Towards a Solution of Higher Airy Structures in Terms of Tau Functions and Matrix Models

Robert Maher

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Department of Mathematical and Statistical Sciences

University of Alberta

Abstract

The first part of this thesis presents an exposition of some of the links between integrability, matrix models and Airy structures. We first introduce the notion of an integrable hierarchy and a tau function. We then probe tau functions further by considering matrix integrals. These are typically a good source of $\mathcal{W}$-algebra representations that annihilate tau functions. We subsequently investigate such $\mathcal{W}$-constraints from the point of view of higher Airy structures introduced in [1]. These constraints depend on two parameters, r and s. In the second part of this thesis, we investigate further the r-KW and r-BGW tau functions using an external field matrix model. Through the works of various authors, the r-KW tau function is well understood from these different perspectives. The r-BGW tau function is far less well understood, however. We speculate on the form of the spectral curve for the r-BGW tau function, implying the corresponding Airy structure is given by $s = r - 1$. We subsequently present an explicit calculation of the Virasoro constraints for the 2-BGW tau function using Ward identities for an external field matrix model. We then match this with the (2,1) Airy structure. While this result is already known, the exact details have never been shown. We then generalise this calculation to the 3-BGW tau function by constructing the lowest order cubic mode of the $\mathcal{W}$-constraints for an external field model. While it is still unclear if this exactly matches with the lowest cubic mode from the (3,2) Airy structure, we believe that these two modes do indeed match, after some minor modifications.

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Chapter 1

Introduction

What do we mean when we say "special functions"? This term of course lacks any concrete definition and may mean something slightly different depending on whether one classifies oneself as a mathematician or a physicist. To a physicist, a special function will often arise from a particular physical problem. For example, the Hermite polynomials arise in the solution of a quantum harmonic oscillator; Legendre polynomials and Bessel functions arise as solutions to Laplace's equation in electromagnetism or fluid mechanics, and Airy functions, a generalisation of which will play a prominent role in this thesis, arise in optical problems. If one asked a mathematician to create a list of special functions, however, it would almost certainly additionally include the gamma function, theta functions, the Riemann zeta function, or more generally L-functions, and so on. What is remarkable about these additional functions appearing in the list of the mathematician, is the vast array of applications these functions have. In the cases we have mentioned alone, these applications stretch from number theory, to modular forms, to Riemann surfaces and, importantly for this thesis, to quantum field theory. With this abundant and rich cross-fertilisation, it is little surprise that the special functions of both mathematics and physics have been discovered to obey many different, but equivalent, relationships. More precisely, these functions often obey differential equations, recurrence relations, orthogonality conditions and also have integral representations. Moreover, a vector space of such special functions are often a representation

of some matrix Lie group that encodes the symmetries of these differential equations.

With these characteristics of special functions in mind, the tau functions of integrable systems and string theory play a similar role. They themselves define integrable differential equations. They obey orthogonality conditions, loosely in the form of Hirota equations. They often have integral representations as matrix models. They also encode symmetries of such integrable equations, by virtue of the fact that they are often annihilated by certain representations of a $\mathcal{W}$-algebra. We refer to such equations as $\mathcal{W}$-constraints. They are, furthermore, intricately and fundamentally linked to the theory of Riemann surfaces, both through the moduli space $\bar{\mathcal{M}}_{g,n}$ and through theta functions. Perhaps most incredibly, by a theorem of Kontsevich and Witten, a particular tau function also solves the physical problem of a theory of quantum gravity in two dimensions [2].

Taking this into consideration, the author aims for this thesis to serve two interconnected purposes. The first is to provide a pedagogical review of the subjects of integrable systems, matrix models, topological recursion and Airy structures. The material here is mostly standard and can be found in the vast array of literature on these subjects. However, it is entirely forgivable for one to feel overwhelmed when first learning these areas, not only due to the vastness of the literature, but also for the intricate ways in which they are connected. This is succinctly summarised in the diagram on the next page.

The graph in Figure (1.1) is by no means an exhaustive list of all relations between these areas. For example, if one takes a potential $V(X) = X^2 + N \log X$ in an external field matrix model, one obtains the standard Hermitian matrix model before double scaling [3]. There are also many other rich topics omitted from this diagram entirely, quantum curves being a prime example. See [4] for an introduction to this area.
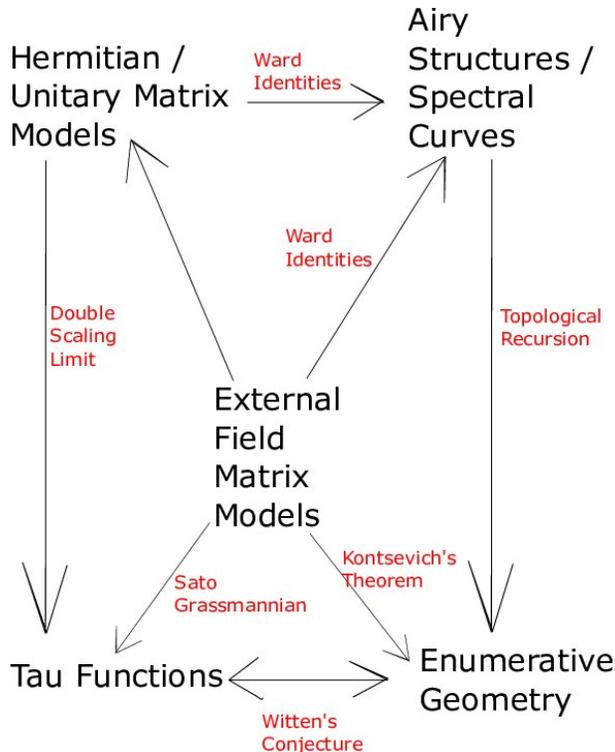
Figure 1.1: A simplified, pictorial representation of the five main areas covered in this thesis. The methods used to establish the links between the five areas are highlighted in red.

The second purpose of the thesis is an original contribution and concerns the equivalence between the r-KdV hierarchy and the $\mathcal{W}$-constraints alluded to above. As reviewed in [5], it is well known that the 2-KdV hierarchy with initial condition $u(t_1, 0, 0, \dots) = t_1$ is equivalent to a set of differential equations $L_k\tau = 0$ where $k \geq -1$, τ is the unique corresponding tau function and the operators L_k form a representation of a subalgebra of the Virasoro algebra. The corresponding tau function in this case is often called the Kontsevich-Witten (KW) tau function. There are other cases one can also consider: it is reasonable to examine a smaller subalgebra of modes, one that begins with $L_0\tau = 0$ instead. In this case, τ is still non-trivial and uniquely specified. This tau function is called the Brézin-Gross-Witten (BGW) tau function. The KW and BGW tau functions, and generalisations thereof, are the two main tau functions considered in this thesis.

It is therefore expected that, roughly speaking, the r-KdV hierarchy is equivalent to a tower of constraints with corresponding operators forming a representation of some $\mathcal{W}$-

algebra. That is, we wish to understand the loose relationship

$$r - \text{KdV} \iff \mathcal{W} - \text{constraints.} \tag{1.1}$$

To approach this problem, we exploit the relationships given in Figure (1.1). In particular, after having defined tau functions in chapter two, in chapter three we turn to Hermitian matrix integral representations of tau functions. In this integral representation, we readily find differential constraints that arise from Ward identities. These satisfy the Virasoro algebra and so are called discrete Virasoro constraints. This proves the forward direction in (1.1). For the converse, it is sufficient to prove that these differential constraints admit a unique solution. For this, however, it is instead better to find different representations of $\mathcal{W}$-algebras rather than considering discrete constraints. Subsequently, in chapter four we move to exploiting the relation between Airy structures and tau functions. It is the formalism of Airy structures that guarantees the existence of a unique solution to certain $\mathcal{W}$-constraints. It should be stressed that these are of a fundamentally different nature to the discrete Virasoro constraints.

In [1], several higher Airy structures were constructed that depend on two parameters, r and s with $r = \pm 1 \mod s$ and $s \in \{1, \ldots, r+1\}$. With $r = 2$, there are two admissible values of s, namely $s = 1$ and $s = 3$. The τ function corresponding to the $(r, s) = (2, 3)$ Airy structure is the KW tau function, while the $(2, 1)$ Airy structure corresponds to the BGW tau function. The generalisation to $r > 2$ for the KW tau function has been well studied in [6, 7] and the corresponding Airy structure is the one given by $s = r + 1$. The generalisation of the BGW case for $r > 2$ is less well understood. In view of the 2-BGW tau function, there are two natural candidates for the Airy structure. We expect the r-BGW tau function to correspond to the Airy structure given by either $s = 1$ or $s = r - 1$. In [1], it is stated that private communications with Di Yang and Chunhui Zhou confirm that the r-BGW tau function corresponds to the $s = 1$ Airy structure. This is not what we find in this thesis, however. Inspired by the work of Kontsevich and Witten that is reviewed in chapter five, we use the following family of Hermitian matrix models with external field to

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probe the r-BGW tau function:

$$\mathcal{F}(\Lambda; r, k) = \int_{H_N} dX e^{\mathrm{Tr}\left(-\frac{X^{r+1}}{r+1} + \Lambda X - k \log X\right)}. \tag{1.2}$$

Here, $k \in \mathbb{R}$, $\Lambda \in H_N$ is the $N \times N$ Hermitian external field matrix and $r \in \mathbb{Z}$ is such that $|r| \geq 2$. Following [8] and [9], if $\mathcal{C}$ is the quasiclassical contribution, that is the leading order behaviour of $\mathcal{F}(\Lambda; r, k)$ in the limit of large Λ, we show that

$$\tau = \frac{\mathcal{F}(\Lambda; r, k)}{\mathcal{C}} \tag{1.3}$$

is a tau function of the $|r|$-KdV hierarchy. In particular, we identify the tau function induced by $\mathcal{F}(\Lambda; -2, N)$ with the 2-BGW tau function.

As part of the original contribution, we use this external field model to speculate that the r-BGW tau function is associated to the spectral curve given by

$$x = \frac{z^r}{r}, \ y = -\frac{1}{z}, \tag{1.4}$$

which we call the r-Bessel curve. As shown in [1], this spectral curve corresponds, in a sense that can be made precise, with the $s = r - 1$ Airy structure. Furthermore, we explicitly calculate the differential constraints for the 2-BGW tau function in some detail and match this with the $(r, s) = (2, 1)$ Airy structure. While the result of this calculation is already known, it has not, as far as the author is aware, been shown in any explicit detail, especially from the external field model perspective. We then generalise this result to the 3-BGW tau function by calculating the lowest order cubic mode which has also never appeared in the literature. However, our calculation is inconclusive and we have thus far been unable to exactly match the differential constraints to the $(r, s) = (3, 2)$ Airy structure. There are many similarities, nevertheless, that are perhaps sufficient to speculate that this cubic mode indeed matches with the $(3, 2)$ Airy structure after some possible valid modifications. We leave the problem of exactly matching these modes to future work.

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Chapter 2

Integrable Systems

Historically, integrability has been a particularly classical subject. It concerns itself with the solvability of differential equations, both of ordinary differential equations (ODE's) and of partial differential equations (PDE's). Despite this, there was a resurgence of interest in integrable systems in the late 1960's with the discovery of soliton solutions to the Korteweg de-Vries equation via the inverse scattering transform. Since then, there has been a plethora of rich and deep applications found, quite notably in algebraic geometry and quantum gravity. Much of the current work on integrable systems comes through quantum integrability, a quantisation of the classical theory of integrable systems. Here, we will only be concerned with the classical theory of integrability. To begin, however, we must accept a potentially very alarming fact: there is no accepted rigorous definition of integrability in all cases. In the 'finite dimensional' case of ODE's, we do have a precise definition of an integrable equation. As we pass into the realm of the 'infinite dimensional' PDE theory, this precision is missing. Yet all is not lost. It is in fact reasonable to expect that there is no exact definition and it is precisely because of these subtleties that we are led to integrable hierarchies. These hierarchies are potentially infinite families of partial differential equations. It is in this context where tau functions naturally appear. These functions encode everything we could wish for in an integrable hierarchy: the differential equations themselves, the conserved quantities, explicit solutions and even the algebro-geometric content of the integrable system.

2.1 Hamiltonian Principles

Here we shall introduce the ideas of Hamiltonian equations which are ubiquitous throughout contemporary physics as well as essential for describing integrable sytems. We first draw from the ideas of classical mechanics as this is where many integrable systems originate. We introduce Lagrangian mechanics and in particular, the concept of an action. We then move to the Hamiltonian formalism and conserved quantities. This is much more suitable from the point of view of integrability. In fact, a unifying theme of modern physics is to consider symmetries of a system rather than trying to solve the equations of motion directly, which is often impossible. Finally, we formulate the abstract theory of integrable equations in terms of symplectic manifolds, removing the need for choices of coordinates and leaving only the intrinsic geometry. This process of axiomisation is in fact a unifying theme of modern mathematics. The aim is to remove all non essential specifics of a certain theory and only capture its fundamental properties, thus leaving a more powerful and generalised theory.

In this section we mainly follow [10–12].

2.1.1 Inspiration from Classical Mechanics

The most intuitive and recognisable starting point for classical mechanics is undoubtedly Newton's second law, stating that the force applied on a body is proportional to the rate of change of momentum. More succinctly,

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}. \tag{2.1}$$

In principle, we can write down Newton's second law for a dynamical system and solve the resulting equations of motion. There are drawbacks to this intuitive approach however. The resulting equations often become ugly for more complicated systems and for coordinate systems other than Cartesian coordinates. It is desirable to have a mathematical framework where we are able to choose a preferred coordinate system whilst retaining simple equations of motion.

This is where the concept of Lagrangian mechanics saves us. Moreover, it will serve as an indispensable tool as we head towards quantum field theory.

Suppose that we have a dynamical system with one degree of freedom. Equivalently, the system can be described by two coordinate functions $\left(q(t), \dot{q}(t)\right)$ that depend on a time variable t where the dot represents differentiation with respect to t. Here, q and $\dot{q}$ are interpreted as the position and velocity in some choice of coordinate system. We can also view $(q, \dot{q})$ as local coordinates of some two dimensional manifold M which we think of as possible solution states for the dynamical system, or 'phase space'.

The Lagrangian in classical mechanics is defined as $\mathcal{L}(q, \dot{q}) := K(q, \dot{q}) - V(q)$ where K and V are the kinetic and potential energies respectively. For simplicity we have assumed that the Lagrangian contains no explicit time dependence.

We wish to obtain the equations of motion from the Lagrangian. For this purpose, define the action functional by,

$$S[q] := \int_{t_0}^{t_1} \mathcal{L}(q, \dot{q}) dt. \tag{2.2}$$

To obtain the equations of motion, we use Hamilton's principle of stationary action on (2.2). The technique of performing variations of integrals to find equations of motion shall be deployed throughout this thesis. Therefore, it will be also be shown here for illustrative purposes. In this process, we also recover the functional derivative which we will employ later.

Let ϵ be arbitrarily small and let $\eta : \mathbb{R} \to \mathbb{R}$ be a function independent of ϵ with $\eta(t_0) = \eta(t_1) = 0$. In other words, we consider variations of q that keep the endpoints fixed. We can make small variations in q so that,

$$S(\epsilon) := S[q + \epsilon\eta(t)] = \int_{t_0}^{t_1} \mathcal{L}(q + \epsilon\eta, \dot{q} + \epsilon\eta') dt. \tag{2.3}$$

Consider the total variation,

$$\Delta S := S(\epsilon) - S(0), \tag{2.4}$$

which reads,

$$\Delta S = \int_{t_0}^{t_1} \mathcal{L}(q + \epsilon\eta, \dot{q} + \epsilon\dot{\eta}) - \mathcal{L}(q, \dot{q})dt. \tag{2.5}$$

We define the functional differential δS as the expansion of ΔS to first order in ϵ,

$$\Delta S = \delta S + O(\epsilon^2). \tag{2.6}$$

In other words, we have,

$$\delta S := \frac{dS}{d\epsilon}\Big|_{\epsilon=0} \cdot \epsilon = \epsilon \int_{t_0}^{t_1} \frac{\partial\mathcal{L}}{\partial q}\eta(t) + \frac{\partial\mathcal{L}}{\partial\dot{q}}\dot{\eta}(t)dt. \tag{2.7}$$

obtained by expanding equation (2.5) to first order. To obtain the dynamical trajectory of a system, we use Hamilton's principle which states,

$$\delta S = 0. \tag{2.8}$$

Equation (2.8) is a very concise form of the classical equations of motion. To be of more use in practical examples, we inspect equation (2.7) again and use integration by parts on the second term in the integral. Observe that the boundary terms vanish since we kept the endpoints fixed. Thus we have,

$$0 = \delta S = \int_{t_0}^{t_1} \left(\frac{\partial\mathcal{L}}{\partial q} - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}}\right)\eta(t)dt. \tag{2.9}$$

for all prescribed variations $\eta(t)$. Since this holds for all admissible $\eta(t)$ we must have that the integrand is zero. In the literature the precise form of this statement is called the fundamental lemma of calculus of variations. Consequently we obtain the Euler-Lagrange equations of motion.

$$\frac{\partial\mathcal{L}}{\partial q} - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}} = 0. \tag{2.10}$$

Note that the Euler-Lagrange equations are completely equivalent to Newton's second law.

Furthermore, we can define the functional derivative, denoted by $\frac{\delta S}{\delta q}$, through,

$$\delta S = \int_{t_0}^{t_1} \frac{\delta S}{\delta q} \eta(t) dt. \tag{2.11}$$

Observe the similarities between the above equation the equation for the differential of a function,

$$dF = \sum_{i=1}^{n} \frac{\partial F}{\partial x_i} dx_i. \tag{2.12}$$

Formally, the sum over the discrete index i in the differential is replaced by an integral over the continuous index t in the functional differential. This idea will become important when we look at the integrability of partial differential equations.

Upon comparing equation (2.11) and equation (2.9) we see that we have a closed expression for the functional derivative in this case,

$$\frac{\delta S}{\delta q} = \frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}}. \tag{2.13}$$

For completeness, we observe that in this case $\mathcal{L}$ only depended on q and $\dot{q}$. We can actually repeat the above exercise for any functional that in general depends not only on q and $\dot{q}$ but for higher derivatives as well. Indeed, suppose that $q^{(n)}$ is the highest derivative to appear in $\mathcal{L}(q, \dot{q}, \ldots, q^{(n)})$. Performing variations similar to the above, we see the general functional derivative is given by,

$$\frac{\delta S}{\delta q} = \sum_{i=0}^{n} (-1)^i \frac{d^i}{dt^i} \frac{\partial \mathcal{L}}{\partial q^{(i)}}, \tag{2.14}$$

which does indeed reduce to equation (2.13) in the case $n = 1$. Note that the alternating sign in equation (2.14) is due to the repeated application of the integration by parts formula.

2.1.2   From Lagrangian to Hamiltonian Formalism

The Lagrangian formalism, though well suited for physical applications, is not particularly convenient for defining integrability. Hence, we shall discuss the Hamiltonian approach which is perhaps more fundamental in this situation.

In the previous section, we considered a system that had one degree of freedom meaning that there were two local coordinates, q and $\dot{q}$. In greater generality suppose we have a dynamical system with n degrees of freedom. This means we have a $2n$ dimensional manifold M with local coordinates $(q, \dot{q}) = (q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n)$. As discussed previously, the dynamical trajectory will be encoded in the Lagrangian $\mathcal{L}(q, \dot{q})$. We define the canonical momenta as

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}. \tag{2.15}$$

It is now more convenient to use the position and momenta (q, p) as local coordinates for the manifold M. In this approach, p and q need not necessarily be interpreted as positions and momenta and instead considered just as geometric data.

The dynamical variables are functions $f : M \to \mathbb{R}$. For simplicity, we assume that $f = f\big(p(t), q(t)\big)$. That is to say, we assume no explicit time dependence in f. For two functions $f : M \to \mathbb{R}$ and $g : M \to \mathbb{R}$, we define the *Poisson bracket* as

$$\{f, g\} := \sum_{i=1}^{n} \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i}. \tag{2.16}$$

Note that the Poisson bracket as defined above is bilinear, anti-commutative,

$$\{f, g\} = -\{g, f\}, \tag{2.17}$$

and satisfies the Jacobi identity

$$\{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\} = 0 \tag{2.18}$$

Furthermore, observe that the coordinate functions satisfy,

$$\{q_i, q_j\} = \{p_i, q_j\} = 0 \text{ and } \{p_i, q_j\} = \delta_{ij}, \tag{2.19}$$

where δ_{ij} is the Kronecker delta. It should be noted here that the above equation is the starting point of canonical quantisation at its most elementary level. In this regime, q_j is lifted to the multiplicative operator q_j, p_j is lifted to $-i\hbar\frac{\partial}{\partial x_j}$ and the Poisson bracket is lifted to the commutator.

We then define the *Hamiltonian* as the Legendre transform of $\mathcal{L}$,

$$\mathcal{H}(q, p) = \sum_{i=1}^{n} p_i \dot{q}_i - \mathcal{L}(q, \dot{q}). \tag{2.20}$$

Whilst the Lagrangian is interpreted as the difference between kinetic and potential energies, the Hamiltonian in classical mechanics, after some calculations, can be seen to be the total energy, or the sum of the kinetic and potential energies.

Given that a Hamiltonian function $\mathcal{H}$ exists for the dynamical system, the time evolution of dynamical variable f is determined by,

$$\dot{f} = \{\mathcal{H}, f\}, \tag{2.21}$$

and so in some sense, the Hamiltonian that represents the total energy 'generates' the time evolution.

If we apply equation (2.20) to the coordinate functions q_i and p_i, in other words set $f = q_i$ and $f = p_i$, we obtain Hamilton's equations of motion,

$$\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i} \text{ and } \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i}. \tag{2.22}$$

The ordinary differential equation that the Hamiltonian describes can then be recovered from Hamilton's equations. This system of differential equations for $q(t)$ and $p(t)$ describes physical trajectories of the dynamical system and are completely equivalent to the Euler-

Lagrange equations (2.10).

Equation (2.21) now allows us to define a first integral.

Definition 2.1.1. A function $f : M \to \mathbb{R}$ which satisfies $\dot{f} = 0$ while equations (2.22) hold is called a *first integral* or alternatively a *constant of motion*. Equivalently, f is a constant of motion if $f\big(q(t), p(t)\big) = C$ where C is a constant when $\big(q(t), p(t)\big)$ are solutions of Hamilton's equations.

Remark. By the very definition of a manifold, the choice of coordinates on M in this definition is not unique, with the relations between them given by transition functions. In what follows we restrict attention to canonical transformations. That is to say, those transformations which preserve the Poisson brackets and hence preserve Hamilton's equations.

Observe that the Hamiltonian $\mathcal{H}$ itself is a first integral since,

$$\dot{\mathcal{H}} = \{\mathcal{H}, \mathcal{H}\} = 0. \tag{2.23}$$

Under the interpretation of the Hamiltonian representing the total energy, equation (2.23) corresponds to the conservation of energy.

Heurstically, Hamilton's equations (2.22) will be exactly solvable if there are 'sufficiently many' constants of motion as each one will, roughly speaking, eliminate an equation in (2.22).

We now make this idea precise with the definition of an integrable system.

Definition 2.1.2. An *integrable system* $(M, f_1, \ldots, f_n)$ is the data of a $2n$ dimensional manifold M together with the existence of n independent functions $f_1, \ldots f_n : M \to \mathbb{R}$ such that $\{f_i, f_j\} = 0$ for all $i, j = 1, \ldots n$.

Remark. Here, independent functions f_i means that the tangent vectors ∇f_i for $i = 1, \ldots n$ are independent in the tangent space at any point in M. This is necessary as if we had a first integral f_1, then without this assumption we could define another first integral as $f_2 = c \cdot f_1$ for any $c \in \mathbb{R}$ and we would have infinitely many first integrals.

If a Hamiltonian function $\mathcal{H}$ exists for a dynamical system, we usually take $f_1 = \mathcal{H}$.

The following theorem due to Arnold and Liouville explains concretely why integrable systems are special in the sense that they can be exactly solved.

Theorem 2.1.3. Let $(M, f_1, \ldots, f_n)$ be an integrable system with Hamiltonian $\mathcal{H} = f_1$. Let

$$M_f = \{(p, q) \in M : f_k(p, q) = c_k \text{ for some } c_k \in \mathbb{R}, \ k = 1, \ldots n\}$$

and assume that this is a connected and compact submanifold of M. Then M_f is diffeomorphic to an n dimensional torus and in a neighbourhood of M_f in M there exists canonical transformations $I_k = I_k(p, q)$ and $\theta_k = \theta_k(p, q)$ that preserve Poisson brackets such that $0 \le \theta_k \le 2\pi$ are local coordinates on M_f. Furthermore Hamilton's equations become $\dot{I}_k = 0$ and $\dot{\theta}_k = \omega_k(I_1, \ldots, I_n)$ for $k = 1, \ldots n$.

Remark. The form of Hamilton's equations in the above theorem means that the system becomes trivial to solve: each I_k is a constant meaning that each ω_k is constant.

The appearance of these so-called *Liouville tori*, M_f, in fact has much to do with the Jacobians of certain Riemann surfaces. We shall briefly return to this point in section 2.2

2.1.3 Coordinate Invariant Definitions - Symplectic Geometry

The formalism introduced thus far has required the choice of local coordinates for the positions and momenta. To properly define an integrable system in greater generality, we will introduce coordinate invariant definitions using the language of differential geometry. Throughout, we let $x \in M$, and let ξ and ω be sections of TM and T^*M respectively. In other words, ξ is a vector field and ω is a differential form. We will also repeatedly use the fact that to any given vector field ξ, there exists a unique integral curve $x(t)$. That is,

$$\dot{x}(t) = \xi(x(t)). \tag{2.24}$$

We also always assume that the manifold M is smooth.

Recall that the *interior product* of a tangent vector $\xi \in T_x M$ and an n form ω is an $n-1$ form $\iota_\xi \omega$ defined by,

$$(\iota_\xi \omega)(\xi_1, \ldots, \xi_{n-1}) = \omega(\xi, \xi_1, \ldots, \xi_{n-1}). \tag{2.25}$$

The *Lie derivative* of ω along ξ is given by

$$L_\xi \omega = d\iota_\xi \omega + \iota_\xi d\omega. \tag{2.26}$$

Definition 2.1.4. Let M be a manifold. A two form ω is called a *symplectic form* if it is non-degenerate and closed. The tuple (M, ω) is called a *symplectic manifold.*

Remark. For convenience, we often just refer to M as a symplectic manifold without explicit reference to the symplectic form ω.

Lemma 2.1.5. If there exists a symplectic form ω on the manifold M, then M is necessarily even dimensional.

Proof. Suppose that $\dim M = n$ where n is odd. Consider the two form ω in local coordinates $(x_1, \ldots x_n)$ as,

$$\omega = \sum_{1 \leq i,j \leq n} \omega_{ij} dx^i \wedge dx^j. \tag{2.27}$$

We can then form the antisymmetric matrix Ω where $(\Omega)_{ij} = \omega_{ij}$. However, since Ω is antisymmetric we have,

$$\det \Omega = \det(-\Omega^{\mathrm{T}}) = (-1)^n \det \Omega = -\det \Omega. \tag{2.28}$$

Thus $\det \Omega = 0$, but this is impossible since ω was assumed non degenerate. $\qquad \square$

The fact that symplectic manifolds are necessarily even dimensional is promising in light of the previous section where we chose M to be even dimensional. To probe the connection between geometry and mechanics further, we make the following definition.

Definition 2.1.6. Let ξ be a vector field on a symplectic manifold M with symplectic form ω. Then ξ is called a *Hamiltonian vector field* if $L_\xi \omega = 0$.

We now construct a correspondence between functions on the symplectic manifold M and Hamiltonian vector fields.

Lemma 2.1.7. There exists a well-defined map from the set of smooth functions on M to Hamiltonian vector fields on M.

Remark. This map is actually a morphism of Lie algebras but we shall not dwell on this matter as we have not yet defined the Lie algebra structure on the space of functions on M.

Proof. Suppose we have a function $\mathcal{H}$. Then we can consider the differential $-d\mathcal{H}$. The minus sign is non essential here but is introduced to agree with the convention in the literature. Since ω is a non degenerate form, this induces an isomorphism on the fibres of TM and T^*M. Therefore, $-d\mathcal{H}$ uniquely determines a vector field $\xi_{\mathcal{H}}$ through,

$$-d\mathcal{H} = \iota_{\xi_{\mathcal{H}}}\omega. \tag{2.29}$$

The vector field $\xi_{\mathcal{H}}$ is Hamiltonian since

$$L_{\xi_{\mathcal{H}}}\omega = d(\iota_{\xi_{\mathcal{H}}}\omega) = -d(d\mathcal{H}) = 0. \tag{2.30}$$

This completes the proof. $\square$

Conversely, suppose ξ is a Hamiltonian vector field. From the definition of the Lie derivative and that $d\omega = 0$ we obtain

$$0 = L_{\xi}\omega = d(\iota_{\xi}\omega) + \iota_{\xi}d\omega = d(\iota_{\xi}\omega). \tag{2.31}$$

Hence, assuming M is a contractible manifold, there exists a function $\mathcal{H} : M \to \mathbb{R}$ such that equation (2.29) holds using Poincaré's lemma. Note that the choice of $\mathcal{H}$ is not unique. This is the obstruction to a bijective correspondence.

The function $\mathcal{H}$ is called a *Hamiltonian* for the system.

Note that the value of $\mathcal{H}$ is constant along the integral curves of $\xi_{\mathcal{H}}$. Indeed, using equation 2.29 and the antisymmetry of ω, we find

$$d\mathcal{H}(\xi_{\mathcal{H}}) = -(\iota_{\xi_{\mathcal{H}}}\omega)(\xi_{\mathcal{H}}) = -\omega(\xi_{\mathcal{H}}, \xi_{\mathcal{H}}) = 0. \tag{2.32}$$

Since $d\mathcal{H} = 0$ along $\xi_{\mathcal{H}}$ we conclude that $\mathcal{H}$ is constant along its integral curves. This corresponds to the conservation of energy in equation (2.23).

In this new formulation of the Hamiltonian formalism of the previous section, we can also recover Hamilton's equations.

Proposition 2.1.8. Let M be a symplectic manifold with symplectic form given locally by $\omega = \sum_{i=1}^{n} dp^i \wedge dq^i$. Let $\mathcal{H}$ be a Hamiltonian of the dynamical system. Then the integral curves of $\xi_{\mathcal{H}}$ are given by,

$$\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i} \text{ and } \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i}.$$

Proof. The integral curves of $\xi_{\mathcal{H}}$ are determined by,

$$\dot{x}(t) = \xi_{\mathcal{H}}(x(t)). \tag{2.33}$$

If we write x in local coordinates $x = (q_1, \ldots, q_n, p_1, \ldots, p_n)$, then the left hand side of equation (2.33) becomes,

$$\dot{x}(t) = \sum_{i=1}^{n} \dot{q}_i \frac{\partial}{\partial q_i} + \dot{p}_i \frac{\partial}{\partial p_i}. \tag{2.34}$$

To evaluate the right hand side of equation (2.33), we use equation (2.29).

We first expand $\xi_{\mathcal{H}}$ in a basis,

$$\xi_{\mathcal{H}} = \sum_{i=1}^{n} f_i \frac{\partial}{\partial q_i} + g_i \frac{\partial}{\partial p_i}, \tag{2.35}$$

for some coefficients f_i and g_i that we wish to calculate. Observe that,

$$\iota_{\xi_{\mathcal{H}}}\omega = \sum_{1 \leq i,j \leq n} f_i(\iota_{\frac{\partial}{\partial q_i}} dp^j \wedge dq^j) + \sum_{1 \leq i,j \leq n} g_i(\iota_{\frac{\partial}{\partial p_i}} dp^j \wedge dq^j). \tag{2.36}$$

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By definition of the dual basis we have that

$$\iota_{\frac{\partial}{\partial p_i}} dp^j = \iota_{\frac{\partial}{\partial q_i}} dq^j = \delta_{ij}, \tag{2.37}$$

and

$$\iota_{\frac{\partial}{\partial p_i}} dq^j = \iota_{\frac{\partial}{\partial q_i}} dp^j = 0. \tag{2.38}$$

Hence we have that,

$$\iota_{\xi_{\mathcal{H}}} \omega = \sum_{i=1}^{n} -f_i dp^i + g_i dq^i. \tag{2.39}$$

Furthermore, we have that,

$$d\mathcal{H} = \sum_{i=1}^{n} \frac{\partial \mathcal{H}}{\partial p_i} dp^i + \frac{\partial \mathcal{H}}{\partial q_i} dq^i. \tag{2.40}$$

Using equation (2.29) we obtain,

$$f_i = \frac{\partial \mathcal{H}}{\partial p_i} \text{ and } g_i = -\frac{\partial \mathcal{H}}{\partial q_i}. \tag{2.41}$$

Now upon comparing equations (2.35) and (2.34) we find the desired Hamilton equations.

□

From the above proposition, for the specific choice of symplectic form $\omega = \sum_{i=1}^{n} dp^i \wedge dq^i$, the Hamiltonian vector field can alternatively be written in matrix form,

$$\xi_{\mathcal{H}} = \Omega \cdot dH = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial \mathcal{H}}{\partial p_i} \\ \frac{\partial \mathcal{H}}{\partial q_i} \end{bmatrix}. \tag{2.42}$$

This choice of ω is certainly not unique. One can indeed consider mappings between symplectic manifolds which need not preserve the symplectic form. Those mappings that do preserve the symplectic form are called symplectomorphisms and these correspond to canonical transformations in the Arnold-Liouville theorem. We will often say that the matrix Ω determines the choice of Poisson structure. Moreover, using the symplectic form we can

also define Poisson brackets in a coordinate invariant way.

Definition 2.1.9. Let f and g be functions on a symplectic manifold M. We define the Poisson bracket by

$$\{f, g\} := \xi_f g, \tag{2.43}$$

where ξ_f and ξ_g are the Hamiltonian vector fields associated with f and g respectively.

Observe that,

$$\xi_f g = dg(\xi_f) = -(\iota_{\xi_g}\omega)(\xi_f) = -\omega(\xi_g, \xi_f) = \omega(\xi_f, \xi_g). \tag{2.44}$$

To be consistent with the previous section, one should check that this does indeed satisfy all the properties of bilinearity, anti-commutativity and the Jacobi identity. Bilinearity and anticommutativity come for free by virtue of the form ω being bilinear and anticommutative. That the Poisson bracket satisfies the Jacobi identity is a consequence of ω being closed. In other words, $d\omega(\xi_f, \xi_g, \xi_h) = 0$ and after some tedious algebra, one recovers the Jacobi identity.

Given a Hamiltonian $\mathcal{H}$, the dynamics of any function $f : M \to \mathbb{R}$ is then governed by,

$$\dot{f} = \xi_{\mathcal{H}} f = \{\mathcal{H}, f\}, \tag{2.45}$$

in exact agreement with the previous section. Again, we say that f is a first integral if $\{\mathcal{H}, f\} = 0$.

Lemma 2.1.10. Given a $2n$ dimensional symplectic manifold M, the maximal number of independent Poisson commuting functions is n.

Proof. Suppose otherwise for a contradiction. Then there exist independent Poisson commuting functions $\mathcal{H}_1, \ldots, \mathcal{H}_n, \mathcal{H}_{n+1}$. Fix a point $x \in M$. Consider the $n + 1$ dimensional vector space,

$$V := \text{span}\{\xi_{\mathcal{H}_1}(x), \ldots, \xi_{\mathcal{H}_n}(x), \xi_{\mathcal{H}_{n+1}}(x)\} \subseteq T_x M. \tag{2.46}$$

Consider also the symplectic orthogonal complement defined by,

$$V^\perp := \{\xi \in T_x M : \omega(\xi, v) = 0 \,\forall v \in V\}. \tag{2.47}$$

This must be $n - 1$ dimensional since $\dim V^\perp = \dim T_x M - \dim V = n - 1$. However, since $\{\xi_{\mathcal{H}_i}, \xi_{\mathcal{H}_i}\} = 0$ Poisson commute for all i and j, we have that $\xi_{\mathcal{H}_i} \in V^\perp$ for each i. Hence $V \subseteq V^\perp$ which is a contradiction. $\qquad \square$

Finally, having constructed coordinate invariant Poisson brackets, we define an integrable system.

Definition 2.1.11. An integrable system is a symplectic manifold M of dimension $2n$ together with functions $\mathcal{H}_1, \ldots, \mathcal{H}_n$ such that $d\mathcal{H}_1 \wedge \cdots \wedge d\mathcal{H}_n \neq 0$ and $\{\mathcal{H}_i, \mathcal{H}_j\} = 0$ for all $i, j = 1, \ldots n$.

If the condition $\{\mathcal{H}_i, \mathcal{H}_j\} = 0$ is satisfied, we say that $\mathcal{H}_i$ and $\mathcal{H}_j$ are in *involution*.

2.2 Integrability in PDE's

The prototypical example of an integrable non-linear PDE comes from the study of weakly dispersive shallow water waves in 1+1 dimensions described by the celebrated Korteweg-de Vries (KdV) equation,

$$u_t = 6uu_x - u_{xxx}, \tag{2.48}$$

where $u = u(x, t)$. There is, moreover, a 2+1 dimensional generalisation of the KdV equation that is crucial in our discussion. This equation reads,

$$3u_{yy} = (4u_t - 6uu_x - u_{xxx})_x, \tag{2.49}$$

and is known as the Kadomtsev-Petviashvili (KP) equation. We remark here that if u has no y dependence in the KP equation, then we obtain that ,

$$4u_t - 6uu_x - u_{xxx} = C \qquad (2.50)$$

for some constant C. Choosing boundary conditions so that u and its derivatives vanish at infinity, we find $C = 0$. This now brings us to an important point. It is possible to rescale x,t and u in the KdV equation so that any of the three coefficients are arbitrary non zero constants. For example, if we rescale (2.50) so that $x \mapsto -x$, $t \mapsto 4t$ and $u \mapsto -u$ we obtain (2.48). The exact numerical value of the coefficients, while important for physical applications as a wave equation, will play no role in this thesis.

There are, of course, much simpler linear PDE's or quasi-linear PDE's of first order, but these can often be exactly solved using elementary methods and so we will not concern ourselves with these cases. To see why the KdV equation deserves to be called integrable, we first heuristically develop the general theory of integrable PDE's before applying it to the specific case of the KdV equation. As such, the first two subsections contain few rigorous definitions or proofs. Its purpose is to introduce a small portion of the vast array of tools in integrability theory. In particular, we focus on Lax equations, symmetries, integrable hierarchies, tau functions, Baker-Akhiezer functions and Hirota bilinear equations. There are many important features of integrable equations that are not discussed in any detail, such as inverse scattering, solitons, R-matrices, monodromies, theta functions and Fay identities to name but a few. See [13] for an in depth discussion on these topics. After having introduced integrable hierarchies and tau functions from a pedagogical perspective, we redefine these terms using the formalism of pseudo-differential operators. This is more convenient for the purposes of this thesis. Here we mainly follow [11–14].

2.2.1 Subtleties in PDE's

What follows in this subsection is a formal description of the foundations of integrability in PDE theory. We rest reassured that it can be given a rigorous functional analytic treatment. See, for example, [13].

The formalism of Poisson structures now carries over to the story of PDE's. Formally, as specified in [11] and [14], we can construct the following dictionary between ODE's and PDE's,

$$\text{ODE's} \longmapsto \text{PDE's}, \tag{2.51}$$

$$x^i(t), \, i = 1, \ldots 2n \longmapsto u(x,t), \, x \in \mathbb{R}, \tag{2.52}$$

$$\sum_{i=1}^{2n} \longmapsto \int_{\mathbb{R}} dx, \tag{2.53}$$

$$\text{Functions } f(x) \longmapsto \text{Functionals } F[u], \tag{2.54}$$

$$\frac{\partial}{\partial x^i} \longmapsto \frac{\delta}{\delta u}. \tag{2.55}$$

We recall that given a functional

$$F[u] = \int_{\mathbb{R}} f(u, u_x, u_{xx}, \ldots, u^{(n)}), \tag{2.56}$$

where $u = u(x,t)$, the functional derivative is given by

$$\frac{\delta F}{\delta u} = \sum_{i=0}^{n} (-1)^i \frac{\partial^i}{\partial x^i} \frac{\partial f}{\partial u^{(i)}}, \tag{2.57}$$

as stated in section 2.1.1. Note that the following observation will be useful. Consider the functional,

$$F[u] := \int_{\mathbb{R}} u(x)\delta(x-y)dx = u(y), \tag{2.58}$$

where δ is the Dirac delta. Then using the formula for the functional derivative we find

$$\frac{\delta u(y)}{\delta u(x)} = \frac{\delta F}{\delta u(x)} = \frac{\partial}{\partial u(x)}\Big(u(x)\delta(x-y)\Big) = \delta(x-y), \tag{2.59}$$

since the integrand of (2.58) does not depend on the derivatives of u.

In light of Definition 1.1.11, for an ODE to be integrable, there must be sufficiently many first integrals, the number of which is specified by exactly the dimension of the symplectic manifold M. However, in the PDE case, the coordinate functions $x^i(t)$ are replaced by a dynamical variable $u(x,t)$. From this point of view, the discrete index i is replaced by the continuous independent variable x. Moreover, the finite dimensional symplectic manifold M is replaced by an *infinite* dimensional space of smooth functions with some suitable boundary conditions. Herein lies the fundamental issue of trying to define what an integrable PDE is: it is clear that there should be an infinite number of first integrals but it is far less clear what the general structure of these first integrals should be.

In analogy with the finite dimensional case in equation (2.16), we define the Poisson bracket of two functionals $F[u]$ and $G[u]$ as,

$$\{F, G\} = \int_{\mathbb{R}^2} \omega(x, y, u) \frac{\delta G}{\delta u(x)} \frac{\delta F}{\delta u(y)} dx dy, \tag{2.60}$$

where $\omega(x, y, u)$ is such that $\{\cdot, \cdot\}$ is antisymmetric and satisfies the Jacobi identity. A canonical choice is,

$$\omega(x, y, u) = \frac{1}{2} \frac{\partial}{\partial x} \delta(x-y) - \frac{1}{2} \frac{\partial}{\partial y} \delta(x-y). \tag{2.61}$$

Here the derivatives of the Dirac delta are distributional derivatives defined in this case by,

$$\int_{\mathbb{R}} f(x) \frac{\partial}{\partial x} \delta(x-a) dx = - \int_{\mathbb{R}} f'(x) \delta(x-a) dx = -f'(a), \tag{2.62}$$

where $a \in \mathbb{R}$ and f is a smooth function that decays to 0 'sufficiently fast' as $x \to \pm\infty$. Here, if a function $f \in C^\infty(\mathbb{R}^n)$ decays to 0 'sufficiently fast', this means that there exists constants $C^{\alpha\beta}$ with $\alpha, \beta \in \mathbb{N}^n$ such that $\sup_{x \in \mathbb{R}^n} |x^\alpha \partial_\beta f| \leq C^{\alpha\beta}$. That is to say, f is a

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Schwartz function.

With this choice of ω, the Poisson bracket (2.60) becomes

$$\{F, G\} = \frac{1}{2} \int_{\mathbb{R}^2} \frac{\delta G}{\delta u(x)} \frac{\delta F}{\delta u(y)} \frac{\partial}{\partial x} \delta(x - y) dx dy - \frac{1}{2} \int_{\mathbb{R}^2} \frac{\delta G}{\delta u(x)} \frac{\delta F}{\delta u(y)} \frac{\partial}{\partial y} \delta(x - y) dy dx \quad (2.63)$$

$$= -\frac{1}{2} \int_{\mathbb{R}^2} \delta(x - y) \frac{\delta F}{\delta u(y)} \frac{\partial}{\partial x} \frac{\delta G}{\delta u(x)} dx dy + \frac{1}{2} \int_{\mathbb{R}^2} \delta(x - y) \frac{\delta G}{\delta u(x)} \frac{\partial}{\partial y} \frac{\delta F}{\delta u(y)} dx dy \quad (2.64)$$

$$= -\frac{1}{2} \int_{\mathbb{R}} \frac{\delta F}{\delta u(x)} \frac{\partial}{\partial x} \frac{\delta G}{\delta u(x)} dx + \frac{1}{2} \int_{\mathbb{R}} \frac{\delta G}{\delta u(x)} \frac{\partial}{\partial x} \frac{\delta F}{\delta u(x)} dx \quad (2.65)$$

$$= \frac{1}{2} \int_{\mathbb{R}} \frac{\delta G}{\delta u(x)} \frac{\partial}{\partial x} \frac{\delta F}{\delta u(x)} dx + \frac{1}{2} \int_{\mathbb{R}} \frac{\delta G}{\delta u(x)} \frac{\partial}{\partial x} \frac{\delta F}{\delta u(x)} dx \quad (2.66)$$

$$= \int_{\mathbb{R}} \frac{\delta G}{\delta u(x)} \frac{\partial}{\partial x} \frac{\delta F}{\delta u(x)} dx, \quad (2.67)$$

where we have used integration by parts and the definition of the distributional derivative of the Dirac delta. Hence we have the result that,

$$\{F, G\} = \int_{\mathbb{R}} dx \frac{\delta G}{\delta u(x)} \frac{\partial}{\partial x} \frac{\delta F}{\delta u(x)}. \quad (2.68)$$

Suppose that we have a Hamiltonian functional $\mathcal{H}[u]$ such that $\frac{\partial \mathcal{H}}{\partial t} = 0$ and that $\mathcal{H}[u]$ is physically interpreted as the total energy of a system. We now *define* Hamilton's equations, in analogy with equation (2.45) as,

$$\frac{\partial}{\partial t} u(x, t) := \{\mathcal{H}, u(x, t)\} = \int_{\mathbb{R}} dy \frac{\delta u(x)}{\delta u(y)} \frac{\partial}{\partial y} \frac{\delta \mathcal{H}}{\delta u(y)} = \frac{\partial}{\partial x} \frac{\delta \mathcal{H}}{\delta u(x)}, \quad (2.69)$$

where we used equation (2.59).

As an example, the total energy of weakly non-linear shallow water waves is given by the Hamiltonian energy functional,

$$\mathcal{H}[u] = \int_{\mathbb{R}} dx \left(\frac{1}{2}(u_x)^2 + u^3 \right). \quad (2.70)$$

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Calculating the functional derivative, one finds

$$\frac{\delta \mathcal{H}}{\delta u} = 3u^2 - u_{xx}, \tag{2.71}$$

and so employing equation (2.69) we have

$$u_t = 6uu_x - u_{xxx}, \tag{2.72}$$

which is exactly the KdV equation.

2.2.2 A Case Study in the Features of Integrable PDE's - The KdV Equation

We have seen that to ask about the integrability of a PDE is already a non-trivial question. Although we cannot rigorously define an integrable PDE, there are multiple defining properties of integrable PDE's. In [15], Hitchin broadly summarises these properties into three interconnecting characteristics:

1. The ability to write down explicit solutions;

2. The existence of enough symmetries;

3. The presence of algebraic geometry.

We will investigate the KdV equation as a specific example in order to illustrate each one of these in turn.

For the first characteristic, the ability to write explicit solutions, one of the most remarkable methods of solving the KdV equation came from Gardner, Greene, Kruskal and Miura [16] and is known as the inverse scattering transform. The details of this method are not important for us, but we shall mention that the crux of inverse scattering is the ability to write down a *Lax pair* for the KdV equation. A Lax pair for a differential equation is a pair of differential operators, L and A such that the differential equation can be written in

the form

$$\frac{\partial L}{\partial t} = [A, L],\tag{2.73}$$

where $[A, L] = AL - LA$ is the commutator of operators. As an example, consider $L = -\partial_x^2 + u$ and $A = -4\partial_x^3 + 6u\partial_x + 3u_x$. Then equation (2.73) is exactly the KdV equation. Indeed, since u commutes with u_x and ∂_x^2 commutes with ∂_x^3 we have

$$[A, L] = [-4\partial_x^3 + 6u\partial_x + 3u_x, -\partial_x^2 + u,] = 6[\partial_x^2, u\partial_x] + 3[\partial_x^2, u_x] + 4[u, \partial_x^3] - 6[u, u\partial_x].\tag{2.74}$$

Evaluating each of these commutators we find

$$[\partial_x^2, u\partial_x] = u_{xx}\partial_x + 2u_x\partial_x^2,\tag{2.75}$$

$$[\partial_x^2, u_x] = u_{xxx} + 2u_{xx}\partial_x,\tag{2.76}$$

$$[u, \partial_x^3] = -3u_x\partial_x^2 - 3u_{xx}\partial_x - u_{xxx},\tag{2.77}$$

$$[u, u\partial_x] = -uu_x.\tag{2.78}$$

Hence, after substituting these commutators back into (2.74) we obtain

$$[A, L] = 6uu_x - u_{xxx}.\tag{2.79}$$

This identity is extremely special as it means the commutator of the two operators L and A is simply a multiplicative operator. Finally, notice that the time dependence in L is introduced through $u(x, t)$ and so $\partial_t L = u_t$ which is again a multiplicative operator. Thus equation (2.73) in this case is indeed the KdV equation.

We can also take a contrasting view of the Lax equation. We assume that there exists a non-zero eigenvalue, λ, of L that does not depend on t, with the corresponding square integrable eigenfunction $\psi \in L^2(\mathbb{R})$. In other words we have,

$$L\psi = \lambda\psi,\tag{2.80}$$

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for the operator L defined as above in the KdV case. We can then ask about the role A plays in the Lax equation for ψ an eigenfunction of L. Differentiating equation (2.80) yields

$$L_t\psi + L\psi_t = \lambda\psi_t. \tag{2.81}$$

Employing the Lax equation we find,

$$AL\psi - LA\psi + L\psi_t = \lambda\psi_t, \tag{2.82}$$

and using the eigenvalue equation we have

$$\lambda A\psi - LA\psi + L\psi_t = \lambda\psi_t. \tag{2.83}$$

We notice here that this equation is satisfied if ψ obeys the Schrödinger equation,

$$A\psi = \frac{\partial\psi}{\partial t}. \tag{2.84}$$

where A plays the role of the Hamiltonian. We will revisit this idea in the next subsection. In this context, the eigenfunction ψ corresponding to an eigenvalue of L is called a *Baker-Akhiezer function*.

We move onto the second characteristic defining integrable equations, namely the appearance of first integrals. Conserved quantities for PDE's take the form of functionals and so are given by,

$$\mathcal{H}_n[u] = \frac{1}{2}\int_{\mathbb{R}} h_n(u, u_x, u_{xx}, \dots)dx, \tag{2.85}$$

with $n \in \mathbb{N}$ and the Hamiltonian densities h_n are polynomials in u and its x derivatives. The factor of $\frac{1}{2}$ is included to agree with convention. For the KdV equation, one finds the following recursion relation:

$$h_1(x,t) = -u(x,t), \;\; h_{n+1} = \frac{\partial h_n}{\partial x} + \sum_{m=1}^{n-1} h_m h_{n-m}. \tag{2.86}$$

See [17] for a detailed proof. For example, the first few can be readily calculated and are given as,

$$h_1 = -u, \tag{2.87}$$

$$h_2 = -\frac{\partial u}{\partial x}, \tag{2.88}$$

$$h_3 = -\frac{\partial^2 u}{\partial x^2} + u^2, \tag{2.89}$$

$$h_4 = \frac{\partial}{\partial x}\left(-\frac{\partial^2 u}{\partial x^2} + 2u^2\right), \tag{2.90}$$

$$h_5 = \frac{\partial}{\partial x}\left(-\frac{\partial^3 u}{\partial x^3} + 2\frac{\partial}{\partial x}u^2 + 2u\frac{\partial u}{\partial x}\right) - \left(\frac{\partial u}{\partial x}\right)^2 - 2u^3. \tag{2.91}$$

Furthermore, every first integral Poisson commutes pairwise,

$$\{\mathcal{H}_i, \mathcal{H}_j\} = 0. \tag{2.92}$$

Not all of the h_n give rise to non trivial conservation laws. For example, h_2 and h_4 are total x derivatives. Hence, assuming u decays to 0 as $x \to \pm\infty$ we have $\mathcal{H}_2 = \mathcal{H}_4 = 0$. It turns out that $\mathcal{H}_n = 0$ for all even $n \in \mathbb{N}$. See [18] for details. We relabel the first integrals as,

$$I_n = \frac{1}{2}\int_{\mathbb{R}} h_{2n+3}(x,t)dx, \tag{2.93}$$

with $n = 0, 1, 2, \ldots$. Here the conserved quantity I_0 represents the conservation of momentum, while we also recognise I_1 as the energy functional $\mathcal{H}[u]$ from equation (2.70). In all previous discussions, a special role was played by the Hamiltonian energy functional as this was the functional that gave rise to the integrable equation. However, from a theoretical point of view, it is more natural to place all first integrals on an equal footing. To this end, we impose that each conserved quantity generates an independent time evolution according to Hamilton's equation. In other words, for $n = 0, 1, 2, \ldots$, we define the flow as,

$$\frac{\partial u}{\partial t_n} := (-1)^n\{I_n, u\} = (-1)^n\frac{\partial}{\partial x}\frac{\delta I_n[u]}{\delta u(x)}, \tag{2.94}$$

for the independent 'time' variables t_n and for the dependent variable $u = u(t_0, t_1, t_2, \dots)$ now depending on an infinite number of variables. The first three equations are

$$u_{t_0} = u_x, \tag{2.95}$$

$$u_{t_1} = 6uu_x - u_{xxx}, \tag{2.96}$$

$$u_{t_2} = 10uu_{xxx} - 20u_xu_{xx} - 30u^2u_x - u_{xxxxx}. \tag{2.97}$$

Equation (2.94) gives rise to an infinite family of partial differential equations. This is what is referred to as an *integrable hierarchy*. In this case, the first equation is trivial, meaning we can identify t_0 and x. The first non trivial equation we recognise as the KdV equation after identifying t_1 and x. Hence we refer to this as the KdV hierarchy. This point is in fact rather general: the equation arising from the energy functional gives its name to the hierarchy. From an applied perspective, the 'lower equations' are more important since they correspond to the conservation of momentum and energy. From a geometric point of view, however, they are all treated on an equal footing.

From an analytic perspective, we are mostly interested in the case that u is a Schwartz function. In particular, u is twice continuously differentiable and therefore satisfies Schwarz's theorem. Thus, for consistency, one should check the compatibility condition,

$$\frac{\partial^2 u}{\partial t_m \partial t_n} = \frac{\partial^2 u}{\partial t_n \partial t_m}. \tag{2.98}$$

We will do this calculation later and in greater generality. There is also perhaps a deeper reason why we require the equality of mixed derivatives and we shall mention this in the next section.

Moreover, there exists a function $\tau(t_0, t_1, \dots)$ such that

$$h_i(u, u_x, u_{xx}, \dots) = -\frac{\partial^2}{\partial t_0 \partial t_{i-1}} \log \tau(t_0, t_1, \dots). \tag{2.99}$$

Details can be found in [19] and references therein. In particular, we find that for h_1 the

above equation reads

$$u = \frac{\partial^2}{\partial x^2} \log \tau(t_0, t_1, \dots), \tag{2.100}$$

after identifying t_0 and x. We will later refer to this function τ as a *tau function* for the KdV hierarchy.

The final defining characteristic of integrability,the presence of algebraic geometry, is perhaps the most mysterious. To clarify this point, we make the following rather curious observation.

Inspired by d'Alembert's solution to the wave equation, we look for a solution of the KdV equation in the form,

$$u(x, t) = -f(x + ct) + \frac{c}{3}, \tag{2.101}$$

where $c \in \mathbb{R}$. This is to be interpreted as a travelling wave solution moving with phase speed c. Letting $f(z) = f(x + ct)$ and $f' = f'(z)$, we obtain,

$$f''' = 12 f f'. \tag{2.102}$$

Upon integrating, we find that,

$$2f'' = 12f^2 - g_2. \tag{2.103}$$

where g_2 is a constant. Multiplying by f' yields,

$$2f'' f' = 12f^2 f' - g_2 f'. \tag{2.104}$$

Integrating again we find,

$$(f')^2 = 4f^3 - g_2 f - g_3, \tag{2.105}$$

where g_3 is a constant. This is exactly the Weierstrass differential equation that parameterises elliptic curves. A family of solutions depending on $\Lambda \subset \mathbb{C}$ is described by the Weierstrass elliptic function

$$\wp_\Lambda(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda \backslash 0} \frac{1}{(z + \omega)^2} - \frac{1}{\omega^2}. \tag{2.106}$$

Here, $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ is a lattice in $\mathbb{C}$ where $\{\omega_1, \omega_2\}$ is an $\mathbb{R}$-basis of $\mathbb{C}$ and

$$g_2 = 60 \sum_{\omega \in \Lambda \backslash 0} \omega^{-4} \text{ and } g_3 = 140 \sum_{\omega \in \Lambda \backslash 0} \omega^{-6}$$

are the Eisenstein series. The elliptic curve is then given by the torus $\mathbb{C}/\Lambda$ and is often called a *spectral curve* of the KdV equation. This torus is exactly the Liouville torus appearing in the Arnold-Liouville theorem. See [20] for details.

Thus,

$$u(x,t) = -\wp_\Lambda(x + ct) + \frac{c}{3} \tag{2.107}$$

is a solution to the KdV equation. To find out what this means, we introduce the *theta function*,

$$\theta(z, \Omega) = \sum_{n \in \mathbb{Z}} e^{i\pi(n^2\Omega + 2nz)}, \tag{2.108}$$

for $\Omega \in \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$. Since Ω has positive imaginary part, the series absolutely converges for all $z \in \mathbb{C}$ and so the theta function is well defined.

It is also well known that, for a lattice given as $\Lambda = \mathbb{Z} + \Omega\mathbb{Z}$, we have

$$\wp_\Lambda(z) = -\frac{d^2}{dz^2} \log \theta\left(z + \frac{1}{2}(1 + \Omega), \Omega\right) + k, \tag{2.109}$$

for some $k \in \mathbb{C}$. The constant k is chosen so that $\wp_\Lambda$ has no constant term in its Laurent expansion. See [21] for details.

By inspecting equations (2.107) and (2.109) we are inspired to look for solutions of the KdV equation of the form,

$$u(x,t) = 2\frac{d^2}{dx^2} \log \tau(x,t), \tag{2.110}$$

for some function τ to be determined. After some algebra, we find the equation

$$\tau\tau_{xxxx} + 3(\tau_{xx})^2 - 4\tau_x\tau_{xx} - (\tau_x)^2 + \tau\tau_{tx} = 0. \tag{2.111}$$

This equation is known as the *Hirota bilinear form* of the KdV equation for the function τ.

Upon comparing equations (2.109), (2.107) and (2.100), it becomes apparent that theta functions and tau functions are intimately connected. More specifically, one can show that theta functions constructed from algebraic curves can play the role of tau functions for some integrable hierarchies. This not only leads to beautiful new method of finding solutions of integrable systems, but also provides an insight into the Schottky problem of characterising Jacobian varieties. Whilst this is a rich and fascinating theory, solutions to integrable equations found using this method depend only on a finite number of the independent time variables t_n. An explicit construction of these solutions and their relation to the Schottky problem can be found in [22]. These solutions are called *stationary solutions*, or *finite gap solutions*. For this thesis, we will not consider these types of solution. Instead, we are interested in formal solutions which depend on an infinite number of times.

2.2.3   From the KP Hierarchy to the r-KdV Hierarchies

In the previous section, we took an analytic and geometric approach to integrable hierarchies using the Hamiltonian formalism. In this section, we redefine the KP and r-KdV hierarchies using algebraic methods, with much greater emphasis on the Lax representation of these equations. This entirely equivalent perspective is much more suitable for our purposes.

The following definitions of the binomial coefficients are useful in what follows.

Define the binomial coefficient $\binom{n}{k}$ for non-negative integer n and integer k as,

$$\binom{n}{k} := \begin{cases} \frac{n!}{k!(n-k)!}, & \text{for } 0 \leq k \leq n \\ 0, & \text{otherwise} \end{cases}$$

and for negative integer n and integer k, define,

$$\binom{n}{k} := \begin{cases} (-1)^k \binom{-n+k-1}{k}, & \text{for } k \geq 0 \\ (-1)^{n-k} \binom{-k-1}{n-k}, & \text{for } k \leq n \\ 0, & \text{otherwise.} \end{cases}$$

These definitions arise from analytic continuation of the binomial coefficients in terms of the gamma function. Crucially for us, they are in agreement with the binomial theorem for the expansion of $(x + y)^n$ with $n \in \mathbb{Z}$ as shown in [23].

Definition 2.2.1. Let ∂ and ∂^{-1} be formal symbols and let $a_i(x)$ be $\mathbb{C}$-valued functions in the variable x with $i \leq N$ for some $N \in \mathbb{Z}$. The set,

$$\mathcal{P}(x, \partial) := \left\{ \sum_{i=-\infty}^{N} a_i(x)\partial^i \right\}, \tag{2.112}$$

is called the set of *pseudo-differential operators.*

We will often use $\mathcal{P}$ instead of $\mathcal{P}(x, \partial)$ in cases with no ambiguity.

This set has the obvious structure of a vector space over $\mathbb{C}$. We can turn this vector space into an algebra according to the rule $\partial\partial^{-1} = \partial^{-1}\partial = 1$ and the generalised Leibniz rule,

$$\partial^n(uv) = \sum_{k=0}^{\infty} \binom{n}{k} \partial^k u \partial^{n-k} v. \tag{2.113}$$

where $\binom{n}{k}$ are the binomial coefficients defined above and $n \in \mathbb{Z}$. Notice the symmetry if we change summation index $k \mapsto n - k$. As the notation suggests, we identify ∂ and ∂^{-1} as formal derivatives and formal integrals respectively. Observe that if $n > 0$, equation (2.113) is nothing other than the product rule for differentiation. Observe also that if $n = -1$, then $\binom{n}{k} = (-1)^k$ and (2.113) is simply iterations of the integration by parts formula.

It is clear that as vector spaces, there is a decomposition,

$$\mathcal{P} = \mathcal{P}_+ \oplus \mathcal{P}_-, \tag{2.114}$$

where $\mathcal{P}_+ := \left\{ \sum_{i=0}^{N} a_i(x)\partial^i \right\}$ and $\mathcal{P}_- := \left\{ \sum_{i=-\infty}^{-1} a_i(x)\partial^i \right\}$ are the subalgebras of differential operators and integral operators respectively. For an operator $Q \in \mathcal{P}$, we write $Q = Q_+ + Q_-$ to represent this decomposition.

Definition 2.2.2. Let $A = \sum_{i=-\infty}^{N} a_i \partial^i \in \mathcal{P}$. The *Adler trace* is a linear form on $\mathcal{P}$ defined

by,

$$\langle A \rangle = \int \text{Res}(A)dx, \tag{2.115}$$

where $\text{Res}(A) := a_{-1}(x)$ is the *residue* of A.

The terminology "trace" is justified in the next proposition.

Proposition 2.2.3. The Adler trace satisfies $\langle AB \rangle = \langle BA \rangle$.

Proof. The proposition follows after a calculation of both the left and right hand sides. See [13, p.340]. $\square$

Definition 2.2.4. Let Q be a pseudo differential operator of the form,

$$Q = \partial + \sum_{i=1}^{\infty} q_{-i}(x)\partial^{-i}. \tag{2.116}$$

Then the *KP hierarchy* is the infinite set of equations for q_{-i} defined by the Lax representation,

$$\partial_{t_k} Q = [B_k, Q] \text{ with } B_k = (Q^k)_+, \tag{2.117}$$

where ∂_{t_k} acts linearly and according to the Leibniz rule.

Remark. We assume the coefficient functions $q_{-i}(x)$ have non-trivial dependence on the time parameters t_k where $k \in \mathbb{N}$. We further assume that the symbols ∂ and ∂^{-1} carry no t_k dependence. This is consistent with the Lax equation since,

$$\partial_{t_k}(\partial^i) = [(\partial^i)_+^k, \partial^i] = 0, \tag{2.118}$$

for all $i \in \mathbb{Z}$. Therefore, $\partial_{t_k}(q_{-i}\partial^{-i}) = (\partial_{t_k} q_{-i})\partial^{-i}$.

The consistency of the above definition relies on the following proposition.

Proposition 2.2.5. The commutator $[B_k, Q]$ is a multiplicative operator for all $k \in \mathbb{N}$.

Rather than give a full proof, we instead provide verification for $k = 1$ and $k = 2$. This does illustrate, on the other hand, the main ideas in the proof. For full details, see [15].

Firstly, for $k = 1$, we have,

$$\partial_{t_1} Q = [(Q)_+, Q] = \left[\partial, \partial + \sum_{i=1}^{\infty} q_{-i} \partial^{-i}\right] = \sum_{i=1}^{\infty} (\partial q_{-i}) \partial^{-i}. \tag{2.119}$$

Now, using the remark above, we find,

$$\partial_{t_1} Q = \partial_{t_1} \left(\partial + \sum_{i=1}^{\infty} q_{-i} \partial^{-i}\right) = \sum_{i=1}^{\infty} (\partial_{t_1} q_{-i}) \partial^{-i}. \tag{2.120}$$

Comparing coefficients of (2.119) and (2.120) we identify t_1 and x up to a constant.

