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Mesoscale Gravity Currents and Cold-pools within a Continuously Stratified Fluid overlying Gently Sloping Topography

by

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Dr. G. E. Swaters

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Date: July 24, 1997

Abstract

It is believed that dense water formation in coastal seas makes a significant contribution to the creation of bottom water in the oceans. Therefore, in order to gain a better understanding of bottom water movement in the world's oceans, it is beneficial to study the dynamics of isolated cold-pools on sloping continental shelves. An analytical model was derived in Swaters & Flierl (1991) and Swaters (1991) that described such a situation. It modelled the low frequency dynamics of mesoscale gravity currents beneath a homogeneous fluid and underlain by a gently sloping bottom that destabilized and then formed isolated cold-pools due to baroclinic instability. The Swaters' model possesses solutions corresponding to pools of dense water which propagate along the shelf. As each cold-pool is transported along the shelf, baroclinic processes generate a cyclonic eddy in the ambient water directly above it. This eddy is depth invariant because of the homogeneity of the slope-water. This differs from numerical simulations of three-dimensional primitive equations in Gawarkiewicz *et al.* (1995) and Jiang *et al.* (1995) that modeled the same phenomena, but found that the eddy was of a truncated cone shape. Here we derive a new model similar to the Swaters' model, where the ambient fluid is three-dimensional, which then allows similar results to be obtained as those found numerically. We will consider the linear stability of gravity currents and what is necessary for destabilization. Then, after establishing a non-canonical Hamiltonian structure, we will find criteria for linear and nonlinear stability. An exact eddy solution will be found that has a zero-wave drag condition imposed. As well, a weakly radiating solution is obtained where there are drag forces present, hence a Rossby wave-wake is radiated from the cold-pool.

Acknowledgements

In my first year of university I had my supervisor, Dr. Gordon Swaters, as a calculus professor. Here, not only did he introduce me to the world of advanced mathematics, but also showed me its beauty and excitement. Years later, he was kind and patient enough to take me on for three consecutive research projects and then as a Masters student. Working with him through these experiences, I began to learn about the world of Geophysical Fluid Dynamics. It is due to his superb enthusiasm and insight that I was first able to appreciate the field. For this, and his financial support, I am extremely grateful.

Through my studies I have had countless discussions with Mateusz Reszka and Richard Karsten which have greatly increased my understanding and appreciation for the field, and for this they will always have my gratitude.

Finally, I want to thank my family and all of my friends, for the richness that they have added to my life. Without them, I would never have found strength and experienced happiness.

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Chapter 1

Introduction

Geophysical Fluid Dynamics (GFD) is the study of fluids where rotation or stratification, or both, are essential factors in determining the evolution of the system. The two mediums which are most studied are the atmosphere and oceans, either individually or coupled together. When considering these fluids on large-scales, the rotation of the earth becomes important and introduces the Coriolis force. Solar heating warms the earths surface and creates the ubiquitous stratification of these fluids, and is the source of most of the energy that is present within.

Within GFD, the two largest terms that appear in the momentum equations are the Coriolis and pressure terms. When these two quantities are in balance, the state is called geostrophy. It is known that large-scale motions are nearly geostrophic, but not exactly. Moreover, it is the slight variations from geostrophy that lead to many interesting phenomena such as weather. Hence to model the atmosphere and oceans it is essential to have these slight variations present. One such model was developed by Charney (1947) and is entitled Quasi-Geostrophy (QG). It is a theoretical model that has had much success in synoptic meteorology and mesoscale oceanography in explaining the onset of baroclinic instability (Charney, 1948, Eady, 1949 and Swaters, 1991). Because of its numerous achievements, QG theory forms a large basis of GFD.

The motivation for the work to follow lies within experimental observations. LeBlond *et al.* (1991) determined that during the summer months, when there is fresh water run-off from the Fraser River due to melted snow, there is an influx into the Strait of Georgia from the Juan de Fuca Strait of cold, well oxygenated water. When the tides are strong, this water is well mixed with the stratified fluid already in the Strait. However, during neap tides, this water enters without much disruption and remains relatively coherent. This forms a current overlying the bottom topography and is slightly denser than the ambient water around it. This density-driven flow is referred to as a gravity current since gravity acts on the system due to the density contrast with the ambient fluid, which is what induces motion into the system. This current is described as being mesoscale because rotational effects cannot be neglected even at leading order.

In the absence of the Coriolis force, gravity would act to pull the current down the shelf. However, in the northern hemisphere, when the Coriolis force is in effect, it deflects the dense water to the right of the down slope direction. It is for this reason that the gravity current can move with the locally deeper water on its left. The upper fluid reacts to the moving lower layer through baroclinic interactions. The velocity at which the cold water travels and the effect it has on the upper fluid are the dynamical properties to be studied in this thesis.

Gravity currents are of interest because they create heat fluxes in the world's oceans and hence contribute to global heat circulation. Since they may also contain pollution emitted from near by regions, they are of interest for environmental reasons. As well, if they are well oxygenated, they can help to support marine life in benthic waters. Knowing the paths that these currents follow could then help in predicting where the marine life will travel to, which would in turn be advantageous to the fishery industry.

These types of currents have also been observed along the Middle Atlantic Bight (Houghton *et al.*, 1982 and Ou & Houghton, 1982). The lower layer which when first observed, appears to be an elongated water mass, i.e. sausage-like, travels along the shelf. Later, it is seen that a small dome-like blob, which we call a cold-pool, breaks off from the main structure. This separation, is the result of a destabilizing process and is believed to be due to baroclinic instability. It is for this reason that it is desirable to develop an analytical model which describes both mesoscale gravity currents and cold-pools and focuses on baroclinic processes. The model derived by G.E. Swaters in Swaters & Flierl (1991) and Swaters (1991), known as the Swaters' model, is such a model.

An alternative is a barotropic model, as was derived in Griffiths *et al.* (1982). This reduced-gravity model studied a mesoscale gravity current beneath an infinitely deep fluid that possessed a horizontal shear-flow, overlying a sloping bottom. When the analytical results were compared to those obtained from experimental observations of a buoyancy current, three significant discrepancies arose. The first being that the unstable modes of the gravity current had infinitesimally small wavenumbers in contrast to those from experimental observations which were finite. As well, the length scale of the instability had no direct relation to that of the originating gravity current. Finally, there was a second branch of instabilities which were dipole-like, which suggested that perhaps baroclinic forces played a significant factor. Since the barotropic mode did not appear by itself, this meant that maybe it was the baroclinic processes that were essential and not the barotropic, as was originally thought. This reasoning was another inspiration for the Swaters' model.

The Swaters' model is a stably stratified two-layer model as depicted in Figure 2.1, except that both fluids are of constant density. To depict a gravity current, the lower layer is chosen to be a coupled front, i.e. there are two incroppings along the bottom topography. The fact that for this profile, the changes in the current height are on the same order as the height itself, is what prevents the lower layer from being QG. The relative height of the lower layer and bottom topography to the height of the upper layer are small quantities. The length scale is the internal deformation radius of the upper layer which is larger than that of the lower layer and hence is an 'intermediate length scale' in the sense of Charney & Flierl (1981). This allows the upper lower to he QG and lower layer to possess a Planetarily Geostrophic balance (Pedlosky, 1984). Both the fluids are geostrophic to leading order, but not necessarily so to second order, which is what allows for ageostrophic effects to occur. Any motion that occurs in the upper layer arises through vortex tube stretching. This model is a baroclinic model in that fast gravity modes are filtered out through the presence of a rigid-lid and the subinertial velocity scaling of the lower layer.

Swaters & Flierl (1991) studied the dynamic and thermodynamic interactions of coherent mesoscale cold-pools with the ambient fluid around it, over a sloping bottom. An exact eddy solution was found to the governing equations, as well as a weakly radiating solution containing a Rossby wave-wake. In both of these solutions, the cold-pool generates an induced cyclone in the upper fluid which is depth invariant due to the homogeneity of the upper fluid.

Swaters (1991) found conditions for linear instability of general gravity current profiles. It is believed that if this type of instability arises, the current will destabilize into isolated cold-pools, as has been observed along the Middle Atlantic Bight. Two necessary conditions for instability were found. These being the presence of a down-sloping side in the current, and sufficiently strong interaction between the two layers. If instability does arise, it will be prominent on the down-sloping side since this is where available potential energy is most easily extracted. Therefore, this instability is asymmetrical with respect to the current. This analysis also found the wavelength and growth rate of the most unstable mode. This mode is of interest since it grows faster than all the others. Hence, the length of the wave behavior observed experimentally, presumably, correspond to the wavelength of the most unstable mode.

When linear instability arises, small perturbations grow. When this growth becomes significantly large, the nonlinear terms become important. They can hinder growth and cause a stabilizing effect which then brings the perturbations back into the linear regime where growth then reoccurs. This is how small, but finite amplitude oscillations can arise. Mooney & Swaters (1996) found an amplitude equation for perturbations of the slightly linearly unstable modes through a weakly nonlinear analysis. This analysis also found approximate soliton solutions, which is suggesting that coherent structures can develop.

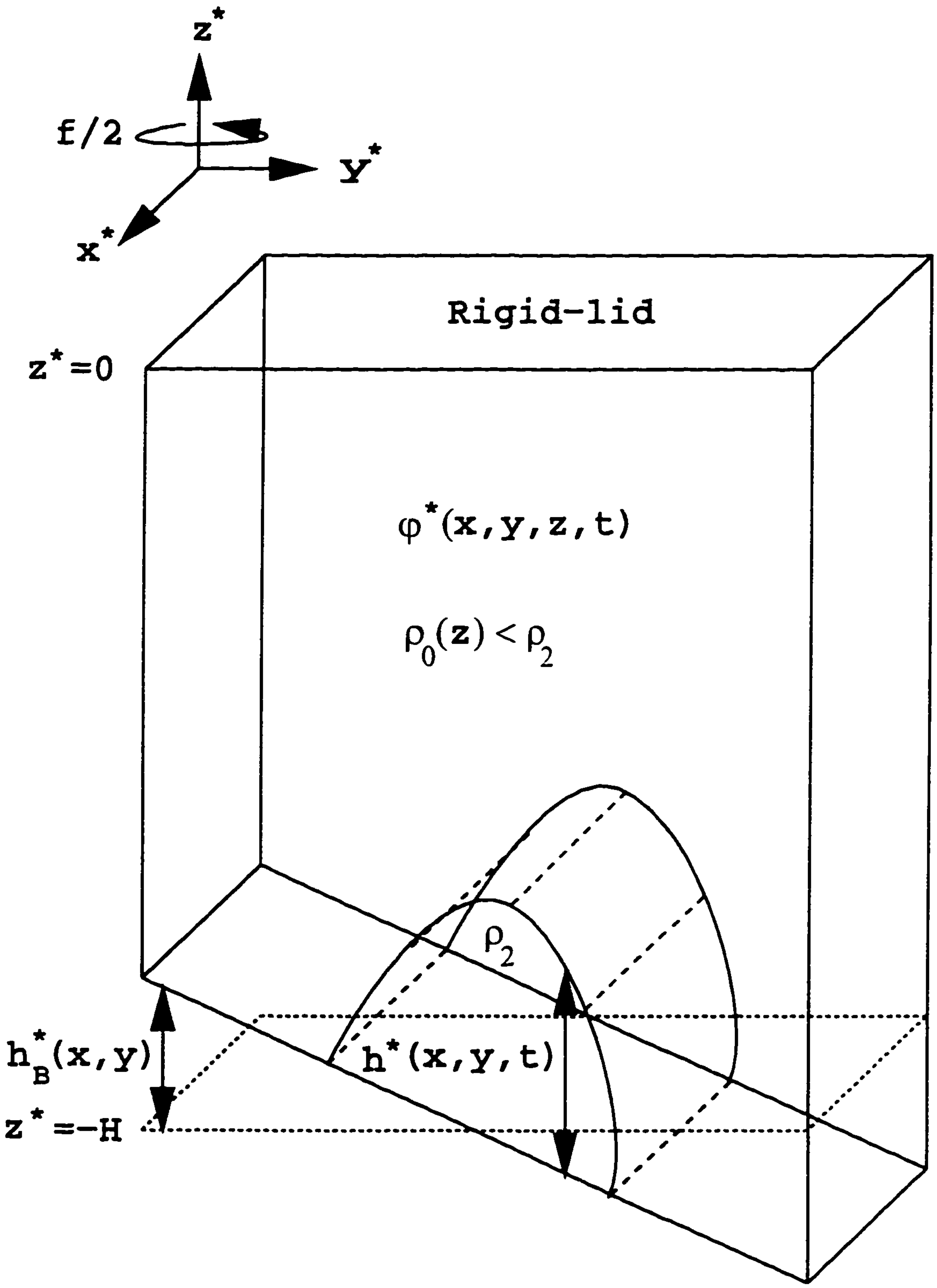
Swaters (1993b) showed that this model possessed a non-canonical Hamiltonian structure. This is important since it adds more credibility to the model because the model equations in their primitive form possess this structure. Through finding bounds of particular functions with respect to certain norms, various sets of conditions can be obtained for both linear and nonlinear stability for the case of a linearly sloping bottom. Karsten & Swaters (1996) determined that the Hamiltonian structure carried through for arbitrarily shaped bottom topography and hence more general stability criteria were established.

Karsten *et al.* (1995) applied the Swaters' model directly to the Strait of Georgia. This gave theoretical predictions of the wavelength of the most unstable mode. This length was larger than the wave behavior observed experimentally, but was much closer than previous theoretical predictions.

The Swaters' model involved scaling the dimensional equations and using asymptotic expansions in order to obtain the governing equations. Recently, numerical simulations of three dimensional primitive equations have been performed to describe gravity currents and cold-pools over a sloping bottom (Gawarkiewicz *et al.*, 1995 and Jiang *et al.*, 1996). Jiang *et al.* (1996) attributes the destabilizing processes to the baroclinic mechanisms put forth in the Swaters' model. These numerical simulations also determined that the propagating cold-pools generate an eddy in the overlying fluid directly above them, but that the eddy is of a truncated cone shape. This is different than the prediction of the Swaters' model, in which the eddy is depth invariant.

In this thesis, a new model will be derived similar to the Swaters' model, but with the added complexity of the upper layer being continuously stratified. This model will support the numerical predictions that the induced eddy above the cold-pool is tapered. As well, within the context of this model, we will study what effect that stratification has upon the instability characteristics of a mesoscale gravity current.

Figure 2.1: Model Geometry



Chapter 2

Derivation of the Model

In GFD, much of the interesting dynamics observed in the atmosphere and oceans, are due to the nonlinear terms present in the primitive momentum equations. These advective terms are very complicated and are what we strive to better understand. They necessarily arise in both conservative and nonconservative models of geophysical flows. We have chosen to neglect dissipative forces in our model since the underlying fluid dynamics seems to be governed by nonlinear processes.

This thesis will study two distinct fluids overlying variable bottom topography and beneath a rigid-lid at mid-latitudes in the northern hemisphere. The presence of a rigid-lid coupled with the velocity scaling, filters out barotropic gravity modes (Kundu, 1990). The upper fluid is QG, with continuous and stable stratification and covers the lower one which is a Planetary Geostrophic fluid (PG) in the form of a mesoscale gravity current, or cold-pool, of homogeneous density and variable height. The lower layer density is slightly larger than then greatest density that occurs in the upper fluid which guarantees that the two fluids together are stably stratified.

In Section 2.1 to 2.3, using appropriate physical principles, we derive equations and boundary conditions that govern both the upper and lower layers. In Section 2.4 we deviate from the model mentioned above and consider a one and a half layer reduced gravity model of a cold-pool over a sloping bottom and beneath a fluid that is assumed to be homogeneous and infinitely deep thereby eliminating any interaction between the two fluids. In this idealized model an along-shelf propagation speed will be determined for the cold-pool, which is called the *Nof speed* (Nof, 1983). We cannot expect that if the upper layer were changed to be a stratified fluid of finite depth the same velocity would arise, but it is expected be of the same order, and hence the Nof speed is the scaling used for the lower layer horizontal velocity field. In Section 2.5 the dimensional equations will be simplified by introducing nondimensional quantities which make the problem more manageable. Then in Section 2.6, asymptotic expansions will be implemented to yield the nondimensional equations that will be studied in the chapters to follow. The final section will show how two of the governing equations can be derived as a consequence of Ertel's theorem using the same scales as previously discussed.

2.1 The Lower Layer Equations

The fluid will be governed by shallow water theory, but rather than simply state the equations, it is informative to derive them from the Euler and continuity equations. To do so it must first be understood that for our regimes of interest, the aspect ratio is much less than one, as is required for the shallow water equations to hold true.

Ultimately, we will study the formation and propagation of cold-pools on continental shelves. We should then consider experimental data found on these pools to give an idea as to what scales they possess. Nof (1985) stated that the length scales of these pools range from 10 to 100 km, whereas their height varies from 20 to 200 m. These two length scales differ by about three orders of magnitude. If we write the horizontal and vertical length scales as L and D respectfully, it follows that for cold-pools

$$\delta_A = \frac{D}{L} \ll 1, \tag{2.1}$$

where δ_A is the aspect ratio.

Secondly, it must be assumed that the cold-pool can be idealized as homogeneous and inviscid. It might be thought that these assumptions are too stringent to describe any real fluid. Yet it has been found that this is not the case since much can be learned about oceanic and atmospheric phenomena by assuming that shallow water theory is applicable (Pedlosky, 1987).

Having gone through the formalities, we can proceed to develop the dimensional shallow water equations. The derivation will follow closely that presented in Chapter 3 of Pedlosky (1987), but first, the environment for our model must be specified. The orientation is such that the x, y and z axes point along the shelf with the deep water on the left, across the shelf towards deeper water and vertically upwards, respectively. The model is situated on an f-plane, and the Coriolis parameter f is equal to $2\Omega \sin \theta_0$ where Ω is the angular velocity of the earth and θ_0 is the reference latitude in which the model is located. Notations that will be used are that subscripts 1 and 2 denote the upper and lower layers respectively. The alphabetical subscripts denote partial derivatives with respect to the variable shown. Bold face will be used to denote a vector, whereas all other quantities are scalars unless otherwise stated.

We begin by applying the conservation of momentum and mass equations. Since the fluid is homogeneous and adiabatic, the internal thermodynamics is decoupled and hence is not a relevant consideration. The lower layer momentum equations, or Euler equations, contain the velocity vector for the lower layer $\mathbf{u}_2(x, y, z, t) = (u, v, w)$, where u, v and w are the velocities in the x, y and z directions respectively. The total pressure and density for this fluid are denoted by $p_2(x, y, z, t)$ and ρ_2. The equation of mass conservation, or continuity equation, reduces to, by homogeneity, the statement that the velocity field is non-divergent. Hence the four equations written in vector form are,

$$\mathbf{u}_{2_t} + (\mathbf{u}_2 \cdot \boldsymbol{\nabla}_3)\, \mathbf{u}_2 + f\ (\hat{\mathbf{e}}_3 \times \mathbf{u}_2) = -\frac{1}{\rho_2} \boldsymbol{\nabla}_3 p_2 - g\hat{\mathbf{e}}_3, \tag{2.2}$$

$$\boldsymbol{\nabla}_3 \cdot \mathbf{u}_2 = 0, \tag{2.3}$$

where $\boldsymbol{\nabla}_3 = (\partial_x, \partial_y, \partial_z)$, g is the magnitude of gravity and $\hat{\mathbf{e}}_3$ the unit vector pointing in the z direction. For future reference, define $\tilde{h}(x, y, t)$ and $h_B(x, y)$ to be the height of the lower layer and bottom topography with respect to some reference height. This implies that $\tilde{h} - h_B$ is the total thickness of the lower layer fluid.

At this stage, a scaling argument needs to be implemented. This procedure determines which terms are relevant by considering relative magnitudes. The terms that are drastically smaller than others in the same equation will then be discarded. This requires that the aspect ratio is small for this fluid, which has already been justified. The continuity equation (2.3), with the characteristic scales written directly beneath each term, is

$$\begin{array}{ccccc} u_x & + v_y & + w_z & = 0, \\ \frac{U}{L} & \frac{U}{L} & \frac{W}{D} & \end{array} \tag{2.4}$$

where U and W are the scales for the horizontal and vertical velocities. W/D cannot be greater than U/L, for then we would need to conclude that it is zero to leading order on account of the fact that there are not any other terms to balance it. Hence the following upper bound is obtained on the vertical velocity scale,

$$W = O\left(\frac{UD}{L}\right), \tag{2.5}$$

where O denotes the order of or smaller than.

Before dealing with the Euler equation, we decompose the total pressure as a sum of hydrostatic and reduced pressures,

$$p_2(x, y, z, t) = -g\rho_2 z + \tilde{p}(x, y, z, t). \tag{2.6}$$

Rewriting the momentum equations in terms of $\tilde{p}$ eliminates the gravity term so that (2.2) and becomes, when written in scaler form

$$u_t + uu_x + vu_y + wu_z - fv = -\frac{1}{\rho_2}\tilde{p}_x, \tag{2.7}$$

$$\frac{U}{T} \quad \frac{U^2}{L} \quad \frac{U^2}{L} \quad \frac{WU}{D} \quad fU = \frac{P}{\rho_2 L}$$

$$v_t + uv_x + vv_y + wv_z + fu = -\frac{1}{\rho_2}\tilde{p}_y, \tag{2.8}$$

$$\frac{U}{T} \quad \frac{U^2}{L} \quad \frac{U^2}{L} \quad \frac{WU}{D} \quad fU = \frac{P}{\rho_2 L}$$

$$w_t + uw_x + vw_y + ww_z = -\frac{1}{\rho_2}\tilde{p}_z, \tag{2.9}$$

$$\frac{W}{T} \quad \frac{UW}{L} \quad \frac{UW}{L} \quad \frac{W^2}{D} = \frac{P}{\rho_2 D}$$

where P and T are scalings for pressure and time. The pressure terms have been written on the right-hand sides of each equation to signify the fact that it is a forcing term whereas all of the terms on the left-hand side, are acceleration terms.

Assuming that pressure has a leading order contribution, requires it be balanced with on of the acceleration terms. Exploiting this balance produces a relationship for P in terms of the other quantities. Multiplying (2.5) by U/D implies

$$\frac{UW}{D} = O\left(\frac{U^2}{L}\right). \tag{2.10}$$

Therefore, the three advective terms in the horizontal momentum equations are $O(U^2/L)$. This implies that the three different scales present on the left-hand side are U/T, U^2/L and fU, which are due to the temporal derivative, advective derivatives and Coriolis term respectively. At this stage it is not known which of these three is largest, but it is known that the pressure gradient must balance the sum. Therefore it is necessary to equate $P/(\rho_2 L)$ with the order of the largest of the three scalings mentioned above,

$$\frac{P}{\rho_2 L} = O\left(\left[\frac{U}{T}, \frac{U^2}{L}, fU\right]_{\max}\right),$$

$$P = O\left(\rho_2 U \left[L/T, U, fL\right]_{\max}\right), \tag{2.11}$$

having multiplied the equation by $\rho_2 L$.

The left-hand side of the vertical momentum equation is the total derivative of the vertical velocity. Using the fact that

$$\frac{W^2}{D} = O(\frac{WU}{L}), \tag{2.12}$$

by (2.5), the relative importance of the left to right-hand sides of (2.9) is

$$\rho_2 \frac{dw/dt}{\tilde{p}_z} = O\left(\frac{\rho_2 \left[W/T, WU/L\right]_{\max}}{P/D}\right). \tag{2.13}$$

Substituting (2.11), (2.1) and (2.5) into (2.13) implies,

$$\begin{aligned} \rho_2 \frac{dw/dt}{\tilde{p}_z} &= O\left(\frac{D\,[W/T, WU/L]_{\max}}{U\,[L/T, U, fL]_{\max}}\right) \\ &= O\left(\frac{\delta_A^2\,[1/T, U/L]_{\max}}{[1/T, U/L, f]_{\max}}\right). \end{aligned} \tag{2.14}$$

If f is larger than $1/T$ and U/L then the right-hand side is less than δ_A^2, hence $O\left(\delta_A^2\right)$. Alternatively, if f is smaller than either of the two terms mentioned above, the right-hand side still reduces to $O\left(\delta_A^2\right)$. This shows that $\tilde{p}_z$ is larger than $\rho_2 dw/dt$ by two orders of magnitude, with respect to δ_A. Therefore, we deduce that $\tilde{p}_z = O\left(\delta_A^2\right)$, which if substituted into the z derivative of (2.6), yields

$$p_{2_z} = -g\rho_2 + O\left(\delta_A^2\right). \tag{2.15}$$

The smallness of the aspect ratio means that δ_A^2 is extremely small which suggests that the acceleration terms in the vertical momentum equation are not influential on the dynamics, thus are ignored. This is equivalent to assuming the lower layer is hydrostatic, which implicitly requires that the slopes associated with the bottom topography are sufficiently small so that the upper bound in (2.5) is not violated.

