convergence in the H^{k-1} norm is linear in ε:

$$\lim_{\varepsilon \searrow 0} \|\mathcal{J}_\varepsilon v - v\|_{H^k} = 0, \tag{4.6}$$

$$\|\mathcal{J}_\varepsilon v - v\|_{H^{k-1}} \leq C\varepsilon \|v\|_{H^k}; \tag{4.7}$$

(v) For all $v \in H^m(\mathbb{R}^3), k \in \mathbb{Z}^+ \cup \{0\}$, and $\varepsilon > 0$,

$$\|\mathcal{J}_\varepsilon v\|_{H^{m+k}} \leq \frac{C_{m,k}}{\varepsilon^k} \|v\|_{H^m}, \tag{4.8}$$

$$\|\mathcal{J}_\varepsilon D^k v\|_{L^\infty} \leq \frac{C_k}{\varepsilon^{3/2+k}} \|v\|_{L^2}. \tag{4.9}$$

To show local existence of approximate solutions, we will need the Picard theorem on a Banach space:

Theorem 4.1.1. Picard theorem on a Banach space (*Majda and Bertozzi Theorem 3.1* [MB02]). Let $O \subset B$ be an open subset of a Banach space B and let $F : O \to B$ be a mapping that satisfies the following parameters:

(i) $F(X)$ maps O to B;

(ii) F is locally Lipschitz continuous, i.e., for any $X \in O$ there exists $L > 0$ and an open neighbourhood $U_X \subset O$ of X such that

$$\|F(\tilde{X}) - F(\hat{X})\|_B \leq L\|\tilde{X} - \hat{X}\|_B \ \text{ for all } \ \tilde{X}, \hat{X} \in U_X. \tag{4.10}$$

Then for any $X_0 \in O$, there exists a time T such that the ordinary differential equation

$$\frac{dX}{dt} = F(X), \qquad X_{t=0} = X_0 \in O, \tag{4.11}$$

has a unique (local) solution $X \in C^1[(-T, T); O]$.

In particular, we will take O to be the whole space $H^k(\mathbb{R}^3)$.

To extend the solutions whose existence is guaranteed by the Picard theorem globally in time, we will need the following theorem about continuing ordinary differential equations on a Banach space:

Theorem 4.1.2. Continuation of an Autonomous ODE on a Banach Space (*Majda and Bertozzi Theorem 3.3* [MB02]). Let $O \subset B$ be an open subset of a Banach space B, and let $F : O \to B$ be a locally Lipschitz continuous operator. Then the unique solution $X \in C^1([0, T); O)$ to the autonomous ODE,

$$\frac{dX}{dt} = F(X), \qquad X_{t=0} = X_0 \in O, \tag{4.12}$$

either exists globally in time, or $T < \infty$ and $X(t)$ leaves the open set O as $t \nearrow T$.

Thus, if our solution remains bounded for all time in H^k, it must exist globally in time.

Afterwards, we need the Aubin-Lions compactness theorem:

Theorem 4.1.3. (*Vicol and Bedrossian Theorem C.6* [VB15]) Let $X \subset Y \subset Z$ be separable, reflexive Banach spaces, such that the embedding $X \subset Y$ is compact, and the embedding $Y \subset Z$ is continuous. Let $T > 0$, and assume that we have a sequence of functions $\{u_n\}_{n \geq 1}$ such that there is an $M > 0$ with $\|u_n(t)\|_X \leq M$ and that $\{u_n\}$ is uniformly equicontinuous on $[0, T]$ with values in Z. Then the sequencce $\{u_n\}$ is pre compact in $C([0, T]; Y)$.

This will give the existence of a limit solution in L^2, and convergence (of a subsequence) to this solution in intermediate norms.

In both the proof of well-posedness and the derivation of Prodi-Serrin conditions, we will use the following form of Grönwall's inequality:

Lemma 4.1.2. (Robinson Lemma 2.8 [Rob01]) Let $x(t) \in \mathbb{R}$ satisfy the differential inequality

$$\frac{d}{dt_+} x(t) \leq g(t)x(t) + h(t).$$

Then

$$x(t) \leq x(0)\exp[G(t)] + \int_0^t \exp[G(t) - G(s)]h(s)ds,$$

where

$$G(t) = \int_0^t g(r)dr.$$

In particular, most times we will have $h(t) = 0$. The exception is for our Prodi-Serrin conditions when $\nabla \cdot H^{ext} \neq 0$.

4.2 Preliminary Lemmas

In this section we present some lemmas that will be used in the proof of our main results later.

4.2.1 Lemmas to Simplify Computation

We begin by proving a generalized version of Young's inequality which will shorten our argument.

Lemma 4.2.1. Let f_i, $i = 1, \ldots, m$ be nonnegative real numbers and a_i be a corresponding family of positive constants such that $\sum_{i=1}^{m} a_i = b$. Then,

$$\prod_{i=1}^{m} f_i^{a_i} \lesssim \sum_{i=1}^{m} f_i^{b}.$$

Proof. We proceed by induction. Suppose $a_i \neq 0$ only for $i = 1, 2$, and $a_1 + a_2 = b$. Then,

$$f_1^{a_1} f_2^{a_2} \lesssim f_1^{(\frac{b}{a_1})a_1} + f_2^{(\frac{b}{a_2})a_2} = f_1^b + f_2^b$$

by Young's inequality, where $\frac{a_1}{b} + \frac{a_2}{b} = \frac{a_1+a_2}{b} = 1$. Assume the result holds when $m = N$. Then, if $m = N+1$ and $\sum_{i\in\{1,\ldots,N+1\}} a_i = b$, we have

$$\prod_{i=1}^{N+1} f_i^{a_i} \lesssim \prod_{i=1}^{N} f_i^{a_i(\frac{b}{b-a_{N+1}})} + a_{N+1}^b = \prod_{i=1}^{N} f_i^{a_i(\frac{b}{\sum_{i=1}^{N} a_i})} + f_{N+1}^b.$$

Clearly $\sum_{i=1}^{N} a_i \left(\frac{b}{\sum_{i=1}^{N} a_i}\right) = b$, so the induction assumption gives

$$\prod_{i=1}^{N+1} f_i^{a_i} \lesssim \sum_{i=1}^{N+1} f_i^{b}.$$

Therefore, by induction, the result holds for all $m \in \mathbb{N}$ (which proves the lemma).

$\square$

To prove the limit of the approximate solutions solves the Shliomis or Rosensweig system, we prove a trivial lemma which allows us to rewrite the difference of bilinear terms in a desirable way.

Lemma 4.2.2. Let $\star : (\cdot, \cdot) \to \cdot \star \cdot$ denote a bilinear operation. Then for any sequences $\{a^\varepsilon\}_{\varepsilon>0}, \{b^\varepsilon\}_{\varepsilon>0}$ with limits a, b, respectively, the following holds:

$$\begin{aligned} a \star b - \mathcal{J}_\varepsilon[\mathcal{J}_\varepsilon a^\varepsilon \star \mathcal{J}_\varepsilon b^\varepsilon] &= (a - a^\varepsilon) \star b + a^\varepsilon \star (b - b^\varepsilon) + (1 - \mathcal{J}_\varepsilon)(a^\varepsilon \star b^\varepsilon) \\ &\quad + \mathcal{J}_\varepsilon[(1 - \mathcal{J}_\varepsilon)a^\varepsilon \star b^\varepsilon] + \mathcal{J}_\varepsilon[\mathcal{J}_\varepsilon a^\varepsilon \star (1 - \mathcal{J}_\varepsilon)b^\varepsilon]. \end{aligned} \tag{4.13}$$

Proof. We perform a simple computation. Denote the terms on the right-hand side of equation (4.13) by A_1, A_2, A_3, A_4, and A_5, respectively. Then

$$\begin{aligned} a \star b - \mathcal{J}_\varepsilon[\mathcal{J}_\varepsilon a^\varepsilon \star \mathcal{J}_\varepsilon b^\varepsilon] &= (a - a^\varepsilon) \star b^\varepsilon + (a^\varepsilon \star b) - \mathcal{J}_\varepsilon[\mathcal{J}_\varepsilon a^\varepsilon \star \mathcal{J}_\varepsilon b^\varepsilon] \\ &= A_1 + a^\varepsilon \star (b - b^\varepsilon) + a^\varepsilon \star b^\varepsilon - \mathcal{J}_\varepsilon[\mathcal{J}_\varepsilon a^\varepsilon \star \mathcal{J}_\varepsilon b^\varepsilon] \\ &= A_1 + A_2 + (1 - \mathcal{J}_\varepsilon)(a^\varepsilon \star b^\varepsilon) + \mathcal{J}_\varepsilon[a^\varepsilon \star b^\varepsilon - \mathcal{J}_\varepsilon a^\varepsilon \star \mathcal{J}_\varepsilon b^\varepsilon] \\ &= A_1 + A_2 + A_3 + \mathcal{J}_\varepsilon[(1 - \mathcal{J}_\varepsilon)a^\varepsilon \star b^\varepsilon + \mathcal{J}_\varepsilon a^\varepsilon \star (1 - \mathcal{J}_\varepsilon)b^\varepsilon] \\ &= A_1 + A_2 + A_3 + A_4 + A_5. \end{aligned} \tag{4.14}$$

$\square$

4.2.2 Integration By Parts

Lemma 4.2.3. (Integration by Parts) Let ∂^k and ∂^l be two particular spatial partial derivatives of order k and l respectively, where $k, l \geq 1$ are integers. Further let $f \in H^k(\mathbb{R}^3)$ and $g \in H^l(\mathbb{R}^3)$. Then

$$\int_{\mathbb{R}^3} \partial^{k-1} f \partial^l g dx = -\int_{\mathbb{R}^3} \partial^k f \partial^{l-1} g dx \tag{4.15}$$

where (abusing notation) $\partial^{k-1}, \partial^{l-1}$ denote particular derivatives of order $k-1$, $l-1$, respectively. In particular, we can integrate by parts with respect to x_1, x_2, or x_3 in such a way that when we sum over partial derivatives of order k later, we get the H^{k-1} norm of the function.

Proof. This is a consequence of the density of $C_0^\infty(\mathbb{R}^3)$ in $H^m(\mathbb{R}^3)$ (say $m \in \mathbb{N}$). Let $f^\varepsilon, g^\varepsilon \in C_0^\infty(\mathbb{R}^3)$ be sequences of functions which converge to f in H^k and g in H^l, respectively. Then (4.15) holds for $f^\varepsilon, g^\varepsilon$ and

$$\begin{aligned}
\left| \int_{\mathbb{R}^3} \partial^{k-1} f \partial^l g dx - \int_{\mathbb{R}^3} \partial^{k-1} f^\varepsilon \partial^l g^\varepsilon dx \right| &= \left| \int_{\mathbb{R}^3} (\partial^{k-1}(f - f^\varepsilon)) \partial^l g + \partial^{k-1} f^\varepsilon (\partial^l (g - g^\varepsilon)) dx \right| \\
&\leq \|f - f^\varepsilon\|_{H^{k-1}} \|g\|_{H^l} + \|f^\varepsilon\|_{H^{k-1}} \|g - g^\varepsilon\|_{H^l} \\
&\to 0 \text{ and similarly,}
\end{aligned}$$

$$\begin{aligned}
\left| \int_{\mathbb{R}^3} \partial^k f \partial^{l-1} g dx - \int_{\mathbb{R}^3} \partial^k f^\varepsilon \partial^{l-1} g^\varepsilon dx \right| &= \left| \int_{\mathbb{R}^3} (\partial^k (f - f^\varepsilon)) \partial^{l-1} g + \partial^k f^\varepsilon (\partial^{l-1} (g - g^\varepsilon)) dx \right| \\
&\leq \|f - f^\varepsilon\|_{H^k} \|g\|_{H^{l-1}} + \|f^\varepsilon\|_{H^k} \|g - g^\varepsilon\|_{H^{l-1}} \\
&\to 0,
\end{aligned}$$

since $f \in H^k(\mathbb{R}^3) \subseteq H^{k-1}(\mathbb{R}^3)$ and $g \in H^l(\mathbb{R}^3) \subseteq H^{l-1}(\mathbb{R}^3)$ and by the density of $C_0^\infty(\mathbb{R}^3)$ in these spaces.

$\square$

Lemma 4.2.4. Let $k \in \mathbb{N}$ be arbitrary, and let ∂^k be a particular partial derivative of order k. Suppose that $\nabla \cdot (M + H) = \nabla \cdot H^{ext} = 0$, $\nabla \times H = 0$, and that $M, H \in H^k(\mathbb{R}^3)$. Then,

$$\int_{\mathbb{R}^3} \partial^k M \cdot \partial^k H dx = -\int_{\mathbb{R}^3} |\partial^k H|^2 dx.$$

Proof. Let $M^\varepsilon, H^\varepsilon \in C_0^\infty(\mathbb{R}^3)$ converge to M, H respectively in $H^k(\mathbb{R}^3)$. The condition $\nabla \cdot H^{ext} = 0$ gives $\nabla \cdot M^\varepsilon = -\nabla \cdot H^\varepsilon$ (in $H^k(\mathbb{R}^3)$). Also, since $\nabla \times H^\varepsilon = 0$, we have $H^\varepsilon = \nabla \phi^\varepsilon$, where ϕ^ε is some potential function. We have:

$$\begin{aligned}
\int_{\mathbb{R}^3} \partial^k M^\varepsilon \cdot \partial^k H^\varepsilon dx = \int_{\mathbb{R}^3} \partial^k M^\varepsilon \cdot \partial^k \nabla \phi^\varepsilon dx &= -\int_{\mathbb{R}^3} \partial^k (\nabla \cdot M^\varepsilon) \partial^k \phi^\varepsilon dx \\
= \int_{\mathbb{R}^3} \partial^k (\nabla \cdot H^\varepsilon) \partial^k \phi^\varepsilon dx &= -\int_{\mathbb{R}^3} \partial^k H^\varepsilon \partial^k (\nabla \phi^\varepsilon) dx \\
= -\int_{\mathbb{R}^3} |\partial^k H^\varepsilon|^2 dx,
\end{aligned}$$

where we have used that partial derivatives commute and that boundary terms vanish. Finally, similar to Lemma 4.2.3, it is easy to check

$$\begin{aligned}
&\left| \int_{\mathbb{R}^3} \partial^k M \cdot \partial^k H dx - \int_{\mathbb{R}^3} \partial^k M^\varepsilon \partial^k H^\varepsilon dx \right| \\
&\qquad \le \|M - M^\varepsilon\|_{H^k} \|H\|_{H^k} + \|M^\varepsilon\|_{H^k} \|H - H^\varepsilon\|_{H^k} \\
&\qquad \to 0,
\end{aligned}$$

and

$$\left| \int_{\mathbb{R}^3} \partial^k H \cdot \partial^k H dx - \int_{\mathbb{R}^3} \partial^k H^\varepsilon \partial^k H^\varepsilon dx \right|$$
$$\leq \|H - H^\varepsilon\|_{H^k} \|H\|_{H^k} + \|H^\varepsilon\|_{H^k} \|H - H^\varepsilon\|_{H^k}$$
$$\to 0.$$

□

Remark. If we instead assume $\nabla \cdot H^{ext} \neq 0$ and $H^{ext} \in H^k(\mathbb{R}^3)$ in Lemma 4.2.4, the result becomes

$$\int_{\mathbb{R}^3} \partial^k M \cdot \partial^k H dx = -\int_{\mathbb{R}^3} |\partial^k H|^2 dx - \int_{\mathbb{R}^3} \partial^k H \cdot \partial^k H^{ext} dx.$$

One can further show,

Lemma 4.2.5. Let $k \in \mathbb{N}$ be arbitrary, and let ∂^k be a particular partial derivative of order k. Suppose $\nabla \cdot (H + M) = \nabla \cdot H^{ext} = 0$, $\nabla \times H = 0$, and that $u, M, H \in H^k(\mathbb{R}^3)$. Then,

$$\int_{\mathbb{R}^3} \frac{\partial(\partial^k M)}{\partial t} \cdot \partial^k H dx = -\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3} |\partial^k H|^2 dx. \tag{4.16}$$

Proof. Modifying the argument in the previous proof in an obvious way, we get

$$\int_{\mathbb{R}^3} \frac{\partial(\partial^k M)}{\partial t} \cdot \partial^k H dx = -\int_{\mathbb{R}^3} \frac{\partial(\partial^k H)}{\partial t} \cdot \partial^k H dx$$
$$= -\int_{\mathbb{R}^3} \frac{1}{2} \partial_t |\partial^k H|^2 dx.$$

By the well-known transport theorem,

$$\frac{d}{dt}\int_{\Omega(t)} |\partial^k H|^2 dx = \int_{\Omega(t)} \partial_t |\partial^k H|^2 + \nabla\cdot(u|\partial^k H|^2)dx$$

Since

$$\nabla\cdot(u|\partial^k H|^2) = |\partial^k H|^2 \underbrace{\nabla\cdot u}_{=0} + u\cdot\nabla|\partial^k H|^2,$$

and the second term vanishes upon integrating by parts (assuming $u, H \in H^k(\mathbb{R}^3)$ is enough), we have

$$\frac{d}{dt}\int_{\Omega(t)} |\partial^k H|^2 dx = \int_{\Omega(t)} \partial_t |\partial^k H|^2 dx,$$

which gives (4.16) when taking $\Omega(t) = \mathbb{R}^3$.

$\square$

Remark. If $\nabla\cdot H^{ext} \neq 0$ and $H^{ext} \in H^k(\mathbb{R}^3)$ in Lemma 4.2.5, then modifying the proof in an obvious way gives,

$$\int_{\mathbb{R}^3} \frac{\partial(\partial^k M)}{\partial t}\cdot\partial^k H dx = -\frac{1}{2}\frac{d}{dt}\int_{R^3} |\partial^k H|^2 dx - \int_{\mathbb{R}^3} \partial^k H \partial_t \partial^k H^{ext} dx.$$

4.2.3 Ning Ju's Lemma

In [Ju06], Ning Ju proposes the following lemma:

Lemma 4.2.6. (*Ning Ju Lemma 2.1* [Ju06]) Suppose that $s > 0$ and $p \in (1, \infty)$. If $f, g \in \mathcal{S}$, the Schwartz class (for a definition, see for instance [Duo01]), then

$$\|\Lambda^s(fg) - f\Lambda^s g\|_{L^p} \leq C(\|\nabla f\|_{L^{p_1}}\|g\|_{H^{s-1,p_2}} + \|f\|_{H^{s,p_3}}\|g\|_{L^{p_4}}$$

$$\|\Lambda^s(fg)\|_{L^p} \leq C(\|f\|_{L^{p_1}}\|g\|_{H^{s,p_2}} + \|f\|_{H^{s,p_3}}\|g\|_{L^{p_4}}$$

with $p_2, p_3 \in (1, \infty)$ such that

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} + \frac{1}{p_4}$$

We will replace the differential operator $\Lambda^s = (-\Delta)^{s/2}$ above with D, and the Bessel potential space norm $\|\cdot\|_{H^{s,p}}$ with $\|D^s \cdot\|_{L^p}$ via the Gagliardo-Nirenberg inequality. We will prove a simplified version (for integer derivatives) which suffices for our analysis. To begin, we prove the following lemma:

Lemma 4.2.7. Let $p, p_1, p_2, p_3, p_4 \in (1, \infty)$ be such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_4} = \frac{1}{p_2} + \frac{1}{p_3}$ and let $\alpha, \beta \in \mathbb{Z}^n_{\geq 0}$ be multi-indices satisfying $|\alpha| + |\beta| = k$. Then for $f, g : \mathbb{R}^n \to \mathbb{R}^n$, there exists $a \in (0, 1)$ such that

$$\|(\partial^\alpha f)(\partial^\beta g)\|_{L^p} \lesssim \|f\|^a_{L^{p_1}}\|D^k f\|^{1-a}_{L^{p_2}}\|g\|^{1-a}_{L^{p_3}}\|D^k g\|^a_{L^{p_4}}. \tag{4.17}$$

Proof. For fixed α, β with $|\alpha| + |\beta| = k$, it suffices to find q_1, q_2, and a such that $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{p}$ and

$$\|\partial^\alpha f\|_{L^{q_1}} \lesssim \|f\|^a_{L^{p_1}}\|\|D^k f\|^{1-a}_{L^{p_2}}, \qquad \|\partial^\beta g\|_{L^{q_2}} \lesssim \|g\|^{1-a}_{L^{p_3}}\|D^k g\|^a_{L^{p_4}},$$

since then (4.17) follows from Hölder's inequality:

$$\|(\partial^\alpha f)(\partial^\beta g)\|_{L^p} \le \|\partial^\alpha f\|_{L^{q_1}} \|\partial^\beta g\|_{L^{q_2}}. \tag{4.18}$$

The Gagliardo-Nirenberg-Sobolev inequality gives the conditions,

$$|\alpha| - \frac{3}{q_1} = \left(-\frac{3}{p_1}\right) a + \left(k - \frac{3}{p_2}\right)(1-a), \qquad |\alpha| \le (1-a)k, \tag{4.19}$$

and

$$|\alpha| - \frac{3}{q_2} = \left(-\frac{3}{p_3}\right) a + \left(k - \frac{3}{p_4}\right)(1-a), \qquad |\beta| \le ak. \tag{4.20}$$

Adding the inequalities from (4.19), (4.20) and recalling $|\alpha| + |\beta| = k$, we find $a = \frac{|\beta|}{k}$. Then the conditions (4.19), (4.20) become

$$\frac{1}{q_1} = \frac{|\alpha|}{k}\frac{1}{p_2} + \frac{|\beta|}{k}\frac{1}{p_1}, \qquad \frac{1}{q_2} = \frac{|\alpha|}{k}\frac{1}{p_3} + \frac{|\beta|}{k}\frac{1}{p_4}.$$

Adding these together gives,

$$\frac{1}{q_1} + \frac{1}{q_2} = \frac{|\alpha|}{k}\left(\frac{1}{p_2} + \frac{1}{p_3}\right) + \frac{|\beta|}{k}\left(\frac{1}{p_1} + \frac{1}{p_4}\right) = \frac{|\alpha|}{k}\frac{1}{p} + \frac{|\beta|}{k}\frac{1}{p}. = \frac{1}{p}$$

$\square$

Next we prove a simplified version of Ning Ju's result for integer derivatives:

Proposition 4.2.1. (Simplified Version of Ning Ju's Lemma 2.1 [Ju06]) : Let $k \in \mathbb{N}$, $p \in (1, \infty)$ and $f, g : \mathbb{R}^n \to \mathbb{R}^n$. Then,

$$\|D^k(fg)\|_{L^p} \leq C(\|f\|_{L^{p_1}}\|D^k g\|_{L^{p_2}} + \|D^k f\|_{L^{p_3}}\|g\|_{L^{p_4}}),$$

for $p_2, p_3 \in (1, \infty)$ satisfying $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}$

Proof. By the Leibniz rule,

$$\|D^k(fg)\|_{L^p} \lesssim \left\| \sum_{\substack{|\alpha|,|\beta| \leq k \\ |\alpha|+|\beta|=k}} (\partial^\alpha f)(\partial^\beta g) \right\|_{L^p} \lesssim \sum_{\substack{|\alpha|,|\beta| \leq k \\ |\alpha|+|\beta|=k}} \left\|(\partial^\alpha f)(\partial^\beta g)\right\|_{L^p}.$$

Then we use Lemma 4.2.7 to bound each term in the sum (for different α, β) to find

$$\|D^k(fg)\|_{L^p} \lesssim \sum_{\substack{|\alpha|,|\beta| \leq k \\ |\alpha|+|\beta|=k}} \|f\|^a_{L^{p_1}}\|D^k f\|^{1-a}_{L^{p_2}}\|g\|^{1-a}_{L^{p_3}}\|D^k g\|^a_{L^{p_4}},$$

where $a = \frac{|\beta|}{k}$. Next, we use Lemma 4.2.1 to separate the terms, and remove dependence of the sum on α, β, a. Finally, we absorb the sum into a constant to finish the proof.

□

Remark. The extreme case of Ning Ju's lemma, $p_1 = p_3 = 1$, $p_2 = p_4 = \infty$ is the well-known calculus inequality from partial differential equations theory, which we will also use later:

Lemma 4.2.8. Calculus Inequality in $\mathbb{R}^3$ (*Majda and Bertozzi Lemma 3.4* [MB02]) For all $k \in \mathbb{Z}^+ \cup \{0\}$, there exists $C > 0$ such that, for all $u, v \in L^\infty \cap H^k(\mathbb{R}^3)$,

$$\|uv\|_{H^k} \leq C(\|u\|_{L^\infty}\|D^k v\|_{L^2} + \|D^k u\|_{L^2}\|v\|_{L^\infty}). \tag{4.21}$$

4.2.4 Norm Bounds

Now we prove that we can bound the L^p and H^k norms of H by those of M, and find an a priori estimate for M.

Lemma 4.2.9. Consider (M, H) from (2.11)-(2.14) or (2.15)-(2.17) with $\nabla \cdot H^{ext} = 0$. Then for all $1 < p < \infty$,

$$\|H\|_{L^p} \lesssim \|M\|_{L^p}.$$

Proof. First note that

$$\begin{cases} \nabla \cdot (H + M) = -\nabla \cdot H^{ext} = 0 \;\text{ gives }\; \nabla \cdot M = -\nabla \cdot H, \\ \nabla \times H = 0 \;\text{ gives }\; H = \nabla\phi, \;\text{ for some scalar function } \phi. \end{cases}$$

Then, $\nabla \cdot M = -\nabla \cdot H = -\nabla \cdot \nabla\phi = -\Delta\phi$, so that

$$H = \nabla(-\Delta)^{-1}(\nabla \cdot M).$$

We want to write H as the convolution of some kernel with M, however a singularity will appear, which will force us to use distributional derivatives (see Majda, Bertozzi [MB02]). Let $\eta \in C_0^\infty(\mathbb{R}^3)$. Then

$$(H, \eta) = -(\nabla \cdot (-\Delta)^{-1} M, \nabla \cdot \eta). \tag{4.22}$$

Recall the Newtonian potential in $\mathbb{R}^3$ given by

$$N_3(x) = \frac{1}{4\pi |x|} \tag{4.23}$$

satisfies

$$(-\Delta)^{-1} M(x) = (N_3 * M)(x),$$

and is locally integrable on $\mathbb{R}^3$. Then we have,

$$\begin{aligned}
\nabla \cdot (-\Delta)^{-1} M &= \int_{\mathbb{R}^3} \nabla_x \cdot N_3(x-y) M(y) dy \\
&= \frac{1}{4\pi} \int_{\mathbb{R}^3} \nabla_x \cdot \left(\frac{1}{|x-y|} M(y) \right) dy \\
&= \frac{1}{4\pi} \int_{\mathbb{R}^3} \left(\nabla_x \frac{1}{|x-y|} \right) \cdot M(y) dy.
\end{aligned}$$

so we can write

$$\nabla \cdot (-\Delta)^{-1} M = \int_{\mathbb{R}^3} K(x-y) \cdot M(y) dy,$$

where

$$K(x) = \nabla \frac{1}{4\pi |x|} = -\frac{1}{4\pi} \frac{x}{|x|^3}. \tag{4.24}$$

Next we want to compute the distributional derivative of K from (4.22).

Since $K \in L^1_{\text{loc}}(\mathbb{R}^3)$, by the dominated convergence theorem and Green's theorem,

$$\begin{aligned}
(H, \eta) &= -\int_{\mathbb{R}^3} \left[\int_{\mathbb{R}^3} K(x-y) \cdot M(y) dy \right] (\nabla \cdot \eta(x)) dx \\
&= -\int_{\mathbb{R}^3} \left[\int_{\mathbb{R}^3} K(x-y)(\nabla \cdot \eta(x)) dx \right] M(y) dy \\
&= -\int_{\mathbb{R}^3} \left[\lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^3 \setminus B_\varepsilon(y)} K_j(x-y) \partial_i \eta_i(x) dx \right] M_j(y) dy \\
&= -\lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^3} M_j(y) \left[\int_{\mathbb{R}^3 \setminus B_\varepsilon(y)} K_j(x-y) \partial_i \eta_i(x) dx \right] dy \\
&= \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^3} M_j(y) \left[\int_{\partial B_\varepsilon(y)} K_j(x-y) n_i \eta_i(x) dS \right] dy \\
&\quad + \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^3} \left[\int_{\mathbb{R}^3 \setminus B_\varepsilon(y)} \partial_i K_j(x-y) \eta_i(x) dx \right] M_j(y) dy.
\end{aligned}$$

For the first term, note that

$$\begin{aligned}
\int_{\partial B_\varepsilon(y)} K_j(x-y) n_i(x) \eta_i(x) dS &= \frac{1}{4\pi\varepsilon^2} \int_{\partial B_\varepsilon(y)} n_j(x) n_i(x) \eta_i(x) dS \\
&\to \frac{\delta_{ij}}{3} \eta_i(y) \quad \text{as } \varepsilon \to 0.
\end{aligned} \tag{4.25}$$

Therefore the first term goes to $\frac{1}{3}(M, \eta)$ by the dominated convergence theorem. Note: in the last step of (4.25), we have used the computation

$$\begin{aligned}
\frac{1}{4\pi\varepsilon^2} \int_{\partial B_\varepsilon(y)} n_j(x)(\eta(x) \cdot n(x)) dS &= \frac{1}{4\pi\varepsilon^3} \int_{\partial B_\varepsilon(y)} (x_j \eta(x)) \cdot n(x) dS \\
&= \frac{1}{4\pi\varepsilon^3} \int_{B_\varepsilon(y)} \nabla \cdot (x_j \eta(x)) dV \\
&= \frac{1}{3} ⨍_{B_\varepsilon(y)} \eta_j(x) dV + \underbrace{\frac{1}{4\pi\varepsilon^3} \int_{B_\varepsilon(y)} x_j \nabla \cdot \eta dV}_{(\star)} \\
&\xrightarrow{\varepsilon \to 0} \frac{1}{3} \eta_j(y),
\end{aligned}$$

where in the second step we have used the divergence theorem, and $(\star)$ has,

$$\left|\frac{1}{4\pi\varepsilon^3}\int_{B_\varepsilon(y)} x_j \nabla\cdot\eta dV\right| \leq \frac{1}{4\pi\varepsilon^3}|\varepsilon| \sup_{x\in B_\varepsilon(y)} |\nabla\cdot\eta| \frac{4\pi\varepsilon^3}{3} \xrightarrow{\varepsilon\to 0} 0.$$

For the second term, we use Fubini's theorem:

$$\begin{aligned}\lim_{\varepsilon\searrow 0}\int_{\mathbb{R}^3}&\left[\int_{\mathbb{R}^3\setminus B_\varepsilon(y)} \partial_i K_j(x-y)\eta_i(x)dx\right] M_j(y)dy \\ &= \lim_{\varepsilon\searrow 0}\int_{\mathbb{R}^3}\left[\int_{\mathbb{R}^3\setminus B_\varepsilon(y)} \partial_i K_j(x-y)M_j(y)dy\right]\eta_i(x)dx.\end{aligned} \tag{4.26}$$

Thus from (4.2.4)-(4.26),

$$\lim_{\varepsilon\searrow 0}\left(\int_{\mathbb{R}^3\setminus B_\varepsilon(y)} \partial_i K_j(x-y)M_j(y)dy, \eta_i(x)\right) = \left(H - \frac{1}{3}M, \eta\right). \tag{4.27}$$

Since η is arbitrary,

$$\lim_{\varepsilon\searrow 0}\int_{\mathbb{R}^3\setminus B_\varepsilon(y)} \partial_i K_j(x-y)M_j(y)dy = \lim_{\varepsilon\searrow 0}\int_{\mathbb{R}^3\setminus B_\varepsilon(y)} \partial_{ij}\left(\frac{1}{4\pi|x-y|}\right)M_j(y)dy$$

exists as a distribution. This is the Cauchy principal-value integral. We denote it by

$$\mathrm{PV}\int_{\mathbb{R}^3}\partial_{ij}\left(\frac{1}{4\pi|x-y|}\right)M_j(y)dy.$$

Therefore,

$$H = \frac{M}{3} + \underbrace{\mathrm{PV}\int_{\mathbb{R}^3}\partial_{ij}\left(\frac{1}{4\pi|x-y|}\right)M_j(y)dy}_{(\star\star)} \tag{4.28}$$

in the sense of distributions. It remains to show regularity of $(\star\star)$.

First we compute,

$$\nabla_x^2 \left(\frac{1}{|x-y|} \right) = \nabla_x \left(\frac{(-1)(x-y)}{|x-y|^3} \right) = -\frac{I_3}{|x-y|^3} + 3\frac{(x-y)\otimes(x-y)}{|x-y|^5}, \tag{4.29}$$

where I_3 is the 3x3 identity matrix. This allows us to write,

$$\mathrm{PV} \int_{\mathbb{R}^3} \partial_{ij} \left(\frac{1}{4\pi|x-y|} \right) M_j(y)dy = \mathrm{PV} \int_{\mathbb{R}^3} R(x-y)M(y)dy = \mathrm{PV}(R * M)(x),$$

where $R(x)$ is a kernel of convolution-type given by

$$R(x) = \frac{1}{4\pi} \left(-\frac{I_3}{|x|^3} + 3\frac{x\otimes x}{|x|^5} \right) = \frac{1}{|x|^3}\Omega(x), \tag{4.30}$$

with $\Omega : \mathbb{R}^n \setminus \{0\} \to \mathbb{C}$ homogeneous of degree 0, given by

$$\Omega(x) = \frac{1}{4\pi} \left(-I_3 + 3\frac{x\otimes x}{|x|^2} \right). \tag{4.31}$$

Then we can write

$$TM(x) = PV \int_{\mathbb{R}^3} \frac{1}{|x-y|^3} \Omega(x-y)M(y)dy. \tag{4.32}$$

We want to show T extends to a bounded linear operator $T : L^p(\mathbb{R}^3) \to L^p(\mathbb{R}^3)$. To accomplish this, we use a theorem from Abels' book:

Theorem 4.2.2. (*Abels Theorem 4.19* [Abe12]) Assume that $k \in L^1_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\})$ satisfies

$$\int_{r<|x|\le 2r} |k(x)|dx \le C_1 \quad \text{for all } r > 0, \tag{4.33}$$

$$\begin{cases} \left| \int_{r<|x|<R} k(x)dx \right| \le C_2 \quad \text{for all} \quad 0 < r < R, \\ \lim_{r\to 0^+} \int_{r<|x|<1} k(x)dx \quad \text{exists}, \end{cases} \tag{4.34}$$

and the Hörmander condition

$$\int_{|x|>2|y|} |k(x-y) - k(x)|dx \le B_K \quad \text{for all } y \in \mathbb{R}^n, \tag{4.35}$$

for some $B_K \in (0, \infty)$. Then

$$Tf(x) = PV \int_{\mathbb{R}^n} k(y)f(x-y)dy \text{ for all } f \in \mathcal{S}(\mathbb{R}^n) \tag{4.36}$$

extends to a bounded linear operator $T : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ for every $1 < p < \infty$.

Remark. In the above, $\mathcal{S}(\mathbb{R}^n)$ is the Schwartz space.

After the proof of *Abels, Theorem 4.19*, the authors remark that for $R(x)$ defined by (4.30), conditions (4.33) and (4.34) are satisfied if and only if

$$\int_{\partial B_1(0)} \Omega(x)d\sigma(x) = 0 \tag{4.37}$$

where σ is the surface measure on the sphere, for $k \in L^1_{\text{loc}}(\mathbb{R}^3)$. We compute

$$\begin{aligned}\int_{\partial B_1(0)} \Omega(x)d\sigma(x) &= \frac{1}{4\pi}\int_{\partial B_1(0)} (I_3 + 3x \otimes x)d\sigma(x) \\ &= \frac{1}{4\pi}\left(-4\pi + 3\left(\frac{4}{3}\pi\right)\right) I = 0,\end{aligned}$$

where we have used the computation

$$\int_{\partial B_1(0)} x \otimes x d\sigma(x) = \int_{\partial B_1(0)} n_i n_j d\sigma(x) = \int_{B_1(0)} \delta^i_j dV = \frac{4}{3}\pi\delta^i_j \tag{4.38}$$

by the divergence theorem. Rather than show the Hörmander condition (4.35), we instead use the following sufficient condition which is much easier to check (see Abels' book [Abe12] Lemma 4.3):

(i) $R \in C^1(\mathbb{R}^3 \setminus \{0\})$,

(ii) $|\nabla R(x)| \leq \dfrac{C}{|x|^4}$.

Indeed, the first condition obviously holds, and a simple computation verifies the second.

By Theorem 4.2.2, we can now bound $(\star\star)$ in equation (4.28) on $L^p(\mathbb{R}^3)$, finally giving

$$\|H\|_{L^p} \leq C\|M\|_{L^p}.$$

$\square$

Corollary 4.2.1. Since T is an operator of convolution-type and therefore commutes with (spatial) derivatives, T also extends to a bounded linear map on H^k, and therefore

$$\|H\|_{H^k} \lesssim \|M\|_{H^k}.$$

$\square$

Remark. If $\nabla \cdot H^{ext} \neq 0$, then $\|H\|_{L^p} \lesssim \|M\|_{L^p} + \|H^{ext}\|_{L^p}$ (and similar similar for H^k norm).

To prove local existence of regularized solutions in the local well-posedness chapter, we will need to bound the time derivative of H.

