On some problems in Random Matrix Theory and Convex Geometry

by

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Abstract

This thesis is devoted to several problems in Random Matrix Theory and Convex Geometry. Its content is based on four papers.

In the first part, we establish an upper estimate for the smallest singular value $s_n(A)$ of a square $n \times n$ random matrix A with i.i.d. zero mean and unit variance entries. The smallest singular value controls the invertibility of the matrix and estimating it is a delicate task. Under additional assumption on the finiteness of the fourth moment of entries, Rudelson and Vershynin showed that $s_n(A)$ is of order $1/\sqrt{n}$. We remove the assumption on the boundedness of the fourth moment, and show an upper bound requiring only two finite moments.

We also study geometric properties of random polytopes generated by rectangular random matrices with entries that are independent symmetric random variables with unit variance satisfying small ball probability condition. In particular, we obtain sharp asymptotic bounds on volume and mean width of such polytopes and their polars. One of the main results shows that with high probability such random polytope contains an intersection of two ℓ_p balls.

The second part of the thesis is concerned with two geometric problems: extensions of the classical Steiner formula and a reverse isoperimetric problem. The classical Steiner formula expresses the volume of the parallel set of a convex body K at distance t as a polynomial in t. Many geometric quantities have found their analogues in L_p Brunn-Minkowski theory which was initiated by Lutwak, but L_p extension of Steiner formula was missing. We establish a tube formula for the L_p affine surface area of the Minkowski outer parallel body for any real parameter p, which provides an extension of the classical Steiner formula to L_p Brunn Minkowski theory. Moreover, a local version of the L_p Steiner formula is also proved, leading to new types of L_p curvature measures.

We address a problem of reversing classical isoperimetric inequality. In particular, we consider convex λ-concave bodies in $\mathbb{R}^n$ (for some $\lambda > 0$), that is, bodies for which at each point at the boundary there is locally an inner supporting ball of radius $1/\lambda$. For such class of bodies we obtain a reverse isoperimetric inequality. We also show that the equality is attained for a sausage body, that is for Minkowski sum of a line segment (or a point) and a ball of radius $1/\lambda$.

Preface

This thesis is based on four published papers: two of them consider problems in Random Matrix Theory and other two answer questions in Convex Geometry.

In particular, Chapter 2 and 3 consist of two papers: K. Tatarko, “An upper bound on the smallest singular value of a square random matrix” which was published in Journal of Complexity 48 (2018), pp. 119–128 and O. Guédon, A. E. Litvak and K. Tatarko, “Random polytopes obtained by matrices with heavy tailed entries” which will appear in Communications in Contemporary Mathematics.

Chapter 4 is a joint work with E. Werner “A Steiner formula in the L_p Brunn Minkowski theory” which was published in Advances in Mathematics 355 (2019), 106772.

The joint work with R. Chernov and K. Drach constitutes Chapter 5 and was published under the title “A sausage body is a unique solution for a reverse isoperimetric problem” in Advances in Mathematics 353 (2019), pp. 431–445.

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Chapter 1

Introduction

In this thesis, we consider several problems in Random Matrix Theory and Convex Geometry.

The central objects of study in Random Matrix Theory are random matrices, that is matrix-valued random variables on the set of all matrices. There are many classical asymptotic results (limit laws) when the dimension of matrices grows to infinity which are concerned with behavior of a spectrum of such matrices. For instance, among others are celebrated Wigner's semicircle law, the Marchenko-Pastur law, Tracy-Widom law, etc. However, the limiting behavior may be of little help for many applications in statistics, computer science, geometric functional analysis that deal with large but fixed dimensions. Questions involving estimates for a high fixed dimensions arise in the non-limit theory (also called non-asymptotic) of random matrices and are concerned with quantitative estimates of some spectral characteristics such as eigenvalues or singular values holding with large probability of success. Many tools used in this theory, such as covering numbers, volumetric bounds, and random projections, are of a geometric nature, which provides a bridge between random matrix theory and geometric functional analysis.

The Brunn-Minkowski theory is at the core of the high-dimensional geometry of convex bodies (compact, convex sets with non-empty interior) that studies properties of convex bodies and geometric invariants associated with them. One of the well-known questions in the theory is isoperimetric problem which long history began in ancient Greece and since then seen tremendous developments and various generalizations. The classical isoperimetric inequality asserts that among all domains with a given surface area, the Euclidean ball has the largest possible volume. In contrast with the classical version, the reverse isoperimetric problem concerns with finding a body of least volume for a given constraint, for example, the surface area. We consider such a problem in Chapter 5.

The classical Steiner formula is one of the most influential results of the Brunn-Minkowski theory. The coefficients in it define so-called intrinsic volumes or quermass-integrals which carry a lot of information about the geometry of convex bodies. For example, volume and surface area are some of them. The notion of affine surface area has been a rich source of fruitful investigations and found its applications in different areas of mathematics. L_p extensions of affine surface area in L_p Brunn-Minkowski theory is not of less importance. They contributed to results on valuations, the affine Plateau problem, approximation of convex bodies by polytopes, solutions of partial differential equations, and many others. In Chapter 4 we present an L_p extension of a classical Steiner formula.

In the following subsections, we introduce main topics of this thesis and discuss them in more details.

1.1 Common Notation and Preliminaries

By $e_1, \dots, e_n$ we denote the canonical basis of $\mathbb{R}^n$ equipped with the canonical inner product $\langle \cdot, \cdot \rangle$. For $1 \le p \le \infty$, the ℓ_p^n-norm is defined for any $x \in \mathbb{R}^n$ by

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p} \quad \text{for} \quad 1 \le p < \infty \qquad \text{and} \qquad \|x\|_\infty = \sup_{i=1,\dots,n} |x_i|.$$

As usual, $\ell_p^n = (\mathbb{R}^n, \|\cdot\|_p)$, and the unit ball of ℓ_p^n is denoted by B_p^n. The unit sphere of ℓ_2^n is denoted by S^{n-1}. The Euclidean ball in $\mathbb{R}^n$ with radius r and center at $x \in \mathbb{R}^n$ is denoted by $B_2^n(x,r)$. We also use $B_2^n(r) := B_2^n(0,r)$ for the Euclidean ball in $\mathbb{R}^n$ of radius r and center at the origin and $B_2^n := B_2^n(0,1)$ for the unit Euclidean ball in $\mathbb{R}^n$.

Given a real number a, we denote by $\lfloor a \rfloor$ the largest integer not exceeding a and by $\lceil a \rceil$ the smallest integer larger than or equal to a. Given a finite set E we denote its cardinality by $|E|$. We also use $\mathrm{vol}_n(K)$ for the volume of a body $K \subset \mathbb{R}^n$ (and, more generally, for the n-dimensional Lebesgue measure of a measurable subset of $\mathbb{R}^n$).

For a given probability space $(\Omega, \mathcal{A}, \mathbb{P})$, we denote by $\mathbb{P}(\cdot)$ and $\mathbb{E}$ the probability and the expectation, respectively. A random variable ξ is a measurable function from a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ to $\mathbb{R}$, i.e., $\xi : \Omega \to \mathbb{R}$. The independent and identically distributed random variables are usually abbreviated as i.i.d random variables.

We will also need the notion of the so-called *Lévy concentration function* $\mathcal{Q}(\xi, \cdot)$ of a (real) random variable ξ which is defined on $(0, \infty)$ as

$$\mathcal{Q}(\xi, t) := \sup_{\lambda \in \mathbb{R}} \mathbb{P}\big(|\xi - \lambda| \le t\big).$$

In other words, the Lévy concentration function measures how likely a random variable ξ enters a small neighborhood. Note that for any centered random variable ξ (i.e., $\mathbb{E}\xi = 0$) with unit variance, there exist $u, v \in (0,1)$ such that

$$\mathcal{Q}(\xi, u) \le v. \tag{1.1}$$

A convex body $K \subset \mathbb{R}^n$ is compact, convex set with non-empty interior.

Given a set $L \subset \mathbb{R}^n$, a convex body $K \subset \mathbb{R}^n$, and $\varepsilon > 0$ we say that a subset $\mathcal{N} \subset \mathbb{R}^n$ is an ε-*net* of L with respect to K if

$$\mathcal{N} \subset L \subset \bigcup_{x \in \mathcal{N}} (x + \varepsilon K).$$

The cardinality of the smallest ε-net of L with respect to K we denote by $N(L, \varepsilon K)$.

The support function $h_K(u)$ of a convex body K is defined as

$$h_K(u) = \sup_{x \in K} \langle x, u \rangle \qquad \text{for } u \in \mathbb{R}^n,$$

and represents the distance from the origin to the support hyperplane H_u of K orthogonal to u if origin is inside K (see Figure 1.1).

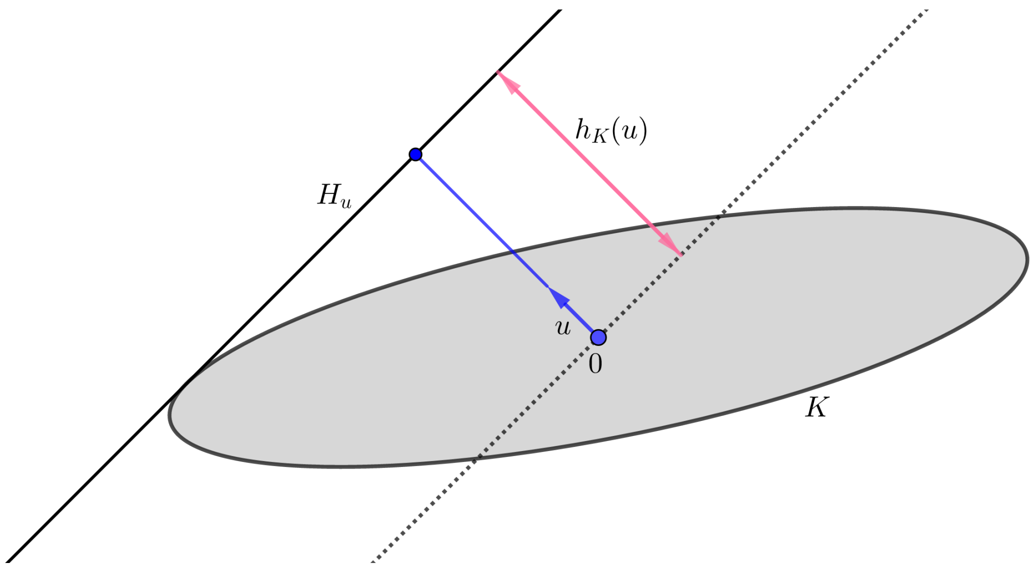

Figure 1.1: The support function $h_k(u)$.

The Minkowski functional $||\cdot||_K$ of K is defined by

$$||x||_K = \inf\{\lambda \geq 0 : \ x \in \lambda K\} \qquad \text{for } x \in \mathbb{R}^n.$$

The polar of K is

$$K^\circ = \{y \in \mathbb{R}^n : \ \langle x, y \rangle \leq 1 \ \text{ for every } \ x \in K\}.$$

Note that $h_K(\cdot) = \|\cdot\|_{K^\circ}$ when origin is contained in K (see [132, p. 53]).

1.2 The smallest singular value

Let $n \leq N$ and A be $N \times n$ real-valued matrix. Then singular values $s_j(A)$ of A are arranged in the non-increasing order $s_1(A) \geq s_2(A) \geq \cdots \geq s_n(A) \geq 0$ and defined as square roots of eigenvalues of the symmetric $n \times n$ positive-semidefinite matrix A^*A where A^* is the adjoint matrix. The matrix A acts as a linear operator from ℓ_2^n to ℓ_2^N. Its operator norm, or spectral norm, is defined as

$$\|A\| = \|A : \ell_2^n \to \ell_2^N\| = \sup_{x \in S^{n-1}} \|Ax\|_2.$$

In particular, the largest and the smallest singular values $s_1(A)$ and $s_n(A)$ can be expressed in terms of the operator norm of A as

$$s_1(A) = \|A\| \quad \text{and} \quad s_n(A) = \frac{1}{\|A^{-1}\|},$$

where A^{-1} is the inverse from the image of A (if $\operatorname{rank} A < n$, then $s_n = 0$).

In Chapter 2 we restrict our attention to symmetric $n \times n$ matrices with real entries. The extremal singular values have been attracting the attention of scientists over many years. A lot is known about the largest singular value, but the smallest singular value is more difficult to analyze. From now on, we focus on the behavior of the smallest singular value.

In 1950s, von Neumann and his collaborators conjectured [111] that the smallest singular value of an $n \times n$ random matrix is of order $1/\sqrt{n}$ with probability close to one. For the case of Gaussian matrices (with independent identically distributed standard normal entries), the conjecture was proven by Edelman [48] and Szarek [143]. Edelman also obtained the limiting distribution of the smallest singular value of a Gaussian matrix. Afterwards, various bounds for the smallest singular value of square matrices have been found under different assumptions. We refer to [39, 85, 90, 120, 128, 147] and references therein. For estimates on the smallest singular value of rectangular matrices, see [85, 129, 149].

Recall that a random variable ξ is called subgaussian if there exists a number $K > 0$ such that $P(|\xi| > t) \leq 2\exp(-t^2/(2K^2))$ for all $t > 0$. Classical examples of subgaussian random variables are Bernoulli, Gaussian and all bounded random variables. Rudelson and Vershynin in [127, 128] confirmed von Neumann's conjecture for the case of square subgaussian matrices. Although in the same papers the authors confirmed conjecture for matrices under fourth moment assumptions on entries, the statements themselves are much weaker than in the subgaussian case. In [128], Rudelson and Vershynin established a lower estimate of the smallest singular value of an $n \times n$ random matrix A with centered subgaussian entries. Namely, they proved that for every $t > 0$

$$\mathbb{P}\left(s_n(A) \geq tn^{-\frac{1}{2}}\right) \geq 1 - Ct - u^n$$

for some constants $C > 0$ and $u \in (0, 1)$. The corresponding upper estimate in [127] says that for a given $t \geq 2$ there are constants $C > 0$ and $u \in (0, 1)$ depending only on the subgaussian moment of entries such that

$$\mathbb{P}\left(s_n(A) \leq tn^{-\frac{1}{2}}\right) \geq 1 - C\frac{\log t}{t} - u^n.$$

Rebrova and Tikhomirov [120] have found a way to bound the smallest singular value from below under only the second moment assumption on the entries of the matrix. They used a special net refinement on the image of the Euclidean unit ball under action of A. Complementing this line of research, I proved in [144] an upper bound on the smallest singular value $s_n(A)$ for a square matrix A with the same assumptions as in [120], this way relaxing assumptions in [127]. More precisely,

Theorem 1.1 (Theorem 1.1 in [144]) *Let $A = (a_{ij})$ be an $n \times n$ matrix whose entries are i.i.d. random variables with $\mathbb{E}a_{ij} = 0$ and $\mathbb{E}a_{ij}^2 = 1$. Then there exist an* *absolute constant* $C > 0$ *such that*

$$\mathbb{P}\left(s_n(A) \le \frac{1}{t^2} n^{-\frac{1}{2}}\right) \ge 1 - Ct - \frac{C}{\sqrt{n}}, \qquad \forall t > 0.$$

Note that together with a lower estimate for $s_n(A)$, this gives a complete answer to the long-standing von Neumann conjecture for distributions with no assumptions on moments higher than 2.

1.3 Random polytopes

Let $n \le N$. We consider a rectangular $N \times n$ matrix $A = (a_{ij})$ which entries are independent symmetric random variables with $\mathbb{E}a_{ij} = 0$ and $\mathbb{E}a_{ij}^2 = 1$ such that in each row the entries are identically distributed satisfying small ball probability estimate. That is, there exist numbers $u, v \in (0, 1)$ such that

$$\sup_{\lambda \in \mathbb{R}} \mathbb{P}\left(|a_{ij} - \lambda| \le u\right) \le v \qquad \text{for all } i, j. \tag{1.2}$$

In Chapter 3 we study the geometry of random polytopes that are determined by A. The random polytope $K_N = A^* B_1^N$ where B_1^N is a cross-polytope, is the absolute convex hull of N rows of A. Such random polytopes have been widely studied, for example, in the Gaussian [61, 101], Bernoulli [60, 85], and heavy-tailed [80] cases. For instance, in [85] it was shown that a random polytope generated by a subgaussian matrix contains an intersection of two ℓ_p balls, while in [62, 78] the inclusion of a multiple of the Euclidean unit ball into a random polytope generated by a Gaussian matrix was found. Hence, the natural question is whether we can find a set associated with row's distribution such that it is contained in a random polytope with high probability.

Yet another motivation is that the theory of compressed sensing studies recovering of sparse vectors from a series of incomplete measurements. It appeared that such problems of recovering sparse vectors are closely connected to results about polytope inclusions as in [78, 80, 85]. It was shown that such quotient properties (inclusions) are responsible for "robustness" in ℓ_1-minimizations — certain efficient methods in compressed sensing. Our goal is to provide such inclusions under weaker moment assumptions on the distribution of entries than Gaussian or subgaussian.

Together with Guédon and Litvak in [70] we showed that K_N contains a large regular body with high probability under assumptions only on the boundedness of second moments of entries.

Theorem 1.2 (Theorem 4.1 in [70]) *There exist positive constants b, M depending only on u and v, and an absolute constant $c > 0$ such that the following holds. Let $N \geq Mn$ and assume that the entries of an $N \times n$ random matrix A are independent symmetric random variables with unit variances satisfying condition* (1.2) *and such that in each row the entries are i.i.d. Then the inclusion*

$$A^* B_1^N \supset \frac{1}{b}\left(B_\infty^n \cap \sqrt{\ln \frac{N}{n}} B_2^n\right)$$

holds with probability at least $1 - \exp(-cn)$.

In the same paper, we also obtained sharp asymptotic bounds for the volumes and the mean widths of K_N and its polar K_N°.

In order to show Theorem 1.2, we need to avoid bounding operator norm of A since our distribution assumptions on the entries of A do not guarantee a good upper bound for it. Indeed, as was shown in [138] (see [88] for quantitative estimates), for operator norm to be bounded one needs to have bounded fourth moment. To this end, we extend the net construction of Rebrova and Tikhomirov in [120]. Namely, we work with rectangular matrices instead of square matrices and find a new net refinement for the image of a given convex body under action of A (not only the image of the Euclidean unit ball). We would like to mention that our paper [70] led to further investigations, see [68].

1.4 Steiner formula

In Chapters 4 and 5 we work with convex bodies in $\mathbb{R}^n$. For a convex body $K \subset \mathbb{R}^n$ and parameter $t > 0$, we define the outer parallel body $K + tB_2^n = \{x + ty : x \in K, y \in B_2^n\}$, where B_2^n is unit n-dimensional Euclidean ball and "+" denotes the Minkowski addition. The Steiner formula gives an expansion of volume $\mathrm{vol}_n(K + tB_2^n)$ as a polynomial of

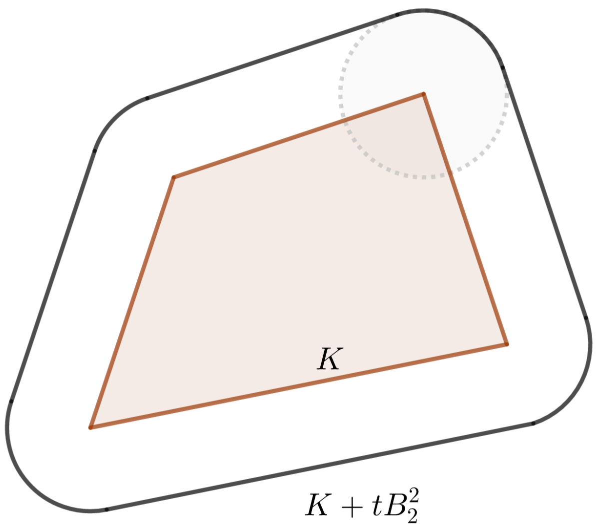

Figure 1.2: An outer parallel body $K + tB_2^2$.

degree at most n in the parameter t :

$$\mathrm{vol}_n(K + tB_2^n) = \sum_{i=0}^{n} \binom{n}{i} W_i(K) t^i, \tag{1.3}$$

where coefficients $W_i(K)$ are called quermassintegrals, important quantities in Brunn-Minkowski theory, which include the volume, the surface area, and the mean width (see Section 4.3.1 for more details). Different generalizations of the Steiner formula were obtained in a number of papers, for example, [159] (smooth manifolds), [50] (sets of positive reach), [116] (Minkowski valuations) and more. We refer readers to [66, 132] for detailed overview of the topic.

Firey in [51] and later Lutwak in [96] initiated an L_p extension of the classical

Brunn-Minkowski theory where many quantities from the classical theory found their L_p analogues. In particular, L_p affine surface area, which is an L_p extension of classical affine surface area, became the focus of investigations of many mathematicians in the areas of convex and differential geometry. It is defined as

$$as_p(K) = \int_{\partial K} \frac{H_{n-1}(x)^{\frac{p}{n+p}}}{\langle x, \nu(x)\rangle^{\frac{n(p-1)}{n+p}}} d\mathcal{H}^{n-1}(x),$$

where ∂K denotes the boundary of K, $\nu(x)$ is the outer unit normal vector at $x \in \partial K$, $H_{n-1}(x)$ is the Gauss curvature at x and $\mathcal{H}^{n-1}$ is the standard surface area measure on ∂K (see Section 4.3.2 for more details). In the case when $p = 0$, $as_0(K)$ is just the volume of the body K, and when $p = \infty$, we get the volume of a polar body of K. In both cases, Steiner formulas are known. Thus, the question is whether we can find all Steiner formulas "in between" of these special cases of p, thus obtaining an L_p extension of Steiner formula via L_p affine surface area. The content of Chapter 4 comes from my work with Werner [145] which provides such generalization.

We recall that the curvature function $f_K(u) : S^{n-1} \to \mathbb{R}_+$ is the reciprocal of the Gauss curvature $H_{n-1}(x)$ at the point $x \in \partial K$ that has u as an outer normal. The body K is of class C^2_+ if K is of class C^2 and the Gauss curvature is nonzero.

Theorem 1.3 (Theorem A in [145]) *Let K be a convex body in $\mathbb{R}^n$ that is C^2_+ and let $t \in \mathbb{R}$ be such that $0 \le t < \min_{u \in S^{n-1}} h_K(u)$. For all $p \in \mathbb{R}$, $p \neq -n$,*

$$as_p(K + tB_2^n) = \sum_{m=0}^{\infty} \left[\sum_{k=m}^{\infty} \binom{\frac{n(1-p)}{n+p}}{k-m} \mathcal{W}_{m,k}\, t^k \right],$$

where

$$\mathcal{W}_{m,k}(K) = \int_{S^{n-1}} f_K(u)^{\frac{n}{n+p}} h_K(u)^{\frac{n(1-p)}{n+p} - k + m} A_p^m \, d\sigma(u)$$

are called L_p Steiner coefficients.

The coefficients A_p^m (see (4.19)) represent a sum of mixed products of the elementary symmetric functions of the principal curvatures $H_i = H_i\left(\bar{\xi}_K(u)\right)$ (see Section 4.3.1), with corresponding multinomial coefficients. To our best knowledge, the coefficients in this formula have not appeared before in the literature.

1.5 Isoperimetry

As already mentioned above, the classical isoperimetric inequality states that if K is an arbitrary domain (connected, open set) in $\mathbb{R}^n$ with volume $\mathrm{vol}_n(K)$ and surface area $\mathrm{vol}_{n-1}(\partial K)$, then

$$\mathrm{vol}_n(K) \leqslant \frac{\mathrm{vol}_{n-1}(\partial K)^{\frac{n}{n-1}}}{n^{\frac{n}{n-1}}\mathrm{vol}_n(B_2^n)^{\frac{1}{n-1}}}. \tag{1.4}$$

It is known that equality in (1.4) holds if and only if K is a ball. In other words, the classical isoperimetric inequality asserts that *among all domains of given surface area, the ball has the largest possible volume.* It also can be restated in the dual way: *among all domains of given volume, the ball has the least surface area.*

Although convex bodies of a given volume may have arbitrarily large surface area if they are very flat, classical inequalities admit reverse forms. Good examples are the reverse Santalo [27] and the reverse Brunn-Minkowski [105] inequalities. In [5, 6] Ball showed that classical isoperimetric inequality can be reversed. In particular, he proved that for any convex body in $\mathbb{R}^n$ there is always an affine transformation such that even though the body originally might be flat, volume of its affine image is no smaller than that of the standard simplex under the surface area constraint.

Instead of considering affine equivalence classes of convex bodies, another natural approach towards reversing the classical isoperimetric inequality is by assuming some curvature constraints on the boundary. This idea was used in [75], where the authors solved an extremal problem for plane curves of not big length and with bounded curvature $k \leq 1$. Various results of the same flavour were obtained over the past years (see [20, 55, 114, 162] and references therein), but most of them regard two- or three-dimensional cases.

With Chernov and Drach in [38], we considered a reverse isoperimetric problem in the class of λ-concave bodies. For a given $\lambda > 0$, a convex body $K \subset \mathbb{R}^n$ is called λ-concave if each of its boundary points supports a tangent ball of radius $1/\lambda$ that locally lies inside the body. Equivalently, for smooth bodies this condition can be reformulated as follows: all principal curvatures $k_i(p)$ are non-negative and uniformly bounded from above by λ, i.e. $0 \leq k_i \leq \lambda$ for all $i \in \{0, \dots, n-1\}$ and $p \in \partial K$.

We obtained a family of reverse quermassintegral inequalities which contains the reverse isoperimetric inequality for λ-concave bodies as a particular case.

Theorem 1.4 (Theorem A in [38]) *Let $K \subset \mathbb{R}^n$ be a convex body. If K is λ-concave for some $\lambda > 0$, then*

$$(k-j)\frac{W_i(K)}{\lambda^i} + (i-k)\frac{W_j(K)}{\lambda^j} + (j-i)\frac{W_k(K)}{\lambda^k} \geqslant 0$$

for every triple (i, j, k) with $0 \leqslant i < j < k \leqslant n$. Moreover, the equality holds if and only if K is a λ-sausage body (the convex hull of two balls of radius $1/\lambda$).

Picking the appropriate indices above, we get

Corollary 1.5 (Theorem B in [38]) *Let $K \subset \mathbb{R}^n$ be a convex body. If K is λ-concave for some $\lambda > 0$, then*

$$\mathrm{vol}_n(K) \geqslant \frac{\mathrm{vol}_{n-1}(\partial K)}{(n-1)\lambda} - \frac{\mathrm{vol}_n(B_2^n)}{(n-1)\lambda^n}.$$

Moreover, the equality holds if and only if K is a λ-sausage body.

Chapter 2

An Upper Bound on the Smallest Singular Value of a Square Random Matrix [1]

2.1 Introduction

The extremal singular values have been attracting the attention of scientists in different disciplines such as mathematical physics or geometric functional analysis. In particular, they play an important role in numerical analysis as the condition number, which is the ratio of the largest to the smallest singular value, is a measure for the worst-case loss of precision in a computational problem. Much is known about the behavior of the largest singular value and we refer the reader to [4, 165]. The study of the behavior of the smallest singular value goes back to von Neumann and his collaborators concerning numerical inversion of large matrices, where they conjectured (see [110, 111]) that the smallest singular value of an $n \times n$ matrix A is of order $n^{-\frac{1}{2}}$ with probability close to one. Estimates of similar type for the case of Gaussian matrices (i.e., matrices with i.i.d.

[1] A version of this chapter was published: K. Tatarko "An upper bound on the smallest singular value of a square random matrix", J. Complexity 48 (2018), pp. 119-128.

standard normal entries) were obtained by Edelman in [48] and Szarek in [139]. For estimates on extremal singular values which were acquired while studying the problem of the approximation of covariance matrices, we refer to [1, 69, 107, 150]. Various bounds for the smallest singular value have been obtained under rather weak assumptions on the rows of the matrix in [79, 113, 163, 164]. For lower bounds on the smallest singular value of random matrices with independent but not identically distributed entries see a recent result by Cook [39].

Rudelson and Vershynin in [127, 128, 129] studied the behavior of the smallest singular value of matrices with i.i.d. subgaussian entries. They showed (see [127, 128]) that the smallest singular value of a square random matrix A with i.i.d. subgaussian entries is of order $n^{-\frac{1}{2}}$. A lower bound for rectangular subgaussian matrices was obtained in [129]. A recent result of Wei (see [153]) provides upper bounds on intermediate singular values of rectangular matrices with subgaussian entries. The corresponding lower bounds were obtained in [125].

Recently, in [120] a new technique was developed, which allowed Rebrova and Tikhomirov to prove a lower bound for $s_n(A)$ of square matrices of order $n^{-\frac{1}{2}}$ under the assumption that the Lévy concentration function of entries of A is bounded. Namely, they showed the small ball probability estimate

$$\forall \varepsilon > 0 : \quad \mathbb{P}\left(s_n(A) \leq \varepsilon n^{-\frac{1}{2}}\right) \ \leq \ C\varepsilon + u^n,$$

where $C > 0$ and $u \in (0,1)$ depend only on the law of a_{11}. Notice that any random variable ξ with $\mathbb{E}\xi = 0$ and $\mathbb{E}\xi^2 = 1$ has a bounded Lévy concentration function, therefore the above statement is valid for matrices with assumptions only on the second moment of entries.

The goal of this chapter is to show that the upper bound on the smallest singular value holds for square matrices with heavy-tailed entries. We prove the following theorem.

Theorem 2.1 *Let $A = (a_{ij})$ be an $n \times n$ matrix whose entries are i.i.d. random variables with $\mathbb{E}a_{ij} = 0$ and $\mathbb{E}a_{ij}^2 = 1$. Then there exists an absolute constant $C > 0$ such that for every $\varepsilon > 0$*

$$\mathbb{P}\left(s_n(A) > \frac{1}{\varepsilon^2} n^{-\frac{1}{2}}\right) \leq C\varepsilon + \frac{C}{\sqrt{n}}.$$

We expect that the dependence on ε can be improved to ε^{-1}, but our proof gives only ε^{-2}.

We now briefly describe the ideas of proof of Theorem 2.1.

To estimate the smallest singular value of a random matrix A we will use the following equivalence, which holds for every $\lambda \geq 0$,

$$s_n(A) \leq \lambda \quad \Longleftrightarrow \quad \exists x \in S^{n-1} : \ \|Ax\|_2 \leq \lambda.$$

We will show that there exists $x \in \mathbb{R}^n$ such that $\|x\|_2 \leq \tau$ and $\|A^{-1}x\|_2 \geq \eta\sqrt{n}$ for some $\tau, \eta > 0$, which implies $s_n(A) \leq \frac{\tau}{\eta\sqrt{n}}$. Let us describe the main difficulty in our proof. It is well-known that $A^{-1}x$ behaves differently depending on the structure of x. We follow [85, 86] and roughly speaking split the unit sphere into two parts consisting of vectors of small dimensions and vectors with bounded ℓ_∞ norm. To deal with vectors of the second type, we use ideas introduced in [128], namely we use the essential least common denominator (see the definition below). Denote by B the transpose of the first $n-2$ columns of matrix A. To show that the essential least common denominator of vectors in the null space of a matrix B has exponential decay with high probability, in [127] the authors used a standard ε-net argument, namely, for a given ε-net $\mathcal{N}$ on a subset $S \subset S^{n-1}$ one has

$$\inf_{y \in S} \|By\|_2 \ \geq \ \inf_{y' \in \mathcal{N}} \Big(\|By'\|_2 \ - \ \|B\|\|y - y'\|_2\Big).$$

This procedure relies on an upper bound for the operator norm $\|B\|$, which is of order $n^{\frac{1}{2}}$

with exponentially high probability under the subgaussian moment assumption on the entries of B. Moreover, as can be seen in [59, 165], one has that $\|B\| \leq C\sqrt{n}$ under the assumption of bounded fourth moments (see [81, 82] for independent but not identically distributed entries). However, in the settings of Theorem 2.1, it is not guaranteed that the operator norm $\|B\|$ has a good upper bound. Moreover, if the fourth moment is unbounded, it is known that $\frac{\|B\|}{\sqrt{n}} \to \infty$ $\big($[4, 138, 165], see also [88] for quantitative estimates$\big)$. To overcome this difficulty, we use a recent technique developed by Rebrova and Tikhomirov in [120]. Starting with a standard ε-net on $S \subset S^{n-1}$ we construct a new net on S which is a $(C\varepsilon\sqrt{n})$-net with respect to the pseudometric $\|B(x-y)\|_2$ with probability close to one. This allows us to circumvent the use of the operator norm $\|B\|$.

2.2 Preliminaries

Recall that for a given metric space X, an ε-net $\mathcal{N}$ in X is a subset of X such that any point of X is within distance at most ε from points of $\mathcal{N}$.

A system $(E_k, F_k)_{k=1}^n$ of vectors $(E_k)_{k=1}^n$ and $(F_k)_{k=1}^n$ in an n-dimensional Hilbert space H is called a biorthogonal system if $\langle E_k, F_s \rangle = \delta_{k,s}$ for all $k, s \in \{1, \ldots, n\}$, where $\delta_{k,s} = 0$ for $k \neq s$ and $\delta_{k,s} = 1$ for $k = s$. The system is called complete if it spans the entire space H. The next proposition contains some well-known properties of biorthogonal systems (see [127], Proposition 2.1).

Proposition 2.2

(i) *Let $(E_k)_{k=1}^n$ be a linearly independent system of vectors in an n-dimensional Hilbert space H. Then there exist unique vectors $(F_k)_{k=1}^n$ such that $(E_k, F_k)_{k=1}^n$ is a complete biorthogonal system in H.*

(ii) *If* $(E_k, F_k)_{k=1}^n$ *is a complete biorthogonal system in* H*, then*

$$\|F_k\|_2 = \frac{1}{\operatorname{dist}(E_k, H_k)} \qquad \textit{for } k \;=\; 1, \ldots, n,$$

where $H_k = \operatorname{span}(E_i)_{i \neq k}$.