There are three vertical boundary conditions that must be introduced, two kinematic at the interface and bottom and the one dynamic condition at the interface. Integrating (2.15) with respect to z from z to $\tilde{h}$, ignoring small terms, and applying the pressure condition at the interface that the pressure at the interface is some function $p_0(x, y, t)$, yields

$$p_2 = g\rho_2\left(\tilde{h} - z\right) + p_0. \tag{2.16}$$

Physically this states that at any particular height, the pressure differs from p_0 by an amount equal to the weight of the unit column directly above it, and which is given by $g\rho_2(\tilde{h} - z)$.

Since the horizontal gradient of the total pressure does not depend on depth, the forcing on the system is depth invariant. This coupled with the assumption that the system initially begins with no z variation, allows us to conclude that the variables in the horizontal momentum equations, u and v do not develop any z dependency, thus

$$u_z = 0 = v_z. \tag{2.17}$$

Substituting (2.17) into equations (2.7) and (2.8) gives

$$u_t + uu_x + vu_y - fv = -\frac{1}{\rho_2} p_{2_x}, \tag{2.18}$$

$$v_t + uv_x + vv_y + fu = -\frac{1}{\rho_2} p_{2_y}. \tag{2.19}$$

Integrating the continuity equation from the bottom h_B to an arbitrary height z, with respect to z, and applying (2.17) yields

$$\int_{h_B}^{z} w_z dz = -\int_{h_B}^{z} u_x + v_y dz,$$

$$w(x, y, z, t) = (h_B - z)(u_x + v_y) + w(x, y, h_B, t). \tag{2.20}$$

To rewrite $w(x, y, h_B, t)$ requires the implementation of the kinematic boundary condition at the bottom of the fluid. This requires that there cannot be any flow through the topography. Mathematically, this is expressed by saying that the velocity vector and the normal vector to the surface must be perpendicular,

$$(u, v, w(x, y, z = h_B, t)) \cdot \nabla (z - h_B) = 0, \tag{2.21}$$

which simplifies to

$$w(x, y, z = h_B, t) = uh_{B_x} + vh_{B_y}. \tag{2.22}$$

Substituting this into (2.20)

$$w(x, y, z, t) = (h_B - z)(u_x + v_y) + uh_{B_x} + vh_{B_y}, \tag{2.23}$$

illustrates how the vertical velocity is completely determined as a function of the horizontal divergence, bottom topography and height level. The dependency on z is strictly linear. However, if the fluid lacked any horizontal divergence, as is true in geostrophy, this dependency would be eliminated and the vertical velocity would be constant. Since our fluids are nearly geostrophic and the bottom topography is gently sloping, only small vertical velocities occur.

The kinematic boundary condition at the interface of the lower layer dictates that the vertical velocity of the fluid at the interface and the interface velocity must be equal since fluid parcels that originate on the interface must remain there always. Mathematically, this is written as

$$w\left(x, y, z = \tilde{h}, t\right) = \left(\partial_t + \mathbf{u}_2 \cdot \boldsymbol{\nabla}\right) \tilde{h}, \tag{2.24}$$

where $\boldsymbol{\nabla} = (\partial_x, \partial_y)$ and the velocity has been redefined to be $\mathbf{u}_2 = (u, v)$. Evaluating (2.23) at $z = \tilde{h}$ and substituting in (2.24) enables the continuity equation to be transformed into

$$\tilde{h}_t + u\left(\tilde{h}_x - h_{B_x}\right) + v\left(\tilde{h}_y - h_{B_y}\right) + \left(\tilde{h} - h_B\right)(u_x + v_y) = 0,$$

or

$$\left(\tilde{h} - h_B\right)_t + \boldsymbol{\nabla} \cdot \left[\mathbf{u}_2\left(\tilde{h} - h_B\right)\right] = 0, \tag{2.25}$$

upon recalling that h_B is time invariant. Introducing the height variable $h = \tilde{h} - h_B$, gives the final equation

$$h_t + \boldsymbol{\nabla} \cdot [\mathbf{u}_2 h] = 0. \tag{2.26}$$

The shallow water equations (2.18), (2.19) and (2.26) rewritten in vector form, with asterisks to denote that these are dimensional equations, are,

$$\mathbf{u}^*_{2_{t^*}} + \left(\mathbf{u}^*_2 \cdot \boldsymbol{\nabla}^*\right)\mathbf{u}^*_2 + f\,\hat{\mathbf{e}}_3 \times \mathbf{u}^*_2 = -\frac{1}{\rho_2}\boldsymbol{\nabla} p^*_2, \tag{2.27}$$

$$h^*_{t^*} + \boldsymbol{\nabla}^* \cdot [\mathbf{u}^*_2 h^*] = 0, \tag{2.28}$$

where p_2 is defined in terms of h through (2.16) and ρ_2 has no asterisks but is a dimensional quantity. Note that in the derivation of these equations, surface tension has been ignored since this is a small scale phenomena and we are concerned strictly with large-scale motion.

We began with four equations and four variables and reduced the system down to three equations and three variables. The quantity that no longer appears is the vertical velocity since (2.23) expresses this explicitly as a function of the other variables. Therefore, given that a solution could be obtained for the above three equations, the vertical velocity is recovered by substituting the solution into (2.23). The shallow water model is two dimensional since the variables that appear in the governing equations are all depth invariant. It should be noted that the vertical velocity does have z dependency.

2.2 The Continuously Stratified Equations

In order to derive the governing equations for the upper layer the conservation of momentum and mass are applied as before. However, another equation is necessary to close the system, for otherwise we would have four equations with five unknowns, which is an insufficient number of equations.

This fifth equation is the statement that the upper layer is also incompressible,

$$\frac{d\rho}{dt} = 0. \tag{2.29}$$

The other four equations are the Euler and continuity equations. As with the lower layer, this fluid will be inviscid which removes any frictional forcing from the momentum equations, so that they become

$$\rho \frac{d\mathbf{u}_1}{dt} + \rho f \ (\hat{\mathbf{e}}_3 \times \mathbf{u}_1) = -\nabla_3 p_1 - g\rho\hat{\mathbf{e}}_3, \tag{2.30}$$

$$\frac{1}{\rho}\frac{d\rho}{dt} + \nabla_3 \cdot \mathbf{u}_1 = 0. \tag{2.31}$$

where $d/dt = (\partial_t + \mathbf{u}_1 \cdot \nabla_3)$ and $\mathbf{u}_1$ is the three dimensional velocity of layer one.

It has already been said that the fluid is incompressible, but we will strengthen this statement by modelling it as a Boussinesq fluid. This assumption is justified by the fact that we are dealing with very small velocities (say 20 cm/s), in relation to the speed of sound (about 1470 m/s), which is what is necessary for an compressible fluid to be Boussinesq (Kundu, 1990). This implies that in the momentum equations, all density terms can be approximated by some constant reference density, which we chose to be the density of the lower layer, except in the buoyancy term. This takes advantage of the fact that the density variation in the upper layer is slight, as is that with respect to the lower layer.

In the conservation of mass equation, the Boussinesq approximation implies that the change in density is small in relation to the density itself, therefore we assume the first term in (2.31) is much smaller than the divergence term. This second condition need not even be assumed, since by (2.29), we know that the total derivative of density is not only small but is precisely zero.

Therefore our governing equations are, rewritten with asterisks to denote the fact that we currently have *dimensional* quantities,

$$(\partial_{t^*} + \mathbf{u}_1^* \cdot \nabla^* + w^* \partial_{z^*})\, \mathbf{u}_1^* + f\ \hat{e}_3 \times \mathbf{u}_1^* = -\frac{1}{\rho_2} \nabla p_1^*, \tag{2.32}$$

$$(\partial_{t^*} + \mathbf{u}_1^* \cdot \nabla^* + w^* \partial_{z^*})\, w^* = -\frac{1}{\rho_2} \partial_{z^*} p_1^* - \frac{g\rho^*}{\rho_2}, \tag{2.33}$$

$$\nabla^* \cdot \mathbf{u}_1^* + w_{z^*}^* = 0, \tag{2.34}$$

$$(\partial_{t^*} + \mathbf{u}_1^* \cdot \nabla^* + w^* \partial_{z^*})\, \rho^* = 0. \tag{2.35}$$

A notational change is that $\mathbf{u}_1^* = (u^*, v^*)$ which separates the horizontal components from the vertical. This is done because the motion in these two directions are qualitatively different. The third equation above, which also applied to the lower layer, is the statement of mass conservation, and in effect, filters out sound waves.

2.3 Boundary Conditions

In addition to deriving the governing equations it is necessary to determine the appropriate kinematic and dynamic boundary conditions. These conditions for the lower layer have already been exploited and hence no longer need be considered. However those for the upper layer have not yet been stated. Before we proceed to establish these conditions it is necessary to choose at what height levels the top and bottom of the upper layer are situated. This is done by having the rigid-lid above the upper layer, coincident with $z^* = 0$, and the bottom situated at $z^* = -H + h^* + h_B^*$ where H is the reference height for the upper layer and the bottom topography is measured with respect to $z^* = -H$. Note that asterisks have been used to denote the dimensional quantities which is the convention that will be maintained throughout the remainder of this thesis unless specified otherwise.

The kinematic condition at the top requires that the fluid does not flow through the rigid-lid and hence that the vertical velocity is zero at that particular height,

$$w^* = 0 \quad \text{at} \quad z^* = 0. \tag{2.36}$$

The bottom of the upper layer is situated at $z^* = -H+h+h_B$. The kinematic boundary condition at the interface is then

$$w^* = (\partial_{t^*} + \mathbf{u}_1^* \cdot \nabla)(h^* + h_B^*) \quad \text{at} \quad z^* = -H + h^* + h_B^*, \tag{2.37}$$

having eliminated H because it is constant.

Recall from Section 2.1 that for the lower layer to be hydrostatic, which is essential for the shallow water equations to be applicable, we must necessarily have small bottom topography. Moreover, it is assumed that the height of the lower layer is small with respect to that of the upper layer in order that QG theory can be applied.

The smallness in the variations of the upper layer from H justifies Taylor expanding (2.37) about $z^* = -H$. Since the right-hand side has no depth variation we need only expand the left-hand side, where (2.37) then becomes

$$w^*(z^* = -H) + \frac{\partial w^*(z = -H)}{\partial z^*}(h^* + h_B^*) + \ldots = (\partial_{t^*} + \mathbf{u}_1^* \cdot \nabla)(h^* + h_B^*). \tag{2.38}$$

The three dots denote terms that are all of higher order. If we select the order of the second term by picking representative values for each variable present within it, the approximation to the above reduces to

$$w^* = (\partial_{t^*} + \mathbf{u}_1^* \cdot \nabla^*)(h^* + h_B^*) + O\left(w^* \left[\frac{h^*}{H}, \frac{h_B^*}{H}\right]\right) \quad \text{at} \quad z^* = -H. \tag{2.39}$$

In the absence of motion, there is a background density field with varies with depth and is denoted by $\rho_0(z^*)$. This density profile is assumed to be in hydrostatic balance with the background pressure field. Note that ρ_0 is a dimensional quantity even though no asterisk is written.

This density field is the source by which the fluid possesses stratification. Any variations in density from this background state are assumed to be small, and are due to the dynamics of the fluid. The total pressure for the upper layer p_1^*, can be decomposed into the hydrostatic pressure due to the background state, and a reduced pressure, caused by the presence of motion,

$$p_1^* = g\int_{z^*}^{0} \rho_0(\xi)\, d\xi + \varphi^*(x^*, y^*, z^*, t^*)\,. \tag{2.40}$$

As previously stated, the lower layer is in hydrostatic balance, which allows us to rewrite the total pressure in the lower layer p_2^* (see (2.6)), as

$$p_2^* = g\int_{-H}^{0} \rho_0(\xi)\, d\xi - g\rho_2(z^* + H) + p^*(x^*, y^*, t^*)\,, \tag{2.41}$$

where p^* is the reduced pressure of the lower layer.

The dynamic boundary condition requires that the total pressures of the upper and lower layers be equal at their interface. This forces the following relation between the reduced pressures φ^* and p^* upon using (2.40) and (2.41),

$$g\int_{-H+h^*+h_B^*}^{0} \rho_0(\xi)\, d\xi + \varphi^*(x^*, y^*, -H + h^* + h_B^*, t^*)$$

$$= g\int_{-H}^{0} \rho_0(\xi)\, d\xi - g\rho_2(h^* + h_B^*) + p^*(x^*, y^*, t^*)\,. \tag{2.42}$$

To simplify this equation, the two terms on the left-hand side need to be Taylor expanded about $z^* = -H$. The expansion for φ^* is,

$$\begin{aligned}
\varphi^*|_{z^*=-H+h^*+h_B^*} &= \varphi^*|_{z^*=-H} + \left.\frac{\partial \varphi^*}{\partial z^*}\right|_{z^*=-H} (h^* + h_B^*) + ... \\
&= \varphi^*|_{z^*=-H} + O\left(\varphi^*\left[\frac{h^*}{H}, \frac{h_B^*}{H}\right]\right),
\end{aligned} \tag{2.43}$$

upon using the scales that appear in the second term to estimate its magnitude.

Expanding the integral in (2.42) in a similar manner, yields

$$\begin{aligned}
g\int_{-H+h^*+h_B^*}^{0} \rho_0(\xi)\, d\xi &= g\int_{-H}^{0} \rho_0(\xi)\, d\xi - g\rho_0(-H)(h^* + h_B^*) \\
&\quad -\frac{g}{2}\frac{d\rho_0}{dz^*}(-H)(h^* + h_B^*)^2 + ... \\
&= g\int_{-H}^{0} \rho_0(\xi)\, d\xi - g\rho_0(-H)(h^* + h_B^*) \\
&\quad + O\left(\varphi^*\left[\frac{h^*}{H}, \frac{h_B^*}{H}\right]\right),
\end{aligned} \tag{2.44}$$

having used the fact that

$$g\frac{d\rho_0}{dz^*}(-H)(h^* + h_B^*)^2 = O\left(\varphi^*\left[\frac{h^*}{H}, \frac{h_B^*}{H}\right]\right). \tag{2.45}$$

The justification of this is reserved until after the scalings are presented. Substituting (2.43) and (2.44) into (2.42) and cancelling the integral that appears on both sides of the equality produces,

$$\varphi^*|_{z^*=-H} - g\rho_0(-H)(h^* + h_B^*) = -g\rho_2(h^* + h_B^*) + p^* + O\left(\varphi^*\left[\frac{h^*}{H}, \frac{h_B^*}{H}\right]\right). \tag{2.46}$$

The reduced gravity is defined to be,

$$g' = g\frac{(\rho_2 - \rho_0(-H))}{\rho_2} > 0, \tag{2.47}$$

which is a relatively small quantity compared to g since the density difference of the two layers is assumed to be slight. This definition allows us to rewrite (2.46) in the following form

$$p^* = \varphi^* + g'\rho_2(h^* + h_B^*) + O\left(\varphi^*\left[\frac{h^*}{H}, \frac{h_B^*}{H}\right]\right) \quad \text{at} \quad z^* = -H. \tag{2.48}$$

The dynamic boundary condition above, relates the dynamic pressure in the lower layer, p^*, to that of the upper layer φ^* evaluated at $z^* = -H$, the density difference between the bottom of the upper layer and the lower layer, gravity, the height of the lower layer and bottom topography, within a certain order of magnitude. Therefore, it is this equation that determines the coupling between the two fluids. The terms that explicitly appear, will be shown to be leading order contributions in the nondimensional equations. This is different from what occurs for buoyancy-driven surface currents where the coupling is in the next order (Flierl, 1984, Swaters & Flierl, 1991).

This section has dealt completely with kinematic and dynamic boundary conditions, which are vertical conditions. There has been no mention as to what horizontal conditions are to be applied; this is because two different scenarios will be considered. The first is where the fluid lies in a periodic channel which is used for the study of mesoscale gravity currents. The other, is where there are no boundary walls are present, which assumes an infinite horizontal domain, which is applied to cold-pools. The reason for this change is that in the study of cold-pools, far field conditions are necessary, which are difficult to apply when channel walls are present.

2.4 The Nof speed

This section will consider a simplified model than that derived previously. We consider the situation where the lower layer has formed a cold-pool of density $\rho + \Delta\rho$ overlying a sloping bottom of slope S and underlying an infinitely deep homogeneous upper layer of density ρ, which is lighter than the lower layer. The upper layer is assumed to be infinitely deep since the baroclinic interactions felt within it are proportional to the ratio between the height of the lower layer to that of the upper layer (Ingersol, 1969). If the limit is taken as the upper layer height goes to infinity it is readily seen that the interaction ratio goes to zero. This situation is called a one and a half layer model since the only effect of the upper layer is to reduce gravity. Note that in this section, since every term is dimensional, for notational convenience, the asterisks will be dropped. Following Nof (1985), solutions will be found to describe isolated cold-pools moving along at possibly time dependent velocities while maintaining their shape.

From (2.27) and (2.28), the governing equations are,

$$\mathbf{u}_{2s_t} + (\mathbf{u}_{2s} \cdot \boldsymbol{\nabla})\, \mathbf{u}_{2s} + f\, \hat{\mathbf{e}}_3 \times \mathbf{u}_{2s} = -\frac{1}{\rho} \boldsymbol{\nabla} p_2, \tag{2.49}$$

$$h_t + \boldsymbol{\nabla} \cdot [\mathbf{u}_{2s} h] = 0, \tag{2.50}$$

where subscript s denotes the horizontal velocity field in the reference frame that is stationary with respect to the upper fluid. We denote the time dependent cold-pool velocities by $C_x(t)$ and $C_y(t)$ directed in the x and y directions respectively.

The following transformation will change our coordinate system to the center of the moving eddy,

$$x = x_s - \int_0^{t_s} C_x(\tau)\, d\tau, \tag{2.51}$$

$$y = y_s - \int_0^{t_s} C_y(\tau)\, d\tau. \tag{2.52}$$

Differentiating these equations with respect to time, yields the following velocity transformation,

$$u_2 = u_{2s} - C_x(t_s), \tag{2.53}$$

$$v_2 = v_{2s} - C_y(t_s). \tag{2.54}$$

The height of the cold-pool, pressure within the cold-pool and time all remain invariant. To transform our equations from the old to the new coordinate system requires that we rewrite the derivatives as follows,

$$\begin{aligned} \frac{\partial}{\partial t_s} &= \frac{\partial x}{\partial t_s}\partial_x + \frac{\partial y}{\partial t_s}\partial_y + \frac{\partial t}{\partial t_s}\partial_t \\ &= -C_x\partial_x - C_y\partial_y + \partial_t, \end{aligned} \tag{2.55}$$

similarly

$$\frac{\partial}{\partial x_s} = \partial_x \qquad \frac{\partial}{\partial y_s} = \partial_y. \tag{2.56}$$

Subestituting (2.53) to (2.56) into the x direction momentum equation of (2.49), gives the following in terms of the new variables,

$$(-C_x\partial_x - C_y\partial_y + \partial_t)[u_2 + C_x] + [u_2 + C_x][u_2 + C_x]_x$$

$$+ [v_2 + C_y][u_2 + C_x]_y - f[v_2 + C_y] + \frac{1}{\rho}p_{2_x} = 0. \tag{2.57}$$

Exploiting the fact that velocity of the cold-pool with respect to the slope-water is assumed to be spatially invariant simplifies the above expression to

$$u_{2_t} + \frac{\partial C_x}{\partial t} + (-C_x\partial_x - C_y\partial_y)u_2 + [u_2 + C_x]u_{2_x} + [v_2 + C_y]u_{2_y}$$

$$-f[v_2 + C_y] + \frac{1}{\rho}p_{2_x} = 0. \tag{2.58}$$

Now cancelling terms from the round bracket with similar terms from the advective derivatives and doing a completely analogous procedure for the other two shallow water equations, translates the old equations into the following new system,

$$u_{2_t} + \frac{\partial C_x}{\partial t} + u_2u_{2_x} + v_2u_{2_y} - f(v_2 + C_y) = -\frac{1}{\rho}p_{2_x}, \tag{2.59}$$

$$v_{2_t} + \frac{\partial C_y}{\partial t} + u_2v_{2_x} + v_2v_{2_y} + f(u_2 + C_x) = -\frac{1}{\rho}p_{2_x}, \tag{2.60}$$

$$h_t + (hu_2)_x + (hv_2)_y = 0. \tag{2.61}$$

The cold-pool being hydrostatic together with the fact that, without loss of generality, the pressure at the interface between the two fluids in this reduced gravity model, can be set to zero, allows us to write the pressure as

$$p_2 = g\Delta\rho\,(h(x,y) - Sy - z)\,. \tag{2.62}$$

Rewriting the momentum equations (2.59) and (2.60) in terms of the reduced pressure and using the fact that since the eddy's shape does not change, u_2 and v_2 are time independent, we obtain

$$\frac{\partial C_x}{\partial t} + u_2 u_{2_x} + v_2 u_{2_y} - f\left(v_2 + C_y\right) = -g' h_x, \tag{2.63}$$

$$\frac{\partial C_y}{\partial t} + u_2 v_{2_x} + v_2 v_{2_y} + f\left(u_2 + C_x - \frac{Sg'}{f}\right) = -g' h_y, \tag{2.64}$$

where it has been used that the reduce gravity for this particular system is $g' = g\Delta\rho/\rho$.

Observe that u_2, v_2 and h are spatially dependent and C_x and C_y are temporally dependent. This allows each of the two equations above to be separated into time and space equations. The temporal equations are

$$\frac{\partial C_x}{\partial t} - fC_y = 0, \tag{2.65}$$

$$\frac{\partial C_y}{\partial t} + fC_x = Sg', \tag{2.66}$$

which is a system of coupled first order ordinary differential equations that can be solved given particular initial conditions.

The steady state solution obtained in Nof (1983), is that which stems from considering the initial conditions $C_x(0) = Sg'/f$ and $C_y(0) = 0$. The solution from Nof (1985), arises from considering the eddy beginning at rest, $C_x(0) = 0 = C_y(0)$. This produced a cycloid solution which oscillates up and down the slope as it moves in the negative x direction at varying speeds. Nof (1985) considered the energetics and stability in an attempt to find which was more physically relevant but concluded that both had equal merit.

There is a problem with Nof's oscillatory solution since it is an inertial oscillation. If we introduce a scaling for the frequency $T = 1/\omega$, (2.65) and (2.66) become,

$$\omega\frac{\partial C_x}{\partial t} - fC_y = 0, \tag{2.67}$$

$$\omega\frac{\partial C_y}{\partial t} + fC_x = Sg'. \tag{2.68}$$

Recall that mesoscale cold-pools are within a subinertial regime, which requires that the Coriolis frequency is much larger than the frequency of oscillation, $\omega \ll f$. Ergo, to leading order, the time derivative terms are not present, which produces essentially the same solutions found in Nof (1983),

$$C_y = 0, \tag{2.69}$$

$$C_x = \frac{Sg'}{f}, \tag{2.70}$$

where the cold-pool travels strictly along the shelf at the Nof speed.

It is remarkable to note that this velocity depends only on the slope, Coriolis term and reduced gravity and is independent of intensity, depth and shape. Nof (1983) also showed that this is the same result that is true for rigid bodies within the same environment. It is true that this situation is very much an idealization, however this gives an indiction as to what the order of magnitude of their speeds are. Hence, in the scalings to follow in the next section, the velocity of the lower layer will be scaled with the Nof speed.

2.5 Nondimensionalization

The governing equations and boundary conditions obtained for our model consist of equations (2.27), (2.28), (2.32) to (2.35), (2.36), (2.39) and (2.48). These equations as they stand are presumably intractable and hence need to be simplified in order that we can analyzed. One classical approach that has been used with much success is scale analysis (Pedlosky, 1987). For each variable that appears in the equations, we assume it is possible to pick a representative scale such that if we divide the variable by that scale, we will be left with an order one quantity. We then rewrite all the equations in terms of the scaled variables and divide through by the largest coefficient that appears in each equation which will eliminate the dimensions and introduce small parameters such as the Rossby number. The appearance of small parameters then allows asymptotic expansions to be used in order to approximate our system with one that is simpler.