For the first non trivial equation, $k = 2$, we have,

$$(Q^2)_+ = \partial^2 + 2q_{-1}, \tag{2.121}$$

and

$$(Q^3)_+ = \partial^3 + 3q_{-1}\partial + 3\partial q_{-1} + 3q_{-2}. \tag{2.122}$$

Thus we find,

$$\partial_{t_2} q_{-1} = \partial^2 q_{-1} + 2\partial q_{-2}, \tag{2.123}$$

$$\partial_{t_2} q_{-2} = \partial^2 q_{-2} + 2\partial q_{-3} + 2q_{-1}\partial q_{-1}, \tag{2.124}$$

$$\partial_{t_3} q_{-1} = \partial^3 q_{-1} + 3\partial^2 q_{-2} + 3\partial q_{-3} + 6q_{-1}\partial q_{-1}. \tag{2.125}$$

Eliminating q_{-2}, q_{-3} between these three equations and renaming $u = -2q_{-1}$, we obtain,

$$3\partial_{t_2}^2 u = \partial(4\partial_{t_3} u + 6u\partial u - \partial^3 u). \tag{2.126}$$

Under the identification $y = t_2$ and $t = t_3$, this is nothing other than the classical KP equation. This justifies the terminology of "KP hierarchy".

Lemma 2.2.6. The flows defined by the KP hierarchy commute. In other words,

$$\partial_{t_i} \partial_{t_j} Q = \partial_{t_j} \partial_{t_i} Q. \tag{2.127}$$

Proof. Using the Lax equation we calculate,

$$\partial_{t_i} \partial_{t_j} Q - \partial_{t_j} \partial_{t_i} Q = \partial_{t_i}[B_i, Q] - \partial_{t_j}[B_i, Q] \tag{2.128}$$

$$= [B_i, [B_j, Q]] - [B_j, [B_i, Q]] = [B_i, [B_j, Q]] + [B_j, [Q, B_i]]. \tag{2.129}$$

Using the Jacobi identity, we find,

$$\partial_{t_i} \partial_{t_j} Q - \partial_{t_j} \partial_{t_i} Q = -[Q, [B_i, B_j]] = 0. \tag{2.130}$$

since $[B_i, B_j] = 0$ $\qquad\square$

The fact that ∂_{t_i} and ∂_{t_j} commute means that each Lax equation $\partial_{t_i} Q = [B_i, Q]$ generates a symmetry of every other equation. In other words, the translation of a solution of $\partial_{t_i} Q = [B_i, Q]$ along the vector field ∂_{t_j} is still a solution. This means we can consider the times t_i as independent coordinates for each i. We will also assume that, as a formal analogue to the Leibniz integral rule, we can commute derivatives and integrals. That is to say, $\partial_{t_k} \partial^{-1} = \partial^{-1} \partial_{t_k}$. See [12] for a proof.

Lemma 2.2.7. If $Q \in \mathcal{P}$ satisfies the Lax equation, $\partial_{t_k} Q = [B_k, Q]$ then so does Q^l for any $l \in \mathbb{N}$. That is, $\partial_{t_k} Q^l = [B_k, Q^l]$

Proof. We proceed by induction. Indeed, the claim is trivially true if $l = 1$. Observe that if $\partial_{t_k} Q^l = [B_k, Q^l]$ then,

$$\partial_{t_k} Q^{l+1} = \partial_{t_k} Q Q^l = (\partial_{t_k} Q) Q^{l+1} + Q \partial_{t_k} Q^l. \tag{2.131}$$

Using the Lax representation and calculating the commutators we find

$$\partial_{t_k} Q^{l+1} = [B_k, Q]Q^l + Q[B_k, Q^l] = [B_k, Q^{l+1}].$$ (2.132)

The lemma now follows by induction. □

The Lax representation is particularly useful because it immediately gives us conserved quantities.

Proposition 2.2.8. The quantities $H_l = \langle Q^l \rangle$ are conserved.

Proof. Let $Q^l = \partial^l + \cdots + \tilde{q}_{-1}(x)\partial^{-1} + \cdots$. Using the linearity of ∂_{t_k} and that $\partial_{t_k}\partial^i = 0$ for all $i \in \mathbb{Z}$ we have

$$\mathrm{Res}\left(\partial_{t_k} Q^l\right) = \partial_{t_k}\tilde{q}_{-1}(x) = \partial_{t_k}\mathrm{Res}(Q^l).$$ (2.133)

Hence, after formally interchanging derivatives and integrals, we have

$$\partial_{t_k}\langle Q^l \rangle = \partial_{t_k}\partial^{-1}\mathrm{Res}(Q^l) = \partial^{-1}\partial_{t_k}\mathrm{Res}\left(Q^l\right) = \partial^{-1}\mathrm{Res}\left(\partial_{t_k} Q^l\right) = \langle \partial_{t_k} Q^l \rangle.$$ (2.134)

Finally, combining (2.134) and Lemma 2.2.7 above, we see

$$\partial_{t_k}\langle Q^l \rangle = \langle \partial_{t_k} Q^l \rangle = \langle [B_k, Q^l] \rangle = 0.$$ (2.135)

by virtue of the cyclicity of the Adler trace. □

In this way, we see that the KP hierarchy is a family of equations in an infinite number of both independent and dependent variables. We now show that the KP hierarchy is, in some sense, universal in that many integrable hierarchies are embedded within the KP hierarchy itself. This is sometimes known as mod-r reduction of the KP hierarchy to the r-KdV hierarchy.

First, suppose $Q = \partial + \sum_{i=0}^{\infty} q_{-i}\partial^{-i}$ satisfies the Lax equation. Then from Lemma 2.2.7, we have that Q^r also satisfies the same Lax equation, $\partial_{t_k} Q^r = [(Q^k)_+, Q^r]$. Fix an integer $r \geq 2$. We impose the condition that Q^r is a differential operator so that $(Q^r)_- = 0$.

This implies that $\partial_{t_k} Q^r$ is also a differential operator since from the Lax equation we have $(\partial_{t_k} Q^r)_- = [(Q^k)_-, Q^r]_- = 0$. Moreover, we restrict attention to when Q^r is a differential operator of the form

$$Q^r = L = \partial^r - \sum_{i=0}^{r-2} u_i \partial^i. \tag{2.136}$$

As a result, given a pseudo-differential operator, the above shows that we can consistently construct a differential operator. Conversely, given a differential operator, we can uniquely reconstruct the pseudo differential operator as the next lemma shows.

Lemma 2.2.9. Fix an integer $r \geq 2$. If $L = \partial^r - \sum_{i=0}^{r-2} u_i \partial^i$ then there exists a unique pseudo differential operator $Q = \partial + \sum_{i=1}^{\infty} q_{-i} \partial^{-i}$ such that $Q^r = L$.

Proof. The proof amounts to computing all q_{-i} recursively in terms of the u_i. See [13]. $\square$

Remark. We denote the pseudo differential operator Q from the above lemma as $L^{\frac{1}{r}}$ and we use these notations interchangeably.

The Lax equations for L now follow from the Lax equations for Q^r as outlined the following proposition.

Proposition 2.2.10. Let $L = \partial^r - \sum_{i=0}^{r-2} u_i \partial^i$. Then the Lax equations ,

$$\partial_{t_k} L = [(L^{\frac{k}{r}})_+, L], \tag{2.137}$$

are consistent for all $k \in \mathbb{N}$.

Proof. First note that given L, by Proposition 2.2.9 we can reconstruct Q so that $Q^r = L$. This means the coefficients of $(Q^k)_+$ are some functions in u_i. Hence equation (2.137) is indeed a system of equations for the u_i. Furthermore, observe that L commutes with Q^k for all $k \in \mathbb{N}$ since $LQ^k = Q^{r+k} = Q^k L$. Consequently, we have,

$$[(Q^k)_+, L] = [Q^k, L] - [(Q^k)_-, L] = -[(Q^k)_-, L]. \tag{2.138}$$

This implies that $[(Q^k)_+, L]$ is a differential operator with the highest derivative having order $r - 2$. Indeed, writing the highest orders of $(Q^k)_- = \tilde{q}_{-1}\partial^{-1} + \cdots$ and $L = \partial^r + \cdots$ we see,

$$[(Q^k)_-, L] = [\tilde{q}_{-1}\partial^{-1} + \cdots, \partial^r + \cdots], \tag{2.139}$$

and so the term in ∂^{r-1} cancels. Therefore, this matches with $\partial_{t_k} L$ implying that (2.137) produces consistent equations for u_i. $\qquad \square$

Equation (2.137) defines an infinite family of equations known as the r-KdV hierarchy.

This allows us to write Lax equations in terms of finitely many functions u_i. For the simplest example, we calculate the first two equations in the 2-KdV hierarchy. The operator L becomes,

$$L = \partial^2 - u. \tag{2.140}$$

We now find the operator $Q = \partial + \sum_{i=1}^{\infty} q_{-1}\partial^{-i}$ such that $Q^2 = L$. To this end, we have

$$Q^2 = \partial^2 + 2q_{-1} + (2q_{-2} + \partial q_{-1})\partial^{-1} + \cdots. \tag{2.141}$$

This means $q_{-1} = -\frac{1}{2}u$, $q_{-2} = \frac{1}{4}\partial u$ and so on. For $k = 1$, we have the trivial equation $\partial_{t_1} L = [Q_+, L]$. In the same way as before, this leads to $\partial_{t_1} = \partial u$ meaning we identify x and t_1. We will return to $k = 2$ in a moment. For $k = 3$, we have,

$$(Q^3)_+ = \partial^3 - \frac{3}{2}u\partial - \frac{3}{4}(\partial u). \tag{2.142}$$

Hence, the Lax equation becomes,

$$\partial_{t_3} u = [\partial^3 - \frac{3}{2}u\partial - \frac{3}{4}(\partial u), \partial^2 - u]. \tag{2.143}$$

If we rescale the variable $t_3 \mapsto 4t_3$, this becomes,

$$\partial_{t_3} u = [-4\partial^3 + 6u\partial + 3\partial u, -\partial^2 + u], \tag{2.144}$$

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which is precisely the Lax representation of the KdV equation shown in the previous section.

For $k = 2n$, we have a trivial equation

$$\partial_{t_{2n}} L = [(Q^{2n})_+, L] = [L^n, L] = 0, \tag{2.145}$$

which implies

$$\partial_{t_{2n}} u = 0. \tag{2.146}$$

This reproduces the fact stated in the previous section that the KdV hierarchy does not depend on even times t_{2n}. Additionally, from Proposition 2.2.8, the quantities $\langle Q^n \rangle$ are conserved in the KP hierarchy. This must also mean that the 2-KdV flows conserve the quantities $\langle L^{\frac{n}{2}} \rangle$ for each $n \in \mathbb{N}$. However, we imposed that $L = Q^2$ is a differential operator. Therefore, if we take $n = 2k$ for $k \in \mathbb{N}$ the conserved quantities are $\langle L^{\frac{2k}{2}} \rangle = \langle L^k \rangle = 0$, again reproducing the triviality of the even first integrals of the KdV hierarchy.

Of course, for the full identification, we need the following theorem which is proved in [12] .

Theorem 2.2.11. The conserved quantities of the 2-KdV hierarchy are Hamiltonians under the same Poisson structure as that of the KdV hierarchy and are in involution with respect to this bracket.

This means the KdV hierarchy coincides, up to rescaling, with the 2-KdV hierarchy. We will explicitly show this in a moment. This justifies the name r-KdV; in general the 2-KdV hierarchy is the ordinary KdV hierarchy whilst r-KdV for $r \geq 3$ are the *generalised higher KdV hierarchies.* Henceforth, we make no distinction between the 2-KdV and KdV hierarchies. The above argument also shows that the solutions of the r-KdV hierarchy do not depend on the times t_{nr} for $n \in \mathbb{N}$.

As a final observation, we use Theorem 2.2.11 to rewrite the KdV hierarchy in a slightly different way that will become useful for the next chapter. In doing so, we will show that

the 2-KdV hierarchy is equivalent to the KdV hierarchy up to a rescaling. Observe that,

$$\frac{\partial L}{\partial t_k} = [(L^{k-1/2})_+, L] = -[(L^{k-1/2})_-, L] = 2\frac{\partial}{\partial x}\text{Res } L^{k-1/2}. \tag{2.147}$$

Setting $R_k[u] = 2\text{Res } L^{k-1/2}$ implies

$$\frac{\partial u}{\partial t_k} = R'_k[u], \tag{2.148}$$

where $R_k[u]$ are the *Gelfand - Dikii* polynomials. We remark that this is consistent with the Poisson structure. Indeed by Theorem 2.2.11 we treat the Adler traces of $L^{k+1/2}$ as Hamiltonians for the KdV hierarchy. Employing equation (2.94) we also have

$$\frac{\partial u}{\partial t_k} = \frac{\partial}{\partial x}\frac{\delta}{\delta u}\int \text{Res}L^{k+1/2}dx = \frac{\partial}{\partial x}\frac{\delta}{\delta u}\int \frac{1}{2}R_{k+1}[u]dx, \tag{2.149}$$

where $R_k[u]$ are the Gelfand-Dikii polynomials as before. Now, to calculate this functional derivative, we introduce the *resolvent*, $G(z) = (z - L)^{-1}$ where z is not an element of the spectrum of L. Here, $G(z)$ can be thought of as a formal series expansion in L, the rigorous justification of which resides in the theory of holomorphic functional calculus. See [24] for example. Through this formal identification, we can define the Green's function $G(z; x, y)$ as

$$(z - L)G(z; x, y) = \delta(x - y). \tag{2.150}$$

In [25], it is then proved that $G(z; x, x)$ and Tr $G(z)$ have the following expansions as $z \to -\infty$ in terms of $R_k[u]$,

$$G(z; x, x) = -\sum_{k=0}^{\infty} R_k[u](-z)^{-k-1/2}, \tag{2.151}$$

and

$$\text{Tr } G(z) = -\sum_{k=0}^{\infty} \int R_k[u](-z)^{-k-1/2}dx. \tag{2.152}$$

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In this proof, one finds the Gelfand - Dikii polynomials are generated by the recursion,

$$\partial R_k = \frac{1}{4}\Big(\partial^3 + 2u\partial + 2(\partial u)\Big)R_{k-1}, \tag{2.153}$$

with $R_k = 0$ for $k \leq -2$, $R_{-1} = 1$ and $R_0 = u$. From equation (2.150) one finds the relation

$$\frac{\delta}{\delta u(x)}\text{Tr } G(z) = -\frac{\partial}{\partial z}G(z; x, x). \tag{2.154}$$

Using the large z expansions we find

$$\frac{\delta}{\delta u}\int R_k[u]dx = -\Big(k - \frac{1}{2}\Big)R_{k-1}[u]. \tag{2.155}$$

Applying (2.155) to (2.149) we find

$$\frac{\partial u}{\partial t_k} = -\frac{1}{2}\Big(k + \frac{1}{2}\Big)R'_k[u], \tag{2.156}$$

in agreement, up to a rescaling of t_k, with equation (2.148). Thus, the the KdV hierarchy is equivalent to the 2-KdV hierarchy.

2.3 The Infinite Dimensional Grassmannian - An Alternate Formulation of Integrability

The language of pseudo differential operators and Lax equations is particularly suitable for neatly describing explicitly the KP and r-KdV hierarchies and their symmetries. We now wish to continue this abstract approach to integrable hiearchies by exploiting properties of an infinite dimensional Grassmannian. It is here we will encounter the most useful definitions of the tau function, Baker - Akhiezer functions and Hirota bilinear equations. This perspective was first initiated in [26] and then further established in [27].

We take two approaches. In the first, we will describe an abelian group action on the

Grassmannian in the spirit of canonical quantisation; the orbits of this abelian group action are in fact the time evolution of the KP hierarchy. Next, in the spirit of second quantisation, we will embed the Grassmannian into a projective exterior space via the Plücker map. We interpret this exterior space as a labelling of occupation states, otherwise known as a Fock space. Here we mainly follow [13, 28, 29].

2.3.1 The Analytic Approach - Canonical Quantisation

We begin with a definition.

Definition 2.3.1. A *KP tau function* is a formal function $\tau = \tau(t) = \tau(t_1, t_2, \cdots)$ in an infinite number of variables t_k satisfying the Hirota bilinear equations,

$$\oint_{\mathcal{C}} e^{\sum_{k=1}^{\infty}(t_k - t'_k)z^k} \tau(t - [z^{-1}])\tau(t' + [z^{-1}])dz = 0, \tag{2.157}$$

which are satisfied identically in the variable $t' = (t'_1, t'_2 \cdots)$. Here the *Sato shifts* are defined through

$$t - [z^{-1}] = (t_1, t_2, \dots) - \left(\frac{1}{z}, \frac{1}{2z^2} \dots\right) \tag{2.158}$$

and we take the contour integral at infinity.

At first glance, this seems a rather curious definition. However, we will illustrate why this is in fact a reasonable definition to make.

Using the residue theorem, the Hirota equations read,

$$\text{Res}_{z=\infty}[\tau(t_1 - \frac{1}{z}, t_2 - \frac{1}{2z^2}, \dots)\tau(t'_1 + \frac{1}{z}, t'_2 + \frac{1}{2z^2}, \dots)e^{z(t_1-t'_1)+z^2(t_2-t'_2)+\cdots}] = 0. \tag{2.159}$$

Technically, it is easier to use the substitutions $t_i = x_i - y_i$ and $t'_i = x_i + y_i$. We can then calculate that equation (2.159) is equivalent to,

$$\text{Res}_{z=\infty}\left\{\left[e^{\sum \frac{1}{k}z^{-k}\frac{\partial}{\partial y_k}}\tau(x-y)\tau(x+y)\right]e^{-2zy_1-2z^2y_2-\cdots}\right\} = 0. \tag{2.160}$$

Indeed, if $\sum \frac{1}{k} z^{-k} \frac{\partial}{\partial y_k}$ is thought of as a vector field, then such a vector field generates translations in the variables y_k.

We introduce the Schur polynomials $h_k(t)$ by means of a generating function,

$$e^{\sum_{j=1}^{\infty} t_j z^j} = \sum_{k=0}^{\infty} h_k(t) z^k. \tag{2.161}$$

For example, the first four are

$$h_0(t) = 1, \quad h_1(t) = t_1, \quad h_2(t) = t_2 + \frac{t_1^2}{2}, \quad h_3(t) = t_3 + t_2 t_2 + \frac{t_1^3}{6} \tag{2.162}$$

Applying this we now have

$$\text{Res}_{z=\infty} \left\{ \left[\sum_i z^{-i} h_i(D_y) \tau(x-y)\tau(x+y) \right] \sum_j z^j h_j(-2y) \right\} = 0, \tag{2.163}$$

where $D_y = \left(\frac{\partial}{\partial y_1}, \frac{1}{2} \frac{\partial}{\partial y_2}, \dots \right)$. This allows us to take the residue so that the only terms to survive are when $-i + j = -1$. Thus we find

$$\sum_{j=0}^{\infty} h_i(-2y) h_{j+1}(D_y) \tau(x-y)\tau(x+y) = 0, \tag{2.164}$$

which we rewrite as,

$$\sum_{j=0}^{\infty} h_i(-2y) e^{\sum y_i \frac{\partial}{\partial w_i}} h_{j+1}(D_w) \tau(x-w)\tau(x+w)|_{w=0} = 0. \tag{2.165}$$

We now expand in y and we read off the lowest order, non-trivial equation as,

$$\left(\frac{\partial^4}{\partial w_1^4} + 3\frac{\partial^2}{\partial w_2^2} - \frac{\partial^2}{\partial w_1 \partial w_3} \right) \tau(x-w)\tau(x+w)|_{w=0} = 0, \tag{2.166}$$

which when written out fully becomes,

$$\tau \tau_{1111} - 4\tau_1 \tau_{111} + 3\tau_{11}^2 + 3\tau \tau_{22} - 3\tau_2^2 - 4\tau \tau_{13} + 4\tau_1 \tau_3 = 0. \tag{2.167}$$

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Inspired by the results of section 1.2, we put $u = (\log \tau)_{11}$ and we obtain,

$$3u_{22} = (4u_3 - 12uu_1 - u_{111})_1, \tag{2.168}$$

which, up to rescaling, is exactly the KP equation of the previous sections. To obtain the higher equations in the hierarchy, one may simply calculate the higher order coefficients in the expansion of equation (2.165).

The Hirota equations can in fact be encoded in a rather elegant way using the geometry of a finite dimensional Grassmannian. This will serve as a useful toy example and motivation for the infinite dimensional generalisation.

Suppose we have a KP tau function $\tau(t)$. Let $z_1, z_2, z_3, z_4 \in \mathbb{C}$. Write

$$\eta_{ij} = (z_i - z_j)\tau(t - [z_i^{-1}] - [z_j^{-1}]), \tag{2.169}$$

for $i, j = 1, 2, 3, 4$. We can view η as being an antisymmetric matrix whose entries are η_{ij}. We can also view η as a two form in $\Lambda^2 \mathbb{C}^4$. From this perspective, for the standard basis e_i of $\mathbb{C}^4$ we write

$$\eta = \sum_{i<j} \eta_{ij} e_i \wedge e_j. \tag{2.170}$$

Now, take the finite dimensional Grassmannian

$$\mathrm{Gr}_2(\mathbb{C}^4) = \{V \text{ a subspace of } \mathbb{C}^4 \text{ and } \dim V = 2\}. \tag{2.171}$$

This Grassmannian can be given the structure of a four dimensional complex manifold. Since $\bigwedge^2 \mathbb{C}^4$ can be identified with antisymmetric 4 x 4 matrices, as a vector space this exterior space has six dimensions. We define the projectivisation of $\bigwedge^2 \mathbb{C}^4$ as,

$$\mathbb{P}(\wedge^2 \mathbb{C}^4) := \bigwedge^2 \mathbb{C}^4 \setminus \{0\} \Big/ \mathbb{C}^*. \tag{2.172}$$

In other words, we factor out by the equivalence relation

$$x \sim y \iff \exists z \in \mathbb{C} \text{ s.t } x = zy. \tag{2.173}$$

Define a map $\Delta : \text{Gr}_2(\mathbb{C}^4) \to \mathbb{P}(\bigwedge^2 \mathbb{C}^4)$ by

$$w = \text{span}(w_1, w_2) \mapsto [w_1 \wedge w_2]. \tag{2.174}$$

This involves a choice of the basis $\{w_1, w_2\}$ for w. However, this map is indeed well defined. Indeed, since any choice of basis will be related by a $\text{GL}(2, \mathbb{C})$ transformation. This scales the wedge product $w_1 \wedge w_2$ by the determinant of the $\text{GL}(2, \mathbb{C})$ transformation. This scale factor is irrelevant in the projectivised space and hence, this map does not depend on the choice of basis as claimed. The map Δ is called the *Plücker embedding*.

Now suppose there exists $w_1, w_2 \in \mathbb{C}^4$ such that $[\eta] = \Delta(\text{span}(w_1, w_2)) = [w_1 \wedge w_2]$ with η defined as above. We must then have $\eta \wedge \eta = 0$. This is called the Plücker relation and written out in component form, it reads,

$$\eta_{12}\eta_{34} + \eta_{13}\eta_{42} + \eta_{14}\eta_{23} = 0. \tag{2.175}$$

Substituting equation (2.169) one finds the *Hirota bilinear difference equation*

$$(z_1 - z_2)(z_3 - z_4)\tau(t - [z_1]^{-1} - [z_2]^{-1})\tau(t - [z_3]^{-1} - [z_4]^{-1}) + (234) + (243) = 0, \quad (2.176)$$

where (234) and (243) are the permutations of the first term. This is in fact equivalent to the Hirota bilinear equation as shown in [29] with a different choice of time variables

$$t'_k = t_k = \frac{1}{k} z_1^{-k} - \frac{1}{k}(z_2^{-k} + z_3^{-k} + z_4^{-k}). \tag{2.177}$$

To get a full description and interpretation of the dynamics of the KP flows, finite Grassmannians are not sufficient. We therefore find it necessary to pass to the so-called Sato

Grassmannian.

Recall a linear operator, $T : X \to Y$ with X and Y Banach spaces, is called *compact* if for any bounded subset $U \subset X$, its image $T(U) \subset Y$ has compact closure. Furthermore, a bounded linear operator $T : X \to Y$ is called *Fredholm* if it has finite dimensional kernel and cokernel.

Definition 2.3.2. Let $\mathcal{H} = L^2(S^1) = \mathcal{H}_+ \oplus \mathcal{H}_-$ where $\mathcal{H}_+ = \text{span}\{z^i\}_{i=0}^{\infty}$ and $\mathcal{H}_- = \text{span}\{z^{-i}\}_{i=1}^{\infty}$. Then we define the infinite dimensional *Sato Grassmannian*,

$$\text{Gr}_{\mathcal{H}_+}(\mathcal{H}) := \{V \subset \mathcal{H} \text{ closed s.t } \pi_+ : V \to \mathcal{H}_+ \text{ Fredholm}; \ \pi_- : V \to \mathcal{H}_- \text{compact}\}, \quad (2.178)$$

where π_+ and π_- are the orthogonal projections onto $\mathcal{H}_+$ and $\mathcal{H}_-$ respectively.

Remark. There is nothing special about the choice $L^2(S^1)$ for the Hilbert space. In principle, we could choose any separable Hilbert space, but to agree with the literature, we will use the space of square integrable functions on the circle.

Notice in particular that $\mathcal{H}_+ \in \text{Gr}_{\mathcal{H}_+}(\mathcal{H})$.

What follows is a brief sketch of how to construct a tau function on this Grassmannian. We will not pursue the rigorous functional analytic framework, although interested readers can consult [27].

Let $\{e_i\}_{i \in \mathbb{Z}}$ be an orthonormal basis for $\mathcal{H}$. For concreteness, choose $e_i = z^{-i-1}$. Although this choice of labelling may seem disturbing, it will be more convenient for later purposes. Let $w = \text{span}\{w_1, w_2 \dots\} \in \text{Gr}_{\mathcal{H}_+}(\mathcal{H})$ where $w_1, w_2 \cdots \in \mathcal{H}$.

Expand now in the orthonormal basis

$$w_i = \sum_{j \in \mathbb{Z}} W_{ji} e_j. \quad (2.179)$$

For $i \in \mathbb{N}$, define the column vectors $\mathbf{W}_i$ by $(\mathbf{W}_i)_j = W_{ji}$. Construct the $\mathbb{Z} \times \mathbb{N}$ matrix, W whose i^{th} column is $\mathbf{W}_i$.

There is a natural Abelian group action on the Grassmannian. Define the Abelian group $\Gamma := \{\gamma(t) = e^{\sum_{j=1}^{\infty} t_i z^i}\}$ where $t = (t_1, t_2, \dots)$ is an infinite sequence containing only finitely many non zero terms. This restriction is necessary to obtain consistent results; we will relax this condition, however, when we consider formal tau functions. The group action $\Gamma \times \mathcal{H} \to \mathcal{H}$ is given by ordinary multiplication, $f \mapsto \gamma(t)f$. This group action induces an action on the Grassmannian in the following way. Let $w \in \mathrm{Gr}_{\mathcal{H}_+}(\mathcal{H})$. We have $\hat{\gamma}(t) \times w \mapsto w(t) = \mathrm{span}\{e^{\sum_{j=1}^{\infty} t_j z^j} w_1, e^{\sum_{j=1}^{\infty} t_j z^j} w_2, \dots, \}$. Equivalently, for $w \in \mathrm{Gr}_{\mathcal{H}_+}(\mathcal{H})$ we can construct the matrix W as above. Then applying the group action outputs the matrix $W(t)$ associated to $w(t)$ defined above. Explicitly, we have $\hat{\gamma}(t) = e^{\sum_{j=1}^{\infty} t_j \Lambda^j}$ where $\Lambda(e_i) := e_{i-1}$ is the shift operator that corresponds to multiplication by z.

It is shown in [27] that the orthogonal projection π_+ differs from the identity by a *trace class operator*. This is a sufficient and necessary condition for the operator π_+ to have a well defined Fredholm determinant. See [27] and references therein for detailed discussions. Given $w \in \mathrm{Gr}_{\mathcal{H}_+}(\mathcal{H})$, we define the function $\tau_w := \det(\pi_+ : w(t) \to \mathcal{H}_+)$ where the determinant here is the Fredholm determinant. Equivalently, one can construct the matrix,

$$W(t) = \begin{bmatrix} W_+(t) \\ W_-(t) \end{bmatrix}, \tag{2.180}$$

associated to $w(t)$ where W_+ is an $\mathbb{N} \times \mathbb{N}$ matrix. One can then think of the Fredholm determinant $\tau_w(t) = \det(P_+ : w(t) \to \mathcal{H}_+)$ as an infinite dimensional matrix determinant $\tau_w(t) = \det W_+(t)$.

As the notation suggests, we have the following theorem.

Theorem 2.3.3. The function $\tau_w(t)$ defined above satisfies the Hirota bilinear equation and so is a KP tau function.

Furthermore, Segal and Wilson proved that the KP flow appears naturally as the orbit of the above group action on the Sato Grassmannian in the following sense.

Theorem 2.3.4. There is a well defined map $w(t) \mapsto u_w(t)$ where $w(t)$ is a point in a connected component of $\mathrm{Gr}_{\mathcal{H}_+}(\mathcal{H})$ and u_w which is a meromorphic function in t_1. Furthermore,

given an initial point $w \in \mathrm{Gr}_{\mathcal{H}_+}(\mathcal{H})$, the function $u_{\hat{\gamma}(t)w}$ is the function obtained by letting u_w flow along the KP flow for a time $t = (t_1, t_2, \dots)$ where only finitely many t_k are non zero.

In light of the above theorem, there is now a natural way to reduce the KP hierarchy to the r-KdV hierarchy. We impose the additional constraint that for special choices of $w \in \mathrm{Gr}_{\mathcal{H}_+}(\mathcal{H})$ we have

$$z^r w \subset w. \tag{2.181}$$

In this way, the multiplication of w by $t_{nr}z^{nr}$ is trivial in the Abelian group action. Thus, the solution $u_{w(t)}$ and the tau function is independent of the times t_{nr} for $n \in \mathbb{N}$ which is the defining characteristic of r-KdV. Furthermore, for completeness, we state the following theorem.

Theorem 2.3.5. Let h_j be the Hamiltonian densities for the KdV hierarchy defined in section 2.2. Then the function $\tau(t)$ defined by $h_j = \frac{\partial^2}{\partial t_0 \partial t_{j-1}} \log \tau(t)$ satisfies the Hirota equations and so is a KP tau function.

Remark. In section 2.2, we defined the hierarchy with the first time given by t_0. In comparison, in thi section we have started at t_1. This is a simple relabelling of times and is unimportant in this discussion.

To prove Theorem 2.3.4, it is more convenient for technical reasons to introduce wavefunctions.

Definition 2.3.6. Suppose $\tau(t)$ is a KP tau function. Define the *wavefunction*, or *Baker-Akhiezer function* as

$$\psi^+(z, t) = e^{\sum_{j=1}^{\infty} t_j z^j} \frac{\tau(t - [z^{-1}])}{\tau(t)}, \tag{2.182}$$

and its adjoint

$$\psi^-(z, t) = e^{-\sum_{j=1}^{\infty} t_j z^j} \frac{\tau(t + [z^{-1}])}{\tau(t)}. \tag{2.183}$$

Note that the Baker - Akhiezer functions allow the Hirota equations to be recast into the more compact form,

$$\oint_{\mathcal{C}} \psi^+(t, z)\psi^-(t', z)dz = 0. \tag{2.184}$$

This is also perhaps a more fundamental perspective. We remark that in this form, the Hirota equations can be viewed as orthogonality conditions if we treat the Baker - Akhiezer functions as orthogonal polynomials. We will return to this perspective when we consider matrix models.

We have already seen that the Hirota equations allow us to reconstruct the equations of the hierarchy from the tau function. We can also recover the hierarchy directly from the Baker-Akhiezer functions as follows.

Proposition 2.3.7. Let Q be the pseudo differential operator given in Definition 2.2.4 in the previous section. Then the Baker-Akhiezer function is an eigenfunction of Q,

$$Q\psi^+(t, z) = z\psi^+(t, z), \tag{2.185}$$

and obeys the Schrödinger equations

$$\partial_{t_i}\psi^+ = (Q^i)_+\psi^+. \tag{2.186}$$

For ease of notation, we let $B_i = (Q^i)_+$. To recover the hierarchy, we make a similar argument as that of the previous section. Differentiating equation (2.185) in the above proposition with respect to t_i we find

$$Q_{t_i}\psi^+ = z\psi^+_{t_i} - Q\psi_{t_i} = B_iQ\psi^+ - QB_i\psi^+. \tag{2.187}$$

These operators are multiplicative and so we recover the Lax equation. Interestingly, we can repeat the same procedure with $B_i\psi = \psi_{t_i}$ and $B_j\psi = \psi_{t_j}$. Cross differentiating and

enforcing the compatibility condition $\frac{\partial^2 \psi^+}{\partial t_i \partial t_j} = \frac{\partial^2 \psi^+}{\partial t_j \partial t_i}$ leads to the zero curvature equations

$$\frac{\partial B_i}{\partial t_j} - \frac{\partial B_j}{\partial t_i} + [B_i, B_j] = 0. \tag{2.188}$$

These equations also contain the equations of the integrable hierarchy.

The same arguments and zero curvature also hold for the r-KdV hierarchy with $L = Q^r$. The only difference is that now ψ^+ is an eigenfunction of L with eigenvalue z^r. The reason for the terminology zero curvature is that equation (2.188) defines a flat connection; this observation represents the beginnings of the relationship between D-modules, quantum cohomology and integrable systems. We shall not dwell on this further, although the interested reader can consult [30].

2.3.2   The Algebraic Approach - Second Quantisation

Recall in the previous subsection that given $w = \mathrm{span}(w_1, w_2) \in \mathrm{Gr}_2(\mathbb{C}^4)$, we defined the Plücker map $\Delta : \mathrm{Gr}_2(\mathbb{C}^4) \to \mathbb{P}(\bigwedge^2 \mathbb{C}^4)$ by $w \mapsto [w_1 \wedge w_2]$. We can define the analogous Plücker map in the infinite dimensional case, $\Delta : \mathrm{Gr}_{\mathcal{H}_+}(\mathcal{H}) \to \mathbb{P}(\bigwedge \mathcal{H})$ by $w = \mathrm{span}(w_1, w_2, \dots) \mapsto [w_1 \wedge w_2 \wedge \cdots]$. There are also analogous Plücker relations which are again equivalent to the Hirota bilinear equations. These however are more subtle to construct in the infinite dimensional case and so we omit the discussion here. Interested readers can see [29].

In the above example, $\bigwedge^2 \mathbb{C}^4$ for the finite dimensional Grassmannian $\mathrm{Gr}_2(\mathbb{C}^4)$ played the role of the Fock space. For the Sato Grassmannian $\mathrm{Gr}_{\mathcal{H}_+}(\mathcal{H})$, however, the natural exterior space to consider is $\bigwedge \mathcal{H}$. We denote $\mathcal{F} := \bigwedge \mathcal{H}$ as the fermionic Fock space. This is fermionic rather than bosonic since we shall see in a moment that $\mathcal{F}$ is a representation of a Clifford algebra.

For concreteness we will take $\mathcal{H} = L^2(S^1)$ as before. Take an orthonormal basis of $\mathcal{H}$ and $\mathcal{H}^*$ as $\{e_i\}$ and $\{e_i^*\}$ respectively such that $e_i^*(e_j) = \delta_{ij}$. Then, for $v \in \mathcal{F}$, define the operators $\psi_j(v) = e_j \wedge v$ and $\psi_j^\dagger(v) = \iota_{e_j^*} v$ where ι is interior multiplication. Explicitly, if

$v = v_1 \wedge \cdots \wedge v_k$ with $v_i \in \mathcal{H}$, we have

$$\iota_{e_j^*}(v) = \sum_{i=1}^{k} (-1)^{i-1} \langle e_j^*, v_i \rangle v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge v_k, \tag{2.189}$$

where $\langle \cdot, \cdot \rangle$ is the pairing between $\mathcal{H}$ and $\mathcal{H}^*$ and $\hat{v}_i$ indicates that v_i is skipped.

We see that these satisfy the Clifford algebra relations

$$\{\psi_i, \psi_j\} = \{\psi_i^\dagger, \psi_j^\dagger\} = 0, \ \{\psi_i, \psi_j^\dagger\} = \delta_{ij}. \tag{2.190}$$

These hold by virtue of the anticommutativity of the exterior product and the alternating sign in the interior multiplication. From this perspective, the vector space $\mathcal{F}$ is a representation for the Clifford algebra generated by ψ_i and $\psi_i^\dagger$.

Define the *charge 0 vacuum* state $|0\rangle = e_{-1} \wedge e_{-2} \wedge \cdots$. This state is also called the Dirac sea at level 0 for reasons that shall become apparent in a moment.

Since we took $e_i = z^{-i-1}$, the vacuum simply corresponds to $\mathcal{H}_+ = \text{span}\{z^i\}_{i \in \mathbb{N}}$. Similarly, we can define the *charge n vacuum* states

$$|n\rangle := \begin{cases} \cdots \wedge e_{-1} \wedge e_0 \wedge e_1 \wedge \cdots \wedge e_{n-1} & n > 0 \\ \cdots \wedge e_{n-3} \wedge e_{n-2} \wedge e_{n-1} & n < 0, \end{cases} \tag{2.191}$$

These states will become useful in the later chapters.

Using the explicit formulae for ψ_i and $\psi_i^\dagger$ we have that

$$\psi_n |n\rangle = |n+1\rangle, \tag{2.192}$$

and

$$\psi_n^\dagger |n+1\rangle = |n\rangle. \tag{2.193}$$

In light of this, for $i \geq 0$, the elements ψ_i are called creation operators, while $\psi_i^\dagger$ are called annihilation operators and vice versa.

Therefore we can alternatively write $|n\rangle$ as

$$|n\rangle = \begin{cases} \psi_{n-1} \cdots \psi_0 \, |0\rangle & n > 0 \\ \psi_n^\dagger \cdots \psi_{-1}^\dagger \, |0\rangle & n < 0, \end{cases} \tag{2.194}$$

Notice also that the vacuum satisfies $\psi_i \, |0\rangle = 0$ for $i < 0$ and $\psi_i^\dagger \, |0\rangle = 0$ for $i \geq 0$.

We remark that the states $\{|n\rangle\}_{n \in \mathbb{Z}}$ do not span the full Fock space $\mathcal{F}$. We could have for example a state $\cdots \wedge e_{-3} \wedge e_{-2} \wedge e_1$. A useful pictorial visualisation of a basis of $\mathcal{F}$ is given by introducing the following notion.

Definition 2.3.8. A *Maya diagram* is a placement of black or white circles along $\mathbb{N}$ such that there exists $N < 0$ and $M > 0$ for which all circles at positions less than N are black, and all circles at positions greater than M are white.

We then have that the set of possible Maya diagrams forms a basis for $\mathcal{F}$. From this perspective, charge n vacuum states correspond to Maya diagrams completely filled with black circles strictly up to the nth position, whilst all other circles are white. A creation operator ψ_i corresponds to filling in a white circle. The analogous statement holds for annihilation operators. This is a convenient description in some physical contexts where the black and white circles are interpreted as electrons and holes respectively. It is for this reason that we labelled the orthonormal basis of $L^2(S^1)$ as $e_i = z^{-i-1}$; electrons occupy the lower, negative states while holes occupy higher, positive states.

(a) Dirac sea of level n (b) One particle one hole

Figure 2.1: Two examples of Maya diagrams [31]. Here, (a) shows the charge n vacuum, while (b) shows the charge n vacuum with a particle created at level $n+1$ and a hole created, or a particle annihilated, at level $n - 1$.

We can similarly define the dual vacuum state

$$\langle 0 | = e_1 \wedge e_2 \wedge \cdots , \tag{2.195}$$

so that it satisfies $\langle 0 | \psi_i^\dagger = 0$ for $i < 0$ and $\langle 0 | \psi_i = 0$ for $i > 0$.

We now define an Abelian group action analogously to the previous subsection. Let $J_i = \sum_{n \in \mathbb{Z}} : \psi_n \psi_{n+i}^\dagger :$ where $: \cdots :$ denotes normal ordering. This means moving all annihilation operators to the right and all creation operators to the left. We also multiply by a factor of -1 each time a creation operator ψ_i is moved past the corresponding annihilation operator $\psi_i^\dagger$. In fact, we see that the normal ordering in J_i is only necessary for $i = 0$.

We define the operators J_i in this way so that they satisfy the Heisenberg algebra $[J_k, J_l] = k\delta_{k,-l}$. Indeed, using the formula $[A, BC] = [A, B]C + B[A, C]$ we calculate

$$[J_k, J_l] = \sum_j [J_k, \psi_j \psi_{j+l}^\dagger] = \sum_j [J_k, \psi_j]\psi_{j+l}^\dagger + \psi_j [J_k, \psi_{j+l}^\dagger]. \tag{2.196}$$

We also have that $[J_k, \psi_m] = \psi_{m-k}$ and $[J_k, \psi_m^\dagger] = \psi_{m-k}^\dagger$. Hence equation (2.196) reads

$$[J_k, J_l] = \sum_j \psi_{j-k} \psi_{j+l}^\dagger - \psi_j \psi_{j+k+l}^\dagger. \tag{2.197}$$

Now, there are issues of when this sum is well defined. By definition, in a given Maya diagram basis element, there are only finitely many circles that differ from the configuration in the vacuum state $|0\rangle$. Hence, when $k \neq -l$, the result of applying $[J_k, J_l]$ on this Maya diagram will result in finitely many terms. Therefore we can shift summation index, $j \mapsto j + k$ in (2.197) to find that $[J_k, J_l] = 0$. However, extra care is needed when $k = -l$ since there could be infinitely many terms. To this end, we rewrite (2.197) as

$$[J_k, J_{-k}] = \sum_j : \psi_{j-k} \psi_{j-k}^\dagger - \psi_j \psi_j^\dagger : + \sum_j \theta(k - j) - \theta(-j), \tag{2.198}$$

where $\theta(k - j) = 1$ for $k - j > 0$ and 0 otherwise. The sum of the normally ordered operators

is now well defined and equal to 0 after a change of index. Hence, we have the Heisenberg algebra, $[J_k, J_l] = k\delta_{k,-l}$ and the J_i are commuting bosons.

Then we can define an action on $\mathbb{P}(\mathcal{F})$ given by

$$v \mapsto e^{\sum_{i=1}^{\infty} t_i J_i} v \text{ where } J_i = \sum_{n \in \mathbb{Z}} : \psi_n \psi_{n+i}^\dagger : . \tag{2.199}$$

for $v \in \mathbb{P}(\mathcal{F})$. Again, only finitely many of the t_i should be non zero. This action is well defined due to similar reasons as above.

To define the tau function, we need a way to encode an initial point in the Sato Grassmannian. To this end, define the vector space

$$\mathfrak{gl}(\infty) := \{c + \sum_{i,j} A_{ij} : \psi_i \psi_j^\dagger | c \in \mathbb{C}, \exists N \in \mathbb{N} \text{ st. } A_{ij} = 0 \text{ for } |i - j| > N\}. \tag{2.200}$$

Lemma 2.3.9. The vector space $\mathfrak{gl}(\infty)$ is an infinite dimensional Lie algebra that has a well defined action on $\mathbb{P}(\mathcal{F})$.

It can be thought of as a Lie algebra of infinite matrices that have finitely many non zero entries. Exponentiating such elements yields elements of the Lie group $\mathrm{GL}(\infty)$ which have the general form, up to a constant factor,

$$g = e^{X_1} e^{X_2} \cdots e^{X_m}. \tag{2.201}$$

where $X_i \in \mathfrak{gl}(\infty)$. To construct the tau function, take $w = \mathrm{span}(w_1, w_2, \dots) \in \mathrm{Gr}_{\mathcal{H}_+}(\mathcal{H})$. Via the Plücker map Δ, this element is mapped to $\Delta(w) = [w_1 \wedge w_2 \wedge \cdots]$. We find an element $g \in \mathfrak{gl}(\infty)$ such that $\Delta(w) = g |0\rangle$. Such a g exists since this action is transitive, although it is not necessarily unique. The corresponding element of the Lie group is given by $\hat{g} = e^{\sum_{i,j \in \mathbb{Z}} A_{ij} : \psi_i \psi_j^\dagger :}$ for some coefficients A_{ij}. In this way we have $w = \hat{g} \mathcal{H}_+$ and moreover, $\mathbb{P}(\mathcal{F})$ can now be viewed as a representation of $\mathrm{GL}(\infty)$. Finally, define the function

$$\tau_w(t) = \langle 0| e^{\sum_{i=1}^{\infty} t_i J_i} \hat{g} |0\rangle . \tag{2.202}$$

Although not immediately obvious, it is true that this construction of the function τ does not in fact depend on the choice of g. This means $\tau_w(t)$ is well defined. We then have the following beautiful theorem due to Sato.

Theorem 2.3.10. The function $\tau_w(t)$ defined above satisfies the Hirota bilinear equation and so is a KP tau function.

Relaxing the condition that almost all $t_k = 0$ yields a formal tau function of the KP hierarchy. This interpretation of the tau function as a vacuum expectation value given by (2.202) is extremely useful in random matrix theory. In particular, this definition will be used extensively when considering the external field matrix integral in chapter six.

Chapter 3

Hermitian Matrix Models

Matrix models, that is to say, integrals over certain spaces of matrices, appear in many branches throughout physics and mathematics. Historically, they first arose in probability theory in describing multivariate random data in [32]. It is no surprise, therefore, that matrix models have found application in quantum mechanics and quantum field theory. Matrix models first came into contact with physics when Wigner studied the energy spectra of atomic nuclei. More recently, there have been important developments in the relationships between matrix models, volumes of moduli spaces of Riemann surfaces and Jackiw-Teitelboim (JT) gravity. Interested readers can consult [33] and [34]. The application of matrix models we will mostly concern ourselves with, however, is the use of formal matrix integrals in studying random surfaces. This idea was inspired by 't Hooft [35] and properly established in [36]. To evaluate formal matrix integrals, one often uses the techniques of Wick's theorem and expansions in terms of fatgraphs. Alternatively, given a suitable gauge invariance, one may also reduce a matrix integral to an integral over the eigenvalues. This technique will be vital in chapter six. Additionally, such matrix models exhibit remarkable geometric and integrable properties. Certain correlation functions of Hermitian matrix models parameterise hyperelliptic curves. We give a derivation of this via Ward identities, the method of which is the main tool of the original calculations of this thesis. Moreover, we will encounter two methods to solve matrix models. The first, topological recursion, is perhaps more rigorous

from a mathematical point of view and we will revisit this in the next chapter. The second method involves the use of orthogonal polynomials. This method is also particularly fruitful, both in general and for the purposes of this thesis, as it allows one to establish a relation between matrix models and tau functions as well as corresponding 'continuous' differential constraints.

3.1 Introduction to Quantum Field Theory

In this section we will give a very brief account of some the main ideas in quantum field theory. We start by introducing the choices required to build a field theory as well as some of the essential objects that are to be computed. For our purposes, we will mainly focus on path integrals, partition functions and correlation functions. We will demonstrate these ideas in the simplest possible case by considering a 'zero dimensional quantum field theory'. This serves a dual purpose. The techniques introduced here, namely that of asymptotic expansions, the Faddeev-Popov determinant and the combinatorics of path integrals, will be used extensively when considering matrix models. Moreover, a basic understanding of quantum field theory is critical to fully appreciate conformal field theory which we will develop in chapter four. Here, we mostly follow the exposition of [37].

3.1.1 An Overview of Quantum Field Theory

Recall that a *semi-Riemannian* manifold (M, g) is a smooth manifold M and a metric g which assigns to every point $p \in M$ a semi-inner product (a non degenerate, symmetric, bilinear form) $g_p : T_pM \times T_pM \to \mathbb{R}$.

For any coordinate chart (U, x) about any $p \in M$, we can define the metric components $g_{ij}(p) = g_p\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)$. Note that we can consider $g_{ij}(p)$ as a function $g_{ij}(p) : U \to \mathbb{R}$. We will require that the metric g is smooth so that the function $g_{ij}(p) : U \to \mathbb{R}$ is smooth. Owing to the fact that g_p is a bilinear map from $T_pM \times T_pM \to \mathbb{R}$ we can write the metric in

coordinates as

$$g = g_{ij} dx^i \otimes dx^j. \tag{3.1}$$

It is common to write g as the line element ds^2.

One can also require that g_p is positive definite. In this case we have a *Riemannian manifold.* As an example of a semi-Riemannian manifold from physics, we have a two dimensional Minkowski space $\mathbb{R}^{1,1} = (\mathbb{R}^2, \eta)$ where in the coordinates (t, x) of $\mathbb{R}^2$ the metric is given by $\eta_{ij} = \text{diag}(-1, 1)$. Hence we have

$$ds^2 = -dt^2 + dx^2. \tag{3.2}$$

For the purposes of quantum field theory, it is technically convenient to only consider Riemannian manifolds. This is not too constraining since we can simply use a Wick rotation, $t \mapsto it$ and $x \mapsto x$ so that the above metric becomes

$$ds^2 = dt^2 + dx^2. \tag{3.3}$$

To build a field theory, we first choose a Riemannian manifold (M, g). Particles are described by *fields*, which are simply functions $\phi : M \to N$ where (N, h) is another Riemannian manifold, often called the target space.

We let $\mathcal{C}$ denote the field configuration space so that $\phi \in \mathcal{C}$. This is typically an infinite dimensional function space. The question of what type of functions we allow is indeed a subtle one. It is not clear whether one should consider all continuous functions or even to allow oneself the luxury of restricting attention to smooth functions. The correct approach comes in the form of renormalisation and we refer the reader to [37] for details. In all examples that we will consider in this thesis, it will be apparent what the field configuration space $\mathcal{C}$ is.

As in classical mechanics, we also need to specify an action which in this case is a

functional $S : \mathcal{C} \to \mathbb{R}$. We often take the action to be of the form

$$S[\phi] = \int_M d^d x \sqrt{|g|} \mathcal{L}(\phi(x), \partial\phi(x), \dots). \tag{3.4}$$

where $|g| = \det g_{ij}$ and $d^d x \sqrt{|g|}$ is the invariant volume form. The classical equations of motion for the field are again given by the Euler-Lagrange equations, or more compactly $\delta S = 0$.

The main objects we want to compute are so-called *path integrals*

$$\int_{\mathcal{C}} \mathcal{D}\phi \, e^{-S[\phi]/\hbar}. \tag{3.5}$$

This integral need not always be convergent; the intuitive idea here is that the integral is computed over a possibly infinite dimensional space $\mathcal{C}$ with some measure $\mathcal{D}\phi \, e^{-S[\phi]/\hbar}$ that suppresses discontinuities. What is worse, however, is that the integration measure $\mathcal{D}\phi$ can be difficult to define in general. For example, one might expect the Lebesgue measure to be a natural candidate. It is, nevertheless, impossible to consistently define a Lebesgue measure on an infinite dimensional space. See [37]. Looking ahead, in all examples that we will consider, the configuration space $\mathcal{C}$ will be finite dimensional and this measure is simply the Lebesgue measure. The general problem of trying to rigorously define path integrals is, therefore, an incredibly difficult one. It is in fact what endows quantum field theory with much of its richness; methods such as regularisation and renormalisation have been developed to deal with these issues. In spite of this, we will not address these subtleties, preferring instead to concentrate on a small number of examples where the path integral is precisely defined.

We remark that equation (3.5) is the path integral on a Riemannian manifold so that the metric has Euclidean signature. If we had instead used a Minkowski space with metric $\mathrm{diag}(-1, -1, \dots, 1, 1)$ the exponential factor would be $e^{iS/\hbar}$. In this case, one would have to compute an oscillatory integral over an infinite dimensional space rather than an exponentially decaying one. It is for this reason we choose the manifold M to be Riemannian and

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not just semi-Riemannian.

The fundamental object of a quantum field theory is the *partition function*, denoted as Z and defined by

$$Z := \int_{\mathcal{C}} \mathcal{D}\phi \, e^{-S[\phi]/\hbar}. \tag{3.6}$$

In addition, we often wish to compute the effect of operator insertions

$$\int_{\mathcal{C}} \mathcal{D}\phi \, e^{-S[\phi]/\hbar} \prod_{i=1}^{n} \mathcal{O}_i[\phi], \tag{3.7}$$

where the $\mathcal{O}_i$ are functionals depending on the field ϕ and its derivatives. We often normalise by the partition function, resulting in the correlation functions

$$\Big\langle \prod_{i=1}^{n} \mathcal{O}_i[\phi] \Big\rangle = \frac{1}{Z} \int_{\mathcal{C}} \mathcal{D}\phi \, e^{-S[\phi]/\hbar} \prod_{i=1}^{n} \mathcal{O}_i[\phi], \tag{3.8}$$

so that $\langle 1 \rangle = 1$. It is for this reason that rescalings of the partition function are often ignored. As the notation suggests, we interpret $\frac{\mathcal{D}\phi}{Z} e^{-S[\phi]/\hbar}$ as a normalised probability density so that the correlation functions $\langle \cdots \rangle$ can be thought of as expectation values.

A useful way of calculating correlation functions comes by introducing a source term. Suppose we are given the action of a theory as $S[\phi]$. We consider an action of the form

$$\tilde{S}[\phi] = S[\phi] + \int_{M} d^d x \, J_i(x) \mathcal{O}_i(x), \tag{3.9}$$

where we have employed Einstein summation convention. The source terms $J_i(x)$ are functions on M. Notice that the partition function Z becomes a functional $Z[J]$. Formally, we find that

$$-\hbar \frac{\delta}{\delta J_i(x)} Z[J] = \int_{\mathcal{C}} \mathcal{D}\phi \, e^{-1/\hbar\left(S[\phi] + \int J_k \mathcal{O}_k\right)} \frac{\delta}{\delta J_i(x)} \left(\int_{M} d^d y \, J_k(y) \mathcal{O}_k(y) \right). \tag{3.10}$$

The functional derivative on the right hand side yields the delta function $\delta^{(d)}(x-y)\delta_{ik}\mathcal{O}_k(y)$

and so we find

$$-\hbar \frac{\delta}{\delta J_i(x)} Z[J] = \int_{\mathcal{C}} \mathcal{D}\phi e^{-S[\phi]/\hbar} \mathcal{O}_i(x) = Z \cdot \langle \mathcal{O}_i(x) \rangle. \tag{3.11}$$

Hence this generalises to

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\cdots\mathcal{O}_n(x_n)\rangle = \frac{(-\hbar)^n}{Z} \frac{\delta^n Z}{\delta J_1(x_1)\delta J_2(x_2)\cdots\delta J_n(x_n)}. \tag{3.12}$$

Here, we have coupled the source functions $J_i(x)$ linearly to the local functionals $\mathcal{O}_i$. We can, however, consider non-linear variations. For example, we can vary the action with respect to the metric. This defines the *energy momentum tensor*

$$T_{ij} = -\frac{1}{2} \frac{1}{\sqrt{|g|}} \frac{\delta S}{\delta g^{ij}}. \tag{3.13}$$

Everything that has been discussed thus far has, mathematically, been from the classical field theory point of view. To transition to quantum field theory, one needs to decide on a procedure for quantisation. The standard quantisation procedure in field theory is known as second quantisation. Here, instead of promoting observables to non-commuting operators as is done in quantum mechanics, we promote the fields themselves to non commuting variables. For example, in the case of the boson on the four dimensional Minkowski spacetime $\mathbb{R}^{3,1}$ with Minkowski metric η, the action is given by

$$S = \int d^4x \mathcal{L} \tag{3.14}$$

where the Lagrangian is given as $\mathcal{L} = \frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$. Upon quantisation, we expand $\phi(x)$ in terms of the Fourier modes

$$\phi(x) = \int \frac{d^3p}{(2\pi)^{3/2}} \left(e^{ip_\mu x^\mu} a_p + e^{-ip_\mu x^\mu} a_p^\dagger \right) \tag{3.15}$$

where $[a_p, a_q] = [a_p^\dagger, a_q^\dagger] = 0$ and $[a_p, a_q^\dagger] = (2\pi)^3\delta^{(3)}(p-q)$. Hence a and $a^\dagger$ are interpreted as annihilation and creation operators respectively. As in the previous chapter, we safely

assume the existence of a vacuum state $|0\rangle$ belonging to some Hilbert space and is such that $a_p |0\rangle = 0$ for all p. There is in fact a more precise relationship between this Hilbert space and the boundaries of the Riemannian manifold M, but we will not need this relationship. Discussion of this idea can be found in [37].

As is often the case in quantum field theory, naive treatments of the mathematics often leads to divergences and infinities. For example, one can calculate the Hamiltonian $\mathcal{H}$ associated to the boson defined by equation (3.15) using a Legendre transformation as in classical mechanics. After some algebra, the Hamiltonian reads

$$\mathcal{H} = \frac{1}{2} \int \frac{d^3 p}{(2\pi)^3} a_p a_p^\dagger + a_p^\dagger a_p. \tag{3.16}$$

Employing the canonical commutation relations, we see that

$$\mathcal{H} = \int \frac{d^3 p}{(2\pi)^3} a_p^\dagger a_p + \frac{1}{2} (2\pi)^3 \delta^{(3)}(0). \tag{3.17}$$

To overcome the singularity of evaluating the delta function at zero, it is usual to define normal ordering. This is the same prescription that was used in the last chapter; it is more natural at the quantum level to consider normally ordered operators such as $: \mathcal{H} :$ where all creation operators are moved to the left. For example, $: a_p a_p^\dagger := a_p^\dagger a_p$ in the bosonic case. In the fermionic case, an extra minus sign is picked up since the creation and annihilation operators satisfy anticommutation relations. Thus, in the example above, the normally ordered Hamiltonian becomes

$$: \mathcal{H} := \frac{1}{2} \int \frac{d^3 p}{(2\pi)^3} a_p a_p^\dagger + a_p^\dagger a_p := \int \frac{d^3 p}{(2\pi)^3} a_p^\dagger a_p. \tag{3.18}$$

Having subtracted off the infinity, this avoids the divergences mentioned above.

3.1.2 Wick's Theorem

Before moving to matrix models, it is useful for pedagogical reasons to first investigate the simplest possible quantum field theory: we shall take the manifold M to be just a point. It is here that we shall introduce Wick's theorem and the diagrammatic expansion of path integrals. Since the manifold M is zero dimensional, there is no notion of coordinates or lengths. Thus there is no metric. The simplest fields are scalars $\phi : \{\star\} \to \mathbb{R}$ and the field configuration space $\mathcal{C}$ becomes $\mathbb{R}$. The path integral measure $\mathcal{D}\phi$ becomes the standard Lebesgue measure, $d\phi$, on $\mathbb{R}$. We then choose an action $S(\phi)$ so that the partition function

$$Z = \int_{\mathbb{R}} d\phi \, e^{-S/\hbar} \tag{3.19}$$

converges. One reason the partition function is especially useful is outlined as follows. Suppose the action is even in ϕ. That is to say, $S = \sum_{k=1}^{N} \lambda_k \frac{\phi^{2k}}{(2k)!}$ where the coefficients λ_k are called the coupling constants. Observe that in this case, the integral defining Z converges absolutely. Taking logarithmic derivatives of the partition function and differentiating under the integral we obtain

$$-\hbar \frac{\partial}{\partial \lambda_l} \log Z = -\frac{\hbar}{Z} \frac{\partial Z}{\partial \lambda_l} = \left\langle \frac{\phi^{2l}}{(2l)!} \right\rangle. \tag{3.20}$$

We are permitted to differentiate under the integral sign since the action was chosen so that the partition function converges absolutely. Consequently, we see that the correlation functions of terms appearing in the action can be easily computed from Z. In fact, the quantity $F := -\log Z$ is often called the *free energy* of the theory.

If all the coupling constants are set to zero, except for the quadratic term $\lambda_1 \frac{\phi^2}{2}$, then we say that the action describes a free field. Free fields are particularly useful in that they can be solved easily. In quantum field theory, often a good strategy is to solve the free field case and then treat more sophisticated actions as a perturbation of the free field case. In this situation, we will often write Z_0 for the free field partition function.

To this end, let us investigate the free field case when M is zero dimensional. Consider

a field $\phi : \{\star\} \to \mathbb{R}^n$ so that $\phi = (\phi_1, \ldots, \phi_n)$. We choose the action to be of the form

$$S(\phi) = \frac{1}{2} M_{ab} \phi^a \phi^b, \tag{3.21}$$

where M_{ab} are the entries of an $n \times n$ positive definite, symmetric matrix. Thus, we write

$$Z = \int_{\mathbb{R}^n} d^n \phi \, e^{-M(\phi,\phi)/2\hbar}, \tag{3.22}$$

where $M(\phi, \phi)$ represents the quadratic form defined by the matrix M and $d^n \phi$ is the Lebesgue measure on $\mathbb{R}^n$. By hypothesis, M is symmetric and so can be diagonalised by an orthogonal transformation. Its eigenvectors, furthermore, can be made orthonormal. It is well known that the Lebesgue measure is invariant under orthogonal transformations. Hence, the integral (3.22) 'decouples' into a product of Gaussian integrals

$$Z = \prod_{i=1}^{n} \int_{\mathbb{R}} d\chi \, e^{-m_i \chi^2 / 2\hbar} = \frac{(2\pi\hbar)^{n/2}}{\sqrt{\det M}}, \tag{3.23}$$

where the m_i are eigenvalues of M so that $\det M = \prod_i m_i$.

We now introduce sources as in the previous subsection. Consider the action

$$S = \frac{1}{2} M(\phi, \phi) + J \cdot \phi, \tag{3.24}$$

where $J \cdot \phi = J_a \phi^a$ is the usual Euclidean inner product. Completing the square we have

$$S = \frac{1}{2} M(\tilde{\phi}, \tilde{\phi}) - \frac{1}{2} M^{-1}(J, J), \tag{3.25}$$

where $\tilde{\phi} = \phi + \frac{1}{2} M^{-1}(J, J)$ are translated coordinates. Since the Lebesgue measure is translationally invariant, we find

$$Z(J) = \int_{\mathbb{R}^n} d^n \phi \, e^{-\frac{1}{\hbar}\left(M(\phi,\phi)/2 + J\cdot\phi\right)} = e^{-\frac{1}{2\hbar} M^{-1}(J,J)} \int_{\mathbb{R}^n} d^n \tilde{\phi} \, e^{-M(\tilde{\phi},\tilde{\phi})/2\hbar} = e^{-\frac{1}{2\hbar} M^{-1}(J,J)} Z_0, \tag{3.26}$$

where Z_0 is the free field partition function with no source terms. In other words, $Z(J)\big|_{J=0} = Z_0$. This now allows for efficient computation of the correlation functions. For example, suppose $P(\phi) : \mathbb{R}^n \to \mathbb{R}^n$ is a polynomial. We wish to calculate $\langle P(\phi) \rangle$. Without loss of generality, by linearity we assume $P(\phi) = \prod_{i=1}^{p} l_i(\phi)$ where $l_i(\phi) = l_i \cdot \phi = l_{ij}\phi^j$ for some constants l_{ij}. Thus, the correlation function we wish to compute is

$$\langle l_1(\phi) \cdots l_p(\phi) \rangle = \frac{1}{Z_0} \int d^n\phi \, e^{-M(\phi,\phi)} \prod_{i=1}^{p} l_i(\phi). \tag{3.27}$$

If p is an odd, then the above integrand is an odd function in at least one component of ϕ. The integral must therefore vanish in this case and so we consider the remaining case $p = 2k$. Thus we have

$$\langle l_1(\phi) \cdots l_{2k}(\phi) \rangle = \frac{1}{Z_0} \int_{\mathbb{R}^n} d^n\phi \prod_{i=1}^{2k} l_i(\phi) e^{-M(\phi,\phi)/2\hbar - J \cdot \phi/\hbar} \bigg|_{J=0}. \tag{3.28}$$

Differentiating with respect to J brings down each ϕ factor and so the right hand side is equal to

$$\frac{(-\hbar)^{2k}}{Z_0} \int_{\mathbb{R}^n} d^n\phi \prod_{i=1}^{2k} l_i \cdot \frac{\partial}{\partial J} \left(e^{-M(\phi,\phi)/2\hbar - J \cdot \phi/\hbar} \right) \bigg|_{J=0}. \tag{3.29}$$

In a similar way to the above, the integral is absolutely convergent and so we exchange differentiation and integration which yields

$$\frac{\hbar^{2k}}{Z_0} \prod_{i=1}^{2k} l_i \cdot \frac{\partial}{\partial J} \left(\int_{\mathbb{R}^n} d^n\phi \, e^{-M(\phi,\phi)/2\hbar - J \cdot \phi/\hbar} \right) \bigg|_{J=0} = \frac{\hbar^{2k}}{Z_0} \prod_{i=1}^{2k} l_i \cdot \frac{\partial}{\partial J} Z(J). \tag{3.30}$$

Using equation (3.26), we have the result

$$\langle l_1(\phi) \cdots l_{2k}(\phi) \rangle = \hbar^{2k} \prod_{i=1}^{2k} l_i \cdot \frac{\partial}{\partial J} \left(e^{-\frac{1}{2\hbar} M^{-1}(J,J)} \right) \bigg|_{J=0}. \tag{3.31}$$

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Consider the special case $k = 1$. The above result becomes

$$\langle l_1(\phi) l_2(\phi) \rangle = \hbar M^{-1}(l_1, l_2),\tag{3.32}$$

or equivalently,

$$\langle \phi^a \phi^b \rangle = \hbar (M^{-1})^{ab}.\tag{3.33}$$

Consequently, in the free theory, the two point correlation function is completely determined by the inverse of the quadratic term in the action. In this context, M^{-1} is called the propagator of the theory. This propagator is interpreted as the response of one operator insertion to another. Later on, we will represent this pictorially using a Feynman diagram.