Lemma 4.2.10. For M, H in (2.15)-(2.17) or (2.11)-(2.14), for all $k \geq 0$,

$$\left\|\frac{dH}{dt}\right\|_{H^k} \leq C\left\|\frac{dM}{dt}\right\|_{H^k}. \tag{4.39}$$

Proof. Recall from Lemma 4.2.9 that we can write

$$H = \frac{M}{3} + TM \tag{4.40}$$

in the sense of distributions, where T is the convolution operator defined by

$$TM := PV \int_{\mathbb{R}^3} \partial_{x_{ij}} \left(\frac{1}{4\pi |x-y|} \right) M(y) dy.$$

We showed in the lemma that T extends to a bounded linear operator $T : L^p(\mathbb{R}^3) \to L^p(\mathbb{R}^3)$ for every $1 < p < \infty$. From (4.40), we can write

$$\frac{dH}{dt} = \frac{1}{3}\frac{dM}{dt} + \frac{d}{dt}(TM) \tag{4.41}$$

(as distributions). Then

$$\begin{aligned} \left\| \frac{d}{dt}(TM) \right\|_{H^k} &= \left\| \frac{d}{dt} PV \int_{\mathbb{R}^3} \partial_{x_{ij}} \left(\frac{1}{4\pi|x-y|} \right) M(y) dy \right\|_{H^k} \\ &= \left\| PV \int_{\mathbb{R}^3} \partial_{x_{ij}} \left(\frac{1}{4\pi|x-y|} \right) \frac{d}{dt} M(y) dy \right\|_{H^k} \\ &\leq C \left\| \frac{dM}{dt} \right\|_{H^k}, \end{aligned}$$

where the inequality holds since, as a convolution operator, T commutes with spatial derivatives and we know it extends to a bounded linear operator on $L^2(\mathbb{R}^3)$. Together with (4.41), this gives

$$\left\| \frac{dH}{dt} \right\|_{H^k} \leq C \left\| \frac{dM}{dt} \right\|_{H^k}. \tag{4.42}$$

□

Now we prove an a priori estimate for M.

Lemma 4.2.11. Suppose $(u, \Omega, M, H) \in C((0,T); H^k(\mathbb{R}^3))$ is a solution to (2.11)-(2.14) with initial data (2.18) in H^k or $(u, M, H) \in C((0,T); H^k(\mathbb{R}^3))$ is a solution to (2.15)-(2.17) with initial data (2.19) in H^k, for $k > \frac{5}{2}$ and $\nabla \cdot H^{ext} = 0$. Then $\|M\|^p_{L^p(\mathbb{R}^3)}$ is uniformly bounded in t on $(0,T)$ for all $2 < p < \infty$.

Proof. First by the Sobolev embedding theorem,

$$u, \Omega, M, H \in L^p(\mathbb{R}^3) \text{ for all } p \in [1, \infty],\ t \in (0,T). \tag{4.43}$$

Consider a radial cut-off function $\phi \in C_0^\infty(\mathbb{R}^3)$ defined by

$$\phi(|x|) := \begin{cases} 1 \text{ if } |x| < 1 \\ 0 \text{ if } |x| > 2 \end{cases},$$

and $\phi(|x|) \in [0,1]$ for $|x| \in (1,2)$. For $R \geq 1$ define

$$\phi_R(|x|) := \phi\left(\frac{|x|}{R}\right) \in C_0^\infty(\mathbb{R}^3).$$

Multiplying the magnetization equation (2.16) or (2.13) by $M|M|^{p-2}\phi_R$ and integrating over $\mathbb{R}^3$ gives (for both models),

$$\begin{aligned}
&\int_{\mathbb{R}^3} (\partial_t M) \cdot M|M|^{p-2}\phi_R dx + \int_{\mathbb{R}^3} [(u \cdot \nabla)M] \cdot M|M|^{p-2}\phi_R dx \\
&\quad = \sigma \int_{\mathbb{R}^3} (\Delta M) \cdot M|M|^{p-2}\phi_R dx - \frac{1}{\tau}\int_{\mathbb{R}^3} |M|^p \phi_R dx \\
&\quad + \frac{\chi_0}{\tau}\int_{\mathbb{R}^3} H \cdot M|M|^{p-2}\phi_R dx.
\end{aligned} \tag{4.44}$$

We have the identities,

$$(\partial_t M) \cdot M|M|^{p-2}\phi_R = \frac{1}{p}\partial_t|M|^p\phi_R, \tag{4.45}$$

and

$$[(u \cdot \nabla)M] \cdot M|M|^{p-2}\phi_R = \frac{1}{p}(u \cdot \nabla)|M|^p\phi_R, \tag{4.46}$$

where the second follows from

$$\begin{aligned} u \cdot \nabla|M|^p = u_i\partial_i|M|^p = pu_i|M|^{p-1}\frac{M_j}{|M|}\partial_i M_j &= [pu_i\partial_i M_j]M_j|M|^{p-2} \\ = p[(u \cdot \nabla)M] \cdot M|M|^{p-2}. \end{aligned}$$

Adding together (4.45) and (4.46) gives

$$(\partial_t M) \cdot M|M|^{p-2}\phi_R + [(u \cdot \nabla)M] \cdot M|M|^{p-2}\phi_R = \frac{1}{p}D_t|M|^p\phi_R. \tag{4.47}$$

Moreover,

$$\begin{aligned} \int_{\mathbb{R}_3} H \cdot M|M|^{p-2}\phi_R dx &\leq \|\phi_R\|_\infty\|H\|_{L^p(\mathbb{R}^3)}\|M^{p-1}\|_{L^{\frac{p}{p-1}}(\mathbb{R}^3)} \\ &\lesssim \|M\|^p_{L^p(\mathbb{R}^3)}, \end{aligned} \tag{4.48}$$

where the first inequality follows from Hölder's inequality and the last from Lemma 4.2.9. Finally by integrating by parts, we obtain

$$\begin{aligned} -\sigma\int_{\mathbb{R}^3}(\Delta M) \cdot M|M|^{p-2}\phi_R dx = \ &\sigma(p-1)\int_{\mathbb{R}^3}|M|^{p-2}|\nabla M|_F^2\phi_R dx \\ &- \sigma\frac{\sigma}{p}\int_{\mathbb{R}^3}|M|^p\Delta\phi_R dx. \end{aligned} \tag{4.49}$$

Here $|\cdot|_F$ denotes the Frobenius matrix norm.

In more detail, we have used,

$$
\begin{aligned}
-\sigma\int_{\mathbb{R}^3}(\Delta M)\cdot M|M|^{p-2}\phi_R dx =& -\sigma\int_{\mathbb{R}^3}\partial_j\partial_j M_i M_i|M|^{p-2}\phi_R dx\\
=&\ \sigma\int_{\mathbb{R}^3}\partial_j M_i\partial_j(M_i|M|^{p-2}\phi_R)dx\\
=&\ \underbrace{\sigma\int_{\mathbb{R}^3}\partial_j M_i\partial_j M_i|M|^{p-2}\phi_R dx}_{(a)}+\underbrace{\sigma\int_{\mathbb{R}^3}\partial_j M_i M_i\partial_j|M|^{p-2}\phi_R dx}_{(b)}\\
&+\underbrace{\sigma\int_{\mathbb{R}^3}\partial_j M_i M_i|M|^{p-2}\partial_j\phi_R dx}_{(c)},
\end{aligned}
$$

where we compute:

$$
\begin{aligned}
(a) &= \sigma\int_{\mathbb{R}^3}|\nabla M|_F^2|M|^{p-2}\phi_R dx,\\
(b) &= (p-2)\sigma\int_{\mathbb{R}^3}\partial_j M_i M_i|M|^{p-4}M_i\partial_j M_i\phi_R dx\\
&= (p-2)\sigma\int_{\mathbb{R}^3}|\nabla M|_F^2|M|^{p-2}\phi_R dx,\\
(c) &= -\sigma\int_{\mathbb{R}^3}|M|^p\Delta\phi_R dx.
\end{aligned}
$$

Using (4.47)-(4.49), we obtain from (4.44)

$$
\begin{aligned}
\int_{\mathbb{R}^3}D_t(|M|^p\phi_R)dx &\lesssim \|M\|_{L^p(\mathbb{R}^3)}^p+\int_{\mathbb{R}^3}|M|^p(\|(u\cdot\nabla)\phi_R\|_{L^\infty(\mathbb{R}^3)}+|\Delta\phi_R|)dx\\
&\lesssim (1+R^{-1}\|u\|_{L^\infty(\mathbb{R}^3)}+R^{-2})\|M\|_{L^p(\mathbb{R}^3)}^p\\
&\lesssim (1+R^{-1}\|u\|_{L^\infty(\mathbb{R}^3)})\|M\|_{L^p(\mathbb{R}^3)}^p.
\end{aligned}
\tag{4.50}
$$

By the transport theorem,

$$\int_{\mathbb{R}^3} D_t(|M|^p \phi_R) dx = \frac{d}{dt}\int_{\mathbb{R}^3} |M|^p \phi_R dx. \tag{4.51}$$

Integrating both sides of (4.50) from some $t_0 \in (0, T)$ to some $t \in (t_0, T)$ using (4.51),

$$\int_{\mathbb{R}^3} |M|^p(t)\phi_R dx - \int_{\mathbb{R}^3} |M|^p(t_0)\phi_R dx \lesssim \int_{t_0}^{t} (1 + R^{-1}\|u\|_{L^\infty})\|M\|_{L^p}^p dt.$$

Then for every

$$R > \max\left\{1, \sup_{s\in[t_0,t]} \|u\|_{L^\infty}(s)\right\}$$

(which is finite since $u \in C((0, T); H^k(\mathbb{R}^3))$ for some $k > 5/2$), we have

$$\int_{\mathbb{R}^3} |M|^p(t)\phi_R dx - \int_{\mathbb{R}^3} |M|^p(t_0)\phi_R dx \lesssim \int_{t_0}^{t} \|M\|_{L^p(\mathbb{R}^3)}^p dt.$$

Since $M \in L^p(\mathbb{R}^3)$, we can take the limit $R \to \infty$ to get

$$\|M\|_{L^p(\mathbb{R}^3)}^p(t) - \|M\|_{L^p(\mathbb{R}^3)}^p(t_0) \lesssim \int_{t_0}^{t} \|M\|_{L^p(\mathbb{R}^3)}^p dt.$$

Finally, using the definition of derivative, we find

$$\frac{d}{dt}\|M\|_{L^p(\mathbb{R}^3)}^p \lesssim \|M\|_{L^p(\mathbb{R}^3)}^p.$$

A standard application of Grönwall's inequality gives the result.

$\square$

Remark. If $\nabla \cdot H^{ext} \neq 0$, then we have instead,

$$\frac{d}{dt}\|M\|_{L^p}^p \lesssim \|M\|_{L^p}^p + \|H^{ext}\|_{L^p}^p, \tag{4.52}$$

and Grönwall's inequality gives,

$$\|M\|_{L^p}^p(t) \lesssim \|M\|_{L^p}^p(0) + e^t \int_0^t \|H^{ext}\|_{L^p}^p(s)ds. \tag{4.53}$$

Then M is uniformly bounded in L^p if $H^{ext} \in L^p(0,t;L^p(\mathbb{R}^3))$ for $p > 2$.

Chapter 5

Local Well-Posedness for Equations of Ferrohydrodynamics

In this chapter, we prove the Shliomis model with Bloch-Torrey magnetization (2.15)-(2.17), and the Rosensweig model with Bloch-Torrey magnetization (2.11)-(2.14) are locally well-posed for classical solutions. While in the literature existence and uniqueness of strong solutions has been considered, to the best of my knowledge, the local well-posedness of classical solutions has not been. We prove that solutions in $C([0,T];H^k(\mathbb{R}^3))$ for $k > 5/2$ are locally well-posed for both models.

5.1 Shliomis Model

First note that throughout this section, we will unapologetically abuse function space notation for the sake of readability. Specifically, by H^k or $H^k(\mathbb{R}^3)$ we mean either $H^k(\mathbb{R}^3)$, $H^k_\sigma(\mathbb{R}^3)\times H^k(\mathbb{R}^3)\times H^k(\mathbb{R}^3)$, or $H^k(\mathbb{R}^3)\times H^k(\mathbb{R}^3)\times H^k(\mathbb{R}^3)$, where $H^k_\sigma(\mathbb{R}^3) := \{f \in H^k(\mathbb{R}^3) : \nabla \cdot f = 0\}$. For any $\varepsilon > 0$, we write

$$\begin{cases} \rho\left(\frac{du^\varepsilon}{dt} + \mathbb{P}\mathcal{J}_\varepsilon[(\mathcal{J}_\varepsilon u^\varepsilon)\cdot\nabla(\mathcal{J}_\varepsilon u^\varepsilon)]\right) - \eta\mathcal{J}_\varepsilon^2\Delta u^\varepsilon \\ \qquad\qquad = \mu_0\mathbb{P}\mathcal{J}_\varepsilon[(\mathcal{J}_\varepsilon M^\varepsilon)\cdot\nabla(\mathcal{J}_\varepsilon H^\varepsilon)] + \frac{\mu_0}{2}\mathcal{J}_\varepsilon[\nabla\times(\mathcal{J}_\varepsilon M^\varepsilon\times\mathcal{J}_\varepsilon H^\varepsilon)], \\ \frac{dM^\varepsilon}{dt} + \mathcal{J}_\varepsilon[(\mathcal{J}_\varepsilon u^\varepsilon)\cdot\nabla(\mathcal{J}_\varepsilon M^\varepsilon)] - \sigma\mathcal{J}_\varepsilon^2\Delta M^\varepsilon \\ \qquad\qquad = \frac{1}{2}\mathcal{J}_\varepsilon[\mathcal{J}_\varepsilon(\nabla\times u^\varepsilon)\times(\mathcal{J}_\varepsilon M^\varepsilon)] - \frac{1}{\tau}(M^\varepsilon - \chi_0 H^\varepsilon) \\ \qquad\qquad\quad -\beta\mathcal{J}_\varepsilon[\mathcal{J}_\varepsilon M^\varepsilon\times(\mathcal{J}_\varepsilon M^\varepsilon\times\mathcal{J}_\varepsilon H^\varepsilon)], \\ \nabla\times H^\varepsilon = 0, \quad \nabla\cdot(H^\varepsilon + M^\varepsilon) = -\nabla\cdot H^{ext}, \end{cases} \tag{5.1}$$

where to eliminate the pressure and inccompressibility condition, we have projected the momentum equation onto the space of divergence functions $H^k_\sigma(\mathbb{R}^3)$ defined above via the Leray projector $\mathbb{P}$. We will refer to equations (5.1) as the mollified Shliomis equations, and for future reference write them as:

$$\begin{aligned} \frac{du^\varepsilon}{dt} &= F_\varepsilon(u^\varepsilon, M^\varepsilon, H^\varepsilon), \\ \frac{dM^\varepsilon}{dt} &= G_\varepsilon(u^\varepsilon, M^\varepsilon, H^\varepsilon), \\ \nabla\times H^\varepsilon = 0, \quad \nabla\cdot(H^\varepsilon + M^\varepsilon) &= -\nabla\cdot H^{ext}, \end{aligned} \tag{5.2}$$

where

$$\begin{aligned}\rho F_\varepsilon(u^\varepsilon, M^\varepsilon, H^\varepsilon) = &-\rho\mathbb{P}\mathcal{J}_\varepsilon[(\mathcal{J}_\varepsilon u^\varepsilon)\cdot\nabla(\mathcal{J}_\varepsilon u^\varepsilon)] + \eta\mathcal{J}_\varepsilon^2\Delta u^\varepsilon \\ &+ \mu_0\mathbb{P}\mathcal{J}_\varepsilon[(\mathcal{J}_\varepsilon M^\varepsilon)\cdot\nabla(\mathcal{J}_\varepsilon H^\varepsilon)] \\ &+ \frac{\mu_0}{2}\mathcal{J}_\varepsilon\nabla\times(\mathcal{J}_\varepsilon M^\varepsilon\times\mathcal{J}_\varepsilon H^\varepsilon),\end{aligned} \tag{5.3}$$

and

$$\begin{aligned}G_\varepsilon(u^\varepsilon, M^\varepsilon, H^\varepsilon) = &-\mathcal{J}_\varepsilon[(\mathcal{J}_\varepsilon u^\varepsilon)\cdot\nabla(\mathcal{J}_\varepsilon M^\varepsilon)] + \sigma\mathcal{J}_\varepsilon^2\Delta M^\varepsilon \\ &+ \frac{1}{2}\mathcal{J}_\varepsilon[\mathcal{J}_\varepsilon(\nabla\times u^\varepsilon)\times(\mathcal{J}_\varepsilon M^\varepsilon)] - \frac{1}{\tau}(M^\varepsilon - \chi_0 H^\varepsilon) \\ &- \beta\mathcal{J}_\varepsilon[\mathcal{J}_\varepsilon M^\varepsilon\times(\mathcal{J}_\varepsilon M^\varepsilon\times\mathcal{J}_\varepsilon H^\varepsilon)].\end{aligned} \tag{5.4}$$

Now we state the main theorem:

Theorem 5.1.1. Suppose $(u_0, M_0, H_0) \in H^k$ for $k > 5/2$ and $\nabla\cdot H^{ext} = 0$. Then there exists $T = (\|u_0\|_{H^k}, \|M_0\|_{H^k}, \|H_0\|_{H^k}) > 0$ such that the Shliomis model with Bloch-Torrey magnetization; problem (2.15)-(2.17), (2.19) admits a unique classical solution $(u, M, H) \in C([0,T]; H^k(\mathbb{R}^3))$.

Proof. We prove Theorem 5.1.1 in multiple steps (labelled for the subsection we will prove them in).

5.1.1 Let $\{\psi_\varepsilon\}$ be a standard family of mollifiers. We show that for any $0 < \varepsilon \le 1$, there exists a time $T_\varepsilon > 0$ and a unique solution

$$u^\varepsilon, M^\varepsilon, H^\varepsilon \in C^1([0,T_\varepsilon); H^k(\mathbb{R}^3))$$

of the mollified Shliomis equations (5.1) with initial conditions

$$u^\varepsilon|_{t=0} = \mathcal{J}_\varepsilon u_0, \qquad M^\varepsilon|_{t=0} = \mathcal{J}_\varepsilon M_0, \qquad H^\varepsilon|_{t=0} = \mathcal{J}_\varepsilon H_0.$$

5.1.2 For each $\varepsilon > 0$, we show the solution from 5.1.1 exists globally in time.

5.1.3 There exists a $T > 0$ such that the family $(u^\varepsilon, M^\varepsilon, H^\varepsilon)$ obeys uniform in ε bounds on $[0, T]$.

5.1.4 $(u^\varepsilon, M^\varepsilon, H^\varepsilon)$ has a subsequence which converges to a limit point (u, M, H) in $C([0, T]; L^2)$.

5.1.5 The function (u, M, H) solves the Shliomis system with Bloch-Torrey magnetization (2.15)-(2.17), (2.19).

5.1.6 (u, M, H) is unique in $C([0, T]; H^k)$.

5.1.7 (u, M, H) belongs to $C([0, T]; H^k)$.

5.1.1 Local Existence of Regularized Solutions

To prove 5.1.1, we first show the existence of regularized solutions $(u^\varepsilon, M^\varepsilon, H^\varepsilon)$ to (5.1) locally in time. To do this, we need an equation for the time derivative of H. Recall from Lemma 4.2.9 we can write $H = \frac{1}{3}M + TM$, where T is a bounded linear operator on L^p and H^k. Therefore,

$$\frac{dH^\varepsilon}{dt} = \frac{1}{3}\frac{dM^\varepsilon}{dt} + \frac{d}{dt}(TM^\varepsilon) := W_\varepsilon. \tag{5.5}$$

Also, in Lemma 4.2.10, we showed $\|W_\varepsilon\|_{H^k} \leq \|G_\varepsilon\|_{H^k}$. With this in mind, note that $(F_\varepsilon, G_\varepsilon, W_\varepsilon) : H^k_\sigma \times H^k(\mathbb{R}^3) \times H^k(\mathbb{R}^3) \to H^k_\sigma \times H^k(\mathbb{R}^3) \times H^k(\mathbb{R}^3)$ since J_ε commmutes with derivatives, $\mathbb{P}$ projects onto divergence free functions, and the divergence of the curl of a vector field is zero. Now, we show $(F_\varepsilon, G_\varepsilon, W_\varepsilon)$

is locally Lipschitz. Let us write

$$\begin{aligned}\rho F_\varepsilon(u^\varepsilon, M^\varepsilon, H^\varepsilon) &= -\rho\mathbb{P}\mathcal{J}_\varepsilon[(\mathcal{J}_\varepsilon u^\varepsilon)\cdot\nabla(\mathcal{J}_\varepsilon u^\varepsilon)] + \eta\mathcal{J}_\varepsilon^2\Delta u^\varepsilon \\ &\quad + \mu_0\mathbb{P}\mathcal{J}_\varepsilon[(\mathcal{J}_\varepsilon M^\varepsilon)\cdot\nabla(\mathcal{J}_\varepsilon H^\varepsilon)] + \frac{\mu_0}{2}\mathcal{J}_\varepsilon[\nabla\times(\mathcal{J}_\varepsilon M^\varepsilon\times\mathcal{J}_\varepsilon H^\varepsilon)] \\ &= \rho F_\varepsilon^1 + \eta F_\varepsilon^2 + \mu_0 F_\varepsilon^3 + \frac{\mu_0}{2}F_\varepsilon^4.\end{aligned} \tag{5.6}$$

Fix $0 < \varepsilon \leq 1$. We bound, following Majda and Bertozzi's book [MB02],

$$\begin{aligned}&\|F_\varepsilon^1(u^1, M^1, H^1) - F_\varepsilon^1(u^2, M^2, H^2)\|_{H^k} \\ &\qquad \leq \|\mathbb{P}\mathcal{J}_\varepsilon[(\mathcal{J}_\varepsilon u^1)\cdot\nabla\mathcal{J}_\varepsilon(u^1-u^2)]\|_{H^k} + \|\mathbb{P}\mathcal{J}_\varepsilon[(\mathcal{J}_\varepsilon(u^1-u^2))\cdot\nabla(\mathcal{J}_\varepsilon u^2)]\|_{H^k} \\ &\qquad \leq C\{\|\mathcal{J}_\varepsilon u^1\|_{L^\infty}\|D^k\mathcal{J}_\varepsilon\nabla(u^1-u^2)\|_{L^2} + \|D^k\mathcal{J}_\varepsilon u^1\|_{L^2}\|J_\varepsilon\nabla(u^1-u^2)\|_{L^\infty} \\ &\qquad\quad + \|\mathcal{J}_\varepsilon(u^1-u^2)\|_{L^\infty}\|D^k\mathcal{J}_\varepsilon\nabla u^2\|_{L^2} + \|D^k\mathcal{J}_\varepsilon(u^1-u^2)\|_{L^2}\|\mathcal{J}_\varepsilon\nabla u^2\|_{L^\infty}\}.\end{aligned} \tag{5.7}$$

Note that we have used the calculus inequality, Lemma 4.2.8. Finally, by (4.8) and (4.9),

$$\|F_\varepsilon^1(u^1, M^1, H^1) - F_\varepsilon^1(u^2, M^2, H^2)\|_{H^k} \leq \frac{C}{\varepsilon^{5/2}}(\|u^1\|_{L^2} + \|u^2\|_{L^2})\|u^1-u^2\|_{H^k}. \tag{5.8}$$

Next,

$$\begin{aligned}\|F_\varepsilon^2(u^1, M^1, H^1) - F_\varepsilon^2(u^2, M^2, H^2)\|_{H^k} &= \|\mathcal{J}_\varepsilon^2\Delta(u^1-u^2)\|_{H^k} \\ &\leq \|\mathcal{J}_\varepsilon^2(u^1-u^2)\|_{H^{k+2}} \\ &\leq \frac{C}{\varepsilon^2}\|u^1-u^2\|_{H^k}.\end{aligned} \tag{5.9}$$

Similar to our first estimate, we find

$$
\begin{aligned}
&\|F^3_\varepsilon(u^1,M^1,H^1) - F^3_\varepsilon(u^2,M^2,H^2)\|_{H^k} \\
&\qquad \le \|\mathbb{P}\mathcal{J}_\varepsilon[(\mathcal{J}_\varepsilon M^1)\cdot\nabla\mathcal{J}_\varepsilon(H^1-H^2)]\|_{H^k} + \|\mathbb{P}\mathcal{J}_\varepsilon[(\mathcal{J}_\varepsilon(M^1-M^2))\cdot\nabla(\mathcal{J}_\varepsilon H^2)]\|_{H^k} \\
&\qquad \le C\{\|\mathcal{J}_\varepsilon M^1\|_{L^\infty}\|D^k\mathcal{J}_\varepsilon\nabla(H^1-H^2)\|_{L^2} + \|D^k\mathcal{J}_\varepsilon M^1\|_{L^2}\|J_\varepsilon\nabla(H^1-H^2)\|_{L^\infty} \\
&\qquad\quad + \|\mathcal{J}_\varepsilon(M^1-M^2)\|_{L^\infty}\|D^k\mathcal{J}_\varepsilon\nabla H^2\|_{L^2} + \|D^k\mathcal{J}_\varepsilon(M^1-M^2)\|_{L^2}\|\mathcal{J}_\varepsilon\nabla H^2\|_{L^\infty}\} \\
&\qquad \le \frac{C}{\varepsilon^{5/2}}\left(\|M^1\|_{L^2}\|H^1-H^2\|_{H^k} + \|H^2\|_{L^2}\|M^1-M^2\|_{H^k}\right).
\end{aligned}
\tag{5.10}
$$

Lastly, we estimate

$$
\begin{aligned}
&\|F^4_\varepsilon(u^1,M^1,H^1) - F^4_\varepsilon(u^2,M^2,H^2)\|_{H^k} \\
&\qquad \le \|\nabla\times(\mathcal{J}_\varepsilon M^1\times\mathcal{J}_\varepsilon(H^1-H^2))\|_{H^k} + \|\nabla\times(\mathcal{J}_\varepsilon(M^1-M^2)\times\mathcal{J}_\varepsilon H^2)\|_{H^k} \\
&\qquad \le C\{\|\mathcal{J}_\varepsilon M^1\|_{L^\infty}\|\mathcal{J}_\varepsilon D^{k+1}(H^1-H^2)\|_{L^2} + \|\mathcal{J}_\varepsilon(H^1-H^2)\|_{L^\infty}\|\mathcal{J}_\varepsilon D^{k+1}M^1\|_{L^2} \\
&\qquad\quad + \|\mathcal{J}_\varepsilon(M^1-M^2)\|_{L^\infty}\|\mathcal{J}_\varepsilon D^{k+1}H^2\|_{L^2} + \|\mathcal{J}_\varepsilon H^2\|_{L^\infty}\|\mathcal{J}_\varepsilon D^{k+1}(M^1-M^2)\|_{L^\infty}\} \\
&\qquad \le \frac{C}{\varepsilon^{5/2}}(\|M^1\|_{L^2}\|H^1-H^2\|_{H^k} + \|H^2\|_{L^2}\|M^1-M^2\|_{H^k}).
\end{aligned}
\tag{5.11}
$$

Altogether, the above estimates give

$$
\begin{aligned}
&\|F_\varepsilon(u^1,M^1,H^1) - F_\varepsilon(u^2,M^2,H^2)\|_{H^k} \\
&\qquad \le C(\varepsilon,\|u^i\|_{L^2},\|M^i\|_{L^2},\|H^i\|_{L^2},\rho,\mu_0,\eta)\times \\
&\qquad\quad \{\|u^1-u^2\|_{H^k} + \|M^1-M^2\|_{H^k} + \|H^1-H^2\|_{H^k}\}.
\end{aligned}
\tag{5.12}
$$

Next, write

$$
\begin{aligned}
G_\varepsilon(u^\varepsilon, M^\varepsilon, H^\varepsilon) = &-\mathcal{J}_\varepsilon[(\mathcal{J}_\varepsilon u^\varepsilon)\cdot\nabla(\mathcal{J}_\varepsilon M^\varepsilon)] + \sigma\mathcal{J}_\varepsilon^2\Delta M^\varepsilon \\
&+ \frac{1}{2}[\mathcal{J}_\varepsilon(\nabla\times u^\varepsilon)]\times(\mathcal{J}_\varepsilon M^\varepsilon) - \frac{1}{\tau}(M^\varepsilon - \chi_0 H^\varepsilon) \\
&- \beta\mathcal{J}_\varepsilon[\mathcal{J}_\varepsilon M^\varepsilon\times(\mathcal{J}_\varepsilon M^\varepsilon\times\mathcal{J}_\varepsilon H^\varepsilon)] \\
= &\, G_\varepsilon^1 + \sigma G_\varepsilon^2 + \frac{1}{2}G_\varepsilon^3 + \frac{1}{\tau}G_\varepsilon^4 + \beta G_\varepsilon^5.
\end{aligned}
\tag{5.13}
$$

We estimate in the same way as above

$$
\begin{aligned}
&\|G_\varepsilon^1(u^1, M^1, H^1) - G_\varepsilon^1(u^2, M^2, H^2)\|_{H^k} \\
&\quad\le \frac{C}{\varepsilon^{5/2}}\left(\|u^1\|_{L^2}\|M^1 - M^2\|_{H^k} + \|M^2\|_{L^2}\|u^1 - u^2\|_{H^k}\right),
\end{aligned}
\tag{5.14}
$$

and

$$
\begin{aligned}
\|G_\varepsilon^2(u^1, M^1, H^1) - G_\varepsilon^2(u^2, M^2, H^2)\|_{H^k} &= \|\mathcal{J}_\varepsilon^2\Delta(M^1 - M^2)\|_{H^k} \\
&\le \|\mathcal{J}_\varepsilon^2(M^1 - M^2)\|_{H^{k+2}} \\
&\le \frac{C}{\varepsilon^2}\|M^1 - M^2\|_{H^k}.
\end{aligned}
\tag{5.15}
$$

For G_ε^3 we estimate (in the same way as for G_ε^1)

$$
\begin{aligned}
&\|G_\varepsilon^3(u^1, M^1, H^1) - G_\varepsilon^3(u^2, M^2, H^2)\|_{H^k} \\
&\quad\le \frac{C}{\varepsilon^{5/2}}\left(\|M^1\|_{L^2}\|u^1 - u^2\|_{H^k} + \|u^2\|_{L^2}\|M^1 - M^2\|_{H^k}\right).
\end{aligned}
\tag{5.16}
$$

Then for G_ε^4, we trivially estimate

$$
\|G_\varepsilon^4(u^1, M^1, H^1) - G_\varepsilon^4(u^2, M^2, H^2)\|_{H^k} \le \|M^1 - M^2\|_{H^k} + \chi_0\|H^1 - H^2\|_{H^k}.
\tag{5.17}
$$

Finally we estimate the trilinear term,

$$
\begin{aligned}
\|G^5_\varepsilon(u^1, M^1, H^1) &- G^5_\varepsilon(u^2, M^2, H^2)\|_{H^k} \\
&\leq \|\mathcal{J}_\varepsilon(M^1 - M^2) \times (\mathcal{J}_\varepsilon M^1 \times \mathcal{J}_\varepsilon H^1)\|_{H^k} \\
&\quad + \|\mathcal{J}_\varepsilon M^2 \times [\mathcal{J}_\varepsilon M^1 \times \mathcal{J}_\varepsilon(H^1 - H^2)]\|_{H^k} \\
&\quad + \|\mathcal{J}_\varepsilon M^2 \times [\mathcal{J}_\varepsilon(M^1 - M^2) \times \mathcal{J}_\varepsilon H^2]\|_{H^k} \\
&\leq (\|\mathcal{J}_\varepsilon M^1\|_{H^k}\|\mathcal{J}_\varepsilon H^1\|_{H^k} + \|\mathcal{J}_\varepsilon M^2\|_{H^k}\|\mathcal{J}_\varepsilon H^2\|_{H^k})\|M^1 - M^2\|_{H^k} \\
&\quad + \|\mathcal{J}_\varepsilon M^1\|_{H^k}\|\mathcal{J}_\varepsilon M^2\|_{H^k}\|H^1 - H^2\|_{H^k} \\
&\leq \frac{C}{\varepsilon^{2k}}\{(\|M^1\|_{L^2}\|H^2\|_{L^2} + \|M^2\|_{L^2}\|H^2\|_{L^2})\|M^1 - M^2\|_{H^k} \\
&\qquad + \|M^1\|_{L^2}\|M^2\|_{L^2}\|H^1 - H^2\|_{H^k}\}.
\end{aligned} \tag{5.18}
$$

Altogether, we have

$$
\begin{aligned}
\|G_\varepsilon(u^1, M^1, H^1) &- G_\varepsilon(u^2, M^2, H^2)\|_{H^k} \\
&\leq C(\varepsilon, \|u^i\|_{L^2}, \|M^i\|_{L^2}, \|H^i\|_{L^2}, \tau, \chi_0, \beta)\times \\
&\quad \{\|u^1 - u^2\|_{H^k} + \|M^1 - M^2\|_{H^k} + \|H^1 - H^2\|_{H^k}\}.
\end{aligned} \tag{5.19}
$$

Also note that W_ε is Lipschitz as T is a bounded linear operator. From (5.12) and (5.19), we have shown $(F_\varepsilon, G_\varepsilon, W_\varepsilon)$ is locally Lipschitz on $H^k_\sigma \times H^k(\mathbb{R}^3) \times H^k(\mathbb{R}^3)$. Thus by the Picard theorem (Theorem 4.1.1), for any $(u_0, M_0, H_0) \in H^k$, there exists a unique solution

$$
(u^\varepsilon, M^\varepsilon, H^\varepsilon) \in C^1([0, T_\varepsilon); H^k), \tag{5.20}
$$

for some $T_\varepsilon > 0$. This proves 5.1.1.

