University of Alberta

Orthogonal Polynomials and Their Applications in Financial and Actuarial Models

by

Hao Li

A thesis submitted to the Faculty of Graduate Studies and Research
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy
in
Mathematical Finance

Department of Mathematical and Statistical Sciences

Spring 2014
Edmonton, Alberta

Abstract

It is well known that the normal return estimation for financial asset prices is defective. In order to find better models to estimate the prices behavior of financial assets, people need probabilistic distribution that can capture fat-tails, non-constant moments, etc. This thesis find some distributions that can be utilized to model the financial asset returns and the actuarial claim sizes, with the help of some orthogonal polynomials. We use the Pearson's differential equation as a way for orthogonal polynomials construction and solution. The generalized Rodrigues formula is used for this goal. Deriving the weight function of the differential equation, we use it as a basic distribution density of variables like financial asset returns or insurance claim sizes. This density function is adjusted using the product with a polynomial, which is expressed as a linear combination of the orthogonal polynomials we find as the solutions of the Pearson's differential equation. Using this method, we create the Polynomial-Normal model, Polynomial-T-Distribution model and some further extensions. We derive explicit formulae for option prices as well as for insurance premiums. The numerical analysis shows that our new models provide a better fit than some previous actuarial and financial models.

Acknowledgements

I want to express my sincere gratitude to my supervisor Dr. Alexander Melnikov, for his continuous support to my studies and research, and for his immense knowledge and thoughtful guidance. I am also grateful to the other members of my thesis committee whose insightful suggestions and comments helped shape the final version of the thesis.

I would like to thank my family, for their understanding, patience and love. I am very grateful to the moral encouragement they provided me when it was most needed.

Contents

1 Introduction and Preliminaries **1**

1.1 Introduction 1

1.2 On modeling of Financial Returns 6

1.3 On the Actuarial Modeling 8

1.4 Outline 9

2 The Pearson's ODE and its Polynomial Solution **11**

2.1 The Pearson's Differential Equation and the Rodrigues Formula 11

2.2 Classification of the Pearson's Differential Equations and the Polynomial Solutions 30

2.2.1 Hermite Polynomial: $s_2(x)$ is a constant 31

2.2.2 Associated Laguerre Polynomial: $s_2(x)$ is a linear function 34

2.2.3 Jocobi Polynomial: $s_2(x)$ is a quadratic function with a positive discriminant 37

2.2.4 Romanovski Polynomial: $s_2(x)$ is a quadratic function with a negative discriminant 40

2.2.5 Bessel Polynomial: $s_2(x)$ is a quadratic function with a zero discriminant 43

3 Polynomial-Normal Model **45**

3.1 Polynomial-Normal Model and Related Considerations 45

3.2 Option Pricing in the framework of Polynomial-Normal Model 51

3.3 VaR And CVaR under the Polynomial-Normal model 61

4 Polynomial-T-Distribution Model 65

4.1 Polynomial-T-Distribution Model and Related Considerations 65

4.2 Option Pricing under Polynomial-T-Distribution Model 72

4.3 Value at Risk under Polynomial-T-Distribution Model 77

5 Model Extensions and Applications 80

5.1 The Polynomial-T-Distribution Model with Asymmetric Adjustor 80

5.2 Orthogonal Polynomials and Their Applications in Actuarial Modeling 87

5.2.1 Polynomial extension of Gamma distribution 87

5.2.2 Polynomial extension of Beta distribution 93

5.2.3 Polynomial extension of Inverse-Gamma distribution 97

6 Numerical Illustrations and Comparison 102

6.1 Numerical Illustrations for Financial Models 102

6.2 Numerical Illustration for Actuarial Models 111

List of Tables

2.1 Different classes of weight function, distributions and polynomials 30

6.1 Max likelihood and BIC value for daily returns of S&P 500 from 1996 to 2010 105

6.2 Max likelihood and BIC value for monthly returns of S&P 500 from 1996 to 2010 105

6.3 optimal parameters, 95% and 90% confidence intervals for the Polynomial-Normal model, fitted by daily return 106

6.4 optimal parameters, 95% and 90% confidence intervals for the Polynomial-Normal model, fitted by monthly return 106

6.5 optimal parameters, 95% and 90% confidence intervals for the Polynomial-T-Distribution model, fitted by daily return 107

6.6 optimal parameters, 95% and 90% confidence intervals for the Polynomial-T-Distribution model, fitted by monthly return 107

6.7 optimal parameters, 95% and 90% confidence intervals for the Polynomial-T-Distribution model with the asymmetric adjustor, fitted by daily return . . . 108

6.8 optimal parameters, 95% and 90% confidence intervals for the Polynomial-T-Distribution model with the asymmetric adjustor, fitted by monthly return . 108

6.9 option prices, $S_0 = 1000$, $r = 0.05$, 1 month expiry 109

6.10 one-day VaR and CVaR values, $S_0 = 1000$, $r = 0.05$ 110

6.11 one-month VaR and CVaR values, $S_0 = 1000$, $r = 0.05$ 110

6.12 parametrization results for the Polynomial-Gamma model, stop-loss premium and probability of ruin 112

6.13 parameterization results for the Polynomial-Beta model, stop-loss premium and probability of ruin 113

6.14 parameterization results for the Polynomial-Inverse-Gamma model, stop-loss premium and probability of ruin 113

List of Figures

3.1 Acceptable skewness-kurtosis region in the Polynomial-Normal model 51

4.1 Acceptable polynomial parameters region in the Polynomial-T-Distribution model 72

6.1 Logarithmic frequency for S&P 500 returns from 1950 104

CHAPTER 1

Introduction and Preliminaries

1.1 Introduction

The main goal of this thesis is to introduce and develop a new method to model the density function of some important variables in financial and actuarial sciences with the help of orthogonal polynomials.

Stochastic models and methods are widely used in comprehensive mathematical finance and actuarial science. For the reason of simplicity and applicability of known mathematical techniques, many popular models in these areas usually start with a number of assumptions and restrictions, which lead to an ideal situation and end up with some fixed distribution for estimated financial and actuarial assets. The Black-Scholes model has become the most well-known model in the analysis of financial asset pricing, with the benefit of its properties in mathematical theory and simplicity in numerical realization. But the shortcomings of Black-Scholes model are obvious - its assumptions on the trading market are too ideal to be possible in the real world (see Black (1989)). Normal distribution for the logarithmic returns of financial assets is one of the most important implications of the Black-Scholes model, and this result is severely doubted by other empirical and theoretical studies. Implications of the normality of financial asset returns, that disagree with the historical data, include the following: extreme returns (more than 3 standard deviations from mean) are only assigned a tiny probability which is nearly neglectable (please see Rachev *et al.* (2005)). Financial returns must have the same probability densities for the same level of prosperity and recession, as a result of the symmetry of the normal distribution and hence, the skewness and kurtosis for financial returns are fixed (see lau *et al.* (1989)). Besides that there are many practical evidences against normality for the financial asset returns. The shape market downs in the equity trading market happening time to time, come up to be one of the worst evidences against the normal assumption. Many financial institutes, especially many of the hedge funds, show distinct non-normal characteristic in their return data. Therefore, other distributions are needed to provide a better estimation for the financial asset returns.

Around such extensions of the Black-Scholes model, we would like to mention first the Gram-Charlier model. This model uses the product of the normal density and a 4^{th} degree polynomial as the financial return density, and thus allows arbitrary skewness and kurtosis (Jarrow and Rudd (1982); Madan and Milne (1994)). This is the first example of using polynomial to generate normal-like distributions to model financial returns. The polynomial used in the Gram-Charlier model is a linear combination of the Hermite polynomial series up to order 4, which is an orthogonal polynomial series based on the normal distribution (see Fedoryuk (2001)). If we extend the Gram-Charlier model by extending the degree of the polynomial used in this model to an arbitrary finite integer N, we arrived at the Polynomial-Normal model (see Li and Melnikov (2012)). The Polynomial-Normal model shows a better ability than the Gram-Charlier model in capturing higher moments besides the skewness and kurtosis. However, there is a restriction for the Gram-Charlier and the Polynomial-Normal model. The polynomials used in these models must be globally positive in order to generate a valid density function. We study this restriction and found the ranges for the valid polynomial parameters that generate a valid density function. The fair prices for the European call/put option, as well as, for the power option and the polynomial options, can be calculated in the framework of this new model. Moreover, the Value at risk (VaR) and conditional Value at Risk (CVaR) can also be determined.

Though the Polynomial-Normal model shows some advantages in capturing non-constant moment parameters, it is not more powerful than the simple normal distribution in estimating extreme returns. The nature of the density tail decay for the Polynomial-normal distribution is still the same as the simple normal distribution, and this property is not consistent with the fact of the fat-tail distribution for financial returns. Some studies focused on tail analysis, claimed that the financial return distribution exhibits power-law behavior (see Gabaix *et al.* (2003)). Considering such an observational result, the Student's T-distribution becomes a nice candidate for financial return densities (see Cassidy *et al.* (2010); Shaw (2011); Fergusson and Platen (2006); Rachev *et al.* (2005)). As a result, we can use the product of the Student's T-distribution density and a polynomial to model the financial return density, thereby improving the estimation on the decay speed of the density tail and the distribution shape around the mean. The polynomial used here is expressed as a linear combination of the Romanovski polynomial series, which is the orthogonal polynomial series based on the corresponding T-distribution. This extension was named as "Polynomial-T-Distribution model"(see Li and Melnikov (2013)). Option pricing and risk measures like VaR and CVaR can be calculated under the framework of this new model.

While we have the first two polynomial model extensions, we found this approach can be elegantly and widely generalized by means of the techniques connected to the famous Pearson's differential equation (see Raposo *et al.* (2007)). This equation is defined as follows:

$$s_2(x)F''(x) + s_1(x)F'(x) + \lambda F(x) = 0, \tag{1.1}$$

where $s_1(x)$ and $s_2(x)$ are polynomials of x with at most first and second degree. The weight function $w(x)$ is defined as the solution of the differential equation

$$(w(x)s_2(x))' = w(x)s_1(x), \tag{1.2}$$

The weight function is found to be a non-polynomial solution of a similar differential e-quation. The weight function plays an important role in exploring the properties of the differential equation. And it can be standardized to be a density function on a certain interval. If

$$\lambda = -ns_1'(x) - \frac{n(n-1)s_2''(x)}{2} \tag{1.3}$$

for some non-negative integer n, there exists a single polynomial solution for the differential equation, and a non-polynomial solution. The polynomial solution is given by the so-called generalized Rodrigues formula (see Raposo *et al.* (2007)), and the non-polynomial solution can be derived from the polynomial solution. The property of orthogonality or partial orthogonality between these polynomial solutions can be proved via the generalized Rodrigues formula. We also provide the general solution for the Pearson's differential equation, in the form of generalized hypergeometric function. This general solution can be reduced to a polynomial if the condition (1.3) holds. The Pearson's differential equations can be splitted into a few classes, which are determined by the quadratic function $s_2(x)$.

When $s_2(x)$ is a constant, the weight function $w(x)$ in (1.2) can be standardized to be the normal density, and the polynomial solution of the Pearson's differential equation (1.1) becomes the Hermite polynomial series. When $s_2(x)$ is a quadratic function of x with a negative discriminant, the weight function can be standardized to be the T-distribution density, and the polynomial solution become the Romanovski polynomial series.

Thus we can generalize our previous approach and considerations by using the product of the standardized weight function $w(x)$, and a linear combination of the polynomial solution series solved by the Pearson's differential equation, to model the density function for random variables like the financial asset returns. The Polynomial-Normal model and the Polynomial-T-Distribution model become special cases of this approach.

Besides the Polynomial-Normal model and the Polynomial-T-Distribution model, we consider and explore other types of the Pearson's differential equations, and the possibility of using the distributions they derive in financial and actuarial modeling. Based on different classes of the Pearson's differential equations, we use the similar method to derive different theoretical models to fit the distribution density of financial asset returns and actuarial claim portfolio amounts.

As we explore many distributions and use them for the purpose of financial and actuarial modeling, a verification for the model extension is needed. We use both historical data and artificial data to test the efficiency of our model extensions. Parametrization methods of different models will be discussed. Maximum likelihood and moment matching are the two main methods we use for parametrization. Advantages and disadvantages of these two parametrization methods are discussed in comparison with each other. We also compare different model extensions and assess the goodness of fitting for these models, based on both historical and artificial data (see Li and Melnikov (2013)). Statistical properties of the parameter estimators are also discussed. Numerical results of option prices and risk measures for financial models are given and compared, as well as the probability of ruin and the stop-loss premium for the actuarial models. The best model will be selected, based on the goodness of fit to different data, and the simplicity of these models.

1.2 On modeling of Financial Returns

The normal distribution, also called Gaussian distribution in memory of the great mathematician Gauss, is frequently used as an assumption for the logarithmic return distribution for varieties of financial assets. It possesses the mathematical properties of the following: first, it is a stable distribution; second, it has a finite mean and variance. Stability of a distribution means that any linear combination of independent random variables following this distribution is still following such a distribution, but may be with different drift and scale parameters. This property helps the researchers to establish financial pricing model using stochastic process, as it is convenient of assume the return distribution over different periods are all normal. The property of finite mean and variance also helps to establish a model that is easily mathematically manipulated. But most studies examining the validity of these normal assumption failed find support for it.

Empirical studies on the stock prices have found that the extreme returns occur more frequently than predicted by the normal assumption. While the stock prices are shifting modestly at some periods, there will be periods when the prices swing up and down dramatically. Some people tried to explain it by the changes of fundamental economic variables, but it seems a better explanation can be given via the heavy-tail distributions (see, for instance, Shiller (1981)). Support on the heavy-tail distributions are found by studies modeling the financial market agents' interactions. One of the well-known studies regarding this aspect is conducted by Arthur *et al.* (1997), who introduced an asset pricing modeling, where different financial market agents continuously adjust their expectation for the financial market. Another explanation was given by Cont and Bouchaud (2000), who studied the herd behaviors of market participators.

People doubted that the financial return is following non-normal stable distributions, which are named as "stable Paretian" or "Levy stable" distributions in some literatures. The stable Paretian distributions draw much attention in this area as it presents heavy tails and skewness while maintaining stability, which is a desirable property for financial return distributions. The earliest published studies regarding the stable Paretian distribution in the financial market context might be given by Mandelbrot (1963), and his research was further developed by Fama (1965). However, empirical studies report that the tails of historical stock returns are heavier than the normal distribution, but thinner than the stable Paretian distributions (see, for instance, Akgiray and Booth (1988)).

While the stable Paretian distribution is found inconsistent with empirical data, people continue to look for other distributions for the proper modeling of the financial returns. The Student's T-distribution has become one of the alternatives, as it presents great similarity with historical data in the density tail. Blattberg and Gonedes (1974) might be the earliest published research using the T-distribution to fit the financial return density. The drawback of the T-distribution in financial return distribution modeling would be its lack of stability, which makes it difficult to find interconnections with diffusion models. However, it excels at fitting the historical data and is still worth our attention. This thesis gives a financial return model extension based on the T-distribution with the name " Polynomial-T-Distribution model", and it will be discussed at Chapter 4.

1.3 On the Actuarial Modeling

Mathematical and statistical methods have been applied in the insurance industry for a long time. Nowadays, the most frequently used methods in actuarial sciences modeling the risks arising from casualty insurance, might be the collective risk model. The first introduction of the collective risk model may date back to the beginning of the last century (see Buhlmann (1997)). In the framework of the collective risk model, the total insurance claim amount is modeled as the sum of the individual claims, the number of which is described by some probabilistic distributions. The Poisson distribution is one of the most frequently used distribution here, as it is resulted from the mechanism of accumulated chances with risk events. Other distribution are also used to model the number of claims sometimes, as they are more reasonable at certain situations. However, to our knowledge, there are no fixed models to estimate the claim severity for the variety of casualty insurance claims.

Different ways are used to estimate the distribution of claim severity, or to estimate the distribution of total claims directly. A few categories of methods that are frequently used are as following:

1. Parametric models. In a typical parametric model, it is assumed that the claim severity or the total claim amount is following a certain distribution, and the parameters are estimated by the ways such as maximum likelihood or moment matching. This is a relatively convenient method. However, its main drawback in practice is that, the accuracy of the estimation

largely depends on how the assumed distribution fits the real data.

2. Non-parametric methods. By the non-parametric methods, the expected claim amount is calculated from historical data, as a function of statistical quantities such as mean and variance. The non-parametric methods avoid inaccuracies caused by distribution assumption. But it is usually hard to find a valid data set that is large enough, since observations dated a long time ago would loose their usefulness to represent the future.

3. Monte Carlo method. This is a good way if the mechanism of the risk event is known or well modeled. However, it usually needs a long calculation time by computer.

We extend our method of polynomial adjusted distribution of model the total claim amount. Our way should be categorized as a parametric method. We have to select a proper basic density function, which is similar to the real density of the total claim amount. The polynomial is used to reduce the difference between our estimation and the real density. The higher degree of polynomial we use, the better level of accuracy we can reach.

1.4 Outline

The thesis contains six chapters. Chapter 1 provides a general description of the problem and the basic idea of our approach. It also tells some historical researches and some basic situations of the fields, and how our approach is related to the previous researches.

Chapter 2 is about the Pearson's differential equation, its polynomial solutions and their related properties. We give a review of the Pearson's differential equation (1.1), its polynomial solution solved by the generalized Rodrigues formula and the related properties. The classification of the differential equations and the corresponding polynomial solution of each type are discussed in the subsections of chapter 2.

Chapter 3 is devoted to the Polynomial-Normal model. The related properties and some considerations regarding to this model are also discussed. The formula for the Value at Risk (VaR) and option prices are also provided, as well as the power option and polynomial option prices.

Chapter 4 discusses the Polynomial-T-Distribution model, and the related properties and considerations. Similar as Chapter 3, we provide the calculation of VaR and the option prices.

Other financial model extensions and actuarial applications will be discussed in Chapter 5. We consider the model generated from other types of Pearson's differential equations. We further extend the Polynomial-T-Distribution model and introduce the asymmetric adjustor for the financial models. Actuarial applications are found through the polynomial model with Gamma, Inverse-Gamma, and Beta distribution. For financial returns modeling, we derive the European option prices and the VaR/CVaR values. For actuarial claim amount modeling, we derive the formulae for the stop-loss premium and probability of ruin at the maturity.

Chapter 6 is devoted to the numerical examples and illustrations. The model extensions for financial return are fitted and parameterized by maximum likelihood method, using historical return data. Goodness of fitting of different models is quantified by the Bayesian Information Criteria (BIC) and compared between different model extensions. Statistical properties such as confidence intervals are found. Option prices and VaR/CVaR are calculated and compared. We use the artificial data to parameterized the actuarial model extensions. Stop-loss premium and probability of ruin are calculated.

CHAPTER 2

The Pearson's Differential Equation and its Polynomial Solution

2.1 The Pearson's Differential Equation and the Rodrigues Formula

This section gives a review of the famous Pearson's differential equation and shows how to use the generalized Rodrigues formula to solve it. The Rodrigues formula was first introduced independently by Rodrigues (1816), Ivory (1824) and Jacobi (1827), to provide a construction of the Legendre polynomials (see Askey, 2005). Later such equation was exploited in more general aspects. Many properties of the polynomials can be recognized using the system of the differential equation and the generalized Rodrigues formula. These properties are extremely important and useful to provide financial and actuarial model extensions.

The Pearson's differential equation is defined as follows:

$$s_2(x)F''(x) + s_1(x)F'(x) + \lambda F(x) = 0, \tag{2.1}$$

where $s_1(x)$ and $s_2(x)$ are polynomials of x with at most first and second degree. The general solution of this differential equation can be expressed as a generalized hypergeometric function. When the parameters of the differential equation satisfy to a certain condition, the solution is reduced to a polynomial. Suppose a polynomial $F(x)$ with degree n is a solution of equation (2.1), the following equation is obtained by eliminating the term x^n at the left hand side of the differential equation (see Raposo *et al.*, 2007)

$$\lambda = -ns_1'(x) - \frac{n(n-1)s_2''(x)}{2}. \tag{2.2}$$

The parameter λ is called the eigenvalue of the Pearson's differential equation (see Raposo *et al.* (2007)). Denote the eigenvalue λ given by (2.2) as λ_n. The following lemma proves the uniqueness of the polynomial solution under certain conditions.

Lemma 2.1. *If the condition (2.2) holds for a non-negative integer n and $\lambda_k \neq \lambda_n$ for any non-negative integer $k < n$, the differential equation (2.2) has a unique polynomial solution with degree n, neglecting scaler multiplications.*

Proof. Denote $s_2(x)$, $s_1(x)$ and $F(x)$ in the differential equation (2.1) as

$$s_2(x) = ax^2 + bx + c, \tag{2.3}$$

$$s_1(x) = px + q, \tag{2.4}$$

$$F(x) = \sum_{k=0}^{\infty} a_k x^k. \tag{2.5}$$

The following relationship is obtained by plugging the above expression and the condition (2.2) in the differential equation (2.1)

$$c(k+2)(k+1)a_{k+2} + (k+1)(bk+q)a_{k+1} + (ak(k-1) + pk - an(n-1) - pn)a_k = 0. \tag{2.6}$$

We note that

$$ak(k-1) + pk - an(n-1) - pn = \lambda_n - \lambda_k \neq 0$$

for any non-negative integer $k < n$. If we let $a_k = 0$ for any $k > n$, the linear system (2.4) is solved as

$$\begin{aligned} a_{n-1} &= \frac{n(b(n-1)+q)}{p+2a(n-1)} a_n, \\ a_k &= \frac{c(k+2)(k+1)a_{k+2} + (k+1)(bk+q)a_{k+1}}{-ak(k-1) - pk + an(n-1) + pn}, \text{ for any } 0 \leq k \leq n-2. \end{aligned}$$

Let a_n be a non-zero constant and we arrive at a unique solution for $F_(x)$. □

Denote this unique solution as $F_n(x)$. The second property of the polynomial series begins from differentiation. Calculating the m^{th} derivative of the equation (2.1) with the polynomial $F_n(x)$, we obtain the following equation:

$$s_2(x)(F_n^{(m)}(x))'' + (s_1(x) + ms_2'(x))(F_n^{(m)}(x))' + (\frac{m(m-1)}{2}s_2''(x) + ms_1'(x) + \lambda)F_n^{(m)}(x) = 0.$$

If we denote

$$s_{1,m}(x) = s_1(x) + ms_2'(x), \tag{2.7}$$

$$\lambda_{n,m} = \frac{m(m-1) - n(n-1)}{2}s_2''(x) + (m-n)s_1'(x) = \lambda_n - \lambda_m, \tag{2.8}$$

then the above differential equation becomes

$$s_2(x)(F_n^{(m)}(x))'' + s_{1,m}(x)(F_n^{(m)}(x))' + \lambda_{n,m}F_n^{(m)}(x) = 0. \tag{2.9}$$

Thus, $F_n^{(m)}(x)$, as a polynomial of degree $n-m$, is the solution of a similar differential equation. Let us introduce the weight function $w(x)$, as a solution of the differential equation (under a proper scale):

$$(s_2(x)w(x))' = s_1(x)w(x). \tag{2.10}$$

It is called as the weight function because it plays the role of a weight function in the inner product and orthogonality between polynomials. The solution of the above equation can be obtained as (see Raposo *et al.*, 2007)

$$w(x) = A\frac{1}{s_2(x)}\exp(\int \frac{s_1(x)}{s_2(x)}dx). \tag{2.11}$$

The weight function is a non-polynomial solution of the Pearson's differential equation connected to the Pearson's differential equation. By differentiating (2.10), we arrive to the next equation

$$s_2(x)w''(x) + (2s_2'(x) - s_1(x))w'(x) + (s_2''(x) - s_1'(x))w(x) = 0.$$

If the constant A is set such that $w(x)$ is positive within the interval $I = (a_1, a_2)$ and the integral of $w(x)$ over I is 1, the weight function $w(x)$ can be further defined as a density function for some random variable on the interval I. With all the above preparation, we are ready to introduce the generalize Rodrigues formula, which gives the explicit solution of the polynomial $F_n(x)$.

The polynomial solution $F_n(x)$ of equation (2.1) and its derivatives are given by the generalized Rodrigues formula presented in the following theorem.

Theorem 2.1. *If $F_n(x)$ is a n^{th}-degree polynomial and the solution of the differential equation (2.1) and $\lambda_{n,m} \neq 0$ for any non-negative integer $m < n$, the m^{th} derivative of $F_n(x)$ should follow the formula below:*

$$F_n^{(m)}(x) = N_{n,m} \frac{1}{w(x)s_2^m(x)} \frac{d^{n-m}}{dx^{n-m}}(w(x)s_2^n(x)), \tag{2.12}$$

$$\textit{where } N_{n,m} = (-1)^m N_{n0} \prod_{k=0}^{m-1} \lambda_{n,k}, \tag{2.13}$$

$$\textit{for any} \quad n \geq m \geq 0, N_{n,0} \in \mathbf{R}.$$

The generalized proof of the Theorem 2.1 is given in Nikiforov (1988). We give a different proof of the formulae (2.12) - (2.13) by induction method.

Proof. Let $F_n^*(x) = w(x)s_2^n(x)$. By calculating the first and second derivatives of $F^*(x)$, we have the following formulae:

$$F_n^{*'}(x) = w(x)s_2^{n-1}(x)(s_1(x)+(n-1)s_2'(x)), \quad (2.14)$$

$$F_n^{*''}(x) = w(x)s_2^{n-2}(x)(s_1(x)+(n-2)s_2'(x))(s_1(x)+(n-1)s_2'(x)) + w(x)s_2^{n-1}(x)(s_1'(x)+(n-1)s_2''(x)). \quad (2.15)$$

From the above formulae, we can verify that

$$s_2(x)F_n^{*''}(x) - (s_1(x)+(n-2)s_2'(x))F^{*'}(x) - (s_1'(x)+(n-1)s_2''(x))F^*(x) = 0. \quad (2.16)$$

Differentiate the above formula for $n-m-2$ times, we have

$$s_2(x)\frac{d^{n-m}}{dx^{n-m}}F_n^*(x) - (s_1(x)+ms_2'(x))\frac{d^{n-m-1}}{dx^{n-m-1}}F_n^*(x) + ((\frac{m(m+1)}{2}-\frac{n(n-1)}{2})s_2''(x)+(m-n+1)s_1'(x))\frac{d^{n-m-2}}{dx^{n-m-2}}F_n^*(x) = 0. \quad (2.17)$$

To prove the theorem, we use the induction method. When $m=n$, $F_n^{(m)}$ is a constant, as a result of the n^{th} differentiation of the n^{th}-degree polynomial $F_n(x)$. Denote $F_n^{(n)}$ as $N_{n,n}$, and this denotation is consistent with the theorem.

