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Alternating p-fold Multilinear Maps in the Presence of an Inner Product Structure

With Special Attention to Symplectic Forms

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Abstract

Let W be a real inner product space and let $\eta : W^p \to \mathbb{R}$ be an alternating p-fold multilinear form on W. A smooth map $\tilde{\eta} : G_p^+(W) \to \mathbb{R}$ is induced from η. The level sets of this map are examined using transformation group theory in the cases of the restricted determinant and the symplectic forms. An intrinsic characterization of a symplectic form with respect to the given inner product structure is developed for this purpose.

Contents

1 Introduction

Let W be an n-dimensional real vector space. Let η be an alternating real-valued p-fold multilinear map, henceforward referred to as a p-form. Then η provides a definition for the oriented p-volume of a parallelepiped spanned by p vectors. Now, on a smooth manifold X, this can be done on each tangent space in a way which varies smoothly with points on X. Such a construction is known as a differential p-form on X and is used in a variety of contexts: de Rham cohomology, algebraic and differential topology, and Hodge theory, to name a few. Thus, any structural properties of differential p-forms may have applications in one or all of these fields. Here, one avenue for increasing such structural understanding is explored.

If W additionally has an inner product structure, it is possible to induce from η a smooth map $\tilde{\eta} : G_p^+(W) \to \mathbb{R}$, where $G_p^+(W)$ is the Grassmann space of oriented p-dimensional subvector spaces of W. How is this done? If $V \in G_p^+(W)$, let $\{v_1, \ldots, v_p\}$ be an orthonormal basis of V such that $(v_1, \ldots, v_p)$ represents a positive orientation on V. Then define

$$\tilde{\eta}(V) := \eta(v_1, \ldots, v_p). \tag{1}$$

Of course, there is a little work involved to ensure that such a map is well defined and smooth. However, it is possible to verify this and even to discover that the original p-form η can be recovered from the induced map. Thus a study of real-valued alternating p-fold multilinear maps on an inner product space is equivalent to the study of the induced maps on the Grassmann manifold.

Now, the Grassmann space is a compact homogeneous manifold. Thus, the smooth map $\tilde{\eta}$ has maximum and minimum values. Furthermore, by Sard's theorem, almost all points are regular. Now, the pre-image under a smooth real-valued function of an interval consisting entirely of regular values is made up of diffeomorphic level sets which, under appropriate group actions, are equivariant. Thus, the critical values of $\tilde{\eta}$ demarcate possible changes in the diffeomorphism classes of level sets.

Furthermore, as a compact homogeneous manifold under $O(n)$, the Grassmann space admits the application of transformation group theory. Suppose that G is a compact Lie group which acts on $G_p^+(W)$ and for which $\tilde{\eta}$ is G-equivariant. Then this group action provides symmetry information about the level sets. Specifically, each level set of $\tilde{\eta}$ is decomposed into orbits of G whose structure can be explicitly calculated. Furthermore, the Slice Theorem provides an explicit fiber bundle structure for a G-equivariant neighbourhood of any orbit. Thus, a local description is provided around each orbit. Moreover, transformation group theory guarantees the existence of a unique maximal orbit type. Orbits of this type, known as principal orbits, come together to form an open dense submanifold of $G_p^+(W)$. Finally, there are G-equivariant maps between principal orbits and non-principal orbits. These facts combined provide a global description of how the orbits of G come together to form $G_p^+(W)$. Moreover, since by the inverse function theorem level sets associated with regular values are submanifolds of $G_p^+(W)$, this analysis can be applied independently to almost every level set. Thus, transformation group theory provides a local and a global description of each regular level set.

As yet, an analysis of this type for all p-forms has not been attempted. As a first step in this direction, this thesis provides an in depth analysis for two types of p-forms. First, largely for illustrative purposes, the restricted determinant form created by defining

$$\eta(v_1, \ldots, v_p) := det[v_1| \ldots |v_p|w_{p+1}| \ldots |w_n] \tag{2}$$

for some fixed $w_{p+1}, \ldots, w_n \in W$ is studied. Thereafter, the more complex task of applying this methodology to the symplectic form, a non-degenerate bilinear form, is completed. Symplectic forms are commonly used in fields such as mechanics and symplectic topology. Thus, this study should be immediately applicable in addition to serving as an example of how general p-forms may be approached.

Summary of Major Results

First, the induced map $\tilde{\eta} : G_p^+(W) \to \mathbb{R}$ was proven to be well-defined and smooth. (2.4). It was also shown that η could be recovered from $\tilde{\eta}$ (2.5).

Next, in the study of the restricted determinant, it was discovered that only 0 and the extremal values are critical (3.7). Thus, all non-zero and non-extremal level sets are equivariantly diffeomorphic. A fiber bundle description of non-extremal level sets was given (3.9) and the structure of orbits in these level sets were determined (3.10) giving sufficient information to apply the Slice theorem and some more global results giving a complete description of each non-critical level set.

In the symplectic case new invariants of the symplectic form were uncovered: the symplectic spectrum $a_1 > \cdots > a_r > 0$ and associated symplectic eigenspaces $W_1, \ldots, W_r$ (4.12). It is worth noting that this spectrum is not completely unknown in the field of symplectic topology. For instance, it is provided as an example in [4]. However, we are unaware of any application of this information. In this study, the symplectic spectrum and eigenspaces show themselves to be intrinsic to the form. The symplectic eigenvalues turned out to be the critical values of $\tilde{\eta}$ (6.5). Furthermore, we find that the maximal subgroup of $O(W)$ which preserves the level sets of η is $U(W_1) \times \ldots \times U(W_r)$ where U refers to a unitary group in terms of a complex structure imposed on W(6.6). The symplectic eigenspaces are also indispensible in the calculation of the manifold structure of orbits under that group action (6.8). It should be noted that this calculation gave a homogeneous structure to the extremal level sets.

2 Framework

Let W be an n-dimensional real vector space with positive definite inner product. Let W^p denote the p-fold cross-product of W and let $\eta : W^p \to \mathbb{R}$ be an alternating p-fold multilinear map on W, henceforward referred to as a p-form. This section provides the framework under which the structural properties of η will be studied.

In the presence of an inner product additional properties come to light. For instance, there is a natural isomorphism between the space of p-forms on W and the space of $(n-p)$-forms on W. To see this, we start by showing that by the universal mapping property of alternating multilinear maps [6, p.57] there is a natural isomorphism between the dual space of the p-fold wedge product of W, $(\bigwedge_p(W))^*$ and the space of p-forms. Thus, a study of p-forms is equivalent to a study of functionals on $\bigwedge_p(W)$.

Define $* : \bigwedge_p(W) \to \bigwedge_{n-p}(W)$ by requiring that for any orthonormal basis, $\{w_1, \ldots, w_n\}$ representing positive orientation,

$$*(w_1 \wedge \cdots \wedge w_p) = w_{p+1} \wedge \cdots \wedge w_n. \tag{1}$$

Since this is required for any positively oriented basis and, in particular, for any orientation preserving re-ordering of a given basis, $*$ is defined on a basis of $\bigwedge_p(W)$ and can be extended linearly. This function, known as the Hodge $*$ operator, provides a natural isomorphism between $\bigwedge_p(W)$ and $\bigwedge_{n-p}(W)$. Clearly, this induces an isomorphism between the dual spaces $\left(\bigwedge_p(W)\right)^*$ and $\left(\bigwedge_{n-p}(W)\right)^*$ which, by the universal mapping property of alternating multilinear maps, induces an isomorphism between the space of p-forms on W and the space of $(n-p)$-forms on W. Thus, it will only be necessary to study p-forms on W for $p \leq \frac{n}{2}$.

Another consequence of the universal mapping property of alternating multilinear maps and the inner product structure on W has much more far-reaching consequences. Any linear functional on $\bigwedge_p(W)$ is defined by its values on a basis of $\bigwedge_p(W)$. Recall the Stiefel manifold of orthonormal p-frames,

$V_p(W)$ which consists of all collections of p orthonormal vectors in W. Now suppose that $(v_1, \ldots, v_p) \in V_p(W)$ and consider $v_1 \wedge \cdots \wedge v_p$. Clearly, the set $\{v_1 \wedge \cdots \wedge v_p \mid (v_1, \ldots, v_p) \in V_p(W)\}$ spans $\bigwedge_p(W)$. Thus, a p-form η on W induces a smooth real valued function $\hat{\eta}$ on $V_p(W)$.

Further recall the Grassmann space of oriented p-dimensional subspaces, $G_p^+(W)$, a quotient space of $V_p(W)$ under the action of SO(p). It turns out, as will be proven in (2.4) that $\hat{\eta}$ factors through the SO(p) action. Thus, we can define $\tilde{\eta} : G_p^+(W) \to \mathbb{R}$ as follows. For any $V \in G_p^+(W)$, choose an orthonormal basis $(v_1, \ldots, v_p)$ which represents the orientation on V. Then

$$\tilde{\eta}(V) := \hat{\eta}(v_1, \ldots, v_p). \tag{2}$$

In fact, as shown in (2.5), given only knowledge of the function $\tilde{\eta}$ on $G_p^+(W)$ induced from a p-form η, we can recover all of the original p-form. Thus, a study of p-forms can be meaningfully reduced to a study of particular smooth maps on $G_p^+(W)$.

In general, it is only necessary to consider level sets of positive values α. Suppose that V is a p-dimensional subspace of W. Denote by V^+ and V^- the two oriented copies of V (choosing positive orientation arbitrarily). Then, since η is alternating, $\tilde{\eta}(V^+) = -\tilde{\eta}(V^-)$. So, for any given $\alpha \in \mathbb{R}$, we can see that $\tilde{\eta}^{-1}(\alpha)$ and $\tilde{\eta}^{-1}(-\alpha)$ will have exactly the same structure. It would be possible to use this argument to justify considering $|\tilde{\eta}|$ on $G_p(W)$. However, in a neighborhood of the 0-level set it may be advantageous to consider $\tilde{\eta}$. Specifically, on $G_p^+(W)$, there is a $\mathbb{Z}_2$ action given by orientation reversal which leaves $\tilde{\eta}^{-1}(0)$ invariant because

$$\tilde{\eta}(V^+) = -\tilde{\eta}(V^-) = 0. \tag{3}$$

On the rest of $G_p^+(W)$, this action maps $\tilde{\eta}^{-1}(\alpha)$ to $-\tilde{\eta}^{-1}(\alpha)$. Thus, it may be possible to express $\tilde{\eta}^{-1}(0)$ as a double cover of some nearby level set. Although this possibility is not explored in this thesis, it is sufficient to affect a minor choice in notation. Furthermore, there exist instances where taking the absolute value involves unnecessarily losing smoothness at 0. Therefore, we continue to study $\tilde{\eta}$ on $G_p^+(W)$ and keep orientation symmetry in mind.

Finally, since $G_p^+(W)$ is compact, $\tilde{\eta}$ achieves a maximum λ_η on $G_p^+(W)$. It is assumed that the $\lambda_\eta = 1$. It is always possible to apply a scaling factor to get this unless $\tilde{\eta} \equiv 0$ and such a trivial case is uninteresting.

Before this analysis is done however, these assertions about how the function $\tilde{\eta}$ can be induced and its properties must be proven. We start with a clarification of the meaning of orientation on a vector space. This requires a recognition of the topological structure of the general linear group $GL(V)$ of a vector space V.

If $\dim(V) = n$, given a basis, GL(V) can be represented by the set of non-degenerate $n \times n$ matrices and, so is isomorphic to a subset of $\mathbb{R}^{n^2}$. Thus, the standard topology on $\mathbb{R}^{n^2}$ induces a topology on GL(V). It is with respect to this topology that we discuss the connectivity of GL(V).

2.1 Definition Let V be a vector space and define the set

$$\mathcal{B} := \{(v_1, \ldots, v_n) \mid \{v_1, \ldots, v_n\} \text{ is a basis of } V\}. \tag{4}$$

Define an equivalence relation on $\mathcal{B}$ by $B_1 \sim B_2$ if and only if the linear transformation which takes B_1 to B_2 is in the connected component of the identity element in GL(V). Then, the **orientation** on V defined by $B \in \mathcal{B}$ is the equivalence class of B under the relation $\sim$.

This definition supports the general practice of using an ordered basis to represent an orientation on V. However, it is also commonly accepted that there are two choices of orientation: positive and negative. This understanding is not implicit in (2.1) and requires a consideration of the number of connected components in GL(V).

2.2 Lemma Let V be a vector space and let GL(V) be the general linear group of V. Then with the standard topology, GL(V) has two connected components.

Proof Given a basis of V, $\mathrm{GL}(V)$ can be represented by

$$\mathcal{B} := \{(v_1, \ldots, v_n) \mid \{v_1, \ldots, v_n\} \text{ is a basis of } V\} \tag{5}$$

and the orthogonal group, $\mathrm{O}(V)$, can be represented by

$$\mathcal{V} := \{B \in \mathcal{B} \mid B \text{ is orthonormal } \}. \tag{6}$$

Then the Gram-Schmidt orthogonalization procedure provides a deformation retraction from $\mathcal{B}$ onto $\mathcal{V}$. Thus, $\mathrm{GL}(V)$ has the same number of components as $\mathrm{O}(V)$ since deformation retractions preserve homotopy sets and, of course, the 0'th homotopy set of a topological space counts its connected components. Therefore, we restrict our attention to $\mathrm{O}(V)$ which is homeomorphic to $\mathrm{O}(n)$ if $n = \dim V$.

Now, if S^p is the p-sphere, there is a fiber bundle $q : \mathrm{O}(p+1) \to \mathrm{S}^p$ with fiber $\mathrm{O}(p)$. Furthermore, any fiber bundle has a long exact sequence of homotopy sets associated with it [1, p.453]. Thus, in particular, the sequence

$$\pi_1(\mathrm{S}^p) \to \pi_0(\mathrm{O}(p)) \xrightarrow{f} \pi_0(\mathrm{O}(p+1)) \to \pi_0(\mathrm{S}^p) \tag{7}$$

is exact. Now, for $p > 1$, since S^p is both connected and simply connected,

$$\pi_1(\mathrm{S}^p) = \pi_0(\mathrm{S}^p) = \{1\}. \tag{8}$$

Thus, $f : \pi_0(\mathrm{O}(p)) \to \pi_0(\mathrm{O}(p+1))$ is a bijection. This means that $\mathrm{O}(p+1)$ has the same number of connected components as $\mathrm{O}(p)$. Therefore, the realization that $\mathrm{O}(1) = \{\pm 1\}$ and that $O(2) \cong S^1 \,\dot{\cup}\, (-1)S^1$ each have two components allows an inductive argument to complete the proof. □

This provides a rigorous definition of orientation on a vector space which coincides with common usage. Henceforward, the term **ordered basis** of V shall refer to an ordered set $B = (v_1, \ldots, v_n)$ such that $\{v_1, \ldots, v_n\}$ forms a basis of V.

Now that orientation on a vector space is an unambiguous concept, it is meaningful to discuss $G_p^+(W)$, the Grassmann space of oriented p-dimensional subspaces of W. The next step is to prove that a p-form η induces a smooth map $\tilde{\eta} : G_p^+(W) \to \mathbb{R}$ from which η can be recovered.

The first stage in this is the realization that the value of a p-form on a set of orthonormal bases of a p-dimensional space V is constant if the bases all represent the same orientation on V.

2.3 Lemma Let V be a p-dimensional vector space and let η be a real-valued alternating p-fold multilinear map on V. Let $\mathcal{B}_1 := (v_1, \ldots, v_p)$ and $\mathcal{B}_2 := (w_1, \ldots, w_p)$ be ordered orthonormal bases of V. If $\mathcal{B}_1$ and $\mathcal{B}_2$ represent the same orientation on V, then

$$\eta(v_1, \ldots, v_p) = \eta(w_1, \ldots, w_p). \tag{9}$$

Proof Now, let det be a determinant function on V which maps ordered orthonormal bases of V to $\{\pm 1\}$.

By definition, η is also a determinant function on V. However, any determinant function on V is a scalar multiple of det [2][p.103]. So let $\lambda \in \mathbb{R}$ be defined by

$$\eta = \lambda \cdot \det . \tag{10}$$

Then supposing, without loss of generality, that $\mathcal{B}_1$ and $\mathcal{B}_2$ both represent the orientation that det maps as positive, we have

$$\begin{aligned} \eta(v_1, \ldots, v_p) &= \lambda \cdot \det(v_1, \ldots, v_p) \\ &= \lambda \cdot 1 \\ &= \lambda \cdot \det(w_1, \ldots, w_p) \\ &= \eta(w_1, \ldots, w_p). \end{aligned} \tag{11}$$

□

This is sufficient to prove that it is possible to induce a smooth function $\tilde{\eta}$ on $G_p^+(W)$ from the p-form η.

2.4 Lemma A real-valued alternating p-fold multilinear function η on an n-dimensional real inner product space W induces a smooth map $\tilde{\eta} : G_p^+(W) \to \mathbb{R}$ such that if $(v_1, \ldots, v_p)$ is an ordered orthonormal basis representing positive orientation on a vector space $V \in G_p^+(W)$,

$$\tilde{\eta}(V) := \eta(v_1, ..., v_p). \tag{12}$$

Proof By lemma (2.3), it is clear that $\tilde{\eta}$ is well-defined.

In order to show that $\tilde{\eta}$ is smooth, we first confirm that η is smooth on W^p. Let $\mathcal{B} := \{w_1, \ldots, w_n\}$ be a basis of W and let

$$\mathcal{A} := \{\{w_{\sigma_1}, \ldots, w_{\sigma_p}\} \mid w_{\sigma_i} \in \mathcal{B} \text{ are distinct elements}\}. \tag{13}$$

Then, since η is multilinear, it is a polynomial on elements of $\mathcal{A}$. However, coordinate directions in W^p are also given by elements of $\mathcal{A}$. Thus, η is smooth on W^p. Furthermore, since $V_p(W)$, the Stiefel space of orthonormal p-frames in W, is an imbedded submanifold of W^p, $\eta|_{V_p(W)}$ is a smooth function on $V_p(W)$.

Let $\pi : V_p(W) \to G_p^+(W)$ be the standard quotient map. By (2.3), $\tilde{\eta}$ factors through π. Thus the following diagram commutes:

$$\begin{array}{ccc} V_p(W) & \xrightarrow{\eta|_{V_p(W)}} & \mathbb{R} \\ {\scriptstyle\pi}\downarrow & \nearrow_{\tilde{\eta}} & \\ G_p^+(W) & & \end{array} \tag{14}$$

Since π is a submersion, the universal property of submersions gives that $\tilde{\eta}$ is smooth as desired. □

It has now been shown that $\tilde{\eta}$ is well-defined and smooth. Structural information of any depth about η will only be achieved through a study of $\tilde{\eta}$ if it can be proven that $\tilde{\eta}$ encompasses all of the information inherent in η. The easiest way to do this is to show that one can recover η from $\tilde{\eta}$.

2.5 Proposition Given a p-form $\eta : W^p \to \mathbb{R}$ and inducing $\tilde{\eta} : G_p^+(W) \to \mathbb{R}$, it is possible to calculate η using only the information provided by $\tilde{\eta}$.

Proof Since η is multilinear, this information is encapsulated in the commutative diagram (14). More explicitly, let $(v_1, \ldots, v_p) \in W^p$ and let V be an oriented p-dimensional subspace of W containing $\{v_1, \ldots, v_p\}$. Let det be the determinant function on V which maps an orthonormal basis of V representing positive orientation to 1. Then use det to calculate the oriented p-volume λ of the parallelepiped generated by $(v_1, \ldots, v_p)$. Thus

$$\eta(v_1, \ldots, v_p) = \lambda \cdot \tilde{\eta}(V). \tag{15}$$

□

In addition, $G_p^+(W)$ is a homogeneous space under $O(W)$. Thus, the action of $O(W)$ on $G_p^+(W)$ provides symmetry information about $G_p^+(W)$. These symmetries can be used to our advantage. Specifically, any subgroup $G < O(W)$ which preserves the level sets will provide similar symmetry information about those level sets.

The overview of transformation group theory provided in Appendix B highlights the importance of the stabilizer of a point under a group action. Thus, at this point, it is reasonable to discuss the stabilizer of an oriented vector space V under an $O(W)$ action.

2.6 Lemma Let W be a real inner product space and let V be an oriented subspace of W. Then the orientation preserving stabilizer of V under the action of SO(W) on W is

$$O(W)_V = SO(V) \times O(V^\perp) \tag{16}$$

where $V^\perp$ is the orthogonal complement of V in W.

Proof First note that there is a natural inclusion of $SO(V)$ into $O(W)$. Observe that if $g \in \mathrm{SO}(V)$, it is possible to extend g to act on W by defining

$$g'(w) := \begin{cases} g(w) & \text{if } w \in V \\ w & \text{if } w \in V^\perp \end{cases} \tag{17}$$

and extending linearly. Therefore, by this method, it is possible to naturally include both $\mathrm{SO}(V)$ and $O(V^\perp)$ into $O(W)$. Furthermore, since $\mathrm{SO}(V)$ fixes $V^\perp$ and $O(V^\perp)$ fixes V, the two groups have trivial intersection. Thus, it is also meaningful to discuss the natural inclusion of $\mathrm{SO}(V) \times O(V^\perp)$ in $O(W)$. Specifically, a pair $(g, h) \in \mathrm{SO}(V) \times O(V^\perp)$ can be used to define a transformation of W by letting

$$k(w) := \begin{cases} g(w) & \text{if } w \in V \\ h(w) & \text{if } w \in V^\perp \end{cases} \tag{18}$$

and extending linearly.

Now it remains to show that this group is the orientation preserving stabilizer of V under the action of $O(W)$. Clearly, $\mathrm{SO}(V)$ is the group of orientation preserving transformations on V and $O(V^\perp)$ fixes V. Thus $\mathrm{SO}(V) \times O(V^\perp)$ certainly is orientation preserving and stabilizes V.

Furthermore, consider $g \in SO(W)$. By definition, g is norm-preserving so, in particular, $g|_V : V \to W$ is norm-preserving. So if g stabilizes V, $g|_V \in O(V)$ and if it additionally preserves orientation on V, $g|_V \in SO(V)$. Thus,

$$O(W)_V = SO(V) \times O(V^\perp). \tag{19}$$

□

We now address the issue of the regularity of points under the map $\tilde{\eta}$. As we shall see, it suffices to consider the regularity of points under $\eta|_{V_p(W)} : V_p(W) \to \mathbb{R}$. Recall the submersion $\pi : V_p(W) \to G_p^+(W)$ given by

$$\pi(v_1, \ldots, v_p) := V := \mathrm{span}\{v_1, \ldots, v_p\} \tag{20}$$

with the orientation defined by $(v_1, \ldots, v_p)$. Thus, we have the following commutative diagram of derivative maps:

$$\begin{array}{ccc} T_{(v_1,\ldots,v_p)}V_p(W) & \xrightarrow{d\hat{\eta}} & T_{\hat{\eta}(v_1,\ldots,v_p)}\mathbb{R} \\ {\scriptstyle d\pi}\downarrow & \nearrow {\scriptstyle d\tilde{\eta}} & \\ T_V G_p^+(W) & & \end{array} \tag{21}$$

Since $\hat{\eta}$ factors through π (2.3), $d\hat{\eta}(\ker(d\pi)) = 0$. Given this and the fact that $d\pi$ is surjective, we get that V is a critical point of $\tilde{\eta}$ if and only if $(v_1, \ldots, v_p)$ is a critical point of $\hat{\eta}$.