We now return to the general case $k \in \mathbb{N}$. Observe that every derivative $\hbar l_i \cdot \frac{\partial}{\partial J}$ in (3.31) brings down a factor $M^{-1}(l_i, J)$ from the exponential. Since we set $J = 0$, we obtain a non zero term only when exactly half of the derivatives $\hbar l_i \cdot \frac{\partial}{\partial J}$ act on the exponential and when the other half removes the factors $M^{-1}(l_i, J)$. In this way, the end result will be a product of M^{-1} that appear k times that depends on the way the derivatives were paired up. More precisely, let Π_{2k} be the set of possible complete pairings of $\{1, 2, \ldots, 2k\}$. Then the above discussion means that

$$\langle l_1(\phi) \ldots l_{2k}(\phi) \rangle = \hbar \sum_{\sigma \in \Pi_{2k}} \prod_{i \in \{1,2,\ldots,2k\}} M^{-1}(l_i, l_{\sigma(i)}).\tag{3.34}$$

The above result is known as *Wick's theorem*. As we shall see, this is an indispensable tool in the theory of matrix models.

3.1.3 Perturbation Theory and Diagrammatic Expansion

Typically, one cannot analytically evaluate integrals such as

$$I(\hbar) = \int_{\mathbb{R}^n} d^n \phi f(\phi) e^{-S(\phi)/\hbar}.\tag{3.35}$$

We may falsely hope that it has a Taylor expansion about the classical limit $\hbar \to 0$. If this were indeed the case, it would have a small disc $D \subset \mathbb{C}$ centred at the origin as its domain of convergence. The action, nevertheless, was chosen so that the integral converges for $\mathrm{Re}(\hbar) > 0$. The integral (3.35) then cannot possibly converge for $\mathrm{Re}(\hbar) < 0$. Consequently, rather than searching for a Taylor expansion, we instead look for an asymptotic expansion. Recall that the series $\sum_{k=0}^{\infty} a_k \hbar^k$ is asymptotic to a function $I(\hbar)$ if

$$\lim_{\hbar \to 0^+} \frac{1}{\hbar^N} \left| I(\hbar) - \sum_{k=0}^{N} a_k \hbar^k \right| = 0, \tag{3.36}$$

for any $N \in \mathbb{N}$. We often write $I(\hbar) \sim \sum_{k=0}^{\infty} a_k \hbar^k$ to denote this asymptotic equivalence as $\hbar \to 0$.

To obtain an asymptotic expansion of (3.35), we use what is referred to as Laplace's method in the literature, or more generally, the method of steepest descent. For simplicity, suppose that in (3.35) we take $n = 1$ and assume $S(\phi)$ has a unique global minimum at ϕ_0. The result known as Watson's lemma then states, roughly speaking, that the dominant behaviour of (3.35) comes from the contribution at ϕ_0. For a thorough discussion on the asymptotic expansion of integrals, see [38]. Hence we calculate the integral about the point ϕ_0. We let $\epsilon \in \left(0, \frac{1}{2}\right)$ and let

$$A(\hbar) = \int_{\phi_0 - \hbar^{1/2-\epsilon}}^{\phi_0 + \hbar^{1/2-\epsilon}} d\phi \, f(\phi) e^{-S(\phi)/\hbar}. \tag{3.37}$$

Watson's lemma ensures that $A(\hbar) \sim I(\hbar)$ in this one dimensional case. Upon changing variables $\chi = \frac{1}{\sqrt{\hbar}}(\phi - \phi_0)$ we find

$$A(\hbar) = \sqrt{\hbar} \int_{-\hbar^{-\epsilon}}^{\hbar^{-\epsilon}} d\chi \, f(\phi_0 + \chi\sqrt{\hbar}) e^{-S(\phi_0 + \chi\sqrt{\hbar})/\hbar}. \tag{3.38}$$

Taylor expanding f and S to lowest order, and enforcing that $S'(\phi_0) = 0$ yields

$$A(\hbar) \sim \sqrt{\hbar} e^{-S(\phi_0)/\hbar} f(\phi_0) \int_{-\hbar^{-\epsilon}}^{\hbar^{-\epsilon}} d\chi e^{-S''(\phi_0)\chi^2/2}. \tag{3.39}$$

Asymptotically, as $\hbar \to 0^+$ we can replace the limits of integration by $\pm\infty$ with a subdominant error. Thus we have the integral

$$A(\hbar) \sim \sqrt{\hbar} e^{-S(\phi_0)/\hbar} f(\phi_0) \int_{-\infty}^{\infty} d\chi e^{-S''(\phi_0)\chi^2/2}. \tag{3.40}$$

Since ϕ_0 is a global minimum, we have that $S''(\phi_0) > 0$ and so we can easily evaluate this Gaussian integral. Hence, we have the final result that

$$I(\hbar) \sim e^{-S(\phi_0)/\hbar} f(\phi_0) \sqrt{\frac{2\pi\hbar}{S''(\phi_0)}}. \tag{3.41}$$

This is the value of the integral had the action just contained a quadratic term. In other words (3.41) is the free field partition function Z_0. If one were to keep higher order terms in the Taylor expansion, then the above discussion also illustrates that $I(\hbar)$ has an asymptotic expansion in $\hbar$,

$$I(\hbar) \sim e^{-S(\phi_0)/\hbar} f(\phi_0) \sqrt{\frac{2\pi\hbar}{S''(\phi_0)}} \left(1 + \sum_{k=1}^{\infty} a_k \hbar^k\right). \tag{3.42}$$

The above expansion is easily generalised to path integrals with multiple fields and in this case, we obtain

$$I(\hbar) \sim (2\pi\hbar)^{n/2} e^{-S(\phi_0)/\hbar} f(\phi_0) \frac{1}{\sqrt{\det(\partial_i \partial_j S)|_{\phi_0}}} \left(1 + \sum_{k=1}^{\infty} a_k \hbar^k\right). \tag{3.43}$$

As an example, consider the action

$$S(\phi) = \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4. \tag{3.44}$$

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We can then calculate the partition function as

$$Z = \int_{\mathbb{R}} d\phi\, e^{-\frac{1}{\hbar}\left(\frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4\right)} = \int_{\mathbb{R}} d\phi\, e^{-m^2\phi^2/2\hbar} \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{-\lambda}{4!\hbar}\right)^n \phi^{4n}. \qquad (3.45)$$

Truncating the Taylor series at finite N means that we obtain an asymptotic expansion,

$$Z \sim \int_{\mathbb{R}} d\phi\, e^{-m^2\phi^2/2\hbar} \sum_{n=0}^{N} \frac{1}{n!}\left(\frac{-\lambda}{4!\hbar}\right)^n \phi^{4n}. \qquad (3.46)$$

Because this is now a finite sum, we can exchange the sum and integral. Changing variables using $x = \frac{m^2}{2\hbar}\phi^2$ we obtain

$$Z \sim \frac{\sqrt{2\hbar}}{m} \sum_{n=0}^{N} \frac{1}{n!}\left(\frac{-\hbar\lambda}{3!m^4}\right)^n \int_0^{\infty} e^{-x} x^{2n+1/2-1} = \frac{\sqrt{2\hbar}}{m} \sum_{n=0}^{N} \frac{1}{n!}\left(\frac{-\hbar\lambda}{3!m^4}\right)^n \Gamma\left(2n + \frac{1}{2}\right), \qquad (3.47)$$

where we recognise the integral representation of the gamma function. Finally, using the well-known values of the gamma function we find

$$Z \sim \frac{\sqrt{2\hbar}}{m} \sum_{n=0}^{N} \left(\frac{-\hbar\lambda}{m^4}\right)^n \frac{1}{(4!)^n n!} \frac{(4n)!}{4^n(2n)!}. \qquad (3.48)$$

This agrees with Wick's theorem as expected: the factor $\frac{(4n)!}{4^n(2n)!}$ is the number of possible ways of pairing $4n$ elements which correspond to the ϕ insertions.

Let us now investigate this from the perspective of Feynman diagrams. With the action given by $S(\phi) = \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4$ the propagator and vertex are shown in Figure 3.1.

To compute asymptotic expansions, we use Wick's theorem and compute all possible graphs. The graphs in question are constructed by joining the ends of the vertex using the propagator which acts as the edges of the graphs. To calculate the partition function, we use only the graphs where every edge has both ends attached to vertex. We let D_n be the set of all such graphs that are labelled and have n vertices. Here, a labelled graph means that each edge and vertex is labelled and so are distinct. Each graph in D_n must have $2n$ edges. For example, the set D_1 contains three graphs which pictorially represents the three

φ_i —————————————— φ_j

(a)

φ_1

φ_4 ————●———— φ_2

(b)

φ_3
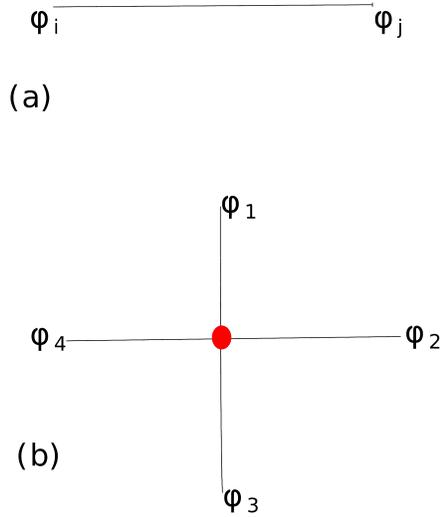
Figure 3.1: Here, (a) is the propagator represented by by a line between two components of the field. Each propagator contributes $-\frac{\hbar}{m^2}$. The diagram (b) is the labelled 4-valent vertex associated to $\langle \phi_1 \phi_2 \phi_3 \phi_4 \rangle$. Each vertex contributes $-\frac{\lambda}{\hbar}$

ways to pair the four ϕ fields.

Therefore, using the propagator and vertex above, each graph belonging to D_n contributes a term proportional to $(-\frac{\hbar\lambda}{m^4})^n$.

However, there is now an issue with overcounting. Joining the vertices in all possible ways implies that D_n contains distinct elements that only differ by labelling, but are otherwise topologically equivalent as unlabelled graphs. For example, the three graphs in Figure 3.2 above in (a) are topologically identical as unlabelled graphs. The partition function is independent of this choice of labelling and as such, we must remove the overcounting.

To this end, notice that the group $G_n = (S_4)^n \rtimes S_n$ acts on D_n by permuting the four fields present in the vertex and the n vertices. Here we emphasise that $\rtimes$ is the semi-direct

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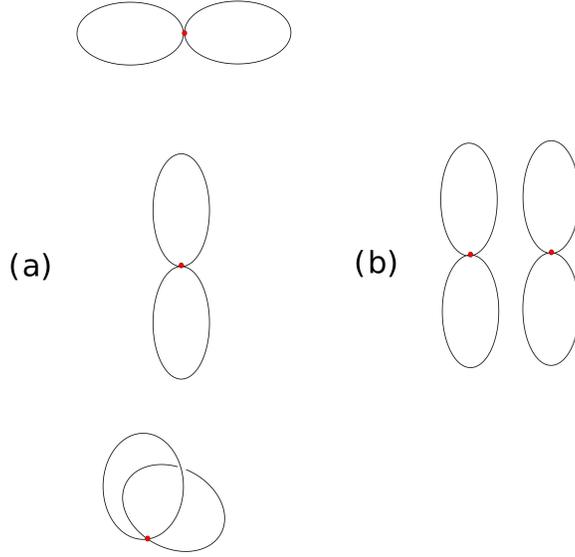
Figure 3.2: Here, (a) is the three graphs in D_1. They are obtained by joining the ends of the vertex with the edge that represents the propagator. We note the top two graphs are called planar graphs since they can be drawn on a flat surface. The third one can be drawn on a torus. The graph in (b) is an element of D_2 that is disconnected.

product rather than the direct product. Thus, summing over the contributions from each graph we find that

$$Z \sim Z_0 \Big(1 + \sum_{n=1}^{N} \big(- \frac{\hbar\lambda}{m^4}\big)^n \frac{|D_n|}{|G_n|}\Big). \qquad (3.49)$$

Going further, there is a more convenient way to calculate the combinatorial factor $\frac{|D_n|}{|G_n|}$. Recall that given the group action G_n on D_n, we can define an orbit Γ as the subset of D_n whose graphs differ only by a relabelling of vertices. Let O_n denote the set of such orbits.

By virtue of the orbit stabiliser theorem we then have that

$$\frac{|D_n|}{|G_n|} = \sum_{\Gamma \in O_n} \frac{1}{\text{Aut } \Gamma}, \tag{3.50}$$

where Aut Γ is the stabiliser of Γ. That is to say, Aut Γ is the automorphism subgroup of G_n that does not change Γ as a labelled graph. For this reason Aut Γ is called the symmetry factor.

Putting this together, we can finally write the asymptotic expansion as

$$\frac{Z}{Z_0} \sim \sum_{n=0}^{N} \left(-\frac{\hbar\lambda}{m^4} \right)^n \sum_{\Gamma \in O_n} \frac{1}{\text{Aut } \Gamma}, \tag{3.51}$$

or alternatively

$$\frac{Z}{Z_0} \sim \sum_{\Gamma} \frac{\hbar^{v(\Gamma)-e(\Gamma)}}{\text{Aut}\Gamma} \frac{(-\lambda)^{v(\Gamma)}}{(m^2)^{e(\Gamma)}}, \tag{3.52}$$

where $v(\Gamma)$ and $e(\Gamma)$ are the number of vertices and edges associated to the graph Γ respectively.

More generally, our theory may involve a different types of fields each associated with a propagator $1/P_a$. Suppose that α labels the type and multiplicity of each field at a particular vertex with coupling constant λ_α. Then the Feynman rules give

$$F(\Gamma) = \prod_{a,\alpha} \frac{(-\lambda_\alpha)^{v_\alpha(\Gamma)}}{(P_a)^{e_\alpha(\Gamma)}}. \tag{3.53}$$

Now, writing $v(\Gamma)$ and $e(\Gamma)$ for the total number of vertices and edges and letting $b = v - e$, we have

$$\frac{Z}{Z_0} \sim \sum_{\Gamma} \frac{1}{|\text{Aut } \Gamma|} \hbar^b F(\Gamma). \tag{3.54}$$

Now, suppose we wished to calculate the free energy

$$W = -\hbar \log Z, \tag{3.55}$$

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which in this case is called the Wilsonian effective action. It turns out that the diagrammatic expansion of W contains only connected Feynman diagrams. Indeed, let $\{\Gamma_j\}$ be the set of all possible connected vacuum graphs we can build using the propagators and vertices of the theory. Define the product $\Gamma_i\Gamma_j$ of two graphs to be their disjoint union. Disconnected graphs are labelled by a set of numbers $\{n_j\}$ that count the number of each graph type Γ_j. The symmetry factor of such a graph is given by

$$|\text{Aut}(\Gamma_1^{n_1}\cdots\Gamma_k^{n_k})| = \prod_{j=1}^{k}(n_j)!|\text{Aut}(\Gamma_j)|^{n_j}. \tag{3.56}$$

The factor of $n_j!$ comes from the symmetry of exchanging any of the n_j copies of the graph Γ_j which leaves the disconnected graph unchanged overall. Now, observe also that

$$F\left(\prod_{j=1}^{k}\Gamma_j^{n_j}\right) = \prod_{j=1}^{k}\left(F(\Gamma_j)\right)^{n_j}, \tag{3.57}$$

and

$$b\left(\prod_{j=1}^{k}\Gamma_j^{n_j}\right) = \sum_{j=1}^{k}n_j b(\Gamma), \tag{3.58}$$

since vertices and propagators contribute multiplicatively to each individual graph. Thus, putting these observations together, we have

$$\frac{Z}{Z_0} \sim \sum_{\Gamma}\frac{1}{|\text{Aut }\Gamma|}\hbar^b F(\Gamma) = \sum_{\{n_j\}}\frac{\hbar^{b\left(\prod_j\Gamma_j^{n_j}\right)}}{\text{Aut}\left((\prod_j\Gamma_j^{n_j})\right)}F\left(\prod_j\Gamma_j^{n_j}\right) \tag{3.59}$$

$$= \sum_{\{n_j\}}\prod_j\frac{\hbar^{n_j b(\Gamma_j)}}{n_j!|\text{Aut}(\Gamma_j)|^{n_j}}\left(F(\Gamma_j)\right)^{n_j} = \prod_j\sum_{n_j=0}^{\infty}\frac{1}{n_j!}\left(\frac{\hbar^{b(\Gamma_j)}}{|\text{Aut}(\Gamma_j)|}\left(F(\Gamma_j)\right)\right)^{n_j} \tag{3.60}$$

$$= \prod_j\exp\left(\frac{\hbar^{b(\Gamma_j)}}{|\text{Aut}(\Gamma_j)|}\left(F(\Gamma_j)\right)\right) = \exp\left(\sum_{\substack{\Gamma\\ \text{connected}}}\frac{\hbar^{b(\Gamma_j)}}{|\text{Aut}(\Gamma_j)|}F(\Gamma_j)\right). \tag{3.61}$$

Thus we find that

$$W \sim W_0 - \hbar \sum_{\substack{\Gamma \\ \text{connected}}} \frac{\hbar^{b(\Gamma_j)}}{|\text{Aut}(\Gamma_j)|} F(\Gamma_j), \tag{3.62}$$

so that the expansion is only over connected diagrams.

3.1.4   The Faddeev-Popov Determinant

Having discussed a perturbative approach to evaluating path integrals, we will now see an exact approach if the path integral in question has 'redundancies'. In what follows, to demonstrate the basic method and idea, we will consider a toy example on the Euclidean plane.

To this end, suppose we have a function $S : \mathbb{R}^2 \to \mathbb{R}$ and assume that it is invariant under the action of $\text{SO}(2)$. That is, S is invariant under rotations about the origin. This rotational invariance implies that $S(x, y) = g(r)$ for some function g and where $r^2 = x^2 + y^2$. Elementary calculus then yields

$$\int_{\mathbb{R}^2} dx dy e^{-S(x,y)/\hbar} = 2\pi \int_{\mathbb{R}_+} dr r e^{-g(r)/\hbar}. \tag{3.63}$$

The factor 2π appears since the original integral is rotationally invariant and, with respect to the Haar measure on $\text{SO}(2)$, we have $2\pi = \text{vol}\big(\text{SO}(2)\big)$. Here $\mathbb{R}_+ = (0, \infty)$ and as a topological space this can be identified with $\mathbb{R}_+ \cong (\mathbb{R}^2 \setminus \{0\})/\text{SO}(2)$.

From the point of view of quantum field theory, the integrand of the right hand side of (3.63) is easier to deal with as the use of the rotational redundancy has been made. However, being a quotient space, $\mathbb{R}_+$ is harder to integrate over than the affine space $\mathbb{R}^2$ that appears on the left hand side of (3.63). Hence, we wish to rewrite the right hand side as integral over $\mathbb{R}^2$ and yet still record the redundancy. The trick for rewriting integrals in this way, first discovered by Faddeev, Popov, Feynman and de Witt, is given as follows.

Suppose that C is a curve emanating from the origin that intersects every circle of constant radius exactly once. More precisely, we let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function and suppose that for all $x \in \mathbb{R}^2$ there exists $R \in \text{SO}(2)$ such that $f(Rx) = 0$. Assume further that f is

non degenerate on the orbit space. In other words, $f(Rx) = f(x)$ if and only if R is the 2×2 identity matrix in SO(2). Given a suitable function f, the curve C can be described by

$$C = \{x \in \mathbb{R}^2 : f(x) = 0\}. \tag{3.64}$$

Of course, a suitable choice for f is also determined by the curve C. The non degeneracy assumption in the definition of f means that f itself is not rotationally invariant. In the quantum field theory formalism, f is called the *gauge fixing function* and the curve C it defines is the *gauge slice*.

Now consider the integral

$$\int_{\mathbb{R}^2} dx\,dy\,\delta(f(x))e^{-S(x,y)/\hbar}. \tag{3.65}$$

Observe that the delta function restricts attention to the gauge slice C. However, the actual value depends on the choice of f for the curve C. For example, if we rescale f by $f \mapsto c \cdot f$ with $c \in \mathbb{R}$, then this defines the same gauge slice but the delta function scales as $\delta(f(x)) \mapsto \frac{1}{|c|}\delta(f(x))$. To rectify this, we introduce the quantity

$$\Delta_f(x) = \frac{\partial}{\partial \theta} f(R_\theta x)\Big|_{\theta=0}. \tag{3.66}$$

Now consider the integral

$$\int_{\mathbb{R}^2} dx\,dy\,|\Delta_f(x)|\delta(f(x))e^{-S(x,y)/\hbar}. \tag{3.67}$$

This integral is now invariant under a rescaling $f \mapsto c(r) \cdot f$ and consequently is independent of the choice of gauge fixing function for the gauge slice C. Moreover, (3.67) is in fact independent of the gauge slice C itself. Indeed, suppose that C_1 and C_2 are two gauge slices defined by the gauge fixing functions f_1 and f_2 respectively. Since C_1 and C_2 intersect every orbit of SO(2) uniquely, we can always map C_1 into C_2 allowing for different rotations at different radii. Thus, we have $f_2(x) \propto f_1(x')$ where $x' = R(r)x$ for some radius dependent

rotation $R(r)$. The proportionality constant here can only depend on r and so using the rescaling invariance, we obtain

$$|\Delta_{f_2}(x)|\delta(f_2(x)) = |\Delta_{f_1}(x')|\delta(f_1(x')). \tag{3.68}$$

The action satisfies $S(x,y) = S(x',y')$ by hypothesis. Finally, one can check that $dxdy = dx'dy'$ explicitly by using the rotation formula

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\alpha(r) & \sin\alpha(r) \\ -\sin\alpha(r) & \cos\alpha(r) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}. \tag{3.69}$$

The integral (3.67) consequently does not depend on the choice of gauge slice C.

To make this more concrete, suppose we chose C to be the zero locus of the function $f(x,y) = y$. In this case we have that $f\big(R(x,y)\big) = y\cos\theta - x\sin\theta$ where R_θ is rotation anticlockwise through θ. Thus we find

$$\Delta_f(x) = \frac{\partial}{\partial\theta}\big(y\cos\theta - x\sin\theta\big)\big|_{\theta=0} = -x \tag{3.70}$$

Hence the integral (3.67) becomes

$$\int_{\mathbb{R}^2} dxdy|x|\delta(y)e^{-S(x,y)/\hbar} = \int_{\mathbb{R}} dx|x|e^{-g(|x|)/\hbar}. \tag{3.71}$$

To see if this agrees with the original integral (3.63), we use the fact that $|x|$ is an even function to write

$$\int_{\mathbb{R}} dx|x|e^{-g(|x|)/\hbar} = 2\int_{\mathbb{R}_+} dxxe^{-g(x)/\hbar}. \tag{3.72}$$

This disagrees with the *radial* part of (3.63) up to the factor of 2. Here, we also need to take account of the fact that the circles intersect $y = 0$ twice. To remedy this we divide by the number of elements in $Z(\mathrm{SO}(2)) = \{\mathbb{1}, -\mathbb{1}\}$, the centre of $\mathrm{SO}(2)$. Thus we can finally write

$$\int_{\mathbb{R}_+} drre^{-g(r)/\hbar} = \frac{1}{|Z(\mathrm{SO}(2))|}\int_{\mathbb{R}^2} dxdy|\Delta_f(x)|\delta(f(x))e^{-S(x,y)/\hbar}. \tag{3.73}$$

Of course in this example, the original integral is trivial to compute. In more sophisticated situations however, rewriting integrals in this way so that we only have to compute integrals over affine spaces with standard measures on $\mathbb{R}^n$ is much simpler.

In general, there may possibly be several gauge fixing functions $f^a(x)$ one for each variable θ^a that parameterises the symmetry group. In this case, we include a factor

$$|\Delta_f(x)| \prod_a \delta(f^a(x)), \tag{3.74}$$

where $\Delta_f(x)$ is the *Faddeev-Popov determinant*,

$$\Delta_f(x) = \det\left(\frac{\partial f^a(R_\theta x)}{\partial \theta^b}\right). \tag{3.75}$$

3.2 Matrix Models as Quantum Field Theories

In the previous section, we heavily focused on the case where the fields ϕ were real valued of the form $\phi : \{\star\} \to \mathbb{R}^n$. Here we will revisit these ideas except that we will now focus on matrix valued fields. Path integrals now become matrix integrals over certain spaces of matrices. These are what are referred to as matrix models. From a physical perspective, this is relevant in certain applications. For example, matrix models can thought of as a zero dimensional Yang-Mills theory as discussed in [39]. Here, we will emphasise the mathematics of Hermitian matrix models where integrals are taken over the space of Hermitian matrices. In principle, we can define many different types of matrix models based on unitary matrices, orthogonal matrices and so on. See [40] for more thorough discussions on this subject. We focus here on Hermitian matrices due to their well-known diagonalisation properties and the result of Brézin, Itzykson, Parisi and Zuber that relates formal Hermitian matrix models to the enumeration of discrete surfaces. We mostly follow [41, 42].

3.2.1 Formal Matrix Integrals

Let H_N be the space of $N \times N$ Hermitian matrices. Let dM be the standard Lebesgue measure on H_N given as

$$dM = \frac{1}{2^{N/2}(\pi t/N)^{N^2/2}} \prod_{i=1}^{N} dM_{ii} \prod_{i<j} d\mathrm{Re}M_{ij} d\mathrm{Im}M_{ij}. \tag{3.76}$$

This measure is normalised so that

$$\int_{H_N} dM e^{-\frac{N}{t}\mathrm{Tr}\, M^2/2} = 1. \tag{3.77}$$

See Appendix A.2.3 for details of this calculation.

As with the previous section, quadratic terms in the action and the corresponding Gaussian integrals will play a crucial role.

The general path integrals we wish to compute are of the form

$$\int_{H_N} dM e^{-\frac{N}{t}\mathrm{Tr}\, V(M)}, \tag{3.78}$$

where $V(M)$ is often referred to as the potential of the matrix model. We will proceed in the same way as we did when considering asymptotic expansions of path integrals. We consider potentials of the form

$$V(M) = \frac{M^2}{2} - \sum_{j=3}^{d} \frac{t_j}{j} M^j. \tag{3.79}$$

We have separated the quadratic term as we interpret the quantity

$$dM e^{-\frac{N}{t}\mathrm{Tr}M^2/2}, \tag{3.80}$$

as a probability measure. Taylor expanding the exponential of the cubic and higher terms,

we define the expectation value

$$A_k := \left\langle \frac{1}{k!} \frac{N^k}{t^k} \Big(\sum_{j=3}^{d} \frac{t_j}{j} \mathrm{Tr}\ M^j \Big)^k \right\rangle_0 = \frac{1}{k!} \frac{N^k}{t^k} \int_{H_N} dM e^{-\frac{N}{t} \mathrm{Tr} M^2/2} \Big(\sum_{j=3}^{d} \frac{t_j}{j} \mathrm{Tr}\ M^j \Big)^k. \qquad (3.81)$$

Here, the subscript zero indicates we are considering expectation values in a Gaussian distribution.

As we saw in the previous section, we do not expect such integrals to converge for all values of t. Hence, we consider asymptotic expansions. In this formal regime, therefore, we may reasonably interchange the integral and sum. Specifically, we have the following lemma.

Lemma 3.2.1. The quantity A_k defined above is given by a polynomial in t,

$$A_k = \sum_{m=k/2}^{[(d-2)k/2]} A_{k,m} t^m. \qquad (3.82)$$

Proof. Observe that by symmetry, an expectation value of an odd monomial vanishes. If the monomial has even degree, $2k$ say, the expectation value is proportion to t^k.

Furthermore, we can decompose $\Big(\sum_{j=3}^{d} \frac{t_j}{j} \mathrm{Tr}\ M^j \Big)^k$ into a finite sum of monomials of the form $\prod_{j=3}^{d} (\mathrm{Tr} M^j)^{n_j}$ where $\sum_{j=3}^{d} n_j = k$. Such a monomial has degree $\sum j n_j$. By the above discussion, this monomial contributes a term proportional to t^m to A_k where

$$m = -k + \frac{1}{2} \sum_{j=3}^{d} j n_j = \frac{1}{2} \sum_{j=3}^{d} (j-2) n_j \geq \frac{1}{2} \sum_{j=3}^{d} n_j = \frac{k}{2}. \qquad (3.83)$$

This gives the desired lower bound for m. Finally, notice that $j \leq d$ and $n_j \leq k$ for all j and hence we have $m \leq (d-2)k/2$. □

Remark. The notion we have defined is that of a formal matrix model. It is also possible to consider convergent matrix models, where one does not in general commute the sum and integral. From the point of view of quantum gravity, formal matrix models are of much greater interest and so we shall exclusively consider the formal integral case.

The previous lemma then allows us to define

$$\tilde{A}_m = \sum_{k=0}^{2m} A_{k,m}. \tag{3.84}$$

It is these quantities $\tilde{A}_m$ which are the integrals arising from the formal exchange of sum and integral in equation (3.81). Hence, we make the following definition.

Definition 3.2.2. The formal integral

$$Z(t) = \int_{H_N} dM\, e^{-\frac{N}{t} \operatorname{Tr} V(M)}, \tag{3.85}$$

is defined as the formal power series

$$Z(t) := \sum_{m=0}^{\infty} \tilde{A}_m t^m, \tag{3.86}$$

in the notations above.

Remark. We have so far only treated Z as a small t asymptotic expansion. However, it is also true that formal matrix integrals always have a $1/N$ expansion as we shall see in a moment.

In summary, formal matrix models $Z(t)$ are given by Taylor expanding the exponential of cubic and higher terms, then subsequently exchanging summation and integration:

$$Z(t) = \prod_{k=3}^{d} \sum_{n_k \geq 0} \int_{H_N} dM \frac{1}{n_k!} \left(\frac{N}{t} \frac{t_k}{k} \operatorname{Tr} M^k \right)^{n_k} e^{-\frac{N}{t} \operatorname{Tr} M^2/2}. \tag{3.87}$$

3.2.2   Wick's Theorem and Matrix Integrals

We recall Wick's theorem that we proved in the previous section.

Theorem 3.2.3. Let A be a positive definite $n \times n$ symmetric matrix. Let $x_1, \ldots, x_n$ be

Gaussian random variables with probability measure

$$d\mu(x_1, \ldots, x_n) = \frac{(2\pi)^{n/2}}{\sqrt{\det A}} e^{-\frac{1}{2}A_{ij}x^i x^j} dx_1 \ldots dx_n. \tag{3.88}$$

Then the propagator $\langle x_i x_j \rangle$ is given by

$$\langle x_i x_j \rangle = (A^{-1})_{ij}. \tag{3.89}$$

Furthermore, the expectation value of an odd number of variables vanishes and

$$\langle x_{i_1} \ldots x_{i_{2m}} \rangle = \sum_{\text{pairings}} \prod_{\text{pairs}(k,l)} (A^{-1})_{i_k i_l}. \tag{3.90}$$

As before, using Wick's theorem we can write expectation values as an expansion in terms of Feynman graphs. An example of a Feynman graph for $\langle x_{i_1}^3 x_{i_2}^5 \rangle$ is shown in Figure 3.3.

There are 105 possible pairings of these vertices. Weighting each of these graphs using the propagator and summing yields $\langle x_{i_1}^3 x_{i_2}^5 \rangle$.

The situation is slightly more complex in the case of matrix models. Here, Feynman graphs will become fat graphs, sometimes known as ribbon graphs. It is therefore instructive to investigate a simple example of the calculation of an expectation value in this context.

Consider the standard Lebesgue measure dM on H_N as before. The Gaussian random variables are $M_{ii}, \text{Re } M_{ij}$ and $\text{Im } M_{ij}$. Consider the expectation value

$$\left\langle \frac{N}{t} \text{Tr } M^4 \right\rangle_0 = \int_{H_N} dM \text{Tr } M^4 e^{-\frac{N}{t} \text{Tr } M^2/2}. \tag{3.91}$$

The quadratic form in this case is given by

$$\frac{N}{t} \sum_{i,j,k,l} \delta_{il}\delta_{jk} M_{ij} M_{kl}. \tag{3.92}$$
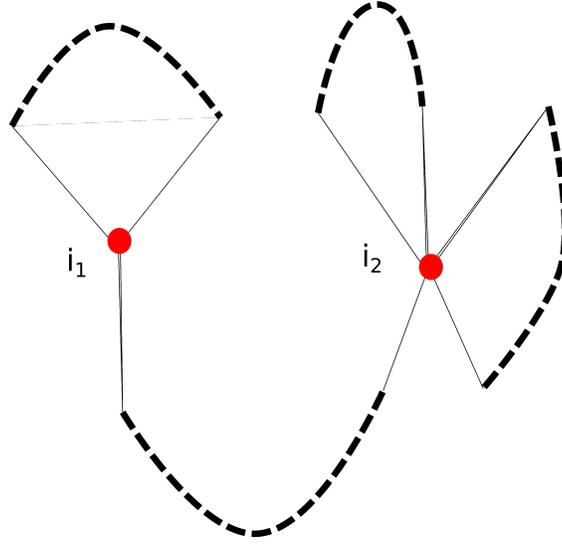
Figure 3.3: The variable $x_{i_1}^3$ is represented by the trivalent vertex i_1 as above and similarly for $x_{i_2}^5$. The solid lines represents the vertices while the dashed line represents the propagator. If the propagator is given as A^{-1}, this graph has weight $(A^{-1})^2_{i_2 i_2}(A^{-1})_{i_1 i_1}(A^{-1})_{i_1 i_2}$.

Hence, the propagator is given by

$$\langle M_{ij} M_{kl} \rangle_0 = \frac{t}{N} \delta_{il} \delta_{kl}. \tag{3.93}$$

Since we wish to calculate $\langle \frac{N}{4t} \mathrm{Tr}\ M^4 \rangle_0$, we consider 4-valent graphs. Now, however, we have two indices for each random variable, M_{ij}. We thus represent the propagator and vertex using fatgraphs as shown in Figure 3.4.
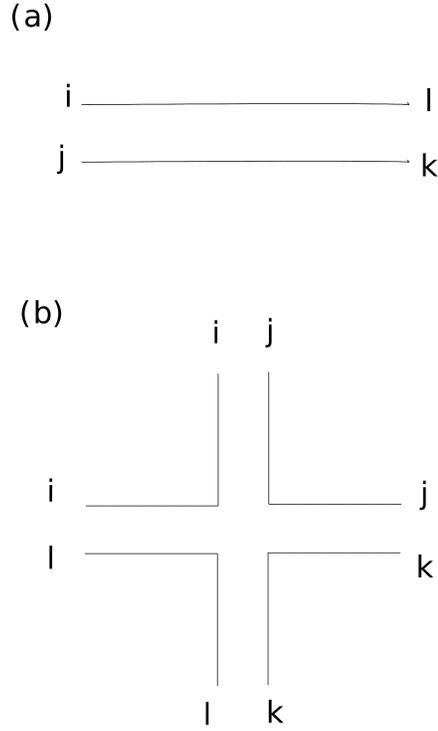
Figure 3.4: Here, (a) is the propagator that act as edges in the fatgraph and (b) is the 4-valent vertex. The fatgraphs are obtained as previously by using the propagator to join the ends of the vertex in all possible ways without twisting the edges.

Using Wick's theorem, we then calculate that

$$\left\langle \frac{N}{4t} \text{Tr } M^4 \right\rangle_0 = \frac{N}{4t} \left\langle \sum_{i,j,k,l} M_{ij} M_{jk} M_{kl} M_{li} \right\rangle_0 \tag{3.94}$$

$$= \frac{N}{4t} \sum_{i,j,k,l} \langle M_{ij} M_{jk} \rangle_0 \langle M_{kl} M_{li} \rangle_0 + \langle M_{ij} M_{li} \rangle_0 \langle M_{jk} M_{kl} \rangle_0 + \langle M_{ij} M_{kl} \rangle_0 \langle M_{jk} M_{li} \rangle_0. \tag{3.95}$$

Substituting the propagator, and performing the sum over indices, we find that

$$\left\langle \frac{N}{4t} \text{Tr } M^4 \right\rangle_0 = \frac{tN^2}{2} + \frac{tN^0}{4}. \tag{3.96}$$

Diagramatically, the first term in the above sum corresponds to the graph shown in Figure

3.5.
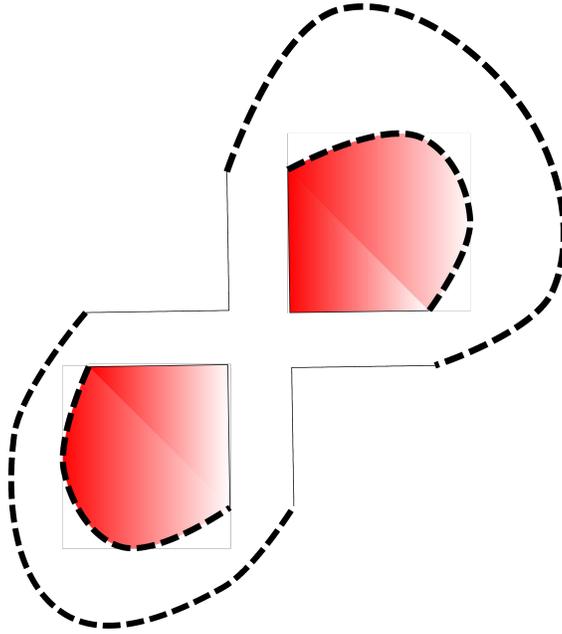
Figure 3.5: As above, the solid and dashed lines represent the vertex and propagator respectively. The red shaded region corresponds to the faces of the graph.

The other graphs are obtained similarly.

Moving to generalise this example, we note that each propagator, and thus each edge e, in a given graph contributes a factor of $\frac{1}{N}$ to the expectation. Furthermore, the factor of N in the exponential contributes another factor of N per vertex v. Finally, due to the form of the propagator, summing over the indices in the above calculation produces a factor of N for each single line the fat graph. The number of single lines is in fact the number of faces f of the graph. Recall that the number of faces of a graph Γ is the number of connected components of the complement when drawn on a surface. Consequently, each face of the

graph contributes a factor N. Hence, for a given graph the total N dependence is given by

$$N^{v-e+f} = N^{\chi}, \tag{3.97}$$

where χ is the Euler characteristic of the given graph. This method for determining the N dependence is often called the set of Feynman rules for the Feynman graphs described. The Euler characteristic of a given graph Γ is also given by $\chi = 2 - 2g$ where g is the smallest genus of a surface Σ such that Γ can be drawn on Σ. This is the essence of the next theorem.

Theorem 3.2.4. Let Z_0 denote the Gaussian Hermitian matrix model

$$Z_0 = \int_{H_N} dM e^{-N\operatorname{Tr} M^2/2}. \tag{3.98}$$

Then the expectation values of this matrix model are given by

$$\frac{1}{\prod_j m_j!} \left\langle \prod_{k=1}^n N \frac{\operatorname{Tr} M^{p_k}}{p_k} \right\rangle_0 = \sum_{\substack{\text{Fatgraphs } \Gamma, \\ n \text{ vertices of valency } p_k}} \frac{1}{|\operatorname{Aut} \Gamma|} N^{\chi(\Gamma)}. \tag{3.99}$$

where m_j is the cardinality of the set $\{k : p_k = j\}$.

Remark. The matrix model defined by Z_0 differs slightly to the previous formal matrix integrals we have considered in that here we have set $t = 1$. From a combinatorial point of view, the variable t counts edges of the graph Γ. This is irrelevant for our purposes; we will mostly be concerned with the existence of a $1/N$ expansion. Henceforth, we shall set $t = 1$ unless otherwise specified.

We note here that the sum above is over unlabelled and possibly disconnected fat graphs. If we briefly consider relabellings of a labelled fat graph, we see that each p_k-valent vertex has p_k rotations of the indices as relabellings. Furthermore, vertices of the same valence can be permuted. The order of the group of relabellings is therefore

$$\prod_{k=1}^m p_k \prod_j m_j! \tag{3.100}$$

Theorem 3.2.4 then follows from the above discussion and an application of the orbit stabiliser theorem in analogy with the previous section. Recall that the partition function of a formal matrix model with potential $V(M)$ is defined as a formal series whose coefficients are Gaussian correlation functions. Hence, upon comparing with the above theorem, we obtain the following result as a corollary.

Corollary 3.2.5. Let $Z(t_3, t_4, \ldots, t_d)$ denote the formal matrix model

$$Z(t_3, t_4, \ldots, t_d) = \int_{H_N} dM e^{-N\mathrm{Tr}\left(M^2/2 + t_3 M^3/3 + \cdots + t_d M^d/d\right)}. \tag{3.101}$$

Then

$$Z(t_3, t_4, \ldots, t_d) = \sum_{\Gamma} \frac{1}{|\mathrm{Aut}\ \Gamma|} N^{\chi(\Gamma)} t_3^{n_3(\Gamma)} \ldots t_d^{n_d(\Gamma)}, \tag{3.102}$$

where $n_k(\Gamma)$ is the number of k-valent vertices in the graph Γ.

In combinatorics, there is a bijection between fat graphs and maps. Here, the term 'map' means the dual of fat graph. Heuristically, gluing vertices using propagators to construct a fatgraph Γ is equivalent to gluing the corresponding k-gons at their sides. This constructs the map Σ corresponding to Γ. In other words, maps are discretised surface whose faces are n-gons. This is illustrated in Figure 3.6.
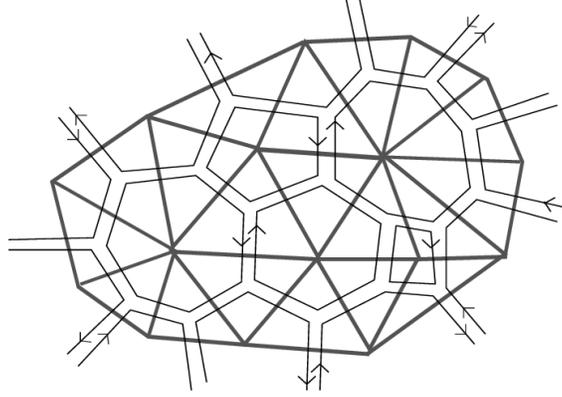
Figure 3.6: The duality between fat graphs and maps. Gluing fat graphs together and taking duals results in a discretised surface. This is what we refer to as a map. The Euler characteristic of the map is then precisely $\chi = 2 - 2g$ where g is the genus of the discrete surface. In this diagram, the surface is discretised into triangles but we can in general discretise a surface using any n-gon [43].

In the duality of graphs Γ and maps Σ, the vertices of Γ become faces of Σ, the faces of Γ become vertices of Σ, and the edges of Γ become edges of Σ. Thus, we can rephrase Theorem 3.2.4 in the following way.

Theorem 3.2.6. The expectation values of the Gaussian Hermitian matrix model are of the form

$$\frac{1}{\prod_j m_j!} \left\langle \prod_{k=1}^{n} N \frac{\text{Tr } M^{p_k}}{p_k} \right\rangle_0 = \sum_{\substack{\text{Maps } \Sigma \\ n, \, p_k- \text{ gons}}} \frac{1}{|\text{Aut } \Sigma|} N^{\chi(\Sigma)}, \tag{3.103}$$

where m_j is the cardinality of the set $\{k : p_k = j\}$.

Corollary 3.2.7. Let $Z(t_3, t_4, \ldots, t_d)$ denote the formal matrix model

$$Z(t_3, t_4, \ldots, t_d) = \int_{H_N} dM e^{-N \text{Tr} \left(M^2/2 + t_3 M^3/3 + \cdots + t_d M^d/d \right)}. \tag{3.104}$$

Then

$$Z_N(t; t_3, t_4, \ldots, t_d) = \sum_{\text{Closed maps } \Sigma} \frac{1}{|\text{Aut } \Sigma|} N^{\chi(\Sigma)} t_3^{n_3(\Sigma)} \ldots t_d^{n_d(\Sigma)}, \tag{3.105}$$

where $n_k(\Sigma)$ is the number of k-gons in the map Σ.

Here, it is important that the map Σ is closed: this is natural from the perspective of closed string theory where we consider compact Riemann surfaces, possibly with punctures. In addition, we observe that the contribution of disconnected maps is the product of contributions from its connected components. We have seen this is the previous section in the context of the Wilsonian effective action. Therefore, if we take the logarithm of Z, this in fact only enumerates connected maps.

Corollary 3.2.8. Let $Z(t_3, t_4, \ldots, t_d)$ denote the formal matrix model as above. Then the free energy is given by

$$F = \log Z_N = \sum_{\text{Closed, connected } \Sigma} \frac{1}{|\text{Aut } \Sigma|} N^{\chi(\Sigma)} t_3^{n_3(\Sigma)} \ldots t_d^{n_d(\Sigma)}. \tag{3.106}$$

Often, we will only be interested in correlation functions that involve sums over connected maps. This is denoted by $\langle \cdots \rangle_c$ so that

$$\frac{1}{\prod_j m_j!} \Big\langle \prod_{k=1}^n N \frac{\text{Tr } M^{p_k}}{p_k} \Big\rangle_c = \sum_{\substack{\text{Connected } \Sigma \\ n \; p_k - \text{ gons}}} \frac{1}{|\text{Aut } \Sigma|} N^{\chi(\Sigma)}. \tag{3.107}$$

3.2.3 Reduction to Eigenvalues

In the previous section, we evaluated formal matrix integrals using perturbative methods such as those described in the beginning of this chapter in the overview of quantum field theory. We also recall that in the overview we saw how we could use Faddeev-Popov techniques to exploit redundancies in the integral and to evaluate such integrals. Here, we will apply those methods of Faddeev-Popov determinants to formal matrix integrals. We follow the exposition of [44].

To employ the Faddeev-Popov technique, we first need to establish gauge invariance of the integral. To this end, we have the following lemma.

Lemma 3.2.9. Consider the map $H_N \to H_N$ given by

$$M \mapsto M' = U^\dagger M U, \tag{3.108}$$

for any fixed $U \in U(N)$. Then the standard Lebesgue measure

$$dM = \prod_{i=1}^{N} dM_{ii} \prod_{i<j} d\mathrm{Re}M_{ij} d\mathrm{Im}M_{ij}. \qquad (3.109)$$

is invariant under this map. That is

$$dM = dM'. \qquad (3.110)$$

Proof. First observe that Tr M^2 is invariant under conjugation. That is,

$$\mathrm{Tr}\ M^2 = \mathrm{Tr}\ (M')^2. \qquad (3.111)$$

We now calculate Tr M^2 in a more convenient form. Indeed, we have

$$\mathrm{Tr}\ M^2 = \sum_{i,j} M_{ij} M_{ji} = \sum_{i,j} M_{ij} M_{ij}^*, \qquad (3.112)$$

where we utilised the fact that M is Hermitian. Now, this can be rewritten as

$$\sum_{i,j} M_{ij} M_{ij}^* = \sum_{i,j} (\mathrm{Re}\ M_{ij})^2 + (\mathrm{Im}\ M_{ij})^2 = \sum_{i} M_{ii}^2 + 2 \sum_{i<j} (\mathrm{Re}\ M_{ij})^2 + (\mathrm{Im}\ M_{ij})^2. \qquad (3.113)$$

In this form, the equality (3.111) becomes

$$\sum_{i} M_{ii}^2 + 2 \sum_{i<j} (\mathrm{Re}\ M_{ij})^2 + (\mathrm{Im}\ M_{ij})^2 = \sum_{i} (M_{ii}')^2 + 2 \sum_{i<j} (\mathrm{Re}\ M_{ij}')^2 + (\mathrm{Im}\ M_{ij}')^2. \qquad (3.114)$$

This identity can be rewritten in the following way. Let

$$\mathbf{M} = (M_{11}, \ldots, M_{nn}, \mathrm{Re}\ M_{12}, \ldots, \mathrm{Re}\ M_{n-1,n}, \mathrm{Im}\ M_{12}, \ldots, \mathrm{Im}\ M_{n-1,n}) \in \mathbb{R}^{n^2}, \qquad (3.115)$$

and similarly for $\mathbf{M}'$. Let $D = \mathrm{diag}(1, \ldots, 1, 2, \ldots, 2)$ where 1 appears n times and 2 appears

$n(n-1)$ times. Then (3.114) can be written as

$$\langle \mathbf{M}, D\mathbf{M} \rangle = \langle \mathbf{M}', D\mathbf{M}' \rangle, \tag{3.116}$$

where $\langle \cdot, \cdot \rangle$ is the standard Euclidean inner product on $\mathbb{R}^{n^2}$. Now, let V be the matrix such that

$$\mathbf{M}' = V\mathbf{M}. \tag{3.117}$$

Then equation (3.116) reads

$$\langle \mathbf{M}, D\mathbf{M} \rangle = \langle \mathbf{M}', D\mathbf{M}' \rangle = \langle V\mathbf{M}, DV\mathbf{M} \rangle = \langle \mathbf{M}, V^T DV\mathbf{M} \rangle. \tag{3.118}$$

It then follows that

$$D = V^T DV. \tag{3.119}$$

Since $\det D \neq 0$, this implies

$$|\det V| = 1. \tag{3.120}$$

Finally, the Jacobian of the transformation $\mathbf{M} \mapsto \mathbf{M}'$ is V itself and so we have

$$dM = dM', \tag{3.121}$$

as desired. $\qquad \square$

The above lemma shows that the Lebesgue measure dM is $\mathrm{U}(N)$ invariant. In addition, the matrix models we are considering have integrands of the form $\mathrm{Tr}\ M^k e^{-N\mathrm{Tr}\ M^2/2}$. By virtue of the cyclicity of the trace, these integrands are also invariant under conjugation by any element in $\mathrm{U}(N)$. Hence, there are redundancies in the theory and so we use the method of Faddeev-Popov determinants.

Suppose that $f(M)$ is a $\mathrm{U}(N)$ gauge invariant function of a Hermitian matrix M. Consider the integral

$$I = \int dM f(M). \tag{3.122}$$

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We note that since M is Hermitian, it can be diagonalised by some unitary matrix U. Denote this diagonal matrix by Λ and the eigenvalues by λ_i. We consider the gauge fixing condition $\delta^{(N^2)}(U\Lambda U^\dagger - M)$. Therefore we have

$$1 = \int \prod_{i=1}^{N} d\lambda_i dU \delta^{(N^2)}(UMU^\dagger - \Lambda)\Delta^2(\lambda), \tag{3.123}$$

where $\Delta^2(\lambda)$ is the Faddeev-Popov determinant and dU is the Haar measure, normalised so that $\int dU = 1$. See Appendix A.2.1 for a discussion of the Haar measure as well as its crucial property of left and right invariance. Substituting this into (3.122) yields

$$I = \int dM \prod_{i=1}^{N} d\lambda_i dU f(M)\delta^{(N^2)}(UMU^\dagger - \Lambda)\Delta^2(\lambda). \tag{3.124}$$

Integrating over M, we obtain

$$I = \int \prod_{i=1}^{N} d\lambda_i dU f(\lambda_i)\Delta^2(\lambda). \tag{3.125}$$

Integrating over U simply yields a constant factor of 1 since the integrand does not depend on U. Hence we have

$$I = \int \prod_{i=1}^{N} d\lambda_i f(\lambda_i)\Delta^2(\lambda). \tag{3.126}$$

To calculate $\Delta(\lambda)$ we consider infinitesimal transformations by writing $U = e^A$ for some A in the Lie algbra of anti-Hermitian matrices $\mathfrak{u}(N)$. We consider the gauge fixing function $F(M) = U\Lambda U^\dagger - M = 0$. Then recalling the discussion in the previous section we have

$$\Delta^2(\lambda) = \det\left(\frac{\delta F(M)}{\delta A}\right)\Big|_{F=0}. \tag{3.127}$$

To calculate the derivative, we expand $F(M)$ to first order in A where $U = e^A$. Using the

Baker-Campbell-Hausdorff formula, we find that

$$(e^A \Lambda e^{-A} - M)_{ij} = ([A, \Lambda] + \cdots)_{ij} = A_{ij}(\lambda_j - \lambda_i) + \cdots \tag{3.128}$$

Note that we only consider the off-diagonal $i \neq j$ elements since $A_{ii} = 0$. Another way of seeing this is that the off-diagonal elements are the unphysical degrees of freedom corresponding to the redundancies in the original integral. Hence, we have

$$\Delta^2(\lambda) = \prod_{i \neq j}(\lambda_i - \lambda_j) = (-1)^{N(N-1)/2} \prod_{i<j}^{N}(\lambda_i - \lambda_j)^2. \tag{3.129}$$

We recognise this as the square of the Vandermonde determinant

$$\Delta(\lambda) = \prod_{i<j}(\lambda_i - \lambda_j) = \det\left([1, \lambda, \lambda^2, \ldots, \lambda^{N-1}]\right), \tag{3.130}$$

where λ is column vector whose i^{th} component is λ_i. The sign in (3.129) is superfluous since rescalings of partition functions are irrelevant. Therefore, we have reduced the original formal matrix integral to an integral over the eigenvalues only,

$$\int dM f(M) = \int \prod_{i=1}^{N} d\lambda_i f(\lambda_i) \Delta^2(\lambda). \tag{3.131}$$

One may alternatively view (3.131) as a change of variables from M_{ij} to λ_i and U_{ij} according to $M = U\Lambda U^\dagger$, where U is a unitary matrix and $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_N)$. One can show that the Jacobian of this transformation is then precisely the square of the Vandermonde determinant and the integral over the normalised 'angular' variables dU again gives 1. See Appendix A.2.2 for the technical details.

3.3 Solution of Matrix Models I - Loop Equations

In this section, we show how Riemann surfaces and spectral curves play a crucial role in the theory of matrix models. This will be done first by saddle point approximations. While not entirely rigorous, this method does give a simple intuitive way of computing the leading order of the large N expansion of matrix models. In doing this, we will also exhibit the so-called one cut solution. This will be the solution that we will focus on for the remaining sections in this chapter. To compute the higher order corrections, we will show how to derive to the full loop equations for the matrix model. We will revisit loop equations and spectral curves in the context of the external field matrix model in chapter six. Here, we will follow the expositions of [41, 45].

3.3.1 Saddle Point Approximation

We first consider a matrix model of the form

$$Z = \int_{H_N} dM e^{N^2 \operatorname{Tr} \tilde{V}(M)}, \tag{3.132}$$

where $\tilde{V}(M) = -\frac{1}{N} V(M)$ for some potential $V(M)$. Notice that this is simply a redefinition of a matrix model with potential $V(M)$ that we considered previously. Reducing this integral to an integral over eigenvalues yields

$$Z = \int \prod_{i=1}^{N} d\lambda_i e^{N^2 V_{\text{eff}}(\lambda)}, \tag{3.133}$$

where the effective potential is given by

$$V_{\text{eff}}(\lambda) = -\frac{1}{N} \sum_{j=1}^{N} V(\lambda_j) + \frac{2}{N^2} \sum_{i<j} \log |\lambda_i - \lambda_j|. \tag{3.134}$$

Now, in the large N limit, the effective potential is $O(1)$. Therefore, in the integral (3.133) we can consider N^2 as playing the role of $\hbar^{-1}$. It is important to note here that we are

allowing N to vary whilst keeping the coupling constants t_k fixed. In the next section, we will consider the case where both t_k and N vary simultaneously.

As we have seen previously, the leading order behaviour as $N \to \infty$ will be controlled by the saddle point of the effective potential. We assume for simplicity that there is only one saddle point.

The saddle point in this case is the solution of the system of N equations $\frac{\partial V_{\text{eff}}}{\partial \lambda_i} = 0$ for $i = 1, \ldots N$. The solution $(\lambda_1, \ldots, \lambda_N)$ satisfies

$$\frac{1}{2} V'(\lambda_i) = \frac{1}{N} \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j}. \tag{3.135}$$

We introduce the discrete resolvent

$$\omega(x) = \frac{1}{N} \text{Tr} \frac{1}{M - x} = \frac{1}{N} \sum_{j=1}^{N} \frac{1}{\lambda_j - x}. \tag{3.136}$$

We now observe that

$$\omega^2(x) - \frac{1}{N} \omega'(x) = \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{1}{(\lambda_i - x)(\lambda_j - x)} - \frac{1}{N^2} \sum_{j=1}^{N} \frac{1}{(\lambda_j - x)^2} \tag{3.137}$$

$$= \frac{1}{N^2} \sum_{\substack{i,j \\ j \neq i}} \frac{1}{(\lambda_i - x)(\lambda_j - x)} = \frac{1}{N^2} \sum_{\substack{i,j \\ i \neq j}} \left(\frac{1}{\lambda_i - x} - \frac{1}{\lambda_j - x} \right) \frac{1}{\lambda_j - \lambda_i}. \tag{3.138}$$

Now, using the symmetry under the interchange of i and j, we find that the right hand side of the above can be written as

$$\frac{2}{N^2} \sum_{i=1}^{N} \frac{1}{\lambda_i - x} \sum_{j \neq i}^{N} \frac{1}{\lambda_j - \lambda_i}. \tag{3.139}$$

Recalling the saddle point condition, this can be rewritten as

$$-\frac{1}{N}\sum_{i=1}^{N}\frac{V'(\lambda_i)}{\lambda_i - x} = -\frac{1}{N}\sum_{i=1}^{N}\frac{V'(x)}{\lambda_i - x} - \frac{1}{N}\sum_{i=1}^{N}\frac{V'(\lambda_i) - V'(x)}{\lambda_i - x} \tag{3.140}$$

$$= -V'(x)\omega(x) - \frac{1}{N}\sum_{i=1}^{N}\frac{V'(\lambda_i) - V'(x)}{\lambda_i - x}, \tag{3.141}$$

and so we thus arrive at the equation

$$\omega^2(x) - \frac{1}{N}\omega'(x) = -V'(x)\omega(x) - \frac{1}{N}\sum_{i=1}^{N}\frac{V'(\lambda_i) - V'(x)}{\lambda_i - x}. \tag{3.142}$$

To take the large N limit, we introduce the distribution of eigenvalues,

$$\rho(\lambda) = \frac{1}{N}\left\langle \text{Tr } \delta(\lambda - M)\right\rangle = \frac{1}{N}\sum_{i=1}^{N}\delta(\lambda - \lambda_i). \tag{3.143}$$

This allows us to rewrite terms according to the standard rule

$$\frac{1}{N}\sum_{i=1}^{N}f(\lambda_i) \xrightarrow{N\to\infty} \int d\lambda f(\lambda)\rho(\lambda). \tag{3.144}$$

We now take the large N limit of (3.142) and look at $O(1)$ terms. In this regime, we drop the $-\frac{1}{N}\omega'(x)$ term. At leading order, we denote $\omega(x)$ as $\omega_0(x)$. In view of the expansion of $\omega(x)$ in terms of graphs, $\omega_0(x)$ is often called the genus zero part, or the planar limit. Thus, (3.142) becomes

$$\omega_0^2(x) + V'(x)\omega_0(x) + \frac{1}{4}P(x) = 0, \tag{3.145}$$

where

$$\omega_0(x) = \int d\lambda\rho(\lambda)\frac{1}{\lambda - x}, \tag{3.146}$$

and

$$P(x) = 4\int d\lambda\rho(\lambda)\frac{V'(x) - V'(\lambda)}{x - \lambda}. \tag{3.147}$$

Then (3.145) is solved as

$$\omega_0(x) = \frac{1}{2}\left(-V'(x) \pm \sqrt{\left(V'(x)\right)^2 - P(x)}\right), \tag{3.148}$$

which defines a hyperelliptic curve. Indeed, defining

$$y = V'(x) - 2\omega_0(x), \tag{3.149}$$

and after choosing the negative root in equation (3.148), we find

$$y^2 = \left(V'(x)\right)^2 - P(x). \tag{3.150}$$

Equation (3.150) is now certainly a hyperelliptic Riemann surface. We remark here that if $V(x)$ is a polynomial of degree n, then $P(x)$ is a polynomial of degree $n-2$. This means there are $2(n-1)$ roots of the polynomial $(V'(x))^2 - P(x)$ and so $\omega_0(x)$ has $2(n-1)$ branch points. Therefore there are $n-1$ cuts along which ω_0 has a discontinuity. To make this explicit, consider the point p lying on a cut $\mathcal{C}$ on the real axis. Then observe that by deforming contours, we have

$$\omega_0(p - i\epsilon) = \int_{\mathbb{R}} d\lambda \frac{\rho(\lambda)}{\lambda + i\epsilon - p} = \int_{\mathbb{R} - i\epsilon} \frac{\rho(\lambda)}{\lambda - p} \tag{3.151}$$

Making a small semi-circular indentation C_ϵ around the point $\lambda = p$ that is traversed anti-clockwise in the lower half plane, we find

$$\omega_0(p - i\epsilon) = \mathrm{P}\int_{\mathbb{R}} \frac{\rho(\lambda)}{\lambda - p} + \int_{C_\epsilon} \frac{\rho(\lambda)}{\lambda - p} \tag{3.152}$$

The second term is readily evaluated as a residue (using the indentation lemma for example) and we obtain

$$\omega_0(p - i\epsilon) = \mathrm{P}\int_{\mathbb{R}} \frac{\rho(\lambda)}{\lambda - p} - i\pi\rho(p) \tag{3.153}$$

Similarly, one finds

$$\omega_0(p + i\epsilon) = \int_{\mathbb{R}+i\epsilon} \frac{\rho(\lambda)}{\lambda - p} = \mathrm{P} \int_{\mathbb{R}} \frac{\rho(\lambda)}{\lambda - p} + i\pi\rho(p) \tag{3.154}$$

Hence, we have the equation

$$\rho(\lambda) = \frac{1}{2\pi i}\big(\omega_0(p + i\epsilon) - \omega_0(p - i\epsilon)\big) \tag{3.155}$$

and so the discontinuity across the cut $\mathcal{C}$ is clear. Equation (3.155) is often called the Sokhotski-Plemelj theorem in the literature. We remark that while this saddle point method is not rigorous, (3.155) has been extensively and precisely studied as a Riemann-Hilbert problem. See [44] and references therein for more details.

In general, these cuts are centered around the extrema of V. Therefore, since we considered the case of a unique extremum, the solution (3.148) is often referred to as the one-cut solution. This corresponds to only considering Gaussian matrix models, exactly as we did in the previous section. The solutions involving multiple extrema of the potential, and therefore multiple cuts, are called multi-cut solutions. We do not consider these cases here and instead refer the reader to the literature.

3.3.2 Loop Equations and Virasoro Constraints

In the previous section, we derived the leading order behaviour of the spectral curve associated to the Hermitian matrix model using a naive saddle point technique. Here, we will rederive this spectral curve using loop equations. From the perspective of quantum field theory, the loop equations that are obeyed by matrix models are nothing but the quantum mechanical equations of motion for that theory. We will also see that loop equations naturally give rise to an infinite set of differential equations for the partition function. In this context, these differential equations are called Virasoro constraints. We mostly follow [41, 46, 47].

Consider the integral

$$Z = \int_{H_N} dM e^{-N\mathrm{Tr}\,V(M)}. \tag{3.156}$$

Suppose that this integral is convergent. It must be invariant under a change of variables $M \mapsto M + \epsilon\delta(M)$ as $\epsilon \to 0$. In the language of quantum field theory, this symmetry is often referred to as a Ward identity. Thus we obtain

$$\int_{H_N} dM e^{-N\mathrm{Tr}\, V(M)} = \int_{H_N} d(M + \epsilon\delta(M)) e^{-N\mathrm{Tr}\, V(M+\epsilon\delta(M))}. \tag{3.157}$$

Expanding this to first order and setting $d(\delta(M)) = J(M)dM$, with $J(M)$ as the Jacobian of this transformation, we have

$$N\langle\mathrm{Tr}\, \left(V'(M)\delta(M)\right)\rangle = \langle J(M)\rangle. \tag{3.158}$$

The notation $V'(M)$ here means one should differentiate $V'(M)$ naively as if it were a function of a single variable. For example, if $V(M) = M^2$, then $V'(M) = 2M$. This is is indeed the correct definition of matrix differentiation in this specific case in order to make the Taylor expansion to first order

$$\mathrm{Tr}\, V\left(M + \epsilon\delta(M)\right) = \mathrm{Tr}\, V(M) + \epsilon\mathrm{Tr}\, \left(V'(M)\delta(M)\right) + \mathcal{O}(\epsilon^2). \tag{3.159}$$

Now, equation (3.158) is valid for any linear combinations of convergent Gaussian integrals. Consequently, the same loop equations also hold for formal matrix integrals. We also observe that from equation (3.158), the loop equations are nothing but integration by parts under the assumption the integrand vanishes at the boundary. This is an example of a Schwinger-Dyson equation in the more general context of quantum field theory.

To calculate the Jacobian factor $J(M)$ in practice, there are several rules one can follow.

1. For $\delta(M) = A$, a constant matrix, $J(A) = 0$.

2. For $\delta(M) = M^l$ we have the split rule: $J(M^l) = \sum_{j=0}^{l-1} \mathrm{Tr}\, (M^j)\mathrm{Tr}\, M^{l-j-1}$.

3. For $\delta(M) = \mathrm{Tr}\, M^l$ we have the merge rule: $J(\mathrm{Tr}\, M^l) = \sum_{j=0}^{l-1} \mathrm{Tr}\, (M^j M^{l-j-1})$.

See Appendix A.1.3 for a derivation of these rules based on matrix differentials.

We note that if we set $\delta(M) = \mathbb{1}$ in equation (3.158), we recover the fact that integral of a total derivative vanishes assuming the integrand vanishes at the boundary.

In summary, there are three different but equivalent ways to interpret the loop equations: the Ward identities arising from symmetry of matrix models under a change of variables; the Schwinger-Dyson equations arising as integration by parts and finally the vanishing of a total derivative. We will frequently switch between these equivalent points of view depending on the convenience of the situation.

