Contributions to Degree Theory, and Applications

By

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Abstract

This thesis is dedicated to the study of topological degree for different classes of monotone maps and applications to nonlinear problems in mathematical analysis and the applied mathematics. Employing topological methods for nonlinear problems in mathematics goes back to the pioneering work of H. Poincaré on the three body problem. The generalization of Brouwer degree for mappings in finite dimensional spaces to compact perturbations of the identity in arbitrary Banach spaces, by J. Schauder and J. Leray, opened up a way to apply such a powerful method to a broad class of complicated nonlinear problems. Further generalizations, including the degree for classes of monotone maps as well as the degree for multi-valued maps have been carried out by several authors in recent decades.

This thesis is divided into two parts. The first part, consisting of Chapters 1 and 2, is about the theoretical aspects of topological degree. The second part, Chapters 3–5, is devoted to the applications of the topological degree in three fields: an integral equation coming from the Doi-Onsager model for liquid crystals; dynamical systems governed by nonlinear ordinary differential equations and finally fully nonlinear elliptic and parabolic partial differential equations.

After a detailed introduction on various topological methods for linear and quasi-linear elliptic problems and presentation of some of its implications for monotone maps and variational problems in Chapter 1, we systematically introduce the concept of finite rank approximation of a map in Chapter 2. This concept enables us to prove the stability of the homotopy class of finite rank approximations for different types of monotone maps including $(S)_+$, pseudo-monotone and maximal monotone maps in separable, locally uniformly convex Banach spaces. Furthermore, we generalize the degree for mappings that are only demi-continuous in a subspace, not necessarily dense, of the focal space. We use this generalization for the Doi-Onsager problem presented in Chapter 3.

The Doi-Onsager problem is a mathematical formulation to model the behaviour of the liquid crystals in terms of the interaction potential field and the temperature of the liquid. In our work on this problem, we solve the problem in dimension $D=2$ and also prove the uniqueness of the isotropic solution for high temperature and the bifurcation of nematic solutions for low temperature in general dimension $D \geq 3$. For a classical application of degree theory, we return in Chapter 4 to the problem of periodic solutions for dynamical systems described in ordinary differential equations. The method that we employ in this chapter is based on the continuation method. For this, we consider a one-parameter family of dynamical systems (dependent on ε) and then prove (under certain conditions) that periodic orbits for $\varepsilon=0$ survive when ε increases to 1. The last chapter of the thesis, Chapter 5 is dedicated to defining a degree for fully nonlinear elliptic and parabolic equations. Even though it is not novel to define a degree for fully nonlinear elliptic equations, our construction can be employed to define a degree for fully nonlinear parabolic equations which, to our knowledge is new.

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Chapter 1

Topological methods

1.1 Introduction

1.1.1 An overview

This thesis is dedicated to the study of a topological invariant called degree. For Banach spaces X, Y and an open bounded set $D \subset X$, assume that $A: \bar{D} \to Y$ is a continuous map. The degree of A in D with respect to $y \in Y$ is generally denoted by $\deg(A, D, y)$ and is invariant under certain type of deformations, in other words, if $h: [0,1] \times \bar{D} \to Y$ is a continuous homotopy satisfying certain conditions on the boundary $[0,1] \times \partial D$ then $\deg(h(t), D, y)$ is independent of t.

The use of topological methods for nonlinear problem goes back to the work of Poincaré on the three body problem, where the existence of periodic orbits could be established by proving the existence of fixed point for the "Poincaré map" for a system of differential equations. The existence of periodic orbits for a dynamical system is one of the places that the topological continuation method is extensively employed. Later, various fixed point theorems in finite dimensional spaces, like Poincaré-Miranda, Poincaré-Bohl and Roth fixed point theorems are unified by the pioneering work of Brouwer in 1912, [12] where a theory of degree for continuous maps is developed. The work of J. Schauder on quasi-linear second order elliptic partial differential equations that leads naturally to the compact perturbation of the identity maps (Schauder maps) opened up a way to generalize the Brouwer degree in infinite dimensional Banach spaces in 1934 [47].

A topological property is one that is invariant under continuous deformations. Such properties are common in mathematics. Here we mention two known results. The first one is the Gauss-Bonnet formula. Let M be an orientable closed hypersurface in $\mathbb{R}^{n+1}$, $\gamma: M \to S^n$ the Gauss map and ω, η the surface form on S^n and M respectively. By the Gauss-Bonnet formula we have

$$\int_M \gamma^*(\omega) = \int_M K\eta. \tag{1.1}$$

On the other hand, by definition

$$\frac{1}{\mathrm{vol}(S^n)} \int_M \gamma^*(\omega) = \deg(\gamma, M) \in \mathbb{Z}. \tag{1.2}$$

If M_t is a continuous deformation of M, since $\deg(\gamma, M_t)$ is invariant (with respect to t), one concludes that the mean Gauss curvature of an orientable closed hypersurface is constant under the continuous deformation.

A more delicate result in this direction is due to M. Atiyah and I. Singer [5] . Assume that (M, g) is a closed Riemannian manifold and (E, π) a vector bundle over M. Let $\mathcal{S}(M)$ denote the vector space of all compactly supported smooth sections $\Gamma: M \to E$. By definition, the map $D: \mathcal{S}(M) \to \mathcal{S}(M)$ is a differential operator if for any $\Gamma \in \mathcal{S}(M)$ it satisfies the property: $\mathrm{supp}\,\{D(\Gamma)\} \subset \mathrm{supp}\{\Gamma\}$. We can make $\mathcal{S}(M)$ a Banach space by introducing the Riemannian metric g on it that is defined as

$$\|\Gamma\| = \sup_{x \in M} g(\Gamma(x), \Gamma(x)). \tag{1.3}$$

An example of a differential operator is the Laplacian operator over $C_0^\infty(M)$. Let $p = p(x, \xi_1, \ldots, \xi_n)$ be an elliptic polynomial in $\xi = (\xi_1, \ldots, \xi_n)$. By the aid of the local chart $(x_1, \ldots, x_n)$ in M we can define the differential operator $p(D)$ as

$$p(D) = p(x, \nabla_{x_1}, \ldots, \nabla_{x_n}), \tag{1.4}$$

where ∇_{x_i} denotes the covariant derivative along the field x_i. The important fact about elliptic operators is that they have a pseudo-inverse, that is, if p is elliptic in terms of $(\xi_1, \ldots, \xi_n)$ at every point $x \in M$ then $p(D)$ is Fredholm. M. Atiyah and I. Singer proved that the index of $p(D)$, defined as

$$\mathrm{ind}(p(D)) = \dim(\ker p(D)) - \dim \mathrm{coker}(p(D)), \tag{1.5}$$

is topologically invariant and is equal to the topological characteristic of M in terms of K-theory. Since elliptic partial differential equations in suitable Banach spaces can be reduced to Fredholm operator equations [3], this theorem plays an important role in the existence theory of elliptic partial differential equations, see [74],[48].

The theory of topological degrees has found applications in various fields including mathematical physics, partial differential equations, mathematical analysis and geometry, as problems in these fields can often be reformulated in terms of abstract equations

$$A(u) = f, \tag{1.6}$$

where usually A belongs to some class of "continuous" functions and u, f belong to some Banach spaces. One most interesting class of mappings that has been extensively studied, is continuous (demi-continuous) maps $A\colon X \to X^*$ (or $A\colon X \to 2^{X^*}$) where X is a separable Banach space equipped with a locally uniformly convex norm and A enjoys a monotonicity property that we describe later. The most classical examples for such class of mappings are the quasi-linear uniformly elliptic partial differential operators

$$A(u)(x) := \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^{\alpha} A_{\alpha}(x, D^{\leqslant m} u), \tag{1.7}$$

defined on on the domain Ω (with $\partial\Omega$ is smooth), $D^{\beta} u = 0$ on $\partial\Omega$ for $0 \leq \beta \leq m - 1$ and A_{α} satisfies some smoothness and monotonicity conditions.

The general degree theoretic procedure to establish the solvability of equation (1.6) is as follows. Assume we have a topological degree for the class of mappings to which A belongs. If there exists a $D \in X$ such that $f \notin A(\partial D)$ and if

$$\deg(A, D, f) \neq 0, \tag{1.8}$$

then equation (1.6) is solvable. For most nonlinear problems, the direct calculation of degree in (1.8) is delicate. To overcome this situation, one can use the fact that the degree is topological. Therefore if $A_t, t \in [0, 1]$ is a continuous deformation of A with respect to t and with the condition $f \notin A_t(\partial D)$ for all $t \in [0, 1]$, then

$$\deg(A, D, f) = \deg(A_t, D, f). \tag{1.9}$$

Now it is enough to choose a deformation such that $A_0 = A$ and such that the degree of A_1 can be easily calculated:

$$\deg(A_1, D, f) \neq 0. \tag{1.10}$$

This method enables us to reduce a complicated nonlinear equation to a simpler one, where the degree of the simpler operator can be computed directly.

From one point of view, the degree theory can be seen as the natural generalization of another topological method called the continuation method which is extensively employed for nonlinear problems after Poincaré. The important drawback of the continuation method lies in its use of the uniqueness of the solution. As a simple example, consider the second order uniformly elliptic operator

$$L(u)(x) := \sum_{|\alpha| \le 2} a_\alpha(x)\, D^\alpha u, \tag{1.11}$$

that is, ther exist $\theta > 0$ such that for all $x \in \Omega$ we have

$$-\sum_{|\alpha|=2} a_\alpha(x)\, \xi^\alpha \ge \theta\, |\xi|^2. \tag{1.12}$$

$a_\alpha(x)$ for $|\alpha| = 2$ forms a uniformly elliptic polynomial in $x \in \Omega$. It is easily verified that the operator $A_t := t\Delta + (1-t)\, L$ for $t \in [0,1]$ is uniformly elliptic. On the other hand, since Δ is an isomorphism from $H^2(\Omega) \cap H_0^1(\Omega)$ onto $L^2(\Omega)$ we can conclude (imposing a simple condition on a_0) that L is an isomorphism.

1.1.2 Overview of degree theory

1.1.2.1 Classical degree

The question of the existence and also the uniqueness of a topological degree is fundamental. To answer this crucial question, one needs to define the degree map in a well formulated (axiomatic) manner. Let X, Y be Banach spaces (finite or infinite), and let $\mathcal{O}(X)$ be the set of all open bounded subsets of X.

Definition 1.1. (classical properties of degree) *The degree of* $A\colon X\to Y$ *in* $D\in\mathcal{O}(X)$ *with respect to* $f\in Y$ *such that* $f\notin A(\partial D)$ *is an integer valued map* deg *that satisfies:*

1. **(reference map)** *There is a reference map* $i\colon X\to Y$ *such that*

$$\deg(i,D,f)=\begin{cases}1 & f\in i(D)\\ 0 & f\notin i(D)\end{cases}$$

2. **(existence of solution)** *If* $\deg(A,D,f)\neq 0$ *then there exists* $u\in D$ *such that* $f=A(u)$.

3. **(domain decomposition)** *If* $D_1,\ D_2\subset D$ *are open disjoint subsets and* $\deg(A,D-(D_1\cup D_2),f)=0$ *then*

$$\deg(A,D,f)=\deg(A,D_1,f)+\deg(A,D_2,f). \tag{1.13}$$

4. **(homotopy invariance)** *If* $h(t), 0\leq t\leq 1$ *is a continuous homotopy of* A *and* f_s *is a continuous path for* $0\leq s\leq 1$ *such that* $f_s\notin h_t(\partial D)$ *then*

$$\deg(h(t),D,f_s)=\deg(A,D,f_0). \tag{1.14}$$

The existence and uniqueness of such a mapping in finite dimensional spaces was completely answered for $A\in C(\mathbb{R}^n)$ by L. Brouwer [12] for the existence part and H. Amann and S. Weiss [4] for the uniqueness part. Further results for Sobolev as well as BMO mappings on Riemannian manifolds were presented later by H. Brezis and L. Nirenberg [9][10],[11].

For the infinite dimensional spaces, it is straightforward to see that the existence of such a mapping is impossible in general for continuous maps. First observe that if $A\colon X\to X$ is a continuous map such that $A\colon\bar{B}_X\to\bar{B}_X$ with no fixed point on ∂B_X then A has a fixed point on B_X provided that degree can be defined. In fact the map $\mathcal{A}(u)=u-A(u)$ has no zero on ∂B_X and then using the homotopy $\mathcal{A}_t(u):=u-tA(u)$ we conclude

$$\deg(\mathcal{A},B_X,0)=\deg(\mathcal{A}_t,B_X,0)=\deg(\mathcal{A}_0,B_X,0)=1.$$

No consider the continuous map A on l_2:

$$A(x) = \left(\sqrt{1-\|x\|^2}, x_1, x_2, \ldots\right), \tag{1.15}$$

that has no fixed point on B_{l_2}. The reason behind the above example is the Kuiper theorem [44] which states that $\mathrm{GL}(H)$ for H a separable Hilbert space is trivial.

As mentioned above, the earliest generalization of the Brouwer degree to infinite dimensional spaces is due to J. Leray and J. Schauder. Schauder in his work on nonlinear elliptic equations figured out that such equations could be transformed to equations involving a completely continuous operator (Schauder map). Furthermore, he could generalize two important results of Brouwer for finite dimensional maps to maps defined on infinite dimensional Banach spaces. First, he proved that a completely continuous operator defined on a bounded convex set of a Banach space has a fixed point. This is now called the Schauder fixed point theorem. The second generalization is the invariance of domain theorem, that the image of an open subset of a Banach space under an injective completely continuous map is open, see [58]. The second result opened up a way to define a degree for Schauder maps. Since every compact map on bounded closed subsets of Banach spaces can be approximated by finite rank maps, Leray and Schauder could define a degree satisfying all classical properties of topological degree listed above for Schauder maps. In particular, they defined a degree for the mapping $A = \mathrm{Id} - K$ on the Banach space X where $K: X \to X$ is a completely continuous operator. Furthermore, for a continuous one parameter family of maps $K = K(\lambda)$, they proved that the defined degree is independednt of λ, [47].

The application of topological method to establish the existence of solution for quasi-linear elliptic equations (that is, the initial work of Schauder) is almost straightforward. Let us illustrate this in the context of second order elliptic quasi-linear equations. Consider the following equation

$$\sum_i a_{ij}(x, u, Du) \frac{\partial^2 u}{\partial x_i \partial x_j} + b(x, u, Du) = 0, \quad x \in \Omega, \tag{1.16}$$

with homogeneous Dirichlet boundary condition $u(x)=0, x\in\partial\Omega$. For the Sobolev space H^m, denote $X=H^2(\Omega)\cap H_0^1(\Omega)$ and assume that smooth functions a_{ij} are uniformly elliptic for all $x\in\Omega$ and $u\in X$, that is, there exists $\theta>0$ such that for all $x\in\Omega$ and $u\in X$ we have

$$\sum_{ij} a_{ij}(a,u,Du)\xi_i\xi_j\geq\theta\,|\xi|^2. \tag{1.17}$$

Define the linear operator $L(u)\colon X\to L^2(\Omega)$ as follows:

$$L(u)(v)=\sum_i a_{ij}(x,u,Du)\,\frac{\partial^2 v}{\partial x_i\partial x_j}+b(x,u,Du). \tag{1.18}$$

If a_{ij} and b are smooth enough, we can solve the equation $L(u)(v)=0$ uniquely. Let $v=g(x,u)$ be the solution of the equation $L(u)(v)=0$. It turns out that g is a completely continuous operator [63] and therefore the existence of a solution for the original equation reduces to the existence of a fixed point of the map $f=\mathrm{Id}-g$ in D.

1.1.2.2 Generalization of Leray-Schauder degree

The first generalization of Leray-Schauder degree is due to M. Nagumo [61] who defined the degree for completely continuous maps on locally convex topological vector spaces instead of Banach spaces. Since for such spaces the notion of boundedness is equivalent to the existence of a norm, he had to redefine the notion of compactness for compact maps. In fact $K\colon X\to Y$ is compact in Nagumo's terminology if $K(X)$ is compact in Y. This is a drawback of the Nagumo's generalization because it is seldom the case that the operator is compact on the whole space.

The existence theory for fully nonlinear elliptic equations has been studied by several authors. Among the important works iun this direction, we could mention the work of F. Browder and R. Nussbaum [18] in which the Leray-Schauder degree was generalized for locally compact perturbations of homeomorphisms. In particular, they considered the map $T\colon\bar{G}\subset X\to Y$, G a bounded and open set and T with the representation $T(u)=S(u,u)$, where for $v\in G$ the map $S_v=S(.,v)\colon\bar{G}\to Y$ is continuous and bijective (then homeomorphism onto its image). Here the map $v\to S_v$ is assumed to be locally compact. They proved that for some types of fully nonlinear elliptic equations, this degree is suitable to establish the existence of a solution [18].

One other direction of generalization of the Leray-Schauder degree is to define a degree on Banach manifolds instead of Banach spaces. A. Tromba in his PhD thesis in 1968 used a generalized version of Sard's theorem due to S. Smale to define a degree for completely continuous maps on Banach manifolds. Afterward, he noted that K. D. Elworthy obtained independently the same results. In their joint book on this subject [33], they defined a degree for compact perturbations of Fredholm maps that is a straightforward generalization of the Schauder map. However the defined degree is not invariant under continuous homotopy. However, the absolute value of the degree is invariant under continuous homotopy.

The main motivation behind the above mentioned generalizations is the simple fact that fully nonlinear elliptic equations with general boundary conditions can not always be transformed to operator equations consisting only of Schauder maps, see [3]. However, an a priori estimate for the solutions of uniformly elliptic equations satisfying the complementing condition (Shapiro condition) on the boundary is proved in [2]. This estimate enables one to prove that uniformly elliptic equations satisfying the complementing condition can be transformed to abstract equations involving Fredholm operators, [2], [3]. Because of this, the study of quasi-linear Fredholm maps became of interest. A quasi-linear Fredholm map is of the form $f(x) = L_x(x) + g(x)$ where g is compact, $x \to L_x$ is continuous and L_x is a linear Fredholm map with index 0. It is not difficult to see that every fully nonlinear uniformly elliptic equation can be reformulated as a quasi-linear Fredholm equation. It is known that defining a degree theory satisfying all classical properties of a topological degree is impossible for such classes of maps, however one of the simplest "degrees" for such class of maps is proposed by Fitzpatrick using the notion of parity number [38],[39].

Yet another direction of generalization is to define a degree for the class of monotone maps. In fact, it is easily shown that nonlinear elliptic equations in divergence form can be formulated (under fairly simple conditions) as abstract equations involving pseudo-monotone or $(S)_+$ maps, see the section on on monotone maps below or for example [74].

It is well known that the Frechet derivative of a C^1 convex functional is monotone. More generally, the map $u \mapsto \partial F(u)$ for a convex functional $F(u)$ is maximal monotone where $\partial F(u)$ denotes the set of sub-gradients of F at u. On the other hand, one important class of equations involves monotone or maximal monotone maps while there is no convex potential for them. From this point of view, having a degree theory for different classes of monotone maps covers equation that cannot be dealt with by classical varitional methods. Note that in the classical variational method, the solvability of a PDE is equivalent to the attainability of the critical value of the some differentiable convex energy functional. The simplest result in this direction is obtained when the functional is convex and coercive. However, many equations in physics and geometry are not the Euler-Lagrange equation of some convex energy functional, see for example N. Goussoub [40]. Therefore these equations cannot be dealt with the classical variational method. On the other hand, many equations with no potential functional can be formulated as abstract equations with pseudo-monotone or $(S)_+$ maps. Thus having a degree theory for such maps will open the way to establish the existence, multiplicity and possible bifurcation of the solutions. A degree theory for the class of monotone maps has been proposed by F. Browder [16], [17], W. Petryshyn [66], J. Berkovits [6], [8], I. Skrypnik [74] and others. The interesting point here is that in addition to elliptic equations, parabolic and hyperbolic equations (not fully nonlinear) could be formulated in terms of monotone maps and then would also enjoy a degree theory.

Here we have to mention a recent result in the direction of variational method reported by N. Goussoub [40]. He can prove that for semi-linear non-self-adjoint elliptic equations there is a potential functional (not necessarily in terms of Euler-Lagrange equation) such that the infimum of the functional is the solution of the focal equation. In fact, the traditional energy functional now is supplemented by the Legendre transformation of the functional. The main task here is to show the new variational energy attains an infimum and then, using Fenchel inequality, this implies that the infimum is the solution of the original problem. It should be noted that, although this method covers many more equations than what traditionally could be dealt by variational methods, it does not provide further information such as bifurcation and multiplicity. As such information is readily included in the framework of degree theory, it is important to develop degree theoretic methods for these equations. In fact, as we will see in this thesis, even fully nonlinear elliptic or parabolic equations can be formulated in terms of $(S)_+$ maps while the variational method fails to apply for such equations.

In the following, we will first briefly go though preliminaries, then give definitions of Brouwer and Leray-Schauder degree, after which we will sketch variational methods, and explain motivations for the study of monotone maps.

1.1.3 Preliminaries

In this section, we fix notation and mention briefly those definitions and theorems that we need in this thesis. All facts mentioned in this section could be found in most books on functional analysis, for example E. Zeidler [83] . Recall that X is locally uniformly convex if for every $x \in S_X$ (S_X denotes the unit sphere in X) and $\varepsilon > 0$ there exist $\delta = \delta(\varepsilon, x)$ such that, for any $y \in S_X$ such that $\|x - y\| \geq \varepsilon$, we have $\left\|\frac{1}{2}x + \frac{1}{2}y\right\| \leq 1 - \delta$. Strong convergence is denoted by $\to$ and weak convergence by $\rightharpoonup$. In the whole of this thesis, Banach spaces are assumed to be separable. It is known that for separable Banach spaces, there is an equivalent norm to make them locally uniformly convex (and in particular strictly convex). In fact we have Kadec-Klee-Asplund theorem, for a proof see [32].

Theorem 1.2. *Let X be a separable Banach spaces, then there exist an equivalent norm on X such that X, X^* are locally uniformly convex.*

While it is generally false that every reflexive Banach space has an equivalent norm that makes it uniformly convex, due to the work of Lindenstruss, Asplund and Trojanski we have the following theorem, see S. Trojanski [75]

Theorem 1.3. *Every reflexive Banach space admits an equivalent locally uniformly convex norm.*

According to the above results, we may assume without loss of generality that the Banach spaces are separable, equipped with a locally uniformly convex norm.

Theorem 1.4. *Assume that X is a locally uniformly convex Banach space, $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$. Then $x_n \to x$.*

For X a Banach space, X^* denotes the topological dual of X and $\langle , \rangle$ denotes the continuous pairing between X, X^*. In most cases H denotes a Hausdorff Hilbert space. If we identify H with H^* then we use the notation $(,)$ both for the inner product in H and the pairing. Recall that the Gelfand triple or evolution triple is the triple (X, H, X^*) such that $X \to H \to X^*$, where embeddings are dense. Compact embeddings are denoted by $\hookrightarrow$ and we call the triple (X, H, X^*) compact if $X \hookrightarrow H \hookrightarrow X^*$. We use extensively a well known theorem in functional analysis that states closed bounded sets are weakly compact in reflexive Banach spaces. The map $T: X \to X$ where $T(x) = x - K(x)$ for K completely continuous is called the Schauder map or a compact vector field. We have the following theorem, see for example L. Nirenberg [64]

Theorem 1.5. *Let $\Omega \subset X$ be a closed bounded set. Then the map $\varphi: \Omega \to Y$ is compact if and only if f is a uniform limit of finite rank mappings $\{f_n\}$, $f_n: \Omega \to Y$; that is, for every $\varepsilon > 0$ there exists $N > 0$ such that for all $n \geq N$ we have $||\varphi_n(x) - \varphi(x)|| < \varepsilon$, $x \in \Omega$.*

Note that the above theorem is not asserting the approximation property for Banach spaces in general which is known to be false [35]. For $\varphi \in L(X, Y)$, the conjugate map $\varphi^*: Y^* \to X^*$ is defined as $\langle \varphi^*(y^*), x \rangle = \langle y^*, \varphi(x) \rangle$. It is easily verified that $\varphi^* \in L(Y^*, X^*)$. It is known (Schauder theorem) that if $\varphi \in L(X, Y)$ is compact then φ^* is compact. In addition to continuity and weak continuity notions, we can define other types of continuity.

Let X, Y be Banach spaces. Then $A: X \to Y$ is called demi-continuous at $x \in X$ if for every sequence $(x_n) \subset X$ that $x_n \to x$, A satisfies $A(x_n) \rightharpoonup A(x)$. The map A: $X \to X^*$ is called monotone if for any pair $x, y \in X$ A satisfies

$$\langle A(x) - A(y), x - y \rangle \geq 0.$$

If the inequality is strict (>0) whenever $x \neq y$, the map is called strictly monotone. The map is called strongly monotone if there exists an increasing function $c: \mathbb{R}_+ \to \mathbb{R}_+$ such that $c(x) = 0$ only for $x = 0$ and

$$\langle A(x) - A(y), x - y \rangle \geq c(\|x - y\|).$$

A is called a map of class $(S)_+$ if for any weakly convergent sequence (x_n), $x_n \rightharpoonup x$ with

$$\limsup_{n \to \infty} \langle A(x_n), x_n - x \rangle \leq 0, \tag{1.19}$$

then $x_n \to x$. It is obvious that every strong monotone map is a map of class $(S)_+$. A is called pseudo-monotone if $x_n \rightharpoonup x$ such that (1.19) holds, then

$$\lim \langle A(x_n), x_n - x\rangle = 0, \tag{1.20}$$

and $A(x_n) \rightharpoonup A(x)$. It is easily verified that if A is pseudo-monotone then for every $\varepsilon > 0$, the map $A_\varepsilon = A + \varepsilon J$ is a map of class $(S)_+$. Note that every demi-continuous monotone map is pseudo-monotone and, also, every demi-continuous map of class $(S)_+$ is pseudo-monotone. The map A is called quasi-monotone if $x_n \rightharpoonup x$ implies that

$$\limsup \langle A(x_n), x_n - x\rangle \geq 0.$$

Obviously every pseudo-monotone map is a quasi-monotone map. We have the following proposition, see [20]

The duality map $J: X \to 2^{X^*}$ in Banach spaces plays an important role in defining degree for monotone maps.

Definition 1.6. *The map $J: X \to 2^{X^*}$ defined as*

$$J(x) = \{w \in X^*, \langle w, x\rangle = \|x\|^2, \|w\| = \|x\|\}, \tag{1.21}$$

is called the duality map.

It is straightforward (using the Hahn-Banach theorem) to check that $J(x)$ is not empty. For locally uniformly convex Banach spaces, it turns out that J is a homeomorphism. In particular we have:

Proposition 1.7. *Assume that X is a locally uniformly convex Banach space, then the duality map J is single valued, bijective and bi-continuous. Furthermore J is strictly monotone and $(S)_+$.*

$T: X \to Y$ is called a Fredholm map if $d = \dim\ker(T)$ and $d' = \dim\operatorname{coker}(T)$ are finite (that the range $\operatorname{Rg}(T)$ is closed follows from the above two assumptions). The analytic index of T is defined as $\operatorname{ind}(T) = d - d'$. As we will see in the last chapter, the analytical index is topological; that is, if F, G are homotopic in the class of Fredholm maps $F \sim G$ then $\operatorname{ind}(F) = \operatorname{ind}(G)$. This implies immediately that the index of Schauder map is 0 since for $t \in [0,1]$ we have

$$\operatorname{ind}(\mathrm{Id} - K) = \operatorname{ind}(\mathrm{Id} - tK) = \operatorname{ind}(\mathrm{Id}) = 0. \tag{1.22}$$

Fixed point theorems play an important role in the existence problem of integral, partial and ordinary differential equations. After Brouwer's celebrated theorem on the fixed point of continuous maps in finite dimensional spaces, the most natural generalization for infinite dimensional spaces was introduced by Tikhonov. If $C \subset X$ is a closed, convex and compact set and A: $C \rightarrow C$ is continuous, then A has a fixed point. Schauder's celebrated theorem on the fixed point of compact vector fields (compact perturbation of the identity map) opened up the way for further generalizations, see for example [61],[13]. Specifically, the work of F. Browder in [13] is of great help in dealing with nonlinear uniformly elliptic PDE's. Further generalizations introduced to encompass the case of nonlinear elliptic problems in divergence form will be considered in detail in next chapters. If $\alpha = (\alpha_1, ..., \alpha_n)$ is a multi-index then D^α denotes the following operator:

$$D^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}.$$

For an integer $m \geq 0$ we define $D^{\leqslant m} = \{D^\alpha, |\alpha| \leq m\}$. For $\Omega \subset \mathbb{R}^n$ a domain with smooth boundary, $W^{m,p}(\Omega)$ denotes the Sobolev space of order m with $D^\alpha u \in L^p(\Omega)$ for $|\alpha| \leq m$. It turns out that $W^{m,p}(\Omega) = H^{m,p}(\Omega)$ the space obtained by the closure of $C^m(\bar{\Omega})$ in the norm of $W^{m,p}(\Omega)$. The space $W_0^{m,p}(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ in $W^{m,p}$ norm and it is easy to prove through the trace theorem that

$$W_0^{m,p}(\Omega) = \{u \in W^{m,p}(\Omega), D^\alpha u|_{\partial\Omega} = 0, |\alpha| \leq m - 1\}.$$

The topological dual of $W_0^{m,p}(\Omega)$, $p > 1$ is the space $W^{-m,q}(\Omega)$ where q is the conjugate number of p, $q = \frac{p}{p-1}$. The map $u \rightarrow u|_{\partial\Omega}$ is continuous from $W^{m,p}(\Omega)$ to $W^{m-1/p,p}(\partial\Omega)$ when $\partial\Omega$ is smooth. $C^{m,\delta}(\Omega)$ for $0 < \delta < 1$ denotes the space of Holder maps of order (m, δ). $C^m(\bar{\Omega})$ denotes the set of all $\varphi \in C^m(\Omega)$ such that $D^\alpha\varphi$ is bounded and uniformly continuous for $0 \leq |\alpha| \leq m$. The norm in $C^m(\bar{\Omega})$ is defined as $\|\varphi\|_\infty = \max_{0 \leq |\alpha| \leq m} \max_{x \in \bar{\Omega}} |D^\alpha \varphi(x)|$. Recall the embedding $C^{m,\lambda}(\bar{\Omega}) \hookrightarrow C^{m,\mu}(\bar{\Omega})$ for $0 < \mu < \lambda \leq 1$. The following are two most important embeddings in Sobolev spaces for bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary:

a) $W^{j+m,p}(\Omega) \hookrightarrow W^{j,q}(\Omega), \quad q < \frac{np}{n - mp}, \quad mp < n,$

b) $W^{j+m,p}(\Omega) \hookrightarrow C^j(\bar{\Omega}), \quad mp > n.$

The most well known examples of monotone, $(S)_+$ or pseudo-monotone maps arising in PDE are divergence form operators $A\colon W^{m,p}(\Omega)\to(W^{m,p}(\Omega))^*$

$$A(u)=\sum_{|\alpha|\leq m}(-1)^m D^\alpha A_\alpha(x,D^m u), \tag{1.23}$$

where A_α satisfies some appropriate conditions for ellipticity, monotonicity and Carathéodory conditions. The fully nonlinear equation $F(x,D^{2m}u)=0$ is uniformly elliptic on Ω and at $u\in W^{m,p}(\Omega)$ if for any $\xi\in\mathbb{R}^n-\{0\}$ and for some $\theta>0$ we have

$$(-1)^m\sum_{|\alpha|=2m}F_\alpha(x,D^{2m}u(x))\,\xi^\alpha\geq\theta\,|\xi|^{2m}, \tag{1.24}$$

where $F_\alpha=\frac{\partial F}{\partial(D^\alpha)}$ and $\xi^\alpha=(\xi_1^{\alpha_1},\ldots,\xi_n^{\alpha_n})$ for $\alpha=(\alpha_1,\ldots,\alpha_n)$.

1.2 Continuation method

1.2.1 Method of Isomorphism

Let X,Y be Banach spaces and let $A\colon X\to Y$ be a continuous map. The general idea of the continuation method is to transform the existence problem of the equation $A(x)=y$ to the existence problem of a "simpler" equation $A_0(x)=y$. For this, usually a continuous homotopy $h\colon[0,1]\times X\to Y$ is defined such that $h(0)=A_0$ is an isomorphism and $h(1)=A$. Under some conditions, it turns out that $h(t)$ is an isomorphism for $t\in[0,1]$. J. Leray in his Ph.D. thesis used the continuation method extensively for nonlinear integral equations [58]. The results of Schauder on compact perturbations of the identity led Leray to develop, in joint with Schauder, a topological degree for Schauder maps.

Recall that isomorphisms between Banach spaces are topological isomorphism due to the Banach isomorphism theorem. On the other hand, the set of all isomorphisms forms an open subset of linear bounded operators with respect to the operator norm topology; that is, if $T\in\operatorname{iso}(X,Y)$ and $\|T-S\|$ is small enough in the operator norm, then $S\in\operatorname{iso}(X,Y)$. The following generalization is well known in the literature and is called the continuation theorem.

Theorem 1.8. *Assume that* $F_0: X \to Y$ *is an isomorphism and* $F: [0,1] \times X \to Y$ *is a deformation with the following properties:*

1. *For each* t, $F(t,.): X \to Y$ *is bounded linear,*

2. $t \to F(t,\,.): X \to Y$ *is continuous; that is, for any sequence* t_n, $t_n \to t$, *the sequence of mappings* $F(t_n,.)$ *converges to* $F(t,.)$ *in the operator topology,*

3. *there exists a* $c > 0$ *independent of* t *such that*

$$\|F(t,x)\|_Y \geq c\,\|x\|_X, \quad x \in X. \tag{1.25}$$

Then $F(1,.) \in \mathrm{iso}(X,Y)$.

The following is a straightforward generalization of the theorem (1.8).

Theorem 1.9. *Let* Ω *be a path connected subset of a topological space and let* F: $\Omega \times X \to Y$ *be a map satisfying the following properties:*

1. *there exists* $\omega_0 \in \Omega$ *such that* $F(\omega_0,.): X \to Y$ *is an isomorphism,*

2. *for each* $\omega \in \Omega$, $F(\omega,.): X \to Y$ *is bounded linear,*

3. $\omega \to F(\omega,.)$ *is continuous; that is, for any sequence* $\omega_n, \omega_n \to \omega$, *the sequence* $F(\omega_n,.)$ *converges to* $F(\omega,.)$ *in the norm topology, and*

4. *there exists* $\omega_0 \in \Omega$ *and* $c > 0$ *such that*

$$\|F(\omega,x)\|_Y \geq c\,\|x\|_X, \quad x \in X. \tag{1.26}$$

Then $F(\omega,.)$, $\omega \in \Omega$ *is an isomorphism.*

Proof. Let $\gamma: [0,1] \to \Omega$ be the path connecting ω_0 to ω. The set $\gamma[0,1]$ is compact in Ω. Define the deformation f: $[0,\,1] \times X \to Y$ as $f(t,\,x) = F(\gamma(t),\,x)$. Since by assumption $f(0,.)$ is an isomorphism and satisfies conditions 1–3 of Theorem (1.8) then $f(1,.) = F(\gamma(1),.) = F(\omega,.)$ is an isomorphism. $\square$

1.2.1.1 Linear elliptic operator

The immediate consequence of the continuation method is the existence of unique solutions for linear uniformly elliptic equations. It is known that linear elliptic equations, under suitable boundary conditions (the Lopatinski-Shapiro condition), are Fredholm, see [3]. In particular the linear uniformly elliptic operator $L: H_0^{2m}(\Omega) \to L^2(\Omega)$ defined as

$$L(u) = \sum_{|\alpha| \leq 2m} a_\alpha(x)\, D^\alpha u, \tag{1.27}$$

where

$$H_0^{2m}(\Omega) = \{u \in H^{2m}(\Omega);\, D^\alpha u|_{\partial\Omega} = 0, 0 \leq |\alpha| \leq m-1\} \tag{1.28}$$

is a Fredholm operator with index 0.

The proof that L in (1.27) is Fredholm is straightforward and is given in the Appendix (A) for $m=1$. We can use this property to prove the following theorem with the aid of the continuation theorem.

Theorem 1.10. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded subset, let a_α be smooth functions on $\bar{\Omega}$, $X = H_0^2(\Omega), Y = L^2(\Omega)$ and let $L: X \to Y$ be a uniformly elliptic operator defined as:*

$$L(u)(x) = \sum_{|\alpha| \leq 2} a_\alpha(x)\, D^\alpha u, \quad x \in \Omega;$$

that is, for some $\theta > 0$

$$-\sum_{|\alpha| = 2} a_\alpha(x)\, \xi^\alpha \geq \theta\, |\xi|^2. \tag{1.29}$$

Furthermore, assume that $\min_\Omega a_0(x) \gg 1$. Then the Dirichlet problem $L(u) = f$ has a unique solution.

Proof. Let $L_0 = L - a_0$ and use the following estimate for uniformly elliptic operators see [2]:

$$\|u\|_X \leq c\, \|L_0(u)\|_Y + C\, \|u\|_Y. \tag{1.30}$$

Hence for the operator L we obtain by a simple calculation that

$$\|L(u)\|_{L^2} \geq C_1\, \|L_0(u)\|_{L^2} + a_0\, C_2\, \|u\|_{L^2}. \tag{1.31}$$

Substituting inequality (1.31) into (1.30) gives

$$\|u\|_X \leq C_1' \,\|L(u)\|_{L^2} - a_0\, C_2' \|u\|_{L^2} + C\, \|u\|_{L^2}. \tag{1.32}$$

Choosing $a_0 > 0$ sufficiently large we obtain for $c > 0$ the following inequality:

$$\|L(u)\|_{L^2} \geq c\, \|u\|_X. \tag{1.33}$$

Let $L_1(u) = \Delta u + a_0 u$ and consider the following convex homotopy

$$L_t = (1-t)L_1 + tL, \quad 0 \leq t \leq 1. \tag{1.34}$$

It is straightforward to verify that the operator L_t is uniformly elliptic, in fact for $\xi = (\xi_1, \ldots, \xi_n)$ we have for some $C > 0$ that

$$(1-t)\sum_{i=1}^{n} \xi_i^2 + t \sum_{|\alpha|=2} a_\alpha(x)\, \xi^\alpha \geq C|\xi|^2. \tag{1.35}$$

Using a similar argument to that used to prove (1.33), one concludes that

$$\|L_t(u)\|_Y \geq c\, \|u\|_X. \tag{1.36}$$

Since L_1 is an isomorphism, then by the theorem (1.8) we conclude that L is an isomorphism. □

1.2.2 Method of homeomorphism

1.2.2.1 Semi-linear elliptic equations

Recall that if the map $f: X \to Y$ is C^1 at $u \in X$ and $D_u f$ is an isomorphism from X to Y, then f is locally homeomorphism. On the other hand, if $D_u f$ is an isomorphism for all $u \in X$, it does not imply that f is a homeomorphism, however, we have the following theorem due to Hadamard. In the appendix we give a proof for the following version of Hadamard's Theorem.

Theorem 1.11. *Assume that for certain $x_0 \in E$ the map $D_{x_0} f$ is an isomorphism and furthermore assume that there exists an $M > 0$ such that $\|D_x f(z)\| \geq M \|z\|$ for all $x, z \in E$. Then $f: E \to Y$ is homeomorphism.*

We can use the above theorem to prove the following theorem. The proof is given in Chapter 5.

Corollary 1.12. *Assume that the function* $f\colon \mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$ *is* C^1 *and* $\Omega\subset\mathbb{R}^n$ *is a bounded domain with smooth boundary. Furthermore, assume that* $\left|\frac{\partial}{\partial x_i} f(x)\right|$ *is sufficiently small for all* $x\in\mathbb{R}\times\mathbb{R}^n$*. Then the operator* $G\colon C_0^{2,\delta}(\Omega)\to C^{0,\delta}(\Omega)$ *defined as*

$$G(u)=\Delta u+f(u,Du),$$

is a homeomorphism.

Remark 1.13. It is possible to generalize the above theorem to one of the following form

$$G(u)=P(u)+f(u,Du),$$

where P is a linear C^1 isomorphism.

Remark 1.14. It is of interest to study equations of the form

$$P(u)+\varepsilon f(u,Du)=g,$$

where P is an isomorphism and $|\varepsilon|$ is sufficiently small. We conjecture that one can use topological methods to show the solutions for ε small survive when $\varepsilon\to 1$.

1.2.2.2 Quasi-linear elliptic equations

As we saw above, under certain conditions, linear differential operators define isomorphisms between corresponding spaces. For nonlinear equations the theory of Fredholm operators (linear and nonlinear) comes into play. As a matter of fact, quasi-linear equations can be reformulated as abstract equations of the Schauder type. To illustrate the process, let us consider the following quasi-linear elliptic equation defined on $X=C_0^{2,\delta}(\bar{\Omega}), Y=C_0^{\delta}(\bar{\Omega})$ for an open bounded subset $\Omega\subset\mathbb{R}^n$:

$$\sum_{|\alpha|\leq 2} a_\alpha(x, D^{\leqslant 1}u)\, D^\alpha u = f, \tag{1.37}$$

where the a_α are continuous functions and satisfy some growth rate condition to guarantees continuity from X to Y. Define the operator

$$F_u(v) = \sum_{|\alpha| \leq 2} a_\alpha(x, D^{\leqslant 1} u) D^\alpha v$$

For fixed u, according to a previous result, the linear operator $L_u(v) := F_u(v) + K_u v$, is an isomorphism from X to Y provided that $K_u > 0$ is sufficiently large. Let us assume that $a_0 \gg 1$ is sufficiently large for $x \in \bar{\Omega}$ and $u \in X$. This implies that for fixed $u \in X$ we can solve in a unique way the following linear equation for $v \in X$:

$$\sum_{|\alpha| \leq 2} a_\alpha(x, D^{\leqslant 1} u)\, D^\alpha v = f. \tag{1.38}$$

Use the Schauder a priori estimate to conclude that the solutions v of (1.38) is bounded for the bounded set of functions $u \in X$. Let us denote the solution as $v = K_f(u) = F_u^{-1}(f)$. First note that F_u is continuous with respect to u and since F_u is a topological isomorphism then $\left\|F_u^{-1}\right\|$ is bounded. For topological isomorphisms F_u, F_w use the relation

$$\left\|F_u^{-1} - F_w^{-1}\right\| \leq \left\|F_u^{-1}\right\| \left\|F_w^{-1}\right\| \|F_u - F_w\|, \tag{1.39}$$

to conclude that K_f is continuous. We show $K_f(D)$ is precompact for arbitrary bounded set $D \subset X$. Choose arbitrary sequence $(v_n) \subset K_f(D)$ and choose the sequence (u_n) such that $v_n = K_f(u_n)$ or equivalently $F_{u_n}(v_n) = f$. Since the embedding X in $C^{1,\delta}(\bar{\Omega})$ is compact then there exists a convergent subsequence (u_{n_k}) of (u_n). By the continuity of K_f we conclude

$$v_{n_k} = K_f(u_{n_k}) \to K_f(u). \tag{1.40}$$

It is straightforward to verify that v is a solution of the equation $F_u(v) = f$. Therefore the existence of a solution for the equation (1.38) reduces to the existence of a fixed point for the equation $u = F_u(u)$.

1.3 Classical degree theory

In this section we review the formulation of Brouwer degree on compact manifolds as well as Leray-Schauder degree of maps on Banach spaces. We follow here L. Nirenberg [64]. The construction of Leray-Schauder degree for mappings defined on Banach manifolds is a little delicate and can be found in [76].

1.3.1 Brouwer degree

There are different approaches to define the Brouwer degree. The original work of Brouwer uses complicated algebraic topology tools, see [29]. E. Heinz [42] introduced an elementary analytic formulation for the Brouwer degree and then P. Lax, Nirenberg and H. Brezis formulated the definition for different classes of mappings, including Sobolev and VMO mapping on Riemannian manifolds, see [64] [9] [11] [10]. Consider A: $M \to N$ where $A \in C^1(M, N)$, M and N are oriented smooth manifolds of dimension n, M is compact and N is a connected manifold without boundary. Let $y \in N$ be a regular value for A; that is, $A^{-1}(y)$ does not contain any critical point. Since y is regular, the elements of $A^{-1}(y)$ are isolated by the inverse function theorem. Because M is compact, it follows that $A^{-1}(y)$ is finite. The crucial assumption that cannot be relaxed is the assumption that $y \notin A(\partial M)$. Now it is possible to define an integer valued mapping $\deg(A, M, y)$ for the mapping $A(M, N)$ in M at y. First note that since ∂M is compact, $A(\partial M)$ is closed in N and then $\mathrm{dist}(y, A(\partial M)) > 0$. Choose a chart about y and a sufficiently small neighbourhood N_y of y in that chart such that $N_y \cap A(\partial M) = \emptyset$. It is possible to choose N_y so small that $A^{-1}(N_y)$ consists of disjoint open neighbourhoods M_{x_i} for x_i such that $A(x_i) = y$ and A is bijective from each M_{x_i} onto N_y. Recall that $\{x_i\}$ is finite (possibly empty). Take an arbitrary n-form ω with the support in N_y such that $\int_N \omega = 1$. Since the manifolds are orientable, the integration is possible by the aid of partition of unity. The pull-back form $A^*\omega$ is a n-form in M with support in each M_{x_i}. The Brouwer degree is defined as follows:

$$\deg(A, M, y) = \int_M A^*\omega. \tag{1.41}$$

In local coordinates, let $(\frac{\partial}{\partial x^j})$ and $(\frac{\partial}{\partial y^j})$ denote the coordinates and let $\mathrm{dx} = dx^1 \wedge ... \wedge dx^n$ and $\mathrm{dy} = dy^1 \wedge ... \wedge dy^n$ be the volume elements in M and N respectively. If ω has the representation $\omega(y) = f(y)\mathrm{dy}$ in N, then $A^*\omega(x)$ is represented as $f(A(x))\, J_A(x)\, \mathrm{dx}$ where J_A denotes the Jacobian of the map A. The degree is then

$$\deg(A, M, y) = \sum_i \int_{M_{x_i}} f(A(x)) J_A(x)\, \mathrm{dx}, \tag{1.42}$$

with

$$\int_{N_y} f(y)\mathrm{dy} = 1. \tag{1.43}$$

Compare the change of variable formula for multiple integrals and the formula for the degree and conclude:

$$\int_{M(x_i)} f(A(x)) J_A(x)\, \mathrm{dx} = \mathrm{sign}(\, J_A(x)) \int_{M(x_i)} f(A(x)) |J_A(x)|\, \mathrm{dx} =$$

$$= \mathrm{sign}\,(J_A(x)) \int_{N(y)} f(y)\mathrm{dy} = \mathrm{sign}\,(J_A(x)).$$

Formally we can take $\omega = \delta(y - y_0)\mathrm{dy}$ where δ is the Dirac function. According to this choice of the form, we obtain

$$\deg(A, M, y) = \sum_{a_i \in A^{-1}(y)} \mathrm{sign}\, J_A(a_i). \tag{1.44}$$

In order to justify the definition, it should be shown that the degree formula is independent of the form ω. The proof relies on the following well known algebraic topological fact (for an elementary proof see Nirenberg [64]).

Proposition 1.15. *If ω is an n form compactly supported on Ω and $\int_\Omega \omega = 0$, then there exists an $(n-1)$-form η compactly supported in Ω such that $\omega = d\eta$.*

The above proposition justifies the definition of degree. In fact if ω_1, ω_2 are two different n-forms with unit volume then $\omega_1 - \omega_2 = d\eta$ for a $n-1$ form and then by Stokes theorem

$$\int_M A^*\omega_1 = \int_M A^*\omega_2 + \int_M d A^*\eta = \int_M A^*\omega_2 + \int_{\partial M} A^*\eta = \int_M A^*\omega_2.$$

Remark 1.16. By (1.44) we conclude that the degree is an integer valued map. In addition if $\int_N \omega \neq 1$ but is still nonzero, we can define $\omega' = \left(\int_N \omega\right)^{-1}\omega$. Then $\int_N \omega' = 1$ and

$$\deg(A, M, y) = \int A^*\omega' = \left(\int_N \omega\right)^{-1} \int_M A^*\omega.$$

Remark 1.17. The regularity restriction can be overcome by Sard's theorem. In fact, due to Sard's theorem, the critical values of continuous functions have Lebesgue measure zero in N and therefore it is possible, for arbitrary $z \in N$, to choose a regular value y arbitrary close to z. Then the degree of A at z is defined as the degree of A at y. The definition is justified then by the homotopy invariance property.

Let $\varphi \in C^1(S^n)$ and dx be the surface element of the unit sphere with respect to a geodesic coordinate system. According to the above formula we can define

$$\deg(\varphi) = \frac{1}{\mathrm{vol}(S^n)} \int_{S^n} J_\varphi \, \mathrm{dx}$$

Because $\partial S^n = \emptyset$, it is plausible to write $\deg(\varphi)$. Now according to the homotopy invariance property, it is completely obvious to define $\deg(\varphi) = 0$ if φ is homotopically trivial.

For the purpose of calculation, it is better to do everything in the embedding space $\mathbb{R}^{n+1}$. For the coordinate $(x^1, \ldots, x^n, x^{n+1})$ define $J = (\varphi, \partial_{x^1}\varphi, \ldots, \partial_{x^n}\varphi)$. Note that $|\varphi| = 1$ and that φ is perpendicular to each $\partial_{x^j}\varphi$ and therefore $J_\varphi = J$.

Remark 1.18. Let $A_1: M_1^{m_1} \to N_1^{n_1}$ and $A_2: M_2^{m_2} \to N_2^{n_2}$ are C^1 maps. Define A as

$$A = (A_1, A_2): M_1 \times M_2 \to N_1 \times N_2,$$

as $A(p, q) = (A_1(p), A_2(q))$. Then for $(y_1, y_2) \in N_1 \times N_2$ when $y_1 \notin A_1(\partial M_1)$ and $y_2 \notin A_2(\partial M_2)$ we have

$$\deg(A, M_1 \times M_2, (y_1, y_2)) = \deg(A_1, M_1, y_1) \deg(A_2, M_2, y_2). \tag{1.45}$$

The most useful property of degree is its invariance under continuous homotopy. Admissible homotopies are defined as follows.

Definition 1.19. (admissible homotopy) *We say f, g: $M \to N$ are homotopic, $f \sim g$ with respect to $y \in N$ if there exists a continuous map h: $[0, 1] \times M \to N$ connecting f to g with $f = h_0$ and $g = h_1$ such that h_t: $M \to N$ belong to $C^1(M, N)$ and $y \notin h(t, z)$ for any $(t, z) \in [0, 1] \times \partial M$.*

Theorem 1.20. *The Brouwer degree satisfies the classical properties of topological degree defined in* (1.1)*.*

We can use the homotopy invariance property to define a degree for continuous maps. If A is a continuous map on M, then for any $\varepsilon > 0$, there is a function $A_\varepsilon \in C^1(M)$ such that $|A - A_\varepsilon| < \varepsilon$. Therefore, there exists $\varepsilon_0 > 0$ such that $\deg(A_\varepsilon, M, y)$ is constant for any $0 < \varepsilon < \varepsilon_0$. This justifies the following definition:

Definition 1.21. (degree for continuous maps) *If A is a continuous map on M, its degree is*

$$\deg(A, M, y) = \lim_{\varepsilon \to 0} \deg(A_\varepsilon, M, y), \tag{1.46}$$

where the A_ε approximate A uniformly on M.

Similarly we can generalize degree for critical values of map A by a well known result by Sard that states that the set of critical values of a continues function has Lebesgue measure zero.

Definition 1.22. *If y is a critical value for A we can choose a sequence y_n of regular values for A such that $y_n \to y$ and then define*

$$\deg(A, M, y) = \lim_{n \to \infty} \deg(A, M, y_n). \tag{1.47}$$

Obviously the homotopy invariance property guarantees the existence of the limit.

Remark 1.23. The Brouwer fixed point theorem easily follows from the degree. In fact if A: $\bar{\mathbb{B}} \to \bar{\mathbb{B}}$ is a continuous map such that $\forall z \in S^{n-1}$, $z \neq A(z)$ then Id and $\mathrm{Id} - A$ are homotopic and therefore

$$\deg(\mathrm{Id} - A, \mathbb{B}, 0) = \deg(\mathrm{Id}, \mathbb{B}, 0) = 1.$$

Another important consequence of the degree that is extensively used is the acute lemma. Let C be an open bounded neighborhood of $0 \in \mathbb{R}^n$ and let $A: \bar{C} \to \mathbb{R}^n$ be a continuous map such that $(A(z), z) > 0$ for $z \in \partial C$. Then there exists a $x \in C$ such that $A(x) = 0$. In fact, the map $B(t) = t\mathrm{Id} + (1-t)A$ has no zero on ∂C, since $t + (1-t)(A(x), x) > 0$. Therefore

$$\deg(B(0), C, 0) = \deg(B(1), C, 0) = \deg(\mathrm{Id}, C, 0) = 1.$$

1.3.1.1 Properties of the Brouwer degree

Here we review some important properties of the Brouwer degree. First, we have the following fundamental fact.

Proposition 1.24. *Let $A: \bar{\mathbb{B}} \to \mathbb{R}^{n+1}$ be a C^1 map on the unit ball in $\mathbb{R}^{n+1}$ and let $\varphi: S^n \to S^n$ such that $A|_{S^n} = \varphi$. Then* $\deg(\varphi) = \deg(A, B, 0)$.