(iii) *If* A *is an* $n \times n$ *invertible matrix, then* $\left(Ae_k, (A^{-1})^t\, e_k\right)_{k=1}^n$ *is a complete biorthogonal system.*

Any random variable ξ with $\mathbb{E}\xi = 0$ and $\mathbb{E}\xi^2 = 1$ satisfies the condition

$$\mathcal{L}(\xi, v) \leq u$$

for some constants $u \in (0,1)$ and $v > 0$ determined by the law of ξ. Therefore, we don't add this constraint to the formulation of our main result Theorem 2.1, but state it only in terms of finiteness of the second moment of entries.

In order to find an upper bound for the smallest singular value $s_n(A)$, we will consider a partition of the sphere into sets of compressible and incompressible vectors. Such an idea to split the sphere into two parts and to use an estimate involving the norm of a matrix, in order to bound the smallest singular value first appeared in [85] and was formalized later (see [128]) in the following definition.

Definition 2.3 *Let* $\delta, \rho \in (0,1)$*. A vector* $x \in \mathbb{R}^n$ *is called* (δn)*-sparse if*

$$|\operatorname{supp}(x)| < \delta n.$$

A vector $x \in S^{n-1}$ *is called compressible if* x *is within Euclidean distance* ρ *from the set of all* δn*-sparse vectors. Otherwise, a vector* $x \in S^{n-1}$ *is called incompressible.*

The sets of compressible and incompressible vectors will be denoted by

$$\mathrm{Comp} = \mathrm{Comp}_n(\delta, \rho) \qquad \text{and} \qquad \mathrm{Incomp} = \mathrm{Incomp}_n(\delta, \rho),$$

respectively.

Since the set of compressible vectors is essentially of the smaller dimension, the following simple result shows that one can find an ε-net on the set of compressible vectors Comp with small cardinality.

Lemma 2.4 *For any $\delta, \rho \in (0, 1]$ a set of compressible unit vectors $\mathrm{Comp}_n(\delta, \rho)$ admits a (2ρ)-net $\mathcal{N}$ of $\mathrm{Comp}_n(\delta, \rho)$ of cardinality*

$$|\mathcal{N}| \leq \left(\frac{e}{\delta}\right)^{\delta n} \left(\frac{5}{\rho}\right)^{\delta n}.$$

Proof. By definition, for every $x \in \mathrm{Comp}_n(\delta, \rho)$ there exist $x' \in S^{n-1}$ such that $|\mathrm{supp}(x')| \leq \delta n$ and $\|x - x'\|_2 \leq \rho$. Thus, to find a (2ρ)-net on a set of compressible vectors, it is enough to find a Euclidean ρ-net on the set of sparse vectors. For a fixed coordinate subspace of dimension δn, the standard volumetric estimate gives a ρ-net of a cardinality at most $(1+\frac{2}{\rho})^{\delta n}$. Applying a union bound over all coordinate subspaces, we have that the set of compressible vectors $\mathrm{Comp}_n(\delta, \rho)$ admits an Euclidean (2ρ)-net of cardinality

$$|\mathcal{N}| \leq \binom{n}{\delta n} \left(1 + \frac{2}{\rho}\right)^{\delta n} \leq \left(\frac{e}{\delta}\right)^{\delta n} \left(\frac{5}{\rho}\right)^{\delta n}.$$

■

We will need a couple of results from [120]. The following theorem allows us to refine a given ε-net $\mathcal{N}$ on a subset of the unit sphere to an $\left(\frac{\varepsilon C}{\delta}\sqrt{n}\right)$-net $\widetilde{\mathcal{N}}$ on the same subset of the sphere with respect to pseudometric $\|A(x-y)\|_2$ with high probability.

Theorem 2.5 ([120], Theorem $A^\star$) *Let $\delta \in (0, \frac{1}{4}]$, $\varepsilon \in (0, \frac{1}{2}]$, $n \geq \frac{1}{4\delta}$, $S \subset S^{n-1}$ be a subset of the sphere, and $\mathcal{N} \subset S$ be an ε-net on S in the Euclidean metric. Then there exists a deterministic subset $\widetilde{\mathcal{N}} \subset S$ with*

$$|\widetilde{\mathcal{N}}| \leq \exp\left(13\delta n \ln \frac{2e}{\delta}\right) |\mathcal{N}|$$

such that for an $n \times n$ random matrix A with i.i.d. zero mean and unit variance entries, with probability at least $1 - 4\exp(-\frac{\delta n}{8})$, the set $\widetilde{\mathcal{N}}$ is an $\left(\frac{\varepsilon C}{\delta}\sqrt{n}\right)$-net on S with respect to the pseudometric $\mathrm{d}(x,y) = \|A(x-y)\|_2$, where $x, y \in S^{n-1}$.

Remark 2.6 One can check that Theorem 2.5 holds for a $(n-2) \times n$ matrix A.

The next lemma gives a strong probability estimate for a fixed unit vector.

Lemma 2.7 ([120], Lemma 4.9) *Let ξ be a random variable with $\mathcal{L}(\xi, \tilde{v}) \leq \tilde{u}$ for some $\tilde{v} > 0$ and $\tilde{u} \in (0,1)$. Then there are $v > 0$ and $u \in (0,1)$ depending only on $\tilde{u}, \tilde{v}$ such that for an $(n-2) \times n$ random matrix A with i.i.d. entries equidistributed with ξ and for any $y \in S^{n-1}$ one has*

$$\mathbb{P}\left(\|Ay\|_2 \leq v\sqrt{n}\right) \leq u^{n-2}.$$

In order to obtain the small ball probability estimate for a random sum, we need the notion of the essential least common denominator. It measures the closeness of the scaled vector $x \in \mathbb{R}^n$ to $\mathbb{Z}^n$. This notion was introduced in [128, 129] (see also [146]) and for more detailed description see [126].

Definition 2.8 *For parameters $\alpha > 0$ and $r \in (0,1)$, the essential least common denominator of a vector $x \in \mathbb{R}^n$ is defined as*

$$\mathrm{LCD}_{\alpha,r}(x) = \inf\left\{t > 0 : \mathrm{dist}(tx, \mathbb{Z}^n) < \min\left(r\|tx\|_2, \alpha\right)\right\}.$$

Then the essential least common denominator of a subspace $H \subset \mathbb{R}^n$ is defined as

$$\mathrm{LCD}_{\alpha,r}(H) = \inf\left\{\mathrm{LCD}_{\alpha,r}(x) : x \in H, \|x\|_2 = 1\right\}.$$

Later we will use this definition with a small constant r, and a small multiple α of $\sqrt{n}$.

The next result gives a small ball probability estimate of a random sum. It is essentially Theorem 3.4 in [127].

Theorem 2.9 *Let $u \in (0,1)$. Let $\xi_1, \ldots, \xi_n$ be i.i.d. zero mean random variables such that $\mathcal{L}(\xi_1, 1) \le u$ and $x = (x_1, \ldots, x_n) \in S^{n-1}$. Then, for every $\alpha > 0$, $r \in (0,1)$ and for every $\varepsilon > 0$ one has*

$$\mathcal{L}\left(\sum_{i=1}^{n} x_i \xi_i, \varepsilon\right) \le \frac{C}{r\sqrt{1-u}}\left(\varepsilon + \frac{1}{\mathrm{LCD}_{\alpha,r}(x)}\right) + Ce^{-2\alpha^2(1-u)},$$

where $C > 0$ is an absolute constant.

In words, the theorem provides useful upper bounds on the small ball probability which depend on the additive structure of the coefficients $x_1, \ldots, x_n$. The less structure the coefficients carry, the more spread the distribution of a random sum is, and the less the small ball probability is.

2.3 Proof of the Theorem 2.1

To prove the boundedness of the smallest singular value of the type

$$s_n(A) \le L n^{-\frac{1}{2}},$$

where $L > 0$ is an absolute constant, it is enough to show that there exists $x \in \mathbb{R}^n$ such that $\|x\|_2 \le \tau$ and $\|A^{-1}x\|_2 \ge \eta n^{-\frac{1}{2}}$ for some $\tau, \eta > 0$.

We follow the ideas of Rudelson and Vershynin in [127]. Consider the columns $X_i = Ae_i$ of a matrix A and the rows $\widetilde{X}_i = (A^{-1})^t e_i$ of an inverse matrix A^{-1}. Let H_i denote the span of all column vectors except the i-th, i.e.

$$H_i = \mathrm{span}\left(X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n\right),$$

and $H_{i,j}$ denote the span of all column vectors except the i-th and j-th $(i < j)$, i.e.

$$H_{i,j} = \mathrm{span}\left(X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_{j-1}, X_{j+1}, \ldots, X_n\right).$$

Let P_1 denote the orthogonal projection in $\mathbb{R}^n$ onto the subspace H_1 and let

$$x = X_1 - P_1 X_1.$$

Then x is orthogonal to H_1. Since our matrix A is invertible and $\dim \ker P_1 = 1$, then we also have that $\|x\|_2 = \operatorname{dist}(X_1, H_1)$.

Note that by Markov's inequality, we have

$$\mathbb{P}(\|x\|_2 > \tau) \leq \frac{\mathbb{E}\|x\|_2^2}{\tau^2}, \qquad \tau > 0. \tag{2.1}$$

Let f_n be a normal vector of the $(n-1)$-dimensional subspace H_1. Then, the vector x can be represented as $x = \langle X_1, f_n \rangle f_n$, and the norm of x is

$$\|x\|_2 = |\langle X_1, f_n \rangle| = \left| \sum_{i=1}^{n} a_{i1} f_n^i \right|.$$

Hence,

$$\mathbb{E}\left| \sum_{i=1}^{n} a_{i1} f_n^i \right|^2 = \mathbb{E}\left(\sum_{i=1}^{n} a_{i1}^2 \left(f_n^i\right)^2 + \sum_{i \neq j} a_{i1} a_{j1} f_n^i f_n^j \right) = \sum_{i=1}^{n} \left(f_n^i\right)^2 \mathbb{E} a_{i1}^2 = 1 \tag{2.2}$$

(this fact also follows from the fact that vector X_1 is isotropic, i.e., $\mathbb{E}(X_1^k X_1^s) = 0$ and $\mathbb{E}(X_1^k)^2 = 1$). Then by (2.1),

$$\mathbb{P}(\|x\|_2 > \tau) \leq \frac{1}{\tau^2}, \qquad \tau > 0.$$

Now we estimate $\|A^{-1}x\|_2$. Note that

$$\|A^{-1}x\|_2 = \|A^{-1}X_1 - A^{-1}P_1 X_1\|_2 = \|e_1 - A^{-1}P_1 A e_1\|_2.$$

Since the vector $P_1 A e_1$ belongs to $\operatorname{span}\{Ae_2, \ldots, Ae_n\}$, then $A^{-1}P_1 A e_1$ is orthogonal to e_1. Therefore, using $P_1\widetilde{X}_1 = 0$ and denoting $Y_k = P_1\widetilde{X}_k,\ k \in \{2,\ldots,n\}$, we obtain

$$\begin{aligned}\|A^{-1}x\|_2^2 = \|e_1\|_2^2 + \|A^{-1}P_1X_1\|_2^2 > \|A^{-1}P_1X_1\|_2^2 &= \sum_{k=1}^{n}\left\langle A^{-1}P_1X_1, e_k\right\rangle^2 \\ = \sum_{k=1}^{n}\left\langle X_1, P_1\widetilde{X}_k\right\rangle^2 &= \sum_{k=2}^{n}\left\langle X_1, Y_k\right\rangle^2. \end{aligned} \tag{2.3}$$

The following lemma provides the relation between families of vectors $(X_k)_{k=2}^n$ and $(Y_k)_{k=2}^n$.

Lemma 2.10 ([127], Lemma 2.1) *If $(X_k, Y_k)_{k=2}^n$ is defined as above, then it is a complete biorthogonal system in H_1.*

The following is a consequence of the uniqueness in Proposition 2.2 (i).

Corollary 2.11 *The system of vectors $(Y_k)_{k=2}^n$ is uniquely determined by the system $(X_k)_{k=2}^n$. In particular, the system $(Y_k)_{k=2}^n$ and the vector X_1 are independent.*

By Proposition 2.2 (ii), we have $\|Y_k\|_2 = \frac{1}{\operatorname{dist}\left(X_k, H_{1,k}\right)}$. Therefore, we can rewrite (2.3) as

$$\|A^{-1}x\|_2^2 \geq \sum_{k=2}^{n}\frac{1}{1/\|Y_k\|_2^2}\left\langle\frac{Y_k}{\|Y_k\|_2}, X_1\right\rangle^2 = \sum_{k=2}^{n}\left(\frac{a_k}{b_k}\right)^2, \tag{2.4}$$

where

$$a_k = \left|\left\langle\frac{Y_k}{\|Y_k\|_2}, X_1\right\rangle\right| \quad \text{and} \quad b_k = \frac{1}{\|Y_k\|_2} = \operatorname{dist}\left(X_k, H_{1,k}\right). \tag{2.5}$$

This reduces our problem to bounding a_k from above and b_k from below. Without loss of generality, we can do it for $k = 2$, since the same argument carries over to any $k \in \{2,\ldots,n\}$

We split the unit sphere into sets of compressible and incompressible vectors. Our next goal is to show that the orthogonal complement $H_{1,2}^{\perp}$ consists of incompressible vectors with high probability. Consider an $(n-2) \times n$ matrix B with columns $X_3,\ldots,X_n$. Since the subspace $H_{1,2}$ is the span of the independent random vectors

$X_3, \ldots, X_n$, we have $H_{1,2}^{\perp} \subset \ker(B)$. We want to show:

$$\forall x \in \text{Comp}: \quad \|Bx\|_2 \geq \lambda\sqrt{n}, \tag{2.6}$$

that is, with high probability compressible vectors do not belong to the kernel of matrix B (the parameter λ will be determined later).

To deal with compressible vectors, we need the following proposition, which is essentially Proposition 5.2 from [120], where it was proved for $n \times n$ matrices. For the sake of completeness, we provide the proof for $(n-2) \times n$ matrices.

Proposition 2.12 *Let ξ be a centered random variable with unit variance such that $\mathcal{L}(\xi, \tilde{v}) \leq \tilde{u}$ for some $\tilde{v} > 0$ and $\tilde{u} \in (0,\, 1)$. Let $n \in \mathbb{N}$ and let Γ be an $(n-2)\times n$ random matrix with i.i.d. entries equidistributed with ξ. Then there are numbers θ, $v > 0$ and $u \in (0,\, 1)$ depending only on $\tilde{u}, \tilde{v}$ such that for $\text{Comp} = \text{Comp}_n(\rho, \rho)$ we have*

$$\mathbb{P}\left(\inf_{x\in\text{Comp}} \|\Gamma y\|_2 < v\sqrt{n}\right) \leq 5u^{n-2}.$$

Proof. The main idea of the proof is to apply the union bound over the set of compressible vectors Comp. In Theorem 2.5 take $\delta \in (0, \frac{1}{4}]$ such that

$$e^{13n\delta \ln \frac{2e}{\delta}} \leq u^{-\frac{n-2}{3}}.$$

Then define the parameter $\rho \in (0, \frac{1}{6}]$ in such a way that

$$\left(\frac{5e}{\rho^2}\right)^{\rho n} \leq u^{-\frac{n-2}{3}} \quad \text{and} \quad \frac{3\rho C}{\delta} \leq \frac{v}{2},$$

where $C > 0$ is a universal constant taken from Theorem 2.5.

By Lemma 2.4, there is a Euclidean (2ρ)-net $\mathcal{N} \subset \text{Comp}$ of cardinality

$$|\mathcal{N}| \leq \left(\frac{5e}{\rho^2}\right)^{\rho n} \;\leq\; u^{-\frac{n-2}{3}}.$$

Now we refine this net using Theorem 2.5, and as a result with probability at least $1 - 4\exp\left(-\frac{\delta n}{8}\right)$ we obtain a $\left(\frac{2\rho C}{\delta}\sqrt{n}\right)$-net $\widetilde{\mathcal{N}} \subset \mathrm{Comp}$ with respect to the pseudometric $\|\Gamma(x-y)\|_2$ which has cardinality $|\widetilde{\mathcal{N}}| \leq u^{-2(n-2)/3}$. In other words, for every $x \in \mathrm{Comp}$ there exists $x' = x'(x) \in \mathrm{Comp}$ such that

$$\|\Gamma(x - x')\|_2 \leq \frac{2\rho C}{\delta}\sqrt{n} \leq \frac{v}{2}\sqrt{n}.$$

Applying the union bound over $\widetilde{\mathcal{N}}$ to the relation from Lemma 2.7, we get

$$\mathbb{P}\left(\|\Gamma x'\|_2 < v\sqrt{n} \text{ for some } x' \in \widetilde{\mathcal{N}}\right) \leq |\widetilde{\mathcal{N}}| u^{n-2} \leq u^{\frac{n-2}{3}}.$$

On the other hand, the construction of $\widetilde{\mathcal{N}}$ implies that

$$\mathbb{P}\left(\inf_{x \in \mathrm{Comp}} \|\Gamma x\|_2 < \inf_{x' \in \widetilde{\mathcal{N}}} \|\Gamma x'\|_2 - \frac{v}{2}\sqrt{n}\right) \leq 4\exp\left(-\frac{\delta n}{8}\right).$$

Therefore,

$$\mathbb{P}\left(\|\Gamma x\|_2 < \frac{v}{2}\sqrt{n} \text{ for some } x \in \mathrm{Comp}\right) \leq u^{\frac{n-2}{3}} + 4\exp\left(-\frac{\delta(n-2)}{8}\right).$$

Taking the maximum of $u^{\frac{1}{3}}$ and $e^{-\frac{\delta}{8}}$ gives the desired result. ∎

The next proposition states that the least common denominator of any incompressible vector in $\mathbb{R}^n$ is of order at least $\sqrt{n}$. This proposition is Lemma 6.1 from [126] (note that the proof does not depend on the parameter α).

Proposition 2.13 *For any parameters $\theta, \rho \in (0,1)$ there are parameters $r, \gamma > 0$ such that for every $\alpha > 0$ any vector $x \in \mathrm{Incomp}_n(\theta, \rho)$ satisfies $\mathrm{LCD}_{\alpha,r}(x) \geq \gamma\sqrt{n}$.*

Recall that B is a $(n-2) \times n$ matrix with columns $X_3, \ldots, X_n$. Since $H_{1,2}^{\perp} \subset \ker(B)$, for the set Comp the following implication holds:

$$\text{if} \qquad \inf_{y \in \mathrm{Comp}} \|By\|_2 > 0 \qquad \text{then} \qquad H_{1,2}^{\perp} \cap \mathrm{Comp} = \emptyset.$$

Applying Proposition 2.12 to the matrix B, we get

$$\mathbb{P}\left(\inf_{y \in \mathrm{Comp}} \|By\|_2 \geq v\sqrt{n} \right) \geq 1 - 5u^{n-2}.$$

Therefore, $H_{1,2}^{\perp} \cap \mathrm{Comp} = \emptyset$ with probability at least $1 - 5u^{n-2}$, or in other words,

$$\mathbb{P}\left(H_{1,2}^{\perp} \cap S^{n-1} \subseteq \mathrm{Incomp} \right) \geq 1 - 5u^{n-2},$$

which means that the subspace $H_{1,2}^{\perp}$ consist of incompressible vectors with probability close to one. By Proposition 2.13, we obtain that for some $u \in (0,1)$

$$\mathbb{P}\left(\mathrm{LCD}_{\alpha,r}\left(H_{1,2}^{\perp} \right) \geq \gamma\sqrt{n} \right) \geq 1 - 5u^{n-2}, \tag{2.7}$$

where α is a small multiple of $\sqrt{n}$.

Recall that the coefficients a_k and b_k were introduced in (2.5). To ensure that the lower bound for b_2 is satisfied with high probability, we condition on $H_{1,2}$ and use Markov's inequality and the fact that X_2 is isotropic (see (2.2)). More precisely, we obtain

$$\mathbb{P}\left(b_2 = \mathrm{dist}\left(X_2, H_{1,2} \right) \geq t \,|\, H_{1,2} \right) \leq \frac{\mathbb{E}\,\mathrm{dist}\left(X_2, H_{1,2} \right)^2}{t^2} \leq \frac{2}{t^2}, \quad t > 0. \tag{2.8}$$

Let $\mathcal{E} = \left\{ \mathrm{LCD}_{\alpha,r}\left(H_{1,2}^{\perp} \right) \geq \gamma\sqrt{n} \text{ and } b_2 < t \,|\, H_{1,2} \right\}$. Combining the two estimates (2.7) and (2.8), we get that

$$\mathbb{P}(\mathcal{E}) \geq 1 - \frac{2}{t^2} - 5u^{n-2}. \tag{2.9}$$

Since we conditioned on the subspace $H_{1,2}$, we may fix a realization of vectors

$(X_j)_{j=2}^n$ for which the statement (2.9) holds. Thus by the uniqueness in Corollary 2.11 the vector Y_2 is also fixed. For convenience, we further consider the normalized vector $Y = \frac{Y_2}{\|Y_2\|_2}$. By Lemma 2.10 we know that $(X_j)_{j=2}^n$ and $(Y_j)_{j=2}^n$ form a biorthogonal system, in particular Y is orthogonal to $(X_j)_{j=3}^n$. Thus, $Y \in H_{1,2}^{\perp}$. Since the event $\mathcal{E}$ in (2.9) holds, we know that

$$\mathrm{LCD}_{\alpha,r}(Y) \geq \gamma\sqrt{n}.$$

Now we proceed to bound the coefficient a_2. Recall that

$$a_2 = |\left\langle Y, X_1 \right\rangle| = |\sum_{k=1}^{n} Y^k X_1^k|$$

and Y^k are coefficients such that $\sum_{k=1}^n \left(Y^k\right)^2 = 1$ and X_1^k are i.i.d. random variables with zero mean and $\mathcal{L}\left(X_1^k, 1\right) < u$ for some $u \in (0,1)$. Applying Theorem 2.9 with $\alpha = c\sqrt{n}$ for some small absolute constant $c > 0$, we obtain for $\varepsilon > 0$, $u \in (0,1)$ and $r \in (0,1)$

$$\begin{aligned} \mathbb{P}_{X_1}\left(a_2 \leq \varepsilon \,|X_2, \ldots, X_n\right) &\leq \widetilde{C}\left(\frac{1}{r\sqrt{1-u}}\left[\varepsilon + \frac{1}{\mathrm{LCD}_{c\sqrt{n},r}(Y)}\right] + e^{-2c^2(1-u)n}\right) \\ &\leq C\left(\varepsilon + \frac{1}{\sqrt{n}} + e^{-c_1 n}\right), \end{aligned} \tag{2.10}$$

where $c_1, C, \widetilde{C} > 0$ are absolute constants. Note that in the above expression all $(X_j)_{j=2}^n$ are fixed and the probability is taken with respect to the random vector X_1.

Now we unfix all random vectors $X_2, \ldots, X_n$. Then,

$$\begin{aligned} \mathbb{P}\left(a_2 \leq \varepsilon \text{ or } b_2 \geq t\right) &= \mathbb{E}_{X_2,\ldots,X_n}\mathbb{P}_{X_1}\left(a_2 \leq \varepsilon \text{ or } b_2 \geq t\right) \\ &= \mathbb{E}_{X_2,\ldots,X_n}\chi_{\mathcal{E}}\mathbb{P}_{X_1}(a_2 \leq \varepsilon \text{ or } b_2 \geq t) + \mathbb{E}_{X_2,\ldots,X_n}\chi_{\mathcal{E}^c}\mathbb{P}_{X_1}(a_2 \leq \varepsilon \text{ or } b_2 \geq t) \\ &\leq \mathbb{P}(a_2 \leq \varepsilon \,|X_2, \ldots, X_n) + \mathbb{P}(\mathcal{E}^c). \end{aligned}$$

Combining the probability estimates in (2.9) and (2.10), we get

$$\mathbb{P}\left(a_2 \le \varepsilon \text{ or } b_2 \ge t\right) \le C\left(\varepsilon + \frac{1}{\sqrt{n}} + e^{-c_1 n}\right) + \left(\frac{2}{t^2} + 5u^{n-2}\right).$$

Repeating this argument for a_k and b_k for $k = 3, \dots, n$, we obtain for any ε, $t > 0$ and $u \in (0,1)$

$$\mathbb{P}\left(\frac{a_k}{b_k} \le \frac{\varepsilon}{t}\right) \le C\left(\varepsilon + \frac{1}{\sqrt{n}} + e^{-c_1 n} + \frac{2}{t^2} + 5u^{n-2}\right) \le C_1\left(\varepsilon + \frac{1}{\sqrt{n}} + \frac{1}{t^2}\right) \tag{2.11}$$

where $C, C_1, c_1 > 0$ are absolute constants.

Now we proceed to estimate the sum of $\left(\frac{a_k}{b_k}\right)^2$ in (2.4):

$$\begin{aligned}
\mathbb{P}\left(\|A^{-1}x\| \le \frac{\varepsilon}{t}\sqrt{n}\right) &\le \mathbb{P}\left(\frac{1}{n}\sum_{k=2}^{n}\left(\frac{a_k}{b_k}\right)^2 \le \frac{\varepsilon^2}{t^2}\right) \\
&\le \mathbb{P}\Big(\exists\, k_1, \dots, k_{\lfloor \frac{n}{2} \rfloor} \in \{2, \dots, n\} \text{ such that} \\
&\qquad \left(\frac{a_{k_i}}{b_{k_i}}\right)^2 \le 2\frac{\varepsilon^2}{t^2} \text{ for all } i \le \left\lfloor \frac{n}{2} \right\rfloor\Big) \\
&= \mathbb{P}\left(\sum_{k=2}^{n} \chi_k \ge \left\lfloor \frac{n}{2} \right\rfloor\right) \le \frac{2}{n}\sum_{k=2}^{n}\mathbb{P}\left(\left(\frac{a_k}{b_k}\right)^2 \le 2\frac{\varepsilon^2}{t^2}\right),
\end{aligned}$$

where we denoted by χ_k the indicator function of the event $\mathcal{E}_k = \left\{\left(\frac{a_k}{b_k}\right)^2 \le 2\frac{\varepsilon^2}{t^2}\right\}$ and in the last step used Markov's inequality. Using the bound in (2.11), we finally obtain

$$\mathbb{P}\left(\|A^{-1}x\|_2 \le \frac{\varepsilon}{t}\sqrt{n}\right) \le 2C_1\left(2\varepsilon + \frac{1}{\sqrt{n}} + \frac{1}{t^2}\right).$$

Together with an estimate in (2.1), we have

$$\begin{aligned}
\mathbb{P}\left(s_n(A) \le \frac{\tau t}{\varepsilon} n^{-\frac{1}{2}}\right) &\ge \mathbb{P}\left(\|x\|_2 \le \tau,\ \|A^{-1}x\|_2 \ge \frac{\varepsilon}{t}\sqrt{n}\right) \\
&\ge 1 - C_2\left(\varepsilon + \frac{1}{\sqrt{n}} + \frac{1}{t^2} + \frac{1}{\tau^2}\right).
\end{aligned}$$

Since the above statement holds for arbitrary $\varepsilon, t, \tau > 0$, the choice $t = \tau = \frac{1}{\sqrt{\varepsilon}}$ gives the desired quantitative estimate in Theorem 2.1. $\blacksquare$

Chapter 3

Random Polytopes Obtained by Matrices with Heavy Tailed Entries [2]

3.1 Introduction

In this section, we deal with rectangular $N \times n$, $(n \leq N)$ random matrices $\Gamma = \{\xi_{ij}\}_{1\leq i\leq N, 1\leq j\leq n}$, where ξ_{ij} are independent symmetric random variables with unit variance satisfying uniform small ball probabilistic estimate. More precisely, in the main theorem we assume that there exist $u, v \in (0,1)$ such that

$$\forall i, j \quad \sup_{\lambda \in \mathbb{R}} \mathbb{P}\big(|\xi_{ij} - \lambda| \leq u\big) \leq v. \tag{3.1}$$

Of course, if variables have a bounded moment $r > 2$, we will have better estimates. We are interested in geometric parameters of the random polytope generated by Γ, that is, the absolute convex hull of rows of Γ. In other words, the random polytope under consideration is $\Gamma^* B_1^N$, where B_1^N is the N-dimensional octahedron (cross-polytope).

[2] A version of this chapter was published: O. Guédon, A. E. Litvak and K. Tatarko, "Random polytopes obtained by matrices with heavy tailed entries", Commun. Contemp. Math.

The Gaussian random polytopes in the case when N is proportional to n have many applications in the Asymptotic Geometric Analysis (see e.g., [61] and [141], and the survey [101]). The Bernoulli case corresponds to 0/1 random polytopes. For their combinatorial properties we refer the reader to [47, 8] (see also the survey [166]). Their geometric parameters have been studied in [60, 85]. In the compressed sensing it was shown that the so-called ℓ_1-quotient property is responsible for robustness in certain ℓ_1-minimizations (see [80] and references therein). More precisely, an $n \times N$ (with $N \geq n$) matrix A satisfies the ℓ_1-quotient property with a constant b relative to a norm $\|\cdot\|$ if for every $y \in \mathbb{R}^n$ there exists $x \in \mathbb{R}^N$ such that $Ax = y$ and $\|x\|_1 \leq b\sqrt{n/\ln(eN/n)}\,\|y\|$, where $\|\cdot\|_1$ denotes the ℓ_1-norm. It is easy to see that geometrically this means

$$B_{\|\cdot\|} \subset b\sqrt{n/\ln(eN/n)}\,AB_1^N,$$

where $B_{\|\cdot\|}$ is the unit ball of $\|\cdot\|$. To prove robustness of noise-blind compressed sensing, the authors of [80] dealt with the norm

$$\|\cdot\| = \max\{\|\cdot\|_2, \sqrt{\ln(eN/n)}\|\cdot\|_\infty\},$$

where $\|\cdot\|_2$ is the standard Euclidean norm and $\|\cdot\|_\infty$ is the ℓ_∞-norm. Theorem 5 in [80] states that assuming that entries of A are symmetric i.i.d. random variables with unit variances, and that they have regular behaviour of all moments till the moment of order $\ln n$, the matrix $A/\sqrt{n}$ has the ℓ_1-quotient property with high probability. Geometrically this means

$$AB_1^N \supset b^{-1}\left(B_\infty^n \cap \sqrt{\ln(N/n)}\,B_2^n\right). \tag{3.2}$$

The work [80] complements results of [85], where this inclusion was proved for random matrices with symmetric i.i.d. entries having at least third bounded moment and such that the operator norm of the matrix is bounded with high probability.

The main purpose of this chapter is to prove such an inclusion with much weaker assumptions on the distribution of the entries. In fact, we require only boundedness of second moments. Thus "robustness" Theorem 8 in [80] holds under much weaker assumptions on the random matrix. Our main result is the following theorem (see Theorem 3.11 for slightly better probability estimates).

Theorem 3.1 *There exist positive constants b, M depending only on u and v and an absolute constant $c > 0$ such that the following holds. Let $N \geq Mn$ and assume that the entries of an $N \times n$ random matrix Γ are independent symmetric random variables with unit variances satisfying condition (3.1) and such that in each row the entries are i.i.d. Then with probability at least $1-\exp(-cn)$ the inclusion (3.2) holds for the matrix $A = \Gamma^*$.*

We use this theorem to study geometric properties of random polytopes $K_N = \Gamma^* B_1^N$ and K_N°, such as behavior of their volumes and mean widths. Our "volume" theorem states the following (see Theorems 3.20 and 3.21 for more precise statements).

Theorem 3.2 *There exist positive constants C_1, C_2 depending only on u and v and absolute positive constants C, c such that for $C_1 n \leq N \leq e^n$ with probability at least $1 - \exp(-cn)$ one has*

$$\mathrm{vol}_n(K_N)^{1/n} \geq C_2 \sqrt{\frac{\ln(N/n)}{n}} \qquad \text{and} \qquad \mathrm{vol}_n(K_N^\circ)^{1/n} \leq \frac{C}{C_2\sqrt{n \ln(N/n)}},$$

where $K_N = \Gamma^* B_1^N$ and the matrix Γ is as in Theorem 3.1. Moreover, the bounds on the volumes are sharp, provided that the Euclidean lengths of the rows of Γ are of order of $\sqrt{n}$ at most.

Our proof of Theorem 3.1 follows the general scheme of [85] with a very delicate change – in [85] there was an assumption that the operator norm of Γ is bounded by $C\sqrt{N}$ with high probability. However it is known that such a bound does not hold in general unless fourth moments are bounded ([137], see also [88] for quantitative bounds). To avoid using the norm of Γ, we use ideas appearing in [120], where the authors constructed a certain deterministic ε-net (in ℓ_2-metric) $\mathcal{N}$ such that $A\mathcal{N}$ is a good net for AB_2^n for most realizations of a square random matrix A. We extend their construction in three directions. First, we work with rectangular random matrices, not only square matrices. Second, we need a net for the image of a given convex body (not only for the image of the unit Euclidean ball). Finally, instead of approximation in the Euclidean norm only, we use approximation in the following norm

$$\|a\|_{k,2} = \left(\sum_{i=1}^{k} (a_i^*)^2 \right)^{1/2}, \tag{3.3}$$

where $1 \le k \le N$ and $a_1^* \ge a_2^* \ge \ldots \ge a_N^*$ is the decreasing rearrangement of the sequence of numbers $|a_1|, \ldots, |a_N|$. This norm appears naturally and plays a crucial role in our proof of inclusion (3.2). The generalization of the net from [120] is a new key ingredient, see Theorem 3.3. We would like to emphasize, that norms $\|\cdot\|_{k,2}$ played an important role in proofs of many results of Asymptotic Geometric Analysis, see e.g. [63, 65, 67]. For the systematic studies of norms $\|\cdot\|_{k,2}$ and their unit balls we refer to [64]. We believe that the new approximation in $\|\cdot\|_{k,2}$ norms will find other applications in the theory. In the last section, we present one more application of Theorem 3.3 – we show that it can be used to estimate the smallest singular value of a tall random matrix – see the discussion at the beginning of Section 3.5.