The actual scales that one picks in nondimensionalizing a model are quite important. It was shown in Section 2.4 with the one and a half layer model that using different time scales gave qualitatively different motions. This is why it is extremely important to know the context in which our model is situated, for otherwise irrelevant conclusions could be made. Since this model is a modification of that found in Swaters & Flierl (1991), the same scales will be used with minor modifications to account for the stratification in the upper layer.

The horizontal length scale which is representative of both layers is denoted with L, whereas the vertical length scale, which is only representative of the upper layer is scaled by H,

$$(x^*, y^*) = L\,(x, y) \quad \text{and} \quad z^* = Hz. \tag{2.71}$$

The height scale for the lower layer, which is much smaller than H, is represented by h_0,

$$h^* = h_0 h, \tag{2.72}$$

and its ratio to the vertical length scale of the model is defined to be

$$\delta = \frac{h_0}{H}. \tag{2.73}$$

As is typical with large-scale motions, it is assumed that $H/L < 1$. This added to the fact that the parameter δ is small, allows us to conclude that the aspect ratio for the lower layer is small, which correlates with the observations stated in Section 2.1.

The horizontal length scale is chosen to be the internal radius of deformation of the upper layer, since this is necessary for the upper layer to be QG

$$L = \frac{\sqrt{g'H}}{f} \quad \Rightarrow \quad B_u = \frac{g'H}{(fL)^2} = 1. \tag{2.74}$$

The second equation above is the definition of the Burger number, which measures the relative importance of stratification versus rotation.

This length scale can be seen as the internal gravity wave speed divided by the inertial frequency. Traditionally, this speed is taken to be the product of the dimensional buoyancy frequency with the height scale. Here, we have chosen to replace the dimensional buoyancy frequency by its dimension, that being $\sqrt{g'/H}$, which when multiplied by H, gives the speed that we have used.

This length scale is much larger then the internal radius of deformation of the lower layer, yet drastically smaller than the circumference of the earth, and hence entitled an 'intermediate length scale' by Charney & Flierl (1981). It is most useful when the essential nonlinearity lies in the continuity equation rather than the momentum equations. This is true since the leading order effect of the momentum equations is geostrophy, a linear relation, whereas the continuity equation of the lower layer (2.28), has a divergence of mass flux present to leading order which is a nonlinear term.

If the Burger number is very small then rotation dominates which yields a two-dimensional system. If the Burger number is very large, that implies once again, that vertical motion is impeded but this time due to strong stratification. Both extremes yield essentially two-dimensional situations which cannot describe certain phenomena due to the lack of physical richness that is only present in a three dimensional system. The only way to capture this richness is to find a balance where rotation and stratification are both important. This occurs when $B_u = 1$, or the length scale is chosen to be the internal radius of deformation (Cushman-Roisin, 1994).

The representative slope parameter s^* for the bottom topography is scaled by the aspect ratio. This ensures that the topographic extent is small so that the shallow water equations are still applicable

$$s^* = \frac{Hs}{L} \quad \text{or} \quad s = \frac{s^* L}{H} = \frac{s^* g'}{f} \frac{fL}{g' H} = \frac{s^* g'}{f} \frac{1}{\sqrt{g' H}}. \tag{2.75}$$

having used (2.74). This form is insightful for it shows that the nondimensional slope parameter s, is a ratio of the Nof speed to the speed of long internal baroclinic gravity waves. Choosing this parameter to be small, which will be justified later, is equivalent to saying that we will consider only slow, subinertial motions in the model, that are generated by vorticity wave processes (Swaters & Flierl, 1991). This filters out the surface and internal gravity waves inherent with a continuously stratified fluid.

The bottom topography is chosen to be

$$h_B^* = s^* L h_B = sHh_B, \tag{2.76}$$

since s^* multiplied by the length scale L is the maximum height the bottom topography can achieve.

For the upper layer to be governed by QG dynamics, it is necessary that the amplitude of the isopycnal deflections are small in comparison to the actual height of the fluid, which requires that both h^* and h_B^* be small quantities compared to H. By considering (2.72) and (2.76), we find that the relations of these two quantities with the vertical height are

$$\frac{h^*}{H} = \delta h \quad \text{and} \quad \frac{h_B^*}{H} = sh_B, \tag{2.77}$$

since h and h_B are order one quantities by assumption. This then implies that δ and s need both be small for QG theory to hold. After the nondimensional equations are derived, experimental data will be considered which shows that in some physical situations, these are indeed small quantities.

The horizontal velocity field in the lower layer will be scaled with the Nof speed

$$\mathbf{u}_2^* = \frac{g' s^*}{f}\mathbf{u}_2 = \frac{g' Hs}{fL}\mathbf{u}_2 = fLs\mathbf{u}_2, \tag{2.78}$$

where (2.75) and (2.73) have been used. This method of simplifying the scaling parameters will be used frequently and the reader will not be continuously reminded of this due to the monotony that would arise from doing so. The Nof speed is chosen because, in the absence of any baroclinic interactions between the two layers, Nof predicted that this is the speed at which an isolated cold-pool would propagate. The reduced pressure field in the lower layer is scaled so that geostrophic balance can be maintained, that is,

$$p^* = (fL\rho_2)\frac{g' s^*}{f}p = (\rho_2 g' Hs)\, p. \tag{2.79}$$

We assume that the motions which arise in the upper layer are produced through interactions with the lower layer. For this reason, an advective time scaling will be implemented using the velocity of the lower layer,

$$t^* = \left(\frac{Lf}{g's^*}\right) t = \left(\frac{L^2 f}{g'Hs}\right) t = \frac{1}{fs} t. \tag{2.80}$$

The velocity scale for the upper layer is determined such that the baroclinic effects generated by vortex tube stretching are as important as relative vorticity. The appropriate relation to accomplish this will be derived in Section 2.6, but for now it is simply stated to be

$$(\nabla^* \times \mathbf{u}_1^*) = O\left(\frac{fh^*}{H}\right), \tag{2.81}$$

where the left and right-hand sides are the relative vorticity and baroclinic stretching scales, respectively. Therefore, the horizontal velocity scaling for layer one can be written, using (2.73), as

$$\mathbf{u}_1^* = \frac{fLh_0}{H}\mathbf{u}_1 = fL\delta\mathbf{u}_1. \tag{2.82}$$

To determine the scaling for the vertical velocity, the kinematic condition at $z^* = -H$ is considered. It requires that

$$w^* = O\left((\partial_{t^*} + \mathbf{u}_1^* \cdot \nabla)(h^* + h_B^*)\right). \tag{2.83}$$

For there to be strong coupling, the vertical velocity must be of the same order as the time rate of change of the lower layer $w^* = O\left(h_{t^*}^*\right)$, which reduces the above to

$$w^* = fsh_0 w = fs\delta H w, \tag{2.84}$$

using (2.73).

Next we chose the dynamic pressure field so that to leading order it can be in geostrophic balance with the velocity field in that layer,

$$\begin{aligned} p_1^* &= g\int_{z^*}^{0} \rho_0(\xi)\, d\xi + \rho_2\delta (Lf)^2 \varphi(x,y,z,t), \\ &= g\int_{z^*}^{0} \rho_0(\xi)\, d\xi + \rho_2\delta g' H\varphi(x,y,z,t). \end{aligned} \tag{2.85}$$

Also, the perturbation density field in the upper layer is chosen in order that it can be in hydrostatic balance with the dynamic pressure field,

$$\rho^* = \rho_0(z^*) + \left(\frac{\rho_2 \delta g'}{g}\right)\rho. \tag{2.86}$$

Given that the scalings have all been chosen, we proceed to obtain the nondimensional equations. Substituting the necessary scalings into (2.27), the momentum equation, in vector form governing the lower layer, becomes,

$$\left[fs\partial_t + \frac{fsL}{L}\boldsymbol{\nabla}\cdot\mathbf{u}_2\right](fLs)\,\mathbf{u}_2 + f^2Ls\;\hat{\mathbf{e}}_3\times\mathbf{u}_2 = -\frac{g'Hs}{L}\boldsymbol{\nabla}p_2. \tag{2.87}$$

Dividing this equation by f^2Ls and recognizing that $\boldsymbol{\nabla}p_2 = \boldsymbol{\nabla}p$, because of (2.41), gives rise to

$$s\left[\partial_t + \boldsymbol{\nabla}\cdot\mathbf{u}_2\right]\mathbf{u}_2 + \hat{\mathbf{e}}_3\times\mathbf{u}_2 = -\boldsymbol{\nabla}p, \tag{2.88}$$

where (2.74) was used to simplify the right-hand side. This demonstrates that to leading order geostrophic balance holds, and the total derivative term, since it is preceded by the small parameter s, is not influential in the leading order dynamics.

The nondimensional continuity equation for the lower layer, after substituting in the scales becomes

$$fsh_0h_t + fsh_0\boldsymbol{\nabla}\cdot[\mathbf{u}_2h] = 0, \tag{2.89}$$

and dividing by fsh_0,

$$h_t + \boldsymbol{\nabla}\cdot[\mathbf{u}_2h] = 0, \tag{2.90}$$

remains virtually unchanged since there are no parameters present. This reflects the fact that variations of h are on the same order as h itself and hence the reason why this fluid cannot be QG.

The upper layer horizontal momentum equations (2.32), after having substituted (2.85) and the appropriate scales, become

$$(fs\partial_t + f\delta\mathbf{u}_1\cdot\boldsymbol{\nabla} + fs\delta w\partial_z)(fL\delta)\,\mathbf{u}_1 + f^{\,2}L\delta\hat{\mathbf{e}}_3\times\mathbf{u}_1,$$

$$= -\frac{1}{L\rho_2}\nabla\left[g\int_{z^*}^{0}\rho_0(\xi)\,d\xi + \rho_2\delta g' H\varphi\right], \tag{2.91}$$

where the first term in the square bracket vanishes due to the horizontal gradient acting upon it and it being only a function of z^*. Then dividing by $f^2L\delta$, the equation simplifies to

$$s\left(\partial_t + \frac{\delta}{s}\mathbf{u}_1\cdot\nabla + \delta w\partial_z\right)\mathbf{u}_1 + \hat{\mathbf{e}}_3\times\mathbf{u}_1 = -\nabla\varphi, \tag{2.92}$$

after using (2.74).

Scaling the vertical momentum equation (2.33) yields

$$(fs\partial_t + f\delta\mathbf{u}_1\cdot\nabla + fs\delta w\partial_z)(fs\delta H)\,w$$

$$= -\frac{1}{\rho_2}\left[-g\rho_0 + \frac{\rho_2 h_0 g'}{H}\varphi_z\right] - \frac{g}{\rho_2}\left[\rho_0 + \frac{\rho_2\delta g'}{g}\rho\right], \tag{2.93}$$

having applied the z^* derivatives. Cancelling the two like terms on the right-hand side, using (2.73) and dividing by $g'\delta$ produces

$$\frac{(fs)^2 H}{g'}\left(\partial_t + \frac{\delta}{s}\mathbf{u}_1\cdot\nabla + \delta w\partial_z\right)w = -\varphi_z - \rho. \tag{2.94}$$

Bringing everything to one side and substituting in (2.74)

$$\left(\frac{sH}{L}\right)^2\left(\partial_t + \frac{\delta}{s}\mathbf{u}_1\cdot\nabla + \delta w\partial_z\right)w + \varphi_z + \rho = 0, \tag{2.95}$$

shows that the total derivative of the vertical velocity is negligible because the aspect ratio is no larger than one and s is much less than one. This is a result similar to that which was found in deriving the shallow water equations, which states that the scaled dynamic pressure and perturbation density are in hydrostatic balance.

The continuity equation for the upper layer (2.34) upon scaling becomes

$$f\delta\nabla\cdot\mathbf{u}_1 + fs\delta w_z = 0. \tag{2.96}$$

Multiplying by the reciprocal of $f\delta$ establishes

$$\nabla \cdot \mathbf{u}_1 + s w_z = 0, \tag{2.97}$$

this being the nondimensional version of the conservation of mass equation. It states that to leading order the horizontal velocity field is non-divergent, which we already know from geostrophy. However, the next order horizontal divergence can be non-zero given that it is balanced with a vertical velocity gradient which gives rise to vortex tube stretching or contraction. It is for this reason that the model is only nearly geostrophic. The fact that to leading order the continuity and geostrophic equations are not independent, is the source of what is known as geostrophic degeneracy and is what requires us to obtain a vorticity equation by considering the next order problem.

The last equation to be considered is the incompressibility equation (2.35) which, after having being scaled, takes the form

$$(fs\partial_t + f\delta \mathbf{u}_1 \cdot \nabla + fs\delta w \partial_z)\left[\rho_0 + \left(\frac{\rho_2 \delta g'}{g}\right)\rho\right] = 0. \tag{2.98}$$

Using the fact that the background density field is a function of z only produces, upon division by fs,

$$\left(\partial_t + \frac{\delta}{s}\mathbf{u}_1 \cdot \nabla + \delta w \partial_z\right)\left(\frac{\rho_2 \delta g'}{g}\right)\rho = -\delta w \frac{d\rho_0}{dz}. \tag{2.99}$$

Dividing by $\rho_2 \delta g'/g$ and separating terms gives

$$\left(\partial_t + \frac{\delta}{s}\mathbf{u}_1 \cdot \nabla\right)\rho + \delta w \partial_z \rho = -\frac{gw}{\rho_2 g'}\frac{d\rho_0}{dz} = -w\frac{H}{\rho_2 - \rho_0(-H)}\frac{d\rho_0}{dz^*}. \tag{2.100}$$

Also, we have recalled the definition of g' from (2.47), and multiplied the right-hand side by one, cleverly disguised as H/H.

Using the following definition for the nondimensional Brunt-Väissälä frequency, or buoyancy frequency,

$$N^2(z) = -\frac{H}{\rho_2 - \rho_0(-H)}\frac{d\rho_0}{dz^*}, \tag{2.101}$$

(2.100) then takes the form,

$$\left(\partial_t + \frac{\delta}{s}\mathbf{u}_1 \cdot \boldsymbol{\nabla}\right)\rho + \delta w \partial_z \rho = N^2(z)\, w. \tag{2.102}$$

Recalling the kinematic condition (2.36), and substituting the scales produces,

$$(f\delta s H)\, w = 0 \quad \text{at} \quad Hz = 0. \tag{2.103}$$

Dividing each equation above by the constants that precede each variable gives,

$$w = 0 \quad \text{at} \quad z = 0. \tag{2.104}$$

The other kinematic condition (2.39) becomes after substituting in the proper scales,

$$(fs\delta H)\, w = (fs\partial_t + f\delta \mathbf{u}_1 \cdot \boldsymbol{\nabla})(\delta H h + sHh_B)$$

$$+O\left(fs\delta H\,[\delta, s]\right) \quad \text{at} \quad Hz = -H, \tag{2.105}$$

where (2.76) has been used. Dividing this equation by $fs\delta H$ and the boundary term by H yields

$$w = (\partial_t + \frac{\delta}{s}\mathbf{u}_1 \cdot \boldsymbol{\nabla})(h + \frac{s}{\delta}h_B) + O(\delta, s) \quad \text{at} \quad z = -1, \tag{2.106}$$

where expanding the differential operator and noting that h_B is time invariant produces

$$w = h_t + \mathbf{u}_1 \cdot \boldsymbol{\nabla}(\frac{\delta}{s}h + h_B) + O(\delta, s) \quad \text{at} \quad z = -1. \tag{2.107}$$

Scaling the dynamic condition (2.48) gives

$$(\rho_2 g' Hs)\, p = (\rho_2 g' \delta H)\, \varphi + g' \rho_2 (\delta H h + sHh_B)$$

$$+O\left(\rho_2 g' \delta H\,[\delta, s]\right) \quad \text{at} \quad Hz = -H, \tag{2.108}$$

which upon dividing the equation by $\rho_2 g' H s$ and the boundary term by H, we get

$$p = \frac{\delta}{s}(\varphi + h) + h_B + O(\delta, s) \quad \text{at} \quad z = -1, \tag{2.109}$$

which completes the nondimensionalization of the model equations. As has been previously stated, for QG theory to govern the upper layer it is necessary that δ and s be small quantities. The question as to whether this has physical significance needs to be directed, for if it does not we cannot expect our model to have applicability.

In Houghton *et al.* (1982), experimental observations for cold-pools overlying gently sloping continental shelves are described where the physical scales are $s^* \approx 1.2$ m/km, $L \approx 15$ km, $H \approx 200-300$ m and $h_0 \approx 30-40$ m. In addition, this suggests an advective time scale, as in Swaters (1991), of $T = fL/(g's^*) \approx 7$ days. Substituting these numbers into (2.75) and (2.73) yield the physical values of $s \approx 4 \times 10^{-2}$ and $\delta \approx 2 \times 10^{-2}$ which are relatively small compared to one, and therefore

$$s \ll 1 \quad \text{and} \quad \delta \ll 1. \tag{2.110}$$

In an attempt to simplify the eight equations and three boundary conditions, the presence of small parameters suggests that asymptotic expansions should be considered. However doing expansions about two variables is rather cumbersome and hence it is desirable to first find a relationship between these two small quantities. By considering the experimental values for s and δ, we observe that they only differ by a factor of two. This means that they have the same order of magnitude which allows us to express one as an order one scalar multiple of the other. This is done through use of the parameter μ

$$\delta = \mu s \quad \text{where} \quad \mu = O(1). \tag{2.111}$$

To justify (2.45), we substitute the scaling for the reduced pressure, (2.77) and (2.111) into the right-hand side to get

$$O(\rho_2 g' H \delta [s, \delta]) = O(\rho_2 g' H \delta^2). \tag{2.112}$$

The left-hand side of (2.45) is rewritten using (2.101), (2.47) and the assumption that $N^2 = O(1)$, to yield precisely what is on the right-hand side of the above equation. Therefore, (2.45) is verified.

Substituting (2.111) into the nondimensional equations derived in this section (2.92), (2.95), (2.97), (2.102), (2.88), (2.90), (2.104), (2.107) and (2.109) gives

$$s(\partial_t + \mu \mathbf{u}_1 \cdot \nabla + \mu s w \partial_z) \mathbf{u}_1 + \hat{\mathbf{e}}_3 \times \mathbf{u}_1 = -\nabla \varphi, \tag{2.113}$$

$$\left(\frac{sH}{L}\right)^2 (\partial_t + \mu \mathbf{u}_1 \cdot \nabla + \mu s w \partial_z) w + \varphi_z + \rho = 0, \tag{2.114}$$

$$\nabla \cdot \mathbf{u}_1 + s w_z = 0, \tag{2.115}$$

$$(\partial_t + \mu \mathbf{u}_1 \cdot \nabla) \rho + \mu s w \rho_z = N^2(z) w, \tag{2.116}$$

$$s(\partial_t + \nabla \cdot \mathbf{u}_2) \mathbf{u}_2 + \hat{\mathbf{e}}_3 \times \mathbf{u}_2 = -\nabla p, \tag{2.117}$$

$$h_t + \nabla \cdot (\mathbf{u}_2 h) = 0, \tag{2.118}$$

and the vertical boundary conditions

$$w = 0 \quad \text{at} \quad z = 0, \tag{2.119}$$

$$w = h_t + \mathbf{u}_1 \cdot \nabla(\mu h + h_B) + O(s) \quad \text{at} \quad z = -1, \tag{2.120}$$

$$p = \mu(\varphi + h) + h_B + O(s) \quad \text{at} \quad z = -1. \tag{2.121}$$

Equation (2.81) stated that the vortex tube stretching term was proportional to fh_0/H which modulo f, is δ. This stretching is an essential component of baroclinicity and will act to destabilize the lower layer flow, whereas it has been shown (Pedlosky, 1987), that the addition of a sloping bottom into a model can act as a stabilizing effect. This gives a physical interpretation for μ, hence forth called the *interaction parameter* (Swaters, 1991), as being the ratio between the destabilizing effect of baroclinicity and the stabilizing effect of topography. This intuitively tells us that for small values of this parameter we should not expect any baroclinic instability, but for large values we should certainly expect instability. This idea will be expanded upon in Chapter 4.

2.6 Derivation of the governing equations

In previous sections, the big 'O' notation has been used to denote 'the order of or smaller than' in the context of scalings. It can also be used to denote two functions having the same order of magnitude. As well, a small 'o' is used to denote that one function has a smaller order of magnitude than another. More explicitly, considering the limit of the quotient of two functions $f(s)$ and $g(s)$ as s approaches zero,

$$\lim_{s \to 0} \frac{f(s)}{g(s)}, \tag{2.122}$$

we see that there are three possible results. The first is that this limit is equal to zero, which signifies that g is of a larger magnitude than f, near zero; we denote this by $f(s) = o(g(s))$ as $s \to 0$. Conversely, if the limit yields an infinite magnitude, be it positive or negative, it is clear that the magnitude of f is larger than that of g, therefore $g(s) = o(f(s))$ as $s \to 0$. The third possibility is that a finite number is achieved, which suggests that the magnitudes of f and g are similar, which is written as $f(s) = O(g(s))$ or $g(s) = O(f(s))$ as $s \to 0$. This does not imply that the functions are equal but it yields an equivalence in an asymptotic sense.

Given an analytic function that depends on a small parameter s, it is possible to write it as a sum of terms each of which has a smaller order of magnitude than all of the terms preceding it. For such a function $f(x;s)$, the expansion about $s = 0$ can be written as

$$f(x;s) = f_0(x) + sf_1(x) + s^2 f_2(x) + \ldots, \tag{2.123}$$

which is an exact equation and the following relationship holds,

$$\lim_{s \to 0} \frac{s^n f_n(x)}{s^m f_m(x)} = 0 \quad \text{if} \quad n < m. \tag{2.124}$$

When finding solutions to our nondimensional equations it is convenient to assume the form as in (2.123), since we hope that solutions to our problems will be analytic.

Hence for each variable, we will assume it can be expressed in this manner. Substituting these expressions into the nondimensional equations, leads to equations that are polynomials in s of infinite order where the coefficients depend on space and time. It is then necessary to equate the coefficients of each s^n for all n, and solve the system of equations that arise. In theory this method is valid. However, finding the exact expansion for all of our variables would be impossible or cumbersome to say the least. Hence we use the fact that s is a small parameter and instead of looking for an exact solution we search for an approximate solution.

If $s = 0$ the first order solution is an exact one, whereas if s is non-zero but small, this solution is accurate within order s. The first order problem by itself is then seen to be a good approximate solution which is what will be sought after. For more information on asymptotic expansions the reader is referred to Zauderer (1989).

This is sufficient motivation to assume solutions of the following form,

$$(\mathbf{u}_1, w, \mathbf{u}_2, \varphi, p, h) \simeq (\mathbf{u}_1, w, \mathbf{u}_2, \varphi, p, h)^{(0)} + s(\mathbf{u}_1, w, \mathbf{u}_2, \varphi, p, h)^{(1)} + o(s), \qquad (2.125)$$

where we have not used an equal sign but instead $\simeq$, to denote asymptotically equal to. The zero superscript denotes the first term in the expansion whereas the one denotes the second, that being a small correction term.