Corollary 1.25. *Assume that $\varphi, \psi: \bar{\mathbb{B}} \to \mathbb{R}^{n+1}$ are C^1 mappings such that $\varphi|_{S^n}$ is homotopic to $\psi|_{S^n}$; that is, there exista a homotopy $h_t: S^n \to R^n$ such that $h_0 = \varphi$ and $h_1 = \psi$ and $0 \notin h_t(S^n)$ for $0 \le t \le 1$. Then*

$$\deg(\varphi, \mathbb{B}^{n+1}, 0) = \deg(\psi, \mathbb{B}^{n+1}, 0). \tag{1.48}$$

The following proposition establishes the existence of continuous extensions of of mappings defined on S^n. The standard proof is based on the Hopf theorem that states that the homotopy class of continuous maps on S^n is $\mathbb{Z}$ and coincides with their degree. We give an alternative proof in Appendix (A).

Corollary 1.26. *Assume that $\varphi: S^n \to S^n$. Then* $\deg(\varphi) = 0$ *if and only if there exists a continuous extension A of φ on $\bar{\mathbb{B}}$, $A: \bar{\mathbb{B}} \to S^n$ such that $A(x) = \varphi(x)$ for $x \in S^n$.*

The following is the fundamental result about the uniqueness of the Brouwer degree. It was presented and proved first in a paper by Amann and Weiss [4]. We will give an elementary proof of the theorem in the Appendix (A).

Theorem 1.27. *Any integer valued map $d_1(A, D, y)$, $y \notin A(\partial D)$ that satisfies the classical properties in (1.20) is equivalent to the Brouwer degree* $\deg(A, D, y)$.

1.3.2 Leray-Schauder degree

J. Leray was the first person who constructed a continuous non-compact mapping defined on the ball with no fixed point. For a simple construction of such map see [56]. Clearly, this example contradicts the well known Brouwer fixed point theorem for continuous maps in finite dimensional spaces.. As is stated above, to have a degree theory that satisfies all the properties in definition (1.1) is impossible. This can be shown using Kuiper's theorem, which states that the space of bounded linear invertible maps on a separable Hilbert space is contractible. Let $H = D_1 \oplus D_2$ be a decomposition for the separable Hilbert space H where $D_1 \subset H$ is a finite dimensional subspace with odd dimension. For the identity map on the unit ball B_H, use (1.45) to write

$$\deg(-\mathrm{Id}, B_H, 0) = \deg(-\mathrm{Id}|_{D_1}, D_1 \cap B_H, 0) \deg(-\mathrm{Id}|_{D_2}, D_2 \cap B_H, 0). \tag{1.49}$$

But due to Kuiper's theorem $-\mathrm{Id} \sim \mathrm{Id}$ on B_H and $D_2 \cap B_H$ and since

$$\deg(\mathrm{Id}, B_H, 0) = \deg(A|_{D_2}, D_2 \cap \mathbb{B}, 0) = 1, \tag{1.50}$$

we conclude The same argument holds for the map $-A$. This implies that

$$\deg(-\mathrm{Id}|_{D_1}, D_1 \cap \mathbb{B}, 0) = 1, \tag{1.51}$$

a contradiction.

The construction of the Leray-Schauder degree for compact vector fields is based on the finite rank approximation of compact maps on bounded sets. Since this approach can be adapted to construct a degree for more general maps, we sketch the main idea here.

Let $A = \mathrm{Id} - K$, for $K: X \to X$ a completely continuous operator and X a Banach space. Recall that K is completely continuous if it is continuous and for every bounded set $D \subset X$, the set $K(D)$ is precompact. Theorem (1.5) in the Preliminaries section states that every completely continuous map can be approximated uniformly on bounded sets by finite rank maps.

Proposition 1.28. *Assume that $K: X \to X$ is a compact map. Then $A = \mathrm{Id} - K$ is closed; that is, the image of closed sets is closed under A.*

Proof. Let $D\subset X$ be closed and let $(y_n)\in(\mathrm{Id}-K)(D)$ be an arbitrary Cauchy sequence with $x_n=K(x_n)+y_n$. Since K is compact, there is a Cauchy sub-sequence $K(x_{n_k})$ and then (x_{n_k}) is a convergent since y_{n_k} is Cauchy. Since D is closed, (x_n) converges in D to some x, and since K is continuous, $K(x_{n_k})$ converges to $K(x)$ and therefore y_{n_k} converges to $x-K(x)$ in $A(D)$ since $x\in D$. Since (y_n) is Cauchy and one of its sub-sequences converges in $A(D)$, then (y_n) converges in $A(D)$. Since (y_n) is arbitrary then $A(D)$ is closed. □

Let Ω be an open bounded subset of X. The above proposition guarantees that if $0\notin A(\partial\Omega)$ then the distance of $A(\partial W)$ to the origin is nonzero. Let $\mathrm{dist}(0, A(\partial\Omega))=r>0$. Then by theorem (1.5) there is a finite rank map K_n such that $0\notin A_n(\partial\Omega)$ for $A_n=\mathrm{Id}-K_n$. Let V_n be the finite dimensional subspace of X such that $\mathrm{range}(K_n(\Omega))\subset V_n$ and define $\Omega_n=\Omega\cap V_n$. The Leray-Schauder degree of the map $A=\mathrm{Id}-K$ at $y=0$ is defined as

$$\deg_{\mathrm{LS}}(A,\Omega,0)=\lim_{n\to\infty}\deg\,(A_n,\Omega_n,0). \tag{1.52}$$

Definition 1.29. *Let $\Omega\subset X$ be a bounded open set, let X be a Banach space and let $K\colon X\to X$ be a completely continuous map. The degree of $A=\mathrm{Id}-K$ in Ω for $y\in X$ is defined as follows provided that $y\notin A(\partial\Omega)$:*

$$\deg_{\mathrm{LS}}(A,\Omega,y)=\deg_{\mathrm{LS}}(A-y,\Omega,0). \tag{1.53}$$

Definition 1.30. *An admissible homotopy for the map $A=\mathrm{Id}-K$ is defined as $A(t)=\mathrm{Id}-K(t)$ where $K\colon[0,1]\times\bar{D}\to X$ is a continuous compact map and $0\notin A(t)(D), t\in[0,1]$.*

In general, $K(t)$ being compact and continuous does not guarantee that K is compact from $[0,1]\times D$ to X. This failure prevents $(\mathrm{Id}-K)([0,1]\times\partial D)$ to be closed in X.

Proposition 1.31. *The degree* (1.52) *is well defined and satisfies all the classical properties of a topological degree.*

The proof is straightforward and can be found in, for example [56] or [64].

1.4 Monotone maps

It turns out that many problems in mathematical physics and geometry can be formulated in terms of equations

$$A(u) = f, \tag{1.54}$$

where A: $X \to X^*$ belongs to some class of continuous functions (demi-, hemi- or weak continuous) and X is a Banach space. The most well-known examples are the elliptic equations in divergence form

$$A(u) = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha A_\alpha(x, D^{\leqslant m} u), \tag{1.55}$$

where $u \in W_0^{m,p}(\Omega)$ and $D^{\leqslant m} u = \{D^\alpha u, |\alpha| \leq m\}$. It is known that the topological dual of $X = W_0^{m,p}$ is $X^* = W^{-m,q}$ with q the Hölder conjugate of p. Under suitable conditions on A_α, A will be a monotone, $(S)_+$ or pseudo-monotone map. The more interesting case is the fully nonlinear elliptic or parabolic equation

$$F(x, D^{\leqslant 2m} u) = 0, \tag{1.56}$$

$$u'(t) + F(t, x, D^{\leqslant 2m} u) = 0, \tag{1.57}$$

for which we defined a degree in the last chapter of this thesis.

It is well known in convex analysis that variational problems can be transformed to functional equations involving monotone maps. In fact the Fréchet derivative of a smooth convex functional is monotone. Therefore having a degree theory for the classes of monotone maps will open the way to establish the existence for variational problems. In addition, in the next chapter we will see how to transform variational inequalities that frequently appear in control theory, optimization and mathematical finance to functional equations involving monotone maps. Let us start with the definition of different types of monotonicity.

For future reference, we mention here the definition of different classes of monotone maps and state the relation between them.

Definition 1.32. *Map $A: G \subset X \to X^*$ is called*

i. monotone *if for all $u, v \in G$, ϕ satisfies $\langle A(u) - A(v), u - v\rangle \geq 0$,* strictly monotone *if it is strictly greater that zero for $u \neq v$ and* strongly monotone *if there exist a non-decreasing function $g(x) > 0$ for $x > 0$ and*

$$\langle A(u) - A(v), u - v\rangle \geq g(\|u - v\|). \tag{1.58}$$

ii. quasi-monotone *if for $u_n \in G$, $u_n \rightharpoonup u$, A satisfies*

$$\text{limsup}\,\langle A(u_n), u_n - u\rangle \geq 0. \tag{1.59}$$

iii. $(S)_+$ *if for $u_n \in G$, $u_n \rightharpoonup u$ and* $\text{limsup}\,\langle A(u_n), u_n - u\rangle \leq 0$, *then $u_n \to u$.*

iv. pseudo-monotone *map if $u_n \in G, u_n \rightharpoonup u$ and* $\text{limsup}\,\langle A(u_n), u_n - u\rangle \leq 0$, *then*

$$\text{limsup}\,\langle A(u_n), u_n - u\rangle = 0. \tag{1.60}$$

If furthermore, $u \in G$ then $A(u_n) \rightharpoonup A(u)$.

The following relation between different classes of monotone maps is clear.

Proposition 1.33. *A strong monotone map is a map of class $(S)_+$. If A is pseudo-monotone then $A_\varepsilon = A + \varepsilon J$ is a map of class $(S)_+$ for every $\varepsilon > 0$. If A is demi-continuous and monotone then A is pseudo-monotone. If A is a demi-continuous map of class $(S)_+$ then it is pseudo-monotone. If A is pseudo-monotone then it is a quasi-monotone map.*

Proposition 1.34. *Assume that X is a Hilbert space and $K: X \to X$ is compact, then $A = \text{Id} - K$ is of class $(S)_+$.*

Proof. Obviously Id is a map of class $(S)_+$, because, if $u_n \rightharpoonup u$, then $\|u\| \leq \text{liminf}\,\|u_n\|$. Therefore, if $\text{limsup}\,\langle u_n, u_n - u\rangle \leq 0$, then $\|u_n\| \to \|u\|$, which in turn in a Hilbert space implies $u_n \to u$. On the other hand, mappings of class $(S)_+$ are invariant under compact perturbations, because if K is compact then $\langle K(u_n), u_n - u\rangle \to 0$. □

Remark 1.35. According to the above proposition, it follows that every degree theory for the class of $(S)_+$ maps will generalize the Leray-Schauder degree in Hilbert spaces.

Definition 1.36. *Map $A: X \to X^*$ is called hemi-continuous at u if for every $v \in X$ it satisfies*

$$A(u+\varepsilon v) \rightharpoonup A(u), \quad (\varepsilon \to 0) \tag{1.61}$$

Proposition 1.37. *Any monotone hemi-continuous map A is demi-continuous and pseudo-monotone.*

Proof. Assume $u_n \to u$. Then by monotonicity we can write for any $z \in X$

$$\langle A(u+\varepsilon z) - A(u_n), u - u_n + \varepsilon z\rangle \geq 0. \tag{1.62}$$

Let us rewrite the above inequality as:

$$0 \leq \langle A(u+\varepsilon z) - A(u_n), u - u_n + \varepsilon z\rangle = \langle A(u+\varepsilon z) - A(u_n), u - u_n\rangle + \\ + \langle A(u+\varepsilon z) - A(u_n), \varepsilon z\rangle.$$

A is locally bounded since it is monotone(see [68]) and then the first term in the right hand side goes to zero when $n \to \infty$. Then we conclude:

$$\operatorname{liminf}\langle A(u+\varepsilon z) - A(u_n), \varepsilon z\rangle \geq 0. \tag{1.63}$$

For $\varepsilon > 0$ we have

$$\operatorname{liminf}\langle A(u+\varepsilon z) - A(u_n), z\rangle \geq 0,$$

and for $\varepsilon \to 0$ we have

$$\operatorname{liminf}\langle A(u) - A(u_n), z\rangle \geq 0.$$

For $\varepsilon < 0$ we have

$$\operatorname{limsup}\langle A(u+\varepsilon z) - A(u_n), z\rangle \leq 0,$$

and for $\varepsilon \to 0$, we have

$$\operatorname{limsup}\langle A(u) - A(u_n), z\rangle \leq 0.$$

We conclude that $\lim\langle A(u) - A(u_n), z\rangle = 0$. Since z is an arbitrary vector in X then $A(u_n) \rightharpoonup A(u)$. Now assume that $u_n \rightharpoonup u$ and

$$\operatorname{limsup}\langle A(u_n), u_n - u\rangle \leq 0. \tag{1.64}$$

Monotonicity implies that

$$\langle A(u_n), u_n - u\rangle \to 0. \tag{1.65}$$

Now consider the following inequality for arbitrary $z \in X$, which holds by the monotonicity

$$\langle A(u+\varepsilon z) - A(u_n), u+\varepsilon z - u_n\rangle \geq 0. \tag{1.66}$$

According to (1.65) we can write

$$\liminf \langle A(u+\varepsilon z) - A(u_n), \varepsilon z\rangle \geq 0, \tag{1.67}$$

which is the same as inequality (1.63). Then $A(u_n) \rightharpoonup A(u)$. This proves that A is pseudo-monotone. □

Here we will show how $(S)_+$ or pseudo-monotone maps come into play in studying nonlinear elliptic equations in divergence form. In the last chapter we generalize these results for fully nonlinear elliptic and parabolic equations. Consider the map $A\colon W_0^{m,p}(\Omega) \to W^{-m,q}(\Omega)$ defined as:

$$A = \sum_{|\alpha|\leq m} (-1)^m D^\alpha A_\alpha(x, D^{\leqslant m} u), \tag{1.68}$$

Under the following conditions on A_α, the map A defined in (1.68) is a continuous $(S)_+$ map. Interpreting ξ as a vector (ζ, η), where η corresponds to $|\alpha| = m$:

1. $A_\alpha(x, \xi)$ is measurable in x for any ξ and it is continuous in ξ for almost all x.
2. For functions a, b in $L_\infty(\Omega)$
$$A_\alpha(x,\xi) \leq a(x) + b(x)\,|\xi|^r, \quad a, b > 0, r \leq p-1. \tag{1.69}$$
3. $\sum_{|\alpha|=m} [A_\alpha(x,\zeta,\eta_1) - A_\alpha(x,\zeta,\eta_2)](\eta_1^\alpha - \eta_2^\alpha) > 0$ for $\eta_1 \neq \eta_2$
4. $\sum_{|\alpha|\leq m} A_\alpha(x,\xi)\xi_\alpha \geq c|\xi|^p - K(x), \quad K \in L_1(\Omega)$

Proposition 1.38. *Assume that A, defined in* (1.68)*, satisfies conditions* $(1-4)$*. Then A is a $(S)_+$ map.*

We first prove the following lemma.

Lemma 1.39. *Under the above setting, let* $|\eta-\eta_0|\geq\delta$ *for some* $\delta>0$*. Then there exists a* $k=k(\delta)>0$ *such that*

$$\sum_{|\alpha|=m}[A_\alpha(x,\zeta,\eta)-A_\alpha(x,\zeta,\eta_0)](\eta^\alpha-\eta_0^\alpha)\geq k(\delta). \tag{1.70}$$

Proof. Otherwise, there exists a sequnce η_n such that $|\eta_n-\eta_0|\geq\delta$ and

$$\sum_{|\alpha|=m}[A_\alpha(x,\zeta,\eta_n)-A_\alpha(x,\zeta,\eta_0)](\eta_n^\alpha-\eta_0^\alpha)\to 0. \tag{1.71}$$

Conditions (2,4) guarantee that $|\eta_n|$ is bounded, since otherwise

$$\sum_{|\alpha|=m}[A_\alpha(x,\zeta,\eta_n)-A_\alpha(x,\zeta,\eta_0)](\eta_n^\alpha-\eta_0^\alpha)\to\infty. \tag{1.72}$$

If $|\eta_n|$ is bounded then there exists a convergent subsequence $\eta_{n_k}\to\eta_1$ with $|\eta_1-\eta_0|\geq\delta$ such that

$$\sum_{|\alpha|=m}[A_\alpha(x,\zeta,\eta_1)-A_\alpha(x,\zeta,\eta_0)](\eta_1^\alpha-\eta_0^\alpha)=0, \tag{1.73}$$

which is impossible according to condition (1). □

Proof. (of proposition 1.38) Let $u_n\rightharpoonup u$ in $W^{m,p}$ and

$$\text{limsup}\sum_{|\alpha|=m}\int_\Omega A_\alpha(x,D^{\leqslant m}u_n)D^\alpha(u_n-u)\leq 0. \tag{1.74}$$

By the compact embedding $W^{m,p}\hookrightarrow W^{m-1,p}$, we conclude that $D^\alpha u_n\to D^\alpha u$ in L^p for $|\alpha|\leq m-1$. Define

$$\Omega_n(\varepsilon)=\left\{x\in\Omega,\sum_{|\alpha|=m}|D^\alpha u_n(x)-D^\alpha u(x)|^p\geq\varepsilon\right\}, \tag{1.75}$$

and

$$\lambda(\Omega_n(\varepsilon))=\sum_{|\alpha|=m}\int_{\Omega_n}[A_\alpha(x,D^{\leqslant m}u_n)-A_\alpha(x,D^{\leqslant m}u)]D^\alpha(u_n-u).$$

According to condition (3) we have liminf $\lambda(\Omega_n)\geq 0$. On the other hand, according to (1.74), we have

$$\text{limsup}\,\lambda(\Omega_n)=\text{limsup}\sum_{|\alpha|=m}\int_{\Omega_n}A_\alpha(x,D^{\leqslant m}u_n)D^\alpha(u_n-u)=$$

$$=\text{limsup}\,\langle A(u_n),u_n-u\rangle\leq 0.$$

Therefore

$$\lim \lambda(\Omega_n) = 0. \tag{1.76}$$

According to the above lemma we conclude that $\lambda(\Omega_n) \geq k(\varepsilon)\,|\Omega_n|$. Then $\lim |\Omega_n| = 0$. We show

$$\lim \int_{\Omega_n} |D^\alpha u_n - D^\alpha u|^p = 0. \tag{1.77}$$

Since $\lim |\Omega_n| = 0$, the above limit is equivalent to

$$\lim \int_{\Omega_n} |D^\alpha u_n|^p = 0. \tag{1.78}$$

According to the condition (4) and the fact that $D^\alpha u_n \to D^\alpha u$ for $|\alpha| \leq m-1$, we have

$$\lim \sum_{|\alpha|=m} \int_{\Omega_n} |D^\alpha u_n|^p = \lim \sum_{|\alpha|\leq m} \int_{\Omega_n} |D^\alpha u_n|^p \leq$$

$$\leq \lim \sum_{|\alpha|\leq m} \int_{\Omega_n} A_\alpha(x, D^{\leqslant m} u_n)\, D^\alpha u_n +$$

$$\lim \int_{\Omega_n} K(x) = \lim \sum_{|\alpha|\leq m} \int_{\Omega_n} A_\alpha(x, D^{\leqslant m} u_n)\, D^\alpha u_n =$$

$$= \lim \sum_{|\alpha|=m} \int_{\Omega_n} A_\alpha(x, D^{\leqslant m-1} u, D^m u_n)\, D^\alpha u_n$$

On the other hand, we have

$$\lim \sum_{|\alpha|=m} \int_{\Omega_n} A_\alpha(x, D^{\leqslant m-1} u, D^m u_n)\, D^\alpha u_n = \lim \lambda(\Omega_n) +$$

$$+ \lim \sum_{|\alpha|=m} \int_{\Omega_n} A_\alpha(x, D^{\leqslant m-1} u, D^m u_n)\, D^\alpha(u).$$

Note that the first term in the right-hand side is zero according to (1.76). The second term is aslo zero according to condition (2). Therefore, for any $\varepsilon > 0$, we conclude that

$$\int_{\Omega_n} |D^\alpha u_n - D^\alpha u| \to 0. \tag{1.79}$$

Since we have

$$\int_\Omega |D^\alpha u_n - D^\alpha u| < \varepsilon\, |\Omega - \Omega_n| + \int_{\Omega_n} |D^\alpha u_n - D^\alpha u|,$$

let $\varepsilon \to 0$ and then we conclude that $D^\alpha u_n \to D^\alpha u$ in L^p. □

It is also possible under weaker conditions that A is a pseudo-monotone mapping. For example, we can introduce the following condition.

Proposition 1.40. *Assume that A is bounded, hemi-continuous, and satisfies conditions (1-3). Then A is pseudo-monotone.*

Proof. By the same argument, we obtaine that $\lim \lambda(\Omega_n) = 0$. It remains only to show that $A(u_n) \rightharpoonup A(u)$. The general technique here is to perturb u to $u - t\,v$ for an arbitrary v and small t and use the fact that A is hemi-continuous. Because A is bounded then $A(u_n) \rightharpoonup h$ for some h, but according to the condition (3) we can write

$$0 \leq \limsup \langle A(u_n) - A(u - t\,v), u_n - u + t\,v\rangle = t\langle h - A(u - t\,v), v\rangle. \tag{1.80}$$

Let t approache to 0^+ (respectively 0^-) and use the hemi-continuity property of A to conclude the claim. □

1.5 The outline of this thesis

Chapter 2 of this thesis is dedicated to the generalized degree for different classes of monotone maps. We start with the concept of finite rank approximations of maps A: $X \to X^*$ (as well as $A: X \to 2^{X^*}$ for multi-valued maps) and present some results in the field of functional analysis. In particular, using the method of approximation by finite rank, we present some results for the solvability of abstract equations with weak continuous maps, the fixed point of multi-valued maps and proofs of some theorems in variational inequalities. We turn then to the defining degree for different classes of monotone maps, including $(S)_+$ maps, pseudo-monotone maps and maximal monotone maps using the introduced approximation by finite rank. Even though the degree is not novel for such maps, our approach is simpler than the one presented in the classical papers by J. Berkovits and F. Browder. In addition, using the defined degree, we will prove some classical theorems in convex analysis. The completely novel work in this chapter is the further generalization of degree for mappings that are not demi or even hemi continuous on the whole space. This generalized degree theory will be applied in Chapter 3 to the study of uniqueness and bifurcation for the Doi-Onsager problem.

In Chapter 3, we study the Doi-Onsager model for multi-phase materials, materials that look like regular liquids in low temperature and then turn into plasma phases when temperature increases. While some partial results, mainly in 1 and 2-dimensional space, have been reported by several authors (for general dimension few results exist), our approach to this problem is within the framework of degree theory and enjoys several advantages. We can solve this problem in the general dimension setting, which does not seem achievable by previous methods. Our method for the case $D=2$ is reported in [62]. In particular, we obtained the following results:

- The isotropic solution is unique for small λ.
- Bifurcation occurs for critical values of λ.
- Two solutions bifurcate at the critical values of λ.
- For S^1, the bifurcation solutions are super-critical and the first solution is stable.

In Chapter 4, we briefly study the field of periodic solutions for nonlinear dynamical systems using classical degree theory. In fact, this chapter is the continuation of the author's previous work in this direction. The study of existence of periodic orbits for dynamical systems was initiated by the pioneering work of Poincaré on the three-body problem. While there are many results in this field, we only present some well known results in this direction along with some of our results for autonomous third order systems. These results show how the periodic orbits of a one parameter family of dynamical systems survive under further deformation of the system itself. In particular we present two main results:

1. Third order system

$$x'''=f(x,x',x'') \tag{1.81}$$

   where f enjoys the parity condition $f(x,-x',x'')=-f(x,x',x'')$

   - $\exists\omega>0$ s.t. $x^{(i)}(t+\omega)=x^{(i)}(t), i=0,1,2,$
   - $\int_0^\omega x(t)\,dt=0$

2. Third order system with no parity condition:

$$x''' + \Lambda x' = \varepsilon f(x, x', x''), \tag{1.82}$$

- the existence of periodic solutions for $\varepsilon \ll 1$.
- the existence of periodic solutions for $\varepsilon = 1$.

The last chapter of this thesis, Chapter 5, is dedicated to the study of fully nonlinear elliptic and parabolic equations. The main work here is done for second order elliptic and parabolic equations, however the method can easily be adapted to higher even order equations. The framework that we work in is due to I. Skrypnik, however our presentation is simpler and self-contained. The main result here is that fully nonlinear elliptic equations can be written as abstract equations involving $(S)_+$ mappings. Then the existence, multiplicity and possible bifurcation of the focal equation are reduced to the obtained abstract equation . Furthermore, we prove that parabolic equations with fully nonlinear elliptic parts can be written also as an equations with $(S)_+$ mapping and this opens up a way to study the existence of the solutions for fully nonlinear evolution equations.

Chapter 2
Generalized degree

2.1 Introduction

This chapter is dedicated to defining a topological degree for different classes of monotone maps including $(S)_+$, pseudo-monotone and maximal monotone maps. The construction is based on a simple notion, the finite rank approximation of A: $X \to X^*$ for X a separable locally uniformly Banach space. Although degrees for such mappings are not novel, see [13], [14], [18], [16], [66], [6],[8], our construction for maps in separable Banach spaces is much simpler. In Section 2.1, we introduce the concept of finite rank approximation and present some results about the multi-valued maximal monotone maps. We use these results to define a degree for multi-valued maximal monotone and monotone maps. In Section 2.2, we use the finite rank approximation to prove some results in the fields of variations inequality. In subsequent sections, we develop a degree theory for a class of operators that is not continuous even on a dense subset of the focal space. We emphasize that such development is new. The obtained results here are particularly useful for our study in Chapter 3 of the Onsager problem.

First we note that the mappings in the class $(S)_+$ generalize the Schauder map in the following sense. Let X be a Hilbert space and $K\colon X \to X$ be compact, the map $A = \mathrm{Id} - K$ is of class $(S)_+$. F. Browder was first to gave a degree theory for $(S)_+$ mapping defined on reflexive Banach spaces, see [13], [14], [18], [16]. Using an embedding theorem due to Browder and Ton ([19]), Berkovits gave a degree for the bounded $(S)_+$ mappings on separable reflexive Banach spaces based on Leray-Schauder degree, see ([8], [6]). Browder and Ton embedding theorem states that for every separable reflexive Banach space X, there exist a separable Hilbert space H and a continuous compact injective map $\varphi\colon H \to X$ such that $\varphi(H)$ is dense in X. For an elementary proof see ([7]). Berkovits' construction is as follows. For a $(S)_+$ map $A\colon X \to X^*$, the map $\varphi\varphi^* A\colon X \to X$ is compact. The degree of A in a open bounded set Ω for $f \in X^*$ is defined as:

$$\deg(A, \Omega, f) = \lim_{\lambda \to \infty} \deg_{\mathrm{LS}}(\mathrm{Id} + \lambda\varphi\varphi^* A, \Omega, \varphi\varphi^* f). \tag{2.1}$$

Further generalization in this direction is the degree for the maximal monotone perturbation of $(S)_+$ mappings. The main application for maximal monotone degree is to prove the existence of solutions for certain evolution equations (see the last chapter of this thesis and also [15], [8], [77], [72]).

In the following section, we introduce the finite rank approximation of maps of the type $A\colon X \to X^*$ (and for the multi-valued maps $A\colon X \to 2^{X^*}$) in a systematic way. As before we assume that X is a separable reflexive locally uniformly Banach space. If A is a single valued bounded demi-continuous $(S)_+$ or pseudo-monotone map, the homotopy class of approximated maps remain stable and this enable us to define a degree for the original map A. When the multi-valued map A is maximal monotone, we replace its finite rank approximations with a continuous single-valued maps. In addition, we show that the homotopy class of such single-valued maps are stable with respect to the dimension of the space in which the original map is reduced. Through this, we can define the degree of maps for the map A.

In addition, we use also the notion of approximation by finite rank to give new proofs for some important theorems in convex analysis. In particular, we prove some theorems about the solvability of equations including weakly continuous maps, the fixed point of monotone maps in Hilbert spaces and also a brief review of basic theorems in variations inequalities.

2.2 Approximation by finite rank maps

2.2.1 The basic notion

For different classes of monotone map, we define a topological degree by the aid of finite rank approximation. Since a well known topological degree is available for maps in the finite dimensional spaces, we define the topological degree for the original map by the aid of the Brouwer degree for its finite rank approximation. The philosophy behind this approach is the same as J. Leray and J. Schauder's definition of a degree for the Schauder maps. To have a reference framework to define the approximations, we use the following embedding theorem due to F. Browder and B. Ton, see [19]. An elementary proof of the theorem can be found in J. Berkovits [7].

Theorem 2.1. *Let X be a separable Banach space, then there exist a Hilbert spaces H such that the embedding $H \to X$ is dense.*

Remark 2.2. It can be show that the above embedding is compact, see [7].

Let X be a separable space, and $\mathcal{H} = \{h^1, h^2, \cdots\}$ an orthonormal basis for H such that $\varphi: H \hookrightarrow X$ is dense and compact. Denote $\mathcal{X} = \{x^1, x^2, \cdots\}$ the image of $\mathcal{H}$ under φ and $X_n = \operatorname{span}\{x^1, x^2, \cdots, x^n\}$ the finite dimensional subspace of X equipped with the inner product $(x^i, x^j)_n = \delta_{ij}$. For the demi-continuous map $A: X \to X^*$, define the finite rank map $A_n: X \to X_n$ called the finite rank approximation of A in X_n as:

$$A_n(u) = \sum_{k=1}^{n} \langle A(u), x^k \rangle x^k. \tag{2.2}$$

Obviously the restriction of A to X_n coincides with A_n in the sense that for $v \in X_n$ and $u \in X$ we have $\langle A(u), v \rangle = (A_n(u), v)$.

This approximation greatly simplifies the proofs in different applications. In sequel, we prove some results of solvability of abstract equations, fixed point theorems and variations inequalities using the approximation by finite rank maps method.

2.2.2 Solvability of equations

When there is a potential for the map $A: X \to X^*$, the existence of the solution for the equation

$$A(u) = f \tag{2.3}$$

can be carried by classical variation method. The method of finite rank approximation enable us to investigate into the existence problem of the equation (2.3) even in the absence of potential functional. Here, we assume that A satisfies two conditions:

1. A is weakly continuous, that is for every weakly convergent sequence (u_n), $u_n \rightharpoonup u$ it implies $A(u_n) \rightharpoonup A(u)$,
2. A is coercive, that is

$$\frac{\langle A(u), u \rangle}{\|u\|} \to \infty, \text{ when } \|u\| \to \infty. \tag{2.4}$$

To be weakly continuous may seem restrictive, however it is easily verified that if L: $X \to X^*$ is a bounded linear operator, then it is weakly continuous. For nonlinear operators, consider A as

$$A(u) = \sum_{|\alpha|=m} a_\alpha(x) D^\alpha u + g(x, D^{\leqslant m-1} u). \tag{2.5}$$

where a_α are smooth enough on the closure of a bounded domain $\Omega \subset \mathbb{R}^n$ and g satisfies some growth rate condition to guarantee the continuity from $W^{m,p}(\Omega)$ into $L^2(\Omega)$. The first part of A is linear and then is weakly continuous. By Sobolev embedding if $u_n \rightharpoonup u$ in $W^{m,p}$ then $g(x, D^{\leqslant m-1} u_n) \to g(x, D^{\leqslant m-1} u)$ in L^2 and then it is continuous. Therefore A is weakly continuous from $W^{m,p}(\Omega)$ into $L^p(\Omega)$.

Theorem 2.3. *Let X be a separable Banach space and A: $X \to X^*$ be a weakly continuous and coercive map. Then $A(u) = f$ has a solution.*

For the proof we use the following lemmas.

Lemma 2.4. *For A a coercive map, there exist $R > 0$ such that*

$$\|u\| \geq R \Rightarrow \langle A(u), u\rangle > \langle f, u\rangle. \tag{2.6}$$

Proof. The proof is straightforward. In fact assuming the contrary, there exist (u_n), $\|u_n\| \geq n$ such that $\langle A(u_n), u_n\rangle \leq \langle f, u_n\rangle$. Then we can write

$$\frac{\langle A(u_n), u_n\rangle}{\|u_n\|} \leq \langle f, \frac{u_n}{\|u_n\|}\rangle \leq \|f\|. \tag{2.7}$$

But A is coercive, that is

$$\frac{\langle A(u_n), u_n\rangle}{\|u_n\|} \to \infty$$

that gives a contradiction.

□

Define A_n: $X_n \to X_n$ as (2.2) and consider the finite dimensional equation

$$A_n(u) = f_n. \tag{2.8}$$

Now we have the following lemma which is extensively used in dealing with the solvability of finite dimensional equations. We give a simple degree theoretic proof for it.

Lemma 2.5. *For any finite dimensional spaces X_n there exist at least one solution of equation* (2.8)*.*

Proof. Let $\mathbb{B}_R$ denote the R-ball in X and $B'_R = B_R \cap X_n$. First note that $f_n \notin A_n(\partial B'_R)$, since otherwise there exist $u \in \partial B'_R$ such that $A_n(u) = f_n$. But $A_n(u) - f_n = A(u) - f$ on the X_n which together with $u \in X_n$ gives $\langle A(u), u\rangle = \langle f, u\rangle$ that is impossible by lemma (2.4). Consider the following convex homotopy:

$$h(t) = t\,\mathrm{Id} + (1-t)(A_n - f_n). \tag{2.9}$$

Note that $0 \notin h(t)(\partial B'_R)$ for $t \in (0,1)$, otherwise for $z \in \partial B'_R$ we have

$$t\,z + (1-t)\,(A_n(z) - f_n) = 0 \Rightarrow A_n\,(z) = f_n + \frac{t}{1-t}\,z. \tag{2.10}$$

The above relation implies in turn

$$\langle A(z), z\rangle = \langle f, z\rangle - \frac{t}{1-t}(z,z) \leq \langle f, z\rangle. \tag{2.11}$$

that is impossible by lemma (2.4). Therefore by degree theoretic argument

$$\deg\,(A_n\,, B'_R, f_n) = \deg\,(\mathrm{Id}, B'_R, 0) = 1, \tag{2.12}$$

and this completes the proof. □

Proof. (of the theorem) By the above lemma, there exist a sequence $(u_n) \subset X$, $u_n \in X_n$ such that $A_n\,(u_n) = f_n$. Since X is reflexive and $\mathbb{B}_R$ is convex then $u_n \rightharpoonup u \in \bar{\mathbb{B}}_R$ in a sub-sequence. A is weakly continuous and then $A(u_n) \rightharpoonup A(u)$. We show $A(u) = f$. Since $(A(u_n))$ is weakly convergent it is bounded, let $M = \limsup \|A(u_n)\|$. The proof when X is separable is straightforward by the following consideration. For any $v \in X$, choose $w \in X_n$ such that

$$\|v - w\| < \frac{\varepsilon}{\|f\| + M}. \tag{2.13}$$

For arbitrary $v \in X$ we can write

$$\langle A(u) - f, v\rangle = \langle A(u) - A(u_n), v\rangle + \langle A(u_n) - f, v - w\rangle. \tag{2.14}$$

Note that $A(u_n) - f$ is zero on X_n. For sufficiently large n, $|\langle A(u) - A(u_n), v\rangle| < \varepsilon$ and also

$$|\langle A(u_n) - f, v - w\rangle| \leq (\|A(u_n)\| + \|f\|)\, \|v - w\| < \varepsilon \tag{2.15}$$

Therefore $|\langle A(u) - f, v\rangle| < \varepsilon$. Now let $\varepsilon \to 0$ and this completes the proof. □

Remark 2.6. The Lax-Milgram lemma in its general form is a direct consequence of the above argument. Assume that $\pi: H \times H \to \mathbb{R}$ is a coercive continuous bi-linear form (not necessarily symmetric). Define the map $A: H \to H$, $u \longmapsto A(u)$ as:

$$(A(u), v) = \pi(u, v), \quad v \in H. \tag{2.16}$$

A is clearly linear continuous and then weakly continuous and also coercive, then by the above argument the equation $A(u) = f$ has a unique solution u_f and therefore

$$\pi(u, v) = (A(u), v) = (f, v), \quad v \in H. \tag{2.17}$$

Coercivity in the whole space is too restrictive. In the following, we follow Y. Dubinskii to introduce the regularization method to overcome this situation, see [31]. Consider the operator $A: X \to X^*$. If A is coercive only on a subspace V of X then the map $i^* A i: V \to V^*$ is coercive. In general assume that there exist subspace $V \subset X$ and a bounded linear map $B: V \to X$ such that $B^* A i: V \to V^*$ is coercive with respect to the norm in X. We have the following theorem:

Theorem 2.7. *Denote $i: V \to X$ the embedding of V into X and assume $B^* A i$ is coercive, that is*

$$\frac{\langle A i(u), B\,(u)\rangle}{\|i(u)\|_X} \to \infty, \ \|i(u)\|_X \to \infty, \tag{2.18}$$

then the equation $A(u) = f$ has a solution up to the kernel of B^*, that is there exist $\rho \in \mathrm{Ker}(B^*)$ such that

$$A u = f + \rho. \tag{2.19}$$

Proof. Since B^*Ai is coercive, then for any $g \in V^*$, there exist $v \in V$ such that $B^*Ai(v) = g$, so if $g = B^*(f)$ then

$$\langle Ai(v), B(u)\rangle = \langle f, B\,(u)\rangle \Rightarrow Ai(v) = f + \rho. \tag{2.20}$$

□

In [31], some evolution equations are given for which the regularization method can be applied. In the last chapter of this thesis, we will consider a parabolic equation and apply this method to justify the applicability of this method for a class of nonlinear problems.

2.2.3 Intermediate value theorem

A wide class of well-known results about the existence of solutions for abstract equations come from the generalization of Bolzano's intermediate value theorem. Let $I = (-1, 1)$ and $f\colon \bar{I} \to \mathbb{R}$ continuous then $f(x) = 0$ is solvable in I if $z f(z)$ does not change sign for $z \in \partial I$. The assumption that $\operatorname{sign}(z\,.f(z))$ is constant for $z \in \partial I$ is known as Bolzano condition. The following proposition is a straightforward generalization of the Bolzano theorem.

Proposition 2.8. *Let $D \subset \mathbb{R}^n$ be an open bounded neighborhood of 0 and $f\colon D \to \mathbb{R}^n$ a continuous map satisfying the Bolzano condition, that is $\operatorname{sign}(z.f(z))$ is constant for $z \in \partial D$. Then $f(x) = 0$ is solvable in D.*

A general variant of the above fact is the following well known proposition that is an immediate corollary of the Hopf's theorem and the following fact we discussed in Chapter (1):

$$\deg\,(f, D, 0) = \deg\,(f, \partial D).$$

We give some other generalization below.

Proposition 2.9. *Let $K \subset X$ be a convex compact set containing 0, that is for $z \in K$ there exist $\mu > 0$ such that $\{tz, |t| \leq \mu\} \subset K$. ∂K, the boundary of K is defined as $z \in K$ such that for all $\varepsilon > 0$, $(1+\varepsilon)z \notin K$. If $f\colon K \subset X \to X^*$ is continuous and satisfies the Bolzano condition on ∂K, then $f(x) = 0$ is solvable on K.*

Proof. For simplicity assume $\langle f(x), x\rangle > 0$ on ∂K. Since ∂K is compact, there exist $r > 0$ such that $\langle f(x), x\rangle \geq r$. Choose ε-net $\{x_k\}_{k=1}^n \subset K$ and $\{y_k\}_{k=1}^n \subset f(K)$ for $0 < \varepsilon < r$. Define the finite dimensional sub-spaces $X_\varepsilon = \mathrm{span}\{x_k\}$ and $Y_\varepsilon = \mathrm{span}\{y_k\}$ and the finite rank map $f_\varepsilon\colon K \to Y_\varepsilon$ as $f_\varepsilon(x) = \sum_k \eta_k(f(x))\, y_k$, where $\{\eta_k\}$ is the partition of unity subordinate to $B_\varepsilon(y_k)$. Since $\|f_\varepsilon(x) - f(x)\| < \varepsilon$, it follows that $\langle f_\varepsilon(x), x\rangle > 0$, for $x \in \partial K$. Therefore the restricted map $f_\varepsilon\colon K \cap X_\varepsilon \to Y_\varepsilon$ is solvable in $K \cap X_\varepsilon$. Let $x_\varepsilon \in K$ such that $f_\varepsilon(x_\varepsilon) = 0$. Choose $\{\varepsilon_n\}, 0 < \varepsilon_n < r$ and $\varepsilon_n \to 0$. The sequence of solutions x_{ε_n} for $f_{\varepsilon_n}(x_{\varepsilon_n}) = 0$ converge in a sub-sequence to some point $x \in K$. It is enough to show $f(x) = 0$. Choose n so large that $\|f_{\varepsilon_n}(x) - f(x)\| < \varepsilon_n/2$ and $\|f(x) - f(x_{\varepsilon_n})\| < \varepsilon/2$. Now it follows

$$\|f(x)\| \leq \|f(x) - f(x_{\varepsilon_n})\| + \|f(x_{\varepsilon_n})\| < \varepsilon_n,$$

and it completes the proof. □

It is possible to generalize the above and relax the assumption on the compactness of K.

Proposition 2.10. *Let X be a separable reflexive Banach space, B the unit ball of X and $f\colon B \to X^*$ a bounded demi-continuous map that satisfies the strong monotonicity condition: $\langle f(u) - f(v), u - v\rangle \geq c(\|u - v\|)$ where c is a non-negative strictly increasing function and $c(0) = 0$. If $\langle f(z), z\rangle > 0$ for $z \in S$, the unit sphere of X, then there exist $x \in B$ such that $f(x) = 0$.*

Proof. Define the approximation $f_n\colon B \cap X_n \to X_n$ as

$$f_n(x) = \sum_{k=1}^{n} \langle f(x), x^k\rangle\, x^k, \tag{2.21}$$

with the inner product on X_n as $(x^i, x^j) = \delta_{ij}$. Obviously $f_n = f$ on X_n. Since $\langle f_n(z), z\rangle > 0$ for $z \in \partial(S \cap X_n)$ then there exist $x_n \in B \cap X_n$ such that $f_n(x_n) = 0$. Let x be a weak limit point of $\{x_n\}$ and then

$$\limsup \langle f(x_n) - f(x), x_n - x\rangle = \limsup \langle f(x_n), -x\rangle.$$

Choose sequence $\{s_n\}, s_n \in X_n$ and $s_n \to x$ then

$$\limsup \langle f(x_n), -x\rangle = \limsup \langle f(x_n), s_n - x\rangle \leq \limsup \|f(x_n)\| \, \|s_n - x\| = 0.$$

According to the monotonicity condition on f, we obtain $x_n \to x$ and then $f(x_n) \rightharpoonup f(x)$. Let $v \in X$ be arbitrary, then for the sequence $\{v_n\}, v_n \in X_n$ and $v_n \to v$ we have

$$\langle f(x), v\rangle = \lim \langle f(x_n), v\rangle = \lim \langle f(x_n), v - v_n\rangle \leq \lim \|f(x_n)\| \, \|x - v_n\| = 0.$$

Since v is arbitrary then $f(x) = 0$. □

More straightforward proof is also possible when we define a degree theory for the class of monotone maps. The condition on the strong monotonicity can be replaced by pseudo-monotonicity or being of class $(S)_+$.

Proposition 2.11. *Let X be a separable reflexive Banach space and $f\colon B \to X^*$ a bounded demi-continuous monotone map. If $\langle f(z), z\rangle > 0$ for $z \in S$ then there exist $x \in B$ such that $f(x) = 0$.*

Proof. The existence of $\{x_n\}$ such that $f(x_n) = 0$ on X_n is established in a similar manner of the previous proposition. For arbitrary y in X_n we can write

$$\langle f(y), y - x_m\rangle = \langle f(y) - f(x_m), y - x_m\rangle \geq 0, \quad m \geq n. \tag{2.22}$$

Let $x \in B$ be a weak limit point of $\{x_n\}$, then for any $y \in X_n$ we have the following inequality:

$$\langle f(y), y - x\rangle \geq 0. \tag{2.23}$$

Let $v \in X$ be arbitrary and $\{v_n\}, v_n \in X_n$ and $v_n \to v$, then

$$\langle f(v), v - x\rangle = \lim \langle f(v_n), v - x\rangle = \lim \langle f(v_n), v - v_n\rangle + \lim \langle f(v_n), v_n - x\rangle \geq 0.$$

Now take $v = x + \varepsilon y$ for $\varepsilon > 0$, then $\langle f(x + \varepsilon y), y\rangle \geq 0$ and by then $\langle f(x), y\rangle \geq 0$. Change y to $-y$ it gives $f(x) = 0$. □

An alternative proof is done by considering the maps $A + \varepsilon J$ for $\varepsilon > 0$ that is strongly monotone and therefore there exist x_ε such that $A(x_\varepsilon) + \varepsilon J(x_\varepsilon) = 0$. Now the existence of a solution for $A(x) = 0$ is achieved by the following fact.

Proposition 2.12. *Let X be a separable reflexive Banach space and $A: X \to X^*$ be a monotone map. In addition assume that for decreasing sequence $\{\varepsilon_n\}, \varepsilon_n > 0$ and $\varepsilon_n \to 0$ there exist bounded set $\{x_n\}$ such that $A(x_n) + \varepsilon_n J(x_n) = 0$ then there exist x_0 such that $x_n \to x_0$.*

Proof. Let x_0 be a weak limit point of $\{x_n\}$. For any fix n and arbitrary $m > n$ we have

$$\begin{aligned}\langle A(x_n) - A(x_m), x_n - x_m\rangle + \langle \varepsilon_n J(x_n) - \varepsilon_m J(x_m), x_n - x_m\rangle =\\ \langle A(x_n) - A(x_m), x_n - x_m\rangle + \varepsilon_m \langle J(x_n) - J(x_m), x_n - x_m\rangle +\\ +(\varepsilon_n - \varepsilon_m)\langle J(x_n), x_n - x_m\rangle = 0.\end{aligned}$$

Since $\varepsilon_n > \varepsilon_m$, we conclude $\langle J(x_n), x_n - x_m\rangle \leq 0$. Therefore $\|x_m\| \geq \|x_n\|$. Let $m \to \infty$ then $\langle J(x_n), x_n - x_0\rangle \leq 0$ them implies $\|x_0\| \geq \|x_n\|$. Therefore $\|x_n\| \to \|x_0\|$ and since E is uniformly convex then $x_n \to x_0$. □

2.2.4 Variational inequality

Variational inequalities are usually obtained by the minimization of convex functionals on a closed convex subsets of Banach spaces. Variation inequalities are fundamental in different fields of applied mathematics from mathematical physics, and optimal control [55] to financial mathematics and bio-mathematics [45]. For a simple example, consider the following problem. As a simple example of the equation that reduces to variational inequality, consider the following equation defined on a open bounded convex subset $\Omega \subset \mathbb{R}^n$:

$$\sum_{|\alpha| \leq 1} (-1)^{|\alpha|} D^\alpha A_\alpha(x, D^{\leqslant 1} u) = f, \tag{2.24}$$

with the boundary condition as

$$u|_{\partial K}\geq 0, \quad \frac{\partial u}{\partial n_A}|_{\partial\Omega}\geq 0, \quad u\frac{\partial u}{\partial n_A}|_{\partial\Omega}=0, \tag{2.25}$$

where

$$\frac{\partial u}{\partial n_A}(x):=\sum_{|\alpha|=1} A_\alpha(x, D^{\leqslant 1}u)n_\alpha(x), \quad x\in\partial\Omega. \tag{2.26}$$

In addition assume that there exist a smooth function $\mathcal{A}(x, D^{\leqslant 1}u)$ such that

$$A_\alpha:=\frac{\partial\mathcal{A}}{\partial\left(D^\alpha u\right)}. \tag{2.27}$$

It turns out that the above problem reduces to a variational inequality [55].

Let φ be a convex l.s.c. map and $K\subset X$ a closed convex subset. The following theorem is the most well known result in the classical variational problem.

Theorem 2.13. *Assume that φ is a l.s.c. convex and coercive functional, that is $\varphi(x)\to\infty$ when $\|x\|\to\infty$, then the minimization problem*

$$\min_{x\in K\subset X}\varphi(x) \tag{2.28}$$

has at least one solution. In addition if φ is strictly convex then the solution is unique.

The proof is straightforward by consideration that φ is weakly lower semi-continuous and the fact that the minimizing sequence is bounded according to the coercive condition. The uniqueness is an immediate consequence of the strict convexity. Variational inequalities come from the minimizing potential function on closed convex subsets as can bee seen by the following proposition.

Proposition 2.14. *Under the above settings, if $\varphi\in C^1(X,\mathbb{R})$ then $\bar{x}$ is a minimizer of the problem (2.28) if and only if*

$$\langle D_{\bar{x}}\varphi, x-\bar{x}\rangle\geq 0, \quad x\in K. \tag{2.29}$$

Proof. One direction is obtained directly as

$$\varphi(x)-\varphi(\bar{x})\geq\langle D_{\bar{x}}\varphi, x-\bar{x}\rangle\geq 0$$

The other direction is obtained by

$$0 \le \varphi(\bar{x} + \varepsilon(x - \bar{x})) - \varphi(\bar{x}) = \langle D\varphi(\bar{x} + \delta(x - \bar{x})), x - \bar{x}\rangle,$$

for $0 < \delta < \varepsilon$. Since φ is C^1 we have $D\varphi(\bar{x} + \delta(x - \bar{x})) \to D_{\bar{x}}\varphi$ when $\varepsilon \to 0$. □

Corollary 2.15. *Assume X is a uniformly convex Banach space and K is a closed convex subset of X, then for every $x \in X$ there exist a unique $P_K(x)$ in K such that*

$$\operatorname{dist}(x, K) = \|x - P_K(x)\|. \tag{2.30}$$

The proof is straightforward by the minimization of $\varphi_x(y) = \|y - x\|$ for $y \in K$. Recall that the duality map $J: X \to X^*$ is the Frechet derivative of $\frac{1}{2}\|x\|^2$ and therefore it implies the following important inequality:

$$\langle J(P_K(x) - x), y - P_K(x)\rangle \ge 0, \quad y \in K. \tag{2.31}$$

Particularly, if X is a Hilbert space then we have

$$\langle P_K(x) - x, y - P_K(x)\rangle \ge 0, \quad y \in K. \tag{2.32}$$

Definition 2.16. *Let X be a reflexive Banach space, $K \subset X$ a closed convex set, $f \in X^*$ and $A: X \to X^*$. For $\bar{x} \in K$, the inequalities of the following type is called a variational inequality:*

$$\langle A(\bar{x}), x - \bar{x}\rangle \ge \langle f, x - \bar{x}\rangle, \quad x \in K. \tag{2.33}$$

Below, we give simple proofs for some important variational inequalities using the approximation by finite rank maps. First we prove the following lemma.

Lemma 2.17. *Assume that X is a separable reflexive Banach space with a uniformly convex norm and $K \subset X$ is a closed convex set. If the map $A: X \to X^*$ is demi-continuous satisfying the following conditions (i),(ii), then there exist $\bar{x} \in K$ such that*

$$\langle A(\bar{x}), y - \bar{x}\rangle \ge 0. \tag{2.34}$$

i. $\langle A(x) - A(y), x - y\rangle \ge c||x - y||^{1+\delta}$ *for* $\delta > 0$, *ii.* $||A(x)|| \leq C\,||x||$, $C > 0$.

Proof. Since X is separable, choose subset $\mathcal{Y} = \{y^1, y^2, \ldots\} \subset K$ that is dense in K. Choose an independent subset of $\mathcal{Y}$ that we still denote by $\mathcal{Y}$ and define the linear sub-spaces $\{Y_n\}$ as $Y_n = \text{span}\{y^1, \ldots, y^n\}$. Define the approximation $A_n: Y_n \to Y_n$ as

$$A_n(u) = \sum_{k=1}^{n} \langle A(u), y^k \rangle\, y^k,$$

with the inner product $(y^i, y^j) = \delta_{ij}$. Obviously $A_n = A$ on Y_n. The a priori estimate for $\bar{x}$, the solution of (2.34) is obtained as follows. According to condition (i) we have

$$||A(\bar{x})||\,||y|| \geq \langle A(\bar{x}), y \rangle \geq \langle A(\bar{x}), \bar{x} \rangle \geq c||\bar{x}||^{1+\delta} - ||A(0)||\,||\bar{x}||,$$

and using condition (ii) we obtain

$$C\,||y|| + ||A(0)|| \geq c||\bar{x}||^{\delta}.$$

Let $y = P_K(0)$ then

$$||\bar{x}|| \leq c^{-1/\delta}(C\,\|P_K(0)\| + \|A(0)\|)^{1/\delta} = r.$$

Therefore the solution lies inside a bounded closed convex set $\Omega = K \cap \bar{B}(r)$. First we show that A_n satisfies inequality (2.34). Let $\Omega_n = \Omega \cap Y_n$, and use the result (2.32) for $K = \Omega_n$, that is

$$\langle P_{\Omega_n}(x) - x, y - P_{\Omega_n}(x) \rangle \geq 0, \quad y \in \Omega_n. \tag{2.35}$$

Let us assume that $\bar{x}_n$, the solution of (2.35) be the projection of some $x \in Y_n$ on Ω_n, that is $\bar{x}_n = P_{\Omega_n}(x)$. Assuming that, we should have $A_n(\bar{x}_n) = P_{\Omega_n}(x) - x$ or equivalently $x = \bar{x}_n - A_n(\bar{x}_n)$ that gives the equation $\bar{x}_n = P_{\Omega_n}(\bar{x}_n - A_n(\bar{x}_n))$. Therefore the solution $\bar{x}_n$ is the fixed point of the map $\varphi(u) = P_{\Omega_n}(u - A_n(u))$ defined on the bounded closed convex subset Ω_n. Since P_{Ω_n} and A_n are continuous functions then $\varphi: \Omega \to \Omega$ is a continuous function and by Brouwer fixed point theorem φ has a fixed point $\bar{x}_n \in \Omega_n$. Since $A_n = A$ on Y_n, it implies that

$$\langle A(\bar{x}_n), x - \bar{x}_n \rangle \geq 0, \quad x \in \Omega_n.$$

Since $\{\bar{x}_n\}$ is bounded in a reflexive Banach space X, then it has a weak limit point, say $\bar{x}\in\Omega$. Now for $\{s_n\}, s_n\in\Omega_n$, and $s_n\to\bar{x}$ we have

$$\limsup\langle A(\bar{x}_n),\bar{x}_n-\bar{x}\rangle=\limsup\langle A(\bar{x}_n),\bar{x}_n-s_n\rangle\leq 0.$$

Since A is strongly monotone $\bar{x}_n\to\bar{x}$ and then since A is demi-continuous we have $A(\bar{x}_n)\rightharpoonup A(\bar{x})$. Therefore we have for any $x\in\Omega_n$

$$\langle A(\bar{x}),x-\bar{x}\rangle=\lim\langle A(\bar{x}_n),x-\bar{x}_n\rangle\geq 0.$$

Now for arbitrary $y\in\Omega$, choose the sequence $\{s_n\}, s_n\in\Omega_n$ and $s_n\to y$ and conclude

$$\langle A(\bar{x}),y-\bar{x}\rangle=\lim\langle A(\bar{x}),s_n-\bar{x}\rangle\geq 0.$$

and this completes the proof. □

Now, by the aid of the above lemma, we give simple proofs for some variational inequalities, see for example [73].