As $\lambda_{n,m} \neq 0$, we arrive to the following induction formula from (2.9)

$$F_n^{(m)}(x) = \frac{-s_2(x)F_n^{(m+2)}(x) - s_{1,m}(x)F_n^{(m+1)}(x)}{\lambda_{n,m}}, \text{ for any } 0 \leq m < n \quad (2.18)$$

When $m = n - 1$, the equation (2.18) becomes

$$\begin{aligned} F_n^{(n-1)}(x) &= \frac{((n-1)s_2'(x) + s_1(x))N_{n,n}}{(n-1)s_2''(x) + s_1'(x)} \\ &= N_{n,n-1}(s_1(x) + (n-1)s_2'(x)) \\ &= N_{n,n-1}\frac{1}{w(x)s_2^{n-1}(x)}\frac{d}{dx}(w(x)s_2^n(x)). \end{aligned} \tag{2.19}$$

The above equation verifies the theorem when $m = n - 1$. If the theorem is satisfied for any $m = n, n-1, \cdots k+1$ where $n - 2 \geq k \geq 0$, we can verify the theorem when $m = k$ as follows, using equations (2.9) and (2.17):

$$\begin{aligned} F_n^{(m)}(x) &= \frac{-1}{\lambda_{n,m}}(s_2(x)F_n^{(m+2)}(x) + (s_1(x) + ms_2'(x))F_n^{(m+1)}(x)) \\ &= \frac{-1}{\lambda_{n,m}}\left(\frac{N_{n,m+2}}{w(x)s_2^{m+1}(x)}\frac{d^{n-m-2}}{dx^{n-m-2}}(w(x)s_2^n(x))\right. \\ &\quad \left. + \frac{N_{n,m+1}(s_1(x) + ms_2'(x))}{w(x)s_2^{m+1}(x)}\frac{d^{n-m-1}}{dx^{n-m-1}}(w(x)s_2^n(x))\right) \\ &= \frac{-N_{n,m+1}}{\lambda_{n,m}w(x)s_2^{m+1}(x)}\left(-\lambda_{n,m+1}\frac{d^{n-m-2}}{dx^{n-m-2}}F_n^*(x)\right. \\ &\quad \left. + (s_1(x) + ms_2'(x))\frac{d^{n-m-1}}{dx^{n-m-1}}F_n^*(x)\right) \\ &= \frac{-N_{n,m+1}}{\lambda_{n,m}w(x)s_2^m(x)}\frac{d^{n-m}}{dx^{n-m}}F_n^*(x) \\ &= \frac{N_{n,m}}{w(x)s_2^m(x)}\frac{d^{n-m}}{dx^{n-m}}(w(x)s_2^n(x)). \end{aligned} \tag{2.20}$$

Therefore, the theorem is proved for every $n \geq m \geq 0$. □

Remark 2.1. If there is a non-negative integer $m < n$ such that $\lambda_{n,m} = 0$, we the following equation by factorizing the equation (2.8)

$$\lambda_{n,m} = (m-n)(\frac{m+n}{2}s_2''(x) + s_1'(x)) = 0.$$

If $s_2''(x) = s_1'(x) = 0$, the differential equation (2.1) is degenerated. The solution of the differential equation becomes the Bessel function if the parameter $\lambda \neq 0$ in the differential equation, or a linear function of x if $\lambda = 0$. We are not going with a detail discussion for this special case.

If $s_2''(x) \neq 0$, we have

$$m + n = -2s_1'(x)/s_2''(x).$$

The solution of the differential equation (2.1) becomes a polynomial with degree m, which is given by the generalized Rodrigues formula (2.12) - (2.13), but using m instead of n.

For simplicity, we let $N_{n,0} = 1$, then the equation (2.12) indicates a unique polynomial. With the explicit formula for the polynomials given, we can explore the orthogonality between the polynomial solutions in the following sense. For any interval $I = (a_1, a_2)$, with $\infty \geq a_1 > a_2 \geq -\infty$, if $w(x)$ is positive on I, we can define the inner product between polynomials as follows:

$$< p, q >= \int_{x=a_1}^{a_2} p(x)q(x)w(x)dx \tag{2.21}$$

for any polynomials p, q with real parameters.

Two polynomials are said to be orthogonal to each other if the inner product between them is zero. The corresponding results about orthogonality of such polynomials are presented in the following theorem. We give it to provide a completeness of exposition of our approach, while similar results can be found in the literature (see, for instance Kirshnamoorthy, 1951).

Theorem 2.2. *If $F_n(x)$ is given by equation (2.12), with $N_{n,0} = 1$, and $s_1(x), s_2(x), w(x)$*

satisfy

$$\lim_{x \to a_1^+} w(x)s_2(x)x^l - \lim_{x \to a_2^-} w(x)s_2(x)x^l = 0, \quad \textit{for any } l = 0, 1, \cdots L, \tag{2.22}$$

then the inner product

$$< F_n, F_m >= \delta_{nm} f_n^2, \tag{2.23}$$

where δ_{nm} *denotes the Kronecker delta notation,* n *and* m *take any non-negative integer under the condition of* $n + m \leq L + 1$*, and*

$$f_n^2 = (-1)^n \frac{N_{n,n}}{N_{n,0}} \int_{x=a_1}^{a_2} w(x)s_2^n(x)dx.$$

Proof. Suppose $n \geq m$, Let $F_m^*(x) = w(x)s_2^m(x)$. Using integration by parts for multiple times, the inner product between F_n and F_m can be calculated as follows:

$$\begin{aligned}
< F_n, F_m > &= \int_{x=a_1}^{a_2} \frac{1}{w(x)} \frac{d^n}{dx^n}(w(x)s_2^n(x)) \frac{d^m}{dx^m}(w(x)s_2^m(x))dx \\
&= \frac{1}{N_{n,0}} \int_{a_1}^{a_2} \frac{d^m}{dx^m}(F_m^*(x))F_n(x)dx \\
&= -\frac{1}{N_{n,0}} \int_{a_1}^{a_2} \frac{d^{m-1}}{dx^{m-1}}(F_m^*(x))F_n'(x)dx + \frac{1}{N_{n,0}} \frac{d^{m-1}}{dx^{m-1}}(F_m^*(x))F_n(x)\bigg|_{a_1}^{a_2} \\
&\cdots \\
&= \frac{(-1)^m}{N_{n,0}} \int_{a_1}^{a_2} F_m^*(x) \frac{d^m}{dx^m}(F_n(x))dx + \sum_{k=0}^{m-1} \frac{(-1)^k}{N_{n,0}} \frac{d^{m-k-1}}{dx^{m-k-1}} F_m^*(x) \frac{d^k}{dx^k} F_n(x)\bigg|_{a_1}^{a_2}
\end{aligned}$$

$$
\begin{aligned}
&= (-1)^m \int_{a_1}^{a_2} \frac{N_{n,m}}{N_{n,0}} \frac{d^{n-m}}{dx^{n-m}}(w(x)s_2^n(x))dx + \\
&\quad \sum_{k=0}^{m-1} \frac{(-1)^k N_{n,k}}{w(x)s_2^k(x)N_{n,0}} \frac{d^{n-k}}{dx^{n-k}}(w(x)s_2^n(x)) \frac{d^{m-k-1}}{dx^{m-k-1}}(w(x)s_2^m(x))\Bigg|_{a_1}^{a_2} \\
&= (-1)^m \int_{a_1}^{a_2} \frac{N_{n,m}}{N_{n,0}} \frac{d^{n-m}}{dx^{n-m}}(w(x)s_2^n(x))dx + \\
&\quad \sum_{k=0}^{m-1} \frac{(-1)^k w(x)s_2(x)}{N_{m,k+1}N_{n,0}} (s_2^k(x)\frac{d^{k+1}}{dx^{k+1}}F_m(x)\frac{d^k}{dx^k}F_n(x))\Bigg|_{a_1}^{a_2}. \qquad (2.24)
\end{aligned}
$$

The second term of the above formula, can be rewritten as the integration of $w(x)s(x)$ times a $m+n-1^{th}$ degree polynomial. If $m+n-1 \leq L$ and the equation (2.23) is satisfied, the second term can be reduced to 0. If $n > m$, the above calculation continues as follows:

$$
\begin{aligned}
< F_n, F_m > &= (-1)^m \int_{x=a_1}^{a_2} \frac{N_{n,m}}{N_{n,0}} \frac{d^{n-m}}{dx^{n-m}}(w(x)s_2^n(x))dx \\
&= (-1)^m \frac{N_{n,m}}{N_{n,0}} \frac{d^{n-m-1}}{dx^{n-m-1}}(w(x)s_2^n(x))dx\Bigg|_{a_1}^{a_2} \\
&= \frac{(-1)^m N_{n,m}w(x)s_2^{m+1}(x)}{N_{n,m+1}N_{n,0}} \frac{d^{m+1}}{dx^{m+1}}(F_n(x))dx\Bigg|_{a_1}^{a_2} \\
&= 0. \qquad (2.25)
\end{aligned}
$$

If $n = m$, the inner product becomes

$$
< F_n, F_m >= (-1)^n \frac{N_{n,n}}{N_{n,0}} \int_{x=a_1}^{a_2} w(x)s_2^n(x)dx. \qquad (2.26)
$$

□

Remark 2.2. The orthogonal polynomials are generated by the Pearson's differential equation only when $s_2(x)$ and $s_1(x)$ satisfy certain conditions. It is also possible that orthogonality is satisfied when the sum of polynomial degrees is bounded by a certain level, which arise the case of finite orthogonality. This is the case when we deal with the Romanovski, Lagurre and Bessel polynomials.

In order to access to the orthogonality between polynomials, the weight function, should be converging to 0 with a certain degree at both of the endpoints of the interval. When this condition is satisfied, the weight function turns out to be integrable on the interval I. Therefore, we can standardize the weight function $w(x)$ and make it a density function on the interval I by multiplication with a proper constant. For example, when $s_2(x)$ is a constant, $w(x)$ can be standardize and becomes the density function of the normal distribution, and $\{F_n(x)\}$ becomes the Hermite polynomial series in this case.

When the full or finite orthogonality is satisfied between the polynomial solutions, we can express $F_n(x)$ in the matrix form presented by the following proposition:

Proposition 2.1. *If $F_n(x)$ is given by equation (2.12) and the condition (2.23) is satisfied for $l = 0, 1, \cdots, L$, then*

$$F_n(x) = A \det \begin{bmatrix} p_1(x) & p_2(x) & \cdots & p_{n+1}(x) \\ m_{1,1} & m_{1,2} & \cdots & m_{1,n+1} \\ m_{2,1} & m_{2,2} & \cdots & m_{2,n+1} \\ \vdots & & \ddots & \vdots \\ m_{n,1} & m_{n,2} & \cdots & m_{n,n+1} \end{bmatrix}, \text{ for } n = 0, 1, \cdots, 1 + \lfloor \frac{L}{2} \rfloor. \tag{2.27}$$

where $p_1(x), p_2(x), \cdots, p_{n+1}(x)$ are any $n+1$ linear independent polynomials of x with the degree no more then n, $q_1(x), q_2(x), \cdots, q_n(x)$ are any n linear independent polynomials of x with that degree no more than $n-1$, and $m_{i,j} = \int_{a_1}^{a_2} q_i(x)p_j(x)w(x)dx$, and A represents *any constant.*

Proof. From the theorem of polynomial orthogonality, we know that $< F_n(x), F_m(x) >= 0$ when $m \neq n$ and $m + n \leq 1 + L$. If we ignore the scaler multiplication, there is only one degree n polynomial that is orthogonal to all polynomials with degree less than n, as $n + (n-1) \leq 1 + L$. The equation (2.27) presents a polynomial with the degree at most n. The inner product between the polynomial (2.27) and $q_i(x)$ is

$$
\begin{aligned}
& \int\limits_{x=a_1}^{a_2} A \det \begin{bmatrix} p_1(x) & p_2(x) & \cdots & p_{n+1}(x) \\ m_{1,1} & m_{1,2} & \cdots & m_{1,n+1} \\ m_{2,1} & m_{2,2} & \cdots & m_{2,n+1} \\ \vdots & & \ddots & \vdots \\ m_{n,1} & m_{n,2} & \cdots & m_{n,n+1} \end{bmatrix} q_i(x) w(x) dx \\
= \quad & A \det \begin{bmatrix} < p_1(x), q_i(x) > & < p_2(x), q_i(x) > & \cdots & < p_{n+1}(x), q_i(x) > \\ m_{1,1} & m_{1,2} & \cdots & m_{1,n+1} \\ m_{2,1} & m_{2,2} & \cdots & m_{2,n+1} \\ \vdots & & \ddots & \vdots \\ m_{n,1} & m_{n,2} & \cdots & m_{n,n+1} \end{bmatrix} \\
= \quad & A \det \begin{bmatrix} m_{i,1} & m_{i,2} & \cdots & m_{i,n+1} \\ m_{1,1} & m_{1,2} & \cdots & m_{1,n+1} \\ m_{2,1} & m_{2,2} & \cdots & m_{2,n+1} \\ \vdots & & \ddots & \vdots \\ m_{n,1} & m_{n,2} & \cdots & m_{n,n+1} \end{bmatrix} \\
= \quad & 0.
\end{aligned}
$$

Therefore, the polynomial (2.27) is orthogonal to any polynomial $q_i(x)$ and as well as to any polynomial with the degree no more than $n-1$, because of the independence of the polynomial series $q_i(x)$. So (2.27) equals $F_n(x)$ if we ignore the scaler multiplication. □

If we let $p_j(x) = q_j(x) = x^j$ in the above formula, we arrive to

$$F_n(x) = A \det \begin{bmatrix} 1 & x & \cdots & x^n \\ m_0 & m_1 & \cdots & m_n \\ m_1 & m_2 & \cdots & m_{n+1} \\ \vdots & & \ddots & \vdots \\ m_{n-1} & m_n & \cdots & m_{2n-1} \end{bmatrix},$$

where $m_k = \int_{a_1}^{a_2} x^k w(x)dx$ is the $\mathrm{k^{th}}$ moment of the distribution defined by $w(x)$.

When the generalized Rodrigues formula (2.12) - (2.13) is satisfied, the polynomial solution $F_n(x)$ can be derived from the recursive formula. The following theorem presents the recursive relation for the polynomial series $F_n(x)$.

Theorem 2.3. *For any positive integer n, if the polynomial series $F_k(x), k = 0, 1, \cdots, n+1$ is given by the generalized Rodrigues formula (2.12) - (2.13) with $s_2(x) = ax^2 + bx + c$ and $s_1(x) = px + q$, the coefficient $N_{k,0} = 1$, and $\lambda_k \neq \lambda_l$ for any $k, l \leq n+1$ and $k \neq l$, then*

$$\begin{aligned} F_{n+1}(x) &= \frac{p+2na-a}{p+na-a}\left((p+2na)x + \frac{2n^2ab - 2nab + 2nbp + pq - 2aq}{p+2na-2a}\right)F_n(x) \\ &+ \frac{n(2na+p)((n-1)(4ac-b^2)(na-a+p) + aq^2 - bpq + cp^2)}{(na-a+p)(2na-2a+p)}F_{n-1}(x). \end{aligned} \tag{2.28}$$

Proof. Following from the generalized Rodrigues formula (2.12), the following equation is derived using the product rule of differentiation

$$\begin{aligned} F_n(x) &= \frac{1}{w(x)}\frac{d^n}{dx^n}(w(x)s_2^n(x)) \\ &= \frac{1}{w(x)}\frac{d^{n-1}}{dx^{n-1}}(w(x)s_2^{n-1}(x)(s_1(x)+(n-1)s_2'(x))). \end{aligned} \tag{2.29}$$

If we use the product rule of differentiation twice on $F_{n+1}(x)$, we arrive to the following equation

$$\begin{aligned} F_{n+1}(x) &= \frac{1}{w(x)}\frac{d^{n+1}}{dx^{n+1}}(w(x)s_2^{n+1}(x)) \\ &= \frac{1}{w(x)}\frac{d^n}{dx^n}(w(x)s_2^n(x)(s_1(x)+ns_2'(x))) \\ &= \frac{1}{w(x)}\frac{d^{n-1}}{dx^{n-1}}\big(w(x)s_2^{n-1}(x) \\ &\quad ((s_1(x)+ns_2'(x))(s_1(x)+(n-1)s_2'(x))+(s_1'(x)+ns_2''(x))s(x)\big). \end{aligned} \tag{2.30}$$

At the mean while, if we use the product rule of differentiation and the Leibniz's formula on $F_{n+1}(x)$, we have

$$\begin{aligned} F_{n+1}(x) &= \frac{1}{w(x)}\frac{d^{n+1}}{dx^{n+1}}(w(x)s_2^{n+1}(x)) \\ &= \frac{1}{w(x)}\frac{d^n}{dx^n}(w(x)s_2^n(x)(s_1(x)+ns_2'(x))) \\ &= \frac{s_1(x)+ns_2'(x)}{w(x)}\frac{d^n}{dx^n}(w(x)s_2^n(x)) \\ &\quad +\frac{n(s_1'(x)+ns_2''(x))}{w(x)}\frac{d^{n-1}}{dx^{n-1}}(w(x)s_2^n(x)) \\ &= (s_1(x)+ns_2'(x))F_n(x)+\frac{n(s_1'(x)+ns_2''(x))}{w(x)}\frac{d^{n-1}}{dx^{n-1}}(w(x)s_2^n(x)). \end{aligned} \tag{2.31}$$

Use a linear combination on the equations (2.29) - (2.31), we arrive at the following

$$\begin{aligned}
&\gamma \cdot (2.29) + \theta \cdot (2.30) + (1-\theta) \cdot (2.31) : \\
&F_{n+1}(x) = ((1-\theta)(s_1(x) + ns_2'(x)) - \gamma)\, F_n(x) + \frac{1}{w(x)} \frac{d^{n-1}}{dx^{n-1}} \big(w(x) s_2^{n-1}(x) \\
&(r(s_1(x) + (n-1)s_2'(x)) + \theta(s_1(x) + ns_2'(x))(s_1(x) + (n-1)s_2'(x)) \\
&+\theta(s_1'(x) + ns_2''(x))s(x) + n(1-\theta)(s_1'(x) + ns_2''(x))s_2(x))\big)\,, \qquad (2.32)
\end{aligned}$$

where γ and θ are predetermined parameters. Our goal here is to find the proper parameters γ and θ to reduce the following polynomial to a constant denoted as A:

$$\begin{aligned}
A &= r(s_1(x) + (n-1)s_2'(x)) + \theta(s_1(x) + ns_2'(x))(s_1(x) + (n-1)s_2'(x)) \\
&\quad +\theta(s_1'(x) + ns_2''(x))s(x) + n(1-\theta)(s_1'(x) + ns_2''(x))s_2(x). \qquad (2.33)
\end{aligned}$$

If the above equation is satisfied for any parameters γ and θ, the second term of the equation (2.32) would become a multiple of $F_{n-1}(x)$ and the recursive relation would be found. Plug the equations $s_2(x) = ax^2 + bx + c$ and $s_1(x) = px + q$ in the equation (2.33), the proper parameters γ and θ are solved as

$$\gamma = \frac{n(2na - a + p)(-bp + 2aq)}{(2na - 2a + p)(na - a + p)}$$
$$\theta = \frac{-na}{p + (n-1)a}.$$

One can verify that if $\lambda_k \neq \lambda_l$ for any $k \neq l$ and $k, l \leq n+1$, the denominators of the above values of γ and θ are non-zero. So the above parameter solutions are valid if the condition holds. Plug these parameters in the equation (2.32), we have the following

$$\begin{aligned} F_{n+1}(x) &= \frac{p+2na-a}{p+na-a}\left((p+2na)x + \frac{2n^2ab-2nab+2nbp+pq-2aq}{p+2na-2a}\right)F_n(x) \\ &+ \frac{n(2na+p)((n-1)(4ac-b^2)(na-a+p)+aq^2-bpq+cp^2)}{(na-a+p)(2na-2a+p)}F_{n-1}(x). \end{aligned}$$

□

The equation (2.1) is a second order ODE, whose solution should be in the form of a linear sum of two different basic solutions. Though Theorem 1 gives one basic solution for the differential equation, one might wonder what the second basic solution is. The following theorem gives the general solution for the differential equation, in the form of a combination of two basic solutions, while the condition (2.2) holds.

Theorem 2.4. *If the function $F(x)$ satisfies the equations (2.1) and (2.2), and the polynomial solution is given by the generalized Rodrigues formula (2.12) - (2.13), then*

$$F(x) = F_n(x)(C_1 + C_2 \int \frac{dx}{w(x)s_2(x)F_n^2(x)}), \tag{2.34}$$

where C_1 and C_2 are constants and $F_n(x)$ is given by the Rodrigues formula in (2.12) and (2.13).

Proof. From the Rodrigues formula, we know that the function $F_n(x)$ is a solution of the differential equation (2.1) - (2.2). Assume the general solution $F(x) = F_n(x)g(x)$. We put this expression in the differential equation (2.1) and the following equation is derived

$$g(x)(s_2(x)F_n''(x) + s_1(x)F_n'(x) + \lambda F_n(x)) +$$
$$2s_2(x)g'(x)F_n'(x) + s_2(x)F_n(x)g''(x) + s_1(x)F_n(x)g'(x) = 0.$$

Because $F_n(x)$ is the solution of the differential equation, the first part of the above formula is 0. The second part ends up with

$$2s_2(x)g'(x)F_n'(x) + s_2(x)F_n(x)g''(x) + s_1(x)F_n(x)g'(x) = 0. \tag{2.35}$$

If $g'(x)$ is not 0, it can be derived as follows

$$\frac{g''(x)}{g'(x)} = -\frac{s_1(x)}{s_2(x)} - \frac{2F_n'(x)}{F_n(x)}, \tag{2.36}$$
$$g'(x) = \exp(-\int \frac{s_1(x)}{s_2(x)} - \frac{2F_n'(x)}{F_n(x)} dx) \tag{2.37}$$
$$= \frac{C_2}{w(x)s_2(x)F_n^2(x)}. \tag{2.38}$$

Then

$$g(x) = C_1 + C_2 \int \frac{dx}{w(x)s_2(x)F_n^2(x)}, \tag{2.39}$$

and

$$F(x) = F_n(x)(C_1 + C_2 \int \frac{dx}{w(x)s_2(x)F_n^2(x)}). \tag{2.40}$$

□

The general solution of the Pearson's differential equation can be written in the form of generalized hypergeometric series. The polynomial solution only exists as a special case for the general solution when the condition (2.2) holds. We explore the general solution as a generalized hypergeometric function in the following discussion. Define the Pochhammer symbol as follows:

$$(x)_n = \prod_{k=1}^{n}(x+k-1) \text{ for any non-negative integer } n \text{ and real number } x. \tag{2.41}$$

A generalized hypergeometric series with the order p and q is defined as follows:

$${}_pF_q(a_1, a_2, \cdots, a_p; b_1, b_2, \cdots, b_q; x) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_p)_k}{(b_1)_k (b_2)_k \cdots (b_q)_k} \frac{x^k}{k!}. \tag{2.42}$$

If one of a_k is a negative integer, the above series become a polynomial as $(a_k)_n = 0$ when $n > -a_k$. Recall the notation of (2.3) - (2.5) for the coefficients and solution of the Pearson's differential equation

$$s_2(x) = ax^2 + bx + c,$$

$$s_1(x) = px + q,$$

$$F(x) = \sum_{k=0}^{\infty} a_k x^k.$$

If the function $s_2(x)$ is not a constant, we can use the proper linear transformation to on the variable x and make $c = 0$ for the function $s_2(x)$. The transformed $s_2(x)$ and $s_1(x)$ may contain complex coefficients after the linear transformation. If $c = 0$ for $s_2(x)$, we plug in the above expressions of $s_2(x)$, $s_1(x)$ and $F(x)$ in the Pearson's differential equation, we have the following relationship for the parameters of $F(x)$:

$$k(kb+q-b)a_k + (a(k-1)(k-2)+p(k-1)+\lambda)a_{k-1} = 0. \quad (2.43)$$

Therefore, if the $jb+q-b \neq 0$ for any integer $j \leq k$, the coefficient a_k can be expressed as

$$\begin{aligned} a_k &= \frac{-a(k-1)(k-2)-p(k-1)-\lambda}{k(kb+q-b)} a_{k-1} \\ &= \prod_{j=1}^{k} \frac{-a(j-1)(j-2)-p(j-1)-\lambda}{j(jb+q-b)} a_0 \\ &= \frac{(1-\alpha_1)_k(1-\alpha_2)_k}{(1)_k(q/b)_k} \left(\frac{-a}{b}\right)^k, \end{aligned} \quad (2.44)$$

where α_1 and α_2 are the two roots of $a(x-1)(x-2)+p(x-1)+\lambda$. The solution of the Pearson's differential equation can be expressed

$$F(x) =_2 F_1(1-\alpha_1, 1-\alpha_2; q/b; \frac{-ax}{b}). \quad (2.45)$$

Remark 2.3. The equation (2.45) can be reduced in two cases. Case 1, if the quadratic function $a(x-1)(x-2)+p(x-1)+\lambda$ is reduced to a linear function or even a constant, the generalized hypergeometric function would be reduced to the form of ${}_1F_2$ or ${}_0F_2$ correspondingly. Case 2, if the $jb+q-b=0$ for some integer j, the generalized hypergeometric function of (2.45) would be reduced to ${}_3F_2(1+j-\alpha_1, 1+j-\alpha_2, 1; 1+j, j+q/b; \frac{-ax}{b})x^j$.

Another special case we want to mention is the case when the function $s_2(x)$ in the Pearson's differential equation (2.1) is a constant. This case will lead to the normal density as the weight function and the Hermite polynomial as the solution of the differential equation. The generalized hypergeometric function (2.45) is not valid in this case. But the polynomial solution can be expressed in the form of the generalized hypergeometric function in another way. We will discuss this method in detail at the section 2.2.

2.2 Classification of the Pearson's Differential Equations and the Polynomial Solutions

With different selections of $s_1(x)$ and $s_2(x)$, we arrive at the different weight function $w(x)$ and the polynomial solution $F_n(x)$. The weight function $w(x)$ can be standardized to be density function for some random variable. The distribution defined by such a density function is called the Pearson distribution (see Pearson (1893, 1895, 1901, 1916)). For example, when $s_2(x) = 1 - x^2$, we arrive at the Beta distribution and the Jacobi polynomials. The Beta distribution is called the type I Pearson distribution. If the additional condition $s_1(0) = 0$ is satisfied, the resulted distribution is the symmetric version of the Beta distribution, which is classified as disIn fact, we can classify the corresponding distributions and polynomials using the discriminant of the second order polynomial $s_2(x)$. After standardization, $s_2(x)$ in the differential equation (2.1) can be reduced to one of five basic forms, which are listed in Table 2.1.