5.1.2 Global Existence of Regularized Solution

For 5.1.2, we prove an energy bound which allows us to continue our solutions from 5.1.1 for all time. Multiplying the mollified momentum equation in (5.1) by u^ε and integrating over $\mathbb{R}^3$, we have

$$\begin{aligned}\frac{\rho}{2}\frac{d}{dt}\|u^\varepsilon\|^2 = -\rho\int_{\mathbb{R}^3} u^\varepsilon \mathbb{P}\mathcal{J}_\varepsilon[(\mathcal{J}_\varepsilon u^\varepsilon)\cdot\nabla(\mathcal{J}_\varepsilon u^\varepsilon)]dx + \eta\int_{\mathbb{R}^3} u^\varepsilon \mathcal{J}_\varepsilon^2 \Delta u^\varepsilon dx \\ + \mu_0\int_{\mathbb{R}^3} u^\varepsilon \mathbb{P}\mathcal{J}_\varepsilon[(\mathcal{J}_\varepsilon M^\varepsilon)\cdot\nabla(\mathcal{J}_\varepsilon H^\varepsilon)]dx \\ + \frac{\mu_0}{2}\int_{\mathbb{R}^3} u^\varepsilon \mathcal{J}_\varepsilon[\nabla\times(\mathcal{J}_\varepsilon M^\varepsilon \times \mathcal{J}_\varepsilon H^\varepsilon)]dx.\end{aligned} \tag{5.21}$$

Using property (4.5) of mollifiers from Lemma 4.1.1 and properties of the Leray projector, we find

$$\int_{\mathbb{R}^3} u^\varepsilon \mathcal{J}_\varepsilon^2 \Delta u^\varepsilon dx = \int_{\mathbb{R}^3} (\mathcal{J}_\varepsilon u^\varepsilon)\Delta(\mathcal{J}_\varepsilon u^\varepsilon)dx = -\int_{\mathbb{R}^3}(\mathcal{J}_\varepsilon \nabla u^\varepsilon)^2 dx,$$

and

$$\begin{aligned}\int_{\mathbb{R}^3} u^\varepsilon \mathbb{P}\mathcal{J}_\varepsilon[(\mathcal{J}_\varepsilon u^\varepsilon)\cdot\nabla(\mathcal{J}_\varepsilon u^\varepsilon)]dx &= \frac{1}{2}\int_{\mathbb{R}^3}(\mathcal{J}_\varepsilon u^\varepsilon)\cdot\nabla(\mathcal{J}_\varepsilon u^\varepsilon)^2 dx \\ &= -\frac{1}{2}\int_{\mathbb{R}^3}(\nabla\cdot\mathcal{J}_\varepsilon u^\varepsilon)(\mathcal{J}_\varepsilon u^\varepsilon)^2 dx = 0.\end{aligned}$$

Next, since $\nabla\times H^\varepsilon = 0$, $\partial_i H_j^\varepsilon = \partial_j H_i^\varepsilon$ so that,

$$\begin{aligned}&\nabla(\mathcal{J}_\varepsilon M^\varepsilon\cdot\mathcal{J}_\varepsilon H^\varepsilon)\cdot\mathcal{J}_\varepsilon u^\varepsilon - (\mathcal{J}_\varepsilon u^\varepsilon\cdot\nabla)\mathcal{J}_\varepsilon M^\varepsilon\cdot\mathcal{J}_\varepsilon H^\varepsilon \\ &\quad= \partial_j(\mathcal{J}_\varepsilon M_i^\varepsilon \mathcal{J}_\varepsilon H_i^\varepsilon)u_j - \mathcal{J}_\varepsilon u_j^\varepsilon \partial_j \mathcal{J}_\varepsilon M_i^\varepsilon \mathcal{J}_\varepsilon H_i^\varepsilon \\ &\quad= \partial_j \mathcal{J}_\varepsilon M_i^\varepsilon \mathcal{J}_\varepsilon u_j^\varepsilon \mathcal{J}_\varepsilon H_i^\varepsilon + \mathcal{J}_\varepsilon M_i^\varepsilon \partial_j \mathcal{J}_\varepsilon H_i^\varepsilon \mathcal{J}_\varepsilon u_j^\varepsilon - \mathcal{J}_\varepsilon u_j^\varepsilon \partial_j \mathcal{J}_\varepsilon M_i^\varepsilon \mathcal{J}_\varepsilon H_i^\varepsilon \\ &\quad= \mathcal{J}_\varepsilon M_i^\varepsilon \partial_i \mathcal{J}_\varepsilon H_j^\varepsilon \mathcal{J}_\varepsilon u_j^\varepsilon = (\mathcal{J}_\varepsilon M^\varepsilon\cdot\nabla\mathcal{J}_\varepsilon H^\varepsilon)\cdot\mathcal{J}_\varepsilon u^\varepsilon.\end{aligned}$$

Integrating by parts again so the first term on the left-hand side above vanishes, equation (5.21) becomes

$$\begin{aligned}\frac{\rho}{2}\frac{d}{dt}\|u^\varepsilon\|_{L^2}^2+\eta\|\nabla\mathcal{J}_\varepsilon u^\varepsilon\|_{L^2}^2=-\mu_0\int_{\mathbb{R}^3}(\mathcal{J}_\varepsilon u^\varepsilon\cdot\nabla\mathcal{J}_\varepsilon M^\varepsilon)\cdot\mathcal{J}_\varepsilon H^\varepsilon dx\\+\frac{\mu_0}{2}\int_{\mathbb{R}^3}\mathcal{J}_\varepsilon u^\varepsilon\nabla\times(\mathcal{J}_\varepsilon M^\varepsilon\times\mathcal{J}_\varepsilon H^\varepsilon)dx.\end{aligned}\tag{5.22}$$

Next, multiplying the magnetization equation in (5.1) by M^ε and integrating over $\mathbb{R}^3$ gives,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|M^\varepsilon\|_{L^2}^2+\int_{\mathbb{R}^3}M^\varepsilon\cdot\mathcal{J}_\varepsilon[(\mathcal{J}_\varepsilon u^\varepsilon)\cdot\nabla(\mathcal{J}_\varepsilon M^\varepsilon)]dx-\sigma\int_{\mathbb{R}^3}M^\varepsilon\mathcal{J}_\varepsilon^2\Delta M^\varepsilon dx\\=-\frac{1}{\tau}\int_{\mathbb{R}^3}M^\varepsilon\cdot(M^\varepsilon-\chi_0H^\varepsilon)dx.\end{aligned}\tag{5.23}$$

Using the same steps as above to deal with the first two terms, and Lemma 4.2.4 for the final term, and multiplying each term by μ_0, equation (5.23) gives

$$\frac{\mu_0}{2}\frac{d}{dt}\|M^\varepsilon\|_{L^2}^2+\mu_0\sigma\|\nabla\mathcal{J}_\varepsilon M^\varepsilon\|_{L^2}^2+\frac{\mu_0}{\tau}\|M^\varepsilon\|_{L^2}^2+\frac{\mu_0\chi_0}{\tau}\|H^\varepsilon\|_{L^2}^2=0.\tag{5.24}$$

Then, multiplying the magnetization equation from (5.1) by $-H^\varepsilon$, and integrating over $\mathbb{R}^3$ gives

$$\begin{aligned}-\int_{\mathbb{R}^3}H^\varepsilon\cdot\frac{dM^\varepsilon}{dt}dx-\int_{\mathbb{R}^3}H^\varepsilon\cdot\mathcal{J}_\varepsilon[(\mathcal{J}_\varepsilon u^\varepsilon)\cdot\nabla(\mathcal{J}_\varepsilon M^\varepsilon)]dx+\sigma\int_{\mathbb{R}^3}H^\varepsilon\mathcal{J}_\varepsilon^2\Delta M^\varepsilon dx\\=-\frac{1}{2}\int_{\mathbb{R}^3}H^\varepsilon\cdot\mathcal{J}_\varepsilon[\mathcal{J}_\varepsilon(\nabla\times u^\varepsilon)\times(\mathcal{J}_\varepsilon M^\varepsilon)]dx+\frac{1}{\tau}\int_{\mathbb{R}^3}H^\varepsilon\cdot(M^\varepsilon-\chi_0H^\varepsilon)dx\\+\beta\int_{\mathbb{R}^3}\mathcal{J}_\varepsilon M^\varepsilon\times(\mathcal{J}_\varepsilon M^\varepsilon\times\mathcal{J}_\varepsilon H^\varepsilon)\cdot\mathcal{J}_\varepsilon H^\varepsilon dx.\end{aligned}\tag{5.25}$$

For the first term we use Lemma 4.2.5 to get the time derivative of $\|H\|_{L^2}$. For the second, we use property (4.5) of mollifiers. For the third term, we integrate by parts, (keeping in mind $\nabla \times H^\varepsilon = 0$ and $\nabla \cdot (H^\varepsilon + M^\varepsilon) = 0$)

$$\int_{\mathbb{R}^3} \mathcal{J}_\varepsilon H^\varepsilon \Delta \mathcal{J}_\varepsilon M^\varepsilon dx = \int_{\mathbb{R}^3} (\mathcal{J}_\varepsilon \nabla H^\varepsilon)^2 dx.$$

We leave the fourth term unchanged. For the fifth term, we use Lemma 4.2.4 again. For the final term, we use the vector triple-product, $A \times (B \times C) = B(A \cdot C) - C(A \cdot B)$, to get

$$\begin{aligned}
\beta \int_{\mathbb{R}^3} & \mathcal{J}_\varepsilon M^\varepsilon \times (\mathcal{J}_\varepsilon M^\varepsilon \times \mathcal{J}_\varepsilon H^\varepsilon) \cdot \mathcal{J}_\varepsilon H^\varepsilon dx \\
&= \beta \int_{\mathbb{R}^3} [\mathcal{J}_\varepsilon M^\varepsilon (\mathcal{J}_\varepsilon M^\varepsilon \cdot \mathcal{J}_\varepsilon H^\varepsilon) - \mathcal{J}_\varepsilon H^\varepsilon (\mathcal{J}_\varepsilon M^\varepsilon \cdot \mathcal{J}_\varepsilon M^\varepsilon)] \cdot \mathcal{J}_\varepsilon H^\varepsilon dx \\
&= \beta \int_{\mathbb{R}^3} (\mathcal{J}_\varepsilon M^\varepsilon \cdot \mathcal{J}_\varepsilon H^\varepsilon)^2 - |\mathcal{J}_\varepsilon H^\varepsilon|^2 |\mathcal{J}_\varepsilon M^\varepsilon|^2 dx \\
&\leq 0,
\end{aligned}$$

by the Cauchy-Schwarz inequality, so we ignore it in the inequality. After multiplying each term in the equation by μ_0 (ignoring the previous one), this gives

$$\begin{aligned}
\frac{\mu_0}{2} \frac{d}{dt} \|H^\varepsilon\|_{L^2}^2 + \mu_0 \sigma \|\nabla \mathcal{J}_\varepsilon H^\varepsilon\|_{L^2}^2 + \frac{\mu_0 (1 + \chi_0)}{\tau} \|H^\varepsilon\|_{L^2}^2 & \\
= \mu_0 \int_{\mathbb{R}^3} [(\mathcal{J}_\varepsilon u^\varepsilon \cdot \nabla)(\mathcal{J}_\varepsilon M^\varepsilon)] \cdot \mathcal{J}_\varepsilon H^\varepsilon dx & \\
- \frac{\mu_0}{2} \int_{\mathbb{R}^3} (\mathcal{J}_\varepsilon M^\varepsilon \times \mathcal{J}_\varepsilon H^\varepsilon) \cdot (\mathcal{J}_\varepsilon \nabla \times u^\varepsilon) dx &
\end{aligned} \tag{5.26}$$

Adding together (5.22), (5.24) and (5.26), and using the scalar triple product $A \cdot (B \times C) = B \cdot (C \times A) = C \cdot (A \times B)$ gives the a priori bound,

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left(\rho\|u^\varepsilon\|_{L^2}^2 + \mu_0\|M^\varepsilon\|_{L^2}^2 + \mu_0\|H^\varepsilon\|_{L^2}^2\right) + \eta\|\nabla u^\varepsilon\|_{L^2}^2 + \frac{\mu_0}{\tau}\|M^\varepsilon\|_{L^2}^2 \\ + \frac{\mu_0(1+2\chi_0)}{\tau}\|H^\varepsilon\|_{L^2}^2 + 2\mu_0\sigma(\|\mathcal{J}_\varepsilon\nabla M^\varepsilon\|_{L^2}^2 + \|\mathcal{J}_\varepsilon\nabla H^\varepsilon\|_{L^2}^2) = 0. \end{aligned} \tag{5.27}$$

This implies

$$\sup_{0\le t\le T}(\rho\|u^\varepsilon\|_{L^2}^2 + \mu_0\|M^\varepsilon\|_{L^2}^2 + \mu_0\|H^\varepsilon\|_{L^2}^2) \le (\rho\|u_0\|_{L^2}^2 + \mu_0\|M_0\|_{L^2}^2 + \mu_0\|H_0\|_{L^2}^2). \tag{5.28}$$

In particular, since all the norms and constants are positive, we have

$$\sup_{0\le t\le T}(\|u^\varepsilon\|_{L^2} + \|M^\varepsilon\|_{L^2} + \|H^\varepsilon\|_{L^2}) \le C(\|u_0\|_{L^2} + \|M_0\|_{L^2} + \|H_0\|_{L^2}). \tag{5.29}$$

Now we use Theorem 4.1.2 to prove our solutions from 5.1.1 exist globally in time. To this end, we prove an a priori bound on the H^k norm of our solution $(u^\varepsilon, M^\varepsilon, H^\varepsilon)$. From the Lipschitz conditions (5.12) and (5.19) with $(u^2, M^2, H^2) \equiv (0,0,0)$,

$$\begin{aligned} &\frac{d}{dt}(\|u^\varepsilon\|_{H^k} + \|M^\varepsilon\|_{H^k} + \|H^\varepsilon\|_{H^k}) \\ &\qquad \le C(\varepsilon, \|u^\varepsilon\|_{L^2}, \|M^\varepsilon\|_{L^2}, \|H^\varepsilon\|_{L^2}, \tau, \chi_0, \beta, \rho, \mu_0, \eta)\times \\ &\qquad\quad \{\|u^\varepsilon\|_{H^k} + \|M^\varepsilon\|_{H^k} + \|H^\varepsilon\|_{H^k}\}. \end{aligned} \tag{5.30}$$

Finally, (5.29) changes dependence of the constant in equation (5.31) from the L^2 norm of $u^\varepsilon, M^\varepsilon, H^\varepsilon$ to the L^2 norm of the initial data:

$$\begin{aligned}\frac{d}{dt}&(\|u^\varepsilon\|_{H^k} + \|M^\varepsilon\|_{H^k} + \|H^\varepsilon\|_{H^k}) \\ &\le C(\varepsilon, \|u_0\|_{L^2}, \|M_0\|_{L^2}, \|H_0\|_{L^2}, \tau, \chi_0, \beta, \rho, \mu_0, \eta)\times \\ &\quad \{\|u^\varepsilon\|_{H^k} + \|M^\varepsilon\|_{H^k} + \|H^\varepsilon\|_{H^k}\}.\end{aligned} \tag{5.31}$$

By Grönwall's lemma, we get

$$\|u^\varepsilon\|_{H^k} + \|M^\varepsilon\|_{H^k} + \|H^\varepsilon\|_{H^k} \le (\|u_0\|_{H^k} + \|M_0\|_{H^k} + \|H_0\|_{H^k})e^{ct}, \tag{5.32}$$

so that by Theorem 4.1.2 our solution can be continued indefinitely in time.

5.1.3 Uniform in ε Bounds

Now we show that for $k > 5/2$, our solution is uniformly bounded in ε on the time interval $[0, T]$ for some $T > 0$. In particular, we prove the bound

$$\begin{aligned}\frac{d}{dt}&(1 + \|u^\varepsilon\|^2_{H^k} + \|M^\varepsilon\|^2_{H^k} + \|H^\varepsilon\|^2_{H^k}) \\ &\le C(1 + \|u^\varepsilon\|^2_{H^k} + \|M^\varepsilon\|^2_{H^k} + \|H^\varepsilon\|^2_{H^k})^2.\end{aligned}$$

For simplicity, and because we won't need a cancellation for the rest of the proof, we hereafter set the constants $\rho, \mu_0, \kappa, \sigma, \zeta, \eta', \lambda', \tau = 1$. For $k > 5/2$, let ∂^k denote a particular derivative of order k. Taking this k-th derivative of the momentum equation from (5.1), multiplying by $\partial^k u^\varepsilon$, and integrating over $\mathbb{R}^3$

gives,

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\partial^k u^\varepsilon\|_{L^2}^2 + \|\partial^k \mathcal{J}_\varepsilon \nabla u\|_{L^2} \\
&\quad = -\int_{\mathbb{R}^3} \mathbb{P}\partial^k(\mathcal{J}_\varepsilon u^\varepsilon \cdot \nabla \mathcal{J}_\varepsilon u^\varepsilon)\cdot \partial^k \mathcal{J}_\varepsilon u^\varepsilon dx + \int_{\mathbb{R}^3} \partial^k(\mathcal{J}_\varepsilon M^\varepsilon \cdot \nabla \mathcal{J}_\varepsilon H^\varepsilon)\cdot \partial^k \mathcal{J}_\varepsilon u^\varepsilon dx \\
&\qquad + \frac{1}{2}\int_{\mathbb{R}^3} \partial^k(\nabla \times (\mathcal{J}_\varepsilon M^\varepsilon \times \mathcal{J}_\varepsilon H^\varepsilon))\cdot \partial^k \mathcal{J}_\varepsilon u^\varepsilon dx \\
&\quad = S_1^\varepsilon + S_2^\varepsilon + S_3^\varepsilon.
\end{aligned} \tag{5.33}
$$

Next, taking the k-th derivative of the magnetization equation of (5.1), multiplying by $\partial^k M^\varepsilon$, and integrating over $\mathbb{R}^3$ gives (integrating by parts and using Lemma 4.2.4),

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\partial^k M^\varepsilon\|_{L^2}^2 + \|\partial^k \mathcal{J}_\varepsilon \nabla M^\varepsilon\|_{L^2}^2 + \|\partial^k M^\varepsilon\|_{L^2}^2 + \|\partial^k H^\varepsilon\|_{L^2}^2 \\
&\quad = -\int_{\mathbb{R}^3} \partial^k(\mathcal{J}_\varepsilon u^\varepsilon \cdot \nabla \mathcal{J}_\varepsilon M^\varepsilon)\cdot \partial^k \mathcal{J}_\varepsilon M^\varepsilon dx \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^3} \partial^k[\mathcal{J}_\varepsilon(\nabla \times u^\varepsilon)\times(\mathcal{J}_\varepsilon M^\varepsilon)]\cdot \partial^k \mathcal{J}_\varepsilon M^\varepsilon dx \\
&\qquad - \int_{\mathbb{R}^3} \partial^k[\mathcal{J}_\varepsilon M^\varepsilon \times (\mathcal{J}_\varepsilon M^\varepsilon \times \mathcal{J}_\varepsilon H^\varepsilon)]\cdot \partial^k M^\varepsilon dx \\
&\quad = S_4^\varepsilon + S_5^\varepsilon + S_6^\varepsilon.
\end{aligned} \tag{5.34}
$$

Then, taking the k-th derivative of the magnetization equation of (5.1), multiplying by $-\partial^k H^\varepsilon$, and integrating over $\mathbb{R}^3$ similarly gives (integrating by parts and using Lemmas 4.2.4 and 4.2.5)

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\partial^k H^\varepsilon\|_{L^2}^2 + \|\partial^k \mathcal{J}_\varepsilon \nabla H^\varepsilon\|_{L^2}^2 + 2\|\partial^k H^\varepsilon\|_{L^2}^2 \\
&\qquad = \int_{\mathbb{R}^3} \partial^k(\mathcal{J}_\varepsilon u^\varepsilon \cdot \nabla \mathcal{J}_\varepsilon M^\varepsilon)\cdot \partial^k \mathcal{J}_\varepsilon H^\varepsilon dx \\
&\qquad - \frac{1}{2}\int_{\mathbb{R}^3} \partial^k[\mathcal{J}_\varepsilon(\nabla \times u^\varepsilon)\times(\mathcal{J}_\varepsilon M^\varepsilon)]\cdot \partial^k \mathcal{J}_\varepsilon H^\varepsilon dx \\
&\qquad + \int_{\mathbb{R}^3} \partial^k[\mathcal{J}_\varepsilon M^\varepsilon \times (\mathcal{J}_\varepsilon M^\varepsilon \times \mathcal{J}_\varepsilon H^\varepsilon)\cdot \partial^k H^\varepsilon dx \\
&\qquad = S_7^\varepsilon + S_8^\varepsilon + S_9^\varepsilon.
\end{aligned} \tag{5.35}$$

We estimate (integrating by parts, using that H^k is an algebra for $k > 3/2$, $\nabla \cdot u^\varepsilon = 0$, and the calculus inequality Lemma 4.2.8),

$$\begin{aligned}
|S_1^\varepsilon| &\lesssim \|\mathcal{J}_\varepsilon u^\varepsilon\|_{H^k}\left|\int_{\mathbb{R}^3}(\partial^k(\mathcal{J}_\varepsilon u^\varepsilon \cdot \nabla \mathcal{J}_\varepsilon u^\varepsilon) - \mathcal{J}_\varepsilon u^\varepsilon \cdot \nabla u^\varepsilon)dx\right| \\
&\le C\|D\mathcal{J}_\varepsilon u^\varepsilon\|_{L^\infty}\|u^\varepsilon\|_{H^k}^2.
\end{aligned} \tag{5.36}$$

For S_2^ε, since $\nabla \times H^\varepsilon = 0$ and $\nabla \cdot (H^\varepsilon + M^\varepsilon) = 0$, we have

$$\begin{aligned}
\nabla \cdot \left[\mathcal{J}_\varepsilon(M^\varepsilon + H^\varepsilon)\otimes \mathcal{J}_\varepsilon H^\varepsilon - \frac{(\mathcal{J}_\varepsilon H^\varepsilon)^2}{2} I\right] &= \partial_i[\mathcal{J}_\varepsilon(M_i^\varepsilon + H_i^\varepsilon)\mathcal{J}_\varepsilon H_j^\varepsilon] - \mathcal{J}_\varepsilon H_j^\varepsilon \partial_i \mathcal{J}_\varepsilon H_j^\varepsilon \\
&= (\mathcal{J}_\varepsilon M_i^\varepsilon + \mathcal{J}_\varepsilon H_i^\varepsilon)\partial_i \mathcal{J}_\varepsilon H_j - \mathcal{J}_\varepsilon H_j^\varepsilon \partial_j \mathcal{J}_\varepsilon H_i^\varepsilon \\
&= \mathcal{J}_\varepsilon M_i^\varepsilon \partial_i \mathcal{J}_\varepsilon H_j^\varepsilon \\
&= (\mathcal{J}_\varepsilon M^\varepsilon \cdot \nabla)\mathcal{J}_\varepsilon H^\varepsilon.
\end{aligned} \tag{5.37}$$

Therefore we estimate (integrating by parts first),

$$\begin{aligned}|S_2^\varepsilon| &\lesssim \|D^k(\mathcal{J}_\varepsilon M^\varepsilon \mathcal{J}_\varepsilon H^\varepsilon + \mathcal{J}_\varepsilon H^\varepsilon \mathcal{J}_\varepsilon H)\|_{L^2}\|D^{k+1}\mathcal{J}_\varepsilon u^\varepsilon\|_{L^2} \\ &\lesssim C(\delta)(\|\mathcal{J}_\varepsilon M^\varepsilon\|_{L^\infty}^2 + \|\mathcal{J}_\varepsilon H^\varepsilon\|_{L^\infty}^2)(\|\mathcal{J}_\varepsilon M^\varepsilon\|_{H^k}^2 + \|\mathcal{J}_\varepsilon H^\varepsilon\|_{H^k}^2) \\ &\quad + \delta\|D^{k+1}\mathcal{J}_\varepsilon u^\varepsilon\|_{L^2}^2,\end{aligned} \tag{5.38}$$

where $\delta > 0$ is some small constant to be determined later.

Similarly, S_3^ε can be estimated as follows,

$$\begin{aligned}|S_3^\varepsilon| &\lesssim \left|\int_{\mathbb{R}^3} \partial^k(\mathcal{J}_\varepsilon M^\varepsilon \times \mathcal{J}_\varepsilon H^\varepsilon)\cdot D^{k+1}u dx\right| \\ &\lesssim \|\mathcal{J}_\varepsilon M^\varepsilon \times \mathcal{J}_\varepsilon H^\varepsilon\|_{H^k}\|D^{k+1}\mathcal{J}_\varepsilon u^\varepsilon\|_{L^2} \\ &\lesssim C(\delta)(\|\mathcal{J}_\varepsilon M^\varepsilon\|_{L^\infty}^2 + \|\mathcal{J}_\varepsilon H^\varepsilon\|_{L^\infty}^2)(\|M^\varepsilon\|_{H^k}^2 + \|H^\varepsilon\|_{H^k}^2) + \delta\|D^{k+1}\mathcal{J}_\varepsilon u\|_{L^2}^2.\end{aligned} \tag{5.39}$$

Next S_4^ε can be estimated by

$$\begin{aligned}|S_4^\varepsilon| &\lesssim \left|\int_{\mathbb{R}^3}[\partial^k(\mathcal{J}_\varepsilon u^\varepsilon\cdot\nabla\mathcal{J}_\varepsilon M^\varepsilon) - \mathcal{J}_\varepsilon u^\varepsilon\cdot\nabla\partial^k\mathcal{J}_\varepsilon M^\varepsilon]\partial^k\mathcal{J}_\varepsilon M^\varepsilon dx\right| \\ &\lesssim \sum_{\substack{i+j=k+1\\ i,j\geq 1}}\int_{\mathbb{R}^3}|D^i\mathcal{J}_\varepsilon u^\varepsilon||D^j\mathcal{J}_\varepsilon M^\varepsilon||D^k\mathcal{J}_\varepsilon M^\varepsilon|dx \\ &\lesssim (\|D\mathcal{J}_\varepsilon u^\varepsilon\|_{L^\infty}\|D^k M^\varepsilon\|_{L^2} + \|D^{k+1}\mathcal{J}_\varepsilon u^\varepsilon\|_{L^2}\|\mathcal{J}_\varepsilon M^\varepsilon\|_{L^\infty})\|M^\varepsilon\|_{H^k} \\ &\lesssim C\|D\mathcal{J}_\varepsilon u^\varepsilon\|_{L^\infty}\|M^\varepsilon\|_{H^k}^2 + C(\delta)\|\mathcal{J}_\varepsilon M^\varepsilon\|_{L^\infty}^2\|M^\varepsilon\|_{H^k}^2 + \delta\|D^{k+1}\mathcal{J}_\varepsilon u^\varepsilon\|_{L^2}^2.\end{aligned} \tag{5.40}$$

where in the first step we have used $\nabla \cdot u = 0$.

Then S_5^ε is controlled by

$$\begin{aligned}
|S_5^\varepsilon| &\lesssim \|\partial^k[\mathcal{J}_\varepsilon(\nabla \times u) \times (\mathcal{J}_\varepsilon M^\varepsilon)]\|_{L^2}\|M^\varepsilon\|_{H^k} \\
&\le (\|D\mathcal{J}_\varepsilon u^\varepsilon\|_{L^\infty}\|D^k M^\varepsilon\|_{L^2} + \|\mathcal{J}_\varepsilon M^\varepsilon\|_{L^\infty}\|D^{k+1}\mathcal{J}_\varepsilon u^\varepsilon\|_{L^2})\|M^\varepsilon\|_{H^k} \\
&\lesssim \|D\mathcal{J}_\varepsilon u^\varepsilon\|_{L^\infty}\|M^\varepsilon\|_{H^k}^2 + C(\delta)\|M^\varepsilon\|_{L^\infty}^2\|M^\varepsilon\|_{H^k}^2 + \delta\|D^{k+1}\mathcal{J}_\varepsilon u^\varepsilon\|_{L^2}^2.
\end{aligned} \tag{5.41}$$

Next, S_6^ε can be controlled by,

$$\begin{aligned}
|S_6^\varepsilon| &\lesssim \|\partial^k(\mathcal{J}_\varepsilon M^\varepsilon \times (\mathcal{J}_\varepsilon M^\varepsilon \times \mathcal{J}_\varepsilon H^\varepsilon))\|_{L^2}\|M^\varepsilon\|_{H^k} \\
&\lesssim (\|\mathcal{J}_\varepsilon M^\varepsilon\|_{L^\infty}\|\mathcal{J}_\varepsilon M^\varepsilon \times \mathcal{J}_\varepsilon H^\varepsilon\|_{H^k} \\
&\quad + \|\mathcal{J}_\varepsilon M^\varepsilon \times \mathcal{J}_\varepsilon H^\varepsilon\|_{L^\infty}\|\mathcal{J}_\varepsilon M^\varepsilon\|_{H^k})\|M^\varepsilon\|_{H^k} \\
&\lesssim \|\mathcal{J}_\varepsilon M^\varepsilon\|_{L^\infty}(\|\mathcal{J}_\varepsilon M^\varepsilon\|_{L^\infty}\|H^\varepsilon\|_{H^k} + \|\mathcal{J}_\varepsilon H^\varepsilon\|_{L^\infty}\|M^\varepsilon\|_{H^k})\|M^\varepsilon\|_{H^k} \\
&\quad + \|\mathcal{J}_\varepsilon M^\varepsilon \times \mathcal{J}_\varepsilon H^\varepsilon\|_{L^\infty}\|M^\varepsilon\|_{H^k}^2 \\
&\le C(\|\mathcal{J}_\varepsilon M^\varepsilon\|_{L^\infty}^2 + \|\mathcal{J}_\varepsilon H^\varepsilon\|_{L^\infty}^2)(\|M^\varepsilon\|_{H^k}^2 + \|H^\varepsilon\|_{H^k}^2).
\end{aligned} \tag{5.42}$$

For S_7^ε, we split the integral into two parts

$$\begin{aligned}
|S_7^\varepsilon| &\lesssim \left|\int_{\mathbb{R}^3} \partial^k[\mathcal{J}_\varepsilon u^\varepsilon \cdot \nabla \mathcal{J}_\varepsilon(M^\varepsilon + H^\varepsilon)]\partial^k \mathcal{J}_\varepsilon H^\varepsilon dx\right| \\
&\quad + \left|\int_{\mathbb{R}^3} \partial^k(\mathcal{J}_\varepsilon u^\varepsilon \cdot \nabla \mathcal{J}_\varepsilon H^\varepsilon)\partial^k \mathcal{J}_\varepsilon H^\varepsilon dx\right| \\
&:= S_{7a}^\varepsilon + S_{7b}^\varepsilon.
\end{aligned} \tag{5.43}$$

The second term can be estimated exactly like S_4^ε (replacing M^ε with H^ε) by

$$|S_{7b}^\varepsilon| \lesssim C\|\nabla \mathcal{J}_\varepsilon u^\varepsilon\|_{L^\infty}\|H^\varepsilon\|_{H^k}^2 + C(\delta)\|\mathcal{J}_\varepsilon H^\varepsilon\|_{L^\infty}^2\|H^\varepsilon\|_{H^k}^2 + \delta\|D^{k+1}\mathcal{J}_\varepsilon u^\varepsilon\|_{L^2}^2.$$

For the first term, note (integrating by parts and using $\nabla\cdot(a\otimes b)=a\cdot\nabla b$ if $\nabla\cdot a=0$, for both terms in the square brackets when appropriate)

$$\begin{aligned}
S_{7a}^{\varepsilon} &= \left|\int_{\mathbb{R}^3}\partial^k[\mathcal{J}_\varepsilon u_i^\varepsilon\partial_i\mathcal{J}_\varepsilon(M_j^\varepsilon+H_j^\varepsilon)]\partial^k\mathcal{J}_\varepsilon\partial_j\phi dx\right| \\
&= \left|\int_{\mathbb{R}^3}\partial^k\partial_i\partial_j[\mathcal{J}_\varepsilon u_i^\varepsilon\mathcal{J}_\varepsilon(M_j^\varepsilon+H_j^\varepsilon)]\partial^k\mathcal{J}_\varepsilon\phi dx\right| \\
&= \left|\int_{\mathbb{R}^3}\partial^k[\partial_j\mathcal{J}_\varepsilon u_i^\varepsilon\mathcal{J}_\varepsilon(M_j^\varepsilon+H_j^\varepsilon)]\partial^k\mathcal{J}_\varepsilon\partial_i\phi dx\right| \\
&= \left|\int_{\mathbb{R}^3}\partial^k[\mathcal{J}_\varepsilon(M^\varepsilon+H^\varepsilon)\cdot\nabla\mathcal{J}_\varepsilon u^\varepsilon]\partial^k\mathcal{J}_\varepsilon H^\varepsilon dx\right| \\
&\lesssim (\|\mathcal{J}_\varepsilon M^\varepsilon+\mathcal{J}_\varepsilon H^\varepsilon\|_{L^\infty}\|D^{k+1}u^\varepsilon\|_{L^2} \\
&\quad +\|D^k(M^\varepsilon+H^\varepsilon)\|_{L^2}\|D\mathcal{J}_\varepsilon u^\varepsilon\|_{L^\infty})\|D^kH^\varepsilon\|_{L^2} \\
&\lesssim C(\delta)(\|\mathcal{J}_\varepsilon M^\varepsilon\|_{L^\infty}^2+\|\mathcal{J}_\varepsilon H^\varepsilon\|_{L^\infty}^2)\|H^\varepsilon\|_{H^k}^2+\delta\|D^{k+1}u^\varepsilon\|_{L^2}^2 \\
&\quad +\|D\mathcal{J}_\varepsilon u^\varepsilon\|_{L^\infty}(\|M^\varepsilon\|_{H^k}^2+\|H^\varepsilon\|_{H^k}^2),
\end{aligned} \tag{5.44}$$

where in the last step we have again used the corollary to Lemma 4.2.9. Finally our estimate for S_7^ε becomes

$$\begin{aligned}
|S_7^\varepsilon| &\le |S_{7a}^\varepsilon|+|S_{7b}^\varepsilon| \\
&\lesssim C(\delta)(\|\mathcal{J}_\varepsilon M^\varepsilon\|_{L^\infty}^2+\|\mathcal{J}_\varepsilon H^\varepsilon\|_{L^\infty}^2)\|H^\varepsilon\|_{H^k}^2+2\delta\|D^{k+1}u^\varepsilon\|_{L^2}^2 \\
&\quad +\|D\mathcal{J}_\varepsilon u^\varepsilon\|_{L^\infty}(\|M^\varepsilon\|_{H^k}^2+\|H^\varepsilon\|_{H^k}^2).
\end{aligned} \tag{5.45}$$

Next we estimate S_8^ε. Similar to S_5^ε, and using the corollary to Lemma 4.2.9,

$$\begin{aligned}
|S_5^\varepsilon| &\lesssim \|\partial^k[\mathcal{J}_\varepsilon(\nabla \times u) \times (\mathcal{J}_\varepsilon M^\varepsilon)]\|_{L^2}\|H^\varepsilon\|_{H^k} \\
&\leq (\|D\mathcal{J}_\varepsilon u^\varepsilon\|_{L^\infty}\|D^k M^\varepsilon\|_{L^2} \\
&\qquad + \|\mathcal{J}_\varepsilon M^\varepsilon\|_{L^\infty}\|D^{k+1}\mathcal{J}_\varepsilon u^\varepsilon\|_{L^2})\|H^\varepsilon\|_{H^k} \\
&\lesssim \|D\mathcal{J}_\varepsilon u^\varepsilon\|_{L^\infty}\|M^\varepsilon\|_{H^k}^2 \\
&\qquad + C(\delta)\|\mathcal{J}_\varepsilon M^\varepsilon\|_{L^\infty}^2\|H^\varepsilon\|_{H^k}^2 + \delta\|D^{k+1}\mathcal{J}_\varepsilon u^\varepsilon\|_{L^2}^2.
\end{aligned} \tag{5.46}$$

Finally, we estimate S_9^ε similar to S_6^ε:

$$\begin{aligned}
|S_9^\varepsilon| &\lesssim \|\partial^k(\mathcal{J}_\varepsilon M^\varepsilon \times (\mathcal{J}_\varepsilon M^\varepsilon \times \mathcal{J}_\varepsilon H^\varepsilon))\|_{L^2}\|H^\varepsilon\|_{H^k} \\
&\lesssim (\|\mathcal{J}_\varepsilon M^\varepsilon\|_{L^\infty}\|\mathcal{J}_\varepsilon M^\varepsilon \times \mathcal{J}_\varepsilon H^\varepsilon\|_{H^k} \\
&\qquad + \|\mathcal{J}_\varepsilon M^\varepsilon \times \mathcal{J}_\varepsilon H^\varepsilon\|_{L^\infty}\|\mathcal{J}_\varepsilon M^\varepsilon\|_{H^k})\|H^\varepsilon\|_{H^k} \\
&\lesssim \|\mathcal{J}_\varepsilon M^\varepsilon\|_{L^\infty}(\|\mathcal{J}_\varepsilon M^\varepsilon\|_{L^\infty}\|H^\varepsilon\|_{H^k} + \|\mathcal{J}_\varepsilon H^\varepsilon\|_{L^\infty}\|M^\varepsilon\|_{H^k})\|H^\varepsilon\|_{H^k} \\
&\qquad + \|\mathcal{J}_\varepsilon M^\varepsilon \times \mathcal{J}_\varepsilon H^\varepsilon\|_{L^\infty}\|M^\varepsilon\|_{H^k}\|H^\varepsilon\|_{H^k} \\
&\leq C(\|\mathcal{J}_\varepsilon M^\varepsilon\|_{L^\infty}^2 + \|\mathcal{J}_\varepsilon H^\varepsilon\|_{L^\infty}^2)(\|M^\varepsilon\|_{H^k}^2 + \|H^\varepsilon\|_{H^k}^2).
\end{aligned} \tag{5.47}$$

Together, estimates (5.36), (5.38), (5.39), (5.40), (5.41), (5.42), (5.45), (5.46), and (5.47) give,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\partial^k u^\varepsilon\|_{L^2}^2 + \|\partial^k M^\varepsilon\|_{L^2}^2 + \|\partial^k H^\varepsilon\|_{L^2}^2) + \|\partial^k \mathcal{J}_\varepsilon \nabla u\|_{L^2} + \|\partial^k \mathcal{J}_\varepsilon \nabla M^\varepsilon\|_{L^2}^2 \\
&\qquad + \|\partial^k \mathcal{J}_\varepsilon \nabla H^\varepsilon\|_{L^2}^2 + \|M^\varepsilon\|_{L^2}^2 + 3\|\partial^k H^\varepsilon\|_{L^2}^2 \\
&\quad \leq C(\delta)(\|\mathcal{J}_\varepsilon M^\varepsilon\|_{L^\infty} + \|\mathcal{J}_\varepsilon H^\varepsilon\|_{L^\infty} + \|\mathcal{J}_\varepsilon M^\varepsilon\|_{L^\infty}^2 + \|\mathcal{J}_\varepsilon H^\varepsilon\|_{L^\infty}^2 + \|D\mathcal{J}_\varepsilon u^\varepsilon\|_{L^\infty}) \\
&\qquad \times (\|u^\varepsilon\|_{H^k}^2 + \|M^\varepsilon\|_{H^k}^2 + \|H^\varepsilon\|_{H^k}^2) + 7\delta\|D^{k+1}\mathcal{J}_\varepsilon u^\varepsilon\|_{L^2}^2.
\end{aligned} \tag{5.48}$$

Choosing $\delta < 1/8$, summing over all derivatives of order k, and using Young's inequality gives,

$$\begin{aligned} &\frac{d}{dt}(1 + \|u^\varepsilon\|_{H^k}^2 + \|M^\varepsilon\|_{H^k}^2 + \|H^\varepsilon\|_{H^k}^2) \\ &\quad \leq C(1 + \|\mathcal{J}_\varepsilon M^\varepsilon\|_{L^\infty}^2 + \|\mathcal{J}_\varepsilon H^\varepsilon\|_{L^\infty}^2 + \|D\mathcal{J}_\varepsilon u^\varepsilon\|_{L^\infty}) \\ &\qquad \times (1 + \|u^\varepsilon\|_{H^k}^2 + \|M^\varepsilon\|_{H^k}^2 + \|H^\varepsilon\|_{H^k}^2). \end{aligned} \tag{5.49}$$

For the terms involving the L^∞-norm of M^ε and H^ε, we use the Sobolev inequality $\|f\|_{L^\infty(\mathbb{R}^3)} \leq C\|f\|_{H^k(\mathbb{R}^3)}$ for $k > 3/2$. For the $\|D\mathcal{J}_\varepsilon u^\varepsilon\|_{L^\infty}$ term, we have by the Gagliardo Nirenberg inequality,

$$\|Du^\varepsilon\|_{L^\infty} \leq C\|D^k u^\varepsilon\|_{L^2}^\alpha \|u\|_{L^\infty}^{1-\alpha},$$

where $\alpha = \frac{2}{2k-3}$ satisfies $\frac{1}{k} \leq \alpha \leq 1$ for all $k > 5/2$. By using Young's inequality, and the Sobolev inequality again, we bound (5.49) by a quadratic:

$$\begin{aligned} &\frac{d}{dt}(1 + \|u^\varepsilon\|_{H^k}^2 + \|M^\varepsilon\|_{H^k}^2 + \|H^\varepsilon\|_{H^k}^2) \\ &\qquad\qquad \leq C(1 + \|u^\varepsilon\|_{H^k}^2 + \|M^\varepsilon\|_{H^k}^2 + \|H^\varepsilon\|_{H^k}^2)^2. \end{aligned} \tag{5.50}$$

Denote $E_\varepsilon(t) := 1 + \|u^\varepsilon\|_{H^k}^2 + \|M^\varepsilon\|_{H^k}^2 + \|H^\varepsilon\|_{H^k}^2$. Then solving inequality (5.50) gives,

$$\sup_{0 \leq t \leq T} E_\varepsilon(t) \leq \frac{E_\varepsilon(0)}{1 - CTE_\varepsilon(0)}. \tag{5.51}$$

In particular, our solution $(u^\varepsilon, M^\varepsilon, H^\varepsilon)$ is uniformly bounded in $C([0,T]; H^k(\mathbb{R}^3))$ for $k > 5/2$ for all $T < \frac{1}{CE_\varepsilon(0)}$.