Substituting (2.125) into (2.113) through to (2.121) yields the following $O(1)$ problem

$$\hat{\mathbf{e}}_3 \times \mathbf{u}_1^{(0)} = -\nabla\varphi^{(0)}, \qquad (2.126)$$

$$\rho^{(0)} = -\varphi_z^{(0)}, \qquad (2.127)$$

$$\nabla \cdot \mathbf{u}_1^{(0)} = 0, \qquad (2.128)$$

$$\left(\partial_t + \mu\mathbf{u}_1^{(0)} \cdot \nabla\right)\rho^{(0)} = N^2 w^{(0)}, \qquad (2.129)$$

$$\hat{\mathbf{e}}_3 \times \mathbf{u}_2^{(0)} = -\nabla p^{(0)}, \qquad (2.130)$$

$$h_t^{(0)} + \nabla \cdot \left[\mathbf{u}_2^{(0)} h^{(0)}\right] = 0, \qquad (2.131)$$

and the vertical boundary conditions

$$w^{(0)} = 0 \quad \text{at} \quad z = 0, \tag{2.132}$$

$$w^{(0)} = h_t^{(0)} + \mathbf{u}_1^{(0)} \cdot \nabla(\mu h^{(0)} + h_B) \quad \text{at} \quad z = -1, \tag{2.133}$$

$$p^{(0)} = \mu\left(\varphi^{(0)} + h^{(0)}\right) + h_B \quad \text{at} \quad z = -1. \tag{2.134}$$

The lower layer equations shall be dealt with first since they are more straightforward. Acting with the vector operation $(\hat{\mathbf{e}}_3 \times)$ on (2.130), gives

$$\hat{\mathbf{e}}_3 \times \left(\hat{\mathbf{e}}_3 \times \mathbf{u}_2^{(0)}\right) = -\hat{\mathbf{e}}_3 \times \nabla p^{(0)},$$

$$\mathbf{u}_2^{(0)} = \hat{\mathbf{e}}_3 \times \nabla p^{(0)}, \tag{2.135}$$

having used the vector identity $\hat{\mathbf{e}}_3 \times (\hat{\mathbf{e}}_3 \times \mathbf{a}) = -\mathbf{a}$. Substituting this into (2.131), the continuity equation for the lower layer becomes

$$h_t^{(0)} + \nabla \cdot \left[h^{(0)}\hat{\mathbf{e}}_3 \times \nabla p^{(0)}\right] = 0. \tag{2.136}$$

Introducing the Jacobian function,

$$\partial\,(a,b) = \left|\frac{\partial\,(a,b)}{\partial(x,y)}\right| = a_x b_y - a_y b_x, \tag{2.137}$$

results in the following identity,

$$\partial\,(a,b) = \nabla \cdot [b\hat{\mathbf{e}}_3 \times \nabla a] = \hat{\mathbf{e}}_3 \times \nabla a \cdot \nabla b. \tag{2.138}$$

The first identity above enables (2.136) to be rewritten as,

$$h_t^{(0)} + \partial\left(p^{(0)}, h^{(0)}\right) = 0. \tag{2.139}$$

Introducing the dynamic condition (2.134), results in the above equation being written in terms of the reduced pressure field in the upper layer,

$$h_t^{(0)} + \mu\partial\left(\varphi^{(0)} + h^{(0)} + \frac{h_B}{\mu}, h^{(0)}\right) = 0 \quad \text{at} \quad z = -1, \tag{2.140}$$

which simplifies to

$$\left[\partial_t + \mu\partial\left(\varphi^{(0)} + \frac{h_B}{\mu}, *\right)\right] h^{(0)} = 0 \quad \text{at} \quad z = -1, \tag{2.141}$$

because the Jacobian is antisymmetric.

Applying the same operator to (2.126), as in (2.135), gives the analogous result

$$\mathbf{u}_1^{(0)} = \hat{\mathbf{e}}_3 \times \nabla\varphi^{(0)}. \tag{2.142}$$

To get an equation for the vertical velocity in terms of the dynamic pressure in the upper layer requires multiplying (2.129) by N^{-2}, then substituting the hydrostatic relation (2.127), to obtain

$$w^{(0)} = -N^{-2}\left(\partial_t + \mu\mathbf{u}_1^{(0)} \cdot \nabla\right)\varphi_z^{(0)}. \tag{2.143}$$

Implementing the geostrophic relationship (2.142) and distributing the differential operator gives

$$w^{(0)} = -N^{-2}\left(\varphi_{zt}^{(0)} + \mu\left(\hat{\mathbf{e}}_3 \times \nabla\varphi^{(0)}\right) \cdot \nabla\varphi_z^{(0)}\right), \tag{2.144}$$

which upon using the second identity in (2.138) brings about

$$w^{(0)} = -N^{-2}\left(\varphi_{zt}^{(0)} + \mu\partial\left(\varphi^{(0)}, \varphi_z^{(0)}\right)\right). \tag{2.145}$$

In the above work, the following identity has been proven,

$$\left(\mathbf{u}_1^{(0)} \cdot \nabla\right) a = \partial\left(\varphi^{(0)}, a\right). \tag{2.146}$$

This identity will be used frequently when such a geostrophic velocity field is given.

It is now possible to translate the boundary conditions in terms of the two variables φ and h. We substitute (2.145) into (2.132) and multiply by the buoyancy frequency, to obtain

$$\left[\partial_t + \mu\partial\left(\varphi^{(0)}, *\right)\right]\varphi_z^{(0)} = 0 \quad \text{at} \quad z = 0, \tag{2.147}$$

which gives the condition in conservation form. Substituting (2.145) evaluated at $z = -1$, into the kinematic condition (2.133), gives

$$h_t^{(0)} + \mathbf{u}_1^{(0)} \cdot \nabla(\mu h^{(0)} + h_B) + N^{-2}\left(\varphi_{zt}^{(0)} + \mu\partial\left(\varphi^{(0)}, \varphi_z^{(0)}\right)\right) = 0 \quad \text{at} \quad z = -1. \tag{2.148}$$

Rewriting the second term using (2.146), reduces the above equation, upon multiplication by N^2 and joining terms, to

$$\left[\partial_t + \mu\partial\left(\varphi^{(0)}, *\right)\right]\left(\varphi_z^{(0)} + N^2\left(h^{(0)} + \frac{h_B}{\mu}\right)\right) = 0 \quad \text{at} \quad z = -1, \tag{2.149}$$

where the fact that the bottom topography term is time invariant has been used.

The equation that determines the evolution of the dynamic pressure of the upper layer was saved for last due to an added complication. This being that the continuity equation and geostrophic equations are not independent at leading order. Geostrophy implies that the horizontal divergence of velocity is zero and hence geostrophic degeneracy arises, as mentioned in Section 2.5. This is dealt with by considering the next order problem, which upon taking the curl, allows us to find the Potential Vorticity (PV) equation.

The horizontal momentum and continuity equations in the upper layer for the $O(s)$ problem are,

$$\left(\partial_t + \mu\mathbf{u}_1^{(0)} \cdot \nabla\right)\mathbf{u}_1^{(0)} + \hat{\mathbf{e}}_3 \times \mathbf{u}_1^{(1)} = -\nabla\varphi^{(1)}, \tag{2.150}$$

$$\nabla \cdot \mathbf{u}_1^{(1)} + w_z^{(0)} = 0. \tag{2.151}$$

Taking the horizontal curl of the horizontal momentum equation, gives a vector in the z direction. The equation takes the following form

$$\left(\nabla \times \mathbf{u}_1^{(0)}\right)_t + \mu\nabla\times\left(\mathbf{u}_1^{(0)} \cdot \nabla\mathbf{u}_1^{(0)}\right) + \nabla\times\left(\hat{\mathbf{e}}_3 \times \mathbf{u}_1^{(1)}\right) = -\nabla \times \nabla\varphi^{(1)}. \tag{2.152}$$

The curl of the gradient of any function is zero, which then eliminates the pressure term from the above equation. Substituting (2.142) into the first term above and using the identity

$$\nabla\times(\mathbf{a}\times\mathbf{b}) = (\nabla\times\mathbf{b})\,\mathbf{a} - (\nabla\times\mathbf{a})\,\mathbf{b}, \tag{2.153}$$

yields

$$\begin{aligned}\left(\nabla\times\mathbf{u}_1^{(0)}\right)_t &= \left(\nabla\times\left(\hat{\mathbf{e}}_3\times\nabla\varphi^{(0)}\right)\right)_t \\ &= \left(\Delta\varphi_t^{(0)}\right)\hat{\mathbf{e}}_3.\end{aligned} \tag{2.154}$$

Applying the curl operator to the advective term in (2.152) and simplifying using (2.153), results in

$$\begin{aligned}\nabla\times\left(\hat{\mathbf{e}}_3\times\mathbf{u}_1^{(1)}\right) &= \left(\nabla\cdot\mathbf{u}_1^{(1)}\right)\hat{\mathbf{e}}_3 - (\nabla\cdot\hat{\mathbf{e}}_3)\,\mathbf{u}_1^{(1)} \\ &= \left(\nabla\cdot\mathbf{u}_1^{(1)}\right)\hat{\mathbf{e}}_3,\end{aligned} \tag{2.155}$$

since $\hat{\mathbf{e}}_3$ is constant.

The second order divergence term can be rewritten using the $O(s)$ continuity equation (2.151),

$$\begin{aligned}\left(\nabla\cdot\mathbf{u}_1^{(1)}\right) &= -w_z^{(0)} \\ &= \left(N^{-2}\left(\varphi_{zt}^{(0)} + \mu\partial\left(\varphi^{(0)},\varphi_z^{(0)}\right)\right)\right)_z \\ &= \left(N^{-2}\varphi_{zt}^{(0)}\right)_z + \mu\left(\partial\left(\varphi^{(0)},N^{-2}\varphi_z^{(0)}\right)\right)_z \\ &= \left(N^{-2}\varphi_{zt}^{(0)}\right)_z + \mu\partial\left(\varphi^{(0)},\left(N^{-2}\varphi_z^{(0)}\right)_z\right) \\ &= \left[\partial_t + \mu\partial\left(\varphi^{(0)},*\right)\right]\left(N^{-2}\varphi_{zt}^{(0)}\right)_z,\end{aligned} \tag{2.156}$$

by substituting in (2.145), expanding terms and using the antisymmetric property of the Jacobian. Using a vector identity translates the second term in (2.152) to

$$\begin{aligned}\nabla\times\left(\mathbf{u}_1^{(0)}\cdot\nabla\mathbf{u}_1^{(0)}\right) &= \mathbf{u}_1^{(0)}\cdot\nabla\left(\nabla\times\mathbf{u}_1^{(0)}\right) + \nabla\cdot\mathbf{u}_1^{(0)}\left(\nabla\times\mathbf{u}_1^{(0)}\right) \\ &= \mathbf{u}_1^{(0)}\cdot\nabla\left(\nabla\times\mathbf{u}_1^{(0)}\right) \\ &= \mathbf{u}_1^{(0)}\cdot\nabla\left(\nabla\times\left(\hat{\mathbf{e}}_3\times\nabla\varphi^{(0)}\right)\right)\end{aligned}$$

$$
\begin{aligned}
&= \mathbf{u}_1^{(0)} \cdot \nabla \left(\Delta\varphi^{(0)}\right) \hat{\mathbf{e}}_3 \\
&= \partial \left(\varphi^{(0)}, \Delta\varphi^{(0)}\right) \hat{\mathbf{e}}_3, \qquad (2.157)
\end{aligned}
$$

having used (2.128) in the second line, (2.142) in the third, the definition $\Delta = (\partial_{xx} + \partial_{yy})$, (2.153) and (2.146).

Substituting (2.154), (2.155), (2.156) and (2.157) into (2.152) produces, upon taking the projection in the z direction, yields

$$
\left[\nabla^2\varphi^{(0)} + \left(N^{-2}\varphi_z^{(0)}\right)_z\right]_t + \mu\partial\left(\varphi^{(0)}, \left[\Delta\varphi^{(0)} + N^{-2}\varphi_z^{(0)}\right]_z\right) = 0,
$$

$$
\left[\partial_t + \mu\partial\left(\varphi^{(0)}, *\right)\right]\left(\Delta\varphi^{(0)} + \left(\varphi_z^{(0)}/N^2\right)_z\right) = 0. \qquad (2.158)
$$

The first term in the conserved quantity is the relative vorticity and the second is a baroclinic stretching term since it stems from the divergence of the second order horizontal velocity field. The latter is ageostrophic because it does not satisfy the geostrophic equations. These two terms are of the same order due to how we have scaled the horizontal velocity field in the upper layer.

In order to determine why this scaling was chosen the way it was, observe that the baroclinic stretching term came from the curl of the Coriolis term and the relative vorticity originated from the curl of the material derivative. Taking the dimensional curl of the dimensional horizontal momentum equation (2.32) and using vector identities, shows that for the relative vorticity and baroclinic stretching terms to be of the same order, the following relation must hold

$$
\nabla^* \times \left[\left(\partial_{t^*} + \mathbf{u}_1^* \cdot \nabla^*\right)\mathbf{u}_1^*\right] = O\left(f\left(\nabla^* \cdot \mathbf{u}_1^*\right)\right). \qquad (2.159)
$$

We choose the time derivative to be representative of the first square bracket and replace the horizontal divergence with the vertical gradient of the vertical velocity, by use of the continuity equation. The scaling for the vertical velocity, as suggested by the dynamic condition,

$$
w^* = O\left(h_{t^*}^*\right) \quad \Longrightarrow \quad w_{z^*}^* = O\left(h_{t^*}^*/H\right), \qquad (2.160)
$$

gives

$$(\nabla^* \times \mathbf{u}_1^*) = O\left(\frac{fh_0}{H}\right), \tag{2.161}$$

after having eliminated the time scalings from both sides. This justifies that (2.81) is the right choice to make for the relative vorticity and baroclinic stretching terms of the same order as is required in QG dynamics.

The four governing equations for the leading order problem, are (2.141) and (2.158) with boundary conditions (2.147) and (2.149). Observe that they are written completely in terms of the dynamic pressure in the upper layer and the height of the lower layer. This leading order problem is more manageable since before, there were eight equations and three boundary conditions, whereas now we have only four equations in total. Given that a solution could be found for this system the other variables could be recovered through (2.127), (2.134), (2.135), (2.142) and (2.145).

The governing equation that determines the evolution of the dynamic pressure and the rigid-lid boundary condition are exactly the same as in QG theory (Pedlosky, 1987), hence this is a QG fluid. However, the boundary condition for the bottom of a QG fluid is replaced by a pair of partial differential equations that couple the evolution of $\varphi_z^{(0)}$ at $z = -1$, with the height of the lower layer.

Since this system of partial differential equations are the equations to be dealt with, there will be a change of notation made, $\varphi^{(0)} \to \varphi$ and $h^{(0)} \to h$,

$$[\partial_t + \mu\partial(\varphi, *)]\left(\Delta\varphi + \left(\varphi_z/N^2\right)_z\right) = 0, \tag{2.162}$$

$$[\partial_t + \mu\partial(\varphi, *)]\,\varphi_z = 0 \quad \text{at} \quad z = 0, \tag{2.163}$$

$$[\partial_t + \mu\partial(\varphi, *)]\left(\varphi_z + N^2\left(h + \frac{h_B}{\mu}\right)\right) = 0 \quad \text{at} \quad z = -1, \tag{2.164}$$

$$\left[\partial_t + \mu\partial\left(\varphi + \frac{h_B}{\mu}, *\right)\right] h = 0 \quad \text{at} \quad z = -1. \tag{2.165}$$

2.7 Potential Vorticity derivation

Having obtained equations (2.162) through to (2.165), the reader could now proceed directly to Chapter 3 without any loss of information. However, it is insightful to derive these equations from a different approach, one not involving asymptotic expansions, but instead that focuses on the conservation of PV.

The Potential Vorticity of an arbitrary fluid is given by,

$$\Pi^* = \frac{(\boldsymbol{\omega}^* + f\hat{\mathbf{e}}_3)}{\rho^*} \cdot \boldsymbol{\nabla}_3^* \lambda, \tag{2.166}$$

which is conserved for each fluid parcel following the motion,

$$\frac{d\Pi^*}{dt^*} = (\partial_{t^*} + u^* \partial_{x^*} + v^* \partial_{y^*} + w^* \partial_{y^*})\, \Pi^* = 0. \tag{2.167}$$

The notation that has been used is that the velocity field is $\mathbf{u}^* = (\, u^*, v^*, w^*)$, $\boldsymbol{\nabla}_3^* = (\partial_{x^*}, \partial_{y^*}, \partial_{z^*})$, $\boldsymbol{\omega}^* = \boldsymbol{\nabla}_3^* \times \mathbf{u}^*$ is the relative vorticity vector and f is the ambient vorticity of the system directed along the z axis and d/dt^* denotes the dimensional material derivative. It is necessary that the fluid is frictionless and that λ satisfies certain properties, the first being that it is a conserved quantity following the motion, i.e. $d\lambda/dt = 0$. If the fluid is barotropic $\boldsymbol{\nabla}\rho \times \boldsymbol{\nabla}p = 0$, this one condition is sufficient. However if the fluid is baroclinic, then it is necessary that λ be strictly a function of pressure and density. This result, known as Ertel's theorem, can be applied to both homogeneous and continuously stratified fluids, however λ must be chosen differently for each (Pedlosky, 1987).

For a frictionless, homogeneous fluid, hence barotropic, λ is picked to be the relative height of the fluid parcel with respect to the total height of the column in which it lies,

$$\lambda = \frac{z^* - h_B^*(x,y)}{h^*(x,y,t)}, \tag{2.168}$$

where the notation is conventional to what has already been defined. To prove this is a conserved quantity, consider (2.23). Realizing that w is nothing more than dz/dt, and substituting in for the horizontal divergence using (2.26), then grouping terms yields,

$$\frac{d(z-h_B)}{dt} - (z-h_B)\frac{1}{h}\frac{dh}{dt} = 0,$$

$$\frac{d}{dt}\left(\frac{z-h_B}{h}\right) = 0, \tag{2.169}$$

where upon introducing dimensional quantities, the above claim is proven.

Since the only component of PV that is non-zero is the z component, (2.166) takes the form

$$\begin{aligned}
\Pi^* &= \frac{([\boldsymbol{\omega}^*]_z + f)}{\rho^*}\frac{\partial}{\partial z^*}\left(\frac{z^*-h_B^*}{h^*}\right) \\
&= \frac{\left(v^*_{x^*} - u^*_{y^*} + f\right)}{\rho^* h^*}\frac{\partial}{\partial z^*}(z^*) \\
&= \frac{v^*_{x^*} - u^*_{y^*} + f}{\rho^* h^*},
\end{aligned} \tag{2.170}$$

where the $[\boldsymbol{\omega}^*]_z$ denotes the z component of vorticity which is $v^*_{x^*} - u^*_{y^*}$. Substituting this into (2.167) yields

$$(\partial_{t^*} + u^*\partial_{x^*} + v^*\partial_{y^*})\left[\frac{v^*_{x^*} - u^*_{y^*} + f}{h^*}\right] = 0, \tag{2.171}$$

after multiplying by the density and observing that (2.170) is depth independent.

For a three-dimensional fluid, λ is chosen to be the density. This can be done since incompressibility dictates that density is a conserved quantity following the motion, and trivially λ is only a function of density. Hence, substituting (2.166) into (2.167), yields,

$$\frac{d}{dt^*}\left[\frac{(\boldsymbol{\omega}^* + f\hat{\mathbf{e}}_3)}{\rho^*}\cdot\nabla_3^*\rho\right] = 0. \tag{2.172}$$

Applying the quotient rule reduces this equation to

$$\frac{1}{\rho^*}\frac{d}{dt^*}\left[(\boldsymbol{\omega}^* + f\hat{\mathbf{e}}_3)\cdot\nabla_3\rho\right] - \frac{(\boldsymbol{\omega}^* + f\hat{\mathbf{e}}_3)\cdot\nabla_3\rho}{(\rho^*)^2}\frac{d\rho^*}{dt^*} = 0, \tag{2.173}$$

which then becomes, by using the conservation of density following the motion (2.35),

$$\frac{d}{dt^*}\left[(\omega^* + f)\cdot\nabla_3\rho\right] = 0. \tag{2.174}$$

This section will first derive (2.171) and (2.174) that resulted from Ertel's theorem by taking the curl of the momentum equations and then introduce the scalings stated in Section 2.5. The appearance of small parameters allows us to Taylor expand to obtain the leading order behavior. The lower layer will be dealt with first since its representative equations are easier to handle.

For future reference, we rewrite the continuity equation (2.28), by solving for the divergence,

$$h^*_{t^*} + (\nabla^*\cdot\mathbf{u}^*_2)\,h^* + \mathbf{u}^*_2\cdot\nabla^* h^* = 0,$$

$$\nabla^*\cdot\mathbf{u}^*_2 = -\frac{1}{h^*}\left(\partial_{t^*} + \mathbf{u}^*_2\cdot\nabla^*\right)h^*, \tag{2.175}$$

which when divided by h^* gives,

$$\begin{aligned}\frac{1}{h^*}\nabla^*\cdot\mathbf{u}^*_2 &= -\left(\frac{1}{h^*}\right)^2\left(\partial_{t^*} + \mathbf{u}^*_2\cdot\nabla^*\right)h^* \\ &= \left(\partial_{t^*} + \mathbf{u}^*_2\cdot\nabla^*\right)\left(\frac{1}{h^*}\right).\end{aligned} \tag{2.176}$$

Applying the horizontal curl operator to the horizontal momentum equation (2.27), gives rise to

$$\nabla^*\times\left[\mathbf{u}^*_{2_{t^*}} + (\mathbf{u}^*_2\cdot\nabla^*)\,\mathbf{u}^*_2 + f\,\hat{\mathbf{e}}_3\times\mathbf{u}^*_2 = -\frac{1}{\rho_2}\nabla^* p^*_2\right], \tag{2.177}$$

which can be simplified term by term. Firstly, the curl of the gradient of any function is zero which implies that the pressure term vanishes. The Coriolis term becomes after using (2.153),

$$f\nabla^*\times(\hat{\mathbf{e}}_3\times\mathbf{u}^*_2) = f\hat{\mathbf{e}}_3\,(\nabla^*\cdot\mathbf{u}^*_2). \tag{2.178}$$

The first term on the left-hand side of (2.177), can be written as

$$\begin{aligned}\nabla^*\times\mathbf{u}^*_{2_{t^*}} &= (\nabla^*\times\mathbf{u}^*_2)_{t^*} \\ &= (\nabla^*\times\mathbf{u}^*_2 + f\hat{\mathbf{e}}_3)_{t^*},\end{aligned} \tag{2.179}$$

since the curl and time derivative operators commute and f is a constant. It is understood that since ∇^* and $\mathbf{u}_2^*$ are two dimensional vectors their curl is in the z direction.

Applying the identity proven in (2.157), allows the curl of the advective term to be expressed as

$$\nabla^* \times ((\mathbf{u}_2^* \cdot \nabla^*)\, \mathbf{u}_2^*) = (\mathbf{u}_2^* \cdot \nabla^*)\, (\nabla^* \times \mathbf{u}_2^* + f\hat{\mathbf{e}}_3) + (\nabla^* \cdot \mathbf{u}_2^*)\, (\nabla^* \times \mathbf{u}_2^*)\,, \tag{2.180}$$

since f is constant. Substituting (2.178), (2.179) and (2.180) into (2.177) gives,

$$(\partial_{t^*} + \mathbf{u}_2^* \cdot \nabla^*)\, (\nabla^* \times \mathbf{u}_2^* + f\hat{\mathbf{e}}_3) + (\nabla^* \cdot \mathbf{u}_2^*)\, (\nabla^* \times \mathbf{u}_2^* + f\hat{\mathbf{e}}_3) = 0. \tag{2.181}$$

Multiplying this equation by $1/h^*$ and substituting (2.176) results in

$$\frac{1}{h^*}\, (\partial_{t^*} + \mathbf{u}_2^* \cdot \nabla^*)\, (\nabla^* \times \mathbf{u}_2^* + f\hat{\mathbf{e}}_3) + (\nabla^* \times \mathbf{u}_2^* + f\hat{\mathbf{e}}_3)\, (\partial_{t^*} + \mathbf{u}_2^* \cdot \nabla^*) \left(\frac{1}{h^*}\right) = 0,$$

$$(\partial_{t^*} + \mathbf{u}_2^* \cdot \nabla^*) \left[\frac{\nabla^* \times \mathbf{u}_2^* + f\hat{\mathbf{e}}_3}{h^*}\right] = 0, \tag{2.182}$$

through the usage of the product rule. This proves that the PV equation for homogeneous fluids derived from Ertel's theorem is valid.

Substituting in the scalings from Section 2.5 in (2.182), gives

$$\begin{aligned} s\, (\partial_t + \mu \mathbf{u}_2 \cdot \nabla) \left[\frac{(fs)\, \nabla \times \mathbf{u}_2 + f\hat{\mathbf{e}}_3}{h_0 h}\right] &= 0, \\ (\partial_t + \mu \mathbf{u}_2 \cdot \nabla) \left[\frac{s\nabla \times \mathbf{u}_2 + \hat{\mathbf{e}}_3}{h}\right] &= 0, \end{aligned} \tag{2.183}$$

upon dividing through by fs/h_0. Since s is a small parameter, the relative vorticity will not be a leading order effect. It is this property, coupled with the fact that the lower layer momentum equation is geostrophic to leading order, that makes this fluid PG. The leading order equation then takes the form

$$(\partial_t + \mu \mathbf{u}_2 \cdot \nabla) \left[\frac{1}{h}\right] = 0. \tag{2.184}$$

Applying the quotient rule, then multiplying this equation by $-h^2$, and rewriting the advective derivative in terms of the Jacobian, as in (2.146), produces

$$(\partial_t + \mu \mathbf{u}_2 \cdot \boldsymbol{\nabla})\,[h] = 0,$$

$$\left[\partial_t + \mu \partial\left(\varphi + \frac{h_B}{\mu}, *\right)\right] h = 0, \tag{2.185}$$

having taken the leading order contribution in the dynamic condition (2.121), which yields precisely the same equation as (2.141).

