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THE UNIVERSITY OF ALBERTA

ON CENTRALITY OF CERTAIN FINITELY GENERATED SOLUBLE GROUPS

by

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled On Centrality of Cerain Finitely Generated Soluble Groups submitted by Shakir H. Nazzal in partial fulfilment of the requirements for the degree of Doctor of Philosophy.

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ABSTRACT

In this thesis, we study three centrality properties for certain classes of finitely generated abelian-by-polycyclic groups. These properties, namely, stuntedness, eremiticity and finite gap number, were first introduced by Lennox and Roseblade in their outstanding paper [3] in 1970. They proved that finitely generated abelian-by-nilpotent groups are stunted, eremitic and of finite gap number, and they asked whether finitely generated abelian by polycyclic groups have such properties.

We show here that finitely generated abelian by polycyclic groups in which the polycyclic quotient is plinth-by-abelian or abelian-by-infinite cyclic with trivial centre, are stunted, eremitic and of finite gap number.

We also show that just non-polycyclic groups are stunted, eremitic and of finite gap number. These groups were first introduced by D.J. Robinson and J.S. Wilson in [11] and they showed that such groups are a special kind of finitely generated abelian-by-polycyclic groups.

For finitely generated abelian-by-polycyclic groups, Lennox and Roseblade proved in [3] that they are *sn*-stunted and *sn*-eremitic, a weaker type of stuntedness and eremiticity. They used in their proof their results mentioned above for finitely generated abelian-by-nilpotent groups. We give here a simpler and unified proof for this result.

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CHAPTER 1

INTRODUCTION AND STATEMENT OF THE MAIN THEOREMS

1.1 Introduction.

The paper "Centrality in Finitely Generated Soluble Groups" by J.C. Lennox and J.E. Roseblade [3] made important contributions to the theory of finitely generated soluble groups. It also raised a whole lot of interesting problems leading to much further research, (see [2],[4],[5],[6],[14],[16]). We are now able to extend their major results proved for finitely generated abelian-by-nilpotent groups to certain class of finitely generated triabelian groups which include the just non-polycyclic groups that were studied by D.J.S. Robinson and J.S. Wilson in [11]. This is made possible mainly by results in [9].

1.2. Definitions and Notation.

In order to give the statement of the main theorems we need the definition of what Roseblade calls in [12] a plinth.

1.2.1 Plinths:

Let G be any group. A plinth for G is a free abelian normal subgroup A of finite positive rank such that no non-trivial subgroup of lower rank is normal in any subgroup of finite index in G. In other words, G and all its subgroups of finite index must act rationally irreducibly on A.

Here is an example of a plinth for a polycyclic group. Let $A = \langle x \rangle \times \langle y \rangle$ be a free abelian group of rank 2 and let the infinite cyclic group $T = \langle t \rangle$ act on A according to the rules

$$x^t = x^2 y \text{ and } y^t xy.$$

It is easily verified that t is an automorphism of A and that A is a plinth for the polycyclic group $\Gamma = T \rtimes A$ the semidirect product of A by T.

It is shown in Lemma 12.1.14 [8] that every infinite polycyclic-by-finite group Γ contains a normal subgroup Γ_0 of finite index which has a plinth A and $\Gamma_0 / c_{\Gamma_0}(A)$ is abelian, where $c_{\Gamma_0}(A)$ is the centralizer of A in Γ_0. We shall say that the group Γ is plinth-by-abelian whenever Γ has a plinth A such that Γ/A is abelian.

1.2.2 Notation:

To simplify the statement of the main theorems we define the class of groups $\mathcal{A}$ as follows:

A group G is in $\mathcal{A}$ if G is a finitely generated soluble group, and contains a normal abelian subgroup M such that $\Gamma = G/M$ is a finite group or an infinite polycyclic group which is metabelian.

The set of all groups $G \in \mathcal{A}$ where Γ is plinth-by-abelian will be denoted by $\mathcal{A}_1$ and the set of those groups in $\mathcal{A}$ where Γ is abelian-by-infinite cyclic with trivial center will be denoted by $\mathcal{A}_2$.

Now let $G \in \mathcal{A}$ and M, Γ as in the definition above, then G acts on M by conjugation with M acting trivially on itself. Thus we have, in fact, M a

Γ-module, and since G is finitely generated and $\Gamma = G/M$ is finitely presented since it is polycyclic, M must be a finitely generated $\mathbb{Z}\Gamma$-module, where $\mathbb{Z}\Gamma$ is the group ring of Γ over $\mathbb{Z}$. Actually M is a Noetherian $\mathbb{Z}\Gamma$-module since $\mathbb{Z}\Gamma$ is a Noetherian ring by the well known theorem of P. Hall in [1], (see Theorem 15.3.3 [10]).

1.2.3. Just non-polycyclic groups:

A soluble group G which is not polycyclic but all its proper quotients, are polycyclic is called a just non-polycyclic group.

It is shown in Theorem 2.1 [11] that every finitely generated soluble group which is not polycyclic has a just non-polycyclic quotient group.

The structure Theorems 2.4, 5.1 and 5.5 of [11] show that a just non-polycyclic group G is either metabelian-by-finite or it is a finite extension of a group $G_0 \in \mathcal{A}_1$. Here is an example of a just non-polycyclic group which is not metabelian-by-finite.

In the example given in section 1.2.1, we have A is a plinth for the polycyclic group $\Gamma = T \rtimes A$, let $M = \mathbb{F}_p A$ the group algebra of A over the field $\mathbb{F}_p$, where p is a prime number. Let A act on M by multiplication and let T act faithfully by conjugation. It is easy to verify that M becomes an $\mathbb{F}_p\Gamma$-module; and the theorem of Bergman, (see p. 194 [11]) shows that every non-zero $\mathbb{F}_p\Gamma$-submodule of M has finite index. It follows from Theorem 2.7(ii) [11] that the corresponding semidirect product of M and Γ is a just non-polycyclic group, which is not metabelian-by-infinite since $T \neq 1$.

1.3 Three centrality concepts.

Now we introduce the three centrality concepts dealt with in [3].

Stuntedness: A group G is said to have upper central height α if and only if $\zeta_\alpha(G) = \zeta_{\alpha+1}(G)$ while $\zeta_\beta(G) < \zeta_{\beta+1}(G)$ for all ordinals $\beta < \alpha$, where $\zeta_\alpha(G)$ is the α-th term of the upper central series of G.

G is said to be centrally stunted, or simply, stunted, of height h if there exists an integer k such that every subgroup of G has central height at most k, and if h is the least such k.

Eremiticity:

A group G is called centrally eremitic, or simply, eremitic of eccentricity e if there exists a positive integer f such that, for any $x \in G$ and any $n > 0$, $c_G(x^n) \leq c_G(x^f)$; and if e is the least such f, where $c_G(x)$ is the centralizer of x in G.

For example, finitely generated abelian-by-nilpotent groups are stunted and eremitic [3]. We shall prove the following results.

THEOREM 1. *If $G \in \mathcal{A}_1$ then G is stunted and eremitic.*

THEOREM 2. *If $G \in \mathcal{A}_2$ then G is stunted and eremitic.*

Finite gap number:

A group G is said to have finite gap number if there exists a positive integer

g such that in any ascending chain

$$c_G(H_1) \leq c_G(H_2) \leq \cdots \leq c_G(H_n) \leq \cdots$$

of centralizers of subgroups $H_1 \geq H_2 \geq \cdots \geq H_n \geq \ldots$ of G there are at most g strict inclusions, or as is sometimes called, at most g gaps.

Finitely generated abelian-by-nilpotent groups, for example, have finite gap number [3]. This result in [3] is given as a corollary to Theorem B [3]. The same proof with only few verbal changes can be used to prove the following as a corollary to Theorems 1 and 2.

THEOREM 3. *If $G \in \mathcal{A}_1$ or $G \in \mathcal{A}_2$ then G has finite gap number.*

As corollaries to these three theorems we shall prove:

COROLLARY 1. *If G is a just non-polycyclic group then G is stunted, eremitic and of finite gap number.*

COROLLARY 2. *If M is a normal abelian subgroup of the finitely generated group G, and if $G/M = \Gamma$ is polycyclic of Hirsch number $h(\Gamma) = 3$ then G is stunted, eremitic and of finite gap number.*

1.4. <u>Modules and Pairs</u>.

Let Γ be any group. We shall usually write a Γ-module M multiplicatively and write the image of an element m of M under the action of x in Γ as m^x.

Effectively, following Lennox and Roseblade in [3], we shall be thinking of the pair (M, Γ) as being embedded in the natural split extension $G = M\Gamma$ with $M \lhd G$ and $M \cap \Gamma = 1$. It then makes sense to use the commutator notation $[m, x] = m^{-1}m^x$ for $m \in M$ and $x \in \Gamma$. We shall write for any ordinal α and any subgroup H of Γ,

$$M_\alpha(H) = M \cap \zeta_\alpha(MH).$$

Thus for a positive integer n, the subgroup $M_n(H)$ of M consists of those $m \in M$ for which $[m, h_1, \ldots, h_n] = 1$ for all n-tuples $h_1, \ldots, h_n$ of elements of H.

1.4.1 Pairs and Stuntedness:

For a Γ-module M, the pair (M, Γ) is said to be stunted if there is an integer $h \geq 0$ such that for all subgroups H of Γ the equality $M_\alpha(H) = M_h(H)$ holds for all $\alpha \geq h$. The least such integer h is called the height of (M, Γ). We also say that the pair (M, Γ) is sn-stunted of height h if the above equality holds for subnormal subgroups H of Γ.

1.4.2. Pairs and Eremiticity:

The pair (M, Γ) is said to be eremitic of eccentricity e if there is an integer $f > 0$ such that for all $n > 0$ and all $x \in \Gamma$ the inclusion $M_1(x^n) \leq M_1(x^f)$ holds, and e is the least such f, where $M_1(x^n) = \{m \in M : [m, x^n] = 1\}$.

The pair (M, Γ) is said to be sn-eremitic of eccentricity e if there is an integer $f > 0$ such that for all $n > 0$ and all $H sn \Gamma$,

$$M_1(H^n) \leq M_1(H^e),$$

and e is the least such f. Here $H sn \Gamma$ means H is subnormal in Γ, and H^n is the subgroup $\langle h^n, h \in H \rangle$ generated by all n-th powers of elements of H.

Now given any group $G \in \mathcal{A}$ we have, by Theorem 15.3.1 [10] that G has max-n, the maximal condition on normal subgroups, and by Lemma 13 [3], $\epsilon(G) < \infty$, where $\epsilon(G)$ is the least upper bound of orders of all torsion elements of G. Thus if (M, Γ) is the pair associated with G, then since Γ is stunted and eremitic by Corollaries A and B [3], we get, by Lemma 17 (i) and (ii) [3], that Theorems 1 and 2 are consequences of

THEOREM 1*. *If M is a Noetherian Γ-module where Γ is a polycyclic group which is plinth-by-abelian, then (M, Γ) is stunted and eremitic.*

THEOREM 2*. *If M is a Noetherian Γ-module where Γ is a polycyclic group which is abelian-by-infinite cyclic with trivial center, then (M, Γ) is stunted and eremitic.*

1.4.3. Pairs and Finite Gap Number

The pair (M, Γ) is said to have a finite gap number if there is a non-negative integer g such that in any sequence

$$M_1(H_1) \leq M_1(H_2) \leq \cdots \leq M_1(H_n) \leq \cdots$$

of M-centralizers of subgroup $H_1 \geq H_2 \geq \cdots \geq H_n \geq \ldots$ of Γ, there are at most g gaps.

In exactly the same way Theorem C [3] is deduced from Theorem C* of [3], we shall deduce Theorem 3 from the more general theorem.

THEOREM 3*. *If M is a Noetherian Γ-module where Γ is polycyclic and if (M, Γ) is eremitic then (M, Γ) has finite gap number.*

1.4.4. Further applications of the isolator property in polycyclic groups.

The proofs of the above theorems depend heavily on the isolator property implied in Proposition 1 [9] for a suitable subgroup of a polycyclic group.

Using this result we can show that the proofs of Theorems E* and F in [3] are considerably simplified and we present at the end of Chapter 2 alternate proofs for them. We state here for easy reference:

THEOREM E* [3]. *If M is a Noetherian Γ-module, where Γ is a polycyclic-by-finite group then the pair (M, Γ) is both sn-stunted and sn-eremitic.*

THEOREM F [3]. *Suppose Δ is a subgroup of the polycyclic group Γ. There is a positive interger $d = d(\Delta, \Gamma)$ such that $|H : H \cap \Delta|$ divides d for all subgroups H of Γ for which $|H : H \cap \Delta|$ is finite.*

1.5. The isolator property

1.5.1 Discussion and Notation

A group G is said to have the strong isolator property if for every subgroup

H of G the set

$$\overline{H} = \{x \in G : x^n \in H \text{ for some } n > 0\}$$

is a subgroup and $|\overline{H} : H| < \infty$ whenever $H / \bigcap_{g \in G} H^g$ is finitely generated. We notice that the last requirement is automatically satisfied when G is polycyclic.

The set $\overline{H}$ is called the isolator of H in G and to emphasize the group G we will sometimes write it as ${}^G\overline{H}$, and H is called isolated in G if $H = {}^G\overline{H}$. We note that $\overline{H}$ is always isolated in G.

A crucial fact about polycylic groups is Proposition 1 [9]. It shows that a polycyclic group always has a normal subgroup of finite index with the strong isolator property.

A very important property of a torsion-free polycyclic group G with the strong isolator property is the fact in Proposition 5 and its corollary, together with the remark at the end of Proposition 6 [9]. In such a group the Fitting subgroup Fitt(G) and all centralizers are isolated in G, also the operators isolator and normalizer commute, i.e. for any subgroup H of G, $\overline{n_G(H)} = n_G(\overline{H})$, where [illegible] H) is the normalizer of H in G. All these facts will be used very often in the proof of the main theorems without further mention.

1.6. Statement of frequently used facts

We list in this section for easy reference the statement of major facts that will be frequently used in the proofs of the main theorems. The name and number of each fact will be retained as it appeared in original paper. Terms and symbols that appeared in these statements are explained wherever they are used in this thesis.

Theorems A^* and B^*, [3]: If Γ is a finitely generated nilpotent group and M is a Noetherian Γ-module then (M, Γ) is stunted and eremitic.