Now, consider the specific matrix model[1]

$$Z = \int_{H_N} dM e^{\sum_{k=0}^{\infty} t_k \text{Tr } M^k}. \tag{3.160}$$

Note that we have absorbed the factor of $-N$ into the definitions of the coupling constants t_k. Now, Z is invariant under the change of variables $M \mapsto M + \epsilon_n M^{n+1}$. Furthermore, there is no explicit factor of $-N$ in Z and so the Ward identity in this case reads

$$\langle \text{Tr } \left(V'(M)\delta(M) \right) \rangle + \langle J(M) \rangle = 0. \tag{3.161}$$

Here, $\delta(M) = M^{n+1}$, the potential is $V(M) = \sum_{k=0}^{\infty} t_k \text{Tr } M^k$ and the Jacobian $J(M)$ is determined using the split rule as

$$J(M) = \sum_{k=0}^{n} \text{Tr} M^k \text{Tr } M^{n-k}. \tag{3.162}$$

Hence (3.161) becomes

$$\int_{H_N} dM \left(\sum_{k=0}^{\infty} k t_k \text{Tr } M^{k+n} + \sum_{k=0}^{n} \text{Tr} M^k \text{Tr } M^{n-k} \right) e^{\sum_{k=0}^{\infty} t_k \text{Tr } M^k}. \tag{3.163}$$

Now, we have already seen how to compute correlation functions by taking derivatives of Z

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[1]Here we should clarify the definition of this matrix model. In the potential, the term with $k = 0$ only contributes a multiplicative constant to Z and so is largely irrelevant at present. The linear term, $k = 1$ should be understood by completing the square, and a change of variables to agree with the convention of Gaussian matrix models introduced previously.

with respect to the coupling constants t_k. Indeed, here we have

$$\langle \text{Tr } M^{a_1} \dots \text{Tr } M^{a_m} \rangle = \int_{H_N} dM \text{Tr } M^{a_1} \text{Tr } M^{a_n} e^{\sum_{k=0}^{\infty} t_k \text{Tr } M^k} = \frac{\partial^n}{\partial t_{a_1} \cdots \partial t_{a_n}} Z. \qquad (3.164)$$

Using this and equation (3.163), we obtain

$$L_n Z = 0, \ n \geq -1, \qquad (3.165)$$

where

$$L_n = \sum_{k=0}^{\infty} k t_k \frac{\partial}{\partial t_{k+n}} + \sum_{k=0}^{n} \frac{\partial^2}{\partial t_k \partial t_{n-k}}. \qquad (3.166)$$

Additionally, we have

$$\frac{\partial Z}{\partial t_0} = \int_{H_N} \text{Tr } \mathbb{1} e^{\sum_{k=0}^{\infty} t_k \text{Tr } M^k} = N Z. \qquad (3.167)$$

Equations (3.165) and (3.167) comprise the discrete Virasoro constraints. They are so named since, as a quick calculation shows, the operators L_n satisfy the Virasoro subalgebra

$$[L_m, L_n] = (m - n) L_{m+n}, \qquad (3.168)$$

where $m, n \geq -1$.

It may be concerning that equation (3.167) contains explicit N dependence. It is often useful to eliminate this explicit N dependence. Indeed, we shall also encounter a situation in chapter six where it is fruitful to eliminate this explicit dependence. Here in this situation, one can take a suitable limit of Z as $N \to \infty$. This limit is known as the double scaling limit and we will briefly sketch its construction in the next section. Looking ahead, if one applies the double scaling limit on the discrete Virasoro constraints above, one obtains a rather interesting representation of the Virasoro algebra. This particular representation will be the main focus of chapter four.

Let us now turn to loop equations in greater generality. Consider the matrix model

$$Z = \int_{H_N} dM e^{-\frac{N}{t}\operatorname{Tr} V(M)}, \qquad (3.169)$$

where we have reintroduced the t dependence for reasons we shall discuss in a moment. We define the connected correlation functions

$$\mathcal{T}_{l_1 \ldots l_k} = \langle \operatorname{Tr} M^{l_1} \ldots \operatorname{Tr} M^{l_n} \rangle_c. \qquad (3.170)$$

Now, since this is a *formal* Gaussian matrix model, we can perform a topological expansion so that

$$\mathcal{T}_{l_1 \ldots l_k} = \sum_{g=0}^{\infty} \mathcal{T}_{l_1 \ldots l_k}^{(g)} \left(\frac{N}{t}\right)^{2-2g-k}. \qquad (3.171)$$

Here, it turns out that $\mathcal{T}_{l_1 \ldots l_k}^{(g)}$ are generating functions that count connected maps of genus g that have k boundaries. The Euler characteristic of such a surface is $\chi_{g,k} = 2 - 2g - k$.

We can then define the resolvent as the corresponding generating function

$$W_k(x_1, \ldots, x_n) = \sum_{l_1, \ldots, l_k=0}^{\infty} \frac{1}{x_1^{l_1+1} \cdots x_k^{l_k+1}} \mathcal{T}_{l_1, \ldots l_k} = \sum_{l_1, \ldots, l_k=0}^{\infty} \left\langle \operatorname{Tr} \frac{M^{l_1}}{x_1^{l_1+1}} \cdots \operatorname{Tr} \frac{M^{l_1}}{x_1^{l_1+1}} \right\rangle_c. \qquad (3.172)$$

We remark that the resolvent is often denoted as

$$W_k(x_1, \ldots, x_k) = \left\langle \operatorname{Tr} \frac{1}{x_1 - M} \cdots \operatorname{Tr} \frac{1}{x_k - M} \right\rangle_c. \qquad (3.173)$$

which agrees formally with the previous definition. Note that the resolvents must be symmetric in their arguments x_i. As above, we can expand the resolvents W_k as

$$W_k(x_1, \ldots, x_k) = \sum_{g=0}^{\infty} \left(\frac{N}{t}\right)^{\chi_{g,k}} W_k^{(g)}(x_1, \ldots, x_k). \qquad (3.174)$$

In this context we take the view that loop equations are given by the vanishing of a total

derivative. Thus observe that loop equations are given by

$$\sum_{i,j} \int_{H_N} dM \frac{\partial}{\partial M_{ij}} \left(G(M)_{ij} e^{-\frac{N}{t} \operatorname{Tr} V(M)} \right) = 0, \tag{3.175}$$

where $G(M)$ is some matrix valued polynomial. If we choose

$$G(M) = M^{l_1} \prod_{n=2}^{k} \operatorname{Tr} M^{l_n}, \tag{3.176}$$

then computing the derivatives using the split and merge rules we find

$$\sum_{j=0}^{l_1-1} \left\langle \operatorname{Tr} M^{\operatorname{Tr}} M^{l_1-1-j} \prod_{i=2}^{k} \operatorname{Tr} M^{l_i} \right\rangle + \sum_{j=2}^{k} l_j \left\langle \operatorname{Tr} M^{l_j+l_1-1} \prod_{\substack{i=2 \\ i\neq j}}^{k} \operatorname{Tr} M^{l_i} \right\rangle \tag{3.177}$$

$$= \frac{N}{t} \left\langle \operatorname{Tr} (M^{l_1} V'(M)) \prod_{i=2}^{k} \operatorname{Tr} M^{l_i} \right\rangle. \tag{3.178}$$

Using the topological expansions of $\mathcal{T}_{l_1,\dots,l_k}$ we can rewrite the above equation as

$$\sum_{j=0}^{l_1-1} \left(\sum_{h=0}^{g} \sum_{J\subset L} \mathcal{T}_{j,J}^{(h)} \mathcal{T}_{l_1-1-j,L\setminus J}^{(g-h)} + \mathcal{T}_{j,l_1-1-j,L}^{(g-1)} \right) + \sum_{j=2}^{k} l_j \mathcal{T}_{l_j+l_1-1,L\setminus\{j\}}^{(g)} = \mathcal{T}_{l_1+1,L}^{(g)} - \sum_{j=3}^{d} t_j \mathcal{T}_{l_1+j-1,L}^{(g)}, \tag{3.179}$$

where d is the degree of the polynomial $V(M)$ and $L = \{l_2,\dots,l_k\}$. We now multiply this by $\prod_{i=1}^{k} \frac{1}{x_i^{l_i+1}}$ and perform the sum over the $l_1,\dots,l_k$. We can perform this summation since these sums are finite for any given power of t. Thus, we obtain the following equation

$$\sum_{h=0}^{g} \sum_{J\subset L} W_{1+|J|}^{(h)}(x_1, J) W_{k-|J|}^{(g-h)}(x_1, L \setminus J) + W_{k+1}^{g-1}(x_1, x_1, L) \tag{3.180}$$

$$+ \sum_{j=2}^{k} \frac{\partial}{\partial x_j} \frac{W_{k-1}^{(g)}(x_1, L \setminus \{x_j\}) - W_{k-1}^{(g)}(L)}{x_1 - x_j} = V'(x_1) W_k^{(g)}(x_1, L) - P_k^{(g)}(x_1, L), \tag{3.181}$$

where

$$P_k^{(g)}(x_1, \ldots, x_k) = \text{Pol}_{x_1}\left(V'(x_1)W_k^{(g)}(x_1, \ldots, x_k)\right) \tag{3.182}$$

$$= -\sum_{j=2}^{d-1} t_{j+1} \sum_{i=0}^{j-1} x_1^j \sum_{l_2,\ldots,l_k=1}^{\infty} \frac{\mathcal{T}_{j-1-i,l_2,\ldots,l_k}^{(g)}}{x_2^{l_2+1} \cdots x_k^{l_k+1}} + t\delta_{g,0}\delta_{k,1}. \tag{3.183}$$

The notation Pol_{x_1} here indicates the polynomial part of the Laurent expansion as $x_1 \to \infty$. Indeed, expanding equation (3.180) as $x_1 \to \infty$, the negative powers of x_1 yield the loop equations. Conversely, due to the definition of $P_k^{(g)}$, the positive powers of x_1 exactly cancel. See [41] for more details.

To make contact with the spectral curve we derived using the saddle point approximation, we first collect the polynomials $P_k^{(g)}$ into a generating function

$$\sum_{g=0}^{\infty} N^{2-2g-k} P_k^{(g)}(x_1, \ldots, x_k) = \left\langle \text{Tr} \frac{V'(x_1) - V'(M)}{x_1 - M} \text{Tr} \frac{1}{x_2 - M} \cdots \text{Tr} \frac{1}{x_k - M} \right\rangle_c. \tag{3.184}$$

Now, considering the planar case, $g = 0$, with one boundary, $k = 1$, the loop equations (3.180) reduce to

$$\left(W_1^{(0)}(x)\right)^2 = V'(x)W_1^{(0)} - P_1^{(0)}(x). \tag{3.185}$$

which is often referred to as the disk amplitude in light of the values of g and k. Loosely speaking, in view of the correlation function P_k defined above, this equation defines the same hyperelliptic curve that was derived in the saddle point approximation. Of course, to fully identify $P_1^{(0)}$ with the curve obtained previously in equation (3.148), further computations are required. These can be found in [41].

As a final observation, we note that the same loop equations can be derived starting from the invariance of the matrix model partition function Z under the change of variables

$$M \mapsto M + \frac{1}{x - M} \prod_{i=1}^{k} \text{Tr} \frac{1}{x_i - M}. \tag{3.186}$$

The analogous split and merge rules for the transformation $M \mapsto M + \epsilon\delta(M)$ are given by

the following formulae.

1. For $\delta(M) = \frac{1}{x-M}$ the split rule is $J(M) = \left(\text{Tr } \frac{1}{x-M}\right)^2$.

2. For $\delta(M) = \text{Tr } \frac{1}{x-M}$ the merge rule is $J(M) = \text{Tr } \left(\frac{1}{x-M}\right)^2$.

The loop equations then follow readily.

3.3.3 Solution of the Loop Equations Via Topological Recursion

In the previous subsection, we derived loop equations for a matrix model with polynomial potential. In doing this, we introduced the quantities $P_n^{(g)}$, although we have not given any indication on how to calculate such quantities. Despite this, there is in fact a method of solving the loop equations recursively without the need to evaluate $P_n^{(g)}$. In this subsection, we will briefly summarise this method, known as topological recursion. This will be generalised in the context of quantum Airy structures in chapter four.

Consider equation (3.185) for the disk amplitude. Analogous to what was done in the saddle point approximation, we define the quantity

$$y(x) = W_1^{(0)}(x) - \frac{1}{2}V'(x). \tag{3.187}$$

In this way, the (3.185) becomes

$$y^2(x) = \frac{1}{4}\left(V'(x)\right)^2 - P_1^{(0)}(x), \tag{3.188}$$

as before. Note that the full loop equation can also be rewritten in terms of $y(x)$ as

$$-2y(x)W_k^{(g)}(x_1, L) = \sum_{h=0}^{g}\sideset{}{'}\sum_{J\subseteq L} W_{1+|J|}^{(h)}(x_1, J)W_{k-|J|}^{(g-h)}(x_1, L \setminus J) + W_{k+1}^{g-1}(x_1, x_1, L) \tag{3.189}$$

$$+ \sum_{j=2}^{k} \frac{\partial}{\partial x_j} \frac{W_{k-1}^{(g)}(x_1, L \setminus \{x_j\}) - W_{k-1}^{(g)}(L)}{x_1 - x_j} + P_{g,n+1}(x, L). \tag{3.190}$$

The prime in the first summations indicates that we exclude the cases $(h, J) = (0, \varnothing)$ and $(h, J) = (g, L)$. It is important to exclude the $W_1^{(0)}$ term here for the recursive structure we will build shortly.

We now state the following crucial result known as the One-Cut Brown's Lemma. This lemma applies to the one-cut solution introduced earlier.

Lemma 3.3.1. There exists a rational polynomial $M(x) \in \mathbb{Q}[x][[t]]$ which is also a formal power series in t, and $a, b \in \mathbb{Q}[[\sqrt{t}]]$ such that

$$y^2(x) = \left(W_1^{(0)}(x) - \frac{1}{2}V'(x)\right)^2 = M^2(x)(x - a)(x - b), \tag{3.191}$$

where

$$a = 2\sqrt{t} + O(t), \; b = -2\sqrt{t} + O(t), \tag{3.192}$$

and

$$M(x) = \frac{V'(x)}{x} + \mathcal{O}(t). \tag{3.193}$$

Furthermore, if $M(x)$ has roots α_i, then

$$\alpha_i = k_i + O(t), \tag{3.194}$$

for some constants $k_i \neq 0$.

For a proof of the lemma, see [41, Lemma 3.1.1]. Consequently, this lemma shows that equation (3.185) does indeed define a genus zero hyperelliptic curve. This Riemann surface, which we recall is referred to as the spectral curve, plays a crucial role in determining the full solution to the matrix model.

To be able to apply the machinery of complex analysis, we parameterise this curve using rational functions

$$x(z) = \frac{a + b}{2} + \frac{a - b}{2}\left(z + \frac{1}{z}\right), \tag{3.195}$$

and

$$y(z) = M\big(x(z)\big)\frac{a-b}{4}\Big(z - \frac{1}{z}\Big), \tag{3.196}$$

where z is a local coordinate on the Riemann sphere $\mathbb{P}^1$. We interpret $x : \mathbb{P}^1 \to \mathbb{P}^1$ as a double covering consisting of two sheets.

Definition 3.3.2. The zeroes of the differential form $dx(z)$ are called ramification points. They are called simple ramification points if they are simple zeroes.

In this case we have

$$dx = \frac{a-b}{b}\Big(1 - \frac{1}{z^2}\Big)dz, \tag{3.197}$$

and so the ramification points are at $z = \pm 1$. We recall that the hyperelliptic involution

$$z \mapsto \sigma(z) = \frac{1}{z}, \tag{3.198}$$

exchanges the two sheets of the double covering x. We then easily observe that

$$x\big(\sigma(z)\big) = x(z), \ y\big(\sigma(z)\big) = -y(z). \tag{3.199}$$

Having given geometrical meaning to the spectral curve, we now want to interpret the correlation functions $W_n^{(g)}$ in this context. For $2g-2+n \geq 1$ we define the multilinear differentials

$$\omega_{g,n}(z_1, \ldots, z_n) := W_n^{(g)}(x_1, \ldots, x_n)dx_1 \ldots dx_n, \tag{3.200}$$

where $x_i := x(z_i)$. Again, this equality holds order by order in the small t expansion. For the remaining two base cases, we define

$$\omega_{0,1}(z) = y(z)dx(z)\big(W_1^{(0)}(x(z)) - \frac{1}{2}V'(x(z))\big)dx(z), \tag{3.201}$$

and

$$\omega_{0,2}(z_1, z_2) = \Big(W_2^{(0)}(x_1, x_2) - \frac{1}{(x_1 - x_2)^2}\Big)dx_1 dx_2. \tag{3.202}$$

We now investigate the pole structure of the differentials $\omega_{g,n}$ For $(g,n) = (0,2)$, we multiply the loop equation (3.189) by $dx_1 dx_2$ to obtain

$$\omega_{0,2}(z_1, z_2) = \frac{dx_1 dx_2}{2y(z_1)} \left(\frac{d}{dx_2} \left(\frac{2y(z_2) + V'(x_1) - V'(x_2)}{2(x_1 - x_2)} \right) + P_2^{(0)}(x_1, x_2) \right) + \frac{dx_1 dx_2}{2(x_1 - x_2)^2}. \tag{3.203}$$

Since $y(\sigma(z_1)) = -y(z_1)$, we have

$$\omega_{0,2}(z_1, z_2) + \omega_{0,2}(\sigma(z_1), z_2) = \frac{dx_1 dx_2}{(x_1 - x_2)^2}. \tag{3.204}$$

Consider the zeroes of $y(z_1)$ which are the roots, α_i, of $M(x_1)$. It follows from (3.203) that $\omega_{0,2}(z_1, z_2)$ can only have poles of the form $1/(x_1 - \alpha_i)$. By virtue of Brown's lemma, however, we recall $\alpha_i = k_i + \mathcal{O}(t)$ for some non zero constant k_i. Thus, in the small t expansion of $\omega_{0,2}$, the constant term has the form

$$\frac{1}{x_1 - \alpha_i} = \sum_{j=0}^{\infty} \frac{k_i^j}{x_1^{j+1}}. \tag{3.205}$$

The coefficient of t^0 is thus an infinite series in $1/x_1$ which cannot be true: each coefficient t^n in the formal expansion of $\omega_{0,2}$ should be a polynomial in $1/x_1$. Hence $\omega_{0,2}$ cannot have poles at α_i.

Moreover, $\omega_{0,2}(z_1, z_2)$, does not have poles at the branch points. Indeed, the branch points are zeroes of $y(z_1)$ but they are also zeroes of $dx(z_1)$.

The only remaining case to consider is when $z_1 = z_2$ and $z_1 = \sigma(z_2)$. As $z_1 \to \sigma(z_2)$ we have $y(z_1) \to -y(z_2)$. Thus, taking the derivative cancels the first and last terms in (3.203) and so $\omega_{0,2}(z_1, z_2)$ has a double pole at $z_1 = z_2$. In fact, it can be proven that there is a unique, up to a multiplicative constant, bilinear differential with a double pole at $z_1 = z_2$ and no other poles and no residues so that

$$\omega_{0,2} = \frac{dz_1 dz_2}{(z_1 - z_2)^2}. \tag{3.206}$$

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This is the statement known as the one-cut lemma for cylinders. See [41] for example. What is quite extraordinary here is that $\omega_{0,2}(z_1, z_2)$ is independent of the original potential $V(x)$. This hints at the much deeper role played by the spectral curve. We will revisit this in chapter four.

Starting from $\omega_{0,1}$ and $\omega_{0,2}$, it is possible to recursively obtain all the correlation functions $\omega_{g,n}$ for $2g - 2 + n \geq 0$. To this end, we prove the following lemma.

Lemma 3.3.3. Let $2g - 2 + n \geq 2$ and $J = \{z_1, \ldots, z_n\}$. Then

$$\omega_{g,n+1}(z_0, J) + \omega_{g,n+1}(\sigma(z_0), J) = 0. \tag{3.207}$$

Proof. We proceed by induction on $-\chi_{g,n} = 2g - 2 + n$. There are two base cases where $-\chi_{g,n} = 1$. For $(g, n) = (1, 1)$, the loop equation (3.189) can be written as

$$-2y(z_0)dx_0\omega_{1,1}(z_0) = -\omega_{0,2}(\sigma(z_0), z_0) + P_1^{(1)}(x_0)dx_0^2. \tag{3.208}$$

Now, the correlation functions are symmetric and so the right hand side is invariant under $z_0 \mapsto \sigma(z_0)$. Thus we find

$$\omega_{1,1}(z_0) + \omega_{1,1}(\sigma(z_0)) = 0. \tag{3.209}$$

Similarly, we find the following loop equation for $(g, n) = (0, 3)$,

$$-2y(z_0)dx_0\omega_{0,3}(z_0, z_1, z_2) = -\omega_{0,2}(z_0, z_1)\omega_{0,2}(\sigma(z_0), z_2) - \omega_{0,2}(\sigma(z_0), z_1)\omega_{0,2}(z_0, z_2) \tag{3.210}$$

$$+dx_0^2\left(dx_1\frac{d}{dx_1}\left(\frac{\omega_{0,2}(\sigma(z_1), z_1)}{x_0 - x_1}\right) + dx_2\frac{d}{dx_2}\left(\frac{\omega_{0,2}(\sigma(z_1), z_2)}{x_0 - x_2}\right) \tag{3.211}$$

$$+P_3^{(0)}(x_0, x_1, x_2)dx_1dx_2\right) \tag{3.212}$$

Again, this is indeed invariant under $z_0 \mapsto \sigma(z_0)$ and so

$$\omega_{0,3}(z_0, z_1, z_2) + \omega_{0,3}(\sigma(z_0), z_1, z_2) = 0. \tag{3.213}$$

Assume now that the result is true for all (g, n) such that $2g - 2 + n < k$ for some $k \in \mathbb{N}$.

With this induction hypothesis, we can write the loop equations as

$$2y(z_0)dx_0\omega_{g,n+1}(z_0, J) = \sum_{h=0}^{g}\sideset{}{'}\sum_{I\subseteq J}\omega_{h,1+|I|}(z_0, I)\omega_{g-h,n-|I|+1}(\sigma(z_0), J\setminus I) + \omega_{g-1,n+2}(z_0, \sigma(z_0), J)$$

(3.214)

$$+dx_0^2\Big(\sum_{i=1}^{|J|}dx_i\frac{d}{dx_1}\Big(\frac{\omega_{g,n}(J)}{x_0-x_i}\Big) - P_{n+1}^{(0)}(x_0,\ldots,x_n)dx_1\ldots dx_n\Big)$$

(3.215)

Again, this is indeed invariant under $z_0 \mapsto \sigma(z_0)$ and so

$$\omega_{g,n+1}(z_0, J) + \omega_{g,n+1}(\sigma(z_0), J) = 0.$$

(3.216)

for $2g - 2 + n = k$. Hence, the lemma follows by induction. $\square$

Let us now examine the pole structure as was done for $\omega_{0,2}$. It suffices to do this in terms of z_0 since the correlation functions are symmetric. Firstly, there cannot be poles at the poles of x_0 since $P_{n+1}^{(g)}$ has degree $d-3$. Secondly, there are now poles at coinciding points $z_0 \to z_i$ by inspecting the loop equation (3.189) and no further poles as $z_0 \to \sigma(z_i)$. Finally, there may be poles at zeroes of $y(x_0)$. As previously, there cannot be poles at α_i, the roots of $M(x_0)$. Indeed then, the only remaining possible poles are the ramification points of x_0. That is to say, $z_0 = \pm 1$.

Let us now return to the loop equations as written in equation (3.214),

$$\omega_{g,n+1}(z_0, J) = \frac{1}{2\omega_{0,1}(z_0)}\sum_{h=0}^{g}\sideset{}{'}\sum_{I\subset J}\omega_{h,1+|I|}(z_0, I)\omega_{g-h,n-|I|+1}(\sigma(z_0), J\setminus I)$$

(3.217)

$$+\frac{\omega_{g-1,n+2}(z_0, \sigma(z_0), J)}{2\omega_{0,1}(z_0)} + \frac{dx_0}{2y(z_0)}\Big(\sum_{i=1}^{|J|}dx_i\frac{d}{dx_1}\Big(\frac{\omega_{g,n}(J)}{x_0-x_i}\Big) - P_{n+1}^{(0)}(x_0,\ldots,x_n)dx_1\ldots dx_n\Big).$$

(3.218)

The last line in this equation has no pole at the ramification points $z_0 = \pm 1$. Therefore, if

one were to take residues of this equation, one would find that this line vanishes entirely. To fully realise this, we introduce the normalised differential of the third kind,

$$\omega^{a-b}(z) = \int_b^a \omega_{0,2}(z',z)dz' = \frac{dz}{z-a} - \frac{dz}{z-b}.$$ (3.219)

Let α be a generic point on $\mathbb{P}^1$. Consider the differential $\omega^{z-\alpha}(z')$. This is a one form in z' on $\mathbb{P}^1$ but can be thought of as a function in z on $\mathbb{P}^1$. It is well known that the sum of residues on any compact Riemann surface is zero and so it follows that

$$\sum_{a\in\{\text{poles}\}} \operatorname{Res}_{w=a} \omega^{w-\alpha}(z_0)\omega_{g,n+1}(w,J) = 0.$$ (3.220)

Going further, we have discussed that the only poles are at $w = z_0$ and the ramification points $w = \pm 1$. Evaluating the residue at $w = z_0$ yields

$$\operatorname{Res}_{w=z_0} \omega^{w-\alpha}(z_0)\omega_{g,n+1}(w,J) = -\omega_{g,n+1}(z_0,J),$$ (3.221)

from which it follows

$$\omega_{g,n+1}(z_0,J) = \sum_{a=\pm 1} \operatorname{Res}_{w=a} \omega^{w-\alpha}(z_0)\omega_{g,n+1}(w,J).$$ (3.222)

Consequently, we obtain the *topological recursion*

$$\omega_{g,n+1}(z_0,J) = \sum_{a=\pm 1} \operatorname{Res}_{w=a} \left[\frac{\omega^{w-\alpha}(z_0)}{2\omega_{0,1}(w)} \left(\sum_{I\subset J}\sum_{h=0}^{g} \omega_{h,1+|I|}(w,I)\omega_{g-h,n-|I|+1}(\sigma(w),J\setminus I) \right.\right.$$ (3.223)

$$\left.\left. +\omega_{g-1,n+2}(w,\sigma(w),J) \right)\right].$$ (3.224)

In this way, we have eliminated the unknown quantities $P_n^{(g)}$ and thus, in principle, solved the matrix model. We will revisit topological recursion from a different viewpoint, independent of matrix models, in the next chapter.

3.4 Solution of Matrix Models II - Integrability

In the previous section, we solved Hermitian matrix models using the loop equations. Here, we will now solve the same matrix models using the technique of orthogonal polynomials. Using this technique brings us naturally to the concept of the double scaling limit. It is here that we will give an outline of the proof that a double scaled Hermitian matrix model with arbitrary even potential is a tau function of the KdV hierarchy. In fact, the spirit of employing orthogonal polynomials will be revisited when discussing the integrability properties of the external field matrix integral. In this section, we follow the outline of [45] and [48].

3.4.1 Orthogonal Polynomials

Consider the integral

$$Z = \int_{H_N} dM e^{-N\mathrm{Tr}\, V(M)} = \int \prod_{i=1}^{N} d\lambda_i \Delta^2(\lambda) e^{-N\sum_i V(\lambda_i)}. \qquad (3.225)$$

In the above and in the following, we denote $\lambda = (\lambda_1, \ldots, \lambda_N)^{\mathrm{T}}$. To evaluate this integral, the idea is to interpret the eigenvalue reduction as some inner product of Vandermonde determinants weighted by the exponential factor. To this end, we wish to write the Vandermonde in a way such that it factorises nicely over the eigenvalues. Consequently, we introduce the monic polynomials, $p_n(x) = x^n + O(x^{n-1})$, in a single variable x of degree n such that

$$(p_n, p_m) = \int_{\mathbb{R}} dx\, p_n(x) p_m(x) e^{-NV(x)} = h_n \delta_{nm}. \qquad (3.226)$$

In other words, the polynomials p_n are orthogonal with respect to the measure $e^{-NV(x)}dx$ in the inner product $(\cdot, \cdot)$. Here, the scalars h_n are taken to be defined by (3.226).

We now rewrite the Vandermonde determinant as

$$\Delta(\lambda) = [1, \lambda, \lambda^2, \ldots, \lambda^{N-1}]. \qquad (3.227)$$

where λ is a column vector with λ_i in the ith row. To rewrite this determinant, to the final column λ^{N-1} we add a finite linear combination of the preceding columns. The overall determinant is unchanged while the last column becomes $p_{N-1}(\lambda)$ where the ith row is $p_{N-1}(\lambda_i)$. We can repeat this with the other columns so that

$$\Delta(\lambda) = [p_0(\lambda), p_1(\lambda), p_2(\lambda^2), \ldots, p_{N-1}(\lambda^{N-1})]. \tag{3.228}$$

We can now evaluate the integral given in (3.225) by expanding the Vandermonde determinant in terms of permutations. Indeed, we have

$$Z = \int_{\mathbb{R}^n} \prod_{i=1}^{N} d\lambda_i \sum_{\sigma, \tau \in S_N} \text{sgn}(\sigma\tau) \prod_{i,j=1}^{N} p_{i-1}(\lambda_{\sigma(i)}) p_{j-1}(\lambda_{\tau(j)}) e^{-N \sum_k V(\lambda_k)}. \tag{3.229}$$

We can reorganise the double product into a single product by setting $i = j$. By virtue of the orthogonality relation, the only terms that contribute are when $\lambda_{\sigma(i)} = \lambda_{\tau(j)}$ so that $\sigma = \tau$. Hence performing the integration, we have

$$Z = \sum_{\sigma, \tau} \text{sgn}(\sigma\tau) \delta_{\sigma, \tau} \prod_{i=1}^{N} h_{i-1} = N! \prod_{i=0}^{N-1} h_i. \tag{3.230}$$

Hence, we have that the free energy is given by, up to an irrelevant additive constant,

$$F = \log Z = \sum_{i=0}^{N-1} \log h_i. \tag{3.231}$$

Alternatively, in terms of $r_i := h_i / h_{i-1}$, we find that the free energy may be written as

$$F = \sum_{i=0}^{N-1} (N - i) \log r_i. \tag{3.232}$$

We will now specialise for simplicity to the case where V is an even, quartic polynomial,

$$V(x) = \frac{x^2}{2} + \frac{t_4 x^4}{4}. \tag{3.233}$$

We will recover relations between the h_i defined above that allow the free energy F to be expressed in terms of t_4 and N only. To achieve that goal, we define the operators Q and P as

$$Qp_n(x) = xp_n(x) = \sum_m Q_{nm}p_m(x), \tag{3.234}$$

and

$$Pp_n(x) = \frac{d}{dx}p_n(x) = \sum_m P_{nm}p_m(x), \tag{3.235}$$

For future reference, we observe that P and Q obey the canonical commutation relation $[P,Q] = 1$. We now calculate the matrix elements Q_{nm} and P_{nm}. Since the p_n are monic, we must have that $Q_{nm} = 0$ if $m > n+1$ and $Q_{n,n+1} = 1$. For the same reason, we have $P_{nm} = 0$ if $m \geq n$ and $P_{n,n-1} = n$. To evaluate the other matrix elements,

$$(p_r, Qp_n) = Q_{nr}h_r = (Qp_r, p_n) = Q_{rn}h_n, \tag{3.236}$$

where we have that Q is self adjoint with respect to this inner product. Now, since Q_{nm} vanishes if $m > n+1$, we have that $Q_{nm} = 0$ if $|m - n| > 1$. From (3.236) and $Q_{n,n+1} = 1$, we compute that

$$Q_{n,n-1} = \frac{h_n}{h_{n-1}} = r_n. \tag{3.237}$$

Finally, the diagonal elements are given as

$$(p_n, Qp_n) = \int_{\mathbb{R}} e^{-NV(x)} xp_n^2(x) = 0, \tag{3.238}$$

by symmetry. Thus, the polynomials p_n satisfy the recursion

$$Qp_n(x) = xp_n(x) = p_{n+1}(x) + r_np_n(x). \tag{3.239}$$

Now, let us compute the matrix elements $P_{n,n-1} = n$ in two different ways. We have

$$(p_{n-1}, Pp_n) = nh_{n-1}. \tag{3.240}$$

On the other hand, we also have

$$(p_{n-1}, Pp_n) = \int_{\mathbb{R}} dx e^{-NV(x)} p_{n-1}(x) \frac{d}{dx} p_n(x). \tag{3.241}$$

Integrating by parts we find

$$(p_{n-1}, Pp_n) = -\int_{\mathbb{R}} dx \frac{d}{dx} \left(e^{-NV(x)} p_{n-1}(x) \right) p_n(x) = N(p_{n-1}, V'(Q)p_n) - (p_n, Pp_{n-1}). \tag{3.242}$$

Note that $P_{n-1,n} = 0$ and so the last term in the above equation vanishes. Hence, by comparing (3.241) and (3.242) and recalling the explicit form of the quartic polynomial V, we find

$$nh_{n-1} = N\left(p_{n-1}, (Q + t_4 Q^3)p_n\right). \tag{3.243}$$

Repeatedly employing the recursion relation (3.239), we obtain

$$\frac{n}{N} h_{n-1} = h_n + t_4 h_{n-1} r_n(r_{n+1} + r_n + r_{n-1}). \tag{3.244}$$

We may alternatively write this as

$$t_4 \frac{n}{N} = t_4 r_n + (t_4)^2 r_n(r_{n+1} + r_n + r_{n-1}). \tag{3.245}$$

For the reader familiar with discrete integrable equations, one may recognise this as the discrete Painlevé I equation.

3.4.2 Double Scaling Limit

Ultimately, we are interested in a continuum limit of the discrete Painlevé I equation analogous to the large N limit of the spectral curves and loop equations of the previous section. One would hope that taking a limit of the discrete Painlevé I equation would recover the standard Painlevé I equation

$$u''(t) + 6u^2(t) = t. \tag{3.246}$$

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However, there are extra subtleties to consider. Indeed, in general the partition function and free energy are not required to be analytic functions of the coupling constant t_4. It may have singular behaviour at a critical point $t_4 = t_c$. Indeed, taking the large N limit, we can view $t_4 r_n := r(z)$ as a function of the variable $z = n/N$ which parameterises the interval $[0, 1]$. Then the discrete Painlevé I equation becomes

$$t_4 z = r(z) + 3r^2(z) =: W(r(z)).$$ (3.247)

We say that the critical point r_c is the solution $dW/dr = 0$ which is $r_c = -1/6$. We then set $t_c = W(r_c) = -1/12$. We note here that it is consistent to ask only for the first derivative of W to vanish. For arbitrary polynomial potentials, one can consider when $W^{(k)}(r_c) = 0$ for all $k < m$ for some natural number m. These are known as multicritical solutions. For detailed discussion on such solutions, we refer the reader to [48]. We recall that the free energy F has a topological expansion

$$F = \sum_{g=0}^{\infty} N^{2-2g} F_g.$$ (3.248)

It can be shown that F_g is not analytic at $t_4 = t_c$. Without showing detailed computations, for $g \geq 2$, it is known that

$$\frac{\partial F_g}{\partial t_4} \sim \frac{5}{4} \frac{2 - 2g}{t_4 - t_c},$$ (3.249)

showing the non-analytic behaviour as $t_4 \to t_c$. These computations are shown explicitly in [49]. One can then ask questions about the behaviour of F_0 as $t_4 \to t_c$. It turns out that computing this limit of F_0 is not just interesting from the point of view of simple asymptotics. Remarkably, it can be argued, as is done in [25], that calculating F_0 in the limit $t_4 \to t_c$ in fact restores the full genus free energy F. We will be more precise in a moment. This 'double scaling' has an important interpretation in terms of maps that we shall discuss in chapter five.

To study the behaviour as $t_4 \to t_c$ and $N \to \infty$, one can introduce small parameters to

the discrete Painlevé equation. For example, we introduce $\epsilon = 1/N$ and $r(z \pm \epsilon) = t_4 r_{n\pm 1}$. Then the discrete Painlevé I equation becomes

$$t_4 r(z) = r(z)\big(1 + 3r(z)\big) + \epsilon^2 r(z) r''(z) + O(\epsilon^4). \tag{3.250}$$

If one were take the naive $N \to \infty$, and $\epsilon \to 0$, one does not obtain the ordinary Painlevé equation. To resolve this, one can introduce another small parameter as we will do shortly. By then balancing the orders of perturbation so that no term is subleading, one does indeed recover the ordinary Painlevé equation. We shall proceed in a more elegant fashion by considering again the operators P and Q and the canonical commutation relation $[P, Q] = 1$.

We first rescale the orthogonal polynomials p_n so that they are orthonormal, $\tilde{p}_n = p_n/\sqrt{h_n}$. In this way, the recursion relation satisfied by p_n becomes

$$Q\tilde{p}_n(x) = x\tilde{p}_n(x) = \sqrt{r_{n+1}}\tilde{p}_{n+1} + \sqrt{r_n}\tilde{p}_{n-1}. \tag{3.251}$$

This is equivalent to

$$Q_{nm} = (\tilde{p}_m, Q\tilde{p}_n) = \sqrt{r_{n+1}}\delta_{m,n+1} + \sqrt{r_n}\delta_{m,n-1}. \tag{3.252}$$

The shift operator $\delta_{m,n+1}$ is generated by d/dn. That is to say

$$e^{\pm d/dn} a_n = a_{n\pm 1}. \tag{3.253}$$

Letting $\epsilon = 1/N$ and passing to the continuum limit $N \to \infty$ we have

$$e^{\pm d/dn} a_n \sim e^{\pm \epsilon d/dz} a(z). \tag{3.254}$$

Thus we have

$$Q \sim \sqrt{r(z)}\big(e^{\epsilon d/dz} + e^{-\epsilon d/dz}\big) = \sqrt{r(z)}\Big(2 + \epsilon^2 \frac{d^2}{dz^2} + \mathcal{O}(\epsilon^4)\Big). \tag{3.255}$$

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Let a be a small parameter whose order is to be determined. We rescale z and $r(z)$ according to

$$t_4 z = t_c(1 - a^2 y), \tag{3.256}$$

and

$$r(z) = r_c(1 - au(y)). \tag{3.257}$$

Substituting (3.257) into (3.255) and expanding to first order in ϵ and a we find,

$$Q \sim \sqrt{-r_c}\left(-2 + au - \epsilon^2 \frac{d^2}{dz^2} + \cdots\right). \tag{3.258}$$

We now observe that

$$\epsilon^2 \frac{d^2}{dz^2} = \left(\frac{t_4}{t_c}\right)^2 \epsilon^2 a^{-4} \frac{d^2}{dy^2}. \tag{3.259}$$

Hence, for the perturbations in (3.258) to have the same order, we require

$$a \sim \epsilon^2 a^{-4}, \tag{3.260}$$

which means that

$$\epsilon = \frac{1}{N} = a^{5/2}. \tag{3.261}$$

Thus, if we let $a \to 0$, this not only controls singular behaviour $r \to r_c$ but also simultaneously computes the $N \to \infty$ limit. However, in the rescaling

$$t_4 z = t_c(1 - a^2 y), \tag{3.262}$$

we see that

$$1 \sim \frac{t_4 - t_c}{t_c} a^{-2} = \frac{t_4 - t_c}{t_c} N^{4/5} =: x^{-1}. \tag{3.263}$$

Keeping the quantity x fixed while taking $N \to \infty$ and $t_4 \to t_c$ defines the so-called double

scaling limit. Hence, for the terms of first order in equation (3.258) for Q we find

$$Q \mapsto \frac{d^2}{dy^2} - u, \tag{3.264}$$

after an appropriate rescaling of u and y. This is exactly the self adjoint Lax operator for the KdV equation. From this perspective, it is perhaps not surprising that performing the same computations for P, we find the continuum limit as

$$P \mapsto \frac{d^3}{dy^3} - \frac{3}{4}u\frac{d}{dy} - \frac{3}{4}\frac{du}{dy}. \tag{3.265}$$

The canonical commutation relation $[P, Q] = 1$ becomes

$$\left[\frac{d^3}{dy^3} - \frac{3}{4}u\frac{d}{dy} - \frac{3}{4}\frac{du}{dy}, \frac{d^2}{dy^2} - u\right] = -\frac{1}{4}u''' + \frac{3}{4}uu' = 1. \tag{3.266}$$

This equation is simply the derivative of the Painlevé I equation after a rescaling of u and y. To summarise, we have seen that the continuum limit of the operators P and Q are the Lax operators that define the KdV equation. There are yet further hints at the KdV equation here. It is well known, and easily verified, that the Painlevé equation

$$u''(z) + 6u^2(z) = 1, \tag{3.267}$$

can be obtained from the KdV equation for $u(x, t)$

$$u_t + 6uu_x + u_{xxx} = 0, \tag{3.268}$$

via the similarity reduction

$$u(x, t) = v\left(x + 3t^2\right) - t. \tag{3.269}$$

Furthermore, the solution $u(y)$ can be obtained from the free energy.

Indeed, the planar limit of the free energy is given by

$$F_0 = \lim_{N \to \infty} \frac{1}{N^2} \log Z = \lim_{a \to 0} \frac{1}{N} \sum_{n=0}^{N} \left(1 - \frac{n}{N}\right) \log r_n = \int_0^1 (1 - z) \log r(z) dz. \qquad (3.270)$$

Considering also that the limit $a \to 0$ restores the higher genus contributions, we find the free energy is given by

$$F = \lim_{a \to 0} \frac{1}{N^2} \int_0^1 (1 - z) \log r(z) dz = \lim_{a \to 0} a^{-5} \int_0^1 (1 - z) \log r(z) dz. \qquad (3.271)$$

Changing variables using (3.256) and (3.257) we find

$$F(x) = \lim_{a \to 0} \int_{a^{-2}}^{x} (y - x) u(y) dy, \qquad (3.272)$$

where x is the fixed double scaling quantity given in (3.263). Taking two derivatives with respect to x using the Leibniz integral rule, we find

$$u(x) = -F''(x). \qquad (3.273)$$

We make a final remark here that we have only dealt with quartic potentials $V(M)$. In general, one can consider arbitrary even potentials. In this case, one has to consider multi-critical solutions. One can perform the same analysis using the Q and P operator formalism. A rather beautiful result is that the commutation relation $[P, Q] = 1$ in the double scaling limit becomes, after a suitable change of variables, exactly the recursion relations obeyed by the Gelfand-Dikii polynomials that define the KdV hierarchy. Furthermore, the analogue of $u(x) = -\partial_x^2 \log Z$ also holds. Consequently, the partition function defined by a matrix model with arbitrary even potential is a tau function for the KdV hierarchy. This gives us a flavour of the relationship between matrix models and tau functions.

Let us now be slightly more general and precise. The free energy in fact has an expansion

in terms of the fixed quantity x as

$$\tilde{F} = \sum_{g \geq 0} x^{2g-2} \tilde{F}_g. \tag{3.274}$$

The quantity $\tilde{F}$ is different to $F = \log Z$ as the cases where $g = 0$ and $g = 1$ are more subtle as F_g do not obey (3.249). There is in fact a more general theorem that we state below [50].

Theorem 3.4.1. Let

$$Z_N = \int_{H_N} dM e^{-N\text{Tr } V(M)}, \tag{3.275}$$

with $V(M)$ an arbitrary potential of the form $V(M) = \sum_{k \geq 0} t_k M^k$. Define the following double scaling limit

$$\mathcal{Z} = \lim_{\substack{\text{d.s. } N \to \infty}} \sqrt{Z_N}. \tag{3.276}$$

Then $\mathcal{Z}$ is a tau function for the KdV hierarchy. Moreover, in the double scaling limit, the discrete Virasoro constraints satisfied by Z_N,

$$\left(\sum_{k=0}^{\infty} k t_k \frac{\partial}{\partial t_{k+n}} + \sum_{k=0}^{n} \frac{\partial^2}{\partial t_k \partial t_{n-k}} \right) Z_N = 0, \tag{3.277}$$

$$\frac{\partial Z}{\partial t_0} = N Z, \tag{3.278}$$

become the 'continuous' Virasoro constraints

$$H_n^{(2,3)} \mathcal{Z} = 0 \tag{3.279}$$

where

$$H_n^{(2,3)} = -\frac{\partial}{\partial x_{n+2}} + \delta_{n,-1} \frac{x_1^2}{2} + \sum_{j=1}^{\infty} \frac{(2n+2j-1)}{2n+3} x_j \frac{\partial}{\partial x_{n+j}} \tag{3.280}$$

$$+ \frac{1}{2} \sum_{j=1}^{n-1} \frac{(2j-1)(2n-2j+1)}{2n-3} \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_{n-j+1}} + \delta_{n,0} \frac{1}{24} \tag{3.281}$$

with $n \geq -1$. These operators furthermore still satisfy the Virasoro algebra $[H_m^{(2,3)}, H_n^{(2,3)}] = (m-n)H_{m+n}^{(2,3)}$ for $m, n \geq -1$.

Remark. The operators $H_n^{(2,3)}$ are so labelled for reasons that will become apparent in the next chapter.

A proof of these continuous Virasoro constraints can be found in [51]. This theorem now gives a way of proceeding with the original aim to show that a function τ is a tau function of the KdV hierarchy if and only if τ is annihilated by a set of differential operators $H_n^{(2,3)}$. Indeed, this theorem proves the 'if' direction for a tau function of the form given by the double scaled limit $\mathcal{Z}$. For the converse, it suffices to show that the differential equations $H_n^{(2,3)}\mathcal{Z}$ specify $\mathcal{Z}$ uniquely. Due to their special form and the Virasoro algebra property, the operators $H_n^{(2,3)}$ do in fact specify $\mathcal{Z}$ uniquely. We will discuss why this is the case in the next chapter.

Chapter 4

Quantum Airy Structures

In the previous section, we have seen that a Hermitian matrix model in the double scaled limit, which is a tau function for the KdV hierarchy, is annihilated by some differential operators. Somewhat surprisingly, these operators are a representation of the Virasoro algebra. This signifies the presence of a conformal field theory. In the first section of this chapter, we review the main components of a conformal field theory on the Riemann sphere, with a view to defining vertex operator algebras. This definition will become important later on in the construction of Airy structures. We will also briefly describe $\mathcal{W}$-algebras here as they will play an important role. In addition, we provide the necessary background to prove Proposition 4.1.14 which will become essential in chapter six. We then will move on to Airy structures and give a sketch proof of their main property involving existence and uniqueness. For completeness, we will also describe the relation between the Eynard-Orantin topological recursion that was encountered in the previous chapter and the generalisation to Airy structures. It is also perhaps worthy of mention that Airy structures are still an active area of research, with the generalisations of the geometric recursion [52] and supersymmetric quantum Airy structures [53] having been recently found. In the final section, we will review a construction of higher quantum Airy structures that will play a crucial role in chapter six.

4.1 A Review of Conformal Field Theory

In the previous two chapters, we saw how to construct both the classical and quantum mechanical equations of motion starting from an action. In chapter two, we then took a different approach, starting instead from the symmetries of the system, the Hamiltonians, and constructing the theory from this viewpoint. Here, we will do much the same for the quantum field theory. The difference is that we will focus on a certain type of symmetry. Specifically, we will require that the action is invariant under conformal transformations. These are the transformations that, roughly speaking, preserve angles. More precisely, we will show that $f : \mathbb{C} \to \mathbb{C}$ is conformal if and only if it is holomorphic and has non vanishing derivative. Requiring this symmetry may seem innocent enough but we shall soon discover that this has many profound consequences. The depth of conformal field theory means that there are many omissions in this discussion such as minimal models, Kac-Moody algebras and the Verlinde formula to name but a few. Readers interested in these subjects can consult [54, 55]. It is these expositions that we follow here.

4.1.1 Action Principles

As was the case in the previous chapter, to specify a quantum field theory we choose a Riemannian manifold M. In two dimensions, we often choose $M \subset \mathbb{R}^2$ with local coordinates given by (x^0, x^1) where x^0 and x^1 are temporal and spatial coordinates respectively. In string theory, we often complexify these coordinates to $z = x^0 + ix^1$ and thus take M to be a Riemann surface Σ. The fields of the theory take the form $X : \Sigma \to N$. These need not be holomorphic and so can depend on the local coordinates z and $\bar{z}$. For example, for a free, massless boson, we choose $N = \mathbb{C}$ and so we have the scalar field $X : \Sigma \to \mathbb{C}$ where $X = X(z, \bar{z})$. The action that describes such a particle is given by

$$S = \int dz d\bar{z} \sqrt{|g|} g^{ij} \partial_i X \partial_j X. \tag{4.1}$$

where $|g| = \det\left(g_{ij}\right)$. Here, we have used Einstein summation convention and the indices i and j stand for z or $\bar{z}$. We note that $g_{ij}g^{jk} = \delta_{ik}$. We choose the specific metric defined by

$$g^{ij} = \begin{bmatrix} 0 & 2z\bar{z} \\ 2z\bar{z} & 0 \end{bmatrix}. \tag{4.2}$$

Under this specific choice, the action becomes

$$S = \int dz d\bar{z}\, \partial_z X \partial_{\bar{z}} X. \tag{4.3}$$

Performing the variation of S in the field X, or directly using the Euler-Lagrange equations, we find that the classical equation of motion is in fact Laplace's equation in complex coordinates,

$$\partial_z \partial_{\bar{z}} X = 0. \tag{4.4}$$

By integrating this equation twice, we easily see the solution decomposes as $X = X(z) + \bar{X}(\bar{z})$. We call $X(z)$ the *chiral part*, or holomorphic part, and $\bar{X}(\bar{z})$ the *anti-chiral part*, or anti-holomorphic part. Very often, we shall restrict attention to the chiral part of the field theory, but completely analogous statements hold for the anti-chiral part. Another form of the solution which will be useful in later chapters is constructed as follows. Let

$$j(z) := i\partial_z X(z), \tag{4.5}$$

and

$$\bar{j}(\bar{z}) := i\partial_{\bar{z}} X(\bar{z}). \tag{4.6}$$

Equation (4.4) then becomes

$$\partial_{\bar{z}} j(z) = 0, \tag{4.7}$$

and similarly for $\bar{j}$. Hence, we expand $j(z)$ as

$$j(z) = \sum_{n \in \mathbb{Z}} j_n z^{n-1}. \tag{4.8}$$

By integrating we find

$$X(z) = x_0 - i j_0 \log z + i \sum_{n \in \mathbb{Z} \backslash \{0\}} \frac{1}{n} j_n z^{-n}. \tag{4.9}$$

This is the explicit solution for the case of the free boson. Upon quantisation, the quantities x_0 and j_n for all $n \in \mathbb{Z}$ are lifted to operators that obey the canonical commutation relation

$$[j_0, x_0] = -i, \tag{4.10}$$

and the Heisenberg algebra

$$[j_n, j_m] = n \delta_{n+m,0}. \tag{4.11}$$

For a proof of these relations, see [56].

We recall that the energy momentum tensor is defined component wise by

$$T_{ij} = \frac{1}{\sqrt{|g|}} \frac{\delta S}{\delta g^{ij}}, \tag{4.12}$$

where $\frac{\delta}{\delta g^{ij}}$ is the functional derivative introduced in the previous chapter. For example, by performing the variation in g, or indeed the definition of the functional derivative, we find that for the free boson,

$$T_{zz} = (\partial_z X)^2, \tag{4.13}$$

and

$$T_{\bar{z}\bar{z}} = (\partial_{\bar{z}} X)^2. \tag{4.14}$$

Here we used,

$$\frac{\partial}{\partial g_{ab}} |g| = -\frac{1}{2} |g| \, g^{ab}, \tag{4.15}$$

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a proof of which can be found in [57].

However, we now return to the definition of S and make the following observation. Suppose that $w = f(z)$ where $f : \mathbb{C} \to \mathbb{C}$ is a conformal transformation. This means that in particular, $\frac{dw}{dz} \neq 0$ for all $z \in \mathbb{C}$. The coordinate $\bar{w}$ transforms in the obvious way as $\bar{w} = f(\bar{z})$ meaning that $\frac{\partial w}{\partial \bar{z}} = \frac{\partial \bar{w}}{\partial z} = 0$. The Jacobian matrix for this transformation is then simply a diagonal matrix whose entries are $\frac{dw}{dz}$ and $\frac{d\bar{w}}{d\bar{z}}$. Thus the scale factor is $\left| \frac{dw}{dz} \right|^2$. Suppose that $X(z, \bar{z})$ is a field such that $X(z, \bar{z}) = X(w, \bar{w})$. Using the chain rule, we note $\partial_z X(z, \bar{z}) = \frac{\partial w}{\partial z} \partial_w X(w, \bar{w})$ since $\frac{\partial \bar{w}}{\partial z} = 0$. Similar reasoning holds for $\partial_{\bar{z}} X(z, \bar{z})$. Hence

$$S = \int dz d\bar{z} \partial_z X \partial_{\bar{z}} X = \int dz d\bar{z} = \frac{\partial w}{\partial z} \partial_w X \frac{\partial \bar{w}}{\partial \bar{z}} \partial_{\bar{w}} X = \int dw d\bar{w} \partial_w X \partial_{\bar{w}} X, \qquad (4.16)$$

Hence, the action defined by (4.3) is indeed invariant under conformal transformations. We shall see the consequences of this in the next section.

4.1.2   The Virasoro Algebra

In the previous section, we started from a specific action and showed that it was conformally invariant. Here, we will not specify an explicit action, but merely assume that it is invariant under conformal transformations. To proceed, we first investigate the algebra of generators of conformal transformations.

Definition 4.1.1. Let (M, g) and (M', g') be two semi-Riemannian manifolds. Let $U \subset M$ and $V \subset M'$ be open. A smooth mapping $\varphi : U \to V$ is called a conformal transformation if there exists a smooth function $\Omega : U \to \mathbb{R}_{\geq 0}$ such that

$$\varphi^* g' = \Omega^2 g, \qquad (4.17)$$

where φ^* is the pull back, $\varphi^* g'(X, Y) = g'(d\varphi(X), d\varphi(Y))$.

In local coordinates we have

$$(\varphi^* g')_{ab}(x) = g'_{ij} \partial_a \phi^i \partial_b \phi^j. \qquad (4.18)$$

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This means ϕ is conformal if

$$\Omega^2 g_{ab} = g'_{ij} \partial_a \varphi^i \partial_b \varphi^j. \tag{4.19}$$

Let $M \subset \mathbb{R}^{p,q}$ and let $X : M \to \mathbb{R}^n$ be a smooth vector field. The integral curves are as usual given by $\dot{\varphi}^X = X(\varphi)$. We recall that there is a unique solution for this ODE given by $\varphi^X : U \subset \mathbb{R} \times M \to M$ subject to the initial condition $\varphi^X(0, x) = x$ for all $x \in M$. This defines a local one parameter group of diffeomorphisms $(\varphi_t^X : M \to M)_{t \in \mathbb{R}}$ where $\varphi_t^X(x) = \varphi^X(t, x)$. In particular, $\varphi_0^X = \mathrm{id}_M$ and

$$\frac{d}{dt}(\varphi_t^X)\Big|_{t=0} = X. \tag{4.20}$$

Definition 4.1.2. Let $M \subset \mathbb{R}^{p,q}$. A vector field $X : M \to \mathbb{R}^n$ is called a conformal Killing vector field if φ_t^X is conformal for all $t \in (-\epsilon, \epsilon)$ for some $\epsilon > 0$.

Theorem 4.1.3. Let $M \subset \mathbb{R}^{p,q}$ be open and let X be a conformal Killing vector field where $X = X^i \partial_i$. Then there exists a function $\kappa : M \to \mathbb{R}$ such that

$$\partial_j X_i + \partial_i X_j = \kappa g_{ij}. \tag{4.21}$$

Proof. Let X be a conformal Killing vector field, φ_t an associated one parameter subgroup and $\Omega_t : M \to \mathbb{R}_{\geq 0}$ such that φ_t^X is conformal,

$$\Omega_t^2(p) g_{ab}(x) = g_{ij}(\varphi_t(x)) \partial_a \varphi_t^i \partial_b \varphi_t^j. \tag{4.22}$$

Recalling that g_{ij} is a constant, we now differentiate with respect to t,

$$\frac{d}{dt}\Omega_t^2(x) g_{ab}(x)\Big|_{t=0} = g_{ij} \partial_a \dot{\varphi}_0^i \partial_b \varphi_0^j + g_{ij} \partial_a \varphi_0^i \partial_b \dot{\varphi}_0^j \tag{4.23}$$

$$= g_{ij} \partial_a X^i \delta_b^j + g_{ij} \partial_b X^j \delta_a^i = \partial_j X_i + \partial_i X_j. \tag{4.24}$$

Hence $\kappa(x) = \frac{d}{dt}\Omega_t^2(x)\Big|_{t=0}$. $\qquad\square$

Remark. Notice that in the case $p = 2$ and $q = 0$, equation (4.21) are simply the Cauchy-

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Riemann equations for X_i. In fact, the converse to this theorem also holds: if there exists $\kappa : M \to \mathbb{R}$ such that (4.21) holds, then X is a conformal Killing field.

Definition 4.1.4. Let $M \subset \mathbb{R}^{p,q}$. A smooth function $\kappa : M \to \mathbb{R}$ is called a conformal Killing factor if there exists a vector field X such that

$$\partial_j X_i + \partial_i X_j = \kappa g_{ij}. \tag{4.25}$$

Corollary 4.1.5. A smooth function $\kappa : M \to \mathbb{R}$ is a conformal Killing factor if and only if

$$(n-2)\kappa_{,ij} + g_{ij}\Delta\kappa = 0, \tag{4.26}$$

where $\Delta = g^{ab}\partial_a\partial_b$ and the comma denotes regular differentiation.

Proof. Suppose that κ is a conformal Killing factor. First, observe that $\partial_i\partial_j(X_{a,b}) = \partial_b\partial_i(X_{a,j})$ and so it follows that

$$\partial_i\partial_j(X_{a,b} + X_{b,a}) - \partial_j\partial_a(X_{i,b} + X_{b,i}) + \partial_a\partial_b(X_{i,j} + X_{j,i}) - \partial_b\partial_i(X_{a,j} + X_{j,a}) = 0. \tag{4.27}$$

However, κ is a conformal Killing factor and so satisfies

$$\partial_i\partial_j(X_{a,b} + X_{b,a}) = \kappa_{,ij}g_{ab}. \tag{4.28}$$

Employing (4.27) we find

$$g_{ab}\kappa_{,ij} - g_{bi}\kappa_{,ja} + g_{ij}\kappa_{,ab} - g_{ja}\kappa_{,bi} = 0. \tag{4.29}$$

Contracting this with g^{ij} reads

$$g_{ab}g^{ij}\kappa_{,ij} - \delta_b^j\kappa_{,ja} + \delta_{ii}\kappa_{,ab} - \delta_a^i\kappa_{,bi} = 0, \tag{4.30}$$

or more simply

$$(n-2)\kappa_{,ij} + g_{ij}\Delta\kappa = 0. \tag{4.31}$$

We do not prove the converse in full generality and restrict instead to the special case $M = \mathbb{R}^{2,0} \cong \mathbb{C}$. This will be of the most interest to us for the later sections. Suppose κ is a function satisfying equation (4.26). Since $n = 2$, this means that

$$\Delta\kappa = 0, \tag{4.32}$$

implying that κ is harmonic. It is a well known fact that, since $\mathbb{R}^2$ is simply connected and κ is harmonic, this determines a holomorphic function $f(x, y) = u(x, y) + iv(x, y)$. Since f is holomorphic, u and v satisfy the Cauchy-Riemann equations, $u_y + v_x = 0$ and $u_x = v_y$. This defines a holomorphic vector field $X = (u, v)$ which is a conformal Killing vector field. □

This provides us with the following characterisation of infinitesimal conformal transformations.

Corollary 4.1.6. Let $M \subset \mathbb{C}$ be open. Then every holomorphic function $\varphi = u + iv : M \to \mathbb{C}$ with nowhere vanishing derivative is a conformal transformation with conformal Killing factor $\Omega^2 = u_x^2 + u_y^2$. Conversely, every conformal transformation is holomorphic.

We remark now on a slight subtlety. We define the conformal group as the group of conformal diffeomorphisms, those conformal transformations which are *globally defined* and invertible. This is a Lie group and we can define the conformal algebra as the Lie algebra of the conformal group under the usual Lie group-Lie algebra correspondence. However, we can also consider the algebra consisting of generators of the infinitesimal conformal transformations. This will contain the conformal algebra as a subalgebra but it may in fact be larger.

As an example, consider the infinitesimal conformal transformations of $\mathbb{C} \cong \mathbb{R}^{2,0}$. From the above corollary, an infinitesimal conformal transformation on an open subset $M \subset \mathbb{C}$ can be written as $z \mapsto z + f(z)$ where f is holomorphic. We do not discount the possibility

of singularities outside of M. On M, we expand f in a convergent Laurent series about zero so that the infinitesimal conformal transformation is $z \mapsto z + \sum_{n \in \mathbb{Z}} \epsilon_n z^n$ where ϵ_n are small, constant parameters. Notice that the infinitesimal generators corresponding to this conformal mapping can be written as

$$\sum_{n \in \mathbb{Z}} \epsilon_n z^{n+1} \frac{d}{dz}. \tag{4.33}$$

Therefore, the algebra over $\mathbb{C}$ consisting of generators of conformal transformations is generated by $L_n = -z^{n+1} \frac{d}{dz}$ for $n \in \mathbb{Z}$. We call this the *Witt algebra* W. It is in fact a Lie algebra since a simple calculation shows that

$$[L_m, L_n] = (m - n) L_{m+n}. \tag{4.34}$$

However, it is shown in [55] that the only generators that give rise to globally defined transformations are L_{-1}, L_0 and L_1. Thus the conformal algebra in this case is generated by these three elements. The corresponding conformal group is the group of Möbius transformations. Notice, however, that in contrast to the conformal algebra in this case, the Witt algebra is infinite dimensional. This is a remarkable feature that only arises in two dimensions and hence the reason we focus on two dimensional conformal field theory.

Now, everything up to this point in the discussion has involved classical field theory. We eventually will need to quantise the system described by a field theory. At the level of symmetries, if one wishes to 'quantise the symmetry algebra', then the concept of central extensions comes into play. See [54] for precise details.

Definition 4.1.7. Let $\mathfrak{a}$ and $\mathfrak{g}$ be Lie algebras such that $\mathfrak{a}$ is abelian. A central extension of $\mathfrak{g}$ by $\mathfrak{a}$ is an exact sequence of Lie algebras

$$0 \to \mathfrak{a} \xrightarrow{\iota} \mathfrak{h} \xrightarrow{\pi} \mathfrak{g} \to 0, \tag{4.35}$$

such that $[X, Y] = 0$ for all $X \in \mathfrak{a}$ and $Y \in \mathfrak{h}$. In other words, $\mathfrak{a}$ is comtained in the centre

of $\mathfrak{h}$. Here we identify $\mathfrak{a}$ as a subalgebra of $\mathfrak{h}$ and $\mathfrak{g}$ as the quotient $\mathfrak{h}/\mathfrak{a}$.

There is always the trivial extension, given by taking $\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{g}$. As the main example, we let $c \in \mathbb{C}$. Then a central extension of the Witt algebra by c is the Virasoro algebra

$$\mathrm{Vir}_c = \mathrm{W} \oplus c \cdot \mathbb{C}, \tag{4.36}$$

equipped with the commutation relations

$$[L_m, L_n] = (m - n)L_{m+n} + \delta_{m+n,0}\frac{c}{12}(m^3 - m). \tag{4.37}$$

This in fact is the unique central extension of the Witt algebra, up to a notion of equivalence for extensions. See [54] for example.

4.1.3 Primary Fields and The State-Operator Correspondence

We recall that we defined a chiral field $\varphi(z)$ as a holomorphic map $\varphi : \Sigma \to \mathbb{C}$ with Σ a Riemann surface.

Definition 4.1.8. Let $\varphi(z)$ be a chiral field and let $f : \mathbb{C} \to \mathbb{C}$ be a conformal transformation. Then φ is called a primary field of conformal dimension h if

$$\varphi(z) = \left(\frac{\partial f}{\partial z}\right)^h \varphi(f(z)). \tag{4.38}$$

For a chiral field φ of conformal dimension h, it is natural to expand this as

$$\varphi(z) = \sum_{n \in \mathbb{Z}} \varphi_n z^{-n-h}. \tag{4.39}$$

Upon quantisation, the modes become operators on some Hilbert space of states. For example, it turns out that the energy momentum tensor has expansion $T(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$ where L_n satisfy the Virasoro algebra.

We now suppose the existence of a vacuum state $|0\rangle$. Given a field $\varphi(z)$, we define the asymptotic in-state as

$$|\varphi\rangle := \lim_{z \to 0} \varphi(z) |0\rangle. \tag{4.40}$$

If we expand $\varphi(z) = \sum_{n \in \mathbb{Z}} \varphi_n z^{-n-h}$ then for this limit to be well defined we must have $\varphi_n |0\rangle = 0$ for $n > -h$ and hence we can write

$$|\varphi\rangle = \varphi_{-h} |0\rangle. \tag{4.41}$$

In fact, for conformal field theories the map $\varphi \mapsto |\varphi\rangle$ is a bijection.

For completeness, we define the Verma module as the set of states

$$\{L_{k_1} L_{k_2} \dots L_{k_n} |0\rangle \,|k_i \leq -2, n \in \mathbb{N}\}. \tag{4.42}$$

Theorem 4.1.9. For every state $|\varphi\rangle$ in the Verma module, there exists a field $F(z) \in \{T(z)\partial T(z), \dots, N(\cdots)\}$ such that

$$|\varphi\rangle = \lim_{z \to 0} F(z) |0\rangle. \tag{4.43}$$

For a proof of this theorem, see [56] and references therein. We remark that under this correspondence, the vacuum state corresponds to the identity operator.