The calculations for the upper layer are more complicated because they involve taking the three-dimensional curl, $(\boldsymbol{\nabla}_3^* \times)$ of the three-dimensional momentum equation. The momentum equations (2.32) and (2.33) can be compactly written as

$$(\partial_{t^*} + \mathbf{u}^* \cdot \boldsymbol{\nabla}_3^*)\,\mathbf{u}^* + f\,\hat{\mathbf{e}}_3 \times \mathbf{u}^* = -\frac{1}{\rho_2}\boldsymbol{\nabla}_3^* p_1^* - \frac{g\rho^*}{\rho_2}\hat{\mathbf{e}}_3, \tag{2.186}$$

where $\mathbf{u}^* = (\mathbf{u}_1^*, w^*)$ and the conservation of mass and energy equations are,

$$\boldsymbol{\nabla}_3^* \cdot \mathbf{u}^* = 0, \tag{2.187}$$

$$(\partial_{t^*} + \mathbf{u}^* \cdot \boldsymbol{\nabla}_3^*)\,\rho^* = 0. \tag{2.188}$$

By considering the expression

$$(\boldsymbol{\nabla}_3^* \times \mathbf{u}^*) \times \mathbf{u}^* + \frac{1}{2}\boldsymbol{\nabla}_3^*(\mathbf{u}^* \cdot \mathbf{u}^*)$$

$$= \begin{pmatrix} w^*(u^*_{z^*} - w^*_{x^*}) - v^*\left(v^*_{x^*} - u^*_{y^*}\right) \\ u^*\left(v^*_{x^*} - u^*_{y^*}\right) - w^*\left(w^*_{y^*} - v^*_{z^*}\right) \\ w^*(u^*_{z^*} - w^*_{x^*}) - v^*\left(v^*_{x^*} - u^*_{y^*}\right) \end{pmatrix} + \begin{pmatrix} u^*u^*_{x^*} + v^*v^*_{x^*} + w^*w^*_{x^*} \\ u^*u^*_{y^*} + v^*v^*_{y^*} + w^*w^*_{y^*} \\ u^*u^*_{z^*} + v^*v^*_{z^*} + w^*w^*_{z^*} \end{pmatrix}$$

$$= \begin{pmatrix} u^*u^*_{x^*} + v^*u^*_{y^*} + w^*u^*_{z^*} \\ u^*v^*_{x^*} + v^*v^*_{y^*} + w^*v^*_{z^*} \\ u^*w^*_{x^*} + v^*w^*_{y^*} + w^*u^*_{z^*} \end{pmatrix}$$

$$= (\mathbf{u}^* \cdot \boldsymbol{\nabla}_3^*)\,\mathbf{u}^*, \tag{2.189}$$

we obtain a vector identity that allows (2.186) to become

$$\mathbf{u}^*_{t^*} + (\nabla^*_3 \times \mathbf{u}^* + f\,\hat{\mathbf{e}}_3) \times \mathbf{u}^* = -\nabla^*_3 \left(\frac{\mathbf{u}^* \cdot \mathbf{u}^*}{2} + \frac{p^*_1}{\rho_2} \right) - \frac{g\rho^*}{\rho_2} \hat{\mathbf{e}}_3. \tag{2.190}$$

Upon taking the three-dimensional curl of (2.190), the buoyancy term becomes

$$\begin{aligned} \nabla^*_3 \times \left(-\frac{g\rho^*}{\rho_2} \hat{\mathbf{e}}_3 \right) &= \frac{g}{\rho_2} \left(-\rho^*_{y^*}, \rho^*_{x^*}, 0 \right) \\ &= \frac{g}{\rho_2}\, \hat{\mathbf{e}}_3 \times \nabla^*_3 \rho^*, \end{aligned} \tag{2.191}$$

and the other term on the right-hand side vanishes

$$\nabla^*_3 \times \nabla^*_3 \left(\frac{\mathbf{u}^* \cdot \mathbf{u}^*}{2} + \frac{p^*_1}{\rho_2} \right) = 0, \tag{2.192}$$

since the curl of the gradient of any function is equal to zero. The first term in (2.190) can be written as

$$\begin{aligned} \nabla^*_3 \times \mathbf{u}^*_{t^*} &= (\nabla^*_3 \times \mathbf{u}^*)_{t^*} \\ &= (\nabla^*_3 \times \mathbf{u}^* + f\hat{\mathbf{e}}_3)_{t^*}, \end{aligned} \tag{2.193}$$

since f is constant.

Upon using the following vector identity (Pedlosky, 1987)

$$\nabla^*_3 \times (\mathbf{a} \times \mathbf{b}) = \mathbf{a}\,(\nabla^*_3 \cdot \mathbf{b}) + (\mathbf{b} \cdot \nabla^*_3)\,\mathbf{a} - \mathbf{b}\,(\nabla^*_3 \cdot \mathbf{a}) - (\mathbf{a} \cdot \nabla^*_3)\,\mathbf{b}, \tag{2.194}$$

the final term then takes the form,

$$\nabla^*_3 \times [(\nabla^*_3 \times \mathbf{u}^* + f\hat{\mathbf{e}}_3) \times \mathbf{u}^*] =$$

$$\begin{aligned} &(\nabla^*_3 \times \mathbf{u}^* + f\hat{\mathbf{e}}_3)\,(\nabla^*_3 \cdot \mathbf{u}^*) + (\mathbf{u}^* \cdot \nabla^*_3)\,(\nabla^*_3 \times \mathbf{u}^* + f\hat{\mathbf{e}}_3) \\ &-\mathbf{u}^*\,(\nabla^*_3 \cdot (\nabla^*_3 \times \mathbf{u}^* + f\hat{\mathbf{e}}_3)) - [(\nabla^*_3 \times \mathbf{u}^* + f\hat{\mathbf{e}}_3) \cdot \nabla^*_3]\,\mathbf{u}^* \end{aligned} \tag{2.195}$$

The third term on the right-hand side vanishes since $f\hat{\mathbf{e}}_3$ is a constant and the divergence of the curl of any vector is equal to zero. The conservation of mass (2.187), causes the first term on the right-hand side to disappear. Equation (2.195) reduces to

$$\nabla_3^* \times [(\nabla_3^* \times \mathbf{u}^* + f\hat{\mathbf{e}}_3) \times \mathbf{u}^*] =$$

$$(\mathbf{u}^* \cdot \nabla_3)(\nabla_3^* \times \mathbf{u}^* + f\hat{\mathbf{e}}_3) - [(\nabla_3^* \times \mathbf{u}^* + f\hat{\mathbf{e}}_3) \cdot \nabla_3^*]\,\mathbf{u}^*. \tag{2.196}$$

The curl of (2.190) then becomes upon using (2.191), (2.192), (2.193) and (2.196),

$$[\partial_{t^*} + \mathbf{u}^* \cdot \nabla_3](\nabla_3^* \times \mathbf{u}^* + f\hat{\mathbf{e}}_3) - [(\nabla_3^* \times \mathbf{u}^* + f\hat{\mathbf{e}}_3) \cdot \nabla_3^*]\,\mathbf{u}^*$$

$$= \frac{g}{\rho_2}\,\hat{\mathbf{e}}_3 \times \nabla_3^* \rho^*. \tag{2.197}$$

Taking the dot product of (2.197) with $\nabla_3^* \rho^*$ contains there terms, one of which is

$$\frac{g}{\rho_2} \nabla_3^* \rho^* \cdot (\hat{\mathbf{e}}_3 \times \nabla_3^* \rho^*) = \frac{g}{\rho_2}\left(-\rho_{y^*}^* \rho_{x^*}^* + \rho_{y^*}^* \rho_{x^*}^*\right) = 0. \tag{2.198}$$

Another term is

$$\nabla_3^* \rho^* \cdot [\partial_{t^*} + \mathbf{u}^* \cdot \nabla_3^*](\nabla_3^* \times \mathbf{u}^* + f\hat{\mathbf{e}}_3). \tag{2.199}$$

In order to rewrite the third term it is necessary to act the gradient operator on (2.188), utilize the product rule, then rearrange terms to obtain the following identity,

$$(\partial_{t^*} + \mathbf{u}^* \cdot \nabla_3^*)\,\nabla_3^* \rho^* = -\nabla_3^* \mathbf{u}^* \cdot \nabla_3^* \rho^*. \tag{2.200}$$

The right-hand side of the equation above should be interpreted as the gradient acting on the velocity vector and this result, then dotted with the gradient of the density field. This identity then implies that the last term can be written as

$$-\nabla_3^* \rho^* \cdot [((\nabla_3^* \times \mathbf{u}^* + f\hat{\mathbf{e}}_3) \cdot \nabla_3)\,\mathbf{u}^*] = -(\nabla_3^* \times \mathbf{u}^* + f\hat{\mathbf{e}}_3) \cdot [\nabla_3^* \mathbf{u}^* \cdot \nabla_3^* \rho^*]$$

$$= (\nabla_3^* \times \mathbf{u}^* + f\hat{\mathbf{e}}_3) \cdot [(\partial_{t^*} + \mathbf{u}^* \cdot \nabla_3^*)\,\nabla_3^* \rho^*]. \tag{2.201}$$

Combining (2.198), (2.199) and (2.201), then using the product rule gives the following equation

$$[\partial_{t^*} + \mathbf{u}^* \cdot \nabla_3^*] \left[(\nabla_3^* \times \mathbf{u}^* + f\hat{\mathbf{e}}_3) \cdot \nabla_3^* \rho^*\right] = 0 \tag{2.202}$$

which is identical to (2.174), therefore the equation obtained for the upper layer from Ertel's theorem is proven.

The vorticity in the vertical direction will be much larger then any vorticity in the horizontal plane and for this reason that we reduce (2.202) to,

$$(\partial_{t^*} + \mathbf{u}_1^* \cdot \nabla^* + w^* \partial_{z^*}) \left[(\nabla^* \times \mathbf{u}_1^* + f) \frac{\partial \rho^*}{\partial z^*}\right] = 0. \tag{2.203}$$

Substituting in the scalings from Section 2.5 produces

$$(sf\partial_t + \delta f \mathbf{u}_1 \cdot \nabla + s\delta f w \partial_z) \left[(f\delta \nabla \times \mathbf{u}_1 + f) \frac{\partial}{\partial z^*} \left(\rho_0 + \left(\frac{\rho_2 \delta g'}{g}\right) \rho\right)\right] = 0, \tag{2.204}$$

dividing through by sf^2 and expanding yields,

$$(\partial_t + \mu \mathbf{u}_1 \cdot \nabla + \delta w \partial_z) \left[\frac{d\rho_0}{dz^*} + \frac{\rho_2 \delta g'}{g} \frac{\partial \rho}{\partial z^*} + \delta \nabla \times \mathbf{u}_1 \frac{d\rho_0}{dz^*} + O\left(\delta^2\right)\right] = 0. \tag{2.205}$$

The $O\left(\delta^2\right)$ terms will be ignored since we only want a leading order equation. The above equation can then be separated as follows,

$$(\partial_t + \mu \mathbf{u}_1 \cdot \nabla) \left[\delta \frac{\rho_2 g'}{g} \frac{\partial \rho}{\partial z^*} + \delta \nabla \times \mathbf{u}_1 \frac{d\rho_0}{dz^*}\right] + (\delta w \partial_z) \frac{d\rho_0}{dz^*} = 0, \tag{2.206}$$

where each term that appears is of order δ, hence this parameter can be cancelled. Also, multiplying by $gH/(\rho_2 g')$ and recalling the definition for the buoyancy frequency gives,

$$(\partial_t + \mu \mathbf{u}_1 \cdot \nabla) \left[\frac{\partial \rho}{\partial z} - N^2 \nabla \times \mathbf{u}_1\right] - w\left(N^2\right)_z = 0. \tag{2.207}$$

By manipulating the incompressibility equation (2.188), the vertical velocity is seen to satisfy to leading order,

$$w = N^{-2} (\partial_t + \mu \mathbf{u}_1 \cdot \nabla) \rho, \tag{2.208}$$

which when substituted into (2.207) produces,

$$(\partial_t + \mu \mathbf{u}_1 \cdot \boldsymbol{\nabla}) \left[\frac{\partial \rho}{\partial z} - \frac{(N^2)_z \rho}{N^2} - N^2 \boldsymbol{\nabla} \times \mathbf{u}_1 \right] = 0. \tag{2.209}$$

Now dividing the equation by $-N^2$, rewriting the buoyancy term and using the product rule produces

$$(\partial_t + \mu \mathbf{u}_1 \cdot \boldsymbol{\nabla}) \left[\boldsymbol{\nabla} \times \mathbf{u}_1 - \frac{\partial}{\partial z} \left(\frac{\rho}{N^2} \right) \right] = 0. \tag{2.210}$$

Substituting in the hydrostatic relation, which comes from the leading order expression of (2.114) and (2.146), allows us to rewrite the equation in terms of the dynamic pressure in the upper layer,

$$(\partial_t + \mu \mathbf{u}_1 \cdot \boldsymbol{\nabla}) \left[\Delta \varphi + \frac{\partial}{\partial z} \left(\frac{\varphi_z}{N^2} \right) \right] = 0, \tag{2.211}$$

$$[\partial_t + \mu \partial (\varphi, *)] \left(\Delta \varphi + \frac{\partial}{\partial z} \left(\frac{\varphi_z}{N^2} \right) \right) = 0, \tag{2.212}$$

which is the same equation as in (2.158), and hence concludes the chapter.

Chapter 3

Linear stability analysis

This chapter will study density-driven currents that do not vary in the along-shelf direction, within a periodic channel domain of width L and length $2x_R$,

$$\Omega = \{(x, y, z), \; -x_R < x < x_R, \; 0 < y < L, \; -1 < z < 0\}, \tag{3.1}$$

where Ω_H denotes the horizontal extent of the channel, $\partial\Omega_H$ its boundary and Ω_V a vertical cross-section running along the y axis.

The channel walls are assumed to be rigid, which implies that $v_1 = 0 = v_2$ on $y = 0$ and L since we do not allow for any normal flow through these boundaries. Translating this using the geostrophic relations (2.126) and (2.130) yield

$$\varphi_x = 0 \quad \text{on} \quad y = 0, L, \tag{3.2}$$

$$h_x = 0 \quad \text{on} \quad y = 0, L. \tag{3.3}$$

Another set of boundary conditions needs to be established along the channel walls. This is done by integrating the x directional momentum equation (2.113) over the x direction, applying the periodic boundary conditions and evaluating the resulting expression at $y = 0$ or L, to give

$$s \int_{-x_R}^{x_R} (u_{1_t} + \mu s w u_z)_{y=0,L} \; dx = 0. \tag{3.4}$$

The leading order expression of this equation then states,

$$\frac{d}{dt}\int_{-x_R}^{x_R} \varphi_y\big|_{y=L}\, dx = \frac{d}{dt}\int_{-x_R}^{x_R} \varphi_y\big|_{y=0}\, dx = 0, \tag{3.5}$$

after having substituted in the geostrophic relation (2.126). These equations above are the statements that the circulation is conserved along each channel wall, and are referred to as Kelvin's circulation theorem.

Defining the following averaging operator along the x direction

$$\langle * \rangle = \frac{1}{2x_R}\int_{-x_R}^{x_R} *\, dx, \tag{3.6}$$

allows us to deduce, that

$$\left\langle f(y,z,t)\,\frac{\partial}{\partial x}\left(g(x,y,z,t)\right)\right\rangle = 0, \tag{3.7}$$

upon integrating and using the periodicity condition.

Throughout this chapter it will be assumed that the bottom topography depends only on y. Since the equations that arise cannot be solved in all generality, this dependency will occasionally be chosen to be linear in order that solutions can be obtained. Due to the scaling already implemented, the canonical choice is $h_B = -y$. For similar reasons, the buoyancy frequency will at some stage be chosen to be a constant rather than a general function of z.

3.1 Linearization of the governing equations

Consider an ordinary frictionless pendulum with the string replaced with a stiff metal rod. Assuming the rod can swing completely around, there are two stationary positions, they correspond to the bobs lowest and highest points. Perturbing the former creates oscillations about that equilibrium position with an amplitude no larger than that of the initial perturbation. However, no matter how small of an initial force perturbs the later situation, it will always cause oscillations of much larger amplitude. Hence, upon considering how they react to perturbations, it is natural to call the lowest and highest positions stable and unstable equilibria, respectively. An important consequence is that since nature intrinsically possesses disturbances, no matter how small, the pendulum can never be naturally observed to be resting at its highest point.

Since it is desirable to determine which steady solutions to our model could occur in nature, it is necessary to do a linear stability analysis. This introduces infinitesimal perturbations to particular basic states. When these perturbed quantities occur quadratically, they can be ignored due to their relative smallness. This has the net effect of ignoring nonlinear terms, which then allows a stability analysis to be applied to the remaining linear equations.

Two different types of results can be obtained from this analysis. The first is where the perturbations either persist or die away, but do not grow. These are formally referred to as neutrally and asymptotically stable, respectively. The second is that the steady state solution is unstable, which means that the perturbations necessarily grow in amplitude. Simply because the linear theory predicts growth, it should not be concluded that the system will continually grow indefinitely. The growth will always reach a stage where their magnitudes can no longer be classified as being infinitesimal, and hence the linear theory is no longer applicable. In this region, the nonlinear terms may cause dampening so that the amplitude then decays, which is how finite amplitude oscillations can arise. To study the nature of these oscillations it would be necessary to do a weakly nonlinear analysis where the perturbations reach a point of being small, but finite. This will not be tackled within this thesis but suggests future work that can be done.

The basic steady states to be studied are

$$h = h_0(y), \tag{3.8}$$

$$\varphi = \varphi_0(y, z), \tag{3.9}$$

which satisfy the governing nonlinear equations and boundary conditions because of their lack of x and t dependency.

Upon considering the geostrophic relations, it is seen that no cross channel velocities can occur for these states. Hence, these basic states are parallel shear flows that can be sheared in the cross channel and vertical directions. The former requires that the flow has kinetic energy, whereas the latter means the flow possesses *both* kinetic and potential energies.

The question of stability reduces to asking whether the perturbations are able to extract energy from the mean flow and hence grow in amplitude. When the perturbations only extract kinetic energy, this type of instability is called barotropic. However, when potential energy is necessarily released along with kinetic energy, this is baroclinic instability (Griffiths *et al.*, 1982).

The perturbed system, is given by

$$h = h_0(y) + h'(x, y, t), \tag{3.10}$$

$$\varphi = \varphi_0(y, z) + \varphi'(x, y, z, t), \tag{3.11}$$

where

$$|h'| \ll 1 \quad \text{and} \quad |\varphi'| \ll 1, \tag{3.12}$$

the bars denoting the absolute value.

The nonlinear governing equations are, after replacing (2.164) with (2.164)$-N^2$(2.165),

$$[\partial_t + \mu\partial(\varphi, *)]\left(\Delta\varphi + \left(\varphi_z/N^2\right)_z\right) = 0, \tag{3.13}$$

$$[\partial_t + \mu\partial(\varphi, *)]\,\varphi_z = 0 \quad \text{at} \quad z = 0, \tag{3.14}$$

$$[\partial_t + \mu\partial(\varphi, *)]\,\varphi_z + N^2 h_{B_y}(\varphi + h)_x = 0 \quad \text{at} \quad z = -1, \tag{3.15}$$

$$[\partial_t + \mu\partial(\varphi, *)]\,h - h_{B_y}h_x = 0 \quad \text{at} \quad z = -1. \tag{3.16}$$

Substituting (3.10) and (3.11) into (3.13) to (3.16) gives, after dropping the primes,

$$[\partial_t - \mu\varphi_{0_y}\partial_x]\left(\Delta\varphi + \left(N^{-2}\varphi_z\right)_z\right) + \mu\varphi_x\left(\Delta\varphi_0 + \left(N^{-2}\varphi_{0_z}\right)_z\right)_y$$

$$+\mu\partial\left(\varphi, \Delta\varphi + \left(N^{-2}\varphi_z\right)_z\right) = 0, \tag{3.17}$$

$$\left[\partial_t - \mu\varphi_{0_y}\partial_x\right]\varphi_z + \mu\varphi_x\left(\varphi_{0_z}\right)_y + \mu\partial\left(\varphi,\varphi_z\right) = 0 \quad \text{at} \quad z=0, \tag{3.18}$$

$$\left[\partial_t - \mu\varphi_{0_y}\partial_x\right]\varphi_z + \mu\varphi_x\left(\varphi_{0_z}\right)_y + N^2 h_{B_y}\left(\varphi + h\right)_x + \mu\partial\left(\varphi,\varphi_z\right) = 0 \quad \text{at} \quad z=-1, \tag{3.19}$$

$$\left[\partial_t - \mu\varphi_{0_y}\partial_x\right]h + \mu\varphi_x\left(h_0\right)_y - h_{B_y}h_x + \mu\partial\left(\varphi,h\right) = 0 \quad \text{at} \quad z=-1. \tag{3.20}$$

Equation (3.12) then allows us to conclude that the nonlinear terms will be much smaller than the remaining terms, at least initially, and are hence ignored. As well, defining $U_0(y,z) = -\varphi_{0_y}$ to be the along channel velocity of the mean flow in the upper fluid allows the linear stability equations to be written as

$$\left[\partial_t + \mu U_0\partial_x\right]\left(\Delta\varphi + \left(N^{-2}\varphi_z\right)_z\right) - \mu\varphi_x\left(\Delta U_0 + \left(N^{-2}U_{0_z}\right)_z\right) = 0, \tag{3.21}$$

$$\left[\partial_t + \mu U_0\partial_x\right]\varphi_z - \mu\varphi_x U_{0_z} = 0 \quad \text{at} \quad z=0, \tag{3.22}$$

$$\left[\partial_t + \mu U_0\partial_x\right]\varphi_z + \varphi_x\left(N^2 h_{B_y} - \mu U_{0_z}\right) + N^2 h_{B_y}h_x = 0 \quad \text{at} \quad z=-1, \tag{3.23}$$

$$\left[\partial_t + \mu U_0\partial_x\right]h + \mu\varphi_x h_{0_y} - h_{B_y}h_x = 0 \quad \text{at} \quad z=-1. \tag{3.24}$$

3.2 Perturbation energetics

The linear equations (3.21) to (3.24), describe the evolution of perturbations in space and time. These equations allow for the construction of an energy equation that describes the time evolution of the perturbation energy. From this, necessary conditions can be obtained where the perturbation energy increases, ergo instability. This onset of instability is possible through either barotropic or baroclinic processes. Since the purpose of this model is to describe baroclinic instability arising from the interaction of the two fluids, U_0 will be chosen so that this goal can be achieved.