3.2 Notations

Given an integer $k \in \{1, \ldots, N\}$, we denote by $X_{k,2}$ the normed space $\mathbb{R}^N$ equipped with the norm $\|\cdot\|_{k,2}$ defined by (3.3). The unit ball of $X_{k,2}$ is denoted by $\mathbf{B}_{k,2}$. Note that for $k = N$ we have $\|a\|_{k,2} = \|a\|_2$ and that for any $k \le N$ and any $a \in \mathbb{R}^N$,

$$\|a\|_{k,2} \le \|a\|_2 \le \sqrt{\frac{N}{k}}\,\|a\|_{k,2} \qquad \text{or, equivalentely,} \qquad B_2^N \subset \mathbf{B}_{k,2} \subset \sqrt{\frac{N}{k}}\,B_2^N.$$

Given integers $\ell \ge k \ge 1$, we denote $[k] = \{1, 2, \dots, k\}$ and $[k, \ell] = \{k, k+1, \dots, \ell\}$. Given points $x_1, \dots, x_k$ in $\mathbb{R}^m$ we denote their convex hull by $\operatorname{conv}\{x_i\}_{i\le k}$ and their absolute convex hull by $\operatorname{abs\,conv}\{x_i\}_{i\le k} = \operatorname{conv}\{\pm x_i\}_{i\le k}$. Given $\sigma \subset [m]$ by P_σ we denote the coordinate projection onto $\mathbb{R}^\sigma = \{x \in \mathbb{R}^m \,|\, x_i = 0 \text{ for } i \notin \sigma\}$.

A ± 1 random variable taking values 1 and -1 with probability $1/2$ is called a Rademacher random variable.

In this chapter we are interested in rectangular $N \times n$ matrices $\Gamma = \{\xi_{ij}\}_{\substack{1\le i\le N\\ 1\le j\le n}}$, with $N \ge n$, where the entries are real-valued random variables on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$. We will mainly consider the following model of matrix Γ:

$$\begin{cases} \forall i, j \quad \xi_{ij} \text{ are independent, symmetric and } \mathbb{E}\xi_{ij}^2 = 1, \\ \text{in each row the entries are identically distributed.} \end{cases} \tag{3.4}$$

At the beginning of Section 3.4, we will also assume that the entries of Γ satisfy a uniform small ball estimate. If $\xi_{ij} \sim \mathcal{N}(0,1)$ are independent Gaussian random variables we say that Γ is *a Gaussian random matrix*.

3.3 Construction of a good deterministic net

In this section we present a key result. Let T be a subset of $\mathbb{R}^n$, we aim at constructing a deterministic net such that for every general random operator $\Gamma : \mathbb{R}^n \to \mathbb{R}^N$, with overwhelming probability, the image of the net by the random operator Γ is a good approximation of ΓT. We show that we can quantify this approximation by almost any norm $\|\cdot\|_{k,2}$ defined in (3.3). For integers $1 \le n \le N$ and for $0 \le \delta \le 1$, set

$$F(\delta, n, N) = \begin{cases} (32\delta N/n)^n & \text{if } \ \delta \ge n/(2N), \\ (en/(\delta N))^{4\delta N} & \text{if } \ \delta \le n/(2N). \end{cases} \tag{3.5}$$

Theorem 3.3 *Let $n \in [N]$, $0 \le \delta \le 1$, $0 < \varepsilon \le 1$. Let $k \in [N]$ such that $k \ln(eN/k) \ge n$. Let T be a non-empty subset of $\mathbb{R}^n$ and denote $M := N(T, \varepsilon B_\infty^n)$. Then there exists a set $\mathcal{N} \subset T$ and a collection of parallelepipeds $\mathcal{P}$ in $\mathbb{R}^n$ such that*

$$\max\{|\mathcal{N}|, |\mathcal{P}|\} \le M\, F(\delta, n, N)\, e^{\delta N}.$$

Moreover, for any random matrix Γ satisfying assumption (3.4)*, with probability at least $1 - e^{-k \ln(eN/k)} - e^{-\delta N/4}$, one has*

$$\begin{cases} \forall x \in T \ \exists y \in \mathcal{N} \quad \textit{such that} \quad \|\Gamma(x-y)\|_{k,2} \le C\varepsilon \sqrt{\dfrac{kn}{\delta} \ln\left(\dfrac{eN}{k}\right)}, \\ \forall x \in T \ \exists P \in \mathcal{P} \quad \textit{such that} \quad x \in P \quad \textit{and} \ \ \Gamma P \subset \Gamma x + C\varepsilon \sqrt{\dfrac{kn}{\delta} \ln\left(\dfrac{eN}{k}\right)}\ \boldsymbol{B}_{k,2}, \end{cases}$$

where $C \ge 1$ is an absolute constant.

Remark 3.4 This result extends Theorem A and Corollary A from [120], where the authors considered the case of square matrices, $T = S^{n-1}$ and $k = N$, which corresponds to the approximation of Γx in the Euclidean norm.

3.3.1 Basic facts about covering numbers and operator norms of random matrices

We begin by recalling some classical estimates for covering numbers that will be used later. It is well known that for any two centrally symmetric bodies K and L in $\mathbb{R}^m$ and any $\varepsilon > 0$ there exists an ε-net $\mathcal{N}$ of L with respect to K with cardinality

$$|\mathcal{N}| \le \mathrm{vol}_m((2/\varepsilon)L + K)/\mathrm{vol}_m(K) \tag{3.6}$$

(see e.g. Lemma 4.16 in [117]). In particular, if $K = L$ are centrally symmetric bodies in $\mathbb{R}^m$ (or if L is the boundary of a centrally symmetric body K) then $|\mathcal{N}| \le (1+2/\varepsilon)^m$.

Lemma 3.5

a) For every $\varepsilon \in (0, 1/\sqrt{m}]$

$$N(B_2^m, \varepsilon B_\infty^m) \le \big(7/(\varepsilon\sqrt{m})\big)^m$$

and for every $\varepsilon \in (1/\sqrt{m}, 1]$

$$N(B_2^m, \varepsilon B_\infty^m) \le \big(17\varepsilon^2 m\big)^{1/\varepsilon^2}.$$

b) For $J \subset [m]$, let $S^J = \{x \in \mathbb{R}^J \mid \|x\|_2 = 1\}$. For every $\varepsilon \in (0,1)$ and every integer $k \le m$, there exists a finite set $\mathcal{N} \subset \cup_{|J|=k} S^J$ such that

$$\begin{cases} |\mathcal{N}| \le \exp\left(k\ln(3/\varepsilon) + k\ln(em/k)\right), \\ \forall J \subset [m] \text{ with } |J| = k \;\; \forall y \in S^J \quad \exists z \in \mathcal{N} \cap S^J \text{ such that } \|y - z\|_2 \le \varepsilon. \end{cases} \tag{3.7}$$

Proof. a) Note that for every $m \ge 1$ one has $(1/\sqrt{m})B_\infty^m \subset B_2^m$ and $\mathrm{vol}_m(B_2^m) \le (2\pi e/m)^{m/2}$. Therefore, by (3.6), we obtain for every $\varepsilon \le 1/\sqrt{m}$

$$N(B_2^m, \varepsilon B_\infty^m) \le \frac{\mathrm{vol}_m\left(\frac{2}{\varepsilon}B_2^m + B_\infty^m\right)}{\mathrm{vol}_m(B_\infty^m)} \le \left(\frac{3}{\varepsilon}\right)^m \frac{\mathrm{vol}_m(B_2^m)}{\mathrm{vol}_m(B_\infty^m)} \le \left(\frac{3\sqrt{\pi e}}{\varepsilon\sqrt{2m}}\right)^m.$$

This implies the first bound. For the second bound note that for every $x \in B_2^n$ the number of coordinates of x larger than ε is at most $1/\varepsilon^2$. Thus every $x \in B_2^n$ can be presented as $x = y + z$, where the cardinality of support of y is at most $1/\varepsilon^2$, $z \in \varepsilon B_\infty^n$, and supports of y and z are mutually disjoint. Therefore, it is enough to cover B_2^σ by $\varepsilon B_\infty^\sigma$ for all $\sigma \subset [n]$ with $|\sigma| = m := \lfloor 1/\varepsilon^2 \rfloor$. Using the above bound we obtain

$$N(B_2^n, \varepsilon B_\infty^n) \leq \binom{n}{m} \left(\frac{3\sqrt{\pi e}}{\varepsilon\sqrt{2m}} \right)^m \leq \left(\frac{3en\sqrt{\pi e}}{\varepsilon m\sqrt{2m}} \right)^m,$$

which implies the desired result as $m \leq 1/\varepsilon^2$.

b) Fix $\varepsilon \in (0,1)$. For any fixed $J \subset [m]$ of cardinality k, we cover S^J by an ε-net (of points in S^J) of cardinality at most $(1 + 2/\varepsilon)^k \leq (3/\varepsilon)^k$ and we take the union of these nets over all sets J of cardinality k. We conclude using that $\binom{m}{k} \leq (em/k)^k$. ■

The next lemma is a classical consequence of estimates for covering numbers for evaluating operator norms of random matrices.

Lemma 3.6 *Let $B = \{b_{ij}\}_{\substack{1 \leq i \leq N \\ 1 \leq j \leq n}}$ be a fixed $N \times n$ matrix. Let $k \in [N]$ be such that $k \ln \frac{eN}{k} \geq n$. Let ε_{ij} be i.i.d. Rademacher random variables. Denote $B_\varepsilon = \{\varepsilon_{ij} b_{ij}\}_{\substack{1 \leq i \leq N \\ 1 \leq j \leq n}}$. Then for every $t \geq 1$ one has*

$$\mathbb{P}\left(\|B_\varepsilon : \ell_\infty^n \to X_{k,2}\| \geq 6\, t \sqrt{k \ln\left(\frac{eN}{k}\right)} \max_{i \leq N} \|R_i(B)\|_2 \right) \leq e^{-t^2 k \ln(eN/k)},$$

where $R_i(B)$, $i \leq N$, are the rows of B.

Proof. Observe that for any $a \in \mathbb{R}^N$, we have

$$\|a\|_{k,2} = \sup_{\substack{J \subset [N] \\ |J| = k}} \sup_{b \in S^J} \sum_{i=1}^N a_i b_i.$$

Given $x \in \{\pm 1\}^n$, $y \in S^{N-1}$, consider the following random variable,

$$\xi_{x,y} = \sum_{j=1}^n \sum_{i=1}^N \varepsilon_{ij} b_{ij} x_j y_i.$$

Since $e^x + e^{-x} \le 2\exp(x^2/2)$ for every real x, we observe for $\lambda > 0$,

$$\mathbb{E}\exp\left(\lambda \sum_{j=1}^{n}\sum_{i=1}^{N} \varepsilon_{ij} b_{ij} x_j y_i\right) = \prod_{j=1}^{n}\prod_{i=1}^{N} \mathbb{E}\exp\left(\lambda \varepsilon_{ij} b_{ij} x_j y_i\right) \le \exp\left(\frac{\lambda^2}{2}\sum_{i=1}^{N} y_i^2 \|R_i(B)\|_2^2\right)$$
$$\le \exp\left(\frac{\lambda^2}{2}\max_{i\le N}\|R_i(B)\|_2^2\right).$$

Therefore, using the Laplace transform of $\xi_{x,y}$, we deduce that for any $u > 0$,

$$\mathbb{P}\left(\xi_{x,y} > u \max_{i\le N}\|R_i(B)\|_2\right) \le e^{-u^2/2}.$$

Note that

$$\|B_\varepsilon : \ell_\infty^n \to X_{k,2}\| = \sup_{x\in\{\pm1\}^n} \sup_{\substack{J\subset[N]\\ |J|=k}} \sup_{y\in S^J} \xi_{x,y}. \tag{3.8}$$

Now we apply the classical net argument. Let $\mathcal{N}$ be the net defined by (3.7) with $\varepsilon = 1/2$. Then

$$\mathbb{P}\left(\sup_{x\in\{\pm1\}^n}\sup_{z\in\mathcal{N}} \xi_{x,z} \ge u \max_{i\le N}\|R_i(B)\|_2\right) \le 2^n |\mathcal{N}| e^{-u^2/2}$$
$$\le 2^n \exp\left(-\frac{u^2}{2} + k\ln 6 + k\ln(eN/k)\right).$$

Taking $u = 3t\sqrt{k\ln(eN/k)}$ and using $k\ln(eN/k) \ge n$, we get for every $t \ge 1$,

$$\mathbb{P}\left(\sup_{x\in\{\pm1\}^n}\sup_{z\in\mathcal{N}} \xi_{x,z} \ge 3t\sqrt{k\ln(eN/k)}\,\max_{i\le N}\|R_i(B)\|_2\right) \le e^{-t^2 k\ln(eN/k)}.$$

By definition of $\mathcal{N}$, for any $J \subset [N]$ of cardinality k and $y \in S^J$, there exists $z \in \mathcal{N}\cap S^J$ such that $\|z - y\|_2 \le 1/2$, hence, by the triangle inequality,

$$\sup_{x\in\{\pm1\}^n} \sup_{\substack{J\subset[N]\\ |J|=k}} \sup_{y\in S^J} \xi_{x,y} \le 2 \sup_{x\in\{\pm1\}^n}\sup_{z\in\mathcal{N}} \xi_{x,z}.$$

This completes the proof of the lemma. ∎

3.3.2 Auxiliary statements

By $\mathcal{D}_n$ we denote the set of all $n \times n$ diagonal matrices whose diagonal entries belong to the set $\{1\} \cup \{2^{-2^k}\}_{k \geq 0}$. The following theorem was proved in [120] in the square case. However the proof works as well in the rectangular case. One just needs to repeat the proof of Proposition 2.7 there for $N \times n$ matrices, to combine it with Remark 2.8 following the proposition, and to substitute the upper bound on the expectation with a probability bound using Markov's inequality.

Theorem 3.7 *Let $\Gamma = \{\xi_{ij}\}_{1 \leq i \leq N, 1 \leq j \leq n}$ be an $N \times n$ random matrix on a probability space Ω. Assume that entries of Γ are independent centered random variables with unit variances and that in each row the entries are identically distributed. Let $\delta \in (0, 1]$. Then there exists a random matrix D_Γ on Ω taking values in $\mathcal{D}_n$ such that*

(i) for every $\omega \in \Omega$, $D_\Gamma(\omega)$ depends only on the realization $\{|\xi_{ij}(\omega)|\}_{1 \leq i \leq N, 1 \leq j \leq n}$,

(ii) for every $\omega \in \Omega$ one has

$$\|R_i(\Gamma(\omega) D_\Gamma(\omega))\|_2 \leq C\sqrt{n/\delta},$$

(iii)

$$\mathbb{P}\left(\det D_\Gamma \leq e^{-4\delta N}\right) \leq e^{-\delta N},$$

where C is an absolute positive constant.

As in [120], Theorem 3.7 has important consequences. It allows us to construct, with high probability, a diagonal matrix D such that the volume of DB_∞^n remains big enough and such that, according to Lemma 3.6, we have a good control of the operator norm of ΓD from ℓ_∞^n to $X_{k,2}$. Comparing to [120], Lemma 3.6 simplifies significantly the proof and allows to extend Theorem 3.1 from [120] to the case of rectangular matrices and to approximations with respect to $\|\cdot\|_{k,2}$ norms.

Theorem 3.8 *Let $1 \le n \le N$ be integers, $\delta \in (0,1]$. Let $k \in [N]$ such that $k \ln \frac{eN}{k} \ge n$. Let Γ be an $N \times n$ random matrix satisfying the hypothesis* (3.4). *Then*

$$\mathbb{P}\left(\exists D \in \mathcal{D}_n: \ \det D \ge e^{-\delta N} \ \ \textit{and} \ \ \|\Gamma D : \ell_\infty^n \to X_{k,2}\| \le C\sqrt{\frac{kn}{\delta}\ln\left(\frac{eN}{k}\right)}\right)$$
$$\ge 1 - e^{-\delta N/4} - e^{-k\ln(eN/k)},$$

where C is a positive absolute constant.

Proof. Let D_Γ be the matrix given by Theorem 3.7. By property (iii) of D_Γ it is enough to prove that

$$\mathbb{P}\left(\|\Gamma D : \ell_\infty^n \to X_{k,2}\| \le C\sqrt{\frac{kn}{\delta}\ln\left(\frac{eN}{k}\right)}\right) \ge 1 - e^{-k\ln(eN/k)}.$$

Consider two probability spaces – the original one $(\Omega, \mathbb{P}_\omega)$, where the matrix Γ is defined, and the auxiliary space $(E, \mathbb{P}_\varepsilon)$, where $E := \{-1,1\}^{N\times n}$ and $\mathbb{P}_\varepsilon$ is the uniform probability on E. Given a matrix $A = \{a_{ij}\}_{1\le i\le N, 1\le j\le n}$ and $\varepsilon \in E$, denote $A_\varepsilon = \{\varepsilon_{ij}a_{ij}\}_{1\le i\le N, 1\le j\le n}$. Since entries of Γ are symmetric, for every fixed $\varepsilon \in E$ the matrix Γ_ε has the same distribution on Ω as Γ. By property (i) of D_Γ, we have $D_\Gamma = D_{\Gamma_\varepsilon}$ for every fixed $\varepsilon \in E$. Therefore, since D_Γ is diagonal, we have for every $\varepsilon \in E$

$$(\Gamma D_\Gamma)_\varepsilon = \Gamma_\varepsilon D_\Gamma = \Gamma_\varepsilon D_{\Gamma_\varepsilon}.$$

Then, by property (ii) of D_Γ from Theorem 3.7, there exists an absolute positive constant C_1 such that for every $i \le N$ and every $(\omega, \varepsilon) \in \Omega \times E$,

$$\|R_i\left((\Gamma(\omega)D_\Gamma(\omega))_\varepsilon\right)\|_2 \le C_1\sqrt{n/\delta}.$$

Fixing $\omega \in \Omega$ and applying Lemma 3.6 to the matrix $B = \Gamma(\omega)D_\Gamma(\omega)$, we obtain that for every fixed $\omega \in \Omega$ one has

$$\mathbb{P}_\varepsilon \left(\|\Gamma_\varepsilon(\omega) D_\Gamma(\omega) : \ell_\infty^n \to X_{k,2}\| > 6C_1 \sqrt{\frac{kn}{\delta} \ln\left(\frac{eN}{k}\right)} \right) \leq e^{-k \ln (eN/k)}.$$

Using that Γ_ε has the same distribution as Γ and the Fubini theorem, we obtain

$$\begin{aligned}
&\mathbb{P}_\omega \left(\|\Gamma D_\Gamma : \ell_\infty^n \to X_{k,2}\| > 6C_1 \sqrt{\frac{kn}{\delta} \ln\left(\frac{eN}{k}\right)} \right) = \\
&= \mathbb{P}_\varepsilon \mathbb{P}_\omega \left(\|\Gamma_\varepsilon(\omega) D_\Gamma(\omega) : \ell_\infty^n \to X_{k,2}\| > 6C_1 \sqrt{\frac{kn}{\delta} \ln\left(\frac{eN}{k}\right)} \right) \\
&\leq e^{-k \ln (eN/k)}.
\end{aligned}$$

■

As in Lemma 3.11 from [120], we need to estimate the cardinality of the set of diagonal matrices in $\mathcal{D}_n$ with not so small determinant.

Lemma 3.9 *Let $n, N \geq 1$ be integers, $\delta \in (0, 1]$ and*

$$Q := \{D \in \mathcal{D}_n : \ \det D \geq \exp(-\delta N)\}.$$

Then $|Q| \leq F(\delta, n, N)$, where $F(\delta, n, N)$ is defined by formula (3.5).

Proof. Note that if $D \in \mathcal{D}_n$ and d_1, ..., d_n its diagonal elements then for every $k \geq 0$ the set

$$Q_D(k) = \left\{ i \leq n \mid d_i = 2^{-2^k} \right\}$$

has cardinality at most $m_k := \min\{n, \lfloor 2^{-k} 2\delta N \rfloor\}$. Thus there are at most

$$\sum_{\ell=0}^{m_k} \binom{n}{\ell} \leq \left(\frac{en}{m_k}\right)^{m_k}$$

choices of $\sigma_k \subset [n]$, where matrices from $\mathcal{D}_n$ may have such coordinates. Note also that the trivial bound for the number of subsets is 2^n. Denote $a := 4\delta N/n$. Note that $m_k \le n/2$ if and only if $2^k \ge a$.

Case 1. $a \ge 2$. Set $m := \lfloor \log_2 a \rfloor \ge 1$. By above we have

$$|Q| \le \prod_{k<m} 2^n \prod_{k \ge m} \left(\frac{en}{m_k}\right)^{m_k} \le 2^{nm} \prod_{k\ge m} \left(\frac{en}{2\delta N}\right)^{2\delta N/2^k} \prod_{k \ge m} 2^{2k\delta N/2^k}$$
$$\le a^n \left(\frac{2e}{a}\right)^{4\delta N/a} 2^{2\delta N(2m+1)/2^m} \le (2e)^n\, a^{4\delta N/2^m}\, 2^{4\delta N/2^m} \le (8a)^n.$$

Case 2. $a \le 2$. Similarly we have

$$|Q| \le \prod_{k\ge 0} \left(\frac{en}{m_k}\right)^{m_k} \le \prod_{k \ge 0} \left(\frac{en}{2\delta N}\right)^{2\delta N/2^k} \prod_{k\ge 0} 2^{2k\delta N/2^k} \le \left(\frac{en}{2\delta N}\right)^{4\delta N} 2^{3\delta N},$$

which implies the desired result. ■

3.3.3 Proof of Theorem 3.3

Let Q be as in Lemma 3.9. Note that every $D \in Q$ is diagonal with reciprocal of integers on the diagonal. Therefore, there exists a set $\mathcal{N}_D \subset T$ of cardinality

$$|\mathcal{N}_D| \le N(T, \varepsilon D B_\infty^n) \le N(T, \varepsilon B_\infty^n) N(B_\infty^n, DB_\infty^n) \le M \det D^{-1} \le M e^{\delta N}$$

which satisfies that for any $x \in T$ there exists $y \in \mathcal{N}_D$ such that $x - y \in \varepsilon D B_\infty^n$. Let

$$\mathcal{P} = \left\{ y + \varepsilon D B_\infty^n : \ D \in Q, y \in \mathcal{N}_D \right\}.$$

Then, by Lemma 3.9, $|\mathcal{P}| \le M e^{\delta N} F(\delta, n, N)$ and for any $x \in T$ and for any $D \in Q$ there exists $P = y_{x,D} + \varepsilon D B_\infty^n \in \mathcal{P}$ such that $x \in P$.

Theorem 3.8 implies that with probability at least $1 - e^{-k\ln(eN/k)} - e^{-\delta N/4}$ there exists $D \in Q$ such that

$$\Gamma(\varepsilon D B_\infty^n) \subset C\,\varepsilon \sqrt{\frac{kn}{\delta} \ln\left(\frac{eN}{k}\right)}\, \mathbf{B}_{k,2}.$$

Therefore, for such D,

$$\Gamma(x - y_{x,D}) \in \Gamma(\varepsilon D B_\infty^n) \subset C\,\varepsilon \sqrt{\frac{kn}{\delta} \ln\left(\frac{eN}{k}\right)}\, \mathbf{B}_{k,2},$$

hence,

$$\Gamma(P) \subset \Gamma x + \Gamma(y_{x,D} - x) + \Gamma(\varepsilon D B_\infty^n) \subset \Gamma x + 2C\,\varepsilon \sqrt{\frac{kn}{\delta} \ln\left(\frac{eN}{k}\right)}\, \mathbf{B}_{k,2}.$$

This proves the existence of a "good" collection $\mathcal{P}$.

Finally, let $\mathcal{P}'$ be the set of all $P \in \mathcal{P}$ such that $P \cap T \neq \emptyset$. For every $P \in \mathcal{P}'$ choose an arbitrary $z_P \in P \cap T$ and let $\mathcal{N} = \{z_P\}_{P \in \mathcal{P}'}$. By above, for every $x \in T$ there exists $D \in Q$ and $P = y_{x,D} + \varepsilon D B_\infty^n \in \mathcal{P}$ such that $x \in P$, in particular $P \in \mathcal{P}'$, and

$$\Gamma(P) \subset \Gamma x + 2C\,\varepsilon \sqrt{\frac{kn}{\delta} \ln\left(\frac{eN}{k}\right)}\, \mathbf{B}_{k,2}.$$

Thus, $\Gamma z_P \in \Gamma x + 2C\,\varepsilon \sqrt{\frac{kn}{\delta} \ln\left(\frac{eN}{k}\right)}\, \mathbf{B}_{k,2}$. This implies the desired result. $\square$

Remark 3.10 We apply Theorem 3.3 for $T \subset t B_2^n$, $t \geq 1$, $\varepsilon \leq 1/\sqrt{n}$, and $\delta \geq n/(2N)$ so that, $F(\delta, n, N) = (32\delta N/n)^n$. Then Theorem 3.3 combined with Lemma 3.5 implies that there exists $\mathcal{N} \subset T$ with cardinality at most

$$\left(\frac{224\delta t N}{\varepsilon n^{3/2}}\right)^n e^{\delta N}$$

such that with probability at least $1 - e^{-k\ln(eN/k)} - e^{-\delta N/4}$ one has

$$\forall x \in T \ \exists y \in \mathcal{N} \quad \text{such that} \quad \Gamma(x-y) \in C\,\varepsilon\sqrt{\frac{kn}{\delta}\ln\left(\frac{eN}{k}\right)}\,\mathbf{B}_{k,2}.$$

3.4 Geometry of Random Polytopes

In this section, we study some classical geometric parameters associated to random polytopes of the form $K_N := \Gamma^* B_1^N$, where $\Gamma = \{\xi_{ij}\}_{1\le i\le N, 1\le j\le n}$ is an $N\times n$ random matrix. In other words, K_N is the absolute convex hull of the rows of Γ. We provide estimates on the asymptotic behavior of the volume and the mean widths of K_N and its polar. In this section, the random operator Γ satisfies the hypothesis (3.4): the random variables ξ_{ij} are independent symmetric with unit variances such that in each row of Γ the entries are identically distributed. Moreover, we assume that the random variables ξ_{ij} satisfy a uniform small ball probability condition which means that we can fix $u, v \in (0,1)$ such that

$$\forall i,j \quad \sup_{\lambda\in\mathbb{R}} \mathbb{P}\big(|\xi_{ij} - \lambda| \le u\big) \le v.$$

3.4.1 Inclusion Theorem

We start by showing that for an $N\times n$ random matrix Γ satisfying conditions described above, the body $K_N = \Gamma^* B_1^N$ contains a large "regular" body with high probability.

Theorem 3.11 *Let $\beta \in (0,1)$. There are two positive constants $M = M(u,v,\beta)$ and $C(u,v,\beta)$ which depend only on u, v, β and an absolute constant $c > 0$, such that the following holds. For every positive integers n, N satisfying $N \ge Mn$ one has*

$$\mathbb{P}\left(K_N \supset C(u,v,\beta)\left(B_\infty^n \cap \sqrt{\ln(N/n)}\,B_2^n\right)\right) \ge 1 - 4\exp\left(-cn^\beta N^{1-\beta}\right).$$

Remark 3.12 It is known that for a Gaussian random matrix one has

$$\mathbb{P}\left(K_N \supset C\sqrt{\beta \ln(N/n)}\ B_2^n\right) \geq 1 - 3\exp\left(-cn^\beta N^{1-\beta}\right),$$

where C, c are absolute positive constants (see e.g. [62]). Moreover, the probability estimate cannot be improved. Indeed, for a Gaussian random matrix and $\beta \in (0, c'')$ one has

$$\mathbb{P}\left(K_N \supset C'\sqrt{\beta \ln(N/n)} B_2^n\right) \leq 1 - \exp\left(-c' n^\beta N^{1-\beta}\right),$$

where $C', c' > 0$ and $0 < c'' \leq 1$ are absolute constants.

Since $B_\infty^n \subset \sqrt{n} B_2^n$, Theorem 3.11 has the following consequence.

Corollary 3.13 *Under the assumptions and notations of Theorem 3.11, for $Mn < N \leq e^n$ one has*

$$\mathbb{P}\left(K_N \supset C(u,v,\beta)\sqrt{\frac{\ln(N/n)}{n}}\ B_\infty^n\right) \geq 1 - 4\exp\left(-cn^\beta N^{1-\beta}\right).$$

In fact, our proof of Theorem 3.11 gives that if

$$N \geq n \max\left\{\exp(4C_v/\beta), \left(\frac{C\ln(e/(1-\beta))}{c_{uv}\,(1-\beta)}\right)^{1/(1-\beta)}\right\},$$

where $C > 1$ is an absolute positive constant, $c_{uv} = cuv\sqrt{1-v}$ is the constant from Lemma 3.14 below, and $C_v = 5\ln(2/(1-v))$, then

$$\mathbb{P}\left(K_N \supset \frac{c_{uv}}{2\sqrt{2}}\ (B_\infty^n \cap RB_2^n)\right) \geq 1 - 4\exp\left(-\frac{n^\beta N^{1-\beta}}{40}\right) \tag{3.9}$$

with $R = \sqrt{\beta \ln(N/n)/C_v}$. Note that $K_N = \text{abs conv}\{x_j\}_{j\leq N}$, where $x_j = \Gamma^* e_j$ are the columns of Γ^*. Hence for every $z \in \mathbb{R}^n$,

$$h_{K_N}(z) = \sup_{j\leq N} |\langle z, x_j\rangle| = \|\Gamma z\|_\infty.$$

Let $L = c_{uv}(B_\infty^n \cap RB_2^n)$. To prove (3.9), we show that

$$\mathbb{P}\left(\exists z \in \partial L^\circ \ : \ \|\Gamma z\|_\infty < \frac{1}{4}\right) \leq 4 \exp\left(-\frac{n^\beta N^{1-\beta}}{40}\right). \tag{3.10}$$

The proof of this statement will be divided into two steps. First, we will show an individual estimate for a fixed $z \in \partial L^\circ$. Then we use the net introduced in Theorem 3.3 to get a global estimate for any point of this net, using that this net is a subset of ∂L°. A crucial point is that this net is a good covering of $\Gamma(\partial L^\circ)$ in $\|\cdot\|_{k,2}$-metric.

3.4.1.1 Basic facts about small ball probabilities

The following lemma is a consequence of Rogozin's theorem [123] that was used for example in [120] (see Lemma 4.7 there).

Lemma 3.14 *Let $\xi_1, ..., \xi_m$ be independent random variables satisfying (1.1) with the same $u, v \in (0,1)$. Then for every $x \in S^{m-1}$ one has*

$$\mathcal{Q}\big(\sum_{i=1}^m x_i \xi_i, c_{uv}\big) \leq v,$$

where $c_{uv} = cuv\sqrt{1-v}$ and $c \in (0,1]$ is an absolute constant.

Remark 3.15 If we have a bounded moment of order larger than 2, then we could use a consequence of the Paley-Zygmund inequality, which also provides a lower bound on the small ball probability of a random sum. The following statement was proved in [87, Lemma 3.1] following the lines of [85, Lemma 3.6] with appropriate modifications to deal with centered random variables (rather than symmetric):

Let $2 < r \leq 3$ and $\mu \geq 1$. Suppose $\xi_1, \ldots, \xi_m$ are independent centered random variables such that $\mathbb{E}|\xi_i|^2 \geq 1$ and $\mathbb{E}|\xi_i|^r \leq \mu^r$ for every $i \leq m$. Let $x = (x_i) \in \ell_2$ be such that $\|x\|_2 = 1$. Then for every $\lambda \geq 0$

$$\mathbb{P}\left(\Big|\sum_{i=1}^m \xi_i x_i\Big| > \lambda\right) \geq \left(\frac{1-\lambda^2}{8\mu^2}\right)^{r/(r-2)}. \tag{3.11}$$

Proof of Lemma 3.14. Fix $x \in S^{m-1}$. We clearly have $\mathcal{Q}\big(x_i\xi_i, |x_i|u\big) \le v$ for every $x_i \neq 0$. Applying Theorem 1 of [123] to random variables $x_i\xi_i$, $i \le m$, we observe there exists and absolute constant $C \ge 1$ such that for every $w \ge u\|x\|_\infty/2$,

$$\mathcal{Q}\big(\sum_{i=1}^m x_i\xi_i, w\big) \le \frac{Cw}{\sqrt{\sum_{i=1}^m |x_i|^2 u^2\big(1 - \mathcal{Q}(x_i\xi_i, |x_i|u)\big)}} \le \frac{Cw}{u\sqrt{1-v}}.$$

Take $w = uv\sqrt{1-v}/C$. If $\|x\|_\infty \le 2v\sqrt{1-v}/C$ then $w \ge u\|x\|_\infty/2$. Therefore for such x we have

$$\mathcal{Q}\left(\sum_{i=1}^m x_i\xi_i, w\right) \le v.$$

If there exists $\ell \le m$ such that $|x_\ell| > 2v\sqrt{1-v}/C$, then we have

$$\mathcal{Q}\left(\sum_{i=1}^m x_i\xi_i, w\right) \le \mathcal{Q}\big(x_\ell\xi_\ell, w\big) = \mathcal{Q}\big(\xi_\ell, w/|x_\ell|\big) \le \mathcal{Q}\big(\xi_\ell, u\big) \le v,$$

which completes the proof. ∎

3.4.1.2 The individual small ball estimate

To prove Theorem 3.11 we need to extend a result by Montgomery-Smith [108], which originally was proved for Rademacher random variables. Note that this lemma does not require any conditions on the moments of random variables.

Lemma 3.16 *Let ξ_i, $i \le n$, be independent symmetric random variables satisfying condition (1.1). Let $\alpha \ge 1$ and $L = c_{uv}(B_\infty^n \cap \alpha B_2^n)$, where c_{uv} is the constant from Lemma 3.14. Then for every non-zero $z \in \mathbb{R}^n$ one has*

$$\mathbb{P}\left(\sum_{i=1}^n \xi_i z_i > h_L(z)\right) > ((1-v)/2)^{5\alpha^2}.$$

We postpone the proof of this lemma to the end of this section. Note that if our variables satisfy $1 \le \mathbb{E}\xi_i^2 \le \mathbb{E}|\xi_i|^r \le \mu^r$ for some $r > 2$ then using (3.11) and repeating the proof of Lemma 4.3 from [85] we could consider $L = (1-\delta)(B_\infty^n \cap \alpha B_2^n)$ and estimate the corresponding probability from below by $\exp(-C_{\mu,\delta,r}\alpha^2)$, where $C_{\mu,\delta,r}$ depends only on μ, δ, r.