Conversely, given any primary field $\varphi(z)$, we can construct an infinite family of descendant fields by taking derivatives and normally ordered products with derivatives of $T(z)$. This set of fields is called the conformal family

$$[\varphi(z)] = \{\varphi, \partial\varphi, \partial^2\varphi, N(T\varphi)\cdots\}, \tag{4.44}$$

where we shall define the normal ordering $N(\cdots)$ in section 4.1.4.

Furthermore, for every field F there exists a corresponding state $|\Phi\rangle = L_{k_1} L_{k_2} \dots L_{k_n} \varphi_{-h} |0\rangle$ with $k_i \leq -1$. For example, given a chiral field $\varphi(z) = \sum \varphi_n z^{-n-h}$, the corresponding state

is $|\varphi\rangle = \varphi_{-h}|0\rangle$. Furthermore, the operator corresponding to $L_{-1}|\varphi\rangle$ is $\partial_z\varphi$.

4.1.4   The Operator Product Expansion and Normal Ordering

Consider the coordinates (x^0, x^1) that parameterise a cylinder. If we complexify as $z = x^0 + ix^1$ then we identify $z \sim z + 2\pi i$. To simplify, we map this cylinder to the complex plane under the conformal map $w = e^z = e^{x^0}e^{ix^1}$. Thus, x^0 controls the radial direction, while x^1 controls the angular direction.

This becomes important when considering products of operators evaluated at different points. In quantum field theory, there is a notion of time ordering within correlation functions. Roughly speaking, this means the fields should be written from right to left in the order of increasing (proper) time coordinate. After mapping to the complex plane however, the time variable becomes a radial variable. Thus in conformal field theory we have a notion of radial ordering of fields that we define as

$$\mathcal{R}\Big(A(z)B(w)\Big) := \begin{cases} A(z)B(w) & |z| > |w| \\ B(w)A(z) & |w| > |z| \end{cases}. \tag{4.45}$$

For example, we have

$$\oint dz[A(z), B(w)] = \oint_{|z|>|w|} dz\, A(z)B(w) - \oint_{|w|>|z|} dz\, B(w)A(z) = \oint_{C(w)} \mathcal{R}\Big(A(z)B(w)\Big). \tag{4.46}$$

This uses the relation of contours shown in Figure 4.1.
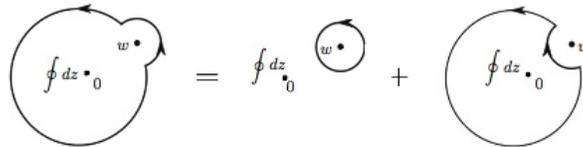
Figure 4.1: Relation between the contour $|z| > |w|$ on the left hand side and the contours $C(w)$ and $|z| < |w|$ on the right hand side [58].

Unless otherwise stated, we will always assume that products of fields are radially ordered

and usually omit the $\mathcal{R}$ to ease notation.

This brings us to the operator product expansion of fields. In general, the idea is that the product of fields evaluated at nearby points can be approximated by a sum of fields evaluated at one of the points. More precisely, if I is a countable index set, then the OPE is

$$A(z)B(w) = \sum_{k \in I} c_k(z - w)C^k(w), \tag{4.47}$$

where the coefficients $c_i(z - w)$ are functions depending only on the distance between where the fields A and B are evaluated and C^i are fields. This product should be understood as an equality of correlation functions; we have omitted the $\langle \cdots \rangle$ for convenience.

Theorem 4.1.10. Let $\varphi(z)$ be a chiral field. Then φ is a primary field of conformal dimension h if and only if it has the following operator product expansion with the energy momentum tensor $T(z)$,

$$T(z)\varphi(w) = \frac{h}{(z - w)^2}\varphi(w) + \frac{1}{z - w}\partial_w\varphi(w) + \cdots, \tag{4.48}$$

where $\cdots$ represents the non singular terms.

Corollary 4.1.11. Let $\varphi(z)$ be a chiral field. Let $T(z)$ be the energy momentum tensor with Laurent expansion $T(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$. Then φ is a primary field of conformal dimension h with Laurent expansion $\varphi(z) = \sum_{n \in \mathbb{Z}} \varphi_n z^{-n-h}$ if and only if φ_n satisfies the following commutation relation

$$[L_m, \varphi_n] = \big((h - 1)m - n\big)\varphi_{m+n}. \tag{4.49}$$

Proof. Expand $T(z)$ in a Laurent series $T(z) = \sum_{m \in \mathbb{Z}} L_m z^{-n-2}$ so that its Fourier modes are given by

$$L_m = \oint \frac{dz}{2\pi i} z^{m+1} T(z). \tag{4.50}$$

Similarly expand the field φ as $\varphi(w) = \sum_{n \in \mathbb{Z}} \varphi_n w^{-n-h}$ so that

$$\varphi_n = \oint \frac{dw}{2\pi i} w^{n+h-1} \varphi(w). \tag{4.51}$$

Now, using equation (4.46) for the radial ordering we have

$$[L_m, \varphi_n] = \oint \frac{dz}{2\pi i} z^{m+1} \oint \frac{dw}{2\pi i} w^{n+h-1} [T(z), \varphi(w)] \tag{4.52}$$

$$= \oint \frac{dw}{2\pi i} w^{n+h-1} \oint_{C(w)} \frac{dz}{2\pi i} z^{m+1} \mathcal{R}\Big(T(z), \varphi(w)\Big). \tag{4.53}$$

We now employ the operator product expansion, ignoring the holomorphic terms since these will integrate to zero. Hence we have

$$[L_m, \varphi_n] = \oint \frac{dw}{2\pi i} w^{n+h-1} \oint_{C(w)} \frac{dz}{2\pi i} z^{m+1} \Big(\frac{h}{(z-w)^2} \varphi(w) + \frac{1}{z-w} \partial_w \varphi(w)\Big). \tag{4.54}$$

Evaluating these integrals using the residue theorem yields

$$[L_m, \varphi_n] = \oint \frac{dw}{2\pi i} w^{n+h-1} \Big((m+1)h w^m \varphi(w) + w^{m+1} \partial_w \varphi(w)\Big). \tag{4.55}$$

Recognising the equation for φ_{m+n} and integrating the second term by parts yields

$$[L_m, \varphi_n] = \Big((m+1)h - (m+n+h)\Big)\varphi_{m+n} = \Big((h-1)m - n\Big)\varphi_{m+n}, \tag{4.56}$$

as required. $\qquad\square$

Using this corollary we can now loosely define $\mathcal{W}$ algebras. Thus far, we have considered algebras generated by the modes of chiral currents which have conformal dimension $h = 1$. We have also considered the Virasoro algebra which is generated by the modes of the chiral energy momentum tensor which is a chiral field of conformal dimension $h = 2$. We therefore define a $\mathcal{W}$-algebra as being generated by both the modes of the Virasoro algebra, L_n and also generated by the modes of chiral primary fields of conformal dimension $h \geq 3$. Now, suppose we have a $\mathcal{W}$-algebra, $\mathcal{W}^{(h_1,\ldots,h_n)}$, that is generated by modes of chiral primary fields $\mathcal{W}^{h_i}(z)$ for $i = 0, 1 \ldots, n$. Each field $\mathcal{W}^{h_i}(z)$ has conformal dimension h_i. We decompose the field $\mathcal{W}^{h_i}(z)$ as

$$\mathcal{W}^{h_i}(z) = \sum_{k \in \mathbb{Z}} \mathcal{W}_k^{h_i} z^{-k-h_i}. \tag{4.57}$$

The first commutation relations are

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}, \tag{4.58}$$

$$[L_m, \mathcal{W}_n^{h_i}] = ((h_i - 1)m - n)\mathcal{W}_{m+n}^{h_i}. \tag{4.59}$$

The first of these commutation relation is the statement that the modes L_n generate the Virasoro algebra, while the second is the statement that $\mathcal{W}^{h_i}(z)$ is a chiral primary field of dimension h_i. The commutation relations of the modes of the higher fields are more involved and depend on the specific fields in the algebra. For example, see [55].

Returning to the operator product expansion, such a notion gives rise to a concept of normal ordering. In the previous chapters, we saw a prescription of normal ordering for the modes of a field. For a field $\varphi(z)$ that has conformal dimension h and Laurent expansion $\varphi(z) = \sum_{n \in \mathbb{Z}} \varphi_n z^{-n-h}$, then φ_n with $n > -h$ are annihilation operators and φ_n with $n \leq -h$ are creation operators. We now give an alternative description of normal ordering in terms of the operator product expansion.

Definition 4.1.12. Suppose A and B are fields and let $\mathrm{sing}\Big(A(z)B(w)\Big)$ be the singular part of the operator product expansion of $A(z)B(w)$. Then the normal ordering $N\big(A(z)B(w)\big)$ is defined as

$$N\big(A(z)B(w)\big) = A(z)B(w) - \mathrm{sing}\Big(A(z)B(w)\Big). \tag{4.60}$$

The quantity $: AB : (w)$ is defined as

$$N(AB)(w) = \lim_{z \to w} N\big(A(z)B(w)\big). \tag{4.61}$$

For free fields, this definition of normal ordering coincides with the prescription in terms of creation and annihilation operators. This is a consequence of Wick's theorem. Hence for free fields, we use the notation $N(\cdots)$ and $: \cdots :$ interchangeably.

Let us now calculate a specific example of the normal ordering procedure that will be

used in chapter six. Consider the free bosonic field derived earlier in this section

$$X(z) = x_0 - ij_0 \log z + \sum_{n \in \mathbb{Z} \backslash \{0\}} \frac{j_n}{n} z^{-n}, \tag{4.62}$$

with $[j_0, x_0] = -i$ and $[j_n, j_m] = n\delta_{n+m,0}$. We wish to calculate

$$: e^{iX(z)} :: e^{iX(w)} : . \tag{4.63}$$

We note that operators of the form $e^{\phi(z)}$ are often called vertex operators in the literature. To proceed, we first prove the following lemma.

Lemma 4.1.13. Suppose that, given a field $A(z)$, we can write A as $A(z) = A^+(z) + A^-(z)$ where A^+ and A^- denote the creation and annihilation parts respectively. Suppose that $B(w)$ is given similarly and such that $[A, B]$ is a multiple of the identity operator. Then

$$: e^A :: e^B := e^{[A^-, B^+]} : e^{A+B} : . \tag{4.64}$$

Proof. Using the prescription of normal ordering we write

$$: e^A :: e^B := e^{A^+} e^{A^-} e^{B^+} e^{B^-}, \tag{4.65}$$

where we have put all creation operators to the left. Now, using the Baker-Campbell-Hausdorff formula, we observe that

$$e^{A^-} e^{B^+} = e^{[A^-, B^+]} e^{B^+} e^{A^-}, \tag{4.66}$$

where $e^{[A^-, B^+]}$ is a scalar since $[A, B]$ was assumed to be a multiple of the identity. It is for this same reason that there are no higher nested commutators in (4.66). Applying this to the previous equation, we find

$$e^{A^+} e^{A^-} e^{B^+} e^{B^-} = e^{[A^-, B^+]} e^{A^+} e^{B^+} e^{A^-} e^{B^-} = e^{[A^-, B^+]} e^{A^+ + B^+} e^{A^- + B^-}. \tag{4.67}$$

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The last equality here is valid since A^+ and B^+ commute, and similarly for A^- and B^-. However, by definition,

$$e^{A^+ + B^+} e^{A^- + B^-} =: e^{A+B} :, \tag{4.68}$$

and so the lemma follows. $\qquad\square$

Here we take $A(z) = iX(z)$ and $B(w) = iX(w)$ so that

$$A^- = j_0 \log z - \sum_{n>0} \frac{j_n}{n} z^{-n}, \tag{4.69}$$

and

$$B^+ = ix_0 - \sum_{m<0} \frac{j_m}{m} w^{-m}. \tag{4.70}$$

Here j_0 is an annihilation operator and we treat x_0 as a creation operator due to the commutation relation $[j_0, x_0] = -i$. Thus calculating the commutator $[A^-, B^+]$ we find

$$[A^-, B^+] = i(\log z)[j_0, x_0] + \sum_{\substack{n>0 \\ m<0}} \frac{1}{nm} \frac{1}{z^n w^m} [j_n, j_m]. \tag{4.71}$$

Employing the canonical commutation relations we obtain

$$[A^-, B^+] = \log z + \sum_{\substack{n>0 \\ m<0}} \frac{1}{m} \frac{1}{z^n w^m} \delta_{m+n,0} = \log z - \sum_{n>0} \frac{1}{n} \left(\frac{w}{z}\right)^n, \tag{4.72}$$

where we have assumed $|w| < |z|$ so that the above series converges. We recognise this series as the Taylor expansion of the logarithm. Thus

$$[A^-, B^+] = \log z + \log\left(1 - \frac{w}{z}\right) = \log(z - w). \tag{4.73}$$

Hence, using the above lemma, we find that

$$: e^{iX(z)} :: e^{iX(w)} := (z - w) : e^{iX(z) + iX(w)} : . \tag{4.74}$$

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This example readily generalises to the following proposition.

Proposition 4.1.14. Let $X(z)$ be the field describing a free, chiral boson defined above. Then

$$\prod_{j=1}^{N} : e^{iX(z_j)} := \prod_{m<n}^{N} (z_m - z_n) : e^{\sum_{j=1}^{N} iX(z_j)} : . \tag{4.75}$$

Proof. This follows by an induction argument. Indeed, we have already shown the base case. Now, assume that the proposition is true for some k. Then

$$: e^{iX(z_{k+1})} : \prod_{j=1}^{k} : e^{iX(z_j)} := \prod_{m<n}^{k} (z_m - z_n) : e^{iX(z_{k+1})} :: e^{\sum_{j=1}^{k} iX(z_j)} : . \tag{4.76}$$

Now using the definition of normal ordering as before, the right hand side yields

$$\prod_{m<n}^{k} (z_m - z_n) e^{iX^+(z_{k+1})} e^{iX^-(z_{k+1})} e^{\sum_{j=1}^{k} iX^+(z_j)} e^{\sum_{j=1}^{k} iX^-(z_j)}. \tag{4.77}$$

The terms in the sum in the exponential all commute and so this can be split as a product of exponentials. Now, using Lemma 4.1.13, commuting $e^{iX^-(z_{k+1})}$ with each $e^{iX^+(z_j)}$ produces a factor $(z_j - z_{k+1})$. Thus we have

$$\prod_{j=1}^{k+1} : e^{iX(z_j)} := \prod_{m<n}^{k+1} (z_m - z_n) : e^{\sum_{j=1}^{k+1} iX(z_j)} : . \tag{4.78}$$

Thus, the proposition follows by induction. $\qquad\square$

To prove this proposition, one can also use Wick's theorem as shown in [59].

At this point, we have stated the most important properties of a conformal field theory: the Virasoro algebra, the energy momentum tensor, the Fock space of states, the state-operator correspondence, the operator product expansion and the concept of normal ordering. It will be useful for later sections to introduce here the notion of a vertex operator algebra. Vertex operator algebras were first introduced as an attempt to axiomise chiral conformal field theory, although their use has transcended this purpose and they are now a subject of research in their own right.

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Definition 4.1.15. A vertex operator algebra is a quadruple $(V, Y, \mathbb{1}, \omega)$ such that the following axioms hold.

- The Fock space V is a $\mathbb{Z}$-graded vector space

$$V = \bigsqcup_{k \in \mathbb{Z}} V_k, \tag{4.79}$$

where $\deg v = k$ for $v \in V_k$, and is such that $\dim V_k$ is finite for every k and $V_k = 0$ for sufficiently negative k.

- The state operator correspondence $Y \in (\text{End } V)[[z, z^{-1}]]$ is a linear map given by

$$v \mapsto Y(v, z) = \sum_{n \in \mathbb{Z}} v_n z^{-n-1}, \tag{4.80}$$

for all $v \in V$ and where the creation and annihilation operators v_n belong to End V. Moreover, for every $u, v \in V$ there exists $N \in \mathbb{N}$ such that the Borcherds identity, or equivalently the operator product expansion, holds,

$$(z - w)^N Y(u, z) Y(v, w) = (z - w)^N Y(v, z) Y(u, w). \tag{4.81}$$

- The vacuum state $\mathbb{1} \in V_0$ satisfies $Y(\mathbb{1}, z) = \mathbb{1}_V$.

- The conformal state ω gives rise to the chiral energy momentum tensor

$$Y(\omega, z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}, \tag{4.82}$$

where $L_0 v = (\deg v) \cdot v$ and L_n generate the Virasoro algebra with central charge c. Furthermore,

$$Y(L_{-1} v, z) = \frac{d}{dz} Y(v, z), \tag{4.83}$$

for all $v \in V$.

We will revisit this definition when we construct higher quantum Airy structures.

4.2 Airy Structures

The concept of Airy structures will now allow us to prove the existence and uniqueness of the solution to the differential equations $H_i Z = 0$ that we have encountered previously. We motivate this problem by first giving a brief sketch of classical Airy structures. Canonical quantisation will then produce quantum Airy structures which will indeed lead to non-trivial solutions of such differential equations. We will also explain the particularly fruitful relationship between this procedure and the well known Eynard-Orantin recursion. Here, we follow the discussions of [60–62].

4.2.1 Kontsevich - Soibelman Recursion

We begin with a definition.

Definition 4.2.1. Let (V, ω) be a symplectic vector space over $\mathbb{C}$ of dimension $2d$ with symplectic form ω. A Lagrangian subspace L is a subspace with $\dim L = d$ and $\omega|_L = 0$.

Given a symplectic vector (V, ω), we let f, g be two smooth functions V. Recall that the Poisson bracket is defined as

$$\{f, g\} = \omega(df, dg) = \omega_{ij} \partial_i f \partial_j g. \tag{4.84}$$

Here and throughout the reaming sections of this chapter, we employ Einstein summation convention. Recall also, that we can always find Darboux coordinates (x_i, y_i) for $i = 1, \ldots, d$ such that

$$\omega = dy_i \wedge dx_i. \tag{4.85}$$

Notice that coordinates satisfy the canonical relations

$$\{x_i, x_j\} = \{y_i, y_j\} = 0, \tag{4.86}$$

and

$$\{y_i, x_j\} = \delta_{ij}. \tag{4.87}$$

From this form, we see that the simplest way to define a Lagrangian subspace is to set $y_i = 0$, or equivalently $x_i = 0$, for every i. Now consider the, at most quadratic, Hamiltonians

$$H_i = y_i - \frac{1}{2}A^i_{jk}x_j x_k - B^i_{jk}x_j y_k - \frac{1}{2}C^i_{jk}y_j y_k, \tag{4.88}$$

with $i = 1, \ldots, d$ and $A^i_{jk}, B^i_{jk}, C^i_{jk} \in \mathbb{C}$. The vanishing of the Hamiltonians $H_i = 0$ then defines a subspace. For example, if we have $A_{ijk} = B^k_{ij} = C^{jk}_i = 0$ for every i, j and k, then we are indeed left with the Lagrangian subspace defined by $y_i = 0$ as above. In general, however, the subspace defined by $H_i = 0$ need not necessarily be Lagrangian. Nevertheless, the following proposition is proven in [61].

Proposition 4.2.2. Let H_i be defined as above. The subspace defined by $H_i = 0$ for all $i = 1, \ldots, d$ is Lagrangian if and only if the H_i are closed under the Poisson bracket. That is

$$\{H_i, H_j\} = f^k_{ij}H_k, \tag{4.89}$$

where $f^k_{ij} \in \mathbb{C}$ are the structure constants.

This motivates the following definition.

Definition 4.2.3. Let (V, ω) be a symplectic vector space and let (x^i, y_i) be Darboux coordinates as above. A *classical Airy structure* is a sequence of at most quadratic polynomials H_i of the form

$$H_i = y_i - \frac{1}{2}A^i_{jk}x^j x^k - B^i_{jk}x^j y_k - \frac{1}{2}C^{jk}_i y_j y_k, \tag{4.90}$$

with $A^i_{jk}, B^i_{jk}, C^i_{jk} \in \mathbb{C}$ such that

$$\{H_i, H_j\} = f^k_{ij}H_k. \tag{4.91}$$

Let us now consider a quantisation of these classical Airy structures. As was mentioned

in the chapter two, the procedure of canonical quantisation involves lifting Poisson brackets to commutators and coordinates to operators. Here, we will take the operators

$$\hat{x}_i = x_i, \tag{4.92}$$

and

$$\hat{y}_i = \hbar\frac{\partial}{\partial x_i}, \tag{4.93}$$

where $\hbar$ is a formal constant. We observe that

$$[\hat{y}_i, \hat{x}_j] = \hbar\delta_{ij}, \tag{4.94}$$

which is simply the quantisation of equation (4.87). Upon attempting to quantise the classical Airy structure (4.90), one immediately notices that there is an ambiguity in the ordering of the B terms. Reordering these terms using the commutation relation results in an extra constant term. This motivates the following definition of a *quantum* Airy structure.

Definition 4.2.4. Let V be a vector space over $\mathbb{C}$, not necessarily finite dimensional. Let $(x_i)_{i \in I}$ be coordinates with respect to a basis $(e_i)_{i \in I}$. Let $\hbar$ be a formal constant. Define the *Weyl algebra* $\mathcal{W}_V^\hbar$ by

$$\mathcal{W}_V^\hbar = \frac{\mathbb{C}[\hbar]\langle\{x_i, \partial_{x_i} | i \in I\}\rangle}{[\hbar\partial_{x_i}, x_i] = \hbar}. \tag{4.95}$$

Let $A_{jk}^i, B_{jk}^i, C_{jk}^i, D^i \in \mathbb{C}$ be scalars indexed by the set I. A sequence $(L_i)_{i \in I} \subset \mathcal{W}_V^\hbar$ is called a *quantum Airy structure* if the L_i are of the form

$$L_i = \hbar\partial_{x_i} - \frac{1}{2}A_{jk}^i x_j x_k - \hbar B_{jk}^i x_j \partial_{x_k} - \frac{1}{2}\hbar^2 C_{jk}^i \partial_{x_j}\partial_{x_k} - \hbar D^i, \tag{4.96}$$

such that they form a Lie algebra

$$[L_i, L_j] = \hbar f_{ij}^k L_k. \tag{4.97}$$

Remark. We will not concern ourselves about convergence issues of these infinite series.

Indeed, we shall only consider quantum Airy structures that define tau functions and matrix integrals as formal series in any case.

By the symmetry of $x_j x_k$ and $\partial_{x_j} \partial_{x_k}$ terms under the interchange of j and k, we can assume that A and C have zero antisymmetric part. That is, $A^i_{jk} = A^i_{kj}$ and $C^i_{jk} = C^i_{kj}$. The condition that the quantum Airy structure (L_i) forms a Lie algebra places several constraints on A, B, C, D and f. For example, substituting the explicit form of L_i into both sides of (4.97) and comparing coefficients of ∂_{x_k} and x_k, we find

$$f^k_{ij} = B^i_{jk} - B^i_{kj}, \tag{4.98}$$

and

$$A^i_{jk} = A^j_{ik}. \tag{4.99}$$

We observe that A is fully symmetric in its indices. These relations allow us to eliminate f^k_{ij} in the remaining equations. Indeed, inspecting coefficients of the constant term yields

$$B^i_{ja} D^a + C^i_{ab} A^j_{ab} = (i \leftrightarrow j), \tag{4.100}$$

where $(i \leftrightarrow j)$ means the same expression as the left hand side with i and j swapped.

Identification of the terms $x_k x_l$, $x_k \partial_{x_l}$ and $\partial_{x_k} \partial_{x_l}$ yield the following three relations

$$B^i_{ja} A^a_{kl} + B^i_{ka} A^j_{al} + B^i_{la} A^j_{ak} = (i \leftrightarrow j), , \tag{4.101}$$

$$B^i_{ja} B^a_{kl} + B^i_{ka} B^j_{al} + C^i_{la} A^j_{ak} = (i \leftrightarrow j), \tag{4.102}$$

and

$$B^i_{ja} C^a_{kl} + C^i_{ka} B^j_{al} + C^i_{la} B^j_{ak} = (i \leftrightarrow j). \tag{4.103}$$

We assign a grading on $\mathcal{W}^\hbar_V$ such that $\deg(x_i) = \deg(\hbar \partial_{x_i}) = 1$ and $\deg(\hbar) = 2$. With this grading, the elements L_i are at most quadratic. The Weyl algebra $\mathcal{W}^\hbar_V$ naturally acts as differential operators on $C^\infty(V)$, the smooth functions on V. In analogy with the Lagrangian

subspaces defined by classical Airy structures, we can ask the question whether there exists a unique solution to $L_i Z = 0$ for all i. The answer to this question is given as affirmative by the next theorem.

Theorem 4.2.5. Given a family of differential equations $L_i Z = 0$, $i \in I$ where the operators L_i are a quantum Airy structure $(L_i)_{i \in I}$, there exists a unique formal solution

$$Z = \exp(F) = \exp \Big(\sum_{g=0}^{\infty} \sum_{n=1}^{\infty} \frac{\hbar^{g-1}}{n!} \sum_{i_1,\dots,i_m \in I} F_{g,n}(i_1,\dots,i_n) x_{i_1} \cdots x_{i_n} \Big), \qquad (4.104)$$

where $F_{g,n}(i_1,\dots,i_n)$ are scalars and completely symmetric in the indices $i_1,\dots,i_n$. Explicitly, $F_{0,1}(i) = F_{0,2}(i,j) = 0$ for all i and j,

$$F_{0,3}(i,j,k) = A^i_{jk}, \quad F_{1,1}(i) = D^i, \qquad (4.105)$$

and for $\chi_{g,n} := 2g - 2 + n \geq 2$,

$$F_{g,n}(i_1, J) = \sum_{m=2}^{n} B^{i_1}_{i_m,a} F_{g,n-1}(a, J \setminus \{i_m\}) \qquad (4.106)$$

$$+ \frac{1}{2} C^{i_1}_{ab} \Big(F_{g-1,n+1}(a,b,J) + \sum_{\substack{J_1 \sqcup J_2 = I \\ h_1 + h_2 = g}} F_{h_1,1+|J_1|}(a, J_1) F_{h_2,1+|J_2|}(b, J_2) \Big). \qquad (4.107)$$

where $J = \{i_2,\dots,i_n\}$ is an $(n-1)$ tuple of indices in I.

Proof. We assume the existence of such a solution and we prove uniqueness. We substitute the expression (4.157) for Z into $L_i Z = 0$. We decompose F in two stages. As suggested by the existence of a solution we write

$$F = \sum_{g=0}^{\infty} \hbar^{g-1} F_g. \qquad (4.108)$$

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By comparing coefficients of $\hbar^g$ in $e^{-F}L_i e^F = 0$ we calculate

$$\partial_{x_i} F_g = \delta_{g,0} \frac{1}{2} A^i_{jk} x_j x_k + B^i_{jk} x_j \partial_{x_k} F_g + \frac{1}{2} C^i_{jk} \left(\partial_{x_j} \partial_{x_k} + \sum_{g_1+g_2=g} \partial_{x_j} F_{g_1} \partial_{x_k} F_{g_2} \right) + \delta_{g,1} D^i. \quad (4.109)$$

We now decompose further according to (4.157),

$$F_g = \sum_{n=1}^{\infty} \frac{1}{n!} F_{g,n}(i_1, \ldots, i_n) x_{i_1} \cdots x_{i_n}. \quad (4.110)$$

Now, we fix $i_2, \ldots, i_n \in I$ and compare coefficients of $\frac{x_{i_2} \cdots x_{i_n}}{(n-1)!}$. By the existence hypothesis, we have that $F_{g,n}$ is symmetric and so we find

$$F_{g,n}(i_1, \ldots, i_n) = \delta_{g,0} \delta_{n,3} \frac{1}{2} A^{i_1}_{i_2,i_3} + \delta_{g,1} \delta_{n,1} D^{i_1} + \sum_{m=2}^{n} B^{i_1}_{i_m,a} F_{g,n-1}(a, J \setminus \{i_m\}) \quad (4.111)$$

$$+ \frac{1}{2} C^{i_1}_{ab} \left(F_{g-1,n+1}(a, b, J) + \sum_{\substack{J_1 \sqcup J_2 = I \\ h_1+h_2=g}} F_{h_1,1+|J_1|}(a, J_1) F_{h_2,1+|J_2|}(b, J_2) \right). \quad (4.112)$$

For $(g, n) = (0, 1)$ and $(g, n) = (0, 2)$ we see that

$$0 = -F_{0,1}(i_1) + C^i_{ab} F_{0,1}(a) F_{0,1}(b), \quad (4.113)$$

and

$$0 = -F_{0,2}(i_1, i_2) + B^{i_1}_{i_2,a} F_{0,1}(a) + C^{i_1}_{ab} F_{0,1}(a) F_{0,2}(b, i_2). \quad (4.114)$$

As we take $F_{0,1} = F_{0,2} = 0$, these equations are trivially satisfied. For $(g, n) = (0, 3)$ we find

$$0 = -F_{0,3}(i_1, i_2, i_3) + A^{i_1}_{i_2,i_3} + B^{i_1}_{i_2,a} F_{0,2}(a, i_3) + B^{i_1}_{i_3,a} F_{0,2}(a, i_2) \quad (4.115)$$

$$+ \frac{1}{2} C^{i_1}_{ab} \left(F_{0,1}(a) F_{0,2}(b, i_2, i_3) + F_{0,2}(a, i_2) F_{0,2}(b, i_3) \right). \quad (4.116)$$

This reduces to

$$F_{0,3}(i_1, i_2, i_3) = A^{i_1}_{i_2,i_3}. \quad (4.117)$$

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Similarly, for $(g, n) = (1, 1)$ we find

$$F_{1,1}(i_1) = D^{i_1}. \tag{4.118}$$

It is furthermore true in general that isolating $F_{g,n}(i_1, \ldots, i_n)$ for $\chi_{g,n} \geq 2$ gives the desired result. This proves uniqueness.

The above argument can also be used to prove the existence of a solution of the form (4.104). There is, nevertheless, an added subtlety in that there is still the issue of showing that $F_{g,n}(i_1, \ldots, i_n)$ constructed above is symmetric in the indices $i_1, \ldots, i_n$. The symmetry is not immediately obvious since at first glance, i_1 seems to play a special role in equation (4.111). It is clear that $F_{g,n}$ is symmetric in the indices $i_2, \ldots, i_n$. Thus it suffices to prove symmetry in i_1 and i_2. Here, we will only give a sketch of the proof.

We have that $F_{0,3}(i_1, i_2, i_3) = A^{i_1}_{i_2, i_3}$ is symmetric since $A^{i_1}_{i_2, i_3}$ is fully symmetric in its indices as justified earlier. We now examine the case where $\chi_{g,n} = 2$ which means that either $(g, n) = (0, 4)$ or $((g, n) = (2, 1)$. Consider $(g, n) = (0, 4)$. From equation (4.111) we find that

$$F_{0,4}(i_1, i_2, i_3, i_4) = B^{i_1}_{i_2, a} F_{0,3}(a, i_3, i_4) + B^{i_1}_{i_3, a} F_{0,3}(a, i_2, i_4) + B^{i_1}_{i_4, a} F_{0,3}(a, i_2, i_3). \tag{4.119}$$

Employing $F_{0,3}(i, j, k) = A^i_{jk}$ which is fully symmetric, we obtain

$$F_{0,4}(i_1, i_2, i_3, i_4) = B^{i_1}_{i_2, a} A^a_{i_3, i_4} + B^{i_1}_{i_3, a} A^{i_2}_{a, i_4} + B^{i_1}_{i_4, a} A^{i_2}_{a, i_3}. \tag{4.120}$$

We recognise the right hand side of this equation as the relation (4.101) which is indeed symmetric in i_1 and i_2. Thus $F_{0,4}$ is symmetric. Similarly, for $(g, n) = (2, 1)$ we find

$$F_{2,1}(i_1, i_2) = B^{i_1}_{i_2, a} D^a + \frac{1}{2} C^{i_1}_{ab} F_{0,3}(a, b, i_2) = B^{i_1}_{i_2, a} D^a + \frac{1}{2} C^{i_1}_{ab} A^a_{b, i_2}. \tag{4.121}$$

We recognise this as the left hand side of the relation (4.100) which is indeed invariant under the interchange of i_1 and i_2. Thus, for $\chi_{g,n} = 2$, we have shown that $\chi_{g,n}$ is symmetric. The

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general result relies on proof by induction on $\chi_{g,n}$. We omit the technical details as they are rather tedious. A full proof is given in [60]. □

Remark. Consider the equation defining $F_{g,n}$. If $\chi_{g,n} = 2g - 2 + n$ defines the Euler characteristic on the left hand side, the right hand side contains terms of Euler characteristic $\chi_{g,n} - 1$. Hence this is indeed a recursive formula on $\chi_{g,n}$. Here, we make a minor abuse of vocabulary by calling $2g - 2 + n$ the Euler characteristic. The true Euler characteristic $\tilde{\chi}_{g,n}$ is given as $\tilde{\chi}_{g,n} = -\chi_{g,n}$. This terminology is standard in the literature however.

The previous theorem now proves that every Airy structure has a unique solution Z of the form (4.104). We can now reasonably ask questions about whether this formal series contains any enumerative information as a generating function, and if so, precisely what this geometric information is. To give a hint of an answer to these questions, we will make a brief foray to Eynard-Orantin topological recursion.

4.2.2 Eynard - Orantin Recursion

One may notice several structural similarities in the proof of Theorem 4.2.5 and the solution of Hermitian matrix models via loop equations. This is indeed no coincidence. We place topological recursion seen previously in a more general context, independent of matrix models and referring only to the spectral curve. We will allow the curve to have arbitrary genus, although we will still restrict attention to curves with simple ramification points. This topological recursion has since been generalised to include curves with ramification points of arbitrary order [63]. Starting from the spectral curve is perhaps more natural since we recall the topological recursion formula given the previous chapter was independent of the potential of the matrix model. Lastly, we will state the relationship, without proof, between the Eynard-Orantin recursion and Kontsevich-Soibelman recursion. We follow [60, 64].

Definition 4.2.6. An *algebraic spectral curve* is a quadruple (Σ, x, y, B) consisting of a compact Riemann surface Σ, two meromorphic functions x and y on Σ, and a symmetric, meromorphic, bilinear differential B on $\Sigma \times \Sigma$ whose only singularity is a double pole given in local coordinates as $z_1 \to z_2$ with no residue. If $\Sigma = \mathbb{P}^1$, the spectral curve is called rational.

The bilinear differential B is sometimes called the Bergmann kernel and is in fact unique, up to a multiplicative scalar, if we also enforce the normalisation,

$$\oint_{\mathcal{A}_i} B(z_1, z_2) = 0. \tag{4.122}$$

Here, $\{\mathcal{A}_i, \mathcal{B}_i | i = 1, \ldots g\}$ is a symplectic basis of cycles. See [65]. For rational curves such as the hyperelliptic curve considered in the previous chapter, one can always find a rational parameterisation $x(z)$ and $y(z)$ where z is a local coordinate on $\mathbb{P}^1$. That is to say, $x(z)$ and $y(z)$ are rational functions of z. Recall that (simple) ramification points a_i are the (simple) zeroes of $dx(z)$. We assume that $dy(z)$ does not vanish at a_i. This means in a neighbourhood of $x(a_i)$, we have the square root behaviour seen from the Taylor expansion,

$$y(z) \sim y(a_i) + C\sqrt{x(z) - x(a_i)}. \tag{4.123}$$

For this reason, a_i are often also called branch points.

We now give a family of examples of spectral curves which are the main curves that will become important later on. In all of these examples, we choose $\Sigma = \mathbb{P}^1$ and the bilinear differential as

$$B(z_1, z_2) = \frac{dz_1 dz_2}{(z_1 - z_2)^2}. \tag{4.124}$$

The spectral curves that will play a crucial role are indexed by (r, s) with $r \in \mathbb{Z}_{\geq 2}$ and $s = \{1, \ldots, r+1\}$ such that $r = \pm 1 \mod s$. The curves themselves are parameterised by

$$x = \frac{z^r}{r}, \; y = -\frac{1}{z^{r-s}}. \tag{4.125}$$

Note that $s = r + 1$ is always valid. In this case, we have the *r-Airy curve*

$$x = \frac{z^r}{r}, \; y = -z. \tag{4.126}$$

In the situation that $r = 2$, we call this simply the Airy curve. Similarly, $s = r - 1$ is always

valid and we have the *r-Bessel curve*

$$x = \frac{z^r}{r}, \quad y = -\frac{1}{z}. \tag{4.127}$$

Again in the situation $r = 2$, we call this simply the Bessel curve. The nomenclature will be justified later.

We now concentrate on the case of the Airy curve, $(r, s) = (2, 3)$. In this situation, inspired by the solution of matrix models, we proceed to define the *Eynard-Orantin* recursion.

Definition 4.2.7. Let $\iota(z) = -z$ be a holomorphic involution that is locally defined. The *recursion kernel* $K(z, w)$ is defined by

$$K(z, w) = -\frac{\int_{\iota(w)}^{w} B(z, z')}{\big(y(w) - y(\iota(w))\big)dx(w)}. \tag{4.128}$$

Definition 4.2.8. Let (Σ, x, y, B) be a spectral curve with simple ramification points a_i. Define the following meromorphic forms

$$\omega_1^{(0)}(z) := -y(z)dx(z), \tag{4.129}$$

and

$$\omega_2^{(0)}(z_1, z_2) := B(z_1, z_2). \tag{4.130}$$

For $2g - 2 + n \geq 0$, we define the multilinear differentials via the Eynard-Orantin recursion,

$$\omega_{n+1}^{(g)}(z_0, J) = \sum_i \operatorname*{Res}_{z \to a_i} K(z_0, z) \bigg(\sum_{h=0}^{g} \sideset{}{'}\sum_{I \subseteq J} \omega_{1+|I|}^{(h)}(z, I) \omega_{n+1-|I|}^{(g-h)}(\iota(z), J \setminus I) + \omega_{n+2}^{g-1}(z, \iota(z), J) \bigg), \tag{4.131}$$

where $J = \{z_1, \ldots, z_n\}$. The prime on the summation indicates that, as before in section 3.3.3, we exclude the cases $(h, I) = (0, \varnothing)$ and $(h, I) = (g, J)$.

We also define the 'stable' free energies as

$$\tilde{F}_g = \omega_0^{(g)} := \frac{1}{2 - 2g} \sum_i \operatorname{Res}_{z \to a_i} \Phi(z) \omega_1^{(g)}(z), \tag{4.132}$$

for $g \geq 2$. Here $\Phi(z)$ is any function defined locally near a_i and is such that $d\Phi = ydx$. The residue in the above equation means that the $\tilde{F}_g$ do not depend on this choice of Φ. See [64] and references therein for a proof of this fact. The definitions for $\tilde{F}_0$ and $\tilde{F}_1$ are slightly more involved and we refer the reader to [64]. These free energies are often called symplectic invariants. One can reparameterise the curve Σ via a symplectomorphism. We recall that a symplectomorphism between two spectral curves (Σ, x, y) and (Σ', x', y') is a holomorphic map $\Sigma \to \Sigma'$ that preserves the symplectic form $dx \wedge dy = dx' \wedge dy'$. To see why the F_g are called symplectic invariants, we note that the Bergmann kernel only depends on the complex structure of Σ and not on the functions x and y. Furthermore, the recursion kernel $K(z, w)$ only depends on x and y through the combination

$$[y(w) - y(\iota(w))]dx(w). \tag{4.133}$$

If this combination is left unchanged, the $\tilde{F}_g$ are left unchanged. In particular, the $\tilde{F}_g$ are invariant under the following transformations.

- $y \mapsto y + R(x)$ for some rational function $R(x)$.

- $y \mapsto \lambda y$ and $x \mapsto x/\lambda$ for some $\lambda \in \mathbb{C}^*$.

- $x \mapsto \frac{ax+b}{cx+d}$ and $y \mapsto \frac{(cx+d)^2}{ad-bc}y$.

It is readily checked that these transformations are all symplectomorphisms. Despite this, to generate the full group of symplectomorphisms, one would also need to check the invariance of $\tilde{F}_g$ under the transformation $x \mapsto y$ and $y \mapsto -x$. This was apparently shown in [66]. However, a simple counterexample to this statement was constructed in [67] showing that the quantities $\tilde{F}_g$ are not invariant under the $x - y$ transformation for all spectral curves. In

general this is still poorly understood. However, spectral curves arising from matrix models do typically obey this $x - y$ invariance. See [68]. Therefore, given a Hermitian matrix model

$$Z = \int_{H_N} dM e^{-\frac{N}{t} \mathrm{Tr}\, V(M)}, \tag{4.134}$$

with free energy topological expansion,

$$F = \log Z = \sum_{g=0}^{\infty} \left(\frac{N}{t}\right)^{2-2g} F_g, \tag{4.135}$$

the quantities F_g typically coincide with the $\tilde{F}_g$'s calculated via topological recursion applied to the spectral curve Σ associated to Z. For a thorough discussion, see [64].

4.2.3 Correspondence Between Eynard-Orantin and Kontsevich-Soibelman Recursions

Having defined Eynard-Orantin and Kontsevich-Soibelman recursion, we can now briefly tie the two together. In this section we omit many of the proofs, although they are somewhat standard and can be found in [60].

Let (Σ, x, y, B) be a spectral curve. Let $p \in \Sigma$ be in a neighbourhood of a ramification point r. We define a meromorphic one form

$$\xi_{k,r}(p) = \operatorname*{Res}_{q \to r} \int_r^q B(\cdot, p) \frac{(2k+1)dz(q)}{z^{2k+2}(q)}, \tag{4.136}$$

with $k \in \mathbb{N}$

Lemma 4.2.9. Let $\omega_{g,n}(J)$ for $2g + n \geq 2$ be a multilinear differential calculated from the Eynard-Orantin recursion from the spectral curve (Σ, x, y, B). Let $\{\xi_{k_i,r_i}(p_i)\}$ be a set of differentials defined as above at each ramification point r_i. Then there exists a unique

decomposition of $\omega_{g,n}(J)$ as a finite sum with

$$\omega_{g,n}(J) = \sum_{r_i,k_i} W_{g,n} \begin{bmatrix} r_1,\dots,r_n \\ k_1,\dots,k_n \end{bmatrix} \prod_{i=1}^{n} \xi_{k_i,r_i}(p_i). \tag{4.137}$$

A proof is given in [60]. We then define the dual basis

$$\xi_{k,r}^*(p) = \frac{z^{2k+1}}{2k+1}, \quad \operatorname*{Res}_{p\to r} \xi_{k,r}(p)\xi_{l,r'}^* = \delta_{kl}\delta_{rr'}. \tag{4.138}$$

We now define

$$\theta(p) := -\frac{2}{\big(y(p)-y(\iota(p))\big)dx(p)}. \tag{4.139}$$

We expand θ as

$$\theta(p) = \sum_{m\geq -1} t_{m,r} z^{2m}(p) \frac{1}{dz(p)}. \tag{4.140}$$

We can then define an expansion of B when p_1 and p_2 are in a neighbourhood of ramification points r_1 and r_2 respectively:

$$B(p_1,p_2) = \left(\frac{\delta_{r_1 r_2}}{\big(z(p_1)-z(p_2)\big)^2} + \sum_{l_1,l_2\geq 0} \varphi_{0,2} \begin{bmatrix} r_1,\ r_2 \\ l_1,\ l_2 \end{bmatrix} z^{l_1}(p_1)z^{l_2}(p_2) \right) dz(p_1)dz(p_2). \tag{4.141}$$

. We then construct

$$A^{(k_1,r_1)}_{(k_2,r_2),(k_3,r_3)} = \operatorname*{Res}_{q\to r_1} \left(\xi_{k_1,r_1}^*(q)d\xi_{k_2,r_2}^*(q)d\xi_{k_3,r_3}^*(q)\theta(q) \right), \tag{4.142}$$

$$B^{(k_1,r_1)}_{(k_2,r_2),(k_3,r_3)} = \operatorname*{Res}_{q\to r_1} \left(\xi_{k_1,r_1}^*(q)d\xi_{k_2,r_2}^*(q)\xi_{k_3,r_3}(q)\theta(q) \right), \tag{4.143}$$

$$C^{(k_1,r_1)}_{(k_2,r_2),(k_3,r_3)} = \operatorname*{Res}_{q\to r_1} \left(\xi_{k_1,r_1}^*(q)\xi_{k_2,r_2}(q)\xi_{k_3,r_3}(q)\theta(q) \right), \tag{4.144}$$

$$D_{(k,r)} = \delta_{k,0} \left(\frac{1}{2} t_{-1,r} \varphi_{0,2} \begin{bmatrix} r,\ r \\ 0,\ 0 \end{bmatrix} + \frac{1}{8} t_{0,r} \right) + \frac{1}{24} \delta_{k,1} t_{-1,r}. \tag{4.145}$$

We then have the following proposition.

Proposition 4.2.10. Let $F_{g,n}\big((k_1,r_1),\dots,(k_n,r_n)\big)$ be the free energies computed by the

Kontsevich-Soibelman recursion from the A, B, C and D tensors above. Then

$$F_{g,n}\big((k_1, r_1), \ldots, (k_n, r_n)\big) = W_{g,n} \begin{bmatrix} r_1, \ldots, r_n \\ k_1, \ldots, k_n \end{bmatrix}, \tag{4.146}$$

where the quantities $W_{g,n}$ are calculated from the Eynard-Orantin recursion as in the above lemma.

As an example of this construction, consider the Airy curve,

$$x = \frac{z^2}{2}, \ y = -z, \tag{4.147}$$

with the unique ramification point $z = 0$. The corresponding ξ and ξ^* basis is

$$\xi_k^*(z) = \frac{z^{2k+1}}{2k+1}, \ \xi_k(z) = \frac{(2k+1)dz}{z^{2k+2}}, \tag{4.148}$$

while θ is given by

$$\theta(z) = \frac{1}{z^2 dz}. \tag{4.149}$$

In particular, the coefficients $\varphi_{0,2} \begin{bmatrix} r_1, \ r_2 \\ l_1, \ l_2 \end{bmatrix}$ all vanish. The above construction of A, B, C and D then yield

$$A_{jk}^i = \delta_{i=j=k=1}, \tag{4.150}$$

$$B_{jk}^i = \frac{2k-1}{2i-1} \delta_{i+j-2,k}, \tag{4.151}$$

$$C_{jk}^i = \frac{(2j-1)(2k-1)}{2i-1} \delta_{i,j+k+1}, \tag{4.152}$$

$$D^i = \frac{1}{24} \delta_{i,2}. \tag{4.153}$$

We will revisit this quantum Airy structure from the point of view of vertex operator algebras in the next section.

4.3 Higher Airy Structures

4.3.1 Construction from the Heisenberg Vertex Operator Algebra

We will now generalise quantum Airy structures to the so-called higher Airy structures. In the definition of a quantum Airy structure, we restricted attention to elements of the Weyl algebra of degree two. To generalise, therefore, we remove this restriction and consider elements of degree higher than two. We recall that Weyl algebra $\mathcal{W}_V^\hbar$ on a vector space V is given by

$$\mathcal{W}_V^\hbar = \frac{\mathbb{C}[\hbar]\langle\{x_i, \partial_{x_i} | i \in I\}\rangle}{[\hbar\partial_{x_i}, x_i] = \hbar}, \tag{4.154}$$

and we assign a grading on $\mathcal{W}_V^\hbar$ such that $\deg(x_i) = \deg(\hbar\partial_{x_i}) = 1$ and $\deg(\hbar) = 2$.

Definition 4.3.1. Let V be a vector space over $\mathbb{C}$ and $r \in \mathbb{Z}_{\geq 2}$. Let $\mathcal{W}_V^\hbar$ be the Weyl algebra and define a grading on $\mathcal{W}_V^\hbar$ as above. A sequence of differential operators $(H_i)_{i\in I}$ is called an *r-Airy structure* if H_i is of the form

$$H_i = \hbar\partial_{x_i} - P_i, \tag{4.155}$$

where $P_i \in \mathcal{W}_V^\hbar$ is not necessarily homogeneous and contains terms of degree between 2 and r. Moreover, we require that the left ideal generated by the H_i,

$$[H_i, H_j] = \hbar f_{ij}^k H_k, \tag{4.156}$$

with $f_{ij}^k \in \mathcal{W}_V^\hbar$, is a Lie subalgebra.

Remark. We note that if we have a 2-Airy structure, that is, those which were considered previously, then f_{ij}^k must be scalars. In general however, we allow generic $f_{ij}^k \in \mathcal{W}_V^\hbar$. This is important for the structure of a $\mathcal{W}$-algebra. As was the case with 2-Airy structures, we will mostly be interested in r-Airy structures that are also representations of $\mathcal{W}$-algebras.

Due to the way these higher Airy structures are defined, we again have the following existence and uniqueness theorem due to Kontsevich and Soibelman.

Theorem 4.3.2. Given a family of differential equations $H_i Z = 0$, $i \in I$ where the operators H_i are a higher quantum Airy structure $(H_i)_{i \in I}$, there exists a unique formal solution

$$Z = \exp(F) = \exp\left(\sum_{g=0}^{\infty}\sum_{n=1}^{\infty} \frac{\hbar^{g-1}}{n!} \sum_{i_1,\dots,i_m \in I} F_{g,n}(i_1,\dots,i_n) x_{i_1} \cdots x_{i_n}\right), \tag{4.157}$$

where $F_{g,n}(i_1,\dots,i_n)$ are scalars, completely symmetric in the indices $i_1,\dots,i_n$ and $F_{0,1}(i) = F_{0,2}(i,j) = 0$ for all i,j.

We omit the proof of this theorem as the combinatorics are somewhat tedious but it follows the same recursive argument as Theorem 4.2.5. See [61].

In [1], it is shown how to construct many different examples of higher Airy structure starting from a Lie algebra $\mathfrak{g}$. This is done from the point of view of vertex operator algebras. Recall that a vertex operator algebra as defined in the first section is a quadruple $(V, Y, \mathbb{1}, \omega)$ consisting of the $\mathbb{Z}$ graded vector space of states V, the state-operator map $Y : V \to (\text{End } V)[[z, z^{-1}]]$, the vacuum state $\mathbb{1}$, and the chiral energy momentum tensor ω such that

$$Y(\omega, z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}. \tag{4.158}$$

A specific example of a vertex operator algebra that is of the most interest to us is the algebra corresponding to a single chiral free boson.

Definition 4.3.3. The Heisenberg vertex operator algebra $(V, Y, \mathbb{1}, \omega)$ is a vertex operator algebra such that the following additional axioms hold.

- There exists $b \in V_1$ such that if the modes b_n are given by

$$Y(b, z) = \sum_{\in \mathbb{Z}} b_n z^{-n-1}, \tag{4.159}$$

then the modes generate the Heisenberg algebra

$$[b_m, b_n] = m\delta_{m+n,0}. \tag{4.160}$$

- For $v = b_{-n_1} \cdots b_{-n_k} \cdot \mathbb{1}$ we have

$$Y(v, z) =: \left(\frac{1}{(n_1 - 1)!} \left(\frac{d}{dz} \right)^{n_1 - 1} Y(b, z) \right) \cdots \left(\frac{1}{(n_k - 1)!} \left(\frac{d}{dz} \right)^{n_k - 1} Y(b, z) \right) :, \quad (4.161)$$

where $: :$ denotes normal ordering as previously.

- The conformal state ω is given by

$$\omega = \frac{1}{2} b_{-1} b_1. \quad (4.162)$$

With this definition, one can show that the generators of the Virasoro algebra in the Heisenberg vertex operator algebra can be written as

$$L_n = \frac{1}{2} \sum_{k \in Z} : b_{n-k} b_k : . \quad (4.163)$$

One can in fact define more general Heisenberg vertex operator algebras that are associated to a Lie algebra $\mathfrak{g}$. Definition 4.3.3 we have given above is the Heisenberg vertex operator algebra associated to $\mathfrak{g} = \mathfrak{gl}_2$. See [1] for more details.

Vertex operator algebras are in fact very useful for constructing quantum Airy structures. The broad recipe, as outlined in [1], is as follows.

1. Construct a *twisted module* $\mathcal{T}$ (see [1] and references therein) of the Heisenberg vertex operator algebra associated to a Lie algebra $\mathfrak{g}$.

2. From $\mathfrak{g}$, construct the $\mathcal{W}$ algebra, $\mathcal{W}(\mathfrak{g})$. This is defined as the $\mathcal{W}$ algebra generated by chiral primary fields W^i with conformal dimensions $d_i + 1$ where d_i are the Dynkin exponents of $\mathfrak{g}$. See for example [69].

3. The algebra $\mathcal{W}(\mathfrak{g})$ can be realised as sub vertex operator algebra of the Heisenberg vertex operator algebra. Upon restricting $\mathcal{T}$ to $\mathcal{W}(\mathfrak{g})$, one finds that this an untwisted module. Furthermore, the modes of the generators of $\mathcal{W}(\mathfrak{g})$ can be realised as dif-

ferential operators acting on the space of formal series of countably many variables x_i.

4. Choose a subalgebra of the algebra of modes that satisfy the Lie subalgebra condition (4.156).

5. If possible perform a dilaton shift on the modes so that they are of the correct form for a higher Airy structure. The dilaton shift here will be a conjugation of the modes by $\exp\left(\frac{\partial}{\partial x_s}\right)$ so that the form of the commutator is unchanged.

Rather than give a full account of the technical details here, we refer the reader to [1]. We will however, give a flavour of this construction by considering the most basic non-trivial example where $\mathfrak{g} = \mathfrak{gl}_2$. This will produce a 2-Airy structure. We construct explicit representations of the Heisenberg algebra in both the twisted and untwisted case. For the untwisted case, consider the representation

$$b_0 = 0, \ b_{-k} = \frac{1}{\sqrt{2\hbar}} k x_k, \ b_k = \sqrt{2\hbar}\frac{\partial}{\partial x_k}, \tag{4.164}$$

for $k \geq 1$. It is readily verified that $[b_m, b_n] = m\delta_{m+n,0}$. Substituting these into equation (4.163) for L_n then gives

$$L_n = \sum_{k=1}^{\infty} k x_k \frac{\partial}{\partial x_{k+n}} + \frac{\hbar}{2} \sum_{k=1}^{n} \frac{\partial}{\partial x_k} \frac{\partial}{\partial x_{n-k}}. \tag{4.165}$$

We recognise these as the discrete Virasoro constraints we encountered in the previous chapter. In this form, these are not Airy structures since there is no linear term ∂_{x_k}. Despite this, we can shift variables $x_2 \mapsto x_2 + \frac{1}{2}$ to obtain an Airy structure. Due to the absence of A and D terms, nevertheless, this is a trivial Airy structure. This is the reason why we need to consider twisted representations.

To proceed, therefore, we consider twisted modules. See [1] and references therein for a technical discussion of twisted modules. Here we shift the indices to be half integers while

still generating the Heisenberg algebra. More precisely, we define

$$b_{-k} = \frac{1}{\sqrt{2\hbar}} k x_{k+1/2}, \; b_k = \sqrt{2\hbar} \frac{\partial}{\partial x_{k+1/2}}. \tag{4.166}$$

where $k \in \mathbb{Z}_{\geq 0} + 1/2$ and

$$[b_m, b_n] = m\delta_{m+n,0}, \tag{4.167}$$

for $m, n \in \mathbb{Z} + 1/2$. In this representation, the modes L_n are given by

$$L_n = \frac{1}{2} \sum_{k \in \mathbb{Z}+1/2} : b_{n-k}b_k : + \delta_{n,0}\frac{1}{16}. \tag{4.168}$$

We now restrict to the Virasoro subalgebra generated by $\{L_n\}_{n\geq -1}$. To see why, recall that L_n satisfy

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}, \tag{4.169}$$

Restricting to $m, n \geq -1$ means the term containing the central charge vanishes immediately for $[L_m, L_n]$ with $m, n \geq 0$ due to the $\delta_{m+n,0}$ condition. However, the commutator $[L_{-1}, L_1]$ also vanishes due to the $m^3 - m$ term. Hence, we have chosen a subalgebra of the modes that do indeed satisfy the correct Lie subalgebra condition for a quantum Airy structure.

Written in terms of the formal variables x_k we have

$$L_i = \frac{1}{\hbar}\delta_{i,1}\frac{x_1^2}{4} + \frac{1}{2}\sum_{j=1}^{\infty}(2i + 2j - 5)x_j\frac{\partial}{\partial x_{i+j-2}} + \frac{\hbar}{4}\sum_{j=1}^{i-2}(2j - 1)(2i - 2j - 3)\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_{i-j-1}} + \delta_{i,2}\frac{1}{16}, \tag{4.170}$$

where we have shifted indices $i = n + 2$. We now apply a dilaton shift and a rescaling,

$$H_i = \frac{2\hbar}{2i - 1} \exp\left(-\frac{\partial}{\partial x_2}\right) L_i \exp\left(\frac{\partial}{\partial x_2}\right). \tag{4.171}$$

By the Baker-Campbell-Hausdorff formula, the conjugation by $\exp\left(\frac{\partial}{\partial x_2}\right)$ is equivalent to shifting $x_2 \mapsto x_2 - 1$. The significance of the multiplicative factor will become apparent in a

moment. Hence, the dilaton shifted operators are given by

$$H_i = -\hbar\frac{\partial}{\partial x_i} + \delta_{i,1}\frac{x_1^2}{2} + \hbar\sum_{j=1}^{\infty}\frac{(2i+2j-5)}{2i-1}x_j\frac{\partial}{\partial x_{i+j-2}} \qquad (4.172)$$

$$+\frac{\hbar^2}{2}\sum_{j=1}^{i-2}\frac{(2j-1)(2i-2j-3)}{2i-1}\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_{i-j-1}} + \delta_{i,2}\frac{\hbar}{24}. \qquad (4.173)$$

These operators H_i are now indeed a quantum Airy structure, having the correct form and commutation relations. From this explicit construction we can prove the following crucial theorem.

Theorem 4.3.4. Let (H_i) be the quantum Airy structure defined above. This coincides with the quantum Airy structure corresponding to Eynard-Orantin recursion applied to the Airy curve $y^2 = 2x$.

Proof. This follows by reading off A, B, C and D from H_i defined above. We indeed find

$$A^i_{jk} = \delta_{i=j=k=1}, \qquad (4.174)$$

$$B^i_{jk} = \frac{2k-1}{2i-1}\delta_{i+j-2,k}, \qquad (4.175)$$

$$C^i_{jk} = \frac{(2j-1)(2k-1)}{2i-1}\delta_{i,j+k+1}, \qquad (4.176)$$

$$D^i = \frac{1}{24}\delta_{i,2}. \qquad (4.177)$$

These precisely match with the A, B, C and D corresponding to the Eynard-Orantin recursion applied to the Airy curve $y^2 = 2x$ in in equations (4.150)-(4.153). $\square$

Remark. In the above discussion, we treated only the largest subalgebra of the full Virasoro algebra. That is to say, the algebra generated by $\{L_i\}_{i\geq-1}$. In fact, we can use the same analysis on smaller subalgebras, such as the one generated by $\{L_i\}_{i\geq0}$. In this case, however, one finds that the necessary dilaton shift is conjugation by $\exp\left(\frac{\partial}{\partial x_1}\right)$.

Thus, we have established a 2-Airy structure given by (H_i). This construction has been generalised to r-Airy structures for all integer $r \geq 2$ in [1]. For the purposes of this thesis,

we only need the explicit form of the Airy structures for $r = 2$ and $r = 3$. To explicitly write these in a compact form, we introduce new notation. For $k > 0$, let $J_k = \hbar \partial_{x_k}$ and $J_{-k} = kx_k$. We also assume $J_0 = 0$. Normal ordering is defined as usual where the creation operators, those J_k with $k < 0$, are placed to the left of the annihilation operators, those J_k with $k > 0$. Then the modes of the fields, up to dimension three, generating $\mathcal{W}(\mathfrak{gl}_r)$ read

$$W_k^1 = J_{kr}, \tag{4.178}$$

$$W_k^2 = \frac{1}{2} \sum_{\substack{p_1, p_2 \in \mathbb{Z} \\ p_1 + p_2 = r(k-1)}} (r\delta_{r|p_1} \delta_{r|p_2} - 1) : J_{p_1} J_{p_2} : -\frac{(r^2 - 1)\hbar}{24} \delta_{k,1}, \tag{4.179}$$

$$W_k^3 = \frac{1}{6} \sum_{\substack{p_1, p_2, p_3 \in \mathbb{Z} \\ p_1 + p_2 + p_3 = r(k-2)}} (r^2 \delta_{r|p_1} \delta_{r|p_2} \delta_{r|p_3} - r\delta_{r|p_1} - r\delta_{r|p_2} - r\delta_{r|p_3} + 2) : J_{p_1} J_{p_2} J_{p_3} : \tag{4.180}$$

$$- \frac{(r-2)(r^2-1)\hbar}{24} J_{r(k-2)}. \tag{4.181}$$

Remark. We will often set the terms $\delta_{r|p_i}$ to zero using the constraint that $W_k^1 Z = J_{kr} Z = \frac{\partial Z}{\partial t_{kr}} = 0$ for $k > 0$. While it is not immediately why this is valid, it is nevertheless true. See [1, Lemma 4.11].

These modes are not a quantum Airy structure yet. As before we perform a dilaton shift. However, there are several possible choices for this shift for a given r. Let $s \in \{1, 2 \ldots, r-1, r+1\}$ such that $r = \pm 1 \mod s$. For example, for $r = 2$ we have either $s = 1$ or $s = 3$ while for $r = 3$, there are three possibilities, $s = 1$, $s = 2$ or $s = 4$. Each value of s then corresponds to a different dilaton shift,

$$W_k^i \mapsto H_k^{(i,s)} = \exp\left(-\frac{J_s}{s\hbar}\right) W_k^i \exp\left(\frac{J_s}{s\hbar}\right). \tag{4.182}$$

Again by the Baker-Campbell-Hausdorff formula, this is equivalent to $J_{-s} \mapsto J_{-s} - 1$ in the modes W_k^i. The following theorem is then proved in [1].

Theorem 4.3.5. Let $r \geq 2$ and let $s \in \{1, \ldots, r-1, r+1\}$ such that $r = \pm 1 \bmod s$. Define

$$\alpha^i := i - 1 - \left\lfloor \frac{s(i-1)}{r} \right\rfloor. \tag{4.183}$$

Then the family of differential operators

$$H_k^{(i,s)} = \exp\left(-\frac{J_s}{s\hbar}\right) W_k^i \exp\left(\frac{J_s}{s\hbar}\right), \tag{4.184}$$

with $1 \leq i \leq r$ and $k \geq \alpha^i + \delta_{i,1}$ forms an r-Airy structure on the vector space $V = \oplus_p \mathbb{C}\langle x_p \rangle$ where $(x_p)_{p>0}$ are linear coordinates.

Note that the condition $r = \pm 1 \bmod s$ is necessary for this construction. If one were to take other values of s, the same linear term $\hbar \partial_{x_p}$ would appear in more than one of $H_k^{(i,s)}$ which cannot happen for a higher quantum Airy structure.

Explicitly writing the modes $H_k^{(2,3)}$ in the case $(r,s) = (2,3)$ and rescaling gives

$$H_k^{(2,3)} = -\hbar \frac{\partial}{\partial x_{k+1}} + \delta_{k,0} \frac{x_1^2}{2} + \hbar \sum_{j=1}^{\infty} \frac{(2k+2j-3)}{2k+1} x_j \frac{\partial}{\partial x_{k+j-1}} \tag{4.185}$$

$$+ \frac{\hbar^2}{2} \sum_{j=1}^{k-1} \frac{(2j-1)(2k-2j-1)}{2k+1} \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_{k-j}} + \delta_{k,1} \frac{\hbar}{24}, \tag{4.186}$$

in agreement, up to a shift of indices, with the operators in equation (4.172). However, recalling Theorem 3.4.1, we see that these are precisely the operators that annihilate $\mathcal{Z}$, the double scaled limit of a Hermitian matrix model which is indeed a tau function of the KdV hierarchy. By Theorem 4.3.5, $H_k^{(2,3)}$ is a quantum Airy structure and so $\mathcal{Z}$ is uniquely specified. Therefore, we obtain the following theorem.

Theorem 4.3.6. The formal function $\tau(x_1, x_2, x_3 \ldots)$ is the unique tau function of the KdV hierarchy corresponding to initial condition $u(x_1, 0, 0, \ldots) = x_1$ if and only if $H_k^{(2,3)} \tau = 0$.

Proof. The above discussion proves that $\mathcal{Z}$ is a tau function if and only if $H_k^{(2,3)} \mathcal{Z} = 0$. From the Kontsevich-Soibelman recursion, we see that $\tau = \mathcal{Z} = e^F$ for some function F. The

initial condition is specified by the *string equation*, $H_1^{(2,3)} \tau = 0$. Setting $x_2 = x_3 = \cdots = 0$ in $H_1 \tau = 0$ we find

$$0 = \left(-\frac{\partial}{\partial x_1} + \frac{x_1^2}{2} \right) e^F, \tag{4.187}$$

where we have set $\hbar = 1$. Thus we have

$$-\frac{\partial F}{\partial x_1} + \frac{x_1^2}{2} = 0. \tag{4.188}$$

Differentiating and using the definition of a tau function,

$$u = \frac{\partial^2}{\partial x_1^2} \log \tau = \frac{\partial^2 F}{\partial x_1^2}, \tag{4.189}$$

we find the initial condition $u(x_1, 0, 0, \dots) = x_1$. $\qquad\square$

We recall that in the previous subsection, we saw that the quantum Airy structure $H_k^{(2,3)}$ corresponds to Eynard-Orantin recursion applied to the Airy curve $y^2 = 2x$. Hence we obtain the following corollary.