Corollaries A and B, [3]: Every nilpotent-by-nilpotent-by-finite group with max-n is stunted and eremetic.

Lemma 4, [3]: Let B be a submodule of the Γ-module A. Suppose that the periodic part of B has exponent ϵ. If (B, Γ) and $(A/B, \Gamma)$ are both eremitic then (A, Γ) is eremitic. If (B, Γ) and $(A/B, \Gamma)$ are both stunted then (A, Γ) is stunted.

Lemma 17, [3]: If A is an abelian normal subgroup of the group G and that $\Gamma = G/A$. If Γ and (A, Γ) are both stunted and eremitic and if $\epsilon(A) < \infty$, then G is stunted and eremitic.

Corollaries 16 and 17, [3]: Suppose K is a stunted and eremitic group with $\epsilon(K) < \infty$. If G is a finite extension of K then G is also stunted and eremitic.

Lemma 19, [3]: Suppose Γ is polycyclic-by-finite and $e > 0$. There is an integer $m = n(e, \Gamma)$ such that H/H^e is finite of order dividing m for every subgroup H of Γ.

Corollary 10.5, [7]: Let A be a finitely generated torsion-free abelian group and let G be a group of operators on A with $D_A(G) = 1$. If Q is a faithful prime ideal of KA, then $\bigcap_{g \in G} Q^g = 0$.

CHAPTER 2

PROOFS OF THE MAIN THEOREMS

2.1. Discussion and Layout.

The highly original tool used in [3] was the "Fan Out Lemma" dealing with Γ-modules where Γ is a finitely generated nilpotent group.

Two important properties of Γ were heavily used. These were the isolator property and the normalizer condition of nilpotent groups. The isolator property holds for a suitable subgroup of finite index in any given polycyclic group Γ. This is Proposition 1 [9].

In any finitely generated group G, the normalizer condition is equivalent to nilpotency of G. Thus, a slight modification was needed to the Fan Out Lemma so that it can be used in the proof of some special cases of the main theorems, and two more versions of this lemma were needed to serve in the proof of other cases. We first give the statement of these three Fan Out Lemmas, which we denote by F.O.L., together with some related results in section 2.2. We deduce the proofs for the main theorems from F.O.L. in the rest of this chapter, and in Chapter 3 we present the proofs of the F.O.L..

2.2 Statement of the F.O.L.

2.2.1 F.O.L. I.

Let M be any $J\Gamma$-module, where J is any commutative ring with 1, and Γ is a torsion-free polycyclic group with the strong isolator property, with a normal abelian subgroup A such that $c_\Gamma(A) < \Gamma$.

Suppose that $M_1(\langle x\rangle) \neq 1$ for some $x \notin c_\Gamma(A)$ and if $N = n_\Gamma\langle x\rangle$ then $\Gamma = AN$, where $n_\Gamma(\langle x\rangle)$ is the normalizer of x in Γ.

Let $U = M_1(\langle x\rangle)$ and $Y = U^\Gamma$. Choose a set T of coset representatives of N in Γ so that $T \leq A$. If M is a torsion-free JA-module, then

(i) $Y = Dr_{t\in T} U^t$, where Dr denotes the direct product.

(ii) U is a Noetherian JN-module whenever M is Noetherian $J\Gamma$-module

(iii) $Y_n(H) = Dr_{t\in T_H} U_n^t(H)$, for all $H \leq \Gamma$ and all $n \geq 0$, where

$$Y_n(H) = Y \cap M_n(H) \text{ and}$$

$$T_H = \{t \in T : H \leq N^t\}.$$

In order to use the F.O.L.I we need the following result which is stated as Exercise 1 in [15] page 47.

2.2.2. Lemma. *Let Γ be a torsion-free polycyclic group with the strong isolator property which has a non-central plinth A and Γ/A is abelian. Let $x \notin c_\Gamma(A)$ and $N = c_\Gamma(x)$, then $A \cap N = 1$ and $|\Gamma : AN| < \infty$.*

Proof: Let $X = \langle x \rangle$. Since Γ/A is abelian, $AX \triangleleft \Gamma$ and $[AX, AX] \triangleleft \Gamma$ because $[AX, AX]$ is characteristic in AX. Also $[AX, AX] \leq A$.

Now since $[A, X] \triangleleft A$ and for any a in A, $a^x = a[a, x] \equiv a \bmod [A, X]$, $A/[A, X]$ is centralized by X and since X is cyclic, $AX/[A, X]$ is abelian. Hence $[AX, AX] \leq [A, X]$. We show next that $[A, X] \leq [A, x]$. Let a in A and n any integer, then

$$[a, x^n] = [a, x^{n-1}][a, x]^{x^{n-1}} = [a, x^{n-1}][a^{x^{n-1}}, x] = \dots$$
$$\dots = [a \dots a^{x^{n-1}}, x] \in [A, x].$$

Since it is obvious that $[A, x] \leq [A, X] \leq [AX, AX]$, we get from above that $[A, x] = [A, X] = [AX, AX]$ and therefore $[A, x] \triangleleft \Gamma$ and $[A, x] \leq A$ for any $x \notin c_\Gamma(A)$.

We have $[A, x] \neq 1$ since $x \notin c_\Gamma(A)$ and it is a normal subgroup of Γ contained in the plinth A, hence $|A : [A, x]| < \infty$. Let $* : A \to A/[A, x]$ be the canonical homomorphism, so that $A^* = A/[A, x]$ is a finite abelian group.

Let $C = c_\Gamma(A^*)$ and notice that A^* is a finite group which is Γ-invariant; hence $|\Gamma : C| < \infty$. Now let $\theta : C \to A^*$ be defined by

$$\theta(g) = [x, g]^*, \quad g \in C.$$

Notice that $[x, g] \in A$ since Γ/A is abelian. We show that θ is a homomorphism:

$$[x, g_1 g_2]^* = [x, g_2]^*([x, g_1]^{g_2})^*, \; g_1, g_2 \in C$$
$$= [x, g_2]^*[x, g_1]^*, \text{ since } g_2 \in C$$
$$= [x, g_1]^*[x, g_2]^*, \text{ since } A^* \text{ is abelian.}$$

Thus θ is a homomorphism. We show next that $\ker\theta \leq AN$. Let $g \in \ker\theta$ then $\theta(g) = [x,g]^* = 1^*$ and hence $[x,g] \in [A,x]$. Therefore $[x,g] = [a,x]$ for some $a \in A$.

Now $[x,ga] = [x,a][x,g]^a = [x,a][x,g] = [x,a][a,x] = [x,a][x,a]^{-1} = 1$, hence $ga \in c_\Gamma(x) = N$ and this implies that $g \in NA = AN$ and $\ker\theta \leq AN$. Now $|C : \ker\theta| < \infty$ and $|\Gamma : C| < \infty$ implies that $|\Gamma : \ker\theta| < \infty$ and therefore $|\Gamma : AN| < \infty$.

To show that $A \cap N = 1$, notice that $A \cap N$ is normal in AN which is of finite index in Γ as shown above, and since $A \cap N$ is contained in the plinth A we must have either $A \cap N = 1$ or $A \cap N$ is of finite index in A. However, if $A \cap N \neq 1$ then since N is isolated in Γ we get $A \cap N$ isolated in A, therefore $A \cap N = A$ and hence A centralizes x and $x \in c_\Gamma(A)$ a contradiction. This shows $A \cap N = 1$ as required.

We note here that $c_\Gamma(\langle x\rangle) = n_\Gamma(\langle x\rangle)$ since centralizers are isolated in Γ.

2.2.3 F.O.L. II

Notation:

Let S be any ring with 1, and M any S-module. We define a set $\Pi_S(M)$ of ideals in S as $\Pi_S(M) = \{P : P$ is an ideal of S maximal with respect to ${}^*P \neq 0\}$, where ${}^*P = \{a \in M : aP = 0\}$. It is well known that when S is Noetherian and M is torsion S-module then $\Pi_S(M)$ contains non-zero elements, and when S is commutative $\Pi_S(M)$ consists of prime ideals. All this will be shown in Chapter 3. Now let J be any Noetherian commutative ring with 1, and Γ any group containing a normal abelian subgroup A.

2.2.4 Statement of F.O.L.II

Let M be any $J\Gamma$-module and $P \in \Pi_{JA}(M)$. Let $N = n_\Gamma(P)$ and T a set of coset representatives of N in Γ and let $U = {}^{*}P$ and $Y = U^\Gamma$ then

(i) $Y = Dr_{t \in T} U^t$

(ii) U is a Noetherian JN-module if M is a Noetherian $J\Gamma$ -module

(iii) If N is isolated then for all $H \leq \Gamma$ and all $h \geq 0$ we have

$$Y_n(H) = Dr_{t \in T_H} \bar{U}_n^t(H), \text{ where}$$

$$Y_n(H) = Y \cap M_n(H), \text{ and}$$

$$T_H = \{t \in T : H \leq N^t\}.$$

We remark that parts (i) and (ii) are Lemma 3 and following comment in [12].

In order to use this lemma in the proofs of the main theorems we need a well known property of subgroups of polycyclic groups. We state this in

2.2.5. LEMMA. *If Γ is polycyclic and $H \leq K$ are subgroups of Γ with $|K : H| < \infty$, then there is a subgroup Γ_1 of finite index in Γ such that $\Gamma_1 \cap K = H$.*

Proof: We use induction on $n = |K : H|$. If $n = 1$ then $K = H$ and we take $\Gamma_1 = \Gamma$. Assume $n > 1$ and that the lemma is true for all subgroups

$H_1 \leq K_1$ of Γ with $|K_1 : H_1| < n$. By Theorem 5.4.16 [10] due to Mal'cev, $H = \bigcap\{L : H \leq L \leq \Gamma \text{ and } |\Gamma : L| < \infty\}$. Since $|K : H| = n > 1$ we have $H < K$ and hence there is a subgroup L of finite index in Γ such that $H \leq L$ and $K \not\leq L$. Therefore, $H \leq L \cap K < K$ and it follows that $|K : L \cap K| > 1$ hence $|L \cap K : H| < n$. By the induction hypothesis there is a subgroup Γ_0 of finite index in Γ so that $\Gamma_0 \cap (L \cap K) = H$. Let $\Gamma_1 = \Gamma_0 \cap L$ then $|\Gamma : \Gamma_1| < \infty$ and $\Gamma_1 \cap K = H$ as required.

2.2.6 COROLLARY. *If Γ is polycyclic with the strong isolator property and $H \leq \Gamma$ then there is a subgroup Γ_1 of finite index in Γ containing H as an isolated subgroup. If $H \lhd \Gamma$ then Γ_1 can be chosen to be normal.*

Proof: Since Γ has the strong isolator property, $\overline{H}$ is a subgroup and since Γ is polycyclic $|\overline{H} : H| < \infty$. By the lemma there is a subgroup Γ_1 of finite index in Γ with $H \leq \Gamma_1$ and $\Gamma_1 \cap \overline{H} = H$. Obviously H is isolated in Γ_1.

If $H \lhd \Gamma$, then $H \leq \bigcap_{s \in \Gamma} \Gamma_1^s$ and $\bigcap_{s \in \Gamma} \Gamma_1^s$ is normal and it is of finite index in Γ since Γ_1 is of finite index in Γ.

2.2.7 F.O.L. III

Notation:

Let J be any commutative ring with 1, and let Γ be a torsion-free polycyclic group with the strong isolator property. Let M be any $J\Gamma$-module and define a set $\mathcal{X}(M,\Gamma)$ of subgroups of Γ as follows:

$X \in \mathcal{X}(M,\Gamma)$ if and only if X is maximal with respect to having the properties:

(i) $X sn \Gamma$

(ii) $M_1(X) > 1$

(iii) If $X < H \leq \Gamma$ and H has properties (i) and (ii) then $|H : X| < \infty$.

(iv) $X = \widetilde{X}^n$ for some $n > 0$.

It will be shown in chapter 3 that when Γ contains a non-trivial subnormal subgroup H_0 with $M_1(H_0) > 1$ then $\mathcal{X}(M,\Gamma)$ contains non-trivial elements, and that each one has isolated normalizer in Γ.

2.2.8 Statement of the F.O.L. III

Let $1 \neq X \in \mathcal{X}(M,\Gamma)$ and $N = n_\Gamma(X)$ and let T be a set of coset representatives of N in Γ. Let $U = M_1(X)$ and $Y = U^\Gamma$ then

(i) $Y = Dr_{t\in T} U^t$

(ii) U is a Noetherian N-module if M is a Noetherian Γ-module.

(iii) $Y_n(H) = Dr_{t\in T_H} U_n^t(H)$ for all $H \leq \Gamma$ and $n \geq 0$, where

$$Y_n(H) = Y \cap M_n(H) \text{ and,}$$

$$T_H = \{t \in T : H \leq N^t\}.$$

2.3. Proofs of the Main Theorems.

We give in this section the proofs of the main theorems and we start with a preliminary result and a reduction.

2.3.1 LEMMA. *If (M,Γ) is stunted and eremitic whenever M is a Noetherian $K\Gamma$-module, where K is any field, then (M,Γ) is stunted and eremitic when M is a Noetherian $\mathbb{Z}\Gamma$-module.*

Proof: Let B be the torsion subgroup of M, then M/B is a torsion free abelian group. Since B is characteristic in M, it must be Γ-invariant. Thus B is a Γ-submodule of M which is a torsion $\mathbb{Z}$-module and M/B is a torsion free $\mathbb{Z}$-module. By Lemma 4 [3] we may consider each case separately.

Case I: M is a torsion $\mathbb{Z}$-module.

Since M has finite exponent we can find a finite series of $\mathbb{Z}\Gamma$-submodules of M such that each of its factors is a Noetherian $\mathbb{Z}_p\Gamma$-module for some prime divisor p of the exponent of M depending on the factor. Thus if W is any factor of this series then (W,Γ) is by assumption, stunted and eremitic. By Lemma 4 [3] the pair (M,Γ) is stunted and eremitic as required.

Case II: M is a torsion-free $\mathbb{Z}A$-module.