Table 2.1: Different classes of weight function, distributions and polynomials

$\tilde{s}_2(x)$	$\tilde{s}_1(x)$	$w(x)$	Distributions	Polynomials	Support
1	$-x$	$\exp(-x^2/2)$	Normal	Hermite	$(-\infty, \infty)$
x	$-x+b$	$\exp(-x)x^{b-1}$	Gamma	Laguerre	$(0, \infty)$
$1+x^2$	$ax+b$	$(1+x^2)^{\frac{a}{2}-1}\exp(b\tan^{-1} y)$		Romanovski	$(-\infty, \infty)$
$1-x^2$	$ax+b$	$(1-x)^{(a+b-2)/2}(1+x)^{(b-a-2)/2}$	Beta	Jacobi	$(-1, 1)$
$1-x^2$	$ax+b$	$(x-1)^{(a+b-2)/2}(1+x)^{(b-a-2)/2}$	F	Jacobi	$(1, \infty)$
x^2	$ax+b$	$x^{a-2}\exp(\frac{-b}{x})$	Inverse Gamma	Bessel	$(0, \infty)$

2.2.1 Hermite Polynomial: $s_2(x)$ is a constant

When $s_2(x)$ in the Pearson's differential equation (2.1) is a non-zero constant, we can standardized the differential equation and let $s_2(x) = 1$. Assume $s_1(x) = px + q$, and the weight function can be derived from (2.11) as

$$w(x) = A\exp(\frac{p}{2}(x + \frac{q}{p})^2).$$

However, not all parameters p, q generate orthogonal polynomials. If orthogonality between different polynomial solutions holds, the condition of orthogonality (2.23) must be satisfied. We conclude that only if the parameter $p < 0$, the weight function can be standardized to be a density function, and the generated polynomials are orthogonal to each other.

Remark 2.4. If the parameter $p > 0$ in $s_1(x)$, the weight function cannot be standardized to be a density function on the whole real line. The polynomial solution of the Pearson's differential equation is given by the generalized Rodrigues formula (2.12) - (2.13), but these polynomials are not orthogonal with each other. If the parameter $p = 0$, the generalized Rodrigues formula is not satisfied. The solution of the Pearson's differential equation can be expressed as a linear combination of exponential functions.

Denote the solution of the Pearson's differential equation as $H(x)$. If we assume $p < 0$ in $s_1(x) = px + q$, we can use the proper linear transformation on the variable of x, and further standardized the Pearson's differential equation as

$$H''(x) - xH'(x) + \lambda H(x) = 0. \tag{2.46}$$

From the equation (2.11), the weight function $w(x)$ of the above differential equation is

$$w(x) = A\exp(\frac{-x^2}{2}). \tag{2.47}$$

This weight function is the density function of the standardized normal distribution if we let $A = 1/\sqrt{2\pi}$. If we use the generalized Rodrigues formula (2.12) - (2.13) and let the parameter $N_{n,0} = (-1)^n$, the polynomial solution $H_n(x)$ can be derived as

$$H_n(x) = \frac{(-1)^n}{\exp}(\frac{x^2}{2})\frac{d^n}{dx^n}(\frac{-x^2}{2}). \tag{2.48}$$

We set the parameter $N_{n,0} = (-1)^n$ in the generalized Rodrigues formula for the above equation. With such a parameter setting, the polynomial $H_n(x)$ equates the Hermite polynomial of degree n. The Hermite polynomial series is named after the mathematician Charles Hermite in 1860s, although they are studied earlier by Laplace and Chebyshev. Using the generalized Rodrigues formula (2.12) - (2.13)

$$H_n^{(m)}(x) = n(n-1)\cdots(n-m+1)H_{n-m}(x). \tag{2.49}$$

The recursive formula for the Hermite polynomial series can be obtained by plugging the coefficients of (2.46) in the equation (2.28):

$$H_{n+1}(x) = xH_n(x) - nH_{n-1}(x). \tag{2.50}$$

Some other properties for the Hermite polynomial series can be derived to help with our derivation of the financial model extensions.

Corollary 2.1. *If $H_n(x)$ is given by (2.48) and t is a real number, then*

$$H_n(x+t) = \sum_{k=0}^{n} \binom{n}{k} t^k H_{n-k}(x). \tag{2.51}$$

Proof. We can prove this equation by induction. From the equation (2.48), it is found that $H_0(x) = 1$ and it is consistent with (2.51). Assume the equation is valid when $n = j$. We can calculate the partial derivative of the Hermite polynomial by

$$\frac{\partial}{\partial t} H_{j+1}(x+t) = (j+1)H_j(x+t).$$

Then we can write $H_{j+1}(x+t)$ as an integral plus a proper constant C.

$$\begin{aligned}
H_{j+1}(x+t) &= (j+1)\int H_j(x+t)dt + C \\
&= (j+1)\int \sum_{k=0}^{j} \binom{j}{k} t^k H_{j-k}(x)dt + C \\
&= \sum_{k=0}^{j} \binom{j}{k} \frac{(j+1)t^{k+1}}{k+1} H_{j-k}(x) + C \\
&= \sum_{k=1}^{j+1} \binom{j+1}{k} t^k H_{j+1-k}(x) + C.
\end{aligned}$$

Plug $t = 0$ in the above equation, the constant term is found as $C = H_{j+1}(x)$. Therefore the equation (2.51) is valid when $n = j+1$ and eventually for every non-negative integer n. ∎

The Hermite polynomial can be expressed in the form of a hypergeometric function. We have explored the expression of $H_n(x)$ as a generalized hypergeometric function in the last section. This expression is presented based on the equation (2.43) - (2.45). However, the

Hermite polynomial series is a special case, for which this method does not work. We explore the Hermite polynomial series using the Pearson's differential equation (2.46). From (2.2), we know the solution of the differential equation is a polynomial with degree n if $\lambda = n$. Let

$$H(x) = \sum_{k=0}^{\infty} a_k x^k.$$

Plug this expression in the differential equation (2.46), we have

$$k(k-1)a_k + (-k+2+n)a_{k-2} = 0. \tag{2.52}$$

Therefore, if we let $a_n = 1$ and $a_{n+1} = 0$, we can express a_k as

$$a_k = \begin{cases} \dfrac{(k+1)(k+2)\cdots n}{(-2)(-4)\cdots(k-n)} & \text{if } n-k \text{ is even,} \\ 0 & \text{if } n-k \text{ is odd.} \end{cases} \tag{2.53}$$

So, we can express $H_n(x)$ as

$$\begin{aligned} H_n(x) &= n! \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^m}{(n-2m)!m!2^m} x^{n-2m} \\ &= x^n \, {}_2F_0\left(\frac{-n}{2}, \frac{-n+1}{2}; -2x^{-2}\right). \end{aligned} \tag{2.54}$$

2.2.2 Associated Laguerre Polynomial: $s_2(x)$ is a linear function

If $s_2(x)$ in the Pearson's differential equation (2.1) is a linear function of x and cannot be reduced to a constant, we can standardized the differential equation by linear transformation, and let $s_2(x) = x$. Let us assume $s_1'(x) \neq 0$ for the Pearson's differential equation. Using the proper linear transformation, we can further standardize the differential equation and let $s_1(x) = -x+q$. Denote the solution of the Pearson's differential equation as $L(x)$. Thus, the Pearson's differential equation becomes

$$xL''(x) + (-x+q)L'(x) + \lambda L(x) = 0. \quad (2.55)$$

Remark 2.5. If $s_2(x) = x$ and $s_1'(x) = 0$ in the Pearson's differential equation (2.1), the solution of the differential equation is a linear function of x if $\lambda = 0$, or a combination of the Bessel functions if $\lambda \neq 0$. We shall focus on other non-trivial cases and will not go for further discuss for these special cases.

The weight function $w(x)$ of the differential equation (2.55) can be derived from (2.11) as

$$w(x) = A\exp(-x)x^{q-1}, \quad (2.56)$$

If $q > 0$ and the constant $A = 1/\Gamma(q)$, the weight function $w(x)$ is standardized and becomes the density function of a Gamma distribution with degree q. The supporting interval of this distribution is $(0, +\infty)$. This distribution is also categorized as the type III Pearson distribution. Denote the polynomial solution with degree n of the differential equation (2.55) as $L_{n,q}(x)$. It can be derived from the generalized Rodrigues formula (2.12) - (2.13)

$$L_{n,q}(x) = (-1)^n x^{1-q} e^x \frac{d^n}{dx^n}(x^{n+q-1}e^{-x}). \quad (2.57)$$

We set the parameter $N_{n,0} = (-1)^n$ in the generalized Rodrigues formula for the above equation. The polynomial series $\{L_n(x)\}_n$ are called the associated Laguerre polynomials.

It can be reduced to the Laguerre polynomials if $q = 1$. The associated Laguerre polynomials are called Sonin polynomials in some literatures, as they were studied by the mathematician Nikolay Yakovlevich Sonin. One can verify that the condition of orthogonality (2.23) is satisfied for any positive number l, therefore the polynomials $L_{n,q}(x)$ with different degree n are orthogonal to each other. By the generalized Rodrigues formula (2.12) - (2.13), the derivatives of $L_{n,q}(x)$ can be derived as follows

$$\begin{aligned} L_{n,q}^{(m)}(x) &= (-1)^n x^{1-m-q} e^x \frac{d^{n-m}}{dx^{n-m}} (x^{n+q-1} e^{-x}) \\ &= (-1)^{n-m} L_{n-m,q+m}(x). \end{aligned} \tag{2.58}$$

From the equation (2.28), we can derive the recursive formula for $L_n(x)$.

$$L_{n+1,q}(x) = (x - 2n - q)L_{n,q}(x) - n(n + q - 1)L_{n-1,q}(x). \tag{2.59}$$

If we use the equation (2.43) to study the coefficients of $L_{n+1,q}(x)$, we arrive at the following equation for the polynomial coefficients a_k:

$$k(k + q - 1)a_k + (-k + 1 + n)a_{k-1} = 0.$$

Therefore, if we let $a_n = 1$, we have

$$\begin{aligned} a_k &= \prod_{j=1}^{n-k} \frac{-(n + 1 - j)(n - j + q)}{j}, \\ L_{n+1,q}(x) &= (-1)^n q(q + 1) \cdots (q + n - 1)\ {}_1F_1\ (-n; q; x). \end{aligned} \tag{2.60}$$

Remark 2.6. Please note that the above formula for $F_n(x)$ is valid only when q is not an integer between $-n+1$ and 0. If $0 \geq q \geq -n+1$ and q is an integer, we can instead express $F_n(x)$ as

$$F_{n,q}(x) = (-1)^q q(q+1)\cdots(-1)\binom{n}{1-q}\, {}_1F_1\,(-n+1-q; 2-q; x)x^{q+1}. \tag{2.61}$$

2.2.3 Jocobi Polynomial: $s_2(x)$ is a quadratic function with a positive discriminant

If $s_2(x)$ is a quadratic function with a positive discriminant in the Pearson's differential equation (2.1), we can standardize the differential equation and let $s_2(x) = 1 - x^2$. Let the function $s_1(x) = -(\alpha+\beta+2)x + \beta - \alpha$. Denote the solution of the differential equation as $J(x)$. The Pearson's differential equation becomes

$$(1-x^2)J''(x) - ((\alpha+\beta+2)x + \alpha - \beta)J'(x) + \lambda J(y) = 0. \tag{2.62}$$

The weight function $w(x)$ of the above equation can be calculated from the equation (2.11) as following

$$w(y) = A|1-x|^{\alpha}|1+x|^{\beta}. \tag{2.63}$$

The above $w(x)$ can be standardize to be a density function of a Beta distribution or a F-distribution. If $\alpha, \beta > -1$, we can standardize the weight function $w(x)$ to be the density function of a beta distribution, with the supporting interval $(-1, 1)$. The constant $A = 2^{-\alpha-\beta-1}/B(\alpha+1, \beta+1)$ in this case. The Beta distribution is also categorized as the type I Pearson distribution. If $\alpha > -1$, $\alpha+\beta < -1$ and $A = 2^{-\alpha-\beta-1}/B(\alpha+1, -\alpha-\beta-1)$, the weight function becomes the density function of an F-distribution, with the supporting interval $(1,\infty)$. If $\beta > -1$, $\alpha+\beta < -1$ and $A = 2^{-\alpha-\beta-1}/B(\beta+1,-\alpha-\beta-1)$, the weight function becomes the density function of an F-distribution, with the supporting interval $(-\infty,-1)$. The F-distribution is also categorized as the type VI Pearson distribution, and is also known as the inverted Beta or Beta prime distribution. Denote the polynomial solution of (2.62) with degree n as $J_{n,\alpha,\beta}(x)$. The polynomial solution can be solved by the generalized Rodrigues formula (2.12) - (2.13) as following.

$$\begin{aligned} J_{n,\alpha,\beta}(x) &= \frac{(-1)^n}{n!}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^n}{dx^n}((1-x)^{\alpha+n}(1+x)^{\beta+n}) \\ &= \sum_{k=0}^{n}\binom{n+\alpha}{k}\binom{n+\beta}{n-k}(-1+x)^{n-k}(1+x)^k. \end{aligned} \tag{2.64}$$

We set $N_{n,0} = (-1)^n/n!$ in the generalized Rodrigues formula in the above equation. It is conventional to use such a setting, as the resulted $J_{n,\alpha,\beta}(x)$ becomes consistent with the Jacobi polynomial with degree n. By checking the condition of orthogonality (2.23), we conclude that $J_{n,\alpha,\beta}(x)$ with different n is orthogonal with each other, if $w(x)$ is standardized as a density function of a Beta distribution with degree $\alpha+1$ and $\beta+1$, and orthogonality is defined based on such a distribution. However, if orthogonality between polynomials is defined based on the F-distribution, only finite orthogonality is granted between $J_{n,\alpha,\beta}(x)$ with different n. Precisely speaking, $J_{n,\alpha,\beta}(x)$ and $J_{m,\alpha,\beta}(x)$ are orthogonal to each other if and only if $n \neq m$ and $n+m < -1-\alpha-\beta$.

Some other properties of $J_{n,\alpha,\beta}(x)$ are found useful in the financial model extension discussed later. From the equation (2.64), we derive the property of symmetry for $Jn,\alpha,\beta(y)$ as follows,

$$J_{n,\alpha,\beta}(y) = (-1)^n J_{n,\beta,\alpha}(-y). \tag{2.65}$$

The property of symmetry requires switching the shape parameters for the polynomial $J_{n,\alpha,\beta}(y)$. From the Rodrigues formula, it is found that the derivative of the Jacobi polynomial can be expressed in the form of another Jacobi polynomial.

$$\begin{aligned} J^{(m)}_{n,\alpha,\beta}(x) &= \frac{(-1)^{n-m}}{n!}\prod_{k=0}^{m-1}((n-k)(\alpha+\beta+1+n+k)) \\ &\quad (1-x)^{-\alpha-m}(1+x)^{-\beta-m}\frac{d^{n-m}}{dx^{n-m}}((1-x)^{\alpha+n}(1+x)^{\beta+n}) \\ &= \prod_{k=0}^{m-1}(\alpha+\beta+1+n+k)J_{n-m,\alpha+m,\beta+m}(x). \end{aligned} \tag{2.66}$$

If we plug the coefficients of $s_2(x)$ and $s_1(x)$ of the differential equation (2.62) in the formula (2.28), we arrive at the following recursive formula.

$$\begin{gathered} n(n+\alpha+\beta)(2n+\alpha+\beta-2)J_{n,\alpha,\beta}(x) = \\ (2n+\alpha+\beta-1)\{(2n+\alpha+\beta)(2n+\alpha+\beta-2)x+\alpha^2-\beta^2\}J_{n-1,\alpha,\beta}(x) \\ -4(n+\alpha-1)(n+\beta-1)(2n+\alpha+\beta)J_{n-2,\alpha,\beta}(x). \end{gathered} \tag{2.67}$$

The Jacobi Polynomial can be expressed in the form of a hypergeometric series, around the point ± 1. If we use the transformation by letting $y = (1-x)/2$, the differential equation (2.62) becomes:

$$y(1-y)J''_{n,\alpha,\beta}(y) - ((\alpha+\beta+2)y-(\alpha+1))J'_{n,\alpha,\beta}(y) + \lambda J_{n,\alpha,\beta}(y) = 0. \tag{2.68}$$

Plug the coefficients of the equation (2.68) in the equation (2.45) and we can express

$J_{n,\alpha,\beta}(x)$ as follows

$$J_{n,\alpha,\beta}(x) = {}_2F_1\left(-n, 1+n+\alpha+\beta; \beta-\alpha; y\right) = {}_2F_1\left(-n, 1+n+\alpha+\beta; \beta-\alpha; 1-x\right). \quad (2.69)$$

2.2.4 Romanovski Polynomial: $s_2(x)$ is a quadratic function with a negative discriminant

If $s_2(x)$ is a quadratic function with a negative discriminant in the Pearson's differential equation (2.1), we can standardize the differential equation and let $s_2(x) = 1 + x^2$. Let the function $s_1(x) = px + q$. Denote the solution of the Pearson's differential equation as $R(x)$. The Pearson's differential equation becomes

$$(1 + x^2)R''(x) + (px + q)R'(x) + \lambda R(y) = 0. \quad (2.70)$$

The weight function $w(x)$ of the above equation can be calculated from the equation (2.11) as following

$$w(x) = A(1 + x^2)^{p/2-1} \exp(q \arctan x). \quad (2.71)$$

When $p < 1$, the above weight function can be standardized to a density function. If we let

$$A = \frac{\Gamma(\frac{qi-p}{2} + 1)\Gamma(\frac{-qi-p}{2} + 1)}{2^p \pi \Gamma(-p + 1)},$$

the weight function $w(x)$ in (2.71) is standardized to be a density function. The distribution defined by such a density function is categorized as the type IV Pearson distribution, and it was studied as a generalization of the Student's T-distribution (see Koepf and Masjed-Jamei

(2006)). We note that when the parameter $q = 0$ in the weight function $w(x)$, it is reduced to the density function of a Student's T-distribution, with degree of freedom $1 - p$. The difference between the T-distribution and distribution defined by (2.71) is the term of $\exp(q \arctan x)$. We name this term the asymmetric adjustor for the T-distribution, as it adds asymmetry to the T-distribution. It is important to note that such a distribution does not necessarily has a finite mean and variance. Only when p is negative in the weight function $w(x)$, the mean of the corresponding distribution is finite. Only when $p < -1$ in the weight function, the corresponding distribution has a finite variance. Denote the polynomial solution with degree n of the differential equation (2.70) as $R_{n,p,q}(x)$. Using the generalized Rodrigues formula (2.12) - (2.13), we can derive the polynomial solution $R_{n,p,q}(x)$ of the differential equation (2.70) as following

$$R_{n,p,q}(x) = \frac{(1+x^2)^{-p/2+1} \exp(-q \arctan x)}{n!} \frac{d^n}{dx^n}((1+x^2)^{p/2-1+n} \exp(q \arctan x)). \quad (2.72)$$

We set the parameter $Nn, 0 = 1/n!$ in the generalized Rodrigues formula for the above equation. The polynomial in (2.72) is called the Romanovski polynomial. The Romanovski polynomials were discovered in 1884 by Routh in the form of complexified Jacobi polynomials on the unit circle in the complex plane (see Raposo *et al.* (2007)) and were rediscovered as the orthogonal polynomials on some probability distribution by Romanovski (Romanovsky (1929)). If we check the condition of orthogonality (2.23), we conclude that $R_{n,p,q}(x)$ and $R_{m,p,q}(x)$ are orthogonal with each other based on the weight function (2.71), if $m \neq n$ and $m + n < 1 - p$. Using the generalized Rodrigues formula, we can derive the derivatives of the polynomial $R_{n,p,q}(x)$ as

$$\begin{aligned} R^{(m)}_{n,p,q}(x) &= (1+x^2)^{-p/2+1-m}\exp(-q\arctan x)\frac{d^{n-m}}{dx^{n-m}}((1+x^2)^{p/2+n-1}\exp(q\arctan x)) \\ &= R_{n-m,p+2m,q}(x). \end{aligned} \tag{2.73}$$

If we plug the coefficients of the differential equation in the equation (2.28), the recursive formula of $R_{n,p,q}(x)$ as follows:

$$\begin{aligned} R_{n+1,p,q}(x) &= \left(\frac{(p+2n-1)(p+2n)x}{p+n-1} + \frac{(p+2n-1)(pq-2q)}{(p+n-1)(p+2n-2)}\right)R_{n,p,q}(x) \\ &\quad + \frac{n(2n+p)((2n-2+p)^2+q^2)}{(n-1+p)(2n-2+p)}R_{n-1,p,q}(x). \end{aligned} \tag{2.74}$$

The Romanovski polynomials can be derived by complexifying the Jacobi polynomials on the unit circle. Consider the Pearson's differential equation (2.70) with the variable x in the complex plane. If we use the transformation $y = ix$, the differential equation can be expressed with the variable y as

$$(1-y^2)F''(-iy) - (py+iq)F'(-iy) - \lambda F(-iy) = 0. \tag{2.75}$$

Therefore, using the equation (2.64) and (2.69), we can express the Romanovski polynomial in the form of generalized hypergeometric function as

$$\begin{aligned} F_{n,p,q}(x) &= J_{n,\frac{p+iq}{2}-1,\frac{p-iq}{2}-1}(ix) \\ &= {}_2F_0\,(-n,-1+n+p;-iq;1-ix). \end{aligned} \tag{2.76}$$

2.2.5 Bessel Polynomial: $s_2(x)$ is a quadratic function with a zero discriminant

If the function $s_2(x)$ in the Pearson's differential equation (2.1) is a quadratic function of x with a zero discriminant, we can use the proper linear transformation on x and let $s_2(x) = x^2$. Denote $s_1(x) = px + q$. If $q = 0$, the weight function $w(x)$ and the solution of the differential equation will both be reduced to power function of x. We shall discard this trivial case and assume $q \neq 0$. If we apply the proper linear transformation on x, we can further standardize the differential equation and let $s_1(x) = px + 1$. Denote the solution of the Pearson's differential equation as $B(x)$. The Pearson's differential equation becomes

$$x^2 B''(x) + (px + 1)B'(x) + \lambda B(x) = 0. \tag{2.77}$$

The weight function $w(x)$ can be derived by plug the parameters of the above differential equation in the formula (2.11)

$$w(x) = Ax^{p-2} \exp(-\frac{1}{x}). \tag{2.78}$$

If $p < -1$, the above weight function can be standardize to be a density function by letting $A = 1/\Gamma(1-p)$. The supporting interval of the density function is $(0, \infty)$. The corresponding distribution is call the inverse gamma distribution, as the reciprocal of the random variable with such a density function is following a gamma distribution. This distribution is also categorized as the type V Pearson distribution. Denote the polynomial solution with degree n of the differential equation (2.78) as $B_{n,p}(x)$. Using the generalized Rodrigues formula

(2.12) - (2.13), $B_{n,p}(x)$ can be derive as

$$B_{n,p}(x) = x^{2-p}\exp(\frac{1}{x})\frac{d^n}{dx^n}(x^{p+2n-2}\exp(-\frac{1}{x})). \tag{2.79}$$

We set the parameter $N_{n,0} = 1$ in the generalized Rodrigues formula for the above equation. The derivatives of $B_{n,p}(x)$ can be derived by the generalized Rodrigues formula as well.

$$\begin{aligned} B^{(m)}_{n,p}(x) &= x^{2-2m-p}\exp(\frac{1}{x})\frac{d^{n-m}}{dx^{n-m}}(x^{p+2n-2}\exp(-\frac{1}{x})). \\ &= B_{n-m,p+2m}(x). \end{aligned} \tag{2.80}$$

If we plug the parameters of the differential equation (2.77) in the equation (2.28), we arrive to the recursive formula for the polynomial series $B_{n,p}(x)$ as follows:

$$\begin{aligned} B_{n+1,p}(x) &= (\frac{(p+2n-1)(p+2n)x}{p+n-1} + \frac{(p+2n-1)(p-2)}{(p+n-1)(p+2n-2)})B_{n,p}(x) \\ &+ \frac{n(2n+p)}{(n-1+p)(2n-2+p)}B_{n-1,p}(x). \end{aligned} \tag{2.81}$$

We can express $B_{n,p}$ in form of a generalized hypergeometric function by plug the parameters of the differential equation in the formula (2.45)

$$B_{n,p} = {}_2F_1\ (-n, n-p-1; 1, -x). \tag{2.82}$$

CHAPTER 3

Polynomial-Normal Model

3.1 Polynomial-Normal Model and Related Considerations

Let $\{\Omega, F, P\}$ be a probability space which will be used in models below. Among the models in our discussion, the following standard assumptions are supposed to hold true (see Black and Scholes (1973)):

1. the market consists of two components: a non-risky asset B (bank account) and a risky asset S (stock); both assets are perfectly divisible;
2. the interest rate is a non-negative constant r;
3. the stock pays no dividends;
4. no penalty and restriction on borrowing and short selling;
5. no transaction costs;

6. the market is complete; Q is the unique risk-neutral probability measure.

One well-known market model that satisfies the conditions above is the Black-Scholes model. Under the risk-neutral probability measure Q (martingale measure), the variation of stock price on the time interval $[0, T]$ is restricted by the following equality

$$E_Q(S_T) = S_0 e^{rT}. \tag{3.1}$$

Define the (logarithmic) stock return as $R_T = \ln(S_T/S_0)$ and the moment generating function of R_T as $M(\theta) = E_Q(e^{\theta R_T})$. Thus

$$M(1) = E_Q(e^{R_T}) = E_Q(\frac{S_T}{S_0}) = e^{rT}.$$

In the Black-Scholes model R_T is normally distributed (for given S_0 and T): $R_T \sim N(m, \delta^2)$, where m and δ are the mean and the standard deviation of R_T respectively. Then $M(\theta) = \exp(m\theta + \theta^2\delta^2/2)$ and $M(1) = \exp(m + \delta^2/2) = \exp(rT)$. Therefore

$$m = rT - \frac{\delta^2}{2}. \tag{3.2}$$

Due to the Gaussian nature of the Black-Scholes model, both the skewness and kurtosis of the return are fixed constants no matter what values m and δ take. However, many real-world examples confirm that skewness and kurtosis are different from those provided by the Black-Scholes model (see for instance, Kling and Alles (1994)).