Theorem 2.18. *Let X be a separable reflexive Banach space with a uniformly convex norm, K is a bounded closed convex subset of X, $A\colon X\to X^*$ a demi-continuous map and $f\in X^*$. Then there exist $\bar{x}\in K$ such that:*

$$\langle A(\bar{x}),x-\bar{x}\rangle\geq\langle f,x-\bar{x}\rangle,\quad x\in K, \tag{2.36}$$

if anyone of the following conditions satisfied:

1. *A is a map of class $(S)_+$.*
2. *A is a monotone map. In addition, if A is strictly monotone, that is*

   $$\langle A(x)-A(y),x-y\rangle>0,\quad x\neq y,$$

   then the solution is unique.
3. *A is a pseudo-monotone map.*

Proof.

1. Let $K_n=K\cap Y_n$ and define A_n the finite dimensional approximation of $A-f$ on K_n as

   $$A_n(u)=\sum_{k=1}^{n}\langle A(u)-f,y^k\rangle\,y^k.$$

Theorem implies the existence of $\{\bar{x}_n\} \subset K$ such that

$$\langle A(\bar{x}_n), x - \bar{x}_n\rangle \geq \langle f, x - \bar{x}_n\rangle, \quad x \in K_n$$

Since K is bounded, then $\{\bar{x}_n\}$ has a weak limit point, say $\bar{x} \in K$. Now for $\{s_n\}, s_n \subset K_n$ and $s_n \to \bar{x}$ we have

$\langle A(\bar{x}_n) - f, \bar{x}_n - \bar{x}\rangle \leq \langle A(\bar{x}_n) - f, \bar{x}_n - s_n\rangle + \|A(\bar{x}_n) - f\| \, \|s_n - \bar{x}\| \leq \|A(\bar{x}_n) - f\| \, \|s_n - \bar{x}\|$.

Therefore

$$\langle A(\bar{x}_n), \bar{x}_n - \bar{x}\rangle \leq \langle f, \bar{x}_n - \bar{x}\rangle + \|A(\bar{x}_n) - f\| \, \|s_n - \bar{x}\|.$$

Therefore

$$\text{limsup}\,\langle A(\bar{x}_n), \bar{x}_n - \bar{x}\rangle \leq 0 \Rightarrow \bar{x}_n \to \bar{x}.$$

Since A is demi-continuous then $A(\bar{x}_n) \rightharpoonup A(\bar{x})$. This implies that for any $x \in K_n$ we have

$$\langle A(\bar{x}), x - \bar{x}\rangle \geq \langle f, x - \bar{x}\rangle.$$

Now let $y \in K$, then for $\{s_n\}, s_n \in K_n$ and $s_n \to y$ we have

$$\langle A(\bar{x}) - f, y - \bar{x}\rangle = \lim \langle A(\bar{x}) - f, s_n - \bar{x}\rangle \geq 0,$$

and this completes the proof.

2. Define the map $A^{\varepsilon}(u) = A(u) - f + \varepsilon J(u)$ for $\varepsilon > 0$. Note that A^{ε} is strongly monotone. According to the above theorem, there exist $\bar{x}_{\varepsilon} \in K$ such that

$$\langle A^{\varepsilon}(\bar{x_{\varepsilon}}), x - \bar{x_{\varepsilon}}\rangle \geq 0, \quad x \in K. \tag{2.37}$$

Now let $\varepsilon \to 0$ and we show that $\bar{x}_{\varepsilon} \to \bar{x}$ for some $\bar{x} \in K$. Let $\{\varepsilon_k\}$ is a decreasing sequence $\varepsilon_k \to 0$ then by (2.37) there exist the sequence $\{x_k\}$ such that

$$\langle A(x_k) - f, x - x_k\rangle \geq -\varepsilon_k \langle J(x_k), x - x_k\rangle, \quad x \in K.$$

Fix n and for $m > n$ we have

$$\langle A(x_n) - f, x_m - x_n\rangle \geq -\varepsilon_n \langle J(x_n), x_m - x_n\rangle$$

$$\langle A(x_m) - f, x_n - x_m\rangle \geq -\varepsilon_m \langle J(x_m), x_n - x_m\rangle.$$

Adding the both sides of the above inequalities gives

$$0 \geq \langle A(x_n) - A(x_m), x_m - x_n \rangle \geq \varepsilon_m \langle J(x_n) - J(x_m), x_n - x_m \rangle + \\ +(\varepsilon_n - \varepsilon_m)\langle J(x_n), x_n - x_m \rangle.$$

Let $\varepsilon_m \to 0$ then and let $\bar{x}$ is a weak limit point for $\{x_k\}$, then we have

$$\langle J(x_n), x_n - \bar{x} \rangle \leq 0,$$

that conclude $\|\bar{x}\| \geq \|x_n\|$. This shows that $x_n \to \bar{x}$. For the last part, assume that x_1, x_2 are among the solution set of (2.36). Then we have

$$\langle A(x_1), x_2 - x_1 \rangle \geq \langle f, x_2 - x_1 \rangle, \quad \langle A(x_2), x_1 - x_2 \rangle \geq \langle f, x_1 - x_2 \rangle,$$

then we obtain $\langle A(x_1) - A(x_2), x_1 - x_2 \rangle \leq 0$ that implies $x_1 = x_2$.

3. If A is pseudo-monotone, the map A^ε define in the corollary (2.20) is a map of class $(S)_+$. In fact if $x_n \rightharpoonup \bar{x}$ and

$$\limsup \{\langle A(x_n), x_n - \bar{x} \rangle + \varepsilon \langle J(x_n), x_n - \bar{x} \rangle\} \leq 0,$$

then since liminf $\langle J(x_n),\ x_n - \bar{x} \rangle \geq 0$ then we conclude limsup $\langle A(x_n), x_n - \bar{x} \rangle \leq 0$. A is pseudo-monotone and then $\lim \langle A(x_n), x_n - \bar{x} \rangle = 0$. Therefore limsup $\langle J(x_n), x_n - \bar{x} \rangle \leq 0$ and since J is a map of class $(S)_+$ then $x_n \to \bar{x}$. According to corollary (2.18) there exist $\bar{x} \in K$ such that A^ε satisfies (2.36). Therefore for any $\varepsilon > 0$ there exist $\bar{x}_\varepsilon$ such that

$$\langle A(\bar{x}_\varepsilon) - f, \bar{x}_\varepsilon - y \rangle \leq \varepsilon\ \langle J(\bar{x}_\varepsilon), \bar{x}_\varepsilon - y \rangle, \quad y \in K.$$

Now let $\{\varepsilon_n\}, \varepsilon_n \downarrow 0$, then $\{\bar{x}_n\}$ has a weak limit point say $\bar{x} \in K$. We can write

$$\langle A(\bar{x}_n) - f, \bar{x}_n - \bar{x} \rangle \leq \varepsilon_n\ \langle J(\bar{x}_n), \bar{x}_n - \bar{x} \rangle.$$

Let $\varepsilon_n \to$ then we have

$$\limsup \langle A(\bar{x}_n), \bar{x}_n - \bar{x} \rangle = 0$$

that implies $A(\bar{x}_n) \rightharpoonup A(\bar{x})$. Now we can write

$$\langle A(\bar{x}) - f, y - \bar{x} \rangle = \lim \langle A(\bar{x}_n) - f, y - \bar{x} \rangle \geq -\lim \varepsilon_n\ \langle J(\bar{x}_n), y - \bar{x}_n \rangle = 0.$$

and this completes the proof. □

Remark 2.19. If K is a closed cone at the origin, the solution of the variational inequality $\langle A(x), y-x\rangle \geq 0$ is characterized as:

$$\langle A(\bar{x}), \bar{x}\rangle = 0, \quad \langle A(\bar{x}), y\rangle \geq 0, \quad y \in K.$$

If K is a closed subspace, then the variational inequality is the same of the existence of a unique solution for the equation $A(x)=0$. When $A(u)(v)=\pi(u,v)-(f,v)$ for $u,v\in H$, a bi-linear elliptic form in the Hilbert spaces H, then the result is know as Lax-Milgram lemma.

Remark 2.20. Lions and Stampacchia [49] proved (2.34) where $A\colon H\to H$, H is a Hilbert space and $A(u)(v)=\pi(u,v)-L(v)$ where π is an elliptic continuous bi-linear form and L is a continuous linear functional. The extensions to nonlinear case can be found in [55]. It is easily seen that that the operator A defined as above is a monotone operator and the existence of the solution is obtained immediately form Theorem 2.18.

Up to our knowledge, no author used degree theoretic argument in the field of variational inequality. The requirement here is to convert the variational inequality to an abstract equation in a suitable Banach space. In sequel, we show that variational inequalities can be transformed to abstract equations and the associated maps are of monotone class maps. This enables us to apply the topological method in deal with variational inequalities. Let $K\subset X$ be a closed convex set, X a separable uniformly convex space and $A\colon X\to X^*$ and $f\in X^*$, we want to solve for $u\in K$ the following inequality

$$\langle A(u)-f, v-u\rangle \geq 0, \quad v\in K \tag{2.38}$$

For any $f\in X^*$ consider the following functional:

$$\varphi_f(y)=\frac{1}{2}\|y\|^2-\langle f,y\rangle, \quad y\in K$$

The minimization of φ_f has a solution $T(f)\in K$. The solution is characterized by the following inequality

$$\langle J(T(f)), y-T(f)\rangle \geq \langle f, y-T(f)\rangle, \quad \forall y\in K$$

Proposition 2.21. *The operator $T\colon X^* \to X$ has the following monotonicity property:*

$$\langle f - g, T(f) - T(g)\rangle \geq 0, \quad \forall f, g \in X^*$$

Proof. For $f, g \in X^*$ we have

$$\langle J(T(f)), T(g) - T(f)\rangle \geq \langle f, T(g) - T(f)\rangle$$

$$\langle J(T(g)), T(f) - T(g)\rangle \geq \langle g, T(f) - T(g)\rangle$$

Then the monotonicity follows by the inequality $\langle J(x) - J(y), x - y\rangle \geq 0$ and the summing up the above two inequalities. □

Assume $u \in K$ is the solution of inequality (2.38) then for any $y \in K$ we have

$$\langle J(u), y - u\rangle \geq \langle J(u) + f - A(u), y - u\rangle$$

and then by uniqueness we have

$$u - T(J(u) + f - A(u)) = 0$$

If K is compact, then $T\colon X^* \to K$ is compact and then the Schauder fixed point is applicable. In the general case, we can use the monotonicity property of T to deduce the fixed point of T.

2.3 Multi-valued mappings

In this section, we generalize the notion of approximation by finite rank maps for single valued map to multi-valued maps of the form $T\colon X \to 2^{X^*}$. Two notions of continuity of multi-valued maps are used: lower semi-continuity and upper semi-continuity.

Definition 2.22. *Let X, Y be Banach spaces. The map $T\colon X \to 2^Y$ is called lower semi-continuous at $x \in X$ if for any $y \in T(x)$ and any open neighborhood V of y, there exist an open neighborhood U of x such that for every $u \in U$ it holds $T(u) \cap V \neq \emptyset$. T is called upper semi-continuous at x if for any neighborhood V of $T(x)$, there exist an open neighborhood U of x such that $T(U) \subset V$.*

It is easily seen that the function $H: \mathbb{R} \to 2^{\mathbb{R}}$ defined as

$$H(x) = \begin{cases} \{1\} & x > 0 \\ [0,1] & x = 0 \\ \{0\} & x < 0 \end{cases},$$

is upper semi-continuous every where while it is not lower semi-continuous at $x = 0$. If $B \subset X$ is the unit ball in X, it is easily seen that the map $T: X \to 2^X$ defined below is lower semi-continuous:

$$T(x) = \begin{cases} \{x\} & x \in \bar{B} \\ X & x \notin \bar{B} \end{cases}$$

Remark 2.23. We should emphasize that the lower and upper semi continuity for single valued function is not the same as the above defined notions for multi-valued functions. The notion of semi-continuity can be defined equivalently as follows. The map $T: X \to 2^Y$ is lower semi-continuous if for every open set $V \subset Y$ the set $T^{-1}(V)$ defined below is open:

$$T^{-1}(V) = \{x \in X; T(x) \cap V \neq \emptyset\}.$$

T is called upper semi-continuous if for every open set V the set $T_{-1}(V)$ defined below is open:

$$T_{-1}(V) = \{x \in X; T(x) \subset V\}.$$

If $T: X \to 2^Y$ is a multi-valued function, the graph of T is defined

$$\mathrm{graph}(T) = \{(x, y), y \in f(x)\}. \tag{2.39}$$

We use the notation $[x, y] \in \mathrm{graph}(T)$ to denote that $y \in T(x)$. The following theorem is attributed to E. Michael [60].

Theorem 2.24. (continuous selection) *Assume that X is a paracompact topological space and Y is a Banach space, $T: X \to 2^Y$ a lower semi-continuous map such that for every x, $T(x)$ is non-empty closed and convex, then there exist a continuous single valued function f such that $[x, f(x)] \in \mathrm{graph}(T)$.*

It turns out that being l.s.c. is too restrictive for multi-valued maps while being u.s.c. is the minimum continuity that we need in dealing with non-linear multi valued mappings including maximal monotone maps. For u.s.c. multi valued maps, we have the following theorem [25]. For an alternative proof see [56].

Theorem 2.25. **(ε-continuous sub-graph)** *Assume X, Y are Banach spaces and $T: X \to 2^Y$ an upper semi-continuous map and $T(x)$ is closed and convex for all $x \in X$. Then for any $\varepsilon > 0$, there exist a continuous single valued function $T_\varepsilon: X \to Y$ such that for any x, there exist $z \in X$ and $y \in T(z)$ such that $\|x - z\| < \varepsilon$ and $\|y - T_\varepsilon(x)\| < \varepsilon$.*

Since the construction of T_ε is used in the definition of generalized degree, we briefly explain the proof of the above theorem here. Fix $\varepsilon > 0$, for arbitrary $x \in X$, let $V_\varepsilon(T(x))$ be the ε-neighborhood of $T(x)$. Since $T(x)$ is convex $V_\varepsilon(T(x))$ is convex. Since T is upper semi-continuous, there exist $\delta = \delta(x) > 0$ such that $T(B_\delta(x)) \subset V_\varepsilon(T(x))$. The family of open sets $\mathcal{B} = \{B_\delta(x), x \in X\}$ covers X. Choose a locally finite star refinement $\mathcal{B}_1 = \{U_\alpha\}$ of $\mathcal{B}$. By definition, for $x \in X$, the star of x in $\mathcal{B}_1$ is defined as

$$\mathrm{st}(x, \mathcal{B}_1) = \cup_\alpha \{U_\alpha \ni x, U_\alpha \in \mathcal{B}_1\}.$$

$\mathcal{B}_1$ is a star refinement of $\mathcal{B}$ if for every $x \in X$, $\mathrm{st}(x, \mathcal{B}_1)$ is contained in some $B_\delta(y) \in \mathcal{B}$. In fact for paracompact spaces (consisting of metrizable spaces), every open cover has an open locally finite refinement and then star refinement, see for example S. Willard [80]. Let f_α be a continuous partition of unity subordinate in U_α, then define T_ε as follows:

$$T_\varepsilon(x) = \sum_\alpha f_\alpha(x)\, z_\alpha,$$

where $z_\alpha \in T(U_\alpha)$ is arbitrary.

By the aid of above theorem, we can prove some results by the finite rank approximation of T_ε. First we prove the lemma that is the generalization of classical acute angle property for single valued maps.

Lemma 2.26. *Let $X = \mathbb{R}^n$ and $T: X \to 2^X$ be an upper semi-continuous multi-valued mapping, $T(x)$ is closed and convex for $x \in X$ and for every $z \in S_X$ and $w \in T(z)$ it satisfies the acute angle property, that is $(w, z) > 0$, then $0 \in T(B_X)$.*

Proof. Let $T_{\varepsilon(n)}$ be the continuous single valued $\varepsilon(n)$-approximation of T where $\varepsilon(n)\to 0$ for $n\to\infty$. Choose $\varepsilon(n)$ so small that $(T_{\varepsilon(n)}(z),z)>0$ and then there exist $x_n\in\mathbb{B}_X$ such that $T_{\varepsilon(n)}(x_n)=0$. Let $x_n\to x\in\bar{\mathbb{B}}_X$ in a sub-sequence, then there exist $w_n\in T(y_n)$ such that $\|w_n\|<\varepsilon(n)$ and $\|y_n\|\in(1+\varepsilon(n))\mathbb{B}_X$. It implies that $0\in T(x)$, $x\in\bar{\mathbb{B}}_X$. But according to the acute condition $x\notin S_X$ and then $0\in T(\mathbb{B}_X)$. □

Proposition 2.27. *Let X be a separable reflexive Banach space, T: $X\to 2^{X^*}$ an upper semi-continuous strong monotone map, that is for every $[x,\, h]$, $[y,\, w]\in$ graph(T) it satisfies*

$$\langle h-w, x-y\rangle \geq c(\|x-y\|).$$

In addition $T(x)$ is closed and convex for every $x\in X$ and on S_X it satisfies the acuteness condition, that is $z\in S_X$ and $w\in T(z)$ it satisfies $\langle w,\, z\rangle>0$. Then $0\in T(B_X)$.

Proof. Let T_ε be a single valued continuous ε-approximation of T. By simple calculation we obtain for any $x,y\in X$ the following relation

$$\langle T_\varepsilon(x)-T_\varepsilon(y), x-y\rangle \geq c(\|x_1-y_1\|)-2\varepsilon(\|x-y\|+\|w_1-w_2\|),$$

where the following estimates holds

$$\|T_\varepsilon(x)-w_1\|, \|T_\varepsilon(y)-w_2\|<\varepsilon, \quad \|y-y_1\|, \|x-x_1\|<\varepsilon.$$

Let $\varepsilon_n\to 0$ and $T_{\varepsilon_n}^n$ is the finite rank approximation of T_{ε_n}. Choose ε sufficiently small such that for any finite dimensional spaces X_n the condition $(T_{\varepsilon_n}^n(z),\, z)>0$ holds for $z\in S_{X_n}$. Therefore we have a sequence of $x_n\in\mathbb{B}_{X_n}$ such that $T_{\varepsilon_n}^n(x_n)=0$. Since $T_{\varepsilon_n}^n=T_{\varepsilon_n}$ on X_n, then $T_{\varepsilon_n}(x_n)=0$ on X_n. $x_n\rightharpoonup x\in\mathbb{B}_X$ in a sub-sequence and then for $(v_n), v_n\in X_n$ and $v_n\to x$ we obtain

$$\begin{aligned} c(\|x_n-x\|)+O(\varepsilon_n)\leq \langle T_{\varepsilon_n}(x_n)-T_{\varepsilon_n}(x), x_n-x\rangle = \\ =-\langle T_{\varepsilon_n}(x_n), x-v_n\rangle-\langle T_{\varepsilon_n}(x), x_n-x\rangle = \\ =-\langle T_{\varepsilon_n}(x_n), x-v_n\rangle-\langle w_1, x_n-x\rangle \\ -\langle T_{\varepsilon_n}(x)-w_1, x_n-x\rangle \end{aligned}$$

For $\varepsilon_n \to 0$ we obtain $\limsup c(\|x_n - x\|) \leq 0$ and then $x_n \to x$. Since $T_{\varepsilon_n}(x_n) = 0$ on X_n, then there exist y_n and $w_n \in T(y_n)$ such that $\|x_n - y_n\| < \varepsilon_n$ and $\|T_{\varepsilon_n}(x_n) - w_n\| < \varepsilon_n$. Let $v \in X$, $\|v\| = 1$ be arbitrary, $v_n \in X_n$ and $v_n \to v$, then we have

$$\varepsilon_n > \langle w_n - T_{\varepsilon_n}(x_n), v \rangle = \langle w_n - T_{\varepsilon_n}(x_n), v - v_n \rangle + \langle w_n, v_n \rangle.$$

For $n \to \infty$ we obtain $\langle w_n, v \rangle \to 0$ that is $w_n \rightharpoonup 0$. On the other hand $y_n \to x$ therefore there exist $\delta > 0$ such that $w_n \in V_\delta(T(x))$ and since $V_\delta(T(x))$ is convex then $0 \in \operatorname{cl} V_\delta(T(x))$. Since it is possible to choose δ arbitrary small then $0 \in T(x)$. According to the acuteness condition $x \notin S(X)$ and then $0 \in T(B_X)$. □

The proof of the following proposition is similar to the previous one.

Proposition 2.28. *Let X be a separable uniformly convex Banach space, $T: X \to 2^{X^*}$ an upper semi-continuous monotone map, that is for every $[x, h], [y, w] \in \operatorname{graph}(T)$ it satisfies*

$$\langle h - w, x - y \rangle \geq 0.$$

In addition $T(x)$ is closed and convex for every $x \in X$ and on S_X it satisfies the acuteness condition, that is $z \in S_X$ and $w \in T(z)$ it satisfies $\langle w, z \rangle > 0$. Then $0 \in T(B_X)$.

2.3.0.1 Maximal monotone maps

Definition 2.29. *$T: X \to 2^{X^*}$ is maximal monotone if for any pair $(x, \varphi) \in X \times X^*$ such that*

$$\langle f - \varphi, u - x \rangle \geq 0, \quad \text{for all } (u, f) \in \operatorname{graph}(T) \Rightarrow (x, \varphi) \in \operatorname{graph}(T).$$

It is equivalent to say that the graph is maximal in the set theory sense. The maximal monotone map T is bounded if for any bounded sequence $(u_n) \subset X$, there exist a bounded sequence $(w_n), w_n \in T(x_n)$.

Proposition 2.30. *Let X be a separable reflexive Banach space, $T: X \to 2^{X^*}$ maximal monotone, then T is norm to weak* upper semi-continuous and for every $x \in X$, $T(x)$ is closed and convex.*

Proof. Assume $x_n \to x$, since T is locally bounded in the interior of $D(T) = X$, then $\{T(x_n)\}$ is bounded. Choose $w_n \in T(x_n)$ arbitrary, then w_n weakly converge in a subsequence (that we still show as w_n) to some $w \in X^*$. For arbitrary $[u, h] \in \text{graph}(T)$ we have $\langle h - w_n, u - x_n \rangle \geq 0$, and then $\langle h - w, u - x \rangle \geq 0$ that is $[x, w] \in \text{graph}(T)$. For arbitrary x and $w_1, w_2 \in T(x)$, let $z_t = t w_1 + (1 - t) w_2$ for $t \in [0, 1]$. For any pair $[y, h] \in \text{graph}(T)$ we have

$$\langle h - z_t, y - x \rangle = t \, \langle h, w_1, y - x \rangle + (1 - t) \langle h - w_2, y - x \rangle \geq 0,$$

and then $[x, z_t] \in \text{graph}(T)$. This shows that $T(x)$ is convex. If $w_n \to w$, $(w_n) \subset T(x)$, then for arbitrary $[y, h] \in \text{graph}(T)$ we can write

$$0 \leq \langle h - w_n, y - x \rangle \to \langle h - w, y - x \rangle,$$

that implies $[x, w] \in \text{graph}(T)$ and then $T(x)$ is closed. □

Now we define a finite rank approximation of a maximal monotone map T. For any finite dimensional subspace $X_n \subset X$ with basis $\mathcal{X} = \{x^1, \ldots, x^n\}$, the finite rank map T_n of the maximal monotone map T is defined as

$$T_n(u) = \bigcup_{w \in T(u)} \sum_{k=1}^{n} \langle w, x^k \rangle x^k. \tag{2.40}$$

Proposition 2.31. *Let X be a separable reflexive Banach spaces, T: $X \to 2^{X^*}$ maximal monotone. Then T_n is a monotone upper semi-continuous map, and for every $x \in X$, $T_n(x)$ is closed and convex.*

Proof. For every $w \in T(u)$, let $w^n \in T_n(u)$ defined as

$$w^n = \sum_{k=1}^{n} \langle w, x^k \rangle x^k.$$

Note that for every $y \in X_n$, we have the equality $(w^n, y) = \langle w, y \rangle$. This fact justifies that the multi-valued map $T_n: X_n \to 2^{X_n}$ is monotone since for every pair $w^n \in T_n(u)$, $h^n \in T_n(v)$ we have

$$(h^n - w^n, v - u) = \langle h - w, v - u \rangle \geq 0.$$

Since T is norm to weak* upper semi-continuous, for arbitrary convergent sequence (u_α) in X_n, $u_\alpha \to u$, the arbitrary sequence (w_α), $w_\alpha \in T(u_\alpha)$, converges weakly in a sub-sequence (w_β) to some $w \in T(u)$. Now for every $y \in X_n$, $\langle w_\beta, y\rangle \to \langle w, y\rangle$. But we have the equalities

$$\langle w_\beta, y\rangle = (w_\beta^n, y), \quad \langle w, y\rangle = (w^n, y).$$

This implies that $w_\beta^n \to w^n$ and therefore $T_n: X_n \to 2^{X_n}$ is an upper semi-continuous multi-valued monotone map. In addition, assume that (w_i^n) is a Cauchy sequence, $w_i^n \in T_n(x)$. In fact we have

$$w_i^n = \sum_{k=1}^{n} \langle w_i, x^k\rangle x^k.$$

Since $T(x)$ is bounded (every maximal monotone map is locally bounded), then (w_i) is bounded and then $w_i \rightharpoonup w$ in a sub-sequence for some $w \in X^*$. This implies that $\langle w_i, x^k\rangle \to \langle w, x^k\rangle$. But for arbitrary $[y, h] \in \mathrm{graph}(T)$ we have

$$0 \le \langle h - w_i, y - x\rangle \to \langle h - w, y - x\rangle,$$

and then $[x, w] \in \mathrm{graph}(T)$ and therefore $w^n \in T(x)$. □

Using the above results and the construction of finite rank approximation, we give a simple proof for the following important theorem in convex analysis. When the map A in the following theorem is the duality map J, part (ii) of the theorem is proved by R. Rockafellar and parts (i),(iii),(iv) are proved by F. Browder by the aid of convex analysis method. Our proof is based on the idea of choosing a continuous single valued function T_n^ε for each T_n that exist according to the Theorem (2.25).

Theorem 2.32. *Assume $T: X \to 2^{X^*}$ is a bounded, maximal monotone map and A: $X \to X^*$ is a continuous, coercive and strongly monotone map, then the following statements hold:*

i. $T_\varepsilon = T + \varepsilon A$ for $\varepsilon > 0$ is a map of class $(S)_+$ in the sense that if $u_n \rightharpoonup u$ and for some $w_n \in T(u_n)$ it satisfies the condition

$$\limsup \langle w_n + \varepsilon A(u_n), u_n - u\rangle \le 0, \tag{2.41}$$

then $u_n \to u$.

ii. The map T_ε *is onto* X^*,

iii. For $u_1, u_2 \in X$ *and* $u_1 \neq u_2$ *we have*

$$T_\varepsilon(u_1) \cap T_\varepsilon(u_2) = \emptyset, \tag{2.42}$$

iv. $T_\varepsilon^{-1}: X^* \times (0,\infty) \to X$ *is well defined and is continuous.*

Proof.

i. Assume $u_n \rightharpoonup u$ and for a sequence $(w_n), w_n \in T(u_n)$ the condition (2.41) holds. for $w \in T(u)$ we can write

$$\text{limsup}\,\langle w_n + \varepsilon A(u_n), u_n - u\rangle = \text{limsup}\,\{\langle w_n - w, u_n - u\rangle + \varepsilon\langle A(u_n), u_n - u\rangle\}.$$

But for the right hand side we have

$$\text{limsup}\,\{\langle w_n - w, u_n - u\rangle + \varepsilon\langle A(u_n), u_n - u\rangle\} \geq \varepsilon\text{limsup}\,\langle A(u_n), u_n - u\rangle.$$

Since A is a $(S)_+$ map, we conclude $u_n \to u$.

ii. Since T is bounded, then for any $u \in X$ and $u_n \to u$, there exists a bounded sequence (w_n) such that $w_n \in T(u_n)$. X^* is reflexive, then there exists a subsequence, that we still show by w_n that weakly converges to some w, that is $w_n \rightharpoonup w \in X^*$. We show $[u, w] \in \text{graph}(T)$. For arbitrary $[x, h] \in \text{graph}(T)$ we have:

$$\langle h - w, x - u\rangle = \lim\,\langle h - w_n, x - u_n\rangle \geq 0,$$

therefore $[u, w] \in \text{graph}(T)$. Next we show T_ε is coercive in the following sense:

$$\forall\,[u, w] \in G(T), \quad \frac{\langle w + \varepsilon A(u), u\rangle}{\|u\|} \to \infty, \|u\| \to \infty. \tag{2.43}$$

In order to prove it, we can write for $[0, h] \in \text{graph}(T)$

$$\frac{\langle w + \varepsilon A(u), u\rangle}{\|u\|} = \frac{1}{\|u\|}\langle w - h, u\rangle + \frac{1}{\|u\|}\langle h, u\rangle + \varepsilon\left\langle A(u), \frac{u}{\|u\|}\right\rangle \geq$$

$$\geq \varepsilon\left\langle A(u), \frac{u}{\|u\|}\right\rangle - \|h\| \to \infty.$$

The last part comes from the fact that A is coercive. This implies that for $f \in X^*$ the solution set of the inclusion equation

$$f \in T_\varepsilon(u) \tag{2.44}$$

is located inside $\mathbb{B}_R$ for some $R > 0$. Define the following finite rank approximation on X_n

$$T_\varepsilon^n = T_n + \varepsilon A_n, \tag{2.45}$$

where T_n is defined in (2.40). Therefore the solution set of the following inclusion equation

$$f_n \in T_\varepsilon^n(u_n) \tag{2.46}$$

is located inside $\mathbb{B}_R^n = \mathbb{B}_R \cap X_n$, where as before f_n is defined in the basis $\mathcal{X} = \{x^1, x^2, \ldots\}$ as follows:

$$f_n = \sum_{k=1}^{n} \langle f, x^k \rangle x^k.$$

The solution set of the inclusion equation (2.46) is non-empty according to the proposition (2.26). In fact the map

$$\mathcal{A}_n = T_n + \varepsilon A_n - f_n,$$

is acute on ∂B_R^n and $0 \in \mathcal{A}_n(u_n)$ for some $u_n \in B_R^n$. Now, since $u_n \in \mathbb{B}_R$ and X is reflexive then $u_n \rightharpoonup u$ in a sub-sequence for some $u \in \bar{\mathbb{B}}_R$. In order to show the existence of a solution for the inclusion $f \in T_\varepsilon(u)$, we show that $u_n \to u$. Choose $w_n \in T(u_n)$ such that $w_n + \varepsilon A(u_n) = f_n$, then

$$\langle w_n + \varepsilon A(u_n), u_n - u \rangle = \langle f_n, u_n \rangle - \langle w_n + \varepsilon A(u_n), u \rangle.$$

Since $f_n \to f$ and $u_n \rightharpoonup u$, $\langle f_n, u_n \rangle \to \langle f, u \rangle$ and then

$$\limsup \{\langle f_n, u_n \rangle - \langle w_n + \varepsilon A(u_n), u \rangle\} = \limsup \langle f - w_n - \varepsilon A(u_n), u \rangle.$$

Choose $z_n \in X_n$ such that $z_n \to u$ then according to the relation

$$\langle f - w_n - \varepsilon A(u_n), z_n \rangle = 0,$$

we can write

$$\langle f - w_n - \varepsilon A(u_n), u \rangle = \langle f - w_n - \varepsilon A(u_n), u - z_n \rangle \to 0. \tag{2.47}$$

Therefore $u_n \to u$ because T_ε is a map of class $(S)_+$. We have established so far the following relationships

$$u_n \to u, \quad w_n \rightharpoonup w, \quad [u, w] \in \operatorname{graph}(T) \tag{2.48}$$

For arbitrary $z \in X$, we can write

$$\langle f - w - \varepsilon A(u_n), z\rangle = \langle f - w_n - \varepsilon A(u_n), z\rangle + \langle w_n - w, z\rangle,$$

and then

$$\operatorname{limsup} \langle f - w - \varepsilon A(u_n), z\rangle = \operatorname{limsup} \langle f - w_n - \varepsilon A(u_n), z\rangle \tag{2.49}$$

Now, for $(z_n), z_n \in X_n$ and $z_n \to z$ we have

$$\langle f - w_n - \varepsilon A(u_n), z\rangle = \langle f - w_n - \varepsilon A(\,u_n), z - z_n\rangle \to 0. \tag{2.50}$$

Since z is arbitrary we deduce

$$f \in (T + \varepsilon A)(u).$$

and this completes the proof.

iii. If $h \in (T + \varepsilon A)(u_1)$ and $h \in (T + \varepsilon A)(u_2)$, then both pairs $[u_1, h - \varepsilon A(u_1)]$ and $[u_2, h - \varepsilon A(u_2)]$ belong to graph(T) and then

$$-\varepsilon \langle A(u_1) - A(u_2), u_1 - u_2\rangle \geq 0 \tag{2.51}$$

that is possible only if $u_1 = u_2$.

iv. According to parts (i)-(ii), $T + \varepsilon A$ is onto and the map $(T + \varepsilon A)^{-1}$ is well defined. Assume $h_n \to h$, and $\varepsilon_n \to \varepsilon$ where $h_n \in (T + \varepsilon_n A)(X)$. Since $T + \varepsilon A$ is onto, there exist $x_n \in X$ such that $h_n \in (T + \varepsilon_n A)(x_n)$. We show that (x_n) is bounded. In fact if $w_n \in T(x_n)$ and $w_0 \in T(0)$, then

$$\langle h_n, x_n\rangle = \langle w_n, x_n\rangle + \varepsilon_n \langle A(x_n), x_n\rangle = \langle w_n - w_0, x_n\rangle + \langle w_0, x_n\rangle + \varepsilon_n \langle A(x_n), x_n\rangle \geq$$
$$\geq \langle w_0, x_n\rangle + \langle A(0), x_n\rangle + \varepsilon_n\, c \|x_n\|^\alpha$$

that is impossible if $\|x_n\| \to \infty$. Therefore x_n are bounded and then $x_n \rightharpoonup x$ in a sub-sequence that we still denote by subscript n. Now consider

$$\langle h_n, x_n - x\rangle = \langle w_n, x_n - x\rangle + \varepsilon_n \langle A(x_n), x_n - x\rangle \geq \langle w, x_n - x\rangle + \varepsilon_n \langle A(x_n), x_n - x\rangle \tag{2.52}$$

where $w \in T(x)$. This implies

$$\limsup \langle A(x_n), x_n - x \rangle \leq 0$$

and then $x_n \to x$. We show $h \in (T + \varepsilon A)(x)$. Since A is continuous, for any $[y, w] \in G(T)$ we have

$$\langle h - \varepsilon A(x) - w, x - y \rangle = \lim \langle h_n - \varepsilon_n A(x_n) - w, x_n - y \rangle \geq 0 \tag{2.53}$$

that implies $h \in (T + \varepsilon A)(x)$.

$\square$

2.4 Degree for mappings of class $(S)_+$

Recall that $A\colon X \to X^*$ is a single valued $(S)_+$ map if for every $x_n \rightharpoonup x$ and

$$\limsup \langle A(x_n), x_n - x \rangle \leq 0,$$

then $x_n \to x$. A is demi-continuous at x if for every sequence (x_n), $x_n \to x$ it implies that $A(x_n) \rightharpoonup A(x)$. A is bounded if it maps bounded sets to bounded sets. It is easily seen that a fairly large class of elliptic operators, including operators in divergence form are of type $(S)_+$, see [6] and [14]. Also, fully nonlinear uniformly elliptic operators defined on a Hilbert space can be reduced to a map of class $(S)_+$, see [74]. In this section we assume that $A\colon X \to X^*$ is a bounded demi-continuous map of class $(S)_+$ where X is separable locally uniformly convex Banach space. In the definition of degree here and in the next section, we extensively use the notion of finite rank approximation introduced in this chapter. This simplifies the degree construction and the proofs of its classical properties considerably.

Let us repeat the definition of finite rank approximation here. Assume that H is a separable Hilbert space such that $\varphi\colon H \to X$ is injective compact and dense in X. Let $\mathcal{H} = \{h^1, h^2, \ldots\}$ be an orthonormal basis for H and $\mathcal{X} = \{x^1, x^2, \ldots\}$ be the image of $\mathcal{H}$ under φ, that is $x^i = \varphi(h^i)$. Define the finite rank map $A_n\colon X \to X_n = \operatorname{span}\{x^1, \ldots, x^n\}$ as

$$A_n(x) = \sum_{k=1}^{n} \langle A(x), x^k \rangle\, x^k.$$

The inner product defined in X_n is defined as $(x^i, x^j) = \delta_{i,j}$ where $\delta_{i,j}$ denotes the Kronecker delta. First note that for $A|_{X_n} = A_n|_{X_n}$ that is for any $y \in X_n$ we have

$$\langle A(x), y\rangle = (A_n(x), y). \tag{2.54}$$

In fact for $y = \sum y_k x^k$ we obtain

$$\left\langle A(x), \sum y_k x^k \right\rangle = \sum \langle A(x), x^k\rangle\, y_k = \sum \langle A(x), x^k\rangle (x^k, y) = (A_n(x), y).$$

Now we can define the degree of A by the aid of its finite rank approximation. Note that this the same method that Leray and Schauder defined their degree for compact perturbations of the identity. For the Leray and Schauder case, the finite rank approximation is obtained through the partition of unity and the uniform approximation of a compact map by a sequence of finite rank maps. Let $A: X \to X^*$ be a bounded demi-continuous map and $D \subset X$ an open bounded set. Denote $D_n = D \cap X_n$. We have the following simple fact.

Lemma 2.33. *Under the above setting, assume $0 \notin A(\partial D)$ when A is a map of class $(S)_+$, there exist $N_0 > 0$ such that $0 \notin A_n(\partial D_n)$ for $n \geq N_0$.*

Proof. Assuming the contrary, there exist a sequence $z_n \in \partial D_n$ such that $0 = A_n(z_n)$. Since X is reflexive and D is bounded, then the sequence (z_n) converges weakly in a sub-sequence to some z. Since $A(z_n) = 0$ on X_n, then we have

$$\langle A(z_n), z_n - z\rangle = -\langle A(z_n), z\rangle.$$

Since X is separable, choose the sequence $\{s_n\}$, $s_n \in X_n$ and $s_n \to z$. Since A is bounded, we can write

$$\langle A(z_n), z\rangle = \langle A(z_n), z - s_n\rangle \leq \|A(z_n)\|\, \|z - s_n\| \to 0.$$

Since A is a map of class $(S)_+$ then $z_n \to z \in \partial D$ and then by demi-continuity $A(z_n) \rightharpoonup A(z)$. Let $v \in X$ arbitrary and $(v_n), v_n \in X_n$ and $v_n \to v$, then

$$\langle A(z), v\rangle = \lim \langle A(z_n), v\rangle = \lim \langle A(z_n), v - v_n\rangle = 0,$$

a contradiction! □

Using the above lemma we can define a degree for the $(S)_+$ map $A: X \to X^*$.

Definition 2.34. *Assume that X is a separable reflexive Banach space with a uniformly convex norm, $D \subset X$ an open bounded set and $A : \bar{D} \to X^*$ a bounded demi-continuous map of class $(S)_+$. If $0 \notin A(\partial D)$ then we can define the following degree for the map A in D for $0 \in X^*$:*

$$\deg(A, D, 0) = \lim_{n \to \infty} \deg_B(A_n, D_n, 0). \tag{2.55}$$

where $\deg_B$ denotes the usual Brouwer degree.

Theorem 2.35. *Degree defined in (2.55) is well defined, that is the Brouwer degree in the right hand side of (2.55) is stable for sufficiently n.*

Proof. Note that A_n is a continuous map since A is demi-continuous and demi-continuity coincide with the continuity in finite dimensional spaces. According to the above lemma, $0 \notin A_n(\partial D)$ and in particular $0 \notin A_n(\partial D_n)$, therefore the Brouwer degree in the right hand side of (2.55) is well defined. It remains to show that the defined degree is stable for sufficiently large n. Define $B_{n+1} : X_{n+1} \to X_{n+1}$ as follows for $y \in D_{n+1}$

$$B_{n+1}(y) = A_n(y) + (x^{n+1}, y) x^{n+1}$$

Note that $B_{n+1}(y) = 0$ only if $\langle x^{n+1}, y \rangle = 0$ or equivalently $y \in X_n$. This implies that

$$\deg_B(B_{n+1}, D_{n+1}, 0) = \deg_B(A_n, D_n, 0). \tag{2.56}$$

Now consider the following convex homotopy

$$h_{n+1}(t) = (1 - t) A_{n+1} + t B_{n+1}, \quad t \in [0, 1].$$

We first show that h is an admissible homotopy. Obviously $0 \notin h_{n+1}(t)(\partial D_{n+1})$ for $t = 0$. For $t = 1$, $h(1) = B_{n+1}$ and $B_{n+1}(z) = 0$ for $z \in \partial D_{n+1}$ implies that $z \in \partial D_n$ and $A_n(z) = 0$ that is impossible for sufficiently large n. For $t \in (0, 1)$ assume that there exist a sequence (t_n), $t_n \in (0,\ 1)$ and a sequence (z_n), $z_n \in \partial D_n$ such that $h_n(t_n)(z_n) = 0$. According to the definition of B_n, the above equality implies $A(z_n) = 0$ on X_{n-1} and also

$$\langle A(z_n), x^n \rangle = \frac{t_n}{1 - t_n} (x^n, z_n). \tag{2.57}$$

Since (z_n) is bounded and X is reflexive, then $\{z_n\}$ converges weakly in a subsequence to some vector z. Now

$$\langle A(z_n), z_n - z\rangle = -\frac{t_n}{1-t_n}(x^n, z_n)^2 - \langle A(z_n), z\rangle. \tag{2.58}$$

Since $A(z_n) = 0$ on X_{n-1}, by the density property we obtain $\langle A(z_n), z\rangle \to 0$. Therefore

$$\text{limsup}\, \langle A(z_n), z_n - z\rangle \leq 0.$$

and then $z_n \to z \in \partial D$ since A is a $(S)_+$ map. A is demi-continuous and then $A(z_n) \rightharpoonup A(z)$. For arbitrary $v \in Y$ we have

$$\langle A(z), v\rangle = \lim \langle A(z_n), v\rangle = \lim \langle A(z_n), v - v_n\rangle = 0, \tag{2.59}$$

for a sequence (v_n), $v_n \in X_n$ and $v_n \to v$, that is a contradiction! By the homotopy invariance property of Brouwer degree and the relations $h(0) = A_{n+1}$, $h(1) = B_{n+1}$ we can write

$$\deg_B(A_{n+1}, D_{n+1}, 0) = \deg_B(B_{n+1}, D_{n+1}, 0).$$

According to (2.56) we conclude that for sufficiently large n the following relation holds:

$$\deg(A_{n+1}, D_{n+1}, 0) = \deg(A_n, D_n, 0). \tag{2.60}$$

Therefore the degree in the right hand side of (2.55) is stable. □

Proposition 2.36. *Degree defined in* (2.55) *is independent of the basis $\mathcal{X}$, that is if $\mathcal{H}' = \{h_1', h_2', ...\}$ is another orthonormal basis for H and $\mathcal{X} = \{x_1', x_2', ...\}$ is the image of $\mathcal{H}'$ under φ then the degrees are the same.*

Proof. First note that if $\{x_1', ..., x_n'\}$ is any basis for X_n and

$$A_n'(x) = \sum_{k=1}^{n} \langle A(x), x_k'\rangle x_k',$$

then $A_n(x) = A_n'(x)$ on X_n if $(x_i', x_j') = \delta_{i,j}$. Now assume $X_n = \text{span}\{x_1, ..., x_n\}$ and $X_m' = \text{span}\{x_1', ..., x_m'\}$ for n, m sufficiently large such that the Brouwer degree is stable in (2.55). Denote $Y = \text{span}\{x_1, ...x_n, x_1', ..., x_m')$. Define A_{nm} in Y in a similar way and also $D_Y = D \cap Y$, then since the degree is stable, we have

$$\deg_B(A_n, D_n, 0) = \deg_B(A_{nm}, D_{nm}, 0) = \deg_B(A_m, D_m, 0),$$

and this comp lets the proof. □

Now we show that degree defined in (2.55) satisfies all classical properties of a topological degree. For the reference map, we choose the duality map J. As proved in the first chapter, if X is a uniformly convex Banach space, then $J: X \to X^*$ is bicontinuous and a map of class $(S)_+$. Let $D \subset X$ is an open bounded domain and $0 \notin \partial D$, then $0 \notin \partial D_n$ for any n. Since J is a $(S)_+$ map, we can define

$$\deg(J, D, f) = \lim_{n \to \infty} \deg(J_n, D_n, 0). \tag{2.61}$$

If $0 \notin D$ then $0 \notin J_n(D_n)$ since

$$J_n(x) = \sum_{k=1}^{n} \langle J(x), x^k \rangle \, x^k,$$

and just take a basis with $x^1 = x/\|x\|$ and then $\langle J(x), x^1 \rangle = \|x\| \neq 0$. Therefore the equation $J_n(x) = 0$ does not have any solution in D and we conclude

$$\deg(J, D, 0) = 0.$$

If $0 \in D$ then the only solution on the equation $J_n(x) = 0$ for $x \in D_n$ is $x = 0$. The homotopy $h_n(t) = tJ_n + (1-t)\mathrm{Id}$ on X_n has the property that $0 \notin h_n(t)(\partial D)$ and therefore

$$\deg_B(J_n, D_n, 0) = \deg(\mathrm{Id}, D_n, 0) = 1.$$

The above argument proved the following proposition:

Proposition 2.37. *Let $D \subset X$ is an open bounded domain and $0 \notin \partial D$ then*

$$\deg(J, D, 0) = \begin{cases} 1 & \text{if } 0 \in D \\ 0 & \text{if } 0 \notin D \end{cases}$$

The invariance property of degree under admissible homotopy is one property that should be satisfied by all defined degree. The following definition defines an admissible homotopy for the maps of class $(S)_+$ followed by a proposition that establishes the invariance under admissible homotopy.

Definition 2.38. *The map* $\mathcal{A}\colon \bar{D} \times [0,1] \to X^*$ *where* $D \subset X$ *is an open bounded set is called an admissible homotopy of class* $(S)_+$ *if* $0 \notin \mathcal{A}(\partial D, [0,1)$, $\mathcal{A}(x,t)$ *is continuous with respect to* t *and demi-continuous with respect to* x *and furthermore for the convergent sequence* $(t_n), t_n \in [0,1]$, $u_n \rightharpoonup u$ *and*

$$\limsup \langle \mathcal{A}(u_n, t_n), u_n - u \rangle \leq 0 \tag{2.62}$$

then $u_n \to u$.

Proposition 2.39. *Degree defined in* (2.55) *is invariant under the admissible homotopy.*

Proof. Let $\mathcal{A}\colon \bar{D} \times [0,1] \to X^*$ be an admissible homotopy and consider the following map

$$\tilde{\mathcal{A}}\colon \bar{D} \times [0,1] \to X^* \oplus \mathbb{R},\ \tilde{\mathcal{A}}(u,t) = (\mathcal{A}(u,t), t). \tag{2.63}$$

It is easily seen that if $\mathcal{A}$ is demi-continuous of class $(S)_+$ then $\tilde{\mathcal{A}}$ is demi-continuous of class $(S)_+$. We define

$$\deg(\tilde{\mathcal{A}}, D \times [0,1], (0,t)) = \lim_{n \to \infty} \deg_B(\tilde{\mathcal{A}}_n, D_n \times [0,1], (0,t)). \tag{2.64}$$

We show if $(0,t) \notin \tilde{\mathcal{A}}(\partial D, t)$ for all $0 \leq t \leq 1$ then

$$\deg(\tilde{\mathcal{A}}, D \times [0,1], (0,t)) = \deg(\mathcal{A}_t, D, 0). \tag{2.65}$$

for any $t \in [0,1]$ where $\mathcal{A}_t = \mathcal{A}(.,t)\colon D \to X^*$. According to the homotopy invariance property of Brouwer degree we can write for $t \in [0,1]$

$$\deg_B(\tilde{\mathcal{A}}_n, D_n \times [0,1], (0,t)) = \deg(\mathcal{A}_{n,t}, D_n, 0). \tag{2.66}$$

Since according to the definition we have

$$\lim_{n \to \infty} \deg_B(\mathcal{A}_{n,t}, D_n, f_n) = \deg(\mathcal{A}_t, D, f), \tag{2.67}$$

then the claim is proved. □

Proposition 2.40. *If* $\deg(A, D, 0)$ *is non-zero, then the equation* $A(x) = 0$ *has a solution.*

Proof. According to the definition (2.55), the equation $A_n(u) = 0$ has a solution for $u \in D_n$, that is equivalent that $A(u) = 0$ on X_n. Since D is bounded and X is reflexive for any $u \in X$ there exist sequence $(u_n), u_n \in D_n$ such that $u_n \rightharpoonup u \in X$. We show $u \in D$ and $A(u) = 0$. First note that

$$\langle A(u_n), u_n - u\rangle = -\langle A(u_n), u\rangle.$$

Since X is separable we can choose a sequence $z_n \in X_n$ such that $z_n \to u$ and then

$$\begin{aligned}\langle A(u_n), u\rangle &= \langle A(u_n), u - z_n\rangle + \langle A(u_n), z_n\rangle = \\ &= \langle A(u_n), u - z_n\rangle \le \|A(u_n)\| \, \|u - z_n\| \to 0\end{aligned} \tag{2.68}$$

The last statement (approaching to zero) follows from the fact that A is bounded. Since A is of class $(S)_+$ it implies $u_n \to u \in \bar{D}$. On the other hand A is a demi-continuous then $A(u_n) \rightharpoonup A(u)$. Now for arbitrary $w \in X$ choose $w_n \to w$ for $w_n \in X_n$ and then

$$\langle A(u), w\rangle \leftarrow \langle A(u_n), w\rangle = \langle A(u_n), w - w_n\rangle \to 0 \tag{2.69}$$

Since w is arbitrary it implies that $A(u) = 0$. But $0 \notin A(\partial D)$ therefore $u \in D$. □

The last property of degree is domain decomposition and the proof is straight forward by the argument in finite dimensional case. Thus we have proved the following theorem.

Theorem 2.41. *The degree defined by* (2.55) *satisfies all properties of a classical degree theory.*

Now it is straightforward to define the degree of the map A with respect to arbitrary $f \in X^*$ as stated in the following definition.

Definition 2.42. *Let $D \subset X$ be an open bounded set, $A: \bar{D} \to X^*$ is a bounded demi-continuous map of class $(S)_+$ such that $f \notin A(\partial D)$, the degree of A in D with respect to f is defined as*

$$\deg(\mathcal{A}, D, f) = \deg(\mathcal{A} - f, D, 0), \tag{2.70}$$

where the finite rank approximation for f is defined by in terms of basis $\mathcal{X}$ as follows:

$$f_n = \sum_{k=1}^{n} \langle f, x^k\rangle x^k.$$

2.4.1 Some results about the solvability

Here we use the above defined degree to prove the solvability of some abstract equations.

Theorem 2.43. *Assume $A: X \to X^*$ is a bounded, demi-continuous operator of class $(S)_+$ and coercive, that is*

$$\lim \frac{\langle A(u), u\rangle}{\|u\|} \to \infty \, (\|u\| \to \infty) \tag{2.71}$$

then A is onto.

Proof. Since A is coercive we have an a priori estimate for the solutions of the equation $A(u) = f$ for any $f \in X^*$. Choose a sufficiently large ball B_R and the affine homotopy $A_t = tJ + (1-t)A$ and use the homotopy invariance property to reach:

$$\deg(A, B_R, f) = \deg(J, B_R, f) = 1, \tag{2.72}$$

and this completes the proof. □

The following proof is an easy degree theoretic proof for the Lax-Milgram lemma in general case.