As just noted, the tangent vectors encompassed in $\ker(d\pi)$ have no impact on the criticality of $(v_1, \ldots, v_p)$ under $\hat{\eta}$. Geometrically, this is seen through the action of $SO(V)$ on W. There is an orbit through $x := (v_1, \ldots, v_p)$. Since $SO(V)$ is a compact Lie group, $SO(V).x$ is a submanifold of $V_p(W)$(B.7). Thus, $T_x(SO(V).x)$ is a subvector space of $T_x(V_p(W))$. Using an $O(W)$-equivariant Riemann structure on $V_p(W)$,

$$T_x(V_p(W)) = T_x(SO(V).x) \oplus (T_x(SO(V).x))^{\perp}. \tag{22}$$

As just observed, the regularity of V under $\tilde{\eta}$ is entirely dependent on the value of $d\tilde{\eta}$ on $(T_x(SO(V).x)^{\perp}$. The next lemma provides a convenient method for picking a basis on this space.

2.7 Lemma (The Normal Basis Picking Lemma) Let W be an n-dimensional inner product space and let $x := (v_1, \ldots, v_p) \in V_p(W)$. Extend to an orthonormal basis $\{v_1, \ldots, v_p, w_{p+1}, \ldots, w_n\}$ of W. Then the space orthogonal to $T_x(SO(V).x)$ in $T_x(V_p(W))$ has a basis given by

$$\mathcal{A} := \{d_{ij} \mid 1 \leq i \leq p, p+1 \leq j \leq n\} \tag{23}$$

where d_{ij} is the tangent vector given by varying the i'th coordinate of x in the direction w_j; i.e.

$$d_{ij} = \frac{d}{dt}(v_1, \ldots, \cos t\, v_i + \sin t\, w_j, \ldots, v_p)\bigg|_{t=0}. \tag{24}$$

Proof These tangent vectors are clearly linearly independent and clearly orthogonal to $T_x(SO(V).x)$. Thus, it remains to show that they span the normal space. For this, a dimension count is sufficient.

$$\begin{aligned}\dim V_p(W) &= \dim O(n)/O(n-p) \\ &= \frac{1}{2}n(n-1) - \frac{1}{2}(n-p)(n-p-1) \\ &= np - \frac{1}{2}(p^2+p) \\ &= (n-p)p + \frac{1}{2}p(p-1).\end{aligned} \tag{25}$$

Now,

$$\#\mathcal{A} = p(n-p). \tag{26}$$

and

$$\begin{aligned}\dim SO(V).x &= \dim SO(V) \\ &= \frac{1}{2}p(p-1).\end{aligned} \tag{27}$$

□

This lemma allows a corollary which makes identifying critical points of a p-form fairly easy.

2.8 Corollary Let W be an inner product space and let η be a p-form on W. Let $V \in G_p^+(W)$ and let $\mathcal{B} := \{v_1, \ldots, v_p\}$ be an orthonormal basis of V. Then V is a regular point of $\tilde{\eta}$ if and only if there is a $v_i \in \mathcal{B}$ and a $w \in V^\perp$ such that

$$\frac{d}{dt}\tilde{\eta}\left(\operatorname{span}(\{\sqrt{1-t^2}v_i + tw\} \cup (\mathcal{B} \setminus \{v_i\}))\right)\Big|_{t=0} \neq 0. \tag{28}$$

Proof If V satisfies such a condition, then its regularity is apparent. So suppose that V does not satisfy this condition, then define $x = (v_1, \ldots, v_p) \in V_p(W)$. Then, by the normal basis picking lemma (2.7), a basis $\mathcal{A}$ of vectors normal to $T_x(SO(V).x)$ can be picked such that since V does not satisfy (28), $d\hat{\eta}(\mathcal{A}) = \{0\}$. Since $\hat{\eta}$ is constant under an $SO(V)$ action (2.3), $d\hat{\eta}|_{T_x(SO(V).x)} \equiv 0$. Thus,

$$d\hat{\eta} \equiv 0 \tag{29}$$

on $T_x(V_p(W))$. Since $\pi : V_p(W) \to G_p^+(W)$ is a submersion,

$$d\tilde{\eta} \equiv 0. \tag{30}$$

□

2.9 Corollary Let W be an inner product space and let η be a p-form on W. Then, $x := (v_1, \ldots, v_p) \in V_p(W)$ is a critical point of the restriction $\hat{\eta} = \eta|_{V_p(W)}$ if all smooth paths through x which vary one component of x in isolation lie in the level set of x.

Proof This is a consequence of (2.8) □

Given these two corollaries, it should prove relatively straight-forward to determine which points are critical points of a given p-form η. This is important since, as will be shown next, any two level sets between adjacent critical points are diffeomorphic. Furthermore, in the presence of a group action which preserves level sets, this diffeomorphism is equivariant.

2.10 Proposition Let W be an inner product space and let η be a p-form on W which induces a smooth map $\tilde{\eta} : G_p^+(W) \to \mathbb{R}$. Further, let $a, b \in \mathbb{R}$ be two consecutive critical values of $\tilde{\eta}$. Then, for $\alpha, \beta \in]a, b[$, there is a diffeomorphism

$$\tilde{\eta}^{-1}(\alpha) \cong \tilde{\eta}^{-1}(\beta). \tag{31}$$

Furthermore, if G is any group acting on $G_p^+(W)$ which preserves the level sets of $\tilde{\eta}$, then this diffeomorphism is equivariant with respect to that group action.

Proof Let $x \in \tilde{\eta}^{-1}(\alpha)$. Then, by the straightening out lemma, there is a chart about x such that the local representation of the gradient field of $\tilde{\eta}$ on this chart is constant.

In fact, there is such a chart around each point in the level set of α. Since $G_p^+(W)$ is compact, $\tilde{\eta}^{-1}(\alpha)$ is compact and so $\tilde{\eta}^{-1}(\alpha)$ can be covered by a finite number of such charts. Thus, within the union of these charts, a tubular neighborhood of diffeomorphic level sets can be found around $\tilde{\eta}^{-1}(\alpha)$.

Finally, an application of Zorn's lemma will extend this tubular neighborhood to $\tilde{\eta}^{-1}(]a,b[)$. So construct a set of open subintervals of $]a,b[$:

$$\mathcal{A} := \{D \mid \forall \alpha_1 \in D, \tilde{\eta}^{-1}(\alpha_1) \cong \tilde{\eta}^{-1}(\alpha)\} \tag{32}$$

Of course, $\mathcal{A}$ is partially ordered by inclusion. Furthermore, iff $\{D_i\}$ is a chain in $\mathcal{A}$, consider

$$\bar{D} = \bigcup_i D_i. \tag{33}$$

Then $\bar{D}$ is an open subinterval of $]a,b[$. If $\alpha_1 \in \bar{D}$ then there is an i in the index set such that $\alpha_i \in D_i$. So, by definition, $\tilde{\eta}^{-1}(\alpha_1) \cong \tilde{\eta}^{-1}(\alpha)$. Thus, $\bar{D} \in \mathcal{A}$. So, by Zorn's lemma, $\mathcal{A}$ contains a maximal element, call it $]a_1,b_1[$.

Suppose that $]a_1,b_1[\neq]a,b[$. Without loss of generality, suppose that $b_1 \neq b$. Then, by the above argument, a tubular neighborhood around $\tilde{\eta}^{-1}(b_1)$ can be chosen containing diffeomorphic level sets of levels both greater and less than b_1. Suppose that $c < b_1 < d$ are such levels. Then

$$\tilde{\eta}^{-1}(d) \cong \tilde{\eta}^{-1}(c) \cong \tilde{\eta}^{-1}(\alpha). \tag{34}$$

So $]a_1,d[\in \mathcal{A}$ and $]a_1,d[\supset]a_1,b_1[$. Since $]a_1,b_1[$ was supposed to be maximal in $\mathcal{A}$, this provides a contradiction.

This proves that level sets between adjacent critical points are diffeomorphic. So now, consider the situation when the action of a group G preserves level sets. Then, since $\tilde{\eta}$ is equivariant, so is its gradient field. This is sufficient to guarantee the equivariance of the diffeomorphism. □

It is worth noting that since η is alternating, there is also a diffeomorphism between level sets associated with positive and negative values of the same magnitude. This diffeomorphism is not equivariant under any of the group actions considered in this thesis as they are chosen specifically to maintain level sets. However, if we define an action of $\mathbb{Z}_2$ on $G_p^+(W)$ by orientation reversal and on $\mathbb{R}$ as usual, we see that $\tilde{\eta}$ is also $\mathbb{Z}_2$-equivariant. Thus, if H is a group extension of

$\mathbb{Z}_2$ over a level-set preserving group, then $\tilde{\eta}$ is also H-equivariant. With respect to this group action, we get an equivariant diffeomorphism of all level sets associated with values in $]-a, -b[\cup]b, a[$ if a and b are consecutive critical points of $\tilde{\eta}$.

We now have a viable framework to work in. We can use η to induce a smooth map $\tilde{\eta} : G_p^+(W) \to \mathbb{R}$ from which η can be recovered. There is a formidable arsenal of theory which can be applied to such smooth maps including calculus on manifolds which finds diffeomorphisms between level sets associated to neighboring regular values and transformation group theory which, using symmetry information, provides explicit descriptions of each level set.

3 Restricted Determinants

In the previous section, a framework in which p-forms can be analyzed was developed. In this section, this type of analysis will be applied to a specific type of p-form: one known as the restricted determinant. This analysis will be of primary use as an illustration since the restricted determinant is simple enough to lend itself easily to study.

3.1 Definition Let W be an n-dimensional real vector space and let det be a determinant function on W. For $1 \leq p \leq n$, fix vectors $w_{p+1}, \ldots, w_n \in W$. The p-form given by

$$\eta(v_1, \ldots, v_p) := \det[v_1|\ldots|v_p|w_{p+1}|\ldots|w_n] \tag{1}$$

is known as the **restricted determinant** p-form defined by $(w_{p+1}, \ldots, w_n)$.

In the presence of an inner product structure, a determinant function is normally expected to give a unit volume for a unit Euclidean cube.

3.2 Definition Let W be an inner product space. A determinant function det on W is **consistent** with the inner product structure on W if for each orthonormal basis $\mathfrak{B}$ of W,

$$\det\mathfrak{B} = \pm 1. \tag{2}$$

This notion allows an unambiguous definition of a p-dimensional unoriented volume function. Specficially, if $\{v_1, \ldots, v_p\} \subset W$, choose a p-dimensional subspace $V \subset W$ such that

$$\{v_1, \ldots, v_p\} \subset V. \tag{3}$$

Then $< \cdot, \cdot > |_{V \times V}$ is an inner product on V. Let $\det_V$ be a determinant function on V consistent with this inner product. Then define

$$\mathrm{vol}_p(v_1, \ldots, v_p) := |\det_V(v_1, \ldots, v_p)|. \tag{4}$$

Since, up to scalar multiplication, there is only one determinant on any given space and since

$$V = \text{span}\,\{v_1, \dots, v_p\} \tag{5}$$

unless the vectors are linearly dependant, this is well-defined and corresponds with the customary understanding of a p-volume.

The natural first question is whether all p-forms can be expressed as restricted determinants. As will be shown in (3.5), for $1 < p < n-1$, the space of p-forms and the space of restricted determinants are of different dimensions confirming that there are many more p-forms than restricted determinants.

A determinant simply measures the oriented volume of the n-parallelepiped spanned by the vectors $(v_1, \dots, v_p, w_{p+1}, \dots, w_n)$. However, in this case, the last $n-p$ vectors determine a constant $n-p$ dimensional face of that n-parallelepiped. The value of $\eta(v_1, \dots, v_p)$ is thus dependent entirely on the p-dimensional face generated by the vectors v_1 through v_p. In the presence of an inner product, this leads to a geometric interpretation of the value $\eta(v_1, \dots, v_p)$. Let

$$Z := \text{span}\{w_{p+1}, \dots, w_n\} \tag{6}$$

and get an orthogonal decomposition of W:

$$W = M \oplus Z. \tag{7}$$

If $\{w_{p+1}, \dots, w_n\}$ are linearly dependant, $\eta \equiv 0$. In a non-trivial case,

$$\begin{aligned} &\dim Z = n - p \\ \text{and} \qquad &\dim M = p. \end{aligned} \tag{8}$$

Then the value of $|\eta(v_1, \dots, v_p)|$ will represent the p-volume of the parallelepiped created by taking the orthogonal projection of $\{v_1, \dots, v_p\}$ into M. The sign of $\eta(v_1, \dots, v_p)$ is determined by the orientation represented by $(v_1, \dots, v_p, w_{p+1}, \dots, w_n)$ in W. This will be explicitly demonstrated in (3.3).

This characterization of the function of the form leads to a few observations. First, considering the induced function $\tilde{\eta}$ on $G_p^+(W)$, it becomes clear that since, outside of M, the orthogonal projection function onto M is strictly norm-decreasing, the maximal level set of $\tilde{\eta}$ is a single vector space: M with an appropriate choice of orientation. Second, as will be seen in (3.8), the largest subgroup of $O(W)$ which preserves the value of the form is the direct product $SO(M) \times O(Z)$. Third, if $V \in G_p^+(W)$ has $V \cap Z \neq \{0\}$ then $\tilde{\eta}(V) = 0$.

This is the point at which transformation group theory can be applied. One need only describe which orbits lie in each level set and calculate the orbit type of each by means of calculating stabilizers. Then there will be sufficient information to directly apply the theory outlined in Appendix B to describe what the level sets of the function $\tilde{\eta}$ look like and how they are glued together to form $G_p^+(W)$. These calculations will be carried out in (3.9) and (3.10) respectively.

We begin our investigation of restricted determinants by explicitly elucidating which characteristic of a given vector space $V \in G_p^+(W)$ will determine which level set V belongs in. This will provide an understanding of the meaning of $\eta(v_1, \ldots, v_p)$ and will also facilitate later work.

3.3 Proposition Let W be a real n-dimensional oriented inner product space with determinant function consistent with the inner product and let η be the restricted determinant p-form determined by $(w_{p+1}, \ldots, w_n)$. Let

$$Z := \operatorname{span}(w_{p+1}, \ldots, w_n) \tag{9}$$

and let M be the choice of oriented orthogonal complement of Z such that $M \dotplus Z$ has positive orientation in W. Let $\pi_M : W \to M$ be the orthogonal projection function. For $V \in G_p^+(W)$, let $\{v_1, \ldots, v_p\}$ be an orthonormal basis of V and define

$$f(V) := \epsilon \operatorname{vol}_p(\pi_M(v_1), \ldots, \pi_M(v_p)) \tag{10}$$

where ϵ is the sign of the orientation represented by $(\pi(v_1), \ldots, \pi(v_p))$ in the oriented space M. This function is well-defined and if

$$\lambda_\eta := \mathrm{vol}_{n-p}(w_{p+1}, \ldots, w_n) \tag{11}$$

then

$$\tilde{\eta} = \lambda_\eta \cdot f. \tag{12}$$

Proof Let $(v_1, \ldots, v_p)$ be an orthonormal basis of V representing positive orientation. Then for each i, v_i can be decomposed into

$$v_i = x_i + y_i \tag{13}$$

where $x_i \in M$, $y_i \in Z$. Then

$$\begin{aligned} \tilde{\eta}(V) &= \det\left[v_1 | \ldots | v_p | w_{p+1} | \ldots | w_n\right] \\ &= \det\left[x_1 + y_1 | \ldots | x_p + y_p | w_{p+1} | \ldots | w_n\right] \\ &= \det\left[x_1 | \ldots | x_p | w_{p+1} | \ldots | w_n\right] \qquad \text{since } y_i \in Z \\ &= \epsilon \mathrm{vol}_p(x_1, \ldots, x_p) \cdot vol_{n-p}(w_{p+1}, \ldots, w_n). \end{aligned} \tag{14}$$

Let

$$\lambda_\eta := \det|_Z[w_{p+1} | \ldots | w_n]. \tag{15}$$

□

It should be noted that. as a consequence of this proposition, any two sets of vectors $\{w_{p+1}, \ldots, w_n\}$ and $\{Z_{p+1}, \ldots, Z_n\}$ which span the same space induce restricted determinant p-forms which only differ by a scalar multiple. As noted before, it shall be assumed that

$$\lambda_\eta = 1. \tag{16}$$

A consequence of this observation is that it is possible to calculate the dimension of the space of restricted determinants. Except when $p = 1$ or $p = n-1$, this dimension turns out to be smaller than the dimension of p-forms in general. Thus, not all p-forms can be represented as restricted determinants. The proof will depend on the following, purely numerical, lemma.

3.4 Lemma Let $n > 3$. Then, for all $1 < p < n-1$, $\binom{n}{p} > p(n-p) + 1$.

Proof First note that by symmetry, we need only prove this for $p \leq \frac{n}{2}$ since $\binom{n}{p} = \binom{n}{n-p}$. The proof is by induction on p.

Case $p = 2$:

$$\begin{aligned} \binom{n}{2} &= \frac{n!}{2!(n-2)!} \\ &= \frac{n(n-1)}{2} \end{aligned} \tag{17}$$

$\frac{1}{2}n(n-1) > 2n - 3$, for all $n > 3$, which concludes the base case.

Inductive case: assume that $\binom{n}{p} > p(n-p)+1$ for all $p \leq k$ where $k+1 \leq \frac{n}{2}$. We wish to prove $\binom{n}{k+1} > (k+1)(n-k-1)+1$.

$$\begin{aligned} \binom{n}{k+1} &= \frac{n!}{(k+1)!(n-k-1)!} \\ &= \frac{n!}{(k+1)k!\frac{(n-k)!}{n-k}} \\ &= \frac{n-k}{k+1}\binom{n}{k} \\ &> \frac{n-k}{k+1}(k(n-k)+1) \qquad \text{by the inductive hypothesis} \\ &= \frac{k}{k+1}(n-k)^2 + \frac{n-k}{k+1}. \end{aligned} \tag{18}$$

We wish to check whether

$$\frac{k}{k+1}(n-k)^2 + \frac{n-k}{k+1} > (k+1)(n-k-1) + 1. \tag{19}$$

Multiply both sides by $k+1$ to check whether

$$k(n-k)^2 + (n-k) > (k+1)^2(n-k-1) + (k+1). \tag{20}$$

We know, by hypothesis, that $k < \frac{n}{2}$ and, therefore, that $n - k > k + 1$. Thus, it remains to check whether

$$k(n-k)^2 > (k+1)^2(n-k-1). \tag{21}$$

But since the function $f(x) := \frac{x}{(x+1)^2}$ is decreasing on $[1, \infty]$, and $k < n - k - 1$, we have

$$\frac{k}{(k+1)^2} > \frac{n-k-1}{(n-k)^2}. \tag{22}$$

□

We are now in a position to prove that the dimension of the space of p-forms is larger than the dimension of the space of restricted determinants.

3.5 Proposition Let W be a real n-dimensional inner product space and let $1 < p < n - 1$. Then, the dimension of the space of p-forms on W is greater than the dimension of the space of restricted determinants on W.

Proof Consider first the space of p-forms. By the universal mapping property of multilinear maps [6, p. 57], there is a natural isomorphism between the dual space of the p-fold wedge product of W, $(\bigwedge_p(W))^*$ and the space of p-forms. Now, since this is finite dimensional space,

$$\wedge_p(W) \cong (\wedge_p(W))^* \tag{23}$$

so we're looking for $\dim \bigwedge_p(W)$. Let $\{w_1, \ldots, w_n\}$ be a basis for W. Then, a basis for $\bigwedge_p(W)$ is given by $\{w_{i_1} \wedge \cdots \wedge w_{i_p} | 1 \leq i_i < \cdots < i_p \leq n\}$ [6, p. 57]. So to count the basis of $\bigwedge_p(W)$, all one needs to do is count how many different collections of p distinct basis vectors one can get. Basic combinatorics tells us that this is $\binom{n}{p} = \frac{n!}{p!(n-p)!}$.

Now, consider the space of restricted determinants. As was shown in (3.3), each restricted determinant η is determined by a scalar, λ_η, and an $(n-p)$-dimensional vector space, $Z := \text{span}\{w_{p+1}, \ldots, w_n\}$. Thus, the dimension of the space of restricted determinants equals $\dim(G_{n-p}(W)) + 1 = p(n-p) + 1$ [6, p.130].

Applying lemma (3.4), the dimension of the space of p-forms is bigger than the dimension of the space of restricted determinants. □

Thus, although restricted determinants will be a good platform on which to demonstrate methods which can be used on all p-forms, they are certainly only a subset of the set of all p-forms.

Another consequence of (3.3) is that the extremal level sets each consist of a single vector space, M with positive and negative orientations distinguishing between the extrema.

3.6 Theorem Let W be an n-dimensional oriented real inner product space with determinant function and let η be a restricted determinant p-form on W determined by $(w_{p+1}, \ldots, w_n)$. Let Z and M be as before. Then the maximal and minimal level sets of η each contain only one element, M with positive and negative orientations respectively.

Proof As proven in (3.3),

$$\tilde{\eta}(V) = \lambda_\eta \cdot \mathrm{vol}_p\left(\pi_M(v_1), \ldots, \pi_M(v_p)\right) \tag{24}$$

where $(v_1, \ldots, v_p)$ determines an oriented unit p-cube in V.

The vectors $(\pi_M(v_1), \ldots, \pi_M(v_p))$ form a parallelepiped in M. Observe that the volume of a parallelepiped is bounded by the product of the norms of the vectors which define it. How is this calculated? Well, the volume of a parallelepiped determined by vectors $(u_1, \ldots, u_p)$ is calculated inductively. If X_p is a p-parallelepiped formed by the vectors $(u_1, \ldots, u_p)$, then let X_{p-1} be the parallelepiped formed by $(u_1, \ldots, u_{p-1})$. Then

$$\mathrm{vol}_p(X_p) = ||u_p|| \cdot \cos\theta \cdot \mathrm{vol}_{p-1}(X_{p-1}) \tag{25}$$

where θ is the minimum angle between u_p and the space spanned by $(u_1, \ldots, u_{p-1})$. Thus, an inductive argument gives, as desired, that the volume of a parallelepiped is at most the product of the norms of the vectors which define it.

Now, consider the specific parallelepiped, spanned by $(\pi_M(v_1), \ldots, \pi(v_p))$. The projection function is strictly norm-decreasing on any vector not already in M so, the p-volume is strictly less that 1 if even one vector $v_i \notin M$. Furthermore, this p-volume, of course, achieves 1 on the positively oriented unit cube of M.

□

Another consequence of the characterization of η (3.3) is that all the non-extremal and non-zero level sets of a given restricted determinant form $\tilde{\eta}$ on $G_p^+(W)$ are diffeomorphic. In fact, as was shown in (2.10), with respect to the action of a subgroup of $O(W)$ which preserves $\tilde{\eta}$, these level sets are equivariantly diffeomorphic. At this point, however, we apply calculus methods to determine critical points of $\tilde{\eta}$.

3.7 Theorem Let W be an inner product space with determinant consistent with the inner product. Let η be a restricted determinant p-form with $\lambda_\eta = 1$ given by vectors $(w_{p+1}, \ldots, w_n)$. Let

$$Z := \operatorname{span}\{w_{p+1}, \ldots, w_n\}. \tag{26}$$

Then $V \in G_p^+(W)$ is a critical point of $\tilde{\eta}$ if and only if either

$$V = M = Z^\perp, \tag{27}$$

or

$$\dim(V \cap Z) \geq 2. \tag{28}$$

Consequently, the only critical values of $\tilde{\eta}$ are $\{\pm 1, 0\}$.

Proof By (3.6), if $V = M$ then V is extremal and is clearly critical. Now, if $\dim(V \cap Z) \geq 2$, then changing any one vector in V in isolation will clearly leave a vector space with non-trivial intersection with Z. By (3.3) then, such a change remains in the 0-level set. Thus, by a corollary to the normal basis picking lemma, (2.9), V is a critical point of $\tilde{\eta}$.

On the other hand, suppose that $V \neq M$ and $\dim V \cap Z \leq 1$. Let $(v_1, \ldots, v_p)$ be an ordered orthonormal basis of V such that if $\dim V \cap Z = 1$, $v_1 \in Z$ and, in all cases, $v_1 \notin M$.