Form the abelian group $\widetilde{M} = Q \otimes_{\mathbb{Z}} M$, where Q is the field of rational numbers. Then $\widetilde{M}$ can be made into a Q-vector space by defining $(r \otimes m) \cdot q =$

$(rq) \otimes m$ for $r, q \in Q$ and $m \in M$. Also we can define a natural Γ-module structure on $\widetilde{M}$ by

$$(r \otimes m) \cdot g = r \otimes (m \cdot g),\ g \in \Gamma.$$

Since M is a Noetherian $\mathbf{Z}\Gamma$-module, $\widetilde{M}$ is a Noetherian $Q\Gamma$-module, and by assumption $(\widetilde{M}, \Gamma)$ is stunted and eremitic.

Therefore, there is an integer $h \geq 0$ such that for any $H \leq \Gamma$ the ascending series of abelian subgroups $\{\widetilde{M}_n(H)\}$ of $\widetilde{M}$ has at most h strict inculsions, and there is also an integer $e > 0$ such that for any $x \in \Gamma$ the ascending series of abelian subgroups $\{\widetilde{M}_1(x^n)\}$ of $\widetilde{M}$ has at most e strict inclusions.

It follows that the series $\{\widetilde{M}_n(H) \cap (1 \otimes M)\}$ and $\{\widetilde{M}_1(H) \cap (1 \otimes M)\}$ have at most h and e distinct terms respectively. Now, in the notation of Chapter 1 we have

$$\{\widetilde{M}_n(H) \cap (1 \otimes M)\} = (1 \otimes M)_n(H)$$

and

$$\{\widetilde{M}_1(x^n) \cap (1 \otimes M)\} = (1 \otimes M)_1(x^n)$$

Thus in the abelian subgroup $1 \otimes M$ of $\widetilde{M}$ the series of centralizers of any $H \leq \Gamma$, $\{(1 \otimes M)_n(H)\}$ breaks off after at most h steps and for any $x \in \Gamma$, the series $\{(1 \otimes M)_1(x^n)\}$ breaks off after at most e steps.

But M is isomorphic, as an abelian group to $1 \otimes M$. Hence for all $H \leq \Gamma$ and all $n \geq h$, $M_n(H) = M_h(H)$, that is to say (M, Γ) is stunted, and for all $x \in \Gamma$ and $n > 0$ we have $M_1(x^n) \leq M_1(x^e)$ hence (M, Γ) is eremitic as required.

2.3.2 A Reduction

In order to prove Theorems 1^* and 2^* we notice that if (M, Γ) is the pair associated with the group G as defined in 1.2.2, where $G \in \mathcal{A}_1$ or $G \in \mathcal{A}_2$, then in either case $\Gamma = G/M$ is polycyclic. Now if Γ is finite then both Theorems 1^* and 2^* follow from Corollaries 16 and 17 [3], thus we need to prove Theorems 1^* and 2^* only when Γ is infinite polycyclic.

By Theorem 5.4.15 [10], Γ contains a normal subgroup Γ_1 of finite index which is torsion-free, and by Proposition 1 [9], Γ_1 contains a subgroup Γ_0 of finite index which has the strong isolator property. Therefore Γ_0 is of finite index in Γ and it is torsion-free polycyclic with the strong isolator property.

Now, since $|\Gamma : \Gamma_0| < \infty$ it is easy to see that if Γ is plinth-by-abelian then Γ_0 is also plinth-by-abelian, and if Γ is abelian-by-infinite cyclic with trivial center then Γ_0 is abelian-by-infinite cyclic with trivial center.

Moreover, M is still a Noetherian Γ_0-module, and if (M, Γ_0) is stunted and eremitic then, by Corollaries 16 and 17 [3], the pair (M, Γ) will be stunted and eremitic.

Let β be the class of torsion-free polycyclic groups with the strong isolator property and β_1 the subclass of all groups of β which are plinth-by-abelian, and let β_2 be the subclass of β containing groups which are abelian-by-infinite cyclic with trivial center.

It follows from the above discussion that we need to prove Theorem 1^* only for pairs (M, Γ), where M is a Noetherian Γ-module and $\Gamma \in \beta_1$, and Theorem 2^* only for (M, Γ), where $\Gamma \in \beta_2$.

We shall prove both theorems by contradiction. That is, we assume in both theorems, that the pair (M, Γ) is not both stunted and eremitic and we must obtain a contradiction.

Let D be a submodule of M chosen maximal with respect to the property that $(M/D, \Gamma)$ is not both stunted and eremitic. We may assume that D is trivial and we must produce a contradiction. In other words, we assume that for every non-trivial submodule B of M, the pair $(M/B, \Gamma)$ is both stunted and eremitic. Thus (M, Γ) is assumed to be a minimal counter-example.

We remark that for any subgroup Δ of finite index in Γ the pair (M, Δ) is also a minimal counter-example, for then M is a Noetherian Δ-module and (M, Δ) cannot be both stunted and eremitic by Corollaries 16 and 17 [3].

By Lemma 4 [3] we need only show that there is some non-trivial Δ-submodule B of M such that (B, Δ) is stunted and eremitic, where Δ is any subgroup of finite index in Γ. Furthermore, by Lemma 2.3.1 we may assume also that M is a Noetherian $K\Gamma$-module where K is a field.

2.3.3 Proof of Theorem 1^*.

M is a Noetherian $K\Gamma$-module, where K is a field and $\Gamma \in \beta_1$, and (M, Γ) is a minimal counter-example. As remarked in 2.3.2, to establish the theorem, we need only show that there is some non-trivial Δ-submodule B of M such that (B,Δ) is stunted and eremitic, where Δ is any subgroup of finite index in Γ.

Let A be a plinth for Γ such that Γ/A is abelian, then by assumption A is not central, otherwise Γ will be nilpotent and by Theorems A^* and B^* [3], (M,Γ) will be stunted and eremitic, a contradiction.

Let $S = \{X \leq \Gamma : X$ is maximal with respect to having the property $M_1(X) \neq 1\}$. Then by assumption $S \neq \{1\}$. We consider two cases:

<u>Case I</u>: $X \cap c_\Gamma(A) = 1$ for all $X \in S$.

Considering M as a KA-module, we recognize two subcases:

(a). M is a torsion-free KA-module. By assumption there is $1 \neq X \in S$, hence there is $x \notin c_\Gamma(A)$ such that $M_1(\langle x \rangle) \neq 1$. Let $N = c_\Gamma(\langle x \rangle)$, then by Lemma 2.2.2, $A \cap N = 1$ and $|\Gamma : AN| < \infty$. We note that $N = n_\Gamma(\langle x \rangle)$ since centralizers are isolated in Γ. By the remark in 2.3.2 we may assume

$\Gamma = AN$ hence the F.O.L.I holds for the pair (M,Γ).

Therefore, if $U = M_1(\langle x \rangle)$ then by F.O.L.I(ii), U is a Noetherian KN-module. Since $A \cap N = 1$ and Γ/A is abelian, so is N, and it follows that (U,N) is stunted and eremitic by Theorems A^* and B^* [3].

Let $Y = U^\Gamma$ and T a set of coset representatives of N in Γ. We shall show that (Y,Γ) is stunted and eremitic.

Let h be the height of (U,N) and $\grave{e}$ its eccentricity. Then for every $t \in T$ the conjugate pairs (U^t, N^t) are all stunted of same height h and all are eremitic of same eccentricity e. Therefore for $t \in T$,

$$\text{if } H \leq N^t \text{ and } n \geq h \text{ then } U_n^t(H) = U_h^t(H) \tag{1}$$

and

$$\text{if } x \in N^t \text{ and } n > 0 \text{ then } U_1^t(x^n) \leq U_1^t(x^e) \tag{2}$$

From F.O.L.I(iii) we have,

$$\text{for every } H \leq \Gamma \text{ and } n \geq 0,\ Y_n(H) = Dr_{t \in T_H} U_n^t(H) \tag{3}$$

Now (1) and (3) show that $Y_n(H) = Y_h(H)$, hence (Y, Γ) is stunted.

To show that (Y, Γ) is eremitic, we notice that in (3) if $n > 0$ and $U_1^t(x^n) \neq 1$, then $t \in T_{\langle x^n \rangle}$. Therefore $\langle x^n \rangle \leq N_f^t$ and since N^t is isolated we have $\langle x \rangle \leq N^t$, hence $t \in T_{\langle x \rangle}$.

Therefore, (3) can be written as

$$\text{if } n \geq 0 \text{ and } x \in \Gamma \text{ then } Y_1(x^n) = Dr_{t \in T_{\langle x \rangle}} U_1^t(x^n) \tag{3'}$$

Now (2) and (3′) show that $Y_1(x^n) \leq Y_1(x^e)$, hence (Y, Γ) is eremitic as required. Since Y is a non-trivial submodule of M, we have the result established in this case.

(b). M is a torsion KA-module.

We shall use additive notation for the module M. Since KA is a Noetherian and commutative ring with 1 (see Theorem 15.3.3 [10]), and since M is a torsion KA-module, we can find a non-zero prime ideal $P \in \Pi_{KA}(M)$. Let $N = n_\Gamma(P)$, then by Corollary 2.2.6 we can find a subgroup Γ_0 of finite index in Γ containing

N as an isolated subgroup. By the remark in 2.3.2 we may assume $\Gamma = \Gamma_0$ and N is isolated in Γ. If $U = {}^{*}P$, then by F.O.L.II(ii), U is a Noetherian KN-module.

Since $X \cap A = 1$ for all $X \in S$, the ideal P is faithful, that is $(P+1) \cap A = 1$. For if $a \in (P+1) \cap A$ then $a-1 \in P$ and since ${}^{*}P \neq 0$ we can find $0 \neq m \in M$ such that $m(a-1) = 0$. This, in the multiplicative notation, implies that $1 \neq m \in M$ and $m^{a} = m$. Thus $M_1 \langle a \rangle \neq 1$ and $\langle a \rangle \leq A$, therefore $a = 1$.

Now $A \lhd N$, and it is a torsion-free finitely generated abelian group with N acting on it by conjugation. And P is a non-zero prime and faithful ideal of KA which is N-invariant. It follows from Corollary 10.5 [7] p. 242, that $D_A(N) \neq 1$, where

$$D_A(N) = \{a \in A : |N : c_N(\langle a \rangle)| < \infty\}.$$

Since $N \in \beta$ we must have

$$\begin{aligned} D_A(N) &= \{a \in A : N = c_N(\langle a \rangle)\} \\ &= c_A(N) \\ &= c_\Gamma(N) \cap A. \end{aligned}$$

Now $c_\Gamma(N)$ is normal in Γ because $N \lhd \Gamma$, and also $c_\Gamma(N)$ is isolated in Γ. Hence $D_A(N)$ is a non-trivial normal subgroup of Γ which is isolated in the plinth A, therefore $A = D_A(N) = c_A(N)$, and A is central in N.

Now N is a finitely generated nilpotent group, and by Theorems A^{*} and B^{*} [3], the pair (U, N) is stunted and eremitic. By F.O.L.II(iii) and an argument similar to that in part (a), we may conclude that (Y,Γ) is stunted and eremitic, where Y is the non-trivial Γ-submodule $U \cdot K\Gamma$. This establishes the result in this case.

<u>Case II</u>. $X \cap c_\Gamma(A) \neq 1$ for some $X \in \mathcal{S}$. Since $1 \neq (X \cap c_\Gamma(A)) sn \Gamma$ and $M_1(X \cap c_\Gamma(A)) \neq 1$ we can find, by Lemma 3.3.2, a non-trivial subgroup $X_0 \in \mathfrak{X}(M,\Gamma)$. By Corollary 2.2.6 we can find a subgroup Γ_0 of finite index in Γ such that X_0 is an isolated subgroup of Γ_0, and by the remark in 2.3.2 we may assume $\Gamma = \Gamma_0$ and X_0 is isolated in Γ.

We show first that either $X_0 \leq c_\Gamma(A)$ or $A \leq X_0$. Since $X_0 \in \mathfrak{X}(M,\Gamma)$, we can find by Lemma 3.3.3 part(2) a subnormal series from X_0 to Γ which is composed of isolated subgroups, say

$$X_0 \lhd X_1 \lhd X_2 \lhd \cdots \lhd X_{k-1} \lhd X_k = \Gamma.$$

Now $X_{k-1} \lhd \Gamma$, therefore $[A, AX_{k-1}] \leq A \cap X_{k-1}$.
If $[A, AX_{k-1}] = 1$ then $X_{k-1} \leq c_\Gamma(A)$ and hence $X_0 \leq c_\Gamma(A)$.
If $[A, AX_{k-1}] \neq 1$ then since it is normal in Γ and it is contained in the plinth A we have $|A : [A, AX_{k-1}]| < \infty$. Hence $A \cap X_{k-1} = A$ because it is isolated in A. Thus $A \leq X_{k-1}$.

Now if $A \leq X_{k-1}$ then, since $X_{k-2} \lhd X_{k-1}$, we have $[A, AX_{k-2}] \leq A \cap X_{k-2}$ and repeating the above argument we conclude that either $X_{k-2} \leq c_\Gamma(A)$ or $A \leq X_{k-2}$. Continuing the same argument for $X_{k-3}, X_{k-4}, \ldots$ we get, finally to the conclusion that either $X_0 \leq c_\Gamma(A)$ or $A \leq X_0$ as claimed.

If $A \leq X_0$ then $X_0 \lhd \Gamma$. Let $U = M_1(X_0)$ and $N = n_\Gamma(X_0) = \Gamma$. Since U is centralized by X_0, it is a Noetherian Γ/X_0-module and (U, Γ) is stunted and eremitic if and only if $(U, \Gamma/X_0)$ is so. But Γ/X_0 is abelian since $A \leq X_0$, therefore, $(U, \Gamma/X_0)$ is stunted and eremitic by Theorems A^* and B^* [3]. Thus (U, Γ) is stunted and eremitic.

Now assume $X_0 \leq c_\Gamma(A)$, and let $* : N \to N/X_0$ be the canonical homomorphism, so that $N^* = N/X_0$. Since X_0 is isolated in Γ, $N^* \in \beta$. Since $X_0 \leq c_\Gamma(A)$, $A \leq N$. Thus $N \lhd \Gamma$ and $A^* = AX_0/X_0 \cong A/A \cap X_0$. Note that N^*/A^* is abelian. By the choice of X_0, $U_1(H^*) = 1$ for all $1 \neq H^* snN^*$.