Moreover, we can bound the time derivatives $\frac{d}{dt}(u^\varepsilon, M^\varepsilon, H^\varepsilon)$ as follows: From equations (5.1) and from relation (4.39),

$$\begin{aligned}\left\|\frac{du^\varepsilon}{dt}\right\|_{H^{k-2}} + \left\|\frac{dM^\varepsilon}{dt}\right\|_{H^{k-2}} + \left\|\frac{dH^\varepsilon}{dt}\right\|_{H^{k-2}} &\lesssim (\|u^\varepsilon\|_{H^k} + \|M^\varepsilon\|_{H^k} + \|H^\varepsilon\|_{H^k})^3 \\ &\lesssim E_\varepsilon(t)^3 \lesssim E_\varepsilon(0)^3,\end{aligned} \tag{5.52}$$

for $k > 5/2$; in the above we have used either the obvious inequalities, or the Sobolev embedding $H^k \subset H^{k-1}$ on $\mathbb{R}^3$ where appropriate to achieve this bound. Note that if the trilinear term were absent, we could achieve a quadratic bound here. Thus, our solution $(u^\varepsilon, M^\varepsilon, H^\varepsilon)$ is also uniformly bounded in $\mathrm{Lip}\{[0,T]; H^{k-2}(\mathbb{R}^3)\}$ for $k > 5/2$.

Remark. Because we were able to ensure all of the $(k+1)$-th derivatives fell on u^ε in estimates (5.36)-(5.47), this bound can also be achieved for systems without a Bloch-Torrey magnetization term.

5.1.4 Limit Point in $C([0, T]; L^2)$

In this subsection, we show the sequence $(u^\varepsilon, M^\varepsilon, H^\varepsilon)$ has a subsequence which converges to a limit point (u, M, H) in the space $C([0, T]; L^2)$. Our plan is to apply the Aubin-Lions compactness theorem (Theorem 4.1.3) to show the sequence $(u^\varepsilon, M^\varepsilon, H^\varepsilon)$ is precompact in $C([0, T]; H^{k'}(\mathbb{R}^3))$ for $(k-2) \leq k' < k$. Then since $H^{k'}_{\text{loc}}$ is compactly embedded in L^2_σ, L^2, there exists a subsequence which converges to a limit point (u, M, H) in $C([0, T]; L^2)$. In the previous step, bounds (5.50) and (5.52) gave us

$$\{(u^\varepsilon, M^\varepsilon, H^\varepsilon)\} \text{ is uniformly bounded in } C([0, T]; H^k(\mathbb{R}^3)),$$
$$\{(u^\varepsilon, M^\varepsilon, H^\varepsilon)\} \text{ is uniformly bounded in } \mathrm{Lip}([0, T]; H^{k-2}(\mathbb{R}^3)),$$

respectively. In particular, the uniform bound in the Lipschitz space gives $\{(u^\varepsilon, M^\varepsilon, H^\varepsilon)\}$ is uniformly equicontinuous on $[0, T]$ with values in $H^{k-2}(\mathbb{R}^3)$. By the Rellich-Kondrachov compactness theorem, the embedding $H^k \subset H^{k'}$ is locally compact on $\mathbb{R}^3$. Therefore, by Theorem 4.1.3 with $X = H^k, Y = H^{k'}, Z = H^{k-2}$, $(u^\varepsilon, M^\varepsilon, H^\varepsilon)$ is precompact in $C([0, T]; H^{k'})$ for all $(k-2) \leq k' < k$ and therefore has a limit point in $C([0, T]; L^2)$. Moreover the subsequence of $(u^\varepsilon, M^\varepsilon, H^\varepsilon)$ converges to (u, M, H) in $H^{k'}$ for all $(k-2) \leq k' < k$.

5.1.5 Limit of Approximate Solutions Solves Equation

In this subsection, we prove that the limit (u, M, H) solves the Shliomis system. Using the fact that $(u^\varepsilon, M^\varepsilon, H^\varepsilon)$ solves (5.1), we obtain considering an integrated form of the momentum equation in (5.1) using Lemma 4.2.2:

$$
\begin{aligned}
u(t)-&u(0)+\int_0^t \mathbb{P}(u(s)\cdot\nabla u(s))ds-\frac{\eta}{\rho}\int_0^t \Delta u(s)ds\\
&-\frac{\mu_0}{\rho}\int_0^t \mathbb{P}(M(s)\cdot\nabla H(s))ds-\frac{\mu_0}{2\rho}\int_0^t \nabla\times(M(s)\times H(s))ds\\
&=(u(t)-u^\varepsilon(t))-(1-\mathcal{J}_\varepsilon)u(0)+\int_0^t \mathbb{P}((u(s)-u^\varepsilon(s))\cdot\nabla u(s))ds\\
&+\int_0^t \mathbb{P}(u^\varepsilon(s)\cdot\nabla(u(s)-u^\varepsilon(s)))ds+\int_0^t \mathbb{P}(1-\mathcal{J}_\varepsilon)(u^\varepsilon(s)\cdot\nabla u^\varepsilon(s))ds\\
&+\int_0^t \mathbb{P}((1-\mathcal{J}_\varepsilon)u^\varepsilon(s)\cdot\nabla u^\varepsilon)ds+\int_0^t \mathbb{P}(\mathcal{J}_\varepsilon u^\varepsilon(s)\cdot\nabla(1-\mathcal{J}_\varepsilon)u^\varepsilon(s))ds\\
&-\frac{\eta}{\rho}\int_0^t \Delta(u(s)-\mathcal{J}_\varepsilon^2 u^\varepsilon(s))ds-\frac{\mu_0}{\rho}\left\{\int_0^t \mathbb{P}((M(s)-M^\varepsilon(s))\cdot\nabla H(s))ds\right.\\
&+\int_0^t \mathbb{P}(M^\varepsilon(s)\cdot\nabla(H(s)-H^\varepsilon(s)))ds+\int_0^t \mathbb{P}(1-\mathcal{J}_\varepsilon)(M^\varepsilon(s)\cdot\nabla H^\varepsilon(s))ds\\
&\left.+\int_0^t \mathbb{P}((1-\mathcal{J}_\varepsilon)M^\varepsilon(s)\cdot\nabla H^\varepsilon)ds+\int_0^t \mathbb{P}(\mathcal{J}_\varepsilon M^\varepsilon(s)\cdot\nabla(1-\mathcal{J}_\varepsilon)H^\varepsilon(s))ds\right\}\\
&+\frac{\mu_0}{2\rho}\left\{\int_0^t \nabla\times[(M(s)-M^\varepsilon(s))\times H(s)]ds\right.\\
&+\int_0^t \nabla\times[M^\varepsilon(s)\times(H(s)-H^\varepsilon(s))]ds+\int_0^t (1-\mathcal{J}_\varepsilon)\nabla\times(M^\varepsilon(s)\times H^\varepsilon(s))ds\\
&\left.+\int_0^t \mathcal{J}_\varepsilon\nabla\times[(1-\mathcal{J}_\varepsilon)M^\varepsilon(s)\times H^\varepsilon(s)]ds+\int_0^t \nabla\times[\mathcal{J}_\varepsilon M^\varepsilon(s)\times(1-\mathcal{J}_\varepsilon)H^\varepsilon(s)]ds\right\}\\
&=D_1+D_2+D_3+D_4+D_5+D_6+D_7+D_8+D_9+D_{10}+D_{11}+D_{12}\\
&\quad+D_{13}+D_{14}+D_{15}+D_{16}+D_{17}+D_{18}.
\end{aligned}
$$

We take the L^2-norm of both sides of equation (5.1.5), and find for the terms D_1-D_{18}:

First, for D_1,

$$
\begin{aligned}
\|D_1\|_{L^2}&=\|u-u^\varepsilon\|_{L^\infty(0,T;L^2(\mathbb{R}^3))}\\
&\to 0 \text{ as } \varepsilon\to 0.
\end{aligned}
\tag{5.53}
$$

Next,

$$\begin{aligned}\|D_2\|_{L^2} &\leq C\varepsilon\|u_0\|_{H^1} \\ &\to 0 \text{ as } \varepsilon \to 0,\end{aligned} \tag{5.54}$$

by Lemma 4.1.1 property (iv). For D_3 we find,

$$\begin{aligned}\|D_3\|_{L^2} &\leq \int_0^t \|u(s) - u^\varepsilon(s)\|_{L^2}\|\nabla u(s)\|_{L^\infty} ds \\ &\leq T\|u - u^\varepsilon\|_{L^\infty(0,T;L^2)}\|u\|_{L^\infty(0,T;H^k)} \\ &\to 0 \text{ as } \varepsilon \to 0,\end{aligned} \tag{5.55}$$

where we have used the Sobolev embedding $H^k \subset W^{1,\infty}$ (for $k > d/2 + 1$). Next, for D_4,

$$\begin{aligned}\|D_4\|_{L^2} &\leq \int_0^t \|u^\varepsilon(s)\|_{L^2}\|\nabla u(s) - \nabla u^\varepsilon(s)\|_{L^\infty} ds \\ &\leq T\|u^\varepsilon\|_{L^\infty(0,T;L^2)}\|\nabla u - \nabla u^\varepsilon\|_{L^\infty(0,T;L^2)} \\ &\leq T\|u^\varepsilon\|_{L^\infty(0,T;L^2)}\|u - u^\varepsilon\|_{L^\infty(0,T;H^k)} \\ &\to 0 \text{ as } \varepsilon \to 0,\end{aligned} \tag{5.56}$$

where we have used the same Sobolev embedding. For D_5, since $\nabla \cdot u^\varepsilon = 0$,

$$\begin{aligned}\|D_5\|_{L^2} &\leq \int_0^t \|(1 - \mathcal{J}_\varepsilon)\nabla \cdot (u^\varepsilon(s) \otimes u^\varepsilon(s))\|_{L^2} ds \\ &\leq C\varepsilon \int_0^t \|u^\varepsilon(s) \otimes u^\varepsilon(s)\|_{H^k} ds \\ &\leq C\varepsilon T\|u^\varepsilon\|^2_{L^\infty(0,T;H^k)} \\ &\to 0 \text{ as } \varepsilon \to 0,\end{aligned} \tag{5.57}$$

where we have used $\|\cdot\|_{H^1} \leq C\|\cdot\|_{H^{\frac{3}{2}+\kappa}}$ and that H^s is an algebra for $s > 3/2$

and finally that $\|\cdot\|_{H^{\frac{3}{2}+\kappa}} \leq C\|\cdot\|_{H^k}$. Next, for D_6 we have

$$\begin{aligned}
\|D_6\|_{L^2} &\leq C\varepsilon \int_0^t \|u^\varepsilon(s)\|_{H^1}\|\nabla u^\varepsilon(s)\|_{L^\infty} ds \\
&\leq C\varepsilon \int_0^t \|u^\varepsilon(s)\|_{H^1}\|\nabla u^\varepsilon(s)\|_{L^\infty} ds \\
&\leq C\varepsilon T\|u^\varepsilon\|^2_{L^\infty(0,T;H^k)} \\
&\to 0 \text{ as } \varepsilon \to 0.
\end{aligned} \tag{5.58}$$

Similarly for D_7,

$$\begin{aligned}
\|D_7\|_{L^2} &\leq C\varepsilon \int_0^t \|u^\varepsilon(s)\|_{L^\infty}\|u^\varepsilon(s)\|_{H^2} ds \\
&\leq C\varepsilon T\|u^\varepsilon\|^2_{L^\infty(0,T;H^k)} \\
&\to 0 \text{ as } \varepsilon \to 0.
\end{aligned} \tag{5.59}$$

For D_8, we estimate

$$\begin{aligned}
\|D_8\|_{L^2} &\leq C \int_0^t \|\Delta(u(s) - \mathcal{J}_\varepsilon^2 u^\varepsilon(s))\|_{L^2} ds \\
&\leq C \int_0^t \|\Delta(u(s) - u^\varepsilon(s)) + \Delta(1-\mathcal{J}_\varepsilon)u^\varepsilon(s) + \Delta\mathcal{J}_\varepsilon(1-\mathcal{J}_\varepsilon)u^\varepsilon(s)\|_{L^2} ds \\
&\to 0 \text{ as } \varepsilon \to 0.
\end{aligned} \tag{5.60}$$

The terms D_9, D_{10}, D_{12} and D_{13} can be estimated in the same way as D_3, D_4, D_6 and D_7, respectively. Since $\nabla \cdot M^\varepsilon \neq 0$ in general, we estimate D_{11} as follows:

$$\begin{aligned}
\|D_{11}\|_{L^2} &\lesssim \varepsilon \int_0^T \|M^\varepsilon(s)\cdot\nabla H^\varepsilon(s)\|_{H^1} ds \lesssim \varepsilon \int_0^T \|M^\varepsilon(s)\|_{H^{\frac{3}{2}+\kappa}}\|H^\varepsilon(s)\|_{H^{\frac{5}{2}+\kappa}} ds \\
&\overset{\kappa\to 0}{\lesssim} \varepsilon T(\|M^\varepsilon\|^2_{L^\infty(0,T;H^k)} + \|H^\varepsilon\|^2_{L^\infty(0,T;H^k)}) \\
&\to 0 \text{ as } \varepsilon \to 0,
\end{aligned} \tag{5.61}$$

where $\kappa > 0$ was an arbitrarily small positive constant such that $k = 5/2 + \kappa$. (Equivalently, we could send $\kappa \to 0$, and use the appropriate Sobolev inequality.) Using the identity

$$\nabla \times (A \times B) = A(\nabla \cdot B) + B(\nabla \cdot A) + (B \cdot \nabla)A - (A \cdot \nabla)B, \tag{5.62}$$

we estimate D_{14}:

$$\begin{aligned}
\|D_{14}\|_{L^2} &\lesssim \int_0^t (\|M(s) - M^\varepsilon(s)\|_{L^2}\|\nabla H(s)\|_{L^\infty} + \|M(s) - M^\varepsilon(s)\|_{H^1}\|H(s)\|_{L^\infty})ds \\
&\lesssim T\|H\|_{L^\infty(0,T;H^1)}\|M - M^\varepsilon\|_{L^\infty(0,T;H^k)} \\
&\quad + T\|M - M^\varepsilon\|_{L^\infty(0,T;H^1)}\|H\|_{L^\infty(0,T;L^\infty)} \\
&\to 0 \text{ as } \varepsilon \to 0,
\end{aligned} \tag{5.63}$$

and similarly D_{15}:

$$\begin{aligned}
\|D_{15}\|_{L^2} &\lesssim T\|M^\varepsilon\|_{L^\infty(0,T;H^1)}\|H - H^\varepsilon\|_{L^\infty(0,T;H^k)} \\
&\quad + T\|H - H^\varepsilon\|_{L^\infty(0,T;H^1)}\|M^\varepsilon\|_{L^\infty(0,T;L^\infty)} \\
&\to 0 \text{ as } \varepsilon \to 0.
\end{aligned} \tag{5.64}$$

Next, we estimate D_{16}:

$$\begin{aligned}
\|D_{16}\|_{L^2} &\lesssim \varepsilon \int_0^t \|M^\varepsilon(s) \times H^\varepsilon(s)\|_{H^2} ds \\
&\lesssim \varepsilon \int_0^t \|M^\varepsilon(s)\|_{H^2}\|H^\varepsilon(s)\|_{H^2} ds \\
&\leq C\varepsilon T(\|M^\varepsilon\|^2_{L^\infty(0,T;H^k)} + \|H^\varepsilon\|^2_{L^\infty(0,T;H^k)}) \\
&\to 0 \text{ as } \varepsilon \to 0.
\end{aligned} \tag{5.65}$$

Using the same trick as we did for D_{11} (5.61), we find

$$\begin{aligned}\|D_{17}\|_{L^2} &\lesssim \int_0^t \|(1-\mathcal{J}_\varepsilon)M^\varepsilon(s)\times H^\varepsilon(s)\|_{H^{3/2+\kappa}}ds\\ &\lesssim \int_0^t \|(1-\mathcal{J}_\varepsilon)M^\varepsilon(s)\|_{H^{3/2+\kappa}}\|H^\varepsilon(s)\|_{H^{3/2+\kappa}}ds\\ &\lesssim \varepsilon\int_0^t \|M^\varepsilon(s)\|_{H^{5/2+\kappa}}\|H^\varepsilon(s)\|_{H^k}ds\\ &\overset{\kappa\to 0}{\lesssim} \varepsilon T(\|M^\varepsilon\|^2_{L^\infty(0,T;H^k)}+\|H^\varepsilon\|^2_{L^\infty(0,T;H^k)})\\ &\to 0 \text{ as } \varepsilon\to 0.\end{aligned}\tag{5.66}$$

Finally, in exactly the same way as for D_{17}, D_{18} can be estimated by,

$$\begin{aligned}\|D_{18}\|_{L^2} &\lesssim \varepsilon T(\|M^\varepsilon\|^2_{L^\infty(0,T;H^k)}+\|H^\varepsilon\|^2_{L^\infty(0,T;H^k)})\\ &\to 0 \text{ as } \varepsilon\to 0.\end{aligned}\tag{5.67}$$

Therefore the limit solution solves the momentum equation.

Similarly, for the magnetization equation,

$$\begin{aligned}&M(t)-M(0)+\int_0^t u(s)\cdot\nabla M(s)ds-\sigma\int_0^t \Delta M(s)ds\\ &\quad -\frac{1}{2}\int_0^t(\nabla\times u(s))\times M(s)ds+\frac{1}{\tau}(M(s)-\chi_0 H(s))ds\\ &\quad +\beta\int_0^t M(s)\times(M(s)\times H(s))ds\\ &\quad =(M(t)-M^\varepsilon(t))-(1-\mathcal{J}_\varepsilon)M(0)+\int_0^t \mathbb{P}((u(s)-u^\varepsilon(s))\cdot\nabla M(s))ds\\ &\quad +\int_0^t \mathbb{P}(u^\varepsilon(s)\cdot\nabla(M(s)-M^\varepsilon(s)))ds+\int_0^t \mathbb{P}(1-\mathcal{J}_\varepsilon)(u^\varepsilon(s)\cdot\nabla M^\varepsilon(s))ds\\ &\quad +\int_0^t \mathbb{P}((1-\mathcal{J}_\varepsilon)u^\varepsilon(s)\cdot\nabla M^\varepsilon)ds+\int_0^t \mathbb{P}(\mathcal{J}_\varepsilon u^\varepsilon(s)\cdot\nabla(1-\mathcal{J}_\varepsilon)M^\varepsilon(s))ds\\ &\quad +\int_0^t \Delta(M(s)-\mathcal{J}_\varepsilon^2 M^\varepsilon(s))ds-\frac{1}{2}\left\{\int_0^t[\nabla\times(u(s)-u^\varepsilon(s))]\times M(s)]ds\right.\\ &\quad +\int_0^t[\nabla\times u^\varepsilon(s)]\times(M(s)-M^\varepsilon(s))+\int_0^t(1-\mathcal{J}_\varepsilon)[(\nabla\times(u^\varepsilon(s))\times M^\varepsilon(s))]ds\end{aligned}$$

$$
\begin{aligned}
&+ \int_0^t \mathcal{J}_\varepsilon(\nabla \times [(1-\mathcal{J}_\varepsilon)u^\varepsilon(s)] \times M^\varepsilon(s))ds \\
&+ \int_0^t \mathcal{J}_\varepsilon[(\nabla \times \mathcal{J}_\varepsilon u^\varepsilon(s)) \times (1-\mathcal{J}_\varepsilon)M^\varepsilon(s)]ds\Big\} \\
&+ \frac{1}{\tau}\int_0^t [M(s) - M^\varepsilon(s)]ds - \frac{\chi_0}{\tau}\int_0^t [H(s) - H^\varepsilon(s)]ds \\
&+ \beta\Big\{\int_0^t (M(s) - M^\varepsilon(s)) \times (M(s) \times H(s))ds \\
&+ \int_0^t M^\varepsilon(s) \times [M^\varepsilon(s) \times (H(s) - H^\varepsilon(s))]ds \\
&+ \int_0^t M^\varepsilon(s) \times [M^\varepsilon(s) \times (H(s) - H^\varepsilon(s))]ds \\
&+ \int_0^t (1-\mathcal{J}_\varepsilon)M^\varepsilon(s) \times (M^\varepsilon(s) \times H^\varepsilon(s))ds \\
&+ \int_0^t \mathcal{J}_\varepsilon M^\varepsilon \times [(1-\mathcal{J}_\varepsilon)M^\varepsilon(s) \times H^\varepsilon]ds + \int_0^t \mathcal{J}_\varepsilon M^\varepsilon \times [\mathcal{J}_\varepsilon M^\varepsilon \times (1-\mathcal{J}_\varepsilon)H^\varepsilon(s)]ds\Big\} \\
&= F_1 + F_2 + F_3 + F_4 + F_5 + F_6 + F_7 + F_8 + F_9 + F_{10} + F_{11} + F_{12} \\
&\quad + F_{13} + F_{14} + F_{15} + F_{16} + F_{17} + F_{18} + F_{19} + F_{20} + F_{21}
\end{aligned}
$$

In the same way as D_1 and D_2, we bound

$$
\begin{aligned}
\|F_1\|_{L^2} &= \|M - M^\varepsilon\|_{L^\infty(0,T;L^2(\mathbb{R}^3))} \\
&\to 0 \text{ as } \varepsilon \to 0,
\end{aligned} \tag{5.68}
$$

$$
\begin{aligned}
\|F_2\|_{L^2} &\le C\varepsilon\|M_0\|_{L^\infty(0,T;H^k)} \\
&\to 0 \text{ as } \varepsilon \to 0.
\end{aligned} \tag{5.69}
$$

In exactly the same way as $D_3 - D_7$, we estimate

$$
\|F_3\|_{L^2} \le T\|u - u^\varepsilon\|_{L^\infty(0,T;L^2)}\|M\|_{L^\infty(0,T;H^k)} \to 0, \tag{5.70}
$$

$$
\|F_4\|_{L^2} \le T\|u^\varepsilon\|_{L^\infty(0,T;L^2)}\|M - M^\varepsilon\|_{L^\infty(0,T;H^k)} \to 0, \tag{5.71}
$$

$$\|F_5\|_{L^2} \leq C\varepsilon T(\|u^\varepsilon\|^2_{L^\infty(0,T;H^k)} + \|M^\varepsilon\|^2_{L^\infty(0,T;H^k)}) \to 0, \tag{5.72}$$

$$\|F_6\|_{L^2} \leq C\varepsilon T(\|u^\varepsilon\|^2_{L^\infty(0,T;H^k)} + \|M^\varepsilon\|^2_{L^\infty(0,T;H^k)}) \to 0, \tag{5.73}$$

$$\|F_7\|_{L^2} \leq C\varepsilon T(\|u^\varepsilon\|^2_{L^\infty(0,T;H^k)} + \|M^\varepsilon\|^2_{L^\infty(0,T;H^k)}) \to 0. \tag{5.74}$$

In exactly the same way as D_8,

$$\|F_8\|_{L^2} \to 0 \text{ as } \varepsilon \to 0. \tag{5.75}$$

For F_9, we estimate

$$\begin{aligned} \|F_9\|_{L^2} &\lesssim \int_0^t \|u(s) - u^\varepsilon(s)\|_{H^1} \|M(s)\|_{L^\infty} ds \\ &\lesssim T\|u - u^\varepsilon\|_{L^\infty(0,T;H^1)} \|M\|_{L^\infty(0,T;H^k)} \\ &\to 0 \text{ as } \varepsilon \to 0. \end{aligned} \tag{5.76}$$

Next, for F_{10},

$$\begin{aligned} \|F_{10}\|_{L^2} &\lesssim \int_0^t \|\nabla u^\varepsilon(s)\|_{L^\infty} \|M(s) - M^\varepsilon(s)\|_{L^2} ds \\ &\lesssim \int_0^t \|\nabla u^\varepsilon(s)\|_{H^{3/2+\kappa}} \|M(s) - M^\varepsilon(s)\|_{L^2} ds \\ &\overset{\kappa \to 0}{\lesssim} T\|u^\varepsilon\|_{L^\infty(0,T;H^k)} \|M - M^\varepsilon\|_{L^\infty(0,T;L^2)} \to 0 \text{ as } \varepsilon \to 0. \end{aligned} \tag{5.77}$$

Next, we estimate

$$\begin{aligned}
\|F_{11}\|_{L^2} &\lesssim \varepsilon \int_0^t \|(\nabla \times u^\varepsilon(s)) \times M^\varepsilon(s)\|_{H^{3/2+\kappa}} ds \\
&\lesssim \varepsilon \int_0^T \|\nabla \times u^\varepsilon(s)\|_{H^{3/2+\kappa}} \|M^\varepsilon(s)\|_{H^{3/2+\kappa}} ds \\
&\overset{\kappa \to 0}{\lesssim} \varepsilon T(\|u^\varepsilon\|^2_{L^\infty(0,T;H^k)} + \|M^\varepsilon\|^2_{L^\infty(0,T;H^k)}) \\
&\to 0 \text{ as } \varepsilon \to 0.
\end{aligned} \tag{5.78}$$

where $\kappa > 0$ was an arbitrary positive constant as above. Now we estimate

$$\begin{aligned}
\|F_{12}\|_{L^2} &\lesssim \varepsilon \int_0^t \|u^\varepsilon(s)\|_{H^2} \|M^\varepsilon(s)\|_{L^\infty} ds \\
&\lesssim \varepsilon T(\|u^\varepsilon\|^2_{L^\infty(0,T;H^k)} + \|M^\varepsilon\|^2_{L^\infty(0,T;H^k)}) \\
&\to 0 \text{ as } \varepsilon \to 0.
\end{aligned} \tag{5.79}$$

For F_{13}, we have (similar to previous estimates)

$$\begin{aligned}
\|F_{13}\|_{L^2} &\lesssim \varepsilon \int_0^t \|u^\varepsilon(s)\|_{H^k} \|M^\varepsilon(s)\|_{H^k} ds \\
&\lesssim \varepsilon T \|u^\varepsilon\|_{L^\infty(0,T;H^k)} \|M^\varepsilon\|_{L^\infty(0,T;H^k)} \\
&\to 0 \text{ as } \varepsilon \to 0.
\end{aligned} \tag{5.80}$$

We trivially estimate,

$$\|F_{14}\|_{L^2} \leq C\|M - M^\varepsilon\|_{L^\infty(0,T;L^2)} \to 0, \tag{5.81}$$

$$\|F_{15}\|_{L^2} \leq C\|H - H^\varepsilon\|_{L^\infty(0,T;L^2)} \to 0. \tag{5.82}$$

Next, we estimate

$$\begin{aligned}\|F_{16}\|_{L^2} &\lesssim \int_0^t \|M(s)-M^\varepsilon(s)\|_{L^2}\|M(s)\|_{L^\infty}\|H(s)\|_{L^\infty}ds \\ &\lesssim \|M-M^\varepsilon\|_{L^\infty(0,T;L^2)}T(\|M\|^2_{L^\infty(0,T;H^k)}+\|H\|^2_{L^\infty(0,T;H^k)}) \\ &\to 0 \text{ as } \varepsilon\to 0.\end{aligned} \tag{5.83}$$

Similarly,

$$\|F_{17}\|_{L^2} \lesssim T\|M^\varepsilon\|^2_{L^\infty(0,T;H^k)}\|H-H^\varepsilon\|_{L^\infty(0,T;L^2)} \to 0, \tag{5.84}$$

and

$$\|F_{18}\|_{L^2} \lesssim T\|M^\varepsilon\|^2_{L^\infty(0,T;H^k)}\|H-H^\varepsilon\|_{L^\infty(0,T;L^2)} \to 0. \tag{5.85}$$

Finally we estimate,

$$\begin{aligned}\|F_{19}\|_{L^2} &\lesssim \int_0^t \|(1-\mathcal{J}_\varepsilon)M^\varepsilon(s)\|_{L^2}\|M^\varepsilon(s)\|_{L^\infty(0,T;L^\infty)}\|H^\varepsilon(s)\|_{L^\infty(0,T;L^\infty)}ds \\ &\lesssim \varepsilon T\|M^\varepsilon\|_{L^\infty(0,T;H^k)}(\|M^\varepsilon\|^2_{L^\infty(0,T;H^k)}+\|H^\varepsilon\|^2_{L^\infty(0,T;H^k)}) \\ &\to 0 \text{ as } \varepsilon\to 0,\end{aligned} \tag{5.86}$$

and similarly

$$\|F_{20}\|_{L^2} \lesssim \varepsilon T\|M^\varepsilon\|_{L^\infty(0,T;H^k)}(\|M^\varepsilon\|^2_{L^\infty(0,T;H^k)}+\|H^\varepsilon\|^2_{L^\infty(0,T;H^k)}) \to 0, \tag{5.87}$$

and

$$\|F_{21}\|_{L^2} \lesssim \varepsilon T\|M^\varepsilon\|^2_{L^\infty(0,T;H^k)}\|H^\varepsilon\|_{L^\infty(0,T;H^k)} \to 0. \tag{5.88}$$

The above shows that equations (2.15)-(2.17), (2.19) holds in L^2 (we don't need to consider the equation for the time derivative of H since it is a linear function of M. However, we can achieve pointwise equality by noting strong convergence in $C([0,T];H^k)$ for $0 < 7/2 < k$ implies strong convergence in $C([0,T];(C^2(\mathbb{R}^3))^3)$ (note that this is sufficient to guarantee continuity of u_t, M_t, H_t). Therefore (u, M, H) solves the Shliomis model pointwise.

5.1.6 Uniqueness of Limit Solution

Next, we prove uniqueness of the limit solution. Suppose (u^1, M^1, H^1) and (u^2, M^2, H^2) are two solutions with the same initial data (u_0, M_0, H_0). Then they would obey

$$
\begin{aligned}
&\rho\{\partial_t(u^1 - u^2) + \mathbb{P}[(u^1 - u^2)\cdot\nabla u^1 + u^2\cdot\nabla(u^1 - u^2)]\} - \eta\Delta(u^1 - u^2) \\
&\quad = \mathbb{P}\mu_0[(M^1 - M^2)\cdot\nabla H^1] + \mathbb{P}\mu_0[M^2\cdot\nabla(H^1 - H^2)] \\
&\quad + \frac{\mu_0}{2}\nabla\times[(M^1 - M^2)\times H^1] + \frac{\mu_0}{2}\nabla\times[M^2\times(H^1 - H^2)],
\end{aligned} \tag{5.89}
$$

and

$$
\begin{aligned}
&\partial_t(M^1 - M^2) + (u^1 - u^2)\cdot\nabla M^1 + u^2\cdot\nabla(M^1 - M^2) \\
&\quad = \frac{1}{2}[\nabla\times(u^1 - u^2)]\times M^1 + \frac{1}{2}(\nabla\times u^2)\times(M^1 - M^2) \\
&\quad - \frac{1}{\tau}[(M^1 - M^2) - \chi_0(H^1 - H^2)] + \sigma\Delta(M^1 - M^2) \\
&\quad - \beta\{(M^1 - M^2)\times[(M^1\times H^1)] + M^2\times[(M^1 - M^2)\times H^1] \\
&\quad + M^2\times[M^2\times(H^1 - H^2)]\}.
\end{aligned} \tag{5.90}
$$

Taking the L^2-inner product of (5.89) with $u - v$ gives,

$$\begin{aligned}
\frac{\rho}{2}\frac{d}{dt}\|u^1 - u^2\|_{L^2}^2 &= -\rho\mathbb{P}\int_{\mathbb{R}^3}(u^1 - u^2)\cdot[(u^1 - u^2)\cdot\nabla u^1]dx \\
&\quad - \rho\mathbb{P}\int_{\mathbb{R}^3}(u^1 - u^2)\cdot[u^2\cdot\nabla(u^1 - u^2)]dx \\
&\quad + \eta\int_{\mathbb{R}^3}(u^1 - u^2)\cdot\Delta(u^1 - u^2)dx \\
&\quad + \mu_0\mathbb{P}\int_{\mathbb{R}^3}(M^1 - M^2)\cdot\nabla H^1\cdot(u^1 - u^2)dx \\
&\quad + \mu_0\mathbb{P}\int_{\mathbb{R}^3}M^2\cdot\nabla(H^1 - H^2)\cdot(u^1 - u^2)dx \\
&\quad + \frac{\mu_0}{2}\int_{\mathbb{R}^3}\nabla\times[(M^1 - M^2)\times H^1]\cdot(u^1 - u^2)dx \\
&\quad + \frac{\mu_0}{2}\int_{\mathbb{R}^3}\nabla\times[M^2\times(H^1 - H^2)]\cdot(u^1 - u^2)dx \\
&= G_1 + G_2 + G_3 + G_4 + G_5 + G_6 + G_7. \qquad (5.91)
\end{aligned}$$

The first term can be estimated by

$$|G_1| \leq \rho\|\nabla u^1\|_{L^\infty}\|u^1 - u^2\|_{L^2}^2. \qquad (5.92)$$

Since $\nabla\cdot u^2 = 0$, the second term vanishes. Upon integrating by parts, the third term becomes negative so move it to the left-hand side to absorb bad terms together. The fourth term can be bounded as follows:

$$|G_4| \lesssim \mu_0\|\nabla H^1\|_{L^\infty}(\|M^1 - M^2\|_{L^2}^2 + \|u^1 - u^2\|_{L^2}^2). \qquad (5.93)$$

Since $\nabla\times H^i = 0$ for $i = 1, 2$, as above $M^2\cdot\nabla(H^1 - H^2)\cdot(u^1 - u^2) = \nabla(M^2\cdot(H^1 - H^2))\cdot(u^1 - u^2) - (u^1 - u^2)\cdot\nabla M^2(H^1 - H^2)$. The first term vanishes upon integration by parts and we bound the contribution from the second by

$$|G_5| \lesssim \mu_0 \|\nabla M^2\|_{L^\infty} (\|H^1 - H^2\|_{L^2}^2 + \|u^1 - u^2\|_{L^2}^2). \tag{5.94}$$

Integrating by parts to move the derivative to the u terms, we bound

$$|G_6| \lesssim \|H^1\|_{L^\infty} (\|M^1 - M^2\|_{L^2}^2 + \varepsilon \|\nabla(u^1 - u^2)\|_{L^2}^2), \tag{5.95}$$

and similiarly,

$$|G_7| \lesssim \|M^2\|_{L^\infty} (\|H^1 - H^2\|_{L^2}^2 + \varepsilon \|\nabla(u^1 - u^2)\|_{L^2}^2). \tag{5.96}$$

Taking the L^2-inner product of equation (5.90) with $M^1 - M^2$ gives

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|M^1 - M^2\|_{L^2} = &- \int_{\mathbb{R}^3} (u^1 - u^2) \cdot \nabla M^1 \cdot (M^1 - M^2) dx \\
&- \int_{\mathbb{R}^3} u^2 \cdot \nabla (M^1 - M^2) \cdot (M^1 - M^2) dx \\
&+ \int_{\mathbb{R}^3} [\nabla \times u^1] \times (M^1 - M^2) \cdot (M^1 - M^2) dx \\
&+ \int_{\mathbb{R}^3} [\nabla \times (u^1 - u^2)] \times M^2 \cdot (M^1 - M^2) dx \\
&- \|M^1 - M^2\|_{L^2} + \int_{\mathbb{R}^3} (H^1 - H^2) \cdot (M^1 - M^2) dx \\
&+ \int_{\mathbb{R}^3} \Delta (M^1 - M^2) \cdot (M^1 - M^2) dx \\
&- \int_{\mathbb{R}^3} M^2 \times [(M^1 - M^2) \times H^1] \cdot (M^1 - M^2) dx \\
&- \int_{\mathbb{R}^3} M^2 \times [M^2 \times (H^1 - H^2)] \cdot (M^1 - M^2) dx \\
= &\, J_1 + J_2 + J_3 + J_4 + J_5 + J_6 + J_7 + J_8 + J_9.
\end{aligned} \tag{5.97}$$

We estimate,

$$|J_1| \lesssim \|\nabla M^1\|_{L^\infty}(\|u^1 - u^2\|_{L^2}^2 + \|M^1 - M^2\|_{L^2}^2). \tag{5.98}$$

Since u is divergence-free, $J_2 = 0$. Clearly $J_3 = 0$. Next,

$$|J_4| \lesssim \|M^2\|_{L^\infty}(\varepsilon\|\nabla(u^1 - u^2)\|_{L^2}^2 + \|M^1 - M^2\|_{L^2}^2). \tag{5.99}$$

Terms J_5, J_6 and J_7 are positive when moved to the left-hand side, and so will vanish in the inequality. Next, we estimate

$$|J_8| \lesssim \|M^2\|_{L^\infty}\|H^1\|_{L^\infty}\|M^1 - M^2\|_{L^2}^2, \tag{5.100}$$

and finally,

$$|J_9| \lesssim \|M^2\|_{L^\infty}^2(\|H^1 - H^2\|_{L^2}^2 + \|M^1 - M^2\|_{L^2}^2). \tag{5.101}$$

Adding together (5.91) and (5.97), recalling that the time derivative of H is a bounded linear operator away from the time derivative of M, and taking into account estimates (5.92), (5.93), (5.94), (5.95), (5.96), (5.98), (5.99), (5.100), and (5.101), we get

$$\frac{d}{dt}(\|u^1-u^2\|_{L^2}^2+\|M^1-M^2\|_{L^2}^2+\|H^1-H^2\|_{L^2}^2)+\eta\|\nabla(u-u^\varepsilon)\|_{L^2}$$
$$\begin{aligned}&\lesssim(\|\nabla u^1\|_{L^\infty}+\|\nabla H^1\|_{L^\infty}+\|\nabla M^2\|_{L^\infty}+\|H^1\|_{L^\infty}\\&\quad+\|\nabla M^1\|_{L^\infty}+\|M^2\|_{L^\infty}(1+\|H^1\|_{L^\infty}+\|M^2\|_{L^\infty}))\\&\quad(\|u^1-u^2\|_{L^2}^2+\|M^1-M^2\|_{L^2}^2+\|H^1-H^2\|_{L^2}^2)\\&\quad+\varepsilon(\|H^1\|_{L^\infty}+\|M^2\|_{L^\infty})\|\nabla(u^1-u^2)\|_{L^2}^2.\end{aligned} \tag{5.102}$$

Choosing

$$\varepsilon<\frac{1}{\|H^1\|_{L^\infty}+\|M^2\|_{L^\infty}}, \tag{5.103}$$

and using Grönwall's inequality gives uniqueness of solutions (note that the spatial L^∞ norms are bounded in t on some interval $(0,T)$ by the Sobolev embedding theorem and the regularity we showed earlier.