Lemma 3.16 has the following consequence.

Lemma 3.17 *Under assumptions of Lemma 3.16 for every $z \in \mathbb{R}^n$ and every $\sigma \subset [N]$ one has*

$$\mathbb{P}\left(\|P_\sigma \Gamma z\|_\infty < h_L(z)\right) < \exp\left(-|\sigma| \exp(-C_v \alpha^2)\right),$$

where $P_\sigma : \mathbb{R}^N \to \mathbb{R}^\sigma$ is the coordinate projection and $C_v = 5\ln(2/(1-v))$.

Proof. Applying Lemma 3.16 to the $|\sigma| \times n$ random matrix $P_\sigma \Gamma = (\xi_{ij})_{i\in\sigma, j\le n}$ we have for every $z = \{z_j\}_{j=1}^n \in \mathbb{R}^n$ and every $i \in \sigma$

$$\mathbb{P}\left(\sum_{j=1}^n z_j \xi_{ij} < h_L(z)\right) \le 1 - \exp(-C_v\alpha^2) \le \exp\left(-\exp(-C_v\alpha^2)\right).$$

Thus

$$\begin{aligned}\mathbb{P}\left(\|P_\sigma \Gamma z\|_\infty < h_L(z)\right) &= \mathbb{P}\left(\sup_{i\in\sigma}\left|\sum_{j=1}^n z_j\xi_{ij}\right| < h_L(z)\right) \\ &= \prod_{i\in\sigma}\mathbb{P}\left(\left|\sum_{j=1}^n z_j\xi_{ij}\right| < h_L(z)\right) < \exp\left(-|\sigma|\exp(-C_v\alpha^2)\right).\end{aligned}$$

■

We can now state the main individual small ball estimate.

Lemma 3.18 *Let $\beta \in (0,1)$ and define $m = 8\lceil (N/n)^\beta \rceil$ (if the latter number is larger than or equal to $N/4$ we take $m = N$) and $k = \lfloor N/m \rfloor$. Let $L = c_{uv}(B_\infty^n \cap RB_2^n)$, where $R = \sqrt{\beta \ln(N/n)/C_v}$. Then for any $z \in \partial L^o$ one has*

$$\mathbb{P}\left(\frac{1}{\sqrt{k}}\|\Gamma z\|_{k,2} < \frac{1}{2}\right) \le \exp(-0.3\, n^\beta N^{1-\beta}).$$

Proof. Below we assume $m < N/4$ (then $k \ge 4$, hence $km > 4N/5$); the proof in the case $m = N$, $k = 1$ repeats the same lines with simpler calculations. Let $\sigma_1, \ldots, \sigma_k$ be a partition of $[N]$ such that $m \leq |\sigma_i|$ for every $i \leq k$. Then, for any $a \in \mathbb{R}^N$

$$\frac{1}{\sqrt{k}}\|a\|_{k,2} \geq \frac{1}{\sqrt{k}}\left(\sum_{i=1}^{k}\|P_i a\|_\infty^2\right)^{1/2} \geq \frac{1}{k}\sum_{i=1}^{k}\|P_i z\|_\infty,$$

where $P_i = P_{\sigma_i} : \mathbb{R}^N \to \mathbb{R}^{\sigma_i}$ is the coordinate projection. Define $|||\cdot|||$ on $\mathbb{R}^N$ by

$$|||z||| = \frac{1}{k}\sum_{i=1}^{k}\|P_i z\|_\infty$$

for every $z \in \mathbb{R}^N$. Note that if for some $z \in \mathbb{R}^n$ we have $|||\Gamma z||| < h_L(z)/2$ then there exists $I \subset [k]$ of cardinality at least $k/2$ such that for every $i \in I$ one has $\|P_i\Gamma z\|_\infty < h_L(z)$. Applying Lemma 3.17 with $\alpha = R$ (note that $\alpha \geq 2$, by the condition on n and N), we obtain for every $z = \{z_i\}_{i=1}^n \in \mathbb{R}^n$,

$$\begin{aligned}
\mathbb{P}\left(|||\Gamma z||| < h_L(z)/2\right) &\leq \sum_{|I|=[(k+1)/2]} \mathbb{P}\left(\|P_i\Gamma z\|_\infty < h_L(z) \ \text{ for every } \ i \in I\right)\\
&\leq \sum_{|I|=[(k+1)/2]} \prod_{i\in I}\mathbb{P}\left(\|P_i\Gamma z\|_\infty < h_L(z)\right)\\
&\leq \sum_{|I|=[(k+1)/2]} \prod_{i\in I}\exp\left(-|\sigma_i|\exp(-C_v\alpha^2)\right)\\
&\leq \binom{k}{[k/2]}\exp\left(-(km/2)\exp(-C_v\alpha^2)\right)\\
&\leq \exp\left(k\ln 2 - (km/2)\exp(-C_v\alpha^2)\right),
\end{aligned}$$

where $C_v = 5\ln(2/(1-v))$. By our choice of k and m we have $km > 4N/5$, therefore $(km/2)\exp(-C_v\alpha^2) \geq 2N^{1-\beta}n^\beta/5$. We also have $k \leq N^{1-\beta}n^\beta/8$. Thus

$$\mathbb{P}\left(|||\Gamma z||| < h_L(z)/2\right) \leq \exp\left(-0.3\ N^{1-\beta}n^\beta\right).$$

This completes the proof. ■

Finally we prove Lemma 3.16. For a positive integer m, define $|||\cdot|||_m$ on $\mathbb{R}^n$ by

$$|||z|||_m = \sup \sum_{i=1}^{m} \left(\sum_{k \in B_i} |z_k|^2 \right)^{1/2},$$

where the supremum is taken over all partitions $B_1, \ldots, B_m$ of $[n]$. We will need the following lemma, which was essentially proved in [108] (see Lemma 2 there).

Lemma 3.19 *Let $\alpha \geq 1$ and $m \geq 1 + 4\alpha^2$ be an integer. For all $x \in \mathbb{R}^n$ one has*

$$h_{B_\infty^n \cap \alpha B_2^n}(x) \leq |||x|||_m.$$

Proof. Fix $x \in \mathbb{R}^n$ and choose $y \in B_\infty^n \cap \alpha B_2^n$ so that $h(x) = \sum_i x_i y_i$. For every k with $y_k^2 \geq 1/2$ choose $B_{1,k} = \{k\}$. Since $|y| \leq \alpha$ there are at most $2\alpha^2$ such sets. Denote $B := \cup_k B_{1,k}$. Now let z_i denote y_i if $|y_i| \leq 1/\sqrt{2}$ and $z_i = 0$ otherwise. Let $n_0 = 0$ and define $n_0 < n_1 < n_2 < \ldots$ by

$$n_{k+1} = 1 + \sup \left\{ \ell \in [n_k + 1, n - 1] \mid \sum_{i=n_k+1}^{\ell} z_i^2 \leq 1/2 \right\}$$

(if $n_k = n$ we stop the procedure). Denote $B_{2,k} := [n_{k-1} + 1, n_k] \setminus B$. Since $|y| \leq \alpha$ we have at most $2\alpha^2 + 1$ such sets. Moreover, we have

$$\sum_{i \in B_{2,k}} z_i^2 = \sum_{i \in B_{2,k}} y_i^2 \leq 1.$$

Since $y \in B_\infty^n$ and $m \geq 4\alpha^2 + 1$, we obtain

$$h(x) = \sum_{i=1}^{n} x_i y_i \leq \sum_{j=1}^{2} \sum_{k} \left(\sum_{i \in B_{j,k}} x_i^2 \right)^{1/2} \left(\sum_{i \in B_{j,k}} y_i^2 \right)^{1/2} \leq \sum_{j \leq 2, k} \left(\sum_{i \in B_{j,k}} x_i^2 \right)^{1/2} \leq |||x|||_m.$$

■

Proofof Lemma 3.16. We folow the lines of Montgomery-Smith's proof. Let $m =$

$\lceil 1+4\alpha^2 \rceil$. Given $z \in \mathbb{R}^n$, let $m' \le m$ and $B_1, \dots, B_{m'}$ be a partition of $[n]$ such that

$$\forall i \le m' \ \sum_{k\in B_i} |z_k|^2 \neq 0 \qquad \text{and} \qquad |||z|||_m = \sum_{i=1}^{m'} \left(\sum_{k\in B_i} |z_k|^2 \right)^{1/2}.$$

Then, using Lemma 3.19, we have

$$\begin{aligned} p := \mathbb{P}\left(\sum_{i=1}^{n} \xi_i z_i > h_L(z) \right) &\geq \mathbb{P}\left(\sum_{i=1}^{n} \xi_i z_i > c_{uv}\, |||z|||_m \right) \\ &= \mathbb{P}\left(\sum_{i=1}^{m'} \sum_{k\in B_i} \xi_k z_k > c_{uv} \sum_{i=1}^{m'} \left(\sum_{k\in B_i} |z_k|^2 \right)^{1/2} \right) \\ &\geq \mathbb{P}\left(\bigcap_{i\le m'} \left(\sum_{k\in B_i} \xi_k z_k \geq c_{uv}\, (\sum_{k\in B_i} |z_k|^2)^{1/2} \right) \right). \end{aligned}$$

Since ξ_i's are independent we obtain

$$p \geq \prod_{i=1}^{m'} \mathbb{P}\left(\sum_{k\in B_i} \xi_k z_k > c_{uv} \left(\sum_{k\in B_i} |z_k|^2 \right)^{1/2} \right).$$

For $i \le m'$ set

$$f_i = \left(\sum_{k\in B_i} \xi_k z_k \right) \cdot \left(\sum_{k\in B_i} |z_k|^2 \right)^{-1/2}.$$

Using that ξ_i's are symmetric and applying Lemma 3.14 we get

$$\mathbb{P}\left(f_i > c_{uv}\right) = \frac{1}{2}\, \mathbb{P}\left(|f_i| > c_{uv}\right) \geq \frac{1-v}{2}.$$

Therefore,

$$p \geq ((1-v)/2)^{m'} \geq ((1-v)/2)^{m} \geq ((1-v)/2)^{5\alpha^2},$$

which implies the desired result. ∎

3.4.1.3 The global small ball estimate

In this section, we prove Theorem 3.11. As we mentioned after its statement, our goal is to prove (3.10) for $N \geq Mn$, where M depends only on β, u and v.

Let $\beta \in (0,1)$ and, as in Lemma 3.18, define $m = 8\lceil (N/n)^{\beta} \rceil$ and $k = \lfloor N/m \rfloor$ so that $N^{1-\beta} n^{\beta}/10 \leq k \leq N^{1-\beta} n^{\beta}/8$. By the choice of M, we obviously have $k \ln(eN/k) \geq n$. Let $T = \partial L^{\circ}$ and set

$$\delta = 0.1(n/N)^{\beta} \quad \text{ and } \quad \varepsilon = \frac{1}{c_{uv}\sqrt{n}\,\exp((N/n)^{1-\beta}/20)}.$$

Since

$$T \subset L^{\circ} = c_{uv}^{-1}\big(\operatorname{conv} B_1^n \cup (B_2^n/R)\big) \subset c_{uv}^{-1} B_2^n,$$

we use Theorem 3.3 (see Remark 3.10) to construct a set $\mathcal{N} \subset T$ of cardinality at most

$$\left(\frac{224\delta N}{\varepsilon c_{uv} n^{3/2}}\right)^n e^{\delta N}$$

such that with probability at least $1 - e^{-k\ln(eN/k)} - e^{-\delta N/4}$ one has

$$\forall x \in T\ \exists z \in \mathcal{N} \quad \text{ such that } \quad \|\Gamma(x-z)\|_{k,2} \leq C_1\,\varepsilon \sqrt{\frac{kn}{\delta} \ln\left(\frac{eN}{k}\right)}, \tag{3.12}$$

where $C_1 > 0$ is an absolute constant. Since

$$\exp\left(n\ln(224\delta N/n) + n\ln(1/(\varepsilon c_{uv} n^{1/2})) + \delta N - 0.3\, N^{1-\beta} n^{\beta}\right) \leq \exp\left(-0.1\, N^{1-\beta} n^{\beta}\right),$$

provided that $(N/n)^{1-\beta}$ is large enough, and $\mathcal{N} \subset T$, we deduce from Lemma 3.18 that

$$\mathbb{P}\left(\exists z \in \mathcal{N} \,:\, \frac{1}{\sqrt{k}}\|\Gamma z\|_{k,2} < 1/2\right) \leq \sum_{z \in \mathcal{N}} \mathbb{P}\left(\frac{1}{\sqrt{k}}\|\Gamma z\|_{k,2} < 1/2\right) \leq \exp\left(-0.1\, N^{1-\beta} n^{\beta}\right).$$

Let $\overline{\Omega}$ be the subset of Ω, where (3.12) holds. Then, on $\overline{\Omega}$, for every $x \in T$ there exists

$z \in \mathcal{N}$ such that

$$\frac{1}{\sqrt{k}}\|\Gamma z\|_{k,2} \le \frac{1}{\sqrt{k}}\|\Gamma x\|_{k,2} + \frac{1}{\sqrt{k}}\|\Gamma(z-x)\|_{k,2} \le \frac{1}{\sqrt{k}}\|\Gamma x\|_{k,2} + C_1\varepsilon\sqrt{\frac{n}{\delta}\ln\left(\frac{eN}{k}\right)}.$$

$$\le \frac{1}{\sqrt{k}}\|\Gamma x\|_{k,2} + \frac{C_2\sqrt{\left(\frac{N}{n}\right)^{\beta}\ln\left(10e\left(\frac{N}{n}\right)^{\beta}\right)}}{c_{uv}\,\exp((N/n)^{1-\beta}/20)},$$

where C_2 is an absolute positive constant. Since $N \ge Mn$ (for large enough M depending only on u, v and β), we observe

$$c_{uv}^2\exp((N/n)^{1-\beta}/10) > 16\,C_2^2\left(\frac{N}{n}\right)^{\beta}\ln\left(10e\left(\frac{N}{n}\right)^{\beta}\right).$$

Therefore,

$$\mathbb{P}\left(\left\{\omega \in \overline{\Omega}:\ \exists x \in \partial L^o \text{ such that } \frac{1}{\sqrt{k}}\|\Gamma x\|_{k,2} < \frac{1}{4}\right\}\right)$$
$$\le \mathbb{P}\left(\left\{\omega \in \overline{\Omega}:\ \exists z \in \mathcal{N} \text{ such that } \frac{1}{\sqrt{k}}\|\Gamma z\|_{k,2} < \frac{1}{2}\right\}\right) \le \exp\left(-0.1\ N^{1-\beta}n^{\beta}\right).$$

The desired result follows since $h_{K_N}(x) = \|\Gamma x\|_\infty \ge \frac{1}{\sqrt{k}}\|\Gamma x\|_{k,2}$ for every $x \in \mathbb{R}^n$ and since

$$\mathbb{P}(\overline{\Omega}) \ge 1 - e^{-k\ln(eN/k)} - e^{-\delta N/4} \ge 1 - 2\exp(N^{1-\beta}n^{\beta}/40).$$

3.4.2 Volumes and mean widths of K_N and K_N°

In this section we apply the results of the previous subsection to obtain asymptotically sharp estimates for the volumes and the mean widths of K_N and K_N°. We refer to [117] for general knowledge about these parameters. We recall that by Santaló inequality and Bourgain-Milman [27] inverse Santaló inequality there exists an absolute positive constant c such that for every convex symmetric body K one has

$$c^n\mathrm{vol}_n(B_2^n)^2 \le \mathrm{vol}_n(K)\mathrm{vol}_n(K^\circ) \le \mathrm{vol}_n(B_2^n)^2. \tag{3.13}$$

Below we fix constants $M = M(u, v, \beta)$ and $C(u, v, \beta)$ from Theorem 3.11.

We start estimating the volumes of K_N and K_N°. For convenience we separate upper and lower estimates (some bounds require an additional condition on the matrix Γ). Corollary 3.13 and (3.13) imply the following volume estimates for K_N and K_N°.

Theorem 3.20 *Let $Mn < N \leq e^n$, $\beta \in (0,1)$. There exists absolute positive constants C and c such that with probability at least $1 - \exp\left(-cn^\beta N^{1-\beta}\right)$ one has*

$$\mathrm{vol}_n(K_N)^{1/n} \geq 2C(u,v,\beta)\sqrt{\frac{\ln(N/n)}{n}} \qquad \textit{and} \qquad \mathrm{vol}_n(K_N^\circ)^{1/n} \leq \frac{C}{C(u,v,\beta)\sqrt{n\ln(N/n)}}.$$

To prove the remaining bounds on volumes of K_N and K_N° we introduce one more condition on the matrix Γ, namely we require that

$$\mathbb{P}\left(\max_{i\leq N} |\Gamma^* e_i| > \lambda\sqrt{n}\right) \leq p_0 \tag{3.14}$$

for some $0 < p_0 < 1$ and $\lambda \geq 1$. Such condition holds for example when entries of Γ are i.i.d. centered random variables with finite p-th moment for some $p > 4$, provided that $N \leq C_p n^{p/4}$ (this can be proved using Rosenthal's inequality, see Corollary 6.4 in [69]).

The lower bound on $\mathrm{vol}_n(K_N)$ (and the upper bound on $\mathrm{vol}_n(K_N^\circ)$) follows from (3.13) and a well known estimate on the volume of the convex hull of k points ([7], [34], [62]):

Let $2n \leq k \leq e^n$ and $z_1, \ldots, z_k \in S^{n-1}$, then

$$|\mathrm{abs\ conv}\{z_i\}_{i\leq k}|^{1/n} \leq c\sqrt{\ln(k/n)}/n,$$

where $c > 0$ is an absolute constant.

Theorem 3.21 *Let $Mn < N \leq e^n$ and $\beta \in (0,1)$. Assume that the matrix Γ satisfies*

(3.14). There exist absolute positive constants c and C such that one has

$$\mathrm{vol}_n(K_N)^{1/n} \le C\lambda\sqrt{\frac{\ln(N/n)}{n}} \qquad \text{and} \qquad \mathrm{vol}_n(K_N^\circ)^{1/n} \ge c/(\lambda\sqrt{n\ln(N/n)})$$

with probability at least $1-p_0$.

An important geometric parameter associated to a convex body is the (half of) mean width of K° defined by

$$M_K = M(K) = \int_{S^{n-1}} \|x\|_K \, d\nu,$$

where ν is the normalized Lebesgue measure on S^{n-1}. It is well known that there exists a constant $c_n > 1$ ($c_n \to 1$ as $n \to \infty$) such that

$$M_K = \frac{c_n}{\sqrt{n}}\mathbb{E}\|\sum_{i=1}^n e_i g_i\|_K,$$

for every $K \subset \mathbb{R}^n$. The (half of) mean width of K, $M(K^\circ)$, we denote by $M_K^* = M^*(K)$. Observe that

$$M^*(K) = \frac{c_n}{\sqrt{n}}\mathbb{E}\|\sum_{i=1}^n e_i g_i\|_{K^0} = \frac{c_n}{\sqrt{n}}\mathbb{E}\sup_{t\in K}\sum_{i=1}^n t_i g_i = \frac{c_n}{\sqrt{n}}\ell_*(K),$$

where $\ell_*(K) = \mathbb{E}\sup_{t\in K}\sum_{i=1}^n t_i g_i$ is the Gaussian complexity measure of the convex body K. We recall the following inequality, which holds for every convex body K (see e.g. [117])

$$M_K^* \ge \left(\frac{\mathrm{vol}_n(K)}{\mathrm{vol}_n(B_2^n)}\right)^{1/n} \ge 1/M_K. \tag{3.15}$$

Now we calculate the mean widths $M(K_N)$ and $M(K_N^\circ)$.

Theorem 3.22 *Let $Mn < N \le e^n$ and $\beta \in (0,1)$. Then*

$$M(K_N) \le CC^{-1}(u,v,\beta)\left(\sqrt{(\ln(2n))/n} + 1/\sqrt{\ln(N/n)}\right)$$

with probability at least $1-\exp\left(-cn^{\beta}N^{1-\beta}\right)$, *where* C *and* c *are absolute positive constants. Moreover, if the matrix* Γ *satisfies (3.14), then there exists an absolute positive constant* c_1 *such that with probability at least* $1-p_0$ *one has*

$$M(K_N) \geq c_1/(\lambda\sqrt{\ln(N/n)}).$$

Proof. By Theorem 3.11 we have

$$\begin{aligned} M(K_N) &\leq M\left(C(u,v,\beta)\left(B_\infty^n \cap \sqrt{\ln(N/n)}B_2^n\right)\right) \\ &\leq (1/C(u,v,\beta))\left(M\left(B_\infty^n\right)+M\left(\sqrt{\ln(N/n)}B_2^n\right)\right), \end{aligned}$$

which proves the upper bound.

By (3.15) and Theorem 3.21 there exists an absolute positive constant c_1 such that

$$M(K_N) \geq \left(\mathrm{vol}_n(B_2^n)/\mathrm{vol}_n(K_N)\right)^{1/n} \geq c_1/(\lambda\sqrt{\ln(2N/n)}),$$

with probability larger than or equal to $1-p_0$. This proves the lower bound. ∎

Remark 3.23 Note that by Theorem 3.22, for $N \leq \exp(n/\ln n)$ we have

$$M(K_N) \approx 1/\sqrt{\ln(N/n)}.$$

If $N \geq \exp(n/\ln(2n))$ there is a gap between lower and upper estimates. Both estimates could be asymptotically sharp as was shown in [85].

Theorem 3.24 *There exist positive absolute constants* c, c_0, *and* C *such that the following holds. Let* $Mn < N \leq e^n$. *Then*

$$M(K_N^\circ) \geq c_0\sqrt{\ln(N/n)}$$

with probability at least $1-\exp\left(-cn^{\beta}N^{1-\beta}\right)$. *Moreover, assuming that the matrix* Γ

satisfies (3.14), with probability at least $1-p_0$ *one has*

$$M(K_N^\circ) \le C\lambda\sqrt{\ln N}.$$

Proof. By (3.15) we have

$$M(K_N^\circ) \ge \left(\mathrm{vol}_n(B_2^n)/\mathrm{vol}_n(K_N^\circ)\right)^{1/n}.$$

Therefore, the lower bound follows by Theorem 3.20.

Let $G = \sum_{i=1}^n g_i e_i$. Recall that K_N is the absolute convex hull of N vertices $\Gamma^* e_i$. Thus we have

$$M(K_N^\circ) \le \frac{c_1}{\sqrt{n}} \mathbb{E}\|G\|_{K_N^0} = \frac{c_1}{\sqrt{n}} \mathbb{E}\max_{i\le N} \langle G, \Gamma^* e_i\rangle,$$

where c_1 is an absolute constant. Since with probability at least $1-p_0$ we have $|\Gamma^* e_i| \le \lambda\sqrt{n}$ for every $i \le N$, using standard estimate for the expectation of maximum of Gaussian random variables (see, e.g., [117]), we obtain that there is an absolute constant c_2 such that

$$M(K_N^0) \le c_2\lambda\,\sqrt{\ln N}$$

with probability larger than or equal to $1-e^{-n}$. ■

Finally we note that the bounds of Theorem 3.24 are sharp, whenever $\ln N$ and $\ln(N/n)$ are comparable, for example if $N > n^2$. However, when N is close to n we have a gap between upper and lower bounds. Below we provide a better lower bound for $M(K_N^\circ)$ in the case $N \le n^2$, which closes this gap. We will need two more conditions on the matrix Γ, namely

$$\mathbb{P}\left(\|\Gamma\|_{HS} < \sqrt{Nn}/2\right) \le p_1, \tag{3.16}$$

for some $p_1 \in (0,1)$ and where $\|\Gamma\|_{HS}$ denotes the Hilbert–Schmidt norm of Γ; and

$$\mathbb{P}\left(\|\Gamma\| > \mu\sqrt{N}\right) \le p_2, \tag{3.17}$$

for some $p_2 \in (0,1)$, $\mu \geq 1$ and where $\|\Gamma\|$ denotes the operator (spectral) norm of Γ. Both conditions are satisfied for example when entries of Γ are i.i.d. centered random variables with finite p-th moment for some $p > 4$. Indeed, Rosenthal's inequality (see proof of Corollary 6.4 in [69]) implies (3.16) with $p_1 \leq (C_p \mathbb{E}|\xi|^p)/(Nn)^{p/4}$; while Theorem 2.1 combined with Corollary 6.4 in [69] implies (3.17) with $\mu = C_p'$ and

$$p_2 \leq 1/N^{c_p} + (C_p \mathbb{E}|\xi|^p) N / n^{p/4}$$

(to make $p_2 < 1$ we have to ask $C_p \mathbb{E}|\xi|^p N \leq n^{p/4}$). We would like also to note that the proof below works also for $N \leq n^\alpha$ for some $\alpha \in (1,2]$ if we substitute the condition (3.17) with

$$\mathbb{P}\left(\|\Gamma\| > \mu (Nn)^\gamma\right) \leq p_2$$

for some $\gamma \in (0, 1/2)$, which could be the case in the absence of 4-th moment (see for example Corollary 2 in [3] and Remark 2 in [88]). Note also that the condition (3.17) implies (3.14), since $\|\Gamma\| \geq \max_{i \leq N} |\Gamma^* e_i|$.

Theorem 3.25 *Let $\mu \geq 1$, $n \geq 16\mu^2$, and $2n < N \leq n^2$ and assume that the matrix Γ satisfies conditions (3.16) and (3.17) for some $p_1, p_2 \in (0,1)$. Then with probability at least $1 - p_1 - p_2$*

$$M(K_N^\circ) \geq c\sqrt{\ln(n/(8\mu^2))}.$$

Proof. We apply Vershynin's extension [151] of Bourgain-Tzafriri theorem [28]. Denote $A = \|\Gamma^*\|_{HS}$, $B = \|\Gamma^*\|$. Vershynin's theorem implies that there exists $\sigma \subset \{1, \ldots, N\}$ of cardinality at least $A^2/(2B^2)$ such that for all $i \in \sigma$ one has $|\Gamma^* e_i| \geq c_3 A/\sqrt{N}$, where c_3 is an absolute positive constant, and vectors $\Gamma^* e_i$, $i \in \sigma$, are almost orthogonal (up to an absolute positive constant). Since Γ satisfies conditions (3.16) and (3.17), with probability at least $1 - p_1 - p_2$ we have $A \geq \sqrt{Nn}/2$ and $B \leq \mu\sqrt{N}$. Therefore, there exists $\sigma \subset \{1, \ldots, n\}$ of cardinality at least $n/(8\mu^2)$ such that $|\Gamma^* e_i| \geq c_3\sqrt{n}/2$ for $i \in \sigma$

and $\{\Gamma^* e_i\}_{i\in\sigma}$ are almost orthogonal. Then,

$$M(K_N^\circ) \geq \frac{1}{\sqrt{n}}\mathbb{E}\|G\|_{K_N^\circ} = \frac{1}{\sqrt{n}}\mathbb{E}\max_{i\leq N}\langle G, \Gamma^* e_i\rangle \geq \frac{1}{\sqrt{n}}\mathbb{E}\max_{i\in\sigma}\langle G, \Gamma^* e_i\rangle .$$

Since $\{\Gamma^* e_i\}_{i\in\sigma}$ are almost orthogonal, by Sudakov inequality (see, e.g., [117]), the last expectation is greater than $c_4\sqrt{\ln(n/(8\mu^2))}$, where c_4 is an absolute constant. This completes the proof. ∎

3.5 Smallest singular value

In this section we provide a simple short proof of a weaker inclusion, namely, we obtain a lower bound on the radius of the largest ball inscribed into K_N. It is based on a lower bound for the smallest singular value for tall matrices. Although such bounds are known with possibly better constants (see the last remark in [79] or the main theorem of [149]), we would like to emphasize a simple short proof, based on our Theorem 3.3. In fact our proof is close to the corresponding proofs in [85] and [87], however it is somewhat cleaner and it uses Theorem 3.3 instead of a standard net argument via the norm of an operator. We would also like to mention that very recently G. Livshyts has extended such results to rectangular random matrices with arbitrarily small aspect ratio [89].

In this section we assume that the random matrix Γ satisfies conditions described at the beginning of Section 3.4 with fixed $u, v \in (0,1)$. Recall that $c_{uv} = cuv\sqrt{1-v}$ is the constant from Lemma 3.14. It will be also convenient to fix two more constants depending on v,

$$\gamma_1 = \gamma_1(v) := \begin{cases} \sqrt{\ln 2} & \text{if } \ v \geq 1/2, \\ \sqrt{\ln \frac{1}{v}} & \text{if } \ v < 1/2 \end{cases} \qquad \text{and} \qquad \gamma_2 = \gamma_2(v) := \begin{cases} \ln \frac{2}{1+v} & \text{if } \ v \geq 1/2, \\ \ln \frac{1}{2v-v^2} & \text{if } \ v < 1/2. \end{cases}$$

Theorem 3.26 *There exist an absolute constant $C_0 > 1$ such that for $N \geq \left(\frac{C_0}{\gamma_2}\ln\frac{1}{c_{uv}}\right) n$*

one has

$$\mathbb{P}\left(s_n(\Gamma) \le \frac{c_{uv}\sqrt{\gamma_2}}{4\gamma_1}\sqrt{N}\right) \le 3\exp\left(-\min\{2,\gamma_2\}N/8\right).$$

Since

$$h_{\Gamma^* B_1^N}(x) = \|\Gamma^* x\|_\infty \quad \text{and} \quad K_N = \Gamma^* B_1^N \supset \frac{1}{\sqrt{N}}\Gamma^* B_2^N,$$

this theorem immediately implies the following inclusion.

Corollary 3.27 *For $N \ge \left(\frac{C_0}{\gamma_2}\ \ln\frac{1}{c_{uv}}\right) n$ one has*

$$\mathbb{P}\left(K_N \supset \frac{c_{uv}\sqrt{\gamma_2}}{4\gamma_1}\sqrt{N}B_2^n\right) \ge 1 - 3\exp\left(-\min\{2,\gamma_2\}N/8\right).$$

To prove Theorem 3.26 we first provide the individual bounds.

Proposition 3.28 *Let $1 \le n < N$. Then for every $x \in S^{n-1}$ one has*

$$\mathbb{P}\Big(\|\Gamma x\|_2 \le \frac{c_{uv}\sqrt{\gamma}_2}{2\gamma_1}\sqrt{N}\Big) \le \exp\left(-3\gamma_2 N/4\right).$$

Proof. Fix $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ with $\|x\|_2 = 1$. Denote $f_j := |\sum_{i=1}^n \xi_{ji}x_i|$, so that

$$\|\Gamma x\|_2^2 = \sum_{j=1}^N f_j^2.$$

Clearly $f_1, \ldots, f_N$ are independent. Therefore, for any $t, \tau > 0$ one has

$$\begin{aligned}\mathbb{P}\big(\|\Gamma x\|_2^2 \le t^2 N\big) &= \mathbb{P}\bigg(\sum_{j=1}^N f_j^2 \le t^2 N\bigg) = \mathbb{P}\bigg(\tau N - \frac{\tau}{t^2}\sum_{j=1}^N f_j^2 \ge 0\bigg)\\ &\le \mathbb{E}\exp\bigg(\tau N - \frac{\tau}{t^2}\sum_{j=1}^N f_j^2\bigg) = e^{\tau N}\prod_{j=1}^N \mathbb{E}\exp\bigg(-\frac{\tau f_j^2}{t^2}\bigg).\end{aligned}$$

Lemma 3.14 implies that $\mathbb{P}(f_j < c_{uv}) \le v$ for every $j \le N$. Write $\tau = t^2\eta/c_{uv}^2$ for some

$\eta > 0$. Then

$$
\begin{aligned}
\mathbb{E}\exp\Big(-\frac{\tau f_j^2}{t^2}\Big) &= \int_0^1 \mathbb{P}\Big(\exp\Big(-\frac{\eta f_j^2}{c_{uv}^2}\Big) > s\Big)\, ds \\
&= \int_0^{e^{-\eta}} \mathbb{P}\Big(\exp\Big(\frac{\eta f_j^2}{c_{uv}^2}\Big) < \frac{1}{s}\Big)\, ds + \int_{e^{-\eta}}^1 \mathbb{P}\Big(\exp\Big(\frac{\eta f_j^2}{c_{uv}^2}\Big) < \frac{1}{s}\Big)\, ds \\
&\le e^{-\eta} + \mathbb{P}(f_j < c_{uv})(1 - e^{-\eta}) \le e^{-\eta} + v(1 - e^{-\eta}).
\end{aligned}
$$

Choose $\eta = \gamma_1^2 = \ln \max\{2, 1/v\}$. Then the right hand side is $e^{-\gamma_2}$. Therefore

$$
\mathbb{P}\big(\|\Gamma x\|_2^2 \le t^2 N\big) \le e^{\tau N} e^{-\gamma_2 N} = \exp\left(-N(\gamma_2 - t^2\gamma_1^2/c_{uv}^2)\right).
$$

Choosing $t = \sqrt{\gamma_2} c_{uv}/(2\gamma_1)$ we complete the proof. ■

Proof of Theorem 3.26. Let $\delta = \min\{1, \gamma_2/2\}$. Note that $n/(2N) \le \delta \le 1$. Let $C \ge 1$ be the absolute constant from Theorem 3.3. Set

$$
\varepsilon := \frac{c_{uv}\sqrt{\gamma_2 \delta}}{4C\gamma_1\sqrt{n}} < \frac{1}{\sqrt{n}}.
$$

By Theorem 3.3 (see Remark 3.10), applied with $T = S^{n-1}$ and $k = N$, there exists a net $\mathcal{N} \subset B_2^n$ with cardinality at most

$$
\left(\frac{224\delta N}{\varepsilon n^{3/2}}\right)^n e^{\delta N} \le \left(\frac{896 C\gamma_1\sqrt{\delta}N}{c_{uv}\sqrt{\gamma_2}n}\right)^n e^{\delta N}
$$

such that with probability at least $1 - e^{-\delta N/4} - e^{-N}$ one has

$$
\forall x \in B_2^n\ \exists y_x \in \mathcal{N} \quad \text{such that} \quad \Gamma(x - y_x) \in C\varepsilon\sqrt{Nn/\delta}\, B_2^n = (c_{uv}\sqrt{\gamma_2}/(4\gamma_1))\sqrt{N}\, B_2^n.
$$

Condition on the corresponding event, denoted below by Ω_0. Assume that $x \in S^{n-1}$

satisfies $\|\Gamma x\|_2 \leq (c_{uv}\sqrt{\gamma_2}/(4\gamma_1))\sqrt{N}$. Then for the corresponding $y_x \in \mathcal{N}$ we have

$$\|\Gamma y_x\|_2 \leq \|\Gamma x\|_2 + \|\Gamma(y_x - x)\|_2 \leq (c_{uv}\sqrt{\gamma_2}/(2\gamma_1))\sqrt{N}.$$

This implies

$$q_0 := \mathbb{P}\left(\exists x \in S^{n-1} \ : \ \|\Gamma x\|_2 \leq \frac{c_{uv}\sqrt{\gamma_2}}{4\gamma_1}\sqrt{N}\right) \leq \mathbb{P}\left(\Omega_0^c\right) + \mathbb{P}\left(\exists y \in \mathcal{N} \mid \|\Gamma y\|_2 \leq \frac{c_{uv}\sqrt{\gamma_2}}{2\gamma_1}\sqrt{N}\right).$$

Applying Proposition 3.28 and using $\delta \leq \gamma_2/2$,

$$q_0 \leq 2e^{-\delta N/4} + \left(\frac{896C\gamma_1\sqrt{\delta}N}{c_{uv}\sqrt{\gamma_2}n}\right)^n \exp\left(-\gamma_2 N/4\right).$$

Using formulas for c_{uv}, γ_1, γ_2, and δ, it is not difficult to check that there exists an absolute constant $C_1 > 0$ such that

$$\ln \frac{896C\gamma_1\sqrt{\delta}}{c_{uv}\sqrt{\gamma_2}} \leq C_1 \ln \frac{1}{c_{uv}}.$$

Therefore there exists another absolute constant $C_2 > 0$ such that

$$\left(\frac{896C\gamma_1\sqrt{\delta}N}{c_{uv}\sqrt{\gamma_2}n}\right)^n \exp\left(-\gamma_2 N/4\right) \leq \exp\left(-\gamma_2 N/8\right),$$

provided that

$$N/n \geq (C_2/\gamma_2)\ \ln(1/c_{uv}).$$

This completes the proof. ■

Chapter 4

A Steiner formula in the L_p Brunn-Minkowski theory [3]

4.1 Introduction

The Brunn-Minkowski theory which classical building block is also called the theory of *mixed volumes* is the very core of convex geometric analysis. It centers around the study of geometric invariants and geometric measures associated with convex bodies. A central part of the theory is the classical Steiner formula.