Then $\exists w \in W$ such that $\|w\| = 1, w \notin Z$, and $w \perp V$. Define a path $\gamma :]-1, 1[\to G_p^+(W)$ by

$$\gamma(t) := \text{span}\{\sqrt{1-t^2}v_1 + tw, v_2, \ldots, v_p\}. \tag{29}$$

Then

$$\begin{aligned} \tilde{\eta}(\gamma(t)) &= \det\left[(\sqrt{1-t^2}v_1 + tw)|v_2| \ldots |v_p\right] \\ &= \sqrt{1-t^2}\det[v_1|v_2|\ldots|v_p] + t\det[w|v_2|\ldots|v_p]. \end{aligned} \tag{30}$$

So

$$\frac{d}{dt}(\tilde{\eta}(\gamma(t))) = \frac{-t}{\sqrt{1-t^2}}\det[v_1|v_2|\ldots|v_p] + \det[w|v_2|\ldots|v_p]. \tag{31}$$

Thus

$$\begin{aligned} \left.\frac{d}{dt}\right|_{t=0}(\tilde{\eta}(\gamma(t))) &= \det[w|v_2|\ldots|v_p] \\ &\neq 0. \end{aligned} \tag{32}$$

Therefore V is a regular point of $\tilde{\eta}$. □

Since the only critical values are $\{-1, 0, 1\}$ the straightening out lemma (2.10) provides that all of the level sets between 0 and 1 are diffeomorphic. Recall that $\tilde{\eta}(V^+) = -\tilde{\eta}(V^-)$ where V^+ and V^- are two copies of the same vector space with opposing orientations. Thus the α and $-\alpha$ level sets are also diffeomorphic. Thus we conclude that all non-zero and non-extremal level sets are diffeomorphic.

Additional information about the level sets of $\tilde{\eta}$ can be obtained by analyzing their symmetries. Since $G_p^+(W)$ is a homogeneous space under $O(W)$, there is a natural action of $O(W)$ in $G_p^+(W)$. Thus, any subgroup of $O(W)$ for which $\tilde{\eta}$ is equivariant provides symmetry information about the level sets of $\tilde{\eta}$. Furthermore, the largest such subgroup of $O(W)$ provides the most information. It turns out that there is such a maximal subgroup and that it is unique.

3.8 Proposition Let W be an oriented n-dimensional real inner product space with a determinant function consistent with the inner product. Let η be a restricted determinant p-form on W defined by $(w_{p+1}, \ldots, w_n)$ with $\lambda_\eta = 1$ and let $\alpha \in [-1, 1]$. Then $G = SO(M) \times O(Z)$ is the largest subgroup of $O(W)$ whose action on $G_p^+(W)$ can be restricted to an action on $\tilde{\eta}^{-1}(\alpha)$.

Proof First, as shown in (3.3) the value of $\tilde{\eta}(V)$ is dependent on the p-volume of the projection of the unit cube of V onto M. Now, $SO(M) \times O(Z)$ has M as an invariant subspace and is volume preserving on M. Thus, the p-volume of any projection onto M is preserved by $SO(M) \times O(Z)$. Thus, we have confirmed that the action of G on $G_p^+(W)$ does keep $\tilde{\eta}^{-1}(\alpha)$ as an invariant subspace. We now need to prove that it is the largest subgroup to do so.

First, suppose that $\alpha = \pm 1$. Then, by (3.6), $\tilde{\eta}^{-1}(\alpha) = \{M\}$. We know that the orientation preserving stabilizer of M within $O(W)$ is $SO(M) \times O(Z)$ (2.6). Thus, for the extremal level sets, we have confirmed that G is the largest level-set preserving subgroup of $O(W)$.

Now suppose that $\alpha \in\ ]-1, 1[$ and suppose that $H \supseteqq G$ is a maximal subgroup of $O(W)$ whose action preserves $\tilde{\eta}^{-1}(\alpha)$ as an invariant subspace. Let $h \in H$ and let $(v_1, \ldots, v_p)$ be an orthonormal basis of M such that $(v_1, \ldots, v_p, w_{p+1}, \ldots, w_n)$ represents the positive orientation on W.

Then, for every $w \in Z$ with $||w|| = 1$, the α level set contains the vector space

$$V_1 := \operatorname{span}(\alpha v_1 + \sqrt{1 - \alpha^2}\, w, v_2, \ldots, v_p). \tag{33}$$

So. by assumption,

$$\tilde{\eta}(h.V_1) = \tilde{\eta}(V_1) = \alpha \tag{34}$$

So. by the multilinearity of η,

$$\begin{aligned} \alpha &= \eta\left(h.(\alpha \cdot v_1 + \sqrt{1-\alpha^2} \cdot w, v_2, \ldots, v_p)\right) \\ &= \alpha\eta(hv_1, \ldots, hv_p) + \sqrt{1-\alpha^2}\eta(hw, hv_2, \ldots, hv_p) \end{aligned} \tag{35}$$

But

$$V_2 := \operatorname{span}(\alpha v_1 - \sqrt{1-\alpha^2}w, v_2, \ldots, v_p) \tag{36}$$

also belongs to the α-level set. Thus,

$$\alpha = \alpha\eta(hv_1, \ldots, hv_p) - \sqrt{1-\alpha^2}\eta(w, hv_2, \ldots, hv_p). \tag{37}$$

Therefore, for every $w \in Z$,

$$\eta(w, hv_2, \ldots, hv_p) = 0. \tag{38}$$

Furthermore, from (37),

$$\eta(hv_1, \ldots, hv_p) = 1. \tag{39}$$

Thus, since the maximal level set has only one element, $(hv_1, \ldots, hv_p)$ forms an orthonormal basis for M and orients M positively. Thus, by (2.6),

$$h \in SO(M) \times O(Z). \tag{40}$$

□

This result is quite convenient because it means that studying one group action will provide maximal symmetry information both overall and for each independent level set. We now focus attention on an individual level set and calculate how many different kinds of orbits it contains. Since we already know that the extremal level sets are points, we concentrate on non-extremal level sets. Suppose that $\alpha \in \;]-1,1[$. We know from (3.3) that if $\tilde{\eta}(V) = \alpha$ then the projection of the unit cube in V onto M is a p-parallelepiped with oriented p-volume α. The action of $SO(M) \times O(Z)$ preserves not only the p-volume of the projection but also the lengths and relative angles of the projected vectors. Hence, for every different combination of lengths and internal angles which makes a parallelepiped of oriented p-volume α, there is at least one orbit. In fact, in each level set, there is exactly one orbit for each configuration. This is because $SO(M) \times O(Z)$ acts transitively on the set of all vector spaces whose unit cube projections have a given configuration of lengths and angles and which make up a parallelepiped of p-volume α. The difficulty lies in elucidating exactly how many such configurations there are.

To explain this we use the decomposition of $V_p(W)$ into an iterated fiber bundle to describe the α-level set of $\hat{\eta} : V_p(W) \to \mathbb{R}$ as a restricted subbundle of $V_p(W)$ with singularities. More explicitly, we get

$$\begin{array}{lcl}
S^{n-p-1} \hookrightarrow V_p(W) & \supset & L_p(\alpha) := \{(v_1, \ldots, v_p) \mid \eta(v_1, \ldots, v_p) = \alpha\} \\
\quad\quad\quad \downarrow \pi_{p-1} & & \quad\quad\quad \downarrow \\
S^{n-p-2} \hookrightarrow V_{p-1}(W) & \supset & L_{p-1}(\alpha) := \{(v_2, \ldots, v_p) \mid \alpha \leq \mathrm{vol}_{p-1}(\pi_M(v_2, \ldots, v_p)) \leq 1\} \\
\quad\quad\quad \downarrow & & \quad\quad\quad \downarrow \\
\quad\quad\quad \vdots & & \quad\quad\quad \vdots \\
\quad\quad\quad \downarrow & & \quad\quad\quad \downarrow \\
S^{n-k-1} \hookrightarrow V_k(W) & \supset & L_k(\alpha) := \{(v_{p-k+1}, \ldots, v_p) \mid \alpha \leq \mathrm{vol}_k(\pi_M(v_{p-k+1}, \ldots, v_p)) \leq 1\} \\
\quad\quad\quad \downarrow & & \quad\quad\quad \downarrow \\
\quad\quad\quad \vdots & & \quad\quad\quad \vdots \\
\quad\quad\quad \downarrow & & \quad\quad\quad \downarrow \\
S^{n-2} \hookrightarrow V_2(W) & \supset & L_2(\alpha) := \{(v_{p-1}, v_p) \mid \alpha \leq \mathrm{vol}_2(\pi_M(v_{p-1}, v_p)) \leq 1\} \\
\quad\quad\quad \downarrow \pi_1 & & \quad\quad\quad \downarrow \\
S^{n-1} \hookrightarrow V_1(W) & \supset & L_1(\alpha) := \{(v_p) \mid \alpha \leq \|\pi_M(v_p)\| \leq 1\}. \\
\quad\quad\quad \downarrow & & \\
\quad\quad\quad * & &
\end{array} \tag{41}$$

where the fiber of $\pi_{k-1}|_{L_k(\alpha)} : L_k(\alpha) \to L_{k-1}(\alpha)$ is generally the pre-image of either an $(p-k+1)$-annulus or a $(p-k)$-sphere in S^{n-k-1} if $\mathrm{vol}_{k-1}(\pi_M(v_{p-k+2}, \ldots, v_p)) = \alpha$. Formally stated, this gives the following theorem.

3.9 Theorem Let W be a real oriented n-dimensional inner product space with determinant consistent with the inner product. Let η be a restricted determinant p-form on W defined by $(w_{p+1}, \ldots, w_n)$ such that

$$\mathrm{vol}_{n-p}(w_{p+1}, \ldots, w_n) = 1. \tag{42}$$

Recall the orthogonal decomposition

$$W = M \oplus Z \tag{43}$$

derived from η. Let $\hat{\eta} : V_p(W) \to \mathbb{R}$ and $\tilde{\eta} : G_p^+(W) \to \mathbb{R}$ be the smooth functions induced from η.

Now, for $\alpha \in]-1, 1[$ and $1 \leq k \leq p$ define $L_k(\alpha) \subset V_k(W)$ as in (41). For a given $x = (v_{p-k+1}, \ldots, v_p) \in L_k(\alpha)$ define

$$\begin{aligned} Z_k(x) &:= \operatorname{span}\{v_{p-k+1}, \ldots, v_p\} \\ \beta_k(x) &:= \frac{\alpha}{\operatorname{vol}_k(\pi_M(v_{p-k+1}), \ldots, \pi_M(v_p))} \end{aligned} \tag{44}$$

and

$$A_k(x) := \begin{cases} \{v \in Z_k(x) \oplus Z^{\perp} \mid \|v\| \in [\beta_k(x), 1]\} & k < p-1 \\ \{v \in Z_{p-1}(x) \oplus Z^{\perp} \mid \|v\| = \beta_{p-1}(x), \text{ orientation constrained}\} & k = p-1 \end{cases} \tag{45}$$

and, finally, if $Pr_M : W \to M$ is the orthogonal projection function, $S(X)$ is the unit sphere in X and if $\alpha \neq 0$

$$F_k(x) := \operatorname{Pr}_M^{-1}(A_k(x)) \cap S(Z_k(x)^{\perp}) \tag{46}$$

or if $\alpha = 0$ for $k < p-1$ let

$$F_k(x) := S(Z_k(x)^{\perp}) \tag{47}$$

and for $k = p-1$ define

$$F_{p-1}(x) := \begin{cases} S(Z_{p-1}(x)^{\perp}) & \text{if } \operatorname{vol}_{p-1}(\operatorname{Pr}_M(v_2, \ldots, v_{p-1})) = 0 \\ S(Z_{p-1}(x)^{\perp} \cap (Z - Z_{p-1}(x)) & \text{otherwise.} \end{cases} \tag{48}$$

Then, the α-level set of $\hat{\eta}$ is given by the singular iterated fiber bundle

$$L_p(\alpha) \xrightarrow{\pi_{p-1}} L_{p-1}(\alpha) \longrightarrow \cdots \longrightarrow L_2(\alpha) \xrightarrow{\pi_1} L_1(\alpha) \tag{49}$$

with fibers given by $F_k(x)$.

Furthermore, the level set $\tilde{\eta}^{-1}(\alpha)$ on the Grassmannian is the image of $L_p(\alpha)$ under the standard quotient map

$$q : V_p(W) \to G_p^+(W) \tag{50}$$

which has fiber SO(p).

Proof Proof by induction on p.

Base Case: Suppose that $p = 1$.

Let $V \in G_1^+(W)$ and let $u \in V$ be a unit vector representing positive orientation on V. Then, by (3.3), we know that $\tilde{\eta}(V)$ is completely determined by the oriented length of the projection of u onto M. But, in this case, M is a one dimensional space. Thus, there is a single vector $v \in M$ which has the appropriate oriented length α. Therefore,

$$\tilde{\eta}^{-1}(\alpha) = \{u \in S^{n-1} | \pi_M(u) = v\}. \tag{51}$$

That is, $\tilde{\eta}^{-1}(\alpha)$ is the pre-image of a single vector under the projection function $\mathrm{Pr}_1 : S^{n-1} \to M$.

Inductive Case: Now suppose that all non-extremal level sets of restricted determinant $(p-1)$-forms can be described as in the statement of the theorem and let η be a restricted determinant p-form.

First note that by (3.3) if $V_1 \in \tilde{\eta}^{-1}(\alpha)$ then $\forall v \in V_1$, if $\|v\| = 1$ then

$$\|\pi_M(v)\| \geq \alpha. \tag{52}$$

Thus, certainly no elements of the α-level set contain unit vectors which are not elements of $L_1(\alpha)$.

Suppose that $u_p \in L_p(\alpha)$, then, if we can find

$$(u_1, \ldots, u_{p-1}) \subset \mathrm{span}\{u_p\}^{\perp} \tag{53}$$

an orthonormal set such that

$$\det[u_1 | \ldots | u_{p-1} | u_p | w_{p+1} | \ldots | w_n] = \alpha, \tag{54}$$

then

$$V := \mathrm{span}(u_1, \ldots, u_{p-1}, u_p) \in \tilde{\eta}^{-1}(\alpha). \tag{55}$$

But of course, we can define a new restricted determinant $(p-1)$-form κ_{u_p} using vectors $(u_p, w_{p+1}, \ldots, w_n)$. Re-normalizing, we define

$$\eta_{p-1} := \frac{1}{\|\pi_M(u_p)\|}\kappa_{u_p}. \tag{56}$$

A quick check confirms that this is a restricted determinant of norm 1. Then,

$$\eta_{p-1}(u_1, \ldots, u_{p-1}) = \frac{1}{\|\pi_M(u_p)\|}\eta(u_1, \ldots, u_{p-1}, u_p). \tag{57}$$

So we would like the $\frac{\alpha}{\|\pi_M(u_p)\|}$-level set of η_{p-1}. Since $\alpha \leq \|\pi_M(u_p)\|$, such a level set exists and is non-empty. By the inductive hypothesis, it can be described using a tower of $(p-2)$ fiber bundles which clearly intersects $(\text{span}(u_p))^{\perp}$, an $(n-1)$-dimensional space, non-trivially. □

Now that we have elucidated exactly which orbits belong in each level set, it only remains to calculate exactly what each orbit looks like. This is done, as is described in (B.7), by choosing a point on any given orbit and calculating its stabilizer under the action of $G = SO(M) \times O(Z)$. The first order of business is to determine which subspaces of M and Z can be related within themselves and still stabilize V.

Let W be an inner product space and let M and V be vector subspaces of W of equal dimension. Let $\pi_M := W \to M$ be the orthogonal projection function. This can be used to define a decomposition of V into mutually orthogonal subspaces. Let

$$\beta_1 := \max\{\|\pi_M(v)\| \mid \|v\| = 1, v \in V\} \tag{58}$$

and let

$$V_1 := \text{span}\{v \in V \mid \|v\| = 1 \text{ and } \|\pi_M(v)\| = \beta_1\}. \tag{59}$$

Now, inductively, if mutually orthogonal subspaces $V_1, \ldots, V_i$ have been chosen, define

$$\beta_{i+1} := \max\{\|\pi_M(v)\| \mid \|v\| = 1, v \in (V_1 \oplus \cdots \oplus V_i)^{\perp} \cap V\}. \tag{60}$$

Then let

$$V_{i+1} := \text{span}\{v \in (V_1 \oplus \cdots \oplus V_i)^{\perp} \cap V \mid \|v\| = 1 \text{ and } \|\pi_M(v)\| = \beta_{i+1}\}. \quad (61)$$

This gives a decomposition

$$V = V_1 \oplus \cdots \oplus V_r. \quad (62)$$

3.10 Theorem Let W be a real oriented n-dimensional inner product space with determinant and let η be a restricted determinant p-form on W determined by the orthonormal vectors $(w_{p+1}, \ldots, w_n)$. Let

$$Z := \text{span}\{w_{p+1}, \ldots, w_n\}. \quad (63)$$

Let M be the orthogonal complement of Z in W. Let $V \in G_p^+(W)$ have the decomposition

$$V = V_1 \oplus \cdots \oplus V_r. \quad (64)$$

defined above. Further, define

$$\begin{aligned} M_\Omega &:= V^{\perp} \cap M, \\ Z_\Omega &:= V^{\perp} \cap Z. \end{aligned} \quad (65)$$

Then

$$H := (\mathsf{SO}(V_1) \times \ldots \times \mathsf{SO}(V_r)) \times (O(M_\Omega) \times O(Z_\Omega)) \quad (66)$$

can be naturally imbedded into $G := SO(M) \times O(Z)$ as a subgroup and this imbedding is the stabilizer of V under the action of G on $G_p^+(W)$. Thus, the orbit structure is given by

$$G.V \cong G/H \quad (67)$$

Proof By construction, H has a natural imbedding in $O(W)$. Thus, it must be shown that this imbedded subgroup of $O(W)$ is also a subgroup of G.

First consider $SO(V_i)$ for some $1 \leq i \leq r$. Any vector $v \in V_i$ can be decomposed into

$$v = x_v + y_v \tag{68}$$

where $x \in M$ and $y \in Z$. Furthermore, by construction, for any unit vector $v \in V_i$, $\|x_v\| = \beta_i$ and $\|y_v\| = \sqrt{1-(\beta_i)^2}$. This makes it clear that SO(V_i) is naturally imbedded as a subgroup of G. Since M_Ω and Z_Ω are subspaces of M and Z respectively and are orthogonal to V, it should be easy to see that their orthogonal groups are subgroups of G. Therefore, since all of these spaces are mutually orthogonal, the natural imbedding of H into $O(W)$ makes H a subgroup of G.

It remains to prove that H is the stabilizer of V under the action of G, or, by (2.6) that $H = \left(SO(V) \times O(V^\perp)\right) \cap G$. Clearly, H is a subgroup of this intersection. Now suppose that $g \in G - H$ and consider the effect of g on V. First, suppose that for some i, g fails to maintain V_i as an invariant subspace. Then, there exists some unit vector $v_i \in V_i$ such that $g.v_i \notin V_i$. But, since $g \in G = SO(M) \times O(Z)$, it preserves norms on M and Z, and

$$\|\pi_M(g.v_i)\| = \beta_i. \tag{69}$$

By construction, V_i contains the set of all unit vectors in V whose projection onto M have norm β_i. Therfore, $g.v_i \notin V$. Thus, the stabilizer of V must maintain each V_i as an invariant subspace. Finally, suppose that g fails to maintain either M_Ω or Z_Ω as invariant subspaces. Well then, since g is a non-degenerate linear transformation, and it stabilizes both Z and M, it must affect one of the V_i's. This, as proven above, is inconsistent with the behaviour of the stabilizer of V. Thus, H is the stabilizer of V under the action of G on $G_p^+(W)$. The final statement is an application of (B.7). □

With this final theorem, we now have sufficient information to completely describe the level sets of any given restricted determinant p-form. We know from

(3.7) that all the non-extremal level sets but 0 are diffeomorphic. We know from (3.9) which orbits exist within any given level set. Finally, from (3.10), we know the stabilizer of a point in each orbit. This information, together with that in Appendix B, gives us complete information on the structure of each orbit and how they're glued together to form each level set.

4 Symplectic Algebra

At this point a general theory giving structural information about level sets of the functions induced from any p-form η on an inner product space W would be appropriate. However, at this time, results of this generality seem difficult to attain and have not been seriously attempted. Instead, we now specialize to the symplectic form. This form is sufficiently similar to the restricted determinant to allow accessibility and yet non-trivial enough to present a hopefully representative richness of results. Furthermore, symplectic forms are used extensively in multiple fields such as mechanics and symplectic topology. Thus, structural results about this form may have immediate application.

4.1 Definition A bilinear alternating map $\eta : W \times W \to \mathbb{R}$ is called **symplectic** if the map $\phi : W \to W^*$ given by $(\phi(v_1))(v_2) = \eta(v_1, v_2)$ is an isomorphism.

4.2 Example The standard determinant on $\mathbb{R}^2$, $\det_2$ is a symplectic form.

In fact, by definition, all symplectic forms on a 2-dimensional space are determinant functions. A standard theorem in Symplectic Algebra generalizes this observation. It states that for every symplectic form η, it is possible to choose a basis $\mathcal{B}$, called here a **standard basis**, such that

$$\eta = \bigoplus_n \det_2. \tag{1}$$

In most of the work to date on symplectic forms, one simply chooses a standard basis and works with the standard form. However, there is no guarantee that a given symplectic form has an orthonormal standard basis. Since we are working in an environment with an inner product structure and we intend to make use of the natural action of $O(W)$ on $G_2^+(W)$, an orthonormal basis is a great convenience. Thus, we present in (4.12) a modification of this theorem which provides for orthonormality. In essence, it is proven that it is possible to find an orthonormal basis with respect to which η is broken into scaled standard chunks. Explicitly,

$$\eta = \bigoplus_{i=1}^{r} a_i(\bigoplus_{d_i} \det{}_2). \tag{2}$$

The constants $a_1 > \cdots > a_r > 0$ and the orthogonal decomposition

$$W = W_1 \oplus \cdots \oplus W_r \tag{3}$$

such that

$$\eta|_{W_i} = a_i \bigoplus_{d_i} \det{}_2 \tag{4}$$

are invariants of η.

4.3 Definition For a given symplectic form η on an inner product space W, the numbers $a_1, \ldots, a_r$ given in (4.12) will be referred to as the **symplectic eigenvalues** of η. The set of such eigenvalues will be known as the **symplectic spectrum** of η. The spaces $W_1, \ldots, W_r$ also given in (4.12) will be called the **symplectic eigenspaces** of η.

As will be shown in a later section, this symplectic spectrum is not simply a computational convenience. The eigenspaces associated with it provide a means to characterize orbits within level sets and the symplectic eigenvalues constitute the critical values of $\tilde{\eta}$.

Once all the assertions in this development are proven, analysis of the symplectic form will proceed in two steps. First, we will consider symplectic forms whose standard basis happens to be orthonormal. Second, we will consider the more general case in which η is a scaled sum of standard chunks.

We now start proving the above assertions by formalizing the bijective relationship between skew-symmetric matrices and alternating bilinear forms. This is a useful tool in later analysis.

4.4 Lemma [3, p.361] Let W be an n-dimensional real vector space and let $\mathcal{B} = \{w_1, \ldots, w_n\}$ be a basis of W. Then there is a bijective relationship between the set of real skew-symmetric $n \times n$ matrices, $M_n(\mathbb{R})$, and the set of real-valued alternating bilinear 2-forms on W.

