University of Alberta

Modeling, analysis and design of networked control systems

by

Jing Wu ©

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Abstract

This dissertation investigates modeling, analysis and design of networked control systems (NCSs), which are of great interest due to the advantages presented, but also are technically complex because of the interaction between the plant dynamics and the discrete and sometimes random network environment. Three relevant issues are considered: packet dropouts, network-induced delays, and sampling.

Firstly, Markovian jump linear systems with delays are briefly introduced. Stochastic stability, stochastic stabilization and H_∞ controller design are addressed with new model transformations and zero equations, and less conservative results are derived. Moreover, how delays affect the stochastic stability of the resulting closed-loop systems is discussed. An approach for calculating the maximum delay while guaranteeing the system stability is also presented.

Secondly, NCSs with packet dropouts are studied. Markov chains are introduced to describe historical behaviors of packet dropouts, the definition of which is different from those in the existing references. By this new definition, general models are derived under both single- and multiple-packet transmission protocols; furthermore, state feedback controllers are designed in the two cases to stabilize the resulting closed-loop systems.

Thirdly, NCSs with network-induced delays are investigated, where the delays are modeled by Markov chains. To compensate the delayed data, a model predictive control (MPC) method is introduced. The control scheme is characterized as a constrained optimization problem of the worst-case quadratic cost over an infinite horizon at each sampling instant. Stochastic stability conditions with and without input/output constraints are then developed in terms of linear matrix inequalities.

Finally, control design for a class of sampled-data systems with variable sampling rates is studied. The sampling rate is time-varying and bounded that brings new

challenge in modelings. To solve this problem, a prediction period is introduced, by which signals both at and between sampling instants are considered for controller design. An modified MPC approach is formulated based on the minimization of a finite horizon quadratic cost with a terminal weighting matrix.

To my parents
for their endless love and encouragement throughout the years.

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I am also deeply indebted to my other coauthors of the papers upon which this dissertation is based: joint work with Professor Long Wang and Professor Shengyuan Xu led to the main results in Chapter 2. I would also like to thank Dr. Huijun Gao, Dr. Chengnian Long and Miss Yuanyuan Zou for their discussion on the variable sampling period problem in Chapter 5. I benefited enormously from many hours of stimulating discussions with them, and I also owe a great deal for their support and friendship.

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Chapter 1

Introduction

Networked control systems (NCSs) are spatially distributed control systems, in which the communication between sensors, actuators, and controllers is accomplished through a shared bandlimited digital network. The study of NCSs is an interdisciplinary research area, combining both control and communication theories. Most NCS research has focused on two areas: control design and communication protocol design. Advanced controller design is desirable to guarantee the control Quality of Performance (QoP), whereas a proper message transmission protocol is necessary to guarantee the network Quality of Service (QoS). Because of the integral link between network and control in an NCS, it is important to consider network and control parameters simultaneously to assure both network QoS and control QoP. However, due to the complexity of communication networks and different definitions of communication protocols, few general results for NCSs have been derived. The goal of this dissertation is to provide general modeling, controller analysis and design tools for NCSs.

There are many control applications configured as NCSs. In general, these applications can be categorized in two general configurations as follows:

Direct structure: NCSs in the direct structure are composed of a controller and a remote system containing a physical plant, sensors and actuators. The controller and the plant are physically placed at different locations and are directly linked by a data network as illustrated in Figure 1.1. The control signal is encapsulated in a frame or a packet and sent to the plant via the network. The plant then returns the system output to the controller by putting the sensor measurement into a frame

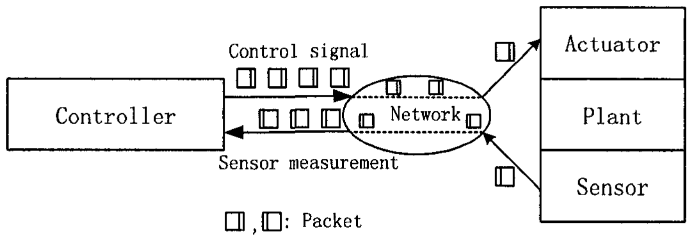

Figure 1.1: NCS setup in direct structure

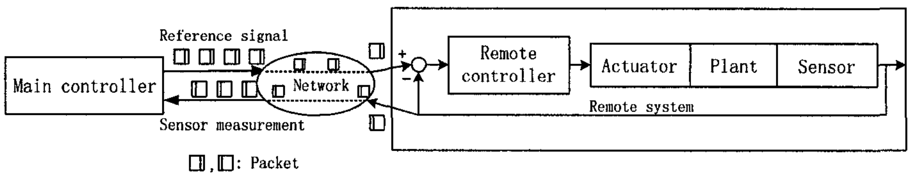

Figure 1.2: NCS setup in the hierarchical structure

or a packet as well. In a practical implementation, multiple controllers can be implemented in a single hardware unit to manage multiple NCS loops in the direct structure, e.g., a distance learning lab [70] and a DC motor speed control system [92].

Hierarchical structure: A basic hierarchical structure consists of a main controller and a remote closed-loop system as depicted in Figure 1.2. Periodically, the main controller computes and sends the reference signal in a frame or a packet via a network to the remote system. The remote system then processes the reference signal to perform local closed-loop control and returns the sensor measurement to the main controller for networked closed-loop control. The networked control loop usually has a longer sampling period than the local control loop since the remote controller is supposed to satisfy certain performance requirements with current reference signal before it processes the newly arrival reference signal. Similar to the direct structure, the main controller can also be implemented to handle multiple networked control loops for several remote systems. This structure is widely used in several applications, including mobile robots [93] and teleoperation [91].

The use of either the direct structure or the hierarchical structure is based on application requirements and preferences of designers. In this dissertation, we mainly focus on the analysis and design of NCSs in the direct structure. Nevertheless, the control and analysis methodologies for the direct structure can be applied to the hierarchical structure by treating the remote closed-loop system as a plant.

1.1 Motivation

The implementation of distributed control can be traced back at least to the early 1970s when Honeywell's Distributed Control System (DCS) was introduced. Control modules in a DCS are loosely connected because most of the real-time control tasks (sensing, calculation, and actuation) are carried out within individual modules. Only on/off signals, monitoring information, and alarm information are transmitted on the serial network. Today, with the help from application specific integrated circuit (ASIC) chip design and significant price drops in silicon, sensors and actuators can be equipped with a network interface. They become independent nodes on a real-time control network, which make more wider distribution possible. Hence, a steady increase has occurred for the past few decades in the use of network-based real-time control systems in a wide variety of fields. Examples include nuclear power plant control [17], industrial manufacturing control [66], and space navigation and guidance [61].

By taking advantage of the network, many good qualities are introduced, e.g., lower cost, reduced weight and power, simpler installation and maintenance, flexibility, and so on. However, the insertion of the communication network in the feedback control loop makes the analysis and design of an NCS complex and brings many new challenges. The first challenge is network-induced delays. They are inevitable not only due to limited bandwidth, but also due to overhead in the communicating nodes and in the network. These delays can be constant, bounded, or even random, depending on the network protocols adopted and the chosen hardware. It is well known in control systems that time delays can degrade a system's performance and even cause instability. As an effect of this, conventional control theories with many ideal assumptions, such as synchronized control and no-delay sensing and actuation, must be reevaluated before they can be applied to NCSs. The second challenge is packet dropouts. Typically, they result from transmission errors in physical network links (which is far more common in wireless than in wired networks) or from buffer overflows due to congestion. Long transmission delays sometimes result in packet reordering, which essentially amounts to a packet dropout if the receiver discards "outdated" arrivals. Reliable transmission protocols, such as Transmission Control Protocol (TCP), guarantee the eventual delivery of packets. However, these protocols are not always appropriate for NCSs since the retransmission of old data will bring packet disordering and only hold for a limited time. Moreover, when the number of packet dropouts is large enough, the system can be considered as an open-loop system, which may be dangerous for unstable systems. Thus, how such packet dropouts affect the performance of an NCS is an issue that must be considered. Other issues, such as control problems for varying sampling interval, synchronization, scheduling, band-limited channels, etc., are also challenging topics in NCSs, all of which stimulate researchers' endless interests.

This dissertation is primarily written from a control perspective. It attempts to systematically address several key issues in NCSs, develop general methods for system modeling, and design control strategies by using the limited network resource efficiently while maintaining good control system performance. Next, a brief literature review on the existing methods for several issues in NCSs will be given.

1.2 Literature review

1.2.1 NCSs with network-induced time delays

Since an NCS operates over a network, data transfers between the controller and the remote system will induce network delays (namely, sensor-to-controller delay and controller-to-actuator delay) in addition to the controller processing delay. These delays, either constant or time varying, can degrade the performance of control systems designed without considering the delays, and can even destabilize the systems, see [9, 52]. Thus, many researchers have paid their attention to the stability analysis and controller design for NCSs in the presence of network-induced delays.

Halevi and Ray [31] considered a continuous-time plant with a discrete-time controller and analyzed the integrated communication and control system (ICCS) using a discrete-time approach. They studied a clock-driven controller with *mis-synchronization* between the plant and the controller. The system was represented by an augmented state vector that consisted of past values of the plant input and output as well as the current state vectors of the plant and controller. This resulted in a finite-dimensional, time-varying discrete-time model. They also took message rejection and vacant sampling into account.

Nilsson [68] also analyzed NCSs in the discrete-time domain. He further modeled the network delays as three different types, namely, constant, independently random, and random but governed by an underlying Markov chain. Then he solved the LQG optimal control problem for various delay models. Moreover, he pointed out the importance of the time-stamping, which allowed the past information of systems to be known.

In [95], Walsh *et al.* considered a continuous-time plant and a continuous-time controller. The control network was only inserted between the sensor and the controller. They introduced the notion of maximum allowable transfer interval (MATI) τ, that is, the successive sensor messages were separated by at most τ seconds. Their goal was to find the value of τ for which the desired property (e.g., stability) of an NCS was guaranteed to be preserved. They also provided a novel scheduling policy, try-once-discard (TOD), and compared it with the token-ring-type static scheduling method by two experiments.

There is significant amount of work done on NCSs with delays by different approaches [44, 47, 103]. Kim *et al.* [44] used a switched system approach to study the stability of NCSs with H_∞ norm constraints. Krtolica *et al.* [47] and Xiao *et al.* [103] used Markov chains to describe the network-induced random delays, where necessary and sufficient conditions for zero-state mean-square exponential stability were given in [47] and a V-K iteration algorithm to design switching and non-switching controllers for resulting discrete-time closed-loop systems was presented in [103].

It is noticed that in all the references cited above, the total maximum network delay is less than one sampling period. However, in practice, the delay is usually more than one sampling interval, especially if the sampling is fast. Moreover, long time delay may disturb the order of the message received, which brings more challenges for the study of NCSs [29, 36, 55, 60, 99, 111, 113]. In [113], the authors derived two classes of closed-loop systems according to delays less than one sampling period and longer delays, but no stability analysis was made for the resulting closed-loop systems. They also introduced a model-based compensator for network-induced delays, which was extended by Wang *et al.* [99] to the system with unknown and stochastic long delays. Hu *et al.* [36] generalized the results in [68] to the case with longer delays with either full or partial state information. Lincoln *et al.* [55] solved the LQG optimal control problem when the probability distribution of delays was known. Zhang *et al.* [111] obtained necessary and sufficient conditions on the existence of stabilizing controllers with two Markov chains. Goodwin *et al.* [29] and Liu *et al.* [60] presented model predictive control methods to compensate and predict the delayed data.

The robustness and fragility of NCSs with delays have been rarely studied except in [54, 109]: in [54] the external disturbance on the system was considered and in [109] the system parameter uncertainties were discussed. Neither of them considers the uncertainties or disturbances from the point of view of a network.

1.2.2 NCSs with packet dropouts

Data packets through networks suffer not only transmission delay, but also, possibly, transmission loss/packet dropout. The network packet drops occasionally happen in an NCS when there are node failures or message collisions. Although most network protocols are equipped with transmission-retry mechanisms, they can only retransmit packets for a limited time. After this time has expired, the packets are dropped. Furthermore, for real-time feedback control data, such as, sensor measurements and calculated control signals, it may be advantageous to discard the old, untransmitted message and transmit a new packet if it becomes available. Therefore, packet dropout is another important factor in NCSs. How this factor affects the stability and performance of NCSs is an issue we will pay attention to.

Prior work, examining the effect of dropouts on system stability and performance, can be roughly categorized into three types based on the resulting closed-loop systems: switching systems [108], asynchronous dynamical systems (ADSs) [113], and jump linear systems with Markov chains [58, 68, 81, 82, 103]. On the other hand, for the measurement of network quality of service (QoS), the rate of data dropouts is one of the most popular topics that researchers consider [56, 58, 59], where in [56], the output power spectral density (PSD) was expressed as a dropout probability function and a direct way of linking control system performance to network QoS was given; Ling *et al.* [58] then extended the results in [56] by relaxing the assumption that the dropout process was identically independently distributed; the authors also used the PSD method to determine an optimal dropout compensator in [59]. Sinopoli *et al.* [86] and Ling *et al.* [57] discussed the optimal control for NCSs with packet dropouts.

It is noticed that all the stability conditions and controller designs given in the aforementioned references are derived based on the assumption that packet dropout exists only in the sensor-to-controller side. The effect of controller-to-actuator packet dropouts is neglected due to the complicated NCS modeling.

Since the controller-to-actuator packet dropouts are inevitable in the data transmission, more and more efforts have been paid to NCSs with both sensor-to-controller and controller-to-actuator packet dropouts recently. Several results have been obtained [32, 39, 74, 114], where switching systems were introduced to model the NCSs [39, 114] and ADSs were presented to represent the closed-loop systems of NCSs [32, 74]. For the MJLSs approach, few results have been derived. Moreover, the reference discussed packet dropouts in a single-packet transmission, where the status of packet dropouts (namely, dropped or sent successfully) were modeled by either a Bernoulli process or a two-state Markov chain process. As to the multiple-packet transmission, rare results have been formulated. In addition, the history dynamics of packets are seldom discussed.

1.2.3 Other work

There are many other topics addressed on NCSs in the literature, such as scheduling problems [8, 73], quantization problems [22, 23], control with variable sampling rates [42, 94], asynchronization problems [32, 74]. In [8] a rate monotonic scheduling algorithm was proposed to schedule a set of NCSs and an optimal scheduling problem was formulated under both rate-monotonic-schedulability constraints and NCS-stability constraints, whereas the proposed scheduling method in [73] could adjust the sampling period as small as possible, allocate the bandwidth of the network for three types of data (periodic data, sporadic data, and messages), and exchange the transmission orders of data for sensors and actuators.

1.3 Contributions of the dissertation

In Chapter 2, both continuous-time and discrete-time Markovian jump linear systems (MJLSs) with delays are discussed. Based on a new model transformation of delays, a delay-independent condition for robust stochastic stabilization is firstly derived. The H_∞ control law designed guarantees the robust stochastic stability of the system and a prescribed disturbance attenuation level. Secondly, stochastic stability, stabilization and H_∞ control problems for a continuous MJLS are considered. Delay-dependent conditions and corresponding controller designs are given by introducing some new zero equations, by which neither model transformation nor bounding for cross terms is required, and thus the results are less conservative. Moreover, an algorithm for calculating the delay upper bound with respect to system stability is given. Numerical examples show that, under the same initial conditions, our results are less conservative than those in existing references.

The results of this chapter are published in

- J. Wu, T. Chen, and S. Xu, Stochastic stabilization and H_∞ control for discrete jumping systems with time delays. *Asian Journal of Control*, vol. 7, pp. 223-230, 2005.
- J. Wu, T. Chen, and L. Wang, Delay-dependent robust stability and H_∞ control for jump linear systems with delays. *Systems and Control Letters*, vol. 55, pp. 939-948, 2006.

In Chapter 3, the stability and controller design of NCSs with packet dropouts are considered. Both sensor-to-controller and controller-to-actuator packet dropouts are considered and described by two independent Markovian chains, by which the resulting closed-loop NCSs can be transformed to a kind of MJLSs with time delays. The Markov chains in this chapter describe the historical behavior of packet dropouts, which is different from what appears in existing references, where Markov chains were used to model the information on if a packet is dropped or not. System modeling, stability analysis and control design are presented for the NCS with single-packet transmissions, and then extended to the case with multiple-packet transmissions. Finally, how packet dropouts affect the system stability is discussed in the examples.

The results of this chapter are published in

- J. Wu and T. Chen, Design of networked control systems with packet dropouts. *IEEE Transactions on Automatic Control*, vol. 52, No. 7, pp. 1314-1319, 2007.

In Chapter 4, the stabilization problem for a class of NCSs with network-induced delays is investigated. Random but bounded sensor-to-controller and controller-to-actuator network-induced delays are considered, the dynamics of which are modeled by Markov chains. To compensate controller-to-actuator delays, which happen after control decision has been made, a model predictive control method is introduced. Based on the minimization of an upper bound of the worst-case infinite horizon quadratic cost function at each sampling instant, a state feedback predictive controller has been proposed to stabilize the resulting closed-loop NCS. The result is simulated on a classical angular positioning system.

The results in this chapter are mainly from

- J. Wu, L. Zhang, and T. Chen, Model predictive control for networked control systems. Submitted to *International Journal of Robust and Nonlinear Control*, revised, Oct., 2007.

In Chapter 5, the stabilization problem for a class of sampled-data systems with variable sampling rates is addressed. A new stabilizing predictive control law, based on a finite input and state horizon cost with a finite terminal matrix, is proposed for the resulting time-varying discrete linear system, where the behavior at and between sampling instants is considered in the design. The terminal weighting matrix can be obtained by solving a linear matrix inequality (LMI), under which closed-loop stability is guaranteed with input and state constraints. The simulation on a continuously stirred tank reactor (CSTR) system illustrates the effectiveness of our method.

The results in this chapter are mainly from

- J. Wu and T. Chen, Stabilization of sampled-data systems with variable sampling rates. Submitted for publication, Feb., 2008.

1.4 Outline of the dissertation

The outline of the dissertation is as follows:

- Chapter 1 presents the motivation of research, related research issues of NCSs, and the main contributions of this dissertation.
- Chapter 2 discusses Markovian jump linear systems with uncertainties and time delays. Stochastic stability, stochastic stabilization and H_∞ controller design are completely addressed; the methods are used in the modeling, analysis and design of NCSs.
- Chapter 3 introduces general frameworks for the NCSs with packet dropouts, respectively, under single- and multiple-packet transmissions. State feedback controllers are designed to stabilize the resulting closed-loop NCSs.
- Chapter 4 develops a jump linear system to model the NCSs with network-induced time delays. To compensate the delayed data on the sensor-to-controller side as well as on the controller-to-actuator side, the control strategy at each sampling instant is characterized as a constrained delay-dependent optimization problem of the worse-case quadratic cost over an infinite horizon.
- Chapter 5 considers a stabilization problem of a class of sampled-data systems with variable sampling rates. A new stabilizing predictive control law, based on a finite input and state horizon cost with a finite terminal matrix, is proposed for the resulting time-varying discrete linear system. Based on the above modeling and design method, extensions to NCSs with fixed network-induced delays are formulated.
- Chapter 6 gives conclusions and some suggestions for the future work of NCSs.

Notation. Throughout this dissertation, $\Re$ stands for the set of real numbers and $\mathcal{N}$ for the set of nonnegative integers. For a given matrix $A \in \Re^{n\times n}$ and vector $x \in \Re^n$, $\|x\|_A^2 := x^T Ax$. diag$\{A,\ B\}$ is a block diagonal matrix with A and B on the diagonal. For symmetric matrices X and Y, the notation $X \geq Y$ (respectively, $X > Y$) means that the matrix $X - Y$ is positive semi-definite (respectively, positive definite); I is an identity matrix with appropriate dimensions; M^T represents the transpose of the matrix M; $\mathcal{E}\{\cdot\}$ denotes the expectation operator with respect to some probability measure $\mathcal{P}$. Matrices, if not explicitly stated, are assumed to have compatible dimensions.

Chapter 2

Markovian Jump Linear Systems with Time Delays

Markovian jump linear systems (MJLSs) are a special class of hybrid systems with two components in their vector states: the modes and the states. The mode is described by a continuous/discrete Markovian process with a finite state space. The state in each mode is represented by a system of differential/difference equations. This class of system has the advantage of better representing physical systems with abrupt variations, e.g., solar thermal central receivers, economic systems, and so on. Therefore, over the past decades, noticeable achievements have been made on design, filtering and stability analysis of jump linear systems.

In this chapter, we brief introduce this kind of systems with time delays, which will be used to model the networked control systems in the following chapters. The criteria for time delays can be generally classified into two categories: delay-independent and delay-dependent, both of which are discussed in details. The chapter is organized as follows. Section 2.1 addresses the delay-independent robust stability and H_∞ control for discrete time MJLSs with delays. Section 2.2 discusses the delay-dependent robust stability and H_∞ control for continuous MJLSs with the delays. Finally, Section 2.3 gives some concluding remarks.

2.1 Delay-independent robust stability and H_∞ control for MJLSs with delays

2.1.1 Introduction

Markovian jump linear systems (MJLSs) have been paid an increasing interest since they were firstly introduced by Krasovskii and Lidskii [46]. Such systems consist of a set of linear systems with transitions between the models determined by a Markov chain, which takes values in a finite set. Models obtained this way allow analysis and design results developed that are not only valid for approximate models but also hold for a given class of plants, including real processes. Hence, many researchers from the control community have been attracted by those characteristics of MJLSs. Boukas and his coauthors [4, 5, 6, 7, 43, 83, 84, 85] extensively contributed to problems like stability, stabilizability, H_∞ control, guaranteed cost control, and to the robustness of these techniques. Costa and his coauthors [16, 19] also contributed to different problems for this class of systems. Other researchers like Cao and Lam [10, 11], Xu and Chen [104, 105, 106], etc., have made contributions to the class of dynamical linear systems with Markovian jump and time delays. The methods used were mostly based on Lyapunov functions or Lyapunov-Krasovskii functions, which are quite general and lead to a complicated system of Riccati-type partial differential equations/inequalities [19], or a simpler (but likely more conservative) delay-independent [85] and delay-dependent sufficient conditions.

Recently, a new system transformation by considering both the state and state difference was introduced [26] for linear systems. This approach significantly reduces the overdesign entailed in the existing methods since it is based on a model that is equivalent to the original system and fewer bounds are required. The bounds can now be made tighter using the recent (less conservative) bound on cross terms that was introduced in [65]. Extending these ideas to MJLSs from continuous time to discrete time is the focus in this section.

In this section, a new model transformation and a corresponding Lyapunov-Krasovskii function are presented for stability analysis of discrete time-delay MJLSs, which are based on an equivalent augmented model. By solving some linear matrix inequalities (LMIs), sufficient conditions for the stochastic stabilization and $\mathcal{H}_\infty$

disturbance attenuation are derived for the resulting closed-loop discrete uncertain systems with Markovian jumping parameters. The corresponding controller design is also proposed. Finally, numerical examples are considered to illustrate the solvability and effectiveness of the proposed design methods.

2.1.2 Problem statement

A mathematical representation of discrete uncertain system with Markovian jump parameters and time delay is described by equations in a fixed complete probability space $(\Omega, \mathbb{F}, \mathbb{P})$, where Ω is the sample space, $\mathbb{F}$ is the algebra of events and $\mathbb{P}$ is the probability measure defined on $\mathbb{F}$. Suppose the measurement is exact, then we have:

$$\left\{ \begin{array}{rcl} x_{k+1} & = & \tilde{A}(\eta_k)\, x_k + \tilde{A}_d(\eta_k)\, x_{k-d} + B_1(\eta_k)\, w_k + B_2(\eta_k)\, u_k, \\ z_k & = & C(\eta_k)\, x_k + D(\eta_k)\, u_k, \\ x_k & = & \psi_k,\ k \in [-d, 0],\ \eta_0 = r_0,\ k \in Z, \end{array} \right. \tag{2.1}$$

where

$$\tilde{A}(\eta_k) = A(\eta_k) + \Delta A(\eta_k, k), \quad \tilde{A}_d(\eta_k) = A_d(\eta_k) + \Delta A_d(\eta_k, k).$$

$x_k \in \Re^n$ is the state, $u_k \in \Re^m$ is the control input, $z_k \in \Re^p$ is the system output, $w_k \in \Re^q$ is the deterministic disturbance input which belongs to $\mathcal{L}_2[0, \infty)$, r_0 is the initial discrete sequence, ψ_k is the initial condition of the state and the model, $\{\eta_k, k \in Z\}$, is a discrete-time homogeneous Markov chain taking values in a finite set $S = \{1, 2, \cdots, h\}$ with transition probabilities

$$\Pr(\eta_{k+1} = j \,|\eta_k = i) = p_{ij},\ p_i = \Pr(\eta_0 = i), \tag{2.2}$$

where $p_{ij} \geq 0$ for $i,\ j \in S$ and

$$\sum_{j=1}^{h} p_{ij} = 1. \tag{2.3}$$

For $\eta_k = i,\ i \in S,\ A(\eta_k),\ A_d(\eta_k),\ B_1(\eta_k),\ B_2(\eta_k),\ C(\eta_k)$ and $D(\eta_k)$ are known constant matrices of appropriate dimensions, and $\Delta A(\eta_k, k),\ \Delta A_d(\eta_k, k)$ are unknown matrices which represent time-varying parametric uncertainties and are assumed to belong to certain bounded compact sets. Throughout this section, it is assumed that the initial state ψ_k is independent of $\{\eta_k, k \in Z\}$.

The admissible parameter uncertainties are assumed to be of the following forms:

$$\begin{aligned}\Delta A(\eta_k, k) &= M_1(\eta_k)\Delta_1(\eta_k, k)N_1(\eta_k),\\ \Delta A_d(\eta_k, k) &= M_2(\eta_k)\Delta_2(\eta_k, k)N_2(\eta_k),\end{aligned} \tag{2.4}$$

where $M_j(\eta_k)$, $N_j(\eta_k)$, $j = 1, 2$, for any $\eta_k = i$, $i \in S$, are known constant matrices of appropriate dimensions, and $\Delta_j(\eta_k, k)$, $j = 1, 2$, satisfy

$$\Delta_j^T(\eta_k, k)\Delta_j(\eta_k, k) \leq I, \ \forall\, i \in S, \ k \in Z.$$

It is assumed that the elements of $\Delta_j(\eta_k, k)$ are Lebesgue measurable. When $\Delta_j(\eta_k, k) \equiv 0$, then the system in (2.1)-(2.4) is referred to as a nominal jump linear system. It is said to be a free system if $u(t) \equiv 0$ and $w(t) \equiv 0$. We introduce the following definitions.

Definition 2.1 *[10] The free nominal jump discrete-time system in (2.1)-(2.3) is said to be stochastically stable, if for all finite $\psi_k \in \Re^n$ defined on $[-d, 0]$ and initial model r_0, there exists a finite number $\tilde{\Xi}(\psi_k,\ r_0) > 0$ such that*

$$\lim_{N\to\infty} \mathcal{E}\left\{ \sum_{k=0}^{N} \|x_k^T(\psi,\ r_0)\|^2 \middle| \psi_0,\ r_0,\ w = 0 \right\} < \tilde{\Xi}(\psi_k,\ r_0) \tag{2.5}$$

holds.

Definition 2.2 *The jump system in (2.1)-(2.4) with $w(t) \equiv 0$ is said to be robust stochastically stabilizable if for all finite $\psi_k \in \Re^n$ defined on $[-d, 0]$ and initial model r_0, there exist a state feedback control law*

$$u_k = K(\eta_k)x_k \tag{2.6}$$

such that the closed-loop system

$$\left\{\begin{aligned} x_{k+1} &= \Gamma x_k + \tilde{A}_d(\eta_k)x_{k-d} + B_1(\eta_k)w_k,\\ z_k &= [C(\eta_k) + D(\eta_k)K(\eta_k)]x_k,\end{aligned}\right. \tag{2.7}$$

where $\Gamma = \tilde{A}(\eta_k) + B_2(\eta_k)K(\eta_k)$, is stochastically stable.

Definition 2.3 *[106] The system in (2.1)-(2.4) is said to be robustly stochastically stable with disturbance attenuation level $\gamma > 0$ if for all $w_k \in \mathcal{L}_2[0, \infty)$, the system is stochastically stable and the response $\{z_k\}$ satisfies*

$$\|z_k\|_2 = \left\{ \sum_{k=0}^{\infty} \mathcal{E}\left(z_k^T z_k \,|x_0, \eta_0\right) \right\}^{1/2} \leq \gamma \|w_k\|_2. \tag{2.8}$$

Remark 2.1 *These definitions are consistent with those of stochastic stability and stochastic stabilizability of jump discrete-time linear systems without time delays. Under Definition 2.2, stochastic stabilizability of a system means that there exists a state feedback control law which drives the state x from any given initial condition (ψ_k, r_0) asymptotically to the origin, in the mean-square sense, which implies the asymptotic stability of the closed-loop system. In other words, as is obvious from inequality* (2.5), *stochastic stabilizability implies*

$$\lim_{k\to\infty} \mathcal{E}\left\{ \|x_k^T(\psi,\ r_0,\ w=0)\|^2 \middle| \psi_0,\ r_0 \right\} = 0,$$

which is generally called mean square stability.

In this chapter, we will discuss stochastic stabilization and $\mathcal{H}_\infty$ control in the stochastic sense. To obtain our main results, we need the following lemmas.

Lemma 2.1 *(Schur complement) Given constant matrices $A,\ B,\ C$ with appropriate dimensions, where $A = A^T$ and $C = C^T > 0$, then $A + B^T C^{-1} B < 0$ if and only if*

$$\begin{bmatrix} A & B^T \\ B & -C \end{bmatrix} < 0 \ \ or \ \ \begin{bmatrix} -C & B \\ B^T & A \end{bmatrix} < 0$$

Lemma 2.2 *[11] For any vectors $x,\ y \in \Re^n$, matrix $R \in \Re^{n\times n}$ and $R > 0$, the following holds*

$$2x^T y \le x^T R^{-1} x + y^T R y.$$

For notational simplicity, in the sequel, for $\eta_k = i \in S$, we will denote $A(\eta_k) \triangleq A_i$, $A_d(\eta_k) \triangleq A_{di}$, $B_j(\eta_k) \triangleq B_{ji}$, $C(\eta_k) \triangleq C_i$, $D(\eta_k) \triangleq D_i$, $\Delta A(\eta_k, k) \triangleq \Delta A(i,k)$, $\Delta A_d(\eta_k, k) \triangleq \Delta A_d(i,k)$, $M_j(\eta_k) \triangleq M_{ji}$, $N_j(\eta_k) \triangleq N_{ji}$, and $\Delta_j(\eta_k, k) \triangleq \Delta_j(i,k)$, $j = 1,\ 2$.

2.1.3 Delay-independent robust stochastic stability

Firstly, a sufficient condition for the robust stochastic stability of the system in (2.1)-(2.4) with $u(t) \equiv 0$ and $w(t) \equiv 0$, namely the free system, is presented.

Theorem 2.1 *The free jump system is stochastically stable if there exist $Q_{1i} = Q_{1i}^T > 0$, Q_{2i}, Q_{3i}, $\tilde{R} = \tilde{R}^T > 0$, $\varepsilon_{1i} > 0$, $\varepsilon_{2i} > 0$, $i = 1, 2, \cdots, h$, satisfying the* *following LMIs*

$$\begin{bmatrix} -Q_{1i} & * & * & * & * & * \\ T_1 & T_2 & * & * & * & * \\ 0 & \tilde{R}A_{di}^T & -\frac{1}{d}\tilde{R} & * & * & * \\ \Theta_1^T & \Theta_2^T & 0 & \Xi & * & * \\ N_{1i}Q_{1i} & 0 & 0 & 0 & -\varepsilon_{1i}I & * \\ N_{2i}Q_{1i} & 0 & N_{2i}\tilde{R} & 0 & 0 & -\varepsilon_{2i}I \end{bmatrix} < 0, \tag{2.9}$$

where $*$ denotes blocks that are readily inferred by symmetry and

$$\begin{aligned} T_1 &= (A_i + A_{di} - I)\, Q_{1i} - Q_{2i}, \quad \tilde{R} = R^{-1}, \\ T_2 &= -Q_{3i} - Q_{3i}^T + \varepsilon_{1i}M_{1i}M_{1i}^T + \varepsilon_{2i}M_{2i}M_{2i}^T, \\ \Xi &= diag\Big[-\tfrac{1}{p_{11}}Q_{11}, \ \cdots, \ -\tfrac{1}{p_{1h}}Q_{1h}, \ -\tfrac{1}{d}\tilde{R} \Big], \\ \Theta_1 &= \Bigg[\underbrace{Q_{1i} + Q_{2i}^T, \ \cdots, \ Q_{1i} + Q_{2i}^T, \ Q_{2i}^T}_{h+1} \Bigg], \\ \Theta_2 &= \Bigg[\underbrace{Q_{3i}^T \ \cdots \ Q_{3i}^T \ Q_{3i}^T}_{h+1} \Bigg]. \end{aligned} \tag{2.10}$$

Proof: Let

$$y_k = x_{k+1} - x_k. \tag{2.11}$$

Then we have

$$x_{k-d} = x_k - \sum_{t=k-d}^{k-1} y_t. \tag{2.12}$$

By equations (2.11) and (2.12), the system can be rewritten as

$$0 = -y_k + \left(\tilde{A}_i + \tilde{A}_{di} - I\right) x_k - \tilde{A}_{di} \sum_{t=k-d}^{k-1} y_t. \tag{2.13}$$

Let the stochastic Lyapunov-Krasovskii function be

$$V\left(x_k, \eta_k\right) = x_k^T P_{1i} x_k + \sum_{\theta=-d+1}^{0} \sum_{t=k-1+\theta}^{k-1} y_t^T R y_t,$$

where $P_{1i} = P_{1i}^T > 0$, $R = R^T > 0$. Then along the state trajectory of system (2.1), we have

$$\begin{aligned} &\mathcal{E}\left\{ V_{k+1}\left(x_{k+1}, \eta_{k+1}\right) \middle| \, x_k, \eta_k = i \right\} - V_k\left(x_k, \eta_k = i\right) \\ = &\sum_{j=1}^{h} p\left(\eta_{k+1} = j \middle| \, i\right) \left(x_{k+1}^T P_{1j} x_{k+1} + \sum_{\theta=-d+1}^{0} \sum_{t=k+\theta}^{k} y_t^T R y_t \right) \end{aligned}$$

$$
\begin{aligned}
& -x_k^T P_{1i} x_k - \sum_{\theta=-d+1}^{0} \sum_{t=k-1+\theta}^{k-1} y_t^T R y_t \\
= \ & (x_k + y_k)^T \bar{P}_{1i} (x_k + y_k) - x_k^T P_{1i} x_k + d y_k^T R y_k - \sum_{t=k-d}^{k-1} y_t^T R y_t \\
= \ & \chi^T \begin{bmatrix} \bar{P}_{1i} - P_{1i} & 0 \\ 0 & \bar{P}_{1i} + dR \end{bmatrix} \chi - \sum_{t=k-d}^{k-1} y_t^T R y_t + 2 x_k^T \bar{P}_{1i} y_k, \qquad (2.14)
\end{aligned}
$$

where $\chi = \begin{bmatrix} x_k^T & y_k^T \end{bmatrix}^T$, $\bar{P}_{1i} = \sum_{j=1}^{h} p_{ij} P_{1j}$.