Corollary 4.3.7. Let τ be the tau function of the KdV hierarchy corresponding to the initial condition $u(x_1, 0, 0, \dots) = x_1$. Define $\omega_{g,n}$ by the Eynard-Orantin recursion applied to the Airy curve $y^2 = 2x$. Write the $\omega_{g,n}$ using a decomposition over finite sums introduced in section 4.2.3,

$$\omega_{g,n}(J) = \sum_{r_i, k_i} F_{g,n} \begin{bmatrix} r_1, \dots, r_n \\ k_1, \dots, k_n \end{bmatrix} \prod_{i=1}^{n} \xi_{k_i, r_i}(p_i). \tag{4.190}$$

Define the following generating function of the $F_{g,n}$ as in Theorem 4.3.2

$$Z = \exp \left(\sum_{g=0}^{\infty} \sum_{n=1}^{\infty} \frac{\hbar^{g-1}}{n!} \sum_{r_i, k_i} F_{g,n} \begin{bmatrix} r_1, \dots, r_n \\ k_1, \dots, k_n \end{bmatrix} x_{k_1}^{r_1} \cdots x_{k_n}^{r_n} \right). \tag{4.191}$$

Then

$$Z = \tau. \tag{4.192}$$

Taken together, these theorems and corollaries mean that the example of the quantum

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Airy structure corresponding to $(r, s) = (2, 3)$ is well understood from these different perspectives of integrability, matrix models, Airy structures and spectral curves. We shall call the corresponding partition function, or τ function, the 2-Kontsevich-Witten (KW) tau function, for reasons that will become apparent in the next chapter.

Let us summarise these results in the following theorem.

Theorem 4.3.8. The following are equivalent.

1. The function Z is the generating function defined in equation (4.192) by applying topological recursion to the Airy curve.

2. The partition function Z is (the square root of) the double scaled limit of a Hermitian matrix model with arbitrary potential.

3. The function Z is the 2-KW tau function of the KdV hierarchy corresponding to the initial condition $u(x_1, 0, 0, \dots) = \partial_{x_1}^2 \log Z|_{(x_1, 0, 0, \dots)} = x_1$.

4. The function Z is the unique solution to $H_k^{(2,3)} Z = 0$.

Remark. Here, there is a formal parameter in the generating function of 1) that plays the role of the fixed parameter in the double scaling limit in 2), and plays the role of $\hbar$ in 4). For the KdV hierarchy in 3), one can consider ϵ as controlling the dispersion term. For example, Dubrovin [70] distinguishes between the KdV hierarchy starting with

$$u_t = u u_x + \frac{\epsilon^2}{12} u_{xxx}, \tag{4.193}$$

and the $\epsilon = 0$ dispersionless KdV hierarchy with first equation,

$$u_t = u u_x. \tag{4.194}$$

Ultimately, the ϵ dependence in the dispersion term can be eliminated by the rescaling of the times of the hierarchy and so for this thesis we shall not dwell on this matter.

4.3.2 Generalisations of 2-KW

With this explicit construction of quantum Airy structures, we can now see several important generalisations of the case $(r, s) = (2, 3)$. The next simplest case to consider is $(r, s) = (2, 1)$. In this case, the quantum Airy structure in Theorem 4.3.5 reads,

$$H_k^{(2,1)} = -\frac{\partial}{\partial x_{2k-1}} + \frac{1}{2}\sum_{m=1}^{\infty}(2m+1)x_{2m+1}\frac{\partial}{\partial x_{2m+2k-1}} + \frac{1}{4}\sum_{i+j=k-2}\frac{\partial^2}{\partial x_{2i+1}\partial x_{2j+1}} + \frac{1}{16}\delta_{k,1}, \quad (4.195)$$

with $k \geq 1$ and where we have rescaled variables and set $\hbar = 1$. Here, we have also employed the trivial constraints $J_{2k}Z = \frac{\partial Z}{\partial x_{2k}} = 0$ and so eliminated the even variables x_{2k} from the Airy structure. This will be convenient when we recalculate these operators in chapter six. If Z is the corresponding unique solution of this quantum Airy structure, one can ask the same questions as we did in the case $(r, s) = (2, 3)$ about what Z computes.

1. Can one apply topological recursion to some spectral curve and recover $Z = e^F$?

2. Is there a matrix model representation for Z?

3. Is Z a tau function of the 2-KdV hierarchy?

For question 1, it is shown in [71] that one can apply Eynard-Orantin to the Bessel curve introduced earlier,

$$x = \frac{z^2}{2}, y = -\frac{1}{z}, \quad (4.196)$$

and this reproduces the corresponding partition function. For question 2, there is indeed a matrix model representation in terms of a unitary matrix ensemble. We shall discuss this in chapter six. We will also show in chapter six that this a tau function of the 2-KdV hierarchy. In addition, one can readily calculate the initial condition using the same method as before. In this case, we find

$$u(x_1, 0, 0, \dots) = \frac{1}{8(2 - x_1)^2}. \quad (4.197)$$

. We call the corresponding tau function the Brézin-Gross-Witten (BGW) tau function [72].

The case $r \geq 3$ is less well understood. One can ask the three questions as before and one soon discovers that the answers are in fact much more delicate. In this case, one has to suitably generalise the Eynard-Orantin recursion for curves with arbitrary order of ramification. This is established in [63]. For the case $r \geq 3$ and $s = r+1$, one finds that the generating function produced by topological recursion on the r-Airy spectral curve,

$$x = \frac{z^r}{r}, \; y = -z, \tag{4.198}$$

in analogy with Corollary 4.3.7 coincides with the partition function of the quantum Airy structure. This is shown in [1]. One may also ask the question of whether there is a generalised BGW partition function and what the corresponding spectral curve is. In chapter six, we give a definition of a generalised BGW model and we further argue that the corresponding spectral curve is the r-Bessel curve

$$x = \frac{z^r}{r}, \; y = -\frac{1}{z}. \tag{4.199}$$

Additionally, we present evidence that the corresponding quantum Airy structure is of type $(r, s) = (r, r-1)$.

Concerning matrix model representations, there is some evidence to suggest that Hermitian $(r-1)$ multi-matrix models play the role of the partition function for the quantum Airy structure. See [73] for example. The link between double scaled multi-matrix models and the r-KdV hierarchies is still not particularly well understood however.

Finally, with regards to tau functions, it is already known that in the Airy case $(r, s) = (r, r+1)$ the partition function is a tau function of the r-KdV hierarchy. See the works of Adler and van Moerbeke [7], and Faber, Shadrin and Zvonkine [6]. In chapter six, we present computational evidence that the partition function associated to the quantum r-Airy structure of type $(r, r-1)$ is a tau function of r-KdV. In order to do this, rather than considering double scaling limits of multi-matrix models, we embark on a new path. The Eynard-Orantin recursion suggests that the partition function encodes geometric invariants

of spectral curves. We now ask further questions about what specific invariants are computed in this regime for the Airy curve.

Chapter 5

Enumerative Geometry

When one investigates differential equations, much of the richness of the problem arises from finding what space the solution lives in. In other words, we are interested in the 'moduli space' of solutions. Here we take a similar view. The objects in question are not solutions of differential equations, however. Rather we will consider geometric objects, namely Riemann surfaces. We will see that we can define cohomology classes and produce enumerative invariants using such classes. In physics, and in particular string theory, the moduli spaces of Riemann surfaces play a critical role. In fact, the partition of two dimensional gravity is given by a generating function of the enumerative invariants mentioned above. It is this statement that originally led to Witten's conjecture that was later proved by Kontsevich. It is this proof which we shall exploit and generalise in chapter six.

5.1   Informal Introduction to Moduli Spaces

The intuitive general definition of a moduli space is a set whose points represent isomorphism classes of a certain object. We have in fact encountered an example of this notion previously: the finite dimensional Grassmannian $\mathrm{Gr}_n(V)$ is a 'parameter space' whose points are n dimensional subspaces of the vector space V. Consider another example of the moduli space of conics in the complex projective plane $\mathbb{P}^2$. In other words, points in this moduli space

correspond to homogenous, degree two polynomials in three variables. In local coordinates $[x_0, x_1, x_2]$, such a polynomial has the form

$$a_1 x_0^2 + a_2 x_1^2 + a_3 x_2^2 + a_4 x_0 x_1 + a_5 x_0 x_2 + a_6 x_1 x_2 = 0, \tag{5.1}$$

with $a_i \in \mathbb{C}$ and at least one $a_i \neq 0$. There is a redundancy here, nonetheless, as one can multiply the above equation by any non zero scalar $\lambda \in \mathbb{C}^*$ and retain the same conic. Hence, the moduli space of conics in $\mathbb{P}^2$ is given as $(\mathbb{C}^6 \setminus \{0\})/\mathbb{C}^* = \mathbb{P}^5$. These two examples highlight the important fact that there is more structure to moduli spaces than just that of a set. In these examples, the moduli spaces were manifolds but we shall see in this section that one can also have moduli spaces with the structure of an orbifold. We shall not concern ourselves with developing the full, abstract theory of moduli spaces involving stacks and schemes, although interested readers can consult [74] and references therein. Here we prefer to be guided by examples and we concentrate on the most relevant moduli space concerning isomorphism classes of Riemann surfaces. To this end, we closely follow [41].

5.1.1 Moduli Space of Riemann Surfaces

We begin with the definition of an elementary, yet extremely important, moduli space. We note here that we are exclusively interested in the moduli space of complex structures of Riemann surfaces, rather than Kähler moduli. We recall that two dimensional surfaces which are compact, connected and orientable, are characterised up to homeomorphism by their genus g. Henceforth, we will always assume that the Riemann surface Σ is compact, connected and orientable.

Definition 5.1.1. Let $(\Sigma, p_1, \ldots, p_n)$ and $(\Sigma', p_1', \ldots, p_n')$ be two Riemann surfaces with marked points $(p_1, \ldots, p_n)$ and $(p_1', \ldots, p_n')$ respectively. We say that $\phi : \Sigma \to \Sigma'$ is an isomorphism if ϕ is holomorphic, bijective and has holomorphic inverse (that is, ϕ is biholomorphic) and $\phi(p_i) = p_i'$ for all $i = 1, \ldots, n$. If $\Sigma = \Sigma'$ and ϕ is an isomorphism, we call ϕ an automorphism.

We remark that the set of automorphisms of $(\Sigma, p_1, \ldots, p_n)$ forms a group under composition.

Definition 5.1.2. The moduli space of Riemann surfaces of genus g and n distinct marked points is given, as a set, by

$$\mathcal{M}_{g,n} := \{(\Sigma, p_1, \ldots, p_n)\}/ \text{ Isomorphisms,} \tag{5.2}$$

where Σ is a connected, compact and orientable Riemann surface and $(p_1, \ldots, p_n)$ are marked points.

We proceed by illustrating this definition with a few examples. Consider $g = 0$ and $n = 3$. It is known that, up to biholomorphism, the only Riemann surface with $g = 0$ is the Riemann sphere $\mathbb{P}^1$. We have already seen in the discussion of the Virasoro algebra that automorphisms of $\mathbb{P}^1$ are given by the Möbius transformations. It is a well known fact in complex analysis that any three points on $\mathbb{P}^1$ can be mapped to any other three points on $\mathbb{P}^1$. Hence, up to the action of automorphisms, $\mathcal{M}_{0,3} \cong \{(\mathbb{P}^1, 0, 1, \infty)\}$. Hence, $\mathcal{M}_{0,3}$ is a single point, a zero dimensional manifold.

As another example, consider $(g, n) = (0, 4)$. We note that if $(\Sigma, p_1, p_2, p_3, p_4) \in \mathcal{M}_{0,4}$, then via a Möbius transformation, p_1, p_2 and p_3 can be mapped to any other three points on $\mathbb{P}^1$. Therefore, without loss of generality we choose $p_1 = 0, p_2 = 1$ and $p_3 = \infty$. There is still, nevertheless, one free parameter $p_4 \in \mathbb{P}^1$ which is distinct from $0, 1$ and ∞. Hence, we have the result that

$$\mathcal{M}_{0,4} = \mathbb{P}^1 \setminus \{0, 1, \infty\}. \tag{5.3}$$

Observe that $\mathcal{M}_{0,4}$ is a complex one dimensional manifold that is not compact in the topology induced by $\mathbb{P}^1$.

We can generalise the above to see that for $n \geq 3$

$$\mathcal{M}_{0,n} = (\mathbb{P}^1 \setminus \{0, 1, \infty\})^{n-3} \setminus \{(p_4, \ldots, p_n) : p_i = p_j \text{ for } i \neq j\}. \tag{5.4}$$

From here, we see that $\mathcal{M}_{0,n}$ with $n \geq 3$ is a complex manifold that is not compact if $n \geq 4$. Not all of the moduli spaces are manifolds however as we shall discover in the next example.

Consider the moduli space $\mathcal{M}_{1,1}$. It is known that every genus one Riemann surface is isomorphic to an algebraic torus of the form $\mathbb{C}/\Lambda$ where $\Lambda = \alpha_1 \mathbb{Z} + \alpha_2 \mathbb{Z}$ is a lattice with $\alpha_1, \alpha_2 \in \mathbb{C}$. The lattice can be translated so that it has a vertex on the origin. This means every point $z \in \mathbb{C}$ is identified with $z \sim z + m\alpha_1 + n\alpha_2$. We can normalise this torus by setting $\Lambda = \mathbb{Z} + \tau\mathbb{Z}$ with $\tau = \frac{\alpha_1}{\alpha_2}$ and where $\tau \in \mathbb{C}$. In other words, every genus one Riemann surface is a copy of the complex plane modulo the relation $z \sim z+1 \sim z+\tau$. We can further restrict τ so that $\tau \in \mathbb{C}_+ = \{z \in \mathbb{C} : \mathrm{Im}\ z > 0\}$. This is because complex conjugation of τ defines isomorphic tori. Hence every point of τ defines a genus one Riemann surface.
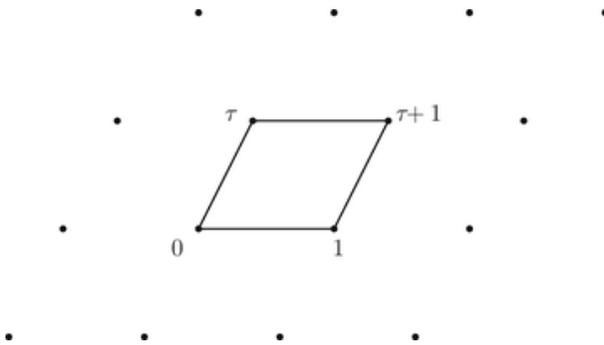
Figure 5.1: A normalised lattice defining a tiling on $\mathbb{C}$ by the parallelogram with lengths 1 and τ. The torus is obtained by identifying edges of the parallelogram [56].

The question remains as to when two lattices generate equivalent tori. Suppose that $\Lambda = \mathbb{Z} + \tau\mathbb{Z}$ and $\Lambda = \mathbb{Z} + \tau'\mathbb{Z}$ are two normalised lattices. If they generate the same torus, then these normalised lattices coincide, meaning that there are integers $a, b, c, d \in \mathbb{Z}$ such that

$$1 = c\tau + d, \tag{5.5}$$

and

$$\tau' = a\tau + b. \tag{5.6}$$

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Thus, an automorphism is given by the Möbius transformation

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}. \tag{5.7}$$

For the inverse map to also have integer coefficients, we require that $ad - bc = \pm 1$. We will enforce $ad - bc = 1$ so that the transformation is orientation preserving. Notice this is simply the requirement that the area of the parallelogram is preserved and we remark that a rescaling of the area corresponds to Kähler moduli rather than complex moduli. Thus the automorphism group is $\mathrm{SL}(2, \mathbb{Z})$. Hence, the orbits of any $\tau \in \mathbb{C}_+$ under $\mathrm{SL}(2, \mathbb{Z})$ consist of an infinite number of points, all corresponding to equivalent tori. Thus we have that

$$\mathcal{M}_{1,1} = \mathbb{C}_+/\mathrm{SL}(2, \mathbb{Z}). \tag{5.8}$$

It is well known that the modular group $\mathrm{SL}(2, \mathbb{Z})$ is generated by the transformations

$$T : \tau \mapsto \tau + 1, \; S : \tau \mapsto -\frac{1}{\tau}. \tag{5.9}$$

Consequently,

$$\mathcal{M}_{1,1} = \mathbb{C}_+/\mathrm{SL}(2, \mathbb{Z}) = \left\{\tau \in \mathbb{C}_+ : |\tau| > 1, -\frac{1}{2} < \mathrm{Re}\,\tau \leq \frac{1}{2}\right\} \cup \left\{e^{i\theta} : \theta \in \left[\frac{\pi}{3}, \frac{\pi}{2}\right]\right\} \tag{5.10}$$

Notice that this is not a manifold due to the action of the modular group $\mathrm{SL}(2, \mathbb{Z})$. This is, however, an orbifold which, roughly speaking, is a space that locally resembles a Euclidean space modulo a group action. This brings us to the notion of stability.

Definition 5.1.3. A marked Riemann surface $(\Sigma, p_1, \ldots, p_n)$ is called stable if it has finite automorphism group and unstable if it has infinite automorphism group.

For example, consider $\mathcal{M}_{0,0}$. We have that the automorphisms of $\mathbb{P}^1$ are exactly the Möbius transformations. Hence $\mathbb{P}^1$ with no marked points is not stable. As another example,

consider $(\mathbb{P}^1, \infty) \in \mathcal{M}_{0,1}$. The subgroup of the Möbius group that fixes ∞ is then

$$\{z \mapsto az + b : a, b \in \mathbb{C}, \, a \neq 0\} \cong \mathbb{C}^* \times \mathbb{C}. \tag{5.11}$$

Hence $(\mathbb{P}^1, \infty)$ is not stable. Similarly, for $(\mathbb{P}^1, 0, \infty) \in \mathcal{M}_{0,2}$ the automorphisms are described by $z \mapsto az$ or $z \mapsto \frac{a}{z}$. Hence the group of automorphisms is still infinite.

Now consider $\mathcal{M}_{0,n}$ for $n \geq 3$. Only the Möbius transformations that map marked points to marked points are automorphisms. This, along with the condition that $ad - bc = 1$, fixes the coefficients for $n \geq 3$. The group of automorphisms is thus some subgroup of the permutation group of labelled points.

Now consider the case $g = 1$ and $n = 0$. Note that the complex plane is invariant under translations if there is no marked point. Hence, $\mathcal{M}_{1,0}$ has an infinite number of automorphisms.

It turns out that we have actually already covered every example of an unstable Riemann surface and we have the following theorem.

Theorem 5.1.4. If a surface has non-negative Euler characteristic, $\chi_{g,n} = 2 - 2g - n > 0$, the surface is unstable. If the surface has $\chi_{g,n} < 0$, then it is stable.

In general, $\mathcal{M}_{g,n}$ with $\chi_{g,n} = 2 - 2g - n < 0$ is locally parameterised by $3g - 3 + n$ coordinates and is an orbifold.

5.1.2 Compactification

As we have already seen, the space $\mathcal{M}_{g,n}$ is not necessarily compact. In general, one can deform a Riemann surface $(\Sigma, p_1, \ldots, p_n)$ such that a cycle is pinched, generating a surface that is not smooth. Alternatively, two or more marked points collide, again creating a singular surface. In this case, via a biholomorphic mapping, we may 'zoom in' on where the marked points collapse. This has the effect of separating the colliding marked points, but disconnecting these points from the rest of the surface. In this limit, the surface becomes singular. In either case, the limit does not belong to $\mathcal{M}_{g,n}$.
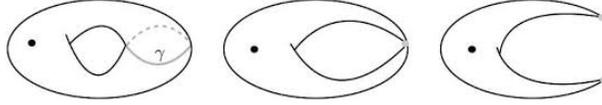
Figure 5.2: Pinched cycle on a Riemann surface - the resulting surface is an element of $\bar{\mathcal{M}}_{0,3}$ [75].

Figure 5.3: Two colliding marked points - the resulting surface is an element of $\bar{\mathcal{M}}_{0,5}$ [76].

Ultimately, we wish to integrate over this moduli space and so we need to identify a suitable compactification. In fact, all that is required is to add these nodal Riemann surfaces to $\mathcal{M}_{g,n}$. Nodal Riemann surfaces are obtained by gluing together surfaces at nodal points such as those shown in Figure 5.3. A stable nodal surface is a nodal Riemann surface where each connected component is stable with nodal points being counted as marked points. For example, each of the three spheres in Figure 5.3 has three marked or nodal points and so each sphere is stable. Hence, we make the following definition the Deligne-Mumford compactification.

Definition 5.1.5. The set of all stable nodal Riemann surfaces $(\Sigma, p_1, \ldots, p_n)$, modulo automorphisms, and with p_i non nodal points, is called the Deligne-Mumford compactification $\bar{\mathcal{M}}_{g,n}$.

We remark that, in general, this construction is neither a manifold nor an orbifold. It is in fact a *stack*. We shall not dwell on the subtleties here and refer the reader to [74]. This space is indeed compact. Let us verify this through two examples. Consider $\mathcal{M}_{1,1}$. From Figure 5.2 we can consider this degenerate torus as a surface of $g = 0$ with three marked points. We identify this torus as the point $i\infty$ and add it to $\mathbb{C}_+$. There are no other ways to obtain a nodal Riemann surface. Therefore, we have $\partial \mathcal{M}_{1,1} = \{i\infty\} = \mathcal{M}_{0,3}$ and so

$$\bar{\mathcal{M}}_{1,1} = \mathcal{M}_{1,1} \cup \mathcal{M}_{0,3}. \tag{5.12}$$

As another example, consider $\mathcal{M}_{0,4} = \mathbb{P}^1 \setminus \{0, 1, \infty\}$. Here, we have fixed three of the marked points to be $p_1 = 0, p_2 = 1$ and $p_3 = \infty$ leaving a distinct marked point $p_4 \neq 0, 1, \infty$. The limiting case corresponds to $p_4 \to 0$, $p_4 \to 1$ and $p_4 \to \infty$. The limit $p_4 \to 0$, that is $p_4 \to p_1$, corresponds to $(\Sigma, p_1, p_2, p_3, p_4)$ splitting into two spheres, one containing p_1 and p_4 with the other containing p_2 and p_3. These spheres are joined by a nodal point so that each sphere is stable. Hence this limit corresponds to $\mathcal{M}_{0,3} \times \mathcal{M}_{0,3}$. The same reasoning applies to $p_4 \to 1$ and $p_4 \to p_3$. Hence for the boundary of $\mathcal{M}_{0,4}$ we obtain three copies of $\mathcal{M}_{0,3} \times \mathcal{M}_{0,3}$:

$$\partial\mathcal{M}_{0,4} = (\mathcal{M}_{0,3} \times \mathcal{M}_{0,3}) \cup (\mathcal{M}_{0,3} \times \mathcal{M}_{0,3}) \cup (\mathcal{M}_{0,3} \times \mathcal{M}_{0,3}). \tag{5.13}$$

Now, $\mathcal{M}_{0,3} \times \mathcal{M}_{0,3}$ is simply a point. Hence the three copies $\mathcal{M}_{0,3} \times \mathcal{M}_{0,3}$ we identify as $\{0\}, \{1\}$ and $\{\infty\}$. In this way, we have

$$\bar{\mathcal{M}}_{0,4} = \mathbb{P}^1 \setminus \{0, 1, \infty\} \cup \partial\mathcal{M}_{0,4} = \mathbb{P}^1, \tag{5.14}$$

which is certainly compact in the standard topology.

5.2 Curvature and Chern Classes

Having defined the moduli space $\bar{\mathcal{M}}_{g,n}$, we now wish to define appropriate cohomology classes that we can integrate over $\bar{\mathcal{M}}_{g,n}$. To this end, we define the Chern classes. There are many equivalent ways to define Chern classes. For example, one may use the correspondence between divisors and line bundles, in addition to Poincaré duality, to define the first Chern class of a line bundle. Alternatively, one can also view the first Chern class as the connecting morphism in a long exact sequence of sheaf cohomology groups. Here, we take the view that Chern classes can be defined from a curvature form of vector bundles over manifolds. The reason for this approach is two-fold. It is firstly closer in spirit to theories of two dimensional quantum gravity where we integrate curvature forms to produce invariants. We shall discuss

in the next section. Moreover, this construction only uses local definitions and so readily extends to vector bundles over orbifolds. Consequently, we shall establish the theory of Chern classes for vector bundles over manifolds and argue that the analogous construction for orbifolds is no different. We closely follow [77].

5.2.1 Connections and Curvature

We assume that all vector bundles $\pi : E \to M$ are holomorphic where M is a complex manifold.

Furthermore, we denote the sections of E as $\Gamma(E)$ and recall that $\Gamma(E)$ is a $C^\infty(M)$ module defined by $(fs)_p = f(p)s_p$ for all $f \in C^\infty(M)$ and $s \in \Gamma(E)$.

Definition 5.2.1. Let E be a vector bundle over M. For any $m \in \mathbb{N}$, an E-valued $m-$form is a section of the vector bundle $\bigwedge^m(T^*M) \otimes E$.

We will use the notation $\Omega^m(M; E) = \Gamma\left(\bigwedge^m(T^*M) \otimes E\right)$. We will often omit M here when the manifold is clear from the context. Note also that $\Omega^0(E) = \Gamma(E)$. When E is the trivial bundle, we will write $\Omega^m(M)$.

Definition 5.2.2. Let E be a vector bundle. A connection ∇ on E is a $\mathbb{C}$-linear homomorphism $\nabla : \Omega^0(E) \to \Omega^1(E)$ such that the the following Leibniz rule is satisfied

$$\nabla(fs) = df \otimes s + f\nabla s, \tag{5.15}$$

for all $f \in C^\infty(M)$ and $s \in \Omega^0(E)$.

As an example, the exterior derivative satisfies the above properties and so we call $\nabla = d$ the trivial connection.

Definition 5.2.3. Let $\omega \in \Omega^k(E)$ and let $s \in \Gamma(E)$. For ∇ a connection on E, the covariant derivative d^∇ is a $\mathbb{C}$-linear homomorphism $d^\nabla : \Omega^k(E) \to \Omega^{k+1}(E)$ defined by

$$\omega \otimes s \mapsto d\omega \otimes s + (-1)^k \omega \wedge \nabla s. \tag{5.16}$$

Remark. For $k = 0$, we have $d^\nabla = \nabla$ after identifying the tensor product with multiplication.

Observe that for any $f \in C^\infty(M)$ we have $(f\omega) \otimes s = \omega \otimes (fs)$. Hence we need to first check that d^∇ is well-defined. Indeed, using the definition,

$$d^\nabla\big((f\omega) \otimes s\big) = d(f\omega) \otimes s + (-1)^k f\omega \wedge \nabla s = df \wedge \omega \otimes s + f d\omega \otimes s + (-1)^k f\omega \wedge \nabla s, \quad (5.17)$$

while on the other hand

$$d^\nabla\big(\omega \otimes fs\big) = d\omega \otimes fs + (-1)^k \omega \wedge \nabla(fs) = d\omega \otimes fs + (-1)^k \omega \wedge (df \otimes s + f\nabla s). \quad (5.18)$$

Recall that for α a k-form and β an l-form we have $\alpha \wedge \beta = (-1)^{kl}\beta \wedge \alpha$. Hence, equations (5.17) and (5.18) coincide. Using, the covariant derivative we can define the curvature form.

Definition 5.2.4. Let E be a vector bundle with connection ∇. Then the curvature F_∇ of E is a map $F_\nabla : \Omega^0(E) \to \Omega^2(E)$ given by $F_\nabla = (d^\nabla \circ d^\nabla)(s) = d^\nabla\big(\nabla(s)\big)$ for all $s \in \Omega^0(E)$.

The curvature cannot be interpreted as a differential operator as it does not satisfy the Leibniz rule. It is instead "$\Omega^0(E)$ linear".

Lemma 5.2.5. For any local function f on M and $s \in \Gamma(E)$, the curvature satisfies $F_\nabla(fs) = fF_\nabla(s)$

Proof. Using the definitions we have

$$F_\nabla(fs) = d^\nabla\big(\nabla(fs)\big) = d^\nabla\big(df \otimes s + f\nabla s\big) \qquad (5.19)$$

$$= d(df) - df \wedge \nabla s + df \wedge \nabla s + f d^\nabla\big(\nabla(s)\big) = fF_\nabla(s), \qquad (5.20)$$

as required. □

Remark. The intuitive idea behind "$\Omega^0(E)$ linear" can be made precise if one considers F_∇ as a morphism of sheaves. We will not do this however. The crux here is that F_∇ can be

considered as a section of $\bigwedge^2 T^*M \otimes \text{End}(E)$ where $\text{End}(E) = \text{Hom}(E, E) \cong E^* \otimes E$ is the endomorphism bundle. This observation is justified by the lemma above.

To make the above remark perhaps more explicit, we outline an alternative, local construction for F_∇.

Lemma 5.2.6. Let E be a vector bundle of rank r over the manifold M. Then any connection ∇ can locally be written as $\nabla = d + A$ where d is the trivial connection and A is a matrix of one forms.

Proof. Let $U \subset M$ be an open subset. Let $(e_1, \ldots, e_r)$ be a local frame corresponding to a locally trivialised domain $E|_U \cong U \times \mathbb{C}^r$. Observe that there is always the trivial connection $d : \Omega^0(E|_U) \to \Omega^1(E|_U)$ on the trivialised domain. Furthermore, any other connection on $E|_U$ is simply the trivial connection $\nabla = d$ scaled by some matrix of one forms A. In other words we have $\nabla e_i = A_i^j e_j$ with $A \in \Omega^1(\text{End}(E))$. Here, we have employed Einstein summation convention. Now, an E-valued 0 form can be written as a finite sum of terms of the form $\omega \otimes s$ with $\omega \in \Omega^0(E)$ and $s = s^i e_i \in \Gamma(E)$ where s^i are smooth $\mathbb{C}$ valued functions. By the $\mathbb{C}$-linearity of ∇, it suffices to prove the lemma for the element $\omega \otimes s$. Hence, we calculate that

$$\nabla(\omega \otimes s) = \nabla(s^i \omega \otimes e_i) \tag{5.21}$$

$$= d(s^i \omega) \otimes e_i + s^i \omega \wedge \nabla e_i \tag{5.22}$$

$$= d(s^j \omega) \otimes e_j + s^i \omega \wedge A_i^j e_j \tag{5.23}$$

$$= \left(d(s^j \omega) + (A_i^j s^i) \wedge \omega \right) e_j \tag{5.24}$$

$$= (d + A \wedge \cdot)(\omega \otimes s). \tag{5.25}$$

Hence, $\nabla = d + A$. $\qquad\square$

We remark that the proof of the above lemma used no particular property of Ω^0 and so the same proof holds for d^∇. In particular, $d^\nabla = d + A$. Consequently, for any $s \in \Gamma(E)$ we

can locally write the curvature as

$$F_\nabla(s) = (d + A \wedge \cdot)(ds + As) \tag{5.26}$$

$$= d(ds) + d(As) + A \wedge ds + A \wedge As \tag{5.27}$$

$$= (dA)s - A \wedge ds + A \wedge ds + A \wedge (As) \tag{5.28}$$

$$= (dA + A \wedge A)s. \tag{5.29}$$

Therefore, we have

$$F_\nabla = dA + A \wedge A, \tag{5.30}$$

or explicitly in components

$$(F_\nabla)_{ij} = d(A_{ij}) + A_{ik} \wedge A_{kj}. \tag{5.31}$$

A connection ∇ on E induces the following exterior covariant derivative on the endomorphism bundle $\mathrm{Hom}(E, E)$ given as

$$(d^{\nabla^{\mathrm{Hom}}}\omega)s = [d^\nabla, \omega]s = d^\nabla(\omega s) - \omega \wedge d^\nabla s. \tag{5.32}$$

for $\omega \in \Omega^k(\mathrm{End}(E))$ and $s \in \Gamma(E)$. This is simply a suitable form of the Leibniz rule. From here, we now prove the Bianchi Identity.

Proposition 5.2.7. The curvature satisfies

$$d^{\nabla^{\mathrm{Hom}}} F_\nabla = 0. \tag{5.33}$$

Proof. Let $s \in \Omega^0(E)$. We calculate $(d^\nabla)^3 s$ in two ways. First note that

$$(d^\nabla)^3 s = (d^\nabla)^2\left(d^\nabla s\right) = F_\nabla \wedge d^\nabla s. \tag{5.34}$$

However, we can also write

$$(d^\nabla)^3 s = d^\nabla \left((d^\nabla)^2 s\right) = d^\nabla(F_\nabla \wedge s). \tag{5.35}$$

Using the relation (5.32) we find

$$d^\nabla(F_\nabla \wedge s) = (d^{\nabla^{\text{Hom}}} F_\nabla)s + F_\nabla \wedge d^\nabla s. \tag{5.36}$$

Hence, equating these two calculations yields

$$F_\nabla \wedge d^\nabla s = (d^{\nabla^{\text{Hom}}} F_\nabla)s + F_\nabla \wedge d^\nabla s, \tag{5.37}$$

for all s. Therefore, $d^{\nabla^{\text{Hom}}} F_\nabla = 0$. $\qquad\qquad\square$

5.2.2 Chern - Weil Theory

Having defined connections and curvature, we use these to construct topological invariants. Chern - Weil theory in fact gives rise to a number of topological invariants such as Euler classes and Todd classes. For our purposes, however, we will focus exclusively on Chern classes. For this, we first introduce some notions in linear algebra.

Definition 5.2.8. Let V be a vector space over $\mathbb{C}$ and let P be some multilinear map $P : V \times \cdots \times V \to \mathbb{C}$. We define the polarised form $\tilde{P}$ of the map P as the homogeneous polynomial $\tilde{P}(B) = P(B, B, \ldots, B)$.

Let $\text{GL}_n(\mathbb{C})$ denote the Lie group of invertible $n \times n$ matrices and $\mathfrak{gl}_n(\mathbb{C})$ be its Lie algebra of $n \times n$ matrices. We restrict attention to the complex vector space $V = \mathfrak{gl}_n(\mathbb{C})$.

Definition 5.2.9. A symmetric map $P : \mathfrak{gl}_n(\mathbb{C}) \times \cdots \times \mathfrak{gl}_n(\mathbb{C}) \to \mathbb{C}$ is said to be invariant if

$$P(C^{-1}B_1 C, \ldots, C^{-1}B_k C) = P(B_1, \ldots, B_k), \tag{5.38}$$

for any $C \in \text{GL}_n(\mathbb{C})$ and $B_1, \ldots, B_k \in \mathfrak{gl}_n(\mathbb{C})$.

Lemma 5.2.10. Let P be a symmetric multilinear map $P : \mathfrak{gl}_n(\mathbb{C}) \times \cdots \times \mathfrak{gl}_n(\mathbb{C}) \to \mathbb{C}$. If P is also invariant then we have

$$\sum_{j=1}^{k} P(B_1, \ldots, B_{j-1}, [B, B_j], B_{j+1}, \ldots, B_k), \tag{5.39}$$

for all $B, B_1, \ldots, B_k \in \mathfrak{gl}_n(\mathbb{C})$

Proof. Take $C = e^{tB} \in \mathrm{GL}_n(\mathbb{C})$. Differentiate (5.38) with respect to t and set $t = 0$. This yields the desired result. $\qquad\square$

Proposition 5.2.11. Let $\pi : E \to M$ be a vector bundle of rank r. Let P be a symmetric, multilinear invariant map on $\mathfrak{gl}_n(\mathbb{C})$. Then for any $m = i_1 + \cdots + i_k$ the map P induces a well defined k-linear map

$$P : \left(\wedge^{i_1} M \otimes \mathrm{End}(E) \right) \times \cdots \times \left(\wedge^{i_k} M \otimes \mathrm{End}(E) \right) \to \wedge^m M, \tag{5.40}$$

defined by $P(\alpha_1 \otimes t_1, \ldots, \alpha_k \otimes t_k) = (\alpha_1 \wedge \cdots \wedge \alpha_k) P(t_1, \ldots, t_k)$.

Proof. Notice that the above definition makes sense if if we choose a local trivialisation $\pi^{-1}(p) \cong \mathbb{C}^r$. This does not actually depend on the choice of local trivialisation since P is an invariant map by hypothesis. $\qquad\square$

This map then also induces a multilinear map

$$P : \Omega^{i_1}(\mathrm{End}(E)) \times \cdots \Omega^{i_k}(\mathrm{End}(E)) \to \Omega^m(M), \tag{5.41}$$

in the notation of the previous subsection and where we recall $\Omega^m(M)$ is the space of sections of $\bigwedge^m T^*M$. Note that this map is in general no longer symmetric. If we restrict to forms of even degree nonetheless, we retain the symmetry of P. In particular, we can apply these invariant, multilinear maps to the curvature form F_∇. Ultimately, these will generate invariant characteristic classes for any vector bundle. First, we need the following proposition.

Proposition 5.2.12. Let P be the map $P : \Omega^{i_1}(\text{End}(E)) \times \cdots \Omega^{i_k}(\text{End}(E)) \to \Omega^m(M)$ defined as above. Let $\gamma_j \in \Omega^{i_j}(\text{End}(E))$. Then P satisfies

$$dP(\gamma_1, \ldots, \gamma_k) = \sum_{j=1}^{k} (-1)^{\sum_{l=1}^{j-1} i_l} P(\gamma_1, \ldots, \nabla^{\text{Hom}}(\gamma_j), \ldots, \gamma_k). \tag{5.42}$$

Proof. Working locally, we write $\nabla = d + A$ where A is the matrix of one forms for the connection ∇. The connection induced on $\text{End}(E)$ we write as $\nabla^{\text{Hom}} = d + A$ where A acts as the operator $\gamma \mapsto [A, \gamma]$ as in the previous subsection. Employing the Leibniz formula for the exterior derivative, we calculate

$$dP(\gamma_1, \ldots, \gamma_k) = \sum_{j=1}^{k} (-1)^{\sum_{l=1}^{j-1} i_l} P(\gamma_1, \ldots, \nabla^{\text{Hom}}(\gamma_j), \ldots, \gamma_k) \tag{5.43}$$

$$= \sum_{j=1}^{k} (-1)^{\sum_{l=1}^{j-1} i_l} P(\gamma_1, \ldots, (\nabla - A)(\gamma_j), \ldots, \gamma_k). \tag{5.44}$$

By linearity we can write the right hand side as

$$\sum_{j=1}^{k} (-1)^{\sum_{l=1}^{j-1} i_l} P(\gamma_1, \ldots, \nabla^{\text{Hom}}(\gamma_j), \ldots, \gamma_k) - \sum_{j=1}^{k} (-1)^{\sum_{l=1}^{j-1} i_l} P(\gamma_1, \ldots, A(\gamma_j), \ldots, \gamma_k). \tag{5.45}$$

By Lemma 5.2.10, we see that the second term vanishes and so the proposition follows. $\square$

Corollary 5.2.13. Let E be a vector bundle of rank r, with a connection ∇ and associated curvature form F_∇. Let P be any invariant, symmetric $k-$multilinear polynomial map on $\mathfrak{gl}_r(\mathbb{C})$. Then the induced, polarised form $\tilde{P}(F_\nabla) \in \Omega^{2k}(M)$ is closed.

Proof. This follows from the above proposition and the Bianchi identity, since

$$d\tilde{P}(F_\nabla) = dP(F_\nabla, \ldots, F_\nabla) = \sum_{j=1}^{k} (-1)^{\sum_{l=1}^{j-1} i_l} P(F_\nabla, \ldots, \nabla^{\text{Hom}}(F_\nabla), \ldots, F_\nabla). \tag{5.46}$$

Since P is homogeneous and symmetric, we then have $d\tilde{P}(F_\nabla) = 0$. $\square$

We now define the Chern forms and Chern classes. Define the homogeneous, polarised

polynomials $\{\tilde{P}_k\}$ with deg $\tilde{P}_k = k$ by

$$\det(\mathbb{1} + B) = 1 + \tilde{P}_1(B) + \cdots + \tilde{P}_r(B). \tag{5.47}$$

Since the determinant is invariant under conjugation, $\tilde{P}_k$ is also invariant.

Definition 5.2.14. Let E be a vector bundle of rank r and endow E with a connection ∇. The kth Chern form is defined by

$$c_k(E, \nabla) := \tilde{P}_k\Big(\frac{i}{2\pi}F_\nabla\Big). \tag{5.48}$$

By virture of Corollary 5.2.13 this form actually defines a cohomology class in the de Rham cohomology. Hence, define the chern class as

$$c_k(E) := \Big[\tilde{P}_k\big(\frac{i}{2\pi}F_\nabla\big)\Big] \in H^{2k}(M, \mathbb{C}). \tag{5.49}$$

Furthermore, we have the following important proposition.

Proposition 5.2.15. The Chern class is independent of the connection on the vector bundle E.

Proposition 5.2.15 and Corollary 5.2.13 are the essence of the Chern - Weil Theorem, providing a cohomology class that is independent of the choice of connection. The contents of this theorem are in fact slightly more general but we shall not need this and so we refer the interested reader to [77].

5.3   The Kontsevich - Witten Theorem

In general, when one has a notion of curvature, one can produce useful geometric information by integrating this curvature. Broadly speaking, we will see two manifestations of this idea. Firstly, the partition function of quantum gravity in two dimensions can be given as a generating function of certain intersection numbers. On the other hand, one may consider

the Einstein-Hilbert action part of which is the integral of the Ricci scalar curvature. To be consistent, these two approaches must yield the same partition function.

For the first approach, we merely state the results following [2]. For a thorough discussion of the results stated here, see [78].

In string theory, the moduli space of stable curves $\bar{\mathcal{M}}_{g,n}$ naturally arises since one 'sums over topologies'. One can consider *tautological line bundles* $\mathcal{L}_i$ over $\bar{\mathcal{M}}_{g,n}$ where the fibre of each point $(\Sigma, p_1, \ldots, p_n) \in \bar{\mathcal{M}}_{g,n}$ is the cotangent space $T^*_{p_i}\Sigma$. We can the define the following intersection numbers.

Definition 5.3.1. Let $\pi_i : \mathcal{L}_i \to \bar{\mathcal{M}}_{g,n}$ be the ith tautological line bundle. Define the ψ classes as $\psi_i := c_1(\mathcal{L}_1) \in H^2(\bar{\mathcal{M}}_{g,n}, \mathbb{C})$ where c_1 is the first Chern class. Recall the dimension of $\bar{\mathcal{M}}_{g,n}$ is given as $d_{g,n} = \dim \bar{\mathcal{M}}_{g,n} = 3g - 3 + n$. Then define the following intersection numbers.

$$\langle \tau_{k_1} \cdots \tau_{k_n} \rangle_{g,n} := \begin{cases} \int_{\bar{\mathcal{M}}_{g,n}} \psi_1^{k_1} \wedge \cdots \wedge \psi_n^{k_n}, & \sum_i k_i = d_{g,n} \\ 0, & \text{otherwise} \end{cases} \tag{5.50}$$

Remark. One should be slightly cautious of this definition since we have not defined either the first Chern class, or integration, over orbifolds or stacks. We rest reassured, however, that all these notions extend naturally from manifolds. See [74] for a more rigorous discussion on intersection numbers.

We note here that these intersection numbers are in fact the Gromov-Witten invariants of a zero dimensional manifold. See [79]. With this in hand, we can form a generating function to encode these intersection numbers. One specific choice of generating function, nevertheless, is rather special.

Theorem 5.3.2. The free energy of two dimensional quantum gravity is given by

$$F(t) = \log Z(t) = \sum_{g,n \geq 0} \sum_{k_1, \ldots, k_n} \langle \tau_{k_1} \cdots \tau_{k_n} \rangle_{g,n} \prod_{i=1}^{n} \frac{t_i^{k_i}}{k_i!}, \tag{5.51}$$

where t_i are coupling constants.

A thorough discussion can be found in [78].

For the second approach, we calculate the partition function as

$$Z \sim \sum_{\text{topologies}} \int \mathcal{D}g\mathcal{D}X e^{-S}, \tag{5.52}$$

where $X : \Sigma \to M$ is a d-dimensional 'matter' field and g is a metric on the Riemann surface Σ which acts as the world sheet in string theory.

In two dimensional gravity, we consider a theory with no matter degrees of freedom which is equivalent to a string propagating in the manifold M with dimension $d = 0$. In this case the partition function reads

$$Z = \sum_{h \geq 0} \int \mathcal{D}g e^{-\gamma A + \beta \chi}, \tag{5.53}$$

where the cosmological term is given by the area,

$$A = \int \sqrt{g}, \tag{5.54}$$

and χ is given by

$$\chi = \frac{1}{4\pi} \int \sqrt{g} R. \tag{5.55}$$

Here, R is the Ricci scalar curvature. We remark here that this action does indeed lead to the Einstein field equations. See [57] for a proof and more thorough discussion on general relativity. In two dimensions, the quantity χ is given as $\chi = 2 - 2h$ by the Gauss-Bonnet theorem, where h is the genus of Σ. Thus, classically in two dimensions, gravity is rather trivial since the action is fixed for a given Riemann surface. In string theory, nevertheless, this is non-trivial as we are performing a sum over Riemann surfaces to calculate Z. Physicists often refer to quantum fluctuations, which change the genus of such a Riemann surface, meaning the partition function does indeed contain non-trivial information.

In general, the integral over worldsheet metrics is difficult to compute. However, a novel approach was established in [80–82], in which the surface Σ is discretised allowing the integral to be evaluated more readily. After calculating the discrete analogue, we will eventually take

a continuum limit to recover the original integral. We recall the construction of maps as discrete surfaces in chapter three in the following figure.
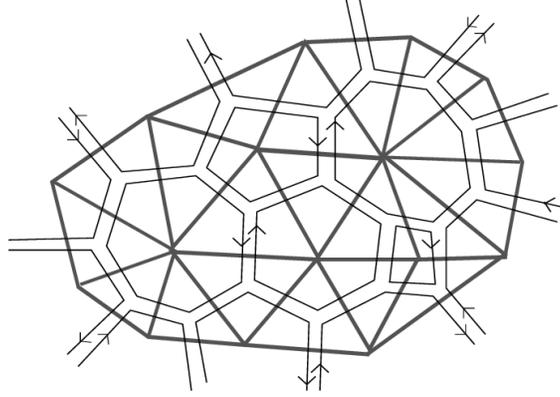
Figure 5.4: A random triangulation of a surface with each face dual to a trivalent vertex.

Now, consider a surface that is discretised using triangles, similar to that of the above figure. For simplicity, we will enforce that all triangular faces are equilateral. We will later argue that this restriction is unnecessary and so remove it. Since we assume all triangles are equilateral, for a vertex i with $N_i > 6$ incident triangles, there is a negative curvature at i. Similarly, for a vertex with $N_i < 6$ there is positive curvature and zero curvature for $N_i = 6$. The discrete form of the Ricci scalar R_i at vertex i is then

$$R_i = \sum \frac{2\pi(6 - N_i)}{N_i}. \tag{5.56}$$

Moreover, let V, E, F be the total number of vertices, edges and faces of the surface. Then we have the relations $2E = \sum_i N_i$, and $3F = 2E$. The second relation follows from the observation that each triangle has three edges and each edge is shared by two faces. Hence, to move from an action on a Riemann surface to an action on a discrete surface, we use the replacements

$$\int \sqrt{g}R \mapsto \sum_i \frac{4\pi(6 - N_i)}{N_i} = 4\pi\left(V - \frac{1}{2}F\right) = 4\pi(V - E + F) = 4\pi\chi, \tag{5.57}$$

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and

$$A = \int \sqrt{g} \mapsto \sum_i \frac{N_i}{3}. \tag{5.58}$$

The factor of $1/3$ in equation (5.58) for the area removes overcounting since each triangular face has three vertices. Thus, the integral over metrics $\int \mathcal{D}g$ summed over Riemann surfaces may be replaced by a sum over random triangulations in the discrete case,

$$\sum_{h \geq 0} \int \mathcal{D}g \mapsto \sum_{\substack{\text{random} \\ \text{triangulations}}}. \tag{5.59}$$

In this procedure, triangles do not play a special role and the above discussion may be repeated for any n-gon. However, we have already seen how to compute sums over random discrete surfaces using Hermitian matrix models. In particular, we recall that the free energy has a topological expansion,

$$F = \log Z = \sum_{h \geq 0} N^{2-2h} F_h \tag{5.60}$$

where

$$F_h = \sum_{\substack{\text{closed, connected} \\ \text{maps } \Sigma}} \frac{1}{\text{Aut}(\Sigma)} t_3^{n_3(\Sigma)} \cdots t_d^{n_d(\Sigma)}. \tag{5.61}$$

Now, to recover the continuous integral $\int \mathcal{D}g$ over all possible geometries, we take a continuum limit. Now, one could take a naive limit of a large number of triangles so that we obtain a good approximation for a Riemann surface. However, one must also send the area of each polygon to zero so that the total area is finite. More precisely, we recall from chapter three that Z is not analytic at all values of t_3. Indeed, there is some critical value t_c at which Z diverges. For simplicity, assume all coupling constants vanish except $t_3 \neq 0$. Then from (5.58), the total area is proportional to the number of faces so that

$$\langle A \rangle = \langle n_3 \rangle = t_3 \frac{\partial \log Z_0}{\partial t_3} \sim \frac{1}{t_c - t_3}. \tag{5.62}$$

Thus, intuitively, if we take $N \to \infty$ and $t_3 \to t_c$, the total area will remain finite. This is

exactly the double scaling limit that was discussed in chapter three. In addition, from Theorem 3.4.1 we know that the double scaled limit of a Hermitian matrix model with aribtrary potential is a KdV tau function that corresponds to initial condition $u(t_1, 0, 0, \dots) = t_1$. Thus, the partition function of two dimensional gravity is solved by this KdV tau function. We note here that, at present, there is no explicit proof that the procedure we have outlined for defining continuous surfaces from discrete surfaces coincides with the original continuous integral over metrics and Riemann surfaces. There is, nonetheless, evidence to suggest that this is indeed a reasonable procedure. See, for example, [80–82] and references therein.

These two approaches, one based on topological field theory and the other based on a discretising sums over worldsheets and metrics, should agree. This is the basis of Witten's conjecture, now commonly known as the Kontsevich-Witten theorem.

Theorem 5.3.3. Let $F(t)$ be the function

$$F(t) = \sum_{g,n \geq 0} \sum_{k_1, \dots, k_n} \langle \tau_{k_1} \cdots \tau_{k_n} \rangle_{g,n} \prod_{i=1}^{n} \frac{t_i^{k_i}}{k_i!}, \tag{5.63}$$

where $\langle \tau_{k_1} \cdots \tau_{k_n} \rangle_{g,n}$ are the intersection numbers defined above. Let $\tau(T)$ be the KdV tau function corresponding to the initial condition $u(T_1, 0, 0, \dots) = T_1$. with

$$T_{2i-1} = \frac{1}{(2i-1)!!} t_i. \tag{5.64}$$

Then

$$F(T) = \log \tau(T). \tag{5.65}$$

To date, this theorem has received a number of proofs. Indeed the work of Kontsevich [2] was the first to prove the above theorem utilising a new type of matrix integral. Furthermore, the proof of Okounkov and Pandharipande [83] is based on properties of Hurwitz numbers. On the other hand, Mirzakhani's proof [84] is based on the Riemannian geometry of moduli spaces and is related to the argument of Eynard and Orantin [85] using topological recursion. It is Kontsevich's original proof, however, that will be of most interest to us.

Chapter 6

Hermitian Matrix Models with an External Field

In order to prove Theorem 5.3.3, in [2] Kontsevich considered an entirely different type of matrix model where an external field matrix Λ was introduced into a model with cubic potential. Indeed, this breaks the $U(N)$ gauge 'symmetry' or, as in chapter three, what is better referred to as a redundancy. It is this fact that allows the cubic Kontsevich matrix model to be identified as both a tau function of the KdV hierarchy and generating function for intersection numbers. This proves Witten's conjecture. Furthermore, as stated in [46], the Kontsevich matrix model is an explicit representation of the double scaled limit of the Hermitian matrix models that were considered in chapter three. Thus, the corresponding continuum Virasoro constraints can be extracted efficiently from this matrix model, rather than appealing to double scaling limits which are in general more difficult. One can also allow for more general potentials in the action. A particular case of interest is when the potential has rational derivatives instead of simply a polynomial. When the potential is $V(X) = 1/X$, we identify this as the BGW tau function as is done in [9]. We show, in some detail, how the corresponding Virasoro constraints are extracted. Finally, we suggest a generalisation of the BGW partition function and, by calculating the $\mathcal{W}$-algebra constraints, present evidence concerning which Airy structure this generalised BGW partition function

corresponds to.

6.1 External Field Matrix Models as Tau Functions

As mentioned above, one can consider external field models with arbitrary potentials $V(X)$. For example, external fields models with arbitrary polynomial potential are connected to $\mathcal{N} = 2$ supersymmetric Landau-Ginzburg theories [3]. There have also been more recent and exciting developments in the formation of an "M-theory of matrix models" [86, 87]. Broadly speaking, in this paradigm external field matrix models are a fundamental building block of tau functions which are *the* special functions of string theory. Here, we consider only the simplest case, restricting attention to potentials of the form $V(X) = -X^{r+1}/(r+1)$. These are the models that give rise to tau functions for the r-KdV hierarchies.

6.1.1 Determinant Representations

In order to identify matrix models coupled to an external field with tau functions, we first rewrite each in some determinant representation. This is not the only approach we could take: a rather beautiful perspective is outlined in [40] where $\mathcal{F}_V$ is treated using the orthogonal polynomial method of chapter three. In this procedure, one finds that, roughly speaking, the orthogonality conditions for such polynomials implies that $\mathcal{F}_V$ obeys the Hirota equations, thus proving that $\mathcal{F}_V$ gives rise to a KP tau function. Here, however, we take a different approach and we broadly follow the discussions given in [3, 9, 88].

We begin with a definition.

Definition 6.1.1. Let H_N be the space of $N \times N$ Hermitian matrices with Lebesgue measure dX. Let $\Lambda \in H_N$. We say that Λ is an *external field matrix*. Let $r \in \mathbb{Z}$ where $|r| \geq 2$. The *external field matrix model* is the formal Hermitian matrix model

$$\mathcal{F}_V(\Lambda; r, k) = \int_{H_N} dX\, e^{\mathrm{Tr}\left(V(X) + \Lambda X - k \log X\right)}, \tag{6.1}$$

with $k \in \mathbb{Z}$ and the potential V given by

$$V(X) = -\frac{X^{r+1}}{r+1}. \tag{6.2}$$

To ease notation, we will often drop the explicit dependence on r and k, preferring instead to write $\mathcal{F}_V(\Lambda)$. If $V(X) = -X^3/3$ and $\mathcal{F}_V$ has no logarithmic part in the exponential, that is we choose $r = 2$ and $k = 0$, we refer to this simply as the Kontsevich model.

This definition deserves some important clarification. In general, a Hermitian matrix need not be invertible or have non-zero eigenvalues. One may therefore be disturbed at the meaning of the terms $\log X$ and X^{r+1} in this definition for $r \leq -2$. For $r \geq 2$ we call this the monomial case while for $r \leq -2$, we call this the *antimonomial* case agreeing with terminology introduced in [9]. The logarithm is not particularly troublesome since, formally, we have the identity $\mathrm{Tr} \, \log X = \log \det X$. Thus, the external field matrix model can be rewritten as

$$\mathcal{F}_V(\Lambda) = \int_{H_N} dX (\det X)^{-k} e^{\mathrm{Tr} \left(V(X) + \Lambda X \right)}. \tag{6.3}$$

Concerning the potential, to give meaning to expressions such as $1/X$, we first find the saddle point X_0 of the action

$$S(X) = \mathrm{Tr} \left(V(X) + \Lambda X \right). \tag{6.4}$$

Here, we treat the logarithmic term as part of the measure rather than the action. We will expand and clarify this point in section 5.3. We recall from chapter three that, due to the trace, we can naively differentiate the above expression by treating X as if it were a single variable. Thus, X_0 satisfies

$$V'(X_0) + \Lambda = 0. \tag{6.5}$$

Having found the saddle point, we then change variables $X \mapsto X + X_0$ in the external field matrix integral. Performing a Taylor expansion of $V(X + X_0)$ then yields an infinite series in the antimonomial case. Due to the judicious choice of X_0, the linear term in the expansion of $S(X)$ will vanish. This leaves a Gaussian Hermitian matrix model that can, in principle,

be evaluated using the methods of chapter three. The external field matrix models are quite different in their nature to those formal matrix integrals considered in chapter three however. Indeed, the potential and logarithmic pieces are invariant under unitary gauge transformations due the cyclicity of the trace and multiplicative property of the determinant respectively. We also recall Lemma 3.2.9 that proved the unitary invariance of the Lebesgue measure dX. Nevertheless, the presence of the external field Λ breaks the unitary gauge symmetry. Despite this, suppose we diagonalised Λ so that $D_\Lambda = U^\dagger \Lambda U$ for some diagonal matrix $D_\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_N)$. Then using the change of variables $X \mapsto UXU^\dagger$ in the external field matrix model (6.1), we find

$$\mathcal{F}_V(\Lambda) = \int_{H_N} dX e^{\operatorname{Tr}\left(V(X) + \Lambda UXU^\dagger - k \log X\right)} = \mathcal{F}_V(D_\Lambda). \tag{6.6}$$

This means that $\mathcal{F}_V(\Lambda)$ can in fact be considered as function of the N eigenvalues $\lambda = (\lambda_1, \ldots, \lambda_N)$ rather than a function of the N^2 entries of Λ. This will often drastically simplify the calculations we will consider later.

When dealing with external field matrix integrals, the following Harish-Chandra-Itzykson-Zuber (HCIZ) formula is especially useful.

Theorem 6.1.2. Let $A, B \in H_N$ with eigenvalues a_i and b_i for $1 \leq i \leq N$ such that $a_i \neq a_j$ and $b_i \neq b_j$ for $i \neq j$. Let dU be the Haar measure on U(N). Then

$$\int dU e^{\operatorname{Tr} AUBU^\dagger} = K_N \cdot \frac{\det\left(e^{a_i b_j}\right)}{\Delta(a)\Delta(b)}, \tag{6.7}$$

for some constant K_N and where

$$\Delta(a) = \prod_{i<j}(a_i - a_j), \tag{6.8}$$

is the Vandermonde determinant.

Proof. We follow the proof given by Brézin and Hikami in [89]. Consider the Laplacian

operator

$$L = \frac{\partial^2}{\partial A_{jk} \partial A_{kj}}. \tag{6.9}$$

See Appendix A.1.1 for the definition of this matrix derivative. Here, we employ Einstein summation convention. The eigenfunctions of L are plane waves

$$L\left(e^{\mathrm{Tr}\, BA}\right) = \mathrm{Tr}\,(B^2)e^{\mathrm{Tr}\, BA}. \tag{6.10}$$

Note that the eigenvalue is unchanged if we replace $B \mapsto UBU^\dagger$. The eigenvalue is also invariant under the continuous superposition of waves. Taken together, these observations imply that

$$L\Psi_B(A) = \mathrm{Tr}\,(B^2)\Psi_B(A), \tag{6.11}$$

where

$$\Psi_B(A) = \int dU e^{\mathrm{Tr}\, AUBU^\dagger}. \tag{6.12}$$

We observe that Ψ_B is in fact a function of the eigenvalues a_j of A. Indeed, in a similar way as before, suppose $V \in \mathrm{U}(N)$ diagonalises A so that $\mathrm{diag}\,(a_1,\ldots,a_N) = D_A = V^\dagger A V$. We then consider the change of variables $U \mapsto VU$. It is well known that the Haar measure dU is both left and right invariant. Thus, using the cyclicity of the trace,

$$\Psi_B(A) = \int dU e^{\mathrm{Tr}\, V^\dagger AVUBU^\dagger} = \int dU e^{\mathrm{Tr}\, D_A UBU^\dagger}. \tag{6.13}$$

Since we can arbitrarily relabel eigenvalues, Ψ_B must also be a symmetric function of the a_j. We reduce the second derivative $\frac{\partial^2}{\partial A_{jk}\partial A_{kj}}$ to derivatives with respect to the eigenvalues using the formula

$$\frac{\partial^2}{\partial A_{ij}A_{ji}}\psi_B = \frac{1}{\Delta(a)}\frac{\partial^2}{\partial a_k\partial a_k}\big(\Delta(a)\psi_B\big). \tag{6.14}$$

We shall clarify this formula Appendix A.1.5. The eigenvalue equation for L then reduces to

$$\frac{1}{\Delta(a)}\frac{\partial^2}{\partial a_j\partial a_j}\big(\Delta(a)\psi_B\big) = \mathrm{Tr}\,B^2\psi_B. \tag{6.15}$$

Now, note that $\Delta(a)\Psi_B$ is totally antisymmetric in a_j. Because of this property, it is well known that $\Delta(a)\Psi_B$ can be obtained as a Slater determinant of the elementary eigenfunctions $\psi_j(a_k) = e^{b_j a_k}$ which satisfy $\psi_j''(a_k) = b_j^2 \psi_j(a_k)$ and $\sum_j b_j^2 = \mathrm{Tr}\, B^2$. That is to say,

$$\Delta(a)\Psi_B \propto \det\left(e^{b_j a_k}\right). \tag{6.16}$$

Finally, consider that Ψ_B is symmetric under the interchange of A and B. Moreover, the right hand side of (6.16) is symmetric under the interchange of A and B since the determinant is unchanged by taking the transpose. Therefore in (6.16) the left hand side must be proportional to $1/\Delta(b)$ leading to

$$\Psi_B = K_N \cdot \frac{\det\left(e^{a_j b_k}\right)}{\Delta(a)\Delta(b)}, \tag{6.17}$$

as desired. $\qquad\square$

The proof of the HCIZ formula we have presented only used elementary methods and masks the much deeper nature of this result. The original proof given in [90] is perhaps richer and relies on the Duistermaat-Heckman theorem. We shall not dwell in this matter however.

Using the HCIZ formula, one can now reduce the external field matrix model to an integral over eigenvalues.

Proposition 6.1.3. Let $\mathcal{F}_V(\Lambda)$ be the external field matrix model defined above where Λ has eigenvalues λ_i. Then

$$\mathcal{F}_V = \frac{C_N}{\Delta(\lambda)} \int \prod_{i=1}^N dx_i \Delta(x) e^{\sum_{j=1}^N V(x_j) + \lambda_j x_j - k \log x_j}, \tag{6.18}$$

for some constant C_N.

Proof. We first use the change of variables $X = U D_X U^\dagger$ where $D_X = \mathrm{diag}\,(x_1, \ldots, x_N)$. We recall that in chapter three, we found that the Lebesgue measure transforms as $dX =$

$\Delta^2(x) \prod_i dx_i dU$. Using this substitution, we find that,

$$\mathcal{F}_V = C'_N \int \prod_{i=1}^{N} dx_i \Delta^2(x) e^{\sum_{j=1}^{N} V(x_j) - k \log x_j} \int dU e^{\Lambda U D_X U^\dagger}, \tag{6.19}$$

where C'_N is a combinatorial proportionality constant. The exact value of C'_N is irrelevant since we recall that overall rescalings of $\mathcal{F}_V$ are irrelevant. Now utilising the HCIZ formula, we obtain

$$\mathcal{F}_V = \frac{C''_N}{\Delta(\lambda)} \int \prod_{i=1}^{N} dx_i e^{\sum_{j=1}^{N} V(x_j) - k \log x_j} \Delta(x) \det\left(e^{\lambda_i x_j}\right), \tag{6.20}$$

with $C''_N = K_N C_N$. Expanding the determinant in terms of permutations of the symmetric group S_N we find

$$\mathcal{F}_V = \frac{C''_N}{\Delta(\lambda)} \sum_{\sigma \in S_N} \text{sgn}\,(\sigma) \int \prod_{i=1}^{N} dx_i e^{\sum_{j=1}^{N} V(x_j) + k \log x_j} \Delta(x) \prod_{k=1}^{N} e^{\lambda_k x_{\sigma(k)}}. \tag{6.21}$$

We now make the change of variables $x'_k = x_{\sigma(k)}$. This implies $\text{sgn}(\sigma)\Delta(x) = \Delta(x')$. Inserting this into (6.21) gives a sum of $N!$ equal terms. After dropping the primes, the expression for $\mathcal{F}$ reads

$$\mathcal{F}_V = \frac{C_N}{\Delta(\lambda)} \int \prod_{i=1}^{N} dx_i e^{\sum_{j=1}^{N} V(x_j) + \lambda_j x_j - k \log x_j} \Delta(x), \tag{6.22}$$

with $C_N = N! C''_N$. $\qquad \square$

Corollary 6.1.4. Omitting unessential constant factors in the external field matrix integral $\mathcal{F}_V$, we have

$$\mathcal{F}_V = \frac{\det_{ab} \Phi_{a-k}(\lambda_b)}{\Delta(\lambda)} \tag{6.23}$$

where

$$\Phi_a(\lambda_b) = \int dx\, x^{a-1} e^{V(x) + \lambda_b x} \tag{6.24}$$

Proof. The follows readily from the previous theorem and recalling that

$$\Delta(x) = \det \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots x_1^{N-1} \\ 1 & x_2 & x_2^2 & \cdots x_2^{N-1} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x_N & x_N^2 & \cdots x_N^{N-1} \end{bmatrix}, \tag{6.25}$$

is an equivalent expression for the Vandermonde determinant. $\qquad\square$

It is this form of $\mathcal{F}_V$ given in the above corollary that allows us to establish the connection between external field matrix integrals and the KP hierarchy.