Remark. Because none of the previous steps required the use of a Bloch-Torrey magnetization term, we have also shown the existence of a limiting solution which is unique in $C([0,T];H^k)$, and solves the Shliomis system.

5.1.7 Regularity of Limit Solution

Finally, we show $(u,M,H)\in C([0,T];H^k)$. To accomplish this, we first prove weak continuity; $(u,M,H)\in C([0,T];H^k \text{ weak})$. Then we show continuity of the H^k norm of (u,M,H). Together, these imply $(u,M,H)\in C([0,T];H^k)$.

We begin by proving weak continuity, i.e. $\langle u(t),\phi\rangle_{L^2}+\langle M(t),\psi\rangle_{L^2}+\langle H(t),\upsilon\rangle_{L^2}$ is continuous in time for $\phi,\psi,\upsilon\in H^{-k}(\mathbb{R}^3)$. Fix test functions $\phi,\psi,\upsilon\in$

$H^{-k}(\mathbb{R}^3)$. Then there exist $\phi', \psi', \upsilon' \in H^{-k+2}$ (which is dense in H^{-k} and dual to H^{k-2}) such that,

$$\begin{aligned} &\|\phi - \phi'\|_{H^{-k}} + \|\psi - \psi'\|_{H^{-k}} + \|u - u'\|_{H^{-k}} \\ &\qquad\qquad \leq \frac{\varepsilon}{4 \sup_{t\in(0,T)}(\|u(t)\|_{H^k} + \|M(t)\|_{H^k} + \|H(t)\|_{H^k})}. \end{aligned} \tag{5.104}$$

For s to be determined later, we write

$$\begin{aligned} &|\langle u(t) - u(s), \phi\rangle_{L^2} + \langle M(t) - M(s), \psi\rangle_{L^2} + \langle H(t) - H(s), \upsilon\rangle_{L^2}| \\ &\quad = |\langle u(t) - u(s), \phi'\rangle_{L^2} + \langle u(t) - u(s), \phi - \phi'\rangle_{L^2} \\ &\qquad + \langle M(t) - M(s), \psi'\rangle_{L^2} + \langle M(t) - M(s), \psi - \psi'\rangle_{L^2} \\ &\qquad + \langle H(t) - H(s), \upsilon'\rangle_{L^2} + \langle H(t) - H(s), \upsilon - \upsilon'\rangle_{L^2}| \\ &\quad \leq \|u(t) - u(s)\|_{H^{k-2}}\|\phi'\|_{H^{-k+2}} + 2\sup_{t\in(0,T)}\|u(t)\|_{H^k}\|\phi - \phi'\|_{H^{-k}} \\ &\qquad + \|M(t) - M(s)\|_{H^{k-2}}\|\psi'\|_{H^{-k+2}} + 2\sup_{t\in(0,T)}\|M(t)\|_{H^k}\|\psi - \psi'\|_{H^{-k}} \\ &\qquad + \|H(t) - H(s)\|_{H^{k-2}}\|\psi'\|_{H^{-k+2}} + 2\sup_{t\in(0,T)}\|H(t)\|_{H^k}\|\upsilon - \upsilon'\|_{H^{-k}} \\ &\quad \leq (\|u(t) - u(s)\|_{H^{k-2}} + \|M(t) - M(s)\|_{H^{k-2}} + \|H(t) - H(s)\|_{H^{k-2}}) \\ &\qquad\quad \times (\|\phi'\|_{H^{-k+2}} + \|\psi'\|_{H^{-k+2}} + \|\upsilon'\|_{H^{-k+2}}) \\ &\qquad + (\|\phi - \phi'\|_{H^{-k}} + \|\psi - \psi'\|_{H^{-k}} + \|u - u'\|_{H^{-k}}) \\ &\qquad\quad \times 2\sup_{t\in(0,T)}(\|u(t)\|_{H^k} + \|M(t)\|_{H^k} + \|H(t)\|_{H^k}). \end{aligned}$$

Since $(u^\varepsilon, M^\varepsilon, H^\varepsilon) \in \mathrm{Lip}([0,T]; H^{k-2})$, there exists $\tau_\varepsilon > 0$ such that

$$\begin{aligned}\|u^\varepsilon(t) - u^\varepsilon(s)\|_{H^{k-2}} + \|M^\varepsilon(t) - M^\varepsilon(s)\|_{H^{k-2}} + \|H^\varepsilon(t) - H^\varepsilon(s)\|_{H^{k-2}} \\ \leq \frac{\varepsilon}{2(\|\phi'\|_{H^{-k+2}} + \|\psi'\|_{H^{-k+2}} + \|\upsilon'\|_{H^{-k+2}})},\end{aligned} \tag{5.105}$$

for $t, s \in [0,T]$ such that $|t-s| < \tau_\varepsilon$. Choosing s in this way, from estimates (5.104) and (5.192) we get

$$|\langle u(t) - u(s), \phi\rangle_{L^2} + \langle M(t) - M(s), \psi\rangle_{L^2} + \langle H(t) - H(s), \upsilon\rangle_{L^2}| < \varepsilon, \tag{5.106}$$

so that our solution is uniformly weakly continuous.

Remark. This step also holds without the addition of a Bloch-Torrey magnetization term.

Finally, we show $\|u(t)\|_{H^k} + \|M(t)\|_{H^k} + \|H(t)\|_{H^k}$ is continuous. We will need to make use of the Bloch-Torrey magnetization term here. First recall the bound we proved earlier (5.51)

$$\sup_{0\leq t\leq T} E_\varepsilon(t) \leq \frac{E(0)}{1 - CTE(0)} = E(0) + \frac{CTE(0)^2}{1 - CTE(0)}, \tag{5.107}$$

where $E_\varepsilon(t) = 1 + \|u^\varepsilon\|_{H^k}^2 + \|M^\varepsilon\|_{H^k}^2 + \|H^\varepsilon\|_{H^k}^2$ and for clarity we have relabelled $E_\varepsilon(0) = E(0) = 1 + \|u_0\|_{H^k}^2 + \|M_0\|_{H^k}^2 + \|H_0\|_{H^k}^2$. Let us similarly define $E(t) := 1 + \|u\|_{H^k}^2 + \|M\|_{H^k}^2 + \|H\|_{H^k}^2$. Since for fixed t, $\limsup_{\varepsilon\to 0} E_\varepsilon(t) \geq E(t)$, the bound (5.107) gives

$$\sup_{0\leq t\leq T} E(t) - E(0) \leq \frac{CTE(0)^2}{1 - CTE(0)}. \tag{5.108}$$

This implies $\limsup_{t\to 0^+} E(t) \le E(0)$, or in particular,

$$\limsup_{t\to 0^+}(\|u\|_{H^k}^2 + \|M\|_{H^k}^2 + \|H\|_{H^k}^2) \le \|u_0\|_{H^k}^2 + \|M_0\|_{H^k}^2 + \|H_0\|_{H^k}^2. \quad (5.109)$$

On the other hand, since $(u, M, H) \in C([0,T]; H^k \text{ weak})$,

$$\liminf_{t\to 0^+}(\|u\|_{H^k}^2 + \|M\|_{H^k}^2 + \|H\|_{H^k}^2) \ge \|u_0\|_{H^k}^2 + \|M_0\|_{H^k}^2 + \|H_0\|_{H^k}^2. \quad (5.110)$$

Thus since the norms are positive,

$$\lim_{t\to 0^+}(\|u\|_{H^k} + \|M\|_{H^k} + \|H\|_{H^k}) = \|u_0\|_{H^k} + \|M_0\|_{H^k} + \|H_0\|_{H^k}, \quad (5.111)$$

so that we have strong right-continuity at $t = 0$.

Next, we prove strong right-continuity on $(0, T)$. Suppose for contradiction that there exists $t_0 \in (0, T]$ such that $\|u(t)\|_{H^k} + \|M(t)\|_{H^k} + \|H(t)\|_{H^k}$ is not right-continuous. Next, recall estimate (5.48) had extra terms on the left-hand side which were ignored in the subsequent estimate (5.50). Without neglecting these terms, the estimate reads

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|u^\varepsilon\|_{H^k}^2 + \|M^\varepsilon\|_{H^k}^2 + \|H^\varepsilon\|_{H^k}^2) + \|\mathcal{J}_\varepsilon \nabla u\|_{H^k}^2 + \|\mathcal{J}_\varepsilon \nabla M^\varepsilon\|_{H^k}^2 + \|\mathcal{J}_\varepsilon \nabla H^\varepsilon\|_{H^k}^2 \\
&\qquad \le C(1 + \|u^\varepsilon\|_{H^k}^2 + \|M^\varepsilon\|_{H^k}^2 + \|H^\varepsilon\|_{H^k}^2)^2.
\end{aligned} \quad (5.112)$$

In particular, this implies (performing the same estimate for the limit solution)

$$\int_0^T \|\nabla u(t)\|_{H^k}^2 dt, \int_0^T \|\nabla M(t)\|_{H^k}^2 dt, \text{ and } \int_0^T \|\nabla H(t)\|_{H^k}^2 dt \quad (5.113)$$

are bounded. Therefore, the limit solution $(u, M, H) \in L^2(0, T; H^{k+1})$, which guarantees that for a.e. $\tau \in (0, t_0)$,

$$(u(\cdot, \tau), M(\cdot, \tau), H(\cdot, \tau)) \in H^{k+1}(\mathbb{R}^3). \tag{5.114}$$

Fix τ. By repeating the argument in subsection 5.1.3, at the higher regularity level (H^{k+1}), we find

$$\exists\, T_\tau \geq \frac{1}{C_0(\|u(\tau)\|^2_{H^{k+1}} + \|M(\tau)\|^2_{H^{k+1}} + \|H(\tau)\|^2_{H^{k+1}})} > 0 \tag{5.115}$$

such that $(u, M, H) \in L^\infty(\tau, \tau + T_\tau; H^{k+1}) \cap C([\tau, \tau + T_\tau]; H^k)$. Since the solution is not right-continuous at t_0, we must have $T_\tau < t_0 - \tau$. Thus

$$\|u(\tau)\|^2_{H^{k+1}} + \|M(\tau)\|^2_{H^{k+1}} + \|H(\tau)\|^2_{H^{k+1}} \gtrsim \frac{1}{t_0 - \tau}. \tag{5.116}$$

But then since $t_0 \in (0, T)$,

$$\int_0^T (\|u(\tau)\|^2_{H^{k+1}} + \|M(\tau)\|^2_{H^{k+1}} + \|H(\tau)\|^2_{H^{k+1}}) d\tau = \infty. \tag{5.117}$$

This is a contradiction to $(u, M, H) \in L^2(0, T; H^{k+1})$.

Strong left-continuity on $(0, T]$ can be proven in a very similar way. Suppose $\exists t_0 \in (0, T]$ where left-continuity does not hold. For a.e. $\tau \in (0, t_0]$,

$$(u(\cdot, \tau), M(\cdot, \tau), H(\cdot, \tau)) \in H^{k+1}(\mathbb{R}^3). \tag{5.118}$$

Fix τ. As before,

$$\exists\ T_\tau = \frac{1}{C_0(\|u(\tau)\|^2_{H^{k+1}} + \|M(\tau)\|^2_{H^{k+1}} + \|H(\tau)\|^2_{H^{k+1}})} > 0 \tag{5.119}$$

such that $(u, M, H) \in L^\infty(\tau, \tau + T_\tau; H^{k+1}) \cap C([\tau, \tau + T_\tau]; H^k)$. Since the solution is not left-continuous at t_0, this gives $\tau + T_\tau < t_0 \Rightarrow T_\tau < t_0 - \tau$. Therefore,

$$\|u(\tau)\|^2_{H^{k+1}} + \|M(\tau)\|^2_{H^{k+1}} + \|H(\tau)\|^2_{H^{k+1}} > \frac{1}{t_0 - \tau}, \tag{5.120}$$

which gives the same contradiction.

5.2 Rosensweig Model

The proof of local well-posedness of classical solutions for the Rosensweig model with Bloch-Torrey magnetization will be essentially the same as for the Shliomis model. Therefore, we will omit the details of various estimates which are no different from those which appear in the proof in the previous subsection. Again we will abuse the notation H^k so that it either means $H^k(\mathbb{R}^3)$, $H^k(\mathbb{R}^3)^4$, or $H^k_\sigma(\mathbb{R}^3) \times H^k(\mathbb{R}^3)^3$. As we did for the Shliomis model, we write the mollified Rosensweig model with the momentum equation projected onto the space of divergence free functions. This is a system of ordinary differential equations on the Banach space $H^k_\sigma(\mathbb{R}^3) \times H^k(\mathbb{R}^3) \times H^k(\mathbb{R}^3) \times H^k(\mathbb{R}^3)$.

$$\begin{cases} \rho\left(\frac{d}{dt}u^\varepsilon + \mathbb{P}\mathcal{J}_\varepsilon[(\mathcal{J}_\varepsilon u^\varepsilon \cdot \nabla(\mathcal{J}_\varepsilon u^\varepsilon)]\right) \\ \qquad = (\eta+\zeta)\mathcal{J}_\varepsilon^2(\Delta u^\varepsilon) + \mu_0\mathbb{P}\mathcal{J}_\varepsilon[(\mathcal{J}_\varepsilon M^\varepsilon)\cdot\nabla(\mathcal{J}_\varepsilon H^\varepsilon)] + 2\zeta\mathcal{J}_\varepsilon\nabla\times(\mathcal{J}_\varepsilon\Omega^\varepsilon), \\ \rho\kappa\left(\frac{d}{dt}\Omega^\varepsilon + \mathcal{J}_\varepsilon[(\mathcal{J}_\varepsilon u^\varepsilon\cdot\nabla(\mathcal{J}_\varepsilon\Omega^\varepsilon)]\right) \\ \qquad = \eta'\mathcal{J}_\varepsilon^2\Delta\Omega^\varepsilon + (\eta'+\lambda')\mathcal{J}_\varepsilon^2\nabla(\nabla\cdot\Omega^\varepsilon) + \mu_0\mathcal{J}_\varepsilon[\mathcal{J}_\varepsilon M^\varepsilon\times\mathcal{J}_\varepsilon H^\varepsilon] \\ \qquad\quad +2\zeta(\mathcal{J}_\varepsilon\nabla\times(\mathcal{J}_\varepsilon u^\varepsilon) - 2\Omega^\varepsilon), \\ \frac{d}{dt}M^\varepsilon + \mathcal{J}_\varepsilon[(\mathcal{J}_\varepsilon u^\varepsilon)\cdot\nabla(\mathcal{J}_\varepsilon)M^\varepsilon)] = \sigma\mathcal{J}_\varepsilon^2\Delta M^\varepsilon + \mathcal{J}_\varepsilon[\mathcal{J}_\varepsilon\Omega^\varepsilon\times\mathcal{J}_\varepsilon M^\varepsilon] \\ \qquad\qquad -\frac{1}{\tau}(M^\varepsilon - \chi_0 H^\varepsilon), \\ \nabla\times H^\varepsilon = 0, \nabla\cdot(H+M) = -\nabla\cdot H^{ext}. \end{cases} \tag{5.121}$$

We alternatively write (moving terms and dividing by constants where necessary) the momentum, angular momentum, and magnetization equations from (5.121) as

$$\frac{du^\varepsilon}{dt} = F_\varepsilon(u^\varepsilon, \Omega^\varepsilon, M^\varepsilon, H^\varepsilon), \tag{5.122}$$

$$\frac{d\Omega^\varepsilon}{dt} = U_\varepsilon(u^\varepsilon, \Omega^\varepsilon, M^\varepsilon, H^\varepsilon), \tag{5.123}$$

$$\frac{dM^\varepsilon}{dt} = G_\varepsilon(u^\varepsilon, \Omega^\varepsilon, M^\varepsilon, H^\varepsilon), \tag{5.124}$$

for later convenience. Now we state the main theorem:

Theorem 5.2.1. Suppose that $(u_0, \Omega_0, M_0, H_0) \in H^k(\mathbb{R}^3)$ for $k > 5/2$ and $\nabla\cdot H^{ext} = 0$. Then there exists $T = T(\|u_0\|_{H^k}, \|\Omega_0\|_{H^k}, \|M_0\|_{H^k}, \|H_0\|_{H^k}) > 0$ such that problem (2.11)-(2.14), (2.18) admits a unique classical solution $(u, \Omega, M, H) \in C([0,T]; H^k)$.

Proof. We prove Theorem 5.2.1 in multiple steps (labelled for the subsection they will be proved in).

5.2.1 Let $\{\psi_\varepsilon\}$ be a standard family of mollifiers. We show that for any $\varepsilon > 0$, there exists a time $T_\varepsilon > 0$ and a unique solution

$$u^\varepsilon, \Omega^\varepsilon, M^\varepsilon, H^\varepsilon \in C([0, T_\varepsilon); (H^k(\mathbb{R}^3))^4)$$

of the mollified Rosensweig equations (5.121) with initial conditions

$$u^\varepsilon|_{t=0} = \mathcal{J}_\varepsilon u_0, \quad \Omega^\varepsilon|_{t=0} = \mathcal{J}_\varepsilon \Omega_0, \quad M^\varepsilon|_{t=0} = \mathcal{J}_\varepsilon M_0, \quad H^\varepsilon|_{t=0} = \mathcal{J}_\varepsilon H_0.$$

5.2.2 For each $\varepsilon > 0$, we show the solution from 5.2.1 exists globally in time.

5.2.3 There exists a $T > 0$ such that the family $(u^\varepsilon, \Omega^\varepsilon, M^\varepsilon, H^\varepsilon)$ obeys uniform in ε bounds on $[0, T]$.

5.2.4 $(u^\varepsilon, \Omega^\varepsilon, M^\varepsilon, H^\varepsilon)$ has a subsequence which converges to a limit point (u, Ω, M, H) in $C([0, T]; L^2)$.

5.2.5 The function (u, Ω, M, H) solves (2.11)-(2.14), (2.18).

5.2.6 (u, Ω, M, H) is unique in $C([0, T]; H^k)$.

5.2.7 (u, Ω, M, H) belongs to $C([0, T]; H^k)$.

5.2.1 Local Existence of Regularized Solutions

To prove 5.2.1, we first show the existence of regularized solutions $(u^\varepsilon, \Omega^\varepsilon, M^\varepsilon, H^\varepsilon)$ to (5.121) locally in time. Again we define W_ε (equation

(5.5)), and use it in the same way as for the Shliomis model:

$$\frac{dH^\varepsilon}{dt} = \frac{1}{3}\frac{dM^\varepsilon}{dt} + \frac{d}{dt}(TM^\varepsilon) := W^\varepsilon. \tag{5.125}$$

Note that $(F_\varepsilon, U_\varepsilon, G_\varepsilon, W_\varepsilon) : H^k_\sigma \times H^k \times H^k \times H^k \to H^k_\sigma \times H^k \times H^k \times H^k$ since $\mathcal{J}_\varepsilon$ commutes with derivatives, $\mathbb{P}$ projects onto divergence free functions, and the divergence of the curl of a vector is zero. Next, we show that $(F_\varepsilon, U_\varepsilon, G_\varepsilon, W_\varepsilon)$ defined by (5.122), (5.123),(5.124), (5.125) is uniformly Lipschitz continuous. We write

$$\begin{aligned}
\rho F_\varepsilon(u^\varepsilon, \Omega^\varepsilon, M^\varepsilon, H^\varepsilon) &= -\rho\mathbb{P}\mathcal{J}_\varepsilon[(\mathcal{J}_\varepsilon u^\varepsilon)\cdot\nabla(\mathcal{J}_\varepsilon u^\varepsilon)] + (\eta+\zeta)\mathcal{J}_\varepsilon^2(\Delta u^\varepsilon) \\
&\quad + \mu_0\mathbb{P}\mathcal{J}_\varepsilon[(\mathcal{J}_\varepsilon M^\varepsilon)\cdot\nabla(\mathcal{J}_\varepsilon H^\varepsilon)] + 2\zeta\mathcal{J}_\varepsilon\nabla\times(\mathcal{J}_\varepsilon\Omega^\varepsilon) \\
&= \rho F_\varepsilon^1 + (\eta+\zeta)F_\varepsilon^2 + \mu_0 F_\varepsilon^3 + 2\zeta F_\varepsilon^4,
\end{aligned}$$

$$\begin{aligned}
\rho\kappa U_\varepsilon(u^\varepsilon, \Omega^\varepsilon, M^\varepsilon, H^\varepsilon) &= -\rho\kappa\mathcal{J}_\varepsilon[(\mathcal{J}_\varepsilon u^\varepsilon)\cdot\nabla(\mathcal{J}_\varepsilon\Omega^\varepsilon)] + \eta'\mathcal{J}_\varepsilon^2(\Delta u^\varepsilon) \\
&\quad + (\eta'+\lambda')\mathcal{J}_\varepsilon^2\nabla(\nabla\cdot\Omega^\varepsilon) + \mu_0\mathcal{J}_\varepsilon[\mathcal{J}_\varepsilon M^\varepsilon \times \mathcal{J}_\varepsilon H^\varepsilon] \\
&\quad + 2\zeta(\mathcal{J}_\varepsilon\nabla\times(\mathcal{J}_\varepsilon u^\varepsilon) - 2\Omega^\varepsilon) \\
&= \rho\kappa U_\varepsilon^1 + \eta' U_\varepsilon^2 + (\eta'+\lambda')U_\varepsilon^3 + \mu_0 U_\varepsilon^4 + 2\zeta U_\varepsilon^5,
\end{aligned}$$

and

$$\begin{aligned}
G_\varepsilon(u^\varepsilon, \Omega^\varepsilon, M^\varepsilon, H^\varepsilon) &= -\mathcal{J}_\varepsilon[(\mathcal{J}_\varepsilon u^\varepsilon)\cdot\nabla(\mathcal{J}_\varepsilon M^\varepsilon)] + \sigma\mathcal{J}_\varepsilon^2(\Delta u^\varepsilon) \\
&\quad + \mathcal{J}_\varepsilon[\mathcal{J}_\varepsilon\Omega^\varepsilon \times \mathcal{J}_\varepsilon M^\varepsilon] - \frac{1}{\tau}(M^\varepsilon - \chi_0 H^\varepsilon) \\
&= G_\varepsilon^1 + \sigma G_\varepsilon^2 + G_\varepsilon^3 + \frac{1}{\tau}G_\varepsilon^4.
\end{aligned}$$

As before, we bound each term following Majda and Bertozzi's book [MB02]. In exactly the same way as for the Shliomis model, we bound

$$\|F_\varepsilon^1(u^1,\Omega^1,M^1,H^1)-F_\varepsilon^1(u^2,\Omega^2,M^2,H^2)\|_{H^k} \le \frac{C}{\varepsilon^{5/2}}(\|u^1\|_{L^2}+\|u^2\|_{L^2})\|u^1-u^2\|_{H^k}, \tag{5.126}$$

$$\|F_\varepsilon^2(u^1,\Omega^1,M^1,H^1)-F_\varepsilon^2(u^2,\Omega^2,M^2,H^2)\|_{H^k} \le \frac{C}{\varepsilon^2}\|u^1-u^2\|_{H^k}, \tag{5.127}$$

and

$$\begin{aligned}\|F_\varepsilon^3(u^1,\Omega^1,&M^1,H^1)-F_\varepsilon^3(u^2,\Omega^2,M^2,H^2)\|_{H^k}\\ &\le \frac{C}{\varepsilon^{5/2}}(\|M^2\|_{L^2}\|H^1-H^1\|_{H^k}+\|H^2\|_{L^2}\|M^1-M^2\|_{H^k}).\end{aligned} \tag{5.128}$$

For F_ε^4, we bound

$$\begin{aligned}\|F_\varepsilon^4(u^1,\Omega^1,&M^1,H^1)-F_\varepsilon^4(u^2,\Omega^2,M^2,H^2)\|_{H^k}\\ &\le \|\mathcal{J}_\varepsilon\nabla\times(\Omega^1-\Omega^2)\|_{H^k} \le \frac{C}{\varepsilon}\|\Omega^1-\Omega^2\|_{H^k},\end{aligned} \tag{5.129}$$

where we have used that $\mathcal{J}_\varepsilon$ commutes with derivatives and mollifier property (4.8).

So altogether for F_ε, by (5.126), (5.127), (5.128), and (5.129), we have

$$\begin{aligned}\|F_\varepsilon(u^1,\Omega^1,&M^1,H^1)-F_\varepsilon(u^2,\Omega^2,M^2,H^2)\|_{H^k}\\ &\le C(\varepsilon,\|u^i\|_{L^2},\|M^i\|_{L^2},\|H^i\|_{L^2},\rho,\eta,\zeta,\mu_0)\times\\ &\quad\{\|u^1-u^2\|_{H^k}+\|\Omega^1-\Omega^2\|_{H^k}+\|M^1-M^2\|_{H^k}+\|H^1-H^2\|_{H^k}\}.\end{aligned} \tag{5.130}$$

Next, for U_ε, we bound in the same way as for F_ε^1 (5.126), F_ε^2 (5.127), and

F_ε^4 (5.129),

$$\begin{aligned}\|U_\varepsilon^1(u^1, \Omega^1, M^1, H^1) - U_\varepsilon^1(u^2, \Omega^2, M^2, H^2)\|_{H^k} \\ \leq \frac{C}{\varepsilon^{5/2}}(\|u^1\|_{L^2}\|\Omega^1 - \Omega^2\|_{H^k} + \|\Omega^2\|_{L^2}\|u^1 - u^2\|_{H^k}),\end{aligned} \tag{5.131}$$

$$\|U_\varepsilon^2(u^1, \Omega^1, M^1, H^1) - U_\varepsilon^2(u^2, \Omega^2, M^2, H^2)\|_{H^k} \leq \frac{C}{\varepsilon^2}\|\Omega^1 - \Omega^2\|_{H^k}, \tag{5.132}$$

and

$$\begin{aligned}\|U_\varepsilon^5(u^1, \Omega^1, M^1, H^1) - U_\varepsilon^5(u^2, \Omega^2, M^2, H^2)\|_{H^k} \\ \leq \frac{C}{\varepsilon}\|u^1 - u^2\|_{H^k} + 2\|\Omega^1 - \Omega^2\|_{H^k}.\end{aligned} \tag{5.133}$$

In the same way as U_2^ε (5.132), we bound

$$\|U_\varepsilon^3(u^1, \Omega^1, M^1, H^1) - U_\varepsilon^3(u^2, \Omega^2, M^2, H^2)\|_{H^k} \leq \frac{C}{\varepsilon^2}\|\Omega^1 - \Omega^2\|_{H^k}. \tag{5.134}$$

Finally we bound

$$\begin{aligned}&\|U_\varepsilon^4(u^1, \Omega^1, M^1, H^1) - U_\varepsilon^4(u^2, \Omega^2, M^2, H^2)\|_{H^k} \\ &\qquad \leq \|\mathcal{J}_\varepsilon H^1\|_{H^k}\|M^1 - M^2\|_{H^k} + \|\mathcal{J}_\varepsilon M^2\|_{H^k}\|H^1 - H^2\|_{H^k} \\ &\qquad \leq \frac{C}{\varepsilon^k}(\|H^1\|_{L^2}\|M^1 - M^2\|_{H^k} + \|M^2\|_{L^2}\|H^1 - H^2\|_{H^k}),\end{aligned} \tag{5.135}$$

where we have again appealed to the mollifier property (4.9). Altogether, estimates (5.131), (5.132), (5.134), (5.135), and (5.133) allow us to bound for

U_ε,

$$
\begin{aligned}
\|U_\varepsilon(u^1,\Omega^1,&M^1,H^1)-U_\varepsilon(u^2,\Omega^2,M^2,H^2)\|_{H^k}\\
&\le C(\varepsilon,\|u^i\|_{L^2},\|\Omega^i\|_{L^2},\|M^i\|_{L^2},\|H^i\|_{L^2},\rho,\kappa,\eta',\lambda',\zeta,\mu_0)\times\\
&\quad\{\|u^1-u^2\|_{H^k}+\|\Omega^1-\Omega^2\|_{H^k}+\|M^1-M^2\|_{H^k}+\|H^1-H^2\|_{H^k}\}.
\end{aligned}
\tag{5.136}
$$

Next we bound the G_ε terms. In the same way as U_ε^1 (5.131) and U_ε^2 (5.132) and U_ε^4 (5.135),

$$
\begin{aligned}
&\|G_\varepsilon^1(u^1,\Omega^1,M^1,H^1)-G_\varepsilon^1(u^2,\Omega^2,M^2,H^2)\|_{H^k}\\
&\qquad\le\frac{C}{\varepsilon^{5/2}}(\|u^1\|_{L^2}\|M^1-M^2\|_{H^k}+\|M^2\|_{L^2}\|u^1-u^2\|_{H^k}),
\end{aligned}
\tag{5.137}
$$

$$
\|G_\varepsilon^2(u^1,\Omega^1,M^1,H^1)-G_\varepsilon^2(u^2,\Omega^2,M^2,H^2)\|_{H^k}\le\frac{C}{\varepsilon^2}\|M^1-M^2\|_{H^k},\tag{5.138}
$$

and

$$
\begin{aligned}
&\|G_\varepsilon^3(u^1,\Omega^1,M^1,H^1)-G_\varepsilon^3(u^2,\Omega^2,M^2,H^2)\|_{H^k}\\
&\qquad\le\frac{C}{\varepsilon^k}(\|M^1\|_{L^2}\|\Omega^1-\Omega^2\|_{H^k}+\|\Omega^2\|_{L^2}\|M^1-M^2\|_{H^k}).
\end{aligned}
\tag{5.139}
$$

Finally, we trivially bound

$$
\|G_\varepsilon^3(u^1,\Omega^1,M^1,H^1)-G_\varepsilon^3(u^2,\Omega^2,M^2,H^2)\|_{H^k}\le\|M^1-M^2\|_{H^k}+\chi_0\|H^1-H^2\|_{H^k}.\tag{5.140}
$$

Together estimates (5.137), (5.138), (5.139), and (5.140) give

$$\begin{aligned}\|G_\varepsilon(u^1,\Omega^1,&M^1,H^1) - G_\varepsilon(u^2,\Omega^2,M^2,H^2)\|_{H^k}\\ &\le C(\varepsilon,\|u^i\|_{L^2},\|\Omega^i\|_{L^2},\|M^i\|_{L^2}\|_{L^2},\sigma,\tau,\chi_0)\times\\ &\quad \{\|u^1-u^2\|_{H^k}+\|\Omega^1-\Omega^2\|_{H^k}+\|M^1-M^2\|_{H^k}+\|H^1-H^2\|_{H^k}\}.\end{aligned} \tag{5.141}$$

Again note that W_ε is Lipschitz since T is a bounded linear operator. Thus by the Picard theorem (Theorem 4.1.1), there exists a unique solution

$$(u^\varepsilon,\Omega^\varepsilon,M^\varepsilon,H^\varepsilon)\in C^1([0,T_\varepsilon);H^k) \tag{5.142}$$

for some $T_\varepsilon > 0$. This proves 5.2.1.