By Lemma 2.2 and equation (2.13), we have the following inequality transformation:

$$
\begin{aligned}
2x_k^T \bar{P}_{1i} y_k \ = \ & 2 \begin{bmatrix} x_k^T & y_k^T \end{bmatrix} \begin{bmatrix} \bar{P}_{1i} & P_{2i}^T \\ 0 & P_{3i}^T \end{bmatrix} \begin{bmatrix} y_k \\ 0 \end{bmatrix} \\
= \ & 2\chi^T P_i^T \left\{ \begin{bmatrix} y_k \\ \left(\tilde{A}_i + \tilde{A}_{di} - I\right) x_k - y_k \end{bmatrix} - \begin{bmatrix} 0 \\ \tilde{A}_{di} \end{bmatrix} \sum_{t=k-d}^{k-1} y_t \right\} \\
= \ & 2\chi^T P_i^T \begin{bmatrix} 0 & I \\ \tilde{A}_i + \tilde{A}_{di} - I & -I \end{bmatrix} \chi - 2 \sum_{t=k-d}^{k-1} \chi^T P_i^T \begin{bmatrix} 0 \\ \tilde{A}_{di} \end{bmatrix} y_t \\
\leq \ & 2\chi^T P_i^T \begin{bmatrix} 0 & I \\ \tilde{A}_i + \tilde{A}_{di} - I & -I \end{bmatrix} \chi + d\chi^T P_i^T \begin{bmatrix} 0 \\ \tilde{A}_{di} \end{bmatrix} R^{-1} \begin{bmatrix} 0 \\ \tilde{A}_{di} \end{bmatrix}^T P_i \chi \\
& + \sum_{t=k-d}^{k-1} y_t^T R y_t,
\end{aligned}
$$

where

$$
P_i = \begin{bmatrix} \bar{P}_{1i} & 0 \\ P_{2i} & P_{3i} \end{bmatrix}.
$$

Substituting the above inequality into equation (2.14), we have

$$
\begin{aligned}
& \mathcal{E}\left\{ V_{k+1}(x_{k+1}, \eta_{k+1}) | \, x_k, \eta_k = i \right\} - V_k(x_k, \eta_k = i) \\
\leq \ & \chi^T \left\{ \begin{bmatrix} \bar{P}_{1i} - P_{1i} & 0 \\ 0 & \bar{P}_{1i} + dR \end{bmatrix} + 2P_i^T \begin{bmatrix} 0 & I \\ \tilde{A}_i + \tilde{A}_{di} - I & -I \end{bmatrix} \right. \\
& \left. + dP_i^T \begin{bmatrix} 0 \\ \tilde{A}_{di} \end{bmatrix} R^{-1} \begin{bmatrix} 0 \\ \tilde{A}_{di} \end{bmatrix}^T P_i \right\} \chi \\
= \ & -\chi^T \Psi \chi. \qquad (2.15)
\end{aligned}
$$

Thus by Lyapunov stability theory, the system is stochastically stable if there exists a matrix $\Psi > 0$ such that

$$
\mathcal{E}\left\{ V_{k+1}(x_{k+1}, \eta_{k+1}) | \, x_k, \eta_k = i \right\} - V_k(x_k, \eta_k = i) \leq -\lambda_{\min}(\Psi) \|\chi\|^2
$$

holds. By Schur complement, inequality (2.15) can be rewritten as the following inequality:

$$\Pi = -\Psi = \begin{bmatrix} \Lambda & P_i^T \begin{bmatrix} 0 \\ \tilde{A}_{di} \end{bmatrix} \\ * & -\frac{1}{d}R \end{bmatrix} < 0. \tag{2.16}$$

Here

$$\begin{aligned} \Lambda &= \begin{bmatrix} \bar{P}_{1i} - P_{1i} & 0 \\ 0 & \bar{P}_{1i} + dR \end{bmatrix} + P_i^T \begin{bmatrix} 0 & I \\ \tilde{A}_i + \tilde{A}_{di} - I & -I \end{bmatrix} \\ &+ \begin{bmatrix} 0 & \left(\tilde{A}_i + \tilde{A}_{di} - I\right)^T \\ I & -I \end{bmatrix} P_i. \end{aligned}$$

Pre-multiplying and post-multiplying (2.16) by diag$\begin{bmatrix} Q_i^T & I \end{bmatrix}$ and diag$\begin{bmatrix} Q_i & I \end{bmatrix}$, respectively with

$$Q_i = \begin{bmatrix} Q_{1i} & 0 \\ Q_{2i} & Q_{3i} \end{bmatrix} = \begin{bmatrix} P_{1i}^{-1} & 0 \\ -P_{3i}^{-1}P_{2i}P_{1i}^{-1} & P_{3i}^{-1} \end{bmatrix}$$

and using Schur complement again, we can derive the inequality in (2.9). Thus the proof is completed. ■

2.1.4 Delay-independent robust $\mathcal{H}_\infty$ control

Then our attention will be focused on the design of a state feedback controller such that the corresponding closed-loop system is robustly stochastically stable with a disturbance attenuation level $\gamma > 0$.

Theorem 2.2 *Consider the closed-loop system in* (2.7) *and the performance index function in* (2.8)*. The system is robustly stochastically stabilizable with disturbance attenuation level* $\gamma > 0$ *if there exist matrices* $Q_{1i} = Q_{1i}^T > 0$, Q_{2i}, Q_{3i}, Y_i, $\tilde{R} = \tilde{R}^T > 0$ *and scalars* $\varepsilon_{1i} > 0$, $\varepsilon_{2i} > 0$, $i = 1, 2, \cdots, h$, *such that the following LMI holds:*

$$\tilde{\Lambda} = \begin{bmatrix} \Lambda_{11} & * \\ \Lambda_{21} & \Lambda_{22} \end{bmatrix} < 0. \tag{2.17}$$

Here $*$ *denotes blocks that are readily inferred by symmetry and*

$$\Lambda_{11} = \begin{bmatrix} -Q_{1i} & * & * \\ \bar{T}_1 & T_2 & * \\ 0 & \tilde{R}A_{di}^T & -\frac{1}{d}\tilde{R} \end{bmatrix},$$

$$
\begin{aligned}
\Lambda_{21} &= \begin{bmatrix} \Theta_1^T & \Theta_2^T & 0 \\ N_{1i}Q_{1i} & 0 & 0 \\ N_{2i}Q_{1i} & 0 & N_{2i}\tilde{R} \\ 0 & B_{1i} & 0 \\ C_iQ_{1i}+D_iY_i & 0 & 0 \end{bmatrix}, \\
\Lambda_{22} &= diag\begin{bmatrix} \Xi & -\varepsilon_{1i}I & -\varepsilon_{2i}I & -\gamma^2 I & -I \end{bmatrix}, \\
\bar{T}_1 &= (A_i + A_{di} - I)\, Q_{1i} - Q_{2i} + B_{2i}Y_i.
\end{aligned}
$$

T_2, Θ_1, Θ_2, Ξ, $\tilde{R}$ are expressed in equation (2.10). *In this case, the state feedback control law is given by* (2.6) *with*

$$
K_i = Y_i Q_{1i}^{-1}. \tag{2.18}
$$

Proof: Similarly, we have the following equation

$$
0 = -y_k + \left(\tilde{A}_i + \tilde{A}_{di} - I + B_{2i}K_i\right) x_k - \tilde{A}_{di}\sum_{t=k-d}^{k-1} y_t + B_{1i}w_k.
$$

By the same Lyapunov-Krasovskii function, we derives

$$
\begin{aligned}
& E\left\{V_{k+1}\left(x_{k+1},\eta_{k+1}\right)\middle| x_k,\eta_k = i\right\} - V_k\left(x_k,\eta_k = i\right) + z_k^T z_k - \gamma^2 w_k^T w_k \\
\leq\ & \chi^T\left\{\begin{bmatrix} \Phi_i & 0 \\ 0 & \bar{P}_{1i}+dR \end{bmatrix} + 2P_i^T\begin{bmatrix} 0 & I \\ \tilde{A}_i + \tilde{A}_{di} + B_{2i}K_i - I & -I \end{bmatrix}\right. \\
& \left. +dP_i^T\begin{bmatrix} 0 \\ \tilde{A}_{di} \end{bmatrix} R^{-1}\begin{bmatrix} 0 \\ \tilde{A}_{di} \end{bmatrix}^T P_i\right\}\chi + 2\chi^T P_i^T\begin{bmatrix} 0 \\ B_{1i} \end{bmatrix} w_k - \gamma^2 w_k^T w_k \\
=\ & -\delta^T\tilde{\Psi}\delta,
\end{aligned} \tag{2.19}
$$

where $\Phi_i = \bar{P}_{1i} - P_{1i} + (C_i + D_iK_i)^T(C_i + D_iK_i)$, $\delta = \begin{bmatrix} \chi^T & w_k^T \end{bmatrix}^T$ and

$$
\tilde{\Psi} = \begin{bmatrix} \Psi + 2P_i^T\begin{bmatrix} 0 & 0 \\ B_{2i}K_i & 0 \end{bmatrix} & * & * \\ \begin{bmatrix} 0 & B_{1i}^T \end{bmatrix} P_i & -\gamma^2 I & * \\ \begin{bmatrix} C_i + D_iK_i & 0 \end{bmatrix} & 0 & -I \end{bmatrix} > 0. \tag{2.20}
$$

Here $*$ denotes blocks that are readily inferred by symmetry. By Schur complement, pre-multiplying and post-multiplying (2.20) by diag $\left[Q_i^T,\ I,\ I\right]$ and diag $[Q_i,\ I,\ I]$, respectively, we get the results in Theorem 2.2. Since the steps of proof are similar to that of Theorem 2.1, the details are omitted. ■

Corollary 2.1 *Consider the nominal closed-loop system in* (2.7) *and the performance index function in* (2.8) *with* $A_d(\eta_k) = 0$. *The system is stochastically stabilizable with disturbance attenuation level* $\gamma > 0$ *if there exist matrices* $Q_{1i} = Q_{1i}^T > 0$, Q_{2i}, Q_{3i}, *and* Y_i, $i = 1, 2, \cdots, h$, *such that the following LMI holds:*

$$\begin{bmatrix} -Q_{1i} & * & * & * & * \\ \tilde{T}_1 & -Q_{3i} - Q_{3i}^T & * & * & * \\ \tilde{\Theta}_1^T & \tilde{\Theta}_2^T & \tilde{\Xi} & * & * \\ 0 & B_{1i} & 0 & -\gamma^2 I & * \\ C_i Q_{1i} + D_i Y_i & 0 & 0 & 0 & -I \end{bmatrix} < 0. \tag{2.21}$$

Here $*$ *denotes blocks that are readily inferred by symmetry and*

$$\begin{aligned} \tilde{T}_1 &= (A_i - I) Q_{1i} - Q_{2i} + B_{2i} Y_i, \\ \tilde{\Xi} &= diag\left[\begin{array}{ccc} -\frac{1}{p_{11}} Q_{11} & \cdots & -\frac{1}{p_{1h}} Q_{1h} \end{array} \right], \\ \tilde{\Theta}_1 &= \left[\underbrace{\begin{array}{ccc} Q_{1i} + Q_{2i}^T & \cdots & Q_{1i} + Q_{2i}^T \end{array}}_{h} \right], \\ \tilde{\Theta}_2 &= \left[\underbrace{\begin{array}{ccc} Q_{3i}^T & \cdots & Q_{3i}^T \end{array}}_{h} \right]. \end{aligned} \tag{2.22}$$

In this case, the state feedback control law is given by (2.6) *with*

$$K_i = Y_i Q_{1i}^{-1}. \tag{2.23}$$

The proof is similar to the proof of Theorem 2.2, hence omitted.

Remark 2.2 *By Corollary 2.1, a better disturbance attenuation level* γ *can be obtained than the results in [16]* (*Theorem 2*). *In our method there is only one constraint and the optimal solution is based on* $\sum_{j=1}^{N} p_{ij} P_j$, *while the optimal solution in [16] is subject to two constraints and the second constraint,* $L_i^{-1} \geq \sum_{j=1}^{N} p_{ij} P_j$, *may increase the overdesign of the system. This is illustrated in the example given in the following section. It should be noted that the optimal disturbance attenuation level* γ *in our method can be solved by LMIs with the corresponding constraint conditions. So the optimal* $\mathcal{H}_\infty$ *performance index algorithm can be given as follows:*

$$\left\{ \begin{array}{l} \min \ \gamma^2 \\ s.t. \ (2.17) \ and \ (2.10) \ or \ (2.21) \ and \ (2.22) \end{array} \right., \tag{2.24}$$

which can be easily solved by mincx$(\cdot)$ *in the LMI toolbox.*

Remark 2.3 *If $h = 1$, then the nominal jumping linear system in (2.1)-(2.3) is changed to a traditional linear discrete system. The theorems and corollary still hold for this case and the results can be simplified.*

2.1.5 Numerical examples

To illustrate the proposed results, two examples are considered, where the Markovian jump system has two modes, namely, $h = 2$.

Example 2.1: Consider the following uncertain time-delay system with Markovian jumping parameters. For mode 1, the system matrices are given by

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0.1 & 0.5 \end{bmatrix}, \; A_{d1} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.02 \end{bmatrix}, \; B_{11} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \; B_{21} = \begin{bmatrix} 1 \\ 0.1 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 0.8 & 0.5 \end{bmatrix}, \; D_1 = 1,$$

$$M_{11} = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}, \; N_{11} = \begin{bmatrix} 0.01 & 0.02 \end{bmatrix}, \; M_{21} = \begin{bmatrix} -0.03 \\ 0.05 \end{bmatrix}, \; N_{21} = \begin{bmatrix} 0.03 & 0.04 \end{bmatrix}.$$

For mode 2, the system matrices are given by

$$A_2 = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.6 \end{bmatrix}, \; A_{d2} = \begin{bmatrix} -0.2 & 0.05 \\ 0.02 & 0.03 \end{bmatrix}, \; B_{12} = \begin{bmatrix} 0.2 \\ 1 \end{bmatrix}, \; B_{22} = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix},$$

$$C_2 = \begin{bmatrix} 0.1 & 0.5 \end{bmatrix}, \; D_2 = 1,$$

$$M_{12} = \begin{bmatrix} -0.1 \\ 0.2 \end{bmatrix}, \; N_{12} = \begin{bmatrix} 0 & 0.01 \end{bmatrix}, \; M_{22} = \begin{bmatrix} 0.3 \\ -0.05 \end{bmatrix}, \; N_{22} = \begin{bmatrix} 0.12 & 0.04 \end{bmatrix}.$$

Assume that the transition probability matrix is

$$p = \begin{bmatrix} 0.5 & 0.5 \\ 0.2 & 0.8 \end{bmatrix},$$

the time-delay is $d = 2$. Then by Theorem 2.2, the optimal attenuation level γ and corresponding controller gains are

$$\gamma = 0.0011, \; K_1 = \begin{bmatrix} -0.8000 & -0.5000 \end{bmatrix}, \; K_2 = \begin{bmatrix} -0.1000 & -0.5000 \end{bmatrix},$$

and other related parameters are as follows:

$$\begin{aligned} \varepsilon_{11} &= 2.4997e+006, \; \varepsilon_{12} = 3.1642e+006, \\ \varepsilon_{21} &= 2.6989e+006, \; \varepsilon_{22} = 4.2256e+006. \end{aligned}$$

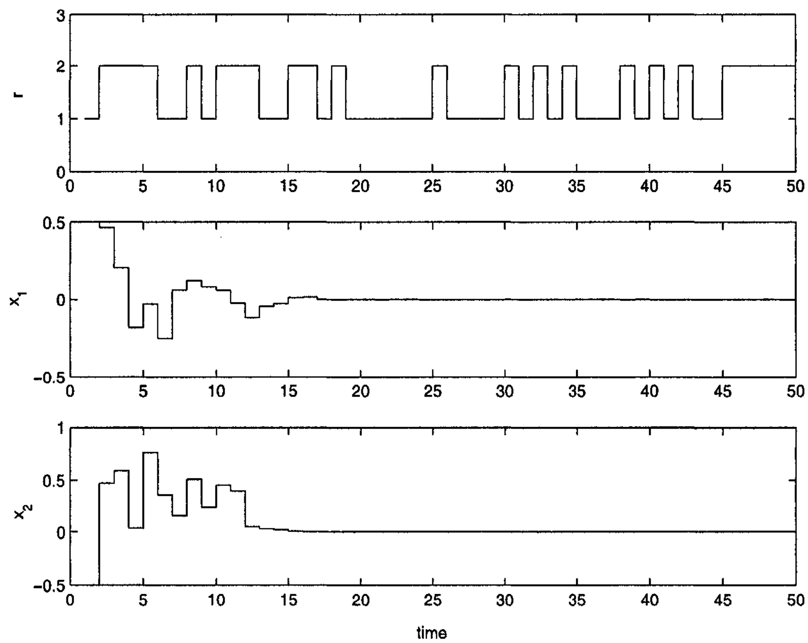

Figure 2.1: Jump disturbance and closed-loop state responses (Example 2.1)

Simulation indicates that the states always converge to zero under any given initial conditions. Here, suppose the initial conditions are

$$x_k = \begin{bmatrix} 0.5 & -0.5 \end{bmatrix}^T, \ k \in \begin{bmatrix} -d, & 0 \end{bmatrix}, \ r_0 = 1,$$

and the simulation results are shown in Figure 2.1.

Example 2.2: This example is borrowed from [16], which is a discrete-time jump linear system without time-delay and uncertainties. The system matrices are given as

$$\begin{aligned} A_1 &= A_2 = \begin{bmatrix} 0.9974 & 0.0539 \\ -0.1078 & 1.1591 \end{bmatrix}, \ B_{21} = \begin{bmatrix} 0.0013 \\ 0.0539 \end{bmatrix}, \\ B_{22} &= \begin{bmatrix} 0.0013 \\ 0.1078 \end{bmatrix}, \ B_{11} = B_{12} = \begin{bmatrix} 1 & 0 \\ 1 & 0.1 \end{bmatrix}, \\ C_1 &= C_2 = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, \ D_1 = D_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \end{aligned}$$

with the transition probability matrix given by

$$p = \begin{bmatrix} 0.7 & 0.3 \\ 0.2 & 0.8 \end{bmatrix}.$$

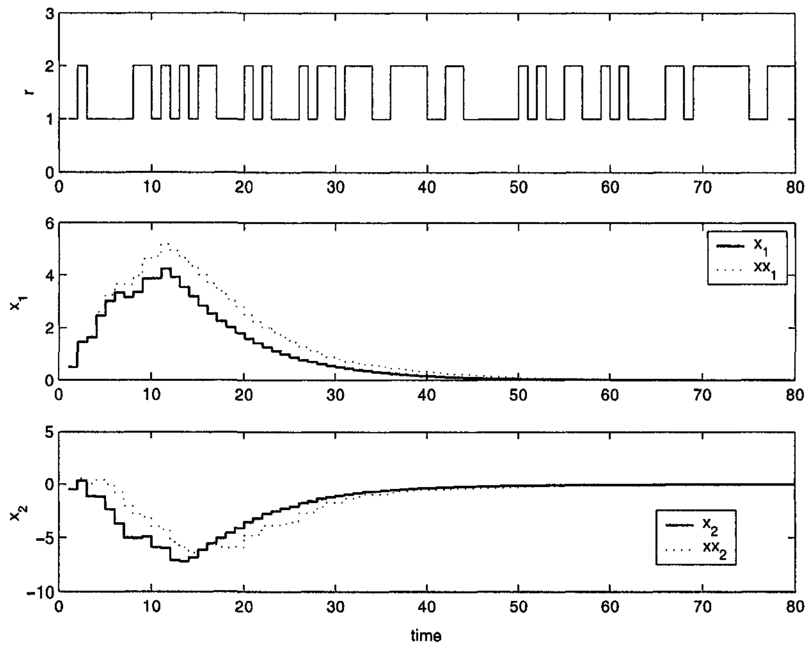

Figure 2.2: Jump disturbance and closed-loop state responses (Example 2.2)

Then by Corollary 2.1, we have the following optimal performance index and the corresponding controllers

$$\gamma = 55.4546,\ K_1 = \left[\begin{array}{cc} -17.3951 & -13.3547 \end{array}\right],\ K_2 = \left[\begin{array}{cc} -11.0488 & -8.0204 \end{array}\right],$$

while in [16] the optimal $\mathcal{H}_\infty$ performance index $\gamma = 66.0984$. This confirms Remark 2.2 that our method reduces the overdesign entailed in [16]. In fact, both the state information and state difference information are used in our method; so the sufficient condition for stability is less conservative. Suppose the initial conditions are

$$x_k = \left[\begin{array}{cc} 0.5 & -0.5 \end{array}\right]^T,\ k \in \left[\begin{array}{cc} -d, & 0 \end{array}\right],\ r_0 = 1.$$

Figure 2.2 depicts the state responses of the closed-loop system by our method (x_1, x_2) and the method in [16] (xx_1, xx_2). It is easy to see the convergence rate in our method is better than that of [16].

2.2 Delay-dependent robust stability and H_∞ control for MJLSs with delays

In section 2.1, the stability and controller design have been discussed for a discrete MJLS, where the conditions hold for any time delay $d \in [0, \infty)$, namely, delay-independent. In this section, delay-dependent conditions for stability and corresponding controller design will be derived for a continuous MJLS, which are dependent on the length of delays. The method can be readily extended to the discrete MJLS in (2.1). For the conciseness of the dissertation, the extensions are omitted.

2.2.1 Introduction

To improve system performance as well as reduce system overdesign and cost, less conservative results or algorithms are always expected. Since delay-dependent criteria make use of information on the length of delays, they are usually less conservative than delay-independent ones, especially when the time delays are small. Thus more and more attention has been paid on delay-dependent stability conditions recently [6, 14, 15, 28, 30, 33, 65, 72, 101]. In [6], delay-dependent stability conditions were obtained based on a first-order model transformation. Since additional eigenvalues are introduced, the transformed system is not equivalent to the original system. In [30], a neutral model transformation was presented, where no additional eigenvalues were needed; but an additional assumption was required to obtain the stability condition for the system. In [72], a new model transformation was introduced to guarantee the equivalence of the transformed system and the original system; it also obtained a less conservative inequality by introducing a free matrix. The method was further extended to a more general form in [65]. But the work in [65, 72] only replaced some delay terms $x(t-\tau)$ by the Leibniz-Newton formula to derive the stability condition. Since all delay terms affect the result, which one should be replaced is difficult to decide. References [14, 15, 28] combined a descriptor model transformation with Park and Moon's inequalities to yield a new transformed system; however, they were also based on the substitution for some $x(t-\tau)$, and did not entirely overcome the conservatism of the methods in [65, 72]. References [33, 101] introduced some zero equations to reduce the conservatism induced by model transformations; therefore, the results were least conservative. But both references considered only the stability of a class of linear time-delay systems; the system performance analysis and controller synthesis were not studied.

In this section, we focus on both delay-dependent stability analysis and H_∞ control synthesis for a class of jump linear time-delay systems. By some zero equations, which are similar to those in [33, 101], sufficient conditions for robust stochastic stability and stochastic stabilization are derived in the form of LMIs, where no model transformation is needed. Note that in computing the derivative of our Lyapunov function, both the state and its derivative are maintained, by which substitution and bounding for cross terms are not needed. Thus the results obtained are less conservative and over-design is avoided to some extent.

2.2.2 Problem statement

Given a probability space $(\Omega, \mathbb{F}, \mathbb{P})$, where Ω is the sample space, $\mathbb{F}$ is the algebra of events and $\mathbb{P}$ is the probability measure defined on $\mathbb{F}$, $\{\eta_t, t \geq 0\}$ is a homogeneous, finite-state Markovian process with right continuous trajectories and taking values in a finite set $S = \{1, 2, \cdots, s\}$ with generator $\Lambda = (\lambda_{ij})$. The transition probability from mode i at time t to mode j at time $t + \Delta$, $i, j \in S$, is

$$\Pr\left(\eta_{t+\Delta t} = j \middle| \eta_t = i\right) = \begin{cases} \lambda_{ij}\Delta t + o\left(\Delta t\right), & i \neq j, \\ 1 + \lambda_{ij}\Delta t + o\left(\Delta t\right), & i = j, \end{cases} \tag{2.25}$$

where $\Delta t > 0$, $\lim_{\Delta t \to 0}\left(o\left(\Delta t\right)/\Delta t\right) = 0$ and the transition probability rates satisfy $\lambda_{ij} \geq 0$ for $i, j \in S$, $i \neq j$ and $\lambda_{ii} = -\sum_{j=1, j\neq i}^{s} \lambda_{ij}$. We consider a class of stochastic uncertain systems over the space $(\Omega, \mathbb{F}, \mathbb{P})$ described by

$$\begin{cases} \dot{x}\left(t\right) = \left[A\left(\eta_t\right) + \Delta A\left(\eta_t, t\right)\right] x\left(t\right) + \left[A_\tau\left(\eta_t\right) + \Delta A_\tau\left(\eta_t, t\right)\right] x\left(t - \tau\right) \\ \qquad + \left[B\left(\eta_t\right) + \Delta B\left(\eta_t, t\right)\right] u\left(t\right) + B_w\left(\eta_t\right) w\left(t\right), \\ z\left(t\right) = \begin{bmatrix} C\left(\eta_t\right) x\left(t\right) + D_w\left(\eta_t\right) w\left(t\right) \\ C_\tau\left(\eta_t\right) x\left(t - \tau\right) \\ D\left(\eta_t\right) u\left(t\right) \end{bmatrix}, \\ x\left(t\right) = \psi\left(t\right), \ -\tau \leq t \leq 0, \eta_0 = r_0, \end{cases} \tag{2.26}$$

where $x(t) \in \Re^n$ is the state vector, $u(t) \in \Re^m$ is the control input, $z(t) \in \Re^r$ is the system output, $w(t) \in \Re^q$ is the deterministic disturbance input which belongs to $\mathcal{L}_2[0, \infty)$. Here $\mathcal{L}_2[0, \infty)$ stands for the space of square integrable vector functions over the interval $[0, \infty)$. $A\left(\eta_t\right)$, $A_\tau\left(\eta_t\right)$, $B_w(\eta_t)$, $B(\eta_t)$, $C\left(\eta_t\right)$, $C_\tau\left(\eta_t\right)$, $D_w\left(\eta_t\right)$

and $D(\eta_t)$ are known matrices of appropriate dimensions, $\Delta A(\eta_t, t)$, $\Delta A_\tau(\eta_t, t)$, $\Delta B(\eta_t, t)$ are unknown matrices which represent time-varying parametric uncertainties and are assumed to belong to certain bounded compact sets to be defined later. The quantity τ is the constant time delay of the state in the system which satisfies $0 \le \tau \le h$. $\psi(t)$ is a vector-valued initial condition of the continuous state of the mode. For notational simplicity, in the sequel, for $\eta_k = i \in S$, we will denote $A(\eta_t)$ by A_i, $\Delta A(\eta_t, t)$ by $\Delta A_i(t)$, and so on.

The admissible parameter uncertainties are assumed to be of the following forms:

$$\left[\ \Delta A_i(t),\ \ \Delta A_{\tau i}(t),\ \ \Delta B_i(t)\ \right] = H_i \Delta_i(t) \left[\ E_{1i}\ \ E_{2i}\ \ E_{3i}\ \right]. \tag{2.27}$$

Here H_i, E_{1i}, E_{2i} and E_{3i} are known real constant matrices with appropriate dimensions and the elements of $\Delta_i(t)$ are Lebesgue measurable for any $\eta_t \in S$ satisfying

$$\Delta_i^T(t)\, \Delta_i(t) \le I,\ \forall t \ge 0. \tag{2.28}$$

Let the Lyapunov-Krasovskii function be

$$V(x, t, \eta_t) = x^T(t)\, P(\eta_t)\, x(t) + \int_{t-\tau}^{t} x^T(\sigma) Q x(\sigma) d\sigma + \int_{-\tau}^{0} \int_{t+\theta}^{t} \dot{x}(\sigma) R \dot{x}(\sigma) d\sigma d\theta, \tag{2.29}$$

where $P(\eta_t) = P^T(\eta_t) > 0$, $Q = Q^T > 0$, and $R = R^T > 0$ are to be determined. We introduce the following definitions.

Definition 2.4 *The free jump system ($u(t) \equiv w(t) \equiv 0$) in (2.25)-(2.28) is said to be* robustly stochastically stable *if for all finite $\psi(t)$ defined on $[-\tau, 0]$ and initial mode r_0, there exists a finite number $\tilde{\Xi}(\psi(\cdot), h, r_0) > 0$ such that*

$$\lim_{N \to \infty} \left\{ \int_0^N \mathcal{E} \left\| x(\psi, h,\ t) \right\|^2 dt \right\} < \tilde{\Xi}(\psi(\cdot), h, r_0) \tag{2.30}$$

holds for all admissible uncertainties satisfying (2.27)-(2.28), where $\mathcal{E}$ is the statistical expectation operator.

Definition 2.5 *The system in (2.25)-(2.28) is said to be* robustly stochastically stable with disturbance attenuation level $\gamma > 0$ *if for all $w(t) \in \mathcal{L}_2[0, \infty)$, the system is robustly stochastically stable and the response $\{z(t)\}$ under zero initial condition, i.e., $\psi = 0$, satisfies*

$$\mathcal{E}\left[\int_0^\infty z^T(t)\, z(t)\, dt\right] \le \gamma^2 \left[\int_0^\infty w^T(t)\, w(t)\, dt\right]. \tag{2.31}$$

Definition 2.6 *The jump system in (2.25)-(2.28) is said to be* robustly stochastically stabilizable with disturbance attenuation level $\gamma > 0$ *if there exists a state feedback control law*

$$u(t) = K(\eta_t)\, x(t) \tag{2.32}$$

such that the resulting closed-loop system satisfies the inequality in (2.31).

In this section, delay-dependent techniques will be investigated for testing robust stability and solving robust stabilization and robust H_∞ control problems. Our purpose is to develop criteria for stochastic stability and stabilization of the system in (2.25)-(2.28), examine its robustness and design appropriate H_∞ state feedback controllers that guarantee stochastic stability with a prescribed performance γ.

2.2.3 Delay-dependent robust stability and stabilization

We will consider the stability and stabilization of the system in (2.25)-(2.28) with $w(t) \equiv 0$. First we introduce the following zero equation which will be used in our main results:

$$\Xi_1 = 2\left[x^T(t)\, Y_i + x^T(t-\tau)\, T_i\right]\left[x(t) - x(t-\tau) - \int_{t-\tau}^{t} \dot{x}(\sigma)d\sigma\right] = 0, \tag{2.33}$$

where Y_i and T_i are unknown constant matrices with appropriate dimensions. On the other hand, for any semi-positive-definite (SPD) matrix

$$X = \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix} \geq 0,$$

we have

$$\Xi_2 = h\xi^T(t)X\xi(t) - \int_{t-\tau}^{t} \xi^T(t)X\xi(t)d\sigma \geq 0, \tag{2.34}$$

where

$$\xi(t) = \begin{bmatrix} x^T(t) & x^T(t-\tau) \end{bmatrix}^T.$$

It is easy to see that equations (2.33)-(2.34) are always satisfied.

Theorem 2.3 *The free jump system in (2.25)-(2.28) is robustly stochastically stable for any constant time delay τ satisfying $0 \leq \tau \leq h$, if there exist $P_i = P_i^T > 0$, $Q = Q^T > 0$, $R > 0$, $\alpha_i > 0$, a symmetric SPD matrix*

$$X = \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix} \geq 0$$

and appropriately dimensioned matrices M_{1i}, M_{2i}, M_{3i}, Y_i and T_i such that the following LMIs are satisfied:

$$\Theta_1 = \begin{bmatrix} X_{11} & X_{12} & Y_i \\ * & X_{22} & T_i \\ * & * & R \end{bmatrix} \geq 0, \tag{2.35}$$

$$\Theta_2 = \begin{bmatrix} \Pi_{11} & \Pi_{12} & \Pi_{13} & M_{1i}H_i & \alpha_i E_{1i}^T \\ * & \Pi_{22} & \Pi_{23} & M_{2i}H_i & \alpha_i E_{2i}^T \\ * & * & \Pi_{33} & M_{3i}H_i & 0 \\ * & * & * & -\alpha_i I & 0 \\ * & * & * & * & -\alpha_i I \end{bmatrix} < 0. \tag{2.36}$$

Here $*$ denotes a block that is readily inferred by symmetry and

$$\begin{aligned} \Pi_{11} &= Q + \sum_{j=1}^{s} \lambda_{ij} P_j + hX_{11} + M_{1i}A_i + A_i^T M_{1i}^T + Y_i + Y_i^T, \\ \Pi_{12} &= hX_{12} + M_{1i}A_{\tau i} + A_i^T M_{2i}^T + T_i^T - Y_i, \\ \Pi_{13} &= -M_{1i} + P_i + A_i^T M_{3i}^T, \\ \Pi_{22} &= -Q + hX_{22} + M_{2i}A_{\tau i} + A_{\tau i}^T M_{2i}^T - T_i - T_i^T, \\ \Pi_{23} &= -M_{2i} + A_{\tau i}^T M_{3i}^T, \\ \Pi_{33} &= -M_{3i} - M_{3i}^T + hR, \\ \alpha_i &= \varepsilon_i^{-1}. \end{aligned} \tag{2.37}$$

Proof: The weak infinitesimal operator $\Im_t^x[\cdot]$ of the stochastic process $\{x(t), \eta_t, t \geq 0\}$, acting on $V(x, t, \eta_t)$ at the point $\{t, x, \ \eta_t = i\}$, is given by

$$\Im_t^x[V] = \frac{\partial V}{\partial t} + \dot{x}^T(t) \left.\frac{\partial V}{\partial x}\right|_{\eta_t = i} + \sum_{j=1}^{s} \lambda_{ij} V(t, \ x, \ i, \ j).$$

Then we have

$$\begin{aligned} \Im_t^x[V] &= x^T(t)Qx(t) - x^T(t-\tau)Qx(t-\tau) + \tau\dot{x}(t)R\dot{x}(t) - \int_{t-\tau}^{t} \dot{x}(\sigma)R\dot{x}(\sigma)d\sigma \\ &\quad + \dot{x}^T(t) P_i x(t) + x^T(t) P_i \dot{x}(t) + \sum_{j=1}^{s} \lambda_{ij} x^T(t) P_j x(t). \end{aligned} \tag{2.38}$$

Introducing the following equation

$$\begin{aligned} \Xi_3 &= 2\left[x^T(t)M_{1i} + x^T(t-\tau)M_{2i} + \dot{x}^T(t)M_{3i}\right]\left[-\dot{x}(t) + (A_i + \Delta A_i(t))x(t)\right. \\ &\quad \left. + (A_{\tau i} + \Delta A_{\tau i}(t))x(t-\tau)\right] = 0, \end{aligned} \tag{2.39}$$

where M_{1i}, M_{2i}, and M_{3i} are unknown constant matrices with appropriate dimensions, it is obvious to see this free equation is obtained by the free jump system in (2.26). Adding equations (2.33), (2.34) and (2.39) to equation (2.38), we have the following inequality:

$$\begin{aligned}
\Im_t^x[V] &\leq \Im_t^x[V] + \Xi_1 + \Xi_2 + \Xi_3 \\
&\leq x^T(t)Qx(t) - x^T(t-\tau)Qx(t-\tau) + h\dot{x}(t)R\dot{x}(t) - \int_{t-\tau}^{t} \dot{x}(\sigma)R\dot{x}(\sigma)d\sigma \\
&\quad + \dot{x}^T(t)P_i x(t) + x^T(t)P_i\dot{x}(t) + \sum_{j=1}^{s}\lambda_{ij}x^T(t)P_j x(t) + \Xi_1 + \Xi_2 + \Xi_3 \\
&= \bar{\xi}^T(t)\begin{bmatrix} \bar{\Pi}_{11} & \bar{\Pi}_{12} & \bar{\Pi}_{13} \\ * & \bar{\Pi}_{22} & \bar{\Pi}_{23} \\ * & * & \bar{\Pi}_{33} \end{bmatrix}\bar{\xi}(t) - \int_{t-\tau}^{t}\zeta^T(\sigma)\Theta_1\zeta(\sigma)d\sigma \\
&= \bar{\xi}^T(t)\bar{\Theta}_2\bar{\xi}(t) - \int_{t-\tau}^{t}\zeta^T(\sigma)\Theta_1\zeta(\sigma)d\sigma, \qquad (2.40)
\end{aligned}$$

where

$$\bar{\xi}(t) = \begin{bmatrix} \xi^T(t) & \dot{x}^T(t) \end{bmatrix}^T, \quad \zeta(\sigma) = \begin{bmatrix} x^T(t) & x^T(t-\tau) & \dot{x}^T(\sigma) \end{bmatrix}^T,$$

and

$$\begin{aligned}
\bar{\Pi}_{11} &= Q + \sum_{j=1}^{s}\lambda_{ij}P_j + hX_{11} + \left(A_i^T + \Delta A_i^T\right)M_{1i}^T + M_{1i}\left(A_i + \Delta A_i\right) + Y_i + Y_i^T, \\
\bar{\Pi}_{12} &= hX_{12} + M_{1i}(A_{\tau i} + \Delta A_{\tau i}) + (A_i^T + \Delta A_i^T)M_{2i}^T + T_i^T - Y_i, \\
\bar{\Pi}_{13} &= -M_{1i} + P_i + \left(A_i^T + \Delta A_i^T\right)M_{3i}^T, \\
\bar{\Pi}_{22} &= -Q + hX_{22} + M_{2i}\left(A_{\tau i} + \Delta A_{\tau i}\right) + \left(A_{\tau i}^T + \Delta A_{\tau i}^T\right)M_{2i}^T - T_i - T_i^T, \\
\bar{\Pi}_{23} &= -M_{2i} + \left(A_{\tau i}^T + \Delta A_{\tau i}^T\right)M_{3i}^T, \\
\bar{\Pi}_{33} &= -M_{3i} - M_{3i}^T + hR.
\end{aligned}$$

By the proof of the Lyapunov stability theory in [11], we know that the system is stochastically stable if there exist $\Theta_1 > 0$ and $\bar{\Theta}_2 < 0$ such that

$$\Im_t^x[V] < \lambda_{\max}(\bar{\Theta}_2)\left\|\bar{\xi}(t)\right\|_2^2 - \lambda_{\min}(\Theta_1)\int_{t-\tau}^{t}\|\zeta(\sigma)\|_2^2\,d\sigma < 0$$

holds. Then by Lemma 1 [101] and Schur complement, $\bar{\Theta}_2 < 0$ can be easily obtained from inequality (2.36). Thus the proof is completed. ■

Remark 2.4 *We can easily derive equation (2.30) in Definition 2.4 from $\Im_t^{\pi}[V] < 0$; the details are given in [11, 12]. For simplicity, we choose to omit the proof here. Note that by substituting equation (2.39) to inequality (2.40), the process of deriving the sufficient condition is simplified. The LMIs we obtained do not include any terms containing the product of the Lyapunov matrices and the system matrices; hence they are easily solvable, and no iteration is needed.*

If the system mode set $S = \{1\}$, the jump linear system is simplified into a general linear system. Then we have the following simplified result.