Proof Let η be an alternating bilinear form on W. Define the matrix A_η by

$$(A_\eta)_{ij} := \eta(w_i, w_j). \tag{5}$$

Then

$$\begin{aligned} (A_\eta)_{ij} &= \eta(w_i, w_j) \\ &= -\eta(w_j, w_i) \qquad \text{since } \eta \text{ is alternating} \\ &= -(A_\eta)_{ji}. \end{aligned} \tag{6}$$

Thus,

$$A_\eta = -(A_\eta)^T \tag{7}$$

so A is skew-symmetric.

Now, suppose that A is a skew-symmetric $n \times n$ matrix. Create an alternating bilinear form η_A by defining

$$\eta_A(w_i, w_j) := A_{ij}. \tag{8}$$

Since a bilinear form is determined by its behaviour on a basis, this is sufficient to define η_A. A quick check confirms that since A is skew-symmetric, η_A is alternating. □

The matrix A_η is referred to as the **matrix representing η with respect to the basis $\mathcal{B}$**. The next step is to investigate what characterizes the matrices representing symplectic forms.

4.5 Lemma [3, p.365] Let η be an $\mathbb{R}$-valued alternating bilinear form on a real inner product space W. Then, for any given basis $\mathcal{B} = (w_1, \ldots, w_n)$, the matrix A_η representing η with respect to $\mathcal{B}$ is non-degenerate if and only if η is symplectic.

Proof Suppose that η is symplectic. Then, by definition, the map $\phi : W \to W^*$ given by

$$(\phi(v_1))(v_2) := \eta(v_1, v_2) \tag{9}$$

is an isomorphism. We will prove that the matrix which represents this isomorphism with respect to the basis $\mathcal{B}$ of W and its dual basis, $\mathcal{B}^*$ of W^*, is $-A_\eta$. Once this has been established, since ϕ is an isomorphism, it will be clear that A_η is non-degenerate.

Now,

$$(\phi(v_1))(v_2) = v_1^T \cdot A_\eta \cdot v_2. \tag{10}$$

But $\phi(v_1) \in W^*$ is a functional and so it makes sense to say that

$$(\phi(v_1))^T = v_1^T \cdot A_\eta \tag{11}$$

which, because A_η is skew-symmetric, gives that

$$\phi(v_1) = -A_\eta \cdot v_1. \tag{12}$$

Thus, since ϕ is an isomorphism, A_η is non-singular.

Now suppose that A is a skew-symmetric non-degenerate $n \times n$ matrix. Then there is the alternating bilinear form η_A derived from A as in (4.4). Since η_A is an alternating bilinear form, it induces a map $\phi : W \to W^*$ and, as shown above, this map has a matrix representation, $-A$. Since $-A$ is non-degenerate, ϕ is an isomorphism, and thus, η_A is symplectic. □

It is now established that symplectic forms can be represented by skew-symmetric non-degenerate matrices. This has a couple of consequences. First, examining the matrices involved shows that the direct sum of symplectic forms is a symplectic form. This will be immediately used in proving that there are symplectic forms which cannot be expressed as restricted determinants. A second consequence is that the dimension of W is even.

4.6 Corollary Let W_1 and W_2 be real vector spaces of dimensions m and n respectively. Let $\eta : (W_1)^2 \to \mathbb{R}$ and $\xi : (W_2)^2 \to \mathbb{R}$ be symplectic forms. Then $\eta \oplus \xi : (W_1 \oplus W_2)^2 \to \mathbb{R}$ given by

$$(\eta \oplus \xi)((a_1u_1 + b_1v_1), (a_2u_2 + b_2v_2)) := \eta(a_1u_1, a_2u_2) + \xi(b_1v_1, b_2v_2) \qquad (13)$$

is, again, a symplectic form.

Proof Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be, respectively, bases for W_1 and W_2. Then, with respect to these bases, η and ξ can be represented by matrices A_η and B_ξ, (4.5). Now, the concatenation of $\mathcal{B}_1$ and $\mathcal{B}_2$ is a basis $\mathcal{B}$ for $W_1 \oplus W_2$. Consider the matrix $C := A_\eta \oplus B_\xi$. This matrix is clearly skew-symmetric and non-degenerate. Thus, by (4.5), with respect to the basis $\mathcal{B}$, this represents a symplectic form on $W_1 \oplus W_2$. Clearly, this symplectic form is the same as that defined above as $\eta \oplus \xi$. □

This result can be directly applied to prove that the set of symplectic forms is not simply a subset of the set of restricted determinants. This will ensure that the work in the next two sections is not simply a well disguised repetition of the work done in the previous section.

4.7 Corollary There are symplectic forms which cannot be expressed as restricted determinants.

Proof

First, consider the determinant function on $\mathbb{R}^2$, $\det_2 : \mathbb{R}^2 \to \mathbb{R}$ given by

$$\det{}_2(u, v) := \det[u|v]. \qquad (14)$$

As shown in example (4.2), $\det_2$ is a symplectic form on $\mathbb{R}^2$. Thus, by (4.6),

$$\eta := \bigoplus_n \det{}_2 \qquad (15)$$

is a symplectic form on $\mathbb{R}^{2n}$.

Now, let $V_1 := \text{span}\{e_1, e_2\}$ with the orientation represented by (e_1, e_2), and let $V_2 := \text{span}\{e_3, e_4\}$ with the orientation represented by (e_3, e_4). Clearly,

$$\tilde{\eta}(V_1) = \tilde{\eta}(V_2) = 1. \qquad (16)$$

Furthermore, the matrix representing η with respect to the standard basis, J_{2n}, is the direct sum of orthogonal matrices and, as such, is orthogonal and therefore norm-preserving. Thus, for any pair of orthonormal vectors, (u, v),

$$|\eta(u, v)| = |u^T J_{2n} v| \leq 1. \tag{17}$$

Thus, η achieves a maximum of 1 and V_1 and V_2 are two distinct elements of the maximal level set of $\tilde{\eta}$. As was shown in (3.6), the maximal level set of a restricted determinant is a single point on the Grassmannian.

Thus, η cannot be expressed as a restricted determinant. □

Next, in the final corollary to (4.5), an application of [3, p.162] gives that symplectic forms only occur on even dimensional spaces. This is important because, as part of our analysis, we intend to impose a complex structure on W which is only possible if W is even-dimensional.

4.8 Corollary Let η be a symplectic form on W, an n-dimensional vector space. Then n is even.

Proof As shown in (4.4) and in (4.5), if $\mathcal{B} = \{w_1, \ldots, w_n\}$ is a basis of W, then there is a $n \times n$ skew-symmetric non-degenerate matrix A_η representing η. Suppose that n is odd.

$$\begin{aligned} \det(A_\eta) &= \det((A_\eta)^T) \\ &= \det(-A_\eta) \qquad \text{(since } A_\eta \text{ is skew-symmetric)} \\ &= (-1)^n \det(A_\eta) \\ &= -\det(A_\eta) \qquad (n \text{ is odd}). \end{aligned} \tag{18}$$

Therefore,

$$\det(A_\eta) = 0. \tag{19}$$

But A_η is non-degenerate. Therefore, n must be even. □

A more important consequence of the fact that forms can be represented by matrices follows. Simple algebra will show that any group G for which a bilinear form η is G-equivariant must be part of the centralizer of the matrix A_η representing η with respect to some basis. Therefore, a matrix which represents a symplectic form gives criteria which helps specify the maximal subgroup G of $O(W)$ for which that symplectic form is G-equivariant. It is then possible to use the action of that group to elicit symmetry information from the level sets.

4.9 Proposition [3, p.380] Let η be a bilinear form on the real $2n$-dimensional inner product space W. Let $\mathcal{B} = (w_1, \ldots, w_{2n})$ be a basis for W and let A_η be the matrix representing η with respect to $\mathcal{B}$. Let G be a subgroup of $O(W)$ which acts diagonally on W^2 and trivially on $\mathbb{R}$ such that η is G-equivariant. Then G is a subgroup of the centralizer of A_η.

Proof Let $g \in G$. Then for every u and $v \in W$,

$$\eta(u, v) = \eta(g.u, g.v). \tag{20}$$

But, by (4.4),

$$\eta(u, v) = u^T A_\eta v \tag{21}$$

and

$$\eta(g.u, g.v) = (g.u)^T A_\eta (g.v). \tag{22}$$

Thus, for each u and $v \in W$,

$$u^T A_\eta v = (g.u)^T A_\eta (g.v). \tag{23}$$

Therefore, this is specifically true for the basis vectors of W. So

$$\begin{aligned} (A_\eta)_{ij} &= w_i^T A_\eta w_j \\ &= (g.w_i)^T A_\eta (g.w_j) \\ &= w_i^T \left(g^T A_\eta g\right) w_j \\ &= (g^T A_\eta g)_{ij}. \end{aligned} \tag{24}$$

Thus, for each $g \in G$,

$$A_\eta = g^T A_\eta g. \tag{25}$$

□

This proposition puts a comparitively easily measured restriction on which group actions will preserve a given bilinear form. The imposition of an inner product structure on W allows the natural action of $O(W)$ on $G_2^+(W)$. Thus, this proposition gives a characterization of the subgroups of $O(W)$ which will preserve a given symplectic form. However, as matters stand, the task is still unmanageably large: each different symplectic form is represented with respect to a standard basis by a different matrix which has a different centralizer. Even the information that symplectic forms are only represented by skew-symmetric non-degenerate matrices does not sufficiently reduce the number of centralizers that would have to be studied. Suppose, however, that under a suitable choice of basis, the matrix representing η could be of a standard form. Then, centralizers would also be of a standard form and some general theory could be presented. In order to achieve such a goal, first we must investigate the effect of changing basis on the representative matrix.

4.10 Lemma [3, p.363] Let $\mathcal{A} = (v_1, \ldots, v_{2n})$ and $\mathcal{B} = (w_1, \ldots, w_{2n})$ be two oriented bases for W. Let A_η and B_η be the matrices representing a binlinear form η with respect to these bases. Let S be the linear transformation mapping basis $\mathcal{A}$ to $\mathcal{B}$. Then,

$$A_\eta = S^T B_\eta S. \tag{26}$$

Proof Recall that for any vectors $u_1, u_2 \in W$ expressed with respect to basis $\mathcal{B}_2$,

$$\eta(u_1, u_2) = u_1^T B_\eta u_2. \tag{27}$$

Let $[v]_{\mathcal{A}}$ and $[v]_{\mathcal{B}}$ denote the column matrices of a vector $v \in V$ in terms of the bases $\mathcal{A}$ and $\mathcal{B}$ respectively. Then clearly,

$$[v]_{\mathcal{B}} = S[v]_{\mathcal{A}}. \tag{28}$$

Then, for any $v_i, v_j \in \mathcal{A}$,

$$\begin{aligned}(A_\eta)_{ij} &= [v_i]^T_{\mathcal{A}} A_\eta [v_j]_{\mathcal{A}} \\ &= \eta(v_i, v_j) \\ &= [v_i]^T_{\mathcal{B}} B_\eta [v_j]_{\mathcal{B}} \\ &= (S[v_i]_{\mathcal{A}})^T B_\eta (S[v_j]_{\mathcal{A}}) \\ &= [v_i]^T_{\mathcal{A}} (S^T B_\eta S)[v_j]_{\mathcal{A}} \\ &= (S^T B_\eta S)_{ij}.\end{aligned} \tag{29}$$

□

Thus, matrices representing a symplectic form transform by transpose conjugation by the change of basis matrix. With this information, it is now possible to continue on to prove that with an appropriate choice of basis, any symplectic form can be represented by a direct sum of determinants.

4.11 Theorem **(Cannonical Symplectic Form)** [3, p.377] Let W be a real $2n$-dimensional vector space. Then, for any symplectic form η on W, there is a choice of basis $\mathcal{B}$ of W with respect to which

$$\eta = \bigoplus_n \det{}_2. \tag{30}$$

Proof Proof by induction on the dimension of W.

Case 1: $\dim(W) = 2$

By definition η is a determinant function on W and up to scalar multiplication, there is only one determinant function on a given vector space [2, p. 103]. So let $\{u, v\}$ be a basis of V such that

$$\eta(u, v) = 1. \tag{31}$$

Then, with respect to this basis

$$\eta = \det_2. \tag{32}$$

Case 2: Inductive Case

Assume that for every symplectic form η_m on a space of dimension $2m < 2n$, there is a basis with respect to which $\eta_m = \bigoplus_m \det_2$. Now, choose a non-zero $w_1 \in W$. Recall that by definition, associated to η there is an isomorphism $\phi : W \to W^*$ such that

$$(\phi(v_1))(v_2) := \eta(v_1, v_2) \tag{33}$$

for all $v_1, v_2 \in W$. So $\phi(w_1) \neq 0 \in W^*$. Therefore, there exists a w_2 such that

$$(\phi(w_1))(w_2) = \eta(w_1, w_2) = +1. \tag{34}$$

Now, define

$$W_1 := \text{span}(w_1, w_2) \tag{35}$$

and

$$W_2 := \ker(\phi(w_1)) \cap \ker(\phi(w_2)). \tag{36}$$

Now

$$w_1 \in \ker(\phi(w_1)) \text{ but } w_1 \notin \ker(\phi(w_2)) \tag{37}$$

and

$$w_2 \in \ker(\phi(w_2)) \text{ but } w_2 \notin \ker(\phi(w_1)). \tag{38}$$

Thus,

$$\dim W_2 = 2n - 2. \tag{39}$$

Therefore it can be seen that

$$W = W_1 \oplus W_2. \tag{40}$$

This decomposition naturally induces a decomposition of the dual space

$$W^* = W_1^* \oplus W_2^*. \tag{41}$$

Now, ϕ is non-degenerate on W and, by construction,

$$\phi(W_1) = W_1^*. \tag{42}$$

Also, by construction

$$\phi(W_2) = W_2^*. \tag{43}$$

Therefore, $\phi|_{W_2}$ is non-degenerate and so, by (4.5), $\eta|_{W_2}$ is a symplectic form on W_2. By the inductive hypothesis, there is a basis $\{w_3, \ldots, w_{2n}\}$ of W_2 with respect to which $\eta|_{W_2} = \bigoplus_{n-1} \det_2$. Thus, since the decomposition of W was direct, $\mathcal{B}_2 = \{w_1, w_2, w_3, \ldots, w_{2n}\}$ is a basis of W. Finally, it is easily seen that with respect to this basis $\eta = \bigoplus_n \det_2$. □

For our purposes, this representation of a symplectic form is not sufficient. This is because a standard basis is not necessarily orthonormal and the representation of groups such as $\mathrm{O}(W)$ with respect to a non-orthonormal basis is difficult. Therefore, although the cannonical symplectic form is ubiquitous in the study of symplectic forms, for our purposes a similar theorem with more restrictive conditions has been developed. In addition to facilitating the application of tranformation group theory to the level sets of symplectic forms, this theorem also highlights some invariants of symplectic forms in relation to an inner product structure.

4.12 Theorem **(Orthonormal Cannonical Symplectic Form)** Given a symplectic form η on a $2n$-dimensional real inner product space W, there exists an orthogonal decomposition of W into $W_1 \oplus \cdots \oplus W_r$ with a composite orthonormal basis $(w_{11}, \ldots, w_{1d_1}, \ldots, w_{r1}, \ldots w_{rd_r})$ with respect to which

$$\eta = \bigoplus_{i=1}^{r} a_i \left(\bigoplus_{d_i} \det{}_2 \right) \tag{44}$$

for some real numbers $a_1 > a_2 > \cdots > a_r > 0$. Furthermore, these numbers and the vector space decomposition are uniquely determined by η.

Proof Let $\mathcal{B}_1$ be an orthonormal basis of W and let B_1 be the matrix representing η with respect to $\mathcal{B}_1$. Then if $\mathcal{A}_1$ is a different orthonormal basis of W with matrix A_1 representing η with respect to $\mathcal{A}_1$ and S is a matrix representing the change of basis we have

$$\begin{aligned} A_1 &= S^T B_1 S \\ &= S^{-1} B_1 S \qquad \text{since } S \text{ is orthogonal.} \end{aligned} \tag{45}$$

Thus, B_1 transforms to A_1 under a change of orthonormal basis identically when viewed as representing a linear transformation or as representing a symplectic form. With this in mind, we apply linear algebra to achieve the desired result.

Outline of Proof

Step 1. Prove B_1 represents a normal operator. This will provide two distinct results. First, it will give an intrinsic orthogonal decomposition of $W = W_1 \oplus \cdots \oplus W_r$. Second, it will allow the decomposition of the minimal polynomial of B_1 into distinct irreducible factors.

Step 2. Show that each irreducible factor of the minimal polynomial of B_1 is of the form $p_1 = t^2 + a_i^2$.

Step 3. Observe that $\bigoplus_{i=1}^{r} a_i(\bigoplus_{d_i} \det_2)$ is represented by a matrix which represents a normal operator with the same characteristic polynomial as B_1.

Then, since normal operators with the same characteristic polynomial are orthogonally equivalent [3, p. 357], the proof will be complete.

Step 1: We start by proving that B_1 commutes with its adjoint B_1^* and thus represents a normal operator.

Let v and w be any vectors in W. Then

$$\begin{aligned} < B_1 v, w > &=< w, B_1 v > \\ &= \eta(w, v) \\ &= -\eta(v, w) \\ &= - < v, B_1 w > \\ &=< v, -B_1 w > . \end{aligned} \tag{46}$$

So $B_1^* = -B_1$ which, of course, commutes with B_1. Thus, B_1 is a normal operator.

Now, by the primary decomposition theorem for normal operators [3, p.348], the decomposition of the minimal polynomial of B_1 into distinct irreducible monic factors is of the form

$$P = p_1 \cdot \cdots \cdot p_r \tag{47}$$

with a corresponding decomposition of W given by

$$W = W_1 \oplus \cdots \oplus W_r, \tag{48}$$

where $W_i :=$ ker $(p_i(B_1))$ and W_i and W_j are mutually orthogonal. Furthermore, p_i is the minimal polynomial of $B_1|_{W_i}$.

Step 2: Now, consider the factors of the minimal polynomial of B_1. We first prove that each of these factors is quadratic. Suppose otherwise. Choose i such that $p_i = t - c_i$ for some $c_i \in \mathbb{R}$ and choose a unit eigenvector $v \in W$

associated with c_i. Then

$$\begin{aligned} 0 &= \eta(v, v) \\ &= v^T B_1 v \\ &=< v, B_1 v > \\ &=< v, c_i v > \\ &= c_i. \end{aligned} \tag{49}$$

But B_1 is non-singular and thus has no 0 eigenvalues. Thus, each irreducible factor of the minimal polynomial must be quadratic. Next, we check that each factor is of the form

$$p_i = t^2 + a_i^2. \tag{50}$$

To understand this, consider $(B_1)^2$. Since B_1 is skew-symmetric, $(B_1)^2$ is symmetric. Thus, $(B_1)^2$ is self-adjoint and has an orthonormal basis of eigenvectors [3, p.314]. Let the eigenvalues of $(B_1)^2$ be $c_1, \ldots, c_{2n}$. Then,

$$g := (t - c_1) \cdot \cdots \cdot (t - c_{2n}) \tag{51}$$

is the characteristic polynomial for $(B_1)^2$. Thus.

$$h := (t^2 - c_1) \cdot \cdots \cdot (t^2 - c_{2n}) \tag{52}$$

annihilates B_1. So the minimal polynomial of B_1 divides h. Furthermore, since p has no linear terms in its factorization, each $c_i < 0$. So, to summarize, the minimal polynomial p of B_1 has a decomposition given by

$$p = (t^2 + a_1^2) \cdot \cdots \cdot (t^2 + a_r^2). \tag{53}$$

Step 3: Now that the composition of the minimal polynomial of B_1 has been confirmed, return to the decomposition of W into mutually orthogonal factors:

$$W = W_1 \oplus \cdots \oplus W_r \tag{54}$$

where p_i is the minimal polynomial for $B_1|_{W_i}$ for each i. Since B_1 is normal, so is $B_1|_{W_i}$. Now, two normal matrices with the same characteristic polynomial are orthonormally equivalent [3, p. 357]. Thus, let us consider the characteristic polynomial of $B_1|_{W_i}$. Since the minimal polynomial of $B_1|_{W_i}$ is $p_i = t^2 + a_i^2$, the characteristic polynomial is $p_i^{\frac{\dim W_i}{2}}$. Another normal matrix on W_i with this same characteristic polynomial is the $(\dim W_i) \times (\dim W_i)$ matrix

$$a_i J_{2d_i} = \begin{bmatrix} 0 & a_i & & & & \\ -a_i & 0 & & & & \\ & & \ddots & & & \\ & & & \ddots & & \\ & & & & 0 & a_i \\ & & & & -a_i & 0 \end{bmatrix} \tag{55}$$

which represents the symplectic form

$$\eta_i := a_i \bigoplus_{d_i} \det{}_2. \tag{56}$$

So, for each i, there is an orthonormal basis for W_i with respect to which

$$\eta|_{W_i} = a_i \bigoplus_{d_i} \det{}_2. \tag{57}$$

Since the spaces W_i are mutually orthogonal, concatenate these bases to form a basis of W with respect to which

$$\eta = \bigoplus_{i=1}^{r} a_i (\bigoplus_{d_i} \det{}_2). \tag{58}$$

□

Thus, for each symplectic form η there is an orthonormal basis with respect to which η is represented by a matrix in standard form, A_η. Furthermore, as shown in (4.9), any group G acting on $G_2^+(W)$ such that $\tilde{\eta}$ is equivariant will belong to the centralizer of A_η. Thus, this representing matrix provides information which will help in the analysis of the symmetry of the level sets. Furthermore, the symplectic spectrum characterizes the form. Such a characterization will certainly be helpful in determining which level sets contain certain orbits and, as it turns out, the symplectic eigenvalues are the critical values of $\tilde{\eta}$. These invariants may even generalize to give some sort of characterization of a differential symplectic form on a manifold equipped with a Riemann structure.

5 Standard Symplectic Form

In the previous section, all preliminary work toward analysing symplectic forms was completed. In particular, it was shown in (4.11) that for each symplectic form η, there is a basis referred to as a standard basis with respect to which

$$\eta = \bigoplus_{n} \det_2. \tag{1}$$

Unfortunately, not all standard bases are orthonormal and, as discussed in the previous section, without an orthonormal reference basis it is difficult to represent the special orthogonal group. Therefore, the analysis of the symplectic form will be broken into two sections: this one in which there is an orthonormal standard basis and the next one in which the more general case is considered.

Recall that symplectic forms only arise on even dimensional spaces (4.8). Thus, throughout this section, assume that W is a $2n$-dimensional real inner product space. Thus, we are able to impose a complex structure on W. We will show in (5.1) that with the existence of an orthonormal standard basis there is an intrinsic way of intertwining the symplectic form η and the inner product on W to achieve a complex structure on W. Full details of the resulting complexification of W and its implications to the classical Lie groups of W are discussed in Appendix C.

The outline of this section is fairly typical of each analysis section in this thesis. First, a characterization of the properties of a vector space V which affect the value of $\tilde{\eta}(V)$ will be developed. This will require, as discussed, a well-defined complex structure on W. The next step is to discover which points of the image of $\tilde{\eta}$ are regular values. Since any interval which contains no critical values has a pre-image in which the level sets are diffeomorphic and, in fact, are equivariantly diffeomorphic under any group action which preserves level sets, explicit knowledge of the critical values gives the maximum number of different diffeomorphism classes of level sets. Next, using information about the form, the maximal subgroup G of $O(W)$ for which $\bar{\eta}$ is G-equivariant is calculated. This group acts on $G_2^+(W)$ and transformation group theory, as outlined in Appendix B. is applied. By this theory, the only two pieces of information remaining to completely understand the structure of the level sets are the number of different orbits in a level set and their structure. This last is given by the stabilizer of any point on an orbit.