If (U, N^*) is not both stunted and eremitic, then we shall show that U contains an N^*-submodule $\mathcal{V}$ such that $(\mathcal{V}, N^*)$ is stunted and eremitic. We consider two cases:

(a). U is a torsion-free KA^*-module. If $U_1(x^*) = 1$ for all $1 \neq x^* \in N^*$ then (U, N^*) is stunted and eremitic, thus we assume that $1 \neq x^* \in N^*$ such that $U_1(\langle x^* \rangle) \neq 1$. Since $U_1(H^*) = 1$ for all $1 \neq H^* snN^*$, we have $x^* \notin c_{N^*}(A^*)$. Let $V = U_1(\langle x^* \rangle)$ and $W^* = c_{N^*}(\langle x^* \rangle)$.

We first show that $A^* \cap W^* = 1$. Assume not, and let $Z^* = A^* \cap W^* \neq 1$, so that $[Z^*, x^*] = 1$. Since $1 \neq Z^* \leq A^* \cong A/A \cap X_0$, we can find a subgroup Z of A such that $Z^* \cong Z/A \cap X_0$ and therefore $[Z, x] \leq A \cap X_0$, where x is an element of N in the inverse image of x^*. Now A^* is torsion free and $A^* \neq 1$, therefore $h(A \cap X_0) < h(Z)$.

Let $\phi_x : Z \to A \cap X_0$ be defined as $\phi_x(z) = [z, x]$ for every $z \in Z$. It is easy to see that ϕ_x is a homomorphism and since $h(A \cap X_0) < h(Z)$ we must have $\ker\phi_x \neq 1$. Therefore, there is $1 \neq a \in A$ such that $[a, x] = 1$. Hence $1 \neq a \in \zeta_1(\langle A, x\rangle) \cap A$. But $\langle A, x\rangle \lhd \Gamma$ since Γ/A is abelian, therefore $\zeta_1(\langle A, x\rangle) \lhd \Gamma$ and also it is isolated. Thus $\zeta_1(\langle A, x\rangle) \cap A$ is a non-trivial normal subgroup of Γ which is isolated in the plinth A, hence $A = A \cap \zeta_1(\langle A, x\rangle)$ and $A \leq \zeta_1(\langle A, x\rangle)$. It follows that $x \in c_\Gamma(A) \cap N = c_N(A)$, and $x^* \in c_N(A)^* \leq c_{N^*}(A^*)$ a contradiction. Thus we do have $A^* \cap W^* = 1$ and therefore W^* is abelian.

Next we show that $|N^* : A^*W^*| < \infty$. Let $x \in N$ as above, then $x \notin c_N(A)$ and hence $x \notin c_\Gamma(A)$. Let $C = c_\Gamma(\langle x\rangle)$, then by Lemma 2.2.2, $A \cap C = 1$ and $|\Gamma : AC| < \infty$. Therefore $|N : N \cap AC| < \infty$ and since $A \leq N$ we have $N \cap AC = A(N \cap C)$. But $N \cap C = c_N(\langle x\rangle)$, so we have $|N^* : A^*(c_N(\langle x\rangle))^*| < \infty$. However, $c_N(\langle x\rangle)^* \leq c_{N^*}(x^*) = W^*$, therefore $|N^* : A^*W^*| < \infty$ as required.

We may assume $N^* = A^*W^*$ and use the F.O.L.I for the pair (U, N^*). By F.O.L.I(ii), V is a Noetherian W^*-module, and since W^* is abelian, the pair (V, W^*) is stunted and eremitic. By F.O.L.I(iii) and a similar argument as in case I(a) we may conclude (V^{N^*}, N^*) is stunted and eremitic. Here we let $\mathcal{V} = V^{N^*}$.

(b). U is a torsion KA^*-module. We shall use additive notation for the module U.

Let $0 \neq P \in \Pi_{KA^*}(U)$, then P is a prime ideal of KA^*, and since $U_1(H^*) = 1$ for all $1 \neq H^* sn N^*$, P is faithful. Let $V = {}^*P$ and $W^* = n_{N^*}(P)$. We may assume that W^* is isolated in N^*. By F.O.L.II(ii), V is a Noetherian W^*-module.

Now $A^* \lhd W^*$ and P is a non-zero prime and faithful ideal of KA^* which is W^*-invariant, and since A^* is a torsion-free finitely generated abelian group, it follows by Corollary 10.5 [7] p. 242, that $D_{A^*}(W^*) \neq 1$. Since $N^* \in \beta$, $D_{A^*}(W^*) = c_{A^*}(W^*)$. Let $Z^* = D_{A^*}(W^*)$, then $1 \neq Z^* \leq A^* \cong A/A \cap X_0$, and we can find a subgroup Z of A such that $A \cap X_0 < Z$ with $h(A \cap X_0) < h(Z)$ and $[W, Z] \leq A \cap X_0$, where W is the inverse image in N of W^*.

Let $w \in W$ and define $\phi_w : Z \to A \cap X_0$, by

$$\phi_w(z) = [z, w] \text{ for all } z \in Z.$$

Then ϕ_w is clearly a homomorphism and since $h(A \cap X_0) < h(Z)$, $\ker\phi_w \neq 1$. Therefore, there is $1 \neq z \in A$ such that $[z, w] = 1$, hence $1 \neq z \in \zeta_1(\langle A, w \rangle) \cap A$. Since $\langle A, w \rangle \lhd \Gamma$ we have $\zeta_1(\langle A, w \rangle) \cap A$ is a non-trivial normal subgroup of Γ which is isolated in the plinth A, hence $A = A \cap \zeta_1(\langle A, w \rangle)$ and $w \in c_\Gamma(A) \cap N$. Since this is true for any $w \in W$, we have $W \leq c_\Gamma(A) \cap N = c_N(A)$ and therefore $W^* \leq c_\Gamma(A)^* \leq c_{N^*}(A^*)$.

Thus A^* is central in W^* and since W^*/A^* is abelian, W^* is nilpotent. It follows by Theorems A^* and B^* [3] that (V, W^*) is stunted and eremitic. By F.O.L.II(iii) and an agrument similar to that in case I(a) we may conclude that (V^{N^*}, N^*) is stunted and eremitic. Here too, we let $\mathcal{V} = V^{N^*}$.

Thus far, we have shown that in each case there is an N^*-submodule $\mathcal{V}$ of U such that $(\mathcal{V}, N^*)$ is stunted and eremitic. Since $N^* = N/X_0$ and X_0 centralizes U, $\mathcal{V}$ is an N-submodule of U and $(\mathcal{V}, N)$ is stunted and eremitic.

Now $\mathcal{V}$ is an N-submodule of U hence $\mathcal{V}^{\Gamma} = Dr_{t \in T} \mathcal{V}^{t}$ by F.O.L.III(i) and it follows from F.O.L.III(iii) that if $\widetilde{Y} = \mathcal{V}^{\Gamma}$ then

$$\widetilde{Y}_n(H) = Dr_{t \in T_n} \mathcal{V}_n^t(H) \text{ for all } H \leq \Gamma \text{ and } n \geq 0. \qquad (*)$$

Since $(\mathcal{V}, N)$ is stunted and eremitic, (*) shows as in case I(a), that $(\widetilde{Y}, \Gamma)$ is stunted and eremitic and this establishes the result in this case, and with it, the proof of Theorem 1*.

2.3.4 <u>Proof of Theorem 2*</u>.

M is a Noetherian $K\Gamma$-module, where $\Gamma \in \beta_2$, and (M, Γ) is a minimal counter-example. We need only show that there is some non-trivial Δ-submodule B of M such that (B, Δ) is stunted and eremitic, where Δ is any subgroup of finite index in Γ.

If A is a normal abelian subgroup such that Γ/A is infinite cyclic then by assumption A is not central, otherwise Γ will be nilpotent and by Theorems A^* and B^*, the pair (M, Γ) will not be a counter-example.

Since centralizers in Γ are isolated, $A = c_{\Gamma}(A)$, because Γ/A is infinite cyclic.

Let $S = \{X \leq \Gamma : X$ is maximal with respect to having $M_1(X) \neq 1\}$, then by assumption $S \neq \{1\}$. We consider two cases:

<u>Case I</u>: $X \cap A = 1$ for all $X \in S$.

We have two subcases:

(a). M is a torsion-free KA-module. By assumption there is $x \notin c_{\Gamma}(A) = A$ such that $M_1(\langle x \rangle) \neq 1$. Let $N = c_{\Gamma}(\langle x \rangle)$, then $|\Gamma : AN| < \infty$

since Γ/A is infinite cyclic. We may assume $\Gamma = AN$ and use F.O.L.I for the pair (M, Γ). Let $U = M_1(\langle x \rangle)$ then by F.O.L.I(ii) we have U is a Noetherian N-module.

We show that $A \cap N = 1$. Assume $a \in A \cap N$, then $a \in \zeta_1(\langle A, x \rangle)$. But $\zeta_1(\langle A, x \rangle) = \zeta_1(\Gamma)$ since $|\Gamma : \langle A, x \rangle| < \infty$ and $\Gamma \in \beta_2$. However $\zeta_1(\Gamma) = 1$ by assumption, hence $A \cap N = 1$ and N is abelian.

By Theorems A^* and B^* [3], the pair (U, N) is stunted and eremitic and by F.O.L.I(iii) and a similar argument as in case I(a) of Theorem 1^*, we may conclude that (U^Γ, Γ) is stunted and eremitic. This establishes the result in this case.

(b). M is torsion KA-module. We use here the additive notation for the module M.

Since KA is a Noetherian commutative ring we can find an ideal $0 \neq P \in \Pi_{KA}(M)$. Then P is prime and by the assumption that $A \cap X = 1$ for all $X \in S$, we have P faithful. Let $N = n_\Gamma(P)$, since Γ has a tirival center and $\Gamma \in \beta$ we must have $D_A(\Gamma) = 1$ where,

$$
\begin{aligned}
D_A(\Gamma) &= \{a \in A : |\Gamma : c_\Gamma(\langle a \rangle)| < \infty\} \\
&= \{a \in A : \Gamma = c_\Gamma(\langle a \rangle)\} \\
&= \zeta_1(\Gamma) \cap A.
\end{aligned}
$$

Now A is a torsion-free finitely generated abelian group with Γ acting on it by conjugation and P is a non-zero prime and faithful ideal of KA, then since

$D_A(\Gamma) = 1$, it follows by Corollary 10.5 [7] p. 242, that P can not be orbital, that is $|\Gamma : N| = \infty$. Since $A \leq N$ and Γ/A is infinite cyclic, $N = A$, an isolated abelian subgroup.

Let $U = {}^{\bullet}P$, by F.O.L.II(ii) U is a Noetherian N-module and since N is abelian the pair (U, N) is stunted and eremitic by Theorems $A^{\bullet}$ and $B^{\bullet}$ [3]. By F.O.L.II(iii) and an argument similar to that in case I(a) of Theorem $1^{\bullet}$ we may conclude that (U^{Γ}, Γ) is stunted and eremitic, thus establishing the result in this case.

Case II $X \cap A \neq 1$ for some $X \in S$.

We shall use additive notation for the module M. Since Γ has the maximal condition, there is a maximal element in the set $\{X \cap A : X \in S\}$. Let $X_0 = X \cap A$ be such a maximal element, then by assumption $X_0 \neq 1$, and $M_1(X_0) \neq 0$. We may assume X_0 is isolated in Γ.

Let $I = \sum_{x \in X_0}(x-1)KA$, then I is an ideal of KA and ${}^{\bullet}I \geq M_1(X_0) \neq 0$. Now KA is a Noetherian and commutative ring hence we can find an ideal P of KA such that $P \geq I$ and P is maximal with respect to having ${}^{\bullet}P \neq 0$. Thus $P \in \Pi_{KA}(M)$ and P is a non-zero prime ideal of KA. Moreover, $X_0 = (P+1) \cap A$ for it is clear that $X_0 \leq (P+1) \cap A$.

Assume $X_0 < (P+1) \cap A$ and let $P^+ = (P+1) \cap A$. Since $M_1(P^+) \neq 1$ we can find $X \in S$ such that $P^+ \leq X$. But then $X \cap A \supseteq P^+ > X_0$ contradicting the choice of X_0. Let $N = n_\Gamma(P)$ then $((P+1) \cap A) \lhd N$, hence $X_0 \lhd N$, and we also have $A \leq N$. If we let $U = {}^{*}P$, then by F.O.L.II(ii) U is a Noetherian N-module.

If $N = A$ then N is abelian and isolated in Γ, and by Theorems A^{*} and B^{*} [3], the pair (U, N) is stunted and eremitic. By F.O.L.II(iii) and a similar argument as in case I(a) of Theorem 1^{*}, we may conclude that (U^{Γ}, Γ) is stunted and eremitic and the result is established in this case.

Now assume $A < N$. Thus $|\Gamma : N| < \infty$, and we may assume $\Gamma = N$. Since X_0 centralizes U, it is a Noetherian N/X_0-module. Let $* : N \to N/X_0$ be the canonical homomorphism, then $N^{*} = N/X_0$ and $N^{*} \in \beta$ since X_0 is assumed to be isolated in Γ. Moreover, since $X_0 \leq A$, $A^{*} = A/X_0$ and N^{*}/A^{*} is infinite cyclic. We show that N^{*} has trivial center. Assume $Z^{*} = \zeta_1(N^{*}) \neq 1$, then $Z^{*} \cap A^{*} \neq 1$ and if Z is the inverse image in N of $Z^{*} \cap A^{*}$, then $X_0 < Z$ and $Z \leq A$ and $[Z, N] \leq X_0$. Furthermore, $h(X_0) < h(Z)$ since N^{*} is torsion-free and $Z^{*} \cap X^{*} \neq 1$.

Let $y \in N$ and define the mapping $\phi_y : Z \to X_0$, by

$$\phi_y(z) = [z, y], \text{ for all } z \in Z.$$

Then ϕ_y is clearly a homomorphism and since $h(X_0) < h(Z)$, we have $\ker\phi_y \neq 1$. Hence there is $1 \neq z \in A$ such that $[z, y] = 1$ and this implies $1 \neq z \in \zeta_1(\langle A, y\rangle)$. Since $A < N$ we may take y such that $y \in N \setminus A$, thus $|\Gamma : \langle A, y\rangle| < \infty$ since Γ/A is infinite cyclic and since $\Gamma \in \beta$ we have $\zeta_1(\Gamma) = \zeta_1(\langle A, y\rangle)$. However, $\zeta_1(\Gamma) = 1$ by assumption and we have got a contradiction. This implies that $N^{*} \in \beta_2$.