To provide a better model of the stock prices, the Black-Scholes model may be generalized with the help of other distributions. It turns out that the Gram-Charlier series expansion presents a particularly good example (see Jarrow and Rudd (1982); Abken *et al.* (1996)). Applying the Gram-Charlier series to model R_T under Q, the density function of R_T under Q can be expressed in the following way

$$f_{R_T}(x) = h(\frac{x-m}{\delta})p(\frac{x-m}{\delta})/\delta, \tag{3.3}$$

where $h(x) = \exp(-x^2/2)/\sqrt{2\pi}$ is the density function of the standard normal distribution $N(0,1)$. The function $p(x)$ is a linear combination of Hermite polynomials

$$\begin{aligned} p(x) &= 1 + \frac{\xi}{6}H_3(x) + \frac{\kappa-3}{24}H_4(x), \\ H_k(x) &= (-1)^k\frac{d^k h(x)}{dx^k}/h(x). \end{aligned}$$

Denote P_R the set of all polynomials on real line R. Let $L : P_R \to P_R$ be an operator such that $L(l(x)) = -(l(x)h(x))'/h(x)$ for any polynomial $l(x)$. Using this we can rewrite Hermite polynomials as follows $H_k(x) = L^k(1_R(x))$, where $1_R(x) \equiv 1$ for any $x \in R$. Let us define the function $\tilde{p}(x) = 1 + \xi x^3/6 + (\kappa-3)x^4/24$, then $p(x) = \tilde{p}(L)(1_R)(x)$.

The probabilistic nature of parameters m and δ remains the same as $E_Q(R_T) = m$, and $Var_Q(R_T) = \delta^2$. It can also be determined from (3.3) that $E_Q((x-m)^3) = \delta^3\xi$ and $E_Q((x-m)^4) = \delta^4\kappa$. So ξ and κ are the skewness and excess kurtosis of R_T respectively.

Generalizing the equality (3.2), we can find that in the model

$$m = rT - \frac{\delta^2}{2} - \ln(\tilde{p}(\delta)). \tag{3.4}$$

The Hermite polynomial $H_k(x)$ is introduced at Chapter 2, as the polynomial solution of the Pearson's differential equation when $s_2(x) = 1$ and $s_1(x) = -x$. The Hermite polynomial series is a very common type of orthogonal polynomials in the theory of probability and functional analysis, featuring a number of useful properties (see Fedoryuk (2001)). We recall a few useful properties of the Hermite polynomial as below:

$$\begin{aligned} H_k'(x) &= kH_{k-1}(x), \\ \int_{-\infty}^{\infty} h(x)H_k(x)H_k(x)dx &= k!\sqrt{2\pi}, \\ H_k(x+y) &= \sum_{j=0}^{k} \binom{k}{j} x^{k-j} H_j(y). \end{aligned}$$

Being an extension of the Black-Scholes model, the Gram-Charlier model allows for arbitrary skewness and kurtosis while the higher moments of the stock return are constants determined by input parameters. The Gram-Charlier model is further expanded using the Polynomial-Normal distribution.

Assume the density function of R_T is given by (3.3), where $p(x) = \sum_{k=0}^{N} b_k H_k(x) = \tilde{p}(L)(1_R)(x)$ and $\tilde{p}(x) = \sum_{k=0}^{N} b_k x^k$. Parameter $b_0 = 1$ because $f_{R_T}(x)$ is a density function. Assume for simplicity that $b_1 = b_2 = 0$ since there exists a unique solution in the parameter set $(m, \delta, b_3, \cdots, b_N)$. Similar to the Gram-Charlier model, $E_Q(R_T) = m$ and $Var_Q(R_T) = \delta^2$. The equation (3.4) also applies here for the value of m in the risk-neutral measure.

We call this model the Polynomial-Normal model. When the polynomial degree $N = 0$, the model becomes the Black-Scholes model. When $N = 4$, the model becomes the Gram-

Charlier model. The Polynomial-Normal model provides a further extension of the above models with the association of arbitrary polynomial. We also notice that the normal distribution is the standardized weight function of the following Pearson's differential equation

$$H''(x) - xH'(x) + \lambda H(x) = 0. \tag{3.5}$$

The Hermite polynomial is the polynomial solution of the above differential equation. Therefore, the model (3.3) is using the product of the weight function and a linear combination of the polynomial solutions of a specific Pearson's differential equation, to fit the density function of the financial return. This method could be extended to other classes of Pearson's differential equations. These further extensions will be discussed at the later chapters.

Considering the Polynomial-Normal model, one must be careful regarding the sign of f_{R_T}. Generally this function is not globally positive. In other words, the function $f_{R_T}(x)$ defined by (3.3) is positive for every x, and therefore is a valid p.d.f., only for parameters b_i in a special region of $\mathbb{R}^N$.

In the Gram-Charlier Model ($N = 4$), Barton and Dennis (1952) obtained the parameter conditions for positivity with the help of a numerical method. Later Jondeau and Rockinger (1998) obtained the border of this region using an analytical method. In the following discussion we provide the region of (ξ, κ) that ensures positivity for the Gram-Charlier distribution, and this method is also extended to the case of a general Polynomial-Normal distribution.

In the Gram-Charlier distribution, the density function of $\ln(S_T/S_0)$ is $f_{R_T}(x) = h(y)p(y)/\delta$ with $y = (x - m)/\delta$. We have to find a region of (ξ, κ) where the following condition is satisfied

$$p(y) = 1 + \frac{\xi}{6} H_3(y) + \frac{\kappa - 3}{24} H_4(y) \geq 0 \text{ for every } y\,.$$

Let us specify the border of this region. If a point (ξ_0, κ_0) is on the border, the function $z = p(y)$ should be tangent to the line $z = 0$ on the $y - z$ plane. It is because any small change of (ξ, κ) would make the function above or below the y-axis at a neighborhood of tangent point, and affect the existence of a negative $p(y)$ value. Therefore, the equation $p(y) = 0$ should have a root of multiplicity 2 or higher. Denote this root by y_0, then (ξ_0, κ_0) should be the solution of following system of linear equations

$$\begin{cases} \left(1 + \frac{\xi_0}{6} H_3(y) + \frac{\kappa_0 - 3}{24} H_4(y)\right)\Big|_{y=y_0} = 0, \\ \frac{d}{dy}\left(1 + \frac{\xi_0}{6} H_3(y) + \frac{\kappa_0 - 3}{24} H_4(y)\right)\Big|_{y=y_0} = 0. \end{cases}$$

Solving these equations, we have

$$(\xi(y), \kappa(y)) = \left(\frac{-24 H_3(y)}{4H_3^2(y) - 3H_4(y)H_2(y)}, 3 + \frac{72 H_2(y)}{4H_3^2(y) - 3H_4(y)H_2(y)}\right).$$

Figure (3.1) below shows the parametric curve $(\xi(y), \kappa(y))$ and the shaded region represents the acceptable area that generates a positive density function.

The similar method can be used for a general Polynomial-Normal distribution where $p(x) = \sum_{k=0}^{N} b_k H_k(x)$. Any parameter set $(b_3, b_4, \cdots, b_N)$ that makes $p(x) = 0$ and $p\prime(x) = 0$ for some x would correspond to a point on the border if 0 is the global minimum of the polynomial. Denote $\mathcal{A} = \{(b_3, b_4, \cdots, b_N) : p(x) = p\prime(x) = 0 \text{ for some } x\}$. The set $\mathcal{A}$ generates a complicated manifold in $\mathbb{R}^N$ dividing the space into many parts; the part containing $(0, 0, \cdots, 0)$ is the region we are looking for.

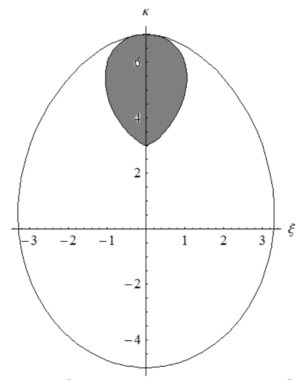


Figure 3.1: Acceptable skewness-kurtosis region in the Polynomial-Normal model

3.2 Option Pricing in the framework of Polynomial-Normal Model

This subsection demonstrates how to calculate call, put, power and polynomial option prices in the Polynomial-Normal model. We also provide the analysis of sensitivities to model parameters (the Greeks).

In a complete market any option can be hedged by a specific strategy and the initial price of the option must be equal to the initial investment to avoid arbitrage. The price of the option turns out to be the discounted expected payoff under the risk-neutral measure.

Theorem 3.1. *In the Polynomial-Normal model the price of a call option is given by*

$$\begin{aligned} C = & - Ke^{-rT}(\Phi(-D_2) + \sum_{k=1}^{n} b_k H_{k-1}(D_2)h(D_2)) \\ & + S_0\Phi(-D_1) + \frac{S_0 h(D_1)}{\tilde{p}(\delta)} \sum_{k=1}^{n} b_k \sum_{j=1}^{k} \binom{k}{j} H_{j-1}(D_1)\delta^{k-j}, \end{aligned} \quad (3.6)$$

where

$$D_1 = (\ln(\frac{K}{S_0}) - rT - \frac{\delta^2}{2} + \ln(\tilde{p}(\delta)))/\delta, \tag{3.7}$$

$$D_2 = D_1 + \delta. \tag{3.8}$$

Proof. To prove the formula we recall some properties of the Hermite polynomials:

$$\int_{x=-\infty}^{+\infty} h(x)H_i(x)H_j(x)dx = \begin{cases} 0 & \text{if } i \neq j, \\ j! & \text{if } i = j, \end{cases}$$

$$H_n{}'(x) = nH_{n-1}{}'(x),$$

$$H_n(x+t) = \sum_{k=0}^{n} \binom{n}{k} t^k H_k(x).$$

The proof of the above properties of the Hermite polynomials can be found at Chapter 2.

In the Polynomial-Normal model the risk-neutral option price is determined, as the discounted expected payoff of the option with respect to the risk-neutral measure. Below we calculate the risk-neutral price C using the equations (3.9) - (3.9) and integration by parts.

$$\begin{aligned} C &= e^{-rT}E_Q(S_T - K)^+ \\ &= e^{-rT}\int_{-\infty}^{\infty}(S_0e^x - K)^+ h(\frac{x-m}{\delta})p(\frac{x-m}{\delta})\frac{1}{\delta}dx \\ &= S_0e^{-rT}\int_{D_2}^{\infty}(e^{\delta+m} - \frac{K}{S_0})h(y)p(y)dy \\ &= S_0e^{-rT}\int_{D_2}^{\infty}e^{\delta+m}h(y)p(y)dy - Ke^{-rT}\int_{D_2}^{\infty}h(y)p(y)dy \end{aligned}$$

$$
\begin{aligned}
&= -Ke^{-rT}(\Phi(y) - \sum_{k=1}^{n} b_k H_{k-1}(y)h(y))\Big|_{D_2}^{+\infty} + S_0 e^{-rT+\frac{\delta^2}{2}+m} \int_{D_2}^{\infty} h(y-\delta)p(y)dy \\
&= -Ke^{-rT}(\Phi(-D_2) + \sum_{k=1}^{n} b_k H_{k-1}(D_2)h(D_2)) + S_0 e^{-\ln(\tilde{p}(\delta))} \int_{D_1}^{\infty} h(z)p(z+\delta)dz \\
&= -Ke^{-rT}(\Phi(-D_2) + \sum_{k=1}^{n} b_k H_{k-1}(D_2)h(D_2)) \\
&\quad + \frac{S_0}{\tilde{p}(\delta)} \int_{D_1}^{\infty} h(z)(1 + \sum_{k=1}^{n} b_k \sum_{j=0}^{k} \binom{k}{j} H_j(z)\delta^{k-j})dz \\
&= -Ke^{-rT}(\Phi(-D_2) + \sum_{k=1}^{n} b_k H_{k-1}(D_2)h(D_2)) \\
&\quad + \frac{S_0}{\tilde{p}(\delta)} \left\{ \Phi(-D_1) + \sum_{k=1}^{n} b_k \left(\delta^k \Phi(-D_1) + h(D_1) \sum_{j=1}^{k} \binom{k}{j} H_{j-1}(D_1)\delta^{k-j} \right) \right\} \\
&= -Ke^{-rT}(\Phi(-D_2) + \sum_{k=1}^{n} b_k H_{k-1}(D_2)h(D_2)) \\
&\quad + S_0\Phi(-D_1) + \frac{S_0 h(D_1)}{\tilde{p}(\delta)} \sum_{k=1}^{n} b_k \sum_{j=1}^{k} \binom{k}{j} H_{j-1}(D_1)\delta^{k-j}.
\end{aligned}
$$

□

In addition, put-call parity can be used in order to derive the explicit formula for the put price:

$$
\begin{aligned}
P = &- Ke^{-rT}(-\Phi(D_2) + \sum_{k=1}^{n} b_k H_{k-1}(D_2)h(D_2)) \\
&- S_0\Phi(D_2) + \frac{S_0 h(D_1)}{\tilde{p}(\delta)} \sum_{k=1}^{n} b_k \sum_{j=1}^{k} \binom{k}{j} H_{j-1}(D_1)\delta^{k-j}.
\end{aligned} \tag{3.9}
$$

Sensitivity of option price is usually measured with the Greeks (See Hull (2011)) which are defined as derivatives with respect to the model parameters.

In the Black-Scholes formula, call and put option prices depend on 5 parameters: initial price, time to maturity, interest rate, volatility and strike price. In the Gram-Charlier and Polynomial-Normal models, these prices also depend on the polynomial parameters. With the option formulae at hand, we can calculate the Greeks simply by differentiating the price. Using the above theorem, we arrive at the following expressions for delta ($\Delta = \partial C/\partial S_0$):

$$\Delta_{\text{call}} = \Phi(-D_1) + \frac{S_0 h(D_1)}{\tilde{p}(\delta)} \sum_{k=1}^{n} b_k \sum_{j=1}^{k} \binom{k}{j} H_{j-1}(D_1)\delta^{k-j}, \tag{3.10}$$

$$\Delta_{\text{put}} = -\Phi(D_1) + \frac{S_0 h(D_1)}{\tilde{p}(\delta)} \sum_{k=1}^{n} b_k \sum_{j=1}^{k} \binom{k}{j} H_{j-1}(D_1)\delta^{k-j}. \tag{3.11}$$

Note that $\Delta_{\text{call}} - \Delta_{\text{put}} = 1$ because of put-call parity. Other Greeks can be calculated in a similar way.

Let us demonstrate our approach for other type of options.

An α-power call is a call option which pays the asset price raised to the power α ($\alpha > 0$) less the strike price. An α-power put is a put option which pays the strike price less the asset price raised to the power α ($\alpha > 0$). The payoff function of an α-power call with maturity T and strike price K is $g = (S_T^\alpha - K)^+$, and the payoff function for the corresponding put option is $g = (K - S_T^\alpha)^+$. A polynomial option is an option whose payoff at maturity is a specific polynomial of the asset price, provided the payoff is positive. The payoff function is $g = (l(S_T))^+$, where $l(x)$ is a polynomial specified by the option.

These kinds of exotic options present a powerful tools for investors, since the payoff has a non-linear relationship with the stock price. Power and polynomial options are used in practice: for instance, squared LIBOR notes in the interest rate market, compensation options for top managers ,etc (see, for example, Macovschi and Quittard-Pinon (2006)). Another example involving polynomial options may be found in portfolio choice theory (see Bertrand and Prigent (2005)).

The fair price of a power option equals the expected value of the discounted payoff with respect to the risk-neutral measure. The pricing of the power option is presented in the following theorem.

Theorem 3.2. *In the Polynomial-Normal model, the fair prices of call/put power options are given by*

$$\begin{aligned} C = & - Ke^{-rT}(\Phi(-D_2) + \sum_{k=1}^{N} b_k H_{k-1}(D_2)h(D_2)) \\ & + S_0^{\alpha} e^{-rT+(\alpha\delta)^2/2+\alpha m}(\Phi(-D_1)\tilde{p}(\alpha\delta) + \sum_{k=1}^{N} b_k \sum_{j=1}^{k} \binom{k}{j} H_{j-1}(D_1)(\alpha\delta)^{k-j}), \end{aligned}$$

and

$$\begin{aligned} P = & - Ke^{-rT}(-\Phi(D_2) + \sum_{k=1}^{N} b_k H_{k-1}(D_2)h(D_2)) \\ & + S_0^{\alpha} e^{-rT+(\alpha\delta)^2/2+\alpha m}(-\Phi(D_1)\tilde{p}(\alpha\delta) + \sum_{k=1}^{N} b_k \sum_{j=1}^{k} \binom{k}{j} H_{j-1}(D_1)(\alpha\delta)^{k-j}). \end{aligned}$$

Here $D_2 = (\ln K/\alpha - \ln S_0 - m)/\delta$ *and* $D_1 = D_2 - \alpha\delta$.

Proof. Again, the fair price of the power option can be derived as the discounted expected payoff of the option, with respect to the risk-neutral measure. Below we calculate the power option price C using the equations (3.9) - (3.9) and integration by parts.

$$\begin{aligned}
C &= e^{-rT} E_Q ({S_T}^{\alpha} - K)^+ \\
&= e^{-rT} \int_{-\infty}^{\infty} ({S_0}^{\alpha} e^{\alpha x} - K)^+ h(\frac{x-m}{\delta}) p(\frac{x-m}{\delta}) \frac{1}{\delta} dx \\
&= {S_0}^{\alpha} e^{-rT} \int_{D_2}^{\infty} (e^{\alpha\delta + \alpha m} - \frac{K}{{S_0}^{\alpha}}) h(y) p(y) dy \\
&= {S_0}^{\alpha} e^{-rT} \int_{D_2}^{\infty} e^{\alpha\delta + \alpha m} h(y) p(y) dy - K e^{-rT} \int_{D_2}^{\infty} h(y) p(y) dy \\
&= -K e^{-rT} (\Phi(y) - \sum_{k=1}^{N} b_k H_{k-1}(y) h(y)) \Big|_{D_2}^{+\infty} + {S_0}^{\alpha} e^{-rT + \frac{(\alpha\delta)^2}{2} + \alpha m} \int_{D_2}^{\infty} h(y - \alpha\delta) p(y) dy \\
&= -K e^{-rT} (\Phi(-D_2) + \sum_{k=1}^{N} b_k H_{k-1}(D_2) h(D_2)) + {S_0}^{\alpha} e^{-rT + \frac{(\alpha\delta)^2}{2} + \alpha m} \int_{D_1}^{\infty} h(z) p(z + \alpha\delta) dz \\
&= -K e^{-rT} (\Phi(-D_2) + \sum_{k=1}^{N} b_k H_{k-1}(D_2) h(D_2)) \\
&\quad + {S_0}^{\alpha} e^{-rT + \frac{(\alpha\delta)^2}{2} + \alpha m} \int_{D_1}^{\infty} h(z) (1 + \sum_{k=1}^{N} b_k \sum_{j=0}^{k} \binom{k}{j} H_j(z) (\alpha\delta)^{k-j}) dz \\
&= -K e^{-rT} (\Phi(-D_2) + \sum_{k=1}^{N} b_k H_{k-1}(D_2) h(D_2)) + {S_0}^{\alpha} e^{-rT + \frac{(\alpha\delta)^2}{2} + \alpha m} \\
&\quad \left\{ \Phi(-D_1) + \sum_{k=1}^{N} b_k \left((\alpha\delta)^k \Phi(-D_1) + \sum_{j=1}^{k} \binom{k}{j} H_{j-1}(D_1) (\alpha\delta)^{k-j} \right) \right\} \\
&= -K e^{-rT} (\Phi(-D_2) + \sum_{k=1}^{N} b_k H_{k-1}(D_2) h(D_2)) \\
&\quad + {S_0}^{\alpha} e^{-rT + \frac{(\alpha\delta)^2}{2} + \alpha m} \left\{ \Phi(-D_1) \tilde{p}(\alpha\delta) + \sum_{k=1}^{N} b_k \sum_{j=1}^{k} \binom{k}{j} H_{j-1}(D_1) (\alpha\delta)^{k-j} \right\}.
\end{aligned}$$

□

If $p(x) = 1$, the model is essentially reduced to the Black-Scholes case, and the prices (3.12), (3.12) have the following form (see Macovschi and Quittard-Pinon (2006)):

$$\begin{aligned} C &= -Ke^{-rT}\Phi(-D_2) + S_0^{\alpha}e^{-rT+(\alpha\delta)^2/2+\alpha m}\Phi(-D_1), \\ P &= Ke^{-rT}\Phi(D_2) - S_0^{\alpha}e^{-rT+(\alpha\delta)^2/2+\alpha m}\Phi(D_1). \end{aligned}$$

To simplify the calculation of polynomial option prices, we introduce the notion of a gap power option. The payoff function of this option is given by

$$g = \begin{cases} S_T^{\alpha} - K & \text{if } S_T > \lambda, \\ 0 & \text{if } S_T \leq \lambda. \end{cases}$$

We can price this option using the same technique as before. Denote the risk neutral price of this option by $C(S_0, \alpha, K, \lambda)$.

Theorem 3.3. *The gap power option price can be calculated as*

$$C(S_0, \alpha, K, \lambda) = \beta(S_0, \alpha, \lambda) - \gamma(K, \alpha, \lambda), \tag{3.12}$$

where

$$\begin{aligned}\beta(S_0,\alpha,\lambda) &= S_0^{\alpha}e^{-rT+(\alpha\delta)^2/2+\alpha m}\Bigg(\Phi(-D_1)\\ &+\sum_{k=1}^{K}b_k((\alpha\delta)^k\Phi(-D_1)+\sum_{j=1}^{k}\binom{k}{j}H_{j-1}(D_1)(\alpha\delta)^{k-j})\Bigg).\end{aligned}$$

represents the present value of S_T^{α}*, and*

$$\gamma(K,\lambda) = Ke{-}rT(\Phi(-D_2) - \sum_{k=1}^{N} b_k H_{k-1}(D_2)h(D_2))$$

represents the present value of the strike price, with

$$\begin{aligned}D_2 &= (\ln(\lambda/S_0)-m)/\delta,\\ D_1 &= D_2-\alpha\delta.\end{aligned}$$

Proof. We know that

$$\begin{aligned}C(S_0,\alpha,K,\lambda) &= e^{-rT}E_Q((S_T{}^{\alpha}-K)I(S_T>\lambda))\\ &= e^{-rT}E_Q(S_T{}^{\alpha}I(S_T>\lambda)) - Ke^{-rT}Q(S_T>\lambda).\end{aligned}$$

Let $\beta(S_0,\alpha,\lambda) = e^{-rT}E_Q(S_T{}^{\alpha}I(S_T>\lambda))$ and $\gamma(K,\alpha,\lambda) = Ke^{-rT}Q(S_T>\lambda)$. Using the properties of Hermite polynomials (3.9) - (3.9) and integration by parts, we can calculate $\beta(S_0,\alpha,\lambda)$ and $\gamma(K,\alpha,\lambda)$ as follows

$$
\begin{aligned}
\beta(S_0,\alpha,\lambda) &= e^{-rT}E_Q({S_T}^{\alpha}I(S_T>\lambda)) \\
&= e^{-rT}\int\limits_{\ln(\lambda/S_0)}^{\infty} {S_0}^{\alpha}e^{\alpha x}h(\frac{x-m}{\delta})p(\frac{x-m}{\delta})\frac{1}{\delta}dx \\
&= {S_0}^{\alpha}e^{-rT}\int\limits_{D_2}^{\infty} e^{\alpha\delta+\alpha m}h(y)p(y)dy \\
&= {S_0}^{\alpha}e^{-rT+\frac{(\alpha\delta)^2}{2}+\alpha m}\int\limits_{D_2}^{\infty} h(y-\alpha\delta)p(y)dy \\
&= {S_0}^{\alpha}e^{-rT+\frac{(\alpha\delta)^2}{2}+\alpha m}\int\limits_{D_1}^{\infty} h(z)p(z+\alpha\delta)dz \\
&= {S_0}^{\alpha}e^{-rT+\frac{(\alpha\delta)^2}{2}+\alpha m}\int\limits_{D_1}^{\infty} h(z)(1+\sum_{k=1}^{N}b_k\sum_{j=0}^{k}\binom{k}{j}H_j(z)(\alpha\delta)^{k-j})dz \\
&= {S_0}^{\alpha}e^{-rT+\frac{(\alpha\delta)^2}{2}+\alpha m}\Bigg(\Phi(-D_1) \\
&\quad +\sum_{k=1}^{N}b_k((\alpha\delta)^k\Phi(-D_1)+\sum_{j=1}^{k}\binom{k}{j}H_{j-1}(D_1)(\alpha\delta)^{k-j})\Bigg) \\
&= {S_0}^{\alpha}e^{-rT+\frac{(\alpha\delta)^2}{2}+\alpha m}\left(\Phi(-D_1)\tilde{p}(\alpha\delta)+\sum_{k=1}^{N}b_k\sum_{j=1}^{k}\binom{k}{j}H_{j-1}(D_1)(\alpha\delta)^{k-j}\right),
\end{aligned}
$$

and

$$
\begin{aligned}
\gamma(K,\alpha,\lambda) &= Ke^{-rT}Q(S_T>\lambda) \\
&= Ke^{-rT}\int\limits_{\ln(\lambda/S_0)}^{\infty} h(\frac{x-m}{\delta})p(\frac{x-m}{\delta})\frac{1}{\delta}dx \\
&= Ke^{-rT}\int\limits_{D_2}^{\infty} h(y)p(y)dy
\end{aligned}
$$

$$
\begin{aligned}
&= Ke^{-rT}(\Phi(y) - \sum_{k=1}^{N} b_k H_{k-1}(y)h(y))\Big|_{D_2}^{+\infty} \\
&= Ke^{-rT}(\Phi(-D_2) + \sum_{k=1}^{N} b_k H_{k-1}(D_2)h(D_2)).
\end{aligned}
$$

□

Let the payoff function of the option be $g = (l(S_T))^+$, where $l(x) = \sum_{i=1}^{n} c_i x^i$ is a polynomial specified by the option. Since a polynomial is a linear sum of power functions, this option can be seen as a linear sum of gap power options. Assume $l(x) > 0$ in the region $(x_1, x_2) \cup (x_3, x_4) \cup \cdots \cup (x_{n-1}, x_n)$, where x_1 and x_n can be $\pm\infty$, and other x_i are roots of $l(x)$. Thus the price of such a polynomial option is given as follows:

$$
\begin{aligned}
C &= e^{-rT} E_Q(g(S_T))^+ \\
&= e^{-rT} E_Q(\sum_{i=1}^{n} c_i S_T^i) I((x_1, x_2) \cup \cdots \cup (x_{n-1}, x_n)) \\
&= \sum_{j=1}^{n/2} \sum_{i=1}^{n} c_i(\beta(S_0, i, x_{2j-1}) - \beta(S_0, i, x_{2j})). \qquad (3.13)
\end{aligned}
$$

The proof of (3.13) is a corollary of the formula for the gap power option.