We begin by multiplying (3.22) by $N^{-2}\,\varphi|_{z=0}$ and integrating over the horizontal extent of the model,

$$\begin{aligned}\int_0^L \left\langle N^{-2}\varphi\varphi_{zt}\right\rangle_{z=0} dy &= -\int_0^L \left\langle \mu N^{-2}\left(U_0\varphi\varphi_{zx} - \frac{1}{2}\left(\varphi\right)_x U_{0_z}\right)\right\rangle_{z=0} dy \\ &= -\int_0^L \left\langle \mu N^{-2}U_0\varphi\varphi_{zx}\right\rangle_{z=0} dy, \end{aligned} \tag{3.25}$$

having used (3.7) and (3.9). Similarly, multiplying (3.23) by $N^{-2}\ \varphi|_{z=-1}$ yields

$$\begin{aligned}\int_0^L \left\langle N^{-2}\varphi\varphi_{zt}\right\rangle_{z=-1} dy &= -\int_0^L \left\langle \mu N^{-2}U_0\varphi\varphi_{zx} + h_{B_y}\varphi h_x\right\rangle_{z=-1} dy \\ &\quad -\frac{1}{2}\int_0^L \left\langle (\varphi)_x \left(h_{B_y} - \mu N^{-2}U_{0_z}\right)\right\rangle_{z=-1} dy \\ &= -\int_0^L \left\langle \mu N^{-2}U_0\varphi\varphi_{zx} - h_{B_y}\varphi_x h\right\rangle_{z=-1} dy. \quad (3.26)\end{aligned}$$

Combining (3.25) and (3.26) produces

$$\int_0^L \left\langle N^{-2}\left(\varphi\varphi_{zt} + \mu N^{-2}U_0\varphi\varphi_{zx}\right)\right\rangle_{z=-1}^{z=0} dy = \int_0^L \left\langle h_{B_y}\varphi_x h\right\rangle_{z=-1} dy. \quad (3.27)$$

Multiplying (3.21) by φ and integrating over the volume Ω gives

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial t}\iint\limits_{\Omega_V} \left\langle \nabla\varphi\cdot\nabla\varphi + N^{-2}\varphi_z^2 \right\rangle dydz &= \int_0^L \left\langle N^{-2}\left(\varphi\varphi_{zt} + \mu U_0\varphi\varphi_{zx}\right)\right\rangle_{z=-1}^{z=0} dy \\ &\quad -\mu\iint\limits_{\Omega_V} \left\langle \nabla\varphi_x\cdot\nabla\left(U_0\varphi\right) + N^{-2}\varphi_{xz}\left(U_0\varphi\right)_z \right\rangle dydz \\ &\quad +\iint\limits_{\Omega_V} \left\langle \nabla\cdot\left(\varphi\nabla\varphi_t\right) + \mu\nabla\cdot\left(U_0\varphi\nabla\varphi_x\right)\right\rangle dydz, \quad (3.28)\end{aligned}$$

after integrating by parts and applying the following identity twice,

$$\nabla\cdot(a\mathbf{b}) = a\nabla\cdot\mathbf{b} + \nabla a\cdot\mathbf{b}. \quad (3.29)$$

The divergence theorem allows us to prove that two of the terms on the right-hand side of (3.28) vanish. Applying this theorem, expanding the contour integral and then applying the boundary conditions (3.1) and (3.5), results in,

$$\begin{aligned}\iint\limits_{\Omega_V} \left\langle \nabla\cdot\left(\varphi\nabla\varphi_t\right)\right\rangle dydz &= \frac{1}{2x_R}\int_{-1}^0 \int\limits_{\partial\Omega_H} \varphi\nabla\varphi_t\cdot\mathbf{n}\ ds\ dz \\ &= \frac{1}{2x_R}\int_{-1}^0 \left\{\int_0^L \left[\varphi\varphi_{xt}\right]_{x=x_R} - \left[\varphi\varphi_{xt}\right]_{x=-x_R} dy\right.\end{aligned}$$

$$+ \varphi|_{y=L} \int_{-x_R}^{x_R} [\varphi_{yt}]_{y=L} \, dx - \varphi|_{y=0} \int_{-x_R}^{x_R} [\varphi_{yt}]_{y=0} \, dx \Big\} \, dz$$

$$= 0. \tag{3.30}$$

The other term in (3.28) that can be written using the divergence theorem is,

$$\iint\limits_{\Omega_V} \langle \nabla \cdot (U_0 \varphi \nabla \varphi_x) \rangle \; dydz = \frac{1}{2x_R} \int_{-1}^{0} \int\limits_{\partial \Omega_H} U_0 \varphi \nabla \varphi_x \cdot \mathbf{n} \; ds \; dz$$

$$= \frac{1}{2x_R} \iint\limits_{\Omega_V} [U_0 \varphi \varphi_{xx}]_{x=x_R} - [U_0 \varphi \varphi_{xx}]_{x=-x_R} \; dydz$$

$$+ \int_{-1}^{0} \langle \varphi \varphi_{yx} \rangle_{y=L} - \langle \varphi \varphi_{yx} \rangle_{y=0} \, dz, \tag{3.31}$$

where again, the first two terms vanish by periodicity, which leaves upon using integration by parts and (3.2)

$$\begin{aligned} \iint_{\Omega_V} \nabla \cdot (U_0 \varphi \nabla \varphi_x) \, dydz &= \int_{-1}^{0} \Big\langle [\varphi_x \varphi_y]_{y=0} - [\varphi_x \varphi_y]_{y=L} \Big\rangle \, dz \\ &= 0. \end{aligned} \tag{3.32}$$

Substituting (3.27), (3.30) and (3.32) into (3.28) and expanding the remaining terms by use of the product rule, yields,

$$\frac{1}{2} \frac{\partial}{\partial t} \iint\limits_{\Omega_V} \Big\langle \nabla \varphi \cdot \nabla \varphi + N^{-2} \varphi_z^2 \Big\rangle \; dydz = -\mu \iint\limits_{\Omega_V} \langle U_0 \nabla \varphi_x \cdot \nabla \varphi \rangle \; dydz$$

$$-\mu \iint\limits_{\Omega_V} \Big\langle \varphi \nabla \varphi_x \cdot \nabla U_0 + N^{-2} U_0 \varphi_{xz} \varphi_z \Big\rangle \; dydz$$

$$-\mu \iint\limits_{\Omega_V} \Big\langle N^{-2} \varphi \varphi_{xz} U_{0z} \Big\rangle \; dydz + \int_0^L \langle h_{B_y} h \varphi_x \rangle_{z=-1} \, dy. \tag{3.33}$$

The first and third terms in the right-hand side are each equal to zero by (3.7), which reduces the equation to

$$\frac{1}{2} \frac{\partial}{\partial t} \iint\limits_{\Omega_V} \Big\langle \nabla \varphi \cdot \nabla \varphi + N^{-2} \varphi_z^2 \Big\rangle \; dydz = \mu \iint\limits_{\Omega_V} \langle \varphi_x \varphi_y U_{0_y} \rangle \; dydz$$

$$+\mu \iint\limits_{\Omega_V} \left\langle N^{-2}\varphi_x\varphi_z U_{0_z} \right\rangle \, dydz - \int_0^L \langle h_{B_y} h\varphi_x \rangle_{z-1} \, dy, \tag{3.34}$$

after integrating by parts repeatedly.

The left-hand side of this equation is the time rate of change of the total energy of layer one, which is a sum of the kinetic and potential energies. Instability requires that the perturbation energy increases with time and hence that the right-hand side is positive.

There are seen to be three potential sources for instability. The first and second terms are barotropic and baroclinic since they depend on the horizontal and vertical shear of the mean flow, respectively. The former extracts kinetic energy whereas the latter extracts available potential energy which exists due to the presence of a horizontal density gradient which is governed by the thermal wind relation. Both these instabilities are intrinsic to the continuously stratified QG fluid as explained in Pedlosky (1987).The third term in (3.34) is due to the existence of a lower layer. In this particular case the total perturbation energy is able to grow through the baroclinic interactions of the two layers.

Studying the model with all of the three terms being non-zero is difficult. Hence, in order to simplify this situation and focus on the instability due to the interaction of the two layers, the internal sources of instability for the slope-water are eliminated by choosing $U_0 = 0$ everywhere, i.e. the upper layer is at rest.

Substituting this into (3.34) along with the geostrophic relation yields,

$$\frac{\partial}{\partial t} \iint\limits_{\Omega_V} \left\langle \nabla\varphi \cdot \nabla\varphi + (\varphi_z/N)^2 \right\rangle \, dydz = -2 \int_0^L \langle v_1 h h_{B_y} \rangle_{z=-1} \, dy. \tag{3.35}$$

If a positive (negative) h anomaly is associated with a cold (warm) anomaly in the upper layer, (3.35) states that for instability to arise, $\langle v_1 h h_{B_y} \rangle_{z=-1}$ must be negative at some y values. This translates to saying that there is a net heat flow in the same direction as the topographic gradient. Conversely, a sufficient condition for stability is that heat never flows in the same direction as the topographic gradient, since this guarantees that $\langle v_1 h h_{B_y} \rangle_{z=-1}$ is positive for all y.

Substituting $U_0 = 0$ into the linear equations (3.21) to (3.24), gives,

$$\left(\Delta\varphi + \left(N^{-2}\varphi_z\right)_z\right)_t = 0, \tag{3.36}$$

$$\varphi_{zt} = 0 \quad \text{at} \quad z = 0, \tag{3.37}$$

$$\varphi_{zt} + N^2 \left(\varphi_x + h_x\right) h_{B_y} = 0 \quad \text{at} \quad z = -1, \tag{3.38}$$

$$h_t + \mu\varphi_x h_{0_y} - h_{B_y} h_x = 0 \quad \text{at} \quad z = -1. \tag{3.39}$$

Substituting (3.39) into the right-hand side of (3.35) produces,

$$-2\int_0^L \langle \varphi_x h h_{B_y} \rangle_{z=-1} \, dy = \int_0^L \frac{h_{B_y}}{\mu h_{0_y}} \left\langle \left(h^2\right)_t - h_{B_y} \left(h^2\right)_x \right\rangle dy. \tag{3.40}$$

Property (3.7) used along with the fact that h_0 and h_B are both invariant with respect to the along channel direction, shows that the last term in the equation above vanishes. Therefore substituting (3.40) into the energy equation (3.35) yields,

$$\frac{\partial}{\partial t}\left[\iint_{\Omega_V} \left\langle \nabla\varphi \cdot \nabla\varphi + (\varphi_z/N)^2 \right\rangle \, dydz - \int_0^L \frac{h_{B_y}}{\mu h_{0_y}} \left\langle h^2 \right\rangle \, dy\right] = 0. \tag{3.41}$$

Since $\mu > 0$ by construction, the condition that $h_{B_y} h_{0_y} < 0$ for all y, is sufficient to ensure the basic state is stable to all disturbances. For instability to arise, it is necessary that there is some y for which $h_{B_y} h_{0_y} > 0$.

Therefore, it is the region where both the bottom topography and gravity current profile slope downwards, that the available potentially energy can most easily be extracted by the perturbation. This results in an asymmetrical instability with respect to the current. It should not be thought that this condition guarantees instability, since that is clearly not the case.

These conditions are exactly the same as those presented for the two-dimensional analogue of this model in Swaters (1991). Hence, the stratification of the upper layer does not alter the criteria for linear stability when the upper layer is motionless.

Quasi-Geostrophic instability criteria often requires that there is a local extrema in the PV in the along channel direction (Pedlosky, 1987). This is not the case in our situation, since the PV for the lower layer is $1/h$ and clearly the gradient of this with respect to y being equal to zero, does not correspond to instability.

3.3 Normal Mode Solutions

When considering the stability of a steady state solution, some perturbations could yield instability while others might not. If even just one unstable perturbation arose, it would suffice sufficient to make the whole system unstable. This is why when finding sufficient conditions for stability, it is necessary to guarantee stability for all types of perturbations.

This section uses the method of normal modes to obtain the same results as those in the previous section. The essence of normal mode analysis is to assume that the disturbances are wavelike in nature, and hence can be written as a sum of modes using a complete orthonormal basis for the function space (Drazin & Reid, 1981). Using the superposition principle for linear differential equations (Zauderer, 1989), every disturbance can be decomposed into modes, each of which satisfies the linear equation. To guarantee stability, it is necessary to find sufficient conditions under which all the modes are stable. Whereas, finding even just one unstable mode is sufficient to conclude that the system is unstable.

Since we want to consider waves travelling down the channel, the solution of (3.21) to (3.24) is assumed to be of the following form

$$\varphi = \psi(y,z)\exp\left[ik(x-ct)\right] + c.c, \tag{3.42}$$

$$h = \eta(y)\exp\left[ik(x-ct)\right] + c.c., \tag{3.43}$$

k and c are the along channel wavenumber and complex phase speeds and c.c. denotes the complex conjugate. Formally, we require $k = n\pi/x_R$ with $n = 0, 1, 2, \ldots$ However, in practice, we assume $x_R \to \infty$ so that any k can be realized. Since we want (3.42) and (3.43) to be bounded for large magnitudes of x, it is necessary to assume that k is strictly real; without less of generality we assume it to be positive.

However the phase speed is complex since we have to allow for the possibility that the system can become unstable in time. Hence, we decompose the phase speed as

$$c = c_R + ic_I, \tag{3.44}$$

where c_R and c_I are real parameters. Therefore, the exponents that appear in (3.42) and (3.43), take the form

$$\exp\left[ik\left(x - c_R t\right)\right] \exp\left[kc_I t\right]. \tag{3.45}$$

It is for this reason that it is said that the method of normal modes takes a Fourier Transform in space and a Laplace transform in time (Drazin and Reid, 1981).

Equation (3.45) shows that if c_I is positive, the solution is unstable as time increases. It can be shown that if (φ, h, c) is a solution to (3.21) through to (3.24), then so is (φ^*, h^*, c^*). Therefore, if c_I is negative for some mode, there is an associated mode, that being its complex conjugate, where c_I is positive. Hence $c_I \neq 0$ is a sufficient condition for linear instability. If $c_I = 0$, this situation is said to be neutrally stable. The quantity kc_I determines how fast the instability grows and is called the growth rate.

We initially assume $U_0(y, z)$ is not necessarily zero in order to obtain general stability criteria. Then when it is appropriate, this function will be set to zero as was done in Section 3.2. Substituting (3.42) and (3.43) into the no normal flow condition (3.2) and linear equations (3.21) through to (3.24), yields

$$\psi = 0 \quad \text{on} \quad y = 0 \text{ and } L. \tag{3.46}$$

$$\eta = 0 \quad \text{on} \quad y = 0 \text{ and } L. \tag{3.47}$$

and

$$[\mu U_0 - c]\left(\left(N^{-2}\psi_z\right)_z + \psi_{yy} - k^2\psi\right) - \mu\left[U_{0_{yy}} + \left(N^{-2}U_{0_z}\right)_z\right]\psi = 0, \tag{3.48}$$

$$[\mu U_0 - c]\,\psi_z - \mu U_{0_z}\psi = 0 \quad \text{at} \quad z = 0, \tag{3.49}$$

$$[\mu U_0 - c]\,\psi_z + \left[N^2 h_{B_y} - \mu U_{0_z}\right]\psi + N^2 h_{B_y}\eta = 0 \quad \text{at} \quad z = -1, \tag{3.50}$$

$$\left[\mu U_0 - c - h_{B_y}\right]\eta + \mu h_{0_y}\,\psi|_{z=-1} = 0. \tag{3.51}$$

having divided each equation by ik.

Assuming that $\mu U_0 - c - h_{B_y} \neq 0$, allows (3.51) to be manipulated in order to gain an expression for η,

$$\eta = -\frac{\mu h_{0_y}\,\psi|_{z=-1}}{\mu U_0 - c - h_{B_y}}, \tag{3.52}$$

which when substituted into (3.50) yields

$$[\mu U_0 - c]\,\psi_z + \left[N^2 h_{B_y} - \mu U_{0_z} - \frac{\mu N^2 h_{B_y} h_{0_y}}{\mu U_0 - c - h_{B_y}}\right]\psi = 0 \quad \text{at} \quad z = -1. \tag{3.53}$$

The function ψ is obtained by solving the system, (3.48), (3.49) and (3.53), where then η is recovered by substituting this solution into (3.52). Unfortunately, this system of equations to be solved is not separable. Moreover, its not Sturm-Liouville since $N^2 h_{0_y} < 0$, which means that we cannot expect all the eigenvalues to be positive.

To obtain an energy equation, we multiply (3.49) by $N^{-2}(0)\psi^*(0)/(\mu U_0(0) - c)$ and integrate over the y domain, where the $*$ superscript denotes complex conjugate,

$$\int_0^L \left[N^{-2}\psi^*\psi_z\right]_{z=0} dy = \int_0^L \left[\frac{\mu N^{-2}}{\mu U_0 - c} U_{0_z} |\psi|^2\right]_{z=0} dy, \tag{3.54}$$

and $|\psi|^2 = \psi^*\psi$ is the square of the modulus of ψ. Multiplying (3.53) by the same term as before, but evaluated at $z = -1$, and integrating over y gives,

$$\begin{aligned}\int_0^L \left[N^{-2}\psi^*\psi_z\right]_{z=-1} dy &= \int_0^L \left[\frac{1}{\mu U_0 - c}\left(\mu N^{-2} U_{0_z} - h_{B_y}\right)|\psi|^2\right]_{z=-1} dy \\ &+ \int_0^L \left[\frac{1}{\mu U_0 - c}\frac{\mu h_{B_y} h_{0_y}}{\mu U_0 - c - h_{B_y}}|\psi|^2\right]_{z=-1} dy. \end{aligned} \tag{3.55}$$

Now, multiplying (3.48) by $-\psi^*/\left(\mu U_0(0) - c\right)$ and integrating over y and z produces

$$\iint_{\Omega_V} N^{-2}|\psi_z|^2 + |\psi_y|^2 + \left(k^2 + \frac{\mu}{\mu U_0 - c}\left[U_{0_{yy}} + \left(N^{-2}U_{0_z}\right)_z\right]\right)|\psi|^2 \, dydz$$

$$= \int_0^L \left[N^{-2}\psi^*\psi_z\right]_{z=-1}^{z=0} dy + \int_{-1}^{0} \left[\psi^*\psi_y\right]_{y=0}^{y=L} dz, \tag{3.56}$$

upon using integration by parts with respect to y and z. Since ψ satisfies (3.46), then so must ψ^*, which implies that the second set of terms on the right-hand side of the line directly above vanishes.

Substituting (3.54) and (3.55) into (3.56) then gives

$$\iint_{\Omega_V} N^{-2}|\psi_z|^2 + |\psi_y|^2 + k^2|\psi|^2 \; dydz$$

$$= \int_0^L \left[\frac{\mu N^{-2}}{\mu U_0 - c} U_{0_z} |\psi|^2\right]_{z=-1}^{z=0} dy - \iint_{\Omega_V} \frac{\mu}{\mu U_0 - c}\left[U_{0_{yy}} + \left(N^{-2}U_{0_z}\right)_z\right]|\psi|^2 \; dydz$$

$$+ \int_0^L \left[\left(\frac{h_{B_y}}{\mu U_0 - c} - \frac{1}{\mu U_0 - c}\frac{\mu h_{B_y} h_{0_y}}{\mu U_0 - c - h_{B_y}}\right)|\psi|^2\right]_{z=-1} dy. \tag{3.57}$$

This equation is the analogous to that of (3.34), but in the context of normal mode solutions. The first row of terms on the right-hand side possesses terms that are proportional to gradients in U_0 and hence are the barotropic and baroclinic source terms for the QG fluid (Pedlosky, 1987). As was done in the previous section, these instabilities are filtered out by choosing $U_0 = 0$, which reduces (3.57) to

$$c\int_{-1}^{0}\int_0^L N^{-2}|\psi_z|^2 + |\psi_y|^2 + k^2|\psi|^2 \; dydz$$

$$+ \int_0^L \left[h_{B_y}\left(N^{-2} + \frac{\mu h_{0_y}}{c + h_{B_y}}\right)|\psi|^2\right]_{z=-1} dy = 0. \tag{3.58}$$

In order to simply the expressions to follow it is advantageous to make the following definition

$$Q = \int_{-1}^{0}\int_0^L N^{-2}|\psi_z|^2 + |\psi_y|^2 + k^2|\psi|^2 \; dydz, \tag{3.59}$$

which is strictly positive for nontrivial perturbations.

Multiplying the numerator and denominator of the second term in the round bracket, in (3.58) by $(c^* + h_{B_y})$, allows (3.58) to be decomposed into real and imaginary parts, which are, respectively,

$$c_R Q + \int_0^L h_{B_y} \left(1 + \frac{\mu\,(c_R + h_{B_y})\,h_{0_y}}{\left|c + h_{B_y}\right|^2}\right) |\psi|^2_{z=-1} \; dy = 0, \tag{3.60}$$

$$c_I \left[Q - \int_0^L \frac{\mu h_{0_y} h_{B_y}}{\left|c + h_{B_y}\right|^2} |\psi|^2_{z=-1} \; dy\right] = 0. \tag{3.61}$$

Neutral stability must occur if the bracket in (3.61) is non-zero, which certainly holds true if $h_{0_y} h_{B_y} \leq 0$. Conversely for instability, it is necessary that there is some y for which $h_{0_y} h_{B_y} > 0$. These results coincide to those obtained in the previous section, by considering the energetics of the perturbations. This is significant since before, the perturbations were assumed to be arbitrary, whereas in this section it has been assumed that the disturbances are wave-like. Either case yields essentially the same conclusions. Hence assuming wave-like perturbations does not affect the linear stability criteria of the model.

Assuming instability and choosing the bottom topography to be $h_B = -y$, allows (3.61) to be rearranged to

$$|c - 1|^2 = -\frac{\mu}{Q} \int_0^L h_{0_y} |\psi|^2_{z=-1} \; dy. \tag{3.62}$$

Which if substituted into (3.60), brings about,

$$c_R = \frac{1}{Q} \int_0^L |\psi|^2_{z=-1} \; dy + \frac{\mu\,(c_R - 1)}{Q\,|c-1|^2} \int_0^L h_{0_y} |\psi|^2_{z=-1} \; dy,$$

$$c_R = \frac{1}{2} + \frac{1}{2Q} \int_0^L |\psi|^2_{z=-1} \; dy. \tag{3.63}$$

Since we are assuming instability, the necessary condition that $h_{0_y} < 0$ for some y, allows us to define some real parameter γ that satisfies

$$\min_{y \in (0,L)} h_{0_y} = -\gamma^2 < 0, \tag{3.64}$$

From this, the following relation is readily seen

$$-h_{0_y} \leq \gamma^2. \tag{3.65}$$

Substituting (3.65) into (3.62) yields

$$|c-1|^2 \le \frac{\mu\gamma^2}{Q}\int_0^L |\psi|^2_{z=-1}\, dy,$$

$$|c-1|^2 \le (2c_R - 1)\,\mu\gamma^2, \tag{3.66}$$

having used (3.63). Expanding (3.66) gives

$$(c_R-1)^2 - (2c_R-1)\,\mu\gamma^2 + c_I^2 \le 0$$

$$\left(c_R - 1 - \mu\gamma^2\right) + c_I^2 \le \mu\gamma^2\left(\mu\gamma^2+1\right). \tag{3.67}$$

This establishes a semicircle theorem for our problem (Drazin and Reid, 1981), which is important since it presents an upper bound on c_I and hence the growth rate, in terms of γ and μ

$$c_I^2 \le \mu\gamma^2\left(\mu\gamma^2+1\right). \tag{3.68}$$

Observe that taking the limit as μ goes to zero yields that c_I also approaches zero, which implies stability and supports the idea that μ is a measure of the instability present in the system. Taking the limit as $\gamma \to 0$ also implies a stable situation, which should be expected since this implies $h_{0_y} = 0$, which is known to be a sufficient condition for stability.

3.4 Topographic Rossby wave solutions

The basic state under consideration is $U_0 = 0$ and $h_0(y)$, which corresponds to the upper layer being motionless. This means that any motion that develops in the upper layer is due to its interaction with the lower layer. The interaction causes vortex tube stretching or contraction to occur, which generates relative vorticity, as can be seen through PV conservation.

A question that arises is, what type of waves will the perturbations that occur in the slope-water develop into? The modes are described by assuming the simplest situation, that being $h = 0$, $h_B = -y$ and $N^2 =$ constant. For this situation, (3.36) through to

(3.39) reduce to the linearized QG equations (LeBlond & Mysak, 1978),

$$\Delta\varphi + N^{-2}\varphi_{zz} = 0, \tag{3.69}$$

$$\varphi_z = 0 \quad \text{at} \quad z = 0, \tag{3.70}$$

$$\varphi_{zt} - N^2\varphi_x = 0 \quad \text{at} \quad z = -1, \tag{3.71}$$

upon integrating the first two equations with respect to time and eliminating the spatial function of integration since we will be implementing the method of normal modes. The lower layer equation vanishes.

The normal mode solution for any positive integer n

$$\varphi = \Phi(z) \sin\left(\frac{n\pi y}{L}\right) \exp\left[ik(x - ct)\right] + c.c., \tag{3.72}$$

simplifies the boundary conditions on the channel walls to

$$\varphi = 0 \quad \text{on} \quad y = 0, L, \tag{3.73}$$

which are satisfied by (3.72). Substituting (3.72) into (3.69) to (3.71) implies

$$\Phi_{zz} - \lambda^2\Phi = 0, \tag{3.74}$$

$$\Phi_z = 0 \quad \text{at} \quad z = 0, \tag{3.75}$$

$$\Phi_z + \frac{N^2}{c}\Phi = 0 \quad \text{at} \quad z = -1, \tag{3.76}$$

where the new parameter satisfies $\lambda^2 = N^2\left(k^2 + \frac{n^2\pi^2}{L^2}\right)$.