Corollary 2.2 *The free system in (2.25)-(2.28) with $i \in S = \{1\}$ is robustly stable for any constant time delay τ satisfying $0 \le \tau \le h$ if there exist $P = P^T > 0$, $Q = Q^T > 0$, $R > 0$, $\alpha > 0$, a symmetric SPD matrix*

$$X = \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix} \ge 0,$$

and appropriately dimensioned matrices M_1, M_2, M_3, Y and T such that the following LMIs are satisfied:

$$\Theta_0 = \begin{bmatrix} X_{11} & X_{12} & Y \\ * & X_{22} & T \\ * & * & R \end{bmatrix} \ge 0,$$

$$\Theta_{00} = \begin{bmatrix} \Pi_{110} & \Pi_{120} & \Pi_{130} & M_1 H & \alpha E_1^T \\ * & \Pi_{220} & \Pi_{230} & M_2 H & \alpha E_2^T \\ * & * & \Pi_{330} & M_3 H & 0 \\ * & * & * & -\alpha I & 0 \\ * & * & * & * & -\alpha I \end{bmatrix} < 0.$$

Here $*$ denotes a block that is readily inferred by symmetry and

$$\begin{aligned} \Pi_{110} &= Q + hX_{11} + M_1 A + A^T M_1^T + Y + Y^T, \\ \Pi_{120} &= hX_{12} + M_1 A_\tau + A^T M_2^T + T^T - Y, \\ \Pi_{130} &= -M_1 + P + A^T M_3^T, \\ \Pi_{220} &= -Q + hX_{22} + M_2 A_\tau + A_\tau^T M_2^T - T - T^T, \\ \Pi_{230} &= -M_2 + A_\tau^T M_3^T, \\ \Pi_{330} &= -M_3 - M_3^T + hR, \\ \alpha &= \varepsilon^{-1}. \end{aligned}$$

Next we will present a solution to the robust stabilization problem for the system in (2.25)-(2.28) with $w(t) \equiv 0$. In order to obtain an LMI solution, we have to restrict ourselves to the case of $M_{1i} = M_{2i} = M_{3i}$, $i \in S$, in the free equation in (2.39), where M_{1i}^{-1} exists. Then we have the following theorem.

Theorem 2.4 *The jump system in (2.25)-(2.28) with $w(t) \equiv 0$ is robustly stochastically stabilizable for any constant time delay τ satisfying $0 \leq \tau \leq h$ if there exist $\tilde{P}_i = \tilde{P}_i^T > 0$, $\tilde{Q} = \tilde{Q}^T > 0$, $\tilde{R} > 0$, $\varepsilon_i > 0$, a symmetric SPD matrix*

$$\tilde{X} = \begin{bmatrix} \tilde{X}_{11} & \tilde{X}_{12} \\ \tilde{X}_{12}^T & \tilde{X}_{22} \end{bmatrix} \geq 0$$

and appropriately dimensioned matrices $\tilde{M}_{1i}$, N_i, $\tilde{Y}_i$ and $\tilde{T}_i$ such that the following LMIs hold:

$$\tilde{\Theta}_1 = \begin{bmatrix} \tilde{X}_{11} & \tilde{X}_{12} & \tilde{Y}_i \\ * & \tilde{X}_{22} & \tilde{T}_i \\ * & * & \tilde{R} \end{bmatrix} \geq 0, \tag{2.41}$$

$$\tilde{\Theta}_2 = \begin{bmatrix} \tilde{\Pi}_{11} & \tilde{\Pi}_{12} & \tilde{\Pi}_{13} & \varepsilon_i H_i & \tilde{M}_{1i} E_{1i}^T + N_i^T E_{3i}^T \\ * & \tilde{\Pi}_{22} & \tilde{\Pi}_{23} & \varepsilon_i H_i & \tilde{M}_{1i} E_{2i}^T \\ * & * & \tilde{\Pi}_{33} & \varepsilon_i H_i & 0 \\ * & * & * & -\varepsilon_i I & 0 \\ * & * & * & * & -\varepsilon_i I \end{bmatrix} < 0. \tag{2.42}$$

Here

$$\begin{aligned}
\tilde{\Pi}_{11} &= \tilde{Q} + \sum_{j=1}^{s} \lambda_{ij} \tilde{P}_j + h\tilde{X}_{11} + A_i \tilde{M}_{1i}^T + \tilde{M}_{1i} A_i^T + \tilde{Y}_i + \tilde{Y}_i^T + B_i N_i + N_i^T B_i^T, \\
\tilde{\Pi}_{12} &= h\tilde{X}_{12} + A_{\tau i} \tilde{M}_{1i}^T + \tilde{M}_{1i} A_i^T + \tilde{T}_i^T - \tilde{Y}_i + N_i^T B_i^T, \\
\tilde{\Pi}_{13} &= -\tilde{M}_{1i}^T + \tilde{P}_i + \tilde{M}_{1i} A_i^T + N_i^T B_i^T, \\
\tilde{\Pi}_{22} &= -\tilde{Q} + h\tilde{X}_{22} + A_{\tau i} \tilde{M}_{1i}^T + \tilde{M}_{1i} A_{\tau i}^T - \tilde{T}_i - \tilde{T}_i^T, \\
\tilde{\Pi}_{23} &= -\tilde{M}_{1i} + \tilde{M}_{1i} A_{\tau i}^T, \\
\tilde{\Pi}_{33} &= -\tilde{M}_{1i} - \tilde{M}_{1i}^T + h\tilde{R}.
\end{aligned}$$

In this case, the control law is given by

$$K_i = N_i \tilde{M}_{1i}^{-T}. \tag{2.43}$$

Proof: With the memoryless state feedback control law $u(t) = K_i x(t)$, where the matrix $K_i \in \Re^{m \times n}$ is to be designed, the resulting closed-loop system becomes

$$\dot{x}(t) = \left(\bar{A}_i + \Delta \bar{A}_i\right) x(t) + \left[A_{\tau i} + \Delta A_{\tau i}\right] x(t - \tau),$$

where

$$\bar{A}_i = A_i + B_i K_i, \quad \Delta\bar{A}_i = \Delta A_i + \Delta B_i K_i.$$

Hence, the result follows immediately by applying the proof of Theorem 2.3, representing A_i by $\bar{A}_i$, ΔA_i by $\Delta\bar{A}_i$, pre-multiplying and post-multiplying the resulting inequality (2.36) by

$$\mathrm{diag}\left\{ \begin{array}{ccccc} M_{1i}^{-1} & M_{1i}^{-1} & M_{1i}^{-1} & \varepsilon_i I & \varepsilon_i I \end{array} \right\}$$

and

$$\mathrm{diag}\left\{ \begin{array}{ccccc} M_{1i}^{-T} & M_{1i}^{-T} & M_{1i}^{-T} & \varepsilon_i I & \varepsilon_i I \end{array} \right\}$$

respectively, and setting

$$\begin{aligned} \tilde{Q} &= M_{1i}^{-1} Q M_{1i}^{-T}, \ \ \tilde{R} = M_{1i}^{-1} R M_{1i}^{-T}, \ \ \tilde{P}_i = M_{1i}^{-1} P_i M_{1i}^{-T}, \\ \tilde{Y}_i &= M_{1i}^{-1} Y_i M_{1i}^{-T}, \ \ \tilde{T}_i = M_{1i}^{-1} T_i M_{1i}^{-T}, \ \ \tilde{X}_{11} = M_{1i}^{-1} X_{11} M_{1i}^{-T}, \\ \tilde{X}_{12} &= M_{1i}^{-1} X_{12} M_{1i}^{-T}, \ \ \tilde{X}_{22} = M_{1i}^{-1} X_{22} M_{1i}^{-T}, \ \ N_i = K_i M_{1i}^{T}. \end{aligned}$$

Thus the proof is completed. ■

Remark 2.5 *Theorems 2.3 and 2.4 provide delay-dependent conditions for robust stability and robust stabilization of uncertain time-delayed jump linear systems. Corollary 2.2 simplifies the results to a general linear system. These results do not need any system transformation, do not require any parameter tuning, and can be tested numerically very efficiently by using standard LMI techniques. Note that in Theorem 2.4, we restrict the results to the case of* $M_{1i} = M_{2i} = M_{3i}$, $i \in S$, *which are the free weighting matrices used to express the relationship of the terms* $\dot{x}(t)$, $x(t)$ *and* $x(t-\tau)$ *in the free equation. Moreover, the Leibniz-Newton formula in (2.33) is also employed to make the criterion delay-dependent. Another advantage is that the problem of finding the largest* τ *in the context of Theorem 1 can be computed by solving the following quasi-convex optimization problem in* $X_{11}, X_{12}, X_{22}, P_i$, Q, R, α_i, M_{1i}, M_{2i}, M_{3i}, Y_i, T_i *and* $\nu = \frac{1}{h}$*:*

$$\begin{aligned} &\min \quad \nu > 0 \\ &s.t. \left\{ \begin{array}{l} X = \left[\begin{array}{cc} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{array} \right] \geq 0, \ P_i = P_i^T > 0, \ Q = Q^T > 0, \\ R > 0, \ \alpha_i > 0, \ \textit{inequality } (2.35), \ \mathcal{B} > 0 \textit{ and } \mathcal{A} < \nu\mathcal{B}, \end{array} \right. \end{aligned} \tag{2.44}$$

where

$$
\begin{aligned}
\mathcal{A} &= \begin{bmatrix} X_{11} & X_{12} & 0 & 0 & 0 \\ * & X_{22} & 0 & 0 & 0 \\ * & * & R & 0 & 0 \\ * & * & * & 0 & 0 \\ * & * & * & * & 0 \end{bmatrix}, \\
\mathcal{B} &= \begin{bmatrix} -\delta_{11} & -\delta_{12} & -\Pi_{13} & -M_{1i}H_i & -\alpha_i E_{1i}^T \\ * & -\delta_{22} & -\Pi_{23} & -M_{2i}H_i & -\alpha_i E_{2i}^T \\ * & * & -\delta_{33} & -M_{3i}H_i & 0 \\ * & * & * & \alpha_i I & 0 \\ * & * & * & * & \alpha_i I \end{bmatrix}, \\
\delta_{11} &= Q + \sum_{j=1}^{s} \lambda_{ij} P_j + M_{1i}A_i + A_i^T M_{1i}^T + Y_i + Y_i^T, \\
\delta_{12} &= M_{1i}A_{\tau i} + A_i^T M_{2i}^T + T_i^T - Y_i, \\
\delta_{22} &= -Q + M_{2i}A_{\tau i} + A_{\tau i}^T M_{2i}^T - T_i - T_i^T, \\
\delta_{33} &= -M_{3i} - M_{3i}^T.
\end{aligned}
$$

A similar optimization problem for Theorem 2.4 can also be obtained by re-arranging inequalities (2.41)-(2.42). We omit it here for saving space. Note that the above problem has the form of a generalized eigenvalue problem and can be solved efficiently using the LMI algorithm "GEVP" [5].

2.2.4 Delay-dependent robust H_∞ Control

Next, we will focus on the design of a delay-dependent robust H_∞ controller for the system in (2.25)-(2.28). In order to solve this problem, we first consider the problem of robust H_∞ performance analysis for the unforced system, namely $u(t) \equiv 0$. Assume the initial condition is zero; then we have the following theorems.

Theorem 2.5 *Given a scalar $h > 0$, the system in (2.25)-(2.28) is robustly stochastically stable with disturbance attenuation γ for any constant time delay τ satisfying $0 \le \tau \le h$, if there exist $P_i = P_i^T > 0$, $Q = Q^T > 0$, $R > 0$, $\alpha_i > 0$, a symmetric SPD matrix*

$$X = \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix} \ge 0$$

and appropriately dimensioned matrices M_{1i}, M_{2i}, M_{3i}, Y_i and T_i such that the *inequality in (2.35) and the following LMI are satisfied:*

$$\Theta_3 = \begin{bmatrix} \hat{\Pi}_{11} & \Pi_{12} & \Pi_{13} & M_{1i}H_i & \alpha_i E_{1i}^T & \hat{\Pi}_{16} \\ * & \hat{\Pi}_{22} & \Pi_{23} & M_{2i}H_i & \alpha_i E_{2i}^T & M_{2i}B_{wi} \\ * & * & \Pi_{33} & M_{3i}H_i & 0 & M_{3i}B_{wi} \\ * & * & * & -\alpha_i I & 0 & 0 \\ * & * & * & * & -\alpha_i I & 0 \\ * & * & * & * & * & \hat{\Pi}_{66} \end{bmatrix} < 0. \tag{2.45}$$

Here $*$ denotes a block that is readily inferred by symmetry, and

$$\begin{aligned} \hat{\Pi}_{11} &= \Pi_{11} + C_i^T C_i, \quad \hat{\Pi}_{16} = C_i^T D_{wi} + M_{1i}B_{wi}, \quad \hat{\Pi}_{22} = \Pi_{22} + C_{\tau i}^T C_{\tau i}, \\ \hat{\Pi}_{66} &= -\gamma^2 I + D_{wi}^T D_{wi}, \quad \alpha_i = \varepsilon_i^{-1}. \end{aligned}$$

Proof: Similarly, we introduce the following free equation

$$\begin{aligned} \bar{\Xi}_3 &= 2\left[x^T(t)M_{1i} + x^T(t-\tau)M_{2i} + \dot{x}(t)M_{3i}\right]\left[-\dot{x}(t) + (A_i + \Delta A_i(t))x(t)\right. \\ &\quad \left. + (A_{\tau i} + \Delta A_{\tau i}(t))x(t-\tau) + B_{wi}w(t)\right] = 0. \end{aligned} \tag{2.46}$$

By the same Lyapunov-Krasovskii function, we derive

$$\begin{aligned} J &= E\int_0^\infty \left(z^T(t)z(t) - \gamma^2 w^T(t)w(t)\right)dt \\ &= E\int_0^\infty \left(z^T(t)z(t) - \gamma^2 w^T(t)w(t) + \Im_t^x[V]\right)dt - E\{V(x,\infty,i)\} \\ &\le \int_0^\infty \left(z^T(t)z(t) - \gamma^2 w^T(t)w(t) + \Im_t^x[V]\right)dt \\ &= \int_0^\infty \left[\phi^T(t)\tilde{\Theta}_3\phi(t) - \int_{t-\tau}^t \zeta^T(\sigma)\Theta_1\zeta(\sigma)d\sigma\right]dt, \end{aligned} \tag{2.47}$$

where $\phi(t) = \begin{bmatrix} x^T(t) & x^T(t-\tau) & \dot{x}^T(t) & w^T(t) \end{bmatrix}^T$ and

$$\tilde{\Theta}_3 = \begin{bmatrix} \bar{\Pi}_{11} + C_i^T C_i & \bar{\Pi}_{12} & \bar{\Pi}_{13} & \hat{\Pi}_{16} \\ * & \bar{\Pi}_{22} + C_{\tau i}^T C_{\tau i} & \bar{\Pi}_{23} & M_{2i}B_{wi} \\ * & * & \bar{\Pi}_{33} & M_{3i}B_{wi} \\ * & * & * & \hat{\Pi}_{66} \end{bmatrix}.$$

Using Lemma 1 in [101] and Schur complement, we can easily obtain $\tilde{\Theta}_3 < 0$ from inequality (2.45). With inequality (2.35), we get $J < 0$. By Definition 2.5, we know that the system in (2.25)-(2.28) is robustly stochastically stable with disturbance attenuation γ. Thus the proof is completed. ■

Theorem 2.6 *Given a scalar $h > 0$, the system in (2.25)-(2.28) is robustly stochastically stabilizable with disturbance attenuation γ for any constant time delay τ satisfying $0 \leq \tau \leq h$ if there exist $\tilde{P}_i = \tilde{P}_i^T > 0$, $\tilde{Q} = \tilde{Q}^T > 0$, $\tilde{R} > 0$, $\varepsilon_i > 0$, a symmetric SPD matrix*

$$\tilde{X} = \begin{bmatrix} \tilde{X}_{11} & \tilde{X}_{12} \\ \tilde{X}_{12}^T & \tilde{X}_{22} \end{bmatrix} \geq 0$$

and appropriately dimensioned matrices $\tilde{M}_{1i}$, N_i, $\tilde{Y}_i$ and $\tilde{T}_i$ such that the inequality in (2.41) and the following LMI are satisfied:

$$\begin{bmatrix} \Phi_{11} & \Phi_{12} & \Phi_{13} & \Phi_{14} & \varepsilon_i H_i & \Phi_{16} & \Phi_{17} \\ * & \Phi_{22} & \Phi_{23} & B_w & \varepsilon_i H_i & \tilde{M}_{1i} E_{2i}^T & \Phi_{27} \\ * & * & \Phi_{33} & B_w & \varepsilon_i H_i & 0 & 0 \\ * & * & * & \Phi_{44} & 0 & 0 & 0 \\ * & * & * & * & -\varepsilon_i I & 0 & 0 \\ * & * & * & * & * & -\varepsilon_i I & 0 \\ * & * & * & * & * & * & -I \end{bmatrix} < 0. \tag{2.48}$$

Here

$$\begin{aligned}
\Phi_{11} &= \tilde{Q} + \sum_{j=1}^{s} \lambda_{ij} \tilde{P}_j + h\tilde{X}_{11} + A_i \tilde{M}_{1i}^T + \tilde{M}_{1i} A_i^T + N_i^T B_i^T + B_i N_i + \tilde{Y}_i + \tilde{Y}_i^T, \\
\Phi_{12} &= h\tilde{X}_{12} + A_{\tau i} \tilde{M}_{1i}^T + \tilde{M}_{1i} A_i + N_i^T B_i^T + \tilde{T}_i^T - \tilde{Y}_i, \\
\Phi_{13} &= -\tilde{M}_{1i}^T + \tilde{P}_i + \tilde{M}_{1i} A_i^T + N_i^T B_i^T, \\
\Phi_{14} &= B_w + \tilde{M}_{1i} C_i^T D_{wi}, \\
\Phi_{16} &= \tilde{M}_{1i} E_{1i}^T + N_i^T E_{3i}^T, \\
\Phi_{17} &= \begin{bmatrix} N_i^T D_i^T & \tilde{M}_{1i} C_i^T & 0 \end{bmatrix}, \\
\Phi_{22} &= -\tilde{Q} + h\tilde{X}_{22} + A_{\tau i} \tilde{M}_{1i}^T + \tilde{M}_{1i} A_{\tau i}^T - \tilde{T}_i - \tilde{T}_i^T, \\
\Phi_{23} &= -\tilde{M}_{1i}^T + \tilde{M}_{1i} A_{\tau i}^T, \quad \Phi_{27} = \begin{bmatrix} 0 & 0 & \tilde{M}_{1i} C_{\tau i}^T \end{bmatrix}, \\
\Phi_{33} &= -\tilde{M}_{1i} - \tilde{M}_{1i}^T + h\tilde{R}, \quad \Phi_{44} = -\gamma^2 I + D_{wi}^T D_{wi}.
\end{aligned}$$

Moreover, the controller gain can be given by

$$K_i = N_i \tilde{M}_{1i}^{-T}. \tag{2.49}$$

Proof: The proof is similar to the proof of Theorem 2.4, hence omitted. ■

Remark 2.6 *Theorems 2.5 and 2.6 provide delay-dependent methods for robust H_∞ analysis and robust H_∞ synthesis, respectively, for a class of uncertain linear time-delayed jump systems. Note that using the methods of Theorems 2.5 and 2.6, the* *problems of finding the largest h for a given γ, or the smallest γ for a given h can be easily solved without the need of explicitly tuning any parameters. For instance, the smallest γ for a given h obtainable from Theorem 2.6 can be determined by solving the following convex optimization problem:*

$$\min \quad \gamma^2$$
$$s.t. \left\{ \begin{array}{l} \tilde{X} = \begin{bmatrix} \tilde{X}_{11} & \tilde{X}_{12} \\ \tilde{X}_{12}^T & \tilde{X}_{22} \end{bmatrix} \geq 0, \tilde{P}_i = \tilde{P}_i^T > 0, \tilde{Q} = \tilde{Q}^T > 0, \\ \tilde{R} > 0, \varepsilon_i > 0 \textit{ and inequalities } (2.41) \textit{ and } (2.48). \end{array} \right.$$

These results can also be reduced to the case of linear time-delay systems. For example, for the following simplified linear system

$$\left\{ \begin{array}{l} \dot{x}(t) = [A + \Delta A(t)]\, x(t) + [A_\tau + \Delta A_\tau(t)]\, x(t-\tau) + Bu(t) + B_w w(t) \\ z(t) = col\{Cx(t),\ Du(t)\} \end{array} \right. \tag{2.50}$$

the simplified results for robust H_∞ control can be given as below.

Corollary 2.3 *Given a scalar $h > 0$, the simplified system in (2.50) is robustly stabilizable with disturbance attenuation γ for any constant time delay τ satisfying $0 \leq \tau \leq h$ if there exist $\tilde{P} = \tilde{P}^T > 0$, $\tilde{Q} = \tilde{Q}^T > 0$, $\tilde{R} > 0$, $\varepsilon > 0$, a symmetric SPD matrix*

$$\tilde{X} = \begin{bmatrix} \tilde{X}_{11} & \tilde{X}_{12} \\ \tilde{X}_{12}^T & \tilde{X}_{22} \end{bmatrix} \geq 0$$

and appropriately dimensioned matrices $\tilde{M}_1$, N, $\tilde{Y}$ and $\tilde{T}$ such that the inequality in (2.41) and the following LMI holds:

$$\begin{bmatrix} \bar{\Phi}_{11} & \bar{\Phi}_{12} & \bar{\Phi}_{13} & B_w & \varepsilon H & \tilde{M}_1 E_1^T & \bar{\Phi}_{17} \\ * & \bar{\Phi}_{22} & \bar{\Phi}_{23} & B_w & \varepsilon H & \tilde{M}_1 E_2^T & 0 \\ * & * & \bar{\Phi}_{33} & B_w & \varepsilon H & 0 & 0 \\ * & * & * & -\gamma^2 I & 0 & 0 & 0 \\ * & * & * & * & -\varepsilon I & 0 & 0 \\ * & * & * & * & * & -\varepsilon I & 0 \\ * & * & * & * & * & * & -I \end{bmatrix} < 0.$$

Here

$$\begin{aligned} \bar{\Phi}_{11} &= \tilde{Q} + h\tilde{X}_{11} + A\tilde{M}_1^T + \tilde{M}_1 A^T + N^T B^T + BN + \tilde{Y} + \tilde{Y}^T, \\ \bar{\Phi}_{12} &= h\tilde{X}_{12} + A_\tau \tilde{M}_1^T + \tilde{M}_1 A + N^T B^T + \tilde{T}^T - \tilde{Y}, \\ \bar{\Phi}_{13} &= -\tilde{M}_1^T + \tilde{P} + \tilde{M}_1 A^T + N^T B^T, \\ \bar{\Phi}_{17} &= \begin{bmatrix} N^T D^T & \tilde{M}_1 C^T \end{bmatrix}, \\ \bar{\Phi}_{22} &= -\tilde{Q} + h\tilde{X}_{22} + A_\tau \tilde{M}_1^T + \tilde{M}_1 A_\tau^T - \tilde{T} - \tilde{T}^T, \end{aligned}$$

$$\bar{\Phi}_{23} = -\tilde{M}_1^T + \tilde{M}_1 A_\tau^T,$$
$$\bar{\Phi}_{33} = -\tilde{M}_1 - \tilde{M}_1^T + h\tilde{R}.$$

Moreover, the controller gain can be given by

$$K = N\tilde{M}_1^{-T}.$$

2.2.5 Numerical examples

Some examples are used to demonstrate that the methods presented for delay-dependent conditions are effective and are an improvement over the existing methods.

Example 2.3: Consider the nominal free jump system with $w(t) = u(t) = 0$, $\Delta A_i = \Delta A_{\tau i} = 0$, and the following parameters as used in [43, 12]:

$$A_1 = \begin{bmatrix} 0.5 & -1 \\ 0 & -3 \end{bmatrix}, \quad A_{\tau 1} = \begin{bmatrix} 0.5 & -0.2 \\ 0.2 & 0.3 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} -5 & 1 \\ 1 & 0.2 \end{bmatrix}, \quad A_{\tau 2} = \begin{bmatrix} -0.3 & 0.5 \\ 0.4 & -0.5 \end{bmatrix}$$

The initial condition is assumed to be $x(t) = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$ and $r_0 = 1$ for $-\tau \le t \le 0$. The generator matrix of the stochastic process η_t is

$$\lambda = \begin{bmatrix} -\lambda_1 & \lambda_1 \\ \lambda_2 & -\lambda_2 \end{bmatrix}$$

When $\lambda_1 = 7$ and $\lambda_2 \le 6$, the result of [43] cannot be applied for stability. When $\lambda_1 = 7$ and $\lambda_2 = 6$, based on the result of [11], the system is found to be delay-independent stable. If we decrease λ_2 further, e.g., $\lambda_1 = 7$ and $\lambda_2 = 3$, the result of [11] cannot guarantee system stability. But Theorem 1 in [12] can be used to obtain a feasible solution with $\tau \le h = 0.404$. Moreover, by Theorem 2.3 of our results, we can obtain a feasible solution with $\tau \le h = 0.7316$, which is much larger than that of [12]. The state trajectories are shown in Figure 2.3 when $h = 0.7316$. By this example we can see that our stability criterion gives a less conservative result than those obtained by the methods in [43, 11, 12].

Example 2.4: Consider the free uncertain time-delay system, namely $w(t) = u(t) = 0$, $\eta_t = i$, $i \in S = \{1\}$, where

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, \quad A_\tau = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}, \quad H = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}, \quad E_1 = E_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Table 2.2: The disturbance attenuation γ due to different h

	$h = 0.3$	$h = 0.2$
[18]	$\gamma = 1.95$	$\gamma = 0.66$
Our result	$\gamma = 0.1642$	$\gamma = 0.1093$

Table 2.3: The disturbance attenuation γ and controller gain with same h

	h	$h = 0.999$	Controller gain K	Iteration
[25]	1.28	$\gamma = 0.1287$	$[0,\ -1.0285 \times 10^6]$	No
[50]	1.25	$\gamma = 0.1015$	$[3.6828,\ -827.0898]$	74
Ours	1.4075	$\gamma = 0.6331$	$[-0.0000,\ -4.8400]$	No

Consider the following linear time delay system:

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_\tau = \begin{bmatrix} -1 & -1 \\ 0 & -0.9 \end{bmatrix}, \quad B_w = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad H = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}, \quad E_1 = E_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

When $D = 0$, comparing Theorem 3.4 in [18] with Corollary 2.3 in our results, we have Table 2.2, by which we can see that for a given time delay, the disturbance attenuation γ obtained by our method is smaller than that of [18].

On the other hand, when $\Delta A = \Delta A_\tau = 0$, $D = 0.1$, we obtain Table 2.3 by comparing Theorem 3.1 in [25], Theorem 4.1 in [50] with Corollary 2.3 in our section.

The state trajectories and the output are shown in Figure 2.4 and Figure 2.5, where the disturbance is a uniformly distributed random signal over $[-1,\ 1]$, and the initial condition of the states is $[1,\ 0.03]^T$. Table 2.3 shows that the disturbance attenuation γ we obtained is larger than those from other references for this particular example. But our controller still gives good performance under the same disturbance, and the state trajectories are very similar to those of the other references, see Figure 2.4. It is worthy to note that our controller gain is much smaller than the other two and less overshoot in the output trajectory is observed (Figure 2.5). Furthermore, no iteration is needed for calculation and our upper bound of the time delay for stability is the largest.

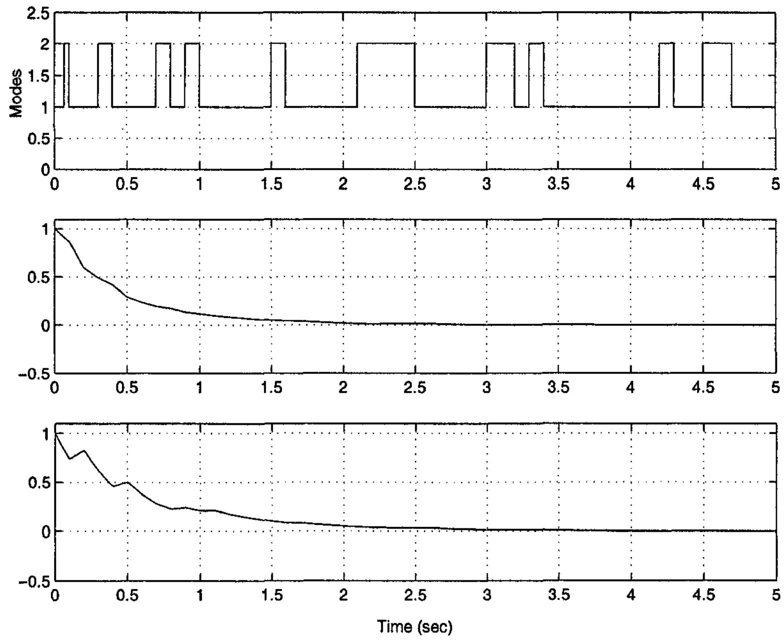

Figure 2.3: The state trajectories of the free jumping system (Example 2.3)

Table 2.1: The upper bound of the time delay for robust stability

Delay bound	[67]	[98]	[51]	[18]	Our result
h	0.1958	0.2010	0.3199	0.4437	2.3970

By comparing the robust stability criterion of Corollary 2.2 with those of [18, 51, 67, 98] for the above system, we have Table 2.1. Hence, for this example, the robust stability criterion we derived for linear time-delay systems is less conservative than those reported in [18, 51, 67, 98]. In addition, we also compare Corollary 2.2 with the results in [101]: The delay bound we obtained is 1.1491, which is slightly larger than that in [101] with constant time delays, 1.1490. However, in [101], only the stability problem is discussed; the stabilization problem and performance analysis were not considered.

Example 2.5: This example illustrates that better control performance can be obtained by our methods. Let us consider the disturbance rejection ability first.

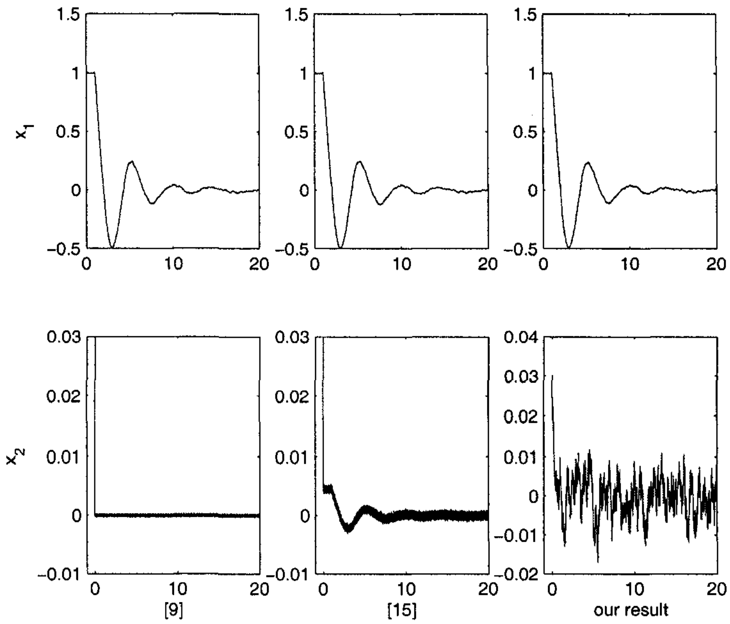

Figure 2.4: The state trajectories (Example 2.5)

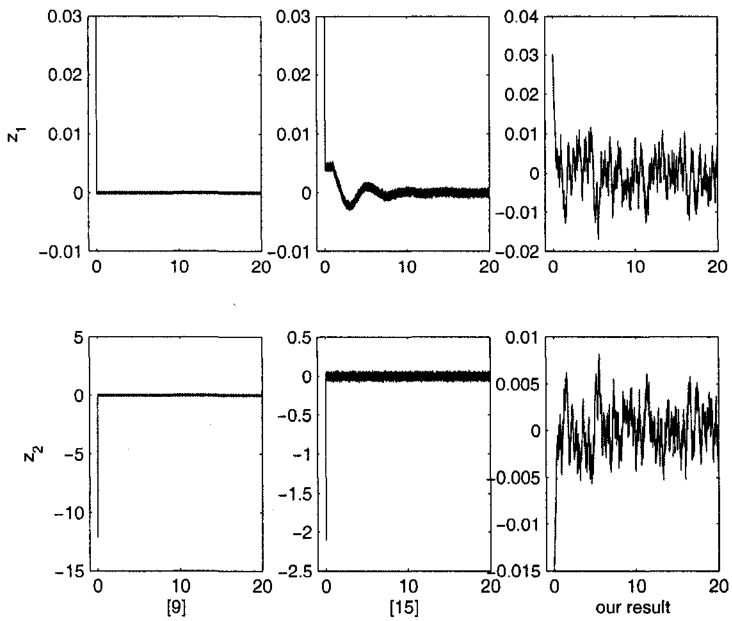

Figure 2.5: The outputs (Example 2.5)

2.3 Summary

In this chapter, some delay-independent sufficient conditions are developed firstly for the stochastic stabilizability and $\mathcal{H}_\infty$ control problems of a class of MJLSs with norm-bounded parameter uncertainties and constant time delays. Secondly, new delay-dependent conditions for robust stochastic stability and stabilization of MJLSs with time delay are derived, where none of model transformation, bounding for cross terms and substitution is needed. The H_∞ controller guarantees the robust stability of the delayed jump linear system, while providing a certain level of disturbance attenuation. Moreover, an algorithm for calculating the delay upper bound for system stability is given. All the results are presented in terms of standard LMIs, which are very easy to be solved in Matlab. Numerical examples illustrate the effectiveness of our methods.

Chapter 3

Design of NCSs with Packet Dropouts

In this chapter, we consider the problem of modeling, design and analysis of networked control systems (NCSs) with packet dropouts. When the history behavior of packet dropouts are described by independent Markovian chains, the problem of finding a controller to stabilize the closed-loop NCSs can be cast as that for some kind of MJLSs.

The chapter is organized as follows. Section 3.1 introduces the problem and presenting the relevant prior work. Section 3.2 presents the basic preliminary of our setup. Section 3.3 and Section 3.4 provide the modeling of NCSs with packet dropouts in single- and multiple-packet transmissions, respectively. According to the resulting NCSs, their stochastic stabilities and controller designs are then discussed. Section 3.5 validates the effectiveness of our results by two numerical examples. Finally, Section 3.6 gives some concluding marks.

3.1 Introduction

In networked control systems (NCSs), control loops are closed through real-time networks. Such networked systems bring new functionalities that were not available in the past, such as low cost, reduced system wiring, simple system diagnosis and maintenance, and increased system agility. However, the insertion of communication networks in feedback control loops makes the NCS analysis and synthesis complex, see [47, 64, 115] and the references therein, where much attention has been paid to the delayed data packets of an NCS due to network transmissions. In fact, data packets through networks suffer not only transmission delays, but also, possibly, transmission loss/packet dropout [110, 111]; the latter is a potential source of instability and poor performance in NCSs because of the critical real-time requirement in control systems. How such packet dropout affects stability and performance of NCSs is an issue focused in this chapter.

There are many results on NCSs with packet dropouts [58, 68, 81, 82, 103, 108, 113], where all the stability conditions and controller designs are derived based on the assumption that packet dropout exists only on the sensor-to-controller (S/C) side. The effect of controller-to-actuator (C/A) packet dropouts is neglected due to the complicated NCS modeling and analysis. However, C/A packet dropouts are not only inevitable in the transmission but also the main factor for system instability and poor performance. Moreover, the controller cannot compensate these dropouts in time since they happen after the control decisions have been made, which brings risks for open-loop control or inappropriate compensation. Thus, more and more attention have been focused on NCSs with both S/C and C/A packet dropouts, and some results were obtained [32, 39, 74, 87, 114]. In [32, 74], ADSs were introduced to model NCSs with packet dropouts on both S/C and C/A sides. The controller gain in [32] is obtained by solving bilinear matrix inequalities (BMIs) with rate constraints on the occurrence of events, while in [74] the controller gain is chosen in advance by a pole placement method considering rate constraints on the occurrence of discrete states of a dynamical system. In [39, 114], switching systems were used to model NCSs, where the controller in [39] was event-driven and the controller in [114] was time-driven. In [87], a linear system with stochastic variables was discussed to describe NCSs with both-side packet dropouts, where the linear/nonlinear LQG optimal controllers were designed to minimize a cost function according to the Transmission Control Protocol (TCP) and User Datagram Protocol (UDP). Those references discuss packet dropouts in the single-packet transmission. As to the jump linear system approach, to our best knowledge, no work has been done at present for modeling NCSs with both S/C and C/A packet dropouts histories simultaneously. Note that packet dropouts defined in the aforementioned references have two cases, dropped or sent successfully, which are modeled as a Bernoulli or a two-state Markov chain process.