3.3 VaR And CVaR under the Polynomial-Normal model

Value at Risk (VaR) and Conditional Value at risk (CVaR) are very important risk measures, which probabilistically estimates the potential risk of the certain financial assets. VaR and CVaR are wildly used by banks, security firms, commodity merchants and other financial institutions. For a given financial portfolio, probability λ and a time period with the length T, VaR is usually defined as the maximum loss value within the $1-\lambda$ left-sided confidence interval, or the $1-\lambda$-quantile of the loss value equivalently, where the loss value on a future financial asset price is the expected asset value (S_T) less the actual asset value (S_T). CVaR is the average of VaR_ξ over the significance level ξ, where ξ is taking the value in $(0, \lambda)$. Mathematically, these two measures can be defined as following (see Rockafellar (2002)):

$$VaR_\lambda(S_T) = \; = \; S_0(e^{rT} - e^{R_\lambda}), \tag{3.14}$$

$$CVaR_\lambda(S_T) \; = \; \frac{1}{\lambda}\int_{-\infty}^{\lambda} VaR_\xi(S_T)d\xi, \tag{3.15}$$

where λ is the significance level of the value at risk, and R_λ is the λ-quantile of R_T. The CVaR can also be interpreted as the conditional expected loss value, given that the loss value is greater than its $1-\lambda$-quantile. The equivalence of these definitions would be proved in the following theorem, as well as the VaR and CVaR value under the Polynomial-Normal model.

Theorem 3.4. *The CVaR value defined by (4.25) can be expressed as*

$$CVaR_\lambda(S_T) = S_0 e^{rT} - \frac{S_0}{\lambda} \int_{-\infty}^{R_\lambda} e^x dF_{R_T}(x), \tag{3.16}$$

where $F_{R_T}(x)$ is the cumulative distribution function (c.d.f.) of R_T. This expression is regardless of the distribution of the returning financial asset value. If the density of the underlying stock is given by (3.3), where the polynomial $p(x) = 1 + \sum_{k=3}^N b_k H_k(x)$, the VaR and CVaR value of the underlying stock are as follows:

$$VaR_\lambda(S_T) = S_0(e^{rT} - e^{R_\lambda}), \tag{3.17}$$

$$\begin{aligned} CVaR_\lambda(S_T) &= S_0 e^{rT} - \frac{S_0 e^{\delta^2/2+m}}{\lambda} \tilde{p}(\delta) \Phi(\frac{R_\lambda - m - \delta^2}{\delta}) \\ &- \sum_{k=3}^N b_k \left(h(\frac{R_\lambda - m - \delta^2}{\delta}) \sum_{j=1}^k \binom{k}{j} H_{j-1}(\frac{R_\lambda - m - \delta^2}{\delta}) \delta^{k-j} \right), \end{aligned} \tag{3.18}$$

where R_λ is the λ-quantile of R_T and

$$\lambda = \Phi(\frac{R_\lambda - m}{\delta}) - h(\frac{R_\lambda - m}{\delta}) \sum_{k=3}^N b_k H_{k-1}(\frac{R_\lambda - m}{\delta}). \tag{3.19}$$

Proof. According to the definition, VaR_λ is the loss of the underlying financial asset when R_T is at its λ-quantile. Therefore (4.24) is satisfied. The cumulative probability in (4.26) can be calculated as following:

$$\begin{aligned}
\lambda &= \int_{x=-\infty}^{R_\lambda} f_{R_T}(x)dx \\
&= \int_{y=-\infty}^{\frac{R_\lambda - m}{\delta}} h(y)p(y)dy \\
&= \left[\Phi(y) + \sum_{k=3}^{N}(-1)^k b_k \frac{d^{k-1}}{dy^{k-1}} h(y)\right]\Bigg|_{-\infty}^{\frac{R_\lambda - m}{\delta}} \\
&= \Phi(\frac{R_\lambda - m}{\delta}) - h(\frac{R_\lambda - m}{\delta}) \sum_{k=3}^{N} b_k H_{k-1}(\frac{R_\lambda - m}{\delta}).
\end{aligned}$$

From the CVaR definition (4.25), the integral can be further derived in the following way, with the help of the integration by parts:

$$\begin{aligned}
CVaR_\lambda(S_T) &= \frac{1}{\lambda}\int_0^{\lambda} VaR_\xi(S_T)d\xi \\
&= \frac{1}{\lambda}\int_{-\infty}^{R_\lambda} VaR_{F_{R_T}(y)}(S_T)dF_{R_T}(y) \\
&= \frac{1}{\lambda}\int_{-\infty}^{R_\lambda} S_0(e^{rT} - e^y)dF_{R_T}(y) \\
&= \frac{1}{\lambda}(S_0 e^{rT}\lambda - S_0 \int_{-\infty}^{R_\lambda} e^y dF_{R_T}(y)) \\
&= S_0 e^{rT} - \frac{S_0}{\lambda}(\int_{-\infty}^{R_\lambda} e^y dF_{R_T}(y)).
\end{aligned}$$

So, the equation (3.16) is satisfied, and therefore under the Polynomial-Normal model, $CVaR_\lambda$ can be calculated as

$$\begin{aligned}
CVaR_\lambda(S_T) &= S_0e^{rT} - \frac{S_0}{\lambda}\int_{-\infty}^{R_\lambda} e^x f_{R_T}(x)dx \\
&= S_0e^{rT} - \frac{S_0}{\lambda}\int_{-\infty}^{\frac{R_\lambda - m}{\delta}} e^{\delta + m} h(y)p(y)dy \\
&= S_0e^{rT} - \frac{S_0 e^{\delta^2/2+m}}{\lambda}\int_{-\infty}^{\frac{R_\lambda - m}{\delta}} h(y-\delta)p(y)dy \\
&= S_0e^{rT} - \frac{S_0 e^{\delta^2/2+m}}{\lambda}\int_{-\infty}^{\frac{R_\lambda - m - \delta^2}{\delta}} h(z)p(z+\delta)dz \\
&= S_0e^{rT} - \frac{S_0 e^{\delta^2/2+m}}{\lambda}\int_{-\infty}^{\frac{R_\lambda - m - \delta^2}{\delta}} h(z)(1 + \sum_{k=3}^{N} b_k \sum_{j=0}^{k}\binom{k}{j} H_j(z)\delta^{k-j})dz \\
&= S_0e^{rT} - \frac{S_0 e^{\delta^2/2+m}}{\lambda}\tilde{p}(\delta)\Phi(\frac{R_\lambda - m - \delta^2}{\delta}) \\
&\quad - \sum_{k=3}^{N} b_k \left(h(\frac{R_\lambda - m - \delta^2}{\delta}) \sum_{j=1}^{k}\binom{k}{j} H_{j-1}(\frac{R_\lambda - m - \delta^2}{\delta})\delta^{k-j} \right).
\end{aligned}$$

Hence, the equation (3.18) is satisfied.

□

CHAPTER 4

Polynomial-T-Distribution Model

4.1 Polynomial-T-Distribution Model and Related Considerations

While the Gram-Charlier model and its extension focused on producing a better estimation of skewness, kurtosis and other moment parameters for financial asset returns, they made no improvements on the tail density estimation. To overcome the shortcomings of tail underestimation in the Black-Scholes model, fat-tail distributions were used to describe financial asset returns. Being a fat-tail distribution with normal-like shape, the Student's T-distribution was used first to model financial asset returns at early 1970s (see Blattberg and Gonedes (1974)). Comparisons between Student's T-distribution and other distributions were made thereafter using historical financial data, and in some cases it provided a better fit than other distributions (see Platen and Sidorowicz (2007)). The density function of the standard Student's T-distribution with ν degree of freedom can be expressed as

$$\mathcal{T}(x) = \frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\nu\pi}\Gamma(\frac{\nu}{2})}(1 + \frac{x^2}{\nu})^{-\frac{\nu+1}{2}}, \tag{4.1}$$

Define $\Phi_T(x)$ as the cumulative probability function (c.d.f.) of the standard Student's T-distribution function, then we have

$$\Phi_T(x) = \frac{1}{2} + \frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\nu\pi}\Gamma(\frac{\nu}{2})}\ {}_2F_1\ (\frac{1}{2}, \frac{\nu+1}{2}, \frac{3}{2}; -\frac{x^2}{\nu})x, \tag{4.2}$$

where ${}_2F_1(x)$ is the hypergeometric function. We will use the following function as the density function of R_T:

$$f_{R_T}(x) = \frac{1}{\delta}\mathcal{T}(\frac{x-m}{\delta}), \tag{4.3}$$

where ν is the degree of freedom, m and δ are the drift and scale parameters respectively. The corresponding model is called as the Student's T-distribution model.

A significant difference between the Black-Scholes and the Student's T-distribution models is that the expected stock value diverges in the Student's T-distribution model. This result is also true for other distributions with a power-law decay. Cassidy *et al.* (2010) used an upper limit to cap the density function to justify the model. Similar methods can be used here for the model parameters' estimation. For any real number S_c, we have the following equation:

$$\begin{aligned}
E(S_T) &= E(S_0 e^{R_T}) \\
&= S_0 \int_{-\infty}^{\infty} e^x f_{R_T}(x)dx \\
&= S_0 \int_{-\infty}^{S_c+m} e^x, f_{R_T}(x)dx + S_0 \int_{S_c+m}^{\infty} e^x f_{R_T}(x)dx \\
&= S_0 \int_{-\infty}^{S_c+m} e^x f_{R_T}(x)dx + S_0 R(S_c),
\end{aligned}$$

where

$$R(S_c) = \int_{S_c+m}^{\infty} e^x f_{R_T}(x)dx. \tag{4.4}$$

The term of $R(S_c)$ is seen as the remainder of the integral subject to truncation, where S_c is large and reasonable selected. Let us denote $E^c(\cdot)$ the partial expectation $E(\cdot I_{\{S_T<S_c+m\}})$. Then the expected value of S_T can be estimated as follows:

$$E(S_T) \approx S_0 \int_{-\infty}^{S_c+m} e^x.f_{R_T}(x)dx = S_0 E^c(e^{R_T}). \tag{4.5}$$

If the density function (4.3) is used to fit the risk-neutral measure density, the estimate $S_0 e^{rT} = E(S_T) \approx E^c(S_T)$ can be applied. In this case, the drift parameter m can be estimated in the following way:

$$
\begin{aligned}
e^{rT} = E_Q(\frac{S_T}{S_0}) &\approx \int_{-\infty}^{S_c+m} e^x f_{R_T}(x)dx = \int_{-\infty}^{\frac{S_c}{\delta}} e^{\delta y+m}\mathcal{T}(y)dy = e^m \int_{-\infty}^{\frac{S_c}{\delta}} e^{\delta y}\mathcal{T}(y)dy, \\
\text{and hence, } m &\approx \ln\left(\frac{e^{rT}}{\int_{-\infty}^{\frac{S_c}{\delta}} e^{\delta y}\mathcal{T}(y)dy}\right). \qquad (4.6)
\end{aligned}
$$

Let p_c denote the probability that the stock price S_T is larger than $S_c + m$. Then

$$
p_c = P(S_T > S_c + m) = \int_{S_c+m}^{\infty} f_{R_T}(x)dx = \int_{\frac{S_c}{\delta}}^{\infty} \mathcal{T}(y)dy. \qquad (4.7)
$$

Solving S_c from p_c by (4.7), each selection of p_c will end up with a unique upper bound value S_c. With an appropriate significance probability p_c, it is reasonable to truncate $R(S_c)$ to be zero and therefore avoid a diverging integration. A value of p_c close to 0 will diminish the effect of truncation and generate more accurate results. Though the integration in (4.6) diverges as p_c approaches to 0, it will not be a problem until the value of p_c is very small. In our numerical example in Chapter 6, the results would make sense for a p_c as small as 10^{-100}.

Using the Student's T-distribution to model the logarithmic return gives better tails fitting to empirical data, but it does not give arbitrary skewness or kurtosis. The skewness of the Student's T-distribution is zero, while empirical data suggest a negative skewness. Excessive kurtosis of the Student's T-distribution, is a constant $3(\nu - 2)/(\nu - 4)$. However, we cannot assess the tail decay speed and the kurtosis at the same time with the Student's

T-distribution. To provide further improvements of the model, we use the product of a polynomial and the Student's T-distribution density as the density function of R_T:

$$f_{R_T}(x) = \frac{1}{\delta}\mathcal{T}(\frac{x-m}{\delta})p(\frac{x-m}{\delta}), \tag{4.8}$$

where $p(x)$ is the N$^{\text{th}}$ degree polynomial. Let us represent $p(x)$ in the form (see Kirshnamoorthy (1951)):

$$\begin{aligned} p(x) &= \sum_{k=0}^{N} b_k R_k(x), & (4.9)\\ \text{where } R_k(x) &= (1+\frac{t^2}{\nu})^{\frac{\nu+1}{2}} \frac{d^k}{dx^k}\{\frac{1}{(1+\frac{t^2}{\nu})^{\frac{\nu+1}{2}-k}}\}. & (4.10)\end{aligned}$$

The polynomials $R_k(x)$ are called the Romanovski Polynomial Series. These polynomials are discussed at Chapter 2 as the solution of the Pearson's differential equation. They are the polynomial solutions of the following Pearson's differential equation

$$(1+\frac{x^2}{\nu})R''(x) + \frac{(1-\nu)x}{\nu}R'(x) + \lambda R(x) = 0, \tag{4.11}$$

for which the weight function turns out to be the density function of the T-distribution presented at the equation (4.1). This is another example of using the product of the weight function and a linear combination of polynomial solution of a specific Pearson's differential equation, to fit the density function of the financial asset return. The model (4.8) - (4.10) is called the Polynomial-T-Distribution model (see Li and Melnikov (2013)).

The linear parameter must satisfy $b_0 = 1$ to be a valid density function. We assume

$b_1 = b_2 = 0$ so that the expectation and variance of the return are solely determined by the drift and scale parameters. Without b_1 and b_2, we can find that $E(R_T) = m$ and $Var(R_T) = \nu\delta^2/(\nu - 2)$.

We can estimate the mean m as in case of the Student's T-distribution model with respect to the risk-neutral measure:

$$m \approx \ln\left(\frac{e^{rT}}{\int_{-\infty}^{\frac{S_c}{\delta}} e^{\delta y}\mathcal{T}(y)p(y)dy}\right). \tag{4.12}$$

The significance probability p_c becomes

$$p_c = \int_{\frac{S_c}{\delta}}^{\infty} \mathcal{T}(y)p(y)dy. \tag{4.13}$$

We can see that the Student's T-distribution model is a particular case when we use a trivial polynomial. When the degree of freedom ν approaches infinity, the Student's T-distribution is reduced to the normal distribution, and the Polynomial-T-Distribution model is reduced to the Gram-Charlier model and the Polynomial-Normal model. We note also that $R_n(x)$ will approach to Hermite polynomial of the same degree. Taking the limit of R_k in (2.18) as $\nu \to \infty$, we have

$$\lim_{\nu\to\infty} R_n(x) = e^{\frac{x^2}{2}}\frac{d^n}{dx^n}e^{-\frac{x^2}{2}}, \tag{4.14}$$

which turns out to be the Hermite polynomial of degree n.

Remark 4.1. With a finite degree of freedom ν, the decay speed of the density tail becomes

$\nu - N$, where N is the degree of the polynomial in (4.9). The existence of finite moments for R_T can also be decided by this decay speed: for any $k < \nu - N$, the k^{th} moment of R_T exists. But existence of a finite kurtosis or even variance is remaining as an open question (see Shaw (2011)).

Another property of the Polynomial-T-Distribution model is the sign of the polynomial $p(x)$. The function $f_{R_T}(x)$, presented as the product of the Student's density function and the polynomial $p(x)$, ought to be globally positive. Thus global positivity of $p(x)$ is necessary. Only when $\{b_k\}_{k=3,\cdots,N}$ resides in a special region of the parameter space $\mathbb{R}^{N-2}$, the positivity condition is satisfied. In the Gram-Charlier model and the Polynomial-Normal model, this region for positive density functions was found. We may use a similar method for the Polynomial-T-Distribution model. If the parameters $\{b_k\}_{k=3,\cdots,N}$ are on the border of this region, $p(x)$ should be tangent to the x-axis. Therefore, $p(x)$ should have a multiple root x_0, satisfying the linear equations:

$$\begin{cases} p(x_0) = 0 \\ p'(x_0) = 0 \end{cases} \tag{4.15}$$

Solving equations (4.15) for any real x_0, we span a solution set into a $(N-3)$-dimensional manifold, which is indicating the border of the positivity region. The part including origin is the region generating valid density functions.

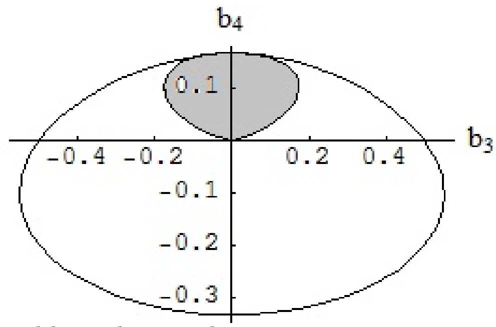


Figure 4.1: Acceptable polynomial parameters region in the Polynomial-T-Distribution model

4.2 Option Pricing under Polynomial-T-Distribution Model

The main difficulty of option pricing under the Polynomial-T-distribution model is the divergence of expected stock prices. It may lead to infinite option prices if the standard pricing method is applied. Cassidy *et al.* (2010) exploited a truncation method to price options in the framework of the Student's T-distribution model. This result was derived in an integral form, called Gosset formula. However, it is not an explicit solution because the integral is non-elementary. We try to show how to solve the problem, in the framework of the Polynomial-T-Distribution model.

Assume that the call or put European option has a time to maturity T, strike price K, in a market of constant interest rate r, and are placed regarding a stock $\{S\}_t$ whose risk-neutral logarithmic return density is given by (4.8). The fair prices of European options should always equate the discounted expected payoff of the underlying option, regarding the risk-neutral measure. Thus the put option price is given as:

$$\begin{aligned} P &= e^{-rT} E_Q (K - S_T)^+ \\ &= e^{-rT} \int_{x=-\infty}^{\ln(K)-\ln(S_0)} K f_{R_T}(x) dx - S_0 e^{-rT} \int_{x=-\infty}^{\ln(K)-\ln(S_0)} e^x f_{R_T}(x) dx. \end{aligned} \tag{4.16}$$

Due to the put-call parity, the call option price would be the following:

$$C = P + S_0 - K e^{-rT}. \tag{4.17}$$

In section 2, we estimated m using a truncation method, by setting

$$S_0 \approx e^{-rT} E_Q^c (S_T).$$

Plugging this expression instead of S_0 in the put-call parity, we have

$$\begin{aligned} C &\approx P + e^{-rT} E_Q^c (S_T) - e^{-rT} K \\ &= e^{-rT} E_Q ((K - S_T) I_{\{S_T < K\}}) + e^{-rT} E_Q (S_T I_{\{S_T < S_c + m\}}) - e^{-rT} K \\ &= e^{-rT} E_Q (S_T I_{\{S_c + m > S_T > K\}}) - e^{-rT} E_Q (K I_{\{S_T > K\}}). \end{aligned}$$

By using estimated expected stock prices and put-call parity, we therefore avoid calculating a divergent integral in call option, but draw it for put option price instead. The put option price is given in the following theorem.

Theorem 4.1. *If the density of logarithmic stock return is defined by (4.8), the European* *put option price with time to maturity T and strike price K is given as*

$$\begin{aligned}
P &= Ke^{-rT}\Phi_T(\frac{\ln(K)-\ln(S_0)-m}{\delta}) \\
&+S_0e^{m-rT}\frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\nu\pi}\Gamma(\frac{\nu}{2})}\sum_{k=0}^{N}b_k(-\delta)^k\sqrt{\nu}\Phi^*(\delta\sqrt{\nu},k-\frac{\nu+1}{2},\frac{\ln(K)-\ln(S_0)-m}{\delta\sqrt{\nu}}) \\
&+\frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\nu\pi}\Gamma(\frac{\nu}{2})}\sum_{k=3}^{N}b_k\left(S_0e^{m-rT+\delta y}\sum_{l=1}^{k}(-\delta)^{l-1}\frac{d^{k-l}}{dy^{k-l}}(1+\frac{y^2}{\nu})^{k-\frac{1+\nu}{2}}\right. \\
&\left.+Ke^{-rT}\frac{d^{k-1}}{dy^{k-1}}(1+\frac{y^2}{\nu})^{k-\frac{1+\nu}{2}}\right)\Bigg|_{y=\frac{\ln(K)-\ln(S_0)-m}{\delta}}, \qquad (4.18)
\end{aligned}$$

where

$$\Phi^*(\xi,k,x)=\int_{-\infty}^{x}e^{\xi y}(1+y^2)^k dy \text{ ,for any } \xi<0, k\in\mathbb{R}. \qquad (4.19)$$

Proof. The first term in (4.16) can be calculated by basic integration as follows:

$$\begin{aligned}
& e^{-rT}\int_{x=-\infty}^{\ln(K)-\ln(S_0)} Kf_{R_T}(x)dx \\
=\ & Ke^{-rT}\int_{y=-\infty}^{\frac{\ln(K)-\ln(S_0)-m}{\delta}} \mathcal{T}(y)p(y)dy \\
=\ & Ke^{-rT}\frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\nu\pi}\Gamma(\frac{\nu}{2})}\int_{y=-\infty}^{\frac{\ln(K)-\ln(S_0)-m}{\delta}}((1+\frac{y^2}{\nu})^{-\frac{\nu+1}{2}}+\sum_{k=3}^{N}b_k\frac{d^k}{dy^k}(1+\frac{y^2}{\nu})^{k-\frac{\nu+1}{2}})dy
\end{aligned}$$

$$
\begin{aligned}
&= Ke^{-rT}\Phi_T(\frac{\ln(K)-\ln(S_0)-m}{\delta}) \\
&\quad +Ke^{-rT}\frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\nu\pi}\Gamma(\frac{\nu}{2})}\sum_{k=3}^{N} b_k \frac{d^{k-1}}{dy^{k-1}}(1+\frac{y^2}{\nu})^{k-\frac{\nu+1}{2}})\Bigg|_{y=\frac{\ln(K)-\ln(S_0)-m}{\delta}}. \qquad (4.20)
\end{aligned}
$$

To calculate the second term in (4.16), we derive the following equation with integration by parts for k times:

$$
\begin{aligned}
&\int_{-\infty}^{x} e^{\delta y+m}\frac{d^k}{dy^k}(1+\frac{y^2}{\nu})^{k-\frac{\nu+1}{2}}dy \\
&= e^{m+\delta y}\sum_{l=1}^{k}(-\delta)^{l-1}\frac{d^{k-l}}{dy^{k-l}}(1+\frac{y^2}{\nu})^{k-\frac{1+\nu}{2}}\Bigg|_x + (-\delta)^k\int_{-\infty}^{x} e^{\delta y+m}(1+\frac{y^2}{\nu})^{k-\frac{\nu+1}{2}}dy \\
&= e^{m+\delta y}\sum_{l=1}^{k}(-\delta)^{l-1}\frac{d^{k-l}}{dy^{k-l}}(1+\frac{y^2}{\nu})^{k-\frac{1+\nu}{2}}\Bigg|_x \\
&\quad +(-\delta)^k e^m\sqrt{\nu}\Phi^*(\delta\sqrt{\nu}, k-\frac{\nu+1}{2}, \frac{x}{\sqrt{\nu}}). \qquad (4.21)
\end{aligned}
$$

Applying now (4.21), we simplify the second part of (4.16) as follows:

$$
\begin{aligned}
& S_0e^{-rT} \int\limits_{x=-\infty}^{\ln(K)-\ln(S_0)} e^x f_{R_T}(x)dx \\
= \quad & S_0e^{-rT} \int\limits_{y=-\infty}^{\frac{\ln(K)-\ln(S_0)-m}{\delta}} e^{\delta y+m}\mathcal{T}(y)p(y)dy \\
= \quad & S_0e^{m-rT}\frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\nu\pi}\Gamma(\frac{\nu}{2})} \int\limits_{y=-\infty}^{\frac{\ln(K)-\ln(S_0)-m}{\delta}} e^{\delta y}((1+\frac{y^2}{\nu})^{-\frac{\nu+1}{2}} + \sum_{k=3}^{N} b_k \frac{d^k}{dy^k}(1+\frac{y^2}{\nu})^{k-\frac{\nu+1}{2}})dy \\
= \quad & S_0e^{m-rT}\frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\nu\pi}\Gamma(\frac{\nu}{2})}\Bigg\{\sqrt{\nu}\Phi^*(\delta\sqrt{\nu}, -\frac{\nu+1}{2}, \frac{\ln(K)-\ln(S_0)-m}{\delta\sqrt{\nu}}) \\
& +\sum_{k=3}^{N} b_k \Bigg\{ e^{\delta y}\sum_{l=1}^{k}(-\delta)^{l-1}\frac{d^{k-l}}{dy^{k-l}}(1+\frac{y^2}{\nu})^{k-\frac{1+\nu}{2}}\Bigg|_{y=\frac{\ln(K)-\ln(S_0)-m}{\delta}} \\
& +(-\delta)^k\sqrt{\nu}\Phi^*(\delta\sqrt{\nu}, k-\frac{\nu+1}{2}, \frac{\ln(K)-\ln(S_0)-m}{\delta\sqrt{\nu}})\Bigg\}\Bigg\}. \qquad (4.22)
\end{aligned}
$$

Combining both (4.20) and (4.22), we arrive to the formula (4.18). □

Remark 4.2. Option prices depend on a few parameters, and sensitivity of the option prices over these parameters is crucially important in problems like hedging portfolios. These sensitivities is often measured in terms of Greek letters, which are defined by derivatives of option prices over corresponding parameters. In the Black-Scholes model, option prices are determined from underlying stock prices, strike prices, volatility, time to maturity and interest rate. In the Polynomial-T-Distribution model, it also depends on degree of freedom of the distribution and the polynomial parameters. With option prices at hand, we can calculate the Greek letter by differentiating the price over particular parameters. Sensitivity with regards to stock price ($\Delta_{put} = \partial P/\partial S_0$, $\Delta_{call} = \partial C/\partial S_0$) can be calculated as follows:

$$\begin{aligned}\Delta_{put} &= e^{m-rT}\frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\nu\pi}\Gamma(\frac{\nu}{2})}\left\{\sqrt{\nu}\Phi^*(\delta\sqrt{\nu},-\frac{\nu+1}{2},\frac{\ln(K)-\ln(S_0)-m}{\delta\sqrt{\nu}})\right.\\ &+\sum_{k=3}^{N}b_k\left\{e^{\delta y}\sum_{l=1}^{k}(-\delta)^{l-1}\frac{d^{k-l}}{dy^{k-l}}(1+\frac{y^2}{\nu})^{k-\frac{1+\nu}{2}}\right|_{y=\frac{\ln(K)-\ln(S_0)-m}{\delta}}\\ &\left.\left.+(-\delta)^k\sqrt{\nu}\Phi^*(\delta\sqrt{\nu},k-\frac{\nu+1}{2},\frac{\ln(K)-\ln(S_0)-m}{\delta\sqrt{\nu}})\right\}\right\}\end{aligned} \tag{4.23}$$

and $\Delta_{call} = \Delta_{put} + 1$.