We first recall facts established in chapters two and four about tau functions and conformal field theory. We recall from chapter two that a tau function for the KP hierarchy can be written in the form

$$\tau_g(t) = \langle 0| : e^{\sum_{n \in \mathbb{Z}} t_n J_n} : g |0\rangle. \tag{6.26}$$

Here, we recall the free, chiral fermionic fields

$$\psi(z) = \sum_{m \in \mathbb{Z}} \psi_m z^m, \quad \psi^\dagger(z) = \sum_{m \in \mathbb{Z}} \psi_m^\dagger z^{-m}, \tag{6.27}$$

and the corresponding current

$$J(z) = \partial\varphi(z) = \sum_{m \in \mathbb{Z}} J_m z^{-m-1}, \tag{6.28}$$

with modes

$$J_m = \sum_{i \in \mathbb{Z}} : \psi_i \psi_{m+i}^\dagger : . \tag{6.29}$$

We recall the chiral bosonic field from chapter four,

$$\varphi(z) = ix_0 + J_0 \log z - \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{1}{n} J_n z^{-n}, \tag{6.30}$$

where we have multiplied by a factor of i for convenience. The fermionic fields $\psi(z)$ and $\psi^\dagger(z)$ that appear the expression for τ are in fact given by the boson-fermion correspondence

$$\psi(z) =: e^{\varphi(z)} :, \tag{6.31}$$

and similarly for $\psi^\dagger(z)$. We also recall that vacuum states are defined by

$$\psi_m \left|0\right\rangle = 0, \tag{6.32}$$

for $m < 0$ and

$$\psi^\dagger_m \left|0\right\rangle = 0, \tag{6.33}$$

for $m \geq 0$. Finally, the charge N vacuum $\left|N\right\rangle$ is defined by

$$\left|N\right\rangle = \begin{cases} \psi_{N-1} \cdots \psi_0 \left|0\right\rangle & N > 0 \\ \psi^\dagger_N \cdots \psi^\dagger_{-1} \left|0\right\rangle & N < 0. \end{cases} \tag{6.34}$$

Furthermore, it can be shown, see [3] for example, that

$$\left\langle N\right| e^{iMx_0} = \left\langle N - M\right|, \tag{6.35}$$

for any $M \in \mathbb{Z}$.

We now find an expression for the tau function (6.26) by calculating the expectation value $\left\langle N\right| \psi(\lambda_N) \cdots \psi(\lambda_1) g \left|0\right\rangle$ in two different ways. Firstly we rewrite the tau function with the aid of the Miwa parameterisation for the time variables

$$t_n = \frac{1}{n} \sum_i \frac{1}{\lambda_i^n}. \tag{6.36}$$

In the free boson representation, $J(z) = \partial \varphi(z)$, we have

$$\sum t_n J_n = \sum_i \sum_n \frac{1}{n \lambda_i^n} \varphi_n = \sum_i \varphi(\lambda_i). \tag{6.37}$$

We now recall Proposition 4.1.14 which in this case reads

$$: e^{\sum_i \varphi(\lambda_i)} := \frac{1}{\prod_{i>j}(\lambda_i - \lambda_j)} \prod_i : e^{\varphi(\lambda_i)} := \frac{1}{\Delta(\lambda)} \prod_i \psi(\lambda_i). \tag{6.38}$$

Note that in Proposition 4.1.14 there was an extra factor of $\sqrt{-1}$ which has been absorbed into the definition of φ here. Hence, using the above equation and (6.35), we have

$$\langle N | \psi(\lambda_N) \cdots \psi(\lambda_1) | 0 \rangle = \Delta(\lambda) \langle 0 | : e^{\sum_{n \in \mathbb{Z}} t_n J_n} : g | 0 \rangle = \Delta(\lambda) \tau_g(t). \tag{6.39}$$

On the other hand, since the tau function in equation (6.26) is the expectation value of a Gaussian operator, we may apply Wick's theorem from chapter three. Namely, that a correlation function of the form

$$C_g(\alpha_1, \beta_1, \dots) = \langle 0 | \prod_i \psi^\dagger(\alpha_i) \psi(\beta_i) g | 0 \rangle, \tag{6.40}$$

may be expressed as an appropriate sum of correlation functions of pairings:

$$C_g(\alpha_1, \beta_1, \dots) = \det C_g(\alpha_i, \beta_j), \tag{6.41}$$

where

$$C_g(\alpha_i, \beta_j) = \langle 0 | \psi^\dagger(\alpha_i) \psi(\beta_j) g | 0 \rangle. \tag{6.42}$$

We consider a fermionic representation in the times

$$t_n = \frac{1}{n} \sum_{i=1}^N \frac{1}{\lambda_i^n} - \frac{1}{\tilde{\lambda}_i^n}. \tag{6.43}$$

To relate this to the bosonic representation, we take the limit $\tilde{\lambda}_i \to \infty$ for all i. Equivalently, one may replace the vacuum $\langle 0 |$ with $\langle N |$ where

$$\langle N | \sim \langle 0 | \, \psi^\dagger(\infty)(\psi^\dagger)'(\infty) \cdots (\psi^\dagger)^{(N-1)}(\infty). \tag{6.44}$$

The primes here denote the derivative with respect to z. Thus, we have

$$\langle N | \, \psi(\lambda_N) \cdots \psi(\lambda_1) \, | 0 \rangle = \langle 0 | \, \psi^\dagger(\infty)(\psi^\dagger)'(\infty) \cdots (\psi^\dagger)^{(N-1)}(\infty)\psi(\lambda_N) \cdots \psi(\lambda_1) \, | 0 \rangle \tag{6.45}$$

Applying Wick's theorem, we obtain,

$$\langle N | \, \psi(\lambda_N) \cdots \psi(\lambda_1) \, | 0 \rangle = \det \phi_i(\lambda_j), \tag{6.46}$$

where the functions ϕ_i have asymptotic behaviour

$$\phi_i(\lambda_j) \sim \langle 0 | \, (\psi^\dagger)^{(i-1)}(\infty)\psi(\lambda_j)g \, | 0 \rangle \sim \lambda_j^{i-1}\left(1 + O\left(\frac{1}{\lambda_j}\right)\right) \tag{6.47}$$

as $\lambda_j \to \infty$. Upon comparing equations (6.46) and (6.39) we find that

$$\tau_g(t) = \frac{\det \phi_i(\lambda_j)}{\Delta(\lambda)}, \tag{6.48}$$

where $\phi_i(\lambda_j)$ has asymptotics prescribed by equation (6.47).

In fact, the converse is also true: given any function τ of the form given in equation (6.48) with prescribed asymptotics (6.47), then τ is a tau function of the KP hierarchy. Hence, the specific form of the functions ϕ_i will play no role in the discussion that follows; we only require that ϕ_i have the correct asymptotics. Let us summarise, and generalise, this discussion in the following theorem.

Theorem 6.1.5. A function $\tau(\lambda)$ with $\lambda \in \mathbb{C}^N$ is a tau function of the KP hierarchy with times

$$t_n = \frac{1}{n}\sum_{i=1}^{N}\frac{1}{\lambda_i^n}, \tag{6.49}$$

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if and only if τ has the form

$$\tau(\lambda) = \frac{\det_{(ab)}(\lambda^k \phi_{a-k}(\lambda_b))}{\Delta(\lambda)}, \tag{6.50}$$

where ϕ_a has asymptotic behaviour

$$\phi_a(\lambda_b) \sim \lambda_b^{a-1}\left(1 + O\left(\frac{1}{\lambda_b}\right)\right), \tag{6.51}$$

as $\lambda_b \to \infty$.

Proof. We have already shown the forward direction. Conversely, given a function of the form (6.50), one needs to show that this is indeed a KP τ function. In other words, one would need to show that (6.50) satisfies the Hirota equations of chapter two. Nonetheless, this is rather technical and unenlightening, and so we omit the proof. Precise details can be found in [8]. $\qquad\square$

Remark. We remark that, for finite N, only a subset of the times $\{t_n : n \in \mathbb{N}\}$ are independent. For example, if $N = 2$, then it is readily checked that $3t_1 t_2 - t_1^3 = 2t_3$. The tau function in the case for finite N should then correspond to a tau function of the truncated KP hierarchy that was briefly mentioned in chapter two. In this regime, solutions have non trivial dependence only on a finite number of times. To obtain a tau function for the full KP hierarchy with all t_n independent, one should therefore take the limit $N \to \infty$. We shall not concern ourselves with such subtleties as we are primarily interested in the equations that are satisfied by tau functions such as the Virasoro constraints. We will see that the same equations are obeyed for both finite N as well as in the limit $N \to \infty$.

We notice the striking similarity between the determinant expressions of a general KP tau function and the external field matrix integral $\mathcal{F}_V$ given in Corollary 6.1.4. Indeed, by virtue of the previous theorem, we need only check the asymptotics of

$$\Phi_{a-k}(\lambda_b) = \int dx \, x^{a-k-1} e^{V(x)+\lambda_b x}, \tag{6.52}$$

as $\lambda_b \to \infty$. To this end, we consider the two cases where V is a monomial and V is an antimonomial. There is a slight subtlety in the antimonomial case which we will treat separately.

6.1.2 Monomial Potential

To obtain the asymptotics of $\Phi_{a-k}(\lambda_b)$ as $\lambda_b \to \infty$, we use the method of steepest descent. The saddle point x_0 is defined through

$$V'(x_0) + \lambda_b = 0. \tag{6.53}$$

Since V is a monomial, the limit $\lambda_b \to \infty$ implies that $x_0 \to \infty$. This is not the case if V is an antimonomial and hence we see why we distinguish the two cases. Changing variables $x \mapsto x + x_0$ and expanding $V(x + x_0)$ to second order we find

$$\Phi_{a-k}(x_0) \sim x_0^{a-k-1} e^{V(x_0) - V'(x_0)x_0} \int dx\, e^{V''(x_0)x^2/2 + \cdots}. \tag{6.54}$$

The second derivative $V''(x_0)$ is negative in this case and so we can integrate the Gaussian to obtain

$$\Phi_{a-k}(x_0) \sim \frac{x_0^{-k} e^{V(x_0) + \lambda_b x_0} x_0^{a-1}}{\sqrt{V''(x_0)}} \left(1 + O\left(\frac{1}{x_0}\right) \right) = \frac{x_0^{-k} e^{V(x_0) - V'(x_0)x_0}}{\sqrt{V''(x_0)}} \left(x_0^k \phi_{a-k} \right) \tag{6.55}$$

where we enforced

$$\phi_{a-k}(x_0) \sim x_0^{a-k-1} \left(1 + O\left(\frac{1}{x_0}\right) \right) \tag{6.56}$$

Hence, we see that $\mathcal{F}_V$ as defined in Definition 6.1.1 is not yet a KP tau function as it does not have the correct asymptotics. In spite of this, we can suitably normalise $\mathcal{F}_V$ by factoring out the leading order, or 'quasiclassical', contribution which we will denote as $\mathcal{C}_V$.

To calculate the leading behaviour $\mathcal{C}_V$, we return to the external field matrix integral

$$\mathcal{F}_V(\Lambda) = \int_{H_N} dX\, e^{\mathrm{Tr}\left(V(X) + \Lambda X - k \log X \right)}. \tag{6.57}$$

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To calculate this perturbatively when all eigenvalues of Λ are large, we still use steepest descent but we use more vigorous methods to calculate the quadratic factor. Suppose that, as before, X_0 satisfies the matrix equation

$$V'(X_0) + \Lambda = 0. \tag{6.58}$$

We then perform the change of variables $X \mapsto X + X_0$ in $\mathcal{F}_V$ and Taylor expand V as

$$\text{Tr } V(X + X_0) = \text{Tr}\Big(V(X_0) + V'(X_0)X + \frac{1}{2}H_V(X_0)X^2 + \cdots\Big). \tag{6.59}$$

where $H_V(X_0)$ is the Hessian of the function $X \mapsto \text{Tr } V(X)$ evaluated at X_0. We will show how to explicitly calculate this Hessian matrix in section 5.4 since the evaluation of the Hessian becomes a more delicate issue here. We will discuss why this is the case in Appendix A.1.1. Returning to the steepest descent method, inserting this Taylor expansion into $\mathcal{F}_V$ after the change of variables reads

$$\mathcal{C}_V = \int_{H_N} dX \det(X_0)^{-k} e^{\text{Tr } \left(V(X_0) + \frac{1}{2}H_V(X_0)X^2 + \Lambda X_0\right)}. \tag{6.60}$$

Notice that again there is no linear term in X in the exponential by construction, since X_0 was defined by equation (6.58).

Integrating the Gaussian and omitting constant factors, we find that

$$\mathcal{C}_V = \frac{e^{\text{Tr } \left(-V'(X_0)X_0 + V(X_0)\right)}}{\det^k(X_0) \det^{1/2}\left(H_V(X_0)\right)}. \tag{6.61}$$

We remark here that the determinant of the Hessian appears in exactly the same way as in equation (3.23): one diagonalises $H_V(X_0)$ and orthonormalises its eigenvectors, in which case the integral decouples into a product of independent Gaussian integrals. Thus, with suggestive notation we define the partition function τ_V as

$$\tau_V := \frac{\mathcal{F}_V}{\mathcal{C}_V}. \tag{6.62}$$

This function τ_V has the correct asymptotics for large eigenvalues of Λ. Thus, we have shown that τ_V is a tau function for the KP hierarchy. In fact, there is a slightly stronger statement. The functions

$$\phi_a(x_0) \sim x_0^{a-1}\left(1 + O\left(\frac{1}{x_0}\right)\right), \tag{6.63}$$

label points $w = \text{span } \{\phi_a : a \in \mathbb{Z}\}$ in the Sato Grassmannain $\text{Gr}_{\mathcal{H}_+}(\mathcal{H})$. For this reason they are often called basis vectors. We recall from chapter two that there is a group action on this Grassmannian with the action depending on an infinite set of flow parameters t_n. We recall further that there is a bijection between points in the orbit of this action and solutions of the KP hierarchy evaluated at times t_n. We also saw that the reduction to the r-KdV hierarchy was achieved by imposing the additional constraint that $x_0^r w \subset w$ for all $w \in \text{Gr}_{\mathcal{H}_+}(\mathcal{H})$. Written explicitly, this condition means that there exists $Q_{ab} \in \mathbb{C}$ independent of x_0 such that

$$x_0^p \phi_a(x_0) = \phi_{a+p}(x_0) + \sum_{b=1}^{a+p-1} Q_{ab}\phi_b(x_0). \tag{6.64}$$

With the potential $V(x) = -x^{p+1}/(p+1)$, the functions

$$\Phi_a(\lambda_b) = \int dx\, x^{a-1} e^{\lambda_b x - \frac{x^{r+1}}{(r+1)}}, \tag{6.65}$$

in fact satisfy this recursion. Indeed, one can consider the invariance of this integral with respect to the change of variables $x \mapsto x+\epsilon$ in a similar way to the Ward identities of chapter three. Thus we find

$$\int dx\, x^{a-1} e^{\lambda_b x - \frac{x^{r+1}}{(r+1)}} = \int dx \left(x^{a-1} + (a-1)x^{a-2}\epsilon\right)\left(1 + \lambda_b \epsilon - x^p \epsilon\right) e^{\lambda_b x - \frac{x^{r+1}}{(r+1)}} + O(\epsilon^2). \tag{6.66}$$

Cancelling the zero order terms, to leading order in ϵ we obtain

$$\lambda_b \Phi_a(\lambda_b) = \Phi_{a+p}(\lambda_b) - (a-1)\Phi_{a-1}(\lambda_b). \tag{6.67}$$

Using the saddle point condition $V'(x_0) + \lambda_b = 0$ we see we have precisely the reduction

condition (6.64). Hence, the partition function τ_V defined above for the monomial potential $V(x) = -\frac{x^{r+1}}{r+1}$ is not only a tau function of the KP hierarchy, but is also a tau function of the r-KdV hierarchy.

6.1.3 Antimonomial Potential

As we have remarked before, there is an added subtlety in the case of a potential $V(X) = -\frac{X^{r+1}}{r+1}$ for $r \le -2$. Considering the saddle point equation

$$V'(X_0) + \Lambda = 0, \tag{6.68}$$

we find that $X_0 = \Lambda^{-1/r}$. To relate this to tau functions of the KP hierarchy, we take the limit where all eigenvalues of Λ are large as previously. If one were to use the steepest descent method by performing a Taylor expansion of $V(X + X_0)$, one soon finds a series in increasing powers of X_0^{-1}. However, in the limit of large Λ, X_0^{-1} is also large. Consequently, in the steepest descent method, truncating the Taylor expansion at quadratic order leads to exponentially large error terms.

There are two ways to resolve this issue. The first is to make a change of variables $X \mapsto Y = 1/X$ in the external field model $\mathcal{F}_V$. The measure transforms as

$$dX = d\left(\frac{1}{Y}\right) = \frac{dY}{(\det Y)^{2N}}, \tag{6.69}$$

Hence, we have the following alternative representation for the antimonomial external field matrix integral,

$$\mathcal{F}_V = \int_{H_N} dY\, e^{\mathrm{Tr}\left(V(\frac{1}{Y}) + \frac{\Lambda}{Y} - N \log Y\right)}, \tag{6.70}$$

The method of steepest descent can now be applied as normal.

The second way to correctly calculate the asymptotic expansion is, rather than expanding around the saddle point, we can instead simply change variables $X \mapsto X + X_0^{-1}$. This is the approach we adopt presently. Consider again the functions $\Phi_{a-k}(\lambda_b)$ in the previous

subsection, suitably normalised with $\mathcal{C}_V$. The saddle point x_0 satisfies $\lambda_b = x_0^{-1/r}$. Expanding around $l := x_0^{-1}$ we find

$$\Phi_{a-k}(l) = l^{1+1/r} e^{-\frac{r}{r+1} l^{-r-1}} \int dx \, x^{a-k-1} e^{-\frac{x^{r+1}}{r+1} + \lambda_b x} \sim l^{k-a}\left(1 + O\Big(\frac{1}{l}\Big)\right). \tag{6.71}$$

This is not the correct asymptotic behaviour. However, we can relabel the basis vectors Φ_{a-k} so that the determinant representation of $\mathcal{F}_V$ is unchanged. We relabel basis vectors according to $a \mapsto N - a + 1$ so that

$$\tilde{\Phi}_a(l) = \Phi_{N-a-k+1}(l) \sim l^{a-1+k-N}\left(1 + O\Big(\frac{1}{l}\Big)\right). \tag{6.72}$$

To obtain correct asymptotics and therefore to obtain a tau function for the antimonomial model, we choose $k = N$. Thus, we have shown the external field matrix model does indeed give rise to a KP tau function. Similarly to the monomial model, for a potential of the form $V(x) = -\frac{x^{r+1}}{r+1}$ with $r \leq -2$, the corresponding tau function is a tau function for the $|r|$-KdV hierarchy. Indeed, when deriving the recursion relation (6.67), we used that V is monomial only when calculating the saddle point x_0. In other words, we retain the same recursion relation

$$\lambda_b \Phi_a(\lambda_b) = \Phi_{a+r}(\lambda_b) - (a - 1)\Phi_{a-1}(\lambda_b). \tag{6.73}$$

The saddle point is given by $\lambda_b = x_0^{-|r|}$. We rewrite this in the variable $l = x_0^{-1}$ as $\lambda_b = l^{|r|}$. Hence, we obtain the recursion relation in the l variable,

$$l^{|r|} \Phi_a(l) = \Phi_{a-|r|}(l) - (a - 1)\Phi_{a-1}(l). \tag{6.74}$$

Relabelling $a \mapsto -a$ and setting $\phi_a = \Phi_{-a}$ we find

$$l^{|r|} \phi_a(l) = \phi_{a+|r|}(l) + (a + 1)\phi_{a+1}(l), \tag{6.75}$$

which is again an appropriate reduction condition. Thus, we arrive at the following theorem.

Theorem 6.1.6. Let $\mathcal{F}_V(\Lambda)$ be the external field matrix model. Let

$$\tau_V := \frac{\mathcal{F}_V}{\mathcal{C}_V}, \tag{6.76}$$

where

$$\mathcal{C}_V = \frac{e^{\mathrm{Tr}\left(-V'(X_0)X_0 + V(X_0)\right)}}{\det^k(X_0)\det^{1/2}\left(H_V(X_0)\right)}. \tag{6.77}$$

is the leading order behaviour of $\mathcal{F}_V$ when the eigenvalues of Λ are large. For $V(X) = -\frac{X^{r+1}}{r+1}$ with $|r| \geq 2$, and setting $k = N$ if $r < 0$, the formal series τ_V is a tau function for the $|r|$-KdV hierarchy.

Remark. The proof we have presented that τ_V is a tau function does not rely on the specific form of the potential V. Thus, a generalised Kontsevich model with arbitrary potential is also a τ function of the KP hierarchy. It is only necessary to restrict to external field matrix models to obtain tau functions for the r-KdV hierarchies.

6.2 The 2-Spin Partition Functions and Generalisations

Having obtained a family of tau functions depending on a potential $V(X) = -X^{r+1}/(r+1)$, we now temporarily focus on the simplest cases of $|r| = 2$. These are often called the 2-spin partition functions. In chapter four, we constructed two Airy structures at quadratic order. One Airy structure was of type (2,3) and we called the unique corresponding partition the 2-KW partition function. The other Airy structure was of type (2,1) and we called the unique corresponding partition function the 2-BGW partition function. Here, we will discuss these partition functions from a different point of view using the external field matrix models. In section 6.4 we will show that these two different perspectives, as Airy structures and external field matrix models, are in fact equivalent. In exploring these two different points of view, we will be able to readily suggest generalisations to the higher r-spin partition functions in terms of the external field matrix models.

6.2.1 Monomial Potential - 2-KW and Enumerative Geometry

We follow the discussion outlined in [2] and the exposition of [41]. We first consider, not an external field matrix model *a priori*, but rather a Gaussian Hermitian matrix model with measure

$$d\mu_0(X) = e^{-N\operatorname{Tr}\Lambda X^2} dX. \tag{6.78}$$

where Λ is assumed to be diagonal. In other words, turning off Einstein summation convention, we may write

$$\Lambda_{ij} = \left(\frac{\lambda_i + \lambda_j}{2}\right)\delta_{ij}. \tag{6.79}$$

We have included an extra factor of N in the exponential. One may simply set $N = 1$ without causing issues but we explicitly include it here to label terms in the generating functions that we will construct in a moment. If we rewrite the trace as

$$\operatorname{Tr}\Lambda X^2 = \sum_{i,j}\frac{(\lambda_i + \lambda_j)}{2}X_{ij}X_{ji}, \tag{6.80}$$

then recalling the discussion of Wick's theorem in chapter three, we find the propagator as

$$\langle M_{ij}M_{kl}\rangle_0 = \frac{1}{\lambda_i + \lambda_k}\frac{1}{N}\delta_{il}\delta_{jk}. \tag{6.81}$$

We can also calculate that

$$\int_{H_N} d\mu_0(X) = \left(\frac{\pi}{N}\right)^{N^2/2}\prod_{i,j}(\lambda_i + \lambda_j)^{-1/2}. \tag{6.82}$$

Now, consider the normalised cubic matrix model

$$Z_{\text{2KW}} = \frac{1}{\int_{H_N} d\mu_0(X)}\int_{H_N} d\mu_0(X)e^{-N\operatorname{Tr} X^3/3} = \frac{1}{\int_{H_N} d\mu_0(X)}\int_{H_N} dX\, e^{N\operatorname{Tr}\left(-X^3/3 - \Lambda X^2\right)}. \tag{6.83}$$

Suppose we now change variables in the numerator so that $X \mapsto X - \Lambda$. Thus we find

$$Z_{\text{2KW}} = \frac{e^{\frac{2}{3}\text{Tr }\Lambda^3}}{\int_{H_N} d\mu_0(X)} \int_{H_N} dX e^{\text{Tr }\left(-X^3/3 + \Lambda^2 X\right)}. \tag{6.84}$$

We now rescale the external field, $\tilde{\Lambda}^{1/2} = \Lambda$. The remaining integral is simply the Kontsevich integral $\mathcal{F}_V(\tilde{\Lambda})$ with cubic potential ($r = 2$) and no logarithmic term ($k = 0$). Furthermore, the multiplicative factor in Z_{2KW}, dependent only on $\tilde{\Lambda}$, is in fact nothing other than the quasiclassical term $\mathcal{C}_V^{-1}$. Thus, by the previous section, we identify Z_{2KW} as a tau function for the KdV hierarchy with time variables given by the Miwa parameterisation

$$t_k = \frac{1}{k}\text{Tr }\Lambda^{-k} = \frac{1}{k}\text{Tr }\tilde{\Lambda}^{-k/2}. \tag{6.85}$$

Thus, we consider the cubic matrix model

$$Z_{\text{2KW}} = \frac{1}{\int_{H_N} d\mu_0(X)} \int_{H_N} d\mu_0(X) e^{-N\text{Tr }X^3/3} = \frac{1}{\int_{H_N} d\mu_0(X)} \int_{H_N} dX e^{\text{Tr }\left(-X^3/3 - \Lambda X^2\right)}. \tag{6.86}$$

We recall from chapter three that Z_{2KW} enumerates trivalent fatgraphs. Thus, using the form of the propagator in equation (6.81) we have

$$Z_{\text{2KW}} = \sum_n \frac{1}{n!} \sum_{\substack{\text{Fatgraphs }\Gamma \\ n \text{ faces}}} \sum_{a_1,\dots,a_n} \frac{1}{|\text{Aut }\Gamma|} N^{v-e} \prod_{\text{Edges }(i,j)} \frac{1}{(\lambda_{a_i} + \lambda_{a_j})}. \tag{6.87}$$

where v and e are the number of vertices and edges of a graph Γ respectively. Here the faces are labelled giving rise to the $1/n!$ automorphism factor and a_i are indices on the ith face. We also recall that the free energy, $F = \log Z_{\text{2KW}}$ only generates connected graphs. Furthermore, for a connected graph with associated genus g and n faces, we have the equality between Euler characteristics

$$v - e + n = 2 - 2g. \tag{6.88}$$

Hence, we may write

$$\log Z_{2\mathrm{KW}} = \sum_{g=0}^{\infty} N^{2-2g} F_g, \tag{6.89}$$

where

$$F_g = \sum_n \frac{1}{n!} \sum_{\substack{\text{Connected Fatgraphs } \Gamma \\ n \text{ faces, genus } g}} \sum_{a_1,\ldots,a_n} \frac{1}{|\mathrm{Aut}\ \Gamma|} N^{-n} \prod_{\text{Edges } (i,j)} \frac{1}{(\lambda_{a_i} + \lambda_{a_j})}. \tag{6.90}$$

The equality (6.89) holds in the sense of an asymptotic series at large Λ. From here, we use a result of Kontsevich [2]. This is the non-trivial part of the proof of Witten's conjecture.

Theorem 6.2.1. Let $d_{g,n} = 3g - 3 + n$ and $\chi_{g,n} = 2 - 2g - n$. Let the intersection numbers $\langle \tau_{d_1} \ldots \tau_{d_n} \rangle$ be defined as in the previous chapter:

$$\langle \tau_{d_1} \cdots \tau_{d_n} \rangle_{g,n} := \begin{cases} \int_{\bar{\mathcal{M}}_{g,n}} \psi_1^{d_1} \wedge \cdots \wedge \psi_n^{d_n}, & \sum_i d_i = d_{g,n} \\ 0, & \text{otherwise} \end{cases}. \tag{6.91}$$

Then the generating function of intersection numbers can be computed as a weighted sum of fat graphs. More precisely,

$$\sum_{d_1+\cdots+d_n=d_{g,n}} \prod_{i=1}^n \frac{(2d_i - 1)!!}{\lambda_i^{2d_i+1}} \langle \tau_{d_1} \cdots \tau_{d_n} \rangle_{g,n} = 2^{-\chi_{g,n}} \sum_{\substack{\text{Connected } \Gamma \\ n \text{ faces, genus } g}} \frac{1}{|\mathrm{Aut}\ \Gamma|} \prod_{\text{Edges } (i,j)} \frac{1}{(\lambda_{a_i} + \lambda_{a_j})}. \tag{6.92}$$

where the ith face in Γ is associated to the variable λ_i.

Thus, we find that

$$F_g = \sum_n \frac{2^{\chi_{g,n}}}{n!} N^{-n} \sum_{d_1+\cdots+d_n=d_{g,n}} \sum_{a_1,\ldots,a_n} \prod_{i=1}^n \frac{(2d_i + 1)!!}{(2d_i + 1)\lambda_{a_i}^{2d_i+1}} \langle \tau_{d_1} \cdots \tau_{d_n} \rangle_{g,n}. \tag{6.93}$$

The sum over the indices a_i can be performed by introducing the Miwa variables

$$t_j = \frac{1}{j} \mathrm{Tr}\ \Lambda^{-j}. \tag{6.94}$$

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Thus, we obtain

$$F_g = \sum_n \frac{2^{\chi_{g,n}}}{n!} N^{-n} \sum_{d_1 + \cdots + d_n = d_{g,n}} \prod_{i=1}^n (2d_i + 1)!! t_{2d_i+1} \langle \tau_{d_1} \cdots \tau_{d_n} \rangle_{g,n}. \tag{6.95}$$

Note that the sum over n starts from $n = 3$ since there are no trivalent fatgraphs with $n = 1$ or $n = 2$ faces. Consequently, the formal generating functions F_g for the intersection numbers coincide with the formal expansion of the external field matrix model at large Λ. We have seen, however, that the external field matrix model with $r = 2$ and $k = 0$ gives rise to a tau function of the 2-KdV hierarchy. We therefore have the following theorem, now commonly known as the Kontsevich-Witten theorem.

Theorem 6.2.2. Witten's conjecture is true. Namely that the intersection numbers on $\bar{\mathcal{M}}_{g,n}$ can be arranged into a generating function that is a tau function for the KdV hierarchy.

In this discussion, we took $r = 2$ and $k = 0$ in the external field matrix integral to obtain a cubic potential. It is reasonable therefore to ask about generalisations of the Kontsevich-Witten theorem for all integer k and r such that $|r| \geq 2$. This is partially answered in the following theorem.

Theorem 6.2.3. Let $r \geq 2$ be an integer and set $k = 0$. Then the following hold.

1. Let $\bar{\mathcal{M}}_{g,n}^{1/r}$ be the moduli space of r-spin structures (see [6]). There exist intersection numbers on $\bar{\mathcal{M}}_{g,n}^{1/r}$ that can be arranged in a generating function $Z_{r\text{KW}}$ such that $Z_{r\text{KW}}$ is a tau function for the r-KdV hierarchy.

2. The function $Z_{r\text{KW}}$ can be represented as a normalised monomial Kontsevich matrix integral with $k = 0$ and potential $V(X) = -\frac{X^{r+1}}{r+1}$. The function $Z_{r\text{KW}}$ is also the double scaled limit of a Hermitian $(r-1)$-multimatrix model.

3. The function $Z_{r\text{KW}}$ can be recovered from the Bouchard-Eynard topological recursion of [63] applied to the r-Airy curve $x = z^r/r$, $y = -z$. Moreover, Z satisfies the $\mathcal{W}$-algebra constraints corresponding to the $(r, s) = (r, r + 1)$ Airy structure. That is to say, $H_k^{(i,r+1)} Z_{r\text{KW}} = 0$ for $1 \leq i \leq r$.

Statement 1) has been proven in [6]; statement 2) is shown in [7]. For statement 3), see [1] and [6].

We mention that another generalisation is to allow for non zero k. In this case for $r = 2$, the corresponding partition partition function computes open intersection numbers and is a tau function for the *open* 2-KdV hierarchy. For $r > 2$, it is expected that $Z_{r\mathrm{KW}}$ again computes open intersection numbers. See [91] and [1] for more details.

6.2.2 Antimonomial Potential - 2-BGW and Unitary Matrix Models

In the case of a monomial potential we have seen that $\mathcal{F}_V$ can be identified with a double scaled limit of Hermitian matrix model. This gives rise to the r-KW tau function for the r-KdV hierarchy. One may ask the same question for the antimonomial potential about whether it is related to other matrix models.

In [72], Brézin and Gross considered a different type of matrix model in the context of lattice gauge theories. This was further developed by Gross and Witten in [92]. This matrix model is an integral over unitary matrices

$$Z_{2\mathrm{BGW}} := \int_{\mathrm{U}(N)} dU\, e^{\mathrm{Tr}\left(UA^\dagger + U^\dagger A\right)}. \tag{6.96}$$

where A is an arbitrary complex matrix where dU is the Haar measure on the unitary group $\mathrm{U}(N)$. The equations of motion read

$$\frac{\partial}{\partial A^\dagger_{ji}} \frac{\partial}{\partial A_{kj}} Z = \delta_{ik} Z. \tag{6.97}$$

This follows since the left hand side produces $UU^\dagger = I$. We now introduce the Hermitian matrix $\Lambda := AA^\dagger$. Using the chain rule, one can prove that

$$\frac{\partial}{\partial A^\dagger_{ji}} \frac{\partial}{\partial A_{kj}} = N \frac{\partial}{\partial \Lambda_{ki}} + \Lambda_{ms} \frac{\partial}{\partial \Lambda_{mi}} \frac{\partial}{\partial \Lambda_{ks}}. \tag{6.98}$$

See Appendix A.1.2 for a derivation.

Hence, in the variables Λ_{ik}, the equations of motion are

$$\left(N\frac{\partial}{\partial\Lambda_{ki}} - \delta_{ik} + \Lambda_{ms}\frac{\partial}{\partial\Lambda_{mi}}\frac{\partial}{\partial\Lambda_{ks}}\right)Z = 0. \tag{6.99}$$

We now turn to the external field matrix model with potential $V(X) = 1/X$. To find the equation of motion in this case, note that the integral of a total derivative is zero, assuming appropriate boundary conditions at infinity. This implies the following equation

$$0 = \int_{H_N} dX\frac{\partial}{\partial X_{ab}}\left(X_{cd}X_{ef}e^{\mathrm{Tr}\left(\frac{1}{X}+\Lambda X - k\log X\right)}\right). \tag{6.100}$$

We have included the extra factor of $X_{cd}X_{ef}$ as this ensures we obtain only positive powers of X in the equation of motion. One can also omit this extra factor and just consider the invariance of Z under a change of variables. One then has to use integration by parts to achieve the same equation of motion. This is the approach adopted in the case of a monomial potential in [93]. To simplify the notation slightly, we let

$$S = \mathrm{Tr}\,\left(\frac{1}{X} + \Lambda X - k\log X\right). \tag{6.101}$$

Now, equation (6.100) reads

$$0 = \int_{H_N} dX\left(\delta_{ac}\delta_{bd}X_{ef} + \delta_{ae}\delta_{bf}X_{cd} - X_{cd}X_{ef}X_{ba}^{-2} + X_{cd}X_{ef}\Lambda_{ba} - kX_{cd}X_{ef}X_{ba}^{-1}\right)e^{S}. \tag{6.102}$$

Here we have used

$$\frac{\partial X_{ii}^{-1}}{\partial X_{ab}} = -X_{ba}^{-2}, \tag{6.103}$$

and also

$$\frac{\partial\mathrm{Tr}\log X}{\partial X_{ab}} = X_{ba}^{-1}. \tag{6.104}$$

See Appendix A.1.1 for details.

Setting $a = c$ and $b = f$ yields

$$0 = \int_{H_N} dX \left((2N - k)X_{ed} - \delta_{ed} + X_{ad}X_{eb}\Lambda_{ba} \right) e^S. \tag{6.105}$$

Rewriting this now in terms of derivatives of Λ yields the matrix valued equation of motion

$$\left((2N - k)\frac{\partial}{\partial \Lambda_{de}} - \delta_{de} + \Lambda_{ba}\frac{\partial}{\partial \Lambda_{be}}\frac{\partial}{\partial \Lambda_{da}} \right) Z = 0. \tag{6.106}$$

Thus, if we choose $k = N$, we see that the equations of motion (6.99) for the unitary matrix model and the equations of motion (6.106) for the external field matrix model with $r = -2$ agree exactly. Thus, up to the quasiclassical term $\mathcal{C}_V$, we identify $\mathcal{F}_{1/X}$, with the BGW matrix model $Z_{2\text{BGW}}$. Let us summarise these results in the following theorem.

Theorem 6.2.4. The following hold.

1. There exist intersection numbers on $\overline{\mathcal{M}}_{g,n}$ that can be arranged in a generating function $Z_{2\text{BGW}}$ such that $Z_{2\text{BGW}}$ is a tau function for the $2-$KdV hierarchy.

2. The function $Z_{2\text{BGW}}$ can be represented as a normalised Kontsevich matrix integral with $k = N$ and potential $V(X) = 1/X$. The function $Z_{2\text{BGW}}$ can also be represented as a unitary matrix model.

3. The function $Z_{2\text{BGW}}$ can be recovered from Eynard-Orantin topological recursion applied to the Bessel curve $x = z^2/2$, $y = -1/z$. Moreover, $Z_{2\text{BGW}}$ satisfies the Virasoro constraints corresponding to the $(r, s) = (2, 1)$ Airy structure.

We have proven the equivalence between $\mathcal{F}_{1/X}$ and $Z_{2\text{BGW}}$ in statement 2. From section 6.1, we can use the external field matrix model representation to identify $Z_{2\text{BGW}}$ as a tau function of the 2-KdV hierarchy. To further identify $Z_{2\text{BGW}}$ as a generating function of some intersection numbers, as the partition function of the (2,1) Airy structure and as the partition function of the Bessel curve, see [71] and [94]. This theorem now motivates us to make the following definition.

Definition 6.2.5. Let $r \leq -2$ and let

$$\mathcal{F}_V = \int_{H_N} dX \, e^{\text{Tr} \, (-X^{r+1}/(r+1) + \Lambda X - N \log X)}, \tag{6.107}$$

be the external field model with antimonomial potential. We define the generalised r-BGW partition function as

$$Z_{r\text{BGW}} := \frac{\mathcal{F}_V}{\mathcal{C}_V}, \tag{6.108}$$

where $\mathcal{C}_V$ is the leading order behaviour of $\mathcal{F}_V$.

We have already shown that $Z_{r\text{BGW}}$ is a tau function of the r-KdV hierarchy. However, it is not immediately clear which Airy structure $Z_{r\text{BGW}}$ should correspond to. In light of $Z_{2\text{BGW}}$ as the partition function of the (2,1) Airy structure, there are two natural candidates for $Z_{r\text{BGW}}$: we expect this corresponds to either the $(r, s) = (r, 1)$ Airy structure, or the $(r, s) = (r, r - 1)$ Airy structure. In the next two sections, we present evidence that $Z_{r\text{BGW}}$ corresponds to the $(r, s) = (r, r - 1)$ Airy structure.

6.3 Spectral Curves from External Field Matrix Models

In Theorem 6.2.3, we have identified the r-Airy curve, $x = z^r/r$, $y = -z$ as the spectral curve which reproduces the r-KW tau function under the application of topological recursion. We have also seen the spectral curve corresponding to the BGW tau function is the $2-$Bessel curve $x = z^2/2$, $y = -1/z$. The works of [71] and [1] have proven both of these statements using topological recursion and without explicit reference to the external field matrix integral. Here, we aim to give an outline of a direct approach to finding the spectral curve from the monomial Kontsevich matrix model. This will lead us to a conjectural relationship between the external field matrix model with antimonomial potential and the r-Bessel curve $x = z^r/r$, $y = -1/z$.

Consider first the case of $\mathcal{F}_V$ with $r = 2$ and $k = 0$. The leading order behaviour of the

curve corresponding to this matrix model is the 2-Airy curve $x = z^2/2$, $y = -z$. Indeed, we summarise the results of [85]. We consider the matrix model

$$\mathcal{F}_V = \int_{H_N} dX e^{\frac{N}{t} \mathrm{Tr}\, V(X) + \Lambda X}, \tag{6.109}$$

with $V(X) = -X^3/3$. As before in chapter three, we introduce the resolvent

$$W_1(x_1) := \left\langle \mathrm{Tr}\, \frac{1}{x_1 - X} \right\rangle = \sum_{g=0}^{\infty} \left(\frac{N}{t}\right)^{2-2g} W_1^{(g)}, \tag{6.110}$$

and the quantity

$$P_1(x_1, x_2) = \left\langle \mathrm{Tr}\, \frac{V'(x_1) - V'(X)}{x_1 - X} \mathrm{Tr}\, \frac{1}{x_2 - \Lambda} \right\rangle = \sum_{g=0}^{\infty} \left(\frac{N}{t}\right)^{2-2g} P_1^{(g)}. \tag{6.111}$$

These equalities hold as formal power series in t, as in chapter three. To derive loop equations, one can use the invariance of the matrix integral under a change of variable

$$X \mapsto X + \frac{\epsilon}{x_1 - X} \frac{1}{x_2 - \Lambda}. \tag{6.112}$$

The loop equations reduce to the spectral curve,

$$x_2(x_1) = V'(x_1) - W_1^{(0)}(x_1), \tag{6.113}$$

with

$$W_1^{(0)} = P_1^{(0)}\big(x_1, x_2(x_1)\big). \tag{6.114}$$

For $r = 2$, the quantity $W_1^{(0)}$ is found to be

$$W_1^{(0)}(x) \sim \frac{t}{N} \mathrm{Tr}\, \frac{1}{x - \Lambda}. \tag{6.115}$$

Thus, in the limit of large Λ, this term is subdominant and the leading order behaviour

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of the spectral curve is given by

$$x_2 = -x_1^2. \tag{6.116}$$

One can rescale the coordinates via a symplectic transformation $x_2 = -2^{1/3}x$ and $x_1 = -2^{-1/3}y$ so that the symplectic form $dx_1 \wedge dx_2 = dy \wedge dx$ is unchanged. In these new coordinates, equation (6.116) becomes

$$x = \frac{1}{2}y^2, \tag{6.117}$$

which is indeed the Airy curve for $r = 2$.

We now hypothesise about the case for general $r \geq 2$ by making the following observation. One can use the same change of variables as in equation (6.112) to achieve the same form for the spectral curve as above

$$x_2 = V'(x_1) - W_1^{(0)}(x_1). \tag{6.118}$$

Without explicitly calculating $W_1^{(0)}(x_1)$, from the above discussion it is perhaps reasonable to suppose that this term is subleading. With this assumption, the spectral curve becomes

$$x_2 = V'(x_1) = -x_1^r. \tag{6.119}$$

To make contact with the r-Airy curve, we can rescale the variables $x_2 = ax$ and $x_1 = \frac{1}{a}y$ with $a = -r^{1/(r+1)}$ so that 6.119 becomes

$$x = \frac{(-1)^r}{r}y^r. \tag{6.120}$$

which is indeed the r-Airy curve. Note that this rescaling is indeed valid since it leaves the symplectic form unchanged. This reaffirms, from the point of view of matrix models, that the r-KW tau function can be recovered from the r-Airy curve and that the corresponding Airy structure is $(r, s) = (r, r + 1)$.

We now turn to the antimonomial external field matrix model in the case $r = -2$. In this situation, we consider the external field matrix model

$$\mathcal{F}_{1/X} = \int_{H_N} dX e^{\text{Tr } 1/X + \Lambda X - N \log X}, \tag{6.121}$$

which we associate with the BGW tau function. As noted in [95], the same Ward identities hold for any potential V with rational derivative. In order to use the results of the discussion above, therefore, we include the logarithmic term in the potential. That is to say, differing from our previous conventions, we set $V(X) = 1/X - N \log X$. One can use the same change of variables as in equation (6.112) to achieve the same form for the spectral curve as above

$$x_2 = V'(x_1) - W_1^{(0)}(x_1), \tag{6.122}$$

with

$$W_1^{(0)} = P_1^{(0)}\big(x_1, x_2(x_1)\big), \tag{6.123}$$

as before. The calculation of $W_1^{(0)}(x_1)$ is now only slightly more complicated and can still be computed. For example, the case of a matrix model with external field and a logarithmic term in the potential was considered in [96], albeit in a slightly different context. In this case, the contribution of the logarithmic term to $W_1^{(0)}(x_1) = P_1^{(0)}(x_1, x_2)$ is proportional to $\sum_{i=1}^{N} \frac{1}{x_2 - \lambda_i}$. This term is again subleading and so we are again left with the spectral curve

$$x_2 = V'(x_1) = -\frac{1}{x_1^2} - \frac{N}{x_1}. \tag{6.124}$$

To eliminate the N dependence, we can make a symplectic transformation $x_1 = -2y$ and $x_2 = -x/2 + N/(2y)$. This gives the spectral curve

$$xy^2 = \frac{1}{2}, \tag{6.125}$$

which is indeed the 2-Bessel curve. Notice that the logarithmic term in the matrix model

in fact plays no role in the Bessel curve. This reaffirms, from the point of view of matrix models, that the BGW tau function can be recovered from the Bessel curve and that the corresponding Airy structure is of the type $(r, s) = (2, 1)$.

One can perhaps also view this result in a slightly different way. In the previous discussion, we included the logarithm in the potential and after applying symplectic transformations, we found that the logarithm does not contribute to the leading order behaviour. However, we can also obtain the spectral curve whereby we treat the logarithmic term separately to the potential at the outset. Indeed, as noted in [9], this is self consistent provided that

$$\frac{1}{x_0} < \lambda, \tag{6.126}$$

for large λ. Here, as usual, λ is an eigenvalue of Λ and x_0 is a saddle point given by

$$-\frac{1}{x_0^2} + \lambda = 0. \tag{6.127}$$

Condition (6.126) can be readily found by performing the Taylor expansions of the logarithm and potential. This means that, to leading order, the logarithm only contributes a multiplicative factor of $\det(X_0)^{-N}$. This factor is unaffected by the change of variables used to derive the Ward identity. Furthermore, the higher order terms arising from the logarithm are subleading in the remaining Gaussian integral. Thus, as far as the spectral curve is concerned, one may treat the logarithm separately from the potential. In doing this, one finds the spectral curve

$$x_2 = V'(x_1) = -\frac{1}{x_1^2}. \tag{6.128}$$

Again, after suitable symplectic transformations, we arrive at the Bessel curve

$$xy^2 = \frac{1}{2}. \tag{6.129}$$

Consequently, this readily generalises to the more general case with $r \leq -2$ in which in the

spectral curve is given by

$$x_2 = V'(x_1) = -\frac{1}{x_1^{|r|}}. \tag{6.130}$$

After symplectic transformations, we find the $|r|-$Bessel curve

$$xy^{|r|} = \frac{(-1)^r}{|r|}. \tag{6.131}$$

This then suggests that the r-BGW tau function corresponds to the partition function of the $(r,s) = (r, r-1)$ Airy structure. Indeed, we will explicitly calculate part of the constraint algebra for the $3-$BGW partition function in the following section.

6.4 $\mathcal{W}$ Algebras from External Field Matrix Models

In contrast to the previous sections, we start with the 2-BGW partition function and explicitly calculate the Virasoro constraints from the antimonomial external field matrix model representation. We show this calculation in some detail. We do indeed find that the resulting constraint algebra matches with the $(2,1)$ Airy structure. While this result is already well-known, the calculations have not, as far as the author is aware, been shown explicitly. It is perhaps useful to show explicit calculations in this context since there are many dangers if one is unaware of the delicate subtleties of matrix calculus. Moreover, we compare the results of this method using the external field matrix integral, to the results that are known for the $(2,3)$ and $(3,4)$-Airy structures for the KW tau functions in order to further demonstrate the effectiveness of this approach.

6.4.1   2-BGW

Recall from section 5.2 that we have already found an equation of motion for the BGW τ partition function

$$\left(N\frac{\partial}{\partial\Lambda_{de}} - \delta_{de} + \Lambda_{ba}\frac{\partial}{\partial\Lambda_{be}}\frac{\partial}{\partial\Lambda_{da}} \right)\mathcal{F} = 0. \tag{6.132}$$

However, this contains explicit dependence on N, the size of the matrix. Consequently, we wish to write the Ward identity in a more covariant way. In other words, it is fruitful to eliminate the dependence on N, as these continuum Virasoro constraints should not depend on such a choice. To this end, notice that we can write

$$0 = \int_{H_N} dX \frac{\partial}{\partial X_{ab}} e^{\mathrm{Tr}\left(\frac{1}{X} + \Lambda X - N \log(X)\right)},\tag{6.133}$$

as the integro-differential equation

$$\left[\Lambda_{ba} - N\left(\frac{\partial}{\partial \Lambda}\right)_{ab}^{-1} - \left(\frac{\partial}{\partial \Lambda}\right)_{ab}^{-2}\right] \cdot \mathcal{F} = 0.\tag{6.134}$$

Here, much like the notation of pseudo-differential operators introduced in chapter two, the term $\left(\frac{\partial}{\partial \Lambda}\right)_{ab}^{-1}$ is to be interpreted as integration with respect to the variable Λ_{ab}. We now differentiate this equation to obtain a second order differential equation.

Contracting the above with $\frac{\partial}{\partial \Lambda_{ia}}$ yields

$$\left[\Lambda_{ba} \frac{\partial}{\partial \Lambda_{ia}} - \left(\frac{\partial}{\partial \Lambda}\right)_{ib}^{-1}\right] \cdot \mathcal{F} = 0.\tag{6.135}$$

Contracting again with $\frac{\partial}{\partial \Lambda_{ji}}$ yields

$$\left[\frac{\partial}{\partial \Lambda_{ji}}\left(\Lambda_{ba} \frac{\partial}{\partial \Lambda_{ia}}\right) - \delta_{jb}\right] \cdot \mathcal{F} = 0.\tag{6.136}$$

upon which calculating the derivative implies

$$\left[\Lambda_{ba} \frac{\partial}{\partial \Lambda_{ji}} \frac{\partial}{\partial \Lambda_{ia}} + \delta_{bj} \frac{\partial}{\partial \Lambda_{ii}} - \delta_{bj}\right] \cdot \mathcal{F} = 0,\tag{6.137}$$

or more simply

$$\left[\Lambda_{ba}\left(\frac{\partial^2}{\partial \Lambda^2}\right)_{ja} + \delta_{bj} \frac{\partial}{\partial \Lambda_{ii}} - \delta_{bj}\right] \cdot \mathcal{F} = 0.\tag{6.138}$$

It should be noted here that to make contact with equation (6.132), one can take equation (6.135) and contract it with $\frac{\partial}{\partial \Lambda_{bj}}$ instead. If one were to use (6.132), however, one finds that

it is more difficult to match the end result with the Airy structures given in [1].

This is a system of N^2 equations for the N^2 matrix entries of Λ. To calculate the full Virasoro constraints, one cannot assume Λ is in general diagonal. However we can reduce this to a system of exactly N equations in N variables by using the fact that Z is actually a function of the eigenvalues $\lambda_1, \ldots \lambda_N$ of Λ.

To reduce equation (6.138) to a differential equation of the eigenvalues of Λ, we follow the standard procedure outlined in [93]. Let Λ have an orthonormal eigenbasis $\{|\phi_\mu\rangle\}$ so that

$$\Lambda |\phi_\mu\rangle = \lambda_\mu |\phi_\mu\rangle, \tag{6.139}$$

where $\mu = 1, \ldots N$.

Let $\{|a\rangle\}$ be an arbitrary, fixed orthonormal basis. Then since Λ is Hermitian, $\langle\phi_\mu|a\rangle = U_{\mu a}$ where U is a unitary matrix. The formulae for the first and second derivatives are then

$$\frac{\partial}{\partial \Lambda_{ab}} = \sum_{\mu=1}^{N} \langle b|\phi_\mu\rangle \langle\phi_\mu|a\rangle \frac{\partial}{\partial \lambda_\mu}, \tag{6.140}$$

and

$$\left(\frac{\partial^2}{\partial \Lambda^2}\right)_{ab} = \sum_{\mu=1}^{N} \langle b|\phi_\mu\rangle \left[\frac{\partial^2}{\partial \lambda_\mu^2} + \sum_{\nu \neq \mu} \frac{1}{\lambda_\mu - \lambda_\nu}\left(\frac{\partial}{\partial \lambda_\mu} - \frac{\partial}{\partial \lambda_\nu}\right)\right] \langle\phi_\mu|a\rangle. \tag{6.141}$$

We recognise the terms in the second derivatives as arising from second order time independent perturbation theory. Here, for simplicity, we use Einstein summation convention for the Roman alphabet, but not for the Greek alphabet.

We take each term in equation (6.138) in turn. The aim now is to accurately remove the inner products appearing in equations (6.141) and (6.140) which involve a choice of basis. In this way, we obtain equations of motion depending only on the eigenvalues of Λ.

After diagonalising Λ, the first term reads,

$$\Lambda_{ba}\left(\frac{\partial^2 \mathcal{F}}{\partial \Lambda^2}\right)_{ja} = \sum_{\mu,\alpha=1}^{N} \langle b|\phi_\alpha\rangle \, \delta_{\alpha c} \lambda_\alpha \, \langle \phi_c|a\rangle \, \langle a|\phi_\mu\rangle \left[\frac{\partial^2 Z}{\partial \lambda_\mu^2} + \sum_{\nu \neq \mu} \frac{1}{\lambda_\mu - \lambda_\nu}\left(\frac{\partial \mathcal{F}}{\partial \lambda_\mu} - \frac{\partial \mathcal{F}}{\partial \lambda_\nu}\right)\right] \langle \phi_\mu|j\rangle$$

(6.142)

$$= \sum_{\mu,\alpha=1}^{N} \langle b|\phi_\alpha\rangle \, \lambda_\alpha \, \langle \phi_\alpha|a\rangle \, \langle a|\phi_\mu\rangle \left[\frac{\partial^2 \mathcal{F}}{\partial \lambda_\mu^2} + \sum_{\nu \neq \mu} \frac{1}{\lambda_\mu - \lambda_\nu}\left(\frac{\partial \mathcal{F}}{\partial \lambda_\mu} - \frac{\partial \mathcal{F}}{\partial \lambda_\nu}\right)\right] \langle \phi_\mu|j\rangle$$

(6.143)

$$= \sum_{\mu,\alpha=1}^{N} \langle b|\phi_\alpha\rangle \, \lambda_\alpha \delta_{\alpha\mu} \left[\frac{\partial^2 \mathcal{F}}{\partial \lambda_\mu^2} + \sum_{\nu \neq \mu} \frac{1}{\lambda_\mu - \lambda_\nu}\left(\frac{\partial \mathcal{F}}{\partial \lambda_\mu} - \frac{\partial \mathcal{F}}{\partial \lambda_\nu}\right)\right] \langle \phi_\mu|j\rangle$$

(6.144)

$$= \sum_{\mu=1}^{N} \langle b|\phi_\mu\rangle \left[\lambda_\mu \frac{\partial^2 \mathcal{F}}{\partial \lambda_\mu^2} + \sum_{\nu \neq \mu} \frac{\lambda_\mu}{\lambda_\mu - \lambda_\nu}\left(\frac{\partial \mathcal{F}}{\partial \lambda_\mu} - \frac{\partial \mathcal{F}}{\partial \lambda_\nu}\right)\right] \langle \phi_\mu|j\rangle \,.$$

(6.145)

The second term in (6.138) reads,

$$\delta_{bj}\frac{\partial \mathcal{F}}{\partial \Lambda_{ii}} = \sum_{\mu=1}^{N} \langle b|\phi_\mu\rangle \, \langle \phi_\mu|j\rangle \sum_{\nu=1}^{N} \langle i|\phi_\nu\rangle \, \langle \phi_\nu|i\rangle \, \frac{\partial \mathcal{F}}{\partial \lambda_\nu}$$

(6.146)

$$= \sum_{\mu=1}^{N} \langle b|\phi_\mu\rangle \, \langle \phi_\mu|j\rangle \sum_{\nu=1}^{N} \delta_{\nu\nu} \frac{\partial \mathcal{F}}{\partial \lambda_\nu}$$

(6.147)

$$= \sum_{\mu=1}^{N} \langle b|\phi_\mu\rangle \left[\sum_{\nu=1}^{N} \frac{\partial \mathcal{F}}{\partial \lambda_\nu}\right] \langle \phi_\mu|j\rangle \,.$$

(6.148)

The third term in (6.138) is simply

$$-\delta_{bj} = \sum_{\mu=1}^{N} \langle b|\phi_\mu\rangle \, [-1] \, \langle \phi_\mu|j\rangle \,.$$

(6.149)

Equation (6.138) now reads

$$\sum_{\mu=1}^{N} \langle b|\phi_\mu\rangle \, f(\lambda_\mu) \, \langle \phi_\mu|j\rangle = 0,$$

(6.150)

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where $f(\lambda_\mu)$ are the terms in the square brackets of (6.145), (6.148) and (6.149),

$$f(\lambda_\mu) = \lambda_\mu \frac{\partial^2 \mathcal{F}}{\partial \lambda_\mu^2} + \sum_{\nu \neq \mu} \frac{\lambda_\mu}{\lambda_\mu - \lambda_\nu} \left(\frac{\partial \mathcal{F}}{\partial \lambda_\mu} - \frac{\partial \mathcal{F}}{\partial \lambda_\nu} \right) + \sum_{\nu=1}^{N} \frac{\partial \mathcal{F}}{\partial \lambda_\nu} - 1. \tag{6.151}$$

Let us now rewrite equation (6.150) using the linearity of the inner product

$$\left\langle b \left| \sum_{\mu=1}^{N} \langle \phi_\mu | j \rangle \, f(\lambda_\mu) \right| \phi_\mu \right\rangle = 0. \tag{6.152}$$

Since $\langle b |$ was an arbitrary basis and the inner product is non degenerate then we must have

$$\sum_{\mu=1}^{N} \langle \phi_\mu | j \rangle \, f(\lambda_\mu) \, | \phi_\mu \rangle = 0. \tag{6.153}$$

Since the ϕ_μ are linearly independent, we have that

$$\sum_{\mu=1}^{N} \langle \phi_\mu | j \rangle \, f(\lambda_\mu) = 0. \tag{6.154}$$

Similar reasoning now shows that we must have $f(\lambda_\mu) = 0$ for each μ. This finally yields

$$\lambda_\mu \frac{\partial^2 \mathcal{F}}{\partial \lambda_\mu^2} + \sum_{\nu \neq \mu} \frac{\lambda_\mu}{\lambda_\mu - \lambda_\nu} \left(\frac{\partial \mathcal{F}}{\partial \lambda_\mu} - \frac{\partial \mathcal{F}}{\partial \lambda_\nu} \right) + \sum_{\nu=1}^{N} \frac{\partial \mathcal{F}}{\partial \lambda_\nu} - 1 = 0, \tag{6.155}$$

for each μ. This now agrees with the Schwinger-Dyson equations found in [97]. In applying this equation, we usually take $\mu = 1$.

We now turn to calculating the leading order contribution where we recall

$$\mathcal{C}_V = \frac{e^{\mathrm{Tr}\left(-V'(X_0)X_0 + V(X_0) \right)}}{\det^N(X_0) \det^{1/2}\left(H_V(X_0) \right)}. \tag{6.156}$$

To calculate the Hessian, $H_V(X_0)$ of the function $X \mapsto \mathrm{Tr}\, V(X)$, we use a result proven in [98].

Proposition 6.4.1. Let ϕ be a twice differentiable real valued function of an $n \times n$ matrix. Then the following relationship holds

$$d^2\phi(X) = \text{Tr } BdXCdX \iff H_\phi(X) = \frac{1}{2}K_{nn}(B^{\mathrm{T}} \otimes C + C^{\mathrm{T}} \otimes B), \qquad (6.157)$$

where $\otimes$ is the Kronecker product between matrices and K_{nn} is an orthogonal matrix.

Now consider the case $\phi(X) = \text{Tr } V(X) = \text{Tr } X^{-1}$. Calculating the first differential we find

$$d\phi(X) = d\,\text{Tr } X^{-1} = \text{Tr } d(X^{-1}) = -\text{Tr } X^{-1}dX X^{-1}. \qquad (6.158)$$

Here we have used $d(X^{-1}) = -X^{-1}(dX)X^{-1}$. Calculating the second differential we find

$$d^2\phi(X) = -\text{Tr } d(X^{-1})(dX)X^{-1} - \text{Tr } X^{-1}(dX)d(X^{-1}) \qquad (6.159)$$

$$= 2\text{Tr } X^{-1}(dX)X^{-1}(dX)X^{-1} = 2\text{Tr } X^{-2}(dX)X^{-1}(dX). \qquad (6.160)$$

Therefore, the Hessian becomes

$$H_\phi(X) = K_{nn}\left((X^{-2})^{\mathrm{T}} \otimes X^{-1} + (X^{-1})^{\mathrm{T}} \otimes X^{-2}\right). \qquad (6.161)$$

In the case $V(X) = X^{-1}$, we have $X_0 = \Lambda^{-1/2}$. and so

$$\det H_V(X_0) = \pm\det\left(\Lambda^{\mathrm{T}} \otimes \Lambda^{1/2} + (\Lambda^{1/2})^{\mathrm{T}} \otimes \Lambda\right). \qquad (6.162)$$

In general, we cannot assume that Λ is diagonal for this calculation. However, the determinant in equation (6.162) can be written as a function of the trace of powers of Λ. These quantities are invariant under diagonalisation of Λ. That is to say, the determinant of the Hessian depends only on the eigenvalues of Λ. See appendix A.3 for a more detailed discussion on this issue. Consequently to evaluate (6.162), we may assume Λ is diagonal to massively simplify the calculation.

Now, up to a constant factor we have

$$\det H_V(X_0) = \prod_{i,j} (\lambda_i \sqrt{\lambda_j} + \sqrt{\lambda_i} \lambda_j). \tag{6.163}$$

Inserting the above into equation (6.61) we find

$$\mathcal{C} = \frac{e^{2 \sum_i \lambda_i^{1/2}}}{\prod_{i,j} \sqrt{\sqrt{\lambda_i} + \sqrt{\lambda_j}}}. \tag{6.164}$$

The advantage in this method of calculating $\mathcal{C}_V$ is that it provides an easy way of calculating the quadratic factor that generalises well to all monomial potentials $V(X)$.

Now, we have seen in section 5.1 that the BGW tau function is given as

$$\tau = \frac{\mathcal{F}}{\mathcal{C}}. \tag{6.165}$$

Now we substitute this, with the expression (6.164) for $\mathcal{C}$ into equation (6.155) in the case $N = 2$. After dividing by $\lambda_1 \mathcal{C}$, with the help of Python, we find the equation

$$\frac{\partial^2 \tau}{\partial \lambda_1^2} + a_1 \frac{\partial \tau}{\partial \lambda_1} + a_2 \frac{\partial \tau}{\partial \lambda_2} + a_3 \tau = 0, \tag{6.166}$$

where the coefficients have Taylor expansion

$$a_1 = \frac{1}{2\lambda_1} + \frac{2}{\sqrt{\lambda_1}} + \sum_{n=1}^{\infty} \frac{\lambda_2^{2n-1}}{\lambda_1^{2n+1}}, \tag{6.167}$$

$$a_2 = -\sum_{n=1}^{\infty} \frac{\lambda_2^n}{\lambda_1^{n+1}}, \tag{6.168}$$

as $\lambda_1 \to \infty$. The coefficient a_3 is given exactly as

$$a_3 = \frac{1}{16\lambda_1^2}. \tag{6.169}$$

We now employ the Miwa parameterisation as suggested in section 5.2

$$t_k = \frac{1}{k}\left(\frac{1}{\lambda_1^{k/2}} + \frac{1}{\lambda_2^{k/2}}\right) \tag{6.170}$$

for $k \in \mathbb{N}$. Thus, using the chain rule we rewrite the derivatives as

$$\frac{\partial}{\partial\lambda_1} = \sum_{k=1}^{\infty} -\frac{1}{2}\frac{1}{\lambda_1^{k/2+1}}\frac{\partial}{\partial t_k} \tag{6.171}$$

$$\frac{\partial}{\partial\lambda_2} = \sum_{k=1}^{\infty} -\frac{1}{2}\frac{1}{\lambda_2^{k/2+1}}\frac{\partial}{\partial t_k} \tag{6.172}$$

$$\frac{\partial^2}{\partial\lambda_1^2} = \sum_{k=1}^{\infty} \frac{1}{2}\left(\frac{k}{2}+1\right)\frac{1}{\lambda_1^{k/2+2}}\frac{\partial}{\partial t_k} + \sum_{k,l=1}^{\infty} \frac{1}{4}\frac{1}{\lambda_1^{k/2+l/2+2}}\frac{\partial^2}{\partial t_k\partial t_l}. \tag{6.173}$$

At order $O\left(1/\lambda_1^2\right)$ we find

$$\left(-\frac{\partial}{\partial t_1} + \frac{1}{16} + \frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{\lambda_2^{k/2}}\frac{\partial}{\partial t_k}\right)\tau = 0. \tag{6.174}$$

To proceed, we eliminate the λ_2 variable. Since this is the leading order equation, we are free to make the replacement $kt_k \sim \frac{1}{\lambda_2^{k/2}}$ in which case we obtain

$$\left(-\frac{\partial}{\partial t_1} + \frac{1}{16} + \frac{1}{2}\sum_{k=1}^{\infty}kt_k\frac{\partial}{\partial t_k}\right)\tau = 0. \tag{6.175}$$

In addition, we have shown that τ is a tau function of the 2-KdV hierarchy and thus only depends on odd times. Applying this reasoning to the above equation, one sees that this is exactly

$$H_1^{(2,1)}\tau = 0, \tag{6.176}$$

in the notation of chapter four in equation (4.195). To calculate the higher orders, one must take into account the corrections from the leading order replacement $kt_k \sim \frac{1}{\lambda_2^{k/2}}$. More

precisely, one can take equation (6.174) and instead 'add zero ' so that

$$\frac{1}{\lambda_1^2}\left(-\frac{\partial}{\partial t_1} + \frac{1}{16} + \frac{1}{2}\sum_{k=1}^{\infty}\left(\frac{1}{\lambda_1^{k/2}} + \frac{1}{\lambda_2^{k/2}}\right)\frac{\partial}{\partial t_k} - \frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{\lambda_1^{k/2}}\frac{\partial}{\partial t_k}\right)\tau = 0, \tag{6.177}$$

where have written the $O(1/\lambda_1^2)$ explicitly as a multiplicative prefactor. Now, the final term in this equation does not contribute at this order and so we again arrive at $H_1^{(2,1)}\tau = 0$. However, this term does need to be taken into account at higher orders. For example, without this correction term, the Schwinger-Dyson equation at $O(1/\lambda_1^{5/2})$ reads

$$-\frac{\partial\tau}{\partial t_2} + \frac{1}{2}\frac{\partial\tau}{\partial t_1} = 0. \tag{6.178}$$

However, we must also take into consideration the correction term $-\frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{\lambda_1^{k/2+2}}\frac{\partial}{\partial t_k}$ at $O(1/\lambda_1^{5/2})$. This contributes $-\frac{1}{2}\frac{\partial\tau}{\partial t_1}$ which cancels with the term in (6.178) and thus we obtain

$$\frac{\partial\tau}{\partial t_2} = 0 \tag{6.179}$$

as expected. Continuing in this manner, one finds at half integer powers of λ_1^{-1} the constraints $\frac{\partial\tau}{\partial t_{2k}} = 0$. At integer powers of λ_1^{-1}, one does indeed find $H_k^{(2,1)}\tau = 0$.