In this chapter, Markov chains are introduced to describe S/C and C/A packet dropouts. The Markov chains here describe the quantity of packet dropouts between current time k and its latest successful transmission instead of only the information on if a packet is dropped or not, which is different from what were in the above references. By this definition, the number of states of Markov chains is larger than two and the history of packet dropouts can be seen clearly. Under consideration of network packet size constraints, new models of NCSs with packet dropouts are presented according to the single-packet and multiple-packet transmissions. By augmenting the state vector, the resulting closed-loop system can be transformed to a standard jump linear system with time delays, which enables us to apply the results of jump linear systems to the analysis and synthesis of such NCSs. Sufficient conditions for stochastic stability are given and corresponding controller design steps are provided. Examples are finally given to show the effectiveness of our method.

3.2 Problem formulation

Consider the NCS setup with data packet dropouts in Figure 3.1, where sensors, controllers and actuators are clock-driven. The linear time-invariant (LTI) plant we consider here is:

$$x(k+1) = \Phi x(k) + \Gamma u(k), \tag{3.1}$$

where $x(k) \in \Re^{\bar{n}}$ is the state, and $u(k) \in \Re^{\bar{m}}$ is the input. Φ and Γ are known real constant matrices with appropriate dimensions. Suppose buffers are long enough to hold all the packets arrived, which will be picked up according to the last-in-first-out

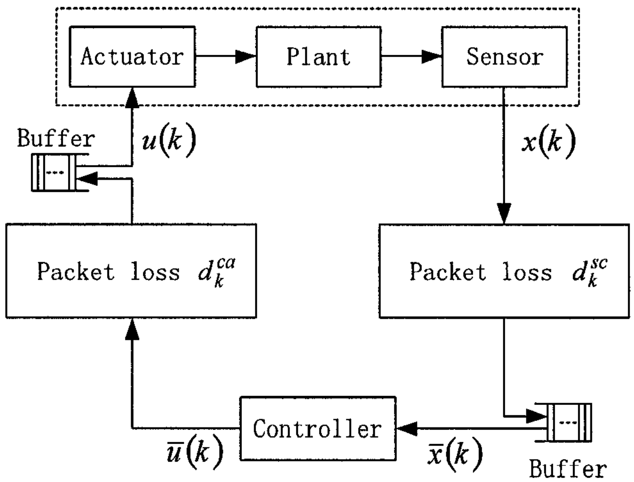

Figure 3.1: An NCS with data packet dropout via state feedback

rule. For example, when a sensor data $x(k)$ is lost, the controller will read out the most recent data $x(k-1)$ from the buffer and utilize it as $\bar{x}(k)$ to calculate the new control input, which will be sent to the plant; otherwise, the new sensor data $x(k)$ will be saved to the buffer and used by the controller as $\bar{x}(k)$. Thus for the buffers, we have:

$$u(k) = \begin{cases} \bar{u}(k) & \text{if transmitted successfully,} \\ u(k-1) & \text{otherwise,} \end{cases} \tag{3.2}$$

$$\bar{x}(k) = \begin{cases} x(k) & \text{if transmitted successfully,} \\ \bar{x}(k-1) & \text{otherwise.} \end{cases} \tag{3.3}$$

Moreover, due to the bandwidth and packet size constraints of the network, the packet transmission can be classified into two types, single- and multiple-packet transmissions. By this classification, we will have two new NCS models for the setup in Figure 3.1, which are given in the following sections.

3.3 Modeling and controller design of NCSs with single-packet transmissions

In this section, system modeling, stability analysis and controller design are considered for the NCSs with single-packet transmissions. The single-packet transmission means that data is lumped together into one network packet and transmitted at the same time. This type of transmission is suitable for networks with large packet sizes, e.g., Ethernet which can hold a maximum of 1500 bytes of data in a single packet.

3.3.1 Modeling NCSs with single-packet transmissions

Assume that d_k^{sc} is the quantity of packets dropped at time k on the S/C side, which is calculated from the current time k to the last successful transmission (happened at time $k - d_k^{sc}$), d_k^{ca} is the packet quantity dropped on the C/A side between the current time k and its last successful transmission at time $k - d_k^{ca}$, and both of them are bounded. Thus we have

$$0 \le d_k^{sc} \le d_1, \quad 0 \le d_k^{ca} \le d_2,$$

where d_1 and d_2 are non-negative integers. We model d_k^{sc} and d_k^{ca} as two homogeneous independent Markov chains, which take values in $S_1 = \{0,\ 1, \cdots,\ d_1\}$ and $S_2 = \{0,\ 1, \cdots,\ d_2\}$ with the generators $\bigwedge_1 = (\rho_{ij})$ and $\bigwedge_2 = (\lambda_{mn})$, respectively. The transition probabilities of d_k^{sc} (jumping from mode i to j) and d_k^{ca} (jumping from mode m to n) are defined by

$$\rho_{ij} = \Pr(d_{k+1}^{sc} = j | d_k^{sc} = i), \quad \lambda_{mn} = \Pr(d_{k+1}^{ca} = n | d_k^{ca} = m), \tag{3.4}$$

where $\rho_{ij} \ge 0,\ i,\ j \in S_1$, $\lambda_{mn} \ge 0,\ m,\ n \in S_2$, and $\sum_{j=0}^{d_1} \rho_{ij} = 1,\ \sum_{n=0}^{d_2} \lambda_{mn} = 1$. It is obvious that the transition probabilities satisfy

$$\rho_{ij} = 0 \ \text{ if } j \ne i+1 \ \&\ j \ne 0, \quad \lambda_{mn} = 0 \ \text{ if } n \ne m+1 \ \&\ n \ne 0,$$

which can be derived by packet dropout definitions. Assume the state feedback control law is

$$\bar{u}(k) = F(d_k^{sc})\bar{x}(k), \tag{3.5}$$

where $F(d_k^{sc})$ is a set of controllers and will be designed based on d_k^{sc}. Substituting Eqs. (3.2) and (3.5) into the system in (3.1), we have the following closed-loop system

$$x(k+1) = \begin{cases} \Phi x(k) + \Gamma F(d_k^{sc})\bar{x}(k) & \text{if } d_k^{ca} = 0, \\ \Phi x(k) + \Gamma u(k-1) & \text{otherwise } d_k^{ca} > 0. \end{cases} \tag{3.6}$$

Note that $\bar{x}(k) = x(k - d_k^{sc})$, which can be easily derived by iterations based on Eq. (3.3). To simplify the expression of the closed-loop system, we introduce a function

$\alpha(\cdot)$ to combine the above closed-loop system as

$$x(k+1) = \Phi x(k) + \alpha(d_k^{ca})\Gamma u(k-1) + [1-\alpha(d_k^{ca})]\Gamma F(d_k^{sc})x(k-d_k^{sc}), \quad (3.7)$$

$$u(k) = \alpha(d_k^{ca})u(k-1) + [1-\alpha(d_k^{ca})]F(d_k^{sc})x(k-d_k^{sc}), \quad (3.8)$$

where

$$\alpha(d_k^{ca}) = \begin{cases} 1, & d_k^{ca} > 0, \\ 0, & d_k^{ca} = 0. \end{cases}$$

Remark 3.1 The value of $\alpha(\cdot)$ depends on whether the designed control signal is successfully transmitted or not (namely, $d_k^{ca} = 0$ or $d_k^{ca} > 0$), instead of how many designed control signals are dropped (the value of d_k^{ca}). This classification can simplify the modeling of the closed-loop system since the control input $u(k)$ will not be updated no matter what value $d_k^{ca} > 0$ will be. That is, the control signal $u(k)$ will be the same when $d_k^{ca} = 1, 2, 3, \cdots, d_2$. Another advantage of this classification is to avoid introducing this unknown d_k^{ca} in the augmented state vectors and controller design. Thus, we replace $u(k)$ with Eq. (3.2) instead of $u(k) = \bar{u}(k - d_k^{ca})$, the deriving method being the same as the iteration method for $\bar{x}(k)$.

Concatenating plant and controller state vectors to obtain a global vector $z(k) = [x^T(k)\ u^T(k-1)]^T$ by Eqs. (3.7)-(3.8), we can obtain the closed-loop system for the NCS with single-packet transmissions in Figure 3.1 as:

$$\begin{aligned} z(k+1) &= \begin{bmatrix} \Phi & \Gamma\alpha(d_k^{ca}) \\ 0 & \alpha(d_k^{ca}) \end{bmatrix} \begin{bmatrix} x(k) \\ u(k-1) \end{bmatrix} \\ &\quad + \begin{bmatrix} (1-\alpha(d_k^{ca}))\Gamma F(d_k^{sc}) & 0 \\ (1-\alpha(d_k^{ca}))F(d_k^{sc}) & 0 \end{bmatrix} \begin{bmatrix} x(k-d_k^{sc}) \\ u(k-d_k^{sc}-1) \end{bmatrix} \\ &= A(d_k^{ca})z(k) + B(d_k^{ca}, d_k^{sc})z(k-d_k^{sc}). \end{aligned} \quad (3.9)$$

Remark 3.2 The resulting closed-loop system in (3.9) is a jump linear system with two modes (d_k^{sc} and d_k^{ca}) and one mode-dependent time-varying delay d_k^{sc}, where their transitions are described by two Markov chains, which give the history behavior of S/C and C/A packet dropouts, respectively. This also enables us to apply the results of jumping linear systems with time-delays to the analysis and synthesis of such NCSs.

Before proceeding, we need the following definition.

Definition 3.1 *[100] The free nominal jump discrete-time system in (3.9) is said to be stochastically stable, if for all finite* $z_k = \varphi \in \Re^{\bar{n}+\bar{m}}$ *defined on* $k \in [-d_1, 0]$ *and initial model* d_0^{sc}, d_0^{ca}, *there exists a finite number* $\tilde{\Xi}(\varphi,\ d_0^{sc},\ d_0^{ca}) > 0$ *such that*

$$\lim_{N\to\infty} \mathcal{E}\left\{ \sum_{k=0}^{N} \|z_k\|^2 \middle| \varphi, d_0^{sc}, d_0^{ca} \right\} < \tilde{\Xi}(\varphi, d_0^{sc}, d_0^{ca}) \tag{3.10}$$

holds, where $\mathcal{E}$ *is the statistical expectation operator.*

3.3.2 Stability analysis and controller design of NCSs with single-packet transmissions

In this subsection, a sufficient condition on the stochastic stability of the system in (3.9) with single-packet transmissions is derived. For notational simplicity, in the sequel, for $d_k^{sc} = i \in S_1$, $d_k^{ca} = m \in S_2$, we denote $A(d_k^{ca}) \triangleq A_m$, $B(d_k^{sc}, d_k^{ca}) \triangleq B_{i,m}$, and

$$\begin{aligned} \underline{\rho} &= \min\{\rho_{ii}, i \in S_1\}, \quad \underline{d_1} = \min\{d_k^{sc}, k \in \mathbb{Z}\}, \\ \mu &= 1 + (1-\underline{\rho})(d_1 - \underline{d_1}),\ K = (I + \Gamma^T\Gamma)^{-1} \begin{bmatrix} \Gamma^T & I \end{bmatrix}. \end{aligned}$$

Then we have the following theorem.

Theorem 3.1 *The system in (3.9) is stochastically stable if there exist* $X_{i,m} > 0$, $Q > 0$, *and* $\Psi_{i,m}$ *such that the following LMI*

$$\begin{bmatrix} -X_{i,m} & 0 & X_{i,m}A_m^T\Theta & X_{i,m} \\ * & -Q & \Psi_{i,m}\Theta & 0 \\ * & * & -\Omega & 0 \\ * & * & * & -\frac{1}{\mu}Q \end{bmatrix} < 0 \tag{3.11}$$

holds for all $i, j \in S_1$ *and* $m, n \in S_2$, *where* $*$ *denotes blocks that are readily inferred by symmetry and*

$$\begin{aligned} \Theta &= \underbrace{[\sqrt{\lambda_{m1}\rho_{i1}}I,\ \cdots,\ \sqrt{\lambda_{mn}\rho_{ij}}I,\ \cdots,\ \sqrt{\lambda_{md_2}\rho_{id_1}}I]}_{d_1\cdot d_2}, \\ Q &= R^{-1},\quad \Omega = diag[X_{1,1}, \cdots, X_{j,n}, \cdots, X_{d_1,d_2}]. \end{aligned} \tag{3.12}$$

Here $d_1 d_2$ *is the number of matrices. The control law we have is*

$$u(k) = \begin{cases} u(k-1) & d_k^{ca} > 0, \\ K\Psi_{i,0}^T Q^{-1} \begin{bmatrix} I & 0 \end{bmatrix}^T x(k - d_k^{sc}) & d_k^{ca} = 0. \end{cases}$$

Proof: Let the Lyapunov-Krasovskii function be

$$\begin{aligned} V(z(k),k) &= z^T(k)P(d_k^{sc},d_k^{ca})z(k)+\sum_{\tau=k-d_k^{sc}}^{k-1} z^T(\tau)Rz(\tau) \\ &\quad +(1-\underline{\rho})\sum_{\theta=-d_1+1}^{-\underline{d_1}}\sum_{\tau=k+\theta}^{k-1} z^T(\tau)Rz(\tau), \end{aligned}$$

where $P(d_k^{sc},\ d_k^{ca}) = P^T(d_k^{sc},\ d_k^{ca}) > 0$, $R = R^T > 0$ are to be determined. Then we have

$$\begin{aligned} & \mathcal{E}\{V(z(k+1),k+1)|V(z(k),k)\} - V(z(k),k) \\ \leq\ & z^T(k+1)\bar{P}(i,m)z(k+1) + z^T(k)[\mu R - P(i,m)]z(k) - z^T(k-i)Rz(k-i) \\ & + \sum_{j=0,j\neq i}^{d_1} \rho_{ij}[\sum_{\tau=k-d_1+1}^{k-1} z^T(\tau)Rz(\tau) - \sum_{\tau=k-\underline{d_1}+1}^{k-1} z^T(\tau)Rz(\tau)] \\ & -(1-\underline{\rho})\sum_{\tau=k-d_1+1}^{k-\underline{d_1}} z^T(\tau)Rz(\tau) \\ \leq\ & \delta(k)^T W(k,i,m)\delta(k), \end{aligned} \tag{3.13}$$

where

$$\begin{aligned} \delta(k) &= \left[\ z^T(k)\ z^T(k-i)\ \right]^T,\ \bar{P}(i,m) = \sum_{n=0}^{d_2}\sum_{j=0}^{d_1} \lambda_{mn}\rho_{ij}P(j,n), \\ W(k,i,m) &= \begin{bmatrix} -P(i,m)+\mu R + A_m^T\bar{P}(i,m)A_m & A_m^T\bar{P}(i,m)B_{i,m} \\ * & -R + B_{i,m}^T\bar{P}(i,m)B_{i,m} \end{bmatrix}. \end{aligned}$$

Let $X_{i,m} = P^{-1}(i,m)$, $\Psi_{i,m} = R^{-1}B_{i,m}^T$, $Q = R^{-1}$. By Schur complement and Eqs. (3.11)-(3.12), we know that $W(k,i,m) < 0$. Thus we have

$$\begin{aligned} & \mathcal{E}\{V(z(k+1),k+1)|V(z(k),k)\} - V(z(k),k) \\ \leq\ & -\eta_{min}[-W(k,i,m)]\delta^T(k)\delta(k) \leq -\beta z^T(k)z(k), \end{aligned} \tag{3.14}$$

where $\eta_{min}[-W(k,i,m)]$ denotes the minimal eigenvalue of $-W(k,i,m)$ and $\beta = \inf\{\eta_{min}[-W(k,i,m)],\ i \in S_1,\ m \in S_2\}$. From Eq. (3.14), we derive that for any $N \geq 1$,

$$\mathcal{E}\{V(z(N+1),d_{N+1}^{sc},d_{N+1}^{ca})\} - \mathcal{E}\{V(z(0),d_0^{sc},d_0^{ca})\} \leq -\beta\sum_{k=0}^{N}\mathcal{E}\|z^T(k)z(k)\|.$$

This yields, for any $N \geq 1$,

$$\sum_{k=0}^{N} \mathcal{E}\|z^T(k)z(k)\| < \frac{1}{\beta}\mathcal{E}\{V(z(0), d_0^{sc}, d_0^{ca})\},$$

$$\Rightarrow \lim_{N\to\infty} \sum_{k=0}^{N} \mathcal{E}\|z^T(k)z(k)\| < \frac{1}{\beta}\mathcal{E}\{V(\varphi, d_0^{sc}, d_0^{ca})\} = \widetilde{\Xi}(\varphi, d_0^{sc}, d_0^{ca}).$$

By Definition 3.1, the system in (3.9) is stochastically stable. In addition, by $\Psi_{i,m} = R^{-1}B_{i,m}^T$, we have

$$\begin{aligned} \Psi_{i,m} &= (1-\alpha(d_k^{ca}))Q \begin{bmatrix} \Gamma F_i & 0 \\ F_i & 0 \end{bmatrix}^T = (1-\alpha(d_k^{ca}))Q \begin{bmatrix} I \\ 0 \end{bmatrix} F_i^T \begin{bmatrix} \Gamma^T & I \end{bmatrix} \\ \Rightarrow & \quad \begin{cases} \Psi_{i,0} = Q \begin{bmatrix} I \\ 0 \end{bmatrix} F_i^T \begin{bmatrix} \Gamma^T & I \end{bmatrix}, & \alpha(d_k^{ca}) = 0 \Leftrightarrow d_k^{ca} = m = 0, \\ \Psi_{i,m} = 0, & \alpha(d_k^{ca}) = 1 \Leftrightarrow d_k^{ca} = m > 0, \end{cases} \end{aligned} \tag{3.15}$$

then the controller F_i in (3.5) is obtained by pre-multiplying $[\ I \quad 0\ ]Q^{-1}$ and post-multiplying $[\ \Gamma^T\ I\ ]^T$ to both sides of Eq. (3.15). Note that $\Gamma^T\Gamma + I$ is of full rank. The proof is completed. ■

3.4 Modeling and controller design of NCSs with multiple-packet transmissions

From the above section, we know that the single-packet transmission is usually used for networks with enough capacity for large-size packets. However, some packet-switched networks can only carry limited information in a single packet due to packet size constraints, e.g., DeviceNet which has a maximum 8-byte data field in each packet. Thus, in such networks, the multiple-packet transmission is widely used, where sensor or actuator data is transmitted in separate network packets and may not arrive at the controller and plant simultaneously.

3.4.1 Modeling NCSs with multiple-packet transmissions

Due to the characteristic given above, the NCSs are modeled with $M + N$ ($M + N > 2$) Markov chains. Split the plant state into M separate packets as $x(k) = [X_1^T(k),\ X_2^T(k),\ \cdots,\ X_M^T(k)]^T$ and the controller output into N separate packets as $\bar{u}(k) = [\bar{U}_1^T(k),\ \bar{U}_2^T(k),\ \cdots,\ \bar{U}_N^T(k)]^T$, where

$$\begin{aligned} X_1(k) &= [x_1^T(k),\ x_2^T(k),\ \cdots,\ x_{r_1}^T(k)]^T, \\ X_2(k) &= [x_{r_1+1}^T(k),\ x_{r_1+2}^T(k),\ \cdots,\ x_{r_2}^T(k)]^T, \\ &\vdots \\ X_M(k) &= [x_{r_{M-1}+1}^T(k),\ x_{r_{M-1}+2}^T(k),\ \cdots,\ x_{\bar{n}}^T(k)]^T, \end{aligned} \tag{3.16}$$

and $1 \leq r_1 < \cdots < r_{M-1} \leq \bar{n}$, then we have the corresponding input of the controller $\bar{x}(k)$ as

$$\begin{aligned} \bar{x}(k) &= [\bar{X}_1^T(k),\ \bar{X}_2^T(k),\ \cdots,\ \bar{X}_M^T(k)]^T, \\ \bar{X}_1(k) &= [\bar{x}_1^T(k),\ \bar{x}_2^T(k),\ \cdots,\ \bar{x}_{r_1}^T(k)]^T, \\ &\vdots \\ \bar{X}_M(k) &= [\bar{x}_{r_{M-1}+1}^T(k),\ \bar{x}_{r_{M-1}+2}^T(k),\ \cdots,\ \bar{x}_{\bar{n}}^T(k)]^T. \end{aligned}$$

Similar definitions are used for the signals $\bar{u}(k)$ and $u(k)$, and are omitted. For the sake of simplicity, in this section we assume two-packet transmission on the S/C side and single-packet transmission on the C/A side; i.e., $M = 2$ and $N = 1$. Thus

$$\bar{x}(k) = \begin{bmatrix} \bar{X}_1(k) \\ \bar{X}_2(k) \end{bmatrix} = \begin{bmatrix} X_1(k - d_{1k}^{sc}) \\ X_2(k - d_{2k}^{sc}) \end{bmatrix}, \quad u(k) = U_1(k) = \bar{U}_1(k - d_k^{ca}),$$

where we set $r_1 = r$, d_{1k}^{sc} and d_{2k}^{sc} reflect the quantities of S/C packet dropouts in channels 1 and 2, and d_k^{ca} reflects the quantity of C/A packet dropouts. Their transition probabilities are given by

$$\begin{aligned} \rho_{ij} &= \Pr(d_{1(k+1)}^{sc} = j | d_{1k}^{sc} = i),\ \rho_{ij} \geq 0,\ i, j \in S_{11} = \{0,\ 1, \cdots,\ d_{11}\}, \\ \pi_{pq} &= \Pr(d_{2(k+1)}^{sc} = q | d_{2k}^{sc} = p),\ \pi_{pq} \geq 0,\ p, q \in S_{12} = \{0,\ 1, \cdots,\ d_{12}\}, \\ \lambda_{mn} &= \Pr(d_{k+1}^{ca} = n | d_k^{ca} = m),\ \lambda_{mn} \geq 0,\ m,\ n \in S_2 = \{0,\ 1, \cdots,\ d_2\} \end{aligned} \tag{3.17}$$

with $\sum_{j=0}^{d_{11}} \rho_{ij} = 1$, $\sum_{q=0}^{d_{12}} \pi_{pq} = 1$, and $\sum_{n=0}^{d_2} \lambda_{mn} = 1$. Similarly as in the single-packet transmission case, the closed-loop system in this case can be modeled by

$$\begin{aligned} z(k+1) &= \begin{bmatrix} \Phi & \Gamma\alpha(d_k^{ca}) \\ 0 & \alpha(d_k^{ca}) \end{bmatrix} z(k) \\ &+ \begin{bmatrix} (1 - \alpha(d_k^{ca}))\Gamma F_{\bar{m}\times r}(d_{1k}^{sc})[I_r \ \ 0_{r\times(\bar{n}-r)}] & 0 \\ (1 - \alpha(d_k^{ca}))F_{\bar{m}\times r}(d_{1k}^{sc})[I_r \ \ 0_{r\times(\bar{n}-r)}] & 0 \end{bmatrix} z(k - d_{1k}^{sc}) \\ &+ \begin{bmatrix} (1 - \alpha(d_k^{ca}))\Gamma F_{\bar{m}\times(\bar{n}-r)}(d_{2k}^{sc})[0_{(\bar{n}-r)\times r} \ \ I_{\bar{n}-r}] & 0 \\ (1 - \alpha(d_k^{ca}))F_{\bar{m}\times(\bar{n}-r)}(d_{2k}^{sc})[0_{(\bar{n}-r)\times r} \ \ I_{\bar{n}-r}] & 0 \end{bmatrix} z(k - d_{2k}^{sc}) \\ &= A(d_k^{ca})z(k) + B_1(d_k^{ca}, d_{1k}^{sc})z(k - d_{1k}^{sc}) + B_2(d_k^{ca}, d_{2k}^{sc})z(k - d_{2k}^{sc}), \end{aligned} \tag{3.18}$$

where the control law to be designed is

$$F(d_k^{sc}) = [F_{\bar{m}\times r}(d_{1k}^{sc}) \quad F_{\bar{m}\times(\bar{n}-r)}(d_{2k}^{sc})]. \tag{3.19}$$

Remark 3.3 It is easy to see that the complexity of the closed-loop models depends on M and N: The larger M and N are, the more complicated models we obtain. However, we remark that this complication will not affect our method for stability analysis and controller design; only the mathematical calculation is complicated.

Definition 3.2 *[100] The free nominal jump discrete-time system (3.18) is said to be stochastically stable, if for all finite $z_k = \varphi \in \Re^{\bar{n}+\bar{m}}$ defined on $k \in [-\max(d_{11}, d_{12}), 0]$ and initial model d_{10}^{sc}, d_{20}^{sc} d_0^{ca}, there exists a finite number $\tilde{\Xi}(\varphi,\ d_{10}^{sc},\ d_{20}^{sc},\ d_0^{ca}) > 0$ such that*

$$\lim_{N\to\infty} \mathcal{E}\left\{ \sum_{k=0}^{N} \|z_k\|^2 \middle| \varphi, d_{10}^{sc},\ d_{20}^{sc},\ d_0^{ca}, \right\} < \tilde{\Xi}(\varphi, d_{10}^{sc}, d_{20}^{sc}, d_0^{ca}) \tag{3.20}$$

holds, where $\mathcal{E}$ is the statistical expectation operator.

3.4.2 Stability analysis and controller design of NCSs with multiple-packet transmissions

In this subsection, the stability of NCSs with multiple-packet transmission is investigated. Denote

$$\begin{aligned}
\underline{\rho} &= \min\{\rho_{ii},\ i \in S_{11}\}, \qquad \underline{\pi} = \min\{\pi_{pp},\ p \in S_{12}\}, \\
\underline{d_{11}} &= \min\{d_{1k}^{sc} = i,\ k \in \mathbb{Z}\}, \ \ \underline{d_{12}} = \min\{d_{2k}^{sc} = p,\ k \in \mathbb{Z}\}, \\
\mu_1 &= 1 + (1-\underline{\rho})(d_{11} - \underline{d_{11}}), \ \ \mu_2 = 1 + (1-\underline{\pi})(d_{12} - \underline{d_{12}}).
\end{aligned}$$

Then we have the following theorem.

Theorem 3.2 *The system in (3.18) is stochastically stable if there exist $X_{i,p,m} > 0$, $Q_1 > 0$, $Q_2 > 0$, $\Psi_{1,i,m}$ and $\Psi_{2,p,m}$ such that the following LMI*

$$\begin{bmatrix}
-X_{i,p,m} & 0 & 0 & X_{i,p,m}A_m^T\Theta_1 & X_{i,p,m} & X_{i,p,m} \\
* & -Q_1 & 0 & \Psi_{1,i,m}\Theta_1 & 0 & 0 \\
* & * & -Q_2 & \Psi_{2,p,m}\Theta_1 & 0 & 0 \\
* & * & * & -\Omega_1 & 0 & 0 \\
* & * & * & * & -\frac{1}{\mu_1}Q_1 & 0 \\
* & * & * & * & * & -\frac{1}{\mu_2}Q_2
\end{bmatrix} < 0 \tag{3.21}$$

holds for all $i,\ j \in S_{11},\ p,\ q \in S_{12}$ *and* $m,\ n \in S_2$*, where* $*$ *denotes blocks that are readily inferred by symmetry and*

$$\Theta_1 = \underbrace{[\sqrt{\rho_{i1}\pi_{p1}\lambda_{m1}}I,\ \cdots,\ \sqrt{\rho_{id_{11}}\pi_{pd_{12}}\lambda_{md_2}}I]}_{d_{11}\cdot d_{12}\cdot d_2},$$

$$Q_1 = R_1^{-1},\ Q_2 = R_2^{-1},\ \Omega_1 = diag[X_{1,1,1}, \cdots, X_{j,q,n}, \cdots, X_{d_{11},d_{12},d_2}]. \quad (3.22)$$

Here $d_{11}d_{12}d_2$ *is the number of matrices. In this case, the control law is*

$$\begin{aligned} F_{\bar{m}\times r}(d_{1k}^{sc}) &= K\Psi_{1,i,0}^T Q_1^{-1} \begin{bmatrix} \begin{bmatrix} I_r \\ 0_{(\bar{n}-r)\times r} \end{bmatrix} \\ 0 \end{bmatrix}, \\ F_{\bar{m}\times(\bar{n}-r)}(d_{2k}^{sc}) &= K\Psi_{2,p,0}^T Q_2^{-1} \begin{bmatrix} \begin{bmatrix} 0_{r\times(\bar{n}-r)} \\ I_{\bar{n}-r} \end{bmatrix} \\ 0 \end{bmatrix}. \end{aligned} \quad (3.23)$$

Proof: Define a Lyapunov-Krasovskii function as

$$\begin{aligned} V(z(k),k) &= z^T(k)P(d_{1k}^{sc}, d_{2k}^{sc}, d_k^{ca})z(k) + \sum_{\tau=k-d_{1k}^{sc}}^{k-1} z^T(\tau)R_1 z(\tau) \\ &+(1-\underline{\rho}) \sum_{\theta=-d_{11}+1}^{-\underline{d_{11}}} \sum_{\tau=k+\theta}^{k-1} z^T(\tau)R_1 z(\tau) + \sum_{\tau=k-d_{2k}^{sc}}^{k-1} z^T(\tau)R_2 z(\tau) \\ &+(1-\underline{\pi}) \sum_{\theta=-d_{12}+1}^{-\underline{d_{12}}} \sum_{\tau=k+\theta}^{k-1} z^T(\tau)R_2 z(\tau), \end{aligned}$$

where $P(d_{1k}^{sc},\ d_{2k}^{sc},\ d_k^{ca}) = P^T(d_{1k}^{sc},\ d_{2k}^{sc},\ d_k^{ca}) > 0,\ R_1 = R_1^T > 0,\ R_2 = R_2^T > 0$ are to be determined. Then the proof follows a similar procedure as that for the single-packet dropout case; hence is omitted. ■

Remark 3.4 Theorem 3.2 gives a sufficient condition for the stochastic stability of NCSs with packet dropouts, which are described by 3 Markov chains. This method can also be extended to the case of NCSs with both dropouts and networked-induced delays.

3.5 Numerical examples

To demonstrate the effectiveness of the methods presented, two examples are presented. We compare our method with the traditional pole placement method, which shows that the packet dropouts do affect the system stability. For simplicity, we just do this comparison in single-packet transmission case.

Example 3.1: (Single-packet transmission case) Consider the following discrete time system

$$\Phi = \begin{bmatrix} -0.7 & 2 \\ 0 & -1.5 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} -0.03 \\ -1 \end{bmatrix}. \tag{3.24}$$

It is clear that the discrete-time system is unstable.

By pole placement method, we have the controller gain $K = [-0.9684,\ 3.6292]$ (closed-loop poles are $\{0.7,\ 0.7,\ 0\}$), which is designed without considering the packet dropouts. Suppose the initial condition is $z(0) = [0.1,\ 0.1,\ 0]^T$, we have (a) and (b) in Figure 3.2, from which we can see that the system in (3.9) can be stabilized by this pole placement controller when there is no packet dropout. However, this method fails when there are packet dropouts, see (c)-(f) in Figure 3.2; Figure 3.2 (c) and (d) give the changes in packet dropout numbers. $d_k^{sc} = i,\ i \in \{0,\ 1,\ 2\}$ means i packets on the S/C side are dropped during the transmission. Similarly, $d_k^{ca} = m,\ m \in \{0,\ 1\}$ means that m packets on the C/A side are dropped. Figure 3.2 (e) and (f) are the state trajectories of the system in (3.9), by which it is easy to see that packet dropouts destroy the system stability. In fact, if we place the eigenvalues closer to the origin, the state trajectories will converge finally but with poor dynamics, e.g., when the poles are placed at $\{0.3,\ 0.3,\ 0\}$, the state trajectories are oscillatory; when the poles are with magnitude less than 0.3, then the state trajectories start converging but with poor ripples in our example.

By these two sequences, we can calculate their transition probability matrices as

$$\Lambda_1 = \begin{bmatrix} 0.5385 & 0.4615 & 0 \\ 0.8333 & 0 & 0.1667 \\ 1 & 0 & 0 \end{bmatrix}, \quad \Lambda_2 = \begin{bmatrix} 0.75 & 0.25 \\ 1 & 0 \end{bmatrix}.$$

Then by Theorem 3.1 of our method, we have the following controllers

$$F_0 = [-0.0451 \quad -1.5041],\ F_1 = [-0.0902 \quad -3.0081],\ F_2 = [-0.1354 \quad -4.5122].$$

With same initial conditions, we have (g)-(h) in Figure 3.2, by which we know that the system in (3.9) can be stabilized by our designed controller. So our method is more effective.

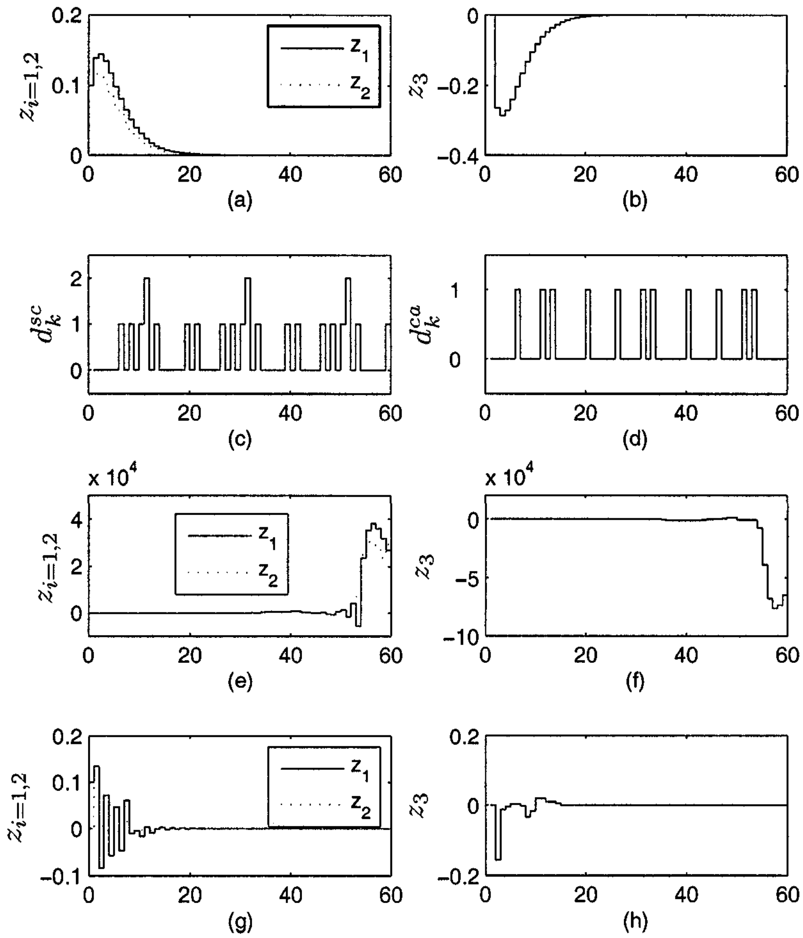

Figure 3.2: The state response with single-packet transmissions (Example 3.1): (a)-(b) with zero packet loss by pole placement method; (c)-(d) packet dropout history; (e)-(f) with packet loss by pole placement method; (g)-(h) with packet loss by our method.

Example 3.2: (Multiple-packet transmission case) This example demonstrate that our method is also feasible for the multi-packet dropout case. The system is

$$\Phi = \begin{bmatrix} -1.43 & 0.1 \\ 0 & -0.3 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} -2.5 \\ -0.15 \end{bmatrix}.$$

For simplicity, we suppose $d_{1k}^{sc} \in \{0,\ 1,\ 2\}$, $d_{2k}^{sc} \in \{0,\ 1\}$, and $d_k^{ca} \in \{0,\ 1\}$. Their

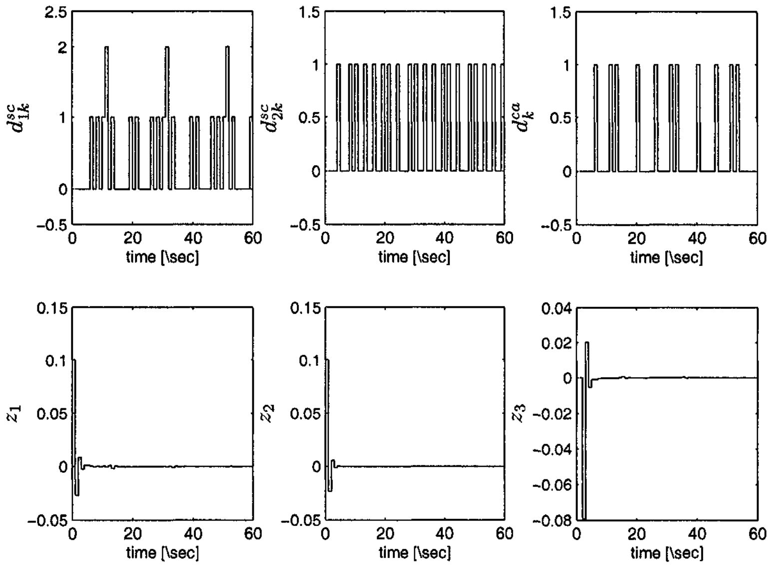

Figure 3.3: The state response with multi-packet transmissions (Example 3.2)

transition probability matrices for d^{sc}_{1k}, d^{sc}_{2k}, d^{ca}_{k} are

$$\Lambda_1,\ \Lambda_1' = \begin{bmatrix} 0.4615 & 0.5385 \\ 1 & 0 \end{bmatrix},\ \Lambda_2,$$

respectively, where Λ_1, Λ_2 are the same as those in Example 1. There are two channels, namely, $X_1 = x_1(k)$, $X_2 = x_2(k)$. Then by Theorem 3.2, we can obtain the control law as

$$F_{10} = -0.3966,\ F_{11} = -0.4231,\ F_{12} = -0.4495,\ F_{20} = -0.0265,\ F_{21} = -0.0280,$$

where F_{1i}, $i = \{0,1,2\}$, belong to the controller set $F_{\bar{m}\times r}(d^{sc}_{1k})$; F_{2p}, $p = \{0,1\}$, belong to the controller set $F_{\bar{m}\times(\bar{n}-r)}(d^{sc}_{2k})$. The state trajectories versus the time is shown in Figure 3.3, which illustrates that the controllers we designed can guarantee the stochastic stability of the NCSs with multiple-packet transmissions.