Other Greek letters can be calculated similarly.

4.3 Value at Risk under Polynomial-T-Distribution Model

This section is devoted to calculations of VaR and CVaR in the Polynomial-T-Distribution model. As VaR and CVaR are usually calculated as percentiles of the predictive distribution for the future financial returns, an appropriate tail estimation for financial returns becomes critically important in calculating these measures. According to the fat-tail property in the returns, a normal distribution would severely underestimate VaR and CVaR.

VaR and CVaR can be interpreted as quantile and partial expected value of the loss variable. The definition and some basic properties of VaR and CVaR is given in Chapter 3 by the equations (3.14) - (3.16). The following theorem gives the VaR and CVaR value in the Polynomial-T-Distribution model.

Theorem 4.2. *If the density of the underlying stock is given by (4.8) - (4.10), the VaR and CVaR value of the underlying stock should be as follows:*

$$
\begin{aligned}
VaR_\lambda &= S_0(e^{rT} - e^{R_\lambda}), \qquad (4.24)\\
CVaR_\lambda &= S_0 e^{rT} - \frac{S_0 e^m \Gamma(\frac{\nu+1}{2})}{\lambda\sqrt{\nu\pi}\Gamma(\frac{\nu}{2})}\left\{\sqrt{\nu}\Phi^*(\delta\sqrt{\nu}, -\frac{\nu+1}{2}, \frac{R_\lambda - m}{\delta\sqrt{\nu}})\right.\\
&+\sum_{k=3}^{N} b_k \left\{ e^{\delta y}\sum_{l=1}^{k}(-\delta)^{l-1}\frac{d^{k-l}}{dy^{k-l}}(1+\frac{y^2}{\nu})^{k-\frac{1+\nu}{2}}\Big|_{y=\frac{R_\lambda - m}{\delta}}\right.\\
&\left.\left.+(-\delta)^k\sqrt{\nu}\Phi^*(\delta\sqrt{\nu}, k-\frac{\nu+1}{2}, \frac{R_\lambda - m}{\delta\sqrt{\nu}})\right\}\right\}, \qquad (4.25)
\end{aligned}
$$

where R_λ is the λ-quantile of R_T and

$$
\lambda = \Phi_T(\frac{R_\lambda - m}{\delta}) + \frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\nu\pi}\Gamma(\frac{\nu}{2})}\sum_{k=3}^{N} b_k \frac{d^{k-1}}{dy^{k-1}}(1+\frac{y^2}{\nu})^{k-\frac{1+\nu}{2}}\Big|_{y=\frac{R_\lambda - m}{\delta}}. \qquad (4.26)
$$

Proof. VaR_λ is the loss of the underlying financial asset when R_T is at its λ-quantile. Therefore (4.24) is satisfied. The cumulative probability in (2.49) can be calculated as following:

$$
\begin{aligned}
\lambda &= \int_{x=-\infty}^{R_\lambda} f_{R_T}(x)dx\\
&= \int_{y=-\infty}^{\frac{R_\lambda - m}{\delta}} \mathcal{T}(y)p(y)dy\\
&= \frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\nu\pi}\Gamma(\frac{\nu}{2})}\int_{y=-\infty}^{\frac{R_\lambda - m}{\delta}}((1+\frac{y^2}{\nu})^{-\frac{\nu+1}{2}} + \sum_{k=3}^{N} b_k \frac{d^k}{dy^k}(1+\frac{y^2}{\nu})^{k-\frac{\nu+1}{2}})dy\\
&= \Phi_T(\frac{R_\lambda - m}{\delta}) + \frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\nu\pi}\Gamma(\frac{\nu}{2})}\sum_{k=3}^{N} b_k \frac{d^{k-1}}{dy^{k-1}}(1+\frac{y^2}{\nu})^{k-\frac{\nu+1}{2}}\Big|_{y=\frac{R_\lambda - m}{\delta}}.
\end{aligned}
$$

As $CVaR_\lambda$ is the conditional expected loss, it can be rewritten as follows using integration by parts

$$\begin{aligned}
CVaR_\lambda &= S_0 e^{rT} - \frac{S_0}{\lambda} \int_{-\infty}^{R_\lambda} e^x f_{R_T}(x)dx \\
&= S_0 e^{rT} - \frac{S_0}{\lambda} \int_{y=-\infty}^{\frac{R_\lambda - m}{\delta}} e^{\delta y + m} \mathcal{T}(y)p(y)dy \\
&= S_0 e^{rT} - \frac{S_0 e^m \Gamma(\frac{\nu+1}{2})}{\lambda\sqrt{\nu\pi}\Gamma(\frac{\nu}{2})} \int_{y=-\infty}^{\frac{R_\lambda - m}{\delta}} e^{\delta y}((1+\frac{y^2}{\nu})^{-\frac{\nu+1}{2}} + \sum_{k=3}^{N} b_k \frac{d^k}{dy^k}(1+\frac{y^2}{\nu})^{k-\frac{\nu+1}{2}})dy \\
&= S_0 e^{rT} - \frac{S_0 e^m \Gamma(\frac{\nu+1}{2})}{\lambda\sqrt{\nu\pi}\Gamma(\frac{\nu}{2})} \left\{ \sqrt{\nu}\Phi^*(\delta\sqrt{\nu}, -\frac{\nu+1}{2}, \frac{R_\lambda - m}{\delta\sqrt{\nu}}) \right. \\
&\quad + \sum_{k=3}^{N} b_k \left\{ e^{\delta y} \sum_{l=1}^{k} (-\delta)^{l-1} \frac{d^{k-l}}{dy^{k-l}} (1+\frac{y^2}{\nu})^{k-\frac{1+\nu}{2}} \right|_{y=\frac{R_\lambda - m}{\delta}} \\
&\quad \left. \left. + (-\delta)^k \sqrt{\nu}\Phi^*(\delta\sqrt{\nu}, k-\frac{\nu+1}{2}, \frac{R_\lambda - m}{\delta\sqrt{\nu}}) \right\} \right\}.
\end{aligned}$$

□

CHAPTER 5

Generalization of Polynomially Extended Models with Financial and Actuarial Applications

5.1 The Polynomial-T-Distribution Model with Asymmetric Adjustor

This chapter is devoted to the further extension of the polynomial models in finance, as well as applications in actuarial science. The leading idea of our approach is to standardize the weight function of the Pearson's differential equation and further use this function for the basic density function construction. We use the product of this basic density function and a linear combination of the corresponding orthogonal polynomials to fit the density function of the variables of interest, like logarithmic financial returns or insurance portfolio claims. The Polynomial-Normal model and the Polynomial-T-Distribution model become special cases of this method.

The Polynomial-Normal model shows better ability than the Black-Scholes model in capturing moment parameters of logarithmic financial returns (see Li and Melnikov (2012)). In the Polynomial-Normal model, the density function of R_T is modeled as the product of the normal density and a linear combination of the Hermite polynomials - the orthogonal polynomials based on the normal distribution. If we let $s_2(x) = 1$ and $s_1(x) = -x$ in the Pearson's differential equation, the differential equation becomes

$$F''(x) - xF'(x) + \lambda F(x) = 0. \tag{5.1}$$

The standardized weight function $w(x)$ of the equation (5.1) becomes the normal density and the polynomial solution $F_k(x)$ of (5.1) is the Hermite polynomial as given by the generalized Rodrigues formula (2.12) - (2.13). In the Polynomial-Normal model, the density function $f_{R_T}(x)$ can be expressed as

$$f_{R_T}(x) = \frac{1}{\delta} w(\frac{x-m}{\delta}) p(\frac{x-m}{\delta}), \tag{5.2}$$

$$p(x) = 1 + \sum_{k=3}^{N} b_k F_k(x). \tag{5.3}$$

Combining the above two equations and the generalized Rodrigues formula, we have the following

$$f_{R_T}(x) = \frac{w(y)}{\delta} + \sum_{k=3}^{N} \frac{b_k}{\delta} \frac{d^k}{dx^k}(w(x)s_2^k(x))\bigg|_{y=\frac{x-m}{\delta}}. \tag{5.4}$$

The Polynomial-T-Distribution model handles the fat-tail property of the logarithmic financial returns and the non-constant moment parameters simultaneously, and thus presents a nice estimate of the distribution density for the logarithmic stock returns (Li and Melnikov (2013)). In the Polynomial-T-Distribution model, we use the above equations (5.2) - (5.4) to model the density function of R_T, where $w(x)$ is the standardized weight function and $F_k(x)$ is the k^{th} order polynomial solution of the following Pearson's differential equation

$$(1+x^2)F''(x) + (1-\nu)xF'(x) + \lambda F(x) = 0. \tag{5.5}$$

The weight function $w(x)$ of the above differential equation turns out to be the density function of the Student's T-distribution with degree of freedom ν, and the polynomial solutions $F_k(x)$ become the Romanovski polynomials. This is our second example of distributions generated from the Pearson's differential equation, used in the propose of financial returns modeling.

A further extension of the approach can be done as follows. The coefficient of the term $F'(x)$ in (5.5) can be generalized as a full linear function of x. In this case, the differential equation (5.5) becomes the full form of the Pearson's differential equation, and can be written as follows

$$(1+x^2)F''(x) + (px+q)F'(x) + \lambda F(x) = 0. \tag{5.6}$$

The weight function $w(x)$ of equation (5.6) can be calculated using equation (2.11):

$$w(x) = A(1+x^2)^{\frac{p}{2}-1}\exp(q\arctan x). \tag{5.7}$$

When $p < 1$ in formula (5.7), $w(x)$ becomes a density function of a distribution defined on the whole real line, where the constant A should be properly selected. We call the additional term $\exp(q \arctan x)$ the *asymmetry adjustor*, because it helps to capture the asymmetry of the density function of R_T. Finally, we arrive to the following expression for A (Koepf and Masjed-Jamei (2006)):

$$A = \frac{\Gamma(\frac{qi-p}{2}+1)\Gamma(\frac{-qi-p}{2}+1)}{2^p \pi \Gamma(-p+1)}.$$

Following from the generalized Rodrigues formula (2.12) - (2.13), the polynomial solution of the differential equation (5.6) is given as

$$R_k(x) = (1+x^2)^{-\frac{p}{2}+1} \exp(-q \arctan x) \frac{d^k}{dx^k}((1+x^2)^{\frac{p}{2}-1+k} \exp(q \arctan x)). \tag{5.8}$$

Similarly, we can use a combination of weight functions and orthogonal polynomials to determine the density function f_{R_T} of logarithmic stock return R_T and find

$$f_{R_T}(x) = \frac{1}{\delta} w(\frac{x-m}{\delta}) p(\frac{x-m}{\delta}), \tag{5.9}$$

$$\text{where } p(x) = 1 + \sum_{k=3}^{N} b_k R_k(x). \tag{5.10}$$

Combining the equations (5.8) - (5.10), we have

$$f_{R_T}(x) = \frac{A}{\delta}(1+y^2)^{\frac{p}{2}-1} e^{q \arctan y} + \frac{A}{\delta} \sum_{k=3}^{N} b_k \frac{d^k}{dy^k} \left((1+y^2)^{\frac{p}{2}+k-1} e^{q \arctan y} \right) \Bigg|_{y=\frac{x-m}{\delta}} . \tag{5.11}$$

Due to the asymmetry adjustor, the above density function is more powerful in capturing the asymmetry of distribution of R_T, in comparison with the Polynomial-T-Distribution model. We name this model the Polynomial-T-Distribution model with the asymmetric adjustor. When the parameter $q = 0$ in the equation (5.11), the model is reduced to the Polynomial-T-Distribution model with degree of freedom $1 - p$.

Similarly to the Polynomial-T-Distribution model, one must be careful about the sign of the polynomial $p(x)$ and the diverging integral when calculating the expected stock prices. The polynomial $p(x)$ used in (5.11) must be globally positive in order to generate a valid density function. Only when the polynomial parameters $\{b_k\}_{k=3,\cdots,N}$ resides in a special region of the parameter space, the positivity condition is satisfied. We use the similar method in the previous models to determine the region of positivity: if the parameters $\{b_k\}_{k=3,\cdots,N}$ are on the border of this region, $p(x)$ should be tangent to the x-axis. Therefore, $p(x)$ should have a multiple root x_0, satisfying the linear equations:

$$\begin{cases} p(x_0) & = 1 + \sum_{k=3}^{N} b_k R_k(x_0) = 0, \\ p'(x_0) & = \sum_{k=3}^{N} b_k R'_k(x_0) = 0 \end{cases}.$$

Solving $\{b_k\}_{k=3,\cdots,N}$ from the above linear equation system for any real x_0, we find the set of solutions in the parameter space $\mathbb{R}^{N-2}$. The set of solutions span a $N-3$ dimensional manifold, and indicates the border of the positivity region. The part including origin is the region generating valid density functions.

Because the density function (5.11) defines a fat-tailed distribution, one will find the integral diverges when calculating the expected stock prices. We use the same method of truncation as we used in the Polynomial-T-Distribution to deal with the problem (see Cassidy *et al.*

(2010)). For any real number S_c, we have the following equation:

$$\begin{aligned} E(S_T) &= E(S_0 e^{R_T}) \\ &= S_0 \int_{-\infty}^{\infty} e^x f_{R_T}(x)dx \\ &= S_0 \int_{-\infty}^{S_c+m} e^x f_{R_T}(x)dx + S_0 \int_{S_c+m}^{\infty} e^x f_{R_T}(x)dx \\ &= S_0 \int_{-\infty}^{S_c+m} e^x f_{R_T}(x)dx + S_0 R(S_c), \end{aligned}$$

where

$$R(S_c) = \int_{S_c+m}^{\infty} e^x f_{R_T}(x)dx. \tag{5.12}$$

When S_c is large and reasonable selected, we can ignore the reminder term $R(S_c)$ in the expected stock prices calculation. If the density function (5.11) is used to fit the risk-neutral measure density, the estimate $S_0 e^{rT} = E(S_T) \approx E^c(S_T)$ can be applied. In this case, the drift parameter m can be estimated in the following way:

$$
\begin{aligned}
e^{rT} = E_Q(\frac{S_T}{S_0}) &\approx \int_{-\infty}^{S_c+m} e^x f_{R_T}(x)dx \\
&= e^m \int_{-\infty}^{\frac{S_c}{\delta}} e^{\delta y} w(y)p(y)dy, \\
\text{Hence, } m &\approx \ln\left(\frac{e^{rT}}{\int_{-\infty}^{\frac{S_c}{\delta}} e^{\delta y} w(y)p(y)dy} \right). \qquad (5.13)
\end{aligned}
$$

Let p_c denote the probability that the stock price S_T is larger than $S_c + m$. Then

$$
p_c = P(S_T > S_c + m) = \int_{S_c+m}^{\infty} f_{R_T}(x)dx = \int_{\frac{S_c}{\delta}}^{\infty} w(y)p(y)dy. \qquad (5.14)
$$

The idea for our estimation here is to assign a reasonable value for p_c, and solve S_c and m by the equations (5.13) and (5.14). With the proper parametrization for the risk-neutral measure, we can estimate the European call and put option prices by the following integrals:

$$
\begin{aligned}
C &= \int_{\ln(K/S_0)}^{S_c+m} (S_0 e^x - K) f_{R_T}(x)dx \\
&= \int_{\frac{\ln(K/S_0)-m}{\delta}}^{S_c/\delta} (S_0 e^{\delta y+m} - K)\tilde{w}(y)p(y)dy, \qquad (5.15) \\
P &= \int_{-\infty}^{\frac{\ln(K/S_0)-m}{\delta}} (K - S_0 e^{\delta y+m})\tilde{w}(y)p(y)dy. \qquad (5.16)
\end{aligned}
$$

Please see Chapter 6 for data processing and numerical illustration for the Polynomial-T-Distribution model with the asymmetric adjustor.

5.2 Orthogonal Polynomials and Their Applications in Actuarial Modeling

5.2.1 Polynomial extension of Gamma distribution

The section develops our approach in risk estimation with respect to insurance claim portfolios. In most cases, it is extremely difficult to evaluate the true distribution of the insurance claim portfolio. However, it is possible to evaluate the moments of the claim amount distribution per claim and the whole portfolio by historical data. Therefore, it is reasonable and convenient to use a basic distribution to fit the distribution of the portfolio insurances claim, and use a combination of polynomials to adjust the distribution according to different moment parameters. For instance, the Gamma distribution and its corresponding polynomial model was explored and used to estimated the density function of portfolio claims amount (see Bowers, 1966). Let us begin with the collective risk model defined on some probability space $\{\Omega, F, P\}$. Denote the number of claims for a given insurance contract portfolio as M, and let it follows a Poisson distribution, whose mean is denoted as η. Then the probability mass of M is given as

$$P(M = k) = e^{-\eta} \frac{\eta^k}{k!}. \tag{5.17}$$

Denote each claim amount as $X_k, \quad k = 1, 2, \cdots M$, and the total claim amount as $X =$

$\sum_{k=1}^{M} X_k$. The claim amount X_k are supposed to be i.i.d. distributed and independent with the claim count M. We are interested in the distribution of the total claim $X = X_1 + X_2 + \cdots + X_M$. By estimating the moments of X_k by observation, we can calculate the estimated moments of X. Denote the j^{th} moment of X_k as $\bar{\mu}_j = E(X_k^j)$ and the j^{th} moments of X as μ_j. The first few μ_j are given as

$$\begin{aligned}
\mu_1 &= \eta\bar{\mu}_1, \\
\mu_2 &= \eta\bar{\mu}_2 + \eta^2\bar{\mu}_1^2, \\
\mu_3 &= \eta\bar{\mu}_3 + 3\eta^2\bar{\mu}_2\bar{\mu}_1 + \eta^3\bar{\mu}_1^3, \\
\mu_4 &= \eta\bar{\mu}_4 + 4\eta^2\bar{\mu}_3\bar{\mu}_1 + 3\eta^2\bar{\mu}_2^2 + 6\eta^3\bar{\mu}_2\bar{\mu}_1^2 + \eta^4\bar{\mu}_1^4, \\
\mu_5 &= \eta\bar{\mu}_5 + 5\eta^2\bar{\mu}_4\bar{\mu}_1 + 10\eta^2\bar{\mu}_3\bar{\mu}_2 + 10\eta^3\bar{\mu}_3\bar{\mu}_1^2 + 15\eta^3\bar{\mu}_2^2\bar{\mu}_1 + 10\eta^4\bar{\mu}_2\bar{\mu}_1^3 + \eta^5\bar{\mu}_1^5.
\end{aligned}$$

Now we can implement the polynomial models to fit the distribution of X. Let us consider the Pearson's differential equation

$$xF''(x) + (q - x)F'(x) + \lambda F(x) = 0. \tag{5.18}$$

The standardized weight function generated by this differential equation directly follows from the formula (2.11):

$$w(x) = \frac{1}{\Gamma(q)} x^{q-1} \exp(-x). \tag{5.19}$$

The polynomial solution of the equation (5.18) is the Lagurre polynomials:

$$\begin{aligned} L_n(x) &= x^{-q+1}e^x \frac{d^n}{dx^n}(x^{n+q-1}e^{-x}) \\ &= \sum_{k=0}^{n}(-1)^k \frac{\Gamma(n+q)\Gamma(n+1)}{\Gamma(k+q)\Gamma(k+1)\Gamma(n-k+1)} x^k. \end{aligned} \tag{5.20}$$

The formula (5.20) follows from the generalized Rodrigues formula (2.12) - (2.13). We use the above $w(x)$ as a basic density function, and use the product of $w(x)$ and a linear combination of $L_n(x)$ to model the density function of the total claim of an insurance portfolio. The density function is modeled as

$$f_X(x) = \frac{1}{\delta} w(\frac{x}{\delta})(1 + \sum_{k=3}^{N} b_k L_k(\frac{x}{\delta})). \tag{5.21}$$

The terms of L_1 and L_2 are ignored, in order to determine a unique parameter set of q, δ and $\{b_k\}_{k\geq 3}$ by the observed moments. For simplicity, we put $b_0 = 1$ and $b_1 = b_2 = 0$.

Let us denote b_k^* as the parameters found via the moment matching method. Denote $\hat{p}(x) = 1 + \sum_{k=3}^{N} \hat{b}_k L_k(x)$ as the estimation of the polynomial $p(x)$, $\hat{q}$ and $\hat{\delta}$ as the estimation of the shape and scale parameters. Hence, the parameters b_k^* should satisfy the following equations

$$\begin{aligned} \mu_1 &= \hat{\delta}\hat{q}, \\ \mu_2 &= \hat{\delta}^2\hat{q} + (\hat{\delta}\hat{q})^2, \\ \mu_k &= \sum_{j=0}^{k} \binom{k}{j} \hat{\delta}^k b_j^* j! (\hat{q})_j^2. \end{aligned}$$

To provide a global positivity of polynomials and to make other estimations reasonable, we use the following process to determine the estimation of $\hat{p}(x)$:

$$\text{Denote } \tau = \min\left\{0, \min_{x\in(0,\infty)}(1+\sum_{k=3}^{N} b_k^*(x))\right\}, \tag{5.22}$$

$$\text{and } \hat{p}(x) = 1+\sum_{k=3}^{N}\hat{b}_k L_k(x) = 1+\sum_{k=3}^{N}\frac{b_k^*}{1-\tau}L_k(x). \tag{5.23}$$

Using this procedure, we will arrive to a globally positive polynomial, as well as a positive density function $f_X(x)$. Denote ρ_X the stop-loss premium of the total claim, which is expected total claim value given that it exceeds its mean. Mathematically, the stop-loss premium is defined as follows:

$$\rho_X = E(X|X > E(X)). \tag{5.24}$$

The following theorem gives the value of the stop-loss premium ρ_X under the above modeling.

Theorem 5.1. *If the density of total claim size X is given by (5.21), then the stop-loss premium ρ_X is determined by the formula*

$$\rho_X = (\frac{q}{e})^q \frac{\delta}{\Gamma(q)}(1+\sum_{k=3}^{N}\frac{b_k q L_k''(q)}{k(k-1)}). \tag{5.25}$$

Proof. The mean of X is already calculated as δq. Using the generalized Rodrigues formula (2.12) - (2.13) and integration by parts, we can calculate ρ_X as follows:

$$
\begin{aligned}
\rho_X &= \int_{\delta q}^{\infty} (x-\delta q) f_X(x) dx \\
&= \int_{\delta q}^{\infty} \frac{x-\delta q}{\delta \Gamma(q)} e^{\frac{-x}{\delta}} (\frac{x}{\delta})^{q-1} p(\frac{x}{\delta}) dx \\
&= \int_{q}^{\infty} \frac{\delta(y-q)}{\Gamma(q)} e^{-y} y^{q-1} p(y) dy \\
&= \int_{q}^{\infty} \frac{\delta(y-q)}{\Gamma(q)} e^{-y} y^{q-1} dy + \sum_{k=3}^{N} b_k \int_{q}^{\infty} \frac{\delta(y-q)}{\Gamma(q)} e^{-y} y^{q-1} L_k(y) dy \\
&= -\frac{\delta e^{-y} y^q}{\Gamma(q)} \Big|_q^{\infty} + \sum_{k=3}^{N} b_k \int_{q}^{\infty} \frac{\delta(y-q)(-1)^k}{\Gamma(q)} \frac{d^k}{dy^k} (e^{-y} y^{q-1+k}) dy \\
&= (\frac{q}{e})^q \frac{\delta}{\Gamma(q)} + \sum_{k=3}^{N} \frac{b_k \delta (-1)^k}{\Gamma(q)} \Bigg((y-q) \frac{d^{k-1}}{dy^{k-1}} (e^{-y} y^{q-1+k}) \Big|_q^{\infty} \\
&\quad - \int_{q}^{\infty} \frac{d^{k-1}}{dy^{k-1}} (e^{-y} y^{q-1+k}) dy \Bigg) \\
&= (\frac{q}{e})^q \frac{\delta}{\Gamma(q)} + \sum_{k=3}^{N} \frac{b_k \delta (-1)^{k+1}}{\Gamma(q)} \frac{d^{k-2}}{dy^{k-2}} (e^{-y} y^{q-1+k}) \Big|_q^{\infty} \\
&= (\frac{q}{e})^q \frac{\delta}{\Gamma(q)} - \sum_{k=3}^{N} \frac{b_k \delta}{\Gamma(q) k(k-1)} e^{-y} y^{q+1} L_k''(y) \Big|_q^{\infty} \\
&= (\frac{q}{e})^q \frac{\delta}{\Gamma(q)} (1 + \sum_{k=3}^{N} \frac{b_k q L_k''(q)}{k(k-1)}).
\end{aligned}
$$

□

Assume the premium income is collected at the rate of $c+\theta$, continuously over time. The collecting rate of c offsets the expected rate of claim payment, while the rate of θ is the price for taking the risk, and/or making a profit for the insurance company (see Beekman, 1968). Denote Z_0 as the original portfolio reserve at time 0 and Z_T as the portfolio value at time

T. If there is no the interest and there are no any other investment incomes, the portfolio value change at $[0, T]$ is expressed in the following equation:

$$Z_T = Z_0 + (c + \theta)T - X. \tag{5.26}$$

The probability of ruin at time T is defined as

$$\psi_T = P(Z_T < 0) = P(X > Z_0 + (c + \theta)T). \tag{5.27}$$

Theorem 5.2. *If the density of total claim size X is given by (5.21), then the probability of ruin ψ_T is determined by the formula*

$$\psi_T = \frac{1}{\Gamma(q)}(\Gamma(q, \frac{Z_0 + \theta T}{\delta} + q) - \sum_{k=3}^{N} \frac{b_k e^{-y} y^q}{k} L'_k(\frac{Z_0 + \theta T}{\delta} + q). \tag{5.28}$$