Corollary 2.44. *Assume $\pi: H \times H \to \mathbb{R}$ is a continuous bi-linear form that is not necessarily symmetric and $\pi(u,u) \geq r\|u\|^2$, then for every $f \in H$, there exist a unique u_0 such that*

$$\pi(u_0, v) = (f, v), \quad \forall v \in H.$$

Proof. Define the map $A: H \to H$, $u \longmapsto \mathcal{A}(u)$ as

$$(A(u), v) = \pi(u, v), \quad \forall v \in H. \tag{2.73}$$

A is a linear continuous map. Assume $u_n \rightharpoonup u$ and

$$\operatorname{limsup} \langle A(u_n), u_n - u\rangle \leq 0.$$

We have then

$$\operatorname{limsup} \langle A(u_n), u_n - u\rangle = \operatorname{limsup} \langle A(u_n - u), u_n - u\rangle \geq r\|u_n - u\|^2. \tag{2.74}$$

that shows $u_n \to u$ and then A is a map of class $(S)_+$. By the above degree theoretic method we conclude A is onto. □

The following proposition is a possible generalization of Brouwer fixed point theorem for infinite dimensional spaces.

Proposition 2.45. *Assume $A: \bar{\mathbb{B}} \subset X \to \bar{\mathbb{B}}$ is a bounded continuous map defined on the unit ball of a separable uniformly convex Banach spaces X and $-JA$ is of class $(S)_+$ then there exist $u \in \bar{\mathbb{B}}$ such that $A(u) = u$.*

Proof. Define $T_t = J - t\,JA$ for $t \in [0,1]$. It is seen that T_t is a $(S)_+$ map since if $u_n \rightharpoonup u$ and

$$\limsup \langle T_t(u_n), u_n - u\rangle \leq 0,$$

then according to the monotonicity of J we obtain

$$t \limsup \langle -JA(u_n), u_n - u\rangle \leq \limsup \{\langle J(u_n), u_n - u\rangle + t\langle -JA(u_n), u_n - u\rangle\} \leq 0.$$

Since $-JA$ is of class $(S)_+$ then $u_n \to u$. It is also clear that $J \neq t\,JA$ on $\partial\mathbb{B}$ for $t \in [0,1)$ since otherwise for $t \in [0,1)$ and for some $z \in \partial\mathbb{B}$ it will be

$$J(z) = t\,J\mathcal{A}(z) \Rightarrow 1 = t\langle JA(z), z\rangle. \tag{2.75}$$

But $A(z) \in \bar{\mathbb{B}}$ and therefore $\|JA(z)\| \leq 1$. Therefore

$$\langle JA(z), z\rangle \leq 1 < \frac{1}{t}, \quad t \in [0,1).$$

If for $t = 1$ $J(z) = JA(z)$ then

$$\langle JA(z), z\rangle = 1 \Rightarrow A(z) = z,$$

since $\langle J(x), y\rangle < \|x\|\,\|y\|$ if $x \neq y$. In this case there exist $z \in \partial\mathbb{B}$ such that $A(z) = z$. Otherwise $A(z) \neq z$ for $z \in \partial\mathbb{B}$, then $J \neq tJA$ on $\partial\mathbb{B}$ for $t \in [0,1]$. Moreover the map T_t is an admissible homotopy and therefore the degree is well defined. Hence

$$\deg(T_t, \mathbb{B}, 0) = \deg(J, \mathbb{B}, 0) = 1, \tag{2.76}$$

that implies the existence of $u \in \mathbb{B}$ such that $J(u) = JA(u)$ that gives in turn $A(u) = u$. □

The following corollary is an immediate consequence of the above proposition.

Corollary 2.46. *Assume $A: \mathbb{B} \subset H \to H$ is a bounded continuous map defined on the unit ball $\mathbb{B}$ of the Hilbert space H and $-A$ is of class $(S)_+$ then there exist $u \in \bar{\mathbb{B}}$ such that $A(u) = u$.*

2.5 Degree for monotone maps

In this section, we define a degree for monotone and maximal monotone operators that enable us to prove some classical theorems by degree theoretic argument. Maximal monotone operators play an important role for the existence of nonlinear evolution equation $u'(t) + F(t, D^{\leqslant 2m}u) = 0$ where the term $u'(t)$ turns out to be a maximal monotone operator in a suitable Banach space [83]. For simplicity we assume that $D(T)$, the effective domain of the maximal monotone map $T: X \to 2^{X^*}$ is the whole of X. First we define the degree for single valued monotone maps.

2.5.1 Degree for single valued monotone maps

Let X be a separable locally uniformly convex Banach space, $D \subset X$ an open bounded set, $A: X \to X^*$ a demi-continuous monotone map such that $0 \notin \mathrm{cl}\, A(\partial D)$. Let $r = \mathrm{dist}(0, \mathrm{cl}\ A(\partial D))$. Obviously $r > 0$ since the minimizer sequence (y_n), $y_n \in \mathrm{cl}\, A(\partial D)$ is Cauchy and then converge to some point in $\mathrm{cl}\, A(\partial D)$. Choose

$$\delta_0 = \frac{r}{\max \|z\|}, \quad z \in \partial D, \tag{2.77}$$

then for $0 < \delta < \delta_0$ the map $A^\delta = A + \delta J$ has the property $0 \notin A^\delta(\partial D)$, in fact for every $z \in \partial D$ we have

$$\|A(z) + \delta J(z)\| \geq \|A(z)\| - \delta \|z\| \geq r - \delta \|z\| > 0.$$

In addition A^δ is a map of class $(S)_+$, since

$$\mathrm{limsup}\, \{\langle A(x_n), x_n - x\rangle + \delta \langle J(x_n), x_n - x\rangle\} \geq \delta\, \mathrm{limsup}\, \langle J(x_n), x_n - x\rangle,$$

and J is a $(S)_+$ map.

Definition 2.47. *Let X be a separable locally uniformly convex Banach space, $D \subset X$ an open bounded set, $A\colon X \to X^*$ a demi-continuous monotone map such that $0 \notin \operatorname{cl} A(\partial D)$. The degree for the map A in D for 0 is defined for $0 < \delta < \delta_0$ as*

$$\deg(A, D, 0) = \lim_{\delta \to 0} \deg\left(A^\delta, D, 0\right) = \lim_{\delta \to 0} \lim_{n \to \infty} \deg_B\left(A_n^\delta, D_n, 0\right). \tag{2.78}$$

Since A^δ is a $(S)_+$ map, the degree in the right hand side of (2.78) is well defined. However, the stability of the degree under the limit of δ needs justification. For δ_1, $\delta_2 < \delta_0$ the homotopy

$$h(t) = A_n + (t\delta_1 + (1-t)\delta_2) J_n,$$

is admissible and $h(t)(z) \neq 0$ for $z \in \partial D$ and therefore

$$\deg_B\left(A_n^{\delta_1}, D_n, 0\right) = \deg_B\left(A_n^{\delta_2}, D_n, 0\right).$$

Since the degree of finite rank is stable with respect to n, it is stable with respect to δ according to the above homotopy. We need to justify that the definition of degree for the monotone map A satisfies all classical properties of a topological degree.

Proposition 2.48. *If $\deg(A, D, 0) \neq 0$ then there exist a solution for the equation $A(x) = 0$.*

Proof. The equation $A^\delta(x) = 0$ has a solution $x(\delta) \in D$ according to the properties of degree for $(S)_+$ mappings. Let δ_n be a decreasing sequence and $\delta_n \to 0$, then according to the proposition (2.12) $x(\delta_n)$ converges to some point $x_0 \in \bar{D}$. Since A is demi-continuous then $A(x_n) \rightharpoonup A(x_0)$, Let $y \in X$ arbitrary then

$$\langle A(x_0), y\rangle \leftarrow \langle A(x_n), y\rangle = -\delta_n \langle J(x_n), y\rangle \to 0,$$

and then $A(x_0) = 0$. But $x_0 \notin \partial D$ then $x_0 \in D$. □

Definition 2.49. *Let $\tilde{A}\colon [0,1] \times \bar{D} \to X^*$ be a map such that for every $t \in [0,1]$, the map $\tilde{A}(t)\colon X \to X^*$ is a monotone map, $\tilde{A}(t,x)$ is continuous with respect to t and demi-continuous with respect to x and $0 \notin \tilde{A}([0,1])(\partial D)$. $\tilde{A}$ is called admissible homotopy for monotone maps.*

Proposition 2.50. *The degree defined in* (2.78) *is invariant under admissible homotopy for monotone maps.*

Proof. We only need to show that $\tilde{A}(t)+\delta J$ is a admissible homotopy of class $(S)_+$. If fact for $x_n \rightharpoonup x$ and $t_n \to t$, we can write

$$\begin{aligned}\left\langle \tilde{A}(t_n,x_n),x_n-x\right\rangle+\delta\langle J(x_n),x_n-x\rangle&=\left\langle \tilde{A}(t_n,x_n)-\tilde{A}(t_n,x),x_n-x\right\rangle+\\&\left\langle \tilde{A}(t_n,x),x_n-x\right\rangle+\delta\langle J(x_n),x_n-x\rangle\geq\\&\geq\left\langle \tilde{A}(t_n,x),x_n-x\right\rangle+\delta\langle J(x_n),x_n-x\rangle.\end{aligned}$$

Now since $\tilde{A}(t,x)$ is continuous with respect to t, then for $t_n \to t$ we obtain $\left\langle \tilde{A}(t_n,x), x_n-x\right\rangle \to 0$ and therefore

$$\operatorname{limsup}\left\langle \tilde{A}(t_n,x_n),x_n-x\right\rangle+\delta\langle J(x_n),x_n-x\rangle=\delta\operatorname{limsp}\langle J(x_n),x_n-x\rangle.$$

This implies that $\tilde{A}(t)+\delta J$ is an admissible homotopy of class $(S)_+$. Therefore

$$\deg\left(\tilde{A}(t),D,0\right)=\lim_{\delta\to 0}\deg\left(\tilde{A}^{\delta}(t),D,0\right),$$

and the degree in the right hand side is independent of t. □

The decomposition of domain property is straightforward to verify and it comes from the same property of $(S)_+$ mappings.

Proposition 2.51. *If $D=D_1\cup D_2$ and $D_1\cap D_2=\emptyset$, and if $A\colon \bar{D}\to X^*$ is demicontinuous monotone map such that $0\notin \operatorname{cl} A(\partial D)$ then*

$$\deg(A,D,0)=\deg(A,D_1,0)+\deg(A,D_2,0). \tag{2.79}$$

Proof. Since open sets D_1,D_2 are disjoint then $\partial D_1\subset\partial D$ and $\partial D_2\subset\partial D$, therefore $\partial D\supset\partial D_1\cup\partial D_2$. Since D is the union of D_1,D_2 this implies that $\partial D=\partial D_1\cup\partial D_2$. In a similar manner we conclude that $\operatorname{cl}(A(\partial D))=\operatorname{cl}(A(\partial D_1))\cup\operatorname{cl}(A(\partial D_2))$. Therefore if $0\notin \operatorname{cl} A(\partial D)$ then $0\notin\operatorname{cl} A(\partial D_1)$ and $0\notin \operatorname{cl} A(\partial D_2)$ and then $0\notin A^{\delta}(\partial D_1)$ and $0\notin A^{\delta}(\partial D_2)$. We can write then

$$\deg(A^{\delta},D,0)=\deg(A^{\delta},D_1,0)+\deg(A^{\delta},D_2,0), \tag{2.80}$$

and all degree in the above relationship are stable under the limit for δ, and this gives the relationship (2.79). □

Remark 2.52. To simplify the related calculations, we can replace J_n with Id map according to the following considerations. Since the set $\mathcal{X}=\{x^1, x^2..., \}$ is arbitrary for the finite rank approximation, we can choose $\mathcal{X}$ such that $\langle J(x^i),\, x^j\rangle = \delta_{i,j}$. Choose x^1 arbitrary with $\|x^1\| = 1$. Let X_1 be the topological complement of x^1, that is $X=\{x^1\}\oplus X_1$. Now choose $f_1\in X^*$ such that $\langle f_1, x^1\rangle=1$ with $\|f_1\|=1$ and $\langle f_1, y\rangle=0$ for $y\in X_1$. Since X is uniformly convex, it is clear that $f_1=J(x^1)$. Let $x^2\in X_1$, $\|x^2\|=1$ and $X=\{x^1\}\oplus\{x^2\}\oplus X_2'$. Define $X_2=\{x^1\}\oplus X_2'$ and $f_2\in X^*$ such that $\langle f_2, x^2\rangle=1$, $\|f_2\|=1$ and $\langle f_2, y\rangle=0$ for $y\in X_2$. Clearly $f_2=J(x^2)$. If we continue this process, we obtain a set such that $\langle J(x^i), x^j\rangle=\delta_{i,j}$.

Definition 2.53. *Assume that X is a separable uniformly convex Banach space, $D\subset X$ an open bounded set and for $f\in X^*$, $f\notin \mathrm{cl}\, A(\partial D)$. Degree of A in D with respect to f is defined*

$$\deg(A, D, f)=\deg(A-f, D, 0). \tag{2.81}$$

Remark 2.54. Since for every $f\in X^*$, the map $A_f = A - f$ is a demi-continuous monotone map if A is demi-continuous monotone and if $f\notin \mathrm{cl}\ A(\partial D)$ then $0\notin \mathrm{cl}\, A_f(\partial D)$, the definition is justified. It is straightforward to check that the degree defined for arbitrary $f\in X^*$ satisfies all classical properties of a topological degree. By the aid of this definition, the another type of homotopy invariance property can be established. If $f(t), 0\le t\le 1$ is a path in X^* and $\tilde{A}(t,x)$ is a admissible homotopy such that $f(t)\notin \mathrm{cl}\ \tilde{A}([0,\,1],\,\partial D)$ for any $t\in[0,\,1]$, then the following degree is independent of t. The justification follows from the fact that $\tilde{A}_f(t,x)=\tilde{A}(t,x)-f(t)$ is an admissible homotopy.

2.5.2 Degree for maximal monotone maps

Now we can define a degree for maximal monotone maps. First a simple proposition.

Proposition 2.55. *Assume that X is a separable locally uniformly convex Banach space, $D\subset X$ an open bounded set and $A\colon X\to 2^{X^*}$ a maximal monotone map. The map $A^\delta = A+\delta J$ for $\delta>0$ is a map of class $(S)_+$ in the following sense: if $x_n \rightharpoonup x$ and if there exist $w_n\in A^\delta(x_n)$ such that*

$$\limsup \langle w_n, x_n - x\rangle \leq 0,$$

then $x_n\to x$. If $0\notin \mathrm{cl}\ A(\partial D)$ then there exist $\delta_0>0$ such that $0\notin A^\delta(\partial D)$ for $0<\delta<\delta_0$.

Clearly for the given sequence (w_n) and the monotonicity property we have the inequality

$$\limsup \langle w_n, x_n - x\rangle \geq \delta \limsup \langle J(x_n), x_n - x\rangle,$$

that in turn implies A^δ is a $(S)_+$ map. Since $\mathrm{cl}\, A(\partial D) = \mathrm{cl}\{w; w\in A(z), z\in \partial D\}$, then there exist $r>0$ such that $\mathrm{dist}(0, \mathrm{cl}\, A(\partial D)) = r$. For any $z\in\partial D$ and $w\in A(z)$ we have

$$\|w+\delta J(z)\| \geq \|w\| - \delta\ \|z\| \geq r - \delta\ \|z\|,$$

and it is enough to choose δ_0 as defined in (2.77).

Proposition 2.56. *Let A_n^δ denotes the finite rank approximation of the map A^δ. There exist $N>0$ such that for $n\geq N$, $0\notin A_n^\delta(\partial D_n)$.*

Proof. Otherwise, assume $z_n\in\partial D_n$ and $w^n\in A_n(z_n)$ such that $w^n+\delta J_n(z_n)=0$. Here w^n is the finite rank approximation of some $w_n\in A(z_n)$, that is

$$w^n = \sum_{k=1}^{n} \langle w_n, x^k\rangle\, x^k.$$

Since ∂D is bounded, $z_n \rightharpoonup z$ in a sub-sequence. Let $(v_n), v_n\in X_n$ and $v_n\to z$ then

$$0 = (w^n, z_n - v_n) + \delta\big(J_n(z_n), z_n - v_n\big) = \langle w_n, z_n - v_n\rangle + \delta\langle J(z_n), z_n - v_n\rangle,$$

This implies that

$$0 = \limsup \langle w_n, z_n - z\rangle + \delta\langle J(z_n), z_n - z\rangle \geq \delta \limsup \langle J(z_n), z_n - z\rangle,$$

and then $z_n \to z \in \partial D$. This implies that $A(z) + \delta J(z) = 0$ that is impossible if $0 < \delta < \delta_0$. □

Let $r_n = \operatorname{dist}\big(0, A_n^\delta(\partial D_n)\big) > 0$ and then define $0 < \varepsilon_n < r_n$ such that $\varepsilon_n \to 0$. According to proposition (2.31), it is possible to define a ε_n-continuous single valued map $A_{n,\varepsilon_n}^\delta$ of A_n^δ, that is $A_{n,\varepsilon_n}^\delta = A_{n,\varepsilon_n} + \delta J_n$, where A_{n,ε_n} is the ε_n-continuous single valued map of the finite rank approximation A_n. Since A_n has n-components $A_n = ({A^1}_n, \ldots, A_n^n)$, then the ε-continuous selection $A_{n,\varepsilon}$ has components $A_{n,\varepsilon} = (A_{n,\varepsilon}^1, \ldots, A_{n,\varepsilon}^n)$.

Definition 2.57. *Assume that X is a separable locally uniformly convex Banach space, $D \subset X$ an open bounded set and $A: X \to 2^{X^*}$ a maximal monotone map such that $0 \notin \operatorname{cl} A(\partial D)$. The degree of A in D with respect to $0 \in X^*$ is defined as*

$$\deg(A, D, 0) = \lim_{\delta \to 0} \lim_{n \to \infty} \deg_B\big(A_{n,\varepsilon_n}^\delta, D_n, 0\big). \tag{2.82}$$

Since A_{n,ε_n} may be not monotone, the stability of the degree in the right hand side should be justified for both n, δ.

Proposition 2.58. *The degree define in the relation* (2.82) *is stable with respect to n and δ.*

Proof. The stability of the degree with respect to n is established through the homotopy invariance with respect to n. Consider $A_{n,\varepsilon_n}^\delta$ and $A_{m,\varepsilon_m}^\delta$ for sufficiently large n, m and $m = n+1$. Define the map $B_m^\delta(x) = A_{n,\varepsilon_m}^\delta(x) + (x^m, x)\, x^m$ for $x \in X_m$. Obviously

$$\deg_B\big(A_{n,\varepsilon_n}^\delta, D_n, 0\big) = \deg_B\big(B_m^\delta, D_m, 0\big).$$

Now for sufficiently large n we show that

$$\deg_B\big(A_{m,\varepsilon_m}^\delta, D_m, 0\big) = \deg_B\big(B_m^\delta, D_m, 0\big).$$

We show that the following convex homotopy is admissible

$$h_m(t) = (1-t)\, A_{m,\varepsilon_m}^\delta + t\, B_m.$$

Clearly $0 \notin h(t)(\partial D_m)$ for $t = 0, 1$. For $t \in (0,1)$ assume that there exist $t_m \in (0,1)$ and a sequence (z_m), $z_m \in \partial D_m$ such that $h_m(t_m)(z_m) = 0$. According to the construction of $h_m(t)$ we obtain

$$0 = h_m(t_m)(z_m) = A^{\delta}_{n,\varepsilon_m}(z_m) + (1 - t_m)\, A^{\delta,m}_{m,\varepsilon_m}(z_m) + t_m\,(x^m, z_m)x^m.$$

The above relation implies $A^{\delta}_{n,\varepsilon_m}(z_m) = 0$ and

$$A^{\delta,m}_{m,\varepsilon_m}(z_m) = -\frac{t_m}{1 - t_m}\,(x^m, z_m)x^m.$$

Since ∂D is bounded then $z_m \rightharpoonup z$ in a sub-sequence. Choose the sequence (v_m), $v_m \in X_m$ and $v_m \to z$ and obtain

$$\big(A^{\delta}_{m,\varepsilon_m}(z_m), z_m - v_m\big) = -\frac{t_m}{1 - t_m}\,|(x^m, z_m)|^2 - \frac{t_m}{1 - t_m}\,(x^m, z_m)(x^m, v_m).$$

On the other hand when $m \to \infty$ $(x^m, v_m) \to 0$ since $v_m \to v$. Also there exist $w^m_m \in A_m(z_m)$ such that $\|w^m_m - A_{m,\varepsilon_m}(z_m)\| < \varepsilon_m$ and then we can write for some $w_m \in A(z_m)$

$$\limsup\,\langle w_m + \delta J(z_m), z_m - v\rangle = \limsup\,\big(A^{\delta}_{m,\varepsilon_m}(z_m), z_m - v_m\big).$$

Therefore we obtain

$$\limsup\,\langle w_m + \delta J(z_m), z_m - v\rangle \leq 0,$$

and then $z_m \to z$ that is impossible since $0 \notin A^{\delta}(\partial D)$. This justifies the stability with respect to n. The stability of the degree definition with respect to δ is easily follows from the fact that for any $0 < \delta < \delta_0$ we have $0 \notin A^{\delta}_{n,\varepsilon_n}(\partial D_n)$. □

Definition 2.59. *Let* $\tilde{A}\colon [0,1] \times X \to 2^{X^*}$ *be a map such that for any* $t \in [0,1]$, *the map* $\tilde{A}(t)\colon X \to 2^{X^*}$ *is a maximal monotone map. In addition assume that* $\tilde{A}(t,x)$ *is continuous with respect to* t *and* $0 \notin \mathrm{cl}\,\tilde{A}([0,1])(\partial D)$. $\tilde{A}$ *is called admissible homotopy for maximal monotone maps.*

Proposition 2.60. *The degree defined in* (2.82) *satisfies all classical properties of topological degree.*

Proof. Assume that $\deg(A, D, 0) \neq 0$, then there exist $z_n \in D$ such that $A_{n,\varepsilon_n}^{\delta_n}(z_n) = 0$ for $\delta_n \to 0$ and $\varepsilon_n \to 0$. This implies that there exist $w_n^n \in T_n(z_n)$ such that $\|w_n^n + \delta_n\, J_n(z_n)\| < \varepsilon_n$. An argument completely similar to the proposition (2.12) implies that $z_n \to z \in \bar{D}$. Since A is norm to weak* upper semi-continuous, then for every $w_n \in A(z_n)$, w_n weakly converges in a sub-sequence to some $w \in A(z)$. Let $y \in X$ arbitrary, $(v_n), v_n \in X_n$ and $v_n \to y$, then

$$\langle w, y\rangle \leftarrow \langle w_n, v_n\rangle = -\delta_n \langle J(z_n), v_n\rangle - \varepsilon_n (A_{n,\varepsilon_n}(z_n) - w_n^n, v_n\rangle \to 0,$$

that is $w = 0$. However, it is impossible that $z \in \partial D$ and therefore $z \in D$. This implies that $0 \in T(z)$. According to the definition of the admissible homotopy for maximal monotone map, it implies that $\deg_B\left(\tilde{A}_{n,\varepsilon_n}^{\delta}(t), D_n, 0\right)$ is independent of t since for sufficiently large n and $t \in [0, 1]$, $0 \notin \tilde{A}_{n,\varepsilon_n}^{\delta}(t)(\partial D)$. The stability of the degree with respect to n, δ implies that the degree $\deg\left(\tilde{A}(t), D, 0\right)$ is independent of t. The decomposition of the domain is also justified similar to one we have done for the degree of monotone maps. □

Remark 2.61. Similar to the degree for monotone maps, we can define the degree for maximal monotone map as

$$\deg(A, D, 0) = \lim_{n\to\infty} \deg_B\left(A_{n,\varepsilon_n}^{\delta}, D_n, 0\right), \quad 0 < \delta < \delta_0 \tag{2.83}$$

Also for the calculation the map A_n^{δ} can be replaced with $A_n + \delta\,\mathrm{Id}$ instead of $A_n + \delta J_n$. For $f \in X^*$ arbitrary such that $f \notin \mathrm{cl}\, A^{\varphi}(\partial D)$, we can define the degree of A in D with respect to f as

$$\deg(A, D, f) = \deg(A - f, D, 0). \tag{2.84}$$

Remark 2.62. If φ is a single valued demi-continuous map of class $(S)_+$, we can define the degree $A^{\varphi} = A + \varphi$ as

$$\deg(A^{\varphi}, D, 0) = \lim_{n\to\infty} \deg_B\left(A_{n,\varepsilon_n}^{\varphi}, D_n, 0\right), \tag{2.85}$$

where $A^{\varphi}_{n,\varepsilon_n} = A_{n,\varepsilon_n} + \varphi_n$, A_{n,ε_n} is the continuous single valued ε-approximation of A_n and φ_n is the finite rank approximation of φ provided $0 \notin \mathrm{cl}\, A(\partial D)$. A simple calculation similar to what have done for the maximal monotone maps, shows that the above degree is well defined and satisfies all classical properties of topological degree.

Remark 2.63. If φ is a single valued pseudo-monotone maps, the degree of $A^{\varphi} = A + \varphi$ can be defined as follows

$$\deg(A^{\varphi}, D, 0) = \lim_{n\to\infty} \left(A^{\varphi,\delta}_{n,\varepsilon_n}, D_n, 0\right), \quad 0 < \delta < \delta_0,$$

where here $A^{\varphi,\delta} = A + \varphi + \delta J$. The proof is similar to one carried out in the above.

2.5.3 Degree theoretic proofs for some theorems

In this subsection, we present new proofs of three theorems in convex analysis based by the aid of constructed degree in this chapter. The goal is to show the power of degree method in dealing with non-linear problems in mathematical analysis. The theorems presented below are chosen randomly from convex analysis.

The first theorem is due to F. Browder [20] for the surjectivity of locally bounded monotone maps.

Theorem 2.64. *Assume $A: X \to X^*$ is a demi-continuous monotone map such that A^{-1} is locally bounded, that is for every $f \in X^*$ there exist a bounded $V_f \ni f$ such that $A^{-1}(V_f)$ is bounded. Then A is onto.*

Proof. For any $f \in X^*$, we show that there exist sufficiently large $r = r(f)$ such that:

$$\deg(A, B_r, f) \neq 0.$$

Choose $r > 0$ such that for a neighborhood $V_f \ni f$ we have

$$S_r \cap A^{-1}(V_f) = \emptyset,$$

that is $f \notin \mathrm{cl}\, A(S_r)$. Since

$$\deg(A, B_r, f) = \lim_{\delta} \deg(A + \delta J, B_r, f),$$

it is enough to show that for sufficiently large r and sufficiently small $\delta > 0$ we have

$$\deg(A + \delta J, B_r, f) \neq 0. \tag{2.86}$$

We show $\deg(A + \delta J, B_r, 0) \neq 0$. In fact if $(A + \delta J)(z) = 0$ for $z \in \partial B_r$ then

$$\langle A(z) - A(0), z\rangle + \delta \|z\|^2 + \langle A(0), z\rangle = 0.$$

Since A is monotone, then $\delta \|z\|^2 + \langle A(0), z\rangle \leq 0$ that implies $\delta \|z\| \leq \|A(0)\|$, that is impossible for sufficiently large r. Since $A + \delta J$ is a map of class $(S)_+$, define the homotopy $h(t) = tA + \delta J$. It is easy to see that $0 \notin h(t)(\partial B_r)$ and then

$$\deg(A + \delta J, B_r, 0) = \deg(h(t), B_r, 0) = \delta \deg(J, B_r, 0) \neq 0,$$

since J is the reference map. The proof of (2.86) is completely similar to one presented above. We conclude then

$$\deg(A, B_r, f) \neq 0,$$

and this completes the proof. □

Now we give a degree theoretic proof for a classic proposition given by D. DeFigueirdo [30]:

Proposition 2.65. *X a separable uniformly Banach space, $T: X \to 2^{X^*}$ is a maximal monotone map such that $0 \notin (T + \lambda J)(S_r)$ for ever $\lambda > 0$ and some $r > 0$. Then there exist $u \in \bar{B}_r$ such that $0 \in T(u)$.*

Proof. Assume that $0 \notin T(\bar{B}_r)$. We show first that $0 \notin \operatorname{cl} T(S_r)$. Otherwise there exist $u_n \in S_r$ and $w_n \in T(u_n)$ such that $w_n \to 0$. The sequence (u_n) converges weakly in a sub-sequence (that we show still by u_n) to some $u \in \bar{B}_r$. Claim: $[u, 0] \in \operatorname{graph}(T)$. For any $[x, h] \in \operatorname{graph}(T)$ we have the inequality

$$\langle h, x - u\rangle = \lim \langle h - w_n, x - u_n\rangle \geq 0.$$

Since T is maximal monotone, then the above inequality justifies $[x, 0] \in \operatorname{graph}(T)$ or equivalently $0 \in T(u)$. This contradicts the assumption that $0 \notin T(\bar{B}_r)$. Next we show $0 \notin \operatorname{cl}((1-t)T + tJ)(S_r))$ for $t \in (0, 1]$. Otherwise there exist $t_n \in (0, 1)$, $u_n \in S_r$ and $w_n \in T(u_n)$ such that

$$(1 - t_n)w_n + t_n J(u_n) \to 0.$$

Again for $u_n \rightharpoonup u$ and $t_n \to t$ we obtain by the monotonicity property of T the following

$$\limsup \langle J(u_n), u_n - u \rangle \le 0,$$

that implies $u_n \to u \in S_r$. Claim: $[u, \frac{-t}{1-t} J(u)] \in \mathrm{graph}(T)$. For any $[x, h] \in \mathrm{graph}(T)$ we obtain by the fact $\frac{-t}{1-t} J(u_n) \in T(u_n)$ the following

$$\langle h + \frac{t}{1-t} J(u), x - u \rangle = \lim \langle h + \frac{t}{1-t} J(u_n), x - u_n \rangle \ge 0,$$

that proves the claim. Now we can use degree theoretic argument to obtain

$$\deg(T, B_r, 0) = \deg((1-t)T + t\, J, B_r, 0) = \deg(J, B_r, 0) = 1.$$

The above calculation guarantees that there exist $u \in B_r$ such that $0 \in T(u)$ and this contradicts the assumption $0 \notin T(\bar{B}_r)$. This shows that the assumption $0 \notin T(\bar{B}_r)$ is wrong and then $0 \in T(\bar{B}_r)$. □

The last theorem is a proposition due to DeFigueirdo [30].

Proposition 2.66. *Let X be a separable uniformly convex Banach space and f: $X \to X^*$ a pseudo-monotone map, then* $\mathrm{Rang}(\partial N_r + f) = X^*$ *where $r > 0$ and the map N_r is defined on $\bar{B}_r$ as follows*

$$N_r(x) = \begin{cases} 0 & \text{if } x \in B_r \\ 1 & \text{if } x \in S_r \end{cases}$$

Proof. First it is straightforward to verify that

$$\partial N_r(x) = \begin{cases} 0 & x \in B_r \\ \{\lambda J(x), \lambda \ge 0\} & x \in S_R \end{cases} \tag{2.87}$$

Claim: for every $f_0 \in X^*$, we have the following inequality:

$$\deg(\partial N_r + f - f_0, B_r, 0) \ne 0. \tag{2.88}$$

First we show if $0 \notin (\partial N_r + f - f_0)(\bar{B}_r)$ then

$$0 \notin \mathrm{cl}(\partial N_r + f - f_0)(S_r). \tag{2.89}$$

Otherwise, there exist $u_n \in S_r$ and $w_n \in \partial N_r(u_n)$ such that $w_n + f(u_n) - f_0 \to 0$. But $u_n \rightharpoonup u \in \bar{B}_r$ in a sub-sequence. We prove that $[u,\, f_0 - f(u)] \in \mathrm{graph}(\partial N_r)$. Let $f_0 = w_n + f(u_n) + \varepsilon(n)$ where $\varepsilon(n) \in X^*$ and $\varepsilon(n) \to 0$. For any arbitrary $[x,$ $h] \in \mathrm{graph}(\partial N_r)$ we have the inequality

$\langle h - f_0 + f(u), x - u\rangle = \lim \langle h + f\,(u) - w_n - f(u_n), x - u_n\rangle \geq \lim \langle f(u) - f(u_n),$ $x - u\rangle$.

But

$$0 = \lim \langle w_n + f(u_n) - f_0, u_n - u\rangle \geq \mathrm{limsup}\, \langle f(u_n), u_n - u\rangle \tag{2.90}$$

Since f is pseudo-monotone we obtain $f(u_n) \rightharpoonup f(u)$ and therefore

$$\langle h - f_0 + f\,(u), x - u\rangle \geq 0.$$

This implies that $[u,\, f_0 - f(u)] \in \mathrm{graph}(\partial N_r)$and then that implies $0 \in (\partial N_r + f - f_0)(\bar{B}_r)$ which is impossible by the assumption. Now consider the affine homotopy

$$h(t) = (1 - t)(\partial N_r + f - f_0) + tJ. \tag{2.91}$$

for $t \in (0, 1]$ and we show that

$$0 \notin \mathrm{cl}((1 - t)(\partial N_r + f - f_0) + tJ)(S_r).$$

Otherwise, there exist $u_n \in S_r$, $w_n \in \partial N_r(u_n)$ and $t_n \to t$ such that

$$(1 - t_n)(w_n + f(u_n) - w) + t_n J u_n \to 0.$$

But $u_n \rightharpoonup u \in \bar{B}_r$ in a sub-sequence and we show

$$[u, f_0 - f(u) - \frac{t}{1-t} J(u)] \in \mathrm{graph}(\partial N_r).$$

For any $[x, h] \in \mathrm{graph}(\partial N_r)$ we have

$$\langle h - f_0 + f(u) + \frac{t}{1-t} J\,(u), x - u\rangle =$$

$$\lim \langle h + f(u) + \frac{t}{1-t} J(u) - w_n - f(u_n) - \frac{t_n}{1-t_n} J(u_n), x - u\rangle \geq$$

$$\geq \mathrm{limsup}\, \langle f(u) - f(u_n), x - u\rangle + \mathrm{liminf}\, \langle \frac{t}{1-t} J(u) - \frac{t_n}{1-t_n} J(u_n), x - u\rangle$$

In a similar way it is shown that $u_n \to u \in S_r$ and $f(u_n) \rightharpoonup f(u)$. Therefore we obtain again

$$0 \in (\partial N_r + f + \frac{t}{1-t} J - f_0)(\bar{B}_r).$$

But $(\partial N_r + f + \frac{t}{1-t} J - f_0)(\bar{B}_r) = (\partial N_r + f - f_0)(\bar{B}_r)$ and then $0 \in (\partial N_R + f - f_0)(\bar{B}_R)$ that is impossible. Use degree argument to obtain

$$\deg(\partial N_R + f - f_0, B_r, 0) = \deg(h(t), B_R, 0) = \deg(J, B_r, 0) = 1.$$

Therefore there exist $u \in \bar{B}_r$ such that $w \in \partial N_r(u) + f(u)$. □

2.6 A generalized degree for $(S)_+$ maps

In this section, we present a generalized degree for a class of mappings that are not demi-continuous on the whole space or even on a dense subspace. This new degree theory will be applied to the Onsager problem in Chapter 3. All the mapping have been considered so far were continuous or demi-continuous in the whole space. In the case that the mappings are demi-continuous on a dense sub-space of the focal space can be death with simply through the homotopy invariance. However, in some circumstances, the map $A: X \to X^*$ is not demi-continuous on any open subset of X. The following Onsager operator [65] defined on $L^2(S^1)$ is one of those operators

$$G(u)(\theta) = \left(\int_{S^1} e^{-u(\theta)} d\theta \right)^{-1} \int_{S^1} K(\theta, \theta') \, e^{-u(\theta')} d\theta', \tag{2.92}$$

Here the kernel K is continuous, symmetric, and satisfies

$$\int_{S^1} K(\theta, \theta') \, d\theta' = 0.$$

In fact, we can take the sequence $\{u_n\}$ as

$$u_n(\theta) = \begin{cases} \log(1/n) & 0 < \theta < \frac{1}{n} \\ 0 \end{cases}. \tag{2.93}$$

It is clear that $u_n \longrightarrow 0$ in $L^2(S^1)$ but $G(u_n) \longrightarrow K(\theta, 0)$ and therefore does not converge weakly to $G(0) = 0$.

In all of this section, D is considered as open and bounded set. We start with the finite dimensional case. Let $D\subset\mathbb{R}^n$ be an open bounded set and $A: D\to\mathbb{R}^n$ a map not necessarily continuous. Assume that there exist a subspace $Y\subset\mathbb{R}^n$ such that A is C^1 on $D_Y = D\cap Y$ and $0\notin A(\partial D_Y)$. Let $U\subset\mathbb{R}^n$ be an open neighborhood of 0, $U_Y = U\cap Y$ and ω be a compactly supported form in U_Y such that $\int_Y \omega = 1$. We define the reduced degree of A as follows:

$$d(A, D_Y, 0) = \int_{D_Y} A^*(\omega). \tag{2.94}$$

To complete the definition, we need to define the admissible homotopy for the reduced degree. The class of admissible homotopy is the mappings h: $[0,1]\times\bar{D}\to\mathbb{R}^n$ such that for any $t\in[0,1]$ we have

$$\{z; h(t)(z) = 0\}\subset D_Y - h(t)(\partial D_Y)$$

Degree (2.94) is different with the classical degree of A_Y: $D_Y\to Y$ with respect to the fact that the definition (2.94) is an estimate for the solution of $A(x) = 0\in\mathbb{R}^n$ rather than of $A(x) = 0\in Y$.

2.6.1 Generalized degree in Hilbert space

Now we turn to the infinite dimensional case. We start with the Hilbert space case where the calculations are simpler compare to the Banach space. In fact, our discussion in this section is restricted to defining a reduced degree for the Schauder map $\Gamma = I - G$ in a Hilbert space which is exactly what we need in the next chapter for the Onsager problem.

Let H be a separable Hilbert space with an orthonormal basis $\mathcal{U} = \{u^1, u^2, \cdots\}$ and D be an open bounded subset of H. $H_n\subset H$ is the finite dimensional subspace spanned by $\{u^1, ..., u^n\}$. Assume that there exist the closed convex set $\Omega\subset H$ such that:

- $\{z; \Gamma(z) = 0\}\subset\alpha\Omega, \quad \alpha<1,$
- G is continuous and compact on $H\cap\Omega$,
- $H_n\cap\Omega$ has non-empty interior in H_n.

Denote $D_n = D \cap H_n$ and define the map $G_n: H_n \cap \Omega \to H_n$ as:

$$G_n(x) = \sum_{k=1}^{n} (G(x), u^k)\, u^k.$$

Proposition 2.67. *Assume D is an open bounded subset of H such that $z \neq G(z)$ for $z \in \partial D$, then there exist $N_0 > 0$ such that $z \neq G_n(z)$ for $z \in \partial D_n$, $n \geq N_0$.*

Proof. Otherwise, there exist $z_n \in \partial D_n$ such that $z_n = G_n(z_n)$. Since Ω is bounded and $\{z_n\} \subset \Omega$, then there is a sub-sequence which we do not relabel $z_n \rightharpoonup z$. Since G is compact on Ω, $y_n = G(z_n)$ converges in a sub-sequence to some point say $y \in H$. Also the sequence $\{y_n^n\}$ defined as $y_n^n = G_n(z_n) = \mathrm{Pr}_{H_n} G(z_n)$ converges to y, that is z_n converges to y in a sub-sequence and therefore $y = z$. Since ∂D is closed then $z \in \partial D$ and since $\{z_n\} \subset \Omega$ then $G(z_n) \to G(z) = z$, contradiction! □

Proposition 2.68. *Assume D is an open bounded subset of H such that $z \neq G(z)$ for $z \in \partial D$, then there exist $N_0 > 0$ such that for $n \geq N_0$ we have*

$$\deg(\Gamma_n, D_n, 0) = \deg(\Gamma_{n+1}, D_{n+1}, 0).$$

Proof. Since G is continuous on H_n and $z \neq G_n(z)$ for $z \in \partial D_n$ and $n \geq N_0$, the degree can be defined in the usual sense. Define the homotopy $h: [0,1] \times D_{n+1} \to H_{n+1}$ as

$$h(t)(x) = (1-t)\,\Gamma_{n+1}(x) + t\,A(x),$$

where $A: D_{n+1} \to H_{n+1}$ is defined as

$$A_{n+1}(x) = \Gamma_n(x) + (x, u^{n+1}) u^{n+1}. \tag{2.95}$$

Obviously $0 \notin h(t)(z)$ for $z \in \partial D_{n+1}$, $t = 1, 0$ and also we have

$$\deg(A_{n+1}, D_{n+1}, 0) = \deg(\Gamma_n, D_n, 0).$$

If $\deg(\Gamma_n, D_n, 0)$ is not stable, then there exist a sub-sequence (t_{nk}) that we still denote as (t_n), such that $t_{n+1} \in (0, 1)$ and $z_{n+1} \in \partial D_{n+1}$ and $h(t_{n+1})(z_{n+1}) = 0$. Therefore

$$\Gamma_{n+1}(z_{n+1}) + t_{n+1}\,(G(z_{n+1}), u^{n+1})\, u^{n+1} = 0.$$

Since $\{z_n\} \subset \Omega$ is bounded and G is compact on Ω then $(G(z_{n+1}), u^{n+1}) \to 0$ and then $\Gamma_{n+1}(z_{n+1}) \to 0$ that implies $z_n \to z \in \partial D$ and $\Gamma(z) = 0$, a contradiction! □

Definition 2.69. *Under the above settings, for any open subset $D \subset H$ such that $z \neq G(z)$ for $z \in \partial D$, define*

$$\deg(\Gamma, D, 0) = \lim_{n\to\infty} \deg(\Gamma_n, D_n, 0). \tag{2.96}$$

Definition 2.70. *Let $G\colon [0,1] \times \Omega \to H$ be a continuous map. The map G is called a compact transformation if $G([0,1])$ is compact from Ω to H, and in addition for every bounded $\Omega' \subset \Omega$ and $\varepsilon > 0$ there exist $\delta = \delta(\varepsilon, \Omega')$ such that*

$$|t - s| < \delta \Rightarrow \|G(t)(x) - G(s)(x)\| < \varepsilon, \quad x \in \Omega'.$$

Proposition 2.71. *The definition (2.96) satisfies the homotopy invariance property for the class of compact transformations.*

Proof. Let Γ denotes the time dependent operator: $\Gamma\colon [0,1] \times D \to H$. There exist N_0 such that $0 \notin \Gamma_n(t)(\partial D_n)$ for all $n \geq N_0$. Assume the contrary, then there exist a sequence t_n and $z_n \in \partial D_n$ such that $\Gamma_n(t_n)(z_n) = 0$. Let $\{t_n\}$ converges to $\bar{t}$ in a sub-sequence, since $G(\bar{t})$ is compact then $G(\bar{t})\{z_n\}$ converges in a sub-sequence to some point say z. Thus we can write

$$\begin{aligned} \|z_n - z\| = \|G(t_n)(z_n) - G(\bar{t})(z_n) + G(\bar{t})(z_n) - z\| \leq \\ \leq \|G(t_n)(z_n) - G(\bar{t})(z_n)\| + \|G(\bar{t})(z_n) - z\| = \\ = \|G(t_n)(z_n) - G(\bar{t})(z_n)\| + o(1). \end{aligned}$$

On the other hand, since G is a compact transformation then

$$\|G(t_n)(z_n) - G(\bar{t})(z_n)\| = o(1),$$

that gives $z_n \to z$ in a sub-sequence. Since ∂D is closed then $z \in \partial D \cap \Omega$ a contradiction. This implies that $\deg(\Gamma_n(t), D_n, 0)$ is independent of t. On the other hand

$$\deg(\Gamma_n(t), D_n, 0) = \deg(A_{n+1}(t), D_{n+1}, 0),$$

where $A_{n+1}(t)(u) = \Gamma_n(t)(u) + \langle z, u^{n+1}\rangle u^{n+1}$. Choose n sufficiently large such that $\deg(\Gamma_n(t), D_n, 0)$ is constant with respect to t. If $\deg(\Gamma_n(t), D_n, 0)$ is not stable then there exist s_n and $z_n \in \partial D_n$ such that $(1 - s_n)\,\Gamma_n(t)(z_n) + s_n\, A_n(t)(z_n) = 0$ that give the convergence of $\{z_n\}$ in a sub-sequence to some $z \in \partial D$ such that $\Gamma(t)(z) = 0$ a contradiction! □

Theorem 2.72. *Degree in (2.96) satisfies all classical properties of a topological degree.*

Proof. The homotopy class and the invariance under the homotopy has been established in the above. For the reference map, it is obvious that identity map Id has the property

$$\deg(I, D, 0) = \begin{cases} 1 & 0 \in D \\ 0 & 0 \notin D \end{cases}.$$

The existence of a solution in the case that the degree is nonzero is shown as follows. According to (2.96) if $\deg(G, D, 0) \neq 0$, then $\deg(\Gamma_n, D_n, 0) \neq 0$ for $n \geq N_0$. Due to the properties of classical topological degree, the equation $\Gamma_n(u) = 0$ has a solution say $u_n \in D_n$. Since $\{u_n\} \subset \Omega$ and Ω is bounded then $u_n \rightharpoonup u$. By the compactness of G on Ω we conclude $u_n \to u \in \bar{D}$ and $\Gamma(u) = 0$ and since $0 \notin \Gamma(\partial D)$ then $u \in D$. The domain decomposition property follows immediately from the property of degree for finite dimensional maps. □

2.6.2 Generalized degree in separable Banach space

Let X be a reflexive separable Banach space equipped with a uniformly convex norm, $\Omega \subset X$ be a closed set and $A: X \to X^*$ not necessary a demi-continuous map. Let H be a separable Hilbert space such that the embedding $j: H \to X$ is dense (see [19]) with the orthonormal basis $\mathcal{H} = \{h^1, h^2, \ldots\}$. Let us call the set $\mathcal{X} = \{x^k, x^k = j(h^k)\}$ a system for X and $X_n = \mathrm{span}\{x^1, \ldots, x^n\}$ a finite dimensional space equipped with the inner product $(x^i, x^j) = \delta_{ij}$.

We make the following assumption: there exists a closed subspace $Y \subset X$, generated by a subset of $\mathcal{X}$ and a closed subset $\Omega \subset Y$ with the following properties:

i. for any finite dimensional subspace $Z \subset Y$, the set $\Omega_Z = \Omega \cap Z$ has non-empty interior in Z

ii. $A|_\Omega$ is a bounded, demi-continuous map of class $(S)_+$,

iii. for any finite dimensional subspace $X_n \subset X$ we have $\{z; A(z) = 0 \,\text{on}\, X_n\} \subset \Omega$.

Here we assume that Y is generated by $\mathcal{Y} = \{y^1, y^2, ...\} \subset \mathcal{X}$. The map $A_n: X_n \to X_n$ is defined as follows:

$$A_n(u) = \sum_{k=1}^{n} \langle A(u), x^k \rangle x^k. \tag{2.97}$$

Obviously $A_n = A$ on X_n, that is for all $u, v \in X_n$, $(A_n(u), v) = \langle A(u), v \rangle$. In particular $A_n(u) = 0$ if and only if $A(u) = 0$ on X_n. For each $n \geq 1$, define $Y_{n'} = X_n \cap Y$ where $n' \leq n$ is the dimension of the space $X_n \cap Y$. Denote $\Omega_{n'} = \Omega \cap Y_{n'}$ and since $\Omega_{n'}^\circ$, the interior of $\Omega_{n'}$ in $Y_{n'}$ is non-empty according to the property (i), then for any open bounded subset $D \subset X$, the set $D_{n'} = D \cap \Omega_{n'}^\circ$ is bounded and open in $Y_{n'}$. Now define the map $A_{n,n'}: D_{n'} \to X_n$ the restriction of A_n to $D_{n'}$. If $0 \notin A_{n,n'}(\partial D_{n'})$ then it is possible to define the reduced degree $d(A_{n,n'}, D_{n'}, 0)$ as (2.94), that is for an n'-form ω compactly supported in $U_{n'} \subset U$ where U is an open bounded neighborhood of 0 such that $U \cap A_{n,n'}(\partial D_{n'}) = \emptyset$:

$$d(A_{n,n'}, D_{n'}, 0) = \int_{D_{n'}} A_{n,n'}^* \omega. \tag{2.98}$$

Proposition 2.73. *Let D be an open bounded subset of X and $0 \notin A(\partial D)$, then there exist N_0 such that $0 \notin A_{n,n'}(\partial D_{n'})$ for $n \geq N_0$.*

Proof. Otherwise there exist a sequence $\{z_{n'}\}$, $z_{n'} \in \partial D_{n'}$ such that $A_{n,n'}(z_{n'}) = 0$. Since D is bounded, $z_{n'} \rightharpoonup z$ in a sub-sequence and since $\{z_{n'}\} \subset \Omega$ then $z \in \mathrm{wcl}(\Omega) \subset Y$. Since $A(z_{n'}) = 0$ on X_n, we have

$$\langle A(z_{n'}), z_{n'} - z \rangle = -\langle A(z_{n'}), z \rangle.$$

On the other hand, $z \in Y$ and then there exist $\{s_{n'}\}$, $s_{n'} \in Y_{n'}$ such that $s_{n'} \to z$, therefore

$$\langle A(z_{n'}), z \rangle = \langle A(z_{n'}), z - s_{n'} \rangle \leq \|A(z_{n'})\| \, \|z - s_{n'}\|.$$

Since A is bounded on Ω then $\|A(z_{n'})\|$ is bounded and then $\langle A(z_{n'}), z\rangle \to 0$. Since A is a map of class $(S)_+$ on Ω, then $z_{n'} \to z \in \partial D$ and since A is demi-continuous on Ω then $A(z_{n'}) \rightharpoonup A(z)$. Let $v \in X$ be arbitrary, then for $\{v_n\}, v_n \in X_n$ and $v_n \to v$ we have:

$$\langle A(z), v\rangle = \lim \langle A(z_{n'}), v\rangle = \lim \langle A(z_{n'}), v - v_n\rangle = 0,$$

that is $A(z) = 0$, a contradiction! □

Definition 2.74. *Let D be an open bounded subset of X and $0 \notin A(\partial D)$. For every finite dimensional subspace X_n define the reduced degree for the map $A_{n,n'}: D_{n'} \to X_n^*$ as*

$$d_n = d(A_{n,n'}, D_{n'}, 0), \tag{2.99}$$

where integer d is defined in (2.98).

Proposition 2.75. *There exist N_0 such that d_n in (2.98) is constant for $n \geq N_0$.*

Proof. Choose N_0 sufficiently large such that d_n is well defined. Consider the map $B_{n+1}: D_{n+1} \to X_{n+1}$ as

$$B_{n+1}(u) = A_n(u) + \mathrm{pr}_{n+1}(u) x^{n+1}, \tag{2.100}$$

where $\mathrm{pr}_k(u)$ is the k-th component of u in the representation $u = u_1 x^1 + \cdots + u_k x^k$. The reduced degree for B_{n+1} is denoted as $d(B_{(n+1),(n+1)'}, D_{(n+1)'}, 0)$. Obviously we have:

$$d(A_{n,n'}, D_{n'}, 0) = d(B_{(n+1),(n+1)'}, D_{(n+1)'}, 0). \tag{2.101}$$

Now consider the following convex homotopy $h(t)$:

$$h(t) = (1-t) A_{n+1} + t B_{n+1}. \tag{2.102}$$

We first show that h is an admissible homotopy. Just for the simplicity, let us denote $(n+1)' = m'$. Obviously $0 \notin h(t)(\partial D_{m'})$ for $t = 0, 1$. If the proposition does not hold, then there exist a sequence $z_{m'} \in \partial D_{m'}$ and $t_{m'} \in (0,1)$ such that

$$(1 - t_{m'}) A_{(n+1),m'}(z_{m'}) + t_{m'} B_{(n+1),m'}(z_{m'}) = 0. \tag{2.103}$$

According to the definition of B_{n+1}, the above equality implies $A(z_{m'}) = 0$ on X_n and also

$$\langle A(z_{m'}), x^{n+1}\rangle = \frac{t_{m'}}{1-t_{m'}}\,\mathrm{pr}_{n+1}(z_{m'}).$$

Since $\{z_{m'}\}$ is bounded and X is reflexive, then $\{z_{m'}\}$ weakly converge in a subsequence to some vector z. Now

$$\langle A(z_{m'}), z_{m'} - z\rangle = -\frac{t_{m'}}{1-t_{m'}}(\mathrm{pr}_{n+1}(z_{m'}))^2 - \langle A(z_{m'}), z\rangle.$$

Since $A(z_{m'}) = 0$ on X_n, by the density property we have: $\langle A(z_{m'}), z\rangle \to 0$. Therefore

$$\limsup \langle A(z_{m'}), z_{m'} - z\rangle \leq 0.$$

and then $z_n \to z \in \partial D \cap \Omega$. A is demi-continuous on Ω and then $A(z_{m'}) \rightharpoonup A(z)$. For arbitrary $v \in Y$ we have

$$\langle A(z), v\rangle = \lim \langle A(z_{m'}), v\rangle = 0,$$

a contradiction! □

Definition 2.76. *Under the above setting the degree of A at 0 is defined as the following integer:*

$$d = \deg(A, D, 0) = \lim_{n\to\infty} d_n. \tag{2.104}$$

In order to prove the classical property of topological degree for the above definition, we define the following class of homotopy:

Definition 2.77. *Let A: $[0,1] \times X \to X^*$ be a one parameter family of maps such that $A(t)(.)$ is continuous with respect to t and for each t, $A(t)|_\Omega$ is a bounded demi-continuous map of class $(S)_+$. A is called an admissible homotopy if for any finite dimensional spaces and any $t \in [0,1]$ we have $\{z, A(t)(z) = 0 \;\text{ on}\, X_n\} \subset \Omega$ and also that $A(t)$ is continuous on bounded sets, that is for every bounded set $\Omega' \subset \Omega$ and $\varepsilon > 0$, there is $\delta > 0$ such that*

$$|t-s| < \delta \Rightarrow \|A(t)(x) - A(s)(x)\| < \varepsilon, \forall x \in \Omega'. \tag{2.105}$$

Proposition 2.78. *Assume $0 \notin A(t)(\partial D)$ for all $t \in [0,1]$, then the definition (2.76) satisfies the homotopy invariance property of the classical topological degree.*

Proof. First we show that there exist N_0 such that $0 \notin A_{n,n'}(t)(\partial D_{n'})$ for all $n \geq N_0$ and $t \in [0,1]$. Assume the contrary, then there exist a sequence $t_{n'}$ and $z_{n'} \in \partial D_{n'}$ such that $A_{n,n'}(t_{n'})(z_{n'}) = 0$. Since $\{t_{n'}\}$ converges to some $\bar{t}$ in a sub-sequence and $\{z_{n'}\}$ converges weakly to some z in a sub-sequence and since $A(\bar{t})$ is a map of class $(S)_+$ then

$$\langle A(\bar{t})(z_{n'}), z_{n'} - z\rangle = \langle A(\bar{t})(z_{n'}) - A(t_{n'})(z_{n'}), z_{n'} - z\rangle + \langle A(t_{n'})(z_{n'}), z_{n'} - z\rangle.$$

According to the boundedness property of $A(t)$ in (2.105), it is seen that:

$$\langle A(\bar{t})(z_{n'}) - A(t_{n'})(z_{n'}), z_{n'} - z\rangle \to 0. \tag{2.106}$$

Since $A(t_{n'})(z_{n'})$ is zero on X_n then by the density argument we have $\langle A(t_{n'})(z_{n'}), z_{n'} - z\rangle \to 0$. This implies that $z_{n'} \to z \in \partial D$ and then $A(\bar{t})(z)$ is zero on X, a contradiction! This implies that for large n the degree $d(A_{n,n'}(t), D_{n'}, 0)$ is independent of t. On the other hand for the mapping $B_{n+1}(t)$ defined as

$$B_{n+1}(t)(u) = A_n(t)(u) + \mathrm{pr}_{n+1}(u) x^{n+1}, \tag{2.107}$$

we have

$$d(A_{n,n'}(t), D_{n,n'}, 0) = d(B_{(n+1),m'}(t), D_{m'}, 0). \tag{2.108}$$

Choose n sufficiently large such that $d(A_{n,n'}(t), D_{n'}, 0)$ is constant with respect to t. If $d(A_{n,n'}(t), D_{n'}, 0)$ is not stable then there exist $s_{n'}$ and $z_{n'} \in \partial D_{n'}$ such that $(1 - s_{n'})\, A_n(t)(z_{n'}) + s_{n'}\, B_n(t)(z_{n'}) = 0$ that give the convergence of $\{z_{n'}\}$ in a sub-sequence to some $z \in \partial D$ and $A(t)(z) = 0$ on X, a contradiction! □

The other classical properties of topological degree are straightforward to verify.