Thus, the first order of business is to define a complex structure on W. Any non-degenerate bilinear form ψ on W determines a linear isomorphism $\bar{\psi}$: $W \to W^*$ by

$$(\bar{\psi}(v))(w) := \psi(v, w). \tag{2}$$

This was shown explicitly in the case of symplectic forms in (4.5). Thus, if η is a symplectic form, and σ is the inner product on W, we get two distinct isomorphisms of W and W^*, $\bar{\eta}$ and $\bar{\sigma}$. When η admits an orthonormal standard basis, these two intertwine to produce the complex structure $\bar{\sigma}^{-1} \circ \bar{\eta} : W \to W$.

5.1 Lemma Let W be a vector space with inner product σ and let η be a symplectic form on W which admits an orthonormal standard basis. Then the map

$$\bar{\sigma}^{-1} \circ \bar{\eta} : W \to W \tag{3}$$

is a complex structure on W.

Proof Suppose $\mathcal{B} = \{w_1, \ldots, w_{2n}\}$ is an orthonormal standard basis on W with dual basis $\mathcal{B}^* := \{w_1^*, \ldots, w_{2n}^*\}$ of W^*. Then with respect to these bases, η is represented by J_{2n} and by (4.5) $\bar{\eta} : W \to W^*$ is represented by $-J_{2n}$.

Also since $\mathcal{B}$ is orthonormal, σ is represented with respect to $\mathcal{B}$ by Id_{2n} and so with respect to $\mathcal{B}$ and $\mathcal{B}^*$, $\bar{\sigma}$ is also represented by Id_{2n}.

Thus $\bar{\sigma}^{-1} \circ \bar{\eta}$ is represented by

$$(\mathrm{Id}_{2n}^{-1})(-J_{2n}) = -J_{2n}. \tag{4}$$

Of course,

$$(-\mathrm{J}_{2n})^2 = -\mathrm{Id}_{2n} \tag{5}$$

and so

$$(\bar{\sigma}^{-1} \circ \bar{\eta})^2 = -\mathrm{Id}_W. \tag{6}$$

□

This complex structure on W will be known as the complex structure induced from η and σ and, unless otherwise stated, will be the complex structure in use for the remainder of this section.

We now present a characterization of which properties of a given plane $V \subset W$ determine $\tilde{\eta}(V)$.

5.2 Lemma Let W be a $2n$-dimensional real inner product space and let η be a symplectic form on W which admits an orthonormal standard basis. Impose on W the complex structure map $\hat{\mathrm{J}}$ induced from η and the inner product and let $\tilde{\eta}$ be the smooth map on $G_2^+(W)$ induced from η. Let $V \in G_2^+(W)$ have an ordered orthonormal basis (u, v) representing positive orientation on V and let $\mathbb{C}u$ be the complex span of u. Then the value $-\tilde{\eta}(V)$ is given by the oriented area of the orthogonal projection of the unit square in V spanned by (u, v) onto $\mathbb{C}u$.

Proof Let $\mathcal{B} := \{w_1, \ldots, w_{2n}\}$ be an orthonormal standard basis for η. Now, observe that $\hat{\mathrm{J}}$ maps each basis vector to another (positive or negative) basis vector. Thus, since the basis is orthonormal, $\hat{\mathrm{J}}$ maps each $u \in W$ to a vector perpendicular to u. Therefore, since $\hat{\mathrm{J}}$ is orthogonal, $(\frac{u}{\|u\|}, \frac{\hat{\mathrm{J}}u}{\|u\|})$ forms an orthonormal real basis for the complex line $\mathbb{C}u$ and assigns it an orientation.

The orthogonal projection of the unit square spanned by (u, v) onto $\mathbb{C}u$

will be a rectangle. So we get

$$
\begin{aligned}
\text{area} &= \text{base} \cdot \text{height} \\
&= \|\pi_{\mathbb{C}u}(u)\| \cdot \|\pi_{\mathbb{C}u}(v)\| \\
&= \|u\| \cdot < v, \hat{\mathrm{J}}u > \\
&= v^T \hat{\mathrm{J}} u \qquad (7) \\
&= \eta(v, u) \\
&= -\eta(u, v) \\
&= -\tilde{\eta}(V).
\end{aligned}
$$

□

Note that although (5.2) is stated in terms of a specific basis of V, $\tilde{\eta}(V)$ is independent of basis (2.3). Thus, the oriented area of the orthogonal projection of a unit square in V onto a complex line generated from one of the basis vectors is independent of the choice of unit square.

There is an equivalent way to express this. One could define an angle θ_V to be the oriented angle between a space V and any complex line generated from some $u \in V$. The same argument as above shows that such an angle is well-defined and that $\tilde{\eta}(V) = -\cos\theta_V$. Although intuitively appealing, the approach using angles does not generalize to other situations as well as projected area does.

5.3 Corollary Under the conditions stated in (5.2),

$$\max_{V \in G_2^+(W)} \tilde{\eta}(V) = 1. \qquad (8)$$

Furthermore, this occurs when V is a negatively oriented complex line.

Proof Following the proof of (5.2), it is clear that for $V \in G_2^+(W)$ and an ordered orthonormal basis (u, v) representing positive orientation,

$$\tilde{\eta}(V) = - < v, \hat{\mathrm{J}}u > . \qquad (9)$$

Since $\hat{J}$ is orthogonal, both v and $\hat{J}u$ are unit vectors. Thus,

$$\tilde{\eta}(V) \leq 1 \tag{10}$$

and achieves 1 when $v = -\hat{J}u$.

This last statement is equivalent to stating that V is a negatively oriented complex line. □

Thus, pointwise, the maximal level set is equivalent to complex projective space. It is natural to inquire whether it also has the manifold structure. This question will be answered affirmatively in (5.6).

At this point, the information is available to check which points on $G_2^+(W)$ are regular. This will, as shown in (2.10), provide a set of diffeomorphism classes of level sets. Furthermore, once we have a group whose action preserves level sets, this diffeomorphism will be equivariant. First, observe that since $\{\pm 1\}$ are extremal values of $\tilde{\eta}$ on $G_2^+(W)$, they cannot possibly be regular values. So, it is only necessary to investigate points in $]-1, 1[$.

5.4 Theorem Let W be a $2n$-dimensional real inner product space and let η be a symplectic form on W which admits an orthonormal standard basis. Then, all the points in the interval $]-1, 1[$ are regular values of the induced map $\tilde{\eta}$ on $G_2^+(W)$.

Proof Let $\alpha \in]-1, 1[$ and let $V \in \tilde{\eta}^{-1}(\alpha)$. Let (u, v) be an ordered orthonormal basis of V which represents the given orientation on V. Now, by (5.2), v can be decomposed into

$$-\alpha\left(\hat{J}u\right) + \sqrt{1-\alpha^2}w \tag{11}$$

for some $w \in (\text{span}\{u, Ju\})^{\perp}$. Now construct a curve on $G_2^+(W)$ by

$$\gamma(t) := \text{span}(u, -t\left(\hat{J}u\right) + \sqrt{1-t^2}w) \tag{12}$$

for $t \in]-1,1[$. Then,

$$\gamma(\alpha) = V \tag{13}$$

and

$$\begin{aligned} \tilde{\eta}\,(\gamma(t)) &= \eta(u, -t(\hat{\mathrm{J}}u) + \sqrt{1-t^2}w) \\ &= -t\eta(u, \hat{\mathrm{J}}u) + \sqrt{1-t^2}\eta(u,w) \\ &= -t(-1) + 0 \\ &= t \end{aligned} \tag{14}$$

so

$$\frac{d\tilde{\eta} \circ \gamma}{dt} = 1. \tag{15}$$

Clearly, $\frac{d\tilde{\eta}}{dt}$ is rank 1 for all $t \in]-1,1[$ and, in particular, is rank 1 at V. Thus, all the points in $]-1,1[$ are regular values and all of the non-extremal level sets are diffeomorphic. □

Now that it has been demonstrated that all non-extremal level sets are diffeomorphic, the next step is to find the maximal subgroup of $\mathrm{O}(W)$ which leaves all level sets invariant. As explained in the first section, only subgroups of the orthogonal group of W, $\mathrm{O}(W)$, are considered because the action of $\mathrm{O}(W)$ preserves the definition of $\tilde{\eta}$ as a function on $G_2^+(W)$.

5.5 Proposition Let W be a $2n$-dimensional real inner product space and let η be a symplectic form on W which has an orthonormal standard basis $\mathcal{B}$. Then, defining a trivial action on $\mathbb{R}$, the largest subgroup G of $\mathrm{O}(W)$ for which $\tilde{\eta}$ is G-equivariant is the unitary group on W, $U(W)$.

Proof First, recall that the unitary group is an embedded subgroup of the orthogonal group (C.3).

As proven in (4.9), G must be a subgroup of the commutator of the matrix J_{2n} which represents η with respect to $\mathcal{B}$; ie. $G < GL(W;\mathbb{C})$.

Now, recall that the complex structure on W induced from η and the inner product is represented by J_{2n} with respect to any orthonormal standard basis. By definition, the set of linear transformations on W which commute with the complex scalar i is $GL(W, \mathbb{C})$.

As shown in (C.3), $GL(W, \mathbb{C}) \cap O(W) = U(W)$ and so $G < U(W)$. Conversely, if $A \in U(W)$ then A commutes with J and

$$\begin{aligned} \eta(Au, Av) &= (Au)^T J(Av) \\ &= u^T(A^T JA)v \\ &= u^T(A^{-1}JA)v && \text{since } A \text{ is orthogonal} \\ &= u^T Jv && \text{since } A \text{ commutes with J} \\ &= \eta(u, v). \end{aligned} \tag{16}$$

Thus, every element of $U(W)$ preserves the value of η. Therefore, the largest subgroup of $O(W)$ which preserves η is the unitary group, $U(W)$. □

Next use the action of $U(W)$ on the level sets of $\tilde{\eta}$ to tell us about the structure of these level sets. First, since the extremal level sets are associated with critical points, they need not be diffeomorphic to non-extremal level sets. Therefore, they are treated separately. As observed in (5.3) these level sets are in bijective correspondence with complex projective manifolds. In order to define an embedding of $\mathbb{CP}^{n-1}$ into $G_2^+(W)$ we define

$$\mathbb{CP}^-(W) := \{\text{complex lines in } W \text{ with a preferred orientation opposite to } (u, \hat{J}u)\} \tag{17}$$

and give it the conventional complex projective manifold structure.

5.6 Theorem Let W be a real $2n$-dimensional inner product space. Let η be a symplectic form on W which has an orthonormal standard basis and let $\tilde{\eta} : G_2^+(W) \to \mathbb{R}$ be the smooth map induced from η. Then the inclusion $U(W) \subset$

$O(W)$ induces an imbedding of homogeneous spaces

$$\mathbb{CP}^{-}(W) \hookrightarrow G_2^{+}(W). \tag{18}$$

The image of this imbedding is the maximal level set.

Proof By (5.3) the set of all negatively oriented complex lines is the maximal level set. Thus, we must confirm the manifold and submanifold structure of this set. First, consider the action of $U(W)$ on $G_2^{+}(W)$. It was shown in (5.5) that $\tilde{\eta}$ is $U(W)$-equivariant. Now, by definition, $U(W)$ acts transitively on $\mathbb{CP}^{-}(W)$. Therefore, the maximal level set is a single orbit of the action of $U(W)$ on $G_2^{+}(W)$. Now, as shown in (B.7), any orbit of a compact group action on a manifold X is a submanifold of X. Finally, that same theorem provides the tools to explicitly determine the structure of that orbit given only the stabilizer of a single point on the orbit.

Let V be a negatively oriented complex line. Then, (5.3) gives that $\tilde{\eta}(V) = 1$. We wish to calculate the stabilizer of V under the action of $U(W)$. Now, a consequence of (2.6) is that

$$U(W)_V = (\mathrm{SO}(V) \times \mathrm{O}(V^{\perp})) \cap U(W). \tag{19}$$

But $\mathrm{SO}(V) \times \mathrm{O}(V^{\perp}) \subset \mathrm{O}(W)$ and, as shown in Appendix C,

$$U(W) = \mathrm{GL}(W, \mathbb{C}) \cap O(W). \tag{20}$$

Therefore,

$$U(W)_V = (\mathrm{SO}(V) \times \mathrm{O}(V^{\perp})) \cap \mathrm{GL}(W, \mathbb{C}). \tag{21}$$

Now, by definition, the complex general linear group is the group of linear transformations which commutes with the complex structure map $\hat{\mathrm{J}}$. Since V is a complex line, $\hat{\mathrm{J}}$ fixes V and, therefore, $\hat{\mathrm{J}}$ fixes $V^{\perp}$. Thus, we can consider

$$\hat{\mathrm{J}} = \hat{\mathrm{J}}|_V \oplus \hat{\mathrm{J}}|_{V^{\perp}}. \tag{22}$$

This means that for a transformation $g = (h, k) \in \mathrm{SO}(V) \times \mathrm{O}(V^{\perp})$ to commute with $\hat{\mathrm{J}}$, then h must commute with $\hat{\mathrm{J}}|_V$ and k must commute with $\hat{\mathrm{J}}|_{V^{\perp}}$. Thus,

$$U(W)_V = (\mathrm{SO}(V) \cap \mathrm{GL}(V, \mathbb{C})) \times (\mathrm{O}(V^{\perp}) \cap \mathrm{GL}(V^{\perp}, \mathbb{C})) \,. \tag{23}$$

Finally, as seen in Appendix C,

$$\mathrm{SO}(V) \cap \mathrm{GL}(V, \mathbb{C}) = U(V) \tag{24}$$

and

$$\mathrm{O}(V^{\perp}) \cap \mathrm{GL}(V^{\perp}, \mathbb{C}) = U(V^{\perp}). \tag{25}$$

So, it is possible to conclude that

$$U(W)_V = U(V) \times U(V^{\perp}). \tag{26}$$

By theorem (B.7), this means that

$$\begin{aligned} \tilde{\eta}^{-1}(1) &\cong U(W) / U(V) \times U(V^{\perp}) \\ &\cong \mathbb{CP}^{n-1} \qquad \text{since } V \text{ is a complex line.} \end{aligned} \tag{27}$$

□

The fact that the maximal level set is homogeneous and the next result which states that all other level sets are also homogeneous simplifies matters. This way each level set is simply an orbit and thus gets its submanifold structure directly from the orbit structure. It is worth noting that, as shown in (5.4), non-extremal values are regular. Therefore, the implicit function theorem applies and guarantees that non-extremal level sets are submanifolds of $G_2^+(W)$ regardless of whether or not they are homogeneous.

Prior to the next theorem, a few constructions are needed. For $\theta \in [0, 2\pi]$.

let

$$A_1(\theta) := \begin{bmatrix} \cos\theta & \alpha\sin\theta & -\sqrt{1-\alpha^2}\sin\theta & 0 \\ -\alpha\sin\theta & \cos\theta & 0 & -\sqrt{1-\alpha^2}\sin\theta \\ \sqrt{1-\alpha^2}\sin\theta & 0 & \cos\theta & \alpha\sin\theta \\ 0 & \sqrt{1-\alpha^2}\sin\theta & -\alpha\sin\theta & \cos\theta \end{bmatrix}. \tag{28}$$

Now, let $B \in U(\mathbb{C}^{n-2})$ and define

$$A_2(\theta) := \begin{bmatrix} A_1(\theta) & 0 \\ 0 & B \end{bmatrix} \tag{29}$$

Note that with respect to an orthonormal basis, this represents an embedding of

$$\{A_2(\theta) |\ \theta \in [0, 2\pi]\} \cong U(\mathbb{C}^1) \times U(\mathbb{C}^{n-2}) \tag{30}$$

into $U(W)$. The next proposition will show that each non-extremal level set has the structure of this embedding.

5.7 Remark A dimension check verifies the plausibility of this claim as these level sets are expected to be of co-dimension 1 in $G_2^+(W)$:

$$\begin{aligned} \dim G_2^+(W) - 1 &= 2(2n-2) - 1 \\ &= 4n - 5 \end{aligned} \tag{31}$$

$$\begin{aligned} \dim U(W)/U(\mathbb{C}) \times U(\mathbb{C}^{n-2}) &= n^2 - 1^2 - (n-2)^2 \\ &= 4n - 5 \end{aligned} \tag{32}$$

when viewed as a real manifold.

5.8 Theorem Let W be a $2n$-dimensional real inner product space. Let η be a symplectic form on W which has an orthonormal standard basis and let $\tilde{\eta}$ be the map on $G_2^+(W)$ induced from η. Then each non-extremal level set is homogeneous and is $U(W)$-equivariantly diffeomorphic to $U(W)/U(\mathbb{C}^1) \times U(\mathbb{C}^{n-2})$.

Proof Recall from (5.3) that the extremal values are ± 1. Thus, let $\alpha \in]-1, 1[$. The assertion is proven by choosing a vector space V in the α-level set and considering the orbit $U(W).V$. First, to prove homogeneity, we will show that $U(W).V = \tilde{\eta}^{-1}(\alpha)$. Then, as shown in (B.7), calculating the stabilizer of V will give the homogeneous structure of the level set. Note, calculation of the stabilizer of V is very intensive. A more geometric version of this proof is provided for the general case in the next section.

Since it is irrelevant to the theory which $V \in \tilde{\eta}^{-1}(\alpha)$ is chosen, choose V to be most convenient for calculations. Specifically, for the orthonormal standard basis $\mathcal{B} := \{w_1, \ldots, w_{2n}\}$, let

$$v := \alpha w_2 + \sqrt{1-\alpha^2} w_3 \tag{33}$$

and give

$$V := \operatorname{span}\{w_1, v\} \tag{34}$$

the orientation represented by (w_1, v).

Now, since $\mathcal{B}$ is a standard basis, it is clear that

$$\eta(w_1, w_2) = 1 \tag{35}$$

and that

$$\eta(w_1, w_3) = 0. \tag{36}$$

Thus, by linearity,

$$\begin{aligned} \tilde{\eta}(V) &= \eta(w_1, v) \\ &= \alpha \cdot \eta(w_1, w_2) + \sqrt{1-\alpha^2} \cdot \eta(w_1, w_3) \\ &= \alpha. \end{aligned} \tag{37}$$

The next step is to prove that the level set is homogeneous. Thus, we must confirm that $U(W).V = \eta^{-1}(\alpha)$. Let $V' \in \eta^{-1}(\alpha)$ and let (u, u') be an orthonormal basis of V' representing the given orientation. Consider the projection of the unit square spanned by (u, v) of V' onto the complex line $\mathbb{C}u$. By (5.2), the oriented area of this projection is α. So

$$u' := -\alpha \hat{J} u + \sqrt{1-\alpha^2}\, w \tag{38}$$

for some unit vector $w \in (\mathbb{C}u)^{\perp}$. Since $U(W) = \mathrm{O}(W) \cap \mathrm{GL}(W, \mathbb{C})$, the action of $U(W)$ can map V onto any such V'. Thus, $U(W)$ acts transitively on the level set and the entire level set is a single orbit.

Finally, calculate the structure of the orbit $U(W).V$. For this, as shown in (B.7), we need the stabilizer of V under the action of $U(W)$. By (2.6), this stabilizer is $(\mathrm{SO}(V) \times \mathrm{O}(V^{\perp})) \cap U(W)$. Suppose that $g \in U(W)_V$. Then we first intend to prove that, $g \in U(\mathbb{C}^2) \times U(\mathbb{C}^{n-2})$. For ease of notation, define the subspace

$$\mathbb{C}V = W_4 := \mathrm{span}(w_1, \ldots, w_4) \tag{39}$$

Thus, since g is unitary and stabilizes V, g must stabilize W_4 and $(W_4)^{\perp}$. As seen in the proof of (5.6), this is sufficient to conclude that $g \in U(W_4) \times U(W_4^{\perp})$ as desired.

Now, of course, $U(W_4) \times U(W_4^{\perp})$ acts diagonally on $W_4 \oplus W_4^{\perp}$. Since $V \subset W_4$, this means that the entire group $U(W_4^{\perp})$ is part of the stabilizer of V. Therefore, it remains to discover the conditions on $h \in U(W_4)$ for it to be an element of the stabilizer of V.

Suppose that h stabilizes V. At this point, it is more convenient to do certain calculations in a matrix setting than with linear transformations so let A be the matrix representing $h \in U(W_4)$ with respect to the basis $\{w_1, w_2, w_3, w_4\}$.

It is clear from matrix multiplication that

$$A.w_1 = \begin{bmatrix} A_{11} \\ A_{21} \\ A_{31} \\ A_{41} \end{bmatrix} \tag{40}$$

Since $A.w_1 \in V$,

$$A.w_1 = A_{11}w_1 + \frac{1}{\sqrt{1-\alpha^2}}A_{31}v \tag{41}$$

Note that $|\alpha| \lneqq 1$. Thus,

$$\begin{aligned} A_{41} &= 0 \\ \text{and } A_{21} &= \frac{\alpha}{\sqrt{1-\alpha^2}}A_{31}. \end{aligned} \tag{42}$$

Now, $A \in U(W_4) \subset O(\mathbb{R}^4)$ so the norms of all of the rows and columns are all 1. Specifically, the norm of the first column is 1. So

$$\begin{aligned} 1 &= (A_{11})^2 + \frac{\alpha^2}{1-\alpha^2}(A_{31})^2 + (A_{31})^2 \\ &= (A_{11})^2 + \frac{1}{1-\alpha^2}(A_{31})^2 \end{aligned} \tag{43}$$

So, for notational convenience, let $A_{11} = \cos\theta$ for some $\theta \in [0, 2\pi]$. Then $A_{31} = \sqrt{1-\alpha^2}\sin\theta$. Note that as θ moves through $[0, 2\pi]$, all possible combinations of positive and negative terms occur.