5.2.2 Global Existence of Regularized Solution

To prove 5.2.2, we derive an energy bound that will allow us to continue our approximate solution from 5.1.1 for all time. Taking the L^2 inner product of the momentum equation in (5.121) with u^ε and integrating by parts, we get

$$\begin{aligned}\frac{\rho}{2}\frac{d}{dt}\|u^\varepsilon\|_{L^2}^2+&(\eta+\zeta)\|\mathcal{J}_\varepsilon\nabla u^\varepsilon\|_{L^2}^2\\ &=\int_{\mathbb{R}^3}\mu_0\mathbb{P}\mathcal{J}_\varepsilon u^\varepsilon\cdot[\mathcal{J}_\varepsilon M^\varepsilon\cdot\nabla)\mathcal{J}_\varepsilon H^\varepsilon+2\zeta\mathcal{J}_\varepsilon u^\varepsilon\cdot\nabla\times\mathcal{J}_\varepsilon\Omega^\varepsilon dx.\end{aligned} \tag{5.143}$$

Again since $\nabla\times H^\varepsilon=0$ we use the identity

$$(\mathcal{J}_\varepsilon M^\varepsilon\cdot\nabla\mathcal{J}_\varepsilon H^\varepsilon)\cdot\mathcal{J}_\varepsilon u^\varepsilon=\nabla(\mathcal{J}_\varepsilon M^\varepsilon\cdot\mathcal{J}_\varepsilon H^\varepsilon)\cdot\mathcal{J}_\varepsilon u^\varepsilon-(\mathcal{J}_\varepsilon u^\varepsilon\cdot\nabla)\mathcal{J}_\varepsilon M^\varepsilon\cdot\mathcal{J}_\varepsilon H^\varepsilon$$

(proved in the previous section) to get,

$$\frac{\rho}{2}\frac{d}{dt}\|u^\varepsilon\|_{L^2}^2+(\eta+\zeta)\|\mathcal{J}_\varepsilon\nabla u^\varepsilon\|_{L^2}^2 = \int_{\mathbb{R}^3} -\mu_0\mathbb{P}\mathcal{J}_\varepsilon M^\varepsilon\cdot[\mathcal{J}_\varepsilon u^\varepsilon\cdot\nabla)\mathcal{J}_\varepsilon M^\varepsilon + 2\zeta\mathcal{J}_\varepsilon u^\varepsilon\cdot\nabla\times\mathcal{J}_\varepsilon\Omega^\varepsilon dx. \tag{5.144}$$

Next, taking the L^2 inner product of the angular momentum equation from (5.121) with Ω^ε, we get

$$\frac{\rho\kappa}{2}\frac{d}{dt}\|\Omega^\varepsilon\|_{L^2}^2+\eta'\|\mathcal{J}_\varepsilon\nabla\Omega^\varepsilon\|_{L^2}^2+(\eta'+\lambda')\|\mathcal{J}_\varepsilon\nabla\cdot\Omega^\varepsilon\|_{L^2}^2 = \int_{\mathbb{R}^3}\mu_0\mathcal{J}_\varepsilon\Omega^\varepsilon\cdot[\mathcal{J}_\varepsilon M^\varepsilon\times\mathcal{J}_\varepsilon H^\varepsilon]+2\zeta(\nabla\times\mathcal{J}_\varepsilon u^\varepsilon-2\Omega^\varepsilon)\cdot\mathcal{J}_\varepsilon\Omega^\varepsilon dx. \tag{5.145}$$

Taking the L^2 inner product of the magnetization equation in (5.121) with $\mu_0 M^\varepsilon$ and using Lemma 4.2.4, we get

$$\frac{\mu_0}{2}\frac{d}{dt}\|M^\varepsilon\|_{L^2}^2+\frac{\mu_0}{\tau}(\|\mathcal{J}_\varepsilon M^\varepsilon\|_{L^2}^2+\chi_0\|\mathcal{J}_\varepsilon H^\varepsilon\|_{L^2}^2)=0. \tag{5.146}$$

Finally, taking the L^2 inner product of the magnetization equation in (5.121) with $\mu_0 H^\varepsilon$ and using Lemma 4.2.4 and the scalar triple product $(A\times B)\cdot C = (C\times A)\cdot B$, we get

$$\begin{aligned}\frac{\mu_0}{2}\frac{d}{dt}\|H^\varepsilon\|_{L^2}^2+\frac{\mu_0}{\tau}(1+\chi_0)\|\mathcal{J}_\varepsilon H^\varepsilon\|_{L^2}^2 &= \mu_0\int_{\mathbb{R}^3}[(\mathcal{J}_\varepsilon u^\varepsilon)\cdot\nabla(\mathcal{J}_\varepsilon M^\varepsilon)]\cdot\mathcal{J}_\varepsilon H^\varepsilon dx \\ &\quad -\mu_0\int_{\mathbb{R}^3}(\mathcal{J}_\varepsilon\Omega^\varepsilon\times\mathcal{J}_\varepsilon M^\varepsilon)\cdot\mathcal{J}_\varepsilon H^\varepsilon dx.\end{aligned} \tag{5.147}$$

Adding (5.144), (5.145), (5.146), and (5.147) gives

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\rho\|u^\varepsilon\|_{L^2}^2 + \rho\kappa\|\Omega^\varepsilon\|_{L^2}^2 + \mu_0\|M^\varepsilon\|_{L^2}^2 + \mu_0\|H^\varepsilon\|_{L^2}^2) \\
&+ (\eta+\zeta+1)\|\mathcal{J}_\varepsilon \nabla u^\varepsilon\|_{L^2}^2 + \eta'\|\mathcal{J}_\varepsilon \nabla \Omega^\varepsilon\|_{L^2}^2 + (\eta'+\lambda')\|\mathcal{J}_\varepsilon \nabla\cdot\Omega^\varepsilon\|_{L^2}^2 + \frac{1}{\tau}\|M^\varepsilon\|_{L^2}^2 \\
&+ \frac{\mu_0}{\tau}(1+2\chi_0)\|H^\varepsilon\|_{L^2}^2 \\
&= 2\zeta\int_{\mathbb{R}^3}(\nabla\times\mathcal{J}_\varepsilon\Omega^\varepsilon)\times\mathcal{J}_\varepsilon u^\varepsilon + (\nabla\times\mathcal{J}_\varepsilon u^\varepsilon)\cdot\mathcal{J}_\varepsilon\Omega^\varepsilon dx - 4\|\mathcal{J}_\varepsilon\Omega^\varepsilon\|_{L^2}^2 + \|\mathcal{J}_\varepsilon\nabla u^\varepsilon\|_{L^2}^2 \\
&= 4\zeta\int_{\mathbb{R}^3}(\nabla\times\mathcal{J}_\varepsilon u^\varepsilon)\cdot\mathcal{J}_\varepsilon\Omega^\varepsilon dx - 4\|\mathcal{J}_\varepsilon\Omega^\varepsilon\|_{L^2}^2 + \|\mathcal{J}_\varepsilon\nabla u^\varepsilon\|_{L^2}^2 \\
&= -\zeta\|\mathcal{J}_\varepsilon\nabla\times u^\varepsilon - 2\mathcal{J}_\varepsilon\Omega^\varepsilon\|_{L^2}^2,
\end{aligned}
\tag{5.148}
$$

where we have integrated the first curl term on the right-hand side by parts, and used that since $\nabla\cdot u^\varepsilon = 0$, $\Delta u^\varepsilon = -\nabla\times(\nabla\times u^\varepsilon)$. Moving the final term on the right-hand side to the left-hand side, we finally get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\rho\|u^\varepsilon\|_{L^2}^2 + \rho\kappa\|\Omega^\varepsilon\|_{L^2}^2 + \mu_0\|M^\varepsilon\|_{L^2}^2 + \mu_0\|H^\varepsilon\|_{L^2}^2) \\
&+ (\eta+\zeta+1)\|\mathcal{J}_\varepsilon \nabla u^\varepsilon\|_{L^2}^2 + \eta'\|\mathcal{J}_\varepsilon \nabla \Omega^\varepsilon\|_{L^2}^2 + (\eta'+\lambda')\|\mathcal{J}_\varepsilon \nabla\cdot\Omega^\varepsilon\|_{L^2}^2 + \frac{1}{\tau}\|M^\varepsilon\|_{L^2}^2 \\
&+ \frac{\mu_0}{\tau}(1+2\chi_0)\|H^\varepsilon\|_{L^2}^2 + \zeta\|\mathcal{J}_\varepsilon\nabla\times u^\varepsilon - 2\mathcal{J}_\varepsilon\Omega^\varepsilon\|_{L^2}^2 = 0.
\end{aligned}
\tag{5.149}
$$

This implies

$$
\begin{aligned}
&\sup_{0\le t\le T}(\rho\|u^\varepsilon\|_{L^2}^2 + \rho\kappa\|\Omega^\varepsilon\|_{L^2}^2 + \mu_0\|M^\varepsilon\|_{L^2}^2 + \mu_0\|H^\varepsilon\|_{L^2}^2) \\
&\qquad\qquad \le (\rho\|u_0\|_{L^2}^2 + \rho\kappa\|\Omega_0\|_{L^2}^2 + \mu_0\|M_0\|_{L^2}^2 + \mu_0\|H_0\|_{L^2}^2).
\end{aligned}
\tag{5.150}
$$

Since norms are positive, we have in particular,

$$\begin{aligned}\sup_{0\le t\le T}(\|u^\varepsilon\|_{L^2}+&\|\Omega^\varepsilon\|_{L^2}+\|M^\varepsilon\|_{L^2}+\|H^\varepsilon\|_{L^2})\\&\le C(\|u_0\|_{L^2}+\|\Omega_0\|_{L^2}+\|M_0\|_{L^2}+\|H_0\|_{L^2}).\end{aligned}\tag{5.151}$$

Once again we use the continuation theorem for ODEs on a Banach space (Theorem 4.1.2) to show our ODE exists globally in time. To this end, we prove an a priori bound on the H^k norm of $(u^\varepsilon,\Omega^\varepsilon,M^\varepsilon,H^\varepsilon)$. From the Lipschitz bounds (5.152), (5.136), and (5.141) with $(u^1,\Omega^1,M^1,H^1)=(u^\varepsilon,\Omega^\varepsilon,M^\varepsilon,H^\varepsilon)$ and $(u^2,\Omega^2,M^2,H^2)\equiv(0,0,0,0)$,

$$\begin{aligned}&\frac{d}{dt}(\|u^\varepsilon\|_{H^k}+\|\Omega^\varepsilon\|_{H^k}+\|M^\varepsilon\|_{H^k}+\|H^\varepsilon\|_{H^k})\\&\quad\le C(\varepsilon,\|u^\varepsilon\|_{L^2},\|\Omega^\varepsilon\|_{L^2},\|M^\varepsilon\|_{L^2},\|H^\varepsilon\|_{L^2},\rho,\eta,\zeta,\mu_0,\kappa,\eta',\lambda',\sigma,\tau,\chi_0)\times\\&\qquad\{\|u^1-u^2\|_{H^k}+\|\Omega^1-\Omega^2\|_{H^k}+\|M^1-M^2\|_{H^k}+\|H^1-H^2\|_{H^k}\}.\end{aligned}\tag{5.152}$$

Our previous estimate (5.151) changes the dependence of the constant in (5.152) on $(u^\varepsilon,\Omega^\varepsilon,M^\varepsilon,H^\varepsilon)$ to dependence on the initial data;

$$\begin{aligned}&\frac{d}{dt}(\|u^\varepsilon\|_{H^k}+\|\Omega^\varepsilon\|_{H^k}+\|M^\varepsilon\|_{H^k}+\|H^\varepsilon\|_{H^k})\\&\quad\le C(\varepsilon,\|u_0\|_{L^2},\|\Omega_0\|_{L^2},\|M_0\|_{L^2},\|H_0\|_{L^2},\rho,\eta,\zeta,\mu_0,\kappa,\eta',\lambda',\sigma,\tau,\chi_0)\times\\&\qquad\{\|u^1-u^2\|_{H^k}+\|\Omega^1-\Omega^2\|_{H^k}+\|M^1-M^2\|_{H^k}+\|H^1-H^2\|_{H^k}\}.\end{aligned}\tag{5.153}$$

By Grönwall's lemma, this gives

$$\begin{aligned}\|u^\varepsilon\|_{H^k}+&\|\Omega^\varepsilon\|_{H^k}+\|M^\varepsilon\|_{H^k}+\|H^\varepsilon\|_{H^k}\\&\le(\|u_0\|_{H^k}+\|\Omega_0\|_{H^k}+\|M_0\|_{H^k}+\|H_0\|_{H^k})e^{ct},\end{aligned}\tag{5.154}$$

so our solution can be continued indefinitely in time.

5.2.3 Uniform in ε Bounds

In this subsection, we prove that for every $k > 5/2$, our solution is uniformly bounded in $H^k(\mathbb{R}^3)$. In particular, we prove the bound

$$\begin{aligned}\frac{d}{dt}(1+\|u^\varepsilon\|_{H^k}^2+&\|\Omega^\varepsilon\|_{H^k}^2+\|M^\varepsilon\|_{H^k}^2+\|H^\varepsilon\|_{H^k}^2)\\ &\leq(1+\|u^\varepsilon\|_{H^k}^2+\|\Omega^\varepsilon\|_{H^k}^2+\|M^\varepsilon\|_{H^k}^2+\|H^\varepsilon\|_{H^k}^2)^2.\end{aligned} \tag{5.155}$$

Since we don't need terms to cancel each other in this estimate, we set

$$\rho,\eta,\zeta,\mu_0,\kappa,\eta',\lambda',\tau,\chi_0=1$$

for simplicity. Let ∂^k denote a particular derivative of order k for $k > 5/2$. Taking the k-th derivative of the momentum equation from (5.121), multiplying by $\partial^k u^\varepsilon$, and integrating over $\mathbb{R}^3$ gives

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\partial^k u^\varepsilon\|_{L^2}^2+2\|\partial^k\nabla u^\varepsilon\|_{L^2}^2\\ &\quad-\int_{\mathbb{R}^3}\partial^k(\mathcal{J}_\varepsilon u^\varepsilon\cdot\nabla\mathcal{J}_\varepsilon u^\varepsilon)\cdot\partial^k\mathcal{J}_\varepsilon u^\varepsilon dx+\int_{\mathbb{R}^3}\partial^k(\mathcal{J}_\varepsilon M^\varepsilon\cdot\nabla\mathcal{J}_\varepsilon H^\varepsilon)\cdot\partial^k\mathcal{J}_\varepsilon u^\varepsilon dx\\ &\quad+2\int_{\mathbb{R}^3}\partial^k(\nabla\times\mathcal{J}_\varepsilon\Omega^\varepsilon)\cdot\partial^k\mathcal{J}_\varepsilon u^\varepsilon dx\\ &\quad=R_1^\varepsilon+R_2^\varepsilon+R_3^\varepsilon.\end{aligned} \tag{5.156}$$

Next, taking the k-th derivative of the angular momentum equation from (5.121), multiplying by $\partial^k\Omega^\varepsilon$, and integrating over $\mathbb{R}^3$ gives,

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\partial^k\Omega^\varepsilon\|_{L^2}^2+\|\partial^k\nabla\mathcal{J}_\varepsilon\Omega^\varepsilon\|_{L^2}^2+2\|\partial^k\nabla\cdot\mathcal{J}_\varepsilon\Omega^\varepsilon\|_{L^2}^2+4\|\partial^k\Omega^\varepsilon\|_{L^2}^2\\
&\qquad=-\int_{\mathbb{R}^3}\partial^k(\mathcal{J}_\varepsilon u^\varepsilon\cdot\nabla\mathcal{J}_\varepsilon\Omega^\varepsilon)\partial^k\mathcal{J}_\varepsilon\Omega^\varepsilon dx+\int_{\mathbb{R}^3}\partial^k(\mathcal{J}_\varepsilon M^\varepsilon\times\mathcal{J}_\varepsilon H^\varepsilon)\cdot\partial^k\mathcal{J}_\varepsilon\Omega^\varepsilon dx\\
&\qquad\quad+2\int_{\mathbb{R}^3}\partial^k(\nabla\times\mathcal{J}_\varepsilon u^\varepsilon)\partial^k\mathcal{J}_\varepsilon\Omega^\varepsilon dx\\
&\qquad=R_4^\varepsilon+R_5^\varepsilon+R_6^\varepsilon.
\end{aligned}
\tag{5.157}
$$

Then taking the k-th derivative of the magnetization equation, multiplying by $\partial^k M^\varepsilon$, integrating over $\mathbb{R}^3$, and using Lemma 4.2.4,

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\partial^k M^\varepsilon\|_{L^2}^2+\|\partial^k M^\varepsilon\|_{L^2}^2+\|\partial^k H^\varepsilon\|_{L^2}^2+\|\partial^k\nabla\mathcal{J}_\varepsilon M^\varepsilon\|_{L^2}^2\\
&\qquad=-\int_{\mathbb{R}^3}\partial^k(\mathcal{J}_\varepsilon u^\varepsilon\cdot\nabla\mathcal{J}_\varepsilon M^\varepsilon)\partial^k\mathcal{J}_\varepsilon M^\varepsilon dx+\int_{\mathbb{R}^3}\partial^k(\mathcal{J}_\varepsilon\Omega^\varepsilon\times\mathcal{J}_\varepsilon M^\varepsilon)\cdot\partial^k\mathcal{J}_\varepsilon M^\varepsilon dx\\
&\qquad=R_7^\varepsilon+R_8^\varepsilon.
\end{aligned}
\tag{5.158}
$$

Finally, taking the k-th derivative of the magnetization equation, multiplying by $\partial^k H^\varepsilon$, integrating over $\mathbb{R}^3$, and using Lemma 4.2.4,

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\partial^k H^\varepsilon\|_{L^2}^2+2\|\partial^k H^\varepsilon\|_{L^2}^2+\|\partial^k\nabla\mathcal{J}_\varepsilon H^\varepsilon\|_{L^2}^2\\
&\qquad=\int_{\mathbb{R}^3}\partial^k(\mathcal{J}_\varepsilon u^\varepsilon\cdot\nabla\mathcal{J}_\varepsilon M^\varepsilon)\partial^k\mathcal{J}_\varepsilon H^\varepsilon dx-\int_{\mathbb{R}^3}\partial^k(\mathcal{J}_\varepsilon\Omega^\varepsilon\times\mathcal{J}_\varepsilon M^\varepsilon)\cdot\partial^k\mathcal{J}_\varepsilon H^\varepsilon dx\\
&\qquad=R_9^\varepsilon+R_{10}^\varepsilon.
\end{aligned}
\tag{5.159}
$$

Adding (5.160), (5.157), (5.158), and (5.159) gives

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\partial^k u^\varepsilon\|_{L^2}^2 + \|\partial^k \Omega^\varepsilon\|_{L^2}^2 + \|\partial^k M^\varepsilon\|_{L^2}^2 + \|\partial^k H^\varepsilon\|_{L^2}^2 \\
&\quad + 2\|\partial^k \nabla u^\varepsilon\|_{L^2}^2 + \|\partial^k \nabla \mathcal{J}_\varepsilon \Omega^\varepsilon\|_{L^2}^2 + 2\|\partial^k \nabla \cdot \mathcal{J}_\varepsilon \Omega^\varepsilon\|_{L^2}^2 + 4\|\partial^k \Omega^\varepsilon\|_{L^2}^2 \\
&\quad + \|\partial^k M^\varepsilon\|_{L^2}^2 + 3\|\partial^k H^\varepsilon\|_{L^2}^2 + \|\partial^k \nabla H^\varepsilon\|_{L^2}^2 + \|\partial^k \nabla M^\varepsilon\|_{L^2}^2 \\
&\quad = R_1^\varepsilon + R_2^\varepsilon + R_3^\varepsilon + R_4^\varepsilon + R_5^\varepsilon + R_6^\varepsilon + R_7^\varepsilon + R_8^\varepsilon + R_9^\varepsilon + R_{10}^\varepsilon.
\end{aligned}
\tag{5.160}
$$

We estimate each term $R_1^\varepsilon - R_{10}^\varepsilon$. First, in exactly the same as S_1^ε we bound

$$
|R_1^\varepsilon| \le C\|D\mathcal{J}_\varepsilon u^\varepsilon\|_{L^\infty}\|u^\varepsilon\|_{H^k}^2. \tag{5.161}
$$

Moreover, in exactly the same way as S_2^ε, we bound

$$
|R_2^\varepsilon| \le C(\delta)(\|\mathcal{J}_\varepsilon M^\varepsilon\|_{L^\infty} + \|\mathcal{J}_\varepsilon H^\varepsilon\|_{L^\infty})(\|\mathcal{J}_\varepsilon M^\varepsilon\|_{H^k}^2 + \|\mathcal{J}_\varepsilon H^\varepsilon\|_{H^k}) + \delta\|D^{k+1}\mathcal{J}_\varepsilon u^\varepsilon\|_{L^2}^2. \tag{5.162}
$$

For R_3^ε, we bound

$$
|R_3^\varepsilon| \le C(\delta)\|u^\varepsilon\|_{H^k}^2 + \delta\|D^{k+1}\Omega^\varepsilon\|_{L^2}^2. \tag{5.163}
$$

In the same way as S_4^ε, we estimate

$$
|R_4^\varepsilon| \le C\|D\mathcal{J}_\varepsilon u^\varepsilon\|_{L^\infty}\|\Omega^\varepsilon\|_{H^k}^2 + C(\delta)\|\mathcal{J}_\varepsilon \Omega^\varepsilon\|_{L^\infty}^2\|\Omega^\varepsilon\|_{H^k}^2 + \delta\|D^{k+1}\mathcal{J}_\varepsilon u^\varepsilon\|_{L^2}^2. \tag{5.164}
$$

For R_5^ε, we use Ning Ju's inequality (Proposition 4.2.1) to get,

$$\begin{aligned}|R_5^\varepsilon| &\le (\|\mathcal{J}_\varepsilon M^\varepsilon\|_{L^\infty}\|H^\varepsilon\|_{H^k} + \|\mathcal{J}_\varepsilon H^\varepsilon\|_{L^\infty}\|M^\varepsilon\|_{H^k})\|\mathcal{J}_\varepsilon \Omega^\varepsilon\|_{L^2}^2 \\ &\le \|\mathcal{J}_\varepsilon M^\varepsilon\|_{L^\infty}(\|H^\varepsilon\|_{H^k}^2 + \|\Omega^\varepsilon\|_{H^k}^2) + \|\mathcal{J}_\varepsilon H^\varepsilon\|_{L^\infty}(\|M^\varepsilon\|_{H^k}^2 + \|\Omega^\varepsilon\|_{H^k}^2).\end{aligned} \tag{5.165}$$

We trivially estimate

$$|R_6^\varepsilon| \le C(\delta)\|\Omega^\varepsilon\|_{H^k}^2 + \delta\|D^{k+1}u^\varepsilon\|_{L^2}^2. \tag{5.166}$$

Similar to R_4^ε, we estimate

$$|R_7^\varepsilon| \le C\|D\mathcal{J}_\varepsilon u^\varepsilon\|_{L^\infty}\|M^\varepsilon\|_{H^k}^2 + C(\delta)\|\mathcal{J}_\varepsilon M^\varepsilon\|_{L^\infty}^2\|M^\varepsilon\|_{H^k}^2 + \delta\|D^{k+1}\mathcal{J}_\varepsilon u^\varepsilon\|_{L^2}^2. \tag{5.167}$$

Similar to R_5^ε,

$$\begin{aligned}|R_8^\varepsilon| &\le (\|\mathcal{J}_\varepsilon M^\varepsilon\|_{L^\infty}\|H^\varepsilon\|_{H^k} + \|\mathcal{J}_\varepsilon H^\varepsilon\|_{L^\infty}\|M^\varepsilon\|_{H^k})\|\mathcal{J}_\varepsilon \Omega^\varepsilon\|_{L^2}^2 \\ &\le \|\mathcal{J}_\varepsilon \Omega^\varepsilon\|_{L^\infty}\|M^\varepsilon\|_{H^k}^2 + \|\mathcal{J}_\varepsilon M^\varepsilon\|_{L^\infty}(\|M^\varepsilon\|_{H^k}^2 + \|\Omega^\varepsilon\|_{H^k}^2).\end{aligned} \tag{5.168}$$

Next, R_9^ε can be estimated in exactly the same way as S_7^ε by

$$\begin{aligned}|R_9^\varepsilon| &\lesssim C(\delta)(\|\mathcal{J}_\varepsilon M^\varepsilon\|_{L^\infty}^2 + \|\mathcal{J}_\varepsilon H^\varepsilon\|_{L^\infty}^2)\|H^\varepsilon\|_{H^k} + \delta\|D^{k+1}u^\varepsilon\|_{L^2} \\ &\quad + \|D\mathcal{J}_\varepsilon u^\varepsilon\|_{L^\infty}(\|M^\varepsilon\|_{H^k}^2 + \|H^\varepsilon\|_{H^k}^2).\end{aligned} \tag{5.169}$$

Finally, R_{10}^{ε} can be estimated similarly to R_5^{ε} by

$$
\begin{aligned}
|R_{10}^{\varepsilon}| &\leq (\|\mathcal{J}_{\varepsilon}\Omega^{\varepsilon}\|_{L^\infty}\|M^{\varepsilon}\|_{H^k} + \|\mathcal{J}_{\varepsilon}M^{\varepsilon}\|_{L^\infty}\|\Omega^{\varepsilon}\|_{H^k})\|\mathcal{J}_{\varepsilon}H^{\varepsilon}\|_{L^2}^2 \\
&\leq \|\mathcal{J}_{\varepsilon}\Omega^{\varepsilon}\|_{L^\infty}(\|M^{\varepsilon}\|_{H^k}^2 + \|H^{\varepsilon}\|_{H^k}^2) + \|\mathcal{J}_{\varepsilon}M^{\varepsilon}\|_{L^\infty}(\|\Omega^{\varepsilon}\|_{H^k}^2 + \|H^{\varepsilon}\|_{H^k}^2).
\end{aligned}
\tag{5.170}
$$

Using estimates (5.161), (5.162), (5.163), (5.164), (5.165), (5.166), (5.167), (5.168), (5.169), and (5.170), summing over all derivatives of order k, using the Sobolev inequality $\|f\|_{L^\infty(\mathbb{R}^3)} \leq C\|f\|_{H^s}$ for $s > 3/2$ and the Gagliardo-Nirenberg inequality $\|Df\|_{L^\infty(\mathbb{R}^3)} \leq C\|D^k f\|_{L^2}^{\alpha}\|f\|_{L^\infty}^{1-\alpha}$ for $k > 3/2$, and taking $\delta < 1/6$, equation (5.160) gives

$$
\begin{aligned}
&\frac{d}{dt}(1 + \|u^{\varepsilon}\|_{H^k}^2 + \|\Omega^{\varepsilon}\|_{H^k}^2 + \|M^{\varepsilon}\|_{H^k}^2 + \|H^{\varepsilon}\|_{H^k}^2) \\
&\qquad\qquad \leq (1 + \|u^{\varepsilon}\|_{H^k}^2 + \|\Omega^{\varepsilon}\|_{H^k}^2 + \|M^{\varepsilon}\|_{H^k}^2 + \|H^{\varepsilon}\|_{H^k}^2)^2.
\end{aligned}
\tag{5.171}
$$

Denoting $E_{\varepsilon}(t) := 1 + \|u^{\varepsilon}\|_{H^k}^2 + \|\Omega^{\varepsilon}\|_{H^k}^2 + \|M^{\varepsilon}\|_{H^k}^2 + \|H^{\varepsilon}\|_{H^k}^2$, solving the inequality (5.171) gives,

$$
\sup_{0\leq t\leq T} E_{\varepsilon}(t) \leq \frac{E_{\varepsilon}(0)}{1 - CTE_{\varepsilon}(0)}. \tag{5.172}
$$

In particular, our solution $(u^{\varepsilon}, \Omega^{\varepsilon}, M^{\varepsilon}, H^{\varepsilon})$ is uniformly bounded in $C([0,T]; H^k)$ for $k > 5/2$ for all $T < \frac{1}{CE_{\varepsilon}(0)}$. Moreover, we can bound the time derivatives as follows: From equations (5.121) and from relation (4.39),

$$
\left\|\frac{du^{\varepsilon}}{dt}\right\|_{H^{k-2}} + \left\|\frac{d\Omega^{\varepsilon}}{dt}\right\|_{H^{k-2}} + \left\|\frac{dM^{\varepsilon}}{dt}\right\|_{H^{k-2}} + \left\|\frac{dH^{\varepsilon}}{dt}\right\|_{H^{k-2}} \lesssim E_{\varepsilon}(0)^2, \tag{5.173}
$$

for $k > 5/2$; in the above we have used either the obvious inequalities, or the locally compact embedding $H^k \subset H^{k-1}$ on $\mathbb{R}^3$ where appropriate to achieve this bound. Note that since there is no $M \times (M \times H)$ term in the Rosensweig model, we can achieve a quadratic, rather than a cubic, bound here. Thus, our solution $(u^\varepsilon, M^\varepsilon, H^\varepsilon)$ is uniformly bounded in $\text{Lip}\{[0,T]; H^{k-2}\}$.

Remark. Since all of the $(k+1)$-th derivatives fell on u^ε or Ω^ε in estimates (5.161)-(5.170), the bounds also hold without a Bloch-Torrey magnetization term.

5.2.4 Limit Point in $C([0,T]; L^2)$

In this subsection we show the sequence $(u^\varepsilon, \Omega^\varepsilon, M^\varepsilon, H^\varepsilon)$ has a subsequence which has a limit point (u, Ω, M, H) in the space $C([0,T]; L^2)$. Our goal is to apply the Aubin-Lions compactness theorem (Theorem 4.1.3) to show the sequence

$(u^\varepsilon, \Omega^\varepsilon, M^\varepsilon, H^\varepsilon)$ is precompact in $C([0,T]; H^{k'})$ for $k-2 \leq k' < k$. Then since $H^{k'}$ is compactly embedded in L^2_σ, L^2, there exists a subsequence which converges to a limit point (u, Ω, M, H) in $C([0,T]; L^2)$. In the previous step, bounds (5.171) and (5.173) gave us

$$\{(u^\varepsilon, \Omega^\varepsilon, M^\varepsilon, H^\varepsilon)\} \text{ is uniformly bounded in } C([0,T]; H^k),$$

$$\{(u^\varepsilon, \Omega^\varepsilon, M^\varepsilon, H^\varepsilon)\} \text{ is uniformly bounded in } \text{Lip}([0,T]; H^{k-2}),$$

respectively. In particular, the uniform bound in the Lipschitz space gives $\{(u^\varepsilon, \Omega^\varepsilon, M^\varepsilon, H^\varepsilon)\}$ is uniformly equicontinuous on $[0,T]$ with values in H^{k-2}. By the Rellich-Kondrachov compactness theorem, the embedding $H^k \subset H^{k'}$ is locally compact on $\mathbb{R}^3$. Therefore, by Theorem 4.1.3, with $X = H^k, Y =$

$H^{k'}, Z = H^{k-2}$, $(u^\varepsilon, \Omega^\varepsilon, M^\varepsilon, H^\varepsilon)$ is precompact in $C([0,T]; H^{k'})$ and therefore has a limit point in $C([0,T]; L^2)$. Moreover, the subsequence of $(u^\varepsilon, \Omega^\varepsilon, M^\varepsilon, H^\varepsilon)$ converges strongly to (u, Ω, M, H) in $H^{k'}$ for all $(k-2) \le k' < k$.

5.2.5 Limit Solution Solves Equation

In subsection 5.2.5, we prove that the limit (u, Ω, M, H) solves the Rosensweig system. Using the fact that $(u^\varepsilon, \Omega^\varepsilon, M^\varepsilon, H^\varepsilon)$ solves the mollified Rosensweig equations (5.121), we obtain considering an integrated form of the momentum equation using Lemma 4.2.2,

$$
\begin{aligned}
u(t) - u(0) &+ \int_0^t \mathbb{P}(u(s)\cdot\nabla u(s))ds - \frac{\eta+\zeta}{\rho}\int_0^t \Delta u(s)ds \\
&- \frac{\mu_0}{\rho}\int_0^t \mathbb{P}(M(s)\cdot\nabla H(s))ds - \frac{\mu_0}{2\rho}\int_0^t \nabla\times(M(s)\times H(s))ds \\
&= (u(t) - u^\varepsilon(t)) - (1-\mathcal{J}_\varepsilon)u(0) + \int_0^t \mathbb{P}((u(s)-u^\varepsilon(s))\cdot\nabla u(s))ds \\
&+ \int_0^t \mathbb{P}(u^\varepsilon(s)\cdot\nabla(u(s)-u^\varepsilon(s)))ds + \int_0^t \mathbb{P}(1-\mathcal{J}_\varepsilon)(u^\varepsilon(s)\cdot\nabla u^\varepsilon(s))ds \\
&+ \int_0^t \mathbb{P}((1-\mathcal{J}_\varepsilon)u^\varepsilon(s)\cdot\nabla u^\varepsilon)ds + \int_0^s \mathbb{P}(\mathcal{J}_\varepsilon u^\varepsilon(s)\cdot\nabla(1-\mathcal{J}_\varepsilon)u^\varepsilon(s))ds \\
&- \frac{\eta+\zeta}{\rho}\int_0^t \Delta(u(s) - \mathcal{J}_\varepsilon^2 u^\varepsilon(s))ds - \frac{\mu_0}{\rho}\left\{\int_0^t \mathbb{P}((M(s)-M^\varepsilon(s))\cdot\nabla H(s))ds\right. \\
&+ \int_0^t \mathbb{P}(M^\varepsilon(s)\cdot\nabla(H(s)-H^\varepsilon(s)))ds + \int_0^t \mathbb{P}(1-\mathcal{J}_\varepsilon)(M^\varepsilon(s)\cdot\nabla H^\varepsilon(s))ds \\
&\left.+ \int_0^t \mathbb{P}((1-\mathcal{J}_\varepsilon)M^\varepsilon(s)\cdot\nabla H^\varepsilon)ds + \int_0^s \mathbb{P}(\mathcal{J}_\varepsilon M^\varepsilon(s)\cdot\nabla(1-\mathcal{J}_\varepsilon)H^\varepsilon(s))ds\right\} \\
&+ \int_0^t (1-\mathcal{J}_\varepsilon)\nabla\times\Omega(s)ds + \int_0^t \mathcal{J}_\varepsilon\nabla\times((1-\mathcal{J}_\varepsilon)\Omega(s))ds \\
&+ \int_0^t \mathcal{J}_\varepsilon\nabla\times(\mathcal{J}_\varepsilon(\Omega(s)-\Omega^\varepsilon(s)))ds \\
&= D_1 + D_2 + D_3 + D_4 + D_5 + D_6 + D_7 + D_8 + D_9 + D_{10} + D_{11} + D_{12} \\
&\quad + D_{13} + D_{14} + D_{15} + D_{16} + D_{17}. \qquad (5.174)
\end{aligned}
$$

We take the L^2 norm of both sides of equation (5.178), and estimate the terms. Since the terms $D_1 - D_{14}$ are exactly the same in the proof for the Shliomis model, we only estimate D_{15} - D_{17} here (see (5.53), (5.54), (5.55), (5.56), (5.57), (5.58), (5.59), (5.60), (5.1.5), (5.1.5), (5.61), (5.1.5), (5.1.5), and (5.63) for the estimates of $D_1 - D_{14}$). We bound, using mollifier property (4.8)

$$\|D_{15}\|_{L^2} \leq CT\varepsilon \|\Omega^\varepsilon\|_{L^\infty(0,T;H^k)} \to 0, \tag{5.175}$$

$$\|D_{16}\|_{L^2} \leq CT\varepsilon \|\Omega^\varepsilon\|_{L^\infty(0,T;H^k)} \to 0, \tag{5.176}$$

and

$$\|D_{17}\|_{L^2} \leq CT\|\Omega - \Omega^\varepsilon\|_{L^\infty(0,T;H^k)} \to 0. \tag{5.177}$$

We do the same for the angular momentum equation from (5.121). Using the fact $(u^\varepsilon, \Omega^\varepsilon, M^\varepsilon, H^\varepsilon)$ solves (5.121), we consider an integrated form of the angular momentum equation to obtain

$$\begin{aligned}
\Omega(t) - \Omega(0) &+ \int_0^t (u(s) \cdot \nabla \Omega(s))ds - \frac{\eta'}{\rho\kappa}\int_0^t \Delta\Omega(s)ds \\
&- \frac{(\eta' + \lambda')}{\rho\kappa}\int_0^t \nabla(\nabla \cdot \Omega(s))ds - \frac{\mu_0}{\rho\kappa}\int_0^t M(s) \times H(s)ds \\
&- \frac{2\zeta}{\rho\kappa}\int_0^t \nabla \times u(s) - 2\Omega(s)ds \\
&= (\Omega(t) - \Omega^\varepsilon(t)) - (1 - \mathcal{J}_\varepsilon)\Omega(0) + \int_0^t ((u(s) - u^\varepsilon(s)) \cdot \nabla\Omega(s))ds \\
&+ \int_0^t (u^\varepsilon(s) \cdot \nabla(\Omega(s) - \Omega^\varepsilon(s)))ds + \int_0^t (1 - \mathcal{J}_\varepsilon)(u^\varepsilon(s) \cdot \nabla\Omega^\varepsilon(s))ds \\
&+ \int_0^t ((1 - \mathcal{J}_\varepsilon)u^\varepsilon(s) \cdot \nabla\Omega^\varepsilon)ds + \int_0^t (\mathcal{J}_\varepsilon u^\varepsilon(s) \cdot \nabla(1 - \mathcal{J}_\varepsilon)\Omega^\varepsilon(s))ds \\
&- \frac{\eta'}{\rho\kappa}\int_0^t \Delta(\Omega(s) - \mathcal{J}_\varepsilon^2\Omega^\varepsilon(s))ds - \frac{(\eta' + \lambda')}{\rho\kappa}\int_0^t \nabla[\nabla \cdot (\Omega(s) - \mathcal{J}_\varepsilon^2\Omega^\varepsilon(s))]ds \\
&+ \frac{\mu_0}{\rho\kappa}\left\{\int_0^t (1 - \mathcal{J}_\varepsilon)M(s) \times H(s)ds + \int_0^t \mathcal{J}_\varepsilon((1 - \mathcal{J}_\varepsilon)M(s) \times H(s)ds\right.
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^t \mathcal{J}_\varepsilon(\mathcal{J}_\varepsilon M(s)\times(1-\mathcal{J}_\varepsilon)H(s))ds\\
&+\int_0^t \mathcal{J}_\varepsilon(\mathcal{J}_\varepsilon(M(s)-M^\varepsilon(s))\times\mathcal{J}_\varepsilon H(s))ds\\
&+\int_0^t \mathcal{J}_\varepsilon(\mathcal{J}_\varepsilon M^\varepsilon(s)\times\mathcal{J}_\varepsilon(H(s)-H^\varepsilon(s)))ds\Bigg\}+\frac{2\zeta}{\rho\kappa}\left\{\int_0^t(1-\mathcal{J}_\varepsilon)\nabla\times u(s)ds\right.\\
&+\int_0^t \mathcal{J}_\varepsilon\nabla\times[(1-\mathcal{J}_\varepsilon)u^\varepsilon(s)]ds+\int_0^t \mathcal{J}_\varepsilon\nabla\times[\mathcal{J}_\varepsilon(u(s)-u^\varepsilon(s)]ds\\
&\left.+\int_0^t 2(\Omega(s)-\Omega^\varepsilon(s))ds\right\}\\
&=E_1+E_2+E_3+E_4+E_5+E_6+E_7+E_8+E_9+E_{10}+E_{11}+E_{12}\\
&\quad+E_{13}+E_{14}+E_{15}+E_{16}+E_{17}+E_{18}. \qquad (5.178)
\end{aligned}
$$

All of these can be bounded in a way we have seen before.

Similarly, for the magnetization equation,

$$
\begin{aligned}
M(t)&-M(0)+\int_0^t u(s)\cdot\nabla M(s)ds-\sigma\int_0^t \Delta M(s)ds\\
&-\frac{1}{2}\int_0^t(\nabla\times u(s))\times M(s)ds+\frac{1}{\tau}(M(s)-\chi_0 H(s))ds\\
&+\beta\int_0^t M(s)\times(M(s)\times H(s))ds\\
&=(M(t)-M^\varepsilon(t))-(1-\mathcal{J}_\varepsilon)M(0)+\int_0^t \mathbb{P}((u(s)-u^\varepsilon(s))\cdot\nabla M(s))ds\\
&+\int_0^t \mathbb{P}(u^\varepsilon(s)\cdot\nabla(M(s)-M^\varepsilon(s)))ds+\int_0^t \mathbb{P}(1-\mathcal{J}_\varepsilon)(u^\varepsilon(s)\cdot\nabla M^\varepsilon(s))ds\\
&+\int_0^t \mathbb{P}((1-\mathcal{J}_\varepsilon)u^\varepsilon(s)\cdot\nabla M^\varepsilon)ds+\int_0^s \mathbb{P}(\mathcal{J}_\varepsilon u^\varepsilon(s)\cdot\nabla(1-\mathcal{J}_\varepsilon)M^\varepsilon(s))ds\\
&+\int_0^t \Delta(M(s)-\mathcal{J}_\varepsilon^2 M^\varepsilon(s))ds+\int_0^t(1-\mathcal{J}_\varepsilon)[\Omega(s)\times M(s)]ds\\
&+\int_0^t \mathcal{J}_\varepsilon[(1-\mathcal{J}_\varepsilon)\Omega(s)]\times M(s)ds+\int_0^t \mathcal{J}_\varepsilon(\mathcal{J}_\varepsilon\Omega(s)\times[(1-\mathcal{J}_\varepsilon)M(s)])ds\\
&+\int_0^t \mathcal{J}_\varepsilon(\mathcal{J}_\varepsilon(\Omega(s)-\Omega^\varepsilon(s))\times\mathcal{J}_\varepsilon M(s))ds\\
&+\int_0^t \mathcal{J}_\varepsilon(\mathcal{J}_\varepsilon\Omega^\varepsilon(s)\times\mathcal{J}_\varepsilon(M(s)-M^\varepsilon(s)))ds
\end{aligned}
$$

$$+\frac{1}{\tau}\int_0^t [M(s)-M^\varepsilon(s)]ds - \frac{\chi_0}{\tau}\int_0^t [H(s)-H^\varepsilon(s)]ds$$
$$= F_1+F_2+F_3+F_4+F_5+F_6+F_7+F_8+F_9+F_{10}+F_{11}+F_{12}$$
$$+F_{13}+F_{14}+F_{15}. \tag{5.179}$$

Again we take the L^2 norm of both sides of the equation and estimate the terms on the right-hand side. First note that $F_1, F_2, F_3, F_4, F_5, F_6, F_7, F_8, F_{14}$, and F_{15} can be estimated in the same way as in the proof for the Shliomis model. We bound

$$\|F_9\|_{L^2}, \|F_{10}\|_{L^2}, \|F_{11}\|_{L^2} \le C\varepsilon\|\Omega\|_{L^\infty(0,T;H^k)}\|M\|_{L^\infty(0,T;H^k)} \xrightarrow{\varepsilon\to 0} 0, \tag{5.180}$$

and

$$\|F_{12}\|_{L^2} \le C\|\Omega-\Omega^\varepsilon\|_{L^\infty(0,T;H^k)}\|M\|_{L^\infty(0,T;H^k)} \to 0, \tag{5.181}$$

$$\|F_{13}\|_{L^2} \le C\|M-M^\varepsilon\|_{L^\infty(0,T;H^k)}\|\Omega\|_{L^\infty(0,T;H^k)} \to 0. \tag{5.182}$$

We have shown that equations (2.11)-(2.14), (2.18) hold a.e. in (x,t) (again we don't need to consider the equation for the time derivative of H since it is a linear function of M). However, we can achieve pointwise equality by noting strong convergence in $C([0,T];H^k)$ for $0 < 7/2 < k$ implies strong convergence in $C([0,T];(C^2(\mathbb{R}^3))^4)$. Therefore (u,Ω,M,H) is a classical (pointwise) solution of the Rosensweig model.