Next we show that if $H^* \leq N^*$ and $U_1(H^*) \neq 1$ then $A^* \cap H^* = 1$. Suppose $A^* \cap H^* \neq 1$ for some $H^* \leq N^*$ with $U_1(H^*) \neq 1$. Let $H_0^* = A^* \cap H^*$ then $U_1(H_0^*) \neq 1$ and $H_0^* = H_0/X_0$ for some $H_0 \leq N$ such that $X_0 < H_0 \leq A$. Since $U_1(H_0/X_0) \neq 1$, we have $U_1(H_0) \neq 1$ because X_0 centralizes U. Thus $M_1(H_0) \neq 1$ and $X_0 < H_0 \leq A$. Now we can find some $X \in S$ such that $H_0 \leq X$, hence $X \cap A \geq H_0 > X_0$ contradicting the choice of X_0.

Thus the claim is established and we are in the case where we have U is a Noetherian N^*-module with $N^* \in \beta_2$ and $H^* \cap A^* = 1$ for all $H^* \leq N^*$ such that $U_1(H^*) \neq 1$. This is case I with the pair (M, Γ) is replaced by (U, N^*). Hence the result is established in this case and this completes the proof of Theorem 2^*.

2.3.5 Proof of Theorem 3^*:

The following proof of Theorem 3^* is the same one given in [3] for the proof of Theorem C^* except for few verbal changes. We include it here for the sake of completeness.

By assumption the pair (M, Γ) is eremitic of eccentricity e, say. Let

$$H_1 \geq H_2 \geq \cdots \geq H_n \geq \cdots \tag{1}$$

be any sequence of subgroups of Γ and

$$M_1(H_1) \leq M_1(H_2) \leq \cdots \leq M_1(H_n) \leq \cdots \tag{2}$$

be the sequence of M-centralizers of the subgroups $H_1, H_2, \ldots, H_n, \ldots$ above. Since the Hirsch number h of Γ is finite there can be at most h indices n for which $h(H_n) > h(H_{n+1})$. Hence the chain (1) breaks up into at most $h + 1$ stretches, where in each stretch $H_\alpha \geq H_{\alpha+1} \geq \cdots \geq H_\beta \geq \cdots$ all the terms have same Hirsch number.

Let $m = m(e, \Gamma)$ be the integer of Lemma 19 [3]. We prove Theorem 3* by showing that in any such stretch there are at most $\log_2 m$ gaps in the corresponding stretch of chain (2). We may therefore assume that (1) is one of the stretches with all the terms having equal Hirsch number. In other words, we assume for each $n \geq 1$ that $|H_1 : H_n| < \infty$. We show that (2) has at most $\log_2 m$ gaps. There exists integers r_n such that $H_1^{r_n} \leq H_n$. Hence $M_1(H_n) \leq M_1(H_1^{r_n})$. But e is the eccentricity of the pair (M, Γ) so that $M_1(H_1^{r_n}) \leq M_1(H_1^e)$ and $M_1(H_n) \leq M_1(H_1^e)$. Write $K = H_1^e$, we have

$$M_1(H_n) \leq M_1(K) \text{ for all } n \geq 1 \tag{3}$$

Now if $H_n K = H_{n+1} K$, then $M_1(H_n) \cap M_1(K)$ equals $M_1(H_{n+1}) \cap M_1(K)$. From (3) it follows that $M_1(H_n) = M_1(H_{n+1})$. Therefore gaps in (2) can occur only at places corresponding to gaps in the chain $H_1 K \geq H_2 K \geq \cdots \geq H_n K \geq \cdots$. By Lemma 19 [3], the order of H_1/K is at most m, so there are at most $\log_2 m$ gaps in this chain as required.

Now we deduce Theorem 3 from Theorem 3* in exactly the same way Theorem C is deduced from Theorem C^* in [3].

2.3.6 Proof of Theorem 3:

Suppose $G \in \mathcal{A}_1$ or $G \in \mathcal{A}_2$, let M be a normal abelian subgroup of G such that $\Gamma = G/M$ is polycyclic. Suppose $H_1, H_2, \ldots$ are subgroups of G such that

$$C_1 \leq C_2 \leq \cdots \leq C_n \leq \cdots \tag{1}$$

with $C_n = c_G(H_n)$, $n \geq 1$. From Theorems 1 and 2 we know G is eremitic. Let its eccentricity be e and $m = m(e, \Gamma)$ be the integer of Lemma 19 [3]. We suppose first that

$$C_1 \cap M = C_n \cap M \, ; \, n \geq 1 \tag{2}$$

and that

$$|MC_n : MC_1| < \infty \, ; \, n \geq 1 \tag{3}$$

Since $MC_n/M \cong C_n/C_n \cap M$, (2) and (3) together show that $|C_n : C_1| < \infty$ for all $n \geq 1$. Writing C for $\bigcup_{i=1}^{\infty} C_i$ we deduce as in the proof of Theorem 3* that H_1 centralizes C^e. Hence

$$C^e \leq C_1 \tag{4}$$

It follows $(MC)^e \leq MC_1$. Writing $L = MC/M$ and $K = MC_1/M$ we may say that $L^e \leq K$. From Lemma 19 [3] the order of L/K is at most m. There are, therefore at most $\log_2 m$ gaps in the chain

$$MC_1 \leq MC_2 \leq \cdots \leq MC_n \leq \cdots$$

However if $MC_n = MC_{n+1}$ then (3) together with $C_n \leq C_{n+1}$ show that $C_n = C_{n+1}$. It follows that there are at most $\log_2 m$ gaps in the chain (1). We now deal with the general case, that is, not assuming (2) and (3). Since $C_n \cap M$ is $c_M(H_n)$, it is precisely $M_1(MH_n/M)$. From Theorem 3*, we know that the pair (M, Γ) has a gap number, say g. Therefore in

$$C_1 \cap M \leq C_2 \cap M \leq \cdots \leq C_n \cap M \leq \cdots$$

there are at most g gaps. Further since Γ is polycyclic it has a finite Hirsch number h, say. There are therefore at most h indices n for which $h(MC_n/M) < h(MC_{n+1}/M)$. For other indices r we have $|MC_{r+1} : MC_r| < \infty$. This shows that the chain (1) breaks up into fewer than $(h+1)\cdot(g+1)$ stretches $C_a \leq C_{a+1} \leq \ldots$, where in each stretch the members have the same intersection with M and neighbouring members $C_n \leq C_{n+1}$ satisfy $|MC_{n+1} : MC_n| < \infty$. According to the case with which we have already dealt, none of these stretches can have more than $\log_2 m$ gaps. Hence (1) as a whole has no more $(h+1)(g+1)(\log_2 m + 1)$ gaps. This proves Theorem 3.

2.3.7. Proof of Corollary 1.

If G is just non-polycyclic and M is the Fitting subgroup of G, then by the structure Theorems 2.4 and 2.5 [11], with $\Gamma = G/M$, we have M is a Noetherian Γ-module and either Γ is abelian-by-finite in which case (M, Γ) is stunted and eremitic and is of finite gap number by Theorems A^*, B^* and C^* [3], or else Γ has a normal subgroup Γ_1 of finite index such that $\Gamma_1 = AB$ is a split extension of a plinth A by an abelian group B, and $A = \mathrm{Fitt}(\Gamma_1)$. Also M is a torsion-free $\mathbb{Z}A$-module. By Theorem 1* case I(a) this implies that (M, Γ_1) is stunted and eremitic. It follows from Corollaries 16 and 17 [3] that (M,Γ) is stunted and eremitic. By Theorem 3*, (M,Γ) is of finite gap number and by Lemma 17 (i) and (ii) [3] we have G stunted and eremitic. Hence, by Theorem 3 we have G of finite gap number as required.

2.3.8 Proof of Corollary 2

We have M a Noetherian Γ-module where Γ is polycyclic-by-finite and $h(\Gamma) = 3$.

We can find a subgroup Γ_1 of finite index in Γ so that Γ_1 is torsion-free polycyclic with the strong isolator property. Therefore $h(\Gamma_1) = 3$ and M is a Noetherian Γ_1-module.

Let $F = \mathrm{Fitt}(\Gamma_1)$. If $h(F) = 3$ then Γ is finitely generated nilpotent by-finite and therefore (M,Γ) is stunted and eremitic by Theorems A^* and B^* [3] and it is of finite gap number by Theorem C^*.

We can not have $h(F) = 1$, for if it is so, then since F is isolated in Γ_1 by Prop. 5 [9], F is infinite cyclic. Moreover $\varsigma(F) \leq c_{\Gamma_1}(F) \leq F$, hence $c_{\Gamma_1}(F) = F$. However, this implies that $|\Gamma_1 : F|$ is finite and $h(\Gamma_1) = 1$ a contradiction.

Thus we are left with the possibility that $h(F) = 2$. Since F is torsion-free nilpotent of rank 2, it is abelian and thus F is a plinth for Γ_1. So that Γ_1 is plinth-by-infinite cyclic and it follows by Theorem 1* that (M,Γ_1) is stunted, and eremitic. Hence (M,Γ) has these properties because $|\Gamma : \Gamma_1| < \infty$ and by Theorem 3* it is of finite gap number.

2.3.9 An alternate proof of Theorem E^* [3]:

M is a Noetherian Γ-module where Γ is any polycyclic-by-finite group. We prove the theorem by induction on the Hirsch number of Γ, $h(\Gamma)$. If $h(\Gamma) = 0$ then Γ is finite and (M, Γ) is stunted and eremitic by Corollaries 16 and 17 [3].

We suppose $h(\Gamma) > 0$ and that all pairs (B, Δ) are *sn*-stunted and *sn*-eremitic whenerver B is a Noetherian Δ-module and Δ is polycyclic-by-finite with $h(\Delta) < h(\Gamma)$.

Now Γ has a subgroup Γ_0 of finite index which is torsion-free polycyclic and with the strong isolator property by Prop. 1 [9]. Since M is still a Noetherian Γ_0-module, it will be sufficient from the analogues of Corollaries 16 and 17 [3] to prove that (M, Γ_0) is *sn*-stunted and *sn*-eremitic. Thus we assume $\Gamma = \Gamma_0$. Let D be a submodule of M chosen maximal with respect to the property that $(M/D, \Gamma)$ is not both *sn*-stunted and *sn*-eremitic. We may assume D is trivial and we must produce a contradiction. In other words, we assume that for any non-trivial submodule Y of M the pair $(M/Y, \Gamma)$ is *sn*-stunted and *sn*-eremitic. By the analogue of Lemma 4 [3], we need only show that there is some non-trivial submodule Y of M such that (Y, Γ) is *sn*-stunted and *sn*-eremitic. Thus (M, Γ) is a minimal counter-example and as such there must exist a non-trivial subnormal subgroup H of Γ such that $M_1(H) > 1$.

It follows that $\mathcal{X}(M, \Gamma)$ contains non-trivial elements and the F.O.L.III holds for the pair (M, Γ). Let $1 \neq X \in \mathcal{X}(M, \Gamma)$ and let $N = n_\Gamma(X)$ and

T a set of coset representatives of N in Γ. Let $U = M_1(X)$ and $Y = U^{\Gamma}$. By F.O.L.III(ii) we have U a Noetherian N-module since M is Noetherian Γ-module.

If $N < \Gamma$ then $h(N) < h(\Gamma)$ because N is isolated. It follows by the inductive hypothesis that (U, N) is sn-stunted and sn-eremitic. By an argument similar to that in the proof of Theorem 1* case I, it follows that (Y, Γ) is sn-stunted and sn-eremitic. Hence the result is obtained in this case.

If $N = \Gamma$ then $1 \neq X \lhd \Gamma$ and since Γ is supposed to be torsion-free $h(X) > 0$, hence $h(\Gamma/X) < h(\Gamma)$. Now U is a Noetherian Γ-module and X centralizes U, therefore U is a Noetherian Γ/X-module and (U, Γ) is sn-stunted and sn-eremitic if and only if $(U, \Gamma/X)$ is so. But $(U, \Gamma/X)$ is sn-stunted and sn-eremitic by the inductive hypothesis. This completes the proof of Theorem E^*.

2.3.10 An alternate proof of Theorem F [3].

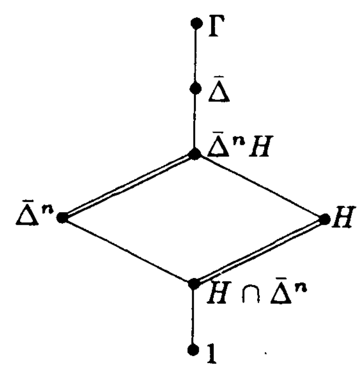
If Γ is finite the result follows trivially, so assume that Γ is infinite polycyclic. By Prop. 1 [9], we can find a normal subgroup Γ_0 of finite index in Γ so that Γ_0 has the strong isolator property.

Let $|\Gamma : \Gamma_0| = r$. If $d(\Delta \cap \Gamma_0, \Gamma_0)$ exists and equals d_0, it is clear that $d(\Delta, \Gamma)$ can be taken as $d_0 r$. In other words we may assume $\Gamma_0 = \Gamma$ and that Γ has the strong isolator property.

If Δ is any given subgroup of Γ then $\overline{\Delta}$ is a subgroup and $|\overline{\Delta} : \Delta| < \infty$. Now if H is any subgroup of Γ with $|H : H \cap \Delta| < \infty$ then we must have

$H \cap \overline{\Delta} = H$, because $H \geq H \cap \overline{\Delta} \geq H \cap \Delta$ and therefore $|H : H \cap \overline{\Delta}| < \infty$. But $\overline{\Delta}$ is isolated in Γ hence $H \cap \overline{\Delta}$ is isolated in H, therefore $H = H \cap \overline{\Delta}$ and $H \leq \overline{\Delta}$.