3.6 Summary

In this chapter, the problem of stability analysis and controller design has been proposed based on a new model of the NCSs with single-packet transmission. In this model, S/C and C/A packet dropout history behaviors are described by two independent Markov chains, the definitions of which are different from those in existing references. Moreover, NCSs with multiple-packet transmission are also investigated. The derived control laws guarantee stochastic stability of the resulting closed-loop systems. Simulation results illustrate the feasibility and effectiveness of our methods.

Chapter 4

Design of NCSs with Network-induced Delays

In Chapter 3, we designed controllers to stabilize the NCSs with packet dropouts, which are one of the main research issues in NCSs. We now proceed to model and study the second main issue in the design of networked control systems: network-induced time delay.

In this chapter, the sensor-to-controller and controller-to-actuator delays are described by Markovian chains, by which the resulting closed-loop NCSs are written as MJLSs with delays. Then we introduce a model predictive control (MPC) strategy to stabilize the resulting closed-loop NCSs. The control scheme is characterized as a constrained delay-dependent optimization problem of the worst-case quadratic cost over an infinite horizon at each sampling instant. A linear matrix inequality approach for the controller synthesis is developed. It is shown that the proposed state feedback model predictive controller guarantees the stochastic stability of the closed-loop system.

The chapter is organized as follows. Section 4.1 presents some relevant prior work and our objective. Section 4.2 introduces the basic setup, closed-loop NCS modeling and the MPC problem formulation. Section 4.3 considers the feasibility of the optimization problem presented in Section 4.2. Section 4.4 discusses the stability of the resulting closed-loop NCS. Section 4.5 provides a numerical example to illustrate the design procedure. Finally, Section 4.6 gives some concluding remarks.

4.1 Introduction

With the development of large-scale or complex industrial systems, communication networks play a more and more important role, by which tremendous amount of information is sensed, processed and transmitted. The data exchanged between these NCS components (sensors, controllers, and actuators) are exposed to stochastic or deterministic delays [113, 111], losses [40, 81, 87], and asynchronization [32, 74], which may degrade performance and even cause instability of the feedback control loops. To solve these problems, various methods and many results have been developed. Network-induced delay, as one of the main issues, has been the focus of attention [36, 47, 54, 69, 103]. In [69], the stability analysis and control design of NCSs were studied when the network-induced delay at each sampling instant is random and less than one sampling time. The results in [69] have been extended to the case with longer delays in [36]. The stability of NCSs was also formulated, respectively, by a hybrid system approach with deterministic delays in [113], by a switched system approach with constant controller gain in [54], and by a jump linear system approach with random delays in [47, 103]. Moreover, some optimization and compensation methods were presented, see [53, 97] and the references therein.

Model predictive control, also known as moving horizon control or receding horizon control, has received much attention in the past decades due to its extensive applications in the control of industrial processes such as distillation and oil fractionation, pulp and paper processing, and so on [75, 76, 77]. Its essence is as follows: at every sampling instant, solutions to an optimization problem over a fixed number of future time instants, known as the time horizon, are obtained; only the first optimal control move is implemented as the current control law; at the next sampling time, the measurement is used to update the state estimate and the same procedure is repeated. This feature renders the MPC approach very appropriate to incorporate the input/output constraints into the on-line optimization as well as compensate time delays, which increases the possibility of its application in the synthesis and analysis of NCSs [29, 37, 60, 90, 102]. In [29], an MPC strategy for multivariable plants was presented. The sensor-to-controller delays were described by stochastic and deterministic quantities, respectively, but controller-to-actuator delays were assumed to be known and fixed; a communication constraint was imposed to restrict that all transmitted data were specified into a region in which the measurements lied and that at any time, only one plant and one actuator were permitted to be addressed. The choice between these alternatives was a function of prior knowledge about the nature of the noise and computational requirements. In [37], two protocols, TCP and UDP, were considered for NCSs with packet losses. The MPC method was used to compensate the packets dropped at the sensor-to-controller side, while zero control was applied when the control packet was lost. In this case, control signal equals to zero when a packet is dropped at the controller-to-actuator side. In [60, 90, 102], modified MPC methods were introduced to compensate the delayed or missing control signals. In [90] both current and future control increment signals were used to update control signals of the plant; an adaptive predictive controller with variable horizon was designed, but no stability was considered. In [60, 102], future control move was chosen from the received control sequences to compensate the delayed control signals. The stability of the system was discussed with the consideration of fixed delay in the sensor-to-controller side and fixed or random delay in the controller-to-actuator side [60] and a constant control gain was designed with the assumption that the delay increases at most 1 at each step [102].

The goal of this chapter is to design a predictive control strategy for an NCS such that at each sampling instant the infinite horizon quadratic objective is minimized while guaranteeing the stochastic stability of the closed-loop system. The actuator will implement most recently received signal directly to the plant and only the first control move will be used. The networked communication delays are assumed to be random and bounded, without loss of generality, which are described by Markovian chains. Delay-dependent conditions for the existence of such controllers are given and an LMI approach is developed. Moreover, a set of corresponding time-varying control laws is presented, which is different from the existing references [60, 90, 102]. A numerical example [45] is given to show the feasibility and efficiency of the proposed method.

4.2 Problem formulation

Consider the networked control setup in Figure 4.1, where the plant is a linear time-

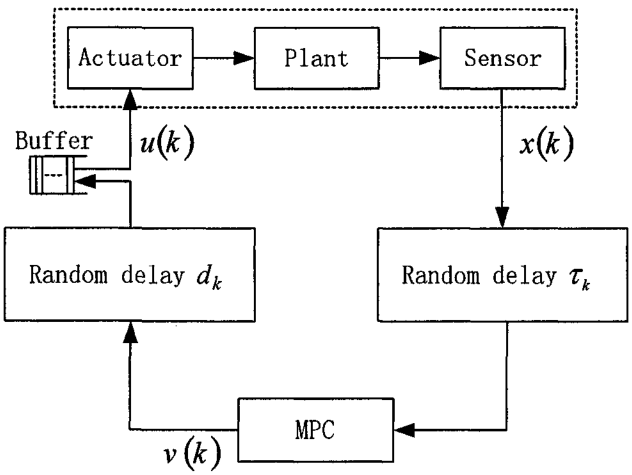

Figure 4.1: The setup of the networked control system

invariant discrete-time system, $\tau_k \geq 0$ is the random time delay from the sensor to the controller, $d_k \geq 0$ is the random time delay from the controller to the actuator, and the controller F_k is to be designed by the MPC method at time instant k. Suppose that the buffer is long enough to hold all the data arrived and that the delay introduced by the buffer can be omitted compared to the network-induced delays. By this assumption, all the past control signals will be available and most recently received control signal will be used by the actuator with the rule of the buffer, namely, first-in-last-out. $v(k)$ is the output of the controller and satisfies

$$v(k) = F_k x(k - \tau_k). \tag{4.1}$$

$u(k)$ is the control input of the plant, which equals to

$$u(k) = v(k - d_k) = F_{k-d_k} x(k - d_k - \tau_{k-d_k}). \tag{4.2}$$

Assume that p is the prediction horizon and q is the control horizon. q control moves $u(k+m|k),\ m = 0, 1, \cdots, q-1$, are computed by minimizing a nominal cost $J_p(k)$ over a prediction horizon p as follows:

$$\min_{u(k+m|k), m=0,1,\cdots,q-1} J_p(k), \tag{4.3}$$

subjecting to some constraints on the control input $u(k+m|k),\ m = 0, 1, \cdots, q-1$ and on the state $x(k+m|k),\ m = 0, 1, \cdots, p$, where $x(k+m|k)$ is the state predicted at time $k+m$ based on the measurements of time k, $x(k|k)$ is the state measured at time k; $u(k+m|k)$ is the corresponding predicted control input at time $k+m$, and $u(k|k)$ is the control move to be implemented at time k.

In this chapter, we consider the case as $p = q = \infty$, which is refereed to as the infinite horizon. Let $\mathcal{F}_k = \sigma\{x_0, \tau_0, d_0, \cdots, x_k, \tau_k, d_k\}$ be the σ-algebra generated by $\{(x_l, \tau_l, d_l), 0 \le l \le k\}$. The quadratic objective is

$$J_\infty(k) = \sum_{m=0}^{\infty} \mathcal{E}\left\{ \left(x(k+m|k)^T Q x(k+m|k) + u(k+m|k)^T R u(k+m|k) \right) |\mathcal{F}_k \right\}, \tag{4.4}$$

where $Q > 0$, $R \ge 0$ are symmetric weighting matrices. It is assumed that both τ_k and d_k are bounded, that is,

$$\underline{\tau} \le \tau_k \le \overline{\tau}, \quad \underline{d} \le d_k \le \overline{d}.$$

Without loss of generality, we assume that $\underline{\tau} = 0$ and $\underline{d} = 0$. Since current time delays are usually correlated with the previous time delays, τ_k and d_k can be modeled by two independent homogeneous Markov chains that take values in $\mathcal{M} = \{0, 1, \ldots, \overline{\tau}\}$ and $\mathcal{N} = \{0, 1, \ldots, \overline{d}\}$, and their transition probability matrices are $\Lambda = [\lambda_{ij}]$ and $\Pi = [\pi_{rs}]$ respectively [69, 103, 111]. That is, τ_k and d_k jump from mode i to j and from mode r to s, respectively, with probabilities λ_{ij} and π_{rs}, which are defined by

$$\lambda_{ij} = \Pr(\tau_{k+1} = j \,|\tau_k = i), \quad \pi_{rs} = \Pr(d_{k+1} = s \,|d_k = r), \tag{4.5}$$

where $\lambda_{ij}, \pi_{rs} \ge 0$ and

$$\sum_{j=0}^{\overline{\tau}} \lambda_{ij} = 1, \quad \sum_{s=0}^{\overline{d}} \pi_{rs} = 1, \tag{4.6}$$

for all $i, j \in \mathcal{M}$ and $r, s \in \mathcal{N}$.

Suppose that the model of the plant is a linear time-invariant discrete-time model as follows

$$x(k+1) = Ax(k) + Bu(k), \tag{4.7}$$

and that the exact measurement of the system state is available at each sampling time k, i.e.,

$$x(k|k) = x(k). \tag{4.8}$$

The controller design scheme can be stated as follows: At each sampling time k,

1. measure the state $x(k)$;

2. compute the state-feedback gain F_k in

$$u(k+m|k) = F_{k-d_{k+m|k}} x(k+m-\tau_{k+m-d_{k+m|k}} - d_{k+m|k}|k) \tag{4.9}$$

such that the performance objective in (4.4) is minimized; $\tau_{k+m-d_{k+m|k}|k}$ is the sensor-to-controller delay, which is an m-step ahead prediction based on the measurement of time k, namely, τ_{k-d_k}, where $d_{k+m|k}$ is the predicted controller-to-actuator delay at time $k+m$;

3. implement the first control move $u(k|k)$, that is,

$$u(k) = u(k|k) = F_{k-d_k} x(k - \tau_{k-d_k} - d_k). \tag{4.10}$$

By the control input given in (4.10), the resulting closed-loop system can be written as

$$x(k+1) = Ax(k) + BF_{k-d_k} x(k - \tau_{k-d_k} - d_k). \tag{4.11}$$

With the modelings of τ_{k-d_k} and d_k as two Markov chains, it can be seen that the system in (4.11) is a Markovian jump linear delay system with two modes, and the time delays are mode-dependent. Furthermore, it can be seen from the 2nd step and the plant model in (4.7) that the predicted state $x(k+m|k)$ satisfies the following difference equation:

$$\begin{aligned} & x(k+m+1|k) \\ = \ & Ax(k+m|k) + Bu(k+m|k) \\ = \ & Ax(k+m|k) + BF_{k-d_{k+m|k}} x(k+m-\tau_{k+m-d_{k+m|k}|k} - d_{k+m|k}|k). \end{aligned} \tag{4.12}$$

The key to solving the MPC problem is to find a way to solve the optimization problem in step 2 at each sampling time k. In the following, we will give the sufficient conditions for the γ−suboptimal problem

$$J_\infty(k) < \gamma \tag{4.13}$$

for a given $\gamma > 0$. At each sampling time, define $X(k+m|k)$ as

$$X(k+m|k) = \begin{bmatrix} x^T(k+m|k) & x^T(k+m-1|k) & \cdots & x^T(k+m-\bar{\tau}-\bar{d}|k) \end{bmatrix}^T.$$

Consider the quadratic function which is given by

$$
\begin{aligned}
& V\left(X(k+m|k), \tau_{k+m-d_{k+m|k}|k}, d_{k+m|k}\right) \\
= & \sum_{t=1}^{4} V_t\left(X(k+m|k), \tau_{k+m-d_{k+m|k}|k}, d_{k+m|k}\right), \qquad (4.14)
\end{aligned}
$$

where

$$
\begin{aligned}
& V_1(X(k+m|k), \tau_{k+m-d_{k+m|k}|k}, d_{k+m|k}) \\
& \quad = x^T(k+m|k) P\left(\tau_{k+m-d_{k+m|k}|k}, d_{k+m|k}\right) x(k+m|k), \\
& V_2(X(k+m|k), \tau_{k+m-d_{k+m|k}|k}, d_{k+m|k}) \\
& \quad = \sum_{\theta=-\tau_{k+m-d_{k+m|k}|k}-d_{k+m|k}+1}^{0} \sum_{l=k+m+\theta-1}^{k+m-1} y^T(l|k) W y(l|k), \\
& V_3(X(k+m|k), \tau_{k+m-d_{k+m|k}|k}, d_{k+m|k}) \\
& \quad = \sum_{l=k+m-\tau_{k+m-d_{k+m|k}|k}-d_{k+m|k}}^{k+m-1} x^T(l|k) S x(l|k), \\
& V_4(X(k+m|k), \tau_{k+m-d_{k+m|k}|k}, d_{k+m|k}) \\
& \quad = (1-\underline{\lambda\pi}) \sum_{\theta=-\overline{\tau}-\overline{d}+1}^{-1} \sum_{l=k+m+\theta}^{k+m-1} \left[x^T(l|k) S x(l|k)\right. \\
& \qquad \left. + y^T(l|k) W y(l|k)(l-k-m-\theta+1)\right],
\end{aligned}
$$

and $y(k+m|k) = x(k+m+1|k) - x(k+m|k)$. $P(\tau_{k+m-d_{k+m|k}|k}, d_{k+m|k})$, W and S are positive definite matrices with appropriate dimensions. At the sampling time k, suppose that the following inequality holds for all $x(k+m|k)$ and $u(k+m|k)$, $m \geq 0$ satisfying (4.12):

$$
\begin{aligned}
& \mathcal{E}\{V(X(k+m+1|k), \tau_{k+m+1-d_{k+m+1|k}|k}, d_{k+m+1|k}) \\
& \quad - V(X(k+m|k), \tau_{k+m-d_{k+m|k}|k}, d_{k+m|k})|\mathcal{F}_k\} \\
\leq & -\mathcal{E}\left\{x(k+m|k)^T Q x(k+m|k) + u(k+m|k)^T R u(k+m|k)|\mathcal{F}_k\right\}. \qquad (4.15)
\end{aligned}
$$

For the control performance $J_\infty(k)$ to be finite, we must have

$$
\mathcal{E}\{V(X(\infty|k), \tau_{\infty|k}, d_{\infty|k})\} = 0.
$$

Thus, from (4.15), we obtain

$$
-V(X(k|k), \tau_{k-d_{k|k}}, d_{k|k}) \leq -J_\infty(k) \qquad (4.16)
$$

Before proceeding, we introduce the following definition and lemma.

Definition 4.1 *[14] The system in (4.11) is stochastically stable if for all finite $x(k) = \varphi$ defined on $[-\bar{\tau} - \bar{d}, 0]$ and initial mode τ_0, d_0, there exists a finite number $\tilde{\Xi}(\varphi, \tau_0, d_0) > 0$ such that*

$$\lim_{N\to\infty} \mathcal{E}\left\{ \sum_{k=0}^{N} \|x(k)\|^2 \middle| \varphi, \tau_0, d_0 \right\} < \tilde{\Xi}(\varphi, \tau_0, d_0) \tag{4.17}$$

holds.

Lemma 4.1 *[27] Let $a \in \Re^{n_a}$, $b \in \Re^{n_b}$ and $M \in \Re^{n_a \times n_b}$. Then, for any matrices $Z \in \Re^{n_a \times n_a}$, $Y \in \Re^{n_a \times n_b}$, and $W \in \Re^{n_b \times n_b}$ satisfying*

$$\begin{bmatrix} Z & Y \\ Y^T & W \end{bmatrix} \geq 0,$$

the following holds

$$-2a^T Mb \leq \begin{bmatrix} a \\ b \end{bmatrix}^T \begin{bmatrix} Z & Y - M \\ Y^T - M^T & W \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}.$$

Theorem 4.1 *Consider the stochastic system in (4.11) and let $x(k|k)$, $x(k-1|k)$, $\ldots$, $x(k - \bar{\tau} - \bar{d}|k)$ be the measured state x at time instant k, $k-1, \cdots, k - \bar{\tau} - \bar{d}$ respectively. Then there exists a state feedback controller in (4.10) such that both (4.13) and (4.15) hold if there exist matrices $P(i,r) > 0$, $P_1(i,r)$, $P_2(i,r)$, $Y(i,r)$ $W > 0$, $S > 0$, $Z(i,r)$ and a scalar $\gamma > 0$ such that the following optimization problem is feasible:*

$$\min \gamma \tag{4.18}$$

subject to

$$\begin{aligned} & x(k|k)^T P(i,r) x(k|k) + \sum_{\theta=-i-r+1}^{0} \sum_{l=k+\theta-1}^{k-1} y(l|k)^T W y(l|k) \\ + & \sum_{l=k-i-r}^{k-1} x^T(l|k) S x(l|k) + (1 - \underline{\lambda\pi}) \sum_{\theta=-\bar{\tau}-\bar{d}+1}^{-1} \sum_{l=k+\theta}^{k-1} \left[x^T(l|k) S x(l|k) \right. \\ + & \left. y^T(l|k) W y(l|k) (l - k - \theta + 1) \right] \leq \gamma. \end{aligned} \tag{4.19}$$

and

$$\begin{bmatrix} Z(i,r) & Y(i,r) \\ * & W \end{bmatrix} \geq 0 \tag{4.20}$$

$$\Theta(i,r)=\begin{bmatrix} \Psi+\begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix} & G^T(i,r)\begin{bmatrix} 0 \\ BF_{k-r} \end{bmatrix}-Y(i,r) \\ * & -S+F_{k-r}^T B^T R B F_{k-r} \end{bmatrix}<0 \tag{4.21}$$

for any $i \in \mathcal{M}$ and $r \in \mathcal{N}$, where $\underline{\lambda} = \min_i \lambda_{ii}$, $\underline{\pi} = \min_r \pi_{rr}$, and

$$\begin{aligned}
\Psi &= \begin{bmatrix} \overline{P}(i,r) - P(i,r) + \rho S & 0 \\ 0 & \overline{P}(i,r) + \mu(i,r) W \end{bmatrix} + (i+r) Z(i,r) \\
&\quad + G^T(i,r)\begin{bmatrix} 0 & I \\ A-I & -I \end{bmatrix} + \begin{bmatrix} 0 & I \\ A-I & -I \end{bmatrix}^T G(i,r) \\
&\quad + \begin{bmatrix} Y(i,r) & 0 \end{bmatrix} + \begin{bmatrix} Y^T(i,r) \\ 0 \end{bmatrix}, \\
\rho &= \left[1 + (1-\underline{\lambda}\underline{\pi})(\overline{\tau}+\overline{d})\right], \quad \overline{P}(i,r) = \sum_{j=0}^{\overline{\tau}} \sum_{s=0}^{\overline{d}} \lambda_{ij}\pi_{rs} P(j,s), \\
\mu(i,r) &= \sum_{j=0}^{\overline{\tau}} \sum_{s=0}^{\overline{d}} \lambda_{ij}\pi_{rs}(j+s) + (1-\underline{\lambda}\underline{\pi}) \frac{(\overline{\tau}+\overline{d}-1)(\overline{\tau}+\overline{d})}{2}.
\end{aligned}$$

Proof: See the Appendix for details. ■

From (4.19)-(4.21), it can be observed that when $r > 0$, F_{k-r} is known, and thus, (4.19)-(4.21) are LMIs. However, when $r = 0$, since F_k is unknown, inequality (4.21) is not linear with unknown variables. In the following theorem, we will give an equivalent LMI condition for (4.21) for any $i \in \mathcal{M}$ and $r \in \mathcal{N}$.

Theorem 4.2 *The matrix inequalities in (4.20)-(4.21) are equivalent to the following LMIs:*

$$\begin{bmatrix} \tilde{Z}_1(i,r) & \tilde{Z}_2(i,r) & 0 \\ * & \tilde{Z}_3(i,r) & \delta(i,r)\widetilde{W} \\ * & * & \widetilde{W} \end{bmatrix} \geq 0 \tag{4.22}$$

$$\begin{bmatrix}
\Theta_{11} & \Theta_{12} & 0 & \Theta_{14} & X_1^T(i,r) & 0 & X(i,r) & X(i,r) \\
* & \Theta_{22} & \Theta_{23} & X_2^T(i,r)T & X_2^T(i,r) & 0 & 0 & 0 \\
* & * & -\widetilde{S} & 0 & 0 & U^T B^T & 0 & 0 \\
* & * & * & \Lambda & 0 & 0 & 0 & 0 \\
* & * & * & * & -\frac{1}{\mu(i,r)}\widetilde{W} & 0 & 0 & 0 \\
* & * & * & * & * & -\widetilde{R} & 0 & 0 \\
* & * & * & * & * & * & -\widetilde{Q} & 0 \\
* & * & * & * & * & * & * & -\frac{1}{\rho}\widetilde{S}
\end{bmatrix} < 0 \tag{4.23}$$

for any $i \in \mathcal{M}$ *and* $r \in \mathcal{N}$, *where* $X(i,r)$, $X_1(i,r)$, $X_2(i,r)$, $\widetilde{W}$, $\widetilde{S}$, U, $\tilde{Z}_1(i,r)$, $\tilde{Z}_2(i,r)$, *and* $\tilde{Z}_3(i,r)$ *are unknown variables, and*

$$\begin{aligned}
\Theta_{11} &= -X(i,r) + (i+r)\tilde{Z}_1(i,r) \\
\Theta_{12} &= -X_1(i,r) + X(i,r)(A - I + \delta(i,r) I)^T + (i+r)\tilde{Z}_2(i,r) \\
\Theta_{14} &= \left[X(i,r) + X_1^T(i,r)\right] T \\
\Theta_{22} &= -X_2(i,r) - X_2^T(i,r) + (i+r)\tilde{Z}_3(i,r) \\
\Theta_{23} &= BU - \delta(i,r)\widetilde{S} \\
U &= F_{k-r}\widetilde{S}, \quad \widetilde{S} = S^{-1}, \quad \widetilde{W} = W^{-1}, \quad \widetilde{R} = R^{-1}, \quad \widetilde{Q} = Q^{-1} \\
T &= \left[\begin{array}{ccccc} \sqrt{\lambda_{i0}\pi_{r0}}I & \cdots & \sqrt{\lambda_{ij}\pi_{rs}}I & \cdots & \sqrt{\lambda_{i\overline{\tau}}\pi_{r\overline{d}}}I \end{array} \right] \\
\Lambda &= diag\left[\begin{array}{ccccc} -X(0,0) & \cdots & -X(j,s) & \cdots & -X(\overline{\tau},\overline{d}) \end{array} \right] \\
Z(i,r) &= \left[\begin{array}{cc} Z_1(i,r) & Z_2(i,r) \\ Z_2^T(i,r) & Z_3(i,r) \end{array} \right] \\
\tilde{Z}(i,r) &= \tilde{G}^T(i,r) Z(i,r) G(i,r) = \left[\begin{array}{cc} \tilde{Z}_1(i,r) & \tilde{Z}_2(i,r) \\ \tilde{Z}_2^T(i,r) & \tilde{Z}_3(i,r) \end{array} \right]
\end{aligned}$$

Moreover, if (4.19), (4.22) and (4.23) are feasible, then

$$F_k = U\widetilde{S}^{-1}.$$

Proof: Pre-multiplying and post-multiplying the matrices diag$\left[\begin{array}{cc} \widetilde{G}^T(i,r) & \widetilde{S} \end{array} \right]$ and diag$\left[\begin{array}{cc} \widetilde{G}(i,r) & \widetilde{S} \end{array} \right]$ to (4.21), where

$$\widetilde{G}(i,r) = \left[\begin{array}{cc} X(i,r) & 0 \\ X_1(i,r) & X_2(i,r) \end{array} \right] = \left[\begin{array}{cc} P^{-1}(i,r) & 0 \\ -P_2^{-1}(i,r)P_1(i,r)P^{-1}(i,r) & P_2^{-1}(i,r) \end{array} \right],$$

then inequality (4.23) can be easily obtained by Schur complement. Here in order to obtain the LMI, it is restricted to the case of $Y(i,r) = \delta(i,r)G^T(i,r)\left[\begin{array}{c} 0 \\ I \end{array} \right]$, where $\delta(i,r)$ is a scalar. Similarly, inequality (4.22) can be derived by pre-multiplying and post-multiplying diag$\left[\begin{array}{cc} \widetilde{G}^T(i,r) & \widetilde{W} \end{array} \right]$ and diag$\left[\begin{array}{cc} \widetilde{G}(i,r) & \widetilde{W} \end{array} \right]$ to (4.20). ■

Note that the optimization problem can be rewritten as minimizing γ subject to (4.19), (4.22) and (4.23) for all $i \in \mathcal{M}$ and $r \in \mathcal{N}$.

If the dimension of system matrix A is small, an augmented MJLS without delays can be used to represent the closed-loop NCS, which can be written as

$$X(k+m+1|k) = (\tilde{A} + \tilde{B}F_{k-d_{k+m|k}}\tilde{E}(\tau_{k+m-d_{k+m|k}|k}, d_{k+m|k})X(k+m|k) \quad (4.24)$$

where

$$\tilde{A} = \begin{bmatrix} A & 0 & \cdots & 0 & 0 \\ I & 0 & \cdots & 0 & 0 \\ 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I & 0 \end{bmatrix} \in \Re^{n(1+\bar{\tau}+\bar{d})\times n(1+\bar{\tau}+\bar{d})}, \ \tilde{B} = \begin{bmatrix} B \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \Re^{n(1+\bar{\tau}+\bar{d})\times n_u},$$

$$\tilde{E}(\tau_{k+m-d_{k+m|k}|k}, d_{k+m|k}) = \begin{bmatrix} 0 & \cdots & 0 & I & 0 & \cdots & 0 \end{bmatrix} \in \Re^{n\times n(1+\bar{\tau}+\bar{d})}$$

and $\tilde{E}(\tau_{k+m-d_{k+m|k}|k}, d_{k+m|k})$ has all elements being zeros except for the $(1+d_{k+m|k}+\tau_{k+m-d_{k+m|k}|k})$th block being identity. Then the quadratic function in (4.14) is simplified to $V_1(X(k+m|k), \tau_{k+m-d_{k+m|k}|k}, d_{k+m|k}, k)$ since no delay exists. Therefore, we have the following theorem:

Theorem 4.3 *Consider the stochastic system in (4.11) and let $x(k|k)$, $x(k-1|k)$, $\ldots$, $x(k-\bar{\tau}-\bar{d}|k)$ be the measured state x at time instant k, $k-1,\cdots,k-\bar{\tau}-\bar{d}$ respectively. Then there exists a state feedback controller in (4.10) such that both (4.13) and (4.15) hold if there exist matrices $\widetilde{X}(i,r)>0$, $\widetilde{U}(i,r)$, and a scalar $\gamma>0$ such that the following optimization problem is feasible:*

$$\min \gamma \tag{4.25}$$

subject to

$$\begin{bmatrix} 1 & X(k|k)^T \\ X(k|k) & \widetilde{X}(i,r) \end{bmatrix} \geq 0 \tag{4.26}$$

and

$$\begin{bmatrix} -\widetilde{X}(i,r) & [\widetilde{X}(i,r)\widetilde{A}^T + \widetilde{U}\widetilde{B}^T]T & \widetilde{X}(i,r)\widetilde{T} & \widetilde{U}^T(i,r)R^{1/2} \\ * & -\widetilde{\Lambda} & 0 & 0 \\ * & * & -\gamma I & 0 \\ * & * & * & -\gamma I \end{bmatrix} < 0 \tag{4.27}$$

for any $i \in \mathcal{M}$, $r \in \mathcal{N}$, where

$$\begin{aligned} \widetilde{X}(i,r) &= \gamma P^{-1}(i,r), \quad \widetilde{T} = \begin{bmatrix} Q^{1/2} & 0_{n\times n(\bar{\tau}+\bar{d})} \end{bmatrix}, \\ \widetilde{\Lambda} &= diag\begin{bmatrix} -\widetilde{X}(0,0) & \cdots & -\widetilde{X}(j,s) & \cdots & -\widetilde{X}(\bar{\tau},\bar{d}) \end{bmatrix}, \\ T &= \begin{bmatrix} \sqrt{\lambda_{i0}\pi_{r0}}I & \cdots & \sqrt{\lambda_{ij}\pi_{rs}}I & \cdots & \sqrt{\lambda_{i\bar{\tau}}\pi_{r\bar{d}}}I \end{bmatrix}. \end{aligned}$$

Moreover, if (4.26) and (4.27) are feasible, then

$$F_{k-r} = \widetilde{U}(i,r)\widetilde{X}^{-1}(i,r)\tilde{E}^T(i,r).$$

Proof: The proof is similar to that in Theorem 4.1. Hence, omitted here. ■

Remark 4.1 *Both Theorem 4.1 and Theorem 4.3 provide the controller design method. Theorem 4.3 is much simpler than Theorem 4.1 in the derivation, while the speed for calculating controller parameters by Theorem 4.3 is much slower than that by Theorem 4.1. The larger the dimension of system matrix A is, the slower the controller parameters are calculated with Theorem 4.3. This is because of the highly increased augmented system matrix in the closed loop.*

For conciseness of the dissertation, the following analysis in this chapter is discussed for Theorem 4.1. All the results are applicable for Theorem 4.3.

4.3 Feasibility analysis

Theorem 4.1 has given the sufficient condition for the existence of the MPC controller at sampling time k. For simplicity, throughout the rest of this section, denote $\tau_{k|k} = \tau_k$, $\tau_{k+1|k+1} = \tau_{k+1}$, $d_{k|k} = d_k$, and $d_{k+1|k+1} = d_{k+1}$.

Theorem 4.4 *If the matrix inequalities in (4.19)-(4.21) are feasible at time k, then they are also feasible for all time $t > k$.*

Proof: Let us assume that the optimization problem in Theorem 4.1 is feasible at the sampling time k. The only LMI in the problem that depends explicitly on the measured state $x(k|k) = x(k)$ of the system is the constraint in (4.19). Inequalities (4.20) and (4.21) can be easily proved by setting the decision variables at time $k+1$ equal to be the optimal values computed at time k since the parameters are independent of the state of $x(k|k)$. Thus, to prove this theorem, we need only to prove that inequality (4.19) is feasible for all the future measured states $X(k+m|k+m) = X(k+m)$, $m \geq 1$. We will show that the theorem holds at the sampling time $k+1$ first.

By $Q > 0$ and $R \geq 0$ in (4.15), we have

$$\begin{aligned}&\mathcal{E}\{V(X(k+m+1|k), \tau_{k+m+1-d_{k+m+1|k}|k}, d_{k+m+1|k})\\&-V(X(k+m|k), \tau_{k+m-d_{k+m|k}|k}, d_{k+m|k})|\mathcal{F}_k\} < 0\end{aligned}$$

if $X(k+m|k) \neq 0$. Iteratively applying this inequality for $m = 0, 1, \cdots$, we obtain

$$\mathcal{E}\{V(X(k+m|k), \tau_{k+m-d_{k+m|k}|k}, d_{k+m|k})|\mathcal{F}_k\} < \mathcal{E}\{V(X(k|k), \tau_{k-d_k}, d_k)|\mathcal{F}_k\}. \quad (4.28)$$

Since the inequalities in (4.19)-(4.21) are feasible at time k, we can easily derive that

$$\mathcal{E}\{V(X(k+m|k), \tau_{k+m-d_{k+m|k}|k}, d_{k+m|k})|\mathcal{F}_k\} < \gamma, \ m \geq 1,$$

for any $\tau_{k+m-d_{k+m|k}|k} \in \mathcal{M}$ and $d_{k+m|k} \in \mathcal{N}$ by the proof of Theorem 4.1. From equations (4.8) and (4.12), it follows that

$$x(k+1|k) = Ax(k) + BF_{k-d_k}x(k - \tau_{k-d_k} - d_k).$$

As a result of (4.11), we obtain that

$$x(k+1|k) = x(k+1|k+1),$$

based on $\mathcal{F}_k$; namely, $X(k+1|k) = X(k+1|k+1)$. Then

$$\mathcal{E}\{V(X(k+1|k+1), \tau_{k+1-d_{k+1|k}|k}, d_{k+1|k})|\mathcal{F}_k\} < \gamma.$$

Therefore

$$\begin{aligned}
& \mathcal{E}\{V(X(k+1|k+1), \tau_{k+1-d_{k+1}}, d_{k+1})|\mathcal{F}_{k+1}\} \\
= \ & \mathcal{E}\{V(X(k+1|k+1), \tau_{k+1-d_{k+1|k}|k}, d_{k+1|k})|\mathcal{F}_{k+1}\} \\
= \ & \mathcal{E}\{\mathcal{E}\{V(X(k+1|k+1), \tau_{k+1-d_{k+1|k}|k}, d_{k+1|k})|\mathcal{F}_k\}|\mathcal{F}_{k+1}\} < \gamma.
\end{aligned}$$

Hence the optimization problem is feasible at time $k+1$. This argument can be continued for times $k+2$, $k+3$, $\cdots$, to complete the proof. ■

4.4 Stability analysis

Now, we are in a position to give our main result on the stability of the model predictive control problem.