Remark. Note that this argument also holds without the addition of a Bloch-Torrey magnetization term to the model.

5.2.6 Uniqueness of Limit Solution

Next we prove uniqueness of the solution above. Suppose $(u^1, \Omega^1, M^1, H^1)$ and $(u^2, \Omega^2, M^2, H^2)$ are two solutions of the Rosensweig system (2.11)-(2.14) with the same initial data $(u_0, \Omega_0, M_0, H_0)$. Then the solutions would obey

$$\begin{aligned}
&\rho\{\partial_t(u^1 - u^2) + \mathbb{P}[(u^1 - u^2)\cdot\nabla u^1 + u^2\cdot\nabla(u^1 - u^2)]\} - (\eta + \zeta)\Delta(u^1 - u^2)\\
&= \mathbb{P}[(M^1 - M^2)\cdot\nabla H^1] + \mathbb{P}\mu_0[M^2\cdot\nabla(H^1 - H^2)] + 2\zeta\nabla\times(\Omega^1 - \Omega^2),
\end{aligned} \tag{5.183}$$

$$\begin{aligned}
&\rho\kappa\{\partial_t(\Omega^1 - \Omega^2) + (u^1 - ^2)\cdot\nabla\Omega^1 + u^2\cdot\nabla(\Omega^1 - \Omega^2)\} - \eta'\Delta(\Omega^1 - \Omega^2)\\
&- (\eta' + \lambda')\nabla(\nabla\cdot(\Omega^1 - \Omega^2)) = \mu_0(M^1 - M^2)\times H^1 + \mu_0 M^2\times(H^1 - H^2)\\
&\qquad\qquad + 2\zeta\nabla\times(u^1 - u^2) - 4\zeta(\Omega^1 - \Omega^2), \text{ and}
\end{aligned} \tag{5.184}$$

$$\begin{aligned}
&\partial_t(M^1 - M^2) + (u^1 - u^2)\cdot\nabla M^1 + u^2\cdot\nabla(M^1 - M^2) + \sigma\Delta(M^1 - M^2)\\
&= (\Omega^1 - \Omega^2)\times M^1 + \Omega^2\times(M^1 - M^2) - \frac{1}{\tau}(M^1 - M^2) + \frac{\chi_0}{\tau}(H^1 - H^2).
\end{aligned} \tag{5.185}$$

Taking the L^2 inner product of (5.183) with $u^1 - u^2$ gives,

$$\begin{aligned}
\frac{\rho}{2}\|u^1 - u^2\|^2_{L^2} = &-\rho\mathbb{P}\int_{\mathbb{R}^3}(u^1 - u^2)\cdot[(u^1 - u^2)\cdot\nabla u^1]dx\\
&- \rho\mathbb{P}\int_{\mathbb{R}^3}(u^1 - u^2)\cdot[u^2\cdot\nabla(u^1 - u^2)]dx\\
&+ (\eta + \zeta)\int_{\mathbb{R}^3}(u^1 - u^2)\cdot\Delta(u^1 - u^2)dx\\
&+ \mu_0\mathbb{P}\int_{\mathbb{R}^3}(M^1 - M^2)\cdot\nabla H^1\cdot(u^1 - u^2)dx\\
&+ \mu_0\mathbb{P}\int_{\mathbb{R}^3} M^2\cdot\nabla(H^1 - H^2)\cdot(u^1 - u^2)dx\\
&+ 2\zeta\int_{\mathbb{R}^3}\nabla\times(\Omega^1 - \Omega^2)\cdot(u^1 - u^2)dx.
\end{aligned} \tag{5.186}$$

The only term here which was not dealt with in the proof for the Shliomis model is the last one. It can be estimated by

$$\left|2\zeta \int_{\mathbb{R}^3} \nabla \times (\Omega^1 - \Omega^2) \cdot (u^1 - u^2) dx\right| \leq C(\|\nabla(\Omega^1 - \Omega^2)\|_{L^2}^2 + \|u^1 - u^2\|_{L^2}^2) \tag{5.187}$$

Next, taking the L^2 inner product of (5.184) with $\Omega^1 - \Omega^2$

$$\begin{aligned}
\frac{\rho\kappa}{2}\|\Omega^1 - \Omega^2\|_{L^2}^2 &= -\rho\kappa\mathbb{P}\int_{\mathbb{R}^3} (\Omega^1 - \Omega^2) \cdot [(u^1 - u^2) \cdot \nabla\Omega^1] dx \\
&\quad - \rho\kappa\mathbb{P}\int_{\mathbb{R}^3} (\Omega^1 - \Omega^2) \cdot [u^2 \cdot \nabla(\Omega^1 - \Omega^2)] dx \\
&\quad + \eta' \int_{\mathbb{R}^3} (\Omega^1 - \Omega^2) \cdot \Delta(\Omega^1 - \Omega^2) dx \\
&\quad + (\eta' + \lambda') \int_{\mathbb{R}^3} \nabla(\nabla \cdot (\Omega^1 - \Omega^2)) \cdot (\Omega^1 - \Omega^2) dx \\
&\quad + \mu_0\mathbb{P}\int_{\mathbb{R}^3} (M^1 - M^2) \times H^1 \cdot (\Omega^1 - \Omega^2) dx \\
&\quad + \mu_0\mathbb{P}\int_{\mathbb{R}^3} M^2 \times (H^1 - H^2) \cdot (\Omega^1 - \Omega^2) dx \\
&\quad + 2\zeta \int_{\mathbb{R}^3} \nabla \times (u^1 - u^2) \cdot (\Omega^1 - \Omega^2) dx \\
&\quad - 4\zeta\|\Omega^1 - \Omega^2\|_{L^2}^2 \\
&= K_1 + K_2 + K_3 + K_4 + K_5 + K_6 + K_7.
\end{aligned} \tag{5.188}$$

Note that integrating by parts gives

$$K_4 = -\|\nabla \cdot (\Omega^1 - \Omega^2)\|_{L^2}^2. \tag{5.189}$$

The rest of the terms $K_1 - K_3, K_5 - K_7$ can be estimated in exactly the same way as previous estimates. Taking the L^2 inner product of (5.185) with

$M^1 - M^2$ gives

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|M^1 - M^2\|_{L^2} = &-\int_{\mathbb{R}^3} (u^1 - u^2)\cdot\nabla M^1\cdot(M^1 - M^2)dx \\ &-\int_{\mathbb{R}^3} u^2\cdot\nabla(M^1 - M^2)\cdot(M^1 - M^2)dx \\ &-\frac{1}{\tau}\|M^1 - M^2\|_{L^2} + \frac{\chi_0}{\tau}\int_{\mathbb{R}^3}(H^1 - H^2)\cdot(M^1 - M^2)dx \\ &+\sigma\int_{\mathbb{R}^3}\Delta(M^1 - M^2)\cdot(M^1 - M^2)dx \\ &+\int_{\mathbb{R}^3}(\Omega^1 - \Omega^2)\times M^1\cdot(M^1 - M^2) \\ &+\int_{\mathbb{R}^3}\Omega^2\times(M^1 - M^2)\cdot(M^1 - M^2)dx.\end{aligned} \tag{5.190}$$

The first five terms above can be estimated exactly as we did for the Shliomis model, and the last two can be estimated in the standard way. Adding equations (5.186), (5.188), and (5.190), using the bounds on $\frac{1}{2}\frac{d}{dt}\|M^1 - M^2\|_{L^2}$ to bound $\frac{1}{2}\frac{d}{dt}\|H^1 - H^2\|_{L^2}$, and using the appropriate estimates (choosing some ε small enough as before) along with Grönwall's inequality gives uniqueness of solutions.

Remark. Because none of the previous steps required the use of a Bloch-Torrey magnetization term, we have also shown the existence of a limiting solution which is unique in $C([0,T];H^k)$, and solves the Rosensweig system.

5.2.7 Regularity of Limit Solution

Finally we show $(u,\Omega,M,H)\in C([0,T];H^k)$. First we prove weak continuity; i.e. that $(u,\Omega,M,H)\in C([0,T];H^k \text{ weak})$. Then we show continuity of the H^k norm of (u,Ω,M,H). These two ingredients show the continuity above.

We begin by proving weak continuity, i.e. $\langle u(t), \phi\rangle_{L^2}+\langle \Omega(t), \theta\rangle_{L^2}+\langle M(t), \psi\rangle_{L^2}+\langle H(t), \upsilon\rangle_{L^2}$ is continuous in time for $\phi, \theta, \psi, \upsilon \in H^{-k}(\mathbb{R}^3)$. Fix test functions $\phi, \theta, \psi, \upsilon \in H^{-k}(\mathbb{R}^3)$ with $\|\phi\|_{H^{-k}} = \|\theta\|_{H^{-k}} = \|\psi\|_{H^{-k}} = \|\upsilon\|_{H^{-k}} = 1$. Then there exist $\phi', \theta', \psi', \upsilon' \in H^{-k+2}$ (which is dense in H^{-k} and dual to H^{k-2}) such that,

$$
\begin{aligned}
&\|\phi - \phi'\|_{H^{-k}} + \|\theta - \theta'\|_{H^{-k}} + \|\psi - \psi'\|_{H^{-k}} + \|u - u'\|_{H^{-k}} \\
&\qquad \leq \frac{\varepsilon}{4 \sup_{t\in(0,T)}(\|u(t)\|_{H^k} + \|\Omega(t)\|_{H^k} + \|M(t)\|_{H^k} + \|H(t)\|_{H^k})}.
\end{aligned} \tag{5.191}
$$

For s to be determined later, we write

$$
\begin{aligned}
&|\langle u(t) - u(s), \phi\rangle_{L^2} + \langle \Omega(t) - \Omega(s), \theta\rangle_{L^2} + \langle M(t) - M(s), \psi\rangle_{L^2} \\
&\qquad + \langle H(t) - H(s), \upsilon\rangle_{L^2}| \\
&\quad = |\langle u(t) - u(s), \phi'\rangle_{L^2} + \langle u(t) - u(s), \phi - \phi'\rangle_{L^2} \\
&\qquad + \langle \Omega(t) - \Omega(s), \theta'\rangle_{L^2} + \langle \Omega(t) - \Omega(s), \theta - \theta'\rangle_{L^2} \\
&\qquad + \langle M(t) - M(s), \psi'\rangle_{L^2} + \langle M(t) - M(s), \psi - \psi'\rangle_{L^2} \\
&\qquad + \langle H(t) - H(s), \upsilon'\rangle_{L^2} + \langle H(t) - H(s), \upsilon - \upsilon'\rangle_{L^2}| \\
&\quad \leq \|u(t) - u(s)\|_{H^{k-2}} \|\phi'\|_{H^{-k+2}} + 2 \sup_{t\in(0,T)} \|u(t)\|_{H^k} \|\phi - \phi'\|_{H^{-k}} \\
&\qquad + \|\Omega(t) - \Omega(s)\|_{H^{k-2}} \|\theta'\|_{H^{-k+2}} + 2 \sup_{t\in(0,T)} \|\Omega(t)\|_{H^k} \|\theta - \theta'\|_{H^{-k}} \\
&\qquad + \|M(t) - M(s)\|_{H^{k-2}} \|\psi'\|_{H^{-k+2}} + 2 \sup_{t\in(0,T)} \|M(t)\|_{H^k} \|\psi - \psi'\|_{H^{-k}} \\
&\qquad + \|H(t) - H(s)\|_{H^{k-2}} \|\psi'\|_{H^{-k+2}} + 2 \sup_{t\in(0,T)} \|H(t)\|_{H^k} \|\upsilon - \upsilon'\|_{H^{-k}} \\
&\quad \leq (\|u(t) - u(s)\|_{H^{k-2}} + \|\Omega(t) - \Omega(s)\|_{H^{k-2}} + \|M(t) - M(s)\|_{H^{k-2}}
\end{aligned}
$$

$$+ \|H(t) - H(s)\|_{H^{k-2}})$$

$$\times (\|\phi'\|_{H^{-k+2}} + \|\theta'\|_{H^{-k+2}} + \|\psi'\|_{H^{-k+2}} + \|\upsilon'\|_{H^{-k+2}})$$

$$+ (\|\phi - \phi'\|_{H^{-k}} + \|\theta - \theta'\|_{H^{-k}} + \|\psi - \psi'\|_{H^{-k}} + \|u - u'\|_{H^{-k}})$$

$$\times 2 \sup_{t\in(0,T)} (\|u(t)\|_{H^k} + \|\Omega(t)\|_{H^k} + \|M(t)\|_{H^k} + \|H(t)\|_{H^k}).$$

Since $(u^\varepsilon, \Omega^\varepsilon, M^\varepsilon, H^\varepsilon) \in \mathrm{Lip}([0,T];; H^{k-2}(\mathbb{R}^3) \times H^{k-2}(\mathbb{R}^3) \times H^{k-2}(\mathbb{R}^3))$, there exists $\tau_\varepsilon > 0$ such that

$$\begin{aligned} \|u^\varepsilon(t) - u^\varepsilon(s)\|_{H^{k-2}} + \|\Omega^\varepsilon(t) - \Omega^\varepsilon(s)\|_{H^{k-2}} + \|M^\varepsilon(t) - M^\varepsilon(s)\|_{H^{k-2}} \\ + \|H^\varepsilon(t) - H^\varepsilon(s)\|_{H^{k-2}} \\ \leq \frac{\varepsilon}{2(\|\phi'\|_{H^{-k+2}} + \|\theta'\|_{H^{-k+2}} + \|\psi'\|_{H^{-k+2}} + \|\upsilon'\|_{H^{-k+2}})}. \end{aligned} \tag{5.192}$$

for $t, s \in [0,T]$ such that $|t-s| < \tau_\varepsilon$. Choosing s in this way, from estimates (5.104) and (5.192) we get

$$|\langle u(t)-u(s),\phi\rangle_{L^2}+\langle \Omega(t)-\Omega(s),\theta\rangle_{L^2}+\langle M(t)-M(s),\psi\rangle_{L^2}+\langle H(t)-H(s),\upsilon\rangle_{L^2}| < \varepsilon, \tag{5.193}$$

so that our solution is uniformly weakly continuous.

Remark. This also holds without the addition of a Bloch-Torrey magnetization term.

Finally, we show $\|u(t)\|_{H^k} + \|\Omega(t)\|_{H^k} + \|M(t)\|_{H^k} + \|H(t)\|_{H^k}$ is continuous. Our proof requires the use of a Bloch-Torrey magnetization term. First recall the bound we proved earlier (5.51)

$$\sup_{0\leq t\leq T} E_\varepsilon(t) \leq \frac{E(0)}{1 - CTE_(0)} = E(0) + \frac{CTE(0)^2}{1 - CTE(0)}, \tag{5.194}$$

where $E_\varepsilon(t) = 1 + \|u^\varepsilon\|_{H^k}^2 + \|\Omega^\varepsilon\|_{H^k}^2 + \|M^\varepsilon\|_{H^k}^2 + \|H^\varepsilon\|_{H^k}^2$ and for clarity we have relabelled $E_\varepsilon(0) = E(0) = 1 + \|u_0\|_{H^k}^2 + \|\Omega_0\|_{H^k}^2 + \|M_0\|_{H^k}^2 + \|H_0\|_{H^k}^2$. Let us similarly define $E(t) := 1 + \|u\|_{H^k}^2 + \|\Omega\|_{H^k}^2 + \|M\|_{H^k}^2 + \|H\|_{H^k}^2$. Since for fixed t, $\limsup_{\varepsilon\to 0} E_\varepsilon(t) \geq E(t)$, the bound (5.194) gives

$$\sup_{0\leq t\leq T} E(t) - E(0) \leq \frac{CTE(0)^2}{1 - CTE(0)}. \tag{5.195}$$

This implies $\limsup_{t\to 0^+} E(t) \leq E(0)$, or in particular,

$$\limsup_{t\to 0^+}(\|u\|_{H^k}^2 + \|\Omega\|_{H^k}^2 + \|M\|_{H^k}^2 + \|H\|_{H^k}^2) \leq \|u_0\|_{H^k}^2 + \|\Omega_0\|_{H^k}^2 + \|M_0\|_{H^k}^2 + \|H_0\|_{H^k}^2. \tag{5.196}$$

On the other hand, since $(u, \Omega, M, H) \in C([0,T]; H^k \text{ weak})$,

$$\liminf_{t\to 0^+}(\|u\|_{H^k}^2 + \|\Omega\|_{H^k}^2 + \|M\|_{H^k}^2 + \|H\|_{H^k}^2) \geq \|u_0\|_{H^k}^2 + \|\Omega_0\|_{H^k}^2 + \|M_0\|_{H^k}^2 + \|H_0\|_{H^k}^2. \tag{5.197}$$

Thus since the norms are positive,

$$\lim_{t\to 0^+}(\|u\|_{H^k} + \|\Omega\|_{H^k} + \|M\|_{H^k} + \|H\|_{H^k}) = \|u_0\|_{H^k} + \|\Omega_0\|_{H^k} + \|M_0\|_{H^k} + \|H_0\|_{H^k}, \tag{5.198}$$

so that we have strong right-continuity at $t = 0$. Next, we show strong right-continuity on $(0,T)$. Suppose for contradiction $\exists t_0 \in (0,T)$ such that right-continuity does not hold. Recall estimate (5.171) had extra terms on the left-hand side which were ignored. Without neglecting these terms, the estimate reads

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}(\|u^\varepsilon\|_{H^k}^2 + +\|\Omega^\varepsilon\|_{H^k}^2 + \|M^\varepsilon\|_{H^k}^2 + \|H^\varepsilon\|_{H^k}^2) + \|\mathcal{J}_\varepsilon \nabla u^\varepsilon\|_{H^k}^2 + \|\mathcal{J}_\varepsilon \nabla \Omega^\varepsilon\|_{H^k}^2 \\
+ \|\mathcal{J}_\varepsilon \nabla M^\varepsilon\|_{H^k}^2 + \|\mathcal{J}_\varepsilon \nabla H^\varepsilon\|_{H^k}^2 \\
\leq C(1 + \|u^\varepsilon\|_{H^k}^2 + \|\Omega^\varepsilon\|_{H^k}^2 + \|M^\varepsilon\|_{H^k}^2 + \|H^\varepsilon\|_{H^k}^2)^2.
\end{aligned} \tag{5.199}$$

In particular, this implies (using the same estimate for the limit equation)

$$\begin{aligned}&\int_0^T \|\nabla u(t)\|_{H^k}^2 dt, \int_0^T \|\nabla \Omega^\varepsilon\|_{H^k}^2 dt + \int_0^T \|\nabla M(t)\|_{H^k}^2 dt,\\ &\text{and } \int_0^T \|\nabla H(t)\|_{H^k}^2 dt\end{aligned} \tag{5.200}$$

are bounded. Therefore, the limit solution $(u, \Omega, M, H) \in L^2(0, T; H^{k+1})$, which guarantees that for a.e. $\tau \in (0, t_0)$,

$$(u(\cdot, \tau), \Omega(\cdot, \tau), M(\cdot, \tau), H(\cdot, \tau)) \in H^{k+1}(\mathbb{R}^3). \tag{5.201}$$

Fix τ. Then

$$\exists\ T_\tau \geq \frac{1}{C_0(\|u(\tau)\|_{H^{k+1}}^2 + \|\Omega(\tau)\|_{H^{k+1}}^2 + \|M(\tau)\|_{H^{k+1}}^2 + \|H(\tau)\|_{H^{k+1}}^2)} > 0$$

such that $(u, , \Omega, M, H) \in L^\infty(\tau, \tau+T_\tau; H^{k+1}) \cap C([\tau, \tau+T_\tau]; H^k)$. In particular, since the solution is not right-continuous at t_0, this gives $T_\tau < t_0 - \tau$. Thus

$$\|u(\tau)\|_{H^{k+1}}^2 + \|\Omega(\tau)\|_{H^{k+1}}^2 + \|M(\tau)\|_{H^{k+1}}^2 + \|H(\tau)\|_{H^{k+1}}^2 \gtrsim \frac{1}{t_0 - \tau}. \tag{5.202}$$

But then since $t_0 \in (0, T)$,

$$\int_0^T (\|u(\tau)\|_{H^{k+1}}^2 + \|\Omega(\tau)\|_{H^{k+1}}^2 + \|M(\tau)\|_{H^{k+1}}^2 + \|H(\tau)\|_{H^{k+1}}^2) d\tau = \infty,$$

which is a contradiction to (5.201). Similarly one can prove strong left-continuity on $(0, T]$ (it was done for the Shliomis and the differences between proving right and left continuity are the same for this proof).

5.3 A Remark about the Bloch-Torrey Magnetization Term

In the above two proofs we highlighted which steps could be done without the use of a Bloch-Torrey magnetization term. In particular, continuity of the norm (used to prove the regularity of our solution) was the only property that ceases to hold in absence of this Bloch-Torrey term. We note that this norm continuity can be proven without a diffusion term in the case of the Euler equation (the argument relies on time-reversibility), and that the argument involving a diffusion term holds for the Navier-Stokes equations. However, for the Rosensweig and Shliomis models without a Bloch-Torrey magnetization term, in which some equations have a diffusion term and others don't, it is unclear whether it is possible to prove norm continuity. It seems impossible to decouple the equations, and the two methods of proof are not compatible in an obvious way.

Chapter 6

Prodi-Serrin type Conditions for Ferrohydrodynamics Models

Now we turn our attention to deriving Prodi-Serrin type conditions for the Shliomis model with Bloch-Torrey magnetization, and Rosensweig model with Bloch-Torrey magnetization. We begin with a discussion of where Prodi-Serrin conditions first arose. Then, we derive conditions for the ferrohydrodynamics models which essentially match the conditions for Navier-Stokes. We first find a condition for the Shliomis model which is exactly the same as for Navier-Stokes. Then we derive conditions for the Rosensweig model, which require an additional (but expected) stipulation for the angular momentum equation.

6.1 Classical Prodi-Serrin Conditions

Before deriving Prodi-Serrin type conditions for the equations of ferrohydrodynamics, we should understand how they arose historically. The prototypical Prodi-Serrin conditions were additional constraints one could impose on a Leray-Hopf weak solution of the Navier-Stokes equations to guarantee that it is smooth. Recall the three-dimensional incompressible Navier-Stokes equations are given by,

$$u_t + u \cdot \nabla u = -\nabla p + \nu \Delta u, \qquad (x,t) \in \mathbb{R}^3 \times (0,\infty), \tag{6.1}$$

$$\nabla \cdot u = 0, \qquad (x,t) \in \mathbb{R}^3 \times (0,\infty), \tag{6.2}$$

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}^3, \tag{6.3}$$

where u is the velocity field, p is the pressure, and ν is the dimensionless viscosity. In 1934, Jean Leray showed (see [LT16] for a translation by Terrell) that for arbitrary $T \in (0,\infty]$, there exists $u(x,t)$ satisfying:

(i) $u \in L^\infty(0,T; L^2(\mathbb{R}^3)) \cap L^2(0,T; H^1(\mathbb{R}^3))$;

(ii) u satisfies (6.1) and (6.2) in the sense of distributions;

(iii) (6.3) holds in the L^2-sense: $\lim_{t \searrow 0} \|u(\cdot,t) - u_0(\cdot)\|_{L^2} = 0$;

(iv) u satisfies the energy inequality

$$\|u(\cdot,t)\|_{L^2}^2 + 2\nu \int_0^t \|\nabla u(\cdot,\tau)\|_{L^2}^2 d\tau \leq \|u_0\|_{L^2}^2$$

for all $0 \leq t \leq T$.

A function $u(x,t)$ satisfying (i)-(iv) is called a Leray-Hopf weak solution for (6.1)-(6.3) in $\mathbb{R}^3 \times [0,T)$. If a Leray-Hopf weak solution is smooth, then it is a classical solution, and furthermore unique (in the class of Leray-Hopf weak solutions). Smoothness can be shown under a variety of additional assumptions (see for example [Str88]). One class of these assumptions are Prodi-Serrin conditions: If a Leray-Hopf weak solution $u(x,t)$ further satisfies

$$u \in L^q(0,T;L^p(\mathbb{R}^3)) \quad \text{with} \quad \frac{2}{q} + \frac{3}{p} \leq 1, \qquad 3 < p \leq \infty, \tag{6.4}$$

then $u(x,t)$ is smooth. Conditions like (6.4) are proven using energy methods techniques.

In this thesis, we consider local-in-time classical solutions (whose well-posedness was shown in the previous chapter) of the Shliomis model with Bloch Torrey magnetization and Rosensweig model with Bloch-Torrey magnetization. Our goal is to extend these to global-in-time solutions via Prodi-Serrin type conditions. To accomplish this, we find an energy estimate for which certain integrability conditions on the solution guarantee its existence up to and beyond every time $T > 0$. In contrast to the classical Prodi-Serrin conditions for the Navier Stokes equations, we don't use these conditions to prove additional regularity or uniqueness. Moreover, we start with a classical, local-in-time solution, instead of a global weak solution. Despite these differences, we will derive our conditions in the same way the conditions for Navier-Stokes are derived, and they will turn out to be essentially the same as (6.4).

6.2 Shliomis System with Bloch-Torrey type Magnetization

In this section, we derive Prodi-Serrin type conditions for the Shliomis model with Bloch-Torrey magnetization. Recall the equations are given by,

$$\begin{cases} \rho(\partial_t u + u\cdot\nabla u) - \eta\Delta u + \nabla p = \mu_0(M\cdot\nabla)H + \frac{\mu_0}{2}\nabla\times(M\times H), \\ \partial_t M + (u\cdot\nabla)M - \sigma\Delta M = \frac{1}{2}(\nabla\times u)\times M - \frac{1}{\tau}(M-\chi_0 H) - \beta M\times(M\times H), \\ \nabla\cdot u = 0, \nabla\times H = 0, \quad \nabla\cdot(H+M) = -\nabla\cdot H^{ext}. \end{cases} \tag{6.5}$$

We have the following theorem:

Theorem 6.2.1. Assume $\nabla\cdot H^{ext} = 0$ and $(u_0, M_0, H_0) \in H^k(\mathbb{R}^3)$ with $k > 5/2$. Then the following holds:

- If a solution $(u, M, H) \in C((0,T); H^k(\mathbb{R}^3))$ of (6.5) satisfies

$$u \in L^q(0,T; L^p(\mathbb{R}^3)) \quad \text{with} \quad \frac{2}{q} + \frac{3}{p} \le 1 \quad \text{and} \quad 3 < p \le \infty,$$

  then the solution exists up to and beyond time T.

Proof. Denote by ∂^k some particular k-th order partial derivative. Taking this k-th order derivative of the momentum equation, dotting with the same

k-th derivative of u, and integrating over space gives:

$$\begin{aligned}\frac{\rho}{2}\frac{d}{dt}\int_{\mathbb{R}^3}(\partial^k u)^2dx+\eta\int_{\mathbb{R}^3}(\partial^k\nabla u)^2dx=&-\rho\int_{\mathbb{R}^3}\partial^k(u\cdot\nabla u)\partial^k udx\\&+\mu_0\int_{\mathbb{R}^3}\partial^k(M\cdot\nabla H)\partial^k udx\\&+\frac{\mu_0}{2}\int_{\mathbb{R}^3}(\partial^k(\nabla\times(M\times H)))\partial^k udx\\=&S_1+S_2+S_3.\end{aligned}\tag{6.6}$$

Next, taking k-th order derivative of the magnetization equation, dotting with $\partial^k M$, and integrating over space gives:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}(\partial^k M)^2dx=\sigma\int_{\mathbb{R}^3}&\partial^k(\Delta M)\partial^k Mdx-\int_{\mathbb{R}^3}\partial^k(u\cdot\nabla M)\partial^k Mdx\\&+\frac{1}{2}\int_{\mathbb{R}^3}\partial^k((\nabla\times u)\times M))\partial^k Mdx\\&-\frac{1}{\tau}\int_{\mathbb{R}^3}\partial^k(M-\chi_0H)\partial^k Mdx\\&-\beta\int_{\mathbb{R}^3}\partial^k(M\times(M\times H))\partial^k Mdx.\end{aligned}$$

This gives, by using Lemma 4.2.4,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}&(\partial^k M)^2dx+\int_{\mathbb{R}^3}(\sigma(\partial^k\nabla M)^2+\frac{1}{\tau}|\partial^k M|^2+\frac{\chi_0}{\tau}|\partial^k H|^2)dx\\&=-\int_{\mathbb{R}^3}\partial^k(u\cdot\nabla M)\partial^k Mdx+\frac{1}{2}\int_{\mathbb{R}^3}\partial^k((\nabla\times u)\times M)\partial^k Mdx\\&\quad-\beta\int_{\mathbb{R}^3}\partial^k(M\times(M\times H))\partial^k Mdx\\&=S_4+S_5+S_6.\end{aligned}\tag{6.7}$$

Then, taking k-th order derivative of the magnetization equation again, this time dotting it with $-\partial^k H$, and integrating over space gives:

$$
\begin{aligned}
-\int_{\mathbb{R}^3} (\partial^k \partial_t M) \cdot \partial^k H dx = & -\sigma \int_{\mathbb{R}^3} \partial^k (\Delta M) \partial^k H dx + \int_{\mathbb{R}^3} \partial^k (u \cdot \nabla M) \partial^k H dx \\
& - \frac{1}{2} \int_{\mathbb{R}^3} \partial^k ((\nabla \times u) \times M) \partial^k H dx \\
& + \frac{1}{\tau} \int_{\mathbb{R}^3} \partial^k (M - \chi_0 H) \partial^k H dx \\
& + \beta \int_{\mathbb{R}^3} \partial^k (M \times (M \times H)) \partial^k H dx.
\end{aligned}
$$

This gives, by using Lemma 4.2.4 and (4.2.5),

$$
\begin{aligned}
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} & (\partial^k H)^2 dx + \int_{\mathbb{R}^3} (\sigma (\partial^k \nabla H)^2 + \frac{(1+\chi_0)}{\tau} (\partial^k H)^2) dx \\
& = \int_{\mathbb{R}^3} \partial^k (u \cdot \nabla M) \partial^k H dx - \frac{1}{2} \int_{\mathbb{R}^3} \partial^k ((\nabla \times u) \times M) \cdot \partial^k H dx \\
& \quad + \beta \int_{\mathbb{R}^3} \partial^k (M \times (M \times H)) \partial^k H dx \\
& = S_7 + S_8 + S_9.
\end{aligned}
\tag{6.8}
$$

Summing equations (6.6)-(6.8) gives,

$$
\begin{aligned}
\frac{1}{2} \frac{d}{dt} & \int_{\mathbb{R}^3} [(\partial^k u)^2 + (\partial^k M)^2 + (\partial^k H)^2] dx \\
& + \int_{\mathbb{R}^3} [(\partial^k \nabla u)^2 + \frac{1}{\tau} (\partial^k M)^2 + \frac{(1+2\chi_0)}{\tau} (\partial^k H)^2] dx \\
& + \int_{\mathbb{R}^3} \sigma (\partial^k \nabla M)^2 + \sigma (\partial^k \nabla H)^2 dx \\
& = S_1 + S_2 + S_3 + S_4 + S_5 + S_6 + S_7 + S_8 + S_9.
\end{aligned}
\tag{6.9}
$$

To prove Theorem 6.2.1, we estimate each term on the right-hand side of (6.9).

First,

$$|S_1| = \left| \int_{\mathbb{R}^3} \partial^k (u \cdot \nabla u) \partial^k u dx \right| = \left| \int_{\mathbb{R}^3} \partial^{k-1} (u \cdot \nabla u) \cdot \partial^{k+1} u dx \right|$$

(from integrating by parts using Lemma 4.2.3). By Hölder's inequality,

$$|S_1| \le \|\partial^{k+1} u\|_{L^2} \|\partial^{k-1}(u \cdot \nabla u)\|_{L^2}.$$

Since $\nabla \cdot u = 0$, we have

$$\partial^{k-1}(u \cdot \nabla u) = \partial^{k-1} \nabla \cdot (u \otimes u)$$

This allows us to use Ning Ju's inequality (Proposition 4.2.1 as follows:

$$|S_1| \lesssim \|u\|_{L^p} \|D^k u\|_{L^q} \|\partial^{k+1} u\|_{L^2} \lesssim \|u\|_{L^p} \|D^k u\|_{L^q} \|D^{k+1} u\|_{L^2},$$

where $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$.

Next, we use the Gagliardo-Nirenberg inequality

$$\|D^k u\|_{L^q} \lesssim \|D^k u\|_{L^2}^{\alpha} \|D^{k+1} u\|_{L^2}^{1-\alpha},$$

where $\frac{1}{q} = \left(\frac{1}{2} - \frac{1}{3}\right)(1-\alpha) + \frac{\alpha}{2} = \frac{1}{6} + \frac{\alpha}{3} \Rightarrow \alpha = 1 - \frac{3}{p}$. The inequality requires $\alpha \in [0,1]$ (but we need $\alpha > 0$ to accomplish our plan). In particular, this forces $3 < p \le \infty$. This gives

$$|S_1| \lesssim \|u\|_{L^p} \|D^k u\|_{L^2}^{\alpha} \|D^{k+1} u\|_{L^2}^{2-\alpha} \quad (\star).$$

Finally, by Lemma 4.2.1,

$$|S_1| \le C(\varepsilon) \|u\|_{L^p}^{\frac{2p}{p-3}} \|D^k u\|_{L^2}^2 + \varepsilon \|D^{k+1} u\|_{L^2}^2. \tag{6.10}$$

Second,

$$|S_2| = \left|\int_{\mathbb{R}^3} \partial^k (M\cdot\nabla H)\partial^k u dx\right| = \left|\int_{\mathbb{R}^3} \partial^{k-1}(M\cdot\nabla H)\partial^{k+1} u dx\right|.$$

Since $\nabla\times H = 0$, $\partial_i H_j = \partial_j H_i$, and using $\nabla\cdot(M+H) = 0$, we have

$$(M\cdot\nabla)H = \nabla\cdot[(M+H)\otimes H - \tfrac{1}{2}H^2 I].$$

Denote by $MH = M\otimes H$ and $HH = H\otimes H - \frac{1}{2}H^2 I$ products of M and H (which obey the product rule and therefore won't affect the proof, but simplify the presentation). We have,

$$\begin{aligned}
|S_2| &\lesssim \left|\int_{\mathbb{R}^3} D^k(MH+HH)\partial^{k+1}u dx\right| \\
&\lesssim \|D^k(MH)\|_{L^2}\|\partial^{k+1}u\|_{L^2} + \|D^k(HH)\|_{L^2}\|\partial^{k+1}u\|_{L^2} \\
&\lesssim (\|M\|_{L^p}\|D^k H\|_{L^q} + \|H\|_{L^p}\|D^k M\|_{L^q} + \|H\|_{L^p}\|D^k H\|_{L^q})\|\partial^{k+1}u\|_{L^2} \\
&\lesssim (\|M\|_{L^p} + \|H\|_{L^p})\|D^k M\|_{L^q}\|\partial^{k+1}u\|_{L^2} \\
&\lesssim (\|M\|_{L^p} + \|H\|_{L^p})\|D^k M\|_{L^2}^{\alpha}\|D^{k+1}M\|_{L^2}^{1-\alpha}\|\partial^{k+1}u\|_{L^2},
\end{aligned}$$

where in the 5-th step we used Gagliardo Nirenberg inequality with $\frac{1}{q} = (\frac{1}{2} - \frac{1}{3})(1-\alpha) + \frac{\alpha}{2} = \frac{1}{6} + \frac{\alpha}{3} = \frac{1}{2} - \frac{1}{p}$ which gives $\alpha = 1 - \frac{3}{p}$. $\alpha\in(0,1]$ gives us $3 < p \le \infty$. Lemma 4.2.1 then allows us to control S_2 by

$$|S_2| \lesssim C(\varepsilon)(\|M\|_{L^p}^{\frac{2p}{p-3}} + \|H\|_{L^p}^{\frac{2p}{p-3}})\|D^k M\|_{L^2}^2 + \varepsilon_1\|\partial^{k+1}u\|_{L^2}^2 + \varepsilon_2\|D^{k+1}M\|_{L^2}^2, \tag{6.11}$$

where $3 < p \le \infty$ and $C(\varepsilon)$ is a constant that depends on ε_1 and ε_2.