Corollary 2.79. *Assume the map $A: X \to X^*$ has the following two properties:*

i. for every finite dimensional subspace X_n, the map A is continuous from X_n to X^*,

ii. for every sequence $\{x_n\} \subset X$ and $A_n(x_n) = 0$, then $x_n \to x$ and $A(x_n) \rightharpoonup A(x)$,

then $\deg(A_n, D_n, 0)$ *is stable for $n \geq N_0$.*

Proof. Choose n sufficiently large that $0 \notin A_n(\partial D_n)$, that is possible, since other wise the sequence $z_n \in \partial D_n$ with the property $A_n(z_n) = 0$ result to $z_n \to z \in \partial D$ and $A(z_n) \rightharpoonup A(z)$. By the density argument, it is obtained $A(z) = 0$, a contradiction. Now define the homotopy $h: [0, 1] \times D_{n+1} \to X_{n+1}$ as follows:

$$h(t, x) = (1 - t)\, A_{n+1}(x) + t\, B(x).$$

If $\deg(A_n, D_n, 0)$ is not stable, then there exist $z_n \in \partial D_n$ and $t_n \in [0, 1]$ such that $h(t_n, z_n) = 0$. This implies again $A_n(z_n) = 0$ and then $z_n \to z \in \partial D$ in a sub-sequence and $A(z) = 0$ a contradiction. □

2.6.3 Schauder maps in Banach spaces and Bifurcation

Consider the Schauder map $A = \mathrm{Id} - G$ defined on the separable uniformly convex Banach space X and $G|_\Omega$ is a continuous compact map for closed subset $\Omega \subset X$. All assumptions made in the previous section extend to hold in this section. By the aid of duality mapping J, define the map $\tilde{A}: X \to X^*$ as $\tilde{A}(u) = J(u) - J(G(u))$ and then the reduced map $\tilde{A}_n$ is defined as

$$\tilde{A}_n(u) = \sum_{k=1}^{n} \langle J(u) - J(G(u)), x^k \rangle x^k. \tag{2.109}$$

Obviously, according to the property of J, $A(u) = 0$ if and only if $\tilde{A}(u) = 0$.

Proposition 2.80. *Assume that $G|_\Omega$ is a continuous and compact map, then $\tilde{A}|_\Omega$ is bounded map of class $(S)_+$.*

Proof. Assume $\{z_n\} \subset \Omega$ and $z_n \rightharpoonup z$ and also

$$\limsup \left\langle \tilde{A}(z_n), z_n - z \right\rangle \leq 0,$$

then since $\langle J(G(z_n)), z_n - z \rangle \to 0$ we have

$$0 \geq \limsup \left\langle \tilde{A}(z_n), z_n - z \right\rangle = \limsup \{ \|z_n\|^2 - \langle J(z_n), z \rangle \} \geq \limsup \{ \|z_n\| (\|z_n\| - ||z|) \},$$

that is $\limsup \|z_n\| \leq \|z\|$ and then $\|z_n\| \to \|z\|$. Since the space is locally uniformly convex, $z_n \to z$. The boundedness of $\tilde{A}$ is easily verified by the compactness of $J \circ G$ on Ω. □

Definition 2.81. *Under the above setting, for any open bounded subset $D \subset X$, the degree of $A\colon \mathrm{Id} - G$ at 0 is defined as*

$$\deg(A, D, 0) = \lim_{n \to \infty} d\big(\tilde{A}_{n,n'}, D_{n'}, 0\big). \tag{2.110}$$

Remark 2.82. If $X = H$, a Hilbert space, then the duality mapping is simply the identity map and the reduced map A_n with respect to an orthonormal basis $\mathcal{X} = \{x^1, x^2, \ldots\}$ is simply defined as:

$$A_n(u) = \sum_{k=1}^{n} \langle u - G(u), x^k \rangle x^k. \tag{2.111}$$

For an illustrative example, the operator $A = I - \lambda G$ where G is the Onsager operator on $L(S^1)$ defined in (2.92) satisfies the conditions of this section if $\mathcal{X} = \{\cos(n\theta)\}_{n=1}^{\infty}$ and $\Omega = \{u \in C(S^1), |u(\theta)|_\infty \le \lambda\, |K|_\infty\}$.

Now consider the one parameter class of operators $A(\lambda, u) = \mathrm{Id} - G(\lambda, u)$ where $G(\lambda, .)|_\Omega$ is a compact continuous map for every λ such that $A(\lambda, 0) = 0$. Further assume that there exist a compact linear continuous map $T\colon X \to X$ such that for all $u \in \Omega$ it satisfies the following:

$$\|\lambda T(u) - G(\lambda, u)\| = o(\|u\|) \tag{2.112}$$

Proposition 2.83. *If $\lambda \notin \sigma(T)$, where $\sigma(T)$ denotes the spectrum of T, then $(\lambda, 0)$ is isolated, that is there exist $\varepsilon > 0$ such that the trivial solution is the unique solution in $(\lambda - \varepsilon, \lambda + \varepsilon) \times B_\varepsilon$.*

Proof. Otherwise, there exist a sequence $\{\lambda_n\}$, $\lambda_n \to \lambda$ and $u_n \to 0$ such that $0 = A(\lambda_n, u_n)$. Since $\lambda \notin \sigma(T)$, then there exist $r > 0$ and $\|u - \lambda T(u)\| \ge r\, \|u\|$. Therefore

$$0 \ge \|u_n - \lambda T(u_n)\| - |\lambda_n - \lambda|\, \|u_n\| - \|\lambda_n T(u_n) - G(\lambda_n, u_n)\| >$$
$$> r\, \|u_n\| - |\lambda_n - \lambda|\, \|u_n\| + o(\|u\|),$$

that is contradiction for sufficiently small $|\lambda - \lambda|$ and u_n.

□

Proposition 2.84. *Assume* $\bar{\lambda} \in \sigma(T)$ *and for* $\lambda < \bar{\lambda} < \mu$ *we have*

$$\operatorname{ind}(\mathrm{Id} - \lambda T, 0)\operatorname{ind}(\mathrm{Id} - \mu T, 0) < 0, \tag{2.113}$$

then $\bar{\lambda}$ *is a bifurcation point.*

Proof. Otherwise $(\bar{\lambda}, 0)$ is an isolated point and therefore there exist $\varepsilon > 0$ such that the trivial solution is the unique solution of $A(t)(u) = 0$ for $t \in (\bar{\lambda} - \varepsilon, \bar{\lambda} + \varepsilon)$ and $u \in B_\varepsilon$. This implies that no solution lies on $\partial B_{\varepsilon/2}$ for $t \in (\bar{\lambda} - \varepsilon, \bar{\lambda} + \varepsilon)$ and then by homotopy invariance property the index $\operatorname{ind}(\mathrm{Id} - tT, 0)$ is constant even when t passes through $\bar{\lambda}$, a contradiction. □

Chapter 3

Doi-Onsager equation

This chapter is dedicated to the Doi-Onsager model in arbitrary dimensional spaces. Up to our knowledge, the complete solution of the problem for dimensions greater than 2 is still open. Our approach is to use the Leray-Schauder degree for the model defined on the unit sphere S^1 and to use the generalized degree we presented in Chapter 2 for $S^D, D \geq 2$. In particular, we prove the uniqueness of the trivial solution for low temperatures and the existence of non-trivial solution (nematic phases) for high temperatures. In addition we study the structure of the bifurcation solution and their stability as well. In our study of this problem, we do not restrict ourselves to the interaction potential kernel suggested by Onsager in his pioneering work [65], but we consider fairly general class of kernels that covers the Onsager kernel too.

In section 1 of this chapter, we give a brief introduction of Doi-Onsager problem and its derivation. In section 2, we reformulate the problem as an operator equation involving a Schauder map. Chapter 3 is dedicated to the main results obtained for the problem on S^1. For $D \geq 3$, we use our generalization of degree that is presented in the previous chapter to calculate the degree of the corresponding operator. The results are given in Section 4 of this chapter.

Our method, that is based on degree argument is simpler and easy to generalize to higher dimensions compared to the method employed by other authors. In fact, the method for $S^D, D > 1$ is completely similar to the one that we used for S^1. Our method also simplifies the study of bifurcation structure of the nematic solutions through the aid of Sattinger's works see [71],[70].

3.1 Introduction

Liquid crystals (LC) are multi-phases materials. In the isotropic phase, molecules have no order in position or in direction and behave like a regular liquid. During cooling down, the isotropic phase changes to the nematic state where molecules arrange into groups with a preferred orientation still without any position order. In 1949, Lars Onsager proposed a mathematical model for the phase transition of equilibria of dilute colloidal solutions of rod-like molecules between the isotropic and nematic phases. As the fluid in both phases is homogeneous, that is the locations of the molecules do not matter, Onsager's theory focuses on a probability density function $f(r)$ over the unit sphere which models distribution of the directions of the rods. Although the original modeling is carried out in $\mathbb{R}^3$, the mathematical formulation can be generalized to $\mathbb{R}^d$ for any dimension $d \geq 2$ in a straightforward manner. In the following we present this generalized version.

Denote by S^{D-1} the unit sphere in $\mathbb{R}^D$. Let $f(r)\colon S^{D-1} \to [0, \infty)$ be the probability density characterizing the directions of the rods, that is

$$P(\text{the rod is along}\, r \in A \subset S^{D-1}) = \int_A f(r)\, d\sigma_D(r) \tag{3.1}$$

where we denote by $\sigma_D(r)$ the volume element on S^{D-1}. As we are modeling "rod-like" molecules with no distinction between the two ends, we can further assume $f(r) = f(-r)$. Consequently the constraints on $f(r)$ are

$$f(r) \geq 0, \quad f(r) = f(-r), \quad \int_{S^{D-1}} f(r)\, d\sigma_D(r) = 1. \tag{3.2}$$

Onsager [65] defined the mean interaction between molecules as follows

$$\mathcal{U}(f)(r) = \lambda \int_{S^{D-1}} K(r, r')\, f(r')\, d\sigma_D(r), \tag{3.3}$$

where $K(r, r') = |r \times r'|$, and the parameter λ can be interpreted as either the concentration of the particles in the carrier fluid or the inverse of the absolute temperature. By the aid of the above interaction field, he obtained the possible phases of the liquid crystal in R^D as the critical points of the following energy functional

$$\mathcal{E}(f) = \int_{S^{D-1}} f(r) \left(\log f(r) + \frac{1}{2} \mathcal{U}(f)(r) \right) d\sigma_D(r). \tag{3.4}$$

The Euler-Lagrange equation is obtained by considering smooth functions η on S^{D-1} with the condition

$$\int_{S^{D-1}} \eta(r)\, d\sigma_D(r) = 0.$$

It is easily obtained that f minimizes (3.4) if it satisfies

$$\int_{S^{D-1}} (\log f(r) + \mathcal{U}(f)(r))\eta(r)\, d\sigma_D(r) = 0.$$

This implies that $V(f) := \log f + \mathcal{U}(f) =$ constant or equivalently

$$f(r) = \beta^{-1} e^{-\mathcal{U}(f)(r)}, \qquad \beta = \int_{S^{D-1}} e^{-\mathcal{U}(f)(r)}\, d\sigma_D(r). \tag{3.5}$$

It is interesting to note that the critical points of (3.4) are steady states of the time dependent Doi equation:

$$\frac{\partial f}{\partial t} = \Delta_r f + \operatorname{div}(f \nabla_r \mathcal{U}(f)), \tag{3.6}$$

where here $f = f(r, t)$. In fact the diffusion-advection term in (3.6) is expressed in terms of V as

$$\Delta_r f + \operatorname{div}(f \nabla_r \mathcal{U}(f)) = \operatorname{div}(f \nabla_r V).$$

Obviously, $\bar{f} = \frac{1}{\mathrm{vol}(S^n)}$ is the trivial solution of (3.5) that corresponds to the uniform distribution of molecules in the liquid without any preferred directional order. This is called an isotropic phase. By approximating the kernel K with simpler analytic mappings, Onsager was able to show a transition in phase when λ increase to a certain level where molecules positioned along a director. This is called a nematic phase.

In general, the kernel $K(r, r')$ in the interaction field (3.3) inherit the following properties:

$$K(r, r') = K(-r, r') = K(r', r) = K(O(r), O(r')), \tag{3.7}$$

for any rotation matrix O. When $D=2$, we can use the natural parametrization of S^1 by the angle $\theta\in[0,2\pi)$ and re-write any kernel satisfying (3.7) as a convolution kernel $K(\theta-\theta')$ for some even function K satisfying $K(\theta+\pi)=K(\theta)$. This reduces the right hand side of (3.3) to a convolution

$$U(f)(\theta)=\lambda\int_0^{2\pi} K(\theta-\theta')\, f(\theta')\, d\theta'. \tag{3.8}$$

The trivial solution in this case is $\bar{f}=(2\pi)^{-1}$. In this case the system (3.5) has the following form

$$f(\theta)=\left(\int_0^{2\pi} e^{-\mathcal{U}(f)(\theta)}\, d\theta\right)^{-1} e^{-\mathcal{U}(f)(\theta)}. \tag{3.9}$$

The interaction kernel that is considered by Onsager in this case reads $K(\theta) = |\sin(\theta)|$. It is it straightforward to check that $K(\theta-\theta')$ satisfies (3.7).

The original kernel considered by Onsager in $\mathbb{R}^3$ is

$$K(r,r')=|r\times r'|=|\sin\gamma|, \tag{3.10}$$

where γ is the angle between r, r'. More quantitative analysis of the system (3.9) with Onsager kernel (3.10) turned out to be difficult. On the other hand there are kernels capturing the qualitative behavior of the solution that are more friendly to mathematical analysis. One such kernel, due to Maier and Saupe, reads

$$K(r,r')=|r.r'|^2-\text{constant}, \tag{3.11}$$

that is usually written as $K(r,r')=\cos^2\gamma$ if the constant is discarded. The advantage in considering (3.11) instead of (3.10) is that the kernel is the eigenvector of Laplace-Beltrami operator on S^{D-1} and then lies in a finite dimensional space [26]. This reduces the infinite dimensional problem (3.5) to a finite dimensional nonlinear system of equations. This reduced system, still highly nontrivial, is nevertheless more tractable than the original system. As a consequence, (3.5) with Maier-Saupe potential has been well understood through brilliant work of many researchers (see [27], [37], [41], [82], [50], [83] for the case $D = 3$; Also see [28], [37], [21] for the case $D=2$, and [79] for the general D-dimensional case.) Inspired by these works, (3.5) with other kernels enjoying similar "dimension- reduction" property has also be analyzed, see e.g. [26].

Fatkullin and Slastikov [37], [36] completely classified the solution of Onsager equation with Maier-Saupe kernel (3.11) and for the antisymmetric kernel $K(r, r') = -r.r'$ on S^1 and S^2. Instead of λ, they presented their results in term of the temperature τ, however since λ and τ are inversely proportional their results hold for the original case. For the system (3.9) with Maier-Saupe kernel (3.11), they obtained the exact nematic solutions as

$$f(\varphi,\theta) = \beta^{-1}\, e^{-r_{1,2}(\tau)(3\cos^2\theta - 1)}, \tag{3.12}$$

and for the anti-symmetric kernel they obtained

$$f(\varphi,\theta) = \beta^{-1}\, e^{-r(\tau)\cos\theta}. \tag{3.13}$$

In addition they presented some results of the stability of the above solutions. Luo et al [53] considered the Maier-Saupe interaction kernel on S^1 and proved that for the potential strength $\lambda \leq 4$ the unique solution is the isotropic solution $\bar{f} = \frac{1}{2\pi}$. The nematic solution will bifurcates when the the crystal liquid cool down or equivalently the potential strength increase to $\lambda > 4$. They also proved that all nematic solutions are obtained by an arbitrary rotation from a π periodic nematic solution. At the same time, Liu et al [51] obtained an explicit solution for (3.5) on S^2 with Maier-Saupe kernel and determined the bifurcation regime for the solution. The solution is of the following form for a director y and constant k and

$$f(x) = k e^{-\eta(x.y)^2}. \tag{3.14}$$

With the Maier-Saupe model (3.11) understood, interest in the original Onsager model (3.10) was resurrected. Much progress has been made in the past few years in the case $D = 2$. In [26], the axisymmetry of all possible solutions is proved, that is, for any solution $f(\theta)$ to (3.5), there is θ_0 such that $f(\theta_0) = f(\theta_0 + \pi)$. It is also proved in [26] that for appropriate λ, there are solutions of arbitrary periodicity. In [79] the authors re-write (3.5) into an infinite system of nonlinear equations for the Fourier coefficients of $f(\theta)$ and calculated numerically the first few bifurcations. Chen et al [26] observed that for even integers $l = 2n$, the the interaction potential

$$U(f)(\theta) = \int_{S^1} \sin^l(\theta - \theta')\, f(\theta')\, d\theta', \tag{3.15}$$

behave completely similar to the Maier-Saupe original potential and can be reduced to a model in finite dimensional space, while for odd $l=2n-1$ the obtained equation will be a nonlinear partial differential equation. By reducing the Onsager equation to a system of ordinary differential equations, they could prove the existence of auxiliary symmetric nematic solution for the Onsager equation on S^1 and for all odd power potential kernel.

More recently, in [52] the authors study the case $D=2$ through cutting-off (3.10) and reducing (3.5) to a finite dimensional system of nonlinear equations, and obtain local bifurcation structure for this finite dimensional approximation. In particular, they used a result of bifurcation by Crandall and Rabinowitz [?] for the general truncated trigonometric kernel

$$K(\theta,\theta')=-\sum_{n=0}^{N} k_n \cos 2n(\theta-\theta'). \tag{3.16}$$

The original Onsager kernel $|\sin\,(\theta-\theta')|$ is approximated by the above kernel for special

$$k_n=\frac{1}{\pi}\left(n^2-\frac{1}{4}\right)^{-1}. \tag{3.17}$$

In this case the problem is reduced to finding the zeros of a finite dimensional nonlinear problem.

In this section we consider the Onsager equation (3.5) on S^1 with fairly general kernel $K(r,r')$ for $r,r'\in S^1$ of the following form

$$K(r,r')=K(|r\times r'|). \tag{3.18}$$

3.2 Reformulation of the problem

We reformulate the system (3.5) into the equation involving a completely continuous operator which we will call Onsager operator. Recall that we need to solve

$$f(r)=\frac{e^{-U(f)(r)}}{\int_{S^{D-1}} e^{-U(f)(r)}\,d\sigma_D(r)}, \quad f(r)=f(-r), \tag{3.19}$$

with

$$U(f)(r) = \lambda \int_{S^{D-1}} K(r, r')\, f(r')\, d\sigma_D(r'), \tag{3.20}$$

where $K(r, r')$ satisfies conditions (3.7). Multiplying both sides of (3.19) by $\lambda K(r, r')$ and integrating over S^{D-1}, we cancel f and reach an equation for the potential $U(r)$.

$$U(r) = \frac{\int_{S^{D-1}} \lambda K(r, r')\, e^{-U(r')}\, d\sigma_D(r')}{\int_{S^{D-1}} e^{-U(r)}\, d\sigma_D(r)}, \quad U(r) = U(-r). \tag{3.21}$$

Note that once (3.21) is solved, $f(r)$ can be recovered from

$$f(r) = \frac{e^{-U(r)}}{\int_{S^{D-1}} e^{-U(r)}\, d\sigma_D(r)}. \tag{3.22}$$

Thus (3.21) is equivalent to the original problem (3.19-3.20).

Further reduction of the problem needs the following lemma which shows that simplification similar to (3.8) can be carried out in the general case.

Lemma 3.1. *Under the symmetry assumptions* (3.7) *on* K*, we have* $K(r,\, r') = F(|r - r'|)$ *for some function* F*. In particular, this gives*

$$\bar{K} = \frac{1}{|S^{D-1}|} \int_{S^{D-1}} K(r, r')\, d\sigma_D(r') = \frac{1}{|S^{D-1}|} \int_{S^{D-1}} K(r, r')\, d\sigma_D(r) \tag{3.23}$$

is a constant.

Proof. All we need to show is that $K(r, r')$ depends only on $|r - r'|$ or equivalently the angle θ between r, r'. It suffices to notice that for any $r, r' \in S^{D-1}$ there is always $T \in O(D)$ such that $T(r) = e_1$ and $T(r') = \cos(\theta)\, e_1 + \sin(\theta)\, e_2$. Consequently

$$K(r, r') = K(e_1, \cos(\theta)\, e_1 + \sin(\theta)\, e_2) = F(|r - r'|).$$

Now we have

$$\int_{S^{D-1}} K(r, r')\, d\sigma_D(r') = \int_{S^{D-1}} K(e_1, r')\, d\sigma_D(r'),$$

which is constant. The second equality in (3.23) follows from the property $K(r, r') = K(r', r)$. □

Now we define

$$\tilde{K}(r,r') = K(r,r') - \bar{K} \tag{3.24}$$

with $\bar{K}$ as defined in (3.23). Define also

$$V(r) = U(r) - \lambda\bar{K}. \tag{3.25}$$

Then we can easily check that (3.21) is equivalent to the following

$$V(r) = \frac{\lambda\int_{S^{D-1}} \tilde{K}(r,r')\, e^{-V(r')}\, d\sigma_D(r')}{\int_{S^{D-1}} e^{-V(r)}\, d\sigma_D(r)}, \quad V(-r) = V(r), \quad \int_{S^{D-1}} V(r)\, d\sigma_D(r) = 0. \tag{3.26}$$

Summarizing the above we reach

Lemma 3.2. *The original problem* (3.5) *is equivalent to the following problem.*

$$V(r) = \frac{\lambda\int_{S^{D-1}} \tilde{K}(r,r')\, e^{-V(r')}\, d\sigma_D(r')}{\int_{S^{D-1}} e^{-V(r)}\, d\sigma_D(r)}, \; V(-r) = V(r), \; \int_{S^{D-1}} V(r)\, d\sigma_D(r) = 0. \tag{3.27}$$

From now on, we will work with (3.27) which can be naturally written as fixed point problem

$$V(r) = \lambda\Gamma(V)(r), \tag{3.28}$$

with the operator Γ defined as

$$\Gamma(V)(r) = \frac{\int_{S^{D-1}} \tilde{K}(r,r')\, e^{-V(r')}\, d\sigma_D(r')}{\int_{S^{D-1}} e^{-V(r)}\, d\sigma_D(r)}. \tag{3.29}$$

3.3 Doi-Onsager equation on S^1

Now we turn to the case $D = 2$, where the natural parametrization of S^1 can be applied to re-write (3.27) as

$$V(\theta) = \lambda\Gamma(V)(\theta) = \frac{\lambda\int_0^{2\pi} \tilde{K}(\theta,\theta')\, e^{-V(\theta')}\, d\theta'}{\int_0^{2\pi} e^{-V(\theta)}\, d\theta}, \quad \int_0^{2\pi} V(\theta)\, d\theta = 0, \;\; V(\theta) = V(\theta+\pi) \tag{3.30}$$

where $\tilde{K} = K - \bar{K}$.

Thanks to the rotational invariance of the kernel and the axisymmetry of the solution proved in [26], we can future require $V(\theta) = V(2\pi - \theta)$. We will make the assumption that the kernel $K(\theta) \in W^{1,\infty}([0,\ 2\pi])$. Note that this assumption is satisfied by all the kernels proposed in the physics literature. The natural function space we will working in is

$$H = \left\{ V(\theta) \in H^1([0, 2\pi]); V(\theta) = V(\theta + \pi)\, a.e.; \int_0^{2\pi} V(\theta) = 0; V(\theta) = V(2\pi - \theta)\, a.e. \right\}. \qquad (3.31)$$

Lemma 3.3. *H is a Hilbert space, the operator Γ in* (3.29) *is into H whenever $K \in W^{1,\infty}([0, 2\pi])$.*

Proof. $H^1([0, 2\pi])$ is a Hilbert space and H with the same inner product of H^1 is a closed subspace of H^1, then H is a Hilbert space. Furthermore we have

$$\Gamma(V)(\theta + \pi) = \frac{\int_0^{2\pi} \tilde{K}(\theta + \pi, \theta')\, e^{-V(\theta')}\, d\theta'}{\int_0^{2\pi} e^{-V(\theta)}\, d\theta} = \frac{\int_0^{2\pi} \tilde{K}(\theta, \theta')\, e^{-V(\theta')}\, d\theta'}{\int_0^{2\pi} e^{-V(\theta)}\, d\theta} = \Gamma(V)(\theta). \qquad (3.32)$$

Similarly we have

$$\int_0^{2\pi} \Gamma(V)(\theta) = \frac{\int_0^{2\pi} \int_0^{\pi} \tilde{K}(\theta + \pi, \theta')\, e^{-V(\theta')}\, d\theta d\theta'}{\int_0^{2\pi} e^{-V(\theta)}\, d\theta} = 0,$$

since

$$\int_0^{2\pi} \tilde{K}(\theta, \theta')\, d\theta = 0.$$

Also we have

$$\Gamma(V)(2\pi - \theta) = \frac{\int_0^{2\pi} \tilde{K}(2\pi - \theta, \theta')\, e^{-V(\theta')}\, d\theta'}{\int_0^{2\pi} e^{-V(\theta)}\, d\theta}.$$

Take $\theta' = 2\pi - \theta''$ and then

$$\Gamma(V)(2\pi - \theta) = \frac{\int_0^{2\pi} \tilde{K}(2\pi - \theta, 2\pi - \theta'')\, e^{-V(\theta'')}\, d\theta''}{\int_0^{2\pi} e^{-V(\theta)}\, d\theta} = \Gamma(V)(\theta)$$

since $V(\theta'') = V(2\pi - \theta'')$ and $\tilde{K}(2\pi - \theta, 2\pi - \theta'') = \tilde{K}(\theta, \theta'')$. For the first condition, we calculate

$$\|\Gamma(V)\|_{L^2}^2 = \int_0^{2\pi} |\Gamma(V)(\theta)|^2 \, d\theta = \frac{\int_0^{2\pi} \left(\int_0^{2\pi} \tilde{K}(\theta, \theta') \, e^{-V(\theta')} d\theta \right)^2}{\left(\int_0^{2\pi} e^{-V(\theta')} d\theta \right)^2} \leq$$

$$\leq 2\pi \left\| \tilde{K} \right\|_{L^\infty}^2 = 2\pi \|K - \bar{K}\|_{L^\infty}^2 < \infty.$$

Similarly since $K \in W^{1,\infty}(\Omega)$ we obtain

$$\left\| \frac{d}{d\theta} \Gamma(V) \right\|_{L^2}^2 \leq 2\pi \|K'\|_{L^\infty}^2 < \infty.$$

and this completes the proof. □

We have re-formulated the problem to finding fixed points for the operator $\lambda\Gamma$ in the space H. In the following we need the following lemma. This is a simple version of a well known inequality called Grüss inequality. The proof given here is elementary and simple.

Lemma 3.4. *Let μ be a probability density measure over a domain Ω. Let $f, g \in L^\infty(\Omega)$, then*

$$\left| \int_\Omega f(x) g(x) \, d\mu - \left(\int_\Omega f(x) \, d\mu \right) \left(\int_\Omega g(x) \, d\mu \right) \right| \leq \|f\|_\infty \|g\|_\infty. \tag{3.33}$$

Proof. Without loss of generality assume $\|f\|_\infty, \|g\|_\infty = 1$. Denote

$$a = \int_\Omega f(x) \, d\mu, \quad b = \int_\Omega g(x) \, d\mu.$$

Claim:

$$\int_\Omega f(x) g(x) \, d\mu - ab \leq 1.$$

If $ab \geq 0$ then the claim is trivial. Without loss of generality assume $b < 0$. For $a > 0$ and $-b > 0$, use the inequality

$$a(-b) \leq \frac{a - b}{2},$$

to obtain

$$\int_\Omega f(x)g(x)\,d\mu - ab \le \int_\Omega f(x)g(x) + 0.5f(x) - 0.5g(x)\,d\mu.$$

Since $-1 \le f(x) \le 1$ and $-1 \le g(x) \le 1$ for almost $x \in \Omega$, then

$$f(x)g(x) + 0.5f(x) - 0.5g(x) \le 1, \quad a.e.$$

and therefore the claim is proved. Claim:

$$\int_\Omega f(x)g(x)\,d\mu - ab \ge -1.$$

If $ab \le 0$ the claim is trivial. Assume $a, b > 0$. Then

$$\int_\Omega f(x)g(x)\,d\mu - ab \ge \int_\Omega f(x)g(x) - 0.5f(x) - 0.5g(x)\,d\mu.$$

In a similar way, it is easy to verify that

$$f(x)g(x) - 0.5f(x) - 0.5g(x) \ge -1, \quad a.e.$$

and then the claim follows. If $a, b < 0$ the proof is similar. □

Remark 3.5. In what follows we often take

$$d\mu_V = \frac{e^{-V(\theta)}\,d\theta}{\int_0^{2\pi} e^{-V(\theta)}\,d\theta}. \tag{3.34}$$

3.3.1 Uniqueness of the trivial solution

The solutions of (3.30) on H are the fixed points of the map $G_\lambda = \mathrm{Id} - \lambda\Gamma$. We show first that G_λ is a Schauder map on H that is Γ is a compact continuous operator on H. In addition we establish an a priori estimate for the solution that is easily obtained by the equation itself. This enables us to use the homotopy invariance property of the degree of G_λ and show it is 1. Then we calculate the index of the possible solutions for small λ and also study the structure of the bifurcation of nematic phases.

According to the conditions on K, we can write the Fourier expansion of $\tilde{K}$ as

$$\tilde{K}(\theta) = -\sum_{k=1}^{\infty} k_m \cos(2m\theta). \tag{3.35}$$

Note that $\tilde{K} = K - \bar{K}$. In this section we prove that the trivial solution $V = 0$ is the unique solution for the problem (3.30) for $0 < \lambda < \lambda_0$, where

$$\lambda_0 = \left(\sum_{m=1}^{\infty} |k_m|\right)^{-1}. \tag{3.36}$$

Remark 3.6. This is the direct generalization of proposition 3.1 b) of [52] to the infinite dimensional case.

Lemma 3.7. *The operator* $\Gamma: H \to H$ *is compact continuous.*

Proof. According to the compact embedding $H^1([0,2\pi]) \hookrightarrow C([0,2\pi])$, if $V_n \xrightarrow{H^1} V$ then there exist $M > 0$ such that $\|V_n\|_\infty \leq M$ and $V_n(\theta) \to V(\theta)$ point-wise. Therefore $e^{-V_n(\theta)} \leq e^M$. Denote

$$d\mu_{V_n}(\theta) = \frac{e^{-V_n(\theta)}\, d\theta}{\int_0^{2\pi} e^{-V_n(\theta)} d\theta},$$

and obtain

$$\|\Gamma(V_n) - \Gamma(V)\|_{L^2} = \left\| \int_0^{2\pi} \tilde{K}(\theta, \theta')(d\mu_{V_n}(\theta') - d\mu_V(\theta')) \right\|_{L^2} \leq \sqrt{2\pi} \|\tilde{K}\|_\infty \int_0^{2\pi} |d\mu_{V_n} - d\mu_V|.$$

But by dominant convergence we have

$$\int_0^{2\pi} e^{-V_n(\theta)} d\theta \to \int_0^{2\pi} e^{V(\theta)}\, d\theta, \tag{3.37}$$

and then we obtain

$$\int_0^{2\pi} |d\mu_{V_n} - d\mu_V| \to 0. \tag{3.38}$$

This shows that $\Gamma(V_n) \xrightarrow{L^2} \Gamma(V)$. Similarly

$$\left\| \frac{d}{d\theta}\Gamma(V_n) - \frac{d}{d\theta}\Gamma(V) \right\|_{L^2} \leq \sqrt{2\pi} \left\| \frac{d}{d\theta}\tilde{K} \right\|_\infty \int_0^{2\pi} |d\mu_{V_n} - d\mu_V| \to 0.$$

This complete the proof that Γ is continuous from H to H. Now we prove that Γ is compact. For this we use the finite rank approximation Γ_n of Γ as follows. Recall that $\Gamma: X \to X$ is compact if and only if Γ is the limit (on the bounded domains) of compact maps $\Gamma_n: X \to X$. Define

$$K_n(\theta) = -\sum_{m=1}^{n} k_m \cos(2m\theta),$$

and also

$$\Gamma_n(V)(\theta) = \frac{\int_0^{2\pi} K_n(\theta-\theta')\, e^{-V(\theta')}\, d\theta'}{\int_0^{2\pi} e^{-V(\theta)}\, d\theta}.$$

It is easily seen that $\Gamma_n(H)$ lies in the space spanned by $\{\cos(2m\theta)\}_{m=1}^n$. For any bounded set $\Omega \subset H$, we obtain

$$\begin{aligned}\|\Gamma_n(V) - \Gamma(V)\|_{H^1,\Omega} \leq \sup_{\Omega} \left\| \frac{\int_0^{2\pi} \left[\tilde{K}(\theta-\theta') - K_n(\theta-\theta')\right] e^{-V(\theta')}}{\int_0^{2\pi} e^{-V(\theta)}} \right\|_{H^1} \leq \\ \leq \sup_{\Omega} \left\| \frac{\int_0^{2\pi} \left[\tilde{K}(\theta-\theta') - K_n(\theta-\theta')\right] e^{-V(\theta')}}{\int_0^{2\pi} e^{-V(\theta)}} \right\|_{L^2} + \\ + \sup_{\Omega} \left\| \frac{\int_0^{2\pi} \left[\tilde{K}(\theta-\theta') - K_n(\theta-\theta')\right] e^{-V(\theta')} V'(\theta')}{\int_0^{2\pi} e^{-V(\theta)}} \right\|_{L^2} \leq \\ \left\|\tilde{K} - K_n\right\|_{\infty} + |\Omega|\, e^{2M} \left\|\tilde{K} - K_n\right\|_{L^2}.\end{aligned}$$

According to the Bessel identity we have $\left\|\tilde{K} - K_n\right\|_{L^2} \to 0$ and also by the assumption $\sum |k_m| < \infty$, we obtain $\left\|\tilde{K} - K_n\right\|_{\infty} \to 0$. This implies that $\|\Gamma_n(V) - \Gamma(V)\|_{H^1,\Omega} \to 0$ uniformly and then $\Gamma: H \to H$ is compact. □

So far we have established that G_λ is a Schauder map. In order to define the Leray-Schauder degree, we establish an a priori bound for the solutions.

Theorem 3.8. *Assume $K \in W^{1,\infty}([0, 2\pi])$ and let*

$$\lambda_0 = \left(\sum_{m=1}^{\infty} |k_m|\right)^{-1}.$$

Let $0 < \lambda < \lambda_0$, then $V = 0$, that is $f = \text{constant}$ is the only solution.

Proof. The proof will now be carried out as follows. First we show the existence of a bounded open set $\Omega \subset H$ such that there is no solution outside Ω. Next we show that $\deg(G_\lambda, \Omega, 0) = 1$. Finally we prove that any possible solution to $G_\lambda = 0$ for $0 < \lambda < \lambda_0$ is isolated with index 1. As in this case the degree is the sum of indices, we know that $V \equiv 0$ is the only solution.

- The existence of a bounded open set $\Omega \subset H$ such that $G_\lambda(V) = 0$ has no solution outside Ω.

  Let $R = \|K\|_{W^{1,\infty}}/\lambda_0$. Then it is easy to see that $\|\lambda\Gamma(V)\|_H \leq CR$ for all $\lambda \in (0, \lambda_0)$. Thus we can take $\Omega = B_{CR}$, the ball centered at the origin with radius CR.

- $\deg(G_\lambda, \Omega, 0) = 1$.

  Introduce the homotopy $h(t) = \mathrm{Id} - t\lambda\Gamma$ with $t \in [0, 1]$. We easily verify that $h(t)(V) = 0$ has no solution on $\partial\Omega$ for all $t \in [0, 1]$. Consequently

$$\deg(G_\lambda, \Omega, 0) = \deg(h(1), \Omega, 0) = \deg(h(0), \Omega, 0) = \deg(\mathrm{Id}, \Omega, 0) = 1.$$

- The solutions are isolated.

  For this we show that if V is a solution of $G_\lambda(V) = 0$, then $DG_\lambda(V)$ is homeomorphism, where $DG_\lambda(V)$ is the Frechet differential of G_λ at V. By standard calculation we obtain

$$D\Gamma(V)(u) = \int_0^{2\pi} \tilde{K}(\theta, \theta')\, d\mu_V(\theta') \int_0^{2\pi} u(\theta')\, d\mu_V(\theta') - \int_0^{2\pi} \tilde{K}(\theta, \theta')\, u(\theta')\, d\mu_V(\theta') \qquad (3.39)$$

  where $d\mu_V$ is defined in (3.34). Since $DG_\lambda(V) = \mathrm{Id} - \lambda D\Gamma(V)$, and $D\Gamma(V)$ is compact, $DG_\lambda(V)$ is a Fredholm operator with index 0. It is enough then to show that $\ker DG_\lambda(V)$ is trivial. Assume that $u \in \ker DG_\lambda(V)$, then

$$u(\theta) = \lambda D\Gamma(V)(u), \qquad (3.40)$$

  where $D\Gamma(V)(u)$ is given in (3.39). According to the lemma (3.4), for every $\theta \in [0, 2\pi]$ we obtain

$$|u(\theta)| \leq \lambda \|\tilde{K}\|_\infty \|u\|_\infty.$$

  Since $0 < \lambda < \lambda_0$, then $\lambda \|\tilde{K}\|_\infty < 1$ and if $u \neq 0$ then we obtain $|u(\theta)| < \|u\|_\infty$ that is impossible for $u \in H$.

- The index of any solution is 1.

We calculate the index of $DG_\lambda(V)$ for the possible solution $G_\lambda(V)=0$. Let

$$\{\phi_n(\theta)\}=\left\{\frac{1}{\sqrt{(4n^2+1)\pi}}\cos(2n\theta)\right\}_{n=1}^{\infty} \tag{3.41}$$

be a basis for H. We calculate $a_{nm}=(D\Gamma(V)(\phi_n),\phi_m)_H$. We have

$$a_{nm}=\int_0^{2\pi} D\Gamma(V)(\phi_n)(\theta)\,\phi_m(\theta)\,d\theta+\int_0^{2\pi}(D\Gamma(V)(\phi_n)(\theta))'\,\phi_m'(\theta)d\theta=$$

$$=\frac{k_m\,b_{nm}(1+4mn)}{\sqrt{1+4n^2}\,\sqrt{1+4n^2}},$$

where

$$b_{nm}=\int_0^{2\pi}\cos(2n\theta)\,d\mu_V(\theta)\int_0^{2\pi}\cos(2m\theta)\,d\mu_V(\theta)-\int_0^{2\pi}\cos(2n\theta)\cos(2m\theta)\,d\mu_V(\theta).$$

Apply lemma (3.4) and obtain $|b_{nm}|\le 1$ and then we reach

$$|a_{nm}|\le|k_m|\frac{1+4mn}{\sqrt{1+4n^2}\,\sqrt{1+4n^2}}\le|k_m|. \tag{3.42}$$

This implies that the eigenvalues of the $DG_\lambda(V)$ are all bounded below by a positive constant. In fact if u is an eigenvector for $DG_\lambda(V)$ with the eigenvalue α, that is $(1-\alpha)\,u=\lambda D\Gamma(V)(u)$, then

$$|1-\alpha||(u,\phi_n)|\le\lambda\,|k_n|\,\|u\|_\infty,$$

and then summing up on n gives

$$|1-\alpha|\,\|u\|_\infty\le\lambda\left\|\tilde{K}\right\|_\infty\|u\|_\infty<\|u\|_\infty.$$

Consequently the index of the solution V of the equation $G_\lambda(V)=0$ is 1 for $0<\lambda<\lambda_0$.

□

Remark 3.9. For the Onsager kernel $K(\theta)=|\sin(\theta)|$ we see that $k_m=\frac{4}{\pi(4m^2-1)}$ and then the above theorem gives $\lambda_0=\frac{\pi}{2}$.

3.3.2 Bifurcation of nematic phases

3.3.2.1 Bifurcation analysis

Changes in the index of the trivial solution $\bar{u}=0$ gives the bifurcation of nematic solutions for the equation. Let $\Gamma: X \to X$ be a continuous compact map, $\Gamma(0)=0$ and the trivial solution $\bar{u}\equiv 0$ of $A_\lambda(u) := u - \lambda\Gamma(u)$ is isolated for $0<|\lambda-\lambda_0|<\delta$, for some $\delta>0$ such that

$$\operatorname{ind}(\bar{u},\lambda)\operatorname{ind}(\bar{u},\mu)<0,$$

for $\lambda<\lambda_0<\mu$. It is clear by Leray-Schauder degree that λ_0 is a bifurcation point of non-trivial solutions. If Γ is linear bounded compact and $\ker(\mathrm{Id}-\lambda_0 L)=1$, then exactly two solutions bifurcate at λ_0. The following theorem generalizes this fact and is standard in bifurcation theory, see e.g. [56], [70].

Theorem 3.10. *Assume $G(\lambda,u)\colon \mathbb{R}\times X\to X$ is compact mapping of the form*

$$G(\lambda,u)=\lambda L+N(\lambda,u),$$

where L is a continuous compact self-adjoint linear map and $N(\lambda,u)=O(\|u\|^2)$ uniformly in a compact interval of λ. In addition assume that L has a simple eigenvalue at λ_0, that is $\dim\ker(\mathrm{Id}-\lambda_0 L)=1$. If trivial solution $\bar{u}$ of $T_\lambda(u)=u-\lambda L(u)=0$ is isolated for $0<|\lambda-\lambda_0|<\delta$ for some $\delta>0$ and

$$\operatorname{ind}(\bar{u},\lambda)\operatorname{ind}(\bar{u},\mu)<0, \qquad \lambda<\lambda_0<\mu,$$

then there exist two non-trivial solutions bifurcating from $\bar{u}$ at λ_0.

The proof of the theorem is based on the fact that $L_\lambda=\mathrm{Id}-\lambda L$ is Fredholm with index zero. Note that L is a completely continuous operator. Since λ_0 is simple eigenvalue of L then $\operatorname{codim}(L_{\lambda_0})=1$. The standard implicit function theorem then gives two solutions of the equation $u=G(\lambda,u)$ of the form $u(\alpha,\lambda(\alpha))$ for positive and negative α where $\alpha=(\phi_0^*,u)$ and ϕ_0^* is the annihilator of the $\operatorname{rang}(L_{\lambda_0})$.

For further bifurcation analysis, we assume that $k_n, n\geq 1$, the Fourier coefficients of $\tilde{K}$ in (3.35) are positive and uniformly decreasing, that is

$$k_1>k_2>k_3>\cdots \tag{3.43}$$

In fact, according to the remark (3.9), the Onsager kernel $K(\theta) = |\sin(\theta)|$ satisfies the above condition. Simple calculations gives

$$(D\Gamma(0)(\phi_n), \phi_m)_H = \begin{cases} 0 & n \neq m \\ \frac{k_n}{2} & n = m \end{cases}.$$

By assumption (3.43), we conclude that

$$\mathrm{ind}(\bar{u}, \lambda) = \begin{cases} 1 & 0 < \lambda < 2k_1^{-1} \\ -1 & 2k_1^{-1} < \lambda < 2k_2^{-1} \end{cases}$$

Similar argument holds for all k_n and we obtain that for $\lambda_n = 2k_n^{-1}$ the index is

$$\mathrm{ind}(\bar{u}, \lambda) = (-1)^n, \quad \lambda_n < \lambda < \lambda_{n+1}. \tag{3.44}$$

We have the following theorem.

Theorem 3.11. *Under the above assumptions, there is a sequence of nematic solutions bifurcating from the trivial solution $\bar{u}$ for the system* (3.30).

Proof. Finite rank approximation of $D\Gamma(0)$ gives critical points $\lambda_n = 2k_n^{-1}$ for $n = 1$, 2, ... At each λ_n, the sign of $\mathrm{Id} - \lambda D\Gamma(0)$ changes from $(-1)^{n-1}$ to $(-1)^n$ when λ passes through λ_n. Since $\bar{u}$, the trivial solution, is isolated for non-critical values of λ that is shown before, we can assign an index to $\bar{u}$ for those non-critical values of λ. In fact for $\mu > \lambda_n$ and $\lambda < \lambda_n$ we have

$$\mathrm{ind}(\bar{u}, \lambda)\, \mathrm{ind}(\bar{u}, \mu) = -1.$$

According to the theorem (3.10), it implies that λ_n are bifurcation points for the isotropic solution. On the other hand, the linearization of $\mathrm{Id} - G$ at $\bar{u} = 0$ gives:

$$L(V)(\theta) = \mathrm{Id} + \frac{\lambda}{2\pi} \int_0^{2\pi} \tilde{K}(\theta, \theta') V(\theta')\, d\theta'. \tag{3.45}$$

$\tilde{K}$ is symmetric and $L - \mathrm{Id}$ is a compact self-adjoint operator and therefore the algebraic multiplicity of its eigenvalues equals its geometrical multiplicity. Obviously $\lambda_n = \frac{2}{k_n}$ is a simple eigenvalue for $L - \mathrm{Id}$ in (3.45) with the eigenfunction $V_n(\theta) = \cos(2n\theta)$. This implies the existence of exactly two nematic solutions that bifurcate from the trivial solution $\bar{u} = 0$ at $\lambda_n = \frac{2}{k_n}$ for $n \geq 1$. □

Now we prove that the system (3.30) under the condition (3.43) has periodic solution with arbitrary period for λ sufficiently large.

Theorem 3.12. *Assume that K enjoys the property* (3.43)*, then for any n, system* (3.30) *has $\frac{\pi}{n}$-periodic solution if $\lambda > \lambda_n$.*

Proof. Instead of the space H in (3.31) consider the space H_n as

$$H_n = \left\{ V(\theta) \in H^1([0, 2\pi]); V(\theta) = V\left(\theta + \frac{\pi}{n}\right) a.e.; \int_0^{2\pi} V(\theta) = 0 \right\}. \tag{3.46}$$

This is the space of $\frac{\pi}{n}$-periodic functions with a basis

$$\{\psi_m^n(\theta)\} = \left\{ \frac{1}{\sqrt{(4n^2m^2+1)\pi}} \cos(2nm\theta) \right\}_{m=1}^{\infty}.$$

It is easy to see that $\Gamma: H_n \to H_n$. In fact we have

$$\Gamma(V)(\theta) = -\beta^{-1} \sum_{m=1}^{\infty} k_m \cos(2m\theta) \int_0^{2\pi} \cos(2m\theta')\, e^{-V(\theta')}\, d\theta'.$$

Since $V(\theta) \in H_n$ then for appropriate v_k^n we can write

$$V(\theta') = \sum_{k=1}^{\infty} \frac{v_k^n}{\sqrt{(4n^2k^2+1)\pi}} \cos(2nk\theta').$$

This implies that $\Gamma(V)(\theta)$ is expandable in terms of ψ_m^n. Repeating calculation in H_n shows that the first bifurcation of the nematic solution $v_1(\theta)$ happens for $\lambda = \lambda_n$. On the other hand, the nth bifurcation solution $u_n(\theta)$ in H occurs at $\lambda = \lambda_n$ and since there are only two nematic solutions bifurcating at λ_n we conclude that $v_1(\theta) = u_n(\theta)$. This establishes that the bifurcation solutions in H at λ_n belong to H_n and therefore $\frac{\pi}{n}$-periodic. □

3.3.2.2 Stability analysis

In addition to the above bifurcation argument, the following result is easily proved by the Leray-Schauder degree, see [70] and [71].