Theorem 4.5 *Let $x(k|k)$ be the measured state $x(k)$ at the sampling instant k, and suppose that the optimization problem in Theorem 4.1 is feasible. Then the closed-loop system (4.11) is stochastically stable by the feasible receding horizon state feedback control law in (4.10), which is obtained from Theorem 4.1.*

Proof: To show the stochastic stability of (4.11), we shall show that the Lyapunov function $V(X(k|k), \tau_{k-d_k}, d_k, k)$ is strictly decreasing. Assume that the optimization problems established in Theorem 4.1 are feasible for time instant $k = 0$. Theorem 4.4 then ensures that these optimization problems are also feasible for all $k > 0$. Denoting the optimal solutions at time k and $k+1$ respectively by $P^k(i,r) > 0$, $P_1^k(i,r)$, $P_2^k(i,r)$, $Y^k(i,r)$, $W^k > 0$, $S^k > 0$, $Z^k(i,r)$, $P^{k+1}(i,r) > 0$, $P_1^{k+1}(i,r)$, $P_2^{k+1}(i,r)$, $Y^{k+1}(i,r)$, $W^{k+1} > 0$, $S^{k+1} > 0$, and $Z^{k+1}(i,r)$, we have

$$\begin{aligned} & \mathcal{E}\{V(X(k+1|k+1), \tau_{k+1-d_{k+1}}, d_{k+1})|\mathcal{F}_{k+1}\} \\ \leq \; & \mathcal{E}\{V(X(k+1|k+1), \tau_{k+1-d_{k+1|k}|k}, d_{k+1|k})|\mathcal{F}_k\} \end{aligned} \tag{4.29}$$

This is because $P^{k+1}(i,r) > 0$, $P_1^{k+1}(i,r)$, $P_2^{k+1}(i,r)$, $Y^{k+1}(i,r)$, $W^{k+1} > 0$, $S^{k+1} > 0$, and $Z^{k+1}(i,r)$ are optimal values at time $k+1$, while $P^k(i,r) > 0$, $P_1^k(i,r)$, $P_2^k(i,r)$, $Y^k(i,r)$, $W^k > 0$, $S^k > 0$ and $Z^k(i,r)$, are only feasible at time $k+1$. By (4.15), we have

$$\begin{aligned} & \mathcal{E}\{V(X(k+1|k), \tau_{k+1-d_{k+1|k}|k}, d_{k+1|k}) - V(X(k|k), \tau_{k-d_k}, d_k)|\mathcal{F}_k\} \\ \leq \; & -\mathcal{E}\{x^T(k|k)Qx(k|k) + x^T(k-\tau_{k-d_k}-d_k)F_{k-d_k}^T R F_{k-d_k} x(k-\tau_{k-d_k}-d_k)|\mathcal{F}_k\} \\ \leq \; & -\mathcal{E}\{x^T(k|k)Qx(k|k)|\mathcal{F}_k\} \end{aligned} \tag{4.30}$$

with $m = 0$. Since the measured state $x(k+1|k+1) = x(k+1)$ equals $Ax(k) + BF_{k-d_k}x(k-\tau_{k-d_k}-d_k)$, $x(k+1|k) = x(k+1|k+1)$. Combing the inequalities (4.29) and (4.30), we have

$$\begin{aligned} & \mathcal{E}\{V(X(k+1|k+1), \tau_{k+1-d_{k+1}}, d_{k+1}) - V(X(k|k), \tau_{k-d_k}, d_k)|\mathcal{F}_k\} \\ \leq \; & -\mathcal{E}\{x^T(k|k)Qx(k|k)|\mathcal{F}_k\} \leq -\lambda_{\min}\{Q\}\mathcal{E}\{\|x(k|k)\|^2|\mathcal{F}_k\}. \end{aligned}$$

From the above inequality, if letting $\epsilon = \lambda_{\min}\{Q\} > 0$ (since $Q > 0$), we can see that for any $N \geq 1$,

$$\mathcal{E}\{V(X(N))|\varphi, \tau_0, d_0\} - \mathcal{E}\{V(X(0))|\varphi, \tau_0, d_0\} \leq -\epsilon \; \mathcal{E}\{\sum_{k=0}^{N} \|x(k|k)\|^2|\varphi, \tau_0, d_0\}$$

or

$$\begin{aligned} \mathcal{E}\{\sum_{k=0}^{N} \|x(k|k)\|^2|\varphi, \tau_0, d_0\} \; & \leq \; \frac{1}{\epsilon}\{\mathcal{E}\{V(X(0))|\varphi, \tau_0, d_0\} - \mathcal{E}\{V(X(N))|\varphi, \tau_0, d_0\}\} \\ & \leq \; \frac{1}{\epsilon}\mathcal{E}\{V(X(0))|\varphi, \tau_0, d_0\} \end{aligned}$$

which implies that

$$\lim_{N\to\infty}\mathcal{E}\{\sum_{k=0}^{N}\|x(k)\|^2|\varphi,\tau_0,d_0\}\leq\frac{1}{\epsilon}\mathcal{E}\{V(X(0))|\varphi,\tau_0,d_0\}=\tilde{\Xi}(\varphi,\tau_0,d_0) \qquad (4.31)$$

From Definition 4.1, the stochastic stability is obtained. ■

Remark 4.2 *It is noted that the LMIs given in (4.19)-(4.21) can be solved numerically efficiently by using standard LMI techniques, and no tuning of parameters are involved. The solution derived is general since it is solved under consideration of all the value of* i *and* r *at each time* k. *Therefore, the model predictive control scheme proposed in Theorem 4.1 makes the on-line implementation possible.*

Our MPC formulation of NCSs can also be extended to the case with input/output constraints. Assume that the output of the linear time-invariant system in (4.7) is given by

$$z(k)=Cx(k)$$

Then we have the following input and output variance constraints:

$$\lim_{m\to\infty}\mathcal{E}\{(u(k+m|k)-\mathcal{E}\{u(k+m|k)\})^T(u(k+m|k)-\mathcal{E}\{u(k+m|k)\})|\mathcal{F}_k\}\leq u_{\max},$$
$$\lim_{m\to\infty}\mathcal{E}\{(z(k+m|k)-\mathcal{E}\{z(k+m|k)\})^T(z(k+m|k)-\mathcal{E}\{z(k+m|k)\})|\mathcal{F}_k\}\leq z_{\max},$$

where $z(k+m|k)=Cx(k+m|k)$ is the predicted output, $u_{\max}>0$ and $z_{\max}>0$. They can be rewritten in LMIs as follows:

$$-z_{\max}^2 I+C\widetilde{S}C^T\leq 0. \qquad (4.32)$$

$$\begin{bmatrix} -u_{\max}^2 I & U \\ U^T & -\widetilde{S}\end{bmatrix}\leq 0, \qquad (4.33)$$

For simplicity, the proof is omitted, which is similar to that in [112]. Thus, we have the following corollary.

Corollary 4.1 *Consider the stochastic system in (4.11) and let* $x(k|k)$, $x(k-1|k)$, $\ldots$, $x(k-\bar{\tau}-\bar{d}|k)$ *be the measured state* x *at time instant* k, $k-1,\cdots,k-\bar{\tau}-\bar{d}$ *respectively. Then there exists a state feedback controller in (4.10) such that both (4.13) and (4.15) hold if there exist matrices* $P(i,r)>0$, $P_1(i,r)$, $P_2(i,r)$, $Y(i,r)$

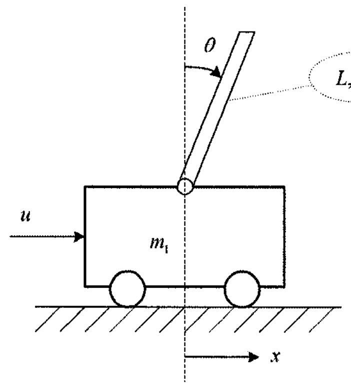

Figure 4.2: Cart and inverted pendulum

$W > 0$, $S > 0$, $Z(i,r)$ and a scalar $\gamma > 0$ such that the following optimization problem holds:

$$\min \gamma$$

subject to (4.19)-(4.21) and (4.32)-(4.33) for all $i \in \mathcal{M}$ and $r \in \mathcal{N}$.

Remark 4.3 *The input and output variance constraints have been often considered for control designs of stochastic systems, see [88] and the reference therein. The variance constraints on the inputs refer to the control energy or power constraints, and the mission requirements can be naturally stated in terms of the variance constraints on the outputs.*

4.5 Numerical examples

In order to demonstrate the proposed model predictive control algorithm, two simple examples will be investigated in this section.

Example 4.1:(Without constraints) Consider the cart and inverted pendulum problem in Figure 4.2, see [111], where m_1 is the cart mass, m_2 is the pendulum mass, L is the length from the point of rotation to the center of gravity of the pendulum, x is the cart position, θ is the pendulum angular position, and u is the input force.

The state variables are

$$x_1 = x,\ x_2 = \dot{x},\ x_3 = \theta,\ x_4 = \dot{\theta}$$

Assume that $m_1 = 1$ kg, $m_2 = 0.5$ kg, $L = 1$ m, and the surface is friction free. The sampling time is $T_s = 0.1$ second, and the random delays exist in $\tau_k \in \{0, 1, 2\}$ and $d_k \in \{0, 1\}$, and their transition probability matrices are given by

$$\Lambda = \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0.3 & 0.6 & 0.1 \\ 0.3 & 0.6 & 0.1 \end{bmatrix}, \; \Pi = \begin{bmatrix} 0.2 & 0.8 \\ 0.5 & 0.5 \end{bmatrix}$$

The controllers are designed using the discretized model, linearized when the pendulum is in the up-position ($\theta = 0$), with a state-space model

$$x(k+1) = Ax(k) + Bu(k)$$

where

$$A = \begin{bmatrix} 1.0000 & 0.1000 & -0.0166 & -0.0005 \\ 0 & 1.0000 & -0.3374 & -0.0166 \\ 0 & 0 & 1.0996 & 0.1033 \\ 0 & 0 & 2.0247 & 1.0996 \end{bmatrix}, \; B = \begin{bmatrix} 0.0045 \\ 0.0896 \\ -0.0068 \\ -0.1377 \end{bmatrix}$$

It is obvious to see that the discretized system is unstable since A has eigenvalues at 1, 1, 1.5569, 0.6423. In [103], this example was considered with the assumption that $d_k = 0$. Comparing our augmented MPC method in Theorem 4.3 with the general linear control method presented in [111], we have the state trajectories of the closed-loop system shown in Figure 4.3, where $x(-3) = x(-2) = x(-1) = x(0) = \begin{bmatrix} 0 & 0 & 0.1 & 0 \end{bmatrix}^T$. The solid lines in Figure 4.3 are the state trajectories by our MPC method. The dash lines are the state trajectories by the method in [111]. By Figure 4.3, we can see that the dynamics by MPC method have better performance and the closed-loop system is stochastically stable.

Example 4.2:(With constraints) Consider a classical angular positioning system in Figure 4.4 [45], which consists of a rotating antenna at the origin of the plane and is driven by an electric motor. The control target is using the motor to rotate the antenna so that it always points to the direction of a moving object in the plane. Assume that the angular position of the antenna θ (rad), the angular position of the moving object θ_r (rad) and the angular velocity of the antenna $\dot{\theta}$ (rad $\cdot\, s^{-1}$) are measurable. The motion of the antenna can be described by the following discrete-time counterparts by discretization, using a sampling time of 0.1s and the Euler's first-order approximation for the derivative

$$x(k+1) = \begin{bmatrix} \theta(k+1) \\ \dot{\theta}(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 0.1 \\ 0 & 1 - 0.1\alpha(k) \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 0.1\kappa \end{bmatrix} u(k), \qquad (4.34)$$

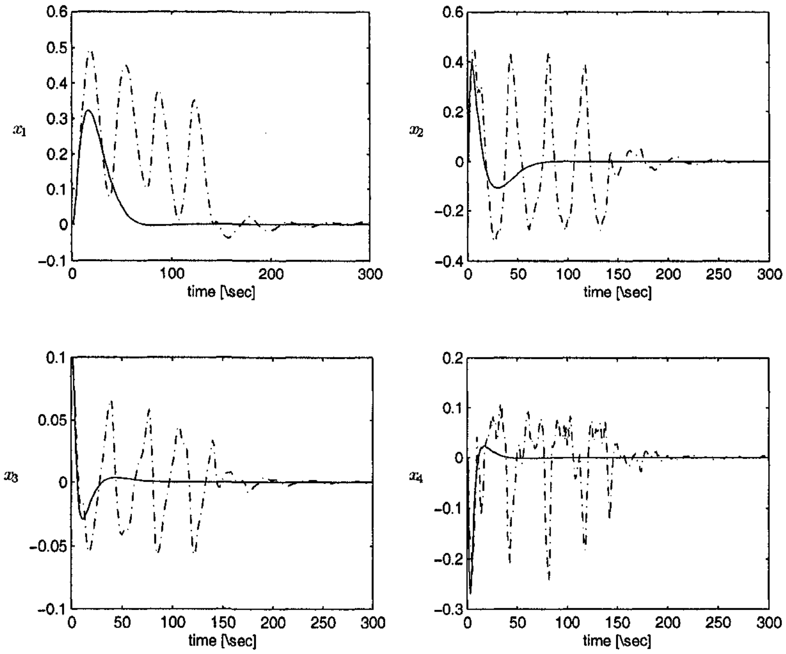

Figure 4.3: The state trajectories of the closed-loop system

where $\kappa = 0.787$ rad$^{-1}V^{-1}s^{-2}$ and $0.1s^{-1} \le \alpha(k) \le 10s^{-1}$. The parameter $\alpha(k)$ is proportional to the coefficient of viscous friction in the rotating parts of the antenna. Assume that $\alpha(k) = 0.1$ and the initially state $x(k) = \begin{bmatrix} 0.01 \\ 0 \end{bmatrix}$, the infinite MPC optimization to be solved at each time k is

$$\min_{u(k+m|k)} J_\infty(k) = \sum_{m=0}^{\infty} \mathcal{E}\{x^T(k+m|k)Qx(k+m|k) + u^T(k+m|k)Ru(k+m|k)|\mathcal{F}_k\}$$

subject to $|u(k+m|k)| \le 0.1V$ with $Q = I_{2\times 2}$ and $R = 0.1$. The system dynamics is shown in Figure 4.5, where Figure 4.5 (a) is the upper bound of the above quadratic function. Figures 4.5 (b) and (c) give the changes of network-induced delays. $\tau_k = i,\ i \in \{0,\ 1,\ 2\}$ means data are delayed by iT_s on the sensor-to-controller side at time k during the transmission, where T_s is the sampling time. Similarly, $d_k = r,\ r \in \{0,\ 1\}$ means that data are delayed by rT_s on the controller-to-actuator side.

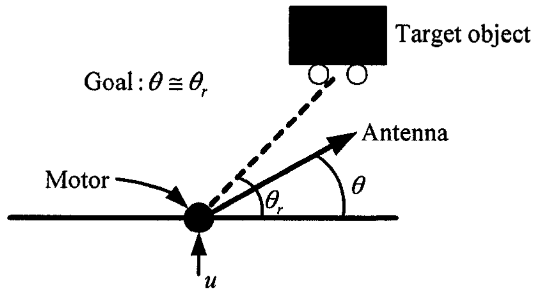

Figure 4.4: The angular positioning system

By these two random serials, we assume their transition probability matrices as

$$\Lambda = \begin{bmatrix} 0.5 & 0.2 & 0.3 \\ 0.4 & 0.5 & 0.1 \\ 0.3 & 0.2 & 0.5 \end{bmatrix}, \quad \Pi = \begin{bmatrix} 0.5 & 0.5 \\ 0.8 & 0.2 \end{bmatrix}.$$

Figure 4.5 (d) is the state trajectories, which show that the closed-loop system is stable by the controller we designed. Moreover, it is easy to see that our designed control sequence, Figure 4.5 (e), lies in the area $[-0.1, 0.1]$.

4.6 Summary

In this chapter, the problem of model predictive control for networked control systems has been studied. Based on the minimization of an upper bound of the worst-case infinite horizon quadratic cost function at each sampling instant, a state feedback predictive controller has been proposed by using the LMI approach. Only the first control move is implemented to the plant. The stochastic delay-dependent stability conditions with and without input/output constraints have been presented respectively for the closed-loop system resulting from the proposed controller. The numerical example shows the effectiveness of our method.

APPENDIX

Proof of Theorem 4.1: Assume that $\tau_{k+m-d_{k+m|k}|k} = i$, $d_{k+m|k} = r$. First, we will show that (4.20) and (4.21) implies (4.15). Following [14, 27] and the methods

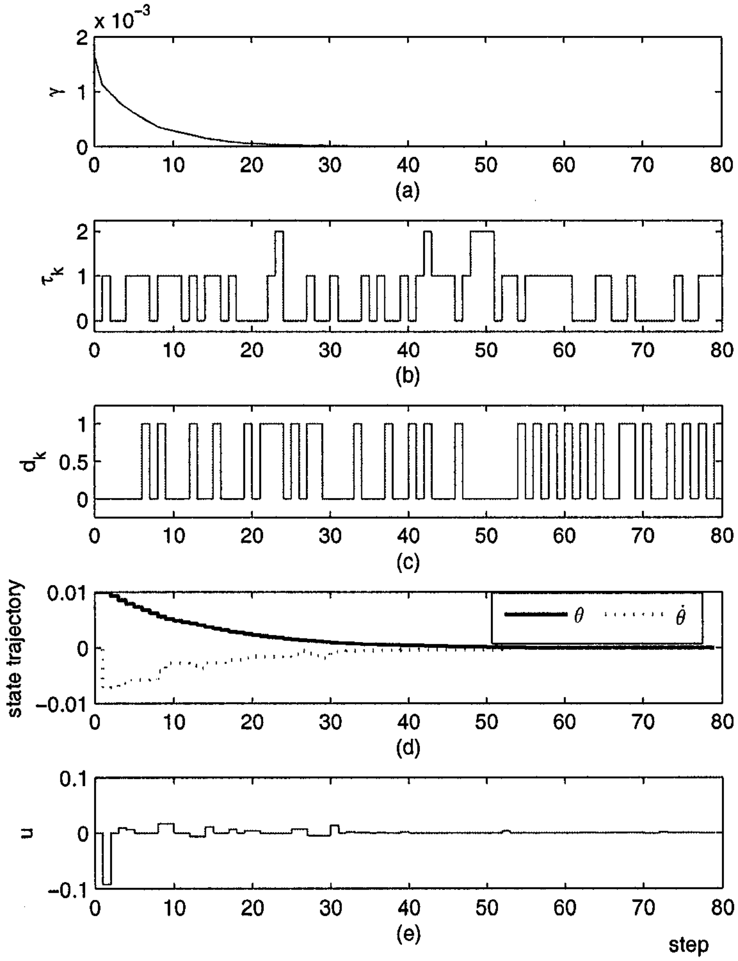

Figure 4.5: The state dynamics of the positioning system

in Chapter 2, we introduce the system transformations:

$$
\begin{aligned}
y(k+m|k) &= x(k+m+1|k) - x(k+m|k), \\
0 &= -y(k+m|k) + (A-I)\,x(k+m|k) \\
&\quad + BF_{k-r}x(k+m-i-r|k), \\
x(k+m-i-r|k) &= x(k+m|k) - \sum_{l=k+m-i-r}^{k+m-1} y(l|k). \qquad (4.35)
\end{aligned}
$$

Notice from (4.12) and (4.35) that

$$\begin{aligned}
&\mathcal{E}\Big\{V_1(X(k+m+1|k),\tau_{k+m+1-d_{k+m+1|k}|k},d_{k+m+1|k}) \\
&\quad -V_1\left(X(k+m|k),\tau_{k+m-d_{k+m|k}|k},d_{k+m|k}\right)|\mathcal{F}_k\Big\} \\
=\;& \mathcal{E}\Big\{\mathcal{E}\Big\{[x(k+m|k)+y(k+m|k)]^T P(\tau_{k+m+1-d_{k+m+1|k}|k},d_{k+m+1|k})[x(k+m|k) \\
&+y(k+m|k)]|\mathcal{F}_k\Big\}-x^T(k+m|k)P(\tau_{k+m-d_{k+m|k}|k},d_{k+m|k})x(k+m|k)|\mathcal{F}_k\Big\} \\
=\;& \mathcal{E}\left\{[x(k+m|k)+y(k+m|k)]^T\overline{P}(i,r)[x(k+m|k)+y(k+m|k)]\right. \\
&\left.-x^T(k+m|k)P(i,r)x(k+m|k)|\mathcal{F}_k\right\} \\
=\;& \mathcal{E}\Big\{x^T(k+m|k)\left[\overline{P}(i,r)-P(i,r)\right]x(k+m|k)+y^T(k+m|k)\overline{P}(i,r) \\
&\times y(k+m|k)+2\eta^T(k+m|k)G^T(i,r)\begin{bmatrix} y(k+m|k) \\ 0\end{bmatrix}\Big|\mathcal{F}_k\Big\},
\end{aligned}$$

where

$$\begin{aligned}
\overline{P}(i,r) &= \sum_{j=0}^{\overline{\tau}}\sum_{s=0}^{\overline{d}}\lambda_{ij}\pi_{rs}P(j,s),\ \eta(k+m|k)=\begin{bmatrix} x(k+m|k) \\ y(k+m|k)\end{bmatrix}, \\
G(i,r) &= \begin{bmatrix} \overline{P}(i,r) & 0 \\ P_1(i,r) & P_2(i,r)\end{bmatrix},
\end{aligned}$$

$P_1(i,r)$ and $P_2(i,r)$ are constant matrices with appropriate dimensions. Thus, by the relation in (4.20), (4.35) and Lemma 4.1, we have

$$\begin{aligned}
&\mathcal{E}\left\{2\eta^T(k+m|k)G^T(i,r)\begin{bmatrix} y(k+m|k) \\ 0\end{bmatrix}\Big|\mathcal{F}_k\right\} \\
\leq\;& \mathcal{E}\Big\{\eta^T(k+m|k)\Big\{G^T(i,r)\begin{bmatrix} 0 & I \\ A-I & -I\end{bmatrix}+\begin{bmatrix} 0 & I \\ A-I & -I\end{bmatrix}^T G(i,r) \\
&+\begin{bmatrix} Y(i,r) & 0\end{bmatrix}+\begin{bmatrix} Y^T(i,r) \\ 0\end{bmatrix}\Big\}\eta(k+m|k)+\sum_{l=k+m-i-r}^{k+m-1}y^T(l|k)Wy(l|k) \\
&+(i+r)\eta^T(k+m|k)Z(i,r)\eta(k+m|k)+2\eta^T(k+m|k) \\
&\times\left(-Y(i,r)+G^T(i,r)\begin{bmatrix} 0 \\ BF_{k-r}\end{bmatrix}\right)x(k+m-i-r|k)\Big|\mathcal{F}_k\Big\}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\mathcal{E}\Big\{V_1\left(X(k+m+1|k),\tau_{k+m+1-d_{k+m+1|k}|k},d_{k+m+1|k}\right) \\
&\quad -V_1\left(X(k+m|k),\tau_{k+m-d_{k+m|k}|k},d_{k+m|k}\right)\Big|\mathcal{F}_k\Big\} \\
\leq\;& \mathcal{E}\Big\{\eta^T(k+m|k)\Big\{\begin{bmatrix} \overline{P}(i,r)-P(i,r) & 0 \\ 0 & \overline{P}(i,r)\end{bmatrix}+G^T(i,r)\begin{bmatrix} 0 & I \\ A-I & -I\end{bmatrix}
\end{aligned}$$

$$+\left[\begin{array}{cc} 0 & I \\ A-I & -I \end{array}\right]^T G(i,r)+\left[\begin{array}{cc} Y(i,r) & 0 \end{array}\right]+\left[\begin{array}{c} Y^T(i,r) \\ 0 \end{array}\right]\Bigg\}\eta(k+m|k)$$

$$+(i+r)\eta^T(k+m|k)Z(i,r)\eta(k+m|k)+\sum_{l=k+m-i-r}^{k+m-1} y^T(l|k)Wy(l|k)$$

$$+2\eta^T(k+m|k)\left(-Y(i,r)+G^T(i,r)\left[\begin{array}{c} 0 \\ BF_{k-r} \end{array}\right]\right)x(k+m-i-r|k)\Bigg|\mathcal{F}_k\Bigg\}. \tag{4.36}$$

For $V_2\left(X(k+m|k),\tau_{k+m-d_{k+m|k}|k},d_{k+m|k}\right)$, we have

$$\begin{aligned}
&\mathcal{E}\Big\{V_2\left(X(k+m+1|k),\tau_{k+m+1-d_{k+m+1|k}|k},d_{k+m+1|k}\right) \\
&\quad -V_2\left(X(k+m|k),\tau_{k+m-d_{k+m|k}|k},d_{k+m|k}\right)|\mathcal{F}_k\Big\} \\
=\ &\mathcal{E}\Big\{\mathcal{E}\Big\{V_2(X(k+m+1|k),\tau_{k+m+1-d_{k+m+1|k}|k},d_{k+m+1|k})|\mathcal{F}_k\Big\} \\
&\quad -V_2(X(k+m|k),\tau_{k+m-d_{k+m|k}|k},d_{k+m|k})\Big|\mathcal{F}_k\Big\} \\
=\ &\mathcal{E}\left\{\sum_{j=0}^{\bar{\tau}}\sum_{s=0}^{\bar{d}}\lambda_{ij}\pi_{rs}\left[\sum_{\theta=-j-s+1}^{0}\ \sum_{l=k+m+\theta}^{k+m} y^T(l|k)Wy(l|k)\right.\right. \\
&\quad \left.\left.-\sum_{\theta=-i-r+1}^{0}\ \sum_{l=k+m+\theta-1}^{k+m-1} y^T(l|k)Wy(l|k)\right]\Bigg|\mathcal{F}_k\right\} \\
=\ &\mathcal{E}\left\{\sum_{j=0}^{\bar{\tau}}\sum_{s=0}^{\bar{d}}\lambda_{ij}\pi_{rs}(j+s)y^T(k+m|k)Wy(k+m|k)\right. \\
&\quad +\left(\lambda_{ii}\sum_{s=0,s\neq r}^{\bar{d}}\pi_{rs}+\pi_{rr}\sum_{j=0,j\neq i}^{\bar{\tau}}\lambda_{ij}+\sum_{s=0,s\neq r}^{\bar{d}}\ \sum_{j=0,j\neq i}^{\bar{\tau}}\lambda_{ij}\pi_{rs}\right) \\
&\quad \times\left[\sum_{\theta=-j-s+1}^{0}\ \sum_{l=k+m+\theta}^{k+m-1} y^T(l|k)Wy(l|k)\right. \\
&\quad \left.\left.-\sum_{\theta=-i-r+1}^{0}\sum_{l=k+m+\theta}^{k+m-1} y^T(l|k)Wy(l|k)\right]-\sum_{l=k+m-i-r}^{k+m-1} y^T(l|k)Wy(l|k)\Bigg|\mathcal{F}_k\right\}.
\end{aligned}$$

Notice that

$$\begin{aligned}
&\mathcal{E}\left\{\sum_{\theta=-j-s+1}^{0}\sum_{l=k+m+\theta}^{k+m-1} y^T(l|k)Wy(l|k)-\sum_{\theta=-i-r+1}^{0}\sum_{l=k+m+\theta}^{k+m-1} y^T(l|k)Wy(l|k)|\mathcal{F}_k\right\} \\
\leq\ &\mathcal{E}\left\{\sum_{\theta=-\bar{\tau}-\bar{d}+1}^{-1}\sum_{l=k+m+\theta}^{k+m-1} y^T(l|k)Wy(l|k)|\mathcal{F}_k\right\}.
\end{aligned}$$

Thus, it can be seen that

$$\begin{aligned}
&\mathcal{E}\Big\{V_2\Big(X(k+m+1|k),\tau_{k+m+1-d_{k+m+1|k}|k},d_{k+m+1|k}\Big) \\
&\quad -V_2\Big(X(k+m|k),\tau_{k+m-d_{k+m|k}|k},d_{k+m|k}\Big)\Big|\,\mathcal{F}_k\Big\} \\
\leq\ &\mathcal{E}\left\{\sum_{j=0}^{\overline{\tau}}\sum_{s=0}^{\overline{d}}\lambda_{ij}\pi_{rs}\,(j+s)\,y^T(k+m|k)\,Wy\,(k+m|k)\right. \\
&+(1-\underline{\lambda\pi})\sum_{\theta=-\overline{\tau}-\overline{d}+1}^{-1}\ \sum_{l=k+m+\theta}^{k+m-1}y^T(l|k)\,Wy\,(l|k) \\
&\left.-\sum_{l=k+m-i-r}^{k+m-1}y^T(l|k)\,Wy\,(l|k)\right|\mathcal{F}_k\Bigg\}
\end{aligned}\tag{4.37}$$

since $1-\lambda_{ii}\pi_{rr}\leq 1-\underline{\lambda\pi}$.

Similarly, for $V_3\Big(X(k+m|k),\tau_{k+m-d_{k+m|k}|k},d_{k+m|k}\Big)$, we have

$$\begin{aligned}
&\mathcal{E}\Big\{V_3\Big(X(k+m+1|k),\tau_{k+m+1-d_{k+m+1|k}|k},d_{k+m+1|k}\Big) \\
&\quad -V_3\Big(X(k+m|k),\tau_{k+m-d_{k+m|k}|k},d_{k+m|k}\Big)\Big|\,\mathcal{F}_k\Big\} \\
=\ &\mathcal{E}\left\{\sum_{j=0}^{\overline{\tau}}\sum_{s=0}^{\overline{d}}\lambda_{ij}\pi_{rs}\sum_{l=k+m+1-j-s}^{k+m}x^T(l|k)\,Sx\,(l|k)\right. \\
&\left.-\sum_{l=k+m-i-r}^{k+m-1}x^T(l|k)\,Sx\,(l|k)\right|\mathcal{F}_k\Bigg\} \\
\leq\ &\mathcal{E}\Big\{x^T(k+m|k)\,Sx\,(k+m|k)-x^T(k+m-i-r|k)\,Sx\,(k+m-i-r|k) \\
&\left.+(1-\underline{\lambda\pi})\sum_{l=k+m+1-\overline{\tau}-\overline{d}}^{k+m}x^T(l|k)\,Sx\,(l|k)\right|\mathcal{F}_k\Bigg\}.
\end{aligned}\tag{4.38}$$

For $V_4\Big(X(k+m|k),\tau_{k+m-d_{k+m|k}|k},d_{k+m|k}\Big)$, we have

$$\begin{aligned}
&\mathcal{E}\Big\{V_4\Big(X(k+m+1|k),\tau_{k+m+1-d_{k+m+1|k}|k},d_{k+m+1|k}\Big) \\
&\quad -V_4\Big(X(k+m|k),\tau_{k+m-d_{k+m|k}|k},d_{k+m|k}\Big)\Big|\,\mathcal{F}_k\Big\} \\
=\ &\mathcal{E}\Bigg\{(1-\underline{\lambda\pi})\sum_{\theta=-\overline{\tau}-\overline{d}+1}^{-1}\ \sum_{l=k+m+1+\theta}^{k+m}\big[y^T(l|k)Wy(l|k)(l-k-m-\theta) \\
&+x^T(l|k)Sx(l|k)\big]-(1-\underline{\lambda\pi})\sum_{\theta=-\overline{\tau}-\overline{d}+1}^{-1}\ \sum_{l=k+m+\theta}^{k+m-1}\big[x^T(l|k)Sx(l|k) \\
&+y^T(l|k)Wy(l|k)(l-k-m-\theta+1)\big]\,\Big|\,\mathcal{F}_k\Big\}
\end{aligned}$$

$$
\begin{aligned}
= \quad & \mathcal{E}\left\{(1-\underline{\lambda\pi})\frac{\left(\overline{\tau}+\overline{d}\right)\left(\overline{\tau}+\overline{d}-1\right)}{2}y^T\left(k+m|k\right)Wy\left(k+m|k\right)\right. \\
& +(1-\underline{\lambda\pi})\left(\overline{\tau}+\overline{d}-1\right)x^T\left(k+m|k\right)Sx\left(k+m|k\right) \\
& -(1-\underline{\lambda\pi})\sum_{\theta=-\overline{\tau}-\overline{d}+1}^{-1}\sum_{l=k+m+\theta}^{k+m-1}y^T\left(l|k\right)Wy\left(l|k\right) \\
& \left.-(1-\underline{\lambda\pi})\sum_{l=k+m-\overline{\tau}-\overline{d}+1}^{k+m-1}x^T\left(l|k\right)Sx(l|k)\right|\mathcal{F}_k\Bigg\}. \qquad (4.39)
\end{aligned}
$$

Combining (4.36), (4.37), (4.38) and (4.39) together, we obtain

$$
\begin{aligned}
& \mathcal{E}\Big\{V\left(X(k+m+1|k),\tau_{k+m+1-d_{k+m+1|k}|k},d_{k+m+1|k}\right) \\
& \quad -V\left(X(k+m|k),\tau_{k+m-d_{k+m|k}|k},d_{k+m|k}\right)\Big|\mathcal{F}_k\Big\} \\
\leq \quad & \mathcal{E}\Big\{\eta^T(k+m|k)\Psi\eta(k+m|k)+2\eta^T(k+m|k)\left(G^T(i,r)\begin{bmatrix}0\\ BF_{k-r}\end{bmatrix}-Y\left(i,r\right)\right) \\
& \times x\left(k+m-i-r|k\right)-x^T\left(k+m-i-r|k\right)Sx\left(k+m-i-r|k\right)|\mathcal{F}_k\Big\} \\
= \quad & \mathcal{E}\left\{\xi\left(k+m|k\right)^T\begin{bmatrix}\Psi & G^T\left(i,r\right)\begin{bmatrix}0\\ BF_{k-r}\end{bmatrix}-Y\left(i,r\right)\\ * & -S\end{bmatrix}\xi\left(k+m|k\right)\right|\mathcal{F}_k\Bigg\}, \qquad (4.40)
\end{aligned}
$$

where

$$
\xi\left(k+m|k\right) = \begin{bmatrix}\eta\left(k+m|k\right)\\ x\left(k+m-i-r|k\right)\end{bmatrix}.
$$

As a result, it can be seen that

$$
\begin{aligned}
& \mathcal{E}\Big\{V\left(X(k+m+1|k),\tau_{k+m+1-d_{k+m+1|k}|k},d_{k+m+1|k}\right) \\
& \quad -V\left(X(k+m|k),\tau_{k+m-d_{k+m|k}|k},d_{k+m|k}\right)|\mathcal{F}_k\Big\} \\
& \quad +\mathcal{E}\left\{x^T(k+m|k)Qx(k+m|k)+u^T(k+m|k)Ru(k+m|k)|\mathcal{F}_k\right\} \\
\leq \quad & \mathcal{E}\left\{\xi^T(k+m|k)\Theta(i,r)\xi(k+m|k)|\mathcal{F}_k\right\}.
\end{aligned}
$$

On the other hand, from (4.16), it follows that (4.13) holds if

$$
V(X(k|k),\tau_{k|k},d_{k|k})\leq\gamma,
$$

that is,

$$
\begin{aligned}
& x(k|k)^T P(\tau_{k-d_k}, d_k) x(k|k) + \sum_{\theta=-\tau_{k-d_k}-d_k+1}^{0} \sum_{l=k+\theta-1}^{k-1} y(l|k)^T W y(l|k) \\
+ & \sum_{l=k-\tau_{k-d_k}-d_k}^{k-1} x^T(l|k) S x(l|k) + (1-\underline{\lambda\pi}) \sum_{\theta=-\overline{\tau}-\overline{d}+1}^{-1} \sum_{l=k+\theta}^{k-1} \big[y^T(l|k) W y(l|k) \\
\times & (l-k-\theta+1) + x^T(l|k) S x(l|k) \big] \leq \gamma.
\end{aligned}
$$

Since (4.19) holds for $i \in \mathcal{M}$ and $r \in \mathcal{N}$, the proof is obtained. ■

Chapter 5

Control Design with Variable Sampling Periods

In Chapter 3 and Chapter 4, We have discussed the stabilizability problem for NCSs with packet dropouts and network-induced time delay respectively, where the results have been derived under the assumption that sensor nodes are sampled regularly at a constant sampling period h. However, in some situations, the sampling period can not be predetermined or may change according to certain system variables, which are time-varying. Therefore, the study of NCSs with variable sampling periods is of theoretical and practical importance.

The chapter is organized as follows. Section 5.1 summarizes several reasons for considering variable sampling period, applications of systems with variable sampling period, and the relevant prior work. Section 5.2 introduces the basic setup and the MPC problem formulation for a class of sampled-data (SD) systems. Section 5.3 presents the stabilization problem for the resulting closed-loop system. Section 5.4 discusses the tracking performance of our SD system with the corresponding control law. Section 5.5 extends the method derived to NCSs with fixed network-induced time delays. Section 5.6 provides the CSTR simulation to illustrate the design procedure. Finally, Section 5.7 gives some concluding remarks.

5.1 Introduction

Sampled-data (SD) control systems are hybrid dynamical systems which usually consist of a continuous-time plant and a digital controller, along with appropriate interface elements (A/D and D/A converters). The study of SD systems is strongly motivated by the fact that modern controllers are typically implemented digitally. Over the past two decades, numerous results have been derived for controller design of SD systems, using three main approaches. The first one is to design a continuous-time controller for the continuous-time plant and discretize the obtained controller for a digital implementation. The second one is to discretize the continuous-time plant and do a discrete-time design. The last one is to design a discrete-time controller directly for the continuous-time plant. These approaches provide sufficient stability conditions for SD systems in the single-rate case [2, 13, 24] and the multi-rate case [35, 63]. It should be pointed out that all the main results of the aforementioned references have been presented under the assumption of a constant or periodic sampling period.

Sampled-data systems with varying sampling rates arise for several reasons:

- the optimal allocation of limited computing resources, namely, central processing unit (CPU) time and communication bandwidth; for example, several control loops share the same CPU of an embedded control system in [21]. When the execution times of all tasks, the number of tasks, or the desired CPU utilization (workload) vary over time, the feedback scheduler will adjust the control loop sampling frequencies to optimize the total control performance while keeping the workload at the desired level. This leads to variations in sampling periods.

- the situation that the sampling rate depends on certain system variables; for instant, the time between consecutive measurements in a brushless DC motor [107] was dependent on the motor speed. That is, the sampling time is velocity dependent and not predetermined. Similar applications are frequently encountered in industrial applications such as the computer hard disk drive [38] and the CD-ROM servo systems [41].