A theory analogous and dual to the Brunn-Minkowski theory, called the theory of *dual mixed volumes* or *dual Brunn-Minkowski theory*, was introduced by Lutwak [93]. The main geometric invariants in the dual Brunn-Minkowski theory are the dual quermassintegrals $\widetilde{W}_i(K)$. As proved in [93], they appear as the coefficients in the Steiner formula of the dual Brunn-Minkowski theory where the Minkowski addition "$+$" of convex bodies is replaced by the radial addition "$\widetilde{+}$" of star bodies. Indeed, for

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a star body K and $t > 0$, we have that

$$\mathrm{vol}_n(K \,\widetilde{+}\, tB_2^n) = \sum_{i=0}^{n} \binom{n}{i} \widetilde{W}_i(K) t^i = \sum_{i=0}^{n} \binom{n}{i} \widetilde{V}_i(K) t^{n-i}.$$

The details are given in Section 4.3.1. Investigations in the dual Brunn-Minkowski theory led to isoperimetric inequalities and kinematic formulas involving dual mixed volumes, as summarized in [56, 132].

A localization of the quermassintegrals gives rise to curvature measures for Borel sets in $\mathbb{R}^n$, respectively area measures for Borel sets on the Euclidean unit sphere. The classical Minkowski problem asks to characterize those measures. Similarly, a localization of the dual quermassintegrals leads to the dual curvature measures [77, 93], and analogously to the dual Minkowski problem. Much work has been devoted to these problems. We refer to, e.g., [24, 25, 26, 36, 77, 118, 140, 170] for background and progress.

An extension of the classical Brunn-Minkowski theory, the L_p *Brunn-Minkowski theory*, was initiated by Lutwak in the groundbreaking paper [96]. This theory evolved rapidly over the last years and due to a number of highly influential works, see, e.g., [57, 71, 72, 73, 91, 92], [103] - [115], [134] - [158], it is now a central part of modern convex geometry. The L_p Brunn-Minkowski theory centers around the study of affine invariants associated with convex bodies. In fact, this theory redirected much of the research about convex bodies from the Euclidean aspects to the study of the affine geometry of these bodies, and some questions that had been considered Euclidean in nature turned out to be affine problems. For example, the famous Busemann-Petty Problem (finally laid to rest in [58, 167, 168]), was shown to be an affine problem with the introduction of intersection bodies by Lutwak in [95]. Central objects in the L_p Brunn-Minkowski theory are the L_p affine surface areas,

$$as_p(K) = \int_{\partial K} \frac{H_{n-1}(x)^{\frac{p}{n+p}}}{\langle x, \nu(x)\rangle^{\frac{n(p-1)}{n+p}}} d\mathcal{H}^{n-1}(x),$$

where $\nu(x)$ denotes the outer unit normal at $x \in \partial K$, the boundary of K, $H_{n-1}(x)$ is the Gauss curvature at x and $\mathcal{H}^{n-1}$ is the usual surface area measure on ∂K.

In this chapter we provide an analogue of the Steiner formula for the L_p affine surface area and s-th mixed p-affine surface area (see Section 4.6) of a Minkowski outer parallel body for any real parameters $p \neq -n$ and s, i.e. we investigate

$$as_p(K + tB_2^n) \quad \text{and} \quad as_{p,s}(K + tB_2^n).$$

A different Steiner formula for the L_p Brunn-Minkowski theory, involving Blaschke sum instead of Minkowski sum, was put forward by Lutwak in [94]. Our new Steiner formula, presented in Theorem 4.1 and 4.4, covers a whole range of Steiner formulas for all $-\infty \leq p \leq \infty$. It includes the classical Steiner formula and the Steiner formula from the dual L_p Brunn-Minkowski theory as special cases.

We call the coefficients in our Steiner formula L_p *Steiner coefficients.* In their general form they do not seem to have appeared before in the literature. Special cases of those include not only the classical quermassintegrals and the dual quermassintegrals, but also variants of the Willmore energy. This is explained in Section 4.5. We also observe a connection of the L_p Steiner coefficients to information theory. Analogously to the curvature measures of the classical theory and the dual curvature measures of the dual Brunn-Minkowski theory, the L_p Steiner coefficients lead to new curvature measures and area measures, respectively. They are discussed in more detail in Section 4.4. The Steiner formula for the s-th mixed L_p affine surface area is treated in Section 4.6.

4.2 Results

For a convex body K in $\mathbb{R}^n$, we define

$$\beta(K) = \min_{u \in S^{n-1}} h_K(u). \tag{4.1}$$

If K is such that its centroid is at the origin, which we will assume throughout, there is $\lambda(K) > 0$ such that $B^n(0, \lambda(K)) \subset K$. As K is bounded, there is $\lambda(K) \leq \Lambda(K) < \infty$ such that $K \subset B^n(0, \Lambda(K))$. Thus,

$$\lambda(K) \leq \beta(K) \leq \Lambda(K).$$

Theorem 4.1 *Let K be a convex body in $\mathbb{R}^n$ that is C^2_+ and let $t \in \mathbb{R}$ be such that $0 \leq t < \beta(K)$. For all $p \in \mathbb{R}$, $p \neq -n$,*

$$as_p(K + tB_2^n) = \sum_{m=0}^{\infty} \left[\sum_{k=m}^{\infty} \binom{\frac{n(1-p)}{n+p}}{k-m} t^k \int_{S^{n-1}} f_K(u)^{\frac{n}{n+p}} h_K(u)^{\frac{n(1-p)}{n+p} - k + m} A_p^m \, d\sigma(u) \right]. \tag{4.2}$$

When integrating over the sphere, we write $\sigma(u) = \mathcal{H}^{n-1}(u)$. The coefficients A_p^m represent a sum of mixed products of the elementary symmetric functions of the principal curvatures $H_i = H_i\left(\bar{\xi}_K(u)\right)$ (see Section 4.3.1 below), with corresponding multinomial coefficients. The detailed statement and local versions of it will be given in Section 4.4. We call the coefficients in the formula (4.2) the L_p Steiner coefficients

$$\mathcal{W}_{m,k}(K) = \int_{S^{n-1}} f_K(u)^{\frac{n}{n+p}} h_K(u)^{\frac{n(1-p)}{n+p} - k + m} A_p^m \, d\sigma(u). \tag{4.3}$$

The expression $f_K(u) h_K(u)^{1-p}$ is called the L_p *curvature function* $f_p(K, \cdot) : S^{n-1} \to [0, \infty)$. With this notation we can write

$$\mathcal{W}_{m,k}(K) = \int_{S^{n-1}} f_p(K, u)^{\frac{n}{n+p}} h_K(u)^{m-k} \; A_p^m \, d\sigma(u).$$

We want to point out that the first coefficient in the expansion (4.2) represents the

L_p affine surface area of the body K,

$$\mathcal{W}_{0,0}(K) = as_p(K).$$

As other special cases, we get mixed affine surface areas, defined in Section 4.6. More properties of the coefficients are discussed in Section 4.5, a connection to information theory among them.

The case $p = 0$ of Theorem A reduces to the classical Steiner formula which involves the classical quermassintegrals $W_i(K)$ (see (4.6) below).

Corollary 4.2 *Let K be a convex body in $\mathbb{R}^n$ that is C^2_+. Then*

$$as_0(K + tB_2^n) = n \sum_{i=0}^{n} \binom{n}{i} W_i(K)\ t^i.$$

In the case $p = \pm\infty$, Theorem A reduces to the Steiner formula from the dual Brunn-Minkowski theory involving Lutwak's dual quermassintegrals $\widetilde{W_i}(K)$ (see (4.8) below).

Corollary 4.3 *Let K be a convex body in $\mathbb{R}^n$ that is C^2_+ and let $t \in \mathbb{R}$ be such that $0 \le t < \beta(K)$. Then*

$$\begin{aligned} as_{\pm\infty}(K + tB_2^n) &= n\ \mathrm{vol}((K + tB_2^n)^\circ) = n\ \widetilde{W}_0((K + tB_2^n)^\circ) \\ &= n\ \sum_{i=0}^{\infty} \binom{-n}{i} \widetilde{W}_{-i}(K^\circ)\ t^i. \end{aligned}$$

Thus Theorem A covers a whole range of Steiner formulas in the L_p Brunn-Minkowski theory for all $-\infty \le p \le \infty$, including the classical case and a case related to the dual Brunn-Minkowski theory.

4.3 Background

4.3.1 Background from differential geometry

For more information and the details in this section we refer to e.g., [56, 132].

Let K be a convex body of class C^2. For a point x on the boundary ∂K of K we denote by $\nu(x)$ the unique outward unit normal vector of K at x. The map $\nu : \partial K \to S^{n-1}$ is called the spherical image map or Gauss map of K and is of class C^1. Its differential is called the Weingarten map. The eigenvalues of the Weingarten map are the principal curvatures $k_i(x)$ of K at x.

The j-th normalized elementary symmetric functions of the principal curvatures are denoted by H_j. They are defined as follows

$$H_j = \binom{n-1}{j}^{-1} \sum_{1\le i_1<\cdots<i_j\le n-1} k_{i_1}\cdots k_{i_j} \tag{4.4}$$

for $j = 1, \ldots, n-1$ and $H_0 = 1$. Note that

$$H_1 = \frac{1}{n-1} \sum_{1\le i\le n-1} k_i$$

is the mean curvature, that is the average of principal curvatures, and

$$H_{n-1} = \prod_{i=1}^{n-1} k_i$$

is the Gauss curvature.

We say that K is of class C^2_+ if K is of class C^2 and the Gauss map ν is a diffeomorphism. This means in particular that ν has a smooth inverse. This assumption is stronger than just C^2, and is equivalent to the assumption that all principal curvatures are strictly positive, or that the Gauss curvature $H_{n-1} \neq 0$. It also means that the differential of ν, i.e., the Weingarten map, is of maximal rank everywhere.

Let K be of class C^2_+. For $u \in \mathbb{R}^n \setminus \{0\}$, let $\xi_K(u)$ be the unique point on the boundary of K at which u is an outward normal vector. The map ξ_K is defined on $\mathbb{R}^n \setminus \{0\}$. Its restriction to the sphere S^{n-1}, the map $\bar{\xi}_K : S^{n-1} \to \partial K$, is called the reverse spherical image map, or reverse Gauss map. The differential of $\bar{\xi}_K$ is called the reverse Weingarten map. The eigenvalues of the reverse Weingarten map are called the principal radii of curvature $r_1, \ldots, r_{n-1}$ of K at $u \in S^{n-1}$.

The j-th normalized elementary symmetric functions of the principal radii of curvature are denoted by s_j. In particular, $s_0 = 1$, and for $1 \leq j \leq n-1$ they are defined by

$$s_j = \binom{n-1}{j}^{-1} \sum_{1 \leq i_1 < \cdots < i_j \leq n-1} r_{i_1} \cdots r_{i_j}. \tag{4.5}$$

Note that the principal curvatures are functions on the boundary of K and the principal radii of curvature are functions on the sphere.

Now we describe the connection between H_j and s_j. For a body K of class C^2_+, we have for $u \in S^{n-1}$ that $\bar{\xi}_K(u) = \nu^{-1}(u)$. In particular, the principal radii of curvature are reciprocals of the principal curvatures, that is

$$r_i(u) = \frac{1}{k_i(\bar{\xi}_K(u))}.$$

This implies that for $x \in \partial K$ with $\nu(x) = u$,

$$s_j = \binom{n-1}{j}^{-1} \sum_{1 \leq i_1 < \cdots < i_j \leq n-1} \frac{1}{k_{i_1}(\bar{\xi}_K(u)) \cdots k_{i_j}(\bar{\xi}_K(u))} = \frac{H_{n-1-j}}{H_{n-1}} \Big(\bar{\xi}_K(u)\Big)$$

and

$$H_j = \frac{s_{n-1-j}}{s_{n-1}} \Big(\nu(x)\Big),$$

for $j = 1, \ldots, n-1$.

The mixed volumes $W_i(K)$ of the classical Steiner formula (1.3) can be expressed with the help of the elementary symmetric functions of the principal curvatures. By definition, $W_0(K) = \mathrm{vol}_n(K)$ and

$$W_i(K) = \frac{1}{n} \int\limits_{\partial K} H_{i-1} d\mathcal{H}^{n-1}, \tag{4.6}$$

for $i = 1, \dots, n$. Moreover, we have the following formulae, which in integral geometry and differential geometry are known as *Minkowskian integral formulae*,

$$\int\limits_{S^{n-1}} s_j d\mathcal{H}^{n-1} = \int\limits_{S^{n-1}} h_K s_{j-1} d\mathcal{H}^{n-1},$$

and

$$\int\limits_{\partial K} H_{j-1} d\mathcal{H}^{n-1} = \int\limits_{\partial K} \langle x, \nu(x) \rangle H_j d\mathcal{H}^{n-1} \tag{4.7}$$

for $j = 1, \dots, n-1$.

The corresponding Steiner formula in the dual Brunn-Minkowski theory is

$$\mathrm{vol}_n(K \,\widetilde{+}\, tB_2^n) = \sum_{i=0}^{n} \binom{n}{i} \widetilde{W}_i(K)\ t^i.$$

For $x, y \in \mathbb{R}^n$, a radial addition $x \,\widetilde{+}\, y$ is defined to be $x + y$ if x and y are on a line through 0, and 0 otherwise. Let K and L be star bodies in $\mathbb{R}^n$, then a radial linear combination $K \,\widetilde{+}\, tL = \{x + ty : x \in K, y \in L\}$ (see Figure 4.1). Moreover, $\rho_{K \widetilde{+} tL}(u) = \rho_K(u) + t\rho_L(u)$ where $\rho_K(u) = \max\{\lambda \geq 0 : \lambda u \in K\}$ is the radial function of K.

The dual mixed volume of the convex bodies K and L that contain 0 in its interior is defined for all real i by

$$\widetilde{V}_i(K, L) = \frac{1}{n} \int\limits_{S^{n-1}} \rho_K(u)^{n-i} \rho_L(u)^i d\sigma(u).$$

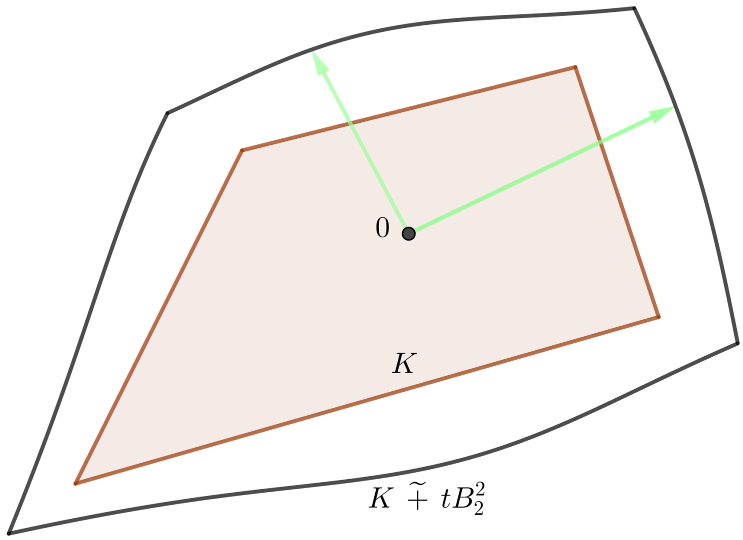

Figure 4.1: An example of a radial linear combination $K \widetilde{+} tB_2^2$.

In particular, if $L = B_2^n$, then

$$\widetilde{W}_i(K) = \widetilde{V}_i(K, B_2^n) = \frac{1}{n} \int_{S^{n-1}} \rho_K(u)^{n-i} d\sigma(u) \tag{4.8}$$

are called *dual quermassintegrals* of order i.

A well known change of integral formula, e.g., [132], which we use frequently, says that for a C_+^2 convex body K and a continuous function $g : \partial K \to \mathbb{R}$

$$\int_{S^{n-1}} g(u) f_K(u) d\sigma(u) = \int_{\partial K} g(x) d\mathcal{H}^{n-1}(x), \tag{4.9}$$

where $u \in S^{n-1}$ and $x \in \partial K$ are related via the Gauss map, i.e., $\nu(x) = u$.

4.3.2 Background from affine geometry

From now on we will always assume that the centroid of a convex body K in $\mathbb{R}^n$ is at the origin. For real $p \neq -n$, we define the L_p affine surface area $as_p(K)$ of K as in [93]

$(p > 1)$ and [135] $(p < 1,\ p \neq -n)$ by

$$as_p(K) = \int\limits_{\partial K} \frac{H_{n-1}(x)^{\frac{p}{n+p}}}{\langle x, \nu(x)\rangle^{\frac{n(p-1)}{n+p}}}\, d\mathcal{H}^{n-1}(x) \tag{4.10}$$

and

$$as_{\pm\infty}(K) = \int\limits_{\partial K} \frac{H_{n-1}(x)}{\langle x, \nu(x)\rangle^{n}} d\mathcal{H}^{n-1}(x). \tag{4.11}$$

In particular, for $p = 0$,

$$as_0(K) = \int\limits_{\partial K} \langle x, \nu(x)\rangle\, d\mathcal{H}^{n-1}(x) = n\, \mathrm{vol}_n(K). \tag{4.12}$$

The case $p = 1$,

$$as_1(K) = \int\limits_{\partial K} H_{n-1}(x)^{\frac{1}{n+1}} d\mathcal{H}^{n-1}(x)$$

is the classical affine surface area which is independent of the position of K in space. For dimensions 2 and 3 and sufficiently smooth convex bodies, its definition goes back to Blaschke [13].

If the boundary of K is sufficiently smooth, then (4.10) and (4.11) can be written as integrals over the boundary $\partial B_2^n = S^{n-1}$ of the Euclidean unit ball B_2^n in $\mathbb{R}^n$,

$$as_p(K) = \int\limits_{S^{n-1}} \frac{f_K(u)^{\frac{n}{n+p}}}{h_K(u)^{\frac{n(p-1)}{n+p}}} d\sigma(u), \tag{4.13}$$

where $f_K(u)$ is the curvature function, i.e. the reciprocal of the Gaussian curvature $H_{n-1}(x)$ at this point $x \in \partial K$ that has u as outer normal. In particular, for $p = \pm\infty$,

$$as_{\pm\infty}(K) = \int\limits_{S^{n-1}} \frac{1}{h_K(u)^n} d\sigma(u) = n\, \mathrm{vol}_n(K^\circ). \tag{4.14}$$

For $p = -n$, the L_{-n} affine surface area was introduced in [103] as

$$as_{-n}(K) = \max_{u \in S^{n-1}} f_K(u)^{\frac{1}{2}} h_K(u)^{\frac{n+1}{2}}. \tag{4.15}$$

4.4 The Steiner formula of the L_p Brunn-Minkowski theory

In this section, we state our main theorems and discuss some of their consequences. The proofs are given in Section 4.7.

4.4.1 The general case

We will need the generalized binomial coefficients. For $\alpha \in \mathbb{R}$ and $k \in \mathbb{N}$, they are defined as

$$\binom{\alpha}{k} = \begin{cases} 1 & \text{if } \ k = 0, \\ 0 & \text{if } \ k < 0, \\ \dfrac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!} & \text{if } k > 0. \end{cases} \tag{4.16}$$

Also, we will need the multinomial coefficients from the multinomial formula

$$(a_1 + \ldots + a_r)^q = \sum_{\substack{i_1,\ldots,i_r \geq 0 \\ i_1+\cdots+i_r=q}} \binom{q}{i_1, \ldots, i_r} a_1^{i_1} \cdot \ldots \cdot a_r^{i_r}, \tag{4.17}$$

where

$$\binom{q}{i_1, i_2, \ldots, i_r} = \frac{q!}{i_1! i_2! \cdots i_r!}$$

is the multinomial coefficient. Note that

$$\binom{q}{i_1, i_2, \ldots, i_r} = 0 \quad \text{if } \ i_j < 0 \text{ or } i_j > q. \tag{4.18}$$

The sum in the multinomial formula is taken over all nonnegative integer indices

$i_1, \ldots, i_r$ such that the sum of all i_j is q. In the case $r = 2$, we get the binomial theorem.

To give the precise statement of Theorem 4.1, we introduce the following coefficients, which are sums of mixed products of the elementary symmetric functions of the principal curvatures, up to some multinomial coefficients. For any real $p \neq -n$,

$$A_p^m = A_p^m\left(\bar{\xi}_K(u)\right) = \tag{4.19}$$
$$\sum_{\substack{i_1,\ldots,i_{n-1}\geq 0\\ i_1+2i_2+\cdots+(n-1)i_{n-1}=m}} \binom{\frac{n}{n+p}}{i_1+\cdots+i_{n-1}} \binom{i_1+\cdots+i_{n-1}}{i_1,\, i_2,\, \ldots,\, i_{n-1}} \prod_{j=1}^{n-1} \left\{ \binom{n-1}{j}^{i_j} H_j^{i_j}\left(\bar{\xi}_K(u)\right) \right\}.$$

For $p = -n$, we have

$$A_{-n}^m = A_{-n}^m(u) = \sum_{\substack{i_1,\ldots,i_{2n}\geq 0\\ i_1+2i_2+\cdots+2ni_{2n}=m}} \binom{\frac{1}{2}}{i_1+\cdots+i_{2n}} \binom{i_1+\cdots+i_{2n}}{i_1,\, \ldots,\, i_{2n}} \prod_{q=1}^{2n} B_q^{i_q} \tag{4.20}$$

and

$$B_q = B_q(u) = \sum_{k+i=q} \left[\binom{n-1}{k} \binom{n+1}{i} \frac{H_k(\bar{\xi}_K(u))}{h_K(u)^i} \right].$$

Recall that the L_p Steiner coefficients are defined by

$$\mathcal{W}_{m,k}(K) = \int_{S^{n-1}} f_K(u)^{\frac{n}{n+p}} h_K(u)^{\frac{n(1-p)}{n+p}-k+m} A_p^m \, d\sigma(u).$$

Theorem 4.4 *Let K be a convex body in $\mathbb{R}^n$ that is C_+^2 and let $t \in \mathbb{R}$ be such that $0 \leq t < \beta(K)$. For all $p \in \mathbb{R}$, $p \neq -n$,*

$$as_p(K + tB_2^n) = \sum_{m=0}^{\infty} \left[\sum_{k=m}^{\infty} \binom{\frac{n(1-p)}{n+p}}{k-m} \mathcal{W}_{m,k}\, t^k \right]. \tag{4.21}$$

In particular,

$$as_1(K+tB_2^n)=\sum_{m=0}^{\infty}\left[\int\limits_{S^{n-1}} f_K(u)^{\frac{n}{n+1}} A_1^m \, d\sigma(u)\right] t^m=\sum_{m=0}^{\infty}\mathcal{W}_{m,m}\, t^m. \tag{4.22}$$

The cases $p=0$ and $p=\pm\infty$ are Corollary 4.2 and Corollary 4.3, respectively.

The next Theorem 4.5 describes the case $p=-n$.

Theorem 4.5 *Let K be a convex body in $\mathbb{R}^n$ that is C_+^2 and let $t\in\mathbb{R}$ be such that $0\le t<\beta(K)$. Then*

$$as_{-n}(K+tB_2^n)=\max_{u\in S^{n-1}} f_K(u)^{\frac{1}{2}} h_K(u)^{\frac{n+1}{2}}\sum_{m=0}^{\infty}A_{-n}^m t^m.$$

Observe that the first coefficient in the expansion is $as_{-n}(K)$.

Remark on the polytopal case

When $K=P$ is a polytope, we denote by vert P the set of its vertices and for $v\in$ vert P,

$$\mathrm{Norm}(v)=\{u\in\mathbb{R}^n:\ \langle u,\, z-v\rangle\le 0 \text{ for all } z\in P\}$$

is the normal cone to P at v, see [132]. Then the following Steiner formula for polytopes holds.

Theorem 4.6 *Let P be a convex polytope in $\mathbb{R}^n$ and let $t\in\mathbb{R}$ be such that $0\le t<\beta(P)$. For all $p\in\mathbb{R}$ such that*

(i) $p\notin[-n,0]$,

$$as_p(P+tB_2^n)=\sum_{m=0}^{\infty}\binom{\frac{n(1-p)}{n+p}}{m}\sum_{v\in vert\,P}\ \int\limits_{u\in Norm(v)} h_P(u)^{\frac{n(1-p)}{n+p}-m} d\sigma(u)\ t^{m+\frac{n(n-1)}{n+p}};$$

(ii) $p\in[-n,0)$, $\qquad as_p(P+tB_2^n)=\infty$;

(iii) $p = 0,$ $\qquad as_0(P + tB_2^n) = n\,\mathrm{vol}_n(P + tB_2^n).$

Note that by definition (4.15), $as_{-n}(P)$ is trivially infinite, since the curvature function of the flat part is infinite. Of course, $as_{-n}(P+tB_2^n)$ is also infinite by the same reasoning.

4.4.2 Local version

In this section, we introduce new curvature and area measures for Borel sets on the boundary of a convex body K and for Borel sets on the Euclidean sphere S^{n-1}, and state local Steiner formula of the L_p Brunn-Minkowski theory.

When K is a C_+^2 convex body, the curvature, respectively area, measures for Borel sets $B \in \mathcal{B}(\mathbb{R}^n)$ and $\omega \in \mathcal{B}(S^{n-1})$ are defined by

$$C_i(K, B) = \int\limits_{\partial K \cap B} H_{n-1-i}(x)\, d\mathcal{H}^{n-1}(x);$$
$$S_i(K, \omega) = \int\limits_{\omega} s_i\, d\sigma(u)$$

for $i = 0, \dots, n-1$, e.g., [132]. Note that for general convex bodies these measures replace the elementary symmetric functions of principal curvatures and the elementary symmetric functions of the principal radii of curvature. In the extreme cases $i = 0$ and $i = n-1$, we obtain $C_{n-1}(K, B) = \mathcal{H}(\partial K \cap B)$ and $S_0(K, \omega) = \sigma(\omega)$. If $B = \mathbb{R}^n$ and $\omega = S^{n-1}$, we get the classical quermassintegrals

$$W_i(K) = \frac{1}{n} C_{n-i}(K, \mathbb{R}^n) = \frac{1}{n} S_{n-i}(K, S^{n-1})$$

which were introduced in (4.6).

Our approach in Theorem 4.4 leads to new curvature and area measures. Namely, for a Borel set $B \in \mathcal{B}(\mathbb{R}^n)$ and for a Borel set $\omega \in \mathcal{B}(S^{n-1})$,

$$\mathcal{C}_{m,k}(K,B) = \int\limits_{\partial K \cap B} \frac{H_{n-1}(x)^{\frac{p}{n+p}}}{\langle x, \nu(x)\rangle^{\frac{n(p-1)}{n+p}+k-m}} A_p^m(x)\, d\mathcal{H}^{n-1}$$

and

$$\mathcal{S}_{m,k}(K,\omega) = \int\limits_{\omega} s_{n-1}(u)^{\frac{n}{n+p}} h_K(u)^{\frac{n(1-p)}{n+p}-k+m} A_p^m(\bar{\xi}(u))\, d\sigma(u)$$

for $k \geq m$ and $m \geq 0$. Taking $B = \mathbb{R}^n$ and $\omega = S^{n-1}$, we recover the L_p Steiner coefficients, i.e.

$$\mathcal{W}_{m,k}(K) = \mathcal{C}_{m,k}(K,\mathbb{R}^n) \quad \text{ and } \quad \mathcal{W}_{m,k}(K) = \mathcal{S}_{m,k}(K,S^{n-1}).$$

When $m = k = 0$, $B = \mathbb{R}^n$ and $\omega = S^{n-1}$, we obtain the L_p affine surface area, i.e.

$$\mathcal{C}_{0,0}(K,\mathbb{R}^n) = \mathcal{S}_{0,0}(K,S^{n-1}) = as_p(K).$$

Now we can state the local Steiner formula for these newly introduced measures.

Theorem 4.7 (Local Steiner formula) *Let K be a convex body in $\mathbb{R}^n$ that is C^2_+, $B \in \mathcal{B}(\mathbb{R}^n)$ and $\omega \in \mathcal{B}(S^{n-1})$. Let t be such that $0 \leq t < \beta(K)$. For all $p \in \mathbb{R}^n$, $p \neq -n$, we have*

$$\mathcal{C}_{0,0}(K + tB_2^n, B) = \sum_{m=0}^{\infty} \sum_{k=m}^{\infty} \binom{\frac{n(1-p)}{n+p}}{k-m} \mathcal{C}_{m,k}(K,B)\, t^k$$

and

$$\mathcal{S}_{0,0}(K + tB_2^n, \omega) = \sum_{m=0}^{\infty} \sum_{k=m}^{\infty} \binom{\frac{n(1-p)}{n+p}}{k-m} \mathcal{S}_{m,k}(K,\omega)\, t^k.$$

We recover the Steiner formula (4.21) of the L_p Brunn-Minkowski theory when $B = \mathbb{R}^n$ or $\omega = S^{n-1}$.

4.5 Properties of the coefficients

In this section, we discuss some properties of the new coefficients which appeared in Theorem 4.4, i.e. the L_p Steiner coefficients,

$$\mathcal{W}_{m,k}(K) = \int\limits_{S^{n-1}} f_K(u)^{\frac{n}{n+p}} h_K(u)^{\frac{n(1-p)}{n+p}-k+m} A_p^m \, d\sigma(u), \qquad k \geq m, \ m \in \mathbb{N} \cup \{0\},$$

where A_p^m is given by (4.19). We will see that the s-mixed p-affine surface areas, defined in Section 4.6, appear as special cases of L_p Steiner coefficients. As it is known that mixed affine surface areas in general are not affine invariant quantities, we cannot expect affine invariance for L_p Steiner coefficients either.

4.5.1 Willmore energy

Firstly, we restrict ourselves to three-dimensional space. Recall that the Willmore energy of a compact surface Σ in $\mathbb{R}^3$ is given by

$$W_E(\Sigma) = \int_\Sigma H_1^2 \, d\mathcal{H}^2,$$

where $H_1 = (k_1 + k_2)/2$ is the mean curvature. The Willmore energy naturally appears in mathematical biology and physics, and has been widely studied. In the 1960s Willmore [160] conjectured the lower bound on the Willmore energy of a torus immersed in $\mathbb{R}^3$. This conjecture was proved only recently in [102]. The choice of the exponent 2 of the mean curvature and dimension $n = 3$ in the definition of the Willmore energy is the proper fit in the context of differential geometry as W_E is invariant under conformal maps. The natural generalization of the type $\int_\Sigma H_1^n \, d\mathcal{H}^{n-1}$ of the Willmore energy to higher dimensional hypersurfaces is called Willmore-Chen functional. When $n = 2$, it coincides with the Willmore energy. The Willmore energy and Willmore-Chen functionals have been studied in [2, 35, 83] and references therein. If one considers integrals of the type $\int_\Sigma H_1^\alpha \, d\mathcal{H}^{n-1}$, then they lose their conformal invariance and become much more difficult to study. For any dimension n, we recover such integrals as the L_p Steiner coefficients in the Steiner formula (4.21).

4.5.2 The case $p = 1$

We start analyzing the coefficients from expansion (4.22). We observe that among them there are mixed affine surface areas which will be introduced in Section 4.6.