Now since A represents a unitary transformation, the first column of A determines the second column (C.1). To recap, thus far we have

$$A = \begin{bmatrix} \cos\theta & -\alpha\sin\theta & \\ \alpha\sin\theta & \cos\theta & A^* \\ \sqrt{1-\alpha^2}\sin\theta & 0 & \\ 0 & \sqrt{1-\alpha^2}\sin\theta & \end{bmatrix} \tag{44}$$

for some as yet unknown two column A^{*}. Now calculate $A.v$:

$$A.v = \begin{bmatrix} \alpha^2 \sin\theta + A_{13}\sqrt{1-\alpha^2} \\ \alpha\cos\theta + A_{23}\sqrt{1-\alpha^2} \\ A_{33}\sqrt{1-\alpha^2} \\ \alpha\sqrt{1-\alpha^2}\sin\theta + A_{43}\sqrt{1-\alpha^2} \end{bmatrix} \tag{45}$$

Since $A.v \in V$, quickly conclude that

$$A.v = (\alpha\sin\theta + A_{13}\sqrt{1-\alpha^2})w_1 + A_{33}v \tag{46}$$

Thus, examining the fourth coordinate,

$$\alpha\sqrt{1-\alpha^2}\sin\theta + A_{43}\sqrt{1-\alpha^2} = 0. \tag{47}$$

Since $\sqrt{1-\alpha^2} \neq 0$, then

$$A_{43} = -\alpha\sin\theta. \tag{48}$$

Now, the norm of the fourth row is 1 so

$$\begin{aligned} 1 &= 0 + (1-\alpha^2)\sin^2\theta + \alpha^2\sin^2\theta + (A_{44})^2 \\ &= \sin^2\theta + (A_{44})^2 \end{aligned} \tag{49}$$

Thus,

$$A_{44} = \pm\cos\theta \tag{50}$$

Furthermore, by the symmetry of the matrices representing $U(W_4)$ imposed by the imbedding of $U(W_4)$ into $O(W_4)$,

$$A_{33} = A_{44} = \pm\cos\theta. \tag{51}$$

Now consider the second entry in $A.v$.

$$\alpha\cos\theta + A_{23}\sqrt{1-\alpha^2} = \alpha A_{33} \tag{52}$$

So.

$$A_{23} = \frac{\alpha}{\sqrt{1-\alpha^2}}(-\cos\theta \pm \cos\theta). \tag{53}$$

We claim that $A_{23} = 0$. Suppose otherwise. Then $\alpha \neq 0$ and $A_{33} = A_{44} = -\cos\theta \neq 0$. Since the norm of the third column must be 1,

$$1 = (A_{13})^2 + \frac{4\alpha^2}{1-\alpha^2}\cos^2\theta + \cos^2\theta + \alpha^2\sin^2\theta \tag{54}$$

So.

$$A_{13} = \pm\sqrt{1 - \frac{1+3\alpha^2}{1-\alpha^2}\cos^2\theta - \alpha^2\sin^2\theta}. \tag{55}$$

One final condition on the rows and columns of A is that they must be mutually orthogonal. Specifically, columns 1 and 3 must be orthogonal. So.

$$\begin{aligned} 0 &= (A_{*1})\cdot(A_{*3}) \\ &= \cos\theta\left(\pm\sqrt{1 - \frac{1+3\alpha^2}{1-\alpha^2}\cos^2\theta - \alpha^2\sin^2\theta}\right) \\ &- (\alpha\sin\theta)(\frac{2\alpha}{\sqrt{1-\alpha^2}}\cos\theta) + \sqrt{1-\alpha^2}\sin\theta(-\cos\theta) + 0 \end{aligned} \tag{56}$$

Since $\cos\theta \neq 0$ by assumption,

$$\begin{aligned} \pm\sqrt{1 - \frac{1+3\alpha^2}{1-\alpha^2}\cos^2\theta - \alpha^2\sin^2\theta} &= \left(\sqrt{1-\alpha^2} + \frac{2\alpha^2}{\sqrt{1-\alpha^2}}\right)\sin\theta \\ 1 - \frac{1+3\alpha^2}{1-\alpha^2}\cos^2\theta - \alpha^2\sin^2\theta &= \left(\frac{1+\alpha^2}{\sqrt{1-\alpha^2}}\right)^2\sin^2\theta \\ 1-\alpha^2-(1+3\alpha^2)\cos^2\theta - \alpha^2(1-\alpha^2)\sin^2\theta &= (1+2\alpha^2+\alpha^4)\sin^2\theta \end{aligned} \tag{57}$$

$$\begin{aligned} 0 &= 1-\alpha^2-(1+3\alpha^2)\cos^2\theta-(1+3\alpha^2)\sin^2\theta \\ &= 1-\alpha^2-(1+3\alpha^2) \\ &= -4\alpha^2 \end{aligned} \tag{58}$$

But the assumption was that $\alpha \neq 0$ leading to a contradiction. Thus,

$$A_{23} = 0 \tag{59}$$

Thus either $\alpha = 0$ or $A_{33} = A_{44} = \cos\theta$. Again, the norm of the third column must be 1 giving

$$1 = A_{13}^2 + 0 + \cos^2\theta + \alpha^2 \sin^2\theta \tag{60}$$

Thus, $A_{13} = \pm\sqrt{1-\alpha^2}\sin\theta$.

To determine the sign of A_{13}, check the orthogonality of the first and third columns.

Case 1: $A_{33} = -\cos\theta$ and $\alpha = 0$.

$$\begin{aligned} 0 &= (A_{*1}) \cdot (A_{*3}) \\ &= \cos\theta(\pm\sqrt{1-\alpha^2}\sin\theta) + 0 + \sqrt{1-\alpha^2}\sin\theta(-\cos\theta) + 0 \end{aligned} \tag{61}$$

So, $A_{13} = +\sqrt{1-\alpha^2}\sin\theta$.

In this case

$$A(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & \cos\theta & 0 & \sin\theta \\ \sin\theta & 0 & -\cos\theta & 0 \\ 0 & \sin\theta & 0 & -\cos\theta \end{bmatrix} \tag{62}$$

But when applied to V, this is an orientation reversing map. Thus, considering V as an oriented vector space, transformations of the above form are not elements of the stabilizer.

Therefore, regardless of whether $\alpha = 0$, $A_{33} = \cos\theta$. Using the same orthogonality argument as above, calculate that

$$A_{13} = -\sqrt{1-\alpha^2}\sin\theta. \tag{63}$$

This gives, as desired,

$$A(\theta) = \begin{bmatrix} \cos\theta & -\alpha\sin\theta & -\sqrt{1-\alpha^2}\sin\theta & 0 \\ \alpha\sin\theta & \cos\theta & 0 & -\sqrt{1-\alpha^2}\sin\theta \\ \sqrt{1-\alpha^2}\sin\theta & 0 & \cos\theta & -\alpha\sin\theta \\ 0 & \sqrt{1-\alpha^2}\sin\theta & \alpha\sin\theta & \cos\theta \end{bmatrix}. \quad (64)$$

Therefore, the stabilizer of V is the embedding of $U(\mathbb{C}^1)\times U(\mathbb{C}^{n-2})$ as desired. As shown in (B.7), the manifold structure of $\tilde{\eta}^{-1}(\alpha)$ is given by

$$\tilde{\eta}^{-1}(\alpha) \cong U(W) \ / \ U(\mathbb{C}^1)\times U(\mathbb{C}^{n-2})\,. \quad (65)$$

□

Therefore, if η is a symplectic form on W which admits an orthonormal standard basis and $\tilde{\eta}$ is the map on $G_2^+(W)$ induced from η, it is possible to completely describe each level set of $\tilde{\eta}$. We know that $\tilde{\eta}$ maps onto $[-1,1]$ and that all non-extremal values are regular which implies that all non-extremal level sets are $U(W)$-equivariantly diffeomorphic and submanifolds of $G_2^+(W)$. We also know that all of the level sets are homogeneous spaces under the action of $U(W)$ on $G_2^+(W)$ and their exact structure. Finally, as described in Appendix B, we can give a complete description of how these different level sets are glued together to form the Grassmann manifold.

6 Symplectic Forms: The General Case

In the previous section, symplectic forms which in the presence of a given inner product have orthonormal standard bases were examined. Although (4.11) does guarantee that for each symplectic form, η, there is a standard basis with respect to which

$$\eta = \bigoplus_{n} \det{}_2 \tag{1}$$

it does not guarantee that such a basis is orthonormal. This poses problems because groups such as $O(W)$ and $U(W)$ are very difficult to characterize with respect to non-orthonormal bases. Rather than attempt this, an orthogonal cannonical symplectic form was developed. Specifically, in (4.12) it was discovered that for each symplectic form there is a unique collection of real numbers $a_1 > \cdots > a_r > 0$ and muliplicities $d_1, \ldots, d_r$ and an orthonormal basis with respect to which

$$\eta = \bigoplus_{i=1}^{r} a_i (\bigoplus_{d_i} \det{}_2). \tag{2}$$

The collection of numbers and multiplicities is called the **symplectic spectrum** of η and any basis satisfying these conditions is called a **symplectic basis** of η. This basis also incorporates an intrinsic decomposition of W into mutually orthonormal subspaces

$$W = W_1 \oplus \cdots \oplus W_r \tag{3}$$

in that the first $2d_1$ vectors form an orthonormal basis for W_1, the next $2d_2$ vectors form an orthonormal basis for W_2, and so on. Therefore, this basis is often denoted by $(w_{1,1}, \ldots, w_{1,2d_1}, \ldots, w_{r,1}, \ldots, w_{r,2d_r})$.

This discovery allows the extension of the results found in the last section relating to symplectic forms which admit an orthonormal standard basis (standard symplectic forms) to all symplectic forms. Thus, this study will follow the same format as the last section. First, a characterization of the behaviour of $\tilde{\eta}(V)$ will be given. Although a complex structure on W can still be induced from η and the inner product structure, it's behaviour is a little less well intertwined with that of η. Thus $\tilde{\eta}(V)$ is not determined by the projection of V onto its complex span as in the general case. Rather, it is determined by those sorts of measurements on each of the projections of V into the various symplectic eigenspaces.

Next, critical points will be identified. In the standard case, these points were the complex lines of W. Analogously, the critical points of a general symplectic form are the complex lines of W which happen to be contained entirely within a single eigenspace. This gives the positive and negative symplectic eigenvalues as the critical values of the function. Since the standard symplectic case has only one symplectic eigenvalue and one symplectic eigenspace, this is a completely analogous result.

Next, the maximal group of $O(W)$ for which $\tilde{\eta}$ is equivariant with respect to a trivial action on $\mathbb{R}$ is determined to be $U(W_1) \times \ldots \times U(W_r)$. Again, as the standard symplectic case only involves one symplectic eigenspace, this is a consistent result. As in the standard case, this gives the extremal level sets a homogeneous manifold structure equivariantly diffeomorphic to complex projective space. However, in the standard case, non-extremal level sets were also homogeneous manifolds. In the more general situation, this result doesn't carry over: non-extremal level sets are made up of multiple orbits. The dissection of the non-extremal level sets will thus be in two phases: first, it will be determined which orbits belong to which level sets and, second, the structure of these orbits will be calculated.

Essentially, an arbritrary symplectic form in the presence of an inner product is the direct sum of scaled orthonormal cannonical chunks. In order to make this understanding meaningful, it is necessary to be able to intrinsically define a cannonical symplectic form given an arbitrary symplectic form and an inner product structure.

6.1 Lemma Let W be an inner product space and let η be a symplectic form on W. For a given symplectic basis of W, define the cannonical symplectic form $S : W \times W \to \mathbb{R}$ by

$$S = \bigoplus_{n} \det{}_2 \tag{4}$$

Then the form S is independent of the symplectic basis used to define it.

Proof Suppose $\mathcal{B}_1$ and $\mathcal{B}_2$ are two symplectic bases for W. Define cannonical symplectic forms S_1 and S_2 in relation to $\mathcal{B}_1$ and $\mathcal{B}_2$ respectively. Then to check whether $S_1 = S_2$, it is sufficient to consider a basis of W, say $\mathcal{B}_1$.

Note that, by construction, $\mathcal{B}_1$ takes only values of 1, 0, or, because it is alternating, -1, on pairs of basis vectors from $\mathcal{B}_1$.

Suppose that $v, w \in \mathcal{B}_1$ and, without loss of generality,

$$S_1(v, w) = 1 \tag{5}$$

Then, since $\mathcal{B}_1$ is a symplectic basis, $v, w \in W_i$ for some i,

$$\eta(v, w) = a_i. \tag{6}$$

Since $\mathcal{B}_2$ is also a symplectic basis and $v, w \in W_i$, and $\eta|_{W_i} = a_i \bigoplus_{d_i} \det_2 = a_i S_2|_{W_i}$,

$$S_2(v, w) = 1. \tag{7}$$

Similarly, if $S_1(v, w) = 0$ and v, w both belong to some W_i,

$$\eta(v, w) = 0 \tag{8}$$

and so

$$S_2(v, w) = 0. \tag{9}$$

Clearly, if $v \in W_i, w \in W_j, i \neq j$ then

$$S_1(v, w) = 0 = S_2(v, w). \tag{10}$$

□

Thus an arbitrary symplectic form in the presence of an inner product intrinsically defines a standard symplectic form on W. Of course, this construction can be restricted to individual symplectic eigenspaces giving $S_i : W_i \times W_i \to \mathbb{R}$. Furthermore, as seen in (5.1) the cannonical symplectic form S together with the inner product structure on W intrinsically define a complex structure map $\hat{J}$ on W. This complex structure will be extensively used through this section.

With these constructions in place, it is possible to characterize the behaviour of η.

6.2 Lemma Let W be a real $2n$-dimensional inner product space. Let η be a symplectic form on W with a symplectic spectrum $a_1 > \cdots > a_r$ and symplectic eigenspaces $W_1, \ldots, W_r$ with orthogonal projection functions $\pi_i : W \to W_i$. Let $\tilde{\eta} : G_2^+(W) \to \mathbb{R}$ be the smooth map induced from η. For each $1 \leq i \leq r$ let S_i be the cannonical symplectic form on W_i induced from η and the inner product.

For a $V \in G_2^+(W)$, let (u, v) be an ordered orthonormal basis for V representing the given orientation. Then

$$\tilde{\eta}(V) = \sum_{i=1}^{r} a_i \cdot S_i(\pi_i(u), \pi_i(v)) \tag{11}$$

Proof By definition,

$$\begin{aligned} \eta(u, v) &= \sum_{i=1}^{r} \eta|_{W_i}(\pi_i(u), \pi_i(v)) \\ &= \sum_{i=1}^{r} a_i \bigoplus_{d_i} \det{}_2(\pi_i(u), \pi_i(v)) \\ &= \sum_{i=1}^{r} a_i S_i(\pi_i(u), \pi_i(v)) \end{aligned} \tag{12}$$

□

6.3 Corollary Let W be an inner product space and let η be a symplectic form on W with symplectic spectrum $a_1 > \cdots > a_r > 0$. Then

$$\max_{V \in G_2^+(W)} \tilde{\eta}(V) = a_1 \tag{13}$$

and $\tilde{\eta}$ attains this maximum on complex lines in W_1 which have the orientation opposite to that given by the complex structure on V.

Proof Let $V \in G_2^+(W)$ and let (u, v) be an ordered orthonormal basis of V representing the given orientation. Then

$$\begin{aligned}
\tilde{\eta}(V) &= \sum_{i=1}^{r} a_i S_i(\pi_i(u), \pi_i(v)) && 6.2 \\
&\leq \sum_{i=1}^{r} a_1 S_i(\pi_i(u), \pi_i(v)) && a_1 \geq a_i \qquad (14) \\
&= a_1 \tilde{S}(V) \\
&\leq a_1 && 5.3
\end{aligned}$$

Furthermore, it is clear that equality is achieved precisely when

$$\pi_i(V) = \{0\} \qquad \forall i > 1 \tag{15}$$

and

$$\tilde{S}(V) = 1. \tag{16}$$

Of course, $\tilde{S}$ achieves its maximum if and only if V is a negatively oriented complex line (5.3). □

The next step is to calculate the critical values of $\tilde{\eta}$. As shown in 2.10, this will confirm that level sets between adjacent symplectic eigenvalues are equivariantly diffeomorphic.

6.4 Theorem Let W be an inner product space with a complex structure map induced from η and the inner product. Let η be a symplectic form on W with symplectic spectrum $a_1 > \cdots > a_r > 0$ and associated symplectic eigenspaces $W_1, \ldots, W_r$. Let $\tilde{\eta} : G_2^+(W) \to \mathbb{R}$ be the smooth map induced from η. Then $\tilde{\eta}(V)$ is critical if and only if

$$V = \mathbb{C}V \subset W_i \qquad \text{for some } i. \tag{17}$$

Proof First we check the critical points. So suppose that

$$V = \mathbb{C}V \subset W_i \tag{18}$$

for some $1 \leq i \leq r$. Now, from the corollary to the normal basis picking lemma (2.9), we know that if it is not possible to vary only one basis vector and get a path along which the derivative of $\tilde{\eta}$ is not zero, then it is not possible to find such a path at all.

However, the combination of (5.6) and (6.2) guarantee that $\tilde{\eta}(V)$ is locally extremal with respect to varying a single vector in isolation. Explicitly, suppose that $(u, \hat{J}u)$ is an orthonormal basis of V and, without loss of generality, assume that V has the conventional orientation of a complex line. Now suppose that $(u, v) \in V_2(W)$. Then

$$v = \alpha \hat{J}u + \beta w \tag{19}$$

for some α, β such that $\alpha^2 + \beta^2 = 1$ and $w \in V^\perp$ with $\|w\| = 1$. So

$$\begin{aligned} \eta(u, v) &= \eta(u, \alpha\hat{J}u + \beta w) \\ &= \alpha\eta(u, \hat{J}u) + \beta(u, v) \\ &= -\alpha a_1 + 0 \\ &> -a_1 \qquad \text{since } \alpha \in [-1, 1] \\ &= \eta(u, \hat{J}u) \end{aligned} \tag{20}$$

Thus, varying a single vector of V in isolation will produce a local minimum.

Thus, V is critical.

Next we consider all other points. This breaks down into two possibilities: either V is not a complex line or V is a complex line but is not contained in any symplectic eigenspace.

Case 1: Suppose $V \neq \mathbb{C}V$.

Let (u, v) be an ordered orthonormal basis of V which represents positive orientation on V. Then define a path $\gamma :]-1, 1[\to G_2^+(W)$ by

$$\gamma(t) := \text{span}\{u, \sqrt{1-t^2}v + t\hat{J}u\} \tag{21}$$

with orientation given by $(u, \sqrt{1-t^2}v + t\hat{J}u)$. Then

$$\begin{aligned} \tilde{\eta}(\gamma(t)) &= \eta(u, \sqrt{1-t^2}v + t\hat{J}u) \\ &= \sqrt{1-t^2}\eta(u, v) + t\eta(u, \hat{J}u) \end{aligned} \tag{22}$$

So

$$\frac{d}{dt}\tilde{\eta}(\gamma(t)) = \frac{-t}{\sqrt{1-t^2}}\eta(u, v) + \eta(u, \hat{J}u) \tag{23}$$

and

$$\begin{aligned} \frac{d}{dt}|_{t=0}\tilde{\eta}(\gamma(t)) &= \eta(u, \hat{J}u) \\ &\neq 0. \end{aligned} \tag{24}$$

Thus, all planes which are not complex lines are regular points of $\tilde{\eta}$.

Case 2: Suppose $V = \mathbb{C}V$ but $V \not\subseteq W_i$ for any i.

Without loss of generality, assume that V is oriented by the pair $(u, \hat{J}u)$ for some arbitrary unit vector $u \in V$.

Define $\pi_i : W \to W_i$ to be the orthogonal projections function. Now, choose i such that

$$\pi_i(u) \neq 0. \tag{25}$$

Let $u_i := \frac{\pi_i(u)}{||\pi_i(u)||}$. Then define a path $\gamma :]-1, 1[\to G_2^+(W)$ by

$$\gamma(t) := \text{span}\left\{u, \sqrt{1-t^2}\hat{J}u + t\hat{J}u_i\right\} \tag{26}$$

with orientation given by $\left(u, \sqrt{1-t^2}\hat{\mathrm{J}}u + t\hat{\mathrm{J}}\right)$. So

$$\tilde{\eta}(\gamma(t)) = \eta\left(u, \sqrt{1-t^2}\hat{\mathrm{J}}u + t\hat{\mathrm{J}}u_i\right) \tag{27}$$

and, as before,

$$\begin{aligned} \frac{d}{dt}|_{t=0}\tilde{\eta}(\gamma(t)) &= \eta\left(u, \hat{\mathrm{J}}u_i\right) \\ &\neq 0. \end{aligned} \tag{28}$$

Thus, all complex lines not contained in any single symplectic eigenspace are regular points of $\tilde{\eta}$. □

6.5 Corollary Let η be a symplectic form with symplectic spectrum $a_1 > \cdots > a_r > 0$. Then the critical values of the induced map $\tilde{\eta}$ are $\{\pm a_i \mid 1 \leq i \leq r\}$.

Proof This is an immediate consequence of (6.4) and (6.2). □

Another consequence of (6.2) is an understanding of what a neighbourhood about these critical points looks like. Of course, if $V = \mathbb{C}V \subset W_1$, then $\tilde{\eta}(V) = \pm a_1$ is extremal (5.3). Thus, these particular planes are local maxima or minima. But what about the complex lines in intermediate symplectic eigenspaces? Suppose $V = \mathbb{C}V \subset W_i$ for some $i > 1$. For any $j \neq i$, choose $w_j \in W_j$, $\|w_j\| = 1$, and define a path

$$\gamma(t) := \operatorname{span}\left(\sqrt{1-t^2}u + tw_j, \sqrt{1-t^2}\hat{\mathrm{J}}u + t\hat{\mathrm{J}}w_j\right). \tag{29}$$

So

$$\begin{aligned} \tilde{\eta}(\gamma(t)) &= (1-t^2)\eta(u, \hat{\mathrm{J}}u) + t^2\eta(w_j, \hat{\mathrm{J}}w_j) \\ &= -a_i + t^2(a_i - a_j). \end{aligned} \tag{30}$$

Depending on the relative values of a_i and a_j, this traces a path on $G_2^+(W)$ on which the value of $\tilde{\eta}$ is always non-decreasing or non-increasing. Thus, the complex lines contained in intermediate symplectic eigenspaces are saddle points. Moreover, the number of directions in which the saddle increases and the number of directions the saddle decreases are given by the collective dimension of all the eigenspaces with symplectic eigenvalues greater or smaller than a_i respectively.

We now begin the study of transformation group symmetries on level sets of $\tilde{\eta}$ by determining the maximal subgroup of $O(W)$ which leaves level sets invariant.

6.6 Proposition Let W be a $2n$-dimensional inner product space. Let η be a symplectic form on W with a symplectic spectrum $a_1 > \cdots > a_r > 0$ and symplectic eigenspaces $W_1, \ldots, W_r$ respectively. Then $U(W_1) \times \ldots \times U(W_r)$ is the largest subgroup of $O(W)$ for which $\tilde{\eta}$ is equivariant with respect to a trivial action on $\mathbb{R}$.

Proof As shown in (6.2), if for each $1 \leq i \leq r$, S_i is the standard symplectic form on W_i induced from $\eta|_{W_i}$ and the inner product, then

$$\tilde{\eta}(V) = \sum_{i=1}^{r} a_i \cdot S_i\left(\pi_i(u, v)\right). \tag{31}$$

Furthermore, as shown in (5.5), for any given i, $U(W_i)$ is the largest subgroup of $O(W_i)$ for which $\tilde{S}_i$ is equivariant. Thus, the action of group $U(W_1) \times \ldots \times U(W_r)$ on $G_2^+(W)$ does preserve the value of $\tilde{\eta}$. It remains to show that this is the largest group to do so. This is done by induction on the number of symplectic eigenspaces. r.

Base Case: Suppose $r = 1$

This reduces to the standard case in which the calculation has already been completed (5.5).

Inductive Case: Suppose the result holds for all symplectic forms with $r - 1$ symplectic eigenspaces.

Consider the maximal level set of $\tilde{\eta}$, the negatively oriented complex lines in W_1 (6.3). Since G keeps this level set invariant,

$$U(W_1) \subset G. \tag{32}$$

Furthermore, since any map involving interchanges between W_1 and $W_1^\perp$ will map at least part of some complex line in W_1 out of W_1, such a map will fail to stabilize the maximal level set. Thus we have

$$U(W_1)\times\ldots\times U(W_r)\subset G\subset U(W_1)\times O(W_1^\perp). \tag{33}$$

Thus, G is of the form

$$U(W_1)\times H \tag{34}$$

for some group H contained in $SO(W_1^\perp)$ and containing $U(W_2)\times\ldots\times U(W_r)$. Therefore, it is reasonable to restrict attention to $\eta|_{W_1^\perp}$. It should be noted that this 2-form is symplectic on $W_1^\perp$ with symplectic spectrum $a_2>\cdots>a_r$ and symplectic eigenspaces $W_2,\ldots,W_r$. Thus, by the inductive hypothesis.

$$H=U(W_2)\times\ldots\times U(W_r). \tag{35}$$

□

Now that there is a group acting on $G_2^+(W)$ for which $\tilde{\eta}$ is equivariant, it is possible to consider the structure of orbits and the types of orbits which belong in a given level set. We now present a characterization of vector spaces which is equivariant and so characterizes orbits. It turns out that there are some conditions on the possible characterizations of orbits in any given level set. Furthermore, this characterization almost completely determines the manifold structure of the orbit.