Since $|\overline{\Delta} : \Delta| < \infty$, we can find a positive integer $n = n(\Delta, \Gamma)$ such that $\overline{\Delta}^n \leq \Delta$. Notice that $\overline{\Delta}^n$ is normal in $\overline{\Delta}$ and it is of finite index since $\overline{\overline{\Delta}^n} = \overline{\Delta}$. Since $|H : H \cap \Delta|$ divides $|H : H \cap \overline{\Delta}^n|$ we need only show that $|H : H \cap \overline{\Delta}^n|$ divides an integer $d = d(\Delta, \Gamma)$.



From the diagram we have $|H : H \cap \overline{\Delta}^n| = |\overline{\Delta}^n H : \overline{\Delta}^n|$ and this number divides $|\overline{\Delta} : \overline{\Delta}^n|$. But $|\overline{\Delta} : \overline{\Delta}^n|$ is an integer that depends only on Δ and Γ. Hence the result.

CHAPTER 3

PROOFS OF THE FAN OUT LEMMAS

3.1 F.O.L. I

Let M be any $J\Gamma$-module, where J is any commutative ring with 1, and Γ is a torsion-free polycyclic group with the strong isolator property, which contains a non-central normal abelian subgroup A.

Suppose that there is some element $x \notin c_\Gamma(A)$ such that $M_1(\langle x\rangle) \neq 1$ and that $\Gamma = AN$, where $N = n_\Gamma(\langle x\rangle)$. Let $U = M_1(\langle x\rangle)$ and $Y = U^\Gamma$. Choose a set T of coset representatives of N in Γ such that $T \leq A$.

If M is torsion-free JA-module, then

(i) $Y = Dr_{t\in T} U^t$

(ii) U is a Noetherian JN-module whenever M is a Noetherian $J\Gamma$-module.

(iii) $Y_n(H) = Dr_{t\in T_H} U_n^t(H)$, for all for all $H \leq \Gamma$ and $n > 0$, where

$$Y_n(H) = Y \cap M_n(H),$$

$$U_n^t(H) = U^t \cap M_n(H), \text{ and}$$

$$T_H = \{t \in T : H \leq N^t\}$$

Proof: We show first that (ii) and (iii) follow from (i).

(a). (ii) follows from (i).

The proof here is exactly the same one used in proving Lemma 3 and the following comment [12], and we include it here for the sake of completeness.

Assume (i) and let V be any JN-submodule of U. Then by (i) $V^{\Gamma} = Dr_{t \in T} V^t$ and this is a direct product, hence

$$V = (U^t \cap V^{\Gamma})^{t^{-1}}.$$

Therefore the correspondence $V \to V^{\Gamma}$ is one-to-one from the set of JN-submodules of U to the set of $J\Gamma$-submodules of M. Since it clearly preserves inclusion we get (ii).

(b). (iii) follows from (i).

The proof here is exactly the same as the one presented in Lemma 10(a) [3] and we include it here for the sake of completeness. The proof is by induction on n. The case $n = 0$ being trivial, we assume $n > 0$ and let $1 \neq \xi \in Y_n(H)$. By (i),

$$Y = Dr_{t \in T} U^t \tag{1}$$

It follows that $\xi = u_1^{t_1} \ldots u_r^{t_r}$ for some non-unit elements $u_1, \ldots, u_r$ of U and distinct elements $t_1, \ldots, t_r$ of T. Let $x \in H$, then $[\xi, x] \in Y_{n-1}(H)$ and by the inductive hypothesis,

$$u_1^{-t_1} \ldots u_r^{-t_r} u_1^{t_1 x} \ldots u_r^{t_r x} \in Dr_{t \in T_H} U^t_{n-1}(H) \tag{2}$$

Suppose if possible that $u^{t_i x} \neq U^{t_j}$ for any $1 \leq j \leq r$. Then $u_i^{t_i x}$ has to be a component element of the direct product in (2). Hence $n > 1$ and $u_i^{t_i x} \in U^t$ for som[illegible] T_H. From (1) we deduce that $U^{t_i x} = U^t$ and hence $t_i x t^{-1} \in n_{\Gamma}(U)$. Observe that we get from (1) also that $n_{\Gamma}(U) = N$. Therefore $t_i x t^{-1} = n$ for some $n \in N$, and since $t \in T_H$ we have $H \leq N^t = N^{nt} = N^{t,x}$. Hence $H \leq N^{t_i}$ because $x \in H$, and we have $U^{t_i x} = U^{t_i}$ for all $x \in H$ a contradiction.

Therefore H permutes $U^{t_1}, \ldots, U^{t_r}$ and it follows that there is a subgroup H_0 of finite index in H that normalizes each of $U^{t_1}, \ldots, U^{t_r}$. Thus $H_0 \leq N^{t_i}$, $i = 1, \ldots, r$. But N is isolated in Γ since $N = n_\Gamma(\langle x \rangle) = c_\Gamma(\langle x \rangle)$, and hence all N^{t_i} are isolated in Γ, therefore $H \leq N^{t_i}$, $i = 1, \ldots, r$, because $|H : H_0| < \infty$. It follows that all $t_1, \ldots, t_r$ are in T_H and from (2) we have for each $1 \leq i \leq r$, $u_i^{-t_i} u_i^{t_i x} \in U^{t_i}_{n-1}(H)$. Hence $u_i^{t_i} \in U^{t_i}_n(H)$ and $u_i \in U_n(H)$, so that $\xi \in Dr_{t \in T_H} U^t_n(H)$ and $Y_n(H) \leq Dr_{t \in T_H} U^t_n(H)$. Since the other inclusion is obvious we get equality as required.

It remains to prove (i). For this we need the following technical lemmas.

3.1.1. Lemma. *Let Γ, A, J, N and T as above and fix $n > 1$ and $t_1, \ldots, t_n$ distinct elements of T. Let X be a cyclic subgroup of Γ such that $X \cap c_\Gamma(A) = 1$ and $S = \{\underline{x} = (x_1, \ldots, x_n);\ x_1, \ldots, x_n$ are all distinct elemtents in $X\}$. Let $I_{\underline{x}}$ be the $n \times n$-matrix over the commutative ring JA defined as*

$$I_{\underline{x}} = \begin{pmatrix} t_1^{x_1} & t_1^{x_2} & \cdots & t_1^{x_n} \\ t_2^{x_1} & t_2^{x_2} & \cdots & t_2^{x_n} \\ \vdots & \vdots & & \\ t_n^{x_1} & t_n^{x_2} & \cdots & t_n^{x_n} \end{pmatrix}.$$

Then $\det I_{\underline{x}} \neq 0$ for some $\underline{x} \in S$.

Proof: Note that no two columns or rows of $I_{\underline{x}}$ are identical since $T \leq A$ and $X \cap c_\Gamma(A) = 1$. Assume $\det I_{\underline{x}} = 0$ for all $\underline{x} \in S$.

Since $\det I_{\underline{x}} = \sum_{\sigma \in S_n} (sgn\,\sigma) t^{x_1}_{\sigma(1)} t^{x_2}_{\sigma(2)} \cdots t^{x_n}_{\sigma(n)}$ is an element of JA with all terms in A and coefficients ± 1, we get $\det I_{\underline{x}} = 0$ only if none of these terms is unique, i.e. for each σ in S_n there is $\phi \neq \sigma$ in S_n such that $t^{x_1}_{\sigma(1)} \ldots t^{x_n}_{\sigma(n)} = t^{x_1}_{\phi(1)} \ldots t^{x_n}_{\phi(n)}$. In particular, we must have for every $\underline{x} \in S$, some $1 \neq \sigma \in S_n$ depending on $\underline{x}$ only, such that if $\underline{x} = (x_1, \ldots, x_n)$ then

$$t_1^{x_1} \ldots t_n^{x_n} = t^{x_1}_{\sigma(1)} \ldots t^{x_n}_{\sigma(n)} \qquad (*)$$

For each $\underline{x} \in S$ choose one of those $1 \neq \sigma \in S_n$ that satisfies (*) and denote it by $\sigma_{\underline{x}}$. Define the map $S \to S_n \setminus \{1\}$ by $\underline{x} \to \sigma_{\underline{x}}$. Now let s be an arbitrary number strictly larger than $n(n! - 1)$. Since X is infinite we can find in X a subset Y of $n \cdot s$ distinct elements. Divide Y into n subsets Y_i of equal size, so that $|Y_i| = s$, $i = 1, 2, \ldots, n$ and the $Y_i's$ are mutually disjoint. Let $\underline{S} = \{\underline{x} \in S : \underline{x} = (x_i, \ldots, x_n);\ x_i \in Y_i;\ i = 1, \ldots, n\}$, then $|\underline{S}| = s^n$.

For each $1 \leq i \leq n$ define on $\underline{S}$ the relation $\overset{i}{\sim}$ by, $\underline{x} \overset{i}{\sim} \underline{y}$ if and only if $\underline{x} = \underline{y}$ or $\underline{x}$ and $\underline{y}$ differ only in the i-th component. This is clearly an equivalence relation. Denote the i-equivalence class containing the element $\underline{x}$ of $\underline{S}$ by $S^i_{\underline{x}}$. Then $|S^i_{\underline{x}}| = s$ and for each $1 \leq i \leq n$, we have s^{n-1} distinct i-equivalence classes.

For <u>fixed</u> $1 \leq i \leq n$ and <u>fixed</u> $\underline{x} \in \underline{S}$ consider the class $S^i_{\underline{x}}$ and note that each $\underline{y} \in S^i_{\underline{x}}$ is associated with a non-identity permutation $\sigma_{\underline{y}} \in S_n$.

We consider the subset $T^i_{\underline{x}}$ of $S^i_{\underline{x}}$ defined as $T^i_{\underline{x}} = \{\underline{y} \in S^i_{\underline{x}} : \sigma_{\underline{y}}(i) \neq i\}$. Since $|S_n \setminus \{1\}| = n! - 1$ we see that $|T^i_{\underline{x}}| \leq n! - 1$, for if $|T^i_{\underline{x}}| > n! - 1$ then there are distinct elements $\underline{y}, \underline{z}$ in $T^i_{\underline{x}}$ so that $\sigma_{\underline{y}} = \sigma_{\underline{z}}$. Let $\sigma = \sigma_{\underline{y}} = \sigma_{\underline{z}}$, then since $\underline{y}$ and $\underline{z}$ are in $T^i_{\underline{x}} \subseteq S^i_{\underline{x}}$ we have $\underline{y}$ and $\underline{z}$ differ only in the i-th component, so if

$\underline{y} = (y_1, \ldots, y_i, \ldots, y_n)$ then $\underline{z} = (y_1, \ldots, z_i, \ldots, y_n)$ and $y_i \neq z_i$. Also by the definition of the map $\underline{x} \to \sigma_{\underline{x}}$ we know that

$$t_1^{y_1} \ldots t_i^{y_i} \ldots t_n^{y_n} = t_{\sigma(1)}^{y_1} \ldots t_{\sigma(i)}^{y_i} \ldots t_{\sigma(n)}^{y_n}$$

and

$$t_1^{y_1} \ldots t_i^{z_i} \ldots t_n^{y_n} = t_{\sigma(1)}^{y_1} \ldots t_{\sigma(i)}^{z_i} \ldots t_{\sigma(n)}^{y_n}.$$

Since all $t_i^x \in A$ for all $x \in X$ and all i, the above two equations show that $(t_i^{-1} t_{\sigma(i)})^{y_i} = (t_i^{-1} t_{\sigma(i)})^{z_i}$. Therefore $\langle y_i z_i^{-1} \rangle$ centralizers $t_i^{-1} t_{\sigma(i)}$. But $\langle y_i z_i^{-1} \rangle$ is of finite index in X because X is cyclic, and since centralizers are isolated in Γ, we must have X centralizing $t_i^{-1} t_{\sigma(i)}$. Thus $t_i^{-1} t_{\sigma(i)} \in N$ and $N t_i = N t_{\sigma(i)}$. But the $t_i's$ are distinct coset representatives of N, hence $t_i = t_{\sigma(i)}$ and this implies $\sigma(i) = i$ which contradicts the choice of $\underline{y}$ and $\underline{z}$ as elements of $T_{\underline{x}}^i$.

This establishes $|T_{\underline{x}}^i| \leq n! - 1$ and since we have s^{n-1}, i-equivalence classes we see that for a fixed i, $1 \leq i \leq n$, the number of $\underline{x} \in \underline{S}$ such that $\sigma_{\underline{x}}(i) \neq i$ is $|\bigcup_{\underline{x}} T_{\underline{x}}^i| \leq s^{n-1}(n! - 1)$. Now, we have n, i-equivalence relations hence the number of elements $\underline{x}$ in $\underline{S}$ such that $\sigma_{\underline{x}}(i) \neq i$ for some $1 \leq i \leq n$ is

$$|\bigcup_i \bigcup_{\underline{x}} T_{\underline{x}}^i| \leq n s^{n-1}(n! - 1).$$

But these are all the elements of $\underline{S}$ because each $\underline{x} \in \underline{S}$ is associated with $\sigma_{\underline{x}} \in S_n \setminus \{1\}$ hence for each $\underline{x} \in \underline{S}$ there must be some $1 \leq i \leq n$ with $\sigma_{\underline{x}}(i) \neq i$. It follows $\underline{S} = \bigcup_{1 \leq i \leq n} \bigcup_{\underline{x}} T_{\underline{x}}^i$ and that $s^n \leq n s^{n-1}(n! - 1)$. Therefore $s \leq n(n! - 1)$ a contradiction. This shows that det $I_{\underline{x}} \neq 0$ for some $\underline{x} \in \underline{S}$ as required.

For the next two results we shall use the additive notation in the module M.

3.1.2 LEMMA. *Let Γ, A, X, N, J, and T as in Lemma 3.1.1, and let M be a $J\Gamma$-module with $M_1(X) \neq 0$.*
Assume that there is a nontrivial relation in M, $\sum_{i=1}^{n} m_i t_i = 0$ for $n > 1$ and $0 \neq m_i \in M_1(X)$ and $t_1, \ldots, t_n$ distinct elements in T. Then there is $r \in JA$ with $r \neq 0$ and $m_i r = 0$ for all $1 \leq i \leq n$.