This eigenvalue problem can be solved to give

$$\Phi = A_n \cosh\left[N\left(k^2 + \frac{n^2\pi^2}{L^2}\right)^{\frac{1}{2}} z\right], \tag{3.77}$$

where A_n is the amplitude of the n-th mode and the dispersion relation is

$$c = N \frac{\coth\left[N\left(k^2 + \frac{n^2\pi^2}{L^2}\right)^{\frac{1}{2}}\right]}{\left(k^2 + \frac{n^2\pi^2}{L^2}\right)^{\frac{1}{2}}}. \tag{3.78}$$

The details of how this solution is obtained will be shown in the next section for a more complicated problem.

Observe that cosh is an even function that is increasing for positive values and hence decreasing for the negative ones. Since Φ takes on its greatest value at $z = -1$, this solution is bottom intensified as remarked in LeBlond & Mysak (1978).

This solution is the same as that in Section 20 of LeBlond & Mysak (1978) and represents topographic Rossby waves on an f-plane. Observe that every term in (3.78) is positive and hence $c > 0$. This signifies that these Rossby waves are retrograde, meaning that they travel with the shallow water on its right.

With the presence of a lower layer the situation is more complicated. However, we expect the upper layer to respond to disturbances in the same qualitative behavior. Hence, disturbances that arise in the upper layer, due to the lower layer, are still described as topographic Rossby waves.

3.5 The linear stability problem for a simple wedge front

The equations that stemmed from assuming a normal mode solution for (3.48) to (3.51), we have not been able to solve in all generality. In order to obtain an analytical solution to these linear equations, we make certain assumptions. The first is that the basic state of the upper fluid, is a state of rest,

$$U_0 = 0. \tag{3.79}$$

In addition, we assume that N and the gradients of the lower layer and bottom topography heights are constant,

$$h_0 = 1 - \frac{\beta}{\mu} y, \tag{3.80}$$

$$h_B = -y. \tag{3.81}$$

This lower layer profile is called a wedge front. The new parameter β, determines the slope of the wedge, that being β/μ.

A simple wedge front profile is chosen since this makes the problem analytically tractable. Certainly, this profile is too stringent for most physical applications, yet it does give insight into the qualitative behavior of the system.

Using the method of separation of variables (Zauderer, 1989), the solution is assumed to be (3.72), where again, the horizontal boundary conditions (3.73) are trivially satisfied. Substituting this, as well as (3.79), (3.80) and (3.81) into (3.48), (3.49) and (3.53) yields,

$$\Phi_{zz} - \lambda^2 \Phi = 0, \tag{3.82}$$

$$\Phi_z = 0 \quad \text{at} \quad z = 0, \tag{3.83}$$

$$\Phi_z + \frac{N^2}{c}\left(1 + \frac{\beta}{1-c}\right)\Phi = 0 \quad \text{at} \quad z = -1, \tag{3.84}$$

where

$$\lambda^2 = N^2\left(k^2 + \left(\frac{n\pi}{L}\right)^2\right). \tag{3.85}$$

The quantity λ^2 is proportional to the sum of the squares of the along channel and cross channel wavenumbers. Notice that the latter is discrete due to the fact that the channel has a finite width. Observe that (3.74) to (3.76) is the limiting case where $\beta = 0$.

The general solution to (3.82) is

$$\Phi = A\cosh(\lambda z) + B\sinh(\lambda z), \tag{3.86}$$

where B must be zero for (3.83) to be satisfied, hence

$$\Phi = A\cosh(\lambda z), \tag{3.87}$$

for A being a free amplitude constant. Substituting this into (3.84) yields the following dispersion relation,

$$\lambda \tanh(\lambda) = \frac{N^2}{c}\left(1 + \frac{\beta}{1-c}\right). \tag{3.88}$$

Defining

$$T_1 = \lambda \tanh(\lambda), \tag{3.89}$$

enables (3.88) to be written as

$$T_1 c^2 - \left(T_1 + N^2\right) c + N^2 (1+\beta) = 0. \tag{3.90}$$

Applying the quadratic formula enables us to solve for the phase speed,

$$\begin{aligned} c &= \frac{T_1 + N^2 \pm \sqrt{(T_1+N^2)^2 - 4T_1 N^2 (1+\beta)}}{2T_1}, \\ &= \frac{T_1 + N^2 \pm \sqrt{(T_1-N^2)^2 - 4\beta N^2 T_1}}{2T_1}. \end{aligned} \tag{3.91}$$

The expression above is known as a 'dispersion relation' since the dependency of c on λ signifies that waves of different wavelengths travel at different speeds, and hence disperse. The only way that this would not occur is if the phase speed was constant for all wavenumbers. The real part of c is always non-negative, which guarantees that the only waves that can exist are those that travel in the positive x direction.

To determine conditions for instability, it is necessary to consider the sign of the discriminant. If it is negative, the phase speed has an imaginary component, and there is at least one unstable mode which implies instability. If the discriminant is non-negative, then stability is assured.

In order to compare this dispersion relation, to that of the original Swaters' model, it is necessary to consider the limit as $\lambda \to 0$, since this corresponds to $N^2 \to 0$, i.e. the homogeneous case. Upon recognizing that

$$\lim_{\lambda \to 0} \frac{\lambda}{\tanh(\lambda)} = 1, \tag{3.92}$$

we conclude that the two functions, in the numerator and denominator, are asymptotically equal. Therefore, for small values of λ, we can replace $\tanh(\lambda)$ by λ itself, in (3.89). Substituting this into (3.91) yields, where $K^2 = k^2 + (n\pi/L)^2$,

$$c = \frac{K^2 + 1 \pm \sqrt{(K^2+1)^2 - 4(1+\beta)K^2}}{2K^2}, \tag{3.93}$$

which no longer depends on the buoyancy frequency and is exactly the same result obtained in Mooney & Swaters (1996). The only difference is that our cross-channel wavenumber is discretized due to the different channel geometry. Thus, the homogeneous limit is contained within (3.93).

Setting the discriminant equal to zero yields the marginal stability curve since in every neighborhood about this curve, there is some wavenumber that results in instability. The marginal stability curve is represented by $\beta_0(\lambda)$, and is

$$\beta_0 = \frac{(T_1 - N^2)^2}{4N^2 T_1}, \tag{3.94}$$

as illustrated in Figure 3.1. The region where $\beta > \beta_0$ and $\beta < \beta_0$, are unstable and stable, respectively.

The point of marginal stability is the critical value of λ that yields the maximum of β_0 that is stable. This point corresponds to a local extrema of the marginal stability curve. However, since this curve is positive definite with a minimum of zero, the point of marginal stability must be where $\beta_0 = 0$, that being

$$\tanh(\lambda_0) = \frac{N^2}{\lambda_0}. \tag{3.95}$$

Because the left and right-hand sides are strictly increasing and decreasing respectively, there can be only one solution to this equation. As $\lambda \to 0^+$, the left and right-hand sides respectively approach zero and positive infinity. These limits are reversed as $\lambda \to \infty$, which implies that there must exist a point of intersection, and by the previous argument, it is unique.

The unstable modes in (3.91) must satisfy the relationship

$$\left(\frac{T_1}{N^2}\right)^2 - 2\,(1+2\beta)\,\frac{T_1}{N^2} + 1 < 0, \tag{3.96}$$

having set the discriminate to be negative, divided by N^4 and rearranged terms. Completing the square and taking the square root then gives

$$(1+2\beta) - 2\sqrt{\beta\,(1+\beta)} < \frac{T_1}{N^2} < (1+2\beta) + 2\sqrt{\beta\,(1+\beta)}. \tag{3.97}$$

Multiplying the numerator and denominator of the left-hand side by the multiplicative conjugate of the numerator gives

$$(1+2\beta) - 2\sqrt{\beta\,(1+\beta)} = \frac{1}{1+2\beta+2\sqrt{\beta\,(1+\beta)}}, \tag{3.98}$$

which when substituted into (3.97) yields

$$\frac{1}{1+2\beta+2\sqrt{\beta\,(1+\beta)}} < \frac{T_1}{N^2} < (1+2\beta) + 2\sqrt{\beta\,(1+\beta)}. \tag{3.99}$$

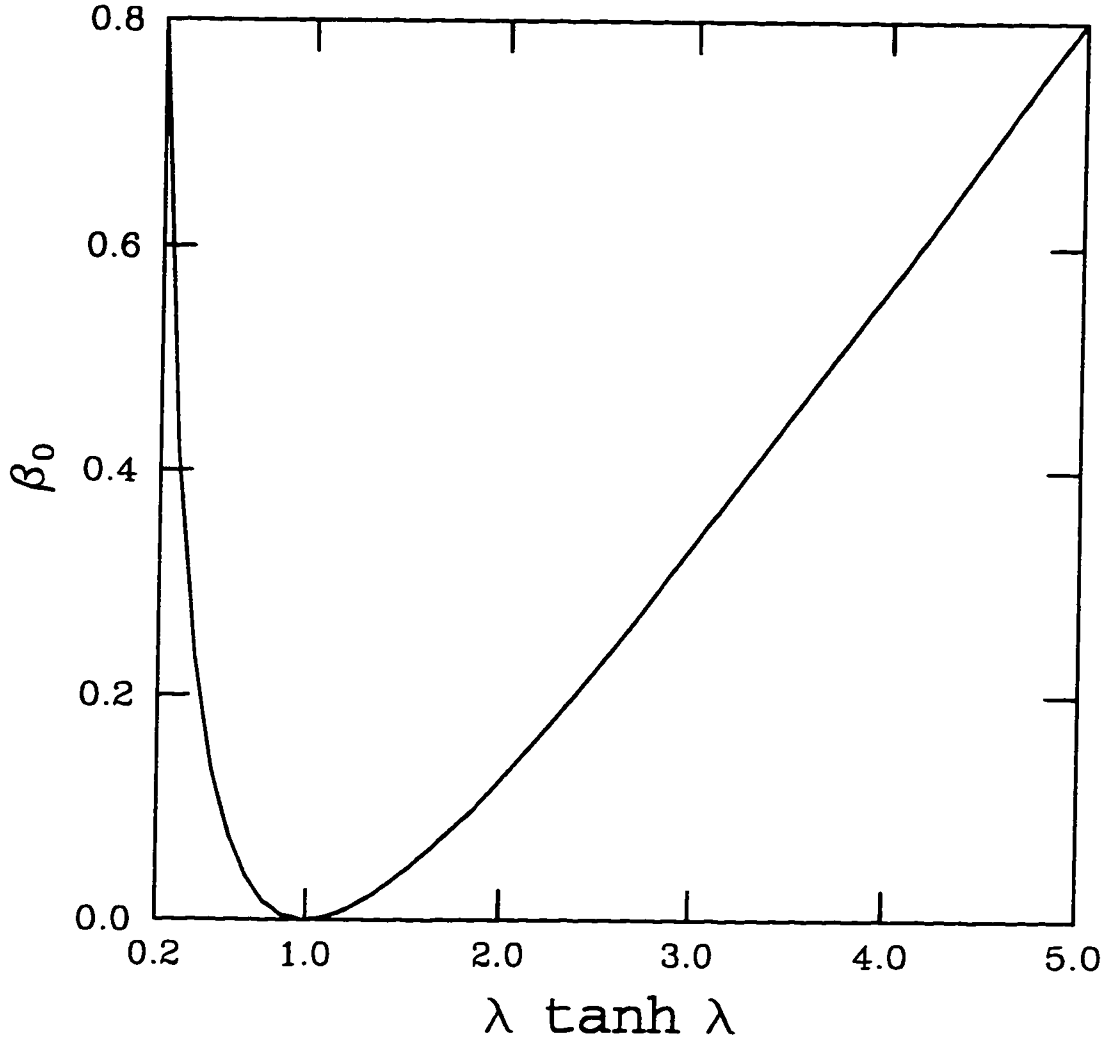
This gives an upper and lower bound on λ, and hence a high and low wavenumber cutoff, which prevents extremely short waves from becoming unstable; a phenomena entitled 'ultraviolet catastrophe' (de Verdiere, 1986). It signifies a shortcoming of a model since short waves, which are not within the regime of consideration, are unstable.

It was demonstrated in (3.45) that the growth rate is given by kc_I. Given the dispersion relation (3.91) for the simple wedge front, it is possible to plot the growth rate for various stratification frequencies in order to determine the effect that stratification has on the growing modes. Finding the imaginary part of the dispersion relation yields the growth rate to be,

$$\sigma = kc_I = k\frac{\sqrt{4\beta N^2 T_1 - (T_1 - N^2)^2}}{2T_1}, \tag{3.100}$$

when the quantity within the square root is positive.

Figure 3.1, Marginal Stability Curve



In Figure 3.2, for the case where $\beta = 0.11$, $\mu = 1.0$, $L = 8.0$ and $n = 1$, the growth rate is plotted for $N = 0.0, 1.0$ and 2.0. The first of these growth rate curves is the same as that in Mooney & Swaters (1996). The maximum growth rates are $kc_I = 0.307$, 0.385 and 0.694, for the $N = 0.0$, 1.0 and 2.0 curves, respectively. The most unstable modes correspond to the along-shelf wavenumbers of $k = 0.93$, 1.25 and 2.43. These modes, since they have the highest growth rates, will eclipse all others, at least initially, and hence are taken to be representative of the growth of the instability. With respect to the length and time scales presented in Chapter 2, the dimensional growth rates and wavelengths, derived as ($\sigma^* = Tkc_I$, $\lambda_l^* = 2\pi L/k$), will be (2.1 days, 101 km), (2.7 days, 75 km) and (4.9 days, 38 km) for the $N = 0.0$, 1.0 and 2.0 curves.

It is clear that with increasing stratification, the maximum growth rate increases and the most unstable mode moves toward higher wavenumbers. This demonstrates that the presence of stratification in the upper fluid enhances the instability and shifts the most unstable mode to smaller wavelengths. A similar result was found in Morgan (1997) where numerical simulations were performed on the primitive equations of a continuously stratified fluid within the context of surface fronts.

This is a significant finding since, when the original Swaters' model was applied to the Strait of Georgia (Karsten & Swaters, 1995), the most unstable mode has a length scale larger than the length scale of the observed eddy features. We speculate that this discrepancy can be accounted for as being due to the lack of stratification in the upper layer of the Swaters' model, which is present in the Strait of Georgia. Applying our continuously stratified model would yield a more accurate prediction, since the most unstable mode would have shorter wavelengths.

For the case when $N = 1.0$, we wish to find the solution for the most unstable mode, that being $k_u = 1.25$. The pressure is obtained by substituting (3.87) into (3.72),

$$\varphi = 2A \sin(\pi y/8) \cosh\left(\sqrt{k_u^2 + (\pi/8)^2} z\right) \exp(k_u c_I t) \cos\left[k_u (x - c_R t)\right]. \tag{3.101}$$

To obtain the perturbation height field of the lower layer, it is first necessary to find η. This is done by substituting (3.80), (3.81) into (3.52), which then gives an expression for η in terms of ψ, where ψ is then obtained by comparing (3.42) with (3.72) and using (3.87). After rewriting the term $(1-c)$ from (3.52), in exponential form, we obtain the following solution

$$h = \frac{2A}{\sqrt{(c_R-1)^2+c_I^2}} \sin(\pi y/8) \cosh\left(\sqrt{k_u^2+(\pi/8)^2}\right)$$

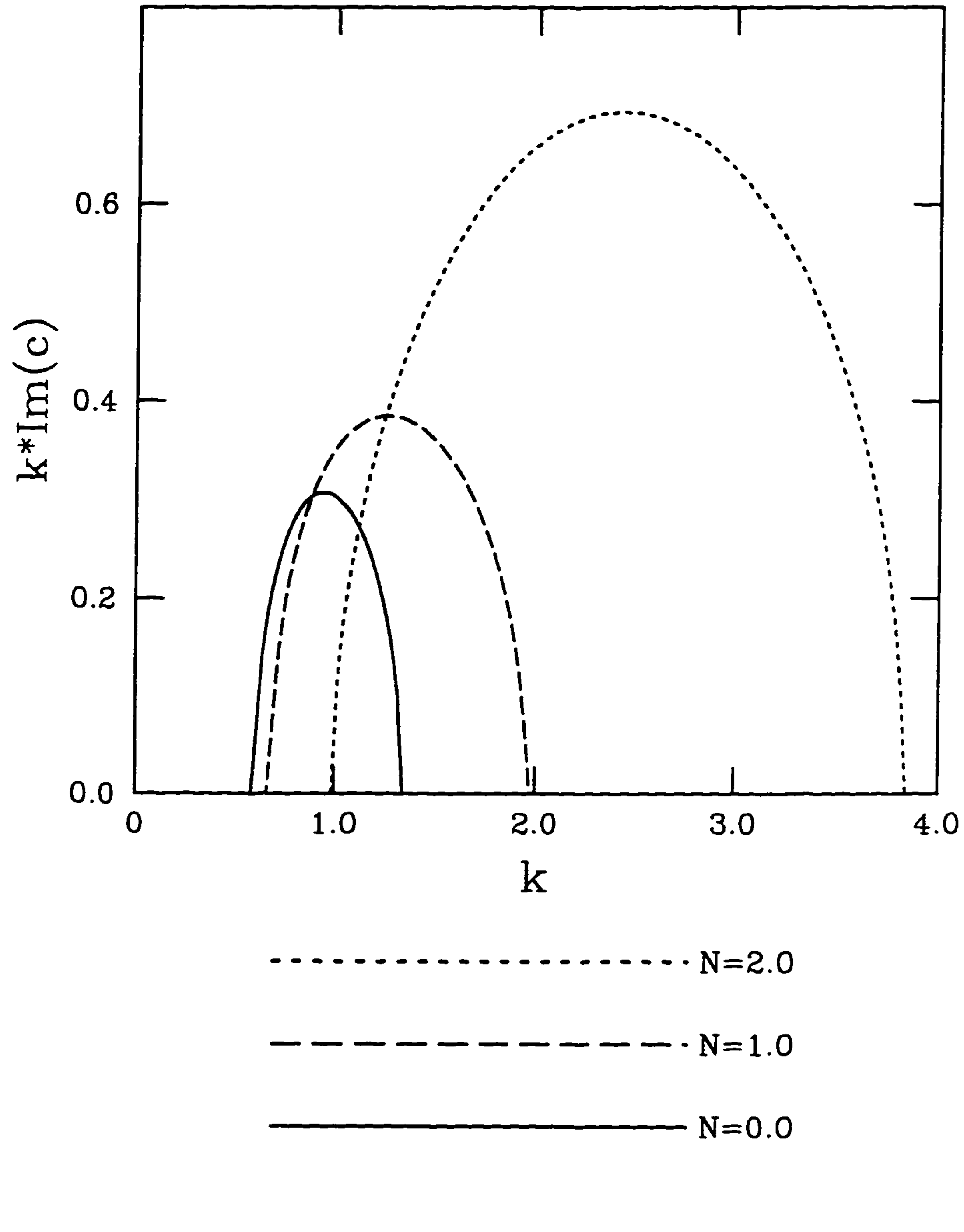
$$\times \exp(k_u c_I t) \cos\left[k_u (x - c_R t) - \arctan\left(\frac{c_I}{1-c_R}\right)\right]. \tag{3.102}$$

This is plotted at $t=0$ with $N=1.0$, $\beta=0.11$, $\mu=1.0$, $L=8.0$, $n=1$, $A=0.005$ and $x_R=5.0$, with a contour separation of 0.01, in Figure 3.3. Adding this perturbation to the steady state (3.80), yields the total height which is illustrated in Figure 3.4 with a contour separation of 0.05. This figure demonstrates how the wedge front initially becomes unstable, with plumes forming on the down sloping side.

Plotting the pressure field (3.101) at $t=0$ for the same parameters as before, at the bottom, middle and top of the upper layer, gives rise to Figures 3.5, 3.6 and 3.7, respectively. These three plots are qualitatively similar, with the only difference being that the range of contours decreases with height. The contour intervals are, 0.004, 0.002 and 0.001 respectively.

Comparing Figure 3.3 with Figures 3.5, 3.6 and 3.7 demonstrates that the perturbation height is shifted in the positive x direction, the shift angle being 1.38 radians. This is important since one of the necessary conditions for baroclinic instability to occur is that there is a phase shift between wave fields of the two layers in the direction of its motion (Cushman-Roisin, 1994), which is indeed satisfied.

Figure 3.2, Growth Rates



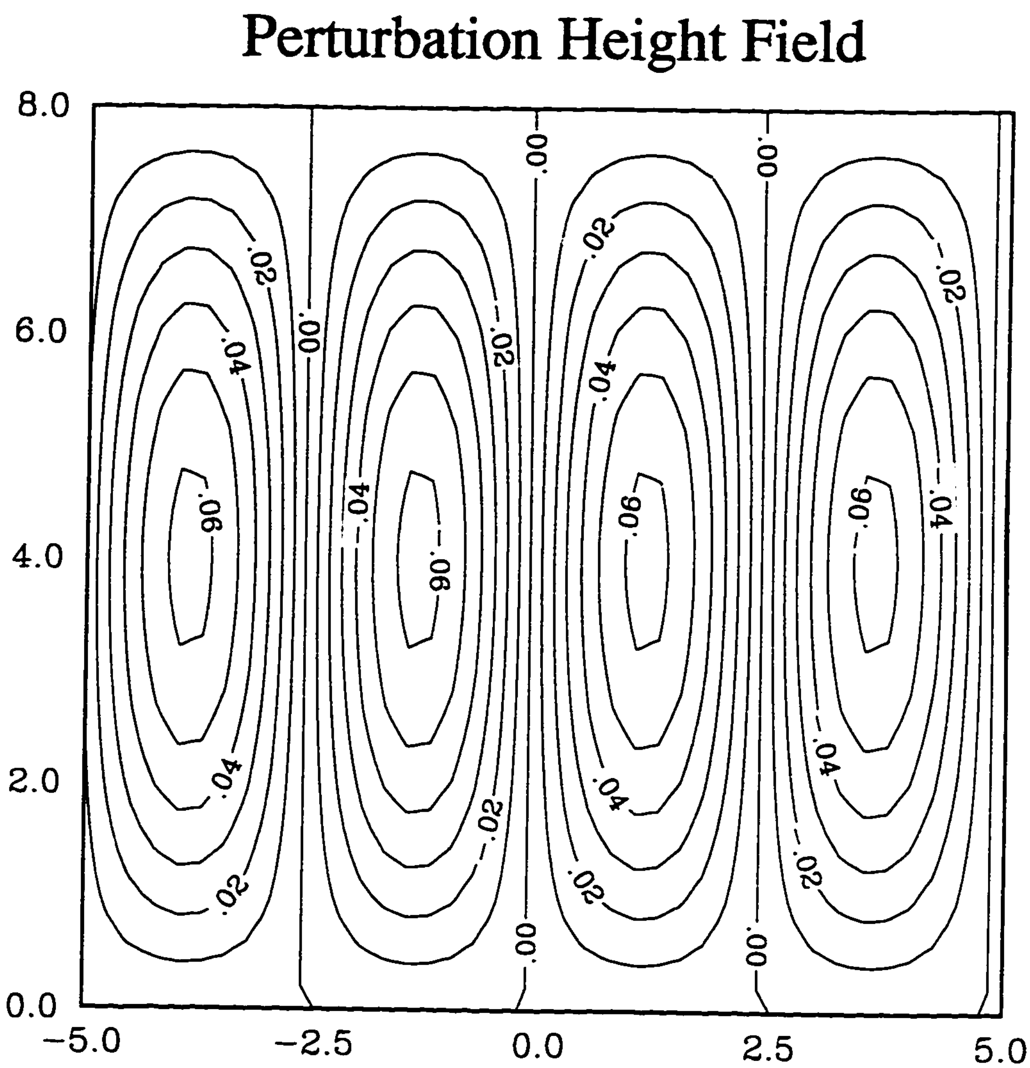


Figure 3.3, Linear stability results for the simple wedge front.