- the global stability of some algorithms, which is guaranteed by using the sampling rate as an extra control variable; for example, digital versions of globally stable adaptive stabilization algorithms are, at best, locally stable due to the incompatibilities between the gain adaptation algorithms and the choice of sampling rates. This fact suggested a wide range of sample interval adaption schemes for stabilizing a single-input-single-output (SISO) system [71].

Therefore, the study of systems with variable sampling rates has begun to gain more and more attention [34, 79, 80, 89]. In [34], the stability and instability of a digital control system were studied for the fixed and non-fixed sampling points, respectively. No general control design method was proposed except a specific controller was given for a specific switched system in the example. In [79], the controller was designed by an LMI gridding approach, which needed a posteriori negative definite checking to guarantee the decay-rate analysis for a whole interval of sampling rates, especially in the case when the grid points were close enough. If the condition was not satisfied, a finer grid sequence was needed to redesign the controller. This means that the fulfillment of the LMIs at the grid points does not give any guarantee that they are also satisfied for all sampling rates, which leads to conservativeness. In [80], a piecewise constant control law was given to stabilize the resulting closed-loop SD system under all possible switching sequences of sampling rates, where the system performance was minimized only by one-step ahead prediction. In [89], three types of controllers were introduced for SD systems with variable sampling period to achieve a certain level of disturbance rejection.

In this chapter, we will consider the stabilization problem for SD systems with variable sampling rates. Different from the existing methods in SD systems with time-varying sampling, we develop a predictive control strategy to stabilize the SD system by a modified MPC approach. At each sampling instant, a finite horizon quadratic objective function with terminal weighting matrix is minimized and an LMI approach for solving the terminal weighting matrix is presented. The optimal predicted control sequence guarantees the stability of the closed-loop system. Our design can incorporate the input/output constraints into the online optimization. Moreover, the signals both at and between sampling instants are considered in the controller design. No posteriori checking is needed. Simulations on a CSTR system

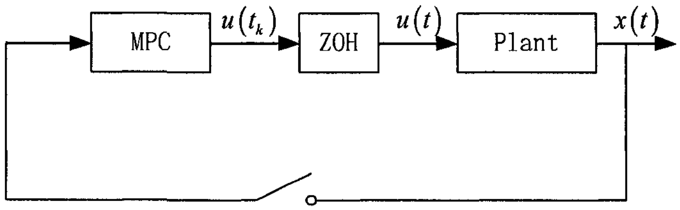

Figure 5.1: Schematic diagram of the sampled-data control system

show the feasibility and efficiency of the proposed method.

5.2 Problem formulation

Consider the sampled-data control setup in Figure 5.1. The continuous time-invariant plant we considered in this chapter is

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{5.1}$$

where $x(t) \in \Re^n$ is the state, $u(t) \in \Re^m$ is the control input, and A, B are system matrices with appropriate dimensions. The sampling instants are $t_k \in \Re,\ k \in \mathcal{N}$ with $t_{k+1} > t_k$, $t_0 = 0$. Then we have the sampling period as $h_k = t_{k+1} - t_k$, which is obviously time-varying. Discretizing the plant at the sampling instants t_k, we get a linear time-varying discrete system as follows:

$$x(t_{k+1}) = e^{Ah_k}x(t_k) + \int_{t_k}^{t_{k+1}} e^{A(t_{k+1}-\sigma)}Bu(\sigma)d\sigma. \tag{5.2}$$

The objective in this chapter is designing a digital state feedback control law to stabilize the system in (5.2), where the controller at time instant t_k depends on the sampling rate h_k, and will be designed by a modified MPC method.

The following assumptions are made throughout the remainder of the chapter:

Assumption 5.1 $\underline{h} \leq h_k = t_{k+1} - t_k \leq \overline{h},\quad \forall k \in \mathcal{N}.$

Assumption 5.2 *Let the period of predictions be* η *and* $h_k = n_k\eta,\ \forall k \in \mathcal{N},\ n_k \geq 1,\ n_k \in \mathcal{N}.$

Remark 5.1 *Assumption 5.1 means that the sampling period does not exceed given bounds, and time-varying. Therefore, the sequence of our sampling rates is less* *conservative than that in [71], where the sequence of sampling rates was assumed to be non-increasing.*

Remark 5.2 *Note that in Assumption 5.2, we use η as the predictive period instead of the real sampling period h_k at time t_k. This is to simplify the design and implementation of the MPC strategy since h_k is unknown at time t_k. Another advantage is that not only the signals at sampling instants, but also the intersampling signals are taken into account on the design. Moreover, by this definition, the sampler works at the period h_k while the zero-order-hold (ZOH) works at the period η, which means our sampler and ZOH are asynchronized.*

Remark 5.3 *The predictive period η always exists since we can find a common divisor for all h_k, $k \in \mathcal{N}$. At least, we can choose the internal clock time increment as η. The smaller the η is, the more the computational burden will be.*

By Assumption 5.1 and Assumption 5.2, the sequence of sampling rates $[h_0,\ h_1,\ \cdots,\ h_k,\ \cdots]$ can be represented as $[n_0\eta, n_1\eta, \cdots, n_k\eta, \cdots]$, where $n_i \geq 1, i \in \mathcal{N}$, are arbitrary positive integers and lie in the set $[a,\ b]$ with $a = \frac{h}{\eta}$ and $b = \frac{\bar{h}}{\eta}$. Assume that p is the prediction horizon, q is the control horizon, and $b \leq p$. q control moves $u(t_k + i\eta|t_k),\ i = 0, 1, \cdots, q-1$, are computed by minimizing a nominal cost $J(x_{t_k}, t_k)$ over a prediction horizon p as follows:

$$\begin{aligned}\min_{u(t_k+i\eta|t_k),\ i=0,1,\cdots,q-1} J(x_{t_k}, t_k) &= \sum_{i=0}^{q-1} \|u(t_k + i\eta|t_k)\|_R^2 + \sum_{i=0}^{p-1} \|x(t_k + i\eta|t_k)\|_Q^2 \\ &+ \|x(t_k + p\eta|t_k)\|_P^2, \end{aligned} \tag{5.3}$$

subject to constraints on the control input $u(t_k + i\eta|t_k),\ i = 0, 1, \cdots, q-1$, and on the state $x(t_k + i\eta|t_k),\ i = 0, 1, \cdots, p$, where $x(t_k + i\eta|t_k)$ is the state predicted at time $t_k + i\eta$ based on the measurements of time t_k, $x(t_k|t_k)$ is the state measured at time t_k; $u(t_k + i\eta|t_k)$ is the corresponding predicted control input at time $t_k + i\eta$, and $u(t_k|t_k)$ is the control move to be implemented at time t_k. $R = R^T > 0$ and $Q = Q^T > 0$ are weighting matrices, $P = P^T > 0$ is the terminal weighting. Besides, we have the terminal constraints

$$u(t_k + i\eta|t_k) = u(t_k + (q-1)\eta|t_k),\ q \leq i \leq p. \tag{5.4}$$

Assume that the exact measurement of the system state is available at each sampling instant t_k, i.e.,

$$x(t_k|t_k) = x(t_k), \tag{5.5}$$

we will have the following predictive equation:

$$x(t_k + (i+1)\eta|t_k) = A_d x(t_k + i\eta|t_k) + B_d u(t_k + i\eta|t_k), \quad i \in \{0, 1, \cdots, p-1\}, \tag{5.6}$$

where

$$A_d = e^{A\eta}, \; B_d = \int_0^\eta e^{A\sigma} d\sigma B.$$

By the discrete-time representation in (5.2) and the predictive model in (5.6), we can easily obtain the predictive state at time t_{k+1} based on the measurements at time t_k as follows:

$$x(t_{k+1}|t_k) = x(t_k + n_k\eta|t_k) = \Phi(n_k)x(t_k|t_k) + \Gamma(n_k)U(t_k), \tag{5.7}$$

where

$$\begin{aligned} \Phi(n_k) &= (A_d)^{n_k}, \;\; \Xi = A_d^{n_k-q}B_d + A_d^{n_k-q-1}B_d + \cdots + B_d, \\ \Gamma(n_k) &= \begin{cases} [\; A_d^{n_k-1}B_d \;\; A_d^{n_k-2}B_d \;\; \cdots \;\; B_d \;\; 0_{n\times m} \;\; \cdots \;\; 0_{n\times m} \;]_{n\times mq}, \; n_k \le q, \\ [\; A_d^{n_k-1}B_d \;\; A_d^{n_k-2}B_d \;\; \cdots \;\; A_d^{n_k-q+1}B_d \;\; \Xi \;]_{n\times mq}, \; n_k > q. \end{cases} \end{aligned}$$

and the control sequence is

$$U(t_k) = \begin{bmatrix} u(t_k|t_k)^T & u(t_k+\eta|t_k)^T & \cdots & u(t_k+(q-1)\eta|t_k)^T \end{bmatrix}^T. \tag{5.8}$$

Then the controller design scheme is: at each sampling time t_k

1. measure the state $x(t_k)$;
2. compute the optimal control sequence $U(t_k)$ with the state feedback controller of the form

   $$u(t_k + i\eta|t_k) = F_i(t_k)x(t_k + i\eta|t_k) \tag{5.9}$$

   such that the performance objective in (5.3) is minimized;
3. implement the first n_k control moves $\begin{bmatrix} u_*^T(t_k|t_k) & \cdots & u_*^T(t_k+(n_k-1)\eta|t_k) \end{bmatrix}^T$, and

   $$u(t_k) = U_{opt}(t_k)(1) = u_*(t_k|t_k). \tag{5.10}$$

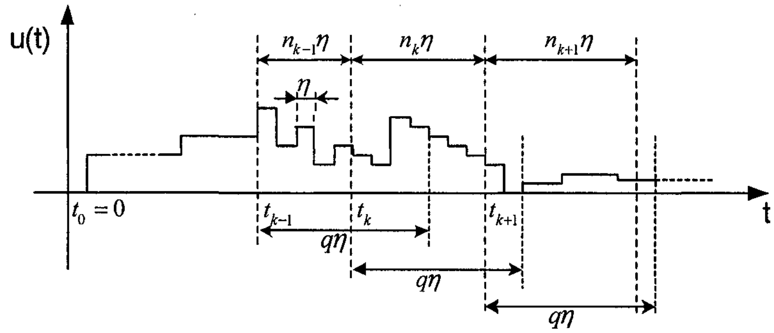

Figure 5.2: Moving horizon strategy

The subscript $*$ means optimal values. The detailed graphical representative of the moving horizon strategy is shown in Figure 5.2. To tackle the MPC problem, the key is to find a solution for the optimization problem in step 2 at each sampling instant t_k.

Remark 5.4 *At each sampling instant t_k, the optimal control sequence $U(t_k)$, namely, q control moves, is obtained. The first n_k control moves are implemented, which is different from the traditional MPC method, where only the first control move is used. This modified method provides enough optimal control signals for systems before the information at the next sampling instant is available. The larger the sampling period is, the more control moves of the optimal control sequence are used.*

For notational simplicity, we denote $F_i(\cdot)$ in (5.9) by F_i. In the following we will give a sufficient condition for the γ-suboptimal problem

$$J(x_{t_k}, t_k) \leq \gamma \tag{5.11}$$

for a given $\gamma > 0$. Throughout this chapter, we assume that the terminal weighting matrix P satisfies the following condition:

Assumption 5.3 *At time t_k, the following inequalities hold for the terminal weighting matrix P in (5.3) and control law $F_{q-i}, i \in \{1, 2, \cdots, n_k\}$ in (5.9):*

$$\Theta_i = -P + (A_d + B_d F_{q-i})^T P (A_d + B_d F_{q-i}) + Q + F_{q-i}^T R F_{q-i} \leq 0, \; i = 1, \cdots, n_k. \tag{5.12}$$

The equation in (5.12) can be transformed to standard LMIs by Schur complement, which can be easily solved.

Remark 5.5 *In this chapter, we will consider the case that the prediction horizon is the same as the control horizon, namely, $p = q = N$, for simplicity. The results can be readily extended to the case when $q < p$, which will not affect our method for stability analysis and controller design. Only the mathematical calculation is more complicated.*

5.3 Stability analysis and controller design for the SD system

In this section, we will show that if we select the terminal weighting matrix satisfying Assumption 5.3, the closed-loop stability for the system in (5.2) with the unconstrained receding horizon control law in (5.8) is guaranteed. Before proceeding, we introduce the following lemma, which will simplify the proof of our theorems.

Lemma 5.1 *Assume that $\{P, F_{N-j}, j = 1, 2, \cdots, n_k\}$ with $P > 0$ satisfy Assumption 5.3. Then the following inequality holds:*

$$
\begin{aligned}
\Xi_{n_k} &= -P + \left[\prod_{j=1}^{n_k}(A_d + B_d F_{N-j})\right]^T P \left[\prod_{j=1}^{n_k}(A_d + B_d F_{N-j})\right] + Q \\
&\quad + F_{N-n_k}^T R F_{N-n_k} + \sum_{i=1}^{n_k-1} \left\{ \left[\prod_{j=1}^{i}(A_d + B_d F_{N-n_k+i-j})\right]^T \right. \\
&\quad \left. \times (Q + F_{N-n_k+i}^T R F_{N-n_k+i}) \left[\prod_{j=1}^{i}(A_d + B_d F_{N-n_k+i-j})\right] \right\} \\
&\leq 0 \qquad (5.13)
\end{aligned}
$$

Proof: See the Appendix.

Theorem 5.1 *Consider the system in (5.2) and let $x(t_k|t_k)$ be the measured state at time t_k. Suppose that Assumption 5.3 is satisfied. If there exist a terminal weighting matrix $X > 0$ and Y_{N-i}, $i = 1, 2, \cdots, n_k$, such that the following optimization problem is feasible:*

$$\min \quad \gamma \qquad (5.14)$$

subject to

$$\begin{bmatrix} -\gamma & x(t_k|t_k)^T & x(t_k|t_k)^T\Phi^T \\ x(t_k|t_k) & -\hat{Q} & 0 \\ \Phi x(t_k|t_k) & 0 & -\Psi - \Gamma\Omega\Gamma^T \end{bmatrix} \leq 0 \tag{5.15}$$

and

$$\begin{bmatrix} -X & XA_d^T + Y_{N-i}^T B_d^T & X & Y_{N-i}^T \\ A_d X + B_d Y_{N-i} & -X & 0 & 0 \\ X & 0 & -\hat{Q} & 0 \\ Y_{N-i} & 0 & 0 & -\hat{R} \end{bmatrix} \leq 0, \tag{5.16}$$

where

$$\begin{aligned} \hat{Q} &= Q^{-1}, \quad \hat{R} = R^{-1}, \quad X = P^{-1}, \\ \Psi &= \mathrm{diag}\{\hat{Q}, \cdots, \hat{Q}, X\}, \quad \Omega = \mathrm{diag}\{\hat{R}, \cdots, \hat{R}\}, \\ \Phi &= \begin{bmatrix} A_d \\ A_d^2 \\ \vdots \\ A_d^N \end{bmatrix}, \quad \Gamma = \begin{bmatrix} B_d & 0 & \cdots & 0 \\ A_d B_d & B_d & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_d^{N-1} B_d & A_d^{N-2} B_d & \cdots & B_d \end{bmatrix}. \end{aligned} \tag{5.17}$$

then the state feedback control law in (5.9) exists and equation (5.11) holds. Moreover, if equations (5.15) and (5.16) are feasible, then the control prediction sequence $U(t_k)$ in (5.8) is

$$U(t_k) = -(\overline{R} + \Gamma^T\overline{Q}\Gamma)^{-1}\Gamma^T\overline{Q}\Phi x(t_k), \tag{5.18}$$

where

$$\overline{Q} = \mathrm{diag}\{Q, \cdots, Q, P\}, \quad \overline{R} = \mathrm{diag}\{R, R, \cdots, R\}. \tag{5.19}$$

Proof: It is easy to see that inequality (5.12) is equivalent to inequality (5.16) by Schur complements, by which the terminal weighting matrix is determined. The cost function in (5.3) with $q = p = N$ can be rewritten as the following equation:

$$\min_{U(t_k)} J(x_{t_k}, t_k) = U^T(t_k)\overline{R}U(t_k) + z^T(t_k)\overline{Q}z(t_k) + x^T(t_k)Qx(t_k), \tag{5.20}$$

where $U(t_k)$ is defined in (5.8) and

$$z(t_k) = \begin{bmatrix} x(t_k + \eta|t_k) \\ x(t_k + 2\eta|t_k) \\ \vdots \\ x(t_k + N\eta|t_k) \end{bmatrix} = \Phi x(t_k|t_k) + \Gamma U(t_k). \tag{5.21}$$

Substituting equation (5.21) to (5.20), we have

$$\begin{aligned} J(x_{t_k}, t_k) &= \|U(t_k) + (\overline{R} + \Gamma^T\overline{Q}\Gamma)^{-1}\Gamma^T\overline{Q}\Phi x(t_k)\|^2_{\overline{R}+\Gamma^T\overline{Q}\Gamma} \\ &\quad + x^T(t_k|t_k)[Q + \Phi^T(\Psi + \Gamma\Omega\Gamma^T)^{-1}\Phi]x(t_k|t_k) \end{aligned}$$

Note that the cost function is quadratic (and always positive) and hence has a unique minimum which can be located by setting the first derivative with respect to $U(t_k)$ to zero [78], that is,

$$\frac{dJ(x_{t_k}, t_k)}{dU(t_k)} = 0 \Rightarrow U(t_k) = -(\overline{R} + \Gamma^T\overline{Q}\Gamma)^{-1}\Gamma^T\overline{Q}\Phi x(t_k) \tag{5.22}$$

and

$$\min_{U(t_k)} J(x_{t_k}, t_k) = x^T(t_k|t_k)[Q + \Phi^T(\Psi + \Gamma\Omega\Gamma^T)^{-1}\Phi]x(t_k|t_k). \tag{5.23}$$

By inequality (5.15) and Schur complements, inequality (5.11) can be easily derived. Thus, the proof is completed. ■

Remark 5.6 *The optimization problem we adopted is with the finite horizon cost and the finite terminal weighting matrix. It is noted that the zero terminal constraint, which has been widely involved for stability in the existing results [3, 48], is not required. Moreover, the proposed method is more flexible than the conventional results, where the cost horizon size was required to be larger than the system order [48] or the number of the unstable modes [75].*

Theorem 5.2 *Let $x(t_k|t_k)$ be the measured state at sampling instant t_k. Suppose that Assumption 5.3 is satisfied. Then the feasible receding horizon control law in (5.18), which stems from the optimization problem in Theorem 5.1 by minimizing the cost function $J(x_{t_k}, t_k)$, asymptotically stabilizes the system in (5.2).*

Proof: To show the asymptotical stability of the system, we shall show that its Lyapunov function is strictly decreasing. Let the Lyapunov function for the closed-loop system be

$$\begin{aligned} V(x_{t_k}, t_k) = J_{opt}(x_{t_k}, t_k) &= \sum_{i=0}^{N-1} \left\{\|x(t_k + i\eta|t_k)\|^2_Q + \|u_*(t_k + i\eta|t_k)\|^2_R\right\} \\ &\quad + \|x(t_k + N\eta|t_k)\|^2_P, \end{aligned} \tag{5.24}$$

where the optimal control sequence at time t_k will be

$$U_{opt}(t_k) = \left[\begin{array}{cccc} u_*^T(t_k|t_k) & u_*^T(t_k+\eta|t_k) & \cdots & u_*^T(t_k+(N-1)\eta|t_k) \end{array} \right]^T. \quad (5.25)$$

We assume that the control sequence of the cost function $J(x_{t_{k+1}}, t_{k+1})$ is defined as

$$\begin{aligned} U(t_{k+1}) &= \left[\begin{array}{l} u(t_{k+1}|t_{k+1}) \\ u(t_{k+1}+\eta|t_{k+1}) \\ \vdots \\ u(t_{k+1}+(N-n_k-1)\eta|t_{k+1}) \\ u(t_{k+1}+(N-n_k)\eta|t_{k+1}) \\ u(t_{k+1}+(N-n_k+1)\eta|t_{k+1}) \\ \vdots \\ u(t_{k+1}+(N-1)\eta|t_{k+1}) \end{array} \right] \\ &= \left[\begin{array}{l} u_*(t_k+n_k\eta|t_k) \\ u_*(t_k+(n_k+1)\eta|t_k) \\ \vdots \\ u_*(t_k+(N-1)\eta|t_k) \\ F_{N-n_k}(t_{k+1})x(t_{k+1}+(N-n_k)\eta|t_{k+1}) \\ F_{N-n_k+1}(t_{k+1})x(t_{k+1}+(N-n_k+1)\eta|t_{k+1}) \\ \vdots \\ F_{N-1}(t_{k+1})x(t_{k+1}+(N-1)\eta|t_{k+1}) \end{array} \right]. \quad (5.26) \end{aligned}$$

Following the idea from [49], the optimal value of $J(x_{t_k}, t_k)$ can be written as

$$\begin{aligned} J_{opt}(x_{t_k}, t_k) &= J(x_{t_{k+1}}, t_{k+1}) + \sum_{i=0}^{n_k-1} \{\|x(t_k+i\eta|t_k)\|_Q^2 \\ &\quad + \|u_*(t_k+i\eta|t_k)\|_R^2\} + M_k, \quad (5.27) \end{aligned}$$

where

$$\begin{aligned} M_k &= \|x(t_k+N\eta|t_k)\|_P^2 - \|x(t_{k+1}+N\eta|t_{k+1})\|_P^2 \\ &\quad - \sum_{i=N-n_k}^{N-1} \{\|x(t_{k+1}+i\eta|t_{k+1})\|_Q^2 + \|u(t_{k+1}+i\eta|t_{k+1})\|_R^2\}. \quad (5.28) \end{aligned}$$

With the control sequence in (5.26), we obtain

$$\begin{aligned} x(t_{k+1}+i\eta|t_{k+1}) &= x(t_k+(n_k+i)\eta|t_k), \quad i = 0, 1, \cdots, N-n_k, \\ x(t_{k+1}+(N-n_k+i)\eta|t_{k+1}) &= \prod_{j=1}^{i}(A_d + B_d F_{N-n_k+i-j})x(t_k+N\eta|t_k), \\ &\quad i = 1, \cdots, n_k. \quad (5.29) \end{aligned}$$

Substituting the above equations to (5.28), it follows that

$$\begin{aligned}
M_k &= x^T(t_k+N\eta|t_k)\left\{P-\left[\prod_{j=1}^{n_k}(A_d+B_dF_{N-j})\right]^T P\left[\prod_{j=1}^{n_k}(A_d+B_dF_{N-j})\right]\right. \\
&\quad -Q-F_{N-n_k}^T RF_{N-n_k}-\sum_{i=1}^{n_k-1}\left\{\left[\prod_{j=1}^{i}(A_d+B_dF_{N-n_k+i-j})\right]^T\right. \\
&\quad \left.\left.\times(Q+F_{N-n_k+i}^T RF_{N-n_k+i})\left[\prod_{j=1}^{i}(A_d+B_dF_{N-n_k+i-j})\right]\right\}\right\}x(t_k+N\eta|t_k) \\
&= -x^T(t_k+N\eta|t_k)\Xi_{n_k}x(t_k+N\eta|t_k).
\end{aligned} \tag{5.30}$$

By Lemma 5.1, we have $M_k \geq 0$. Then it is easy to see from equation (5.27) that

$$J(x_{t_{k+1}},t_{k+1})-J_{opt}(x_{t_k},t_k)\leq -\sum_{i=0}^{n_k-1}\left\{\|x(t_k+i\eta|t_k)\|_Q^2+\|u_*(t_k+i\eta|t_k)\|_R^2\right\}<0. \tag{5.31}$$

Therefore,

$$V(x_{t_{k+1}},t_{k+1})=J_{opt}(x_{t_{k+1}},t_{k+1})\leq J(x_{t_{k+1}},t_{k+1})<J_{opt}(x_{t_k},t_k)=V(x_{t_k},t_k) \tag{5.32}$$

since $V(x_{t_{k+1}},t_{k+1})$ is the optimal value at time t_{k+1} while $J(x_{t_{k+1}},t_{k+1})$ is a feasible value at time t_{k+1}. Thus, the proof is completed. ■

Our MPC formulation of sampled-data systems can be extended to the case with input/output constraints. Generally, the input constraints should be satisfied because of physical limitations inherently in process equipment; while performance specifications impose constraints on the process outputs. Here we consider the Euclidean norm constraints at sampling instant t_k as follows:

$$\begin{aligned}
&\|u(t_k+i\eta|t_k)\|_2 \leq u_{max},\quad i\geq 0,\ t_k\geq 0, \\
&\|y(t_k+i\eta|t_k)\|_2 \leq y_{max},\quad i\geq 0,\ t_k\geq 0,
\end{aligned} \tag{5.33}$$

where $y(\cdot)$ is the system output satisfying $y(t)=Cx(t)$. Then we have the following corollary.

Corollary 5.1 *Consider the system in (5.2) and let $x(t_k|t_k)$ be the measured state at time t_k. Suppose that Assumption 5.3 is satisfied. Then the feasible receding* *horizon control law in (5.18) asymptotically stabilizes the system in (5.2) if there exist a terminal weighting matrix $X > 0$, Y_{N-i}, $i = \{1, 2, \cdots, n_k\}$ and a scalar $\gamma > 0$ such that the following optimization problem is feasible:*

$$\min \quad \gamma$$

subject to (5.15), (5.16) and

$$-y_{\max}^2 I + CXC^T \leq 0, \tag{5.34}$$

$$\begin{bmatrix} -u_{\max}^2 I & Y_{N-i} \\ Y_{N-i}^T & -X \end{bmatrix} \leq 0. \tag{5.35}$$

Proof: The constraints in (5.34) and (5.35) are easily derived from the inequality in (5.33) by the method in [45]. The rest proof is similar to the proof of Theorem 5.2, hence omitted. □

5.4 Tracking performance

In this section, we will enhance the MPC algorithm with a reference tracking scheme. Once the intersampling system responses are estimated, i.e., $y(t_k + i\eta|t_k)$, $i = 1, 2, \cdots, N$, the next step of the predictive control strategy developed here is to predict the required n_k-step-ahead future control signals that will drive the system to track a desired trajectory. We introduce the cost function as

$$\begin{aligned} J(U(t_k), t_k) &= \sum_{i=0}^{N-1} \|u(t_k + i\eta|t_k)\|_R^2 + \sum_{i=1}^{N-1} \|r(t_k + i\eta) - y(t_k + i\eta|t_k)\|_Q^2 \\ &\quad + \|r(t_k + N\eta) - y(t_k + N\eta|t_k)\|_P^2 \end{aligned} \tag{5.36}$$

with $Q = Q^T > 0$, $R = R^T > 0$, and $P = P^T > 0$. $r(\cdot)$ is the reference signal and represented by a state model:

$$\begin{cases} \dot{x}_r(t) = Gx_r(t) \\ r(t) = Hx_r(t) \end{cases}. \tag{5.37}$$

Then we have the following theorem.

Theorem 5.3 *Consider the system in (5.2) and let $x(t_k|t_k)$ be the measured state at time t_k. Then the tracking error of the system in (5.2), with the resulting receding horizon control law*

$$U_{opt}(t_k) = (\overline{R} + \Gamma^T \overline{C}^T \overline{Q}\ \overline{C}\Gamma)^{-1} \Gamma^T \overline{C}^T \overline{Q}(\overline{r} - \overline{C}\Phi x(t_k)), \tag{5.38}$$

will be bounded, if there exist a terminal weighting matrix $X > 0$, Y_{N-i}, $i = 1, 2, \cdots, n_k$, and a scalar $\gamma > 0$ such that the following optimization problem is feasible:

$$\min \quad \gamma \tag{5.39}$$

subject to

$$\begin{bmatrix} -\gamma & (\overline{r} - \overline{C}\Phi x(t_k|t_k))^T \\ (\overline{r} - \overline{C}\Phi x(t_k|t_k)) & -\Psi - \overline{C}\Gamma\Omega\Gamma^T\overline{C}^T \end{bmatrix} \leq 0, \tag{5.40}$$

$$\begin{bmatrix} -X & X\tilde{A}_d^{\;T} + Y_{N-i}^T B_d^T C^T & X & Y_{N-i}^T \\ \tilde{A}_d X + C B_d Y_{N-i} & -X & 0 & 0 \\ X & 0 & -\hat{Q} & 0 \\ Y_{N-i} & 0 & 0 & -\hat{R} \end{bmatrix} \leq 0, \tag{5.41}$$

and (5.34), (5.35), where Ψ, Ω, Φ, Γ, $\overline{Q}$ and $\overline{R}$ are expressed in equations (5.17) and (5.19), and

$$\begin{aligned} \tilde{A}_d &= (CA_dC^T + He^{G\eta}H^T)(CC^T + HH^T)^{-1}, \quad \overline{C} = \text{diag}\{C, C, \cdots, C\} \\ \overline{r} &= \begin{bmatrix} r^T(t_k+\eta) & \cdots & r^T(t_k + N\eta) \end{bmatrix}^T. \end{aligned}$$

Proof: By recasting the system in (5.6) and the discretized reference signal in (5.37), we have

$$\begin{aligned} \xi(t_k + (i+1)\eta|t_k) &= \begin{bmatrix} x_r(t_k + (i+1)\eta) \\ x(t_k + (i+1)\eta|t_k) \end{bmatrix} \\ &= \begin{bmatrix} e^{G\eta} & 0 \\ 0 & A_d \end{bmatrix} \xi(t_k + i\eta|t_k) + \begin{bmatrix} 0 \\ B_d \end{bmatrix} u(t_k + i\eta|t_k) \end{aligned}$$

and

$$\begin{aligned} J(U(t_k), t_k) &= \sum_{i=0}^{N-1} \|u(t_k + i\eta|t_k)\|_R^2 + \sum_{i=1}^{N-1} \|\xi(t_k + i\eta|t_k)\|^2_{[H \;\; -C]^T Q [H \;\; -C]} \\ &\quad + \|\xi(t_k + N\eta|t_k)\|^2_{[H \;\; -C]^T P [H \;\; -C]}. \end{aligned}$$

The rest of the proof is similar to that of Theorem 5.1; thus, omitted here. Note that the strictly decreasing Lyapunov function, which is the same as the cost function in (5.36), guarantees the boundedness of the tracking error. ∎

5.5 Extensions

The method derived for SD systems in this chapter can be extended to the case of networked control systems (NCSs) with network-induced time delays/packet dropouts, where sensors, controllers and actuators are working at variable sampling rate. To show how it works, the NCSs with fixed time delays are discussed as an example in this section.

Assume that the fixed network-induced time delay τ in the transmission satisfies $\tau = d\eta$, $d \in \{1, \cdots, N-1\}$. We have the following prediction equations:

$$
\begin{aligned}
x(t_k + (i+1)\eta | t_k) &= A_d x(t_k + i\eta | t_k) + B_d u(t_k + (i-d)\eta | t_k), i \in \{0, 1, \cdots, N-1\}, \\
x(t_{k+1} | t_k) &= x(t_k + n_k \eta | t_k) = \Phi(n_k) x(t_k | t_k) + \Theta_1 U_{past} + \Theta_2 U(t_k),
\end{aligned}
$$

where A_d, B_d, $\Phi(n_k)$, and $U(t_k)$ are defined in (5.6)-(5.8), and

$$
\begin{aligned}
\Theta_1 &= \begin{cases} [\ A_d^{n_k-1}B_d \quad A_d^{n_k-2}B_d \quad \cdots \quad A_d^{n_k-d}B_d\ ],\ n_k > d, \\ [\ A_d^{n_k-1}B_d, \quad A_d^{n_k-2}B_d, \quad \cdots, \quad B_d\ ],\ n_k \le d, \end{cases} \\
\Theta_2 &= \begin{cases} [\ A_d^{n_k-d-1}B_d \quad \cdots \quad B_d \quad 0_{n\times m} \quad \cdots \quad 0_{n\times m}\ ],\ n_k > d, \\ 0_{n\times mN},\ n_k \le d, \end{cases} \\
U_{past} &= \left[\ u^T(t_k - d\eta | t_k) \quad u^T(t_k - (d-1)\eta | t_k) \quad \cdots \quad u^T(t_k - \eta | t_k)\ \right]^T.
\end{aligned}
$$

Remark 5.7 *Here $n_k > d$ means that the delay is less than the sampling period at time t_k, while $n_k \le d$ means that the delay equals or is longer than the sampling period at time t_k. Since n_k is time-varying, both cases may be included in our prediction equations, which make our model more general.*

The cost function in (5.3) can be written as

$$
\min_{U(t_k)} J(x_{t_k}, t_k) = U^T(t_k)\overline{R}U(t_k) + z^T(t_k)\overline{Q}z(t_k) + x^T(t_k)Qx(t_k), \tag{5.42}
$$

where

$$
z(t_k) = \begin{bmatrix} x(t_k + \eta | t_k) \\ x(t_k + 2\eta | t_k) \\ \vdots \\ x(t_k + N\eta | t_k) \end{bmatrix} = \Phi x(t_k | t_k) + \Gamma_1 U_{past} + \Gamma_2 U(t_k),
$$

$$\Phi = \begin{bmatrix} A_d \\ A_d^2 \\ \vdots \\ A_d^N \end{bmatrix}, \quad \Gamma_1 = \begin{bmatrix} B_d & 0 & \cdots & 0 \\ A_d B_d & B_d & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_d^{d-1} B_d & A_d^{d-2} B_d & \cdots & B_d \\ \vdots & \vdots & \vdots & \vdots \\ A_d^{N-1} B_d & A_d^{N-2} B_d & \cdots & A_d^{N-d} B_d \end{bmatrix},$$

$$\Gamma_2 = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ B_d & 0 & \cdots & 0 & 0 & \cdots & 0 \\ A_d B_d & B_d & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ A_d^{N-d-1} B_d & A_d^{N-d-2} B_d & \cdots & B_d & 0 & \cdots & 0 \end{bmatrix},$$

and $\overline{R}$ and $\overline{Q}$ are defined in (5.19). With the methods presented for SD systems, we have following theorem:

Theorem 5.4 *Consider the system in (5.2) and let $x(t_k|t_k)$ be the measured state at time t_k. Suppose that Assumption 5.3 is satisfied. Then there exists a state feedback control sequence in (5.8) such that equation (5.11) holds if there exist a terminal weighting matrix $X > 0$, Y_{N-i}, $i = \{1, 2, \cdots, n_k\}$ and a scalar $\gamma > 0$ such that the following optimization problem is feasible:*

$$\min \quad \gamma \tag{5.43}$$

subject to

$$\begin{bmatrix} -\gamma & x(t_k|t_k)^T & (\Phi x(t_k|t_k) + \Gamma_1 U_{past})^T \\ x(t_k|t_k) & -\hat{Q} & 0 \\ \Phi x(t_k|t_k) + \Gamma_1 U_{past} & 0 & -\Psi - \Gamma_2 \Omega \Gamma_2^T \end{bmatrix} \leq 0 \tag{5.44}$$

and

$$\begin{bmatrix} -X & X A_d^T + Y_{N-i}^T B_d^T & X & Y_{N-i}^T \\ A_d X + B_d Y_{N-i} & -X & 0 & 0 \\ X & 0 & -\hat{Q} & 0 \\ Y_{N-i} & 0 & 0 & -\hat{R} \end{bmatrix} \leq 0. \tag{5.45}$$

Moreover, if equations (5.44) and (5.45) are feasible, then the control prediction sequence $U(t_k)$ in (5.8) is

$$U(t_k) = -(\overline{R} + \Gamma_2^T \overline{Q} \Gamma_2)^{-1} \Gamma_2^T \overline{Q} (\Phi x(t_k) + \Gamma_1 U_{past}). \tag{5.46}$$

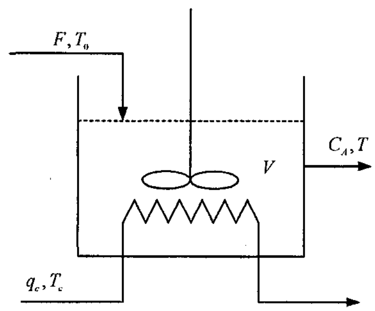

Figure 5.3: A continuous stirred tank reactor

Proof: The proof is similar to that in Theorem 5.1. Thus, omitted here. ■

Remark 5.8 *Theorem 5.4 is obtained under the constraints that the communication delay is fixed and that its value is an multiple integer of prediction period* η*, which bring conservativeness. Loosening those constraints, such as random delays, is significant and difficult. New iterative prediction equations are needed. Same problem exists for NCSs with random packet dropouts. This research direction is quite new and no general results have been derived till now. We enumerate them in the future work.*

5.6 Numerical example

In order to illustrate the proposed model predictive control algorithm for the SD system, a simple example will be investigated. Consider the following single non-isothermal continuously stirred tank reactor (CSTR) [62] in Figure 5.3. The component material balance on the reactant gives

$$V\frac{dC_A}{dt} = F(C_{A0} - C_A) - Vk_0 e^{-E/RT} C_A. \tag{5.47}$$

The energy balance for reacting system is

$$V\rho C_p \frac{dT}{dt} = \rho C_p F(T_0 - T) - \frac{aq_c^{b+1}}{q_c + \frac{aq_c^b}{2\rho_c C_{pc}}}(T - T_{cin}) + (-\Delta H_{rxn}) V k_0 e^{-E/RT} C_A. \tag{5.48}$$

The parameters of the plant and their steady state operating condition used for the CSTR process are reported in Table 5.1, by which we have the linearized equations

Table 5.1: Steady-state operating data

Process variable	Normal operating condition
Reactor concentration (C_{As})	0.265 kmol/m^3
Reactor temperature (T_s)	394 K
Coolant flow rate (q_{cs})	15 m^3/min
Process flow rate (F)	1 m^3/min
CSTR volume (V)	1 m^3
Reaction rate constant (k_0)	10^{10} /min^{-1}
Activation energy term (E/R)	8330.1 K
Heat of reaction ($-\Delta H_{rxn}$)	10^8 cal/kmole
Feed temperature (T_0)	323 K
Inlet coolant temperature (T_{cs})	365 K
Liquid density (ρ)	10^6 g/m^3
Specific heats (C_p)	1 cal/gK
Overall heat-transfer coefficients (UA)	5.34×10^6 cal/K

in deviation variables are as follows [96]:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases} \tag{5.49}$$

where $x = \begin{bmatrix} C_A \\ T \end{bmatrix}$ with C_A the reactor concentration and T the temperature, and $u = \begin{bmatrix} C_{A0} \\ q_c \end{bmatrix}$, with C_{A0} the feed concentration and q_c the coolant flow. A and B depend on the operating condition as follows:

$$A = \begin{bmatrix} -\frac{F}{V} - k_0 e^{-E/RT_s} & -\frac{E}{RT_s^2} k_0 e^{-E/RT_s} C_{As} \\ \frac{-\Delta H_{rxn} k_0 e^{-E/RT_s}}{\rho C_p} & -\frac{F}{V} - \frac{UA}{V\rho C_p} - \Delta H_{rxn} \frac{E}{\rho C_p R T_s^2} k_0 e^{-E/RT_s} C_{As} \end{bmatrix},$$

$$B = \begin{bmatrix} \frac{F}{V} & 0 \\ 0 & -2.098 \times 10^5 \frac{T_s - 365}{V\rho C_p} \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{5.50}$$

Assume that the initial state $x(0) = [0.5 \quad 1]^T$, Q=0.1I, R=I, $N = 10$, $\eta = 0.05$ min, and $h_k \in [0.05, \ 0.5]$ min, the system dynamics under the proposed MPC controller is shown in Figure 5.4. Figure 5.4 (a) is the trajectories of the reactor concentration and temperature, which show that the closed-loop system is stable by the controller we designed. Figure 5.4 (b) is a sequence of the changing sampling periods, and Figure 5.4 (c) is the upper bound of the cost function.