Proof. The fist part of the premium income cT should be equal to the expected claim amount δq. Using the density function (5.21) and the generalized Rodrigues formula (2.12) - (2.13), the probability of ruin is calculated as follows:

$$
\begin{aligned}
\psi_T &= P(X > Z_0 + (c+\theta)T) \\
&= P(X > Z_0 + \delta q + \theta T) \\
&= \int\limits_{Z_0+\delta q+\theta T}^{\infty} f_X(x)dx \\
&= \int\limits_{Z_0+\delta q+\theta T}^{\infty} \frac{1}{\delta\Gamma(q)} \exp(\frac{-x}{\delta})(\frac{x}{\delta})^{q-1} p(\frac{x}{\delta})dx \\
&= \int\limits_{(Z_0+\theta T)/\delta+q}^{\infty} \frac{1}{\Gamma(q)} \exp(-y)y^{q-1} \sum_{k=0}^{N} b_k L_k(y)dy \\
&= \int\limits_{(Z_0+\theta T)/\delta+q}^{\infty} \frac{1}{\Gamma(q)} \sum_{k=0}^{N} (-1)^k b_k \frac{d^k}{dy^k}(e^{-y}y^{q-1+k})dy \\
&= \frac{1}{\Gamma(q)} \left(\Gamma(q, \frac{Z_0+\theta T}{\delta}+q) + \sum_{k=3}^{N}(-1)^k b_k \frac{d^{k-1}}{dy^{k-1}}(e^{-y}y^{q-1+k})dy)\right|_{(Z_0+\theta T)/\delta+q}^{\infty} \\
&= \frac{1}{\Gamma(q)}(\Gamma(q, \frac{Z_0+\theta T}{\delta}+q) - \sum_{k=3}^{N} \frac{b_k e^{-y} y^q}{k} L_k'(\frac{Z_0+\theta T}{\delta}+q).
\end{aligned}
$$

□

5.2.2 Polynomial extension of Beta distribution

We show that the Gamma distribution and its polynomial extension is useful in modeling the density function of the total claims X. The density tail of X would be similar to an exponential distribution in the model based on the Gamma distribution. However, if we have valid evidence that the total claim amount is below a certain limit, it is not appropriate to model the density of X by this model. In this case, we would use a combination of the Beta distribution and a polynomial to model the density function. Consider the following

Pearson's differential equation

$$(1-x^2)F''(x) + (\beta - \alpha - (\alpha+\beta+2)x)F'(x) + \lambda F(x) = 0. \tag{5.29}$$

The standardized weight function generated by this differential equation is

$$w(x) = \frac{(1-x)^\alpha (1+x)^\beta}{2^{\alpha+\beta+1}B(\alpha+1,\beta+1)}. \tag{5.30}$$

The polynomial solution of differential equation is the Jacobi Polynomials, given as follows

$$\begin{aligned} J_k(x) &= \frac{(-1)^k}{k!}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^k}{dx^k}((1-x)^{\alpha+k}(1+x)^{\beta+k}) \\ &= \sum_{j=0}^{k}\binom{k+\alpha}{j}\binom{k+\beta}{k-j}(-1+x)^{k-j}(1+x)^j. \end{aligned} \tag{5.31}$$

Combining the weight function and a polynomial expressed as a linear summation of $J(x)$, we model the density function of total claim X as follows:

$$f_X(x) = \frac{1}{m}w(\frac{x-m}{m})p(\frac{x-m}{m}), \tag{5.32}$$

$$\text{where,} \qquad p(x) = 1 + \sum_{k=3}^{N} b_k J_k(x). \tag{5.33}$$

The drift parameter m is set such that $2m$ is the upper bound of the total claims. Denote $\hat{\alpha}$ and $\hat{\beta}$ as the estimation of the parameters α and β, $\hat{p}(x) = 1 + \sum_{k=3}^{N} b_k J_k(x)$ as the estimation of the polynomial used in the density function, b_k^* as the polynomial parameters found via the moment matching method. The following equations are used to solve the estimation of the parameters $\hat{\alpha}$, $\hat{\beta}$ and $\hat{b}_k$.

$$\begin{aligned}
\mu_1 &= \frac{2m\hat{\beta}+2m}{\hat{\alpha}+\hat{\beta}+2},\\
\mu_2 &= \frac{4m^2(\hat{\beta}+2)(\hat{\beta}+1)}{(\hat{\alpha}+\hat{\beta}+2)(\hat{\alpha}+\hat{\beta}+1)},\\
\mu_k &= \sum_{l=0}^{k}\sum_{j=0}^{l}\sum_{i=0}^{l-j}\binom{k}{l}m^{k-l}\frac{m^l b_j^*(-1)^i 2^{2j}\binom{l}{j}\binom{l-j}{i}B(\hat{\alpha}+j+i+1,\hat{\beta}+l-i+1)}{B(\hat{\alpha}+1,\hat{\beta}+1)},\\
&\quad \text{for any } N>k>2,\\
\tau &= \min\left\{0, \min_{x\in(0,\infty)}(1+\sum_{k=3}^{N} b_k^* J_k(x))\right\},\\
\hat{p}(x) &= 1+\sum_{k=3}^{N}\hat{b}_k L_k(x) = 1+\sum_{k=3}^{N}\frac{b_k^*}{1-\tau}J_k(x).
\end{aligned}$$

With the estimated density function at hand, the stop-loss premium and the probability of ruin for such a model, can be calculated accordingly. The following theorems provide the value of stop-loss premium and probability of ruin.

Theorem 5.3. *If the density function of total claim amount X is defined by (5.30) - (5.33), the stop-loss premium of the total claim is as follows*

$$\begin{aligned}
\rho_X &= \frac{2m(\alpha+1)B(\frac{\beta+1}{\alpha+\beta+2},\alpha+1,\beta+2)-2m(\beta+1)B(\frac{\beta+1}{\alpha+\beta+2},\alpha+2,\beta+1)}{(\alpha+\beta+2)B(\alpha+1,\beta+1)}\\
&\quad +\sum_{k=3}^{N}\frac{(-1)^k b_k}{k!2^{\alpha+\beta+1}B(\alpha+1,\beta+1)}\left(m\frac{d^{k-2}}{dy^{k-2}}((1-y)^{\alpha+k}(1+y)^{\beta+k})\right)\Bigg|_{\frac{\beta-\alpha}{\alpha+\beta+2}} \qquad (5.34)
\end{aligned}$$

Proof. Using the density function (5.32) and integration by parts, the stop-loss premium

ρ_X is calculated as follows

$$
\begin{aligned}
\rho_X &= \int_{\frac{2m(\beta+1)}{\alpha+\beta+2}}^{2m} (x - \frac{2m(\beta+1)}{\alpha+\beta+2}) f_X(x)dx \\
&= \frac{1}{2^{\alpha+\beta+1}B(\alpha+1,\beta+1)} \int_{\frac{\beta-\alpha}{\alpha+\beta+2}}^{1} (my - \frac{m(\beta-\alpha)}{\alpha+\beta+2})(1-y)^{\alpha}(1+y)^{\beta}p(y)dy \\
&= \frac{1}{2^{\alpha+\beta+1}B(\alpha+1,\beta+1)} \\
&\quad \int_{\frac{\beta-\alpha}{\alpha+\beta+2}}^{1} (my - \frac{m(\beta-\alpha)}{\alpha+\beta+2}) \sum_{k=0}^{N} \frac{(-1)^k b_k}{k!} \frac{d^k}{dy^k}((1-y)^{\alpha+k}(1+y)^{\beta+k})dy \\
&= \frac{1}{2^{\alpha+\beta+1}B(\alpha+1,\beta+1)} \left\{ \sum_{k=3}^{N} \frac{(-1)^k b_k}{k!} \left(-m \int_{\frac{\beta-\alpha}{\alpha+\beta+2}}^{1} \frac{d^{k-1}}{dy^{k-1}}((1-y)^{\alpha+k}(1+y)^{\beta+k})dy \right.\right. \\
&\quad \left. + \ (my - \frac{m(\beta-\alpha)}{\alpha+\beta+2}) \frac{d^{k-1}}{dy^{k-1}}((1-y)^{\alpha+k}(1+y)^{\beta+k}) \Big|_{\frac{\beta-\alpha}{\alpha+\beta+2}}^{1} \right) \\
&\quad \left. + \int_{\frac{\beta-\alpha}{\alpha+\beta+2}}^{1} \frac{m(\alpha+1)}{\alpha+\beta+2}(1-y)^{\alpha}(1+y)^{\beta+1} + \frac{m(-\beta-1)}{\alpha+\beta+2}(1-y)^{\alpha+1}(1+y)^{\beta}dy \right\} \\
&= \frac{2m(\alpha+1)B(\frac{\beta+1}{\alpha+\beta+2},\alpha+1,\beta+2) - 2m(\beta+1)B(\frac{\beta+1}{\alpha+\beta+2},\alpha+2,\beta+1)}{(\alpha+\beta+2)B(\alpha+1,\beta+1)} \\
&\quad + \sum_{k=3}^{N} \frac{(-1)^k b_k}{k! 2^{\alpha+\beta+1} B(\alpha+1,\beta+1)} \left(m \frac{d^{k-2}}{dy^{k-2}}((1-y)^{\alpha+k}(1+y)^{\beta+k}) \right) \Big|_{\frac{\beta-\alpha}{\alpha+\beta+2}} .
\end{aligned}
$$

□

Theorem 5.4. *If the density function of total claim amount X is defined by (5.30) - (5.32), the probability of ruin at time T is given as*

$$\psi_T = \frac{1}{2^{\alpha+\beta+1}B(\alpha+1,\beta+1)}\Bigg(B(\frac{Z_0-m+(c+\theta)T}{m},\alpha+1,\beta+1) + \sum_{k=3}^{N}\frac{(-1)^{k+1}b_k}{k!}\frac{d^{k-1}}{dy^{k-1}}((1-y)^{\alpha+k}(1+y)^{\beta+k})\Big|_{\frac{Z_0-m+(c+\theta)T}{m}}\Bigg). \quad (5.35)$$

Proof. Using the density function (5.32) and integration by parts, the probability of ruin ψ_T is calculated as follows

$$\begin{aligned}
\psi_X &= P(X > Z_0 + (c+\theta)T) \\
&= \int_{Z_0+(c+\theta)T}^{2m} f_X(x)dx \\
&= \int_{\frac{Z_0-m+(c+\theta)T}{m}}^{1} \frac{1}{2^{\alpha+\beta+1}B(\alpha+1,\beta+1)}\sum_{k=0}^{N}\frac{(-1)^k b_k}{k!}\frac{d^k}{dy^k}((1-y)^{\alpha+k}(1+y)^{\beta+k})dy \\
&= \frac{1}{2^{\alpha+\beta+1}B(\alpha+1,\beta+1)}\Bigg(B(\frac{Z_0-m+(c+\theta)T}{m},\alpha+1,\beta+1) \\
&\quad + \sum_{k=1}^{N}\frac{(-1)^k b_k}{k!}\frac{d^{k-1}}{dy^{k-1}}((1-y)^{\alpha+k}(1+y)^{\beta+k})\Big|_{\frac{Z_0-m+(c+\theta)T}{m}}^{1}\Bigg) \\
&= \frac{1}{2^{\alpha+\beta+1}B(\alpha+1,\beta+1)}\Bigg(B(\frac{Z_0-m+(c+\theta)T}{m},\alpha+1,\beta+1) \\
&\quad + \sum_{k=3}^{N}\frac{(-1)^{k+1} b_k}{k!}\frac{d^{k-1}}{dy^{k-1}}((1-y)^{\alpha+k}(1+y)^{\beta+k})\Big|_{\frac{Z_0-m+(c+\theta)T}{m}}\Bigg).
\end{aligned}$$

□

5.2.3 Polynomial extension of Inverse-Gamma distribution

We have developed the polynomial extensions based on the Gamma distribution and the Beta distribution to model the insurance portfolio claims. The Gamma distribution is good at estimating portfolio claim amounts following distributions with exponential tails. The Beta distribution is good at estimation portfolio claim amounts with certain upper limits. But other distributions are needed when it comes to the fat-tail claims. The power principle is a common property for the fat-tailed distributions (Rachev *et al.* (2005). Some previous research used the generalized Pareto distribution to model the portfolio claim amounts (Gay (2005)). In this subsection, we introduce a model using a combination of the inverse Gamma distribution and a polynomial to model the density function of insurance portfolio claims. Consider the following Pearson's differential equation

$$x^2 B''(x) + (qx+1)B'(x) + \lambda B(x) = 0. \tag{5.36}$$

The standardized weight function of the above differential equation can be derived from (2.11) as

$$w(x) = \frac{x^{q-2}}{\Gamma(1-q)} \exp(-\frac{1}{x}). \tag{5.37}$$

The polynomial solution of the differential equation (5.36) is the Bessel polynomials, given as follows

$$B_k(x) = x^{-q+2} \exp(\frac{1}{x}) \frac{d^k}{dx^k} (x^{q+2k-2} \exp(-\frac{1}{x})). \tag{5.38}$$

We use the product of the standardized weight function $w(x)$ and a linear combination of $B_k(x)$ to model the density function of the insurance portfolio claim amount

$$\begin{aligned} f_X(x) &= \frac{1}{\delta} w(\frac{x}{\delta}) p(\frac{x}{\delta}), && (5.39) \\ \text{where,} \qquad & p(x) = 1 + \sum_{k=3}^{N} b_k B_k(x). && (5.40) \end{aligned}$$

In the above density function of X, the polynomial degree N must satisfy $N < 1 - q$ to make the integral of $f_X(x)$ converge. Similar to the previous model extensions, We ignore the term of $B_1(x)$ and $B_2(x)$ in the polynomial $p(x)$, in order to find the unique parameter set for δ, q, and b_k via the moment matching method. Denote $\hat{q}$ and $\hat{\delta}$ as the estimation of the shape parameter q and the scale parameter δ, $\hat{p}(x) = 1+\sum_{k=3}^{N} \hat{b}_k B_k(x)$ as the estimation of the polynomial $p(x)$, and b_k^* as the polynomial parameters found via moment matching method. We obtain the moments μ_k from data, and use the following procedure to calculate $\hat{q}$, $\hat{\delta}$ and $\hat{p}(x)$:

$$\begin{aligned} \mu_1 &= -\frac{\hat{\delta}}{\hat{q}}, \\ \mu_2 &= \frac{\hat{\delta}^2}{\hat{q}(\hat{q}+1)}, \\ \mu_k &= \hat{\delta}^k \left(1 + \sum_{j=3}^{k} (-1)^j b_j^* \frac{k(k-1)\cdots(k-j+1)}{-\hat{q}(-\hat{q}-1)\cdots(-\hat{q}-k-j+1)} \right), \\ \tau &= \min \left\{ 0, \min_{x\in(0,\infty)} (1 + \sum_{k=3}^{N} b_k^* B_k(x)) \right\}, \text{ when } N > k > 2, \\ \hat{p}(x) &= 1 + \sum_{k=3}^{N} \hat{b}_k B_k(x) = 1 + \sum_{k=3}^{N} \frac{b_k^*}{1-\tau} B_k(x). \end{aligned}$$

Using the density function of the total claim X, we give the value of stop-loss premium and probability of ruin in the following theorems.

Theorem 5.5. *If the density function of total claim amount X is defined by (5.39) - (5.40), the stop-loss premium of the total claim is as follows*

$$\begin{aligned}\rho_X &= \frac{\delta e^q(-q)^{-q-1}}{\Gamma(1-q)} + \sum_{k=3}^{N} \frac{b_k \delta}{\Gamma(1-q)} \\ &\quad \left(\frac{d^{k-2}}{dy^{k-2}}(y^{q+2k-2}e^{-1/y}) - (y+\frac{1}{q})\frac{d^{k-1}}{dy^{k-1}}(y^{q+2k-2}e^{-1/y}) \right)\Bigg|_{y=-1/q}. \quad (5.41)\end{aligned}$$

Proof. Using integration by parts, we can calculate the stop-loss premium of X as follows

$$\begin{aligned}\rho_X &= \int_{-\delta/q}^{\infty} (x+\frac{\delta}{q}) f_X(x)dx \\ &= \delta \int_{-1/q}^{\infty} (y+\frac{1}{q}) w(y)p(y)dy \\ &= \frac{\delta}{\Gamma(1-q)} \int_{-1/q}^{\infty} (y+\frac{1}{q})(y^{q-2}e^{-1/y} + \sum_{k=3}^{N} b_k \frac{d^k}{dy^k}(y^{q+2k-2}e^{-1/y}))dy \\ &= \frac{\delta}{\Gamma(1-q)} \left(\frac{y^q e^{-1/y}}{q}\Bigg|_{-1/q}^{\infty} + \sum_{k=3}^{N} b_k \int_{-1/q}^{\infty} (y+\frac{1}{q})\frac{d^k}{dy^k}(y^{q+2k-2}e^{-1/y})dy \right) \\ &= \frac{\delta e^q(-q)^{-q-1}}{\Gamma(1-q)} + \sum_{k=3}^{N} \frac{b_k\delta}{\Gamma(1-q)} \\ &\quad \left(\frac{d^{k-2}}{dy^{k-2}}(y^{q+2k-2}e^{-1/y}) - (y+\frac{1}{q})\frac{d^{k-1}}{dy^{k-1}}(y^{q+2k-2}e^{-1/y}) \right)\Bigg|_{y=-1/q}.\end{aligned}$$

□

Theorem 5.6. *If the density function of total claim amount X is defined by (5.30) - (5.32),* *the probability of ruin at time T is given as*

$$\begin{aligned}\psi_X &= \frac{\Gamma(1-q)-\Gamma(1-q,\delta/(Z_0+(c+\theta)T))}{\Gamma(1-q)} \\ &\quad + \sum_{k=3}^{N}\frac{b_k}{\Gamma(1-q)}\frac{d^{k-1}}{dy^{k-1}}(y^{q+2k-2}e^{-1/y})\Big|_{(Z_0+(c+\theta)T)/\delta}. \end{aligned} \tag{5.42}$$

Proof. The probability of ruin at time T can be calculated as follows using basic integration

$$\begin{aligned}\psi_X &= P(X > Z_0+(c+\theta)T) \\ &= \int_{Z_0+(c+\theta)T}^{\infty} f_X(x)dx \\ &= \int_{(Z_0+(c+\theta)T)/\delta}^{\infty} w(y)p(y)dy \\ &= \frac{1}{\Gamma(1-q)}\int_{(Z_0+(c+\theta)T)/\delta}^{\infty}(y^{q-2}e^{-1/y}+\sum_{k=3}^{N}b_k\frac{d^k}{dy^k}(y^{q+2k-2}e^{-1/y}))dy \\ &= \frac{1}{\Gamma(1-q)}\int_{0}^{\delta/(Z_0+(c+\theta)T)} z^{-q}e^{-z}dz+\sum_{k=3}^{N}\frac{b_k}{\Gamma(1-q)}\frac{d^{k-1}}{dy^{k-1}}(y^{q+2k-2}e^{-1/y})\Big|_{(Z_0+(c+\theta)T)/\delta} \\ &= \frac{\Gamma(1-q)-\Gamma(1-q,\delta/(Z_0+(c+\theta)T))}{\Gamma(1-q)} \\ &\quad +\sum_{k=3}^{N}\frac{b_k}{\Gamma(1-q)}\frac{d^{k-1}}{dy^{k-1}}(y^{q+2k-2}e^{-1/y})\Big|_{(Z_0+(c+\theta)T)/\delta}.\end{aligned}$$

□

CHAPTER 6

Numerical Illustrations and Comparison of Different Models

6.1 Numerical Illustrations for Financial Models

This section is devoted to the numerical fitting of our model extensions and their effectiveness. We use the monthly and daily returns for the index of S&P 500 from 1996 to 2010 process for parameter identification.

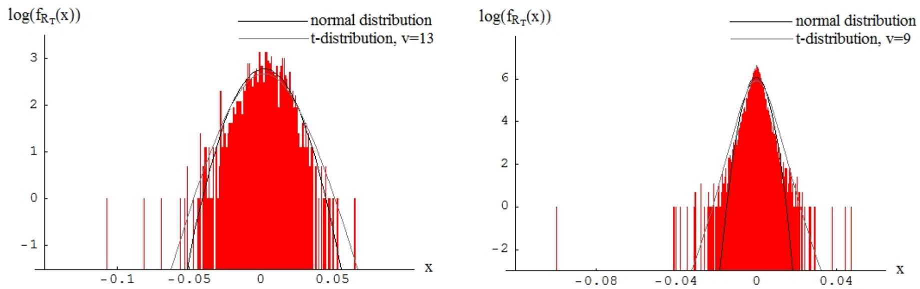
Firstly, We perform a tail analysis to see how the return period affects normality of the return distribution. The normal distribution and the Student's T-distribution represent the normal decay and power decay at tail analysis, respectively. We produce a logarithmic density plot for the normal distribution and the Student's T-distribution, along with the logarithmic frequency of the historical data, to see the approximate decay speed of the density tail.

Secondly, we fit and parameterize the financial model extensions by historical data. The

Black-Scholes model, the Polynomial-Normal model, the T-distribution model and the Polynomial-T-Distribution model and the Polynomial-T-Distribution model with the asymmetric adjustor are considered. We use the maximum likelihood method to find the optimal parameters, and use the Bayesian Information Criteria (BIC) to balance the simplicity and goodness of fit for different models (see Liddle (2007)). The optimal model is found in the sense of largest BIC values. The maximum polynomial degree we use in the financial polynomial models are set as 10. We use the bootstrap method to simulate the 95% and 90% percentage confidence intervals for the optimized parameters, in order to explore the statistical properties of these parameters. We provide the European option prices, the VaR and CVaR value, to estimate the effectiveness of the financial model extensions.

Figure 6.1(a) below shows the logarithmic frequency of the monthly returns beginning from 1950, and figure 6.1(b) shows the logarithmic frequency of daily returns in the same period. We can see the normal distribution severely underestimates the tail density of index return. The Student's T-distribution has a much better estimation on tail densities, but it is not accurate in estimations around the mean. This shortcoming can be overcome by using the Polynomial-T-Distribution model and the asymmetric adjustor. We can also see that the degree of freedom in daily returns is smaller than that in the monthly returns. It means that monthly returns present lighter density tails than monthly return, since the non-normality becomes diluted in the monthly returns.

Optimization over different polynomials for different model extensions indicates that the optimal polynomial for the Polynomial-Normal model is $p(x) = 1 + b_4H_4 + b_6H_6$ for daily data, and $p(x) = 1 + b_3H3 + b_4H_4$ for monthly data, in the sense that it provides the largest BIC value and, therefore, balances the simplicity and goodness of the best fit. The optimal polynomial for the Polynomial-T-Distribution model is $p(x) = 1 + b_4R_4 + b_6R_6 + b_8R_8$ for



(a) monthly return (b) daily return

Figure 6.1: Logarithmic frequency for S&P 500 returns from 1950

daily data, and $p(x) = 1 + b_3R_3 + b_4R_4$ for monthly data. The optimal polynomial for the Polynomial-T-Distribution model with the asymmetric adjustor is $p(x) = 1 + b_4R_4 + b_6R_6 + b_8R_8$ for daily data, and $p(x) = 1 + b_3R_3 + b_4R_4$ for monthly data. The Polynomial-T-Distribution model is the best among different models, because of the larger BIC value. Table 6.1 and Table 6.2 below shows the likelihood value and BIC value for the daily data and monthly data, respectively.

Table 6.1: Max likelihood and BIC value for daily returns of S&P 500 from 1996 to 2010

	Black-Scholes	Polynomial-Normal	T-Distribution	Polynomial-T	P-T with asymmetric adjustor
polynomial	1	$1 + b_4H_4 + b_6H_6$	1	$1 + b_4R_4 + b_6R_6 + b_8R_8$	$1 + b_4R_4 + b_6R_6 + b_8R_8$
degree of freedom ν	∞	∞	22.65	20.566	16.35
decay speed	∞	∞	22.65	12.566	8.35
asymmetric adjustor					0.08
Max Likelihood	14172.1	14447.2	16436.5	16716.5	16716.8
BIC Value	14172.1	14438.9	16432.3	16700.1	16696.2

Table 6.2: Max likelihood and BIC value for monthly returns of S&P 500 from 1996 to 2010

	Black-Scholes	Polynomial-Normal	T-Distribution	Polynomial-T	P-T with asymmetric adjustor
polynomial	1	$1 + b_3H_3 + b_4H_4$	1	$1 + b_3R_3 + b_4R_4$	$1 + b_3R_3 + b_4R_4$
degree of freedom ν	∞	∞	52.843	56.952	15.099
decay speed	∞	∞	52.843	52.952	11.099
asymmetric adjustor					1.17
Max Likelihood	442.2	449.1	545.7	553.1	553.2
BIC Value	442.2	443.9	543.1	545.3	542.8

Table 6.3 below provides the optimal parameters of the polynomial and the 95% and 90% confidence intervals of these parameters, for the Polynomial-Normal model fitted by the daily data. Table 6.4 below gives the corresponding results fitted by the monthly data. These results are obtained by the bootstrap method (see Kling and Alles (1994)).

Table 6.3: optimal parameters, 95% and 90% confidence intervals for the Polynomial-Normal model, fitted by daily return

	optimal parameters	95% CI	90% CI
b_3		(-0.05467,0.10891)	(-0.03123,0.08027)
b_4	0.31291	(0.13997,0.43187)	(0.12598,0.36528)
b_5		(-0.10412,0.08226)	(-0.09011,0.06903)
b_6	0.42810	(-0.00723,0.16515)	(0.00071,0.1512)
b_7		(-0.06196,0.01407)	(-0.06022,0.01190)
b_8		(-0.02791,0.00452)	(-0.02002,0.00188)
b_9		(-0.01501,0.04199)	(-0.01134,0.03670)
b_{10}		(-0.0088,0.00342)	(-0.0063,0.00302)

Table 6.4: optimal parameters, 95% and 90% confidence intervals for the Polynomial-Normal model, fitted by monthly return

	optimal parameters	95% CI	90% CI
b_3	0.14400	(-0.04820,0.23196)	(0.03126,0.21517)
b_4	0.05322	(-0.00145,0.11902)	(0.00093,0.10317)
b_5		(-0.05731,0.03411)	(-0.02619,0.02893)
b_6		(-0.00744,0.00618)	(-0.00580,0.00489)
b_7		(-0.00561,0.00419)	(-0.00358,0.00299)
b_8		(-0.00360,0.00106)	(-0.00297,0.00079)
b_9		(-0.00082,0.00037)	(-0.00064,0.00026)
b_{10}		(-0.00023,0.00041)	(-0.00012,0.00038)

Table 6.5 provides the optimal polynomial parameters and the confidence intervals, for the Polynomial-T-Distribution model fitted by the daily data. Table 6.6 gives the corresponding results fitted by the monthly data.

Table 6.7 provides the optimal polynomial parameters and the confidence intervals, for the Polynomial-T-Distribution model with the asymmetric adjustor, fitted by the daily data.