If $m = 0$, then $A_1^0 = 1$ and we have

$$\mathcal{W}_{0,0}(K) = \int_{S^{n-1}} f_K(u)^{\frac{n}{n+1}} A_1^0 \, d\sigma(u) = \int_{S^{n-1}} f_K(u)^{\frac{n}{n+1}} \, d\sigma(u) = as_1(K).$$

Hence, the first term in (4.22) is the classical affine surface area of a body K.

If $m = l(n-1)$, $l \in \mathbb{N}$, then $A_1^{l(n-1)} = \binom{\frac{n}{n+1}}{l} H_{n-1}^l$. Thus,

$$\begin{aligned} \mathcal{W}_{l(n-1),l(n-1)}(K) &= \int_{S^{n-1}} f_K(u)^{\frac{n}{n+1}} A_1^{l(n-1)} \, d\sigma(u) = \binom{\frac{n}{n+1}}{l} \int_{S^{n-1}} f_K(u)^{\frac{n}{n+1}-l} \, d\sigma(u) \\ &= \binom{\frac{n}{n+1}}{l} \int_{S^{n-1}} f_K(u)^{\frac{n-l(n+1)}{n+1}} \, d\sigma(u) = \binom{\frac{n}{n+1}}{l} as_{1,\, l(n+1)}(K), \end{aligned}$$

where $as_{1,\, l(n+1)}(K)$ is the $l(n+1)$-mixed 1-affine surface area of K. Therefore, in (4.22) we have mixed affine surface areas as coefficients in front of powers of t, which are multiples of $n-1$.

4.5.3 The general case

Now we move to analyzing the coefficients appearing in (4.21). We note that again mixed affine surface areas appear.

If $m=0$, then $A_p^0=1$ and we have

$$\begin{aligned}\mathcal{W}_{0,k}(K) &= \int\limits_{S^{n-1}} f_K(u)^{\frac{n}{n+p}} h_K(u)^{\frac{n(1-p)}{n+p}-k} A_p^0 \, d\sigma(u) = \int\limits_{S^{n-1}} f_K(u)^{\frac{n}{n+p}} h_K(u)^{\frac{n(1-p)}{n+p}-k} \, d\sigma(u)\\ &= as_{p+\frac{k}{n}(n+p),\,-k}(K),\end{aligned}$$

for $k \in \mathbb{N}\cup\{0\}$. When $k=0$, we get the L_p affine surface area $as_p(K)$ of a body K.

If $m=l(n-1),\ l\in\mathbb{N}$, then $A_p^{l(n-1)} = \binom{\frac{n}{n+p}}{l} H_{n-1}^l$. Thus,

$$\begin{aligned}\mathcal{W}_{l(n-1),k}(K) &= \int\limits_{S^{n-1}} f_K(u)^{\frac{n}{n+p}} h_K(u)^{\frac{n(1-p)}{n+p}-k+l(n-1)} A_p^{l(n-1)} \, d\sigma(u) =\\ &= \binom{\frac{n}{n+p}}{l} \int\limits_{S^{n-1}} f_K(u)^{\frac{n}{n+p}-l} h_K(u)^{\frac{n(1-p)}{n+p}-k+l(n-1)} \, d\sigma(u)\\ &= as_{\frac{np+(n+p)(k-ln)}{n-l(n+p)},\,2nl-k}(K).\end{aligned}$$

The L_p Steiner coefficients consist of combinations of mixed products of the elementary symmetric functions of the principal curvatures H_i and the support function. Applying Hölder's inequality, we can bound those integrals from above. We present one typical example and the other cases can be dealt accordingly.

For instance, consider the case when only one symmetric function of the principal curvatures H_j appears in A_p^m, that is, if $m=lj,\ l\in\mathbb{N}$, i.e., all $i_s=0,\ s\neq j$ and $i_j=l$ for $1\le j\le n-2$, then $A_p^{lj} = \binom{\frac{n}{n+p}}{l}\binom{n-1}{j}^l H_j^l$. Using Hölder's inequality, we get an upper bound

$$\begin{aligned}\mathcal{W}_{lj,k}(K) &= \int\limits_{S^{n-1}} f_K(u)^{\frac{n}{n+p}} h_K(u)^{\frac{n(1-p)}{n+p}-k} A_p^{lj} \, d\sigma(u)\\ &= \binom{\frac{n}{n+p}}{l}\binom{n-1}{j}^l \int\limits_{S^{n-1}} f_K(u)^{\frac{n}{n+p}} h_K(u)^{\frac{n(1-p)}{n+p}-k+lj} H_j^l \, d\sigma(u)\\ &\le \binom{\frac{n}{n+p}}{l}\binom{n-1}{j}^l \left(\int\limits_{S^{n-1}} H_j^{2l}\, d\sigma(u)\right)^{\frac{1}{2}} as_{p+\frac{(k-lj)(n+p)}{n},\,2lj-2k-n}(K)^{\frac{1}{2}}.\end{aligned}$$

Applying Hölder's inequality a necessary number of times, we can obtain similar bounds for any term in the expansion (4.21).

The following theorem gives the inequality for the L_p Steiner coefficients which is similar to the inequality for the mixed affine surface areas given in [94, Theorem 3] when the second body is taken to be a ball.

Theorem 4.8 *Let K be a convex body and $i,\ j,\ k \in \mathbb{R}$ such that $m \le i < j < k$. Then for a fixed $m \in \mathbb{N} \cup \{0\}$,*

$$\mathcal{W}_{m,k}(K)^{j-i}\mathcal{W}_{m,i}(K)^{k-j} \ge \mathcal{W}_{m,j}(K)^{k-i}$$

with equality if and only if K is a Euclidean ball.

Proof. We use Hölder's inequality

$$\int\limits_{S^{n-1}} g_1(u)\cdots g_q(u)\,d\sigma(u) \le \prod_{i=1}^{q}\left(\int\limits_{S^{n-1}} g_i(u)^{a_i}\,d\sigma(u)\right)^{\frac{1}{a_i}}$$

with $q=2$ and $a_1 = \frac{k-i}{j-i},\ a_2 = \frac{k-i}{k-j}$. We take

$$\begin{aligned} g_1(u) &= \left(f_p(K,u)^{\frac{n}{n+p}}\, h_K(u)^{\frac{n(1-p)}{n+p}}\, A_p^m\right)^{\frac{1}{a_1}} h_K(u)^{\frac{1}{a_1}(m-k)}, \\ g_2(u) &= \left(f_p(K,u)^{\frac{n}{n+p}}\, h_K(u)^{\frac{n(1-p)}{n+p}}\, A_p^m\right)^{\frac{1}{a_2}} h_K(u)^{\frac{1}{a_2}(m-i)}. \end{aligned}$$

Then using the definition of the L_p Steiner coefficients (4.3), we get the desired result.

Equality holds in Hölder inequality if and only if $g_1^{a_1}$ is proportional to $g_2^{a_2}$. This leads to the condition that the support function of K must be a constant, i.e. $h_K(u) = const.$ Thus, K must be a ball, which follows from the fact that the support function uniquely determines a convex body. ■

As we have shown above that some $\mathcal{W}_{m,k}$ are mixed affine surface areas, they satisfy the inequality of Theorem 4.8.

4.5.4 Connections to information theory

We would like to point out a connection between the L_p Steiner coefficients and information theory. To do so we need some background.

Let (X, μ) be a measure space and let $dP = p\, d\mu$ and $dQ = q\, d\mu$ be measures on X that are absolutely continuous with respect to the measure μ. Then the Rényi divergence of order α, introduced by Rényi [122] for $\alpha > 0$ and $\alpha \neq 1$, is defined by

$$D_\alpha(P\|Q) = \frac{1}{\alpha - 1} \log \int_X p^\alpha q^{1-\alpha} d\mu. \tag{4.23}$$

It is the convention to put $p^\alpha q^{1-\alpha} = 0$, if $p = 0$ or $q = 0$, even if $\alpha < 0$ and $\alpha > 1$. The integrals

$$\int_X p^\alpha q^{1-\alpha} d\mu \tag{4.24}$$

are also called *Hellinger integrals*. See e.g. [84] for those integrals and additional information. Rényi divergences and Hellinger integrals and their related inequalities are important tools in information theory, statistics, probability theory, machine learning and convex geometry, see e.g., [9, 32, 33, 40, 74, 84, 121].

Usually, in the literature, $\alpha \geq 0$. However, we will also consider $\alpha < 0$, provided the expressions exist. Following the ideas of [156], where Rényi divergences for convex bodies K were introduced, we consider the measure space $(\partial K, \mathcal{H}^{n-1})$ and densities p_K, q_K on ∂K,

$$p_K(x) = \frac{H_{n-1}(x)}{\langle x, \nu(x)\rangle^n}, \qquad q_K(x) = \langle x, \nu(x)\rangle. \tag{4.25}$$

Then

$$P_K = p_K\, \mu_K \quad \text{and} \quad Q_K = q_K\, \mu_K \tag{4.26}$$

are measures on ∂K that are absolutely continuous with respect to μ_K. We remark that those measures are the *cone measures* of K and K°, respectively, see e.g. [115].

By the change of integration formula (4.9), the L_p Steiner coefficients can be written as

$$\begin{aligned}\mathcal{W}_{m,k}(K) &= \int\limits_{S^{n-1}} f_K(u)^{\frac{n}{n+p}} h_K(u)^{\frac{n(1-p)}{n+p}-k+m} A_p^m \, d\sigma(u) \\ &= \int\limits_{\partial K} H_{n-1}(x)^{\frac{p}{n+p}} \langle x, \nu(x)\rangle^{\frac{n(1-p)}{n+p}-k+m} A_p^m \, d\mathcal{H}^{n-1}(x) \\ &= \int\limits_{\partial K} \left(\frac{H_{n-1}(x)}{\langle x, \nu(x)\rangle^n}\right)^{\frac{p}{n+p}} \langle x, \nu(x)\rangle^{1-\frac{p}{n+p}} \langle x, \nu(x)\rangle^{m-k} A_p^m \, d\mathcal{H}^{n-1}(x).\end{aligned} \tag{4.27}$$

Thus the L_p Steiner coefficients are weighted (by the weight $\langle x, \nu(x)\rangle^{m-k} A_p^m$) Hellinger integrals of the measures P_K and Q_K, respectively,

$$\frac{1}{\alpha - 1} \log \mathcal{W}_{m,k}(K)$$

are weighted α-Rényi divergences with weight $\langle x, \nu(x)\rangle^{m-k} A_p^m$ and $\alpha = \frac{p}{n+p}$.

4.6 Mixed affine surface areas

For all $p \geq 1$ and all real s, the s-th mixed L_p affine surface area of K is defined in [94]. We use generalization of this definition to all $p \neq -n$ and all real s, which is given in [158] by

$$as_{p,\,s}(K) = \int\limits_{S^{n-1}} f_p(K, u)^{\frac{n-s}{n+p}} d\sigma(u),$$

where $f_p(K, u) = f_K(u) h_K(u)^{1-p}$.

Theorem 4.9 *Let K be a convex body in $\mathbb{R}^n$ that is C^2_+ and let $t \in \mathbb{R}$ be such that $0 \leq t < \beta(K)$. For all $p \in \mathbb{R}$, $p \neq -n$, for all $s \in \mathbb{R}$,*

$$as_{p,\,s}(K + tB_2^n) = \\ \sum_{m=0}^{\infty} \left[\sum_{k=m}^{\infty} \binom{\frac{n-s}{n+p}(1-p)}{k-m} \int\limits_{S^{n-1}} f_K(u)^{\frac{n-s}{n+p}} h_K(u)^{\frac{n-s}{n+p}(1-p)-k+m} A_{p,\,s}^m \, d\sigma(u) \right] t^k,$$

where

$$A^m_{p,s} = A^m_{p,s}\left(\bar{\xi}_K(u)\right) =$$
$$\sum_{\substack{i_1,\dots,i_{n-1}\geq 0\\ i_1+2i_2+\cdots+(n-1)i_{n-1}=m}} \binom{\frac{n-s}{n+p}}{i_1+\cdots+i_{n-1}} \binom{i_1+\cdots+i_{n-1}}{i_1,\, i_2,\, \dots,\, i_{n-1}} \prod_{j=1}^{n-1} \left\{ \binom{n-1}{j}^{i_j} H_j^{i_j} \right\}.$$

When $s = 0$, we recover Theorem 4.4 and when $p = 1$, we get the expansion for the s-mixed affine surface areas.

Remark on the polytopal case

The next theorem gives the corresponding Steiner formula for the s-mixed L_p affine surface area of the polytope P.

Theorem 4.10 *Let P be a convex polytope in $\mathbb{R}^n$ and t be such that $0 \leq t < \beta(P)$. For all $s \in \mathbb{R}$, for all $p \in \mathbb{R}$ such that if $n < s$, then $p \notin (-s, -n]$ and if $n > s$, then $p \notin [-n, -s)$,*

$$as_{p,s}(P + tB^n_2) =$$
$$\sum_{m=0}^{\infty} \binom{\frac{n-s}{n+p}(1-p)}{m} \sum_{v \in vert\ P} \int_{u\in Norm(v)} h_P(u)^{\frac{(n-s)(1-p)}{n+p}-m} d\sigma(u)\ t^{m+\frac{(n-1)(n-s)}{n+p}}.$$

4.7 Proofs

We start by showing that Corollary 4.2 and 4.3 are consequences of Theorem 4.4.

4.7.1 Proof of Corollary 4.2 and 4.3

Proof of Corollary 4.2

By (4.12), we get for the left hand side of (4.21),

$$as_0(K + tB^n_2) = n\mathrm{vol}_n(K + tB^n_2).$$

The right hand side of (4.21) becomes

$$\sum_{m=0}^{\infty}\left[\sum_{k=m}^{\infty}\binom{1}{k-m} t^k \int_{S^{n-1}} f_K(u)h_K(u)^{1-k+m}A_0^m\,d\sigma(u)\right] =$$
$$\sum_{m=0}^{\infty}\left[t^m \int_{S^{n-1}} f_K(u)h_K(u)A_0^m\,d\sigma(u) + t^{m+1}\int_{S^{n-1}} f_K(u)A_0^m\,d\sigma(u)\right],$$

where

$$A_0^m = \sum_{\substack{i_1,\dots,i_{n-1}\geq 0\\ i_1+2i_2+\cdots+(n-1)i_{n-1}=m}} \binom{1}{i_1+\cdots+i_{n-1}}\binom{i_1+\cdots+i_{n-1}}{i_1,\, i_2,\, \dots, i_{n-1}}\prod_{j=1}^{n-1}\left\{\binom{n-1}{j}^{i_j} H_j^{i_j}\right\}.$$

By (4.16), we only get a contribution for A_0^m if either $i_1+i_2+\cdots+i_{n-1}=0$, i.e., all $i_j=0$, or if $i_1+i_2+\cdots+i_{n-1}=1$, which means that $i_j=1$ and $i_k=0$ for all $k\neq j$. Then $m=j$ as the summation is over all $i_1,\dots,i_{n-1}\geq 0$ such that $i_1+2i_2+\cdots+(n-1)i_{n-1}=m$. We also use (4.9) and then we get for the right hand side of (4.21),

$$\begin{aligned}
&\sum_{m=0}^{n-1}\binom{n-1}{m}\left[t^m\int_{\partial K}\langle \nu_K(x),x\rangle H_m\,d\mathcal{H}^{n-1} + t^{m+1}\int_{\partial K} H_m d\mathcal{H}^{n-1}\right]\\
&= \int_{\partial K}\langle \nu_K(x),x\rangle\, d\mathcal{H}^{n-1}(x)\\
&+\sum_{m=1}^{n-1}\binom{n-1}{m}\left[t^m\int_{\partial K} H_{m-1}(x)\,d\mathcal{H}^{n-1}(x) + t^{m+1}\int_{\partial K} H_m(x) d\mathcal{H}^{n-1}(x)\right].
\end{aligned}$$

In the last equality we have used (4.7). Collecting terms and using the recursive identity for binomial coefficients

$$\binom{n}{k} = \binom{n-1}{m} + \binom{n-1}{m-1}$$

for all integers n, m such that $1\leq m\leq n-1$, and the identity (4.6), we get for the right hand side of (4.21)

$$n\mathrm{vol}_n(K) + \sum_{m=1}^{n} \binom{n}{m} \int\limits_{\partial K} H_{m-1}(x)\, d\mathcal{H}^{n-1}(x)\ t^m = n \sum_{m=0}^{n} \binom{n}{m} W_m\ t^m.$$

This shows that the classical Steiner formula is a corollary of Theorem 4.4.

Proof of Corollary 4.3

By (4.11) and (4.8), we get for the left hand side of (4.21),

$$as_{\pm\infty}(K + tB_2^n) = n\ \mathrm{vol}_n((K + tB_2^n)^\circ) = n\ \widetilde{W}_0((K + tB_2^n)^\circ).$$

The right hand side of (4.21) becomes

$$\sum_{m=0}^{\infty} \left[\sum_{k=m}^{\infty} \binom{-n}{k-m} t^k \int\limits_{S^{n-1}} h_K(u)^{-n-k+m} A^m_{\pm\infty}\, d\sigma(u) \right],$$

where

$$A^m_{\pm\infty} = \sum_{\substack{i_1,\dots,i_{n-1} \geq 0 \\ i_1 + 2i_2 + \cdots + (n-1) i_{n-1} = m}} \binom{0}{i_1 + \cdots + i_{n-1}} \binom{i_1 + \cdots + i_{n-1}}{i_1,\ i_2,\ \dots,\ i_{n-1}} \prod_{j=1}^{n-1} \left\{ \binom{n-1}{j}^{i_j} H_j^{i_j} \right\}.$$

We only get a contribution for $A^m_{\pm\infty}$ if $i_1 + i_2 + \cdots + i_{n-1} = 0$, i.e., if $i_j = 0$ for all j. This means that we only get a contribution for $m = 0$. As $A^0_{\pm\infty} = 1$, the right hand side of (4.21) becomes

$$\sum_{k=0}^{\infty} \binom{-n}{k} t^k \int\limits_{S^{n-1}} h_K(u)^{-n-k}\, d\sigma(u).$$

If K is a convex body such that $0 \in \text{int}(K)$, then

$$\rho_{K^\circ}(u) = \frac{1}{h_K(u)} \qquad \text{for all } u \in S^{n-1}.$$

Hence, with (4.8),

$$\begin{aligned}
&\sum_{k=0}^{\infty} \binom{-n}{k} t^k \int_{S^{n-1}} h_K(u)^{-n-k} \, d\sigma(u) = \\
&\sum_{k=0}^{\infty} \binom{-n}{k} t^k \int_{S^{n-1}} \rho_{K^\circ}(u)^{n+k} \, d\sigma(u) \;=\; n \sum_{k=0}^{\infty} \binom{-n}{k} \widetilde{W}_{-k}(K^\circ) \, t^k.
\end{aligned}$$

Before we prove the general case of Theorem 4.4, it will be helpful to treat a special case.

Throughout this chapter, we will also need the facts that are listed next. First, note that it is not difficult to see that the support function of $K + tL$ can be expressed in terms of the support functions of K and L by

$$h_{K+tL}(u) = h_K(u) + t h_L(u) \tag{4.28}$$

(see, for example, [132]). Now we write the expression for the curvature function $f_{K+tB_2^n}(u)$. Recall that the curvature function is reciprocal of the Gauss curvature, that is,

$$f_{K+tB_2^n}(u) = \frac{1}{H_{n-1}(\bar{\xi}_{K+tB_2^n}(u))}.$$

Since $\bar{\xi}_{K+tB_2^n}(u)$ is the point on $\partial(K + tB_2^n)$ that has u as unique outer unit normal, $\bar{\xi}_{K+tB_2^n}(u) = x + tu$, where x is this point on ∂K that has u as unique outer normal, i.e., $x = \bar{\xi}_K(u)$. We will also use the fact that the Gauss curvature $H_{n-1}(x + tu)$ is the product of the principal curvatures $k_1^t(x + tu), \ldots, k_{n-1}^t(x + tu)$. A well-known fact from differential geometry provides the connection between the principal curvatures k_i^t

of the outer parallel body $K + tB_2^n$ and principal curvatures k_i of the body K, namely

$$k_i^t(x+tu) = \frac{k_i(x)}{1+tk_i(x)},$$

for $x \in \partial K$ and u the outer unit normal vector to K at the point x. Therefore, the Gauss curvature of the parallel body is

$$H_{n-1}(x+tu) = \prod_{i=1}^{n-1} k_i^t(x+tu) = \prod_{i=1}^{n-1} \frac{k_i(x)}{1+tk_i(x)} = \frac{H_{n-1}(x)}{\prod_{i=1}^{n-1}(1+tk_i(x))}.$$

Since u is a unit outward normal vector at x to K (and also a unit outer normal vector at $x+tu$ to $K+tB_2^n$), we derive an expression for the curvature function $f_{K+tB_2^n}(u)$,

$$\begin{aligned} f_{K+tB_2^n}(u) &= f_K(u) \prod_{i=1}^{n-1} \Big(1 + tk_i\big(\bar{\xi}_K(u)\big)\Big) = f_K(u) \sum_{k=0}^{n-1} \binom{n-1}{k} H_k(\bar{\xi}_K(u)) t^k \\ &= f_K(u) \left(1 + \sum_{k=1}^{n-1} \binom{n-1}{k} H_k(\bar{\xi}_K(u)) t^k\right). \end{aligned} \tag{4.29}$$

4.7.2 The case when $\frac{n}{n+p}$ is a natural number

We consider case when $\frac{n}{n+p}$ is a natural number, that is

$$\frac{n}{n+p} = l \quad \text{where } l \in \mathbb{N},$$

or equivalently,

$$p = -\frac{n(l-1)}{l}, \quad \text{for } l \in \mathbb{N}.$$

Then $(1-p)\frac{n}{n+p} = l + n(l-1) \in \mathbb{N}$, since $l \in \mathbb{N}$.

Then, by (4.13), (4.28) and (4.29),

$$as_p(K+tB_2^n) = \int\limits_{S^{n-1}} f_K^l(u)\left(1+\sum_{k=1}^{n-1}\binom{n-1}{k}H_k t^k\right)^l (h_k+t)^{l+n(n-1)}\, d\sigma(u)$$

$$= \int\limits_{S^{n-1}} f_K^l(u)\Bigg[\binom{l}{0}+\binom{l}{1}\sum_{k=1}^{n-1}\binom{n-1}{k}H_k t^k+\binom{l}{2}\left(\sum_{k=1}^{n-1}\binom{n-1}{k}H_k t^k\right)^2+\dots$$

$$+ \binom{l}{l}\left(\sum_{k=1}^{n-1}\binom{n-1}{k}H_k t^k\right)^l\Bigg]\cdot$$

$$\left[\binom{l+n(l-1)}{0}h_K^{l+n(l-1)}+\dots+\binom{l+n(l-1)}{l+n(l-1)}t^{l+n(l-1)}\right]d\sigma(u),$$

where we used Taylor series expansion of the curvature term and support function term. Note that $\frac{t}{h_K(u)} \le \frac{t}{\beta(K)} < 1$. Hence, the binomial series for the support function uniformly converges on S^{n-1}. Since K is C_+^2, all curvature expressions are bounded from above and strictly positive, independently of x.

Expanding the summations, we can write the general term corresponding to the power t^m as

$$\sum_{\substack{i_1,\dots,i_{n-1}\ge 0\\ i_1+2i_2+\dots+(n-1)i_{n-1}=m}} \binom{l}{i_1+\dots+i_{n-1}}\binom{i_1+\dots+i_{n-1}}{i_1,\, i_2,\, \dots,i_{n-1}}\prod_{j=1}^{n-1}\binom{n-1}{j}^{i_1}H_j^{i_j}.$$

The multinomial coefficients here are coming from the multinomial formula (4.17).

Therefore, the following gives the general formula for the L_p affine surface area,

$$as_p(K+tB_2^n) =$$
$$\sum_{m=0}^{l(n-1)}\Bigg[\sum_{k=m}^{l+n(l-1)+m}\binom{l+n(l-1)}{k-m}t^k\int\limits_{S^{n-1}} f_K^l(u)h_K^{l+n(l-1)-k+m}\cdot$$
$$\left(\sum_{\substack{i_1,\dots,i_{n-1}\ge 0\\ i_1+2i_2+\dots+(n-1)i_{n-1}=m}} \binom{l}{i_1+\dots+i_{n-1}}\binom{i_1+\dots+i_{n-1}}{i_1,\, i_2,\, \dots,i_{n-1}}\prod_{j=1}^{n-1}\left\{\binom{n-1}{j}^{i_j}H_j^{i_j}\right\}\right)d\sigma\Bigg].$$

The parameter m determines the number of sums inside and varies between 0 and $l(n-1)$, since the highest power of t is $l+n(l-1)+l(n-1)$.

For convenience, we denote the part of the expression under the integral by $A^m_{p,\,s} = A^m_{p,\,s}\left(\bar{\xi}_K(u)\right)$:

$$A^m_{p,\,s} = \sum_{\substack{i_1,\dots,i_{n-1}\geq 0\\ i_1+2i_2+\dots+(n-1)i_{n-1}=m}} \binom{\frac{n-s}{n+p}}{i_1+\dots+i_{n-1}} \binom{i_1+\dots+i_{n-1}}{i_1,\, i_2,\, \dots,\, i_{n-1}} \prod_{j=1}^{n-1} \left\{ \binom{n-1}{j}^{i_j} H_j^{i_j} \right\} \tag{4.30}$$

for any real $p \neq -n$ and s. Note that in the current case $p = \frac{n(1-l)}{l}$, $s = 0$ and the expression $\frac{n}{n+p} = l$. Coefficients $A^m_p = A^m_{p,\,0}$ represent a sum of mixed products of symmetric functions of the principal curvatures H_i (4.4) with corresponding multinomial coefficients.

Thus,

$$as_{\frac{n(1-l)}{l}}(K+tB^n_2) = \sum_{m=0}^{l(n-1)} \left[\sum_{k=m}^{l+n(l-1)+m} \binom{l+n(l-1)}{k-m} t^k \int_{S^{n-1}} f^l_K(u) h_K^{l+n(l-1)-k+m} A^m_{\frac{n(1-l)}{l}}\, d\sigma(u) \right] \tag{4.31}$$

for $l \in \mathbb{N}$.

4.7.3 The case of real $p \neq -n$

The L_p affine surface area of $K+tB^n_2$ is given by (4.13). Using relation (4.29) for the curvature function of the parallel body, we can rewrite it in the following form,

$$as_p(K+tB^n_2) = \int_{S^{n-1}} f_K(u)^{\frac{n}{n+p}} \left(1 + \sum_{k=1}^{n-1} \binom{n-1}{k} H_k(\bar{\xi}_K(u)) t^k \right)^{\frac{n}{n+p}} (h_K(u)+t)^{\frac{n(1-p)}{n+p}}\, d\sigma(u).$$

Using Taylor series expansions of the curvature and support functions terms, we have

$$as_p(K+tB_2^n) =$$
$$\int\limits_{S^{n-1}} f_K(u)^{\frac{n}{n+p}} \sum_{i=0}^{\infty} \left\{ \binom{\frac{n}{n+p}}{i} \left[\sum_{k=1}^{n-1} \binom{n-1}{k} H_k(\bar{\xi}_K(u)) t^k \right]^i \right\} \sum_{j=0}^{\infty} \binom{\frac{n(1-p)}{n+p}}{j} h_K(u)^{\frac{n(1-p)}{n+p}-j} t^j \, d\sigma.$$

Now, analogous to the procedure used in Section 4.7.2, we have the general formula

$$as_p(K+tB_2^n) =$$
$$\sum_{m=0}^{\infty} \left[\sum_{k=m}^{\infty} \binom{\frac{n(1-p)}{n+p}}{k-m} t^k \int\limits_{S^{n-1}} f_K(u)^{\frac{n}{n+p}} h_K(u)^{\frac{n(1-p)}{n+p}-k+m} \cdot \right.$$
$$\left. \left(\sum_{\substack{i_1,\dots,i_{n-1}\geq 0 \\ i_1+2i_2+\cdots+(n-1)i_{n-1}=m}} \binom{\frac{n}{n+p}}{i_1+\cdots+i_{n-1}} \binom{i_1+\cdots+i_{n-1}}{i_1,\ i_2,\ \dots,i_{n-1}} \prod_{j=1}^{n-1} \left\{ \binom{n-1}{j}^{i_j} H_j^{i_j} \right\} \right) d\sigma \right].$$

Using the notation A_p^m defined in (4.30), we can write the expression above in a more compact way:

$$as_p(K+tB_2^n) = \sum_{m=0}^{\infty} \left[\sum_{k=m}^{\infty} \binom{\frac{n(1-p)}{n+p}}{k-m} t^k \int\limits_{S^{n-1}} f_K(u)^{\frac{n}{n+p}} h_K(u)^{\frac{n(1-p)}{n+p}-k+m} A_p^m \, d\sigma(u) \right].$$

We note that the first coefficient in this expansion represents the L_p affine surface area $as_p(K)$ of a body K.

Similarly, an expansion for the s-th mixed p-affine surface area $as_{p,s}(K + tB_2^n)$ is obtained for all real s. In fact, for a convex body K and $0 \leq t < \beta(K)$, one has

$$as_{p,s}(K+tB_2^n) = \sum_{m=0}^{\infty} \left[\sum_{k=m}^{\infty} \binom{\frac{n-s}{n+p}(1-p)}{k-m} t^k \int\limits_{S^{n-1}} f_K(u)^{\frac{n-s}{n+p}} h_K(u)^{\frac{n-s}{n+p}(1-p)-k+m} A_{p,s}^m \, d\sigma(u) \right].$$

Note that the first coefficient in this expansion gives the s-mixed L_p affine affine surface area $as_{p,s}(K)$ of a body K. If $p=1$, this gives the expansion for the s-th mixed affine surface area $as_{1,s}(K+tB_2^n)$ for any real s.

This finishes the proof of Theorem 4.4 and Theorem 4.9.

4.7.4 Proof of Theorem 4.5

The L_{-n} affine surface area of $K+tB_2^n$ is given by

$$as_{-n}(K+tB_2^n) = \max_{u\in S^{n-1}} f_{K+tB_2^n}(u)^{\frac{1}{2}} h_{K+tB_2^n}(u)^{\frac{n+1}{2}}.$$

Using relations (4.29) and (4.28), we can rewrite it as

$$\begin{aligned}
& as_{-n}(K+tB_2^n) = \\
= \quad & \max_{u\in S^{n-1}} f_K(u)^{\frac{1}{2}} h_K(u)^{\frac{n+1}{2}} \left(\left[1 + \sum_{k=1}^{n-1} \binom{n-1}{k} H_k(\bar{\xi}_K(u)) t^k \right] (h_K(u)+t)^{n+1} \right)^{\frac{1}{2}} \\
= \quad & \max_{u\in S^{n-1}} f_K(u)^{\frac{1}{2}} h_K(u)^{\frac{n+1}{2}} \left(1 + \sum_{j=1}^{2n} \sum_{k+i=j} \left[\binom{n-1}{k} \binom{n+1}{i} \frac{H_k(\bar{\xi}_K(u))}{h_K(u)^i} \right] t^j \right)^{\frac{1}{2}}.
\end{aligned}$$

For convenience, we denote the coefficients in front of powers of t as

$$B_j = B_j(u) = \sum_{k+i=j} \left[\binom{n-1}{k} \binom{n+1}{i} \frac{H_k(\bar{\xi}_K(u))}{h_K(u)^i} \right], \qquad 0 \le j \le 2n.$$

Using Taylor series expansion, we obtain

$$\begin{aligned}
as_{-n}(K+tB_2^n) &= \max_{u\in S^{n-1}} f_K(u)^{\frac{1}{2}} h_K(u)^{\frac{n+1}{2}} \left(1 + \sum_{j=1}^{2n} B_j t^j \right)^{\frac{1}{2}} \\
= \quad & \max_{u\in S^{n-1}} f_K(u)^{\frac{1}{2}} h_K(u)^{\frac{n+1}{2}} \sum_{m=0}^{\infty} \left[\sum_{\substack{i_1,\dots,i_{2n}\ge 0 \\ i_1+2i_2+\dots+2ni_{2n}=m}} \binom{\frac{1}{2}}{i_1+\dots+i_{2n}} \binom{i_1+\dots+i_{2n}}{i_1,\,\dots,\,i_{2n}} \prod_{q=1}^{2n} B_q^{i_q} \right] t^m.
\end{aligned}$$

Similarly to (4.30), we can introduce coefficients $A_{-n}^m = A_{-n}^m(u)$ by

$$A_{-n}^m = A_{-n}^m(u) = \sum_{\substack{i_1,\dots,i_{2n}\ge 0 \\ i_1+2i_2+\dots+2ni_{2n}=m}} \binom{\frac{1}{2}}{i_1+\dots+i_{2n}} \binom{i_1+\dots+i_{2n}}{i_1,\,\dots,\,i_{2n}} \prod_{q=1}^{2n} B_q^{i_q}.$$

Then

$$as_{-n}(K+tB_2^n) = \max_{u\in S^{n-1}} f_K(u)^{\frac{1}{2}} h_K(u)^{\frac{n+1}{2}} \sum_{m=0}^{\infty} A_{-n}^m t^m.$$

4.7.5 Proof of Theorem 4.6

For $-n < p < 0$, the exponent $\frac{p}{n+p}$ of the Gauss curvature $H_{n-1}(x)$ in (4.10) is negative. Since for the set $\{y \in \partial(P+tB_2^n) : H_{n-1}(y) = 0\}$ Gauss curvature is equal to zero, we have that $as_p(P+tB_2^n) = \infty$.