Next, we estimate S_3:

$$|S_3| = \left|\frac{1}{2}\int_{\mathbb{R}^3} \partial^{k-1}(\nabla\times(M\times H))\partial^{k+1}u dx\right|.$$

In the same way as S_2, we can estimate

$$|S_3| \lesssim C(\varepsilon)(\|M\|_{L^p}^{\frac{2p}{p-3}} + \|H\|_{L^p}^{\frac{2p}{p-3}})\|D^k M\|_{L^2}^2 + \varepsilon_1\|\partial^{k+1}u\|_{L^2}^2 + \varepsilon_2\|D^{k+1}M\|_{L^2}^2, \tag{6.12}$$

where $3 < p \leq \infty$ and $C(\varepsilon)$ is a constant that depends on ε_1 and ε_2. Next,

$$|S_4| = \left|\int_{\mathbb{R}^3} \partial^k(u\cdot\nabla M)\partial^k M dx\right| = \left|\int_{\mathbb{R}^3} \partial^{k-1}(u\cdot\nabla M)\partial^{k+1} M dx\right|.$$

Since $\nabla \cdot u = 0$,

$$\partial^{k-1}(u\cdot\nabla M) = \partial^{k-1}\nabla\cdot(u\otimes M).$$

Then we can write,

$$|S_4| \lesssim \|D^k(u\otimes M)\|_{L^2}\|\partial^{k+1}M\|_{L^2}.$$

Similar to S_1 and S_2, this allows us to estimate

$$\begin{aligned} |S_4| &\lesssim (\|M\|_{L^p}\|D^k u\|_{L^q} + \|u\|_{L^p}\|D^k M\|_{L^q})\|\partial^{k+1}M\|_{L^2} \\ &\lesssim C(\varepsilon)\|M\|_{L^p}^{\frac{2p}{p-3}}\|D^k u\|_{L^2}^2 + C(\varepsilon)\|u\|_{L^p}^{\frac{2p}{p-3}}\|D^k M\|_{L^2}^2 + \varepsilon_1\|D^{k+1}u\|_{L^2}^2 \\ &\quad + \varepsilon_2\|D^{k+1}M\|_{L^2}^2, \end{aligned} \tag{6.13}$$

where again $3 < p \leq \infty$ and $C(\varepsilon)$ is a constant that depends on ε_1 and ε_2.

Now we estimate S_5:

$$\begin{aligned} |S_5| &= \left|\frac{1}{2}\int_{\mathbb{R}^3} \partial^k((\nabla\times u)\times M)\partial^k M dx\right| \\ &= \left|\frac{1}{2}\int_{\mathbb{R}^3} \partial^{k-1}(M\times(\nabla\times u))\partial^{k+1} M dx\right|. \end{aligned}$$

By Hölder's inequality and the triangle inequality,

$$|S_5| \lesssim \sum_{\substack{m+n=k\\ m,n\geq 1}} \|\partial^m u \partial^n M\|_{L^2}\|\partial^{k+1}M\|_{L^2} + \|\partial^k u M\|_{L^2}\|\partial^{k+1}M\|_{L^2}.$$

The right-hand term can be estimated as before;

$$\|\partial^k u M\|_{L^2}\|\partial^{k+1}M\|_{L^2} \lesssim C(\varepsilon_1)\|M\|_{L^p}^{\frac{2p}{p-3}}\|\partial^k u\|_{L^2}^2 + \varepsilon_1\|\partial^{k+1}M\|_{L^2}^2.$$

where $p \in (3,\infty)$. On the other hand, for the sum we bound each term individually. By Lemma 4.2.7, $\exists\ \alpha \in (0,1)$ such that

$$\|(\partial^m u)(\partial^n M)\|_{L^2} \lesssim \|u\|_{L^p}\|D^k u\|_{L^q}^{1-\alpha}\|M\|_{L^p}^{1-\alpha}\|D^k M\|_{L^q}^{\alpha}.$$

This gives, using Young's inequality,

$$\sum_{\substack{m+n=k\\ m,n\geq 1}} \|\partial^m u \partial^n M\|_{L^2}\|\partial^{k+1}M\|_{L^2} \lesssim (\|u\|_{L^p}\|D^k M\|_{L^q} + \|M\|_{L^p}\|D^k u\|_{L^q})\|\partial^{k+1}M\|_{L^2},$$

which allows us to estimate in the same way as (6.11)

$$|S_5| \lesssim C(\varepsilon)\|u\|_{L^p}^{\frac{2p}{p-3}}\|\partial^k M\|_{L^2}^2 + C(\varepsilon)\|M\|_{L^p}^{\frac{2p}{p-3}}\|\partial^k u\|_{L^2}^2 + \varepsilon_2\|\partial^{k+1}M\|_{L^2}^2 + \varepsilon_1\|\partial^{k+1}u\|_{L^2}^2, \tag{6.14}$$

where $3 < p \leq \infty$ and $C(\varepsilon)$ depends on ε_1 and ε_2.

For S_6, we perform two separate estimates. Firstly, we have

$$|S_6| = \left|\int_{\mathbb{R}^3} \partial^k (M \times (M \times H))\partial^k M dx\right| \leq \|\partial^k (M \times (M \times H))\|_{L^2}\|\partial^k M\|_{L^2}.$$

We want

$$|S_6| \lesssim \|M\|_{L^p}^2\|\partial^k M\|_{L^2}^{2-\alpha}\|\partial^{k+1}M\|_{L^2}^{\alpha} \quad (\star).$$

By Ning Ju's lemma (Proposition 4.2.1),

$$\|\partial^k(M \times (M \times H))\|_{L^2} \lesssim \|M\|_{L^p}\|D^k(M \times H)\|_{L^s} \lesssim \|M\|_{L^p}\|M\|_{L^q}\|D^k H\|_{L^r},$$

where $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=\frac{1}{2}$. Using $\|\partial^l H\|_{L^p} \lesssim \|\partial^l M\|_{L^p}$, and setting $p=q$, this gives,

$$\|D^k(M \times (M \times H))\|_{L^2} \lesssim \|M\|_{L^p}^2\|D^k M\|_{L^r},$$

where now $\frac{1}{r}=\frac{1}{2}-\frac{2}{p}$. Next, we use Gagliardo-Nirenberg to get ($\star$):

$$\|D^k M\|_{L^r} \lesssim \|D^{k+1}M\|_{L^2}^{\alpha}\|D^k M\|_{L^2}^{1-\alpha},$$

where $\frac{1}{r}=(\frac{1}{2}-\frac{1}{3})\alpha+\frac{1-\alpha}{2}=\frac{1}{2}-\frac{\alpha}{3} \Rightarrow \alpha=\frac{6}{p}$. Then $\alpha \in [0,1]$ implies $p \in [6,\infty]$. Finally we use Lemma 4.2.1 to get,

$$|S_6| \lesssim C(\varepsilon)\|M\|_{L^p}^{\frac{2p}{p-3}}\|\partial^k M\|_{L^2}^2+\varepsilon\|\partial^{k+1}M\|_{L^2}^2, \tag{6.15}$$

for $p \geq 6$.

On the other hand,

$$\begin{aligned}|S_6| &= \left|\int_{\mathbb{R}^3}\partial^k(M \times (M \times H))\partial^k M dx\right| = \left|\int_{\mathbb{R}^3}\partial^{k-1}(M \times (M \times H))\partial^{k+1} M dx\right| \\ &\leq \|\partial^{k-1}(M \times (M \times H))\|_{L^2}\|\partial^{k+1}M\|_{L^2}.\end{aligned}$$

We want

$$\|\partial^{k-1}(M \times (M \times H))\|_{L^2} \lesssim \|M\|_{L^p}^2\|\partial^k M\|_{L^2}^{\alpha}\|\partial^{k+1}M\|_{L^2}^{1-\alpha}.$$

By using Ning Ju's inequality (Proposition 4.2.1) twice, we get,

$$\begin{aligned}\|\partial^{k-1}(M \times (M \times H))\|_{L^2} &\lesssim \|M\|_{L^p}\|D^{k-1}(M \times H)\|_{L^s} \\ &\lesssim \|M\|_{L^p}\|M\|_{L^l}\|D^{k-1}H\|_{L^q},\end{aligned}$$

where $\frac{1}{2}=\frac{1}{p}+\frac{1}{s}=\frac{1}{p}+\frac{1}{l}+\frac{1}{q}$. Choosing $p=l$ gives,

$$\|\partial^{k-1}(M \times (M \times H))\|_{L^2} \lesssim \|M\|_{L^p}^2 \|D^{k-1}H\|_{L^q}, \qquad \text{where } \tfrac{2}{p} + \tfrac{1}{q} = \tfrac{1}{2} \quad (\star\star).$$

First recall that $\|\partial^m H\|_{L^q} \lesssim \|\partial^m M\|_{L^q}$. Next, we want to move the derivative term up to ∂^k and ∂^{k+1} terms. First, by the Sobolev embedding theorem,

$$\|D^{k-1}M\|_{L^q} \lesssim \|D^k M\|_{L^r} \text{ with } \tfrac{1}{r} = \tfrac{1}{q} + \tfrac{1}{3} \;\; (\star\star\star).$$

Second, by the Gagliardo-Nirenberg inequality,

$$\|D^k M\|_{L^r} \lesssim \|D^{k+1}M\|_{L^2}^{1-\alpha} \|D^k M\|_{L^2}^{\alpha}, \qquad \text{with } \tfrac{1}{r} = \left(\tfrac{1}{2} - \tfrac{1}{3}\right)(1-\alpha) + \tfrac{\alpha}{2}.$$

Solving for p in terms of α gives us the condition we are looking for. In particular,

$$\tfrac{1}{r} = \tfrac{1}{6} + \tfrac{\alpha}{3} \overset{(\star\star\star)}{\Rightarrow} \tfrac{1}{q} + \tfrac{1}{3} = \tfrac{1}{6} + \tfrac{\alpha}{3} \overset{(\star\star)}{\Rightarrow} \tfrac{1}{2} - \tfrac{2}{p} + \tfrac{1}{3} = \tfrac{1}{6} + \tfrac{\alpha}{3} \Rightarrow \tfrac{2}{p} = \tfrac{2-\alpha}{3} \Rightarrow \tfrac{1}{p} = \tfrac{2-\alpha}{6}.$$

The requirement $\alpha \in (0,1]$ gives $\frac{1}{p} \in \left(\frac{1}{6}, \frac{1}{3}\right) \Rightarrow p \in (3,6]$. However, $(\star\star)$ forces us to choose $p \geq 4$. Altogether, we have,

$$|S_6| \lesssim \|M\|_{L^p}^2 \|D^k M\|_{L^2}^{\alpha} \|D^{k+1}M\|_{L^2}^{2-\alpha}.$$

Finally, by Lemma 4.2.1,

$$|S_6| \lesssim C(\varepsilon)\|M\|_{L^p}^{\frac{2p}{p-3}} \|D^k M\|_{L^2}^2 + \varepsilon \|D^{k+1}M\|_{L^2}^2, \tag{6.16}$$

where $p \in [4,6]$. Putting (6.15) and (6.16) together give us the same inequality with the more general condition $p \geq 4$.

In the same way as S_4,

$$|S_7| = \left| \int_{\mathbb{R}^3} \partial^k (u \cdot \nabla M) \partial^k H dx \right|$$

can be estimated by

$$|S_7| \le C(\varepsilon)\|u\|_{L^p}^{\frac{2p}{p-3}}\|D^k M\|_{L^2}^2 + C(\varepsilon)\|M\|_{L^p}^{\frac{2p}{p-3}}\|D^k u\|_{L^2}^2 + \varepsilon_1\|D^{k+1}u\|_{L^2}^2 + \varepsilon_2\|\partial^{k+1}H\|_{L^2}^2. \tag{6.17}$$

with $3 < p \le \infty$ and $C(\varepsilon)$ depends on ε_1 and ε_2.

In exactly the same way as S_5, we bound

$$\begin{aligned}|S_8| &= \left|\frac{1}{2}\int_{\mathbb{R}^3} \partial^k((\nabla \times u) \times M) \cdot \partial^k H dx\right| \\ &= \left|\frac{1}{2}\int_{\mathbb{R}^3} \partial^{k-1}((\nabla \times u) \times M) \cdot \partial^{k+1} H dx\right|\end{aligned}$$

by

$$\begin{aligned}|S_8| \lesssim\ & C(\varepsilon)\|u\|_{L^p}^{\frac{2p}{p-3}}\|D^k M\|_{L^2}^2 + C(\varepsilon)\|M\|_{L^p}^{\frac{2p}{p-3}}\|D^k u\|_{L^2}^2 + \varepsilon_2\|D^{k+1}M\|_{L^2}^2 \\ &+ \varepsilon_1\|D^{k+1}u\|_{L^2}^2 + \varepsilon_1\|\partial^{k+1}H\|_{L^2}^2,\end{aligned} \tag{6.18}$$

where $3 < p \le \infty$ and $C(\varepsilon)$ depends on ε_1 and ε_2.

In exactly the same way as S_6 ((6.15), (6.16)),

$$|S_9| = \left|\int_{\mathbb{R}^3} \partial^k(M \times (M \times H))\partial^k H dx\right|$$

can be estimated by

$$|S_9| \lesssim C(\varepsilon)\|M\|_{L^p}^{2p/(p-3)}\|D^k M\|_{L^2}^2 + \varepsilon\|D^{k+1}H\|_{L^2}^2, \tag{6.19}$$

where again $p \ge 4$.

Putting everything together and summing over all partial derivatives of order k, we get from (6.9) and (6.10), (6.11), (6.12), (6.13), (6.14), (6.15), (6.16), (6.17), (6.18), (6.19):

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|u\|_{H^k}^2+\|M\|_{H^k}^2+\|H\|_{H^k}^2\right)\\
&\qquad+\|\nabla u\|_{H^k}^2+\|M\|_{H^k}^2+3\|H\|_{H^k}^2+\|\nabla M\|_{H^k}^2+\|\nabla H\|_{H^k}^2\\
&\quad\lesssim C(\varepsilon)(\|u\|_{L^p}^{\frac{2p}{p-3}}+\|M\|_{L^p}^{\frac{2p}{p-3}})\|u\|_{H^k}^2 \qquad (6.20)\\
&\qquad+C(\varepsilon)(\|M\|_{L^p}^{\frac{2p}{p-3}}+\|H\|_{L^p}^{\frac{2p}{p-3}}+\|u\|_{L^p}^{\frac{2p}{p-3}})\|M\|_{H^k}^2\\
&\qquad+\varepsilon_1\|D^{k+1}u\|_{L^2}^2+(\varepsilon_1+\varepsilon_2)\|D^{k+1}H\|_{L^2}^2+\varepsilon_2\|D^{k+1}M\|_{L^2}^2.
\end{aligned}
$$

Choosing $\varepsilon_{1,2}<\frac{1}{2}$, we can absorb the $(k+1)$-th order derivative terms into terms on the left-hand side of (6.20). We then rearrange terms, adding constants when necessary to get

$$
\begin{aligned}
&\frac{d}{dt}\left(\|u\|_{H^k}^2+\|M\|_{H^k}^2+\|H\|_{H^k}^2\right)\\
&\quad\le C(\|M\|_{L^p}^{\frac{2p}{p-3}}+\|H\|_{L^p}^{\frac{2p}{p-3}}+\|u\|_{L^p}^{\frac{2p}{p-3}})(\|M\|_{H^k}^2+\|H\|_{H^k}^2+\|u\|_{H^k}^2).
\end{aligned} \tag{6.21}
$$

Then by Gronwall's inequality,

$$
\mathcal{E}(t)\le C\mathcal{E}(0)\exp\left(\int_0^t\mathcal{K}(s)ds\right),
$$

where

$$
\begin{aligned}
\mathcal{E}(t)&:=(\|u(t)\|_{H^k}^2+\|M(t)\|_{H^k}^2+\|H(t)\|_{H^k}^2),\\
\mathcal{K}(t)&:=(\|M(s)\|_{L^p}^{\frac{2p}{p-3}}+\|H(s)\|_{L^p}^{\frac{2p}{p-3}}+\|u(s)\|_{L^p}^{\frac{2p}{p-3}}).
\end{aligned}
$$

By Lemma 4.2.11 and Lemma 4.2.9, $M,H\in L^p$ for all $p>3$ are uniformly bounded in t. This gives the result.

$\square$

Remark. If $\nabla \cdot H^{ext} \neq 0$, then by Lemmas 4.2.4 and 4.2.5, the theorem still holds under the additional conditions

$$H^{ext}, \partial_t H^{ext} \in L^2 \cap C((0,T); H^k(\mathbb{R}^3)),$$
$$\|H^{ext}\|_{L^p} \text{ is uniformly bounded in on } (0,T).$$

Indeed, we use Grönwall's inequality from Robinson's book [Rob01].

6.3 Rosensweig System with Bloch-Torrey type Magnetization

Recall the Rosensweig system with Bloch-Torrey magnetization,

$$\begin{cases} \rho(\partial_t u + u \cdot \nabla u) - (\eta + \zeta)\Delta u + \nabla p = \mu_0 (M \cdot \nabla) H + 2\zeta \nabla \times \Omega, \\ \rho\kappa(\partial_t \Omega + (u \cdot \nabla)\Omega) - \eta' \Delta \Omega - (\eta' + \lambda')\nabla(\nabla \cdot \Omega) = \mu_0 M \times H + 2\zeta(\nabla \times u - 2\Omega), \\ \partial_t M + (u \cdot \nabla) M - \sigma \Delta M = \Omega \times M - \frac{1}{\tau}(M - \chi_0 H), \\ \nabla \cdot u = 0, \nabla \times H = 0, \quad \nabla \cdot (H + M) = -\nabla \cdot H^{ext}. \end{cases} \tag{6.22}$$

Since we don't have $\Omega \in L^p(\mathbb{R}^3)$ (as we do for M), we need a condition on Ω for this theorem. We prove,

Theorem 6.3.1. Assume $\nabla \cdot H^{ext} = 0$ and $(u_0, \Omega_0, M_0, H_0) \in H^k(\mathbb{R}^3)$ with $k > 5/2$. Then the following holds:

- If a solution $(u, M, \Omega, H) \in C((0,T); H^k(\mathbb{R}^3))$ of (6.22) satisfies

  $$u, \Omega \in L^q(0,T; L^p(\mathbb{R}^3)) \quad \text{with} \quad \frac{2}{q} + \frac{3}{p} \le 1 \quad \text{and} \quad 3 < p \le \infty,$$

  then the solution exists up to and beyond time T.

Proof. Denote by ∂^k some particular k-th order partial derivative. Taking this k-th order derivative of the momentum equation, dotting with the same

k-th derivative of u, and integrating over space gives:

$$
\begin{aligned}
&\frac{\rho}{2}\frac{d}{dt}\int_{\mathbb{R}^3}(\partial^k u)^2 dx + (\eta+\zeta)\int_{\mathbb{R}^3}(\partial^k \nabla u)^2 dx \\
&\qquad\qquad = -\rho\int_{\mathbb{R}^3}\partial^k(u\cdot\nabla u)\partial^k u dx + \mu_0\int_{\mathbb{R}^3}\partial^k(M\cdot\nabla H)\partial^k u dx \\
&\qquad\qquad\quad + 2\zeta\int_{\mathbb{R}^3}(\partial^k\nabla\times\Omega)\partial^k u dx \\
&\qquad\qquad = S_1 + S_2 + R_1.
\end{aligned}
\tag{6.23}
$$

Next, taking k-th order derivative of the Ω equation, multiplying by $\partial^k\Omega$, and integrating over space gives

$$
\begin{aligned}
&\frac{\rho\kappa}{2}\frac{d}{dt}\int_{\mathbb{R}^3}(\partial^k\Omega)^2 dx + \int_{\mathbb{R}^3}\eta'(\partial^k\nabla\Omega)^2 + (\eta'+\lambda')(\partial^k\nabla\cdot\Omega)^2 + 4\zeta|\partial^k\Omega|^2 dx \\
&\quad = -\rho\kappa\int_{\mathbb{R}^3}\partial^k(u\cdot\nabla\Omega)\partial^k\Omega dx + \mu_0\int_{\mathbb{R}^3}\partial^k(M\times H)\partial^k\Omega dx \\
&\quad\quad + 2\zeta\int_{\mathbb{R}^3}\partial^k(\nabla\times u)\partial^k\Omega dx \\
&\quad = R_2 + R_3 + R_4.
\end{aligned}
\tag{6.24}
$$

Next, taking k-th order derivative of the magnetization equation, dotting with $\partial^k M$, and integrating over space gives:

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}(\partial^k M)^2 dx &= \sigma\int_{\mathbb{R}^3}\partial^k(\Delta M)\partial^k M dx - \int_{\mathbb{R}^3}\partial^k(u\cdot\nabla M)\partial^k M dx \\
&\quad - \frac{1}{\tau}\int_{\mathbb{R}^3}\partial^k(M-\chi_0 H)\partial^k M dx + \int_{\mathbb{R}^3}\partial^k(\Omega\times M)\partial^k M dx.
\end{aligned}
$$

This gives by (4.2.4),

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}(\partial^k M)^2dx &+ \int_{\mathbb{R}^3}(\sigma(\partial^k\nabla M)^2 + \frac{1}{\tau}|\partial^k M|^2 + \frac{\chi_0}{\tau}|\partial^k H|^2)dx \\ &= -\int_{\mathbb{R}^3}\partial^k(u\cdot\nabla M)\partial^k M dx + \int_{\mathbb{R}^3}\partial^k(\Omega\times M)\cdot\partial^k M dx \qquad (6.25) \\ &= S_4 + R_5.\end{aligned}$$

Then, taking k-th order derivative of the magnetization equation again, this time dotting it with $-\partial^k H$ and integrating over space gives:

$$\begin{aligned}-\int_{\mathbb{R}^3}(\partial^k\partial_t M)\cdot\partial^k H dx = -\sigma\int_{\mathbb{R}^3}\partial^k(\Delta M)\partial^k H dx + \int_{\mathbb{R}^3}\partial^k(u\cdot\nabla M)\partial^k H dx \\ -\int_{\mathbb{R}^3}\partial^k(\Omega\times M)\partial^k H dx + \frac{1}{\tau}\int_{\mathbb{R}^3}\partial^k(M-\chi_0 H)\partial^k H dx.\end{aligned}$$

Thus by (4.2.4) and (4.2.5),

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}(\partial^k H)^2dx &+ \int_{\mathbb{R}^3}(\sigma(\partial^k\nabla H)^2 + \frac{(1+\chi_0)}{\tau}(\partial^k H)^2)dx \\ &= \int_{\mathbb{R}^3}\partial^k(u\cdot\nabla M)\partial^k H dx - \int_{\mathbb{R}^3}\partial^k(\Omega\times M)\cdot\partial^k H dx \qquad (6.26) \\ &= S_7 + R_6.\end{aligned}$$

Summing equations (6.23)-(6.26) gives,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}&[(\partial^k u)^2 + (\partial^k\Omega)^2 + (\partial^k M)^2 + (\partial^k H)^2]dx \\ &+ \int_{\mathbb{R}^3}\Big[(\eta+\zeta)(\partial^k\nabla u)^2 + \eta'(\partial^k\nabla\Omega)^2 + (\eta'+\lambda')(\partial^k\nabla\cdot\Omega)^2 \\ &\quad + 4\zeta|\partial^k\Omega|^2 + \sigma(\partial^k\nabla M)^2 + \sigma(\partial^k\nabla H)^2 + \frac{1}{\tau}(\partial^k M)^2 \\ &\quad + \frac{(1+2\chi_0)}{\tau}(\partial^k H)^2\Big]dx\end{aligned}$$

$$= S_1 + S_2 + R_1 + R_2 + R_3 + R_4 + S_4 + R_5 + S_7 + R_6. \qquad (6.27)$$

To prove Theorem 6.3.1, we estimate each term on the right-hand side of (6.27). First of all, S_1, S_2, S_4, and S_7 can be estimated as above [(6.10), (6.11), (6.13), (6.17)]. Next, we estimate

$$\begin{aligned}|R_1| &= 2\zeta \left| \int_{\mathbb{R}^3} (\partial^k \nabla \times \Omega)\partial^k u dx \right| = 2\zeta \left| \int_{\mathbb{R}^3} (\partial^{k-1} \nabla \times \Omega)\partial^{k+1} u dx \right| \\ &\lesssim \|D^k \Omega\|_{L^2} \|\partial^{k+1} u\|_{L^2}.\end{aligned}$$

Young's inequality immediately gives

$$|R_1| \lesssim C(\varepsilon)\|D^k \Omega\|_{L^2}^2 + \varepsilon \|\partial^{k+1} u\|_{L^2}^2. \qquad (6.28)$$

Moreover,

$$|R_2| = \rho\kappa \left| \int_{\mathbb{R}^3} \partial^k (u \cdot \nabla \Omega)\partial^k \Omega dx \right| = \left| \int_{\mathbb{R}^3} \partial^{k-1} (u \cdot \nabla \Omega)\partial^{k+1} \Omega dx \right|.$$

Similar to our estimate of S_4, note that since $\nabla \cdot u = 0$,

$$\partial^{k-1}(u \cdot \nabla \Omega) = \partial^{k-1} \nabla \cdot (u \otimes \Omega).$$

This gives,

$$|R_2| \lesssim \|D^k (\Omega \otimes u)\|_{L^2} \|\partial^{k+1} \Omega\|_{L^2},$$

which allows us to estimate in the same way as S_4 (6.13):

$$|R_2| \lesssim C(\varepsilon)\|\Omega\|_{L^p}^{\frac{2p}{p-3}} \|D^k u\|_{L^2}^2 + C(\varepsilon)\|u\|_{L^p}^{\frac{2p}{p-3}} \|D^k \Omega\|_{L^2}^2 + \varepsilon_1 \|D^{k+1} u\|_{L^2}^2 + \varepsilon_2 \|D^{k+1} \Omega\|_{L^2}^2, \qquad (6.29)$$

where $3 < p \leq \infty$.

We compute

$$|R_3| = \mu_0 \left| \int_{\mathbb{R}^3} \partial^k (M \times H) \partial^k \Omega dx \right| \leq \mu_0 \|\partial^k (M \times H)\|_{L^2} \|\partial^k \Omega\|_{L^2}.$$

Applying Ning Ju's inequality (Proposition 4.2.1) gives

$$|R_3| \lesssim (\|M\|_{L^p} \|D^k H\|_{L^q} + \|H\|_{L^p} \|D^k M\|_{L^q}) \|\partial^k \Omega\|_{L^2},$$

where $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$. Next, the Gagliardo Nirenberg inequality gives,

$$\|D^k M\|_{L^q} \lesssim \|D^{k+1} M\|_{L^2}^{\alpha} \|D^k M\|_{L^2}^{1-\alpha},$$

where $\frac{1}{q} = (\frac{1}{2} - \frac{1}{3})\alpha + \frac{1-\alpha}{2} = \frac{1}{2} - \frac{\alpha}{3}$ gives $\frac{1}{p} = \frac{\alpha}{3}$. Then $\alpha \in [0, 1]$ gives $p \in [3, \infty]$. We apply the exact same computations for $\|D^k H\|_{L^q}$. Now we compute,

$$\begin{aligned}
&\|H\|_{L^p} \|\partial^k M\|_{L^2}^{1-\alpha} \|\partial^{k+1} M\|_{L^2}^{\alpha} \|\partial^k \Omega\|_{L^2} \\
&\qquad\qquad \lesssim \|H\|_{L^p}^2 \|D^k M\|_{L^2}^{2(1-\alpha)} \|D^{k+1} M\|_{L^2}^{2\alpha} + \|\partial^k \Omega\|_{L^2}^2 \\
&\qquad\qquad \lesssim C(\varepsilon) \|H\|_{L^p}^{\frac{2p}{p-3}} \|D^k M\|_{L^2}^2 + \|\partial^k \Omega\|_{L^2}^2 + \varepsilon \|D^{k+1} M\|_{L^2}^2.
\end{aligned}$$

We do the same for the $\|M\|_{L^p} \|\partial^k H\|_{L^q}$ term. Finally, we have

$$\begin{aligned}
|R_3| \lesssim\, & C(\varepsilon) \|M\|_{L^p}^{\frac{2p}{p-3}} \|D^k H\|_{L^2}^2 + C(\varepsilon) \|H\|_{L^p}^{\frac{2p}{p-3}} \|D^k M\|_{L^2}^2 + C(\varepsilon) \|\partial^k \Omega\|_{L^2}^2 \\
& + \varepsilon_1 \|D^{k+1} H\|_{L^2}^2 + \varepsilon_2 \|D^{k+1} M\|_{L^2}^2,
\end{aligned} \tag{6.30}$$

where $3 \leq p \leq \infty$ and $C(\varepsilon)$ is a constant that depends on ε_1 and ε_2.

Next we estimate R_4 in the same way as R_1 (6.28):

$$|R_4| = 2\zeta \left| \int_{\mathbb{R}^3} \partial^k (\nabla \times u) \partial^k \Omega dx \right| = 2\zeta \left| \int_{\mathbb{R}^3} \partial^{k-1} (\nabla \times u) \partial^{k+1} \Omega dx \right|$$
$$\lesssim \|D^k u\|_{L^2} \|\partial^{k+1} \Omega\|_{L^2}.$$

By Young's inequality,

$$|R_4| \lesssim C(\varepsilon) \|D^k u\|_{L^2}^2 + \varepsilon \|\partial^{k+1} \Omega\|_{L^2}^2. \tag{6.31}$$

Next, consider

$$|R_5| = \left| \int_{\mathbb{R}^3} \partial^k (\Omega \times M) \cdot \partial^k M dx \right|.$$

In exactly the same way as R_3 (6.30), we can bound

$$|R_5| \lesssim C(\varepsilon) \|M\|_{L^p}^{\frac{2p}{p-3}} \|D^k \Omega\|_{L^2}^2 + C(\varepsilon) \|\Omega\|_{L^p}^{\frac{2p}{p-3}} \|D^k M\|_{L^2}^2 + C(\varepsilon) \|\partial^k M\|_{L^2}^2$$
$$+ \varepsilon_1 \|D^{k+1} \Omega\|_{L^2}^2 + \varepsilon_2 \|D^{k+1} M\|_{L^2}^2, \tag{6.32}$$

where $3 \leq p \leq \infty$ and $C(\varepsilon)$ is a constant that depends on ε_1 and ε_2.

Finally, we bound

$$|R_6| = \left| \int_{\mathbb{R}^3} \partial^k (\Omega \times M) \cdot \partial^k H dx \right|,$$

again in exactly the same way as R_3 (6.30):

$$|R_6| \lesssim C(\varepsilon) \|M\|_{L^p}^{\frac{2p}{p-3}} \|D^k \Omega\|_{L^2}^2 + C(\varepsilon) \|\Omega\|_{L^p}^{\frac{2p}{p-3}} \|D^k M\|_{L^2}^2 + C(\varepsilon) \|\partial^k H\|_{L^2}^2$$
$$+ \varepsilon_1 \|D^{k+1} \Omega\|_{L^2}^2 + \varepsilon_2 \|D^{k+1} M\|_{L^2}^2, \tag{6.33}$$

where $3 \leq p \leq \infty$ and $C(\varepsilon)$ is a constant that depends on ε_1 and ε_2. Now by taking (6.10), (6.11), (6.28), (6.29), (6.30), (6.31), (6.13), (6.32), (6.33), (6.17), and (6.27), and summing over all partial derivatives of order k gives,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\|u\|_{H^k}^2+\|\Omega\|_{H^k}^2+\|M\|_{H^k}^2+\|H\|_{H^k}^2)+(\eta+\zeta)\|\nabla u\|_{H^k}^2+\eta'\|\nabla\Omega\|_{H^k}^2\\ +(\eta'+\lambda')\|\nabla\cdot\Omega\|_{H^k}^2+4\zeta\|\Omega\|_{H^k}^2+\sigma\|\nabla M\|_{H^k}^2+\sigma\|\nabla H\|_{H^k}^2\\ +\frac{1}{\tau}\|M\|_{H^k}^2+\frac{(1+2\chi_0)}{\tau}\|H\|_{H^k}^2\\ \lesssim C(\varepsilon)(\|u\|_{L^p}^{\frac{2p}{p-3}}+\|M\|_{L^p}^{\frac{2p}{p-3}}+\|\Omega\|_{L^p}^{\frac{2p}{p-3}}+1)\|D^k u\|_{L^2}^2\\ +C(\varepsilon)(\|M\|_{L^p}^{\frac{2p}{p-3}}+\|H\|_{L^p}^{\frac{2p}{p-3}}+\|u\|_{L^p}^{\frac{2p}{p-3}}+\|\Omega\|_{L^p}^{\frac{2p}{p-3}}+1)\|\partial^k M\|_{L^2}^2\\ +C(\varepsilon)(\|M\|_{L^p}^{\frac{2p}{p-3}}+\|u\|_{L^p}^{\frac{2p}{p-3}}+1)\|D^k\Omega\|_{L^2}^2\\ +\varepsilon_1\|D^{k+1}u\|_{L^2}^2+(\varepsilon_1+\varepsilon_2)\|D^{k+1}H\|_{L^2}^2+\varepsilon_2\|D^{k+1}M\|_{L^2}^2\\ +(\varepsilon_1+\varepsilon_2)\|\partial^{k+1}\Omega\|_{L^2}^2.\end{aligned} \tag{6.34}$$

Choosing $\varepsilon < \frac{1}{2}$, we can absorb the $(k+1)$-th order derivative terms into terms on the right-hand sice of (6.35). We then rearrange terms, adding constants when necessary to get

$$\frac{d}{dt}\mathcal{E}(t) \leq C\mathcal{K}(t)\mathcal{E}(t), \tag{6.35}$$

where

$$\begin{aligned}\mathcal{E}(t) &:= (\|u(t)\|_{H^k}^2+\|\Omega(t)\|_{H^k}^2+\|M(t)\|_{H^k}^2+\|H(t)\|_{H^k}^2),\\ \mathcal{K}(t) &:= (\|M(t)\|_{L^p}^{\frac{2p}{p-3}}+\|H(t)\|_{L^p}^{\frac{2p}{p-3}}+\|u(t)\|_{L^p}^{\frac{2p}{p-3}}+\|\Omega(t)\|_{L^p}^{\frac{2p}{p-3}}+1).\end{aligned} \tag{6.36}$$

Finally, by Grönwall's inequality,

$$\mathcal{E}(t) \leq \mathcal{E}(0)\exp\left(\int_0^t \mathcal{K}(s)ds\right). \tag{6.37}$$

Again by Lemma 4.2.11 and Lemma 4.2.9, $M, H \in L^p$ for all $p > 3$ are uniformly bounded in t. This gives the result, under the assumptions on u, Ω.

$\square$

Remark. If $\nabla \cdot H^{ext} \neq 0$, then the additional assumption,

$$H^{ext}, \partial_t H^{ext} \in L^2 \cap C((0,T); H^k(\mathbb{R}^3)),$$
$$\|H^{ext}\|_{L^p} \text{ is uniformly bounded in on } (0,T).$$

guarantees the theorem still holds. Again we use the version of Grönwall from Lemma 4.1.2 in Robinson's book [Rob01].

Chapter 7

Conclusion

In this work, we have extended the mathematical literature on Ferrohydrodynamics to include analysis of classical solutions. More specifically, we have proved the local well-posedness of the Rosensweig model with Bloch-Torrey magnetization, and the Shliomis model with Bloch-Torrey magnetization for classical solutions. This included constructing a solution in $C([0,T];H^k)$, showing the solution is unique in this class, and showing the solution changes continuously with respect to the initial data in the topology of this class. Then, we derived Prodi-Serrin type conditions for the solutions we constructed. These conditions (that $u, \Omega \in L^q(0,T;L^p(\mathbb{R}^3))$ with $\frac{2}{q} + \frac{3}{p} \leq 1$ and $3 < p \leq \infty$, where the condition on Ω is disregarded for the Shliomis model) guarantee that the solutions we previously constructed can be continued up to and beyond time T. Moreover, these conditions essentially agree (or exactly agree for the Shliomis model) with the conditions for the Navier-Stokes equations, upon which the Ferrohydrodynamics models are built.

In our proof of local well-posedness, we needed to include the Bloch-Torrey magnetization term $\sigma \Delta M$ in the magnetization equation of both models. Otherwise, we would have been unable to prove regularity of the solution constructed in the proof, since we would have been unable to prove continuity of the norm. The techniques to achieve this continuity for the Euler and Navier-Stokes equations rely on either time-reversibility (for Euler), or a diffusion term (Navier-Stokes). Because it seems impossible to decouple the equations in the ferrohydrodynamics models, using a combination of the techniques likely won't work. Therefore, we leave the problem of showing continuity of the norm for future work. Another potential future work could be to extend the use of our Prodi-Serrin conditions to other classes of solutions (weak solutions or strong solutions), where they may give additional properties. For example, for Leray-Hopf weak solutions of the Navier-Stokes equations, these conditions guarantee smoothness of the solution, and therefore that it is a classical solution, and is unique.

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