Theorem 3.13. *Under the setting of the Theorem 3.10, assume that N is a twice Frechet differentiable operator with respect to λ, u and in addition for the first eigenvalue λ_1 it satisfies the property* $\operatorname{ind}(\bar{u}, \lambda) = 1$ *for* $\lambda < \lambda_1$ *and* $\operatorname{ind}(\bar{u}, \lambda) = -1$ *for* $\lambda > \lambda_1$, *then the bifurcating solution for* $\lambda > \lambda_1$ *is stable and for* $\lambda < \lambda_1$ *is unstable.*

Based on theorem (3.13), we can determine the stability of the obtained nematic solutions in the previous section according to their topological index. If λ_1 is the first bifurcation point, we need an estimate for the value $\tau = \lambda - \lambda_1$. In order to obtain and estimate for τ, we follow the standard Lyapunov-Schmidt process, see for example [70]. For the mapping

$$F(u) = u - \lambda L(u) + N(\lambda, u)$$

assume that λ_1 is a simple eigenvalue of L and then the kernel of L_{λ_1} is a one dimensional subspace of X. Let $u_1 \in \ker(T_{\lambda_1})$ and X_1 be the topological complement of $\{u_1\}$ in X, that is $X = \{u_1\} \oplus X_1$. Since L_{λ_1} is Fredholm with index zero then $\operatorname{codim}(L_{\lambda_1}) = 1$. Let Y_1 denote the range of L_{λ_1} and $X = M \oplus Y_1$ where M is a one dimensional subspace of X. It is easy to see that $u_1 \notin Y_1$, otherwise there exists $v \in X$ such that $u_1 = L_{\lambda_1}(v)$ and then $L^2_{\lambda_1}(v) = 0$ contradicts the fact that λ_1 is simple eigenvalue for L_λ. This implies that X_1 and Y_1 are isomorphic. Let $\phi_1 \in X^*$ be the annihilator of Y_1 such that $\langle \phi_1, u_1 \rangle = 1$. ϕ_1 exists since $u_1 \notin Y_1$. Define the projection P into $\{u_1\}$ as $P(u) = \langle \phi_1, u \rangle\, u_1$. Write the nonlinear map $F(u)$ in terms of $\tau = \lambda - \lambda_1$ as

$$F_\tau(u) = L_{\lambda_1}(u) - \tau L(u) + N(\lambda, u). \tag{3.47}$$

The equivalent system for the equation $F_\tau(u) = 0$ is $P(F_\tau(u)) = 0$ and $Q(F_\tau(u)) = 0$, for $Q = \mathrm{Id} - P$. Since $\langle \phi_1, L_{\lambda_1}(u) \rangle = 0$, (note that $L_{\lambda_1}(u) \in Y_1$) we can write

$$P(F_\tau(u)) = -\tau\, \langle \phi_1, L(u) \rangle u_1 + \langle \phi_1, N(\lambda, u) \rangle = 0. \tag{3.48}$$

But the relation $\langle \phi_1, L_{\lambda_1}(u) \rangle = 0$ gives $\langle \phi_1, L(u) \rangle = \left\langle \phi_1, \frac{1}{\lambda_1} u \right\rangle$. Let $\alpha = \langle \phi_1, u \rangle$, then we can write

$$\frac{-\alpha \tau}{\lambda_1} + \langle \phi_1, N(\lambda, u) \rangle = 0. \tag{3.49}$$

If $N(\lambda, \alpha v) = \alpha^2 N_1(\lambda, v, \alpha)$ then the above equation for $u = \alpha v$ can be written as

$$\tau = \alpha\lambda_1 \langle \phi_1, N_1(\lambda, v, \alpha) \rangle. \tag{3.50}$$

Now we apply the above results to the problem defined in this chapter. The operator $F = \mathrm{Id} - \lambda\Gamma$ where Γ is defined in (3.30) can be rewritten as

$$F_\tau = \mathrm{Id} - \lambda_1 L - \tau L + \lambda N,$$

where $\lambda_1 = 2k_1^{-1}$, $\tau = \lambda - \lambda_1$,

$$L(u) = -\frac{1}{2\pi} \int_0^{2\pi} \tilde{K}(\theta, \theta')\, u(\theta'),$$

and

$$N(u) = L(u) - \Gamma(u).$$

Since L is self-adjoint and λ_1 is a simple eigenvalue of T,

$$\phi_1(\theta) = \mathrm{rang}(\mathrm{Id} - \lambda_1 L)^\perp = \frac{1}{\sqrt{5\pi}} \cos(2\theta),$$

where the constant $\frac{1}{\sqrt{5\pi}}$ is to normalize the norm $(\phi_1, \phi_1) = 1$. It is easily verified that $N(\alpha u) = \alpha^2 N_1(u; \alpha)$, in fact

$$N(u) \sim -\frac{1}{4\pi} \int_0^{2\pi} \tilde{K}(\theta, \theta')\, u^2(\theta') + \frac{1}{12\pi} \int_0^{2\pi} \tilde{K}(\theta, \theta')\, u^3(\theta').$$

According to the relation (3.50) we can write

$$-\frac{\alpha\tau k_1}{2} - \frac{\lambda}{4\pi} \left(\int_0^{2\pi} \tilde{K}(\theta, \theta')\, u^2(\theta'), \phi_0 \right) + \frac{\lambda}{12\pi} \left(\int_0^{2\pi} \tilde{K}(\theta, \theta')\, u^3(\theta'), \phi_0 \right) = 0.$$

Assuming $u \sim \alpha\phi_1$ and $\lambda = \tau + \lambda_1$, the sign of τ determines the stability or non-stability of the first bifurcation solution. The exact analysis with this approach is done by the author in a submitted paper. Here we follow another approach to determine the stability of the first bifurcation solution. The result completely confirms the previous one, however we are still working to give a satisfactory justification for this method. Ignoring the contribution of $\phi_1.\phi_2$ we obtain τ as:

$$\tau = -\frac{2\alpha^2}{k_1} \frac{1}{8 + \alpha^2} \sim -\frac{\alpha^2}{4k_1}.$$

On the other hand we have

$$D\Gamma(V)(\cos(2n\theta)) = -\beta^{-1}\int_0^{2\pi}\tilde{K}e^{-V(\theta')}\cos(2n\theta') + \beta^{-2}\int_0^{2\pi}\tilde{K}e^{-V(\theta')}\int_0^{2\pi}e^{-V(\theta)}\cos(2n\theta).$$

We assume that $V\neq 0$ but sufficiently small. We can assume $e^{-V(\theta)}\sim 1-V(\theta)$ and since $V(\theta)=\alpha\phi_0(\theta)+\alpha\phi(\theta,\alpha)$ where $\phi(\theta,\alpha)\to 0$ for $\alpha\to 0$ then we can assume $V\sim\alpha\phi_0$. Simple calculation then gives

$$DG(V) = \begin{pmatrix} 1-\frac{\lambda k_1(8-\alpha^2)}{16} & \frac{\lambda k_2\,\alpha}{4} & 0 & 0 & \cdots \\ \frac{\lambda k_1\alpha}{4} & 1-\frac{\lambda k_2(8-\alpha^2)}{16} & \frac{\lambda k_3\,\alpha}{4} & 0 & \cdots \\ 0 & \frac{\lambda k_2\,\alpha}{4} & 1-\frac{\lambda k_3(8-\alpha^2)}{16} & \frac{\lambda k_4\,\alpha}{4} & \cdots \\ 0 & 0 & \frac{\lambda k_3\,\alpha}{4} & 1-\frac{\lambda k_4(8-\alpha^2)}{16} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

The sign of $DG(V)$ for V sufficiently small is the index of the solution V. Substitute $\lambda=\tau+\lambda_1$ in $DG(V)$ and then the diagonal is dominant. The first diagonal entry is $1-\frac{\lambda k_1}{2}=\frac{\alpha^2}{8}$ and all other diagonal entries are positive. The determinant of the finite approximation of $DG(V)$ has dominant terms in α^2 as

$$\gamma = \frac{\alpha^2}{4}\prod_{n=2}^{N}\left(1-\frac{\lambda k_n(8-\alpha^2)}{16}\right) - \frac{\lambda^2 k_1 k_2\,\alpha^2}{16}\prod_{n=3}^{N}\left(1-\frac{\lambda k_n(8-\alpha^2)}{16}\right).$$

Assuming $\lambda=\lambda_1$ gives

$$\gamma = \left(1-\frac{2k_2}{k_1}\right)\frac{\alpha^2}{4}\prod_{n=3}^{N}\left(1-\frac{\lambda k_n(8-\alpha^2)}{16}\right).$$

Therefore if we have

$$\left(1-\frac{2k_2}{k_1}\right)>0,$$

then $\gamma>0$ and then sign $DG(V)=+1$. This also shows that the sub or super criticallity depends on the sign of γ that is the same as the sign of $1-2k_2/k_1$. For example for the original Onsager kernel $K(\theta)=|\sin(\theta)|$ we have

$$\left(1-\frac{2k_2}{k_1}\right)=\frac{1}{3}>0,$$

and then in this case $\text{sign}(DG(V)) = +1$ for the first bifurcation solution. This proves that the first nematic solution are stable super-critical. It is possible to repeat this process to obtain the sign of other bifurcating solutions. For example for the bifurcation solutions at λ_2 we can define the operator Γ in the space H_2 and repeat the above process. Similar calculation shows that the bifurcation solutions are super-critical if $1 - \frac{2k_4}{k_2} > 0$, that is true for the Onsager kernel. In general the bifurcation solutions at λ_n are super-critical if $k_n > 2k_{2n}$ and sub-critical of $k_n < 2k_{2n}$. For Onsager kernel all bifurcation solutions at $\lambda_n, n = 1, 2, \ldots$ are super-critical.

The above argument proves the following theorem.

Theorem 3.14. *For problem* (3.30) *with Onsager kernel* $\tilde{K}(\theta) = |\sin\,(\theta)|$*, the bifurcating solution at* $\lambda_n = 2k_n^{-1}$ *are super-critical. Moreover the bifurcation at* λ_1 is stable.

3.4 Higher dimensional Doi-Onsager equation

Here we prove a sufficient condition for the existence of nematic solution for the Onsager equation. We also give an upper bound for λ the guarantees the uniqueness of the trivial solution. Our method is based on a generalized version of the Leray-Schauder degree. In this section we apply our method for the case $D = 2$ to an arbitrary dimension $D \geq 3$ for the following problem

$$u - \lambda G(u) = 0, \tag{3.51}$$

where

$$G(u) = \beta(u)^{-1} \int_{S^{D-1}} \hat{K}(r, r')\, e^{-u(r')}\, d\sigma_D(r'), \tag{3.52}$$

where $\hat{K} = K - \bar{K}$ and β is

$$\beta(u) = \int_{S^{D-1}} e^{-u(r)}\, d\sigma_D(r). \tag{3.53}$$

Here K enjoys the properties

$$K(-r, r') = K(r, -r') = K(O(r), O(r')), \tag{3.54}$$

where O is any rotation map in $\mathbb{R}^D$.

3.4.1 Spherical harmonic

Recall that the spherical harmonics on S^D are the natural generalization of the harmonics on S^2. For the unit mass located at the distance $r = \frac{1}{\rho} > 1$, expansion of the potential V

$$V = \frac{1}{(1+\rho^2 - 2\rho\cos\gamma)^{\frac{D-2}{2}}}, \tag{3.55}$$

in terms of ρ gives

$$V = \sum_{n=0}^{\infty} P_n(D, \cos\gamma)\rho^n, \tag{3.56}$$

where $P_n(D, \cos\gamma)$ are the Gegenbauer polynomial or Legendre polynomial on S^{D-1}. In general, Q_n, a harmonic polynomial of order n in $\mathbb{R}^D$ is defined as the homogeneous polynomial of order n that satisfies the Laplace equation $\Delta Q_n = 0$. It turns out that there exist exactly $N(D, n)$ such polynomials for each n, see [22]

$$N(D, n) = \frac{(2n + D - 2)\,(n + D - 3)!}{(D-2)!\,n!}. \tag{3.57}$$

The restricting Q_n to S^{D-1} are called spherical harmonics of order n and are denoted by $S_{nj}(D, r)$ for $r \in S^{D-1}$ and $1 \leq j \leq N(D, n)$. $P_n(D, t)$ for the variable $t \in [-1, 1]$ are spherical harmonics that are invariant under the rotation of $S^{D-2} \subset S^{D-1}$.

According to the assumption (3.54), we consider the following expansion for K:

$$K(\gamma) = -\sum_{n=1}^{\infty} k_n P_{2n}(D, \cos\gamma) + k_0, \tag{3.58}$$

where γ is the angle between r, r' and furthermore $k_n, n \geq 1$ are positive and form a decreasing sequence, that is for $n \geq 1$

$$k_n > 0, \qquad k_n > k_{n+1} \tag{3.59}$$

Proposition 3.15. *The Onsager kernel*

$$K(r, r') = |r \times r'| = \sin\gamma, \qquad 0 \leq \gamma \leq \pi \tag{3.60}$$

has the expansion (3.58) *with* k_n *satisfying* (3.59).

Proof. The set $P_n(D,\cos\gamma), n\geq 0$ forms a complete system for smooth functions on S^{D-1} that are defined only in terms of γ. Since the Onsager kernel (3.60) is even with respect to γ, the coefficients of the odd terms in the expansion are zero, that is the Onsager kernel has the expansion of the form (3.58). Clearly $k_0>0$ and in order to show that k_n enjoy (3.59) we start with $D=3$. The coefficients k_n for $n\geq 1$ are obtained as

$$k_n=-\frac{(2n+1)}{2}\int_{-1}^{1} P_{2n}(t)\,(1-t^2)^{\frac{1}{2}}\,dt. \tag{3.61}$$

Use the identity:

$$\int_{-1}^{1}\sqrt{1-x^2}\,P_{2n+2}=\frac{4n^2-1}{4(n+1)(n+2)}\int_{-1}^{1}\sqrt{1-x^2}\,P_{2n}, \tag{3.62}$$

to obtain the following recursive formula for $k_n, n>1$:

$$k_{n+1}=\frac{(4n^2-1)(4n+5)}{4(n+1)(n+2)(4n+1)}\,k_n. \tag{3.63}$$

The direct computation shows that $k_1>0$ and then all $k_n>0$ for $n\geq 1$ and they form a decreasing sequence. In general the coefficients k_n are obtained as

$$k_n=-\frac{\sigma_{D-1}\,N(D,n)}{\sigma_D}\int_{-1}^{1} P_{2n}(D,t)\,(1-t^2)^{\frac{D-2}{2}}\,dt. \tag{3.64}$$

Now by the formula

$$P_n(D,t)=C(n,D)\,\frac{d}{dt}\,P_{n+1}(D-2,t), \tag{3.65}$$

one can convert the integral to the case $D=3$. □

3.4.2 Main Result

In sequel, we always assume $D\geq 3$. Recall that we want to find solutions of the equation

$$u-\lambda G(u)=0, \tag{3.66}$$

where G is defined in (3.52). Let $H_0(S^{D-1})$ denotes the space:

$$H_0(S^{D-1}) = \left\{ u \in L^2(S^{D-1}), u(-r) = u(r), \int_{S^{D-1}} u(r)\, d\sigma_D(r) = 0 \right\}.$$

We look for the solutions of (3.66) in H_0. The existence and bifurcation of the solution can be established by the aid of Leray-Schauder degree if G is a completely continuous operator on H_0. However, it is seen that G is not continuous on $H_0(S^{D-1})$. Consider, for example the sequence of functions $u_n(\theta)$ for the case $D=3$ (θ is the angle with the z-axis):

$$u_n(\theta) = \begin{cases} \log\left(2\pi(1-\cos(1/n))\right) & \theta \in \left(0, \frac{1}{n}\right) \\ 0 & \text{otherwise} \end{cases}. \tag{3.67}$$

Obviously $u_n \xrightarrow{L^2(S^2)} 0$, and $G(\lambda, 0) = 0$, however we have

$$G(\lambda, u_n) = \frac{\lambda}{2\pi(1-\cos(1/n))} \int_0^{1/n} \int_0^{2\pi} \hat{K}(\gamma)\, d\sigma_2(r') \nrightarrow 0.$$

We have the following theorem.

Theorem 3.16. *Assume that K belong to the Holder class of maps, then for $\Omega = \{|u(r)| \leq \lambda\ ||K||_\infty\}$, the map G is a continuous compact operator on $H_\Omega = H_0(S^{D-1}) \cap \Omega$.*

Proof. First note that the fixed point set of G has the a priori bound

$$|u(r)| \leq \lambda \|K\|_\infty\, \beta(u)^{-1} \int_{S^{D-1}} e^{-u(r)}\, d\sigma_D(r') = \lambda \|K\|_\infty. \tag{3.68}$$

The continuity of G on H_Ω is easily proved by the dominant convergence theorem. In order to show that G is compact on H_Ω, define the truncated sequence of kernels $\hat{K}_N$ as

$$\hat{K}_N(\gamma) = -\sum_{n=1}^{N} k_n\, P_{2n}(D, \cos\gamma), \tag{3.69}$$

and the finite range operators G_N on H_Ω as

$$G_N(u)(r) = \lambda\beta(u)^{-1}\int_{S^{D-1}} \hat{K}_N(\gamma)\, e^{-u(r')}\, d\sigma_D(r'). \tag{3.70}$$

It is seen that for $u \in H_\Omega$ we have

$$\|G(u) - G_N(u)\|^2 = \lambda^2\, \beta(u)^{-2}\int_{S^{D-1}}\left(\int_{S^{D-1}}(\hat{K}(\gamma) - \hat{K}_N(\gamma))\, e^{-u(r')}\, d\sigma_D(r')\right)^2 d\,\sigma_D(r) \leq$$

$$\leq \max_\gamma \left|\hat{K}(\gamma) - \hat{K}_N(\gamma)\right| 2\,\pi\,\lambda^2 \to 0, (N \to \infty).$$

Therefore G is the uniform limit of a sequence of finite range operators on H_Ω and then is compact on H_Ω. □

3.4.2.1 Generalized degree

Since G is not continuous on $H_0(S^{D-1})$, we generalize the Leray-Schauder degree such that it fit the situation presented in Theorem (3.16). Let H be a separable Hilbert space with an orthonormal basis $\mathcal{H} = \{u^1, u^2, \cdots\}$, $H^n := \text{span}\{u^1, \ldots, u^n\}$ for $n \geq 1$ and $G: H \to H$ a map. The inner products in H and H_n are denoted by $(,)$, $((,))$ respectively. The finite rank approximation of G on H^n is defined as

$$G^n(u) = \sum_{k=1}^{n} (G(x), u^k)\, u^k. \tag{3.71}$$

It is clearly seen that G^n coincides with G on H^n, that is for all $v \in H^n$, we have $(G(u), v) = ((G^n(u), v))$. Assume that there exist a closed bounded set Ω such that:

- $\{z; z - G(z) = 0\} \subset \alpha\Omega, \quad \alpha < 1,$
- G is continuous and compact on $H_\Omega := H \cap \Omega$,
- $H_\Omega^n{:} = H^n \cap \Omega$ has non-empty interior in H^n.

For an open bounded subset $D \subset H$, we denote $D^n := D \cap H_n$, $D_\Omega := D \cap H_\Omega$ and $D_\Omega^n := D_\Omega \cap H_n$.

Proposition 3.17. *Assume D is an open bounded subset of H such that G has no fixed point on ∂D, then there exist $N_0 > 0$ such that for $n \geq N_0$, G_Ω^n has no fixed point on ∂D_Ω^n.*

Proof. Otherwise, there exist $z_n \in \partial D_\Omega^n$ such that $z_n = G^n(z_n)$. Since $(z_n) \subset \Omega$ and G is completely continuous on H_Ω, the sequence $G(z_n)$ converges (in a subsequence that we do not relabel) to some point $z \in H$. Since $G^n(z_n) = \mathrm{Pr}_{H^n} G(z_n)$, we conclude that $G^n(z_n)$ converges to z, that implies in turn the convergence of z_n to $z \in \partial D$. Since Ω is closed, $z \in \Omega$ and therefore $G(z_n)$ converges to $G(z)$. Therefore we conclude the contradiction $z = G(z)$ for some $z \in \partial D$. □

Let use denote the operator $\Gamma\colon H \to H$ as the G perturbation of the identity, that is $\Gamma(u) = u - G(u)$. The finite rank approximation Γ^n of Γ into H^n is defined accordingly.

Proposition 3.18. *Assume D is an open bounded subset of H such that $z \neq G(z)$ for $z \in \partial D$, then there exist $N_0 > 0$ such that for $n \geq N_0$ we have*

$$\deg\left(\Gamma^n, D_\Omega^n, 0\right) = \deg\left(\Gamma^{n+1}, D_\Omega^{n+1}, 0\right). \tag{3.72}$$

Proof. Note that the interior of D_Ω^n is non-empty in H^n. Since G is continuous on H_Ω and G^n has no fixed point on ∂D_Ω^n for $n \geq N_0$, the degree of Γ^n on D_Ω^n can be defined in the usual sense. Define the map A^{n+1} as

$$A^{n+1}(x) = (\Gamma^n(x), (x, u^{n+1}) u^{n+1}). \tag{3.73}$$

Obviously we have

$$\deg\left(A^{n+1}, D_\Omega^{n+1}, 0\right) = \deg\left(\Gamma^n, D_\Omega^{n+1}, 0\right). \tag{3.74}$$

Consider the homotopy $h\colon [0,1] \times D_\Omega^{n+1} \to H^{n+1}$ as

$$h(t)(x) = (1-t)\,\Gamma^{n+1}(x) + t\,A^{n+1}(x). \tag{3.75}$$

It is easily seen that $0 \notin h(t)(z)$ for $z \in \partial D_\Omega^{n+1}$ for $t = 0, 1$. If $\deg\left(\Gamma^n, D_\Omega^n, 0\right)$ is not stable, then there exist a sub-sequence (t_{nk}) that we still denote as (t_n), such that $t_n \in (0,1)$ and $z_n \in \partial D_\Omega^n$ and $h(t_n)(z_n) = 0$. Therefore

$$\Gamma^n(z_n) + t_n\,(G(z_n), u^n)\,u^n = 0.$$

Since $\{z_n\} \subset \Omega$ is bounded and G is compact on H_Ω then $(G(z_n), u^n) \to 0$ and then $\Gamma^n(z_n) \to 0$ that implies $z_n \to z \in \partial D$ and $\Gamma(z) = 0$, a contradiction! □

By the aid of Proposition (3.18), we can define the degree of the map Γ on an open bounded subset D with respect to 0 as follows.

Definition 3.19. *Under the above settings, for any open subset $D \subset H$ such that $z \neq G(z)$ for $z \in \partial D$, define*

$$\deg(\Gamma, D, 0) = \lim_{n \to \infty} \deg(\Gamma^n, D_\Omega^n, 0). \tag{3.76}$$

Proposition 3.20. *If $\deg(\Gamma, D, 0) \neq 0$ then there exist at least one fixed point for the map G in D.*

Proof. By (3.76) it implies the existence of a sequence $(u_n) \subset D_\Omega^n$ such that $u_n = G^n(u_n)$. Since G is compact on H_Ω then the sequence $(G(u_n))$ converges (in a subsequence) to some point $u \in H$. This implies that $G^n(u_n) = \mathrm{Pr}_{H^n} G(u_n)$ converges to u and then $u_n \to u$. Since Ω is closed and G is continuous on H_Ω then $G(u_n) \to G(u)$. $\square$

The homotopy invariance property is established by the following proposition.

Definition 3.21. *Let $G\colon [0,1] \times H_\Omega \to H$ be a continuous map. The map G is called a compact transformation if $G([0,1])$ is compact from H_Ω to H, and in addition for every bounded $\Omega' \subset \Omega$ and $\varepsilon > 0$ there exist $\delta = \delta(\varepsilon, \Omega')$ such that*

$$|t - s| < \delta \Rightarrow \|G(t)(x) - G(s)(x)\| < \varepsilon, \quad x \in \Omega'.$$

Proposition 3.22. *The definition (3.76) satisfies the homotopy invariance property for the class of compact transformations.*

Proof. Let Γ denotes the time dependent operator: $\Gamma\colon [0,1] \times D \to H$. There exist N_0 such that $0 \notin \Gamma^n(t)(\partial D_\Omega^n)$ for all $n \geq N_0$. Assume the contrary, then there exist a sequence t_n and $z_n \in \partial D_\Omega^n$ such that $\Gamma^n(t_n)(z_n) = 0$. Let $\{t_n\}$ converges to $\bar{t}$ in a sub-sequence, since $G(\bar{t})\colon H_\Omega \to H$ is compact then $G(\bar{t})\{z_n\}$ converges in a subsequence to some point say z. Thus we can write

$$\begin{aligned}\|z_n - z\| = \|G(t_n)(z_n) - G(\bar{t})(z_n) + G(\bar{t})(z_n) - z\| \leq \\ \leq \|G(t_n)(z_n) - G(\bar{t})(z_n)\| + \|G(\bar{t})(z_n) - z\| = \\ = \|G(t_n)(z_n) - G(\bar{t})(z_n)\| + o(1).\end{aligned}$$

On the other hand, since G is a compact transformation then

$$\|G(t_n)(z_n) - G(\bar{t})(z_n)\| = o(1),$$

that gives $z_n \to z$ in a sub-sequence. Since ∂D is closed then $z \in \partial D$ a contradiction. This implies that $\deg(\Gamma^n(t), D_n, 0)$ is independent of t. On the other hand

$$\deg(\Gamma^n(t), D_\Omega^n, 0) = \deg\left(A^{n+1}(t), D_\Omega^{n+1}, 0\right), \tag{3.77}$$

where $A^{n+1}(t)(u) = (\Gamma^n(t)(u), (z, u^{n+1})u^{n+1})$. Choose n sufficiently large such that $\deg(\Gamma^n(t), D_\Omega^n, 0)$ is constant with respect to t. If $\deg(\Gamma^n(t), D_\Omega^n, 0)$ is not stable then there exist s_n and $z_n \in \partial D_\Omega^n$ such that $(1 - s_n)\,\Gamma^n(t)(z_n) + s_n\,A^n(t)(z_n) = 0$ that give the convergence of $\{z_n\}$ in a sub-sequence to some $z \in \partial D$ such that $\Gamma(t)(z) = 0$ a contradiction! □

Now consider the one parameter class of operators $\Gamma(\lambda, u) = \mathrm{Id} - G(\lambda, u)$ where $G(\lambda, .)\colon H_\Omega \to H$ is a compact continuous map for every λ such that $\Gamma(\lambda, 0) = 0$. Further assume that there exist a completely continuous linear map $T\colon H \to H$ such that

$$\|\lambda T(u) - G(\lambda, u)\| = o(\|u\|), \qquad u \in \Omega. \tag{3.78}$$

Proposition 3.23. *If $\lambda \notin \sigma(T)$, where $\sigma(T)$ denotes the spectrum of T, then $(\lambda, 0)$ is isolated, that is there exist $\varepsilon > 0$ such that the trivial solution of* (3.66) *is the unique solution in $(\lambda - \varepsilon, \lambda + \varepsilon) \times B_\varepsilon$.*

Proof. Otherwise, there exist a sequence $\{\lambda_n\}, \lambda_n \to \lambda$ and $u_n \to 0$ such that $0 = \Gamma(\lambda_n, u_n)$. Since $\lambda \notin \sigma(T)$, then there exist $r > 0$ and $\|u - \lambda T(u)\| \geq r\,\|u\|$. Therefore

$0 \geq \|u_n - \lambda T(u_n)\| - |\lambda_n - \lambda|\,\|u_n\| - \|\lambda_n\,T(u_n) - G(\lambda_n, u_n)\| > r\,\|u_n\| - |\lambda_n - \lambda|\,\|u_n\| + o(\|u\|)$,

that is contradiction for sufficiently small $|\lambda - \lambda|$ and u_n. □

Proposition 3.24. *Assume $\bar{\lambda} \in \sigma(T)$ and for $\lambda < \bar{\lambda} < \mu$ we have*

$$\mathrm{ind}(\mathrm{Id} - \lambda T, 0)\,\mathrm{ind}(\mathrm{Id} - \mu T, 0) < 0,$$

then $\bar{\lambda}$ is a bifurcation point for (3.66).

Proof. Otherwise $(\bar{\lambda}, 0)$ is an isolated point and therefore there exist $\varepsilon > 0$ such that the trivial solution is the unique solution of $\Gamma(t)(u) = 0$ for $t \in (\bar{\lambda} - \varepsilon, \bar{\lambda} + \varepsilon)$ and $u \in B_\varepsilon$. This implies that no solution lies on $\partial B_{\varepsilon/2}$ for $t \in (\bar{\lambda} - \varepsilon, \bar{\lambda} + \varepsilon)$ and then by homotopy invariance property the index $\mathrm{ind}(\mathrm{Id} - tT, 0)$ is constant even when t passes through $\bar{\lambda}$, a contradiction. □

3.4.2.2 Uniqueness of the isotropic phase

We prove that the isotropic phase is the unique phase for system (?),(?) for sufficiently small values of λ. First the following lemma.

Lemma 3.25. *Let $L: H_0(S^{D-1}) \to H_0(S^{D-1})$ be the map*

$$L(u)(r) = \frac{-1}{\sigma_D} \int_{S^{D-1}} \hat{K}(\gamma)\, u\,(r') d\sigma_D(r'). \tag{3.79}$$

If λ is not a characteristic value for L then $\bar{u} = 0$ is an isolated solution for (3.66) .

Proof. For fixed λ, it is seen by dominant convergence theorem that for $u \in H_{2\Omega}$, we have

$$\|G(u) - L(u)\|_{L^2(S^{D-1})} = o\big(\|u\|_{L^2(S^{D-1})}\big). \tag{3.80}$$

If $\bar{u} = 0$ is not isolated, then choose sequences (λ_n, u_n) such that $u_n = G(u_n)$ and $\lambda_n \to \lambda$ and $u_n \to 0$. But

$$0 = \|u_n - G(u_n)\| \geq \|u_n - \lambda L(u_n)\| - |\lambda - \lambda_n|\, \|L\|\, \|u\| - \|G(u_n) - \lambda_n L(u_n)\|. \tag{3.81}$$

Since λ is not a characteristic value for L then there exist $k > 0$ and $\|u_n - \lambda L(u_n)\| > k\, \|u_n\|$. Take λ_n very close to λ and then

$$k\, \|u_n\| + o(\|u_n\|) < 0, \tag{3.82}$$

a contradiction. □

Theorem 3.26. *Under the above settings, there exist $\lambda_0 > 0$, such that* (3.66) *has the unique solution $\bar{u} \equiv 0$ for $0 < \lambda < \lambda_0$.*

Proof. For $R = \lambda \sqrt{\sigma_D}\, \|K\|_\infty$ the equation

$$u_t - tG\,(u_t) = 0, \tag{3.83}$$

has no solution on the R-sphere $S(R)$ for $t\in[0,1]$. In fact $||u_t||_\infty \le R/\sqrt{\sigma_D}$ and $\|u_t\|_{L^2} \le \mathrm{R}$. By homotopy invariance property of degree we conclude:

$$\deg(\mathrm{Id}-G,\mathbb{B}(r),0)=\deg(\mathrm{Id}-tG,\mathbb{B}(r),0)=\deg(\mathrm{Id},\mathbb{B}(r),0)=+1 \tag{3.84}$$

We show that the index of the trivial solution is $+1$. The function in $H_0(S^{D-1})$ can be expanded in terms of the orthonormal spherical harmonics $\{S_{2nj}(D,r)\}$ for $j=1,\dots,N(D,n)$, in fact for $u\in H_0(S^{D-1})$ we can write

$$u(r)=\sum_{n=1}^{\infty}\sum_{j=1}^{N(D,2n)} u_{nj}\,S_{2nj}(D,r), \tag{3.85}$$

for the appropriate coefficients u_{nj}. With this representation, it is convenient sometimes to write $u=u(u_{nj})$. Calculation of the entries of the Jacobian matrix of G at $u=\bar{u}$ gives entries

$$\frac{\partial}{\partial u_{nj}}G(\bar{u})=-\frac{\lambda}{\sigma_D}\int_{S^{D-1}}\hat{K}(\gamma)\,S_{2nj}(D,r')\,d\sigma_D(r')=$$
$$\frac{\lambda k_n}{\sigma_D}\int_{S^{D-1}}P_{2n}(D,\cos\gamma)\,S_{2nj}(D,r')\,d\,\sigma_D(r')=$$
$$=\frac{\lambda k_n}{N(D,2n)}\,S_{2nj}(D,r).$$

This implies that the infinite Jacobian matrix of $G(\bar{u})$ in the bases $\{S_{nj}\}$ with n even numbers is of the form

$$J_G=\mathrm{diag}\left(\frac{\lambda k_n}{N(D,2n)}\right). \tag{3.86}$$

Now if $\lambda_0\le Dk_1^{-1}$, one conclude that for $0<\lambda<\lambda_o$ we have

$$\mathrm{ind}(\bar{u},\lambda)=1. \tag{3.87}$$

In a similar way, we can calculate the index of any possible solution (not necessary the trivial) of the fixed points of G. In fact for the arbitrary solution u we have

$$\left\langle\frac{\partial}{\partial u_{nj}}G(u),S_{2ml}(D,r)\right\rangle=-\lambda\int_{S^{D-1}}\int_{S^{D-1}}\hat{K}(\gamma)f(r')\,S_{2nj}(D,r')\,\overline{S_{2ml}(D,r)}\,d\sigma_D(r')+$$
$$+\lambda\int_{S^{D-1}}f(r)\,S_{2nj}(D,r)\int_{S^{D-1}}\int_{S^{D-1}}\hat{K}(\gamma)\,f(r')\overline{S_{2ml}(D,r)}\,d\,\sigma_D(r')=$$
$$=\frac{\sigma_D\,\lambda k_m}{N(D,2m)}\int_{S^{D-1}}f(r)\,S_{2nj}(D,r)\,\overline{S_{2ml}(D,r)}-$$
$$-\frac{\sigma_D\,\lambda k_m}{N(D,2m)}\int_{S^{D-1}}f(r)\,S_{2nj}(D,r)\int_{S^{D-1}}f(r)\overline{S_{2ml}(D,r)}\,.$$

Let b_{nj}^{ml} denotes the following expression:

$$b_{nj}^{ml} = \int_{S^{D-1}} f(r)\, S_{2nj}(D,r) \overline{S_{2ml}}\,(D,r) - \int_{S^{D-1}} f(r)\, S_{2nj}(D,r) \int_{S^{D-1}} f(r) \overline{S_{2ml}(D,r)}\,.$$

An estimate, not necessary optimal, for b_{nj}^{ml} using the a priori estimate (3.68) is obtained as:

$$\left|b_{nj}^{ml}\right| \leq \frac{2\, e^{4\lambda ||K||_\infty}}{\sigma_D}. \tag{3.88}$$

This implies that

$$\frac{\partial}{\partial u_{nj}} G(u)(r) = \sum_{m=1}^{\infty} \sum_{l=1}^{N(D,2m)} a_{ml}\, S_{2ml}(D,r),$$

where a_{ml} has the following bound:

$$|a_{ml}| \leq \frac{2\, \lambda k_m\, e^{4\lambda ||K||_\infty}}{N(D,2m)}. \tag{3.89}$$

The following inequality gives a condition for that we have $\mathrm{ind}(u,\, \lambda) = 1$ for any solution of (3.66):

$$\lambda e^{4\lambda ||K||_\infty} \sum_{m=0}^{\infty} \sum_{l=1}^{N(D,2m)} \frac{k_m}{N(D,2m)} = \lambda e^{4\lambda ||K||_\infty} \sum_{m=1}^{\infty} k_m < \frac{1}{2}. \tag{3.90}$$

Using (3.90) we conclude that the index of every possible solution of (3.66) is $+1$ for $0 < \lambda < \lambda_0$ where

$$\lambda_0 = \frac{1}{5} \left\| \hat{K} \right\|_\infty^{-1}. \tag{3.91}$$

Since the degree of G is $+1$ according to (3.84), we conclude that $\bar{u} \equiv 0$ is the unique solution for $0 < \lambda < \lambda_0$. □

Remark 3.27. In the class of axially symmetric solutions, where the functions in $H_0(S^{D-1})$ are symmetric with respect to the rotation of S^{D-2} around any point of S^{D-1}, the computations are simpler. For the fixed $r \in S^{D-1}$, let θ denotes the angel between arbitrary point $r' \in S^{D-1}$ to r, then u can be expanded as

$$u(\theta) = \sum_{n=1}^{\infty} u_n\, P_{2n}(D, \cos\theta). \tag{3.92}$$

In this case G has the simpler form:

$$G(u)(\theta) = \lambda \int_0^\pi \hat{K}(\gamma) g(\theta')\, d\theta', \tag{3.93}$$

where g is defined as

$$g(\theta) = \frac{e^{-u(\theta)} \sin^{D-2}(\theta)}{\int_0^\pi e^{-u(\theta)} \sin^{D-2}(\theta)\, d\theta}. \tag{3.94}$$

The Jacobian matrix entries for the trivial solution is obtained as:

$$\frac{\partial}{\partial u_n} G(\bar{u}) = -\lambda \frac{\sigma_{D-1}}{\sigma_D} \int_0^\pi \hat{K}(\gamma)\, P_{2n}(D, \cos\theta') \sin^{D-2}(\theta') d\theta' = \frac{\lambda k_n}{N(D, 2n)} P_{2n}(D, \cos\theta). \tag{3.95}$$

The calculation for a non-trivial solutions also is carried out as

$$\left\langle \frac{\partial}{\partial u_n} G(u), P_{2m}(D, \cos\theta) \right\rangle = \lambda k_m \left\{ \int_0^\pi g(\theta)\, P_{2n}(D, \cos\theta) P_{2m}(D, \cos\theta)\, d\theta - \right.$$
$$\left. - \int_0^\pi g(\theta)\, P_{2n}(D, \cos\theta) d\theta \int_0^\pi g(\theta) P_{2m}(D, \cos\theta) d\theta \right\}.$$

Simple calculation shows that

$$\left| \left\langle \frac{\partial}{\partial u_n} G(u), P_{2m}(D, \cos\theta) \right\rangle \right| \leq \lambda k_m, \tag{3.96}$$

that gives the following estimation for λ_0

$$\lambda_0 = \left(\sum_{m=1}^{\infty} k_m \right)^{-1}. \tag{3.97}$$

Therefore $\operatorname{ind}(u, \lambda) = +1$ for $0 < \lambda < \lambda_0$, for λ_0 given in (3.97).

3.4.2.3 Bifurcation of nematic phases

According to the Proposition (3.24), we can establish the possible bifurcation of nematic phases from the trivial solution $\bar{u}$ for the Onsager model. Here we consider the subspace of axially symmetric functions of $H_0(S^{D-1})$, the functions that can be expanded as (3.92).

Theorem 3.28. *There exist a sequence of axially symmetric solution of* (3.66) *in* $H_0(S^{D-1})$ *bifurcating from the trivial solution* $\bar{u}$ *at* $\lambda_n = N(D, 2n)\, k_n^{-1}$. *The multiplicity of the bifurcating solutions at each bifurcating pint* λ_n *is exactly 2.*

Proof. λ_n are the eigenvalues of the operator L defined in (3.79). Since $H(\lambda, u) = G(u) - \lambda L(u)$ satisfies the condition (3.78) then according to the Proposition (3.23) the trivial solution is isolated for $\lambda = \lambda_n$. On the other hand, by the calculation (3.95) we conclude that for $0 < \lambda < \lambda_n < \mu$

$$\operatorname{ind}(\bar{u}, \lambda) \operatorname{ind}(\bar{u}, \mu) = -1, \tag{3.98}$$

and the according to the Proposition (3.24) each λ_n is a bifurcation point. Since L is a self-adjoint operator, the algebraic and geometric multiplicity of eigenvalues of L coincide. It is easily seen that the unique eigenfunction (up to normalization) of L at λ_n is $P_{2n}(D, \cos\theta)$ and then λ_n is a simple eigenvalue for L. According to the Theorem 4.2 in [71] we conclude that there exist exactly two solutions bifurcating at each λ_n and furthermore the first bifurcation solution at λ_1 is stable. □

Chapter 4

Periodic solutions

This chapter is dedicated to the study of the existence problem of periodic orbits for nonlinear autonomous and non-autonomous ordinary differential systems. The main tool here is classical degree theory (Brouwer and Leray-Schauder). Although using degree theoretic method in proving the existence of periodic orbits for dynamical system is not novel at all, our approach for third order autonomous nonlinear systems, (where Poincare-Bendixon theorem fails to apply) gives some new insights. The multiplicity and stability of periodic orbits are not discussed in this chapter, however when the periodic orbits are not "critical" the degree gives a lower bound for the multiplicity of the solutions.

The study of periodic solutions for dynamical systems came mainly into consideration after the pioneering works of H. Poincarè on the three body problem. Poincarè's approach to this problem relies on the application of continuation method for the Poincarè map (translation operator). Let us introduce the method briefly in terms of the fixed point theory. For the system $x' = f(t,x)$, let $x_f(t,c)$ denotes the solution with the initial condition c, that is $x_f(0,c) = c$. If f is ω-periodic with respect to t, then the existence of a periodic solution for the above system is equivalent to the existence of $c \in \mathbb{R}^n$ such that $x_f(\omega, c) = c$. In other word, c is a fixed point for the mapping $c \to x_f(\omega, c)$. The existence of such fixed point can be established by Poincarè-Bohl or Brouwer fixed point theorem. This is a classical approach that is extensively employed in this direction.

4.1 Some general results

In this section we review some general results about the existence of periodic solution for the system of first order equations. The explanation here is based on the works of Cronin [29], Halanay [43], Mawhin [57], Zanolin [81], Capietto et al [24] and Lloyd [56]. Consider the system of first order nonlinear system:

$$x' = f(t,x), \quad x \in \mathbb{R}^n, \tag{4.1}$$

where f is smooth and ω-periodic with respect to t. Let $x_f(t;c)$ denote the solution (if any) of the system (4.1) at time t with the initial condition $x(0) = c \in \mathbb{R}^n$. The necessary and sufficient condition for the existence of a ω-periodic solution for the system (4.1) is $x_f(\omega;c) = c$. Define the translation map V_t as

$$V_t(c) = x_f(t;c) - c, \tag{4.2}$$

for values of t that the solution of (4.1) exists. For small $t > 0$ we can write

$$V_t(c) = f(0,c)t + o(t). \tag{4.3}$$

Assume that $D \subset \mathbb{R}^n$ is a bounded open set such that $f(0,c) \neq 0$ for $c \in \partial D$ and that

$$\deg\left(f(0,.), D, 0\right) \neq 0. \tag{4.4}$$

Apply now the homotopy invariance property of Brouwer degree to conclude that for small values of $t > 0$ we have:

$$\deg\left(V_t, D, 0\right) \neq 0.$$

In fact we have for $t > 0$

$$\deg\left(V_t, D, 0\right) = \deg\left(f(0,.) + \frac{o(t)}{t}, D, 0 \right),$$

and for the value of t such that $\left|\frac{o(t)}{t}\right| < \mathrm{dist}(0, f(0, \partial D))$, then

$$\deg\left(f(0,.), D, 0\right) = \deg\left(V_t, D, 0\right).$$

In order to establish the existence of ω-periodic orbits, we need to pose some extra conditions. For this, we assume that the solution of the equation (4.1) is extendable on $[0,\omega]$ and also that no point of ∂D is a ω-returning point, that is $V_t(c)\neq c$ for $c\in\partial D$ and for $t\in[0,\omega]$. Now we can use the invariance property of degree to conclude that

$$\deg(V_\omega, D, 0)\neq 0.$$

The first property of Brouwer degree now guarantees the existence of solution for the equation $V_\omega(c)=0$ and therefore there exist $c\in D$ such that

$$V_\omega(c)=x_f(\omega;c)-c=0.$$

We have just proved the following general theorem.

Theorem 4.1. *Assume the solution of system* (4.1) *is extendable on* $[0,\omega]$ *and in addition there exist* $D\subset\mathbb{R}^n$ *open and bounded such that*

i. $f(0,c)\neq 0$ *for* $c\in\partial D$,

ii. $\deg(f(0,c), D, 0)\neq 0$,

iii. $x_f(t,c)\neq c$ *for* $t\in[0,\omega]$ *and* $c\in\partial D$,

then there exist at least one ω*-periodic solution for the system* (4.1).

The difficult part in the Theorem 4.1 is to find the open set D on which $f(0,c)$ satisfies the imposed conditions. The extendability of the solutions on $[0,\omega]$ can be overcome with the following trick. Let h be the cut off function equal to 1 in B_r for some $r>0$. The solution of the following auxiliary system

$$x'=f(t,x)h(x), \tag{4.5}$$

is extendable without any escape time. In fact use a trick by Lasalle [54] to define

$$V(x)=\frac{1}{2}\sum_i x_i^2,$$

and then

$$\frac{dV}{dt}=\sum_i x_i f_i(t,x)h(x)\leq V+M,$$

for some $M>0$ and use a Gronwall type inequality to conclude the claim. Now if there exist $D\subset B_r$ such that f satisfies conditions $(i),(ii),(iii)$ in the above theorem(since $h\equiv 1$ in B_r) then we conclude the existence of a periodic solution for (4.1).

Here, let us review briefly the work of J. Mawhin and his colleagues [23] [24] [81] in this direction.

Theorem 4.2. *Assume there exists $c\in\mathbb{R}$ and a C^1 function $W:\mathbb{R}^n\to\mathbb{R}$ such that*

i. c is a regular value for W,

ii. $W^{-1}(c)$ is bounded and $\Omega=\{x;W(x)<c\}$ is star shaped domain,

iii. $(\mathrm{grad}(W(z)), f(t,z))<0, \quad z\in V^{-1}(c), t\in[0,\omega]$,

then (4.1) *has a periodic solution.*

Proof. Note that $\partial\Omega=W^{-1}(c)$, since c is a regular value for W. If $V_t(c)=x_f(t;c)$ denote the solution of (4.1) then we show a fixed point for the map V_ω in Ω. Let $z\in\partial\Omega$, since

$$\frac{dW}{dt}(z(t))=(\mathrm{grad}(W(z)), f(t,z))\leq 0,$$

then W is decreasing along the solution of the system (4.1) on $\partial\Omega = W^{-1}(c)$. Therefore, the trajectories of $x_f(t;c)$ for $c\in\bar{\Omega}$ remain in $\bar{\Omega}$. Hence for any t, the map $V_t:\bar{\Omega}\to\bar{\Omega}$ has a fixed point according to the Brouwer fixed point theorem for star shaped domains. In particular for $t=\omega$, we have $V_\omega(c)=c$ and it is a periodic solution for the system (4.1). □

By simple condition on grad $W(z)$, it is possible to make sure that Ω is star-shaped, see Zanolin [81].

4.1.1 Green function and Leray-Schauder degree

Let us illustrate a simple result for the second order equations using Leray-Schauder degree. Consider the following system

$$x''+f(t,x,x')=0, \tag{4.6}$$

that f is ω-periodic and is smooth. We will prove that if the norm of $f - \mu x$ for some $\mu > 0$ in some sense is small, then equation (4.6) has a non-trivial periodic solution. The method is based on the calculation of Green's function with some periodic boundary condition.

Theorem 4.3. *Assume that there exist $c, \lambda > 0$ such that $\lambda\omega < \pi$ and $M \leq \lambda^2 c$ where*

$$M = \max_{t\in[0,\omega]} \max_{\|x\|_\infty < c} |\lambda^2 x - f(t, x, x')|, \tag{4.7}$$

then equation (4.6) has a ω-periodic solution.

Proof. The idea of proof is based on the transformation of the equation (4.6) into an operator equation. Define

$$G(t,s) = \frac{1}{2\lambda(1-\cos(\lambda\omega))} \begin{cases} \sin\lambda(\omega + s - t) + \sin\lambda(t - s) & 0 \leq s < t \leq \omega \\ \sin\lambda(\omega + t - s) + \sin\lambda(s - t) & 0 \leq t < s \leq \omega \end{cases},$$

then we can write the solution of the equation (4.6) as the following integral equation:

$$x(t) = \int_0^\omega G(t,s)[\lambda^2 x(s) - f(s, x(s), x'(s))]\, ds. \tag{4.8}$$

Note that the right hand side in (4.8) is ω-periodic due to the periodicity property of $G(t,s)$. Consider the following convex subset of the the Banach space $\mathcal{B} = C^1[0,\omega]$:

$$B = \{x \in C^1[0,\omega], \|x\| \leq c\}, \quad \|x\| = \max\{|x(t)|_\infty, |x'(t)|_\infty\}. \tag{4.9}$$

Define the operator Γ as:

$$\Gamma x(t) = \int_0^\omega G(t,s)[\lambda^2 x(s) - f(s, x(s), x'(s))]\, ds. \tag{4.10}$$

It is easy to check (by (4.7)) that Γ maps B into itself. In addition Γ is uniformly bounded and equi-continuous and then due to Ascoli-Arzela theorem it is compact. Since B is convex set, Γ has a fixed point in B due to the Leray-Schauder fixed point theorem, that is there exist $x \in B$ such that $\Gamma x = x$ and this is the desired periodic solution. □

Corollary 4.4. *Assume that $f(t,x,x')$ is a bounded function, ω-periodic and $\lambda\omega \neq 2k\pi$, then the equation*

$$x'' + \lambda^2 x = f(t,x,x'),$$

has a ω-periodic solution.

Using fixed point theorems to establish the existence of non-trivial periodic orbits has a drawback. In fact, in order to establish the existence of a non-trivial orbit, one has to prove that the fixed point is non-trivial. One important example of such systems is the following equation, see [60]

$$x'' + \varphi(t)\, f(x,x') = 0, \tag{4.11}$$

where $f(0,0)=0$. As a matter of fact, for any convex domain Ω including the origin, the associated operator of the equation has a fixed point $(x,x')=(0,0)$ in Ω that is a trivial solution. One method to overcome this situation is to calculate the multiplicity of the fixed points.

4.2 A solution of Reissig problem

In this section, we give some sufficient conditions for an extension of Reissig equation (for a review of Reissig equation see [68] and [67])

$$x^{(n)} + \sum_{i=1}^{n-1} \varphi_i\big(t,x,\ldots,x^{(n-1)}\big)x^{(n-i)} + f(x) = p(t). \tag{4.12}$$

Note that the above equation is an extension of the following Ezeilo's equation too:

$$x''' + ax'' + bx' + f(x) = p(t).$$

Lemma 4.5. *Assume there exist $c>0$ such that φ_i, f, p satisfies the following conditions:*

i. $|\varphi_i(t,u_0,\ldots,u_{n-1})| \leq \alpha_i \prod_k |u_i|^{\beta_{k(i)}}$ *where* $\beta_{k(i)} \geq 0$ *and* $\sum_k \beta_{k(i)} = 1$ *for any* $t\in[0,\omega]$ *where* $|u_i| \leq c$*,* *ii.* $|f(u)| < \gamma\,|u|$ *for* $|u| \le c$ *and* $\gamma < \frac{1}{3\omega\max\{1, \omega^{n-1}\}}$,

iii. $|p(t)| \le k, \quad t \in [0, \omega]$.

Then equation (4.12) *has a solution that satisfies the boundary condition*

$$x^{(i)}(0) + x^{(i)}(\omega) = 0, \quad i = 0..., n-1. \tag{4.13}$$

Proof. We use the Green function of the problem $x^{(n)} = \delta(t-s)$ with boundary conditions (4.13) as follows, see [1] as

$$G(t,s) = \begin{cases} \sum a_i t^{n-i} & 0 \le t \le s \le \omega \\ \sum b_i t^{n-i} & 0 \le s \le t \le \omega \end{cases}. \tag{4.14}$$

It turns out that

$$|a_1| = |b_1| = \frac{1}{2(n-1)!}, \quad |a_i|, |b_i| \le \frac{\max\{1, \omega^{i-1}\}}{(n-i)!}, i = 2, ..., n.$$

Define the following bounded subset of $C^n[0,\omega]$:

$$X = \left\{x \in C^n[0, \omega]; \left|x^{(i)}(t)\right|_\omega \le c, i = 0, ..., n-1\right\}, \tag{4.15}$$

where $|u(t)|_\omega = \max_{t \in [0,\omega]} |u(t)|$ and $\|x\| = \max_i \left\{\left|x^{(i)}(t)\right|_\omega\right\}$. Now define the operators $\Gamma = \left(\Gamma^{(i)}\right)$ on X for $i = 0$ to $n-1$ as follows:

$$\Gamma^{(i)}(x)(t) = \int_0^\omega G^{(i)}(t,s) g[s, x(s)]\, ds,$$

where $g[t,x] = p(t) - f(x) - \sum_{i=1}^{n-1} \varphi_i\left(t, x, ..., x^{(n-1)}\right) x^{(n-i)}$ and

$$G^{(i)}(t,s) = \frac{\partial^i G}{\partial t^i}(t,s).$$

Now we obtain the norm of $\Gamma^{(i)}$. Simple calculation shows

$$\int_0^\omega \left|G^{(i)}(t,s)\right| \le 3\omega \max\{1, \omega^{n-i-1}\},$$

and hence we have:

$$\left|\Gamma^{(i)}(x)\right|_\omega \le \int_0^\omega \left|G^{(i)}(t,s)\right| |g[s, x(s)]|_\omega\, ds \le 3M\omega \max\{1, \omega^{n-i-1}\},$$

where $M = |g|_\omega$. Let us find a bound for M.

$$|g|_\omega \le k + \gamma c + \sum_i \sum_k \alpha_i \beta_{k(i)} \big|x^{(i)}\big|_\omega \big|x^{(k)}\big|_\omega \le k + \gamma c + \frac{n\alpha}{2} c^2,$$

where $\alpha = \max\{\alpha_i\}$. The above calculation implies that

$$\|\Gamma(x)\| \le 3\Big(k + \gamma c + \frac{n\alpha}{2} c^2\Big) \omega \max\{1, \omega^{n-1}\}.$$

In order that Γ maps X into itself, it should satisfy the following inequality:

$$\frac{n\alpha}{2} c^2 + (\gamma - \mu) c + k \le 0, \quad \mu = \frac{1}{3\omega \max\{1, \omega^{n-1}\}}. \tag{4.16}$$

If we assume $\gamma < \mu$ and $k < \frac{(\gamma - \mu)^2}{2n\alpha}$, then the above inequality has a positive root c. Therefore Γmaps B into itself. On the other hand, $\Gamma^{(i)}$ are equi-continuous, since G is uniformly continuous with respect to t and it is equi-bounded in subsets of B. Ascoli-Arzela theorem then guarantees that $\Gamma^{(i)}$ are completely continuous. Now Schauder fixed point implies the existence of a solution in B and then $\Gamma(x) = x$ that is the solution of (4.12) with boundary conditions (4.13). □

Theorem 4.6. *In addition to assumption in the above lemma, assume the following parity conditions*

i. p, φ_i *are* 2ω*-periodic functions such that* $p(t-\omega) = -p(t)$,

ii. $\varphi_i\big(t-\omega, -x, \ldots, -x^{(n-1)}\big) = \varphi_i\big(t, x, \ldots, x^{(n-1)}\big)$,

iii. $f(-x) = -f(x)$,

then equation (4.12) *has a* 2ω*-periodic solution.*

Proof. Define the extension

$$z(t) = \begin{cases} x(t) & 0 \le t \le \omega \\ -x(t-\omega) & \omega \le t \le 2\omega \end{cases},$$

and then observe that $z \in C^n[0, 2\omega]$ and $z^{(i)}(0) = z^{(i)}(2\omega)$ for $i = 0, \ldots, n-1$. □

4.3 Periodic solutions for third order systems(I)

For planar systems, the Poincare-Bendixon theorem is extensively in use since in this case the ω-limit set of trajectories consists only equilibrium points and cycles (homoclinic and hetroclinic orbits). In this section, we present a method to establish the existence of periodic solutions for nonlinear third order equations. It is a possible generalization of the works of M. Berger [5] and A. Lazer [46] for non-dissipative second order systems and of B. Mehri and H. Emamirad [34] for dissipative systems for odd dimensional spaces. We use certain parity condition in this section that we will relax in the subsequent section.

Consider the following equation:

$$x''' = h(x, x', x''). \tag{4.17}$$

Lemma 4.7. *Assume that there exist $a, b, \lambda > 0$ such that*

$$M \leq \frac{b\lambda^3}{3\pi}, \tag{4.18}$$

where $M = \text{Max}|\lambda^2 x' - h(x, x', x'')|$ on the following subset of $C^2[0, 3\pi/2\lambda]$:

$$B = \left\{ x \in C^2\left[0, \frac{3\pi}{2\lambda}\right], |x|_\infty \leq a + 2b, |x'|_\infty \leq 2\lambda b, |x''|_\infty \leq 2\lambda^2 b \right\}. \tag{4.19}$$

Then there exist $\pi/2\lambda < T < 3\pi/2\lambda$ such that $x'(0) = x'(T) = 0$.