Next, we consider the case when $p \notin [-n, 0]$. Note that in this case the exponent $\frac{p}{n+p}$ of $H_{n-1}(x)$ in (4.10) is positive. We have that $f_{P+tB_2^n}(u) \neq 0$ only for those $u \in S^{n-1}$ for which $f_{P+tB_2^n}(u) < \infty$ and then $f_{P+tB_2^n}(u)^{\frac{n}{n+p}} = t^{\frac{n(n-1)}{n+p}}$. By this, (4.13) and (4.29),

$$\begin{aligned}
as_p(P+tB_2^n) &= \int\limits_{\{y\in\partial(P+tB_2^n):\, H_{n-1}(y)=0\}} \frac{H_{n-1}(y)^{\frac{p}{n+p}}}{\langle y, \nu(y)\rangle^{\frac{n(p-1)}{n+p}}} d\mathcal{H}^{n-1}(y) \\
&+ \int\limits_{\{y\in\partial(P+tB_2^n):\, H_{n-1}(y)\neq 0\}} \frac{H_{n-1}(y)^{\frac{p}{n+p}}}{\langle y, \nu(y)\rangle^{\frac{n(p-1)}{n+p}}} d\mathcal{H}^{n-1}(y) \\
&= \int\limits_{\{u: f_{P+tB_2^n}(u)<\infty\}} f_{P+tB_2^n}(u)^{\frac{n}{n+p}} h_{P+tB_2^n}(u)^{\frac{n(1-p)}{n+p}} d\sigma(u) \\
&= t^{\frac{n(n-1)}{n+p}} \int\limits_{\{u: f_{P+tB_2^n}(u)<\infty\}} (h_P(u)+t)^{\frac{n(1-p)}{n+p}} d\sigma(u) \\
&= t^{\frac{n(n-1)}{n+p}} \int\limits_{\{u: f_{P+tB_2^n}(u)<\infty\}} h_P(u)^{\frac{n(1-p)}{n+p}} \sum_{j=0}^{\infty} \binom{\frac{n(1-p)}{n+p}}{j} \frac{t^j}{h_P(u)^j} d\sigma(u) \\
&= \sum_{j=0}^{\infty} \binom{\frac{n(1-p)}{n+p}}{j} t^{j+\frac{n(n-1)}{n+p}} \int\limits_{\{u: f_{P+tB_2^n}(u)<\infty\}} h_P(u)^{\frac{n(1-p)}{n+p}-j} d\sigma(u) \\
&= \sum_{j=0}^{\infty} \binom{\frac{n(1-p)}{n+p}}{j} \sum_{v\in \mathrm{vert}\, P} \int\limits_{u\in \mathrm{Norm}(v)} h_P(u)^{\frac{n(1-p)}{n+p}-j} d\sigma(u)\ t^{j+\frac{n(n-1)}{n+p}}.
\end{aligned}$$

As by assumption $t < \beta(P)$, we have for all $u \in S^{n-1}$ that $\frac{t}{h_P(u)} \leq \frac{t}{\beta(P)} < 1$ and, therefore, the above infinite sum converges uniformly. Moreover, for all $u \in S^{n-1}$,

$\lambda(P) \leq h_P(u) \leq \Lambda(P)$. Thus, we can interchange integration and summation, which was done in the second last equality above.

For $p = 0$, we have a different situation, namely,

$$
\begin{aligned}
as_0(P + tB_2^n) &= \int\limits_{\{y \in \partial(P+tB_2^n):\, H_{n-1}(y)=0\}} \frac{1}{\langle y, \nu(y)\rangle^{-1}} d\mathcal{H}^{n-1}(y) \\
&+ \int\limits_{\{y \in \partial(P+tB_2^n):\, H_{n-1}(y)\neq 0\}} \frac{1}{\langle y, \nu(y)\rangle^{-1}} d\mathcal{H}^{n-1}(y) \\
&= \int\limits_{\partial(P+tB_2^n)} \langle y, \nu(y)\rangle \, d\mathcal{H}^{n-1}(y) = n \operatorname{vol}_n(P + tB_2^n).
\end{aligned}
$$

Chapter 5

A sausage body is a unique solution for a reverse isoperimetric problem [4]

5.1 Introduction

Inequality (1.4) has a long and beautiful history, and has been generalized to a variety of different settings (see, for example, surveys [29, 124]). The distinctive point of almost all of these generalizations is that the extreme object is always a ball, as the most symmetric body. On the other hand, the problem can be looked at from a different point of view: under which conditions can one minimize the volume among all domains of a given constraint (such as given surface area, etc.)? Questions of such type are known as reverse isoperimetric problems, and have been actively studied recently.

The naive attempt of minimizing volume among all sets of a given surface area will clearly lead to a trivial result: the n-dimensional volume is zero for every set with empty interior. Therefore, we must consider a family of sets with additional conditions

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imposed in order to obtain a well-posed reverse isoperimetric problem. One of the natural conditions is convexity or strict convexity.

One of the first results on the reverse isoperimetric problem is due to Keith Ball. In his celebrated works [5, 6] he showed that for any convex body K in $\mathbb{R}^n$ there is an affine transformation T such that the volume of $T(K) \subset \mathbb{R}^n$ is no smaller than that of the standard simplex of the same surface area; if the bodies are additionally assumed to be symmetric, then the cube is an extreme object. The equality case in Ball's reverse isoperimetric inequalities was completely settled later by Barthe [10]. Observe that for Ball's approach the minimizers are no longer balls.

Another approach towards obtaining a reverse isoperimetric inequality was recently taken in [114], where the authors provided a lower bound on the area enclosed by a convex curve $\gamma \subset \mathbb{R}^2$ in terms of its length and the area of the domain enclosed by the locus of curvature centers of γ. The authors also showed that equality is attained only for a disk. In this respect, the results in [114] do not follow the philosophy of a reverse isoperimetric problem. See also [162], but again these results, although called 'reverse', do not follow the philosophy of a reverse isoperimetric problem.

At the same time, motivated by the study of strictly convex hypersurfaces in Riemannian spaces (see, for instance, [23, 17, 19]), Borisenko and Drach in a series of papers [20, 21, 43] obtained two-dimensional reverse isoperimetric inequalities for so-called λ-convex curves, i.e. curves whose curvature k, in a weak sense, satisfies $k \geqslant \lambda > 0$. Recently, these results were generalized in [18] for λ-convex curves in Alexandrov metric spaces of curvature bounded below. The result of Borisenko completely settles the reverse isoperimetric problem for λ-convex curves.

λ-convexity is a notion that can be easily transferred to higher dimensions. A convex body in $\mathbb{R}^n$ is λ-convex if the principal curvatures $(k_i)_{i=1}^{n-1}$ of the boundary of the body are uniformly bounded, in a weak sense, by λ, i.e. $k_i \geqslant \lambda > 0$ for all $i \in \{1, \ldots, n-1\}$ (we refer to [23, 22, 44] for various results concerning the geometry of multidimensional λ-convex bodies). It is worth pointing out that the reverse isoperimetric problem for

λ-convex bodies has a non-trivial solution in any dimension, although for dimensions greater than two it is a hard problem that is still widely open (see Subsection 5.4.2).

In this chapter we consider a notion, in a sense, dual to the notion of λ-convexity. In particular, we consider so-called λ-concave bodies in $\mathbb{R}^n$. These are the convex sets such that the principal curvatures of their boundaries satisfy $\lambda \geqslant k_i \geqslant 0$ for all $i \in \{1, \ldots, n-1\}$ (in viscosity sense, see Definition 5.2). For λ-concave bodies we completely solve the reverse isoperimetric problem in any dimension. This is the first result on the reverse isoperimetric problem in $\mathbb{R}^n$, besides the celebrated results of Ball and their various extensions, where the inequality is not restricted to curves or surfaces. Moreover, our methods allow us to prove the full family of sharp inequalities involving quermassintegrals of a convex body.

5.1.1 Further motivation

Part of our motivation, besides previously mentioned work on the reverse isoperimetric problem for λ-convex domains due to Borisenko and Drach [20, 21, 43], came from results on so-called Will's conjecture.

If K is a planar convex body with inradius r, i.e., the radius of the largest ball which is contained in the body, then the inequality

$$\operatorname{vol}_2(K) \leqslant r \operatorname{vol}_1(\partial K) - r^2\pi$$

is called Bonnesen's inradius inequality. Equality holds for the sausage body, that is, the Minkowski sum of a line segment and a circle with radius r. An extension of Bonnesen's inradius inequality to higher dimensions was conjectured by Wills [161] in 1970. He conjectured that

$$\operatorname{vol}_n(K) \leqslant r\operatorname{vol}_{n-1}(\partial K) - (n-1)\, r^n \operatorname{vol}_n(B_2^n),$$

for every convex body $K \subset \mathbb{R}^n$ with inradius r. This conjecture was proven independently by Bokowski [15] and Diskant [42]. Although the same inequality with the circumradius R of K substituting r is not true in dimensions greater than two (see [42, 76]), Bokowski and Heil [16] showed that for higher dimensions, in fact, the inequality sign is reversed:

$$\mathrm{vol}_n(K) \geqslant \frac{2R}{n-1}\mathrm{vol}_{n-1}(\partial K) - \frac{(n+1)R^n}{n-1}\mathrm{vol}_n(B_2^n). \tag{5.1}$$

In [16] inequality (5.1) was obtained as a corollary of the following more general result

Theorem 5.1 ([16]) *For an arbitrary convex body $K \subset \mathbb{R}^n$ with circumradius R, the inequalities*

$$c_{ijk}R^iW_i(K) + c_{jki}R^jW_j(K) + c_{kij}R^kW_k(K) \geqslant 0, \tag{5.2}$$

hold for every $0 \leqslant i < j < k \leqslant n$, where $c_{pqr} = (r-q)(p+1)$.

Here $W_i(K)$ is the quermassintegral of order i of the convex body K, $i \in \{1, \dots, n\}$ (see Section 4.3.1 for details). Quermassintegrals can be viewed as geometric quantities assigned to a convex body that are a higher-dimensional generalization of the integral curvature of a closed curve, and can be explicitly calculated in terms of the principal curvatures k_i of ∂K, provided ∂K is sufficiently smooth (see (4.6)). The quermassintegrals of different order provide a natural embedding of the volume $\mathrm{vol}_n(K)$, the surface area $\mathrm{vol}_{n-1}(\partial K)$ and the volume of the unit ball $\mathrm{vol}_n(B_2^n)$ into the sequence $(W_i(K))_{i=0}^n$ for which (up to a constant) these are respectively, the zeroth, the first, and the n-th element. Therefore, (5.1) is a special case of (5.2) with $i = 0$, $j = 1$ and $k = n$.

The form of the Bokowski–Heil inequality (5.1) inspired the statement of our main result, Theorem 5.4, although we use different techniques for the proof. It appears that, having a natural inclusion of the volume, the surface area and the volume of the unit ball into the sequence of quermassintegrals helps to solve the reverse isoperimetric problem for λ-concave bodies in $\mathbb{R}^n$ for every $n \geqslant 2$.

5.1.2 The main results

Recall that a convex body in the Euclidean space $\mathbb{R}^n$ is a compact convex set with a non-empty interior.

Definition 5.2 (λ-concave body) *For a given $\lambda > 0$, a convex body $K \subset \mathbb{R}^n$ is called* λ-concave *if for every $p \in \partial K$ there exists a ball $B_{1/\lambda,p}$ (called a supporting ball at p) of radius $1/\lambda$ passing through p in such a way that*

$$B_{1/\lambda,p} \cap U(p) \subseteq K \cap U(p) \tag{5.3}$$

for some small open neighborhood $U(p) \subset \mathbb{R}^n$ of p.

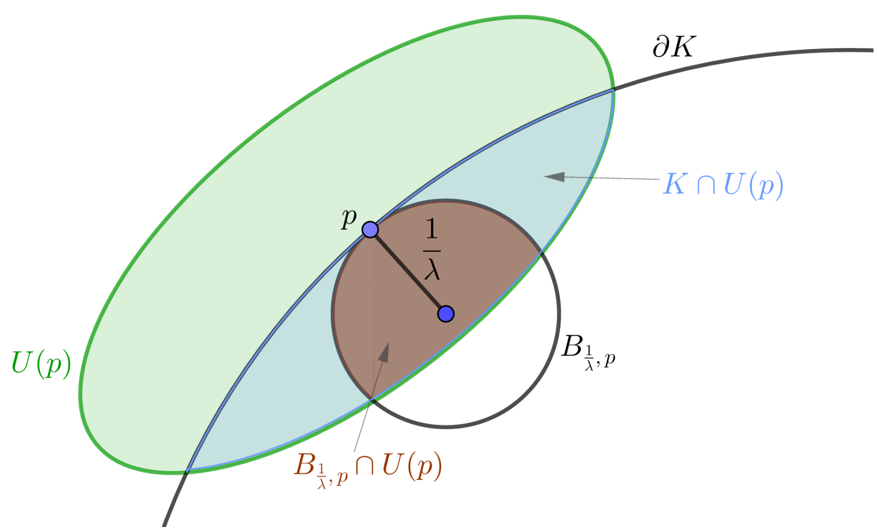

Figure 5.1: A λ-concave body.

Note that since K is assumed to be convex, if K is λ-concave, then a supporting ball is unique at every point. As for the nomenclature, compare it to the notion of λ-convexity (see [22, 19, 44]), for which inclusion (5.3) is reversed (see also the discussion in Subsection 5.4.2).

If the boundary ∂K of a convex body K is at least C^2-smooth, then K is λ-concave if and only if the principal curvatures $k_i(p)$ for all $i \in \{1, \ldots, n-1\}$ are non-negative and

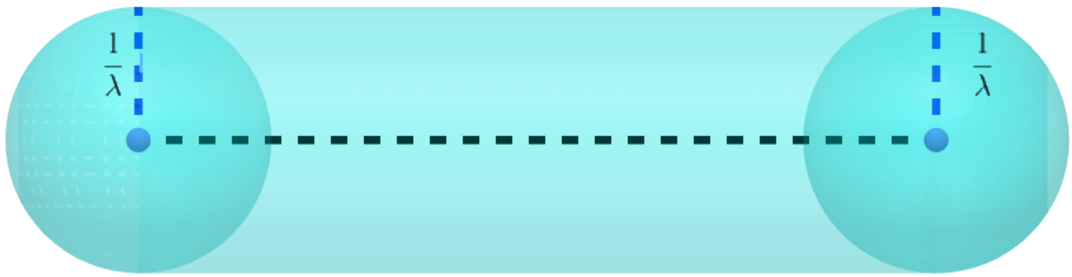

Figure 5.2: A λ-sausage body.

uniformly bounded above by λ, i.e. $0 \leqslant k_i(p) \leqslant \lambda$ for every i and $p \in \partial K$. Equivalently, in the smooth setting λ-concavity can be expressed in terms of uniformly bounded normal curvature. Let $p \in \partial K$ be a point, $v \in T_p\partial K$ be a vector which belongs to the tangent hyperplane at point p, ν be the inward pointing normal to ∂K at p, and $\pi(p, v)$ be the two-dimensional plane through p spanned by v and ν. The *normal curvature* $k_{\mathrm{n}}(p, v)$ of $\partial K \subset \mathbb{R}^n$ at the point p in the direction of v is defined as

$$k_{\mathrm{n}}(p, v) := \kappa(p),$$

where $\kappa(p)$ is the curvature of the planar curve $\partial K \cap \pi(p, v)$ at the point p. Using this notion, a convex body K with smooth boundary is λ-concave if and only if $0 \leqslant k_{\mathrm{n}}(p, v) \leqslant \lambda$ uniformly over p and v. In general, K is λ-concave if the uniform bound on normal curvatures is satisfied in the viscosity sense (see [22, Definition 2.3] for a similar approach).

Definition 5.3 (λ-sausage body) *A* λ-sausage body *in* $\mathbb{R}^n$ *is the convex hull of two balls of radius* $1/\lambda$ *(see Figure 5.2).*

We are now ready to state the main results of this chapter.

Theorem 5.4 (Reverse quermassintegral inequality for λ-concave bodies)

Let $K \subset \mathbb{R}^n$ be a convex body. If K is λ-concave, then

$$(k-j)\frac{W_i(K)}{\lambda^i}+(i-k)\frac{W_j(K)}{\lambda^j}+(j-i)\frac{W_k(K)}{\lambda^k}\geqslant 0 \tag{5.4}$$

for every triple (i,j,k) with $0 \leqslant i < j < k \leqslant n$. Moreover, equality in (5.4) holds if and only if K is a λ-sausage body.

Since $W_0(K)=\mathrm{vol}_n(K)$, $W_1(K)=\mathrm{vol}_{n-1}(\partial K)/n$ and $W_n(K)=\mathrm{vol}_n(B_2^n)$, inequality (5.4) for $i=0$, $j=1$ and $k=n$ immediately implies the following result:

Theorem 5.5 (Reverse isoperimetric inequality for λ-concave bodies) *Let $K \subset \mathbb{R}^n$ be a convex body. If K is λ-concave (for some $\lambda>0$), then*

$$\mathrm{vol}_n(K)\geqslant\frac{\mathrm{vol}_{n-1}(\partial K)}{(n-1)\lambda}-\frac{\mathrm{vol}_n(B_2^n)}{(n-1)\lambda^n}. \tag{5.5}$$

Moreover, equality holds if and only if K is a λ-sausage body.

Theorem 5.4 (and hence Theorem 5.5) for $n=2$ and $n=3$ was first proved using different techniques in [37]. It should be pointed out that Theorem 5.5 for $n=2$ was suggested earlier in [21]; in that paper the authors also prove a similar result on the two-dimensional sphere.

In Section 5.2 we recall some necessary background from convex geometry that will be used in the sequel. In Section 5.3 we provide a proof of the key result (Theorem 5.4). Finally, Section 5.4 contains some further remarks on the reverse problems; in particular, in Subsection 5.4.1 we obtain a so-called reverse isodiametric inequality for λ-concave bodies, and in Subsection 5.4.2 we discuss a connection to the dual problem for λ-convex bodies.

5.2 General background on quermassintegrals and convex geometry

In this section we present some background material and auxiliary lemmas towards the proof of the main result.

As was mentioned above, the Minkowski addition of two convex bodies K and L in $\mathbb{R}^n$ is defined by

$$K + L := \{x + y \colon x \in K,\, y \in L\}.$$

One can rewrite the definition in the following form

$$K + L = \bigcup_{y \in L} (K + y),$$

that is $K + L$ can be viewed as the set that is covered if K undergoes translations by all vectors in L. Since K and L are convex, then $K + L$ is also convex. The Minkowski difference of convex bodies K and L is defined by

$$K - L := \{x \in \mathbb{R}^n \colon L + x \subset K\}.$$

Similarly to the operation of addition, we can rewrite the definition of Minkowski difference in the form

$$K - L = \bigcap_{y \in L} (K - y).$$

For a parameter $t \geqslant 0$, the Minkowski difference $K - tB_2^n$ is called the inner parallel body.

By convention, $W_0(K)$ is equal to the n-dimensional volume of the body. In particular, $nW_1(K)$ is the $n-1$-dimensional volume (surface area) of ∂K, and $nW_n(K) =: s_{n-1}$ is the $n-1$-dimensional volume (surface area) of the unit sphere S^{n-1}. Recall that $s_{n-1}/n = \mathrm{vol}_n(B_2^n)$; hence $W_n(K) = \mathrm{vol}_n(B_2^n)$.

We will need the following generalization of the Steiner formula for inner parallel bodies (see [132, p. 225]):

$$W_q(K - B_2^n) = \sum_{i=0}^{n-q} (-1)^i \binom{n-q}{i} W_{q+i}(K) \tag{5.6}$$

for every $0 \leqslant q \leqslant n$. In particular, for $q = 0$ we have

$$\mathrm{vol}_n(K - B_2^n) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} W_i(K). \tag{5.7}$$

For the later purposes we will also adopt the notation $W_{k,j}(K)$ for the quermassintegral of order j of a convex body K lying in $\mathbb{R}^k$. In particular, such a distinction is needed for the following Kubota formula.

Lemma 5.6 (Kubota formula, [132, p. 301]) *For given $0 < k \leq n-1$, let $G_{n,k}$ be the Grassmann manifold of all k-dimensional linear subspaces of $\mathbb{R}^n$, and let $d\mathbf{P}$ be the probability measure on $G_{n,k}$ which is invariant under the orthogonal group. Then for every convex body K in $\mathbb{R}^n$ and for every integer j with $0 \leq j \leq k$,*

$$\int_{G_{n,k}} W_{k,j}\,(K|P)\,d\mathbf{P} = \frac{\mathrm{vol}_k(B_2^k)}{\mathrm{vol}_n(B_2^n)} W_{n,n-1-k+j}(K),$$

where $K|P$ is the orthogonal projection of K onto the k-dimensional linear subspace $P \in G_{n,k}$.

The Kubota formula allows to run inductive arguments over the dimension of the space provided that the class of convex bodies in question is closed under orthogonal projections. As we will see now, this is exactly the case for λ-concave bodies.

The classical result due to Blaschke implies that local condition (5.3) is in fact global (see [30, 104], and [14] for the original result of Blaschke).

Theorem 5.7 (Blaschke's ball rolling theorem of λ-concave bodies)

Let $K \subset \mathbb{R}^n$ be a λ-concave body. Then

$$B_{1/\lambda,p} \subseteq K$$

for every point $p \in \partial K$ and every supporting ball $B_{1/\lambda,p}$ at p.

From Blaschke's ball rolling theorem it follows that the class of convex λ-concave bodies in $\mathbb{R}^n$ is exactly the class of convex bodies $K \subset \mathbb{R}^n$ such that $K = K_{\mathrm{c}} + B_2^n\left(\frac{1}{\lambda}\right)$ for some convex set K_{c}, where $B_2^n\left(\frac{1}{\lambda}\right)$ is Euclidean ball of radius $\frac{1}{\lambda}$ (see 1.1). This motivates the following definition.

Definition 5.8 (Core of a λ-concave body) *A core of a λ-concave body K is the set $K_{\mathrm{c}} := K - B_2^n\left(\frac{1}{\lambda}\right)$.*

It is easy to see that K_{c} is a convex set in $\mathbb{R}^n$; however, the core is not necessarily λ-concave, even more, K_{c} is not necessarily a convex body in $\mathbb{R}^n$. Recall that the affine hull of a convex set $S \subset \mathbb{R}^n$ is the affine subspace of least dimension that contains S. We will call the dimension of the core (denoted by $\dim K_{\mathrm{c}}$) to be the dimension of the affine hull of K_{c}. Clearly, K_{c} is a convex body if its dimension is n. In these terms, a λ-concave body is a λ-sausage body if and only if the dimension of its core is at most one, and hence it is either a point or a segment.

Let P be a k-dimensional subspace of $\mathbb{R}^n$, and K be a λ-concave body in $\mathbb{R}^n$. Then

$$K|P = \left(K_{\mathrm{c}} + B_2^n\left(\frac{1}{\lambda}\right)\right)|P = K_{\mathrm{c}}|P + B_2^n\left(\frac{1}{\lambda}\right)|P$$

by linearity of orthogonal projections. But $B_2^n\left(\frac{1}{\lambda}\right)|P$ is a ball of radius $1/\lambda$ in P, while $K_{\mathrm{c}}|P$ is some convex set in P. Therefore, $K|P$ is a λ-concave body in P with the core equal to $K_{\mathrm{c}}|P$. These facts prove the following easy, but structurally important, lemma.

Lemma 5.9 (Orthogonal projections of λ-concave bodies) *Orthogonal projections of λ-concave bodies are λ-concave. More precisely, if K is a λ-concave body in $\mathbb{R}^n$ and P*

is a k-dimensional linear subspace of $\mathbb{R}^n$, then $K|P$ is a λ-concave body in P; moreover, if K is a λ-sausage body, then so is $K|P$.

5.3 Proof of the reverse quermassintegral inequality for λ-concave bodies (Theorem 5.4)

In this section we will prove the main result of this paper — Theorem 5.4. Since the left-hand side of (5.4) divided by λ is scale-invariant, without loss of generality we can assume that $\lambda = 1$.

The following lemma is an important step towards the proof of the result. It allows to drastically simplify further computations. The proof partly follows the ideas in [16, Theorem 2].

Lemma 5.10 (On three consecutive indices) *Theorem 5.4 holds true if and only if it holds true for every triple of consecutive indices $(l, l+1, l+2)$ with $0 \leq l \leq n-2$.*

Proof. One direction in this lemma is obvious; so suppose Theorem 5.4 holds true for every triple of consecutive indices. For three consecutive indices $(l, l+1, l+2)$ Theorem 5.4 reads as follows:

$$W_l(K) - 2W_{l+1}(K) + W_{l+2}(K) \geq 0, \tag{5.8}$$

and equality holds if and only if K is a sausage body. Then for any given triple (i, j, k) with $0 \leq i < j < k \leq n$ applying (5.8) repeatedly, we get

$$W_k - W_{k-1} \geqslant W_{k-1} - W_{k-2} \geqslant \cdots \geqslant W_j - W_{j-1} \geqslant \cdots \geqslant W_{i+1} - W_i \tag{5.9}$$

(here we simplify our notation by setting $W_i := W_i(K)$). Estimating the sum of the first $k-j$ and the last $j-i$ differences in (5.9), we get

$$(W_k - W_{k-1}) + \ldots + (W_{j+1} - W_j) \geqslant (k-j)(W_j - W_{j-1}),$$
$$(W_j - W_{j-1}) + \ldots + (W_{i+1} - W_i) \leqslant (j-i)(W_j - W_{j-1}).$$

Performing cancellation and dividing both inequalities by $k-j$ and $j-i$ respectively, we obtain

$$\frac{W_k - W_j}{k-j} \geqslant W_j - W_{j-1} \geqslant \frac{W_j - W_i}{j-i},$$

and hence $(W_k - W_j)/(k-j) \geqslant (W_j - W_i)/(j-i)$. This is equivalent to (5.4); the inequality is proven. If we have equality in (5.4), then we must have equality throughout in (5.9). But equality for a triple of consecutive indices yields that K is a sausage body. This concludes the lemma. ■

Our main approach towards the proof of Theorem 5.4 will be by induction on the dimension of the ambient space. The following two lemmas provide necessary steps to run such an induction.

Lemma 5.11 (Reverse inequality for $(0,1,2)$) *For every $n \geq 2$, if Theorem 5.4 holds for the triple $(1,2,3)$, then it also holds for the triple $(0,1,2)$.*

Proof. This lemma is a consequence of the general Steiner formula for inner parallel bodies. Indeed, by (5.6) for every integer q with $0 \leq q \leq n$ we have

$$W_q(K - B_2^n(1)) = W_q(K_{\mathrm{c}}) = \sum_{i=0}^{n-q} (-1)^i \binom{n-q}{i} W_{q+i}(K) \geq 0. \tag{5.10}$$

Therefore

$$R := \sum_{q=0}^{n-3} \binom{n-3}{q} W_q(K_{\mathrm{c}}) = \sum_{q=0}^{n-3} \binom{n-3}{q} \sum_{i=0}^{n-q} (-1)^i \binom{n-q}{i} W_{q+i} \geq 0. \tag{5.11}$$

Using the simplified notation $W_j = W_j(K)$, we claim that $R = W_0 - 3W_1 + 3W_2 -$

W_3. This can be easily seen by using the formalism of generating functions. To a linear combination of quermassintegrals $\sum_{i=0}^{n} c_i W_i$ we associate the generating function $\sum_{i=0}^{n} c_i x^i$. Using such a formalism, the sum in (5.10) corresponds to the generating function $x^q(1-x)^{n-q}$. Hence, the sum in (5.11) corresponds to the generating function

$$\sum_{q=0}^{n-3} \binom{n-3}{q} x^q (1-x)^{n-q} = (1-x)^3 \cdot \sum_{q=0}^{n-3} \binom{n-3}{q} x^q (1-x)^{n-3-q}$$
$$= (1-x)^3 \cdot (x+1-x)^{n-3} = (1-x)^3.$$

Therefore, $R = W_0 - 3W_1 + 3W_2 - W_3$, as was claimed. But then, since $R \geq 0$, and $W_1 - 2W_2 + W_3 \geq 0$ by the hypothesis of the lemma, it follows that

$$W_0 - 2W_1 + W_2 = R + W_1 - 2W_2 + W_3 \geq 0. \tag{5.12}$$

This is inequality (5.4) for the triple $(0,1,2)$. In order to conclude equality case, assume that $W_0 - 2W_1 + W_2 = 0$. Inequality (5.12) then implies $W_1 - 2W_2 + W_3 = 0$ because R is non-negative (by (5.11)). By hypothesis, Theorem 5.4 holds for the triple $(1,2,3)$, and thus equality case for this triple implies that K is a sausage body. The lemma follows. ■

Recall that the extended notation $W_{k,l}(K)$ stands for the quermassintegral of order l of a convex body K in $\mathbb{R}^k$. For $0 \leq l \leq k-2$, put

$$E_{k,l}(K) := W_{k,l}(K) - 2W_{k,l+1}(K) + W_{k,l+2}(K).$$

The next lemma guarantees that this quantity is always non-negative in dimension n provided that $l > 0$ and that Theorem 5.4 holds in all lower dimensions.

Lemma 5.12 (Reverse inequality for $(l, l+1, l+2)$ with $l \geq 1$) *For a given $n \geq 2$, if Theorem 5.4 holds in $\mathbb{R}^k$ for every k with $2 \leq k < n$, then Theorem 5.4 holds in $\mathbb{R}^n$ for every triple of consecutive indices $(l, l+1, l+2)$ with $1 \leq l \leq n-2$.*

Proof. Let K be a 1-concave body in $\mathbb{R}^n$, and l be an integer satisfying $1 \le l \le n-2$. By the Kubota formula (Lemma 5.6),

$$E_{n,l}(K) = \frac{\mathrm{vol}_n(B_2^n)}{\mathrm{vol}_{n-l}(B_2^{n-l})} \int\limits_{G_{n,n-l}} E_{n-l,0}(K|P)d\mathbf{P}. \tag{5.13}$$

From the bounds on l it follows that $2 \le n-l < n$, and hence the Grassmann manifold $G_{n,n-l}$ is not the trivial one point set $\{\mathbb{R}^n\}$.

By Lemma 5.9, the set $K|P$ is a λ-concave body in the $(n-l)$-dimensional subspace P. By hypothesis of the lemma, Theorem 5.4 holds true for such spaces. Therefore,

$$E_{n-l,0}(K|P) \ge 0 \text{ for every } P \in G_{n,n-l}, \tag{5.14}$$

and for any given P equality holds if and only if $K|P$ is a sausage body. Combining (5.13) and (5.14) we conclude that $E_{n,l}(K) \ge 0$, which is exactly the inequality part in Theorem 5.4 for the triple $(l, l+1, l+2)$ and all 1-concave bodies in $\mathbb{R}^n$.

Let us analyze the equality part of Theorem 5.4 for the triple $(l, l+1, l+2)$. Suppose for a given 1-concave body $K \subset \mathbb{R}^n$ one has $E_{n,l}(K) = 0$. Then $E_{n-l,0}(K|P) = 0$ for almost all $P \in G_{n,n-l}$. By hypothesis, the latter equality implies that $K|P$ is a sausage body, again for almost all $P \in G_{n,n-l}$. Moreover, since

$$K|P = K_{\mathrm{c}}|P + B_2^n(1)|P,$$

and $K|P$ is a sausage body, we conclude that $\dim(K_{\mathrm{c}}|P) \le 1$ for almost all P in the Grassmannian $G_{n,n-l}$. Taking into account that the dimension of each P is at least 2, this gives us that $\dim(K_{\mathrm{c}}) \le 1$. Therefore, K is a 1-sausage body. The lemma is proven. ■

Proof of Theorem 5.4. Due to Lemma 5.10, it is enough to prove the result for every triple of consecutive indices. The claim of Theorem 5.4 is now a consequence of

Lemmas 5.11 and 5.12 by induction on the dimension of the ambient space, which is done as follows.

The case $n = 2$ forms the base of the induction. In $\mathbb{R}^2$ the result is obvious, and is just a restatement of Steiner formula (5.7). Indeed, if K is a 1-concave body in $\mathbb{R}^2$, then

$$0 \leq \mathrm{vol}_2(K - B_2^n) = \mathrm{vol}_2(K_\mathrm{c}) = W_0(K) - 2W_1(K) + W_2(K)$$

by (5.7); this proves the inequality (the triple $(0, 1, 2)$ is the only possible in dimension two). In order to conclude the inequality part, observe that $\mathrm{vol}_2(K_\mathrm{c}) = 0$ implies $\dim K_\mathrm{c} \leq 1$, and hence K is a 1-sausage body.

The inductive step is a combination of Lemmas 5.11 and 5.12. ■

5.4 Concluding remarks

5.4.1 The reverse isodiametric inequality

In this subsection we want to extend our philosophy of a reverse isoperimetric problem to a so-called isodiametric inequality. Recall that a diameter of a convex body $K \subset \mathbb{R}^n$, denoted as $\mathrm{diam}(K)$, is the following quantity:

$$\mathrm{diam}(K) = \max_{p,q \in K} |p - q|.$$

In other words, the diameter is the length of the largest segment that connects two points in K. The classical isodiametric inequality for convex bodies in $\mathbb{R}^n$ asserts that for a given diameter D the ball of radius $D/2$ has the largest volume among all convex bodies of diameter D (see [132, p. 383]).

One simple observation allows us to prove the reverse isodiametric inequality.

Theorem 5.13 (Reverse isodiametric inequality for λ-concave bodies) *Let $K \subset \mathbb{R}^n$ be a convex body. Suppose K is λ-concave, and let $S_\lambda \subset \mathbb{R}^n$ be the λ-sausage body* *with*

$$\operatorname{diam}(K) = \operatorname{diam}(S_\lambda).$$

Then

$$W_i(K) \geqslant W_i(S_\lambda) \tag{5.15}$$

for every $i \in \{0, 1, \ldots, n-1\}$*. Moreover, equality holds if and only if* K *is a* λ*-sausage body.*

Proof. Let p and q be a pair of points in K realizing the diameter of K. It is easy to see that necessarily $p, q \in \partial K$, and moreover, both tangent planes to ∂K at p and q are perpendicular to the segment pq. Therefore, if $B_{1/\lambda,p}$ and $B_{1/\lambda,q}$ are the supporting balls at p resp. q of radius $1/\lambda$, then $B_{1/\lambda,p} \subseteq K$ and $B_{1/\lambda,q} \subseteq K$ (by Blaschke's ball rolling theorem (Theorem 5.7)) and the convex hull of $B_{1/\lambda,p} \cup B_{1/\lambda,q}$ is the λ-sausage body S_λ of diameter $|p-q| = \operatorname{diam}(K)$. Inequality (5.15) and the equality case then follow by monotonicity of quermassintegrals with respect to inclusion (see [132, p. 282]).