Define $\xi: G_2^+(W)\to\{0,1,2,3\}^r$ by

$$\xi_i(V)=\begin{cases}0 & \text{if } \dim\pi_i(V)=0\\ 1 & \text{if } \dim\pi_i(V)=1\\ 2 & \text{if } \dim\pi_i(V)=2 \text{ and } \pi_i(V)=\mathbb{C}\pi_iV\\ 3 & \text{if } \dim\pi_i(V)=2 \text{ and } \pi_i(V)\neq\mathbb{C}\pi_iV\end{cases}. \tag{36}$$

If an action of the group $G=U(W_1)\times\ldots\times U(W_r)$ on $\{0,1,2,3\}^r$ is defined trivially (so that multiplication by any element is the identity map), then we can see that

ξ is a G-equivariant function. This means that each orbit of the action of G on $G_2^+(W)$ will consist of vector spaces with the same characterization. Thus, it is appropriate to discuss the characterization of an orbit.

Now, the characterization of a vector space, in most cases, does not indicate which level set it falls in. In fact, many characterizations are present in all level sets. Consider an example. In the standard symplectic case, $r = 1$ and $a_1 = 1$. In this case, if $n > 2$, (5.6) showed that the extremal level sets were of type $\xi(V) = (2)$ while all the rest were of type $\xi(V) = (3)$: a direct relationship between vector space characterization and orbit type but very little relation between characterization and level set. The following gives the relationship between level set and possible characterizations of vector spaces in that level set.

6.7 Proposition Let W be a real $2n$-dimensional inner product space. If η is a symplectic form on W with symplectic eigenvalues $a_1 > \cdots > a_r > 0$ and associated symplectic eigenspaces $W_1, \ldots, W_r$ then the following restrictions on vector space characterizations exist for $V \in G_2^+(W)$:

1) $\xi(V) = (2, 0, \ldots, 0)$ if and only if $|\tilde{\eta}(V)| = a_1$

2) $\forall i$, if $\xi_j(V) = 2\delta_{ij}, \forall j$, then $|\tilde{\eta}(V)| = a_i$

3) If $\exists k \leq r$ such that $\xi_i(V) \leq 1\ \forall i \leq k$, then $|\tilde{\eta}(V)| \leq a_{k+1}$ where $a_{r+1} := 0$

4) If $\exists k \leq r$ such that $\xi_i(V) = 0\ \forall i \geq k$, and $\forall j \leq k, \xi_i(V) = 2$ or 0 then $\tilde{\eta}(V) \geq a_k$

Furthermore, for each value $\alpha \in [-a_1, a_1]$ and each characterization satisfying the above conditions, there is at least one orbit contained in $\tilde{\eta}^{-1}(\alpha)$ which has that characterization.

Proof This is a simple consequence of propositions (6.2) and (5.6). □

This last proposition gives complete information about which orbits belong to any given level set. Therefore, all that remains is to calculate explicit orbit structures. Recall that, as shown in (B.7), if a compact Lie group G acts on a differentiable manifold X, then for each $x \in X$, the orbit $G.x$ is diffeomorphic to G/G_x where G_x is the stabilizer of x under the action of G. Therefore, to calculate these orbit structures, it is only necessary to calculate stabilizers. It turns out, as shown in the remainder of this section, that these stabilizers are almost entirely dependent on the characterizations of the orbits.

First, a few simple constructions are needed. As always in this section, let η be a symplectic form on a real $2n$-dimensional inner product space W with a symplectic spectrum $a_1 > \cdots > a_r > 0$, associated symplectic eigenspaces $W_1, \ldots, W_r$, orthogonal projection functions $\pi_i : W \to W_i$. Let $\hat{J}$ be the complex structure map on W induced by η and the inner product. Then for each $V \in G_2^+(W)$ and for each $1 \leq i \leq r$ define subspaces of W by

$$X_i(V) := \mathbb{C}(\pi_i(V)). \tag{37}$$

Note that since each W_i is also a complex subspace of W,

$$X_i(V) \subset W_i \tag{38}$$

for each i. Also, the dimension of $X_i(V)$ is entirely dependent on the characterization $\xi_i(V)$ defined earlier. Explicitly,

$$\dim_{\mathbb{R}} X_i(V) = \begin{cases} 0 & \text{if } \xi_i(V) = 0 \\ 2 & \text{if } \xi_i(V) \in \{1, 2\} \\ 4 & \text{if } \xi_i(V) = 3 \end{cases} \cdot \tag{39}$$

6.8 Theorem Let W be a real $2n$-dimensional inner product space and let η be a symplectic form on W with symplectic eigenspaces $W_1, \ldots, W_r$ and orthogonal projection functions $\pi_i : W \to W_i$. Let $\hat{J}$ be the complex structure map on W induced by η and the inner product and let $< \cdot, \cdot >_{\mathbb{C}}$ be the hermitian inner product induced from the real inner product on W and $\hat{J}$. With respect to this hermitian inner product, let

$$G := U(W_1) \times \ldots \times U(W_r). \tag{40}$$

Let $V \in G_2^+(W)$ have characterization $\xi(V) = (\xi_i(V))_i$ and for each $1 \leq i \leq r$ let $X_i(V) := \mathbb{C}(\pi_i(V))$ as defined in (37). Let $\{u, v\}$ be an orthonormal basis of V such that (u, v) represents the given orientation on V.

Then if for each $1 \leq i \leq r$,

$$\begin{aligned} &< \pi_i(u), \pi_i(v) > = 0, \\ &\xi_i(V) \neq 1, \\ \text{and} \quad &\|\pi_i(u)\| = \|\pi_i(v)\|, \end{aligned} \tag{41}$$

then the orbit $G.V$ can be written

$$G.V \cong G \Big/ SO(V) \times \prod_i U(X_i(V)^{\perp}) \tag{42}$$

Should one of the conditions (41) not hold, then

$$G.V \cong G \Big/ C_2 \times \prod_i U(X_i(V)^{\perp}) \tag{43}$$

where C_2 is the cyclic group on two elements embedded naturally as a subgroup of $SO(V)$, and $X_i(V)^{\perp} \subset W_i$.

Proof First, define some notation: if $w \in W$ then $\forall 1 \leq i \leq r$,

$$w_i := \pi_i(w). \tag{44}$$

Since $SO(V)$ can be imbedded in $U(\mathbb{C}V)$, $SO(V) \times \prod_i U(X_i(V)^{\perp})$ is indeed a subgroup of G. From (B.7), it is clear that

$$G.V \cong G/G_V \tag{45}$$

where G_V is the stabilizer of V in G. Thus, it is necessary to calculate

$$G_V = (SO(V) \times O(V^{\perp})) \cap \prod_i U(W_i). \qquad (2.6) \tag{46}$$

Now let $Y(V) := \mathbb{C}V$. Then any unitary transformation which stabilizes V also stabilizes $Y(V)$. Thus,

$$G_V \subseteq U(Y) \times U(Y^\perp) \cap \prod_i U(W_i) \tag{47}$$

and in particular

$$O(V^\perp) \cap \prod_i U(W_i) \subseteq U(Y^\perp) \cap \prod_i U(W_i). \tag{48}$$

But

$$Y^\perp \subset V^\perp \tag{49}$$

and all unitary transformations are orthogonal (C.3). Thus,

$$U(Y^\perp) \subset O(V^\perp). \tag{50}$$

Therefore,

$$\begin{aligned} O(V^\perp) \cap \prod_i U(W_i) &= U(Y^\perp) \cap \prod_i U(W_i) \\ &= \prod_i U(X_i(V)^\perp). \end{aligned} \tag{51}$$

Thus, it remains to calculate

$$SO(V) \cap \prod_i U(W_i). \tag{52}$$

Now, by (C.4)

$$SO(V) \subset U(Y) \tag{53}$$

so

$$\begin{aligned} SO(V) \cap \prod_i U(W_i) &= SO(V) \cap (U(Y) \cap \prod_i U(W_i)) \\ &= SO(V) \cap \prod_i U(X_i(V)) \end{aligned} \tag{54}$$

Now suppose that $g \in SO(V)$. Then $g \in \prod_i U(X_i(V))$ if and only if $g \in \prod_i SO(\pi_i(V))$. By definition, this occurs if and only if it commutes with the orthogonal projection functions $\pi_i : W \to W_i$. Thus, since V is a plane, g is a rotation by some $\theta \in [0, 2\pi]$ and $g \in \prod U(X_i(V))$ if and only if

$$\begin{aligned} g(u_i) &= \pi_i(g(u)) \\ &= \pi_i(\cos\theta u + \sin\theta v) \\ &= \cos\theta(u_i) + \sin\theta(v_i) \end{aligned} \tag{55}$$

and

$$\begin{aligned} g(v_i) &= \pi_i(g(v)) \\ &= -\sin\theta(u_i) + \cos\theta(v_i). \end{aligned} \tag{56}$$

Now, clearly, if $\theta \in \{0, \pi\}$, then $\sin\theta = 0$ and $\cos\theta = \pm 1$. So, in such a circumstance, it is easy to see that g stabilizes V. Therefore,

$$C_2 \times \prod_i U(X_i(V)^{\perp} \cap W_i) \subset G_V \tag{57}$$

for any $V \in G_2^+(W)$.

Now suppose that $V \in G_2^+(W)$ satisfies the conditions (41). For each $\theta \in [0, 2\pi]$, let $g_{\theta i} : \pi_i(V) \to \pi_i(V)$ be defined by

$$\begin{aligned} g_{\theta i}(u_i) &= \cos\theta(u_i) + \sin\theta(v_i) \\ g_{\theta i}(v_i) &= -\sin\theta(u_i) + \cos\theta(v_i) \end{aligned} \tag{58}$$

Since V satisfies (41), $g_{\theta i} \in SO(\pi_i(V))$ thus, $g'_{\theta i} \in U(X_i(V))$. Therefore

$$g := (g'_{\theta i})_i \in \prod_i U(X_i(V)). \tag{59}$$

Clearly, $g \in SO(V)$. Thus, when $V \in G_2^+(W)$ satisfies the conditions given in (3),

$$SO(V) \subset \prod_i U(X_i(V)). \tag{60}$$

Therefore, in this case

$$G_V = SO(V) \times \prod_i U(X_i(V)^\perp \cap W_i). \tag{61}$$

Thus, it remains to show that the conditions given in (3) are necessary. So suppose that $g \in SO(V) \cap \prod_i U(X_i(V))$ is a rotation by θ.

First suppose that $\exists 1 \leq i \leq r$ such that $\xi_i(V) = 1$. Without loss of generality, assume that $\pi_i(u) \neq 0$. Now,

$$SO(\pi_i(V)) = C_2 \tag{62}$$

Now, by assumption, $g \in \prod_i SO(\pi_i(V))$. Thus since π_i is linear,

$$\begin{aligned} g(u_i) &= \cos\theta u_i + \sin\theta v_i = \epsilon u_i \\ \text{and} \qquad g(v_i) &= -\sin\theta u_i + \cos\theta v_i = \epsilon v_i \end{aligned} \tag{63}$$

for $\epsilon \in \{\pm 1\}$. Now, by assumption, u_i and v_i are linearly dependent. Thus it is necessary to perform the mechanical check that the only possible values for θ are 0 and π. So

$$\begin{aligned} \sin^2\theta v_i &= \sin\theta\epsilon u_i - \sin\theta\cos\theta u_i \\ \text{and} \qquad \sin\theta\cos\theta u_i &= \cos^2\theta v_i - \cos\theta\epsilon v_i \end{aligned} \tag{64}$$

Thus,

$$\begin{aligned} \sin^2\theta v_i &= \sin\theta\epsilon u_i - \cos^2\theta v_i + \cos\theta\epsilon v_i \\ \text{and so} \qquad (1 - \cos\theta\epsilon)v_i &= \sin\theta\epsilon u_i \end{aligned} \tag{65}$$

But, from (63), multiplying by ϵ

$$\epsilon\sin\theta u_i = (\epsilon\cos\theta - 1)v_i \tag{66}$$

Thus,

$$\sin\theta = 0. \tag{67}$$

Therefore, g is a rotation through either 0 or π and so $g \in C_2$.

Now suppose that $\pi_i(V)$ is either 0 or 2-dimensional for all $1 \leq i \leq r$. Then, as a rotation, g is orientation preserving on each of the projections $\pi_i(V)$. Therefore, it only remains to discover the conditions under which g is norm-preserving on $\pi_i(V)$.

Suppose that for some $1 \leq i \leq r$, $\pi_i(V)$ is 2-dimensional and $g|_{\pi_i(V)}$ is norm preserving, Now $\{u_i, v_i\}$ forms a basis of $\pi_i(V)$ but u_i and v_i are not necessarily orthogonal. However, by assumption,

$$\begin{aligned} g(u_i) &= \cos\theta u_i + \sin\theta v_i \\ \text{and} \qquad g(v_i) &= -\sin\theta u_i + \cos\theta v_i. \end{aligned} \tag{68}$$

It is a simple matter to check that unless u_i and v_i are mutally orthogonal and have the same norm, such a transformation is only norm-preserving for the trivial rotations of $\theta = 0$ and $\theta = \pi$. This concludes our proof. □

It is worth noting, in particular, that if $V = \mathbb{C}V \subset W_i$ for some i, then by (6.8),

$$\begin{aligned} G.V &= U(W_1) \times \ldots \times U(W_r) \Big/ SO(V) \times U(\pi_i(V)^{\perp}) \times \prod_{j \neq i} U(W_j)) \\ &\cong \mathbb{CP}^{d_i - 1} \end{aligned} \tag{69}$$

where $2d_i = \dim W_i$. Furthermore, G acts transitively on the set of complex lines of a given orientation in each eigenspace. This confirms that each critical level set contains a single homogeneous orbit diffeomorphic to complex projective space.

It is worth noting that the criteria in (41) are G-equivariant since G preserves inner products on each W_i. Thus, these criteria can be applied as easily to orbits as to vector spaces. It is now possible to tie in all the information in this section together with information from transformation group theory giving explicit structural information about level sets. The most important such conclusion giving maximal orbit types follows. It should be noted that according to (6.7), an orbit of this type occurs in each non-extremal level set.

6.9 Theorem Using the same assumptions as (6.8), let $\tilde{\eta}$ be the smooth map on $G_2^+(W)$ induced from η. Then each non-critical level set of $\tilde{\eta}$ has a dense open submanifold consisting of points V which do not satisfy the conditions (41) and which have characterizations

$$\xi_i(V) = \begin{cases} 3 & \text{where } \dim(W_i) > 2 \\ 1 \text{ or } 2 & \text{where } \dim(W_i) = 2 \end{cases} \tag{70}$$

Proof By the implicit value theorem, all non-critical level sets are submanifolds of $G_2^+(W)$. Since the action of $G := \prod_i U(W_i)$ preserves level sets, it is reasonable to consider the action of G on an individual level set $\tilde{\eta}^{-1}(a)$.

Now, a consequence of (B.7) is that orbits of a transformation group action on a manifold X are classified by the conjugacy class of the stabilizer of any point in the orbit: the smaller the stabilizer, the larger the orbit. Further, as noted in (B.12), there is a unique maximal orbit type and orbits of this type come together to form on open dense submanifold of X.

Therefore, it remains to show that the stabilizer of a vector space with characterization given by (70) is smaller than the stabilizer of any other vector space.

Now, first note that by (6.2) and (5.3), vector spaces with that characterization do, indeed, belong to each level set. Finally, by theorem (6.8), the stabilizer of such a vector space V under the action of G is

$$G_V = C_2 \times \prod_i U(X_i(V)^\perp) \tag{71}$$

where $X_i(V)$ is the complex span of $\pi_i(V)$.

Now, the characterization of V is such that $X_i(V)$ is as large as possible for each i. Thus, $U(X_i(V)^\perp)$ is as small as possible for each i. Thus, principal orbits are formed from vector spaces with characterizations given by (70). The result follow from (B.12) □

Thus, for each $V \in G_2^+(W)$, information about the structure of $G.V$ and by (B.12) a G-equivariant map from a principal orbit to $G.V$ follows entirely from the characterization $\xi(V)$ and a small amount of extra symmetry information. This, together with the slice theorem (B.10), is sufficient to completely describe all non-critical level sets of the map $\tilde{\eta}$ induced from a symplectic form η on W.

The information presented thus far gives complete and cohesive structural information about the level sets of $\tilde{\eta}$. In the case of critical orbits, however, it is possible to customize the group action to provide additional symmetry information.

6.10 Proposition Let W be an inner product space and let η be a symplectic form on W with symplectic eigenspaces $W_1, \ldots, W_r$. Let $\hat{\mathrm{J}}$ be the complex structure on W induced from η. For $1 \leq i \leq r$ define

$$C_i := \{V \in G_2^+(W) | V = \mathbb{C}V, V \subset W_i\}. \tag{72}$$

Then the maximal subgroup of $O(W)$ which leaves C_i invariant in $H = U(W_i) \times O(W_i^\perp)$.

Furthermore, if $V \in C_i$,

$$\begin{aligned} H.V &\cong U(W_i) \times O(W_i^\perp) / U(\mathbb{C}^1) \times U(\mathbb{C}^{d_i-1}) \times O(W_i^\perp) \\ &\cong \mathbb{CP}^{d_i-1}. \end{aligned} \tag{73}$$

Proof Recall that for a given i, if $V \in C_i$ then the characterization of V is given by $\xi_i(V) = 2$ and $\xi_j(V) = 0$ for all $j \neq i$. Thus, from (6.7),

$$|\tilde{\eta}(V)| = a_i. \tag{74}$$

Let H be the largest subgroup of $O(W)$ which acts on C_i. Thus, the canonical embedding of $U(W_i)$ into $O(W)$ must be a subgroup of H. Similarly, since if $V \in C_i$,

$$\pi_j(V) = \{0\} \quad \forall j \neq i, \tag{75}$$

any map which only affects $W_i^\perp$ will actually fix C_i. Finally, any map involving interchanges between W_i and $W_i^\perp$ will map at least a part of some complex line in W_i out of W_i and thus fail to stabilize the C_i. Thus, we conclude that $H = U(W_i) \times O(W_i^\perp)$ as desired.

It should be noted that $\tilde{\eta}$ is not H-equivariant. However, as it does act on the critical orbit associated with a_i, it provides extra symmetry information about that orbit. In particular, since the extremal level sets are among these orbits, extra symmetry information has been provided for these level sets.

Now, if $V \in C_i$, the structure of the orbit $H.V$ is determined by the stabilizer of V under the action of H. Since $SO(W_i^\perp)$ fixes every plane in C_i, only the stabilizer of the action of $U(W_i)$ need be calculated. However, this has already been done in the standard symplectic case (5.5). Thus,

$$\begin{aligned} H.V &\cong U(W_i) \times O(W_i^\perp) \ / \ (U(V) \times U(V^\perp \cap W_i)) \times O(W_i^\perp) \\ &\cong \mathbb{CP}^{d_i - 1} \end{aligned} \tag{76}$$

if we recall that $2d_i$ is the dimension of W_i. □

This provides a small amount of extra symmetry information about the extremal level sets and the critical sets at each symplectic eigenvalue.

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A Symplectic Results Summary

Let W be a real $2n$-dimensional inner product space and let $\eta : W^2 \to \mathbb{R}$ be a symplectic form on W.

A.1 Theorem **(Orthonormal Cannonical Symplectic Form (4.12))** Given a symplectic form η on a $2n$-dimensional real inner product space W, there exists an orthogonal decomposition of W into $W_1 \oplus \cdots \oplus W_r$ of dimensions $2d_1, \ldots, 2d_r$ and an orthonormal basis with respect to which

$$\eta = \bigoplus_{i=1}^{r} a_i \left(\bigoplus_{d_i} \det{}_2 \right) \tag{1}$$

for some real numbers $a_1 > a_2 > \cdots > a_r > 0$. Furthermore, these numbers and the vector space decomposition are uniquely determined by η.

A.2 Definition The numbers $a_1, \ldots, a_r$ will be referred to as the **symplectic eigenvalues** of η. The set of such eigenvalues will be known as the **symplectic spectrum** of η. The spaces $W_1, \ldots, W_r$ will be called the **symplectic eigenspaces** of η.

A.3 Lemma **(Calculation of η) (6.2)**

Let W be a real $2n$-dimensional inner product space. Let η be a symplectic form on W with a symplectic spectrum $a_1 > \cdots > a_r$ and symplectic eigenspaces $W_1, \ldots, W_r$ with orthogonal projection functions $\pi_i : W \to W_i$. Let $\tilde{\eta} : G_2^+(W) \to \mathbb{R}$ be the smooth map induced from η. For each $1 \leq i \leq r$ let S_i be the cannonical symplectic form on W_i induced from η and the inner product.

For a $V \in G_2^+(W)$, let (u, v) be an ordered orthonormal basis for V representing the given orientation. Then

$$\tilde{\eta}(V) = \sum_{i=1}^{r} a_i \cdot S_i\left(\pi_i(u), \pi_i(v)\right) \tag{2}$$

A.4 Theorem (Regular and Critical Points) (6.4)

Let W be an inner product space with a complex structure map induced from η and the inner product. Let η be a symplectic form on W with symplectic spectrum $a_1 > \cdots > a_r > 0$ and associated symplectic eigenspaces $W_1, \ldots, W_r$. Let $\tilde{\eta} : G_2^+(W) \to \mathbb{R}$ be the smooth map induced from η. Then $\tilde{\eta}(V)$ is critical if and only if

$$V = \mathbb{C}V \subset W_i \qquad \text{for some } i. \tag{3}$$

A.5 Corollary (Critical Values) (6.5) The critical values of the map $\tilde{\eta}$ induced from a symplectic form are the symplectic eigenvalues: $\{\pm a_i \mid 1 \leq i \leq r\}$.

A.6 Proposition (Maximal Group (6.6)) Let η be a symplectic form on W with a symplectic spectrum $a_1 > \cdots > a_r > 0$, multiplicities d_i and symplectic eigenspaces $W_1, \ldots, W_r$ respectively. Then $U(W_1) \times \ldots \times U(W_r)$ is the largest subgroup of $SO(W)$ for which $\tilde{\eta}$ is G-equivariant.

A.7 Definition (Characterization of Vector Spaces and Orbits) Define $\xi : G_2^+(W) \mapsto \{0, 1, 2, 3\}^r$ by

$$\xi_i(V) = \begin{cases} 0 & \text{if } \dim \pi_i(V) = 0 \\ 1 & \text{if } \dim \pi_i(V) = 1 \\ 2 & \text{if } \dim \pi_i(V) = 2 \text{ and } \pi_i(V) = \mathbb{C}\pi_i(V) \\ 3 & \text{if } \dim \pi_i(V) = 2 \text{ and } \pi_i(V) \neq \mathbb{C}\pi_i(V). \end{cases} \tag{4}$$

A.8 Proposition (Possible Orbit Types in each Level Set) (6.7)

Let W be a real $2n$-dimensional inner product space. If η is a symplectic form on W with symplectic eigenvalues $a_1 > \cdots > a_r > 0$ and associated symplectic eigenspaces $W_1, \ldots, W_r$ then the following restrictions on vector space characterizations exist for $V \in G_2^+(W)$:

1) $\xi(V) = (2, 0, \ldots, 0)$ if and only if $|\tilde{\eta}(V)| = a_1$

2) $\forall i$, if $\xi_j(V) = 2\delta_{ij}, \forall j$, then $|\tilde{\eta}(V)| = a_i$

3) If $\exists k \leq r$ such that $\xi_i(V) \leq 1$ $\forall i \leq k$, then $|\tilde{\eta}(V)| \leq a_{k+1}$ where $a_{r+1} := 0$

4) If $\exists k \leq r$ such that $\xi_i(V) = 0$ $\forall i \geq k$, and $\forall j \leq k, \xi_i(V) = 2$ or 0 then $\tilde{\eta}(V) \geq a_k$

Furthermore, for each value $\alpha \in [-a_1, a_1]$ and each characterization satisfying the above conditions, there is at least one orbit contained in $\tilde{\eta}^{-1}(\alpha)$ which has that characterization.