Proof. Define the operator Γ on B as follows:

$$\Gamma(x)(t) = a + b\cos(\lambda t) + \lambda^{-2}\int_0^t (1 - \cos\lambda(t-s))[\lambda^2 x' - h]\, ds.$$

Under the condition (4.18) it implies that Γ maps B into itself and also that Γ is compact, then there exist a solution $x(t)$ that can be extended on $[0, 3\pi/2\lambda]$. On the other hand

$$x'(\pi/2\lambda) = -\lambda b + \lambda^{-1}\int_0^{\pi/2\lambda} \cos(\lambda s)[\lambda^2 x' - h]\, ds < 0,$$

$$x'(3\pi/2\lambda) = \lambda b - \lambda^{-1}\int_0^{3\pi/2\lambda} \cos(\lambda s)[\lambda^2 x' - h]\, ds > 0.$$

This complete the proof. ☐

Theorem 4.8. *Under the above setting, assume h satisfies the following parity condition*

$$h(x, -x', x'') = -h(x, x', x''),$$

then equation (4.17) *has a $2T$-periodic solution $\varphi(t)$ such that*

$$\int_0^{2T} \varphi(t)\, dt = 0. \tag{4.20}$$

Proof. Using the above lemma, there exist T such that $x'(0) = x'(T) = 0$. Extend the solution on $[0, 2T]$ as

$$\varphi(t) = \begin{cases} x(t) & 0 \leq t \leq T \\ -x(2T - t) & T \leq t \leq 2T \end{cases},$$

It is easily verified that $\varphi(t)$ is a periodic solution of (4.17) satisfying (4.20). ☐

As we have seen in the above, for the existence of T such that x satisfies the condition $x'(0) = x'(T) = 0$, we have used only the simple version of the intermediate value theorem. In the general case when $x \in \mathbb{R}^n$, we have to use a degree theoretic argument. Let x in (4.17) belongs to $\mathbb{R}^n$ for n an odd integer. We need the following fact.

Proposition 4.9. *Assume $D \subset \mathbb{R}^n$ is an open bounded set and f, g: $\bar{D} \to \mathbb{R}^n$ are continuous maps such that $0 \notin f(\partial D)$ and $\|f(z)\| > \|g(z)\|$ for $z \in \partial D$, then*

$$\deg(f, D, 0) = \deg(f + g, D, 0).$$

The proof is straightforward by defining the homotopy $h(t) = f + tg$.

Theorem 4.10. *Consider the system*

$$x''' + h(x, x', x'') = 0, \quad x \in \mathbb{R}^n.$$

If there exist a positive diagonal matrix $\Lambda = \mathrm{diag}(\lambda_i)$ and $a, b > 0$ such that $\lambda_m / \lambda_M \geq 2/3$ and

$$M \leq \frac{\lambda_m^3 b}{3\pi} \min\{\sin(\pi\lambda_m / 2\lambda_M), |\sin(3\pi\lambda_m / 2\lambda_M)|\},$$

where λ_m, λ_M *are minimum and maximum of* λ_i *in* Λ *and* $M = \|\Lambda^2 x' - h\|$ *in* B. *Then there exist* T *such that* $x'(T) = 0$.

For the scalar case, $x \in \mathbb{R}$, some examples are simulated numerically in previous references to verify the obtained results.

Proof. Define the operator Γ on B as

$$\Gamma(x)(t) = C_1 + \cos(\lambda t) C_2 + \Lambda^{-2} \int_0^t (1 - \cos\Lambda(t-s))[\Lambda^2 x' - h]\, ds,$$

where $|C_1|_\infty = a$, $|C_2|_\infty = b$ and conclude that $x(t)$ can be extended $[0, 3\pi/2\lambda_M]$. Now

$$x'(t) = -\Lambda \sin(\Lambda t) C_2 + \Lambda^{-1} \int_0^t \sin\Lambda(t-s)[\Lambda^2 x' - h]\, ds.$$

Define maps

$$f(t, C_2) = -\Lambda \sin(\Lambda t) C_2,$$

$$g(t, C_2) = \Lambda^{-1} \int_0^t \sin\Lambda(t-s)[\Lambda^2 x' - h]\, ds.$$

Since

$$|f(\pi/2\lambda_M, C_2)|_\infty \geq |g(\pi/2\lambda_M, C_2)|_\infty,$$

then since the space is of odd dimension, using the above proposition to conclude:

$$\deg(f(\pi/2\lambda_M) + g(\pi/2\lambda_M), \Omega, 0) = \deg(f(\pi/2\lambda_M), \Omega, 0) = -1$$

where $\Omega = \{C, |C|_\infty < b\}$. Similarly we have

$$|f(3\pi/2\lambda_M, C_2)|_\infty \geq |g(3\pi/2\lambda_M, C_2)|_\infty,$$

and then

$$\deg(f(3\pi/2\lambda_M) + g(3\pi/2\lambda_M), \Omega, 0) = \deg(f(3\pi/2\lambda_M), \Omega, 0) = 1.$$

By the invariance of degree under continuous homotopy we conclude that there exist $C, |C|_\infty = b$ and $T \in (\pi/2\lambda_M, 3\pi/2\lambda_M)$ such that $(f+g)(T, C) = 0$ and this completes the proof. $\square$

Now the existence of a periodic solution can be established by the aid of parity argument as the same we presented for the scalar equation.

4.4 Periodic solutions for third order systems(II)

This section generalizes the previous works of the author for third order systems, see [59]. Consider the following system as a one parameter family of systems of ε:

$$x''' + \Lambda^2 x' = \varepsilon f(x, x', x''), \quad x \in \mathbb{R}^n \tag{4.21}$$

where $\Lambda = \operatorname{diag}(\lambda_i)$ with $\lambda_i > 0$, rational. The solution of (4.21) can be written as

$$x(t, p, \varepsilon) = A + \cos(\Lambda t)\, B + \sin(\Lambda t)\, C + \varepsilon \Lambda^{-2} \int_0^t (\mathrm{Id} - \cos\Lambda(t - s)) f[s, p, \varepsilon]\, ds,$$

p denotes the initial vector of the system and

$$f[s, p, \varepsilon] = f(x(s, p, \varepsilon), x'(s, p, \varepsilon), x''(s, p, \varepsilon)).$$

The necessary and sufficient condition for the existence of a ω-periodic solution for (4.21) is the existence of p such that for $i = 0, 1, 2$ we have

$$x^{(i)}(\omega, p, \varepsilon) - x^{(i)}(0, p, \varepsilon) = 0. \tag{4.22}$$

We can use the implicit function theorem here for sufficiently small ε as follows. Let us define u_i as follows

$$u_i(\omega, p, \varepsilon) = x^{(i)}(\omega, p, \varepsilon) - x^{(i)}(0, p, \varepsilon), \quad i = 0, 1, 2,$$

and denote $U(\omega, p, \varepsilon) = (u_i(\omega, p, \varepsilon))$, $i = 0, 1, 2$. Choose λ such that $\lambda.\lambda_i \in \mathbb{N}$ for $i = 1, ..., n$ that is possible according to the fact that λ_i are rational. Now consider $\omega = 2\pi\lambda + \varepsilon\tau$ where in general $\tau = \tau(\varepsilon)$. Therefore we obtain

$$u_0 = \varepsilon\,\tau\,\Lambda\, C + \varepsilon \Lambda^{-2} \int_0^{2\pi\lambda} (\mathrm{Id} - \cos(\Lambda\, s))\, f[s, p]\, ds + o(\varepsilon),$$

$$u_1 = -\varepsilon\tau\,\Lambda^2 B - \varepsilon\,\Lambda^{-1} \int_0^{2\pi\lambda} \sin(\Lambda\, s)\, f[s, p]\, ds + o(\varepsilon),$$

$$u_2 = -\varepsilon\,\tau\,\Lambda^3 C + \varepsilon \int_0^{2\pi\lambda} \cos(\Lambda\, s) f[s, p]\, ds + o(\varepsilon),$$

where $f[s,p]=f(x_0(s,p),x_0'(s,p),x_0''(s,p))$ and $x_0(t,p)=A+\cos(\Lambda t)\,B+\sin(\Lambda t)\,C$. Define $V_0=(v_0,v_1,v_2)$ as follows:

$$
\begin{aligned}
v_0(A,B,C,\tau) &= \tau\,\Lambda^3 C + \int_0^{2\pi\lambda} (\mathrm{Id}-\cos(\Lambda\, s))\, f[s,p]\,ds,\\
v_1(A,B,C,\tau) &= -\tau\,\Lambda^3 B - \int_0^{2\pi\lambda} \sin(\Lambda s)\, f[s,p]\,ds, \qquad (4.23)\\
v_2(A,B,C,\tau) &= -\tau\,\Lambda^3 C + \int_0^{2\pi\lambda} \cos(\Lambda\, s) f[s,p]\,ds,
\end{aligned}
$$

Hence we have

$$\Lambda^2 U(\omega,p,\varepsilon)=\varepsilon V_0+o(\varepsilon).$$

Now, the existence of a ω-periodic solution for (4.21) reduces to the the existence of p such that $V(\varepsilon,p,\varepsilon)=0$ for $V=\varepsilon^{-1}\Lambda^2 U$. Note that $V(\omega,p,\varepsilon)=V_0(\tau,p)+o(1)$. By implicit function theorem, if there exist (τ_0,A_0,B_0,C_0) such that $V_0(\tau_0,p_0)=0$ and furthermore $J_{V_0}(\tau_0,p_0)$, the Jacobian of V_0 at (τ_0,p_0) is nonzero, then there exist $\varepsilon_0>0$ such that for $|\varepsilon|<\varepsilon_0$ there exist continuous functions $\tau=\tau(\varepsilon)$ and $A=A(\varepsilon)$, $B=B(\varepsilon), C=C(\varepsilon)$ such that $V(\omega(\varepsilon),p(\varepsilon))=0$. This argument proves the following simple theorem.

Theorem 4.11. *Assume following conditions hold:*

i. there exist $(A_0,B_0,C_0)\neq 0$ *and* τ_0 *such that* $V_0(A_0,B_0,C_0,\tau_0)=0$,

ii. the Jacobian of V_0 *with respect to* A_0,B_0,C_0,τ_0 *is nonzero.*

Then there exist $\sigma_0>0$ *such that for all* $|\sigma|<\sigma_0$ *the system* (4.21) *has a periodic solution.*

Remark 4.12. For the following equation

$$\mathcal{L}(x)=g_\sigma(x,x',\cdots,x^{(2n)}), \qquad (4.24)$$

where

$$\mathcal{L}(x)=x^{(2n+1)}+\sum_{i=1}^{n}\alpha_i\, x^{(2(n-i)+1)},$$

let us assume that the eigenvalues of $\mathcal{L}$ are $2n$ distinct pure imaginary $\pm i\,\lambda_1, \cdots, \pm i\,\lambda_n$ and furthermore λ_i/λ_j is rational for every $1 \le i, j \le n$. Then equation (4.24) can be rewritten as the following third order system:

$$x_k''' = -\lambda_1^2\, x_k' + \delta\, x_{k+1}', \qquad k = 1, \cdots, n-1,$$

$$x_n''' = -\lambda_n^2\, x_n' + \frac{1}{\delta^n}\, g_\sigma$$

where $\sigma = \delta^{n+1}$ and then we can apply the above theorem.

The above theorem gives a sufficient condition for the existence of periodic solution for the small parameter perturbations of a linear system. Obviously, the main job here, is to extend the solution to $\varepsilon = 1$. We try to do that via geometric considerations and topological degree.

Remark 4.13. For simplicity, let us consider only the scalar case when $\lambda = 1$ and $C = 0$. Note that $C = 0$ is justifiable because $x'(t)$ should change the sign in order that a periodic orbit exists. The set of equations in this case reduces to the equation for $p = (a, b, \tau)$:

$$V_0(p) = 0, \tag{4.25}$$

where $V_0 = (v_0, v_1, v_2)$ defined below:

$$\begin{aligned} v_0(p) &= \tau b + \int_0^{2\pi} \sin(s)\; f[s, a, b]\, ds, \\ v_1(p) &= \int_0^{2\pi} \cos(s)\; f[s, a, b]\, ds, \\ v_2(p) &= \int_0^{2\pi} f[s, a, b]\, ds. \end{aligned} \tag{4.26}$$

Through simple calculations, the condition of non-zero Jacobian is equivalent to $b \neq 0$ and

$$\int_0^{2\pi} f_x' ds \int_0^{2\pi} \cos(s)\eta ds - \int_0^{2\pi} \cos(s) f_x'\, ds \int_0^{2\pi} \eta ds \neq 0, \tag{4.27}$$

where

$$\eta = f_x' \cos(s) - f_{x'}' \sin(s) - f_{x''}' \cos(s).$$

Now theorem (4.11) is equivalent to the following.

Theorem 4.14. *For* $V_0=(v_0, v_1, v_2)$ *defined in* (4.25)*, assume there are* a_0, b_0, τ_0 *such that* $V_0(p_0)=0$ *and the condition* (4.27) *holds, then there exist* $\sigma_0>0$ *such that for all* $|\sigma|<\sigma_0$ *there are* $\gamma(\sigma)=(a(\sigma), b(\sigma), \tau(\sigma))$ *that satisfy* $U(\gamma(\sigma), \sigma)=0$ *and* $\gamma(\sigma)$ *converges to* $\gamma_0=(a_0, b_0, 0)$ *when* $\sigma\to 0$.

4.4.1 Topological considerations

Note that every solution of equations (4.25) corresponds to a point in $\mathbb{R}^3$ that in turn corresponds to a periodic orbit. When ε varies in (4.21), each point makes a curve $\gamma(\varepsilon)$ in $\mathbb{R}^3$ such that each point on γ corresponds to a periodic solution. The solution of $V_0(p)=0$ with zero Jacobian also corresponds to a periodic orbit and we call them the zero periodic orbits. Following is a simple fact regarding the solution set $\gamma(\varepsilon)$.

Proposition 4.15. *For every solution of* (4.21)*, there is* $\varepsilon_0>0$ *such that for* $|\varepsilon|<\varepsilon_0$ *the path* $\gamma(\varepsilon)$ *is connected without any bifurcation.*

Proof. For a small neighborhood $B_\delta(\gamma_0)$ of γ_0 and $p\in B_\delta(\gamma_0)$, consider the map

$$\varphi(p,\varepsilon)=p-J_{V_0}^{-1}(\gamma_0)\,V(p,\varepsilon).$$

Since $V(p,\varepsilon)$ is C^1 with respect to ε, $||\,V(p,\varepsilon)-V_0(p,\varepsilon)||$ is sufficiently small in C^1 topology for ε small. Therefore $||\,\varphi'||\leq\lambda<1$. Then φ is a contraction mapping and then has a unique fixed point in B_δ that is denoted by $\gamma(\varepsilon)$. Now consider φ as a parameter family functions of ε as $\varphi=\varphi(p,\varepsilon)$ where $p\in B_\delta$. We have

$$||\,\gamma(\varepsilon)-\gamma(0)||\leq\frac{1}{1-\lambda}||\,\varphi(\gamma(\varepsilon),\varepsilon)-\varphi(\gamma(0),0)|| \qquad (4.28)$$

but $\varphi(p,\varepsilon)$ is continuous with respect to ε, then $\gamma(\varepsilon)$ is continuous and continuously converges to $\gamma(0)$. □

Proposition 4.16. *The solutions of* (4.25)*,* (4.27) *occur in pair, that is, if* (a,b) *is a solution then* $(a,-b)$ *is another one. Furthermore, the pair solutions rest on the same orbit and their degree are opposite.*

Proof. The integrals in the condition $V_0(p)=0$ holds equally in the range $(k\pi, (k+2)\pi)$ and in particular in the range $(\pi, 3\pi)$. Change s to $\pi+s$ in V_0 gives the result. It is also obvious that for $\varepsilon=0$ we have $x(0)=a+b$, $x(\pi)=a-b$ so they correspond to the same periodic orbit. Their Jacobian signs are in opposite because:

$$J_{V_0}(a,b,\tau)=b\,(\frac{\partial v_1(0)}{\partial a}\,\frac{\partial v_2(0)}{\partial b}-\frac{\partial v_1(0)}{\partial b}\,\frac{\partial v_2(0)}{\partial a})$$

It is easy to verify that the term in the parenthesis is even with respect to b, so the Jacobian sign is the same as the sign of b and this completes the proof. □

Let P denote the set of all solutions of (4.25), (4.27). It is called the set of non-critical orbits. Note that the set is finite in any closed bounded domain. In fact, P consists of all periodic solutions for $\varepsilon=0$ such that they survive under the deformation with respect to $\varepsilon\ll 1$. We can classify non-critical orbits according to the following definition.

Definition 4.17. *The connected path $\gamma_p(\varepsilon)$ started at $p\in P$ is called a positive branch and denoted by γ_p^+ if $J_{V_0}(p)>0$. The negative branch denoted by γ_p^- is defined similarly when $J_{V_0}(p)<0$. Q denotes the set of all isolated solutions of $V_0=0$ with zero Jacobian. We call the path started at a $q\in Q$ the zero branch and denote it by γ_q.*

We can prove now the following proposition using Brouwer degree.

Proposition 4.18. *P contains only isolated points, that is for each $p\in P$ there is an open bounded neighborhood Ω_p of p such that the Brouwer degree of $V(\varepsilon)$ remains constant on Ω_p for small values of ε.*

Proof. By the finiteness of the solutions (4.25)-(4.27) in a bounded domain, they are isolated (using the inverse function theorem). Thus, for an open bounded neighborhood Ω_p of p, we can define the degree for V_0 as:

$$\deg\,[V_0(p),\Omega_p,0]=\operatorname{sign}(J_{V_0}(p)). \tag{4.29}$$

By the homotopy invariance property of topological degree we have

$$\deg[V(p,\varepsilon),\Omega_p,0]=\deg\,[V_0(p),\Omega_p,0]. \tag{4.30}$$

□

Definition 4.19. *For each $p \in P$ define the following index:*

$$i(V(p,\varepsilon)) = \deg\,[V(p,\varepsilon),\Omega_p,0].$$

The fact that the above index is preserved under small deformations is now clear by equation (4.30). In order to extend γ_p^+,γ_p^- with respect to $\varepsilon \in [0,1]$, we need to assume that for any $\varepsilon \in [0,1]$, the periodic solutions remain bounded that is, there exist $M > 0$ such that

$$\|x\|_\omega = \max|x(t)|_\omega \leq M.$$

From a geometrical point of view, the unperturbed periodic orbits for $\varepsilon = 0$ in (4.21) are deformed into the periodic orbits for $\varepsilon \neq 0$. For ε sufficiently small, this deformation defines a homeomorphism due to the proposition (4.15). For any p, the degree of $V(p,\varepsilon)$ is changed only if some 0-point appear in $\partial\Omega_p$, the boundary of Ω_p. If γ_p is isolated in Ω_p, then they can be extended further with respect to ε provided that no other curve $\gamma_{p'}$ originated at some p' touches $\partial\Omega_p$. But if $\gamma_{p'}$ touches Ω_p for any small neighborhood tube Ω_p of γ_p, then γ_p and $\gamma_{p'}$ have to meet at some point for some ε. We call this point a *resolving point* since the branches resolve in each other and form a zero or a critical branch γ_q. This point is already a solution of the equation $V(q,\varepsilon) = 0$ but with zero index. Note that γ_q branches could not be extended in general. As a simple result of the homotopy invariance property, for a resolving point, the sum of indexes coming in the point must be equal to the indices going out the point. In fact if q is a resolving point at ε_1, since the periodic orbits are bounded by the assumption, we can define a degree for the q and this degree is constant for small changes in $(\varepsilon_1 - \delta, \varepsilon_1 + \delta)$ for sufficiently small $\delta > 0$. Therefore the indices in $(\varepsilon_1 - \delta, \varepsilon_1)$ is equal with indices in $(\varepsilon_1, \varepsilon_1 + \delta)$. Thus, a path that end in any such points may not be extended anymore and this cancels the possibility of increasing ε in (4.21). If the algebraic sum of the outgoing indices is nonzero then there would be one outgoing branch with the nonzero index. It is possible also that some zero branches goes out from the resolving point. Since each point in the curve corresponds to a periodic orbit, the argument shows that how periodic orbits resolve into each other. If resolving occurs for two branches in a same orbit, then the corresponding orbit shrinks into a point with zero index.

Now, we discuss about the transversality of the resolving branches. According to the condition $V(\gamma_p(\varepsilon),\varepsilon)=0$ for the curve $\gamma_p(\varepsilon)$ we can write

$$\frac{d\gamma_p(\varepsilon)}{d\varepsilon}=-J_V^{-1}(\gamma_p(\varepsilon),\varepsilon)\,\frac{\partial V}{\partial\varepsilon}(\gamma_p(\varepsilon),\varepsilon).$$

If two curves γ_p^+, $\gamma_{p'}^-$ resolve into each other at $\varepsilon=\varepsilon_1$ then for the point $\gamma_p^+(\varepsilon_1)=\gamma_{p'}^-(\varepsilon_1)=q$, we have

$$J_V(q,\varepsilon_1)=J_V(q,\varepsilon_1)=0.$$

If $\frac{\partial V}{\partial\varepsilon}(q,\varepsilon)\neq 0$, that geometrically means γ_p^+, $\gamma_{p'}^-$ are not horizontal at q for $\varepsilon=\varepsilon_1$ then the branches γ_p^+, $\gamma_{p'}^-$ meet each other vertically, since

$$\left|\frac{d\gamma_p^+(\varepsilon)}{d\varepsilon}\right|=\left|\frac{d\gamma_{p'}^-(\varepsilon)}{d\varepsilon}\right|=\infty.$$

Their signs are opposite according to the different Jacobian signs for $\varepsilon_1-\delta$ for $\delta\ll 1$.

A *branching* or *bifurcation* point is a point from which two or more branches come out. This is the reverse of the resolving point. Geometrically, it is equivalent to the appearance of a new periodic orbit. A similar transversality condition to one stated above holds for a branching point. The fundamental fact about the periodic orbits is that they disappear if and only if:

1. they degenerate into a point with zero degree
2. they go to infinity for finite values of ε and
3. they resolve into each other and making critical orbits.

In sequel, we will state a theorem that guarantees the possibility of extension of $\gamma_p(\varepsilon)$ to $\varepsilon=1$. We present the theorem for a scalar equation. Consider the following scalar equation:

$$x'''+f(x,x',x'')=0 \tag{4.31}$$

The existence of a periodic orbit for (4.31) is equivalent to the existence of a solution for the following system when $\varepsilon=1$:

$$x'''+x'=\varepsilon(x'-f(x,x',x'')) \tag{4.32}$$

We consider (4.32) as a parametric equation depending on ε for $\varepsilon \in [0, 1]$. Denote $x' - f = g$ and rewrite the solution of the equation (4.32) as follows:

$$x(\varepsilon; a, b, t) = a + b\cos(t) + \varepsilon \int_0^t (1 - \cos(t-s))\, g(s, \varepsilon)\, ds$$

where $g(s,\varepsilon) = g(x_\varepsilon(s), x'_\varepsilon(s), x''_\varepsilon(s))$. Let us denote the mapping $F: [0,1] \times \mathbb{R}^3 \to \mathbb{R}^3$ as

$$F(\varepsilon; a, b, \omega) = \left(x^{(i)}(\varepsilon; a, b, \omega) - x^{(i)}(\varepsilon; a, b, 0)\right), \quad i = 0, 1, 2$$

where as before we consider $\omega = 2\pi + \varepsilon\tau$. Simple calculation gives the following set of equations:

$$\int_0^{2\pi} f(x_0(s), x'_0(s), x''_0(s))\, ds = 0 \tag{4.33}$$

$$\int_0^{2\pi} \cos(s)\, f(x_0(s), x'_0(s), x''_0(s))\, ds = 0 \tag{4.34}$$

and the following inequality that corresponds to $J_F(0; a, b, \tau) \neq 0$

$$\int_0^{2\pi} \int_0^{2\pi} \cos(s)\{h(s)\, f_x(u) - f_x(s)\, h(u)\} ds\, du \neq 0 \tag{4.35}$$

where

$$h(s) = \cos(s)\, f_x(s) - \sin(s)\, f_{x'}(s) - \cos(s) f_{x''}(s)$$

and

$$f_x(s) = \frac{\partial}{\partial x} f(x_0(s), x'_0(s), x''_0(s)),$$

where

$$x_0(s) = a + b\cos(s), x'_0(s) = -b\sin(s), x''_0(s) = -b\cos(s).$$

Similarly $f_{x'}$, $f_{x''}$ are defined. The solution for τ is obtained as follows:

$$\int_0^{2\pi} \sin(s)\, f(x_0(s), x'_0(s), x''_0(s))\, ds = \pi + b\tau.$$

Theorem 4.20. *Assume that equation* (4.31) *has no critical point other than the origin and that there exist a smooth closed surface* Γ *such that for any* $(x_1, x_2, x_3) \in \Gamma$ *and* $\varepsilon \in (0,1]$ *it satisfy the following inequality:*

$$\nabla\Gamma(x_1, x_2, x_3).(x_2, x_3, -(1-\varepsilon)x_2 - \varepsilon f(x_1, x_2, x_3)) < 0 \tag{4.36}$$

In addition, assume that equations (4.33),(4.34) *has* $4k+2$ *number of solutions inside* Γ *such that for them the relation* (4.35) *holds. Then the equation* (4.31) *has at least one periodic solution.*

The proof rests on the extendability of the solution set $(a(\varepsilon), b(\varepsilon), \tau(\varepsilon))$ for $\varepsilon = 1$, that corresponds to the survive of periodic orbits. In fact, we can consider the smooth curve

$$\gamma(\varepsilon) = (a(\varepsilon), b(\varepsilon), \tau(\varepsilon)), \quad \varepsilon \in [0, \varepsilon_0)$$

for each solution of the above conditions. To each curve is assigned an index that is the sign of the Jacobian or equivalently the Brouwer index of each solution. The extendability of curves results from the boundedness of the solution that is guaranteed by the inequality (4.36) and the fact that the solution set contains odd number of pairs and also from the fact that the origin is the only stationary point of the equation. The inequality (4.36) guarantees that orbits remain inside Γ and also that no periodic orbit enters inside Γ from infinity.

Proof. The condition (4.36) guarantees that the solutions originating inside Γ remain always inside and that no closed orbit originating outside could enter Γ. Since Γ does not include any periodic orbit, then since the number of periodic orbits inside Γ is odd, then the degree of F is odd and constant for $\varepsilon \in (0, 1]$, that is some branches $\gamma_p(\varepsilon)$ could be extended to $\varepsilon = 1$. □

It is possible also to impose other conditions for the extendability of the periodic orbits. One alternative condition is obtained by rewriting the equation (4.32) as follows:

$$\begin{cases} x_1' = x_2, \\ x_2' = x_3 = x_1, \\ x_3' = -(1-2\varepsilon)x_2 - \varepsilon f(x_1, x_2, x_3 - x_1) \end{cases}$$

Now define Γ as

$$\Gamma(x_1, x_2, x_3) = \frac{1}{2}\|x\|^2 - R = 0.$$

For the extendability one can impose the following condition on f:

$$x_3(2x_2 - f(x_1, x_2, x_3 - x_1)) < 0.$$

Chapter 5

Fully non-linear equations

In this chapter we study non-linear elliptic and evolution equations and define a degree for fully non-linear parabolic equations. The degree for fully nonlinear elliptic equation is not novel, in fact Y. Y. Li defined a degree for fully nonlinear second order uniformly elliptic equations [48] using the degree that is developed for Fredholm operators by Fitzpatrick [38]. I. V. Skrypnik [74] constructed a degree for fully nonlinear uniformly elliptic PDE's by the aid of degree for $(S)_+$ maps. In fact, for every uniformly elliptic equation, it is possible to define an operator equation consisting of $(S)_+$ maps. We generalize these results for the parameter dependent family of elliptic operator. In particular we use these result to define a degree for fully non-linear parabolic equations.

It is very well known that the non-linear elliptic equations satisfying the complementing conditions (Shapiro condition) can be written as an operator equation of the type quasi-linear Fredholm map, see [38]. On the other hand, Fredholm maps with index 0 can be written as the compact perturbation of homeomorphisms and then having Schauder maps as a subclass. It is possible to define a degree theory for index 0 Fredholm maps possessing all classical properties of a topological degree except the homotopy invariance property, see [38],[39].

Since evolution equations can be formulated as $(S)_+$ perturbation of maximal monotone maps, a degree theory can be defined for non-linear evolution equations if we could formulate a given nonlinear elliptic equations in terms of $(S)_+$ mappings.

5.1 From variational method to monotone maps

The reformulation of a functional equation to a minimization problem became the main subject of the variational method after D. Hilbert's work on Dirichlet problem. When A: $X \rightarrow X^*$ is a potential map, that is, if there exist a potential φ that $A(u) = \varphi'(u)$ for the Frechet derivative φ', then the solvability of the equation

$$A(u) = f \tag{5.1}$$

will reduces to the following minimization problem:

$$\min_u \{\varphi(u) - \langle f, u\rangle\}. \tag{5.2}$$

The most well-known example is when A is a self-adjoint uniformly elliptic operator in divergence form:

$$A(u) = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha f_\alpha(x, D^{\leqslant m} u), \tag{5.3}$$

where $x \in \Omega \subset \mathbb{R}^n$, $D^{\leqslant m} u = \{D^\alpha u, |\alpha| \leq m\}$ and f_α are

$$f_\alpha = \frac{\partial F}{\partial (D^\alpha u)},$$

for some smooth function $F(x, D^{\leqslant m} u)$. Here we consider the map A defined on the space $X := W_0^{m,p}(\Omega)$, that is A: $W_0^{m,p}(\Omega) \rightarrow (W_0^{m,p}(\Omega))^* = W^{-m,q}(\Omega)$ with q the Holder conjugate of p. Recall that the norm of $W^{-m,q}(\Omega)$ is defined as

$$\|u\|_{W^{-m,q}} = \sup_{v \in W_0^{m,p}(\Omega)} \frac{1}{\|v\|_{W_0^{m,p}}} \int_\Omega u\, v. \tag{5.4}$$

In this case the potential φ is defined as

$$\varphi(u) = \int_\Omega F(x, D^{\leqslant m} u(x))\, dx. \tag{5.5}$$

In fact when $u \in W_0^{m,p}(\Omega)$ the Euler-Lagrange equation of the minimization problem

$$\min_{u \in W_0^{m,p}(\Omega)} \int_\Omega F(x, D^{\leqslant m} u(x)) dx - \langle f, u\rangle \tag{5.6}$$

has the form $A(u) = f$. In the above case, the solvability of the equation (5.1) is established by the following standard theorem:

Theorem 5.1. *Assume X is a reflexive Banach space, $\varphi\colon X\to\mathbb{R}$ is a weakly lower semi-continuous function and bounded below by φ_0. If φ is coercive, that is $\lim\varphi(x)\to\infty$ for $\|x\|\to\infty$, then φ attains its infimum at a point $u\in X$. Furthermore u is the solution of equation* (5.1).

In the above theorem, that X is a reflexive Banach space plays an important role, since according to the coercivity condition the minimizing sequence is bounded and then weakly compact. A sufficient condition for the weak lower semi-continuity of φ is given in the following Proposition.

Proposition 5.2. *If $\varphi\colon X\to\mathbb{R}$ is a continuous convex function then φ is weakly lower semi-continuous.*

The convexity of φ is guaranteed by the Theorem below, see for example [69].

Theorem 5.3. *Let $\varphi\colon X\to\mathbb{R}$ be a C^1 function, where X is a Banach space. $\varphi'\colon X\to X^*$, the Frechet derivative of φ is monotone if and only if φ is convex.*

Therefore if the map A in (5.3) is monotone, that is if

$$\sum_{|\alpha|\le m}\int_\Omega [f_\alpha(x,D^{\leqslant m}u(x))-f_\alpha(x,D^{\leqslant m}v(x))]\,[D^\alpha u(x)-D^\alpha v(x)]\ge 0, \tag{5.7}$$

then φ in (5.5) is convex. In order to have (5.7), we restrict that f_α satisfy the following inequality for any pair of vectors $(\xi),(\eta)$:

$$[f_\alpha(x,\xi)-f_\alpha(x,\eta)](\xi_\alpha-\eta_\alpha)\ge 0,\quad |\alpha|\le m. \tag{5.8}$$

Under the above condition, φ is convex. The assumption on the coercivity of φ can not be removed in general. We have the following proposition.

Proposition 5.4. *If $A\colon X\to X^*$ is a continuous potential map that satisfies the coercivity condition in the following sense*

$$\frac{\langle Au,u\rangle}{\|u\|}\to\infty,\ \|u\|\to\infty, \tag{5.9}$$

and the potential φ of A is convex, then

$$\frac{|\varphi(u)|}{\|u\|}\to\infty,\ \|u\|\to\infty. \tag{5.10}$$

Proof. For some $\tau \in (1/2,1)$ we can write

$$\varphi(u)-\varphi(0)=\int_0^1 \langle A(tu)\,,u\rangle dt=\int_0^{1/2} \langle A(tu)\,,u\rangle\,dt+\frac{1}{2}\langle A(\tau u),u\rangle,$$

Since φ is convex, then A is monotone and

$$\langle A(u),u\rangle \geq \langle A(0),u\rangle, \quad \forall u\in X.$$

Therefore

$$\varphi(u)\geq\varphi(0)+\frac{1}{2}\langle A(0),u\rangle+\frac{1}{2\tau}\langle A(\tau u),\tau u\rangle, \quad \tau\in(\frac{1}{2},1)$$

and this completes the proof. □

In order to justify that φ in (5.5) is coercive, we assume the following ellipticity condition for A in (5.3)

$$\sum_{|\alpha|=m} f_\alpha(x,\xi)\xi_\alpha \geq c\,|\xi_m|^p, \;\; c>0, p>1, \tag{5.11}$$

where $\xi_m=(\xi_\alpha)$ and $|\alpha|=m$. Since $u\in W_0^{m,p}(\Omega)$ and Ω is bounded, the norm in $W_0^{m,p}(\Omega)$ is equivalent to

$$\|u\|_{m,p}^p=C\int_\Omega |D^m u(x)|^p\,dx.$$

Use the relation (5.8) to write

$$\langle A(u),u\rangle \geq \sum_{|\alpha|\leq m-1}\int_\Omega f_\alpha(x,0)\,D^\alpha u(x)\,dx+\sum_{|\alpha|=m}\int_\Omega f_\alpha(x,D^{\leqslant m}u)\,D^\alpha u(x)\,dx.$$

Using (5.11) we have

$$\sum_{|\alpha|=m}\int_\Omega f_\alpha(x,D^{\leqslant m}u)\,D^\alpha u(x)\,dx\geq c\|u\|_{m,p}^p.$$

On the other hand, using Hölder inequality we obtain for some $C>0$ the following:

$$\sum_{|\alpha|\leq m-1}\int_\Omega f_\alpha(x,0)\,D^\alpha u(x)\,dx\geq -C\,\|u\|_{m,p}. \tag{5.12}$$

Since $p>1$, the above inequalities prove that A is coercive and then φ is coercive.

The continuity of the map A in (5.3) is not trivial and does not follow from simply the continuity of f_α with respect to its arguments. The following proposition set the conditions under which A is continuous from $W_0^{m,p}$ to $W^{-m,q}$.

Proposition 5.5. *For A defined in (5.3) the following conditions are sufficient that A maps $W_0^{m,p}$ continuously into $W^{-m,q}$:*

i. $f_\alpha(x,\xi)$ is measurable in x for any ξ and it is continuous in ξ for almost all x.

ii. f_α satisfies the following condition for $L_\infty(\Omega)$ functions a,b

$$f_\alpha(x,\xi) \le a(x) + b(x)\,|\xi|^r, \quad a,b>0, r\le p-1. \tag{5.13}$$

Proof. The above two conditions implies that f_α are measurable and also f_α belong to $L_q(\Omega)$. In fact since $u\in W_0^{m,p}$ then for some $C_1,C_2>0$ we have

$$\int_\Omega |f_\alpha(x, D^{\leqslant m}u(x))|^q \le C_1 + C_2 \sum_{|\alpha|\le m} \int_\Omega |D^\alpha(u)|^p < \infty.$$

Assume that $u_n \to u$ in $W_0^{m,p}$, then

$$\|A(u_n) - A(u)\|_{-m,q} = \sup_{\|v\|_{m,p}=1} \langle A(u_n) - A(u), v\rangle \le$$

$$\le \left(\sum_{|\alpha|\le m} \int_\Omega |f_\alpha(x, D^{\leqslant m}u_n) - f_\alpha(x, D^{\leqslant m}u)|^q \right)^{1/q}.$$

To complete the proof we need the following standard result in measure theory. If $u_n \to u$ in $L_p(\Omega)$, then there exist a sub-sequence u_{n_k} such that

1. $u_{n_k} \to u$ point-wise a.e. in Ω
2. $|u_{n_k}| < h$, $h\in L_p(\Omega)$

Now we complete the proof of the continuity of A. Since $u_n\to u$ in $W_0^{m,p}$ then there exist a sub-sequence we denote again by u_n converging point-wise to u and then by the continuity of f_α with respect to ξ we have

$$\lim_{n\to\infty} |f_\alpha(x, D^{\leqslant m}u_n(x)) - f_\alpha(x, D^{\leqslant m}u(x))| \to 0.$$

But according to the second condition on f_α and the property (2) above we can write for $h\in W^{m,p}$

$$|f_\alpha(x,\xi)(u_n-u)| \le |f_\alpha(x,\xi)u_n| + |f_\alpha(x,\xi)u| \leqslant 2a(x) + b(x)\,|\xi(u)|^r +$$

$$+ b(x)|\xi(u_n)|^r \le 2a(x) + b(x)|\xi(u)|^r + b\,|\xi\,(h)|^r.$$

Here for simplicity we used the jet-map ξ to represent $D^{\leqslant m}$ as ξ and

$$f_\alpha(x,\xi)(u) = f_\alpha(x, D^{\leqslant m} u).$$

Then by dominant convergence theorem

$$\lim_{n\to\infty} \int_\Omega |f_\alpha(x,\xi)(u_n-u)|^q\, dx = \int_\Omega \lim_{n\to\infty} |f_\alpha(x,\xi)(u_n-u)|^q\, dx = 0. \tag{5.14}$$

Therefore $A(u_n)\to A(u)$ in $W^{-m,q}(\Omega)$. □

The possibility of variational formulation of equation (5.1) depends on whether A is a potential operator, that is whether there exist a convex function φ such that $A=\varphi'$. This question has been answered for a long time. In fact we have the following theorem, see for example [78].

Theorem 5.6. *Assume $A\colon X\to X^*$ is continuous, then A is a potential operator if and only if its line integral is independent of integration path.*

The line integral for the map $A\colon X\to X^*$ is defined as follows. If $\gamma\colon(\alpha,\beta)\to X$ is a smooth path, the line integral is defined as:

$$\int_\gamma A(u)\, du = \lim_{n\to\infty} \sum_{k=1}^{n} \langle A(u_k), u_k - u_{k-1}\rangle = \int_\alpha^\beta \langle \mathcal{A}(\gamma(t), d\gamma(t)\rangle. \tag{5.15}$$

In particular if $\gamma(t) = u + t(v-u)$ for $t\in[0,1]$ then

$$\int_\gamma A(u)\, du = \int_0^1 \langle A(u+t(v-u)), v-u\rangle\, dt. \tag{5.16}$$

In the case that $A\colon X\to X^*$ is Frechet differentiable, the alternative condition is given in the following theorem.

Theorem 5.7. *Assume $A\colon X\to X^*$ is Frechet differentiable, then A is a potential operator if and only if for every $u, v, w\in X$ the following symmetric condition hold:*

$$\langle A'(u)(v), w\rangle = \langle A'(u)(w), v\rangle, \tag{5.17}$$

where $A'(u)$ is the Frechet differential of A at u.

In general, the reduction of functional equations to a minimization problem is impossible. For example the equation (5.1) when A is not a self-adjoint map cannot be reduced to a variational problem according to the above theorem. It is also well known that the equation of the form

$$\operatorname{div}(T(\nabla u)) = f, \quad \text{on } \Omega,$$

where T is a vector field not coming from a potential function, can not be reduced to a variational problem in the classical sense, see [40]. However, recently N. Ghoussoub [40] defined a type of calculus of variation to reduce equations involving self-dual operators to a minimization problem. In fact, for the equation $-A(u) = \partial\varphi(u)$ when φ is a smooth convex function but $A: X \to X^*$ is not necessary a self-adjoint operator we can define

$$I(u) = \langle A(u), u\rangle + \varphi(u) + \varphi^*(-A(u)), \tag{5.18}$$

where $\varphi^*: X^* \to \mathbb{R}$ is the Legendre transformation of φ that is defined as

$$\varphi^*(p) = \sup_u \{\langle p, u\rangle - \varphi(u)\}. \tag{5.19}$$

By Fenchel duality it turns out that for all $u \in X$ we have

$$\varphi(u) + \varphi^*(-A(u)) \geq \langle -A(u), u\rangle,$$

and $I(u) = 0$ only if $-A(u) = \partial\varphi(u)$. Therefore $I(u) \geq 0$ and $I(u) = 0$ gives the solution of the equation $-A(u) = \partial\varphi(u)$.

5.2 Fredholm maps

In this section we briefly review some known results about Fredholm maps that we will need for our construction of degree for elliptic and parabolic problems.

Definition 5.8. *The map $F \in L(X, Y)$ for X, Y Banach spaces is called Fredholm map and we write $F \in \operatorname{Fred}(X, Y)$ if the following two properties hold:*

i. $\dim\ker(F) = d < \infty$,

ii. $\operatorname{codim} F = \dim F/F(X) = d' < \infty$

The index of F is defined as $\operatorname{ind}(F) = d - d'$.

5.2.1 Properties of Fredholm maps

5.2.1.1 Analytic index is topological

Recall that if $Y \subset X$ are two topological spaces, X/Y denote the topological spaces of equivalent classes $x_1 \sim x_2$ if $x_1 - x_2 \in Y$ and the topology is one that makes the projection $P: X \to X/Y$ continuous. Using the duality relationships between $\ker(F)$ and $\operatorname{coker}(F^*)$, it turns out that F^* is Fredholm and $\operatorname{ind}(F) = -\operatorname{ind}(F^*)$. The most important property of the index of a Fredholm map that we need in this chapter is the following that we give an elementary proof in the appendix.

Theorem 5.9. *Assume that F_t for $t \in [0,1]$ is a continuous family of Fredholm maps then $\operatorname{ind}(F_t)$ is independent of t.*

One important class of Fredholm maps is the class of Schauder maps, the compact perturbations of the identity map. In his works on quasi-linear elliptic equations, Schauder introduced the theory of such maps. Recall that, according to Schauder theorem, if $K: X \to Y$ is compact then $K^*: Y^* \to X^*$ is compact that is proved easily by Ascoli-Arzela theorem. On the other hand, the eigenspace of K associated to any eigenvalue is a finite dimensional space that is easily follows from Riesz lemma. Using these facts and the duality between $\ker(F)$ and $\operatorname{coker}(F^*)$ one can show that the compact vector field $\mathrm{Id} + K$ is Fredholm. Define the family $F_t = \mathrm{Id} + tK$ for $t \in [0,1]$ and use the invariance property of Fredholm index to conclude that $\operatorname{ind}(\mathrm{Id} + K) = 0$. Later, Leray and Schauder developed a topological degree for such maps in a joint paper [47]. The most important generalizations in this direction are the degree for the class of $(S)_+$ developed by F. Browder, [17] and the degree for index zero Fredholm maps developed by P. M. Fitzpatrick and J, Pejsachowiz [38].

5.2.1.2 Fredholm alternative

Theorem 5.10. *Assume $F \in L(X,Y)$ then $F \in \operatorname{Fred}(X,Y)$ if and only if there exist $A \in L(Y,X)$ such that $AF - \mathrm{Id}$ and $FA - \mathrm{Id}$ are compact operators. The index of F is zero if and only if there exist $A \in \operatorname{iso}(Y,X)$ such that $AF - \mathrm{Id}$ and $FA - \mathrm{Id}$ are compact operators.*

Remark 5.11. In order to define a degree for the map $F \in \mathrm{Fred}(X, Y)$, one can use the above theorem to define the map $AF = \mathrm{Id} + K$ where A is an isomorphism. Now we can define the degree of F in the open bounded subset Ω at 0 as

$$\deg(F, \Omega, 0) = \deg_{L.S.}(\mathrm{Id} + K, \Omega, 0). \tag{5.20}$$

This degree is of course dependent on A and since the homotopy class of isomorphisms can be trivial (due to Kuiper theorem) then the degree is not invariant under an arbitrary continuous homotopy.

One corollary of the above theorem is the fact that $F + K$ are Fredholm if F is Fredholm.

Corollary 5.12. *Assume $F \in \mathrm{Fred}(X, Y)$ and $K: X \to Y$ is a compact map, then $F + K$ is a Fredholm map and $\mathrm{ind}(F) = \mathrm{ind}(F + K)$.*

Proof. Since F is Fredholm, there exist $A \in L(Y, X)$ such that $K_1 = AF - \mathrm{Id}$ and $K_2 = FA - \mathrm{Id}$ are compact maps. Now for this A we can write

$$A(F + K) - \mathrm{Id} = K_1 + AK, \tag{5.21}$$

that is compact. Similarly is $(F + K)A - \mathrm{Id}$ is compact that proves $F + K \in \mathrm{Fred}(X, X)$. Use the homotopy $F + tK$ and conclude $\mathrm{ind}(F + tK) = \mathrm{ind}(F) = \mathrm{ind}(F + K)$. $\square$

5.2.1.3 Existence of parametrix

Using Fredholm alternative, the linear Fredholm maps of index 0 can be transformed to an isomorphism by adding a finite rank linear continuous map. Now if $F(t)$ is a continuous family of Fredholm maps on $[0, 1]$, we show that there exist a continuous family of continuous finite rank maps $C(t)$ that makes $F(t)$ an isomorphism. This is called parametrix of Fredholm maps $F(t)$. Here we give an elementary proof for the following parametrix theorem without appealing to any algebraic topological tool.

Theorem 5.13. *Assume $L: I \times X \to Y$ is a continuous map and for any $t \in I$, $L_t \in \mathrm{Fred}(X, Y)$ with $\mathrm{ind}(L_t) = 0$ where $L_t(x) = L(t, x)$. Then there exist a continuous map $C: I \times X \to Y$ such that for any $t \in I$, C_t is a finite rank map and $L_t + C_t \in \mathrm{iso}(X, Y)$.*

In order to prove the theorem (5.13), we need to control the kernel of the family maps L_t. It is done in the following proposition:

Proposition 5.14. *Assume* $L_t, t\in[0,1]$ *is a continuous family of Fredholm maps with* $\ker L_0 = N$*, then there is a continuous family of maps* C_t *with finite rank such that* $\ker(L_t + C_t) = N$.

Proof. Let us denote $I_{t,\varepsilon} = (t-\varepsilon, t+\varepsilon)\cap I$, $N_t = \ker(L_t)$, $R_t = L_t(X)$ and D_t the complement subspace of N_t. By proposition (2.24), for any $t\in I$ there exist $\varepsilon > 0$ such that for all $s\in I_{t,\varepsilon}$, $L_s\in \mathrm{iso}(D_t, R_t)$. Since I is compact, it is covered by finitely many such $I_{t_j,\varepsilon}$ and then we define $D = \cap D_{t_j}$. We show D has finite co-dimension. It is obvious that

$$\operatorname{codim} D = \dim X/D = \dim \operatorname{span}\{X/D_j\} < \infty. \tag{5.22}$$

We note that L_t is injective on D. Let N be the complement subspace of D, $X = D\oplus N$ and π_1, π_2 denote the continuous projection on D, N respectively. Define $C_t(x) = -L_t\circ\pi_2(x)$ and we observe that

$$(L_t + C_t)(x) = L_t\circ\pi_1(x). \tag{5.23}$$

Since L_t is injective on D we conclude that $\ker(L_t + C_t) = \pi_2(X) = N$. Finally, we notice that

$$\operatorname{ind}(L_t + C_t) = \operatorname{ind}(L_t), \tag{5.24}$$

because C_t is a compact map. □

Proof. (of theorem 5.13) By the aid of the proposition (5.14), we can assume that $\ker L_t = N$ is fixed and $\operatorname{ind}(L_t) = 0$. For any $t\in I$, there exist $\varepsilon > 0$ such that for any $s\in I_{t,\varepsilon}$, $L_s\in\mathrm{iso}(D, R_t)$. If we decompose $X = D\oplus N$ and $Y = R_t\oplus N_t'$ then L_s has the representation $L_s = (L_{11}(s)\oplus 0,\, L_{21}(s)\oplus 0)$. Define the isomorphism $j_{t,-s} = (L_{11}(t)\circ L_{11}^{-1}(s)\oplus 0, -L_{21}(s)\circ L_{11}^{-1}(s)\oplus \mathrm{Id})$ then $j_{t,-s}\circ L_s = L_t$. Let C_t be a finite rank map such that $(L_t + C_t)\in \mathrm{iso}(X, Y)$. Note that $C_t\in\mathrm{iso}(N, N_t')$. It implies that $L_s + j_{s,-t}\circ C_t$ is an isomorphism where $j_{s,-t} := j_{t,-s}^{-1}$. We observe that $C_s = j_{s,-t}\circ C_t\colon N\to N_s'$ is an isomorphism. Since I is compact, we can proceed in finitely many steps to construct a continuous finite rank maps of the following form

$$C_t = j_{t,-t_n}\circ j_{t_n,-t_{n-1}}\circ\ldots\circ j_{t_1}\circ C_0. \tag{5.25}$$

Define finally $C_s = j_{s,-t} \circ C_t \circ \pi_2: X \to Y$. □

Proposition 5.15. *Assume L_t is a continuous family of Fredholm maps with index zero, then there exist a continuous family of isomorphisms A_t such that $A_t \circ L_t = \mathrm{Id} - K(t)$ and $L_t \circ A_t = \mathrm{Id} - K'(t)$ where K, K' are compact maps.*

Proof. Let C_t be a continuous family of finite rank maps such that $L_t + C_t = A_t$ is an isomorphism. It is easily seen that $A_{-t} \circ L_t$ and $L_t \circ A_{-t}$ are compact perturbations of the identity map where $A_{-t} = A_t^{-1}$. □

The above proposition holds also when $\mathrm{ind}(L_t) = n \neq 0$.

Proposition 5.16. *Assume $L_t \in L(X, Y)$ is a continuous family of maps, then L_t is a family of Fredholm maps if and only if there exist a continuous family of map $A_t \in L(Y, X)$ such that $A_t \circ L_t = \mathrm{Id} - K_t$ and $L_t \circ A_t = \mathrm{Id} - K_t'$ where K_t, K_t' are compact maps.*

Proof. Assume L_t is a family of Fredholm map, then their indices are constant. Let us modify L_t such that $N \subset X$ is the kernel for any L_t. As we have seen, the index of the modified map is the same as the index of the original map. As we have seen before, there exist $s > 0$ small such that for any $0 \leq t < s$ we have $j_{-t} \circ L_t = L_0$. By proposition (5.10), there exist A_0 such that $A_0 \circ L_0 = \mathrm{Id} - K_0$ and $L_0 \circ A_0 = \mathrm{Id} - K'$. Define $A_t = A_0 \circ j_{-t}$. It is easily seen that $L_t \circ A_t = \mathrm{Id} - K_0$. Proceeding this way we cover I in a finite steps. □

5.2.1.4 Quasi-linear Fredholm maps

Let X, Y be Banach spaces and $X \hookrightarrow X_1$ a compact embedding. The map f: $X \to Y$ is quasi-linear Fredholm if f has the representation $f(x) = L_x(x) + C(x)$ where $x \to L_x$ is continuous and compact from X_1 to $L(X, Y)$ and C is compact. For example, consider the following quasi-linear equation homogeneous Dirichlet problem on $\Omega \subset \mathbb{R}^n$:

$$\sum_{|\alpha| \leq 2} A_\alpha(x, u, Du) D^\alpha u - g(x, u, Du) = 0.$$

where A_α, g are smooth enough. Let $X = H^2(\Omega) \cap H_0^1(\Omega)$ and $Y = L^2(\Omega)$. The map $u \to A_\alpha(., u, Du) D^\alpha$ is continuous and compact, since $u_n \rightharpoonup^Y u$ then $u_n \xrightarrow{H^1} u$ and then $A_\alpha(., u_n, Du_n) \xrightarrow{Y} A_\alpha(., u, Du)$. This implies that $L_{u_n} = A_\alpha(., u_n, Du_n) D^\alpha \to A_\alpha(., u, Du) D^\alpha = L_u$ in $L(X, Y)$. In a similar way, it is clear that g is continuous and compact from X to Y. Impose additional conditions on A_α we conclude that the map $L_u(u)$ is isomorphism and then we can write the equation as $u - L_u^{-1} g[u] = 0$ where $g[u] = g(., u, Du)$. Since g is compact then the equation is written as $(\mathrm{Id} - K)[u] = 0$ with $K = L_u^{-1} g$ compact. Now it is possible to apply the Leray-Schauder degree for the Schauder map $F = \mathrm{Id} - K$.

Similarly it is possible to rewrite the fully nonlinear elliptic equations (under suitable boundary conditions) as the Schauder map. First let us define the non-linear Fredholm maps.

Definition 5.17. *Assume f: $G \subset X \to Y$ is a nonlinear C^1 map, we say f is a nonlinear Fredholm map if $\forall x \in G$, $f_x' \in \mathrm{Fred}(X, Y)$ where f_x' is the Frechet derivative. We define $\mathrm{ind}(f) = \mathrm{ind}(f_x')$ for some arbitrary $x \in G$.*

Proposition 5.18. *The definition (5.17) is well-defined.*

Proof. Choose an arbitrary $x \in G$, for any $y \in G$ let γ: $[0, 1] \to G$ is a path that $\gamma(0) = x$ and $\gamma(1) = y$. As we have seen in proposition (B.2), $\mathrm{ind}(f_x')$ is constant then $\mathrm{ind}(f_x') = \mathrm{ind}(f_y')$. □

The deep result in this direction is the existence of a parametrix for the nonlinear Fredholm map f as follows. The proof of the following theorem can be found in [38].