Proof: The relation, $\sum_{i=1}^{n} m_i t_i = 0$ implies that $\sum_{i=1}^{n} m_i t_i^x = 0$ for all $x \in X$ since $m_i \in M_1(X)$. Let $\underline{x} = (x_1, \ldots, x_n)$, $x_1, \ldots, x_n \in X$, be the n-tuple provided by Lemma 3.1.1, so that det $I_{\underline{x}} \neq 0$. Form the direct sum of n copies of the abelian group M and define a $(JA)_{n\times n}$-module structure on $\bigoplus_{i=1}^{n} M$ in a natural way:

$$(m_1, \ldots, m_n) \cdot (a_{ij}) = (\sum_{i=1}^{n} m_i a_{i1}, \ldots, \sum_{i=1}^{n} m_i a_{in})$$

where $m_i \in M$ and (a_{ij}) is an $n \times n$-matrix over JA.

Now the equations:

$\sum_{i=1}^{n} m_i t_i^{x_j} = 0$, $j = 1, \ldots, n$; can be written as

$$(m_1, \ldots, m_n) \cdot I_{\underline{x}} = (0, 0, \ldots, 0) \tag{1}$$

where $I_{\underline{x}} = \begin{pmatrix} t_1^{x_1} & \cdots & t_1^{x_n} \\ \vdots & & \\ t_n^{x_1} & \cdots & t_n^{x_n} \end{pmatrix}$. Since JA is a commutative ring with 1, the classical adjoint formula holds in $(JA)_{n\times n}$. Multiply both sides of (1) by adj $I_{\underline{x}}$

to get

$$(m_1, \ldots, m_n) \cdot \begin{pmatrix} \det I_{\underline{x}} & & \bigcirc \\ & \ddots & \\ \bigcirc & & \det I_{\underline{x}} \end{pmatrix} = (0, \ldots, 0)$$

Hence if $r = \det I_{\underline{x}}$ then $r \in JA$ and $r \neq 0$ by Lemma 3.1.1 and $m_i r = 0$, $i = 1, \ldots, n$ as required.

3.1.3 Proof of part (i) of F.O.L.I:

We have M a torsion-free JA-module and $U = M_1(X)$ and $Y = U \cdot J\Gamma$. Written additively, $Y = U \cdot J\Gamma = \sum_{t \in T} U \cdot t$. If this sum is not direct then there is a non-trivial relation in M, $\sum_{i=1}^{n} m_i \cdot t_i = 0$; $n > 1$, $0 \neq m_i \in U$ and $t_1, \ldots, t_n$ distinct elements in T. By Lemma 3.1.2 there is $0 \neq r \in JA$ such that $m_i r = 0$ a contradiction. Hence the sum is direct as required.

3.2 F.O.L. II:

We shall adopt the additive notation for the module M throughout this lemma.

DEFINITION. *Let S be any ring with 1, and let M be any S-module. For any subset X of S, define the victim of X in S as* ${}^{\bullet}X = \{m \in M : mX = 0\}$ *and let*

$\Pi_S(M) = \{P : P$ *is an ideal of S maximal with respect to* ${}^{\bullet}P \neq 0\}$

Here are some known properties of $\Pi_S(M)$:

3.2.1 LEMMA.

(1) If $M \neq 0$ and S is Noetherian then $\Pi_S(M) \neq \emptyset$

(2) $\Pi_S(M)$ consists of prime ideals whenever S is commutative.

(3) If S is commutative and $P_1, P_2, \ldots, P_r$ are all different elements of $\Pi_S(M)$ then ${}^{\bullet}P_1 + \cdots + {}^{\bullet}P_r$ is a direct sum of submodules.

Proof: (1) Clear

(2) If I_1, I_2 are any ideals of S containing P such that $I_1 I_2 \subseteq P$ then $({}^{\bullet}P)(I_1 I_2) = 0$. If $({}^{\bullet}P)I_1 = 0$ then ${}^{\bullet}I_1 = 0$ and hence $I_1 = P$ since $P \in \Pi_S(M)$. If $({}^{\bullet}P)I_1 \neq 0$ then since $({}^{\bullet}PI_1)I_2 = 0$ we have ${}^{\bullet}I_2 \neq 0$ and hence $I_2 = P$ since $P \in \Pi_S(M)$.

(3) First we show that ${}^{\bullet}P$ is a submodule of M when S is commutative:

Let $m_1, m_2 \in {}^{\bullet}P$ then $(m_1 + m_2)P = m_1 P + m_2 P = 0$ hence $m_1 + m_2 \in P$. If $r \in S$ and $m \in {}^{\bullet}P$ then $(mr)P = m(rP) = (mP)r = 0$ since S is commutative, hence $mr \in {}^{\bullet}P$.

Let $P_1, \ldots, P_r$ be r distinct elements of $\Pi_S(M)$. Notice that $P_1 < P_1 + P_2P_3 \ldots P_r$ for if not, then, since by (2) P_1 is prime we have $P_1 \geq P_2P_3 \ldots P_r$ hence $P_1 \geq P_i$ for some $2 \leq i \leq r$, and $P_1 = P_i$ because $P_i \in \Pi_S(M)$, and this contradicts the choice of P_1 and P_i.

Now let $m \in {}^\bullet P_1 \cap ({}^\bullet P_2 + \cdots + {}^\bullet P_r)$ then $m = m_2 + \cdots + m_r$, $m_i \in {}^\bullet P_i$, $2 \leq i \leq r$. Also $m \in {}^\bullet(P_1 + P_2P_3 \ldots P_r)$. Since $P_1 < P_1 + P_2P_3 \ldots P_r$ and $P_1 \in \Pi_S(M)$ we must have $m = 0$ and therefore the sum is direct as required.

The Fan Out Lemma II shows that the prime ideals in $\Pi_S(M)$ enjoy some more desirable properties:

3.2.2 <u>F.O.L. II</u>:

Let Γ be any group and A an abelian normal subgroup of Γ. Let $R = J\Gamma$ where J is any commutative ring with 1, and $S = JA$. Let M be any R-module and $P \in \Pi_S(M)$. Let $N = n_\Gamma(P)$ and T a set of coset representatives of N in Γ, then

(i) $({}^\bullet P)R = \sum_{t \in T} {}^\bullet P \cdot t$ and this sum is direct

(ii) ${}^\bullet P$ is Noetherian JN-module whenever M is Noetherian $J\Gamma$-module

(iii) If N is isolated in Γ and $U = {}^\bullet P$, $Y = {}^\bullet P \cdot R$ then for any $H \leq \Gamma$ and $n \geq 0$, we have $Y_n(H) = \sum_{t \in T_H} U_n^t(H)$ and this sum is direct, where $Y_n(H) = \{y \in Y : [y, h_1, \ldots, h_n] = 1$ for all n-tuples $h_1, \ldots, h_n$ in $H\}$ and $T_H = \{t \in T : H \leq N^t\}$.

Remark: Parts (i) and (ii) are Lemma 3 and following comment of Roseblade's paper [12], we include the proof here for the sake of completeness.

Proof: Parts (ii) and (iii) follow from (i) in the same way as in F.O.L.I, the proof is analogous. The proof of (i) as presented in [12] follows:

Notice that for $x \in \Gamma$, ${}^*P \cdot x = {}^*(P^x)$ for if $m \in {}^*P$ then $mx \cdot P^x = mP \cdot x = 0$ hence ${}^*P{\cdot}x \subseteq {}^*(P^x)$. On the other hand, if $m \in {}^*(P^x)$ then $0 = mP^x = mx^{-1}Px$, hence $mx^{-1}P = 0$ and $mx^{-1} \in {}^*P$, therefore $m \in {}^*Px$ as required. From the above we conclude that ${}^*P$ is a JN-module, and since $\Gamma = NT$ we have ${}^*P \cdot R = \sum_{t \in T} {}^*P \cdot t = \sum_{t \in T} {}^*(P^t)$. If $t_1, \ldots, t_r$ are distinct elements of T, with r arbitrary, then $P^{t_1}, \ldots, P^{t_r}$ are all different members of $\Pi_S(M)$. Since S is commutative ring with 1, we have by Lemma 1 part 3 that the sum $\sum_{i=1}^{r} {}^*(P^{t_i})$ is direct, hence $\sum_{t \in T} {}^*(P^t)$ is direct as required.

3.3 F.O.L. III.

We first define the set $\mathcal{X}(M,\Gamma)$ of subgroups of Γ, where M is a $J\Gamma$-module, where J is any commutative ring with 1, and Γ is torsion-free polycyclic group with the strong isolator property. We say that the subgroup X of Γ is in $\mathcal{X}(M,\Gamma)$ if and only if X is maximal with respect to having the following four properties:

(i) $X sn \Gamma$

(ii) $M_1(X) > 1$

(iii) If $X < H \leq \Gamma$ and H has (i) and (ii) then $|H : X| < \infty$

(iv) $X = \overline{X}^n$ for some $n > 0$.

We show that if Γ contains a non-trivial subnormal subgroup H_0 with $M_1(H_0) > 1$, then $\mathcal{X}(M,\Gamma)$ contains non-trivial elements. This will follow from

3.3.1 LEMMA. *If Γ is a torsion-free polycyclic group with the strong isolator property and $H \leq \Gamma$, then H contains a subgroup K which is normal and of finite index in H, also $K = \overline{K}^n$ for some $n > 0$, and $n_\Gamma(K)$ is isolated in Γ.*

Proof: Since Γ has the strong isolator property, $\overline{H}$ is a subgroup of Γ and $|\overline{H} : H| < \infty$. Hence $\overline{H}^n \leq H$ for some positive integer n. Since Γ is torsion free we have $\overline{H}^n \neq 1$ unless $H = 1$. If $\overline{H}^n = H$ we take $K = H$, and if $\overline{H}^n < H$ we take K to be $\overline{H}^n$. It is clear that $\overline{H} = \overline{K}$. Hence $\overline{K}^n = \overline{H}^n = K$ and this is normal and of finite index in $\overline{H}$. But since $K = \overline{H}^n \leq H \leq \overline{H}$ we get K normal and of finite index in H. To show that $n_\Gamma(K)$ is isolated in Γ we use the property that in Γ, $n_\Gamma(\overline{K}) = \overline{n_\Gamma(K)}$. Since $K = \overline{K}^{\circ}$ we have K is fully invariant in $\overline{K}$ and since $\overline{K} \lhd n_\Gamma(\overline{K})$ we get $K \lhd n_\Gamma(\overline{K}) = \overline{n_\Gamma(K)}$. Hence $\overline{n_\Gamma(K)} \leq n_\Gamma(K)$ and we must have equality.

3.3.2 LEMMA. *If Γ contains non-trivial subnormal subgroup H_0 with $M_1(H_0) \neq 1$ then $\mathcal{X}(M,\Gamma)$ contains non-trivial elements.*

Proof: Let d be the largest Hirsch number of any subnormal subgroup with non-trivial centralizer in M, and suppose H is one of such subgroups with $h(H) = d$. So that H has (i), (ii) and (iii) follows immediately. By Lemma 1, H contains a non-trivial subgroup K which is normal of a finite index in H and has property (iv). Thus K has properties (i)-(iv) and the set of subgroup of Γ with the properties (i)-(iv) contain non-trivial elements. Since Γ has the maximal condition there must exist maximal elements in this set that are non-trivial as required.

Now we give some elementary properties of members of $\mathcal{X}(M,\Gamma)$ that will be used very often in the proof of F.O.L. III.

3.3.3 LEMMA. *Let $X \in \mathcal{X}(M,\Gamma)$ then*

(1) $X^\gamma \in \mathcal{X}(M,\Gamma)$ for all $\gamma \in \Gamma$.

(2) There is a (shortest) subnormal series from X to Γ which is composed of isolated subgroups.

(3) If $X \leq \Delta$ and Δ is an isolated and subnormal subgroup of Γ, then

$X \in \mathcal{X}(M, \Delta)$.

Proof: (1) We show that X^γ is maximal with respect to having properties (i)-(iv)

(i) Since $Xsn\Gamma$ we have $X^\gamma sn\Gamma$

(ii) If $1 \neq \alpha \in M_1(X)$ then $1 = [\alpha, x]$ for all $x \in X$ and therefore for all $\gamma \in \Gamma$; $1 = [\alpha, x]^\gamma = [\alpha^\gamma, x^\gamma]$, hence $1 \neq \alpha^\gamma \in M_1(X^\gamma)$.

(iii) If $X^\gamma < H \leq \Gamma$ and H has (i) and (ii) then $H^{\gamma^{-1}}$ has (i) and (ii) by the above argument and $X < H^{\gamma^{-1}}$ hence $|H^{\gamma^{-1}} : X| < \infty$ because $X \in \mathcal{X}(M, \Gamma)$. Therefore $|H : X^\gamma| < \infty$

(iv) Since $X = \overline{X^n}$ for some $n > 0$, we have $X^\gamma = (\overline{X^n})^\gamma = (\overline{X^\gamma})^n = (\overline{X^\gamma})^n$.

Finally to show maximality of X^γ, assume $X^\gamma < H \leq \Gamma$ and H has properties (i)-(iv). Then $X < H^{\gamma^{-1}} \leq \Gamma$ and from what has been shown $H^{\gamma^{-1}}$ has properties (i)-(iv) and this contradicts $X \in \mathcal{X}(M, \Gamma)$ unless $H^{\gamma^{-1}} = \Gamma$ and hence $H = \Gamma$ as required.

(2) Since $Xsn\Gamma$, let $X = X_0 \lhd X_1 \cdots \lhd X_k = \Gamma$ be a shortest series from X to Γ. $X_0 \lhd X_1$ implies $X_1 \leq n(X_0)$, hence $\overline{X_1} \leq \overline{n(X_0)} = n(X_0)$ since $n(X_0)$ is isolated by Lemma 1. Thus $X = X_0 \lhd \overline{X}_1$. For $i = 1, \ldots, k$ we have $X_i \lhd X_{i+1}$ implies $X_{i+1} \leq n(X_i)$, hence $\overline{X}_{i+1} \leq \overline{n(X_i)} = n(\overline{X}_i)$. Therefore $\overline{X}_i \lhd \overline{X}_{i+1}$ and $X = X_0 \lhd \overline{X}_1 \lhd \cdots \lhd \overline{X}_k = \Gamma$ and all $\overline{X}_i$, $i = 1, \ldots, k$ are isolated and obviously this is also a shortest series.