Moreover, in order to assess the tracking capability of the proposed controller based on the MPC scheme, a square wave reference input as shown in Figure 5.5

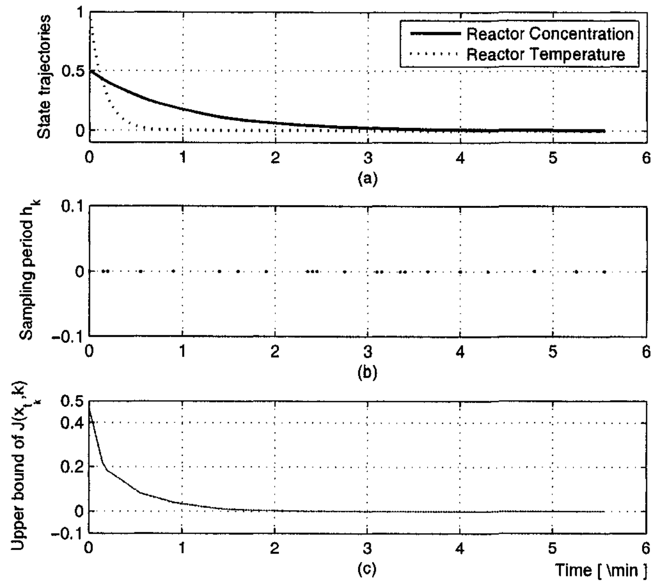

Figure 5.4: State trajectories of the plant

(a) has been introduced. The goal is that the reactor temperature should track the reference waveform and satisfy $392 \leq T \leq 396$ K, by adjusting the coolant flow. The change in coolant flow satisfies $|\Delta q_c| \leq 15$ m^3/min. Assume that $G = 0$, $H = 1$, $C = [0\ 1]$, $Q = 50$, $R = 0.01$ and x_0 is the steady state, then we have Figure 5.5, where Figure 5.5 (a) shows that the output of the plant can track the reference input effectively and Figure 5.5 (b) is a sequence of the time-varying sampling period. Note that the change of the coolant flow in Figure 5.5 (c) is less than 15 m^3/min.

5.7 Summary

In this chapter, the problem of model predictive control for sampled-data systems with variable sampling periods has been studied. Based on the minimization of the finite horizon quadratic cost function at each sampling instant, a state feedback predictive control sequence has been derived. In order to achieve stability, a condition on the finite terminal weighting matrix P, which has some free parameters and can be converted to an LMI, has been proposed. The asymptotical stability conditions with and without input/output constraints have been presented respectively for the

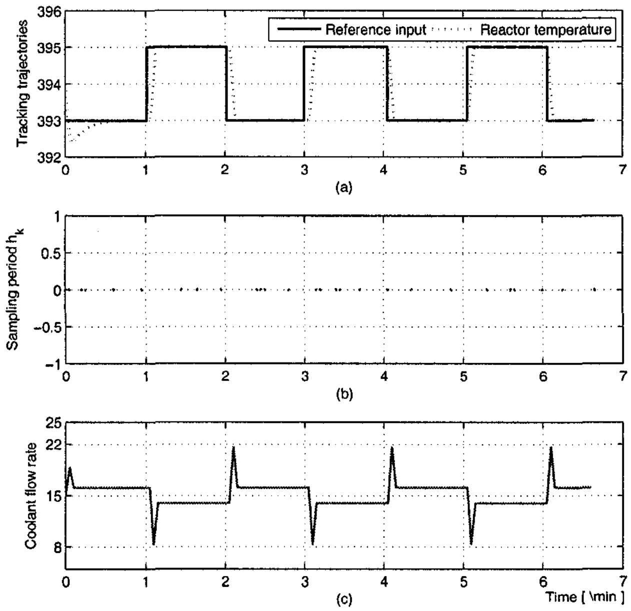

Figure 5.5: Tracking performance of the predictive controller

closed-loop system resulting from the proposed control sequence. Simulation results show the feasibility and effectiveness of our method.

APPENDIX

Proof of Lemma: We prove it by Mathematical Induction method. By Assumption 5.3, we have

$$\begin{aligned}
\Xi_1 &= -P + [(A_d + B_d F_{N-1})]^T P [(A_d + B_d F_{N-1})] + Q + F_{N-1}^T R F_{N-1} = \Theta_1 \leq 0 \\
\Xi_2 &= -P + [(A_d + B_d F_{N-1})(A_d + B_d F_{N-2})]^T P [(A_d + B_d F_{N-1})(A_d + B_d F_{N-2})] \\
&\quad + Q + F_{N-2}^T R F_{N-2} + [(A_d + B_d F_{N-2})]^T (Q + F_{N-1}^T R F_{N-1}) [(A_d + B_d F_{N-2})] \\
&= [(A_d + B_d F_{N-2})]^T \Xi_1 [(A_d + B_d F_{N-2})] + \Theta_2 \leq 0
\end{aligned}$$

Assume that the inequality in (5.13) is true for $n_k = m, m \in \{1, 2, \cdots, n_k - 1\}$, namely,

$$\Xi_m = -P + \left[\prod_{j=1}^{m} (A_d + B_d F_{N-j})\right]^T P \left[\prod_{j=1}^{m} (A_d + B_d F_{N-j})\right] + Q + F_{N-m}^T R F_{N-m}$$

$$+\sum_{i=1}^{m-1}\left\{\left[\prod_{j=1}^{i}(A_d+B_dF_{N-m+i-j})\right]^T(Q+F_{N-m+i}^TRF_{N-m+i})\right.$$
$$\left.\times\left[\prod_{j=1}^{i}(A_d+B_dF_{N-m+i-j})\right]\right\}\leq 0$$

Then we have

$$\begin{aligned}
\Xi_{m+1} &= -P+\left[\prod_{j=1}^{(m+1)}(A_d+B_dF_{N-j})\right]^T P\left[\prod_{j=1}^{(m+1)}(A_d+B_dF_{N-j})\right]+Q \\
&\quad +F_{N-(m+1)}^TRF_{N-(m+1)}+\sum_{i=1}^{(m+1)-1}\left\{\left[\prod_{j=1}^{i}(A_d+B_dF_{N-(m+1)+i-j})\right]^T\right. \\
&\quad \left.\times(Q+F_{N-(m+1)+i}^TRF_{N-(m+1)+i})\left[\prod_{j=1}^{i}(A_d+B_dF_{N-(m+1)+i-j})\right]\right\} \\
&= \Xi_{m+1}+(A_d+B_dF_{N-(m+1)})^TP(A_d+B_dF_{N-(m+1)}) \\
&\quad -(A_d+B_dF_{N-(m+1)})^TP(A_d+B_dF_{N-(m+1)}) \\
&= \Theta_{m+1}+(A_d+B_dF_{N-(m+1)})^T\left\{-P+\left[\prod_{j=1}^{m}(A_d+B_dF_{N-j})\right]^T\right. \\
&\quad \left.\times P\left[\prod_{j=1}^{m}(A_d+B_dF_{N-j})\right]+Q+F_{N-m}^TRF_{N-m}\right\}(A_d+B_dF_{N-(m+1)}) \\
&\quad +\sum_{i=2}^{(m+1)-1}\left\{\left[\prod_{j=1}^{i}(A_d+B_dF_{N-(m+1)+i-j})\right]^T(Q+F_{N-(m+1)+i}^TRF_{N-(m+1)+i})\right. \\
&\quad \left.\times\left[\prod_{j=1}^{i}(A_d+B_dF_{N-(m+1)+i-j})\right]\right\} \\
&= \Theta_{m+1}+(A_d+B_dF_{N-(m+1)})^T\{-P+Q+F_{N-m}^TRF_{N-m} \\
&\quad +\left[\prod_{j=1}^{m}(A_d+B_dF_{N-j})\right]^T P\left[\prod_{j=1}^{m}(A_d+B_dF_{N-j})\right] \\
&\quad +\sum_{i=1+1}^{(m-1)+1}\left\{\left[\prod_{j=1}^{i-1}(A_d+B_dF_{N-m+(i-1)-j})\right]^T(Q+F_{N-m+(i-1)}^TRF_{N-m+(i-1)})\right.
\end{aligned}$$

$$\times \left[\prod_{j=1}^{i-1} (A_d + B_d F_{N-m+(i-1)-j}) \right] \Bigg\} \Bigg\} (A_d + B_d F_{N-(m+1)})$$

$$= [(A_d + B_d F_{N-(m+1)})]^T \Xi_m [(A_d + B_d F_{N-(m+1)})] + \Theta_{m+1} \le 0. \quad (5.51)$$

Thus, the proof is completed. □

Chapter 6

Conclusions and Future Work

This dissertation has presented general models for a class of NCSs, provided corresponding controller designs with the consideration of network and control parameters, formulated and solved optimal control problems to compensate for communication delays/packet dropouts, and validated the analysis and design of NCSs. The main contributions of this dissertation are summarized below:

- The problem of control using MJLSs has been formulated and studied. Earlier approaches exist, for instance, de Souza [18], Niculescu [67], Costa [16], Benjelloun [43], Cao [11, 12], Fridman [25], and Lee [50], but this dissertation gives methods that allow comparison with earlier results within a common structure. The simulations show that the results in this dissertation are less conservative.

- Using the method developed for MJLSs in this dissertation, NCSs with random packet dropouts has been investigated. The modeling of the closed-loop NCS is a nice generalization of MJLSs with time delays, where the Markov chains are used to describe not only the information that if a packet is dropped or not, but also how many packets have been dropped since the last successful transmission. This study has been enriched by extending the case in a single-packet transmission protocol to that in a multiple-packet transmission protocol.

- The stabilization problem of NCSs with communication delays has also been discussed with the developed method for MJLSs. While sensor-to-controller delays can be known by using synchronized clocks and time-stamped packets, the upcoming controller-to-actuator delay is unknown, which makes the compensation difficult. To compensate the delayed data, a standard MPC method has been introduced in the controller design. The control scheme has been characterized as a constrained optimization problem of the worst-case quadratic cost over an infinite horizon at each sampling instant.

- Several reasons for plants with variable sampling rates have been investigated, which motivated the study of SD systems with variable sampling rates in this dissertation. An modified MPC approach has been formulated to stabilize the resulting closed-loop system, where the controller has been obtained based on the minimization of an upper bound of the worse-case finite horizon quadratic cost function with a terminal weighting matrix. The results have been finally extended to NCSs with fixed delays.

The research presented in this dissertation has provided general methods for modeling, analysis and design of NCSs. It lays a foundation for future research efforts in NCSs. We conclude this dissertation by listing some future research directions:

- In Chapter 5, we discussed the control problem for NCSs with variable sampling period under the assumption that the communication delay is zero or fixed. The case that the communication delays are time-varying or random is not discussed. Since the controller-to-actuator delays are unknown and difficult to compensate, how to get a general mathematical model is an important problem needed to be solved at first. New system analysis and controller design are required for NCSs with variable sampling rates. The problem also exists for NCSs with packet dropouts, where the sampling period changes with time. The study for NCSs with variable sampling rate is quite new and many problems are still open. Thus, this research is of importance in both theory and practice.

- Throughout the dissertation, we considered three related issues in NCSs. There are several more interesting problems in the area of NCSs worth investigation, such as problems with system uncertainties, perturbations, noise, packet disordering, scheduling, nonlinearity and so on. How do these issues affect the performance of NCSs? How to design controllers to stabilize the NCSs with these issues?

- The hidden Markov case, where the state of the Markov chain is unknown, is not treated in the dissertation. A method for prediction of controller performance when using an estimated Markov state would be very useful. This problem is very hard. An interesting study would be to try the theory from adaptive control [1, 20].

Bibliography

[1] R. Agrawal, D. Teneketzis, and V. Anantharam. Asymptotically efficient adaptive allocation schemes for controlled Markov chains: finite parameter space. *IEEE Transactions on Automatic Control*, 34(12):1249–1259, 1989.

[2] K. J. Astrom and B. Wittenmark. *Computer Controlled Systems: Theory and Design.* 2nd ed., Englewood Cliffs, NJ: Prentice-Hall, 1990.

[3] A. Bemporad, L. Chisci, and E. Mosca. On the stabilizing property of SIORHC. *Automatica*, 30(12):2013–2015, 1994.

[4] E. K. Boukas and Z. K. Liu. Robust stability and H_∞ control of discrete-time jump linear systems with time-delay: an LMI approach. In *Proceedings of the 39th IEEE Conference on Decision and Control*, pages 1527–1532, 2000.

[5] E. K. Boukas, Z. K. Liu, and G. X. Liu. Delay-dependent robust stability and H_∞ control of jump linear systems with time-delay. In *Proceedings of the 39th IEEE Conference on Decision and Control*, pages 1521–1526, 2000.

[6] E. K. Boukas, Z. K. Liu, and P. Shi. Delay-dependent stability and output feedback stabilisation of Markov jump linear system with time-delay. *IEE Proceedings on Control Theory and Applications*, 149(5):379–386, 2002.

[7] E. K. Boukas and P. Shi. Stochastic stability and guaranteed cost control of discrete-time uncertain systems with Markovian jumping parameters. *International Journal of Robust and Nonlinear Control*, 8(13):1155–1167, 1998.

[8] M. S. Branicky, S. M. Phillips, and W. Zhang. Scheduling and feedback codesign for networked control systems. In *Proceedings of the 41st IEEE Conference on Decision and Control*, pages 1211–1217, 2002.

[9] L. G. Bushnell. Networks and control. *IEEE Control Systems Magazine*, 21(1):22–23, 2001.

[10] Y. Y. Cao and J. Lam. Stochastic stabilizability and H_∞ control for discrete-time jump linear systems with time delay. *Journal of The Franklin Institute*, 336(8):1263–1281, 1999.

[11] Y. Y. Cao and J. Lam. Robust H_∞ control of uncertain Markovian jump systems with time-delay. *IEEE Transactions on Automatic Control*, 45(1):77–83, 2000.

[12] Y. Y. Cao, J. Lam, and L. S. Hu. Delay-dependent stochastic stability and H_∞ analysis for time delay systems with Markovian jumping parameters. *Journal of the Franklin Institute*, 340(6).

[13] T. Chen and B. A. Francis. *Optimal Sampled-Data Control Systems.* Springer, London, 1995.

[14] W. H. Chen, Z. H. Guan, and P. Yu. Delay-dependent stability and H_∞ control of uncertain discrete-time Markovian jump systems with mode-dependent time delays. *Systems & Control Letters*, 52(5):361–376, 2004.

[15] W. H. Chen, J. X. Xu, and Z. H. Guan. Guaranteed cost control for uncertain Markovian jump systems with mode-dependent time-delays. *IEEE Transactions on Automatic Control*, 48(12):2270–2277, 2003.

[16] O.L.V. Costa and R. P. Marques. Mixed H_2/H_∞ control of discrete-time Markovian jump linear systems. *IEEE Transactions on Automatic Control*, 43(1):95–100, 1998.

[17] M. Das, R. Ghosh, B. Goswami, A. Gupta, A. P. Tiwari, R. Balasubrmanian, and A. K. CHandra. Network control system applied to a large pressurized heavy water reactor. *IEEE TRANSACTIONS ON NUCLEAR SCIENCE*, 53(5):2948–2956, 2006.

[18] C. E. de Souze and X. Li. Delay-dependent robust H_∞ control of uncertain linear state-delayed systems. *Automatica*, 35(7):1313–1321, 1999.

[19] J. B. R. Do Val, J. C. Geromel, and O. L. V. Costa. Uncoupled Riccati iterations for the linear quadratic control problem of discrete-time Markov jump linear systems. *IEEE Transactions on Automatic Control*, 43(12):1727–1733, 1998.

[20] T. E. Duncan, B. Pasik-Duncan, and Q. Zhang. Adaptive control of stochastic manufacturing systems with hidden Markovian demands and small noise. *IEEE Transactions on Automatic Control*, 44(2):427–430, 1999.

[21] J. Eker. *Flexible Embedded Control Systems.* PhD thesis, Dept. Automatic Control, Lund Institute of Technology, Lund, Sweden, 1999.

[22] F. Fagnani and S. Zampieri. Stability analysis and synthesis for scalar linear systems with a quantized feedback. *IEEE Transactions on Automatic Control*, 48(9):1569–1584, 2003.

[23] F. Fagnani and S. Zampieri. Quantized stabilization of linear systems: complexity versus performance. *IEEE Transactions on Automatic Control*, 49(9):1534–1548, 2004.

[24] G. F. Franklin, J. D. Powell, and M. Workman. *Digital Control of Dynamic Systems.* 3rd ed., Addison-Wesley, Reading, MA, 1997.

[25] E. Fridman and U. Shaked. A descriptor system approach to H_∞ control of linear time-delay systems. *IEEE Transactions on Automatic Control*, 47(2):253–270, 2002.

[26] E. Fridman and U. Shaked. Parameter dependent stability and stabilization of uncertain time-delay systems. *IEEE Transactions on Automatic Control*, 48(5):861–866, 2003.

[27] H. Gao, L. Lam, C. Wang, and Y. Wang. Delay-dependent output-feedback stabilisation of discrete-time systems with time-varying state delay. *IEE Proceedings on Control Theory and Applications*, 151(6):691–698, 2004.

[28] H. J. Gao and C. H. Wang. Comments and further results on "A descriptor system approach to H-infinity control of linear time-delay systems". *IEEE Transactions on Automatic Control*, 48(3):520–525, 2003.

[29] G. C. Goodwin, H. Haimovich, D. E. Quevedo, and J. S. Welsh. A moving horizon approach to networked control system design. *IEEE Transactions on Automatic Control*, 49(9):1427–1445, 2004.

[30] K. Gu and S. I. Niculescu. Further remarks on additional dynamics in various model transformations of linear delay systems. *IEEE Transactions on Automatic Control*, 46(3):497–500, 2001.

[31] Y. Halevi and A. Ray. Integrated communication and control systems: Part I-analysis. *ASME Journal of Dynamic Systems, Measurement and Control*, 110(4):367–373, 1988.

[32] A. Hassibi, S. P. Boyd, and J. P. How. Control of asynchronous dynamical systems with rate constraints on events. In *Proceeding of the 38th IEEE Conference on Decision and Control*, pages 1345–1351, 1999.

[33] Y. He, M. Wu, J. H. She, and G. P. Liu. Parameter-dependent Lyapunov functional for stability of time-delay systems with polytopic-type uncertainties. *IEEE Transactions on Automatic Control*, 49(5):828–832, 2004.

[34] B. Hu and A. N. Michel. Stability analysis of digital feedback control systems with time-varying sampling periods. *Automatica*, 36(6):897–905, 2000.

[35] L. S. Hu, J. Lam, Y. Y. Cao, and H. H. Shao. A linear matrix inequality (LMI) approach to robust H_2 sampled-data control for linear uncertain systems. *IEEE Transactions on Systems, Man, and Cybernetics–Part B: Cybernetics*, 33(1):149–155, 2003.

[36] S. Hu and W. Zhu. Stochastic optimal control and analysis of stability of networked control systems with long delay. *Automatica*, 39(11):1877–1884, 2003.

[37] O. C. Imer, S. Yuksel, and T. Basar. Optimal control of LTI systems over unreliable communication links. *Automatica*, 42(9):1429–1439, 2006.

[38] A. P. Ioannou, H. J. Xu, and B. Fidan. Identification and high bandwidth control of hard disk drive servo systems based on sampled data measurements. *IEEE Transactions on Control Systems Technology*, 15(6):1089–1095, 2007.

[39] H. Ishii. H_∞ control with limited commnunication and message losses. In *Proceeding of the American Control Conference*, pages 5620–5625, 2006.

[40] Z. P. Jin, V. Gupta, and R. M. Murray. State estimation over packet dropping networks using multiple description coding. *Automatica*, 42(9):1441–1452, 2006.

[41] J. C. Juang and J. S. Sheu. On adaptive speed design in optical disc servo systems. *IEEE Transactions on Control Systems Technology*, 8(6):971–978, 2000.

[42] E. H. Jung, H. H. Lee, and Y. S. Suh. LMI-based output feedback control of networked control systems. In *Proceeding of IEEE International Workshop on Factory Communication Systems*, pages 311–314, 2004.

[43] Benjelloun K. and E. K. Boukas. Mean square stochastic stability of linear time-delay system with Markovian jumping parameters. *IEEE Transactions on Automatic Control*, 43(10):1456–1460, 1998.

[44] D. K. Kim, P. Park, and J. W. Ko. Output-feedback H_∞ control of systems over communication networks using a deterministic switching system approach. *Automatica*, 40(7):1205–1212, 2004.

[45] M. V. Kothare, V. Balakrishnan, and M. Morari. Robust constrained model predictive control using linear matrix inequalities. *Automatica*, 32(10):1361–1379, 1996.

[46] N. N. Krasovskii and E. A. Lidskii. Analytical design of controllers in systems with random attributes. *Automation and Remote Control*, 22:1021–1025, 1961.

[47] R. Krtolica, U. Ozguner, H. Chan, J. Winkelman, and M. Liubakka. Stability of linear feedback systems with random communication delays. *International Journal of Control*, 59(4):925–953, 1994.

[48] W. H. Kwon and A. E. Pearson. On feedback stabilization of time-varying discrete linear systems. *IEEE Transactions on Automatic Control*, 23(3):479–481, 1978.

[49] J. W. Lee, W. H. Kwon, and J. H. Choi. On stability of constrained receding horizon control with finite terminal weighting matrix. *Automatica*, 34(12):1607–1612, 1998.

[50] Y. S. Lee, Y. S. Moon, W. H. Kwon, and P. G. Park. Delay-dependent robust H_∞ control for uncertain systems with a state-delay. *Automatica*, 40(1):65–72, 2004.

[51] X. Li and C. E. de Souza. Criteria for robust stability and stabilization of uncertain linear systems with state delay. *Automatica*, 33(9):1657–1662, 1997.

[52] F. L. Lian, J. Moyne, and D. Tilbury. Performance evaluation of control networks: Ethernet, controlNet, and deviceNet. *IEEE Control Systems Magazine*, 21(1):66–83, 2001.

[53] F. L. Lian, J. Moyne, and D. Tilbury. Optimal controller design and evaluation for a class of networked control systems with distributed constant delays. In *Proceedings of the 2002 American Control Conference*, volume 4, pages 3009 – 3014, 2002.

[54] H. Lin, G. Zhai, and P. J. Antsaklis. Robust stability and disturbance attenuation analysis of a class of networked control systems. In *Proceedings of the 42nd IEEE Conference on Decision and Control*, pages 1182–1187, 2003.

[55] B. Lincoln and B. Bernhardsson. Optimal control over networks with long random delays. In *Proceedings of the 14th International Symposium on Mathematical Theory of Networks and Systems*, pages 1–7, 2000.

[56] Q. Ling and M. D. Lemmon. Robust performance of soft real-time networked control systems with data dropouts. In *Proceedings of the 41st IEEE Conference on Decision and Control*, pages 1225–1230, 2002.

[57] Q. Ling and M. D. Lemmon. Optimal dropout compensation in networked control systems. In *Proceedings of the 42st IEEE Conference on Decision and Control*, pages 670–675, 2003.

[58] Q. Ling and M. D. Lemmon. Soft real-time scheduling of networked control systems with dropouts governed by a Markov chain. In *Proceedings of the American Control Conference*, pages 4845–4850, 2003.

[59] Q. Ling and M. D. Lemmon. Power spectral analysis of networked control systems with data dropouts. *IEEE Transactions on Automatic Control*, 49(6):955–960, 2004.

[60] G. P. Liu, J. X. Mu, D. Rees, and S. C. Chai. Design and stability analysis of networked control systems with random communication time delay using the modified MPC. *International Journal of Control*, 79(4):288–297, 2006.

[61] V. Malyavej, I. R. Manchester, and A. V. Savkin. Precision missile guidance using radar/multiple-video sensor fusion via communication channels with bit-rate constraints. *Automatica*, 42(5):763–769, 2006.

[62] T. E. Marlin. *Process Control: Designing Processes and Control Systems for Dynamic Performance.* McGraw-Hill, New York, 1995.

[63] I. Mizumoto, T. Chen, S. Ohdaira, M. Kumon, and Z. Iwai. Adaptive output feedback control of general mimo systems using multirate sampling and its application to a cart-crane system. *Automatica*, 43(12):2077–2085, 2007.

[64] L. A. Montestruque and P. Antasklis. Stability of model-based networked control systems with time-varying transmission times. *IEEE Transactions on Automatic Control*, 49(9):1562–1572, 2004.

[65] Y. S. Moon, P. Park, W. H. Kwon, and Y. S. Lee. Delay-dependent robust stabilization of uncertain state-delayed systems. *International Journal of Control*, 74(14):1447–1455, 2001.

[66] J. R. Moyne and D. M. Tilbury. The emergence of industrial control networks for manufacturing control, diagnostics, and safety data. *Proceedings of the IEEE*, 95(1):29–47, 2007.

[67] S. I. Niculescu, C. E. de Souza, J. M. Dion, and L. Dugard. Robust stability and stabilization of uncertain linear systems with state delay: single delay

case. In *Proceedings of* IFAC *Symposium on Robust Control Design*, pages 469–474, 1994.

[68] J. Nilsson. *Real-Time Control Systems with Delays*. PhD thesis, Dept. Automatic Control, Lund Institute of Technology, Lund, Sweden, 1998.

[69] J. Nilsson, B. Bernhardsson, and B. Wittenmark. Stochastic analysis and control of real-time systems with random time delays. *Automatica*, 34(1):57–64, 1998.

[70] J. W. Overstreet and A. Tzes. An Internet-based real-time control engineering laboratory. *IEEE Control Systems Magazine*, 19(5):19–34, 1999.

[71] D. H. Owens. Adaptive stabilization using a variable sampling rate. *International Journal of Control*, 63(1):107–117, 1996.

[72] Park P. A delay-dependent stability criterion for systems with uncertain time-invariant delays. *IEEE Transactions on Automatic Control*, 44(4):876–877, 1999.

[73] H. S. Park, Y. H. Kim, D. S. Kim, and W. H. Kwon. A scheduling method for the network-based coontrol systems. *IEEE Transactions on Control Systems Technology*, 10(3):318–330, 2002.

[74] A. Rabello and A. Bhaya. Stability of asynchronous dynamical systems with rate constraints and application. *IEE Proceedings on Control Theory and Applications*, 150(5):546–550, 2003.

[75] J. B. Rawling and K. R. Muske. The stability of constrained receding horizon control. *IEEE Transactions on Automatic Control*, 38(10):1512–1516, 1993.

[76] J. Richalet. Industrial applications of model based predictive control. *Automatica*, 29(5):1251–1274, 1993.

[77] J. Richalet, A. Rault, J. L. Testud, and J. Papon. Model predictive heuristic control: applications to industrial processes. *Automatica*, 14(5):413–428, 1978.

[78] J. A. Rossiter. *Model-Based Predictive Control: a Practical Approach*. CRC Press LLC, Florida, 2003.

[79] A. Sala. Computer control under time-varying sampling period: an LMI gridding approach. *Automatica*, 41(12):2077–2082, 2005.

[80] M. Schinkel and W. H. Chen. Control of sampled-data systems with variable sampling rate. *International Journal of Systems Science*, 37(9):609–618, 2006.

[81] P. Seiler and R. Sengupta. Analysis of communication losses in vehicle control problems. In *Proceedings of the American Control Conference*, pages 1491–1496, 2001.

[82] P. Seiler and R. Sengupta. An H_∞ approach to networked control. *IEEE Transactions on Automatic Control*, 50(3):356–364, 2005.

[83] P. Shi, E. K. Boukas, and R. K. Agarwal. Control of Markovian jump discrete-time systems with norm bounded uncertainty and unknown delay. *IEEE Transactions on Automatic Control*, 44(11):2139–2144, 1999.

[84] P. Shi, E. K. Boukas, and R. K. Agarwal. Kalman filtering for continuous-time uncertain systems with Markovian jumping parameters. *IEEE Transactions on Automatic Control*, 44(8):1592–1597, 1999.

[85] P. Shi, E. K. Boukas, S. P. Shue, and R. K. Agarwal. H_∞ control of discrete-time linear uncertain systems with delayed-state. In *Proceedings of the 37th IEEE Conference on Decision and Control*, volume 4, pages 4551–4552, 1998.

[86] B. Sinopoli, L. Schenato, M. Franceschetti, K. Poolla, and S. Sastry. Time varying optimal control with packet losses. In *Proceedings of the 43th IEEE Conference on Decision and Control*, pages 1938–1943, 2004.

[87] B. Sinopoli, L. Schenato, M. Franceschetti, K. Poolla, and S. Sastry. An LQG optimal linear controller for control systems with packet losses. In *Proceedings of the 44th IEEE Conference on Decision and Control, and the European Control Conference*, pages 458–463, 2005.

[88] R. E. Skelton and M. DeLorenzo. Space structure control design by variance assignment. *Journal of Guidance, Control, and Dynamics*, 8(4):454–462, 1985.

[89] V. Suplin, E. Fridman, and U. Shaked. Sampled-data H_∞ control and filtering: nonuniform uncertain sampling. *Automatica*, 43(6):1072–1083, 2007.

[90] P. L. Tang and C. W. De Silva. Compensation for transmission delays in an ethernet-based control network using variable-horizon predictive control. *IEEE Transactions on Control Systems Technology*, 14(4):707–718, 2006.

[91] T. J. Tarn and N. Xi. Planning and control of Internet-based teleoperation. In *Proceedings of SPIE: Telemanipulator and Telepresence Technologies*, pages 189–193, 1998.

[92] Y. Tipsuwan and M. Y. Chow. Network-based controller adaptation based on QoS negotiation and deterioration. In *The 27th Annual Conference of the IEEE Industrial Electronics Society (IECON 01)*, volume 3, pages 1794–1799, 2001.

[93] Y. Tipsuwan and M. Y. Chow. Gain adaptation of networked mobile robot to compensate QoS deterioration. In *the 28th Annual Conference of the IEEE Industrial Electronics Society (IECON 02)*, volume 4, pages 3146–3151, 2002.

[94] M. Velasco, P. Martí, R. Villà, and J. M. Fuertes. Stability of networked control systems with bounded sampling rates and time delays. In *Proceedings of the 32nd IEEE conference on Industrial Electronics Society*, pages 2417–2422, 2005.

[95] G. C. Walsh, H. Ye, and L. G. Bushnel. Stability analysis of networked control systems. *IEEE Transactions on Control systems and Technology*, 10(3):438–446, 2002.

[96] Z. Y. Wan and M. V. Kothare. An efficient off-line formulation of robust model predictive control using linear matrix inequalities. *Automatica*, 39(5):837–846, 2003.

[97] C. H. Wang, Y. F. Wang, and G. C. Ma. Compensation time-varying delays in networked control systems via jump linear system approach. In *5th World Congress on Intelligent Control and Automation (WCICA)*, volume 2, pages 1343–1347, 2004.

[98] S. S. Wang, B. S. Chen, and T. P. Lin. Robust stability of uncertain time-delay systems. *International Journal of Control*, 46(3):963–976, 1987.

[99] Z. Wang, J. Yang, D. Tan, and X. D. Wang. Compensation for the networked control systems with the long time delays. In *IEEE International Conference on Systems, Man and Cybernetics*, pages 3170–3175, 2003.

[100] J. Wu, T. Chen, and S. Xu. Stochastic stabilization and H_∞ control for discrete jumping systems with time delay. *Asian Journal of Control*, 7(3):223–230, 2005.

[101] M. Wu, Y. He, J. H. She, and G. P. Liu. Delay-dependent criteria for robust stability of time-varying delay systems. *Automatica*, 40(8):1435–1439, 2004.

[102] Y. Xia, G. P. Liu, and D. Rees. Predictive control of networked systems with random delay and data dropout. In *Proceeding of* IEEE *International Conference on Networking, Sensing and Control*, pages 643–648, 2006.

[103] L. Xiao, A. Hassibi, and J. P. How. Control with random communication delays via a discrete-time jump linear system approach. In *Proceeding of the American Control Conference*, pages 2199–2204, 2000.

[104] S. Xu and T. Chen. Robust H_∞ control of uncertain stochastic systems with state delay. *IEEE Transactions on Automatic Control*, 47(12):2089–2094, 2002.

[105] S. Xu, T. Chen, and J. Lam. Robust H_∞ filtering for uncertain Markovian jump systems with mode-dependent time delays. *IEEE Transactions on Automatic Control*, 48(5):900–907, 2003.

[106] S. Xu, J. Lam, and T. Chen. Robust H_∞ control for uncertain discrete stochastic time-delay systems. *Systems & Control Letters*, 51(3):203–215, 2004.

[107] J. Y. Yen, Y. L. Chen, and M. Tomizuka. Variable sampling rate controller design for brushless DC motor. In *Prceedings of 41st IEEE Conference on Decision and Control*, pages 462–467, 2002.

[108] M. Yu, L. Wang, G. M. Xie, and T. G. Chu. Stabilization of networked control systems with data packet dropout via switched system approach. In *2004 IEEE International Symposium on Computer Aided Control Systems Design*, pages 362–367, 2004.

[109] D. Yue, Q. L. Han, and J. Lam. Network-based robust H_∞ control of systems with uncertainty. *Automatica*, 41(6):999–1007, 2005.

[110] D. Yue, Q. L. Han, and C. Peng. State feedback controller design of networked control systems. *IEEE Transactions on Circuits and Systems-II*, 51(11):640–644, 2004.

[111] L. Zhang, Y. Shi, T. Chen, and B. Huang. A new method for stabilization of networked control systems with random delays. *IEEE Transactions on Automatic Control*, 50(8):1177–1181, 2005.

[112] L. Zhang, S. Y. Xu, J. Sheng, T. Chen, and B. Huang. Model predictive control for uncertain stochastic systems. *International Journal of Control (accepted)*, 2006.

[113] W. Zhang, M. S. Branicky, and S. M. Phillips. Stability of networked control systems. *IEEE Control Systems Magazine*, 21(2):84–99, 2001.

[114] W. A. Zhang and L. Yu. Output feedback stabilization of networked control systems with packet dropouts. *IEEE Transactions on Automatic Control*, 52(9):1705–1710, 2007.

[115] P. V. Zhivoglyadov and R. H. Middleton. Networked control design for linear systems. *Automatica*, 39(4):743–750, 2003.