Theorem 5.19. *Assume that X, Y are Banach spaces and $f \in C^1(X, Y)$ is a nonlinear Fredholm map of index 0. There exist a continuous finite rank map C: $X \to L(X, Y)$ such that $f_x' + C_x' \in \mathrm{iso}(X, Y)$ for $x \in X$.*

5.2.1.5 Fredholm maps and $(S)_+$ maps

Let $F\colon \mathrm{Id}-K$ is a Schauder map defined on the Hilbert space H. It is straightforward to show that the map $A\colon H\to H$ defined as $A(u)(\varphi)=(u-K(u),\varphi)_H$ is a map of class $(S)_+$. According to this observation, the Leray-Schauder degree for F is the same as the degree for the map A since $F[u]=0$ if and only if $A(u)=0$ and both F and A have the same finite rank reductions on the finite dimensional spaces.

If $h\colon H\to H$ is a homeomorphism, then the map $G\colon h+K$ for $K\colon H\to H$ compact is a Fredholm map of index 0 and then a degree can be defined for G, see Fitzpatrick and Pejsachowiz [38],[39]. However, the degree defined is not invariant under the continuous homotopy (in fact its absolute value is invariant) and it turns out that it is impossible to have a degree theory for general Fredholm maps satisfying all classical properties of degree theory.

It is easily seen and it is proved in the following proposition that for "uniformly elliptic homeomorphisms" that is defined below, the degree defined by Fitzpatrick and Pejsachowicz is invariant. We have shown in the next section that this case fully covers the fully nonlinear elliptic equation. This consideration fully covers the work of Y. Y. Li [48].

Proposition 5.20. *Let H be a Hilbert space and $h\colon H\to H$ a linear homeomorphism with the property*

$$(h(v),v)_H\geq\theta\,\|v\|^2,\quad \theta>0, v\in H$$

then the map $A=h+K$ for $K\colon H\to H$ compact, is a map of class $(S)_+$.

Proof. Let $v_n\rightharpoonup v$ and

$$\operatorname{limsup}(A(v_n),v_n-v)_H\leq 0. \tag{5.26}$$

Since K is compact then

$$(K(v_n),v_n-v)\to 0,$$

and

$$\operatorname{limsup}\,(A(v_n),v_n-v)=\operatorname{limsup}\,(h(v_n),v_n-v)_H=\operatorname{limsup}\,(h(v_n-v),v_n-v)\geq$$
$$\geq\theta\operatorname{limsup}\|v_n-v\|^2.$$

Now the inequality (5.26) implies that $v_n \to v$. □

5.3 Second order elliptic equations

The very well known Fredholm maps with index 0 are isomorphisms. In Chapter(1) we introduced the continuation method for isomorphisms, however the application of continuation theorem for partial differential equations are very limited. Let us present one straightforward result in this direction. First recall the version of global homeomorphism (Theorem (1.11)) that we have proved in the Appendix. We can use that result to prove the following theorem. L. Nirenberg [64] proved the existence of solution through degree theoretic argument for a more general form of equation (5.27) below. Here we will prove that the associated operator is a homeomorphism.

Theorem 5.21. *Let $\Omega \subset \mathbb{R}^n$ is a bounded domain with the smooth boundary and $f(D^{\leqslant 2}u)$ is C^1 with respect to its arguments such that for all u*

$$\left|\frac{\partial f}{\partial (D^\alpha u)}(D^{\leqslant 2}u)\right| \ll 1, \quad |\alpha| \leq 2. \tag{5.27}$$

Then the operator $G: C_0^{2,\delta}(\Omega) \to C^{0,\delta}(\Omega)$ defined as

$$G(u) = \Delta u + f(D^{\leqslant 2}u),$$

is homeomorphism.

Proof. First we note that Δ: $C_0^{2,\delta}(\Omega) \to C^{0,\delta}(\Omega)$ is an isomorphism and therefore there exist $M > 0$ such that $\|\Delta u\|_{0,\delta} \geq M\,\|u\|_{2,\delta}$. Define the operator $F(u) = f(D^2 u)$. It follows that F is C^1 from $C_0^{2,\delta}(\Omega)$ to $C^{0,\delta}(\Omega)$ and

$$F'(u)(v) = \sum_{|\alpha| \leq} f_\alpha(D^{\leqslant 2}u)\, D^\alpha v, \qquad f_\alpha = \frac{\partial f}{\partial\, (D^\alpha u)}$$

According to the condition (5.27), it follows that $\|F'(u)\|$ is sufficiently small for all $u \in C_0^{2,\delta}(\Omega)$ and therefore the C^1 operator $A = \Delta + F$ is such that there exist $M' > 0$ such that $\|(\Delta + F'(u))(v)\|_{0,\delta} \geq M'\,\|v\|_{2,\delta}$. Now the above theorem guarantees that $D + F$ is homeomorphism. □

In the above proposition, the condition on the norm of $\frac{\partial f}{\partial (D^\alpha u)}$ is too restrictive, however, the condition that the map $f(D^{\leqslant 2}u)$ is compact is more common. In this case the solution for the equation can be treated by Leray-Schauder degree.

5.3.1 Degree theory for the second order problem

Let $\Omega \subset \mathbb{R}^n$ be a domain with $\partial\Omega$ smooth, $n \geq 2$ and $f(x, D^{\leqslant 2}u) = 0$ be an uniformly elliptic second order equation on Ω where f is sufficiently smooth with respect to its arguments. Recall that f is uniformly elliptic at u if there exist $\theta > 0$ such that for $\xi \neq 0$ we have

$$\sum_{|\alpha|=2} -f_\alpha(x, D^{\leqslant 2}u)\, \xi^\alpha \geq \theta\, |\xi|^2,$$

where $f_\alpha = \frac{\partial f}{\partial (D^\alpha u)}$. Consider the space $X = H^{2+n_0}(\Omega) \cap H_0^1(\Omega)$ where $n_0 \geq \left[\frac{n}{2}\right] + 1$ an integer and $Y = H^{n_0}(\Omega)$.

Proposition 5.22. *The map $F: X \to Y$ is bounded.*

Proof. Use the compact embedding $X \hookrightarrow C^{2,\delta}(\bar{\Omega})$ for $\delta \in (0, 1)$ to conclude that $\|u\|_{C^2(\bar{\Omega})} \leq M$ for some $M > 0$ and then there exist $M_1 > 0$ such that $\|F[u]\|_{C^2(\bar{\Omega})} \leq M_1$. This implies in turn that $F[u] \in L^2(\Omega)$. For the multi-index β with $|\beta| = 1$ we have

$$D^\beta F[u] = \partial^\beta f(x, D^{\leqslant 2}u) + \sum_{|\alpha| \leq 2} f_\alpha(x, D^{\leqslant 2}u)\, D^\beta D^\alpha u,$$

where $\partial^\beta f(x,\, D^{\leqslant 2}u)$ is simply the partial derivative of f with respect to x_j with $\beta_j = 1$. Note that there exist $M_\alpha > 0$ such that $\|f_\alpha(., D^{\leqslant 2}u)\|_{C^2(\bar{\Omega})} \leq M_\alpha$ and therefore $\|F[u]\|_{H^1(\Omega)} < \infty$. In the general case, we can use the relationship

$$\frac{d^s}{dx^s} f(u(x)) = \sum a_I(u) \big(u^{(k_1)}\big)^{l_1} \big(u^{(k_2)}\big)^{l_2} \cdots \big(u^{(k_m)}\big)^{l_m},$$

where $k_1 l_1 + \cdots + k_m l_m = s$ and $a_I(u)$ are some functions of u. Using Holder inequality for $u \in X$, a simple calculation gives the following estimation for $1 \leq s \leq n_0$:

$$\|D^s F[u]\|_{L_2} \leq C_1 + C_2 \sum_{j=3}^{2+s} \|D^j u\|_{L_{rj}}, \tag{5.28}$$

where $r_j = \frac{2s}{j-2}$. Using the Nirenberg-Gagliardo inequality

$$\|D^j u\|_{L_{r_j}} \leq C\left\{ \|D^{2+n_0} u\|_{L_2}^{\frac{j-2}{n_0}} \|u\|_{C^2(\bar{\Omega})}^{1-\frac{j-2}{n_0}} + \|u\|_{C^2(\bar{\Omega})} \right\}, \tag{5.29}$$

we conclude that $\|D^s F[u]\|_{L_2} < \infty$. This establishes that $F[u] \in Y$ for $u \in X$. In addition if $\Omega \subset X$ is a bounded domain then inequalities (5.28),(5.29) establish that $F[\Omega]$ is bounded in Y. □

For $u \in X$ define the following linear operator $L_u: X \to Y$:

$$L_u(v) = \sum_{|\alpha| \leq 2} f_\alpha(x, D^{\leqslant 2} u)\, D^\alpha v, \tag{5.30}$$

Proposition 5.23. *For every $u \in X$ the linear map $L_u: X \to Y$ is a Fredholm map of index* 0.

Proof. It is easy to verify that L_u for any fixed $u \in X$ is a linear continuous map from X to Y. We show that for arbitrary $u \in X$, the map L_u can be written as a compact perturbation of a homeomorphism. Since f is elliptic, then L_u is elliptic and we have the following estimate, see [2]:

$$\|v\|_X \leq c\, \|L_u(v)\|_Y + C\, \|v\|_2. \tag{5.31}$$

For $k > 0$ sufficiently large define the compact map $K_u: X \to Y$ as

$$(K_u(v), \varphi)_Y = k(v, \varphi)_2, \tag{5.32}$$

where $k = k(u)$ and then define

$$\tilde{L_u}(v) = L_u(v) + K_u(v). \tag{5.33}$$

The computation for $\tilde{L_u}(v)$ gives

$$\left(\tilde{L_u}(v), \tilde{L_u}(v)\right)_Y = (L_u(v), L_u(v))_Y + k^2(v, v)_2 + 2k(L_u(v), v)_2. \tag{5.34}$$

Simple computations shows that for $C' > 0$ we have

$$(L_u(v), v)_2 + C'\,(v, v)_2 \geq 0,$$

where $C' = C'(u)$. We conclude that for $C_1, C_2 > 0$ we have

$$\left\| \tilde{L}_u(v) \right\|_Y \geq C_1 \, \|L_u(v)\|_Y + C_2 \, k \, \|v\|_2.$$

Substitution this estimate into (5.31) gives

$$\left\| \tilde{L}_u(v) \right\|_Y \geq C_2 \, \|v\|_X, \tag{5.35}$$

that implies $\tilde{L}_u$ is one to one. Define the homotopy for $t \in [0,1]$

$$h_u(t) = t(-\Delta + K) \, \Delta + (1-t) \, \tilde{L}_u \tag{5.36}$$

In a similar argument, it is seen that $\|h_u(t)(v)\|_Y \geq C_3 \|v\|_X$. Since $h_u(1) = -\Delta + K_u$ is isomorphism from X to Y then $h_u(0)$ is isomorphism from X to Y. $\tilde{L}_u$ is topological isomorphism by Banach isomorphism theorem. Using Fredholm alternative theorem $L_u = \tilde{L}_u - K_u$ is Fredholm of index 0. □

The above proposition guarantees that the map $F[u]\colon X \to Y$ is a nonlinear Fredholm map with index 0 since for arbitrary $u \in X$, L_u the linearization of f at u is Fredholm with index 0. On the other hand, since $\tilde{L}_u$ is homeomorphism, then $h_u := F[u] + K_u$ is a local homeomorphism.

Proposition 5.24. *Define $A\colon X \to X^*$ as*

$$\langle A(u), \varphi \rangle = (F[u], L_u(\varphi) + K_u(\varphi))_Y \tag{5.37}$$

If $u_n \rightharpoonup u$ in X and

$$\text{limsup} \, \langle A(u_n), u_n - u \rangle \leq 0,$$

then $u_n \to u$ in X.

Proof. for $\xi_n \in [0,1]$ and $v_n = u_n + \xi_n(u_0 - u_n)$ we can write

$$F[u_n] = f(x, D^{\leqslant 2} u_0) + \sum_{|\alpha| \leq 2} f_\alpha(x, D^{\leqslant 2} v_n) \, D^\alpha(u_n - u) = f(x, D^{\leqslant 2} u_0) + L_{v_n}(u_n - u).$$

Since K_u is compact and $u_n \to u$ in $L^2(\Omega)$ then

$$(f(x, D^{\leqslant 2} u_o), L_{u_n}(u_n - u_0) + K_{u_n}(u_n - u_0))_Y = (f(x, D^{\leqslant 2} u_0), L_{u_n}(u_n - u_0))_Y +$$
$$+ k(f(x, D^{\leqslant 2} u_0), u_n - u_0)_2 \to (f(x, D^{\leqslant 2} u_0), L_{u_n}(u_n - u_0))_Y.$$

Using compact embedding $X \hookrightarrow Y$ we obtain

$$(f(x, D^{\leqslant 2}u_0), L_{u_n}(u_n - u_0))_Y \to (f(x, D^{\leqslant 2}u_0), L_{u_0}(u_n - u_0))_Y \to 0.$$

Therefore we obtain

$$\langle A(u_n), u_n - u\rangle \to (L_{u_n}(u_n - u_0) + K(u_n - u_0), L_{u_n}(u_n - u_0) + K_{u_n}(u_n - u_0))_Y + \\ + \left(\sum_{|\alpha| \leq 2} [f_\alpha(x, D^{\leqslant 2}v_n) - f_\alpha(x, D^{\leqslant 2}u_n)] D^\alpha(u_n - u_0), L_{u_n}(u_n - u_0) \right)_Y .$$

Simple calculations show that

$$\left(\sum_{|\alpha| \leq 2} [f_\alpha(x, D^{\leqslant 2}v_n) - f_\alpha(x, D^{\leqslant 2}u_n)] D^\alpha(u_n - u_0), L_{u_n}(u_n - u_0) \right)_Y \to 0,$$

and therefore we obtain

$$\langle A(u_n), u_n - u\rangle \to \|L_{u_n}(u_n - u_0) + K_{u_n}(u_n - u_0)\|_Y^2.$$

Since $L_{u_n} + K_{u_n}$ is homeomorphism from X to Y, then according to the uniform ellipticity of f we conclude that for some $C > 0$ we have

$$\|L_{u_n}(u_n - u_0) + K_{u_n}(u_n - u_0)\|_Y \geq C\, \|u_n - u\|_X$$

Therefore $\langle A(u_n), u_n - u_0\rangle \to 0$ if and only if $u_n \to u_0$ in X and this completes the proof. □

Proposition 5.25. *The map* $A \colon X \to X^*$ *defined in* (5.37) *is bounded and demi-continuous.*

Proof. Clearly for every $u \in X$, the map $A(u)$ is linear and continuous and then $A(u) \in X^*$. Let $D \subset X$ is bounded, in a similar to the proposition (5.22) we can prove that there exist $M > 0$ such that $\|L_u(\varphi) + K_u(\varphi)\|_Y \leq M$ for $\varphi \in X$ with $\|\varphi\| \leq 1$ and $u \in D$. This establishes that there exist $C > 0$ such that $|\langle A(u), \varphi\rangle| \leq C$ since $|\langle A(u), \varphi\rangle| \leq \|F[u]\|_Y\, \|L_u(\varphi) + K_u(\varphi)\|_Y$. Now using uniformly boundedness theorem, $A(D)$ is bounded and then A is a bounded map. Now let $u_n \to u$ in X, then for arbitrary $\varphi \in X$ we can use estimates in the proposition (5.22) to conclude

$$\langle A(u_n), \varphi\rangle = (f(x, D^{\leqslant 2}u_n), L_{u_n}(\varphi) + K_{u_n}(\varphi))_Y \to (f(x, D^{\leqslant 2}u), L_u(\varphi) + K_u(\varphi))_Y =$$

$\langle A(u), \varphi\rangle$.

This proves that A is a demi-continuous map. □

Proposition 5.26. *Let* $F(t)[u] = f(t, ., D^{\leqslant 2} u)$ *be smooth in* (t, u) *for* $x \in X$ *and* $t \in [0,1]$ *and furthermore there exist* $\theta > 0$ *such that*

$$\sum_{|\alpha|=2} -f_\alpha(t, x, D^{\leqslant 2}u)\, \xi^\alpha \geq \theta\, |\xi|^2, \quad \xi \in \mathbb{R}^n - \{0\}.$$

Then the operator $A: [0,1] \times X \to X^*$ *defined as*

$$\langle A(t)(u), v\rangle = (f(t, x, D^{\leqslant 2}u), L_u(t)(v) + K_u(v))_Y,$$

where

$$L_u(t)(v) = \sum_{|\alpha| \leq 2} f_\alpha(t, x, D^{\leqslant 2}u)\, D^\alpha v,$$

is a admissible homotopy of class $(S)_+$*.*

Proof. The proof is straightforward and can be carried completely similar to the proof of the proposition (5.24). In fact if $t_n \to t$ and $u_n \rightharpoonup u$ in X then

$$\langle A(t_n)(u_n), u_n - u\rangle \to \|L_{u_n}(t_n)(u_n - u_0) + K_{u_n}(u_n - u_0)\|_Y^2. \tag{5.38}$$

According to the uniformly elliptic condition on $f(t, x, D^{\leqslant 2}u)$, we conclude that for some $C > 0$

$$\|L_{u_n}(t_n)(u_n - u_0) + K_{u_n}(u_n - u_0)\|_Y \geq C\, \|u_n - u\|_X,$$

and then the condition

$$\text{limsup}\, \langle A(t_n)(u_n), u_n - u\rangle \leq 0,$$

implies that $u_n \to u$. □

5.3.2 Fully nonlinear second order parabolic equations

In Chapter (2) we saw that if $A: X \to X^*$ a continuous map, $\varphi: V \to X$ the embedding of V into X and $B: V \to X$ a linear bounded map such that$B^* A \varphi$ is coercive then the equation $A(u) = f$ has a solution up to the kernel of B^*, that is there exist $\rho \in \text{Ker}(B^*)$ such that

$$Au = f + \rho. \tag{5.39}$$

Let us consider an example of the above theorem for evolution equations. Let Ω be an open bounded subset of $\mathbb{R}^n$, $Q=[0,\infty)\times\Omega$ and the space $W_{0,\gamma}^{1,m,p}(Q)$ is the Sobolev space weighted with $e^{-\gamma t}$ and zero on the boundary ∂Q up to order $m-1$, that is $D^\alpha u(\partial Q)=0$ for $|\alpha|\leq m-1$. The norm on $W_{0,\gamma}^{1,m,p}$ is

$$\|u\|_{1,m,p,\gamma}^p=\int_\Omega\int_0^\infty\left|\frac{\partial u}{\partial t}\right|^p e^{-\gamma t}+\sum_{|\alpha|=m}\int_\Omega\int_0^\infty|D^\alpha u|^p e^{-\gamma t}. \tag{5.40}$$

It is easy to justify that (5.40) is a norm. In fact we can show there exist $C>0$ such that

$$\int_0^\infty|u|^p e^{-\gamma t}\leq C\int_0^\infty\left|\frac{\partial u}{\partial t}\right|^p e^{-\gamma t}. \tag{5.41}$$

Since $u(0,x)=0$ then

$$\int_0^\infty d(|u|^p e^{-\gamma t})=0.$$

This implies that

$$\int_0^\infty|u|^{p-1}\operatorname{sign}(u)\,\frac{\partial u}{\partial t}\,e^{-\gamma t}=\frac{\gamma}{p}\int_0^\infty|u|^p e^{-\gamma t}. \tag{5.42}$$

But using Holder inequality we can write

$$\int_0^\infty|u|^{p-1}\operatorname{sign}(u)\,\frac{\partial u}{\partial t}\,e^{-\gamma t}\leq\left(\int_0^\infty|u|^p e^{-\gamma t}\right)^{1/q}\left(\int_0^\infty\left|\frac{\partial u}{\partial t}\right|^p e^{-\gamma t}\right)^{1/p}. \tag{5.43}$$

Therefore we obtain

$$\int_0^\infty|u|^p e^{-\gamma t}\leq\frac{p^p}{\gamma^p}\int_0^\infty\left|\frac{\partial u}{\partial t}\right|^p e^{-\gamma t}.$$

Now consider the operator $A_t\colon W_{0,\gamma}^{1,m,p}(Q)\to W_\gamma^{-1,-m,q}(Q)$ as follow:

$$A_t=\left|\frac{\partial}{\partial t}\right|^{p-2}\frac{\partial}{\partial t}+A,$$

where A is defined as

$$A(u)=\sum_{|\alpha|,|\beta|\leq m}(-1)^\alpha D^\alpha(a_{\alpha\beta}(x)\,D^\beta u).$$

Define the linear operator B as

$$B(u)=\frac{\partial u}{\partial t}\,e^{-\gamma t},$$

and then through a simple calculation we obtain

$$\langle A_t\,\varphi(u),B(u)\rangle=\int_{\Omega}\int_0^{\infty}\left|\frac{\partial u}{\partial t}\right|^p e^{-\gamma t}\quad+\frac{\gamma}{2}\langle A(u),u\,e^{-\gamma t}\rangle\geq c\|u\|^p_{1,m,p,\gamma}$$

5.3.2.1 Fully nonlinear parabolic equations

In this section we define a degree for the following equation

$$u'(t)+f(t,x,D^{\leqslant 2}u)=0,\quad t\in(0,T],\tag{5.44}$$

for $u\colon[0,t]\to X$ with the condition $u(0)=0$. f is smooth in (t,x,u) and for every $t\in[0,T]$ the map $f(t,x,D^{\leqslant 2}u)$ is elliptic with respect to u. By the aid of the degree that we define in sequel, we are able to establish the existence of the solution as well as the structure of the possible bifurcation solutions of equation (5.44). The proper space to work in for the equation (5.44) is

$$V=H_0^1(0,T,X)=\{u\colon[0,T]\to X,u(0)=0,u(t),u'(t)\in X\},$$

with the norm

$$\|u\|_V^2=\int_0^T\|u(t)\|_X^2+\|u'(t)\|_X^2\,dt<\infty.$$

The continuous pairing of $V^*=H^{-1}(0,T,X^*)$ and V is denoted by $\langle\langle,\rangle\rangle$. Define the linear operator $\partial_t\colon V\to V^*$ as

$$\langle\langle\partial_t u,v\rangle\rangle=\int_0^T(u'(t),v(t))_{L^2(\Omega)}\,dt.\tag{5.45}$$

Clearly if $v_n\to v$ in V then

$$|\langle\langle\partial_t u,v_n-v\rangle\rangle|\leq\int_0^T|(u'(t),v_n(t)-v(t))_{L^2(\Omega)}|\,dt\leq\int_0^T\|u'(t)\|_{L^2}\|v_n(t)-v(t)\|_{L^2}\leq$$

$$\left(\int_0^T\|u'(t)\|_{L^2}^2dt\right)^{1/2}\left(\int_0^T\|v_n(t)-v(t)\|_{L^2}^2\right)\to 0.$$

We conclude that $\partial_t u \in V^*$ whenever $u \in V$. Furthermore, $\partial_t: V \to V^*$ is continuous, since for $u_n \to u$ in V we can write

$$\|\partial_t u_n - \partial_t u\|_{V^*} = \sup_{\|\varphi_n\|=1, \varphi_n \in V} \langle\langle \partial_t u_n - \partial_t u, \varphi_n \rangle\rangle = \sup_{\|\varphi_n\|=1, \varphi_n \in V} \int_0^T (u_n'(t) - u'(t),$$

$$\varphi_n(t))_{L^2}\, dt \leq$$

$$\leq \sup_{\|\varphi_n\|=1, \varphi_n \in V} \int_0^T \|u_n'(t) - u'(t)\|_{L^2} \|\varphi_n(t)\|_{L^2} \leq \left(\int_0^T \|u_n'(t) - u'(t)\|_{L^2}^2 \right)^{1/2} \to 0$$

In addition ∂_t is monotone, since

$$\langle\langle \partial_t u - \partial_t v, u - v \rangle\rangle = \int_0^T (u'(t) - v'(t), u(t) - v(t))_{L^2} = \|u(T) - v(T)\|_{L^2}^2 - \langle\langle \partial_t u - \partial_t v,$$

$$u - v \rangle\rangle,$$

and therefore

$$\langle\langle \partial_t u - \partial_t v, u - v \rangle\rangle = \frac{1}{2} \|u(T) - v(T)\|_{L^2} \geq 0. \tag{5.46}$$

Now since $\partial_t: V \to V^*$ is continuous and monotone we conclude that ∂_t is maximal monotone.

Now, similar to the previous section, we can define a family of $(S)_+$ operators $A(t): X \to X^*$ for the term $f(t, x, D^{\leqslant 2} u)$, that is

$$\langle A(t)(u), \varphi \rangle = (f(t, x, D^{\leqslant 2} u(t)), L_u(t)(\varphi) + K_u(t)(\varphi))_Y. \tag{5.47}$$

Proposition 5.27. *The operator $A: V \to V^*$ defined as*

$$\langle\langle A(u), v \rangle\rangle = \int_0^T \langle A(t)\, u(t), v(t) \rangle\, dt \tag{5.48}$$

is a bounded, demi-continuous $(S)_+$ operator.

Proof. Let $u_n \to u$ in V. Since $A(t)$ is bounded from X to X^* then using (5.28) there exist $C_1, C_2 > 0$ such that for fixed $v(t) \in X$ with $\|v(t)\|_X \leq 1$ we have

$$|\langle A(t)\, u_n(t), v(t) \rangle| \leq \|A(t)\, u_n(t)\|_{X^*} \leq C_1 + C_2 \|u_n(t)\|_X. \tag{5.49}$$

Since

$$\int_0^T C_1 + C_2 \|u_n(t)\|_X < \infty, \tag{5.50}$$

we conclude that

$$\lim \langle\langle A(u_n), v\rangle\rangle = \lim \int_0^T \langle A(t)\, u_n(t), v(t)\rangle\, dt = \int_0^T \lim \langle A(t)\, u_n(t), v(t)\rangle =$$
$$= \int_0^T \langle A(t)\, u(t), v(t)\rangle = \langle\langle A\,(u), v(t)\rangle\rangle.$$

This establishes that $A: V \to V^*$ is demi-continuous. In addition if U is a bounded subset of V that is

$$\int_0^T \|u(t)\|_X^2 < \infty \quad u \in U, \tag{5.51}$$

then for $v \in V$ we can write

$$\int_0^T |\langle A(t) \quad u(t), \quad v(t)\rangle| \leq \int_0^T \|v(t)\|_X \quad \|A(t) \quad u(t)\|_{X^*} \leq$$
$$\left(\int_0^T \|v(t)\|_X^2\right)^{1/2} \left(\int_0^T \|A(t)\, u(t)\|_{X^*}^2\right)^{1/2} \leq$$
$$\left(\int_0^T \|v(t)\|_X^2\right)^{1/2} \left(\int_0^T (C_1 + C_2 \|u(t)\|_X)^2\right)^{1/2} < \infty.$$

This establishes the boundedness of the map $A: V \to V^*$. In order to prove that A: $V \to V^*$ is a map of class $(S)_+$, assume that $u_n \rightharpoonup u$ in V and

$$\operatorname{limsup} \langle\langle A\, u_n, u_n - u\rangle\rangle \leq 0,$$

or equivalently

$$\operatorname{limsup} \int_0^T \langle A(t)\, u_n(t), u_n(t) - u(t)\rangle \leq 0.$$

According to the relation (5.38), we can write

$$\int_0^T \|L_{u_n}(t)(u_n(t) - u_0(t)) + K_{u_n}(u_n(t) - u_0(t))\|_Y^2 \leq 0. \tag{5.52}$$

According to the uniform ellipticity of $f(t, x, D^{\leqslant 2} u)$, for every $t \in [0, t]$, there exist $C > 0$ such that

$$\|L_{u_n}(t)(u_n(t) - u_0(t)) + K_{u_n}(u_n(t) - u_0(t))\|_Y \geq C\, \|u_n(t) - u(t)\|_X.$$

In turn this implies that

$$\operatorname{limsup} \int_0^T \|u_n(t) - u(t)\|^2 \leq 0, \tag{5.53}$$

or equivalently $u_n \to u$ in V. □

Now the map $\mathcal{A}: \partial_t + A: V \to V^*$ is a map of class $(S)_+$ and therefore a topological degree can be defined for $\mathcal{A}$.

Chapter 6

Conclusion

The homotopy class of different types of monotone maps is stable. This is the main result that the whole of this thesis is based on. Since the homotopy class is stable, the degree construction for such mappings reduces to Brouwer degree on finite dimensional space. Since a broad class of nonlinear problems (and linear) in functional analysis, mathematical physics and convex analysis could be formulated in terms of monotone maps, the existence, multiplicity and bifurcation problems for the equations in those fields could be studied by the aid of such degree theory.

In this thesis, we studied the different constructions of degree for the class of monotone maps and generalized it for mappings missing desired continuity properties that is classically assumed for such mappings. Our construction is based on the direct use of the finite rank approximation that is originally used in the works of F. Browder for uniformly convex Banach spaces. The formulation of finite rank approximation in separable Banach spaces has a simple form that we used it in our study in this thesis. In addition to giving new proofs for some old theorems using the degree arguments, we studied variational inequalities (that is crucial in the field of optimal control theory and mathematical finance) by the aid of topological method and finite rank approximation method. We have shown that the variational inequality could be reduced to equations involving monotone maps.

Returning to the applications of degree theory, we could first study an open problem in the field of mathematical physics, the Doi-Onsager model for liquid crystals. In spite of the previous works in this field, our approach is mainly based on the generalization of Leray-Schauder degree. We have shown that the problem, regardless of the dimension of the embedding space, could be formulated in terms of degree of a map in a fairly simple way that gives the conditions for bifurcation of nematic solutions, the structure of the bifurcation and the uniqueness of the isotropic solution for low temperature of the liquid.

Traditionally, topological degree was employed to establish the existence of periodic solutions for nonlinear dynamical systems. The formulation of this problem as an abstract equation results in a fixed point problem that has been extensively studied by several authors. In our approach to this problem, we considered a one parameter family of nonlinear dynamical systems (with respect to ε) such that the existence of periodic solutions for $\varepsilon = 0$ is obtained by elementary methods in ODE and that for sufficiently small $\varepsilon \ll 1$, is established by the implicit function theorem. Using results in continuation theorem and degree theory, we have shown that the periodic solution (under specific conditions) would survive when ε increases to 1. Since every dynamical system could be reformulated as a one-parameter family of dynamical systems (in an appropriate way), this method enable us to establish the sufficient conditions for which the existence of periodic solution for such a systems can be established.

Fully non-linear uniformly elliptic equations as well as parabolic equations are studied in the last chapter. We followed here the works of I. Skrypnik to define an equivalent formulation operator of class $(S)_+$ for uniformly elliptic equations. Using the construction, we introduced a degree for fully non-linear parabolic equations. Results obtained in this chapter have immediate consequences for optimal control of distributed parameter systems governed by evolution equations and also the existence of periodic solution for partial differential equations.

6.1 Future research

6.1.1 Nonlinear hyperbolic equations

J. Berkovits in [6] proposed a degree for one-dimensional wave equation of the type

$$u_{tt} - u_{xx} + g(x, D^{\leqslant 2}u) = h, \tag{6.1}$$

for $u = u(t, x)$ where g is a map of class $(S)_+$. Imposing some condition on the growth rate of g, he could transform the equation to an equation of class $(S)_+$. In a general setting, he considered the equation of the type

$$L(u) + g(u) = h, \tag{6.2}$$

where u belong to some Hilbert space H and $L: D(L) \to H$ is a linear map with the property ker (L) = ker (L^*) and $\mathrm{Im}(L)$ is closed. Now for the closed subspace $M = D(L) \cap \mathrm{Im}(L)$, he proved that the solution of the equation (6.2) is equivalent to

$$Q(u - KQg(u)) + Pg(u) = (KQ - P)h, \tag{6.3}$$

where $Q: H \to M$ and $P: \mathrm{Id} - Q$ and $K = L_0^{-1}$ where L_0 is the restriction of L to M, that is $L_0: M \to M$. The degree using Galerkin approximation is defined in [10].

Similar to our method for fully nonlinear parabolic equations, we can consider the equation of the type

$$u_{tt} + g(t, x, D^{\leqslant 2}u) = h \tag{6.4}$$

where g is uniformly elliptic, that is, there exist $\theta > 0$ regardless of t such that

$$-\sum_{|\alpha|=2} \frac{\partial g}{\partial D^\alpha} g(t, x, \xi)\, \xi^\alpha \geq \theta\, |\xi|^2.$$

However, in this case the map $u \mapsto u_{tt}$ is not anymore a maximal monotone map. According to [6], we can consider the space $M \subset H$ and formulate the above problem as an equation of the type (6.3). By this method we think we can define a degree for the equation (6.4).

6.1.2 Higher order elliptic and parabolic equations

In the last chapter of this dissertation, we introduced a degree for fully non-linear second order parabolic equations. The construction is completely possible to be done for higher order equations. For the equation

$$u_t + f(t, x, D^{\leqslant 2m} u) = 0, \tag{6.5}$$

one can consider the space $X = H^{2m+n_0}(\Omega) \cap H_0^m(\Omega)$ first to define a map

$$A(t): X \to X^*,$$

that is of class $(S)_+$ if f is uniformly elliptic (regardless of t). That A is a demi-continuous and bounded is a subtle fact established through involved computations. I. Skrypnik [74] did the calculations when f is independent of t. Now, one can consider the space

$$V = \left\{ u, u\colon [0,T] \to X, u(0) = 0, \int_0^T \|u(t)\|_X^2 + \int_0^T \|u'(t)\|_X^2 < \infty \right\}$$

and define the operator $\mathcal{A}\colon V \to V^*$ as defined in the last chapter to show that A is a demi-continuous, bounded map of class $(S)_+$. Y.Y. Li [48], using a different method, proposed a degree for the fully nonlinear second-order elliptic equations and also an argument for the bifurcation of solutions. I believe that, similar argument can be employed (through our construction of degree) for fully non-linear higher order parabolic equation.

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Appendix A

Proposition A.1. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded subset, $a_{\alpha\beta}$ smooth functions on $\bar{\Omega}$ and $F: H_0^1(\Omega) \to H^{-1}(\Omega) = (H_0^1(\Omega))^*$ as defined below is uniformly elliptic:*

$$F(u)(x) = \sum_{|\alpha|,|\beta| \leq 1} (-1)^{|\alpha|} D^\alpha(a_{\alpha\beta}(x)\, D^\beta u), \quad x \in \Omega,$$

that is for any $\xi \in \mathbb{R}^n - \{0\}$ we have

$$\sum_{|\alpha|=2} a_{\alpha\beta}(x) \xi^\alpha \xi^\beta \geq c\, |\xi|^2.$$

Then F is Fredholm of index 0.

Proof. Denote $H = H_0^1(\Omega)$ and let $A: H \to H$ be the map defined as

$$(A(u), v)_H = \langle F(u), v \rangle.$$

Define the bounded bilinear form $\pi: H \times H \to \mathbb{R}$ defined as $\pi(u, v) = (A(u), v)_H$. By Garding inequality we have

$$\pi(u, u) + \lambda \|u\|_{0,2}^2 \geq c \|u\|_{1,2}^2.$$

Define the compact map $C: H \to H$ as $(C(u), v)_H = k(u, v)_{L^2}$ for $k > \lambda$. Therefore the bounded bilinear form

$$\tilde{\pi}(u, v) = \pi(u, v) + k\,(u, v)_{L^2},$$

has the property

$$\tilde{\pi}(u, u) \geq c \|u\|_{1,2}^2 + (k - \lambda)\, \|u\|_{0,2}^2 \geq c \|u\|_{1,2}^2.$$

Therefore $\tilde{\pi}$ is coercive that implies $\tilde{A} = A + C$ is an isomorphism according to Lax-Milgram theorem. Hence $A = \tilde{A} - C$ is Fredholm with index 0. □

Theorem A.2. *Assume that for certain $x_0 \in E$ the map $D_{x_0} f$ is an isomorphism and furthermore assume that there exist $M > 0$ such that $\|D_x f(z)\| \geq M \|z\|$ for all $x, z \in E$ then $f: E \to Y$ is homeomorphism.*

For the proof, we need the following lemma. Let us denote $(D_x f)^{-1} = D_x^{-1} f$:

Lemma A.3. *Assume that* $f\colon E \to Y$ *is a* C^1 *map and for some* $c > 0$ *we have* $\|D_z^{-1} f\| \le c$, $z \in Y$. *Furthermore assume that* f^{-1} *restricted to* $B(f(z), \varepsilon)$ *is a local* C^1 *homeomorphism, then for some* $r > 0$ *we have:*

$$\operatorname{diam} f^{-1}(B(f(z), \varepsilon)) \le r.$$

Proof. For $u, v \in B(f(z), \varepsilon)$ we have for w in the segment $[u, v]$ the following relation:

$$\|f^{-1}(u) - f^{-1}(v)\| \le 2\varepsilon \|D_w f^{-1}\| \le 2\varepsilon \|D_{f^{-1}(w)}^{-1} f\| \le 2\varepsilon c.$$

□

Proof. (of the theorem) For any $x \in E$, let γ is the path that connect x_0 to x. Since $D_{x_0} f$ is an isomorphism, inequality (1.15) implies that $D_x f$ is an isomorphism. Choose y arbitrary, then for $z = D_x^{-1} f(y)$ we have

$$\|D_x f(z)\| \ge M \|z\| \Rightarrow \|y\| \ge M \|D_x^{-1} f(y)\|.$$

Since y is arbitrary then $\|D_x^{-1} f\| \le \frac{1}{M}$. Therefore for any x the norm of $D_x^{-1} f$ is uniformly bounded. First we show that f is surjective. For any $y \in Y$ and $y \neq f(x_0)$, let ω denote the path that connect $f(x_0)$ to y. Let $B(f(x_0), \varepsilon)$ is a ball such that f^{-1} is a local homeomorphism. For $y_1 \in B(f(x_0), \varepsilon) \cap \omega$, choose the ball $B(y_1, \varepsilon_1)$ such that f^{-1} is a local homeomorphism. By the above lemma, the diameter of $f^{-1}(B(y_1, \varepsilon_1))$ is less than $2\varepsilon_1/M$ and this guarantees that the curve $f^{-1}(B(f(x_0), \varepsilon_1) \cap \omega)$ remain bounded in E. Since ω is compact then there exist $x \in E$ such that $f(x) = y$. Next we show that f is injective. Assume $x_1, x_2 \in E$ such that $f(x_1) = f(x_2)$. Let $\gamma_0(t), 0 \le t \le 1$ denotes the path that connect x_1 and x_2, then $\omega_0(t) = f(\gamma_0(t))$ is a loop with the base point $y = \omega_0(0)$. Define the continuous homotopy $\omega_s(t), 0 \le s \le 1$ such that $\omega_s(0) = \omega_s(1) = y$ and $\omega_0(t) = f(\gamma_0(t))$ and $\omega_1(t) = y$. We show that $\gamma_s = f^{-1}(\omega_s)$ is a path that connect x_1 to x_2. For any $t \in (0, 1)$, there exist $s_0 = s_0(t) > 0$ such that $\omega_s(t)$ is a smooth path for $s \in [0, s_0)$ lying in a ball $B(\omega_0(t), \varepsilon)$ homeomorphic to $f^{-1}(B(\omega_0(t), \varepsilon))$. By uniform continuity of $\|D_x^{-1} f\|$ we conclude that γ_{s_0} is also a path that connect x_1 to x_2. Continuing this to $s = 1$ we conclude that $\gamma_1 = f^{-1}(\omega_1)$ is a path that connect x_1 to x_2. But it is impossible since there exist a ball $B(y, \varepsilon)$ that is homeomorphic to $f^{-1}(B(y, \varepsilon)) \ni x_1$. Finally, the continuity of f^{-1} is obtained by inverse function theorem and this completes the proof. □

Theorem A.4. *Assume $\varphi: S^n \to S^n$, then* $\deg(\varphi) = 0$ *if and only if there exist a continuous extension A of φ on $\bar{\mathbb{B}}$, $A: \bar{\mathbb{B}} \to S^n$ that $A(x) = \varphi(x)$ for $x \in S^n$.*

Proof. Assume there exist an extension A for φ, then define the homotopy h_t: $S^n \to S^n$ as follows

$$h_t(x) = A(tx), 0 \leq t \leq 1,$$

then

$$\deg(\varphi) = \deg(h_t) = \deg(h_0) = 0.$$

Now assume $\deg(\varphi) = 0$, we first show that φ is trivial homotopy. If there is not any continuous extension $A: \bar{\mathbb{B}} \to S^n$, then every extension $\tilde{A}: \bar{\mathbb{B}} \to \mathbb{R}^{n+1}$, $\tilde{A}|_{S^n} = \varphi$, has a zero point, that is, there is point $a \in \mathbb{B}$ such that $\tilde{A}(a) = 0$. Since $\deg(\varphi) = 0$ then there should be even number of zero points in $\mathbb{B}$ with alternative index such that $\deg(\tilde{A}, \mathbb{B}^{n+1}, 0) = 0$. In below we show how to construct a homotopy to remove zero points. Choose points α, $\beta \in \mathbb{B}$ such that $\tilde{A}(\alpha) = \tilde{A}(\beta) = 0$ and without loss of generality assume $\alpha = (a, 0, ...0)$ and $\beta = (b, 0, ..., 0)$, $b > a$ and for sufficiently small ε, $\deg(\tilde{A}, \mathbb{B}_\varepsilon(\alpha), 0) = -1$ and $\deg(\tilde{A}, \mathbb{B}_\varepsilon(\beta), 0) = 1$. Now $\tilde{A} \sim \mathrm{Id}^-$ in $\mathbb{B}_\varepsilon(\alpha)$ and $\tilde{A} \sim \mathrm{Id}$ in $\mathbb{B}_\varepsilon(\beta)$ where $\mathrm{Id}^- = \mathrm{diag}(-1, 1, ..., 1)$. Let η, η_α, η_β are partition of unity subordinate in $\mathbb{B} - \{\bar{\mathbb{B}}_{\varepsilon/2}(\alpha) \cup \bar{\mathbb{B}}_{\varepsilon/2}(\beta)\}$, $\mathbb{B}_\varepsilon(\alpha)$ and $\mathbb{B}_\varepsilon(\beta)$ respectively. Define $\tilde{\tilde{A}}(x) = \tilde{A}(x)\ \eta(x) + \mathrm{Id}^-(x - \alpha)\ \eta_\alpha(x) + (x - \beta)\ \eta_\beta(x)$ because $\mathrm{GL}(\mathbb{R}^n)$ has two contractible components. Obviously $\tilde{\tilde{A}} \sim \tilde{A}$. Define the map $f(x^1) = \tilde{\tilde{A}}(x^1, 0, ..., 0)$ on the interval $(a - \varepsilon, b + \varepsilon)$. Define a homotopy $h_t(x^1) = f(x^1) + kt$ for sufficiently large $k > 0$ and $0 \leq t \leq 1$. This shows that there exist a homotopy for $\tilde{A}$ that removes zero points α, β. Therefor we can remove all zero points for $\tilde{A}$ because there are even numbers of zero points. This shows that every extension of φ is homotopic to a map having no zero point in $\mathbb{B}$ and then it is homotopic to a map of $\mathbb{B}$ to S^n. □

Theorem A.5. *Any integer valued map $d_1(A, D, y)$, $y \notin A(\partial D)$ that satisfies classical properties in (1.20) is equivalent to Brouwer degree* $\deg(A, D, y)$.

In order to prove the theorem we need the following lemma.

Lemma A.6. *Assume that A is a linear map on the ball $\bar{\mathbb{B}}_{\varepsilon}(a)$ for $\varepsilon > 0$ sufficiently small and d_1 is an arbitrary degree satisfies properties (1.20). If $J_A(a) < 0$ then $d_1(A, \mathbb{B}_\varepsilon(a), A(a)) = -1$.*

Proof. First note that

$$d_1(A, \mathbb{B}_\varepsilon(a), A(a)) = d_1(A, \mathbb{B}_\varepsilon(0), 0),$$

through considering the homotopy

$$d_1(A, \mathbb{B}_\varepsilon(t), t\,A(a)),$$

Therefore without loss of generality we can assume that $a = 0$. Since $J_A(0) < 0$, there is a homotopy from A to Id^- on $B_\varepsilon(0)$ where Id^{-1} is the identity matrix with -1 on the first entry. We have then

$$d_1(A, \mathbb{B}_\varepsilon(0), 0) = d_1(\mathrm{Id}^-, \mathbb{B}_\varepsilon(0), 0).$$

Consider the following map

$$f\colon \bar{D} \to \mathbb{R}^n,\ f(x^1, \ldots, x^n) = (x^1(x^1 - 1), x^2, \ldots, x^n)$$

where $D = \cup_{0 \leq t \leq 1} \mathbb{B}_\varepsilon(t)$. Obviously $f(x) = 0$ has two solutions $x = 0$, $x = (1, 0, \ldots, 0)$. Therefore by the second property we have

$$d_1(f, D, 0) = d_1(f, \mathbb{B}_\varepsilon(0), 0) + d_1(f, \mathbb{B}_\varepsilon(1), 0).$$

But f is homotopic to Id^- in $\mathbb{B}_\varepsilon(0)$ and is homotopic to Id in $\mathbb{B}_\varepsilon(1)$ for sufficiently small ε because

$$f(x) = f'_x(0)\,x = \mathrm{Id}^-\,x, \qquad x \in \mathbb{B}_\varepsilon,$$

$$f(x) = f'_x(1)\,x = x, \qquad x \in \mathbb{B}_\varepsilon(1).$$

Then by the first property we have

$$d_1(f, D, 0) = d_1(\mathrm{Id}^-, \mathbb{B}_\varepsilon(0), 0) + 1$$

Now consider the homotopy

$$h\colon [0,1] \times D \to \mathbb{R}^n,\ h_t(x) = h(t, x) = (x^1(x^1 - 1) + t(\frac{1}{4} + \varepsilon), x^2, \ldots, x^n).$$

Obviously

$$d_1(f, D, 0) = d_1(h_t, D, 0) = d_1(h_1, D, 0) = 0,$$

that proves the lemma. □

Proof. (of the theorem) Assume $A \in C^1(D) \cap C(\bar{D})$ and $A^{-1}(0) = \{a_i\}_{i=1}^n$, then by the third property of the topological degree we can write for sufficiently small $\varepsilon > 0$:

$$d_1(A, D, y) = \sum_{i=1}^{n} d_1(A, \mathbb{B}_\varepsilon(a_i), y).$$

But for small ε, we have $A(x) = A'(a_i)(x - a_i) + o(\varepsilon)$ that show A is homotopic to $A'(a_i)$ in $\mathbb{B}_\varepsilon(a_i)$. Because a_i are regular points, then $A'(a_i)$ are homotopic to Id or Id^- and by the above lemma

$$d_1(A, D, y) = \deg(A, D, y).$$

If A is continuous only, by approximating it with a C^1 map $\tilde{A}$ on D and using the homotopy invariance property we can write

$$d_1(A, D, y) = d_1(\tilde{A}, D, y) = \deg(\tilde{A}, D, y) = \deg(A, D, y).$$

□

Appendix B

For the proofs, the matrix representation of a linear map is of great help. Let $X = D_1 \oplus D_2$ and $Y = R_1 \oplus R_2$ are complemented spaces and $F\colon X \to Y$ a linear map. Define

$$F_{11} = \pi_1 F|_{D_1}, F_{12} = \pi_1 F|_{D_2}, F_{21} = \pi_2 F|_{D_1}, F_{22} = \pi_2 F|_{D_2}, \tag{B.1}$$

then we can represent write the representation $F = (F_{11} \oplus F_{12},\ F_{21} \oplus F_{22})$ or equivalently the following matrix form:

$$F = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix}. \tag{B.2}$$

For X, Y Banach spaces, $\mathrm{iso}(X, Y)$ denotes the subspaces of isomorphisms in $L(X, Y)$.

Proposition B.1. *Let X, Y be Banach spaces, $T, G\colon X \to Y$ Fredholm maps such that $\|T - G\|$ is sufficiently small. Then* $\mathrm{ind}(F) = \mathrm{ind}(G)$.

Proof. Let $F \in \mathrm{Fred}(X, Y)$, then we can decompose X, Y as $X = D \oplus N$ and $Y = R \oplus N'$ where $N = \ker(F)$ and $R = \mathrm{rang}(F)$ and $\mathrm{ind}(F) = \dim N - \dim N'$. Based on this decomposition, F has a decomposition of the form $(F_{11} \oplus 0,\ 0 \oplus 0)$ where $F_{11} \in \mathrm{iso}(D, R)$. Let $F(t)$ be a continuous small perturbation, then it has a matrix representation

$$F(t) = (F_{11}(t) \oplus F_{12}(t), F_{21}(t) \oplus F_{22}(t)).$$

Since the set of isomorphisms is open, $F_{11}(t)$ is an isomorphism and then $(F_{11}(t))^{-1}\colon = F_{11}^{-1}(t)$ is defined. We define two other isomorphisms $i(t), j(t)$ as $i(t) = (\mathrm{id} \oplus -F_{11}^{-1}(t) F_{12}(t), 0 \oplus \mathrm{id})$ and $j(t) = (\mathrm{id} \oplus 0, -F_{21}(t) F_{11}^{-1}(t) \oplus \mathrm{id})$. We observe that

$$j(t) \circ F(t) \circ i(t) = (F_{11}(t) \oplus 0, 0 \oplus -F_{21}(t) F_{11}^{-1}(t) F_{12}(t) + F_{22}(t)). \tag{B.3}$$

It is easy to see that

$$\ker(F(t)) = \ker j(t)\circ F(t)\circ i(t) = \ker(-F_{21}(t)F_{11}^{-1}(t)F_{12}(t) + F_{22}(t)), \tag{B.4}$$

$$\operatorname{coker}(F(t)) = \operatorname{coker} j(t)\circ F(t)\circ i(t) = \operatorname{coker}(-F_{21}(t)F_{11}^{-1}(t)F_{12}(t) + F_{22}(t)). \tag{B.5}$$

according to the argument in the finite dimensional spaces we conclude

$\dim\ker(-F_{21}(t)F_{11}^{-1}(t)F_{12}(t) + F_{22}(t)) - \dim\operatorname{coker}(-F_{21}(t)F_{11}^{-1}(t)F_{12}(t) + F_{22}(t)) =$

$$= \dim N - \dim N' = \operatorname{ind}(F).$$

□

Now we extend the above proposition to the case when $t\in[0,1]$.

Theorem B.2. *X, Y Banach spaces, $F\in\operatorname{Fred}(X, Y)$ and $F(t)$ for $t\in[0, 1]$ is a continuous homotopy $F_0 = F$ such that for any t, $F(t)\in\operatorname{Fred}(X, Y)$. Then $\operatorname{ind}(F(t)) = \operatorname{ind}(F)$.*

Proof. According to proposition (B.1), there exist sufficiently small $t_0 > 0$ such that $\operatorname{ind}(F(t))$ is constant for $t\in[0,t_0)$. Since $F(t_0)$ is a Fredholm then there exist $\varepsilon > 0$ such that $\operatorname{ind}(F(t_0-\varepsilon)) = \operatorname{ind}(F(t_0))$. But $\operatorname{ind}(F(t_0-\varepsilon)) = \operatorname{ind}(F_0)$ then $\operatorname{ind}(F_0) = \operatorname{ind}(F(t_0))$. We conclude that the set of t for which the index is constant is both open and closed and then it is $[0,1]$. □

There is a nice proof of the following index theorem using invariance property of the index.

Corollary B.3. *X, Y, Z Banach spaces, $F\in\operatorname{Fred}(X,Y)$ and $G\in\operatorname{Fred}(Y,Z)$ then $G\circ F\in\operatorname{Fred}(X,Z)$ and furthermore*

$$\operatorname{ind}(G\circ F) = \operatorname{ind}(F) + \operatorname{ind}(G). \tag{B.6}$$

Proof. It is straightforward to verify that

$$\dim\ker(G\circ F) \leq \dim\ker(F) + \dim\ker(G) < \infty,$$

$$\dim\operatorname{coker}(G\circ F) \leqslant \dim\operatorname{coker}(F) + \dim\operatorname{coker}(G) < \infty.$$

Therefore $G \circ F$ is a Fredholm operator. Now define $A(t)\colon X \oplus Y \to Y \oplus Z$ as

$$A(t) = (\cos(\pi t) F \oplus \sin(\pi t) \mathrm{Id}, -\sin(\pi t) G \circ F \oplus \cos(\pi t) G). \tag{B.7}$$

In fact $A(t)$ is the multiplication of following three Fredholm maps:

$$(\mathrm{Id} \oplus 0, 0 \oplus G)\colon X \oplus Y \to Y \oplus Y, \tag{B.8}$$

$$(\cos(\pi t) \oplus \sin(\pi t), -\sin(\pi t) \oplus \cos(\pi t))\colon Y \oplus Y \to Y \oplus Y,$$

$$(F \oplus 0, 0 \oplus \mathrm{Id})\colon Y \oplus Y \to Y \oplus Z.$$

Therefore $A(t)$ is a homotopy of Fredholm maps on $[0,1]$ and then

$$\mathrm{ind}(A(0)) = \mathrm{ind}(F \oplus 0, 0 \oplus G) = \mathrm{ind}(A(1)) = \mathrm{ind}(0 \oplus \mathrm{Id}, -G \circ F \oplus 0). \tag{B.9}$$

But $\mathrm{ind}(A(0)) = \mathrm{ind}(F) + \mathrm{ind}(G)$ and $\mathrm{ind}(A(1)) = \mathrm{ind}(G \circ F)$. $\square$