A.9 Theorem **(Structure of Orbits) (6.8)** Let W be a real $2n$-dimensional inner product space and let η be a symplectic form on W with symplectic eigenspaces $W_1, \ldots, W_r$ and orthogonal projection functions $\pi_i : W \to W_i$. Let $\hat{\mathrm{J}}$ be the complex structure map on W induced by η and the inner product and let $< \cdot, \cdot >_{\mathbb{C}}$ be the hermitian inner product induced from the real inner product on W and $\hat{\mathrm{J}}$. With respect to this hermitian inner product, let

$$G := U(W_1) \times \ldots \times U(W_r). \tag{5}$$

Let $V \in G_2^+(W)$ have characterization $\xi(V) = (\xi_i(V))_i$ and for each $1 \leq i \leq r$ let $X_i(V) := \mathbb{C}(\pi_i(V))$ as defined in (37). Let $\{u, v\}$ be an orthonormal basis of V such that (u, v) represents the given orientation on V.

Then if for each $1 \leq i \leq r$,

$$\begin{aligned} < \pi_i(u), \pi_i(v) > &= 0, \\ \xi_i(V) &\neq 1, \\ \text{and} \qquad \|\pi_i(u)\| &= \|\pi_i(v)\|, \end{aligned} \tag{6}$$

then the orbit $G.V$ can be written

$$G.V \cong G \Big/ SO(V) \times \prod_i U(X_i(V)^{\perp}) \tag{7}$$

Should one of the conditions (6) not hold, then

$$G.V \cong G \Big/ C_2 \times \prod_i U(X_i(V)^\perp) \tag{8}$$

where C_2 is the cyclic group on two elements embedded naturally as a subgroup of $SO(V)$, and $X_i(V)^\perp \subset W_i$.

A.10 Theorem (Maximal Orbit Type)(6.9) Each non-critical level set of $\tilde{\eta}$ has a dense open submanifold consisting of points V which do not satisfy the conditions given in the previous theorem and which have characterizations

$$\xi_i(V) = \begin{cases} 3 & \text{where } \dim(W_i) > 2 \\ 1 or 2 & \text{where } \dim(W_i) = 2 \end{cases} \tag{9}$$

B Transformation Group Theory

Transformation group theory studies the symmetries on a manifold which is equipped with a group action. Consider, for example, the set of rotations of a sphere about a given axis: the circle action on the sphere. The circle, S^1 can be viewed as the topological quotient space $[0, 2\pi] / 0 \sim 2\pi$. Then for each $g \in S^1$, there is a map $f_g : S^2 \to S^2$ which consists of rotating the sphere by g radians about the given axis.

Now let $x \in S^2$ be fixed and consider the set of all points that x can be rotated to by this action. This is known as the **orbit** of x. Simple visualization shows that, unless x is on the axis of rotation, this orbit is a circle perpendicular to the axis of rotation. So a sphere can be viewed as a collection of parallel circles shrinking down to degenerate circles (points) at the poles.

Thus, the sort of group actions that a manifold supports and the orbits which result say something about the types of symmetry that manifold exhibits. These sorts of symmetries, with various types of groups have proven useful in many fields including crystallography and mathematical physics.

This section will explicitly define group actions on a manifold X and will explain the structure of individual orbits and how they come together to form X. An excellent reference for this material is the first chapter of tom Dieck's book *Transformation Groups* [5].

B.1 Definition A **Topological Group** G is a Hausdorff topological space together with the continuous functions

$$\begin{aligned} \text{multiplication: } \mu &: G \times G \to G \\ \text{and inverse: inv} &: G \to G \end{aligned} \tag{1}$$

such that (G, μ) forms a group.

This combination of topological and algebraic structure is, in and of itself, quite powerful. However, even more results are obtained if one insists on a differentiable structure.

B.2 Definition A **Lie group** is a C^∞ differentiable manifold together with a group structure on that manifold such that the group operations are smooth.

Common examples of Lie groups include $\mathbb{R}$ and, more generally, all vector spaces. Also, the classical groups of transformations on a vector space are Lie groups.

B.3 Definition A smooth **action** of the Lie group G on a smooth manifold X is a smooth map

$$\lambda : G \times X \to X \tag{2}$$

written $\lambda(g, x) = g.x$ such that

$$e.x = x \qquad \forall x \in X \tag{3}$$

and

$$g.(h.x) = (gh).x \qquad \forall g, h \in G, x \in X. \tag{4}$$

To be completely accurate, the above definition actually refers to a **smooth left action** of G on X. Of course, it is possible to similarly define a smooth right action of G on X. This distinction, however, is insignificant since from any action it is possible to derive an action of the other "side". For instance, if $\lambda : G \times X \to X$ is a left action, define $\bar{\lambda} : X \times G \to X$ by $\bar{\lambda}(x, g) = \lambda(g^{-1}, x)$. Since

$$(gh)^{-1} = h^{-1}g^{-1}, \tag{5}$$

$\bar{\lambda}$ satisfies the conditions for a group action. Now, of course, one can observe that if a left action, λ, is by this means transformed into a right action and the resulting action is by similar means transformed into a left action, then the resulting action is the original λ. Thus, we shall suppress the "side" of any action unless it aids in exposition.

B.4 Definition Let a topological group G act continuously on a manifold X and fix $x \in X$. The **orbit** of x under the action of G is

$$G.x := \{y \in X \mid y = g.x \text{ for some } g \in G\} \tag{6}$$

The orbits of an action provide the entry point for any analysis using transformation group theory. They are disjoint, as will be proven next and so divide the manifold X up into separate parts which may be independently analyzed.

B.5 Theorem Let G be a topological group which acts continuously on a manifold X. Then the orbits of the action are disjoint.

Proof Let $x, y \in X$ and assume that $\exists z \in X$ such that

$$z \in G.x \cap G.y \tag{7}$$

So, there exist $g, h \in G$ such that

$$\begin{aligned} & z = g.x \\ \text{and} \quad & z = h.y. \end{aligned} \tag{8}$$

Then

$$\begin{aligned} & x = g^{-1}hy \\ \text{and} \quad & y = h^{-1}gx. \end{aligned} \tag{9}$$

Thus,

$$\begin{aligned} & G.x \subset G.y \\ \text{and} \quad & G.y \subset G.x. \end{aligned} \tag{10}$$

Therefore,

$$G.x = G.y. \tag{11}$$

So it has been shown that if two orbits have a non-trivial intersection, then they are identical. □

We now know what a group action is and that it breaks a manifold up into disjoint orbits. We now need to know two things: first, what these orbits look like, and second, how they are glued together to form the manifold.

In general, the answers to these questions are not entirely known. However, with some compactness and connectedness conditions, it turns out that the orbits are diffeomorphic to quotient groups of G and that they are glued together by G-equivariant maps.

First, however, consider whether a quotient group of a Lie group has the differentiable manifold structure necessary to be a Lie group in its own right. Let G be a Lie group and let H be a subgroup of G. Then consider G/H. If H is a normal subgroup of G then G/H is a group. We can certainly put the quotient topology on G/H to give it a topological structure. However, giving it the manifold structure necessary for it to be a Lie group is less easy.

B.6 Theorem Let H be a closed normal subgroup of the Lie group G. Then the quotient group G/H together with the quotient topology has a differentiable manifold structure such that G/H is a Lie group and the quotient map is a submersion.

Proof See [6, p.120]

Now turn to the structure of the orbits of a G action. Let a Lie group G act smoothly on a manifold X and fix $x \in X$. Let $G_x := \{g \in G | g.x = x\}$ be the stabilizer of x under the action of G. Then G_x is a normal subgroup of G. Also, it is closed since it is the pre-image of a single point under the smooth function $x : G \to X$ given by $x(g) := g.x$. Therefore G/G_x is a Lie group. Since x factors through the quotient map $g : G \to G/G_x$, the universal property of quotient maps gives a continuous bijection f between G/G_x and $G.x$.

Now if G is compact then since the continuous image of a compact space is compact, $G/G.x$ is compact. Further, a bijective map from a compact space to a Hausdorff space is a homeomorphism. Thus, the orbit $G.x$ of a compact Lie group acting on a manifold X is diffeomorphic to the quotient Lie group G/G_x. So each orbit is a homogeneous space.

B.7 Theorem Let G be a compact Lie group which acts smoothly on the differentiable manifold X. Then, for each $x \in X$, the orbit $G.x$ is a submanifold of X and is diffeomorphic to the Lie group G/G_x where G_x is the stabilizer of x under G.

Proof See [5, p.39]

There is now an explicit relationship between orbits and stabilizer groups. Suppose two orbits have stabilizers H and K which are conjugate to each other. It is reasonable to inquire what effect this has on the relative shapes of the orbits. A little thought produces a G-equivariant diffeomorphism $G/H \cong G/K$ in this situation. Thus, orbits with conjugate stabilizers are diffeomorphic: it is said that they are of the same **orbit type**.

Now, of course, there is a partial ordering of conjugacy classes of subgroups of G. This induces a partial ordering of orbit types. After some work, we shall see that if X/G is connected, there is a unique maximal orbit type and that the set of all points on orbits of that type form a dense open submanifold of X.

First, however, we investigate how orbits are glued together. The Slice Theorem will give a precise description of a neighborhood around any given orbit. It turns out that if $G.x$ is the orbit of some $x \in X$ there is an open set $U \subset X$ such that U is a locally trivial fiber bundle with base space $G.x = G/G_x$ (B.7).

To get this result, a few constructions are needed. Suppose G is a compact Lie group. From (B.7), $G.x$ is a submanifold of X. Thus, the tangent space $T_x(G.x)$ is a subvector space of the tangent space T_xX. As such, under a G-equivariant Riemann structure [1, p.304], it has an orthogonal complement denoted by $\nu_x(G.x \subset X)$. This vector space serves as the fiber of the fiber bundle describing our open neighborhood U.

A construction to explicitly describe this fiber bundle is given next.

B.8 Definition Let G be a Lie group and let X and Y be right and left G-spaces respectively. Then the **Borel construction** of X with Y over G is

$$X \times_G Y := X \times Y \ / \ (xg, y) \sim (x, gy) \tag{12}$$

In certain circumstances, this construction is actually a locally trivial fiber bundle.

B.9 Theorem If $\rho : E \to B$ is a G-principal bundle and F is a left G-manifold. then

$$\pi : E \times_G F \to B \tag{13}$$

given by $\pi[s, x] := \rho(s)$ is a locally trivial fiber bundle with fiber F and structure group G.

Proof See [1, p.74]

Now, of course, when G is a Lie group. G_x is a Lie subgroup and we can express G as a (G_x)-principal bundle. Thus, we will be able to apply (B.9) to better understand the Slice Theorem.

B.10 Theorem **(The Slice Theorem)** Let G, a compact Lie group, act smoothly on a differentiable manifold X. Let $x \in X$ and let $A := G.x$. Finally, let

$$\nu_x(A \subset X) := (T_x A)^{\perp T_x X}. \tag{14}$$

Then A has a G-equivariant open neighbourhood U and a G-equivariant diffeomorphism

$$G \times_H \nu_x(A \subset X) \to U \tag{15}$$

Proof See [1, p.306]

This result describes explicitly how orbits of a smooth Lie group action come together to form neighbourhoods on the manifold X. We are now in a position to consider the relationships between various orbit types.

First, using the Slice theorem and induction on the dimension of X, we can conclude that a closed manifold has only finitely many different orbit types.

B.11 Theorem Let G, a compact Lie group, act smoothly on a compact differentiable manifold X. Then X has only finitely many distinct orbit types.

Proof See [5, p.42].

This result simplifies any study of orbits of a compact Lie group action since it guarantees that there is only a finite number of calculations required.

Finally, since we have a partial ordering on the orbit types, it is reasonable to ask whether there is a maximal orbit type (ie. an orbit type whose stabilizer is conjugate to a subgroup of every other stabilizer). As shown next, this is indeed the case when X/G is connected.

B.12 Theorem Let G be a compact Lie group acting smoothly on a differentiable manifold X. Suppose X/G is connected. Then there is a Lie group H of G associated with an orbit of G on X such that $X_{(H)} := \{x \in X \mid G_x \text{ is conjugate to } H\}$ is an open dense submanifold of X and such that every other subgroup K associated to an orbit G of X has H as a conjugate subgroup.

Proof See [5, p.42].

Orbits of this unique maximal type are known as **principal orbits**. A consequence of (B.12) is that there is a G-equivariant map from a principal orbit to any non-principal orbit.

B.13 Corollary Let G be a compact Lie group acting smoothly on a differentiable manifold X and suppose X/G is connected. Let $G.x$ be a principal orbit and let $G.y$ be a non-principal orbit. Then there is a unique G-equivariant map from $G.x$ to $G.y$.

Proof Let $H = G_x$ and let $K = G_y$ be stabilizers. Then, by (B.7),

$$G.x \cong G/H, \tag{16}$$

and

$$G.y \cong G/K. \tag{17}$$

Now, by (B.12), H is conjugate to a subgroup of K. Then basic group theory gives a unique G-equivariant map between G/H and G/K by mapping the equivalence class $[H]$ to $[K]$ and extending G-equivariantly. □

Thus, transformation group theory gives a method of decomposing a manifold, X, into distinct orbits. Under appropriate conditions, the structure of these orbits is known. Furthermore, for any orbit, the Slice theorem provides an explicit description of a neighborhood around that orbit. Also, these orbits are put in correspondance with subgroups of the structure group G. This correspondance provides a definition and a partial ordering of orbit types. Finally, this partial ordering has a unique maximal element and orbits of this type come together to form an open dense submanifold of X. Thus, transformation group theory forms a fairly potent method for describing symmetries on manifolds.

C A Complex Structure on $\mathbb{R}^{2n}$

It is well known that any n-dimensional complex vector space can be viewed as a $2n$-dimensional real vector space. Thus, it is reasonable that there are ways of viewing $2n$-dimensional real vector spaces as n-dimensional complex ones. This appendix briefly explores one way of doing this and touches on the implications of this dual vision to some of the classical Lie groups.

Let W be a real $2n$-dimensional inner product space with an orthonormal basis $\mathcal{B} := \{w_1, \ldots, w_{2n}\}$ and let J be the $2n \times 2n$ matrix:

$$\mathrm{J} := \begin{bmatrix} 0 & -1 & & & \\ 1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & -1 \\ & & & -1 & 0 \end{bmatrix}. \tag{1}$$

Note that with respect to the basis $\mathcal{B}$, J represents an automorphism $\hat{\mathrm{J}}$ on W and that $\hat{\mathrm{J}}^2 = -\mathrm{Id}$. Thus, $\hat{\mathrm{J}}$ can serve as a **complex structure map** on W. This means that it is consistent to define $\hat{\mathrm{J}}$ as "multiplication by the complex scalar i." Note that with this particular construction,

$$i.w_{2k-1} := \hat{\mathrm{J}}(w_{2k+1}) = -w_{2k} \tag{2}$$

and so a basis for W as a complex vector space is $\{w_1, w_3, \ldots, w_{2n-1}\}$. This is one of the two most commonly used methods of putting a complex structure on a real vector space.

Now consider the general linear group $GL(W, \mathbb{C})$ on the complex vector space $W_{\mathbb{C}}$. Since these transformations are complex-linear, they are also real-linear. Thus, $GL(W, \mathbb{C}) \subset GL(W, \mathbb{R})$, the group of linear transformations of W as a real vector space. Now, given the basis $\{w_1, \ldots, w_{2n}\}$, $GL(W, \mathbb{R})$ can be represented by the set of real $2n$x$2n$ non-degenerate matrices. We now characterize the elements of that matrix group which represent elements of $GL(W, \mathbb{C})$.

C.1 Proposition Let W be a real $(2n)$-dimensional real vector space with basis $\{w_1, \ldots, w_{2n}\}$ and let $\hat{\mathrm{J}}$ as defined by (1) be a complex structure map on W. Then if $A \in M_{2n}(\mathbb{R})$ then A represents a complex linear transformation if and only if

$$A_{(2i-1),(2j-1)} = A_{(2i),(2j)} \tag{3}$$

and

$$A_{(2i),(2j-1)} = -A_{(2i-1),(2j)} \tag{4}$$

for all $1 \leq i, j \leq n$.

Proof By definition, a real linear transformation is also a complex linear transformation if and only if it commutes with multiplication by i. Thus, if a matrix A represents $T \in GL(W, \mathbb{R})$, with respect to the basis $\mathcal{B}$, then T is a complex linear transformation if and only if

$$A\mathrm{J} = \mathrm{J}A. \tag{5}$$

Suppose that A commutes with J. Then for any $1 \leq i, j \leq n$,

$$\begin{aligned} (A\mathrm{J})_{(2i-1),(2j-1)} &= (\mathrm{J}A)_{(2i-1),(2j-1)} \\ \sum_{k=1}^{2n} A_{(2i-1),k} \cdot \mathrm{J}_{k,(2j-1)} &= \sum_{l=1}^{2n} \mathrm{J}_{(2i-1),l} \cdot A_{l,(2j-1)} \\ A_{(2i-1),(2j)} &= -A_{(2i),(2j-1)} \end{aligned} \tag{6}$$

since J is non-zero in only one position per row or column.

An identical argument gives the other condition on A. Similarly, the same argument presented in the reverse order proves the converse: that any matrix satisfying the above conditions will commute with J and thus represent a complex-linear transformation. □

In other words, the transformation represented by a $2n \times 2n$ real matrix A is complex linear exactly when it has a decomposition into 2×2 blocks of the form

$\left(\begin{smallmatrix} a_{jk} & -b_{jk} \\ b_{jk} & a_{jk} \end{smallmatrix}\right)$. Conceptually, this block corresponds to the complex number $a_{jk} + i \cdot b_{jk}$ which is entered in the (j,k)'th position of the complex $n \times n$ matrix representing that linear transformation in the group of complex $n \times n$ matrices.

Now suppose that as a real vector space, W has an inner product structure, $< \cdot, \cdot >$. Then the orthogonal group on W, $O(W)$, is defined as the group of transformations on W which preserve the inner product. If $\mathcal{B} := \{w_1, \ldots, w_{2n}\}$ is an orthonormal basis of W, then with respect to $\mathcal{B}$, $O(W)$ is represented by

$$O(2n) := \{A \in M_{2n} \mid A^T A = Id\}. \tag{7}$$

Now, some elements of this group satisfy the conditions given in (C.1) and, thus, represent complex linear transformations. We now define a hermitian inner product structure on $W_{\mathbb{C}}$ such that it is consistent to refer to these particular transformations as unitary.

C.2 Definition Let W be a real $2n$-dimensional inner product space. Let $\{w_1, \ldots, w_{2n}\}$ be an orthonormal basis for W. Let $\hat{J}$ be the complex structure map on W defined above. Then, recall that $\mathcal{B}_1 = \{w_1, w_3, \ldots, w_{2n-1}\}$ forms a complex basis for $W_{\mathbb{C}}$ as a complex vector space. The **induced hermitian inner product** on $W_{\mathbb{C}}$ is the unique hermitian inner product on $W_{\mathbb{C}}$ for which $\mathcal{B}_1$ is an orthonormal basis of $W_{\mathbb{C}}$.

Since an inner product is a bilinear form, it can be defined by its effect on a basis. Therefore, the hermitian inner product is well-defined. Note that this choice of hermitian inner product is dependent on the initial basis $\mathcal{B}$ chosen. However, as shall be shown in (C.3), any change of basis T which is orthonormal and for which the resulting basis still satisfies the condition that

$$\hat{J}(T.w_{2i-1}) = T.w_{2i}, \tag{8}$$

preserves the hermitian inner product. Therefore, this definition of a hermitian inner product is not, in reality, dependent on a given basis. The real and induced hermitian inner products will be denoted $< \cdot, \cdot >_{\mathbb{R}}$ and $< \cdot, \cdot >_{\mathbb{C}}$ respectively.

Now, since J is, by construction, orthogonal, the complex inner product gives the same vector norms that the real inner product does. That is, for each $u \in W$,

$$< u, u >_{\mathbb{R}} = < u, u >_{\mathbb{C}} \tag{9}$$

This observation leads to a characterization of the group of complex linear transformations preserving the hermitian inner product on $W_{\mathbb{C}}$, the unitary group denoted by $U(W)$.

C.3 Proposition Let W be a $2n$-dimensional real inner product space and let $\mathcal{B} := \{w_1, \ldots, w_{2n}\}$ be an orthonormal basis on W. Let $\hat{J}$ be a complex structure map on W as defined above and let $\{w_1, w_3, \ldots, w_{2n-1}\}$ be a basis on $W_{\mathbb{C}}$ induced from $\mathcal{B}$ and $\hat{J}$. Then,

$$U(W) = GL(W, \mathbb{C}) \cap O(W) \tag{10}$$

Proof Suppose that T is a transformation in $U(W)$. Then, clearly, T is complex linear so $T \in GL(W, \mathbb{C})$. Furthermore, since T is unitary, it preserves complex inner products and thus complex norms. But, as previously observed, complex norms and real norms are equal. Thus, T is orthogonal. Therefore,

$$U(W) \subseteq GL(W, \mathbb{C}) \cap O(W) \tag{11}$$

Now suppose that $T \in GL(W, \mathbb{C}) \cap O(W)$. Then T preserves real norms and so T preserves complex norms. However, a linear transformation which preserves norms is inner-product preserving [3, p.300]. Thus, T preserves hermitian inner products. Therefore, since T is complex linear, T is unitary. Thus,

$$U(W) \supseteq GL(W, \mathbb{C}) \cap O(W) \tag{12}$$

□

Finally, we note that elements of $GL(W, \mathbb{C})$ are orientation preserving.

C.4 Proposition Let W be a real $2n$-dimensional inner product space and let $\mathcal{B} := \{w_1, \ldots, w_{2n}\}$ be an orthogonal basis on W. Let $(w_1, \ldots, w_{2n})$ represent a choice of positive orientation on W. Now, let J be the real $2n \times 2n$ matrix defined in (1) and let $\hat{J}$ be the complex structure map on W represented by J with respect to the basis $\mathcal{B}$. Then each transformation in the unitary group $U(W)$ is orientation preserving and

$$U(W) = GL(W, \mathbb{C}) \cap SO(W). \tag{13}$$

Proof First, note that the unitary group of a vector space is a connected Lie group [6, p. 130]. Consider the determinant function on linear transformations. With respect to a given basis, it can be expressed as a polynomial of the coordinate functions of $GL(W, \mathbb{R})$. Thus, it is a continuous function on $GL(W, \mathbb{R})$. Now, the continuous image of a connected set is connected. Thus, $\det(U(W))$ is connected. However, since $U(W) \subset O(W)$ (C.3),

$$\det(U(W)) \subset \{-1, 1\}, \tag{14}$$

and, of course, $\{-1, 1\}$ is disconnected, the determinant function must map $U(W)$ onto either 1 or -1. Now,

$$\mathrm{Id} \in U(W) \tag{15}$$

and

$$\det(\mathrm{Id}) = 1 \tag{16}$$

therefore,

$$\det(U(W)) = \{1\}. \tag{17}$$

Thus, the unitary group is orientation preserving as desired. The final conclusion is a direct result of (C.3). □

Thus, to summarize, given a real $2n$-dimensional inner product space, W, and a complex structure map $\hat{J}$, it is possible to express W as an n-dimensional complex vector space $W_{\mathbb{C}}$. This induces an embedding of the complex general linear group $GL(W, \mathbb{C})$ into the real general linear $GL(W, \mathbb{R})$. Furthermore, the real inner product structure $< \cdot, \cdot >_{\mathbb{R}}$ on W together with the complex structure map $\hat{J}$ induce a hermitian inner product $< \cdot, \cdot >_{\mathbb{C}}$ on $W_{\mathbb{C}}$ for which the imbedding of the unitary group into $GL(W, \mathbb{R})$ satisfies

$$U_W = GL(W, \mathbb{C}) \cap O_W. \tag{18}$$

Finally, the unitary group is not only orthogonal but also orientation preserving.