■

Remark 5.14 Theorem 5.13 implies the following sharp estimate on the i-th quermassintegral $W_i = W_i(K)$ of a 1-concave body K in terms of its diameter $D = \operatorname{diam}(K)$:

$$W_i \geqslant \operatorname{vol}_n(B_2^n) + \frac{n-i}{n}\,(D-2)\operatorname{vol}_{n-1}(B_2^n) \quad \text{for every } i \in \{0, \ldots, n-1\}.$$

The estimate follows by a direct computation of $W_i(S_1)$.

5.4.2 The reverse isoperimetric problem for λ-convex domains

We conclude with a surprising difference between the reverse isoperimetric problems for λ-convex and λ-concave bodies. For simplicity we restrict ourselves to the Euclidean space, although everything written below makes perfect sense for constant curvature spaces and even general Riemannian manifolds (with appropriate adjustments).

Recall that a convex body $K \subset \mathbb{R}^n$ is λ-convex if for every $p \in \partial K$ there exists a ball $B_{1/\lambda,p}$ of radius $1/\lambda$ with the boundary sphere passing through p in such a way that

$$B_{1/\lambda,p} \cap U(p) \supseteq K \cap U(p) \tag{5.16}$$

for some small open neighborhood $U(p) \subset \mathbb{R}^n$ of p (see [19, 44]).

Although λ-convexity and λ-concavity seem to be two notions dual to each other, methods and difficulties in solving the reverse isoperimetric problem in each of these classes are quite distinct. In our paper we completely solved the reverse isoperimetric problem for λ-concave bodies in $\mathbb{R}^n$. At the same time, only partial results are currently available for λ-convex bodies. In particular, the two-dimensional case of the reverse isoperimetric problem for λ-convex curves, as we already mentioned in the introduction, is completely solved, see [18, 20, 21, 43]. For higher dimensions the following conjecture is due to Alexander Borisenko (private communication; see also [44, Subsection 4.7]).

Conjecture 5.15 (Reverse isoperimetric inequality for λ-convex bodies)

A λ-convex lens in $\mathbb{R}^n$, that is an intersection of two balls of radius $1/\lambda$, is the unique body that minimizes the volume among all λ-convex bodies of given surface area.

Remark 5.16 A similar conjecture can be stated for all model spaces of constant curvature. In this case the balls are substituted with convex bodies whose boundary is of constant normal curvature equal to λ.

Apart from the case $n = 2$, so far this conjecture was verified only in $\mathbb{R}^3$ for λ-convex surfaces of revolution, see the research announcement in [45], and [46].

Finally, it is interesting to point out numerous results concerning so-called ball-polyhedra (see, for example, the paper of Bezdek et al. [12] and references therein). A ball-polyhedron is the intersection of finitely many balls of the same radius. Therefore, this is a dual notion to a λ-concave polytope. In our terminology we would call them λ-convex polytopes, and a λ-convex lens is one of them. Following the ideas of Bezdek et al., Fodor, Kurusa and Vígh [53] introduce a notion of r-hyperconvexity, which is $1/r$-convexity in the sense of the definition above. In the same paper the authors prove that a two-dimensional λ-convex lens is a solution of the reverse isoperimetric problem for λ-convex curves in $\mathbb{R}^2$ [53, Theorem 1.3], which was proven earlier in a sharper version in [20]. Besides, Fodor, Kurusa and Vígh [53] state a conjecture (attributed to Bezdek) which in our language asserts that the intersection of all balls of radius $1/\lambda$ containing a pair of given points (a λ-convex spindle) is a unique body with smallest volume among all λ-convex bodies of given surface area. This conjecture is false, at least in $\mathbb{R}^3$, as the results in [45, 46] indicate (it is also not hard to check by a direct comparison of volumes of the conjectural solutions).

Chapter 6

Conclusion

In this thesis, we solved several problems from Random Matrix Theory and Convex Geometry.

In Chapter 2, we obtained non-asymptotic bound for the smallest singular value of a square matrix with i.i.d. heavy-tailed entries. Szarek in [143] showed that intermediate singular values s_ℓ of Gaussian matrix are of order $\frac{n+1-\ell}{\sqrt{n}}$, $\ell = 2, \ldots, n-1$. Earlier mentioned results of Rudelson and Vershynin in [129] and Wei in [153] confirmed the same estimates of singular values s_ℓ for matrices with subgaussian entries. What one can say about behavior of the intermediate singular values s_ℓ, $\ell = 2, \ldots, n-1$ for matrices with i.i.d. heavy-tailed entries? Since the largest singular value can be unbounded for such matrices, we cannot expect to obtain estimates for all s_ℓ. Although, we can ask whether some portion of the intermediate singular values will resemble the same behavior as in Gaussian and subgaussian cases. In Chapter 3, we considered random polytopes which are generated by rectangular matrices with i.i.d. heavy-tailed entries and studied their geometric parameters. We extended known estimates to a much larger class of random polytopes by reducing assumptions on the entries of the matrix.

We established an analogue of the Steiner formula for the L_p surface area in Chapter 4. This provided an L_p extension of the classical Steiner formula. In the future, I would like to generalize this formula to the case of Minkowski sum of two convex bodies

K and L, that is $K + tL$ for some parameter $t > 0$. Theorem 4.1 deals with a special case when $L = B_2^n$. In Chapter 5, we obtained a family of reverse quermassintegral inequalities for λ-concave bodies. The reverse isoperimetric inequality in the class of n-dimensional λ-concave bodies appeared as a particular case. One can consider a similar reverse isoperimetric problem for the notion of λ-convexity which is in some sense a dual notion to λ-concavity. As we mentioned in Section 5.4.2, only partial results are known for λ-convex bodies. Conjecture 5.15 suggests that a λ-convex lens, that is an intersection of two balls of radius $1/\lambda$, is the unique volume minimizer among all λ-convex bodies of given surface area.

Bibliography

[1] R. Adamczak, A. E. Litvak, A. Pajor, N. Tomczak-Jaegermann, *Quantitative estimates of the convergence of the empirical covariance matrix in log-concave Ensembles*, J. Amer. Math. Soc. **23** (2010), 535–561.

[2] J. Arroyo, M. Barros, O. J. Garay, *Willmore-Chen tubes on homogeneous spaces in warped product spaces*, Pacific J. Math. **188** (1999), 201–207.

[3] A. Auffinger, G. Ben Arous, S. Péché, *Poisson convergence for the largest eigenvalues of heavy tailed random matrices,* Ann. Inst. Henri Poincar Probab. Stat. **45** (2009), 589–610.

[4] Z. D. Bai, J. Silverstein, Y. Q. Yin, *A note on the largest eigenvalue of a large-dimensional sample covariance matrix*, J. Multivariate Anal. **26** (1988), 166–168.

[5] K. Ball, *Volume ratios and a reverse isoperimetric inequality*, J. London Math. Soc. **44** (1991), 351–359.

[6] K. M. Ball, *Volumes of sections of cubes and related problems*, In J. Lindenstrauss and V. D. Milman, editors, Israel seminar on Geometric Aspects of Functional Analysis, number 1376 in Lectures Notes in Mathematics, Springer-Verlag, 1989.

[7] I. Bárány, Z. Füredy, *Approximation of the sphere by polytopes having few vertices,* Proc. Amer. Math. Soc. **102** (1988), no. 3, 651–659.

[8] I. Bárány, A. Pór, *On 0-1 Polytopes with many facets,* Adv. Math. **161** (2001), 209–228.

[9] A. R. Barron, L. Györfi, E. C. van der Meulen, *Distribution estimation consistent*

in total variation and in two types of information divergence, IEEE Trans. Inform. Theory **38** (1992), 1437–1454.

[10] F. Barthe, *On a reverse form of the Brascamp-Lieb inequality*, Invent. Math. **134** (1998), 335–361.

[11] A. Basak, M. Rudelson, *Invertibility of sparse non-hermitian matrices*, Adv. Math. **310** (2017), 426–483.

[12] K. Bezdek, Z. Lángi, M. Naszódi, P. Papez, *Ball-polyhedra*, Discrete Comput. Geom. **38** (2007), no. 2, 201–230.

[13] W. Blaschke, *Vorlesungen über Differentialgeometrie II: Affine Differentialgeometrie*, Springer, Berlin, 1923.

[14] W. Blaschke, *Kreis und Kugel*, – de Gruyter, Berlin, 1956.

[15] J. Bokowski, *Eine verschärfte Ungleichung zwischen Volumen, Oberfläche und Inkugelradius im R*, Elem. Math. **28** (1973), 32–44.

[16] J. Bokowski, E. Heil, *Integral representations of quermassintegrals and Bonnesen-style inequalities*, Arch. Math. **47** (1986), 79–89.

[17] A. Borisenko, *Convex sets in Hadamard manifolds*, Diff. Geom. Appl. **17** (2002), 111–121.

[18] A. Borisenko, *Reverse isoperimetric inequality in two-dimensional Alexandrov spaces*, Proc. Amer. Math. Soc. **145** (2017), 4465–4471.

[19] A. Borisenko, K. Drach, *Closeness to spheres of hypersurfaces with normal curvature bounded below*, Sb. Math. **204:11** (2013), 1565–1583.

[20] A. Borisenko, K. Drach, *Isoperimetric inequality for curves with curvature bounded below*, Math. Notes **95:5** (2014), 590–598.

[21] A. Borisenko, K. Drach, *Extreme properties of curves with bounded curvature on a sphere*, J. Dyn. Control Syst. **21:3** (2015), 311–327.

[22] A. Borisenko, E. Gallego, A. Reventós, *Relation between area and volume for λ-convex sets in Hadamard manifolds*, Differential Geom. Appl. **14** (2001), no. 3,

267–280.

[23] A. Borisenko, V. Miquel, *Total curvatures of convex hypersurfaces in hyperbolic space*, Illinois J. Math. **43** (1999), no. 1, 61–78.

[24] K. J. Böröczky, M. Henk, H. Pollehn, *Subspace concentration of dual curvature measures of symmetric convex bodies*, J. Differential Geom. **109** (2018), 411–429.

[25] K. J. Böröczky, E. Lutwak, D. Yang, G. Zhang, *The logarithmic Minkowski problem*, J. Amer. Math. Soc. (JAMS) **26** (2013), 831–852.

[26] K. J. Böröczky, E. Lutwak, D. Yang, G. Zhang, *Affine images of isotropic measures*, J. Differential Geom. **99** (2015), 407–442.

[27] J. Bourgain, V. D. Milman, *New volume ratio properties for symmetric bodies in* $\mathbb{R}^n$, Invent. Math. **88** (1987), no 2, 319–340.

[28] J. Bourgain, L. Tzafriri, *Invertibility of "large" submatrices with applications to the geometry of Banach spaces and harmonic analysis*, Israel J. Math. **57** (1987), 137–224.

[29] Y. D. Burago, V. A. Zalgaller, *Geometric inequalities*, Springer-Verlag, Berlin, 1988.

[30] J. N. Brooks, J. B. Strantzen, *Blaschke's rolling theorem in* $\mathbb{R}^n$, Mem. Amer. Math. Soc. **80** (1989), no 405.

[31] L. A. Caffarelli, *Interior* $W^{2,p}$ *estimates for solutions of the Monge-Ampere equation*, Ann. of Math. **131** (1990), 135–150.

[32] U. Caglar, E. M. Werner, *Divergence for* s*-concave and log concave functions*, Adv. Math. **257** (2014), 219–247.

[33] U. Caglar, E. M. Werner, *Mixed* f*-divergence and inequalities for log-concave functions*, Proc. Lond. Math. Soc. (2) **110** (2015), 271–290.

[34] B. Carl, A. Pajor, *Gelfand numbers of operators with values in a Hilbert space,* Invent. Math. **94** (1988), 479–504.

[35] B. Y. Chen, *Some conformal invariants of submanifolds and their applications*,

Boll. Un. Mat. Ital. **10** (1974), 380–385.

[36] S. Y. Cheng, S. T. Yau, *On the regularity of the solution of the n-dimensional Minkowski problem*, Commun. Pure Appl. Math. **29** (1976), 495–516.

[37] R. Chernov, *Extreme problems for curves and surfaces of bounded curvature*, bachelor thesis, V. N. Karazin Kharkiv National University, Kharkiv, 2016. (Russian)

[38] R. Chernov, K. Drach, K. Tatarko, *A sausage body is a unique solution for a reverse isoperimetric problem*, Adv. Math. **353** (2019), 431–445.

[39] N. A. Cook, *Lower bounds for the smallest singular value of structured matrices*, Ann. Probab. **46(6)** (2018), 3442–3500.

[40] T. Cover, J. Thomas, *Elements of information theory*, second ed., Wiley-Interscience, (John Wiley and Sons), Hoboken, NJ, 2006.

[41] K. R. Davidson, S. J. Szarek, *Local operator theory, random matrices and Banach spaces*, in "Handbook on the Geometry of Banach spaces," V. 1, 317–366, W. B. Johnson, J. Lindenstrauss eds., Elsevier Science 2001.

[42] V. I. Diskant, *A generalization of Bonnesens inequalities*, Soviet Math. Doklady **14** (1973), no. 6, 1728–1731 (translation of Doklady Akad. Nauk SSSR **213** (1973), 519–521).

[43] K. Drach, *About the isoperimetric property of λ-convex lunes on the Lobachevsky plane*, Dopov. Nats. Akad. Nauk Ukr., Mat. Pryr. Tekh. Nauky **11** (2014), 11-15. (Russian; English translation: arXiv:1402.2688)

[44] K. Drach, *Extreme bounds for complete hypersurfaces in Riemannian spaces*, PhD thesis, B. Verkin Institute for Low Temperature Physics and Engineering of the NAS of Ukraine, Kharkiv, 2016. (Russian)

[45] K. Drach, *On a solution of the reverse Dido problem in a class of convex surfaces of revolution*, Dopov. Nats. Akad. Nauk Ukr. Mat. Prirodozn. Tekh. Nauki **4** (2016), 7–12.

[46] K. Drach, *Reverse isoperimetric inequality for strictly convex surfaces of revolution*,

in preparation.

[47] M. E. Dyer, Z. Füredi, C. McDiarmid, *Volumes spanned by random points in the hypercube,* Random Structures Algorithms **3** (1992), 91–106.

[48] A. Edelman, *Eigenvalues and condition numbers of random matrices*, SIAM J. Matrix Anal. Appl., **9** (1988), 543–560.

[49] H. Federer, *Geometric Measure Theory*, Springer, Berlin, 1969.

[50] H. Federer, *Curvature measures*, Trans. Amer. Math. Soc. **93** (1959), 418–491.

[51] W. J. Firey, *p-means of convex bodies*, Math. Scand. **10** (1962), 17–24.

[52] W. J. Firey, *Blaschke sums of convex bodies and mixed bodies*, Proc. Coll. Convexity (Copenhagen, 1965), Københavns Univ. Mat. Inst (1967), 94–101.

[53] F. Fodor, A. Kurusa, V. Vígh, *Inequalities for hyperconvex sets*, Adv. Geom. **16** (2016), no. 3, 337–348.

[54] J.H.G. Fu, *Curvature measures of subanalytic sets*, Amer. J. Math. **116** (1994), 819–880.

[55] A. Gard, *Reverse isoperimetric inequalities in* $\mathbb{R}^3$, PhD Thesis, The Ohio State University, Columbus, 2012.

[56] R. Gardner, *Geometric Tomography*, second edition, Cambridge University Press, New York, 2006.

[57] R. J. Gardner, D. Hug, W. Weil, *The Orlicz-Brunn-Minkowski theory: a general framework, additions, and inequalities*, J. Differential Geom. **97** (2014), 427–476.

[58] R. J. Gardner, A. Koldobsky, T. Schlumprecht, *An analytical solution to the Busemann-Petty problem on sections of convex bodies*, Ann. Math. (2) **149** (1999), 691–703.

[59] S. Geman, *A limit theorem for the norm of random matrices*, Ann. Probab. **8** (1980), 252–261.

[60] A. Giannopoulos, M. Hartzoulaki, *Random spaces generated by vertices of the cube*, Discrete Comp. Geom. **28** (2002), 255–273.

[61] E. D. Gluskin, *The diameter of Minkowski compactum roughly equals to n*, Funct. Anal. Appl. **15** (1981), 57–58 (English translation).

[62] E. D. Gluskin, *Extremal properties of orthogonal parallelepipeds and their applications to the geometry of Banach spaces*, (Russian) Mat. Sb. (N.S.) **136** (178) (1988), no. 1, 85–96; translation in Math. USSR-Sb. **64** (1989), no. 1, 85–96.

[63] E. D. Gluskin, *The octahedron is badly approximated by random subspaces*, Funct. Anal. Appl. **20** (1986), 11–16; translation from Funkts. Anal. Prilozh. **20** (1986), no. 1, 14–20.

[64] Y. Gordon, A. E. Litvak, C. Schuett, E. Werner, *Geometry of spaces between zonoids and polytopes,* Bull. Sci. Math. **126** (2002), 733–762.

[65] Y. Gordon, O. Guédon, M. Meyer, A. Pajor, *Random Euclidean sections of some classical Banach spaces*, Math. Scand. **91** (2002), no. 2, 247–268.

[66] A. Gray, *Tubes. Second edition*, Progress in Mathematics **221**, Birkhäuser Verlag, Basel, 2004.

[67] O. Guédon, *Gaussian version of a theorem of Milman and Schechtman*, Positivity 1 (1997), no. 1, 1–5.

[68] O. Guédon, F. Krahmer, C. Kümmerle, S. Mendelson, H. Rauhut, *On the geometry of polytopes generated by heavy-tailed random vectors*, preprint, arXiv:1907.07258.

[69] O. Guédon, A.E. Litvak, A. Pajor, N. Tomczak-Jaegermann, *On the interval of fluctuation of the singular values of random matrices*, JEMS **19** (2017), 1469–1505.

[70] O. Guédon, A. Litvak, K. Tatarko, *Random polytopes obtained by matrices with heavy tailed entries*, to appear in Commun. Contemp. Math..

[71] C. Haberl, *Blaschke valuations*, Amer. J. of Math. **133** (2011), 717–751.

[72] C. Haberl, F. Schuster, *General L_p affine isoperimetric inequalities*, J. Differential Geom. **83** (2009), 1–26.

[73] C. Haberl, E. Lutwak, D. Yang, G. Zhang, *The even Orlicz Minkowski problem*, Adv. Math. **224** (2010), 2485–2510.

[74] P. Harremoes, F. Topsøe, *Inequalities between entropy and the index of coincidence derived from information diagrams*, IEEE Trans. Inform. Theory **47** (2001), 2944–2960.

[75] R. Howard, A. Treibergs, *A reverse isoperimetric inequality, stability and extremal theorems for plane curves with bounded curvature*, Rocky Mountain J. Math. **25:2** (1995), 635–684.

[76] B. Herz, *Über die Willssche Verallgemeinerung einer Ungleichung von Bonnesen*, Monatsh. Math. **75** (1971), 316–319.

[77] Y. Huang, E. Lutwak, D. Yang, G. Zhang, *Geometric measures in the dual Brunn-Minkowski theory and their associated Minkowski problems*, Acta Math. **216** (2016), 325–388.

[78] B. S. Kashin, *On properties of random sections of an n-dimensional cube*, Vestnik Moskov. Univ. Ser. I Mat. Mekh. 3 (1983) 8–11, English transl. in Moscow Univ. Math. Bull. **38** (1983).

[79] V. Koltchinskii, S. Mendelson, *Bounding the smallest singular value of a random matrix without concentration*, Int. Math. Res. Notices. **23** (2015), 12991–13008.

[80] F. Krahmer, C. Kummerle, H. Rauhut, *A Quotient Property for Matrices with Heavy-Tailed Entries and its Application to Noise-Blind Compressed Sensing*, preprint, arXiv:1806.04261.

[81] R. Latała, *Some estimates of norms of random matrices*, Proc. Amer. Math. Soc. **133** (2005), 1273–1282.

[82] R. Latała, R. van Handel, P. Youssef, *The dimension-free structure of nonhomogeneous random matrices*, Invent. Math. **214** (2018), 1031–1080 .

[83] P. Li, S.T. Yau, *A new conformal invariant and its applications to the Willmore conjecture and the first eigenvalue of compact surfaces*, Invent. Math. **69** (1982), 269–291.

[84] F. Liese, I. Vajda, *On Divergences and Information in Statistics and Information*

Theory, IEEE Trans. Inform. Theory **52** (2006), 4394–4412.

[85] A. E. Litvak, A. Pajor, M. Rudelson, N. Tomczak-Jaegermann, *Smallest singular value of random matrices and geometry of random polytopes*, Adv. Math. **195** (2005), 491–523.

[86] A. E. Litvak, A. Pajor, M. Rudelson, N. Tomczak-Jaegermann, R. Vershynin, *Euclidean embeddings in spaces of finite volume ratio via random matrices*, J. Reine Angew. Math. **589** (2005), 1–19.

[87] A. E. Litvak, O. Rivasplata, *Smallest singular value of sparse random matrices,* Stud. Math. **212** (2012), 195–218.

[88] A. E. Litvak, S. Spektor, *Quantitative version of a Silverstein's result*, GAFA, Lecture Notes in Math. **2116** (2014), 335–340.

[89] G. V. Livshyts, *The smallest singular value of heavy-tailed not necessarily i.i.d. random matrices via random rounding,* to appear in J. Anal. Math..

[90] G. V. Livshyts, K. Tikhomirov, R. Vershynin, *The smallest singular value of inhomogeneous square random matrices*, preprint, arXiv:1909.04219.

[91] M. Ludwig, *Minkowski areas and valuations*, J. Diff. Geom. **86** (2010), 133–162.

[92] M. Ludwig, M. Reitzner, *A classification of $SL(n)$ invariant valuations*, Ann. of Math. **172** (2010), 1223–1271.

[93] E. Lutwak, *Dual mixed volume*, Pacific J. Math. **58**(2) (1975), 531–538.

[94] E. Lutwak, *Mixed affine surface area*, J. Math. Anal. Appl. **125** (1987), 351–360.

[95] E. Lutwak, *Intersection bodies and dual mixed volumes*, Adv. Math. **71** (1988), 232–261.

[96] E. Lutwak, *The Brunn-Minkowski-Firey theory. II. Affine and geominimal surface areas*, Adv. Math. **118** (2) (1996), 244–294.

[97] E. Lutwak, D. Yang, G. Zhang, *The Cramer–Rao inequality for star bodies*, Duke Math. J. **112** (2002), 59–81.

[98] E. Lutwak, D. Yang, G. Zhang, *Volume inequalities for subspaces of L_p*, J. Differ-

ential Geom. **68** (2004), 159–184.

[99] E. Lutwak, D. Yang, G. Zhang, *Moment-entropy inequalities*, Ann. Probab. **32** (2004), 757–774.

[100] E. Lutwak, D. Yang, G. Zhang, *Cramer-Rao and moment-entropy inequalities for Renyi entropy and generalized Fisher information*, IEEE Trans. Inform. Theory **51** (2005), 473–478.

[101] P. Mankiewicz, N. Tomczak-Jaegermann, *Quotients of finite-dimensional Banach spaces; random phenomena.* In:"Handbook in Banach Spaces" Vol II, ed. W. B. Johnson, J. Lindenstrauss, Amsterdam: Elsevier (2003), 1201–1246.

[102] F. C. Marques, A. Neves, *Min-Max theory and the Willmore conjecture*, Ann. of Math. **179** (2) (2014), 683–782.

[103] M. Meyer, E. Werner, *On the p-affine surface area*, Adv. Math. **152** (2000), 288–313.

[104] A. D. Milka, *A certain theorem of Schur–Schmidt*, Ukrain. Geometr. Sb. Vyp. **8** (1970) 95–102.(Russian)

[105] V. D. Milman, *An inverse form of the Brunn-Minkowski inequality, with applications to the local theory of normed spaces*, C. R. Acad. Sci. Paris Sér. I Math. **302** (1986), 25–28.

[106] V.D. Milman, G. Schechtman, *Asymptotic theory of finite-dimensional normed spaces*, Lecture Notes in Math., vol. 1200, Springer, Berlin, 1986.

[107] S. Mendelson, G. Paouris, *On the singular values of random matrices*, Journal of EMS **16** (2014), no.4, 823–834.

[108] S.J. Montgomery-Smith, *The distribution of Rademacher sums*, Proc. Amer. Math. Soc. **109** (1990), no. 2, 517–522.

[109] F. Nazarov, F. Petrov, D. Ryabogin, A. Zvavitch, *A remark on the Mahler conjecture: local minimality of the unit cube*, Duke Math. J. **154** (2010), 419–430.

[110] J. von Neumann, *Collected works. Vol. V: Design of computers, theory of au-*

tomata and numerical analysis, A Pergamon Press Book The Macmillan Co., New York, 1963.

[111] J. von Neumann, H. H. Goldstine, *Numerical inverting of matrices of high order*, Bull. Amer. Math. Soc. **53** (1947), 1021–1099.

[112] H, Nguyen, V. Vu, *Normal vector of a random hyperplane,* IMRN **2018**(6) (2018), 1754–1778.

[113] R. I. Oliveira, *The lower tail of random quadratic forms, with applications to ordinary least squares and restricted eigenvalue properties*, Probability Theory and Related Fields **166** (3-4) (2016), 1175–1194.

[114] S. L. Pan, H. Zhang, *A reverse isoperimetric inequality for convex plane curves*, Beiträge Algebra Geom. **48** (2007), 303–308.

[115] G. Paouris, E. Werner, *Relative entropy of cone measures and L_p-centroid bodies*, Proc. Lond. Math. Soc. **104**(2) (2012), 253–286.

[116] L. Parapatits, F. E. Schuster, *The Steiner formula for Minkowski valuations*, Adv. Math. **230** (2012), 978–994.

[117] G. Pisier, *The Volume of Convex Bodies and Banach Space Geometry*, Cambridge University Press, Cambridge, 1989.

[118] A. V. Pogorelov, *The Minkowski multidimensional problem*, V. H. Winston & Sons, Washington, D.C., 1978.

[119] D. Pokorny, J. Rataj, *Normal cycles and curvature measures of sets with d.c. boundary*, Adv. Math. **248** (2013), 963–985.

[120] E. Rebrova, K. Tikhomirov, *Coverings of random ellipsoids, and invertibility of matrices with i.i.d. heavy-tailed entries*, Israel J. Math. **227** (2018), 507–544.

[121] M. D. Reid, R. C. Williamson, *Information, Divergence and Risk for Binary Experiments*, J. Mach. Learn. Res. **12** (2011), 731–817.

[122] A. Rényi, *On measures of entropy and information*, Proceedings of the 4th Berkeley Symposium on Probability Theory and Mathematical Statistics **1** (1961), 547–

561.

[123] B. A. Rogozin, *On the increase of dispersion of sums of independent random variables*, Teor. Verojatnost. i Primenen **6** (1961), 106–108.

[124] A. Ros, *The isoperimetric problem*, In Global theory of minimal surfaces, volume 2 of Clay Math. Proc., pages 175–209. Amer. Math. Soc., Providence, RI, 2005.

[125] M. Rudelson, *Lower estimates for the singular values of random matrices,* Compt. Rendus Math. de l'Académie des Sciences **342** (2006), no. 4, 247–252.

[126] M. Rudelson, *Lecture notes on non-asymptotic random matrix theory*, AMS Proceedings of Symposia in Applied Mathematics (2014), 83–121.

[127] M. Rudelson, R. Vershynin, *The least singular value of a random square matrix is* $O(n^{-\frac{1}{2}})$, Comptes rendus de l'Académie des sciences - Mathématique **346** (2008), 893–896.

[128] M. Rudelson, R. Vershynin, *The Littlewood-Offord problem and invertibility of random matrices*, Adv. Math. **218** (2008), no. 2, 600–633.

[129] M. Rudelson, R. Vershynin, *The smallest singular value of a random rectangular matrix*, Comm. Pure Appl. Math. **62** (2009), no. 12, 1707–1739.

[130] M. Rudelson, R. Vershynin, *Non-asymptotic theory of random matrices: extreme singular values*, Proc. ICM, Vol. 3 (2010), 1576–1602.

[131] J. R. Sangwine-Yager, *Bonnesen-Style Inequalities for Minkowski Relative Geometry*, Transactions of the American Mathematical Society **307** (1988), no. 1, 373–382.

[132] R. Schneider, *Convex bodies: The Brunn–Minkowski theory*, Second expanded edition, Cambridge press, 2014.

[133] C. Schütt, *Entropy numbers of diagonal operators between symmetric Banach spaces*, J. Approximation Theory **40** (1984), 121–128.

[134] C. Schütt, E. Werner, *The convex floating body*, Math. Scand. **66** (1990), 275–290.

[135] C. Schütt, E. Werner, *Surface bodies and p-affine surface area*, Adv. Math. **187**

(2004), 98–145.

[136] F. Schuster, *Crofton measures and Minkowski valuations*, Duke Math. J. **154** (2010), 1–30.

[137] J.W. Silverstein, *The smallest eigenvalue of a large dimensional Wishart matrix*, Ann. Probab. **13** (1985), 1364–1368.

[138] J. Silverstein, *On the weak limit of the largest eigenvalue of a large dimensional sample covariance matrix*, J. of Multivariate Anal. **30** (1989), 2, 307–311.

[139] S. Smale, *On the efficiency of algorithms of analysis*, Bull. Amer. Math. Soc. (N.S.) **13** (1985), 87–121.

[140] A. Stancu, *On the number of solutions to the discrete two-dimensional L_0-Minkowski problem*, Adv. Math. **180** (2003), 290–323.

[141] S. J.Szarek, *The finite-dimensional basis problem with an appendix on nets of Grassman manifold*, Acta Math. **141** (1983), 153–179.

[142] S. J. Szarek, *Spaces with large distance to ℓ_∞^n and random matrices*, American Journal of Mathematics **112** (1990), 899–942.

[143] S. Szarek, *Condition numbers of random matrices*, J. of Complexity **7** (1991), 131–149.

[144] K. Tatarko, *An upper bound on the smallest singular value of a square random matrix*, J. Complexity **48** (2018), 119–128.

[145] K. Tatarko, E. Werner, *A Steiner formula in the L_p Brunn Minkowski theory*, Adv. Math. **355** (2019), 106772.

[146] T. Tao, V. Vu, *Inverse Littlewood-Offord theorems and the condition number of random discrete matrices*, Ann. Math. **169** (2009), 595–632.

[147] T. Tao, V. Vu, *Smooth analysis of the condition number and the least singular value*, Math. Comp. **79** (2010), no. 272, 2333–2352.

[148] K. Tikhomirov, *The limit of the smallest singular value of random matrices with i.i.d. entries*, Adv. Math. **284** (2015), 1–20.

[149] K. Tikhomirov, *The smallest singular value of random rectangular matrices with no moment assumptions on entries*, Israel J. Math. **212** (2016), no. 1, 289–314.

[150] K. Tikhomirov, *Sample covariance matrices of heavy-tailed distributions*, Int. Math. Res. Notes **2018**(20) (2018), 6254–6289.

[151] R. Vershynin, *John's decompositions: selecting a large part*, Israel J. Math. **122** (2001), 253–277.

[152] R. Vershynin, *Spectral norm of products of random and deterministic matrices*, Probability Theory and Related Fields **150** (2011), 471–509.

[153] F. Wei, *Upper bound for intermediate singular values of random matrices*, JMAA **445** (2017), 1530–1547.

[154] E. Werner, *The p-affine surface area and geometric interpretations*, Rend. Circ. Math. Palermo Serie II **70** (2002), 367–382.

[155] E. Werner, *On L_p-affine surface areas*, Indiana Univ. Math. J. **56**(5) (2007), 2305–2324.

[156] E. Werner, *Rényi Divergence and L_p-affine surface area for convex bodies*, Adv. Math. **230** (2012), 1040–1059.

[157] E. Werner, D. Ye, *New L_p affine isoperimetric inequalities*, Adv. Math. **218**(3) (2008), 762–780.

[158] E. Werner, D. Ye, *Inequalities for mixed p-affine surface area*, Math. Ann. **347** (2010), 703–737.

[159] H. Weyl, *On the volume of tubes*, Amer. J. Math. **61** (1939), 461–472.

[160] T. Willmore, *Note on embedded surfaces*, An. Sti. Univ. "Al. I. Cuza" Iasi Sect. I a Mat. **11** (1965), 493–496.

[161] J. M. Wills, *Zum Verhaltnis von Volumen zur Oberfläche bei konvexen Körpern*, Arch. Math **21** (1970), 557–560.

[162] Y. Xia, W. Xu, J. Zhou, B. Zhu, *Reverse Bonnesen style inequalities in a surface $\mathbb{X}_\varepsilon^2$ of constant curvature*, Sci. China Math. **56** (2013), 1145–1154.

[163] P. Yaskov, *Lower bounds on the smallest eigenvalue of a sample covariance matrix*, Electron. Commun. Probab. **19** (2014), 1–10.

[164] P. Yaskov, *Sharp lower bounds on the least singular value of a random matrix without the fourth moment condition*, Electron. Commun. Probab. **20** (2015), no. 44,

[165] Y.Q. Yin, Z.D. Bai, P.R. Krishnaiah, *On the limit of the largest eigenvalue of the large dimensional sample covariance matrix*, Probab. Theory Rel. **78** (1988), 509–521.

[166] G. M. Ziegler, *Lectures on 0/1 polytopes,* in "Polytopes-Combinatorics and Computation" (G. Kalai and G. M. Ziegler, Eds), 1–44, DMV Seminars, Birkhäuser, Basel, 2000.

[167] G. Zhang, *Intersection bodies and Busemann-Petty inequalities in* $\mathbb{R}^4$, Ann. of Math. **140** (1994), 331–346.

[168] G. Zhang, *A positive answer to the Busemann-Petty problem in four dimensions*, Ann. of Math. **149** (1999), 535–543.

[169] G. Zhang, *New Affine Isoperimetric Inequalities*, ICCM 2007, Vol. II, 239–267.

[170] Y. Zhao, *The dual Minkowski problem for negative indices*, Calc. Var. Partial Differential Equations **56** (2):18 (2017).