In more complicated situations, it is much more difficult to calculate the higher order terms in this perturbation expansion. Fortunately, there is an alternative approach to show that $H_k^{(2,1)}\tau = 0$ for all $k \geq 1$. To agree with the conventions given in the literature, we shift indices of the operators $H_k^{(2,1)}$ so that $L_k = H_{k-1}^{(2,1)}$. One can employ the external field matrix integral to show that the lowest order constraints are $L_0\tau = L_1\tau = L_2\tau = 0$. Now, it is known that the L operators must satisfy the Virasoro algebra from chapter four,

$$[L_m, L_n] = (m - n)L_{m+n}. \tag{6.180}$$

As mentioned before, the central charge term in this case is irrelevant since we are considering the subalgebra where the lowest mode is L_0. We proceed by induction. We assume $L_n\tau = 0$

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for some $n \geq 2$. Using the Virasoro commutation relation above we may write

$$L_{n+1}\tau = \frac{1}{n+1}(L_n L_1 - L_1 L_n)\tau = 0. \tag{6.181}$$

Hence, $L_k\tau = 0$ for all $k \geq 0$.

6.4.2 2-KW and 3-KW

In this subsection, we briefly state the results of Schwinger-Dyson equations and the leading order term $\mathcal{C}$ obtained via the previous method in the cases of the 2-KW and 3-KW partition functions. The results obtained in this approach coincide with those found in the literature and so we omit most of the details here. While this may not lift the cloud of mystery surrounding Proposition 6.4.1, it does at least demonstrate the reasonable efficiency of this method.

For the 2-KW partition function, we consider the matrix model

$$\mathcal{F} = \int_{H_N} dX\, e^{\operatorname{Tr}\,(-X^3/3 + \Lambda X)}. \tag{6.182}$$

This satisfies the following Ward identity

$$0 = \int_{H_N} dX\, \frac{\partial}{\partial X_{ab}} e^{\operatorname{Tr}\left(-X^3/3 + \Lambda X\right)} = \left(\Lambda_{ba} - \left(\frac{\partial}{\partial \Lambda_{ab}}\right)^2\right)\mathcal{F}. \tag{6.183}$$

We can reduce this to a differential equation in terms of the eigenvalues λ_i in the same way as before. This yields

$$\left(\lambda_\mu - \frac{\partial^2}{\partial \lambda_\mu^2} - \sum_{\nu \neq \mu} \frac{1}{\lambda_\mu - \lambda_\nu}\left(\frac{\partial}{\partial \lambda_\mu} - \frac{\partial}{\partial \lambda_\nu}\right)\right)\mathcal{F} = 0. \tag{6.184}$$

To calculate $\mathcal{C}$, we first find the saddle point as

$$X_0 = \Lambda^{1/2}. \tag{6.185}$$

Now consider the case $\phi(X) = \operatorname{Tr} V(X) = \operatorname{Tr} (-X^3/3)$. Calculating the first differential we find

$$d\phi(X) = d \operatorname{Tr} \left(-\frac{X^3}{3} \right) = -\operatorname{Tr} (-X^2)dX. \tag{6.186}$$

Calculating the second differential we find

$$d^2\phi(X) = -2\operatorname{Tr} X(dX)I(dX). \tag{6.187}$$

Therefore, by Proposition 6.4.1, the determinant of the Hessian becomes

$$\det (H_\phi)(X) = \left(X^{\mathrm{T}} \otimes I + I^{\mathrm{T}} \otimes X \right), \tag{6.188}$$

where we have omitted constant factors. Evaluated at $X_0 = \Lambda^{1/2}$, we obtain

$$\det H_V(X_0) = \det \left(\Lambda^{1/2} \otimes I + I \otimes \Lambda^{1/2} \right), \tag{6.189}$$

in exact agreement with what is found in [2]. One now proceeds as above by substituting $\mathcal{F} = \mathcal{C}\tau$ into the Schwinger-Dyson equations in order to find the corresponding Virasoro constraints. As before, one needs only calculate the lowest order modes. In fact, the full tower of constraints in this case is equivalent to the Virasoro commutation relation and the *string equation* $L_{-1}\tau = 0$.

For the 3-KW partition function, we only show how to calculate the quasiclassical contribution $\mathcal{C}$; the corresponding Ward identity should now hopefully be clear to write down, albeit with a more complicated yet standard formula for the third order matrix derivative. We shall explicitly show this in the next section when considering the 3-BGW tau function. We consider the matrix model

$$\mathcal{F} = \int_{H_N} dX e^{\operatorname{Tr} (-X^3/3 + \Lambda X)}. \tag{6.190}$$

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To calculate $\mathcal{C}$, we first find the saddle point as

$$X_0 = \Lambda^{1/3}. \qquad (6.191)$$

Now consider the case $\phi(X) = \text{Tr } V(X) = \text{Tr } -X^4/4$. Calculating the first differential we find

$$d\phi(X) = d\,\text{Tr}\left(-\frac{X^4}{4}\right) = -\text{Tr }(-X^3)dX. \qquad (6.192)$$

Calculating the second differential we find

$$d^2\phi(X) = -2\text{Tr } X^2(dX)I(dX) - \text{Tr } X(dX)X(dX). \qquad (6.193)$$

Therefore, by Proposition 6.4.1, the determinant of the Hessian becomes

$$\det\left(H_\phi\right)(X) = \det\left(X^{2T} \otimes I + I^{\text{T}} \otimes X^2 + X^{\text{T}} \otimes X\right), \qquad (6.194)$$

where we have omitted constant factors. Evaluated at $X_0 = \Lambda^{1/2}$, we obtain

$$\det H_V(X_0) = \det\left(\Lambda^{2/3} \otimes I + I \otimes \Lambda^{2/3} + \Lambda^{1/3} \otimes \Lambda^{1/3}\right), \qquad (6.195)$$

in exact agreement with what is found in [99]. One now proceeds as above by substituting $\mathcal{F} = \mathcal{C}\tau$ into the Schwinger-Dyson equations in order to find the corresponding $\mathcal{W}$ constraints.

6.4.3 3-BGW

We use the same method as before to calculate the Schwinger-Dyson equations and the leading order contribution to $\mathcal{F}$. This has never appeared in the literature to date.

The matrix valued Ward Identity is

$$\left[\Lambda_{ba} - N\left(\frac{\partial}{\partial\Lambda}\right)_{ab}^{-1} - \left(\frac{\partial}{\partial\Lambda}\right)_{ab}^{-3}\right] \cdot \mathcal{F} = 0. \qquad (6.196)$$

Contracting the above with $\frac{\partial}{\partial \Lambda_{ia}}$ yields

$$\left[\Lambda_{ba} \frac{\partial}{\partial \Lambda_{ia}} - \left(\frac{\partial}{\partial \Lambda}\right)_{ib}^{-2}\right] \cdot \mathcal{F} = 0. \tag{6.197}$$

Contracting again with $\frac{\partial}{\partial \Lambda_{ji}}$ yields

$$\left[\frac{\partial}{\partial \Lambda_{ji}} \left(\Lambda_{ba} \frac{\partial}{\partial \Lambda_{ia}}\right) - \left(\frac{\partial}{\partial \Lambda}\right)_{jb}^{-1}\right] \cdot \mathcal{F} = 0. \tag{6.198}$$

Contracting again with $\frac{\partial}{\partial \Lambda_{kj}}$ yields

$$\left[\frac{\partial}{\partial \Lambda_{kj}} \frac{\partial}{\partial \Lambda_{ji}} \left(\Lambda_{ba} \frac{\partial}{\partial \Lambda_{ia}}\right) - \delta_{kb}\right] \cdot \mathcal{F} = 0, \tag{6.199}$$

upon which calculating the derivatives implies

$$\left[\Lambda_{ba} \frac{\partial^3}{\partial \Lambda_{ka}^3} + \frac{\partial}{\partial \Lambda_{kb}} \frac{\partial}{\partial \Lambda_{aa}} + \delta_{kb} \frac{\partial^2}{\partial \Lambda_{aa}^2} - \delta_{kb}\right] \cdot \mathcal{F} = 0. \tag{6.200}$$

To reduce the third order matrix derivative to derivatives of the eigenvalues, we use the formula

$$\frac{\partial^3}{\partial \Lambda_{ka}^3} = \sum_{\mu=1}^{N} \langle a|\phi_\mu\rangle \langle \phi_\mu|k\rangle \left[\frac{\partial^3}{\partial \lambda_\mu^3} + \sum_{\nu \neq \mu} \frac{1}{\lambda_\mu - \lambda_\nu} \left(\frac{\partial}{\partial \lambda_\mu} - \frac{\partial}{\partial \lambda_\nu}\right)\left(2\frac{\partial}{\partial \lambda_\mu} + \frac{\partial}{\partial \lambda_\nu}\right) - \tag{6.201}\right.$$

$$\left.\sum_{\nu \neq \mu} \frac{1}{(\lambda_\mu - \lambda_\nu)^2} \left(\frac{\partial}{\partial \lambda_\mu} - \frac{\partial}{\partial \lambda_\nu}\right) + 2 \sum_{\nu \neq \mu} \sum_{\alpha \neq \nu, \mu} \frac{1}{(\lambda_\mu - \lambda_\alpha)(\lambda_\alpha - \lambda_\mu)} \left(\frac{\partial}{\partial \lambda_\mu} - \frac{\partial}{\partial \lambda_\alpha}\right)\right]. \tag{6.202}$$

Now, equation (6.200) can be reduced in the same way as the 2-BGW case. For each

$\mu = 1, \ldots, N$ this yields

$$\lambda_\mu \frac{\partial^3 \mathcal{F}}{\partial \lambda_\mu^3} + \sum_{\nu \neq \mu} \frac{\lambda_\mu}{\lambda_\mu - \lambda_\nu} \Big(\frac{\partial}{\partial \lambda_\mu} - \frac{\partial}{\partial \lambda_\nu} \Big) \Big(2 \frac{\partial \mathcal{F}}{\partial \lambda_\mu} + \frac{\partial \mathcal{F}}{\partial \lambda_\nu} \Big) - \sum_{\nu \neq \mu} \frac{\lambda_\mu}{(\lambda_\mu - \lambda_\nu)^2} \Big(\frac{\partial \mathcal{F}}{\partial \lambda_\mu} - \frac{\partial \mathcal{F}}{\partial \lambda_\nu} \Big)$$

$$(6.203)$$

$$+ 2 \sum_{\nu \neq \mu} \sum_{\alpha \neq \nu, \mu} \frac{\lambda_\mu}{(\lambda_\mu - \lambda_\alpha)(\lambda_\alpha - \lambda_\mu)} \Big(\frac{\partial}{\partial \lambda_\mu} - \frac{\partial}{\partial \lambda_\alpha} \Big) \mathcal{F}$$

$$(6.204)$$

$$+ \frac{\partial}{\partial \lambda_\mu} \sum_{\nu=1}^{N} \frac{\partial \mathcal{F}}{\partial \lambda_\nu} + \sum_{\nu=1}^{N} \Big[\frac{\partial^2 \mathcal{F}}{\partial \lambda_\nu^2} + \sum_{\alpha \neq \nu} \frac{1}{\lambda_\nu - \lambda_\alpha} \Big(\frac{\partial \mathcal{F}}{\partial \lambda_\nu} - \frac{\partial \mathcal{F}}{\partial \lambda_\alpha} \Big) \Big] - \mathcal{F} = 0.$$

$$(6.205)$$

Note that we shall take $N = 2$ and so the second line in the above vanishes. Hence, for $\mu = 1$, the above becomes

$$\lambda_1 \frac{\partial^3 \mathcal{F}}{\partial \lambda_1^3} + \frac{\lambda_1}{\lambda_1 - \lambda_2} \Big(\frac{\partial}{\partial \lambda_1} - \frac{\partial}{\partial \lambda_2} \Big) \Big(2 \frac{\partial \mathcal{F}}{\partial \lambda_1} + \frac{\partial \mathcal{F}}{\partial \lambda_2} \Big) - \frac{\lambda_1}{(\lambda_1 - \lambda_2)^2} \Big(\frac{\partial \mathcal{F}}{\partial \lambda_1} - \frac{\partial \mathcal{F}}{\partial \lambda_2} \Big) \qquad (6.206)$$

$$+ 2 \frac{\partial^2 \mathcal{F}}{\partial \lambda_1^2} + \frac{\partial^2 \mathcal{F}}{\partial \lambda_1 \partial \lambda_2} + \frac{\partial^2 \mathcal{F}}{\partial \lambda_2^2} + \frac{2}{\lambda_1 - \lambda_2} \Big(\frac{\partial \mathcal{F}}{\partial \lambda_1} - \frac{\partial \mathcal{F}}{\partial \lambda_2} \Big) - \mathcal{F} = 0. \qquad (6.207)$$

For the leading order contribution $\mathcal{C}$, we first calculate the saddle point of $\frac{1}{2X^2} + \Lambda X$ as $X_0 = \Lambda^{-1/3}$. We now calculate the Hessian of the transformation ϕ defined by $X \mapsto \text{Tr} \frac{1}{2} X^{-2}$. We have that

$$d\phi = -\text{Tr} X^{-3} dX, \qquad (6.208)$$

and furthermore

$$d^2 \phi(X) = 2\text{Tr} X^{-1} (dX) X^{-3} (dX) + \text{Tr} X^{-2} (dX) X^{-2} (dX). \qquad (6.209)$$

Consequently, the determinant of the Hessian evaluated at $X_0 = \Lambda^{-1/3}$ of ϕ is

$$\det H_V(X_0) = \pm \det \big(\Lambda^{1/3} \otimes \Lambda + \Lambda \otimes \Lambda^{1/3} + \Lambda^{2/3} \otimes \Lambda^{2/3} \big), \qquad (6.210)$$

which in terms of the eigenvalues of Λ becomes

$$\det H_V(X_0) = \pm \prod_{i,j} \lambda_i^{1/3} \lambda_j + \lambda_i \lambda_j^{1/3} + \lambda_i^{2/3} \lambda_j^{2/3}. \tag{6.211}$$

Hence, according to equation (6.61) in this case we have, up to constant factors,

$$\mathcal{C} = \frac{e^{\frac{3}{2} \sum_i \lambda^{2/3}}}{\sqrt{\prod_{i \neq j} \lambda_i^{1/3} \lambda_j + \lambda_i \lambda_j^{1/3} + \lambda_i^{2/3} \lambda_j^{2/3}}} = \frac{e^{\frac{3}{2} \sum_i \lambda^{2/3}}}{\prod_{i<j} \lambda_i^{1/3} \lambda_j + \lambda_i \lambda_j^{1/3} + \lambda_i^{2/3} \lambda_j^{2/3}}. \tag{6.212}$$

We restrict attention to $N = 2^1$. To calculate the $\mathcal{W}$-constraints, we substitute $\mathcal{F} = \mathcal{C}\tau$ into equations (6.206) and (6.207). After dividing by $\lambda_1 \mathcal{C}$, we find the following differential equation with the help of Python,

$$\frac{\partial^3 \tau}{\partial \lambda_1^3} + a_1 \frac{\partial^2 \tau}{\partial \lambda_1^2} + a_2 \frac{\partial^2 \tau}{\partial \lambda_1 \partial \lambda_2} + a_3 \frac{\partial^2 \tau}{\partial \lambda_2^2} + a_4 \frac{\partial \tau}{\partial \lambda_1} + a_5 \frac{\partial \tau}{\partial \lambda_2} + a_6 g = 0, \tag{6.213}$$

where as $\lambda_1 \to \infty$ the coefficients a_i are given by

$$a_1 = \frac{3}{\lambda_1^{1/3}} + O\left(\frac{1}{\lambda_1}\right), \tag{6.214}$$

$$a_2 = -\frac{\lambda_2}{\lambda_1^2} + O\left(\frac{1}{\lambda_1^3}\right), \tag{6.215}$$

$$a_3 = -\frac{\lambda_2}{\lambda_1^2} + O\left(\frac{1}{\lambda_1^3}\right), \tag{6.216}$$

$$a_4 = \frac{3}{\lambda_1^{2/3}} + O\left(\frac{1}{\lambda_1^{4/3}}\right), \tag{6.217}$$

$$a_5 = \frac{-2\lambda_2^{2/3} - \frac{1}{3}}{\lambda_1^2} + O\left(\frac{1}{\lambda_1^{7/3}}\right), \tag{6.218}$$

$$a_6 = -\frac{1}{9\lambda_2 \lambda_1^2} + O\left(\frac{1}{\lambda_1^{7/3}}\right). \tag{6.219}$$

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[1] As with the 2-BGW case and also noted by Brézin and Hikami in [93], we suspect that $N = 2$ is sufficient for calculating the lowest order modes of the $\mathcal{W}$-constraint algebra.

We now employ the Miwa parameterisation

$$t_k = \frac{1}{k}\left(\frac{1}{\lambda_1^{k/3}} + \frac{1}{\lambda_2^{k/3}}\right), \tag{6.220}$$

for $k \in \mathbb{N}$. Thus, using the chain rule we rewrite the derivatives as

$$\frac{\partial}{\partial \lambda_1} = \sum_{k=1}^{\infty} -\frac{1}{3}\frac{1}{\lambda_1^{k/3+1}}\frac{\partial}{\partial t_k}, \tag{6.221}$$

$$\frac{\partial}{\partial \lambda_2} = \sum_{k=1}^{\infty} -\frac{1}{3}\frac{1}{\lambda_2^{k/3+1}}\frac{\partial}{\partial t_k}, \tag{6.222}$$

$$\frac{\partial^2}{\partial \lambda_1^2} = \sum_{k=1}^{\infty} \frac{1}{3}\left(\frac{k}{3}+1\right)\frac{1}{\lambda_1^{k/3+2}}\frac{\partial}{\partial t_k} + \sum_{k,l=1}^{\infty} \frac{1}{9}\frac{1}{\lambda_1^{k/3+l/3+2}}\frac{\partial^2}{\partial t_k \partial t_l}, \tag{6.223}$$

$$\frac{\partial^2}{\partial \lambda_1 \partial \lambda_2} = \sum_{k,l=1}^{\infty} \frac{1}{9}\frac{1}{\lambda_1^{k/3+1}\lambda_2^{l/3+1}}\frac{\partial^2}{\partial t_k \partial t_l}, \tag{6.224}$$

$$\frac{\partial^2}{\partial \lambda_2^2} = \sum_{k=1}^{\infty} \frac{1}{3}\left(\frac{k}{3}+1\right)\frac{1}{\lambda_2^{k/3+2}}\frac{\partial}{\partial t_k} + \sum_{k,l=1}^{\infty} \frac{1}{9}\frac{1}{\lambda_2^{k/3+l/3+2}}\frac{\partial^2}{\partial t_k \partial t_l}. \tag{6.225}$$

At lowest order, which in this case is $O\left(\frac{1}{\lambda_1^2}\right)$, the term $\frac{\partial^3}{\partial \lambda_1^3}$ will not contribute. Indeed, one readily finds that the lowest order terms in third derivative are of order at least $\frac{1}{\lambda_1^3}$ whereas the coefficient of the third derivative is order one. Consequently, we omit this formula. Inserting these into (6.213) and using the equations for the coefficients, at lowest order, $O\left(\frac{1}{\lambda_1^2}\right)$, we find

$$0 = \sum_{k,l=1}^{\infty} \frac{1}{9}\frac{1}{\lambda_2^{k/3+l/3+1}}\frac{\partial^2 \tau}{\partial t_k \partial t_l} + \frac{2}{3}\sum_{k=1}^{\infty} \frac{1}{\lambda_2^{(k+1)/3}}\frac{\partial \tau}{\partial t_k} - \frac{1}{9\lambda_2}\tau - \sum_{k=1}^{\infty} \frac{1}{9}(k+2)\frac{1}{\lambda_2^{k/3+1}}\frac{\partial \tau}{\partial t_k} - \frac{\partial \tau}{\partial t_1} \tag{6.226}$$

Unlike previously, we keep the coefficients in terms of λ_2. In fact at lowest order, there is an ambiguity in using the simple replacement $kt_k \sim 1/\lambda_2^{k/3}$. For example, we shall see in a moment that terms of the form t_1^3 and t_3 appear in the (3,2) Airy structure. From the Miwa parameterisation, however, these terms are indistinguishable. This ambiguity is fixed when

one considers the higher order correctin terms.

This calculation should now be compared to the Airy structure $H_k^{(3,2)}$ where we obtain $H_1^{(3,2)}$ after a dilaton shift in equation (4.180) from chapter four:

$$H_1^{(3,2)} = \sum_{k,l>0}(k+l+3)t_{k+l+3}\hbar^2\frac{\partial^2}{\partial t_k\partial t_l} + \sum_{\substack{k,l>0\\k+l>3}}klt_kt_l\hbar\frac{\partial}{\partial t_{k+l-3}} + \frac{1}{3}t_1^3 \tag{6.227}$$

$$-2\sum_{p=1}^{\infty}(k+1)t_{k+1}\hbar\frac{\partial}{\partial t_k} + \hbar\frac{\partial}{\partial t_1} - \hbar x_3. \tag{6.228}$$

At leading order we have $kt_k \sim \frac{1}{\lambda_2^{k/3}}$ as $\lambda_1 \to \infty$. Hence in terms of λ_2, the above becomes

$$H_1^{(3,2)} = \sum_{k,l>0}\frac{1}{\lambda_2^{k/3+l/3+1}}\hbar^2\frac{\partial^2}{\partial t_k\partial t_l} + \sum_{\substack{k,l>0\\k+l>3}}\frac{1}{\lambda_2^{k/3+l/3}}\hbar\frac{\partial}{\partial t_{k+l-3}} + \frac{1}{3}\frac{1}{\lambda_2} \tag{6.229}$$

$$-2\sum_{k=1}^{\infty}\frac{1}{\lambda_2^{(k+1)/3}}\hbar\frac{\partial}{\partial t_k} + \hbar\frac{\partial}{\partial t_1} - \frac{\hbar}{3}\frac{1}{\lambda_2}. \tag{6.230}$$

Consider the second sum in equation (6.229). This in fact depends only the combination $k+l$. Thus, we make a change of index by letting $p = k+l-3$. Thus we obtain

$$H_1^{(3,2)} = \sum_{k,l>0}\frac{\hbar^2}{\lambda_2^{k/3+l/3+1}}\frac{\partial^2}{\partial t_k\partial t_l} + \sum_{p=1}^{\infty}C_p\frac{\hbar}{\lambda_2^{p/3+1}}\frac{\partial}{\partial t_p} + \frac{1}{3}\frac{1}{\lambda_2} - 2\sum_{k=1}^{\infty}\frac{\hbar}{\lambda_2^{(k+1)/3}}\frac{\partial}{\partial t_k} + \hbar\frac{\partial}{\partial t_1} - \frac{\hbar}{3}\frac{1}{\lambda_2},$$
$$\tag{6.231}$$

where C_p is the combinatorial factor that counts how many pairs $(k,l) \in \mathbb{N}^2$ there are such that $p = k+l-3$. In fact, $C_p = p+2$.

Thus we obtain

$$H_1^{(3,2)} = \sum_{k,l>0}\frac{\hbar^2}{\lambda_2^{k/3+l/3+1}}\frac{\partial^2}{\partial t_k\partial t_l} - 2\sum_{k=1}^{\infty}\frac{\hbar}{\lambda_2^{(k+1)/3}}\frac{\partial}{\partial t_k} + \frac{1}{3\lambda_2}(1-\hbar) + \sum_{k=1}^{\infty}(k+2)\frac{\hbar}{\lambda_2^{k/3+1}}\frac{\partial}{\partial t_k} + \hbar\frac{\partial}{\partial t_1}.$$
$$\tag{6.232}$$

where we have coloured the terms that have the same form in equations (6.232) and (6.226).

The coefficients of equation (6.232) can be modified by four ways: a rescaling of $t_k \mapsto a_k t_k$ by a parameter a_k depending on k; a choice of $\hbar \neq 1$; multiplying $H_1^{(3,2)}$ by a non zero constant, or addition of combinations of the lower constraints $H_k^{(2,2)} \tau = 0$ and $H_k^{(1,2)} \tau = 0$. This has not yet been done fully and we leave this for future work. While it may be disheartening to learn that even the lowest order modes are rather intricate to calculate, we recall that the commutation relations of the $\mathcal{W}$ algebras constructed in [1] are known. Encouragingly therefore, the lowest order modes are sufficient for reproducing the full tower of constraints, as was the case for the 2-KW and 2-BGW tau functions.

Chapter 7

Conclusions and Future Work

Let us give a quick summary of the main points of this thesis. In chapter two, we defined integrable hierarchies and tau functions, both from the Hamiltonian and Lax equation points of view. It is these Lax equations which allowed us to reduce the KP hierarchy to the r-KdV hierarchies. We also presented an approach to the KP hierarchy using the Sato Grassmannian. This provided an elegant description of tau functions as vacuum expectation values in a certain Fock space. In chapter three, we introduced many of the tools required to understand matrix models: Wick's theorem, diagrammatic expansions, reduction to eigenvalues, Ward identities, spectral curves, topological recursion, orthogonal polynomials and the double scaling limit. In addition, we found a set of discrete Virasoro constraints that advanced the aim of understanding the relationship between $\mathcal{W}$-constraints and r-KdV. Ultimately we were interested in continuous Virasoro constraints and so we slightly changed direction. We subsequently focused further on the topological recursion aspect, introducing Airy structures and the Kontsevich-Soibelman recursion. Furthermore, we related this to the usual Eynard-Orantin recursion applied to spectral curves of simple ramification. We then introduced the crucial construction of higher Airy structures that form representations of certain $\mathcal{W}$-algebras. We denoted these Airy structures as $H_k^{(i,s)}$ with $1 \leq i \leq r$. This construction depended on two parameters (r, s) with $s \in \{1, \ldots, r+1\}$ and $r = \pm 1 \mod s$. From here we saw generalisations to the 2-KW tau function in the form of r-KW and BGW tau functions.

The rest of the thesis was then devoted to gaining a better understanding of the r-BGW tau function, a suitably generalised BGW tau function which has not been considered before in the literature. Inspired by the work of Kontsevich and Witten as reviewed in chapter five, we next considered matrix models with an external field. Using the Grassmannian approach to tau functions, we proved that for special choices of potential, these formal matrix integrals give rise to families of tau functions. We then gave an appropriate definition of the r-BGW tau function in this integral representation. We speculated on the form of the spectral curve corresponding to r-BGW before proceeding to calculate the continuum $\mathcal{W}$-constraints for 3-BGW.

The result of the original calculation of the $H_1^{(3,2)}$ mode was inconclusive in that we have been unable to match this with the higher Airy structures in [1]. Aside from calculation errors, this could be due to a few reasons. As already mentioned, these calculations could match after adding 'zero'. That is, we can add a suitable combination of the lower constraints $H_k^{(2,2)}\tau = 0$ and $H_k^{(1,2)}\tau = 0$. It remains to be seen whether this, along with perhaps a rescaling of the variables, rectifies the disparities between this calculation and the Airy structure. We believe that that are enough similarities, however, to justify further work to match these modes.

Furthermore, another possibility for future work is to consider other integral representations of tau functions that may lead more directly to the desired Airy structure. It may be that one needs to include additional terms in the definition of the generalised Kontsevich integral. Indeed, if one takes a potential in positive powers of X, one obtains the r-KW tau function. This corresponds to the r-Airy structure with $s = r + 1$. Conversely, taking the potential to include inverse powers of X leads to, we suspect, the r-BGW tau function. In [1], it is claimed private communications with Di Yang and Chunhui Zhou show that the corresponding r-Airy structure is given by $s = 1$. However, the results of our speculative calculation of the spectral curve and the more concrete calculation of the $\mathcal{W}$-constraints seem to suggest that the Airy structure for the r-BGW tau function is given by $s = r - 1$ instead.

Nevertheless, the r-KW and r-BGW tau functions do not exhaust all possible values

of s for higher Airy structures. For example, it is known that the $3-$KW tau function corresponds to $s = 4$ while the original calculations of this thesis seem to suggest the 3-BGW tau function corresponds to $s = 2$. This still leaves the remaining possibility of $s = 1$. It is not immediately clear what the appropriate potential in the external field matrix model should be. However, it has recently become known, see [100] for example, that the Kontsevich model giving rise to the 2-KW tau function can be represented as

$$Z_{(r,\tilde{s})} = \int dX \exp\left(\mathrm{Tr}\left(-\frac{X^{r+1}}{r+1} + \Lambda X - \frac{k}{\tilde{s}+1}X^{\tilde{s}+1}\right)\right) \tag{7.1}$$

with $r = 2$ and $k = 0$. Furthermore, the 2-BGW tau function arises from (7.1) with $r = -2, \tilde{s} = -1$ and $k = N$ although it is not immediately clear from this equation how to define $Z_{(-2,-1)}$. Despite this, if one were to differentiate the $X^{\tilde{s}+1}$ term and set $\tilde{s} = -1$, we find that this term is indeed equivalent to the logarithmic term in the external field matrix model. It should therefore be possible to extract the $\mathcal{W}$-constraints in the same way as above. In this representation, the Airy structure may reveal itself more clearly for the 3-BGW tau function than the model we have used in this thesis. Additionally, it should also be possible to tune the parameter $\tilde{s}$ so that one obtains the other possible values of s in the higher Airy structures.

Finally, there may be a more elegant approach to calculating such $\mathcal{W}$-constraints. In [101], Alexandrov used so called Kac-Schwarz operators arising from the Grassmannian description of integrability to find the Airy structure corresponding to the 2-BGW tau function. This rather beautiful approach bypasses the need for the asymptotic analysis and matrix calculus employed in this thesis. Furthermore, it may be hoped that this method generalises well to higher r. Indeed, in our method a matrix derivative of order r appears in the Ward identity for the r-BGW tau function, the calculation of which in terms of eigenvalues quickly becomes computationally cumbersome for larger r. In the appendix we give a derivation of these formulae. Advantageously, such formulae are not required in the Kac-Schwarz operator formalism. The Kac-Schwarz operators also lead to relationships between integrability and quantum curves, an area which we have not even touched in this thesis.

Finally, we return to the figure presented in the introduction.
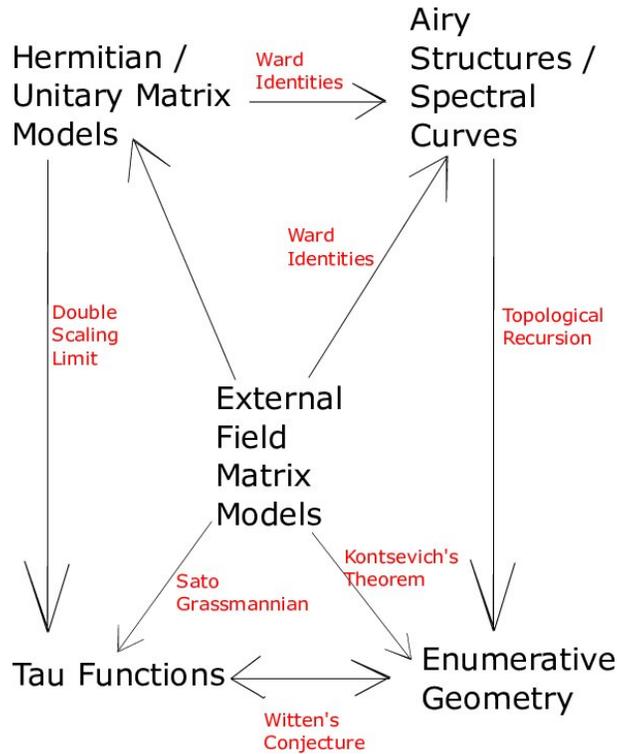
The reader should hopefully by now be convinced that tau functions and matrix models touch many different areas of mathematics. It is hoped that this thesis has served to make this diagram more intelligible, although we again emphasise that it is not an exhaustive description of all topics related to tau functions, matrix models and Airy structures. In fact, one might have need of drawing such a graph on a surface of genus $g > 0$ if one wished to exhaust all the known relationships between these exciting areas!

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Appendix A

Matrix Calculus

A.1 Matrix Differentiation

A.1.1 Matrix Differentials for the BGW Ward Identity I

One should treat matrix differentiation with a heightened sense of caution. In the literature, one often sees the notation $\frac{\partial}{\partial X}$ which can sometimes be misleading. It is often better to explicitly write the indices. Given a fully general matrix function $F : M_{m \times n}(\mathbb{C}) \to M_{p \times q}(\mathbb{C})$, one can naturally consider ordinary differentiation using the identification $M_{m \times n}(\mathbb{C}) \cong \mathbb{C}^{mn}$. Thus, a matrix derivative in its greatest generality is labelled by four indices,

$$\frac{\partial}{\partial X_{kl}} \big(F(X) \big)_{ij}, \tag{A.1}$$

with

$$\frac{\partial X_{ij}}{\partial X_{kl}} = \delta_{ik} \delta_{jl}. \tag{A.2}$$

The algorithm for differentiating matrix functions is then clear: one should write out each entry $F(X)_{ij}$ explicitly in its components X_{kl} and differentiate each entry according to (A.2). However, this can be rather cumbersome even for the simplest examples. Fortunately, in this thesis, we exclusively are interested in the case where $F(X)$ is a scalar function. For example,

consider the function $F : M_{n\times n}(\mathbb{C}) \to \mathbb{C}$ given by

$$F(X) = \text{Tr } \log X. \tag{A.3}$$

We remark that F defined as a power series, need not converge for all matrices X. Indeed, if we were to formally use the identity $\text{Tr } \log X = \log \det X$, we immediately see there is a singularity if X is not invertible. However, for this thesis we are interested only in formal matrix calculus and so we shall not dwell on this matter. We now calculate the two matrix derivatives, (6.103) and (6.104) used in the derivation of the BGW Ward Identity. Calculating the matrix derivative (6.104) we have

$$\frac{\partial \text{Tr} \log X}{\partial X_{ab}} = \frac{\partial \log \det(X)}{\partial X_{ab}} = \frac{1}{\det(X)} \frac{\partial \det(X)}{\partial X_{ab}}. \tag{A.4}$$

To differentiate the determinant, there is a standard rule, known as Jacobi's formula.

Theorem A.1.1. Let $X : \mathbb{C} \to M_{n\times n}(\mathbb{C})$ be a differentiable map. Then

$$\frac{d}{dt} \det X(t) = \text{Tr}\left(\text{adj}(X)\frac{dX}{dt}\right), \tag{A.5}$$

where $\text{adj}(X)$ is the adjugate matrix of X, the transpose of the matrix of cofactors, that is to say, a transposed matrix of minors with appropriate signs.

As a special case, we have the following corollary.

Corollary A.1.2. The following identity is satisfied

$$\frac{\partial}{\partial X_{ij}} \det(X) = \left(\text{adj}^{\text{T}}(X)\right)_{ij}. \tag{A.6}$$

Proof. This follows by setting $t = X_{ij}$. Indeed,

$$\frac{\partial}{\partial X_{ij}} \det(X) = \left(\text{adj}(X)\right)_{ab} \frac{\partial X_{ba}}{\partial X_{ij}} = \left(\text{adj}(X)\right)_{ab} \delta_{bi}\delta_{aj} = \text{adj}(X)_{ji}. \tag{A.7}$$

$\square$

As an aside, it is interesting to note that the Jacobi formula contains the following, perhaps more familiar, identity as a special case.

Corollary A.1.3. For an invertible matrix X, the following identity is satisfied

$$\det e^{tX} = e^{t\operatorname{Tr} X}. \tag{A.8}$$

Proof. Recall the adjugate matrix is given by

$$\operatorname{adj}(Y) = \det(Y)Y^{-1}. \tag{A.9}$$

Thus, using Jacobi's formula we find

$$\frac{d}{dt}\det Y(t) = \det(Y)\operatorname{Tr}\left(Y^{-1}\frac{dY}{dt}\right) \tag{A.10}$$

Setting $Y(t) = e^{tX}$ we find

$$\frac{d}{dt}\det e^{tX} = \det\left(e^{tX}\right)\operatorname{Tr}X \tag{A.11}$$

Solving this ordinary differential equation yields the desired result. $\qquad\square$

Returning to the matrix derivative (A.4), we use Corollary A.1.2 and the relation between the adjugate matrix and the inverse to write

$$\frac{1}{\det X}\frac{\partial}{\partial X_{ab}}\det X = \frac{1}{\det X}\left(\operatorname{adj}^{\mathrm{T}}(X)\right)_{ab} = (X^{-1})_{ba}. \tag{A.12}$$

Thus we have

$$\frac{\partial}{\partial X_{ij}}\operatorname{Tr}\log X = (X^{-1})_{ji}, \tag{A.13}$$

or in matrix form

$$\frac{\partial}{\partial X}\operatorname{Tr}\log X = (X^{-1})^{\mathrm{T}}. \tag{A.14}$$

To calculate this matrix derivative, we were fortunate enough to able use Jacobi's formula.

Now suppose we wished to calculate the matrix derivative

$$\frac{\partial}{\partial X_{ab}} \text{Tr}(X^{-1}). \tag{A.15}$$

This highlights the general difficulty in calculating these derivatives: in each case, one needs to first make use of specific identities, as we did with Jacobi's formula, to simplify the function. There is in fact a more efficient way of computing matrix derivatives through the differential. Given a scalar function $F : M_{n \times n}(\mathbb{C}) \to \mathbb{C}$ the differential can be defined as it is done in standard multivariable calculus,

$$dF = \frac{dF}{\partial X_{ij}} dX_{ij} \tag{A.16}$$

In light of this, we have the following proposition, analogous to Proposition 6.4.1.

Proposition A.1.4. Let ϕ be a differentiable function of an $n \times n$ matrix. Then the following relationship holds

$$d\phi(X) = \text{Tr } A dX \iff \frac{\partial \phi}{\partial X_{ij}} = A_{ji}. \tag{A.17}$$

This is proved in [98]. This simplifies many calculations. Indeed, taking the previous example, we have that

$$d\text{Tr } \log X = \text{Tr } d \log X = \text{Tr } (X^{-1}) \tag{A.18}$$

and hence by the proposition,

$$\frac{\partial}{\partial X_{ij}} \text{Tr} \log X = (X^{-1})_{ji}. \tag{A.19}$$

Similarly for $\text{Tr}(X^{-1})$, we find

$$d\text{Tr}(X^{-1}) = \text{Tr } d(X^{-1}). \tag{A.20}$$

To calculate dX^{-1} we notice that

$$0 = d(I) = d(XX^{-1}) = Xd(X^{-1}) + (dX)X^{-1}, \tag{A.21}$$

meaning that

$$d(X^{-1}) = -X^{-1}(dX)X^{-1}. \tag{A.22}$$

Substituting this into the above we find

$$\text{Tr } d(X^{-1}) = \text{Tr } (-X^{-2}dX), \tag{A.23}$$

by virtue of the cyclicity of the trace. Hence by the proposition we find

$$\frac{\partial}{\partial X_{ij}}\text{Tr}(X^{-1}) = -(X^{-2})^{\text{T}}, \tag{A.24}$$

as desired in (6.103).

Furthermore, it is proven in [98] that the scalar function $F(X) = \text{Tr } V(X)$ has Taylor expansion

$$\text{Tr } V(X + X_0) = \text{Tr } V(X_0) + \text{Tr } \frac{\partial}{\partial X_{ab}}\big(\text{Tr } V(X_0)\big)X + \cdots. \tag{A.25}$$

This expansion and the above proposition help to explain why one may naively differentiate $V(X)$ as $V'(X)$ when Taylor expanding such functions. As in Proposition 6.4.1, however, the higher order terms are not quite so simple.

A.1.2   Matrix Differentials for the BGW Ward Identity II

We recall the Ward identity (6.97) for the unitary matrix model involved a matrix derivative of the form

$$\frac{\partial}{\partial A_{ji}^\dagger}\frac{\partial}{\partial A_{kj}}. \tag{A.26}$$

We now change variables using

$$\Lambda_{ik} = A_{ij}A_{jk}^\dagger. \tag{A.27}$$

Using the chain rule, we can rewrite the equations of motion for $Z_{2\text{BGW}}$ in terms of Λ. Indeed,

$$\frac{\partial}{\partial A_{ji}^\dagger}\frac{\partial}{\partial A_{kj}} = \frac{\partial}{\partial A_{ji}^\dagger}\frac{\partial \Lambda_{qs}}{\partial A_{kj}}\frac{\partial}{\partial \Lambda_{qs}} = \frac{\partial}{\partial A_{ji}^\dagger}A_{js}^\dagger\delta_{qk}\frac{\partial}{\partial \Lambda_{qs}} = \delta_{is}\delta_{jj}\frac{\partial}{\partial \Lambda_{ks}} + A_{js}^\dagger\frac{\partial}{\partial A_{ji}^\dagger}\frac{\partial}{\partial \Lambda_{ks}}. \qquad (A.28)$$

We note that $\delta_{jj} = N$. Using the chain rule again yields

$$N\frac{\partial}{\partial \Lambda_{ki}} + A_{js}^\dagger\frac{\partial \Lambda_{mn}}{\partial A_{ji}^\dagger}\frac{\partial}{\partial \Lambda_{mn}}\frac{\partial}{\partial \Lambda_{ks}} = N\frac{\partial}{\partial \Lambda_{ki}} + A_{js}^\dagger A_{mj}\delta_{in}\frac{\partial}{\partial \Lambda_{mn}}\frac{\partial}{\partial \Lambda_{ks}} = N\frac{\partial}{\partial \Lambda_{ki}} + \Lambda_{ms}\frac{\partial}{\partial \Lambda_{mi}}\frac{\partial}{\partial \Lambda_{ks}}.$$
$$(A.29)$$

Hence we arrive at the stated result (6.98).

A.1.3 Split and Merge Rules for Loop Equations

We recall the split and merge rules used to find the Ward identities of Hermitian matrix models in chapter three. These are best thought of using matrix differentials. To derive the split and merge rules, we calculate

$$(d(M^l))_{ij} = \sum_{n=0}^{l-1}(M^n)_{ia}(dM)_{ak}(M^{l-n-1})_{kj} \qquad (A.30)$$

For the split rule, one should set $i = a$ and $k = j$. For the merge rule, we note the differential commutes with the trace so that

$$d\text{Tr } M^l = \text{Tr }(d(M^l)) = (d(M^l))_{ii} \qquad (A.31)$$

Hence, to obtain the merge rule, we set $i = j$ in (A.30).

A.1.4 Reduction to Eigenvalues for $\mathcal{W}$-Constraints

We now derive formulae for reducing matrix derivatives such as $\frac{\partial}{\partial \Lambda_{ab}}$ to derivatives over the eigenvalues λ_c of the Hermitian matrix Λ. Since Λ is Hermitian, we can find corresponding orthonormal eigenfunctions $|\phi_a\rangle$. Throughout this section, we use Einstein summation convention for the Roman alphabet but not the Greek alphabet. Now, consider the perturbed

eigenvalue equation

$$(\Lambda + d\Lambda)(|\phi_\mu\rangle + |d\phi_\mu\rangle) = (\lambda_\mu + d\lambda_\mu)(|\phi_\mu\rangle + |d\phi_{mu}\rangle). \tag{A.32}$$

Cancelling zero order terms we find

$$(\Lambda - \lambda_\mu)|d\phi_\mu\rangle + (d\Lambda - d\lambda_\mu)|\phi_\mu\rangle = 0. \tag{A.33}$$

Taking the inner product with $|\phi_\mu\rangle$ we find the diagonal matrix elements

$$\langle\phi_\mu| d\Lambda |\phi_\mu\rangle = d\lambda_\mu. \tag{A.34}$$

We note that the transition matrix elements are defined by

$$|\phi_\mu\rangle = \langle c|\phi_\mu\rangle |c\rangle, \tag{A.35}$$

where $\langle\phi_\mu|c\rangle = \langle c|\phi_\mu\rangle^* = U_{\mu c}$ defines a unitary matrix. Using this, we find that (A.34) becomes

$$d\lambda_\mu = \langle\phi_\mu|b\rangle \langle b| d\Lambda |c\rangle \langle c|\phi_\mu\rangle. \tag{A.36}$$

Therefore we obtain the first important formula

$$\frac{\partial\lambda_\mu}{\partial\Lambda_{bc}} = \langle\phi_\mu|b\rangle \langle c|\phi_\mu\rangle. \tag{A.37}$$

To proceed, we take equation (A.33) and take the inner product with $|\phi_b\rangle$ to obtain

$$\langle\phi_\nu|d\phi_\mu\rangle = \frac{1}{\lambda_\mu - \lambda_\nu} \langle\phi_\nu| d\Lambda |\phi_\mu\rangle. \tag{A.38}$$

Hence, we have

$$|d\phi_\mu\rangle = \sum_{\nu\neq\mu} \frac{1}{\lambda_\mu - \lambda_\nu} |\phi_\nu\rangle \langle\phi_\nu| d\Lambda |\phi_\mu\rangle. \tag{A.39}$$

From this, it follows that

$$\frac{\partial}{\partial \Lambda_{ab}} \langle \nu | \phi_\mu \rangle = \sum_{\alpha \neq \mu} \frac{1}{\lambda_\mu - \lambda_\alpha} \langle \nu | \phi_\alpha \rangle \langle \phi_\alpha | a \rangle \langle b | \phi_\mu \rangle. \tag{A.40}$$

Armed with equations (A.37) and (A.40) we are ready to reduce matrix derivatives to eigenvalue derivatives.

Using the chain rule and (A.37) we find

$$\frac{\partial}{\partial \Lambda_{ab}} = \sum_\mu \frac{\partial \lambda_\mu}{\partial \Lambda_{ab}} \frac{\partial}{\partial \lambda_\mu} = \sum_\mu \langle b | \phi_\mu \rangle \langle \phi_\mu | a \rangle \frac{\partial}{\partial \lambda_\mu} \tag{A.41}$$

as stated in equation (6.140). For the second derivative we find

$$\left(\frac{\partial^2}{\partial \Lambda^2} \right)_{ab} = \frac{\partial}{\partial \Lambda_{ac}} \frac{\partial}{\partial \Lambda_{cb}} = \sum_\mu \frac{\partial}{\partial \Lambda_{ac}} \left(\langle c | \phi_\mu \rangle \langle \phi_\mu | b \rangle \frac{\partial}{\partial \lambda_\mu} \right). \tag{A.42}$$

Now, using (A.40) we observe that

$$\sum_\mu \frac{\partial}{\partial \Lambda_{ac}} \left(\langle b | \phi_\mu \rangle \right) \langle \phi_\mu | c \rangle \frac{\partial}{\partial \lambda_\mu} \tag{A.43}$$

$$= \sum_{\alpha \neq \mu} \frac{1}{\lambda_\mu - \lambda_\alpha} \langle b | \phi_\alpha \rangle \langle \phi_\alpha | a \rangle \langle c | \phi_\mu \rangle \langle \phi_\mu | c \rangle \frac{\partial}{\partial \lambda_\mu} \tag{A.44}$$

$$= \sum_{\alpha \neq \mu} \langle b | \phi_\alpha \rangle \langle \phi_\alpha | a \rangle \frac{1}{\lambda_\mu - \lambda_\alpha} \frac{\partial}{\partial \lambda_\mu}, \tag{A.45}$$

where we performed the sum over c to get from the second line to the third. This sum over c produces a $\delta_{\mu\mu}$ which is equal to one for each term in the sum and so is irrelevant. Similarly, we have

$$\frac{\partial}{\partial \Lambda_{ac}} \left(\langle \phi_\mu | c \rangle \right) \langle b | \phi_\mu \rangle \frac{\partial}{\partial \lambda_\mu} = \sum_{\alpha \neq \mu} \langle b | \phi_\alpha \rangle \langle \phi_\alpha | a \rangle \frac{1}{\lambda_\alpha - \lambda_\mu} \frac{\partial}{\partial \lambda_\alpha}. \tag{A.46}$$

Thus, upon relabelling indices $\alpha \mapsto \mu$ we find

$$\left(\frac{\partial^2}{\partial \Lambda^2}\right)_{ab} = \sum_{\mu=1}^{N} \langle b|\phi_\mu\rangle \left[\frac{\partial^2}{\partial \lambda_\mu^2} + \sum_{\nu \neq \mu} \frac{1}{\lambda_\mu - \lambda_\nu}\left(\frac{\partial}{\partial \lambda_\mu} - \frac{\partial}{\partial \lambda_\nu}\right)\right]\langle\phi_\mu|a\rangle. \tag{A.47}$$

exactly as in equation (6.141). This can be systematically repeated in the same way to obtain the higher derivatives.

A.1.5 Reduction to Eigenvalues for the Proof of the HCIZ Formula

In the proof of the HCIZ formula, Theorem 6.1.2, we used the formula

$$\sum_i \left(\frac{\partial^2}{\partial X^2}\right)_{ii} = \sum_{i,j} \frac{\partial^2}{\partial X_{ij} \partial X_{ji}}\psi = \sum_i \frac{1}{\Delta(x)}\frac{\partial^2}{\partial x_i \partial x_i}(\Delta(x)\psi), \tag{A.48}$$

where x_k are the eigenvalues of X. This is in fact nothing other than equation (A.47). To see this, let us rewrite the above equation in a slightly different form. It is well-known that the Vandermonde determinant is a harmonic function. Expanding this using the Leibniz product rule therefore yields

$$\sum_i \frac{\partial^2}{\partial x_i^2} + 2\sum_i \frac{1}{\Delta(x)}\frac{\partial}{\partial x_i}(\Delta(x))\frac{\partial}{\partial x_i}. \tag{A.49}$$

This is in fact nothing other than equation (A.47). Indeed, (A.47) with $a = b = i$ and summing over i yields

$$\sum_i \frac{\partial^2}{\partial x_i^2} + \sum_{i \neq j} \frac{1}{x_i - x_j}\left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}\right) = \sum_i \frac{\partial^2}{\partial x_i^2} + 2\sum_{i<j} \frac{1}{x_i - x_j}\left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}\right). \tag{A.50}$$

Upon comparing equations (A.49) and (A.50), the identity we wish to prove is

$$\sum_{i<j} \frac{1}{x_i - x_j}\left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}\right) = \sum_i \frac{1}{\Delta(x)}\frac{\partial}{\partial x_i}(\Delta(x))\frac{\partial}{\partial x_i}. \tag{A.51}$$

To prove this identity, we compare coefficients of $\frac{\partial}{\partial x_i}$ for each i. However, notice that by the symmetry of the above identity, it suffices to only prove the coefficient of $\frac{\partial}{\partial x_1}$ matches on both sides. To this end notice that

$$\frac{1}{\Delta(x)} \frac{\partial}{\partial x_1} (\Delta(x)) = \frac{1}{\prod_{i<j}(x_i - x_j)} \frac{\partial}{\partial x_1} \Big(\prod_{j\geq 2}(x_1 - x_j) \prod_{1<i<j}(x_i - x_j) \Big)$$

(A.52)

$$= \frac{\prod_{1<i<j}(x_i - x_j)}{\prod_{i<j}(x_i - x_j)} \frac{\partial}{\partial x_1} \Big(\prod_{j\geq 2}(x_1 - x_j) \Big) = \frac{1}{\prod_{j\geq 2}(x_1 - x_j)} \sum_{k\geq 2} \prod_{\substack{j\geq 2 \\ j\neq k}}(x_1 - x_k) = \sum_{k\geq 2} \frac{1}{x_1 - x_k}$$

(A.53)

which is exactly the coefficient of $\frac{\partial}{\partial x_1}$ on the left hand side of (A.51).

A.2 Integration and Measure

A.2.1 The Haar Measure on Compact Lie Groups

We often referred to the Haar measure dU when integrating over the unitary group. We recall that the unitary group $\mathrm{U}(N)$ is a compact Lie group. Here, we give a very brief overview of the Haar measure for compact Lie groups as well as its main properties of left and right invariance.

Let G be a Lie group. It is well known that Lie groups are parallelisable meaning that the tangent bundle is trivial. That is to say $TG \cong G \times \mathfrak{g}$. This implies that G is orientable and we can define volume forms on G. In particular, we wish to choose volume forms that also respect the group structure on G.

Definition A.2.1. Let G be a Lie group and let $L_g : G \to G$ be left multiplication by g. Then a volume form ω on G is said to be left invariant if $L_g^* \omega = \omega$.

Proposition A.2.2. Left invariant volume forms always exist on a Lie group G and are furthermore unique up to a multiplication by a constant.

Proof. Choose any basis of the cotangent space T_e^*G to form a non zero element $\omega_e \in \bigwedge^n T_e^*G$. Define the volume form $\omega_g = L_{g^{-1}}^*\omega_e$. Observe that this volume form is left invariant since

$$L^*g\omega_{gh} = L^*gL_{h^{-1}g^{-1}}^*\omega_e = (L_{h^{-1}g^{-1}} \circ L_g)^*\omega_e = L_{h^{-1}}^*\omega_e = \omega_h. \tag{A.54}$$

To show uniqueness, suppose ω' is any left invariant volume form on the Lie group G. Now, we have that $\dim \bigwedge^n T_e^*G = 1$. This means there exists some $C \neq 0$ such that $\omega_e' = C\omega_e$. Thus, via left invariance we have

$$\omega_g' = L_{g^{-1}}^*\omega_e' = CL_{g^{-1}}^*\omega_e = C\omega_g, \tag{A.55}$$

implying the left invariant volume form is unique up to a multiplicative constant. $\square$

Definition A.2.3. A left invariant volume form on a Lie group G is called a left Haar measure.

By replacing ω with $-\omega$ if necessary, without loss of generality, we may assume that ω is positive with respect to the orientation of G. Thus, if G is compact, for any $f \in C^\infty(G)$ we can define the integral

$$\int_G f(g)\omega(g), \tag{A.56}$$

in the usual way. As stated in chapter three, we can consider the normalised Haar measure where

$$\text{Vol}(G) = \int_G \omega = 1. \tag{A.57}$$

It is convention to write dg for the volume form ω. Left invariance implies that $d(hg) = dg$ for all $h \in G$. Since the Haar measure is unqiue up to a multiplicative constant, this choice of normalisation specifies the Haar measure uniquely. Let us summarise these findings in the following corollary.

Corollary A.2.4. There exists a unique, normalised left invariant Haar measure for al compact Lie groups.

Completely analogously, one can define right Haar measures as right invariant volume forms. Such volume forms are again unique up to the choice of multiplicative constant. However, left Haar measures need not necessarily be right Haar measures.

Lemma A.2.5. Let $R_g : G \to G$ be right multiplication. For all $g \in G$ and any left Haar measure ω, the form $R_g^* \omega$ is left invariant.

Proof. The proof consists of the following one line observation

$$L_h^*(R_g^*\omega) = (R_g \circ L_h)^*\omega = (L_h \circ R_g)^*\omega = R_g^* L_h^*\omega = R_g^*\omega, \qquad (A.58)$$

where we used that left and right multiplication operations commute. $\qquad \square$

Since any two left Haar measures differ only by a constant, there exists a positive constant $\Delta(g)$ such that

$$\omega = \Delta(g) R_g^*\omega. \qquad (A.59)$$

Proposition A.2.6. The function $\Delta : G \to \mathbb{R}^+$ is well-defined and continuous.

Proof. That Δ is well defined is clear from the fact that any two left Haar measures differ only by a constant meaning that Δ does not depend on the choice of ω. It is also clear that Δ is continuous. $\qquad \square$

Definition A.2.7. The function $\Delta : G \to \mathbb{R}^+$ is called the modular function.

The modular function is in fact a Lie group homomorphism, although we shall not prove this here.

Definition A.2.8. A Lie group G is called unimodular if $\Delta \equiv 1$.

Unimodular Lie groups are particularly special: by definition, a Lie group is unimodular if and only if every left Haar measure is also a right Haar measure. Thus, we wish to restrict attention to compact, unimodular Lie groups. Fortunately, we have the following proposition.

Proposition A.2.9. All compact Lie groups are unimodular.

Proof. We have seen that Δ is continuous. Hence, if G is compact, then so is $\Delta(G)$. However, $\Delta(G)$ is also a subgroup of $\mathbb{R}^+$. The only compact subgroup of $\mathbb{R}^+$ is $\{1\}$ and so $\Delta \equiv 1$. $\quad\square$

Corollary A.2.10. The unique normalised left Haar measure on a compact Lie group G is also right invariant.

A.2.2 Diagonalisation and the Lebesgue Measure

In chapter three, we presented a method using Faddeev-Poppov determinants of reducing matrix integrals to integrals over the eigenvalues. Here, we present another way of performing this reduction which is perhaps more efficient. Consider the standard Lebesgue measure

$$dM = \prod_i dM_{ii} \prod_{i<j} d\mathrm{Re}(M_{ij})d\mathrm{Im}(M_{ij}) \tag{A.60}$$

on the Hermitian matrices H_N. Now, suppose M is diagonalised by a unitary matrix U so that $\Lambda = U^\dagger M U$. Then, using $U^\dagger = U^{-1}$ we find that

$$d(U^\dagger) = -U^\dagger(dU)U^\dagger, \tag{A.61}$$

as before. Then observe that

$$dM = d(U\Lambda U^\dagger) = (dU)\Lambda U^\dagger + U(d\Lambda)U^\dagger - U\Lambda U^\dagger(dU)U^\dagger \tag{A.62}$$

$$= U(d\Lambda + U^\dagger dU\Lambda - \Lambda U^\dagger dU)U^\dagger = U(d\Lambda + [U^\dagger dU, \Lambda])U^\dagger \tag{A.63}$$

In particular, expanding around $U = \mathbb{1}$, we find

$$dM = d\Lambda + [U^\dagger dU, \Lambda] \tag{A.64}$$

so that

$$dM_{ii} = d\Lambda_{ii} = d\lambda_i \tag{A.65}$$

and for $i \neq j$,

$$dM_{ij} = (\lambda_i - \lambda_j)dU_{ij}. \tag{A.66}$$

Substituting this into the Lebesgue measure we indeed find the same result as the Faddeev-Poppov determinant procedure,

$$dM = \left(\prod_i d\lambda_i\right) \prod_{i<j}(\lambda_i - \lambda_j)^2 dU. \tag{A.67}$$

A.2.3 Evaluation of Gaussian Matrix Integrals

In equation (3.77), with Lebesgue measure given by

$$dM = \prod_i dM_{ii} \prod_{i<j} d\mathrm{Re}(M_{ij})d\mathrm{Im}(M_{ij}), \tag{A.68}$$

we stated that

$$I = \int_{H_N} dM e^{-\frac{N}{t}\mathrm{Tr}\, M^2/2} = 2^{N/2}\left(\frac{t\pi}{N}\right)^{N^2/2} = 2^{N/2}\left(\frac{\pi t}{N}\right)^{N^2/2}. \tag{A.69}$$

To show this, we use the identity given in equation (3.112),

$$\mathrm{Tr}\, M^2 = \sum_i M_{ii}^2 + 2\sum_{i<j}(\mathrm{Re}\, M_{ij})^2 + (\mathrm{Im}\, M_{ij})^2. \tag{A.70}$$

Thus, the integral (A.69) becomes

$$I = \left(\int_{-\infty}^{\infty} dM_{ii}e^{-\frac{N}{2t}M_{ii}^2}\right)^N \left(\int_{-\infty}^{\infty} d\mathrm{Re}M_{ij}e^{-\frac{N}{2t}(\mathrm{Re}M_{ij})^2}\right)^{\frac{N^2-N}{2}} \left(\int_{-\infty}^{\infty} d\mathrm{Im}M_{ij}e^{-\frac{N}{2t}(\mathrm{Im}M_{ij})^2}\right)^{\frac{N^2-N}{2}}. \tag{A.71}$$

Thus, we obtain

$$I = \left(\frac{2\pi t}{N}\right)^N \left(\frac{\pi t}{N}\right)^{N^2-N} = 2^{N/2}\left(\frac{\pi t}{N}\right)^{N^2/2} \tag{A.72}$$

as required.

A.3 Kronecker Products

In section 6.4, we calculated determinants of the form

$$\det \left(\Lambda \otimes \Lambda^{1/2} + \Lambda^{1/2} \otimes \Lambda \right), \tag{A.73}$$

that arose as determinants of Hessians. See Proposition 6.4.1.

Here, we give a brief overview of the main properties of the Kronecker product $\otimes$ as well as argue that to calculate such a determinant, one can assume that Λ is diagonal. Here we omit proofs but they can be found in [98].

Definition A.3.1. Let A be an $m \times n$ matrix and B be $p \times q$ matrix. Then the Kronecker product $A \otimes B$ is the $mp \times nq$ matrix given by

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix}. \tag{A.74}$$

The Kronecker product satisfies the mixed product property. That is to say,

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD), \tag{A.75}$$

whenever the corresponding products are defined.

As the notation suggests, the Kronecker product is in fact related the abstract tensor product. Indeed, given vector spaces U, V, W_1, W_2, define linear maps $S : U \to W_1$ and $T : V \to W_2$. Let A and B be the matrices that represent the maps S and T with respect to the bases $\{u_i\}$ and $\{v_i\}$. Then the map $S \otimes T : U \otimes V \to W_1 \otimes W_2$ is represented by the Kronecker product $A \otimes B$ with respect to the basis $\{u_i \otimes v_j\}$.

Furthermore, if A has eigenvalues λ_i and B has eigenvalues μ_j, then the eigenvalues of $A \otimes B$ are given by $\lambda_i \mu_j$. Consequently, the trace of a Kronecker product is multiplicative.

That is

$$\text{Tr } A \otimes B = \text{Tr}(A)\text{Tr}(B). \tag{A.76}$$

Now, we argue that in expressions such as (A.73), one can assume Λ to be diagonal. To illustrate this point, we consider 2×2 matrices for simplicity. One can use the Cayley-Hamilton theorem to show that

$$\det A = \frac{1}{2}\big((\text{Tr } A)^2 - \text{Tr}(A^2)\big). \tag{A.77}$$

One can then substitute $A = \Lambda \otimes \Lambda^{1/2} + \Lambda^{1/2} \otimes \Lambda$ to find an expression for (A.73) in terms of trace of powers of Λ. If we focus on the term $\text{Tr}(A^2)$ we find

$$\text{Tr}(A^2) = \text{Tr}\big((\Lambda \otimes \Lambda^{1/2})^2\big) + 2\text{Tr}\big((\Lambda \otimes \Lambda^{1/2})(\Lambda^{1/2} \otimes \Lambda)\big) + \text{Tr}\big((\Lambda^{1/2} \otimes \Lambda)^2\big). \tag{A.78}$$

Using the mixed product property, this can be rewritten as

$$\text{Tr}(A^2) = \text{Tr}(\Lambda^2 \otimes \Lambda) + 2\text{Tr}(\Lambda^{3/2} \otimes \Lambda^{3/2}) + \text{Tr}(\Lambda \otimes \Lambda^2). \tag{A.79}$$

Using the multiplicative property (A.76) we find

$$\text{Tr}(A^2) = \text{Tr}(\Lambda^2)\text{Tr}(\Lambda) + 2\text{Tr}(\Lambda^{3/2})\text{Tr}(\Lambda^{3/2}) + \text{Tr}(\Lambda)\text{Tr}(\Lambda^2). \tag{A.80}$$

It is now clear that $\text{Tr}(A^2)$ depends only the eigenvalues of Λ since all the quantities in (A.80) are invariant under the diagonalisation of Λ. Similar considerations also show that $(\text{Tr } A)^2$ is also invariant under diagonalisation. Hence, when evaluating $\det(A)$, one may take Λ to be diagonal without loss of generality.

Now, in the above discussion, we considered 2×2 matrices only. Similar formulae exist, nevertheless, that express the determinant as powers of traces for matrices of any finite size. In chapter six, we found it was sufficient to consider the case when Λ was a 2×2 matrix. In that case, therefore, one need only consider the analogue of (A.77) in the case of 4×4

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matrices which reads

$$\det A = \frac{1}{24}\left((\text{Tr }A)^4 - 6\text{Tr }(A^2)(\text{Tr }A)^2 + 3(\text{Tr }A^2)^2 + 8\text{Tr}(A^3)\text{Tr}(A) - 6\text{Tr}(A^4)\right). \quad \text{(A.81)}$$

Furthermore, in the above there was no special role for the size of the matrix, or the exponents appearing in the traces. There was also no special role for the particular form of the Hessian matrix that appears in Proposition 6.4.1. Thus, one can again use the mixed product and multiplicative properties of the Kronecker product to show that, in general, one may take Λ to be diagonal.