(3) We show that X as a subgroup of Δ has properties (i)-(iv)

(i) Since $X sn \Gamma$ we have $X sn \Delta$

(ii) $M_1(X) > 1$ since $X \in \mathcal{X}(M, \Gamma)$

(iii) If $X < H \leq \Delta$ and H has (i) and (ii) then since $\Delta sn \Gamma$ we have $H sn \Gamma$ and has (i) and (ii) hence $|H : X| < \infty$.

(iv) Since Δ is isolated in Γ we have ${}^{\Delta}\overline{X} = {}^{\Gamma}\overline{X}$ and it follows that $X = {}^{\Gamma}\overline{X}^n = {}^{\Delta}\overline{X}^n$ for some $n > 0$.

Now to show maximality of X with respect to having properties (i)-(iv) when considered as a subgroup of Δ, assume $X < H \leq \Delta$ and H has properties (i)-(iv). Then since $\Delta sn \Gamma$ we have H, as a subgroup of Γ, satisfying properties (i)-(iii). Property (iv) follows since Δ is isolated in Γ as above. However this contradicts the maximality of X with respect to having propeties (i)-(iv).

3.3.4 The Fan Out Lemma III

The Fan Out Lemma III stated in Chapter two contains three parts, where parts (ii) and (iii) follow from (i) exactly as in the proof of F.O.L.I. So we need only prove part (i). However for the proof of this we will need for the inductive step part (iii) and more consequences that will be included as parts of the F.O.L.III in this section. Thus the following statement of the F.O.L.III contains more items not mentioned in chapter two because they are not used directly in the proof of the main theorems. We shall follow in our proof the same scheme used by Roseblade and Lennox in [3].

3.3.5 Statement of F.O.L.III:

(1) If $X \in \mathcal{X}(M,\Gamma)$ and T is a set of coset representatives of $n_\Gamma(X)$ in Γ then

$$M_1(X)^\Gamma = Dr_{t\in T} M_1(X)^t$$

(2) If $X \in \mathcal{X}(M,\Gamma)$ and B is a submodule of M then $B \cap M_1(X)^\Gamma > 1$ only if $B_1(X) > 1$, where

$$B_1(X) = B \cap M_1(X).$$

3.3.6 Notation

To simplify matters we shall use the letters X, X_1, X_2, ... to denote members of $\mathcal{X}(M,\Gamma)$. Corresponding to these we shall use U, U_1, U_2, ... for the subgroups $M_1(X)$, $M_1(X_1)$, $M_1(X_2)$, ... and Y, Y_1, Y_2, ... to denote the submodules U^Γ, U_1^Γ, U_2^Γ, ... We shall always write N for the normalizer of X in Γ and T for a set of coset representatives of N in Γ. For any $H \leq \Gamma$ we shall write $T_H = \{t \in T : H \leq N^t\}$ and for $n \geq 0$ we write $M_n(H) = \{m \in M : [m, h_1, \ldots, h_n] = 1$ for all n-tuples $h_1, \ldots, h_n$ in $H\}$ and $Y_n(H) = Y \cap M_n(H)$ and $U_n^t(H) = U^t \cap M_n(H)$.

To prove the F.O.L. III as stated above we need the following consequences:

3.3.7 **Lemma.** *If F.O.L.III (1) and (2) hold then so also do*

(3) If $X \in \mathcal{X}(M,\Gamma)$ then $Y_n(H) = Dr_{t\in T_H} U_n^t(H)$ for all $H \leq \Gamma$ and $n \geq 0$.

(4) If X_λ, $\lambda \in \Lambda$ are elements of $\mathcal{X}(M,\Gamma)$ mutually inconjugate in Γ then,

$\langle Y_\lambda, \lambda \in \Lambda\rangle = Dr_{\lambda\in\Lambda} Y_\lambda$.

(5) If $X_1, \ldots, X_r$ are elements of $\mathcal{X}(M,\Gamma)$ mutually inconjugate in Γ, and B is a submodule of M such that $B \cap (Y_1 \cdot Y_2 \ldots Y_r) > 1$, then there is $1 \leq i \leq r$ such that $B_1(X_i) > 1$.

Proof: We divide the proof into three parts

(a). (3) follows from (1) and the proof is analogous to that given in F.O.L.I.

(b). (4) follows from (1) and (2)

We prove this by induction on r to show that if $X_1, \ldots, X_r$ are any elements of $\mathcal{X}(M,\Gamma)$ mutually inconjugate in Γ then

$$\langle Y_1, \ldots, Y_r \rangle = Dr_{1 \leq i \leq r} Y_i.$$

If $r = 1$ then there is nothing to prove, so assume $r > 1$ and $B = Y_2 Y_3 \ldots Y_r$ $= Dr_{2 \leq i \leq r} Y_i$, but $B \cap Y_1 \neq 1$. By (2) we must have $B_1(X_1) > 1$ and since $B = Y_2 Y_3 \ldots Y_r$ is direct, $Y_{i,1}(X_1) > 1$ for some $2 \leq i \leq r$. To simplify the notation we write X for X_i and Y for Y_i. Since $i \geq 2$, X and X_i are inconjugate in Γ and $Y_1(X_1) > 1$. Since (1) holds by assumption then by part (a), so does (3). Thus $Y_1(X_1) = Dr_{t \in T_{X_1}} U_1^t(X_1)$ and it follows that for some $t \in T_{X_1}$ we have $U^t \cap M_1(X_1) = U_1^t(X_1) > 1$. In other words, $M_1(X^t) \cap M_1(X_1) > 1$, and hence $M_1(\langle X^t, X_1 \rangle) > 1$. Since X_1 and X^t are in $\mathcal{X}(M,\Gamma)$ none of them contains the other by the definition of $\mathcal{X}(M,\Gamma)$, so if we write $H = \langle X^t, X_1 \rangle$, then $X_1 < H$ and $X^t < H$. Since Γ has the maximal condition and X_1, X^t are both subnormal in Γ it follows by Theorem 13.1.9 [10], that $H = \langle X_1, X^t \rangle sn\Gamma$. Thus H has properties (i) and (ii) in the definition of $\mathcal{X}(M,\Gamma)$, and properly contains X_1

and X^t, it follows therefore that $|H : X_1|$ and $|H : X^t|$ are both finite and hence $\overline{H} = \overline{X_1} = \overline{X^t}$. Since X_1 and X^t are in $\mathcal{X}(M,\Gamma)$ we can find positive integers m_1 and m so that $X_1 = \overline{X_1}^{m_1}$ and $X^t = \overline{X^t}^{m}$. Let $d = g.c.d.(m_1, m)$ then $X_1 = \overline{X_1}^{m_1} \leq \overline{X_1}^{d} = \overline{H}^{d}$ and similarly $X^t \leq \overline{H}^{d}$. Hence $H = \langle X_1, X^t \rangle \leq \overline{H}^{d}$.

On the other hand, we can find integers s and t so that $d = sm + tm_1$, and it follows that:

$$\begin{aligned}
\overline{H}^{d} &= \langle g^{d}, g \in \overline{H} \rangle \text{ by definition} \\
&= \langle g^{sm} \cdot g^{tm_1}, g \in \overline{H} \rangle \\
&\leq \langle \overline{H}^{m}, \overline{H}^{m_1} \rangle \\
&= \langle \overline{X^t}^{m}, \overline{X_1}^{m_1} \rangle \text{ since } \overline{H} = \overline{X^t} = \overline{X_1} \\
&= \langle X^t, X_1 \rangle = H
\end{aligned}$$

We conclude that $H = \overline{H}^{d}$, $d > 0$ and thus H has properties (i)-(iv) and properly contains X_1 and this contradicts the maximality of X_1 with respect to having these properties. This contradiction shows that (4) is true for any finite subset of Λ, hence (4) is proved.

(c). (5) follows from (1) and (2).

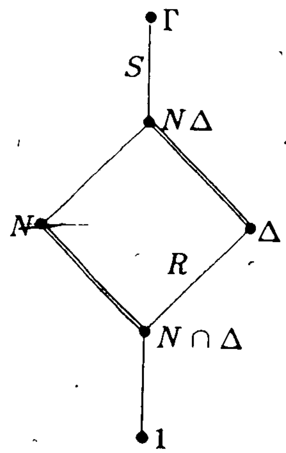
We prove this by induction on r. The case $r = 1$, is just (2). Suppose $r > 1$ and that $B \cap (Y_1 Y_2 \ldots Y_r) > 1$. Let $C = Y_2 Y_3 \ldots Y_r$. We suppose that $B \cap Y_1 = B \cap C = 1$ and produce a contradiction. Since $B \cap (Y_1 C) > 1$ it follows $Y_1 \cap (BC) > 1$, and by (2) we get $(BC)_1(X_1) > 1$. In other words $BC \cap M_1(X_1) > 1$. Since BC is a direct product by assumption, we must have either $B_1(X_1) > 1$ or $C_1(X_1) > 1$. But the former possibility contradicts $B \cap Y_1 = 1$ and the latter contradicts $C \cap Y_1 = 1$ which is known from (4). This is the required contradiction.

3.3.8 Proof of (1) and (2) of F.O.L.III:

We use induction on the Hirsch number of Γ. If $h(\Gamma) = 1$ then $h(X) = h(\Gamma)$ for any $1 \neq X \in \mathfrak{X}(M, \Gamma)$, and hence $|\Gamma : X| < \infty$. So $|\Gamma : N| < \infty$ and because N is isolated we have $N = \Gamma$ and $X \lhd \Gamma$ and (1) and (2) are trivially true in this case. Thus assume $h(\Gamma) > 1$ and that (1) and (2) hold true for all pairs (M, Δ) where $\Delta \leq \Gamma$ and $h(\Delta) < h(\Gamma)$.

Let $X \in \mathfrak{X}(M, \Gamma)$. By Lemma 3.3.3 part (2) there is a subnormal series $X = X_0 \lhd X_1 \trianglelefteq \cdots \lhd X_k = \Gamma$ where X_i, $i = 1, \ldots, k$ are isolated in Γ. If $k = 1$ then $X \lhd \Gamma$ and again (1) and (2) hold trivially. Assume $k > 1$ and let $\Delta = X_{k-1}$, then $X \leq \Delta$ and Δ is normal and isolated in Γ hence $h(\Delta) < h(\Gamma)$ and by Lemma 3.3.3 parts (1) and (3) we have $X^\gamma \in \mathfrak{X}(M, \Delta)$ for all $\gamma \in \Gamma$.
By the induction hypothesis we have both (1) and (2) hold true for the pair (M, Δ). Let R be a set of coset representatives of $n_\Delta(X) = N \cap \Delta$ in Δ. Using F.O.L.III (1) for the pair (M, Δ) we get

$$M_1(X)^\Delta = Dr_{r \in R} M_1(X)^r \qquad (*)$$



Let S be a set of coset representatives of $N\Delta$ in Γ. It is easy to see that

$$\left.\begin{array}{l}\text{the elements } \{X^S, s \in S\} \text{ of } \mathcal{X}(M,\Delta) \\ \text{are all mutually inconjugate in } \Delta\end{array}\right\} \qquad (^{**})$$

For let s_1, s_2 any distinct elements of S and assume for some $\delta \in \Delta$ that $X^{s_1\delta} = X^{s_2}$ then, $s_1\delta s_2^{-1} = s_1 s_2^{-1}(s_2\delta s_2^{-1}) \in N$. But $s_2\delta s_2^{-1} \in \Delta$ since $\Delta \lhd \Gamma$, therefore $s_1 s_2^{-1} \in N\Delta$ and $N\Delta s_1 = N\Delta s_2$ a contradiction. Now, since (1) and (2) hold for the pair (M, Δ), so does (4). Therefore,

$$\langle M_1(X^s)^\Delta, s \in S\rangle = Dr_{s\in S} M_1(X^s)^\Delta \qquad (^{***})$$

Now let $T = RS$. We have T is a set coset representatives of N in Γ. For R itself is a set of coset representatives of N in $N\Delta$, and to see that notice that $NR = N \cdot NR \supseteq N(N \cap \Delta)R = N\Delta$, hence $NR = N\Delta$, and if $r_1 \neq r_2$ are any elements in R then $r_1 r_2^{-1} \notin N$, otherwise $r_1 r_2^{-1} \in N \cap \Delta$ contradicting the fact that R is a set of coset representatives of $N \cap \Delta$ in Δ, thus we have $Nr_1 \neq Nr_1$

as required. Thus T is a coset representatives of N in Γ, and $\Gamma = NT$. Now we have the following:

$$
\begin{aligned}
M_1(X)^{\Gamma} = M_1(X)^{NT} = M_1(X)^{NRS} &= M_1(X)^{\Delta S} \\
&= \langle M_1(X)^{\Delta s}, s \in S\rangle \text{ by definition} \\
&= \langle M_1(X^s)^{\Delta}, s \in S\rangle, \text{ since } \Delta \lhd \Gamma \\
&= Dr_{s \in S} M_1(X^s)^{\Delta} \text{ by } (***) \\
&= Dr_{s \in S} (M_1(X)^{\Delta})^s, \text{ since } \Delta \lhd \Gamma \\
&= Dr_{s \in S} (Dr_{r \in R} M_1(X)^r)^s, \text{ by } (*) \\
&= Dr_{\substack{r \in R \\ s \in S}} M_1(X^{rs}) = Dr_{t \in T} M_1(X)^t
\end{aligned}
$$

Therefore $Y = Dr_{t \in T} U^t$ as required, and since this holds for this particular set of coset representatives of N in Γ it must hold for any other set of coset representatives. This establishes (1).

We now prove (2).

From the inductive hypothesis and Lemma 3.3.3 part (3), it follows that (5) holds for the pair (M, Δ). Therefore, using (**) above we have:

$$
\left.
\begin{aligned}
&\text{If } B \text{ is any } \Delta\text{-submodule of } M \\
&\text{such that } B \cap (\langle M_1(X^s)^{\Delta}, s \leq S\rangle) > 1 \\
&\text{then } B_1(X^s) > 1 \text{ for some } s \in S.
\end{aligned}
\right\} \qquad (****)
$$

Now let B be any Γ-submodule of M and suppose $B \cap Y = 1$. Since B is also a Δ-submodule of M, we have by (***) and (****) that for some $s \in S$, $B_1(X^s) > 1$. But $B^{s^{-1}} = B$ since B is a Γ-module, so transforming with

s^{-1} yields the desired $B_1(X) > 1$. This establishes (2) and the proof of F.O.L.III is complete.

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