Table 6.5: optimal parameters, 95% and 90% confidence intervals for the Polynomial-T-Distribution model, fitted by daily return

	optimal parameters	95% CI	90% CI
b_3		(-0.04417,0.08181)	(-0.03253,0.07237)
b_4	0.42569	(0.11432,0.30547)	(0.12598,0.28528)
b_5		(-0.11212,0.07934)	(-0.09625,0.06537)
b_6	0.43165	(-0.00885,0.16498)	(0.00092,0.14784)
b_7		(-0.06693,0.01392)	(-0.05816,0.01035)
b_8	0.304304	(-0.02598,0.00361)	(-0.01740,0.00197)
b_9		(-0.01427,0.04243)	(-0.01426,0.03596)
b_{10}		(-0.0089,0.00248)	(-0.01063,0.00321)

Table 6.6: optimal parameters, 95% and 90% confidence intervals for the Polynomial-T-Distribution model, fitted by monthly return

	optimal parameters	95% CI	90% CI
b_3	0.11360	(-0.0422,0.23195)	(0.05138,0.21397)
b_4	0.04000	(-0.00113,0.10967)	(0.00108,0.10019)
b_5		(-0.02218,0.02838)	(-0.01937,0.02122)
b_6		(-0.00561,0.00571)	(-0.00464,0.00492)
b_7		(-0.00483,0.00329)	(-0.00387,0.00295)
b_8		(-0.00348,0.00065)	(-0.00291,0.00045)
b_9		(-0.00071,0.00028)	(-0.00064,0.00022)
b_{10}		(-0.00013,0.00046)	(-0.00009,0.00042)

Table 6.8 gives the corresponding results fitted by the monthly data.

We can verify that polynomials with the optimized parameters in our models satisfy the global positivity condition and therefore generate valid density functions. The distribution decay speed for the models fitted with daily data is mostly less than the corresponding decay speed fitted with monthly data, except for the Polynomial-T-Distribution model with the asymmetric adjustor. It is an indication of the diluted normality for the monthly data, as the fitted distribution for the monthly data is closer to the normal distribution. The T-Distribution based models yield a better fit than normal distribution based models, in the sense of much larger likelihood and BIC values. The confidence intervals (see Table

Table 6.7: optimal parameters, 95% and 90% confidence intervals for the Polynomial-T-Distribution model with the asymmetric adjustor, fitted by daily return

	optimal parameters	95% CI	90% CI
b_3		(-0.14217,0.27928)	(-0.10491,0.18898)
b_4	0.24349	(0.09391,0.37131)	(0.12660,0.29003)
b_5		(-0.14607,0.05660)	(-0.10558,0.04910)
b_6	0.08783	(-0.00935,0.17117)	(0.00252,0.14487)
b_7		(-0.03673,0.01188)	(-0.03091,0.00889)
b_8	0.01617	(-0.00440,0.03282)	(-0.00191,0.03097)
b_9		(-0.07310,0.03055)	(-0.03539,0.02897)
b_{10}		(-0.00732,0.00238)	(-0.0686,0.00197)

Table 6.8: optimal parameters, 95% and 90% confidence intervals for the Polynomial-T-Distribution model with the asymmetric adjustor, fitted by monthly return

	optimal parameters	95% CI	90% CI
b_3	0.00457	(-0.00192,0.00893)	(-0.00665,0.00080)
b_4	0.00107	(-0.00083,0.00528)	(0.00023,0.00462)
b_5		(-0.00148,0.00098)	(-0.00131,0.00088)
b_6		(-0.00051,0.00059)	(-0.00046,0.00053)
b_7		(-0.00010,0.00019)	(-0.00008,0.00017)
b_8		(-0.00007,0.00006)	(-0.00005,0.00005)
b_9		(-0.00002,0.00001)	(-0.00002,0.00001)
b_{10}		(-0.00001,0.00001)	(-0.00000,0.00001)

6.3 - Table 6.8) demonstrate that some polynomial parameters are significantly different from 0, and therefore, they should not be ignored statistically. The asymmetric adjustor for the Polynomial-T-Distribution model slightly increases the maximum likelihood value, but decreases the BIC value. It means that the additional goodness of fit provided by the asymmetric adjustor is not enough to afford the increase of model complexity. The optimal polynomials we found also shows that the distribution with daily return is more likely symmetric, as the non-zero parameters are found with even degree. It means that daily returns have similar chances to increase or decrease. The distribution with monthly return becomes less symmetric, as the skewness becomes more statistically important for monthly returns. It indicates a non-zero skewness returns of a longer period.

Assume the original index price and strike price to be 1000 and the market risk-free rate to be 0.05, then we can calculate the one-month option prices as in Table 6.9 below. These prices depend on the significance probability for the T-Distribution model, the Polynomial-T Distribution model and the Polynomial-T-Distribution model with the asymmetric adjustor.

Table 6.9: option prices, $S_0 = 1000$, $r = 0.05$, 1 month expiry

		$K = 950$		$K = 1000$		$K = 1050$	
	p_c	call	put	call	put	call	put
Black-Scholes model		53.95	0	9.84	5.682	0	45.634
Polynomial-Normal model		54.04	0.09	10.489	6.331	0.042	45.676
T-Distribution model	10^{-2}	53.96	0.01	7.324	3.166	-9.998	35.636
	10^{-4}	53.993	0.042	10.784	6.626	0.258	45.892
	10^{-6}	53.993	0.043	10.830	6.672	0.365	45.999
	10^{-8}	53.993	0.043	10.830	6.672	0.365	45.999
Polynomial-T model	10^{-2}	54.028	0.078	7.655	3.497	-10.218	35.416
	10^{-4}	54.158	0.208	10.625	6.467	0.121	45.755
	10^{-6}	54.160	0.21	10.666	6.508	0.231	45.865
	10^{-8}	54.160	0.21	10.666	6.508	0.231	45.865
P-T model with the	10^{-2}	54.035	0.085	7.703	3.545	-9.892	35.742
asymmetric adjustor	10^{-4}	54.162	0.212	10.635	6.477	0.171	45.805
	10^{-6}	54.164	0.214	10.674	6.516	0.252	45.886
	10^{-8}	54.164	0.214	10.674	6.516	0.252	45.886

We can see how option prices vary with different significance probability p_c in the T-Distribution based models. A p_c value of 10^{-2} is obviously not producing sufficient accuracy, as option prices generated are far away from those in other models. When we use p_c values smaller than 10^{-4}, option prices begin to converge to reasonable prices, and show little changes when the p_c value is below 10^{-6}. We regard $p_c = 10^{-6}$ as an acceptable significance probability that leads to reasonable accuracy. We also note that when the value of p_c is below 10^{-160}, option prices begin to diverge and finally end up with a zero call price. This is because in the equations (4.6) and (4.12), the estimation of m will diverge as p_c approaches to 0. The value of p_c that leads to such divergence depends on the data variance. The T-Distribution based models give higher values for the out-of-money options than the Polynomial-Normal and Black-Scholes model. Both the out-of-money options worth 0 in the Black-Scholes model according to the table above, because the return tail density are severely underestimated. The Polynomial-Normal and the T-Distribution model are possibly underestimating tail density too, as the out-of-money option prices generated by these two models are relatively low.

Assume the original index price is 1000 and the market risk-free rate is 0.05, Table 6.10 and Table 6.11 below give the VaR and CVaR values for the different models, fitted by the daily return data and the monthly return data respectively.

Table 6.10: one-day VaR and CVaR values, $S_0 = 1000$, $r = 0.05$

	VaR value			CVaR value		
	$\lambda = 0.05$	$\lambda = 0.02$	$\lambda = 0.01$	$\lambda = 0.05$	$\lambda = 0.02$	$\lambda = 0.01$
Black-Scholes model	9.364	11.664	13.195	11.712	13.723	15.093
Polynomial-Normal model	9.272	14.276	16.674	13.82	17.224	19.045
T-Distribution model	10.216	12.946	14.847	13.077	15.592	17.388
Polynomial-T model	8.655	13.594	17.202	14.572	20.505	25.787
P-T with the asymmetric adjustor	9.282	13.709	16.860	14.401	18.922	23.713

Table 6.11: one-month VaR and CVaR values, $S_0 = 1000$, $r = 0.05$

	VaR value			CVaR value		
	$\lambda = 0.05$	$\lambda = 0.02$	$\lambda = 0.01$	$\lambda = 0.05$	$\lambda = 0.02$	$\lambda = 0.01$
Black-Scholes model	36.135	44.332	49.757	44.482	51.609	56.439
Polynomial-Normal model	39.008	50.236	57.294	50.123	59.367	65.281
T-Distribution model	37.382	46.185	52.124	46.442	54.262	59.667
Polynomial-T model	41.812	55.919	64.291	55.583	66.759	73.764
P-T with the asymmetric adjustor	38.481	53.741	63.363	53.662	66.422	74.672

Comparing Table 6.10 and Table 6.11 above, we see that the T-distribution based models give higher VaR and CVaR values than the normal distribution based models, and the distributions with polynomial adjustments give higher VaR and CVaR values than distributions without polynomial adjustments. The polynomial adjustments present heavier effects than T-distribution tails. The cause of such effect would be the lack of goodness of fit in the Black-Scholes model and the T-Distribution model. The distributions of both the daily and monthly returns could not be fitted well and are underestimated at the lower quantiles by the normal distribution and T-distribution. Polynomial adjustments appears to be effective in adjusting such underestimations. The Polynomial-T-Distribution model gives the best estimates as the T-Distribution tails provides more reasonable estimations than normal tail-s, in additional to the polynomial adjustment. Comparing the Polynomial-T-Distribution model and the Polynomial-T-Distribution model with the asymmetric adjustor, we see that the asymmetric adjustor reduces the VaR and CVaR values slightly. It might be caused by the reduced probability mass assigned to the negative returns when we use the asymmetric adjustor.

6.2 Numerical Illustration for Actuarial Models

For the actuarial modeling, we use artificial data to compare the fitting of different models. The number of claims N and the claim sizes $X_k, k = 1, 2, \cdots, N$ are generated randomly, based on some prior distribution assumptions. The method of moments is exploited for identification of parameters in the models under consideration. The identification procedure is processed on multiple scenarios, which are classified by the prior assumptions on the artificial data.

In the first scenario, we use the Poisson distribution with mean $\eta = 10$, to count the number of claims, while each claim follows a uniform distribution on the interval $[0, 1000]$.

In the second scenario, we again use the Poisson distribution with mean $\eta = 10$, but the claim size each follows an exponential distribution with mean 1000.

In the third scenario, we model the claim number as a Binomial distribution $\mathcal{B}(25, 0.4)$ (Binomial distribution, which indicates the success counts for 25 trials, while the success rate for each trial is 0.4), while each claim follows a uniform distribution on the interval $[0, 1000]$. In all cases, we use polynomials with degrees up to 8.

Table 6.12 shows the parametrization results of our artificial data(the stop-loss premium and the probability of ruin), generated by a Polynomial-Gamma model. Table 6.13 shows the corresponding results for the Polynomial-Beta model. Finally, Table 6.14 displays the corresponding results for the Polynomial-Inverse-Gamma model. In the calculation of the probability of ruin, the additional premium rate θ is set as $\theta = 0.01c$, and the initial reserve is $Z_0 = 2000$.

Table 6.12: parametrization results for the Polynomial-Gamma model, stop-loss premium and probability of ruin

distribution of N distribution of X_k	$Poisson(10)$ $Uniform(0, 1000)$	$Poisson(10)$ $Exponential(1000)$	$Binomial(25, 0.4)$ $Uniform(0, 1000)$
shape parameter	$b = 7.313$	$b = 5.887$	$b = 8.190$
scale parameter	$\delta = 638.82$	$\delta = 776.87$	$\delta = 721.95$
polynomial Parameters	$b_3 = 2.69 \times 10^{-6}$	$b_3 = 3.28 \times 10^{-6}$	$b_3 = 2.39 \times 10^{-6}$
	$b_4 = 3.83 \times 10^{-8}$	$b_4 = 4.81 \times 10^{-8}$	$b_4 = 3.31 \times 10^{-8}$
	$b_5 = 6.77 \times 10^{-10}$	$b_5 = 1.02 \times 10^{-9}$	$b_5 = 5.35 \times 10^{-10}$
	$b_6 = 7.99 \times 10^{-12}$	$b_6 = 1.33 \times 10^{-11}$	$b_6 = 5.96 \times 10^{-12}$
	$b_7 = 7.53 \times 10^{-14}$	$b_7 = 1.38 \times 10^{-13}$	$b_7 = 5.31 \times 10^{-14}$
	$b_8 = 4.90 \times 10^{-16}$	$b_8 = 1.04 \times 10^{-15}$	$b_8 = 3.18 \times 10^{-16}$
stop-loss premium	681.418	909.649	878.024
probability of ruin	0.1219	0.1753	0.1692

Table 6.13: parameterization results for the Polynomial-Beta model, stop-loss premium and probability of ruin

distribution of N distribution of X_k	$Poisson(10)$ $Uniform(0, 1000)$	$Poisson(10)$ $Exponential(1000)$	$Binomial(25, 0.4)$ $Uniform(0, 1000)$
shape parameter	$\alpha = 11.099$	$\alpha = 16.317$	$\alpha = 12.877$
	$\beta = 3.892$	$\beta = 4.160$	$\beta = 4.555$
drift parameter	$m = 7501$	$m = 24326$	$m = 10734$
polynomial Parameters	$b_3 = -1.83 \times 10^{-8}$	$b_3 = -2.94 \times 10^{-6}$	$b_3 = -5.02 \times 10^{-7}$
	$b_4 = 3.61 \times 10^{-7}$	$b_4 = 1.91 \times 10^{-6}$	$b_4 = 1.59 \times 10^{-6}$
	$b_5 = -1.01 \times 10^{-6}$	$b_5 = -6.44 \times 10^{-7}$	$b_5 = -3.01 \times 10^{-7}$
	$b_6 = -1.08 \times 10^{-7}$	$b_6 = 2.91 \times 10^{-7}$	$b_6 = 2.25 \times 10^{-7}$
	$b_7 = 1.73 \times 10^{-7}$	$b_7 = -1.26 \times 10^{-7}$	$b_7 = -4.25 \times 10^{-8}$
	$b_8 = -4.01 \times 10^{-5}$	$b_8 = -2.67 \times 10^{-5}$	$b_8 = -7.85 \times 10^{-5}$
stop-loss premium	645.022	1692.25	863.134
probability of ruin	0.0071	0.012	0.0145

Table 6.14: parameterization results for the Polynomial-Inverse-Gamma model, stop-loss premium and probability of ruin

distribution of N distribution of X_k	$Poisson(10)$ $Uniform(0, 1000)$	$Poisson(10)$ $Exponential(1000)$	$Binomial(25, 0.4)$ $Uniform(0, 1000)$
shape parameter	$q = -8.391$	$q = -7.159$	$q = -9.037$
scale parameter	$\delta = 52375$	$\delta = 87300$	$\delta = 50175$
polynomial Parameters	$b_3 = 0.301$	$b_3 = 1.00$	$b_3 = 0.048$
	$b_4 = 0.342$	$b_4 = -0.74$	$b_4 = -0.053$
	$b_5 = -0.145$	$b_5 = 2.27$	$b_5 = 0.0193$
	$b_6 = -0.985$	$b_6 = -0.47$	$b_6 = 0.1685$
	$b_7 = 0.105$	$b_7 = -0.09$	$b_7 = -0.0277$
	$b_8 = 0.043$	$b_8 = -0.07$	$b_8 = -0.0062$
stop-loss premium	530.85	551.78	670.35
probability of ruin	0.1304	0.0902	0.1772

Bibliography

Abken, P. A., Madan, D. B., Ramamurtie, S. (1996) Estimation of Risk-Neutral and Statistical Densities by Hermite Polynomial Approximation: With Application to Eurodollar Futures Options, Working Paper 96-5, (Federal Reserve Bank of Atlanta).

Abken, P. A., Madan, D. B.; Ramamurtie, S. Pricing S&P 500 Index options using a Hilbert space basis, Working Paper 96-21 (1996), (Federal Reserve Bank of Atlanta).

Akgiray, A. and Booth, G. G. (1988) The Stable-Law Model of Stock Returns. *Journal of Business and Economic Statistics* 6, 51-57.

Andrews, G. E., Askey, R. and Roy, R. (1999) Special functions. Encyclopedia of Mathematics and its Applications. *Cambridge University Press*, 71.

Arthur, W. B., Hollan, J. H., LeBaron, B., Palmer, R. and Tayler, P. (1997) Asset Pricing Under Endogenous Expectations in an Artificial Stock Market. *The Economy as an Evolving Complex System, II*, XXVII of SFI Studies in the Sciences of Complexit. Redwood, CA: Addison-Wesley.

Askey, R. (2005) In Altmann, Simon L.; Ortiz, Eduardo L., Mathematics and social utopias in France: Olinde Rodrigues and his times, History of mathematics, 28, 105-118.

Backus, D., Foresi, S., Wu, L. (2002) Accounting for Biases in Black-Scholes, *Finance*, (provided by EconWPA), No. 0207008.

Bartlett, D. K. (1965) Excess Ratio Distribution in Risk Theory. *Transactions of Society of Actuaries*, 17, 435-453.

Barton, D. E. and Dennis, K. E. (1952) The Conditions under Which Gram-Charlier and Edgeworth Curves are Positive Definite and Unimodal, *Biometrika*, 39 (3-4), 425-427.

Beekman, J. K. (1968) Collective Risk Results, *Transactions of Society of Actuaries*, 20, 182-199.

Bertrand, P., Prigent, J.-L. (2005) Portfolio Insurance Strategies: OBPI versus CPPI, *Finance*, 26 (1), 5-32.

Black, F. (1989) How to Use the Holes in Black-Scholes, *Journal of Applied Corporate Finance*, 1 (4), 67-73.

Black, F., Scholes, M. The Pricing of Options and Corporate Liabilities, *The Journal of Political Economy*, 81 (3), 637-654, (1973).

Blattberg, R. C. and Gonedes, N. A. (1974) Comparison of the Stable and Student Distributions as Statistical Models for Stock Prices, *Journal of Business*, 47 (2), 244-280.

Bowers, N. L. (1966) Expansion of Probability Density Functions as a Sum of Gamma Densities with Applications in Risk Theory, *Transactions of Society of Actuaries*, 18, 125-137.

Buhlmann, H. (1997) The Actuary: the Role and Limitations of the Profession Since the Mid-19th Century, *Astin Bulletin*, 27 (2), 165-171.

Cassidy, D. T., Hamp, M. J. and Ouyed, R. (2010) Pricing European Options with a Log Student's t-distribution: a Gosset Formula, *Physica A: Statistical Mechanics and its Applications* 389 (24), 5736-5748.

Cont, R. and Bouchaud, J. P. (2000) Herd Behavior and Aggregate Fluctuations in Financial Markets. *Macroeconomic Dynamics*, 4, 170-196.

Darolles, S., Laurent, J. (2000) Approximating Payoffs and Pricing Formulas. *Journal of Economic Dynamics & Control*, 24, 1721-1746.

de Andrade, E. X. L.,Dimitrov D. K., Ranga A. S., Unesp I. (1998) Characterization of Generalized Bessel Polynomials in Terms of Polynomial Inequalities, *Journal of Mathematical Analysis and Applications* 211, 538-543.

de Jager, G., Winsen, J. (1992) Curved Option Payoffs, Series working paper, 19, (provided by School of Finance and Economics, University of Technology, Sydney).

Fama, E. F. (1965) Mandelbrot and the Stable Paretian Hypothesis, *Journal of Business*, 36, 420-429.

Fedoryuk, M. V. (2001) Hermite Polynomial, (Editor: Hazewinkel), *M. Encyclopedia of Mathematics, Springer-Verlag.*

Fergusson, K. and Platen, E. (2006) On the Distributiona Characterization of Daily Log-Returns of a World Stock Index, *Applied Mathematical Finance*, 13 (1), 19-38.

Gabaix, X., Gopikrishnan, P., Plerou, V. and Eugene, S. H. (2003) A Theory of Power-Law Distributions in Financial Market Fluctuations, *Nature*, 423 (6937), 267–270.

Gay, R. (2005) Premium Calculation for Fat-Tailed Risk. *Austin Bulletin*, 35 (1), 163-188.

Gopikrishnan, P., Plerou, V., Amaral, L. A. N., Meyer, M., Stanley, H. E. (1999) Scaling of the distribution of fluctuations of financial market indices. *Physical Review E*, 60, 6519C6529.

Hardy, M. (2006) An Introduction to Risk Measures for Actuarial Applications, *SOA Syllabus Study Note*.

Hibbert, A. M., Daigler, R. T. and Dupoyet, B. (2008) A Behavioral Explanation for the Negative Asymmetric Return-volatility Relation. *Journal of Banking & Finance*, 32, 2254-2266.

Hull, J. (2011) Options, Futures and Other Derivative Securities, 8th edition, Prentice-Hall.

Jarrow, R. and Rudd, A. (1982) Approximate Option Valuation for Arbitrary Stochastic Processes, *Journal of Financial Economics*, 10 (3), 347-369.

Jondeau, E. and Rockinger, M. (1998) Estimating Gram-Charlier Expansions with Positive Constraints, *Documents de Travail*, 56, (provided by Banque de France).

Jondeau, E. and Rockinger, M. (2001) Gram-Charlier Densities, *Journal of Economic Dynamics and Controls*, 25 (10), 1457-1483.

Ki, H., Choi, B., Chang, K., Lee, M. (2005) Option Pricing Under Extended Normal Distribution, *Journal of Future Markets*, 25 (9), 845-871.

Kirshnamoorthy, A. S. (1951) On the Orthogonal Polynomials Associated with Student's Distribution, *The Indian Journal of Statistics*, 11 (1), 37–44.

Kling, J. L. and Alles, L. A. (1994) Regularities in the Variation of Skewness in Asset Returns, *Journal of Financial Research*, 17 (3), 427-438.

Knight, J., Satchell, S. Pricing Derivatives Written on Assets with Arbitrary Skewness and Kurtosis. (2000) *Return Distributions in Finance*, Butterworth Heinemann.

Koepf, W. and Masjed-Jamei, M. (2006).A generalization of Student's T-distribution from the viewpoint of special functions, *Integral Transforms and Special Functions*, 17:12, 863-875.

Lau, H., Wingender, J. R. and Lau, A. H. (1989) On Estimating Skewness in Stock Returns. *Management Science*, 35, 9, 1139-1142.

Li, H. and Melnikov, A. (2012) On the Polynomial-Normal Model and Option Pricing. *Advances in Statistics, Probability and Actuarial Science*, 1, 285-302. World Scientific Published, Singapore.

Li, H. and Melnikov, A. (2013) On polynomial extension of t-distribution and its financial applications. *Risk and Decision Analysis*, 4, 255-266.

Liddle, A. R. (2007) Information Criteria for Astrophysical Model Selection. *Monthly Notices of the Royal Astronomical Society: Letters*, 377 (1), 74-78.

Lien, D. (2002) A Note on the Relationships between Some Risk-Adjusted Performance Measures, *The Journal of Future Markets*, 22 (5), 483-495.

Lo, A. W. (2002) The Statistics of Sharpe Ratio, *Financial Analysts Journal*, 58 (4), 36-52.

Macovschi, S. and Quittard-Pinon, F. (2006) On the Pricing of Power and Other Polynomial Options, *The Journal of Derivatives*, 13 (4), 61-71.

Madan, D. and Milne, F. (1994). Contingent Claims Valued and Hedged by Pricing and Investing in a Basis, *Mathematical Finance*. 4 (3), 223-245.

Malik, P., Swaminathan, A. (2012). Derivatives of a Finite Class of Orthogonal Polynomials Related to Inverse Gamma Distribution. *Applied Mathematics and Computation*, 218, 6251-6262.

Mandelbrot, B. B. (1963). The Variation of Certain Speculative Prices. *Journal of Business*, 36, 394-419.

Nikiforov A. F. and Uvarov V. B (1988). Special Functions of Mathematical Physics, Birkhauser Verlag, Basel.

Pearson, K (1893). Contributions to the mathematical theory of evolution. *Proceedings of the Royal Society of London*, 54 (326C330): 329C333.

Pearson, K (1895). Contributions to the mathematical theory of evolution, II: Skew variation in homogeneous material. *Philosophical Transactions of the Royal Society of London*, ARRAY 186: 343C414.

Pearson, K (1901). Mathematical contributions to the theory of evolution, X: Supplement to a memoir on skew variation. *Philosophical Transactions of the Royal Society of London*, Series A, Containing Papers of a Mathematical or Physical Character 197 (287C299): 443C459.

Pearson, K (1916). Mathematical contributions to the theory of evolution, XIX: Second supplement to a memoir on skew variation. *Proceedings of the Royal Society of London*, Series A, Containing Papers of a Mathematical or Physical Character 216 (538C548): 429C457.

Platen, E. and Rendek, R. (2008) Empirical Evidence on Student-t Log-Returns of Diversified World Stock Indices, *Journal of Statistical Theory and Practice*, 2 (2), 233-251.

Platen, E. and Sidorowicz, R. (2007) Empirical evidence on Student-t log-returns of diversified world stock indices. Research paper series, 194, Finance Research Centre, University of Technology, Sydney, Broadway, N.S.W.

Rachev, S. T. (2003) Handbook of Heavy-Tailed Distribution in Finance, North Holland.

Rachev, S. T., Menn, C. and Fabozzi, F. J. (2005) Fat-Tailed and Skewed Asset Return Distributions: Implications for Risk Management, Portfolio Selection, and Option Pricing. John Wiley, New York.

Raposo, A. P., Weber, H. J., Alvarez-Castillo, D. and Kirchbach, M. (2007) Romanovski Polynomials in Selected Physics Problems, *Central European Journal of Physics*, 5 (3), 253-284.

Rockafellar, R. T. and Uryasev, R. Conditional value-at-risk for general loss distributions, *Journal of Banking & Finance*, 26, 1443-1471.

Romanovsky, V. (1929) Calcul des Probabilities-Sur quelques classes nouvelles de polynomes orthogonaux. *Comptes Rendus*, 188, 1023-1025.

Saphores, J. M. The Density of Bounded Diffusions, *Journal of Economics Letters*, 86 (1), 87-93.

Shaw, W. T. (2011) Risk, VaR, CVaR and Their Associated Portfolio Optimizations When Asset Returns Have a Multivariate Student T Distribution. Available at SSRN: http://ssrn.com/abstract=1772731 or doi:10.2139/ssrn.1772731.

Shiller, R. (1981) Do Stock Prices Move Too Much to be Justified by Subsequent Changes in Dividens. *American Economic Review*, 71, 421-436.

Weber, H. J. (2007). Connections Between Real Polynomial Solutions of Hypergeometric-Type Differential Equations with Rodrigues Formula. *Central European Journal of Mathematics*, 5, 415-427.

Weber, H. J. (2007). Connections between Romanovski and other polynomials. *Central European Journal of Mathematics*, 5, 581-595.