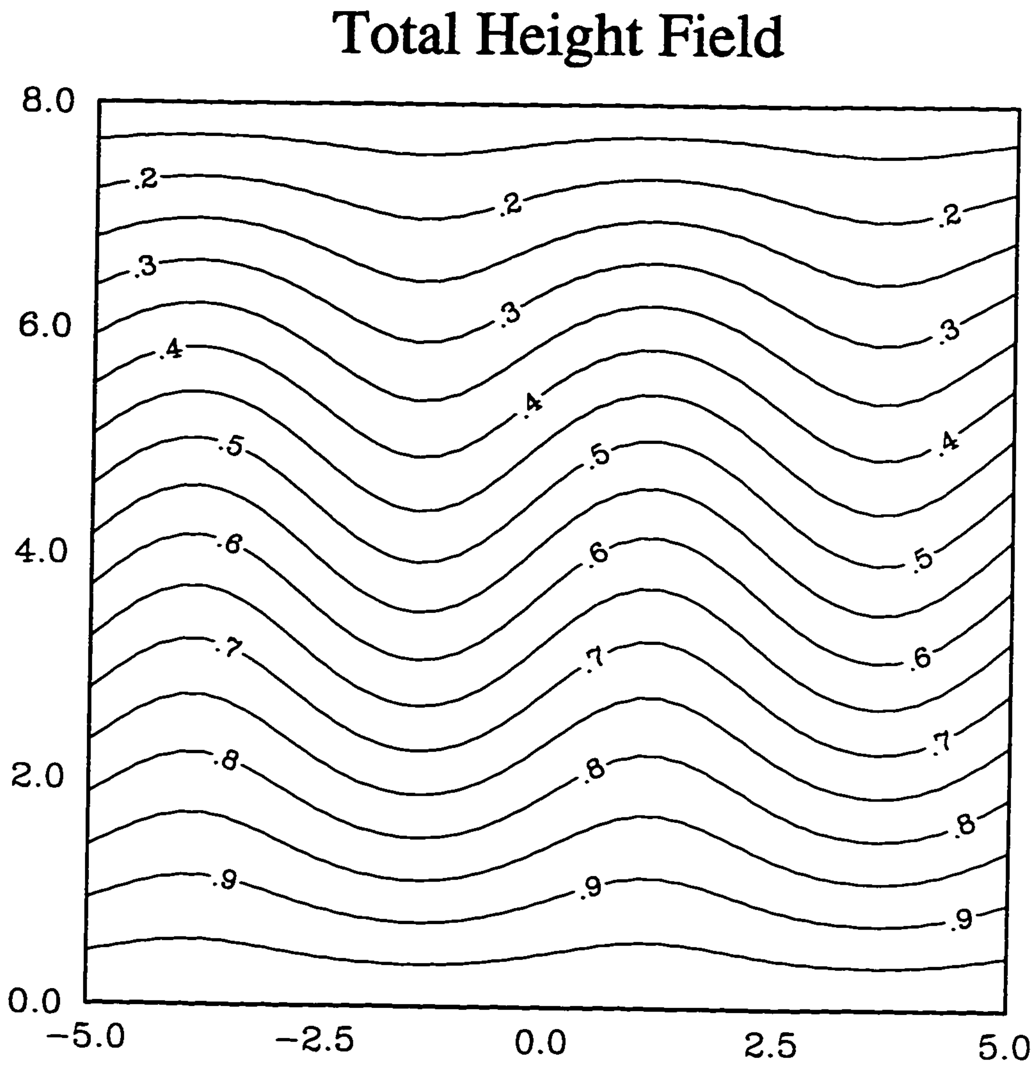


Figure 3.4, Linear stability results for the simple wedge front.

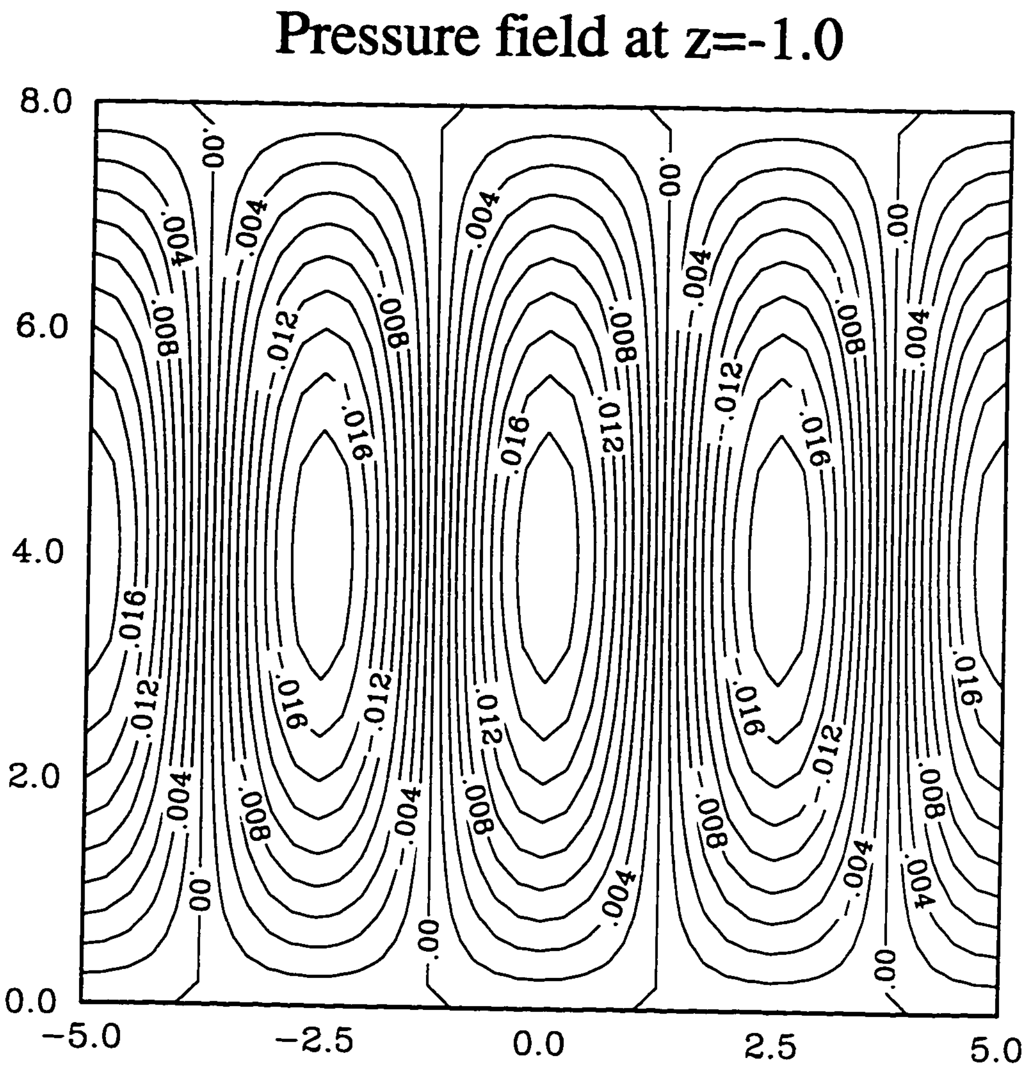


Figure 3.5, Linear stability results for the simple wedge front.

Pressure field at z=-0.5

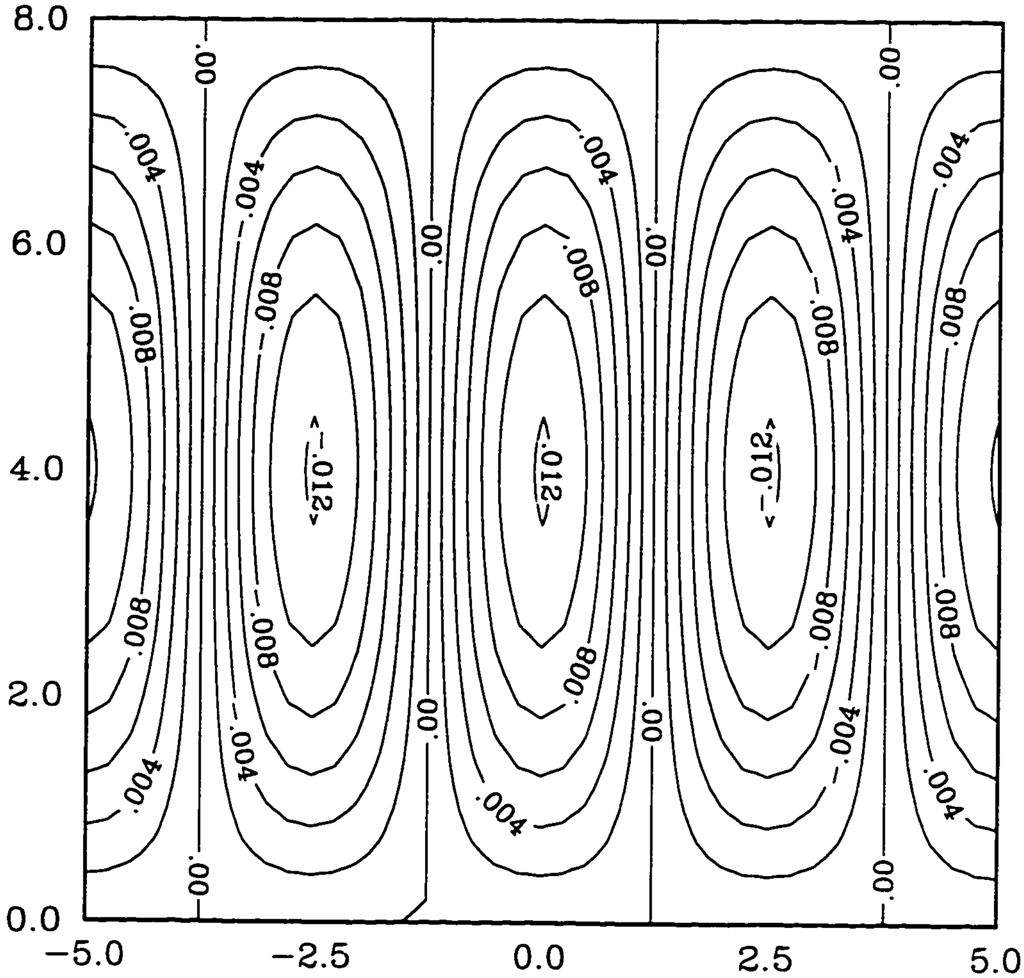


Figure 3.6, Linear stability results for the simple wedge front.

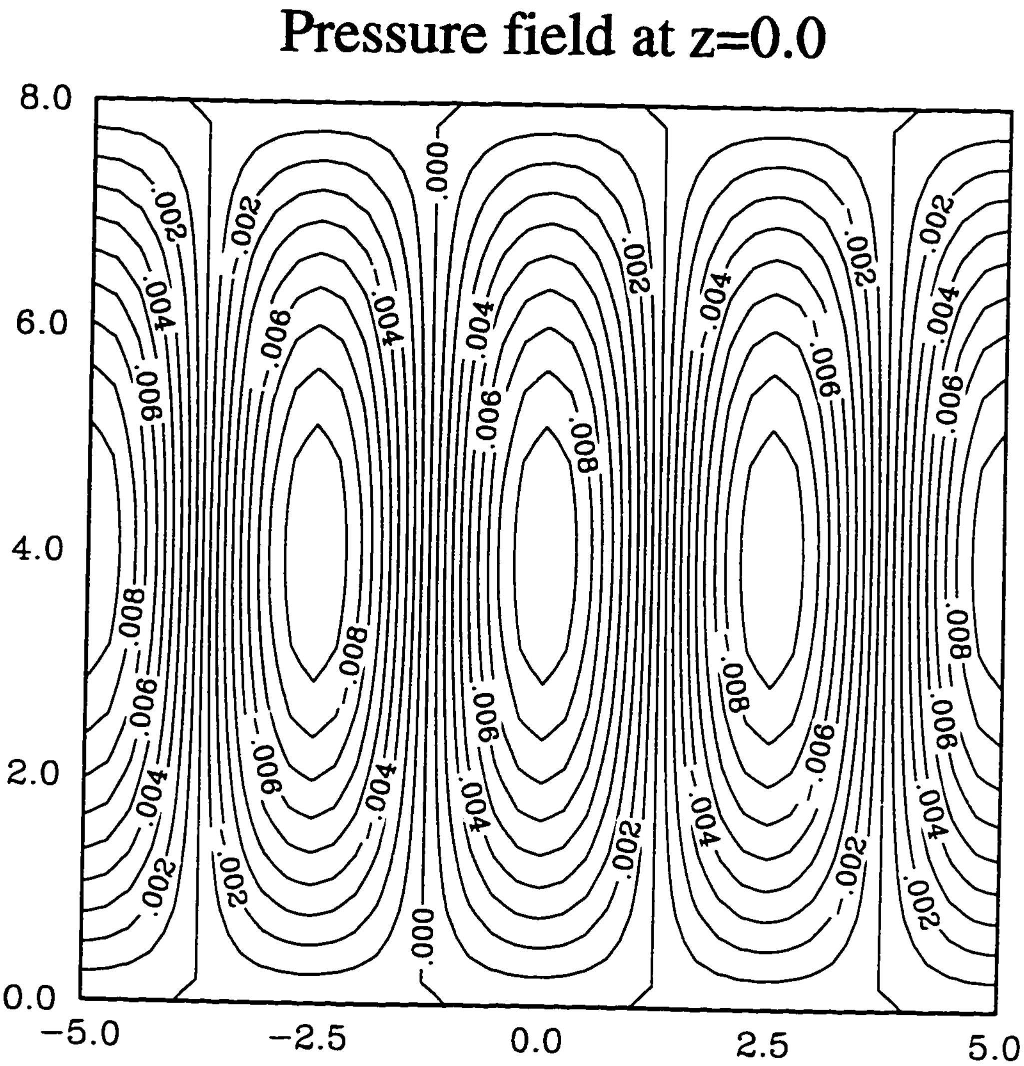


Figure 3.7, Linear stability results for the simple wedge front.

Chapter 4

Hamiltonian Formalism

The origin of Hamiltonian Formalism lies within classical particle mechanics. To see how this structure can be exploited in order to better understand the system, the reader is referred to Goldstein (1980). The extension from discrete systems to continuous ones, as arise in GFD, is a relatively recent event.

The three-dimensional Euler equations for compressible stratified fluids in their conservative forms have been shown to be Hamiltonian (Morrison & Greene, 1980). This is significant since this is the basis system through which virtually all GFD models are derived from. When approximating the Euler equations it would be desirable to maintain this Hamiltonian structure, due to the wealth of information that is derivable from it. However, when doing asymptotic expansions, there is no guarantee that this will occur. Lorenz (1967) found that making the hydrostatic approximation of the Euler equations on a sphere, meant the loss of the conservation of angular momentum, which implies that the Hamiltonian structure itself is violated. In order to salvage the approximation, it was necessary to introduce new terms into the equations so that the conservation law could be maintained.

It will be shown that the model being discussed in this thesis is indeed Hamiltonian. This structure will then be exploited to obtain conserved quantities due to symmetries present in the system. Then stability criteria for steady solutions with respect to infinitesimal perturbations, which upon simplification reduce to the conditions obtained in the linear stability analysis. As well, stability conditions will be found for finite amplitude perturbations that guarantee nonlinear stability.

4.1 Definition of the Hamiltonian structure

For a discrete system the equations of motion are ordinary differential equations, whereas for continuous systems, such as with fluids, partial differential equations are what govern the motion. A system of n partial differential equations can be written abstractly as

$$\Psi\left(\mathbf{q},\, \partial_{x_i}, \partial_t\right) = 0, \tag{4.1}$$

where $\mathbf{q} = (q_1(\mathbf{x}, t), ..., q_n(\mathbf{x}, t))^T$, is a column vector that depends on time t and m independent spatial variables $\mathbf{x} = (x_1, ..., x_m)$, all contained within a region $\Omega \subseteq \mathbf{IR}^m$ that could have a boundary denoted by $\partial\Omega$ (Swaters, 1993b).

For a system to be Hamiltonian, there are three criteria that must be met (Shepherd, 1994 & Swaters, 1993b). The first is that there must be a functional $H(\mathbf{q})$ that is conserved with respect to time, that is called the Hamiltonian. The second requires there to be a matrix J of differential operators such that the equations of motion (4.1), can be written as

$$\mathbf{q}_t = J\frac{\delta H}{\delta \mathbf{q}}. \tag{4.2}$$

The term $\delta H/\delta \mathbf{q}$ denotes the vector variational, or Euler-Lagrange, derivative of H with respect to $\mathbf{q}$, as defined in Gelfand (1963).

The third and final criteria is that the bracket, defined with respect to the appropriate inner product $\langle *_1, *_2 \rangle$, is given by

$$[F, G] = \left\langle \frac{\delta F}{\delta \mathbf{q}}, J\frac{\delta G}{\delta \mathbf{q}} \right\rangle, \tag{4.3}$$

for arbitrary smooth functionals F and G of $\mathbf{q}$. This bracket satisfies the four properties of being skew-symmetric or (anticommutative)

$$[F, G] = -[G, F], \tag{4.4}$$

bilinear

$$[\alpha F + \beta G, H] = \alpha [F, H] + \beta [G, H], \tag{4.5}$$

a derivation

$$[FG, H] = F [G, H] + [F, H] G, \tag{4.6}$$

and finally, satisfies the Jacobi identity

$$[F, [G, H]] + [G, [H, F]] + [H, [F, G]] = 0, \tag{4.7}$$

where Q is another arbitrary smooth functional of $\mathbf{q}$ and α and β are constants.

It should be noted that a fifth condition is usually listed as well, that being self-commutation. However, since this follows directly from skew-symmetry, it is not necessary to prove it separately. If a bracket satisfies these four properties, it has the effect of making the space of functionals into a Lie Algebra and (4.3) is called a Poisson bracket (Arnol'd, 1978). The classical example of an operator that satisfies these properties is the Jacobian, as has been proven in Goldstein (1980). Knowing this, these properties of the Jacobian will be applied without proof, as has already been done.

If the matrix J is nonsingular then the system is called invertible, but if J is singular, as happens when dealing with continuous systems, it is called noninvertible. If the system is noninvertible, this implies that there can exist nontrivial functionals C, that satisfy the equation

$$J \frac{\delta C}{\delta \mathbf{q}} = 0, \tag{4.8}$$

which are called Casimirs. They form the kernel of the J matrix and are important in establishing a variational principle which is essential in obtaining stability theorems.

For any Hamiltonian system, it is known (e.g., Shepherd, 1994) that (4.2) is equivalent to the equation

$$\mathbf{q}_t = [\mathbf{q}, H]. \tag{4.9}$$

Hence, instead of proving that the equations of motion follow from (4.2), it could be proven that (4.9) results from the equations of motion. Within the Hamiltonian framework, (4.2) can also be used to prove the following equation that yields an expression of the total derivative of a functional (Shepherd, 1994),

$$\frac{dF}{dt} = [F, H], \tag{4.10}$$

for any admissible functional F.

4.2 Discrete Hamiltonian Systems

It is insightful to consider the Hamiltonian for a discrete system since it gives a physical interpretation of this quantity. As well, it is useful in understanding how ideas are extended from an invertible to a noninvertible system. The definition of the Hamiltonian for a discrete system is the same as that for the continuous system, except that the functionals must all be replaced by functions, and the variational derivatives are exchanged with partial derivatives.

For a system of N particles, each of mass m_i and position $\mathbf{q}_i$, in three-dimensional space for $i = 1, ..., N$, the Hamiltonian function is defined to be the sum of the kinetic and potential energies of the system. Assuming that the potential energy V, is only a function of the position, the Hamiltonian is

$$H = \sum_{i=1}^{N} \frac{\mathbf{p}_i^2}{2m_i} + V(\mathbf{q}_1, ..., \mathbf{q}_N), \tag{4.11}$$

where the generalized coordinates $\mathbf{q}_i$ are the positions of the particles in three-dimensional space and the generalized momenta are defined to be

$$\mathbf{p}_i = m_i \dot{\mathbf{x}}_i. \tag{4.12}$$

The equations of motion can then be written in symplectic form (Goldstein, 1980),

$$\frac{d\mathbf{u}}{dt} = \mathbf{M}\frac{\partial H}{\partial \mathbf{u}}, \tag{4.13}$$

where $\mathbf{u} = (\mathbf{q}_1, ..., \mathbf{q}_N, \mathbf{p}_1, ...\mathbf{p}_N)$ and

$$\mathbf{M} = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}, \tag{4.14}$$

where I is the $N \times N$ identity matrix. This Hamiltonian system is invertible with phase space of dimension $2N$, which is finite for any number of particles. Writing out (4.13) in component form gives

$$\dot{\mathbf{q}}_i = \frac{\partial H}{\partial \mathbf{p}_i} = \frac{\mathbf{p}_i}{m_i}, \qquad \dot{\mathbf{p}}_i = -\frac{\partial H}{\partial \mathbf{q}_i} = -\frac{\partial V}{\partial \mathbf{q}_i}, \tag{4.15}$$

having computed the partial derivatives of H.

To verify that the Hamiltonian is a conserved quantity it is necessary to find the total derivative of H with respect to time and then substitute in (4.15), as is done below

$$\begin{aligned} \frac{dH}{dt} &= \sum_{i=1}^{N} \left(\mathbf{p}_i \dot{\mathbf{p}}_i + \frac{\partial V}{\partial \mathbf{q}_i} \dot{\mathbf{q}}_i \right) \\ &= \sum_{i=1}^{N} \left(-\mathbf{p} \frac{\partial V}{\partial \mathbf{q}_i} + \frac{\partial V}{\partial \mathbf{q}_i} \mathbf{p}_i \right) \\ &= 0. \end{aligned} \tag{4.16}$$

This proves the first requirement, where the second is shown by considering (4.15). The first equation is simply a restatement of (4.12), which when substituted into the second yields the equations of motion, which are

$$m_i \ddot{\mathbf{q}}_i = -\frac{\partial V}{\partial \mathbf{q}_i}, \tag{4.17}$$

as can be confirmed in Arnol'd (1978). The bracket for this problem is

$$\begin{aligned} [f, g] &= \left(\frac{\partial f}{\partial \mathbf{u}} \right)^T \mathbf{M} \frac{\partial g}{\partial \mathbf{u}} \\ &= \sum_{i=1}^{N} \frac{\partial f}{\partial \mathbf{q}_i} \frac{\partial g}{\partial \mathbf{p}_i} - \frac{\partial f}{\partial \mathbf{p}_i} \frac{\partial g}{\partial \mathbf{q}_i}. \end{aligned} \tag{4.18}$$

The proof that this satisfies the required algebraic properties is presented in Goldstein (1980), and hence will be omitted.

Therefore, any system of particles in such a potential field is proven to be Hamiltonian. Now that we have seen the outlines of a proof of the Hamiltonian structure for the discrete case, it is possible to extend the same ideas for a continuous system.

4.3 Hamiltonian structure for our Model

The geometry to be considered within this chapter is that of a periodic channel as in Chapter 3, see (3.1). Because of this geometry certain horizontal boundary conditions arise. The first is that there cannot be any normal flow through the channel walls. The second states that all the variables and their derivatives take on equal values at $x = -x_R,\ x_R$, due to the fact that these positions are coincident. These conditions can be written as

$$\varphi_x = h_x = 0 \quad \text{on} \quad y = 0, L, \tag{4.19}$$

$$\{\varphi, h\}_{x=x_R} = \{\varphi, h\}_{x=-x_R}, \tag{4.20}$$

where the curly bracket denotes that not only φ and h are periodic, but all of their derivatives as well. The third condition is that circulation is conserved along each channel wall, as in (3.5),

$$\frac{d}{dt}\int_{-x_R}^{x_R} \varphi_y\big|_{y=L}\, dx = \frac{d}{dt}\int_{-x_R}^{x_R} \varphi_y\big|_{y=0}\ dx = 0. \tag{4.21}$$

In order to derive an expression of the horizontal integral of the Jacobian of two functions, we rewrite the expression as

$$\begin{aligned} \iint_{\Omega_H} \partial(f,g)\ dxdy &= \iint_{\Omega_H} \nabla\cdot(g\mathbf{e}_3 \times \nabla f)\ dxdy \\ &= \oint_{\partial\Omega_H} (g\mathbf{e}_3 \times \nabla f)\ \cdot \mathbf{n}\ ds, \end{aligned} \tag{4.22}$$

using the identity

$$\partial(f,g) = \nabla\cdot(g\mathbf{e}_3 \times \nabla f), \tag{4.23}$$

and the divergence theorem. The domain in consideration demands that the boundary integral in (4.22) takes the form

$$\iint\limits_{\Omega_H} \partial(f,g)\ dxdy = \int_0^L [gf_y]_{x=x_R} - [gf_y]_{x=-x_R}\ dy$$

$$+\int_{-x_R}^{x_R} [gf_x]_{y=L} - [gf_x]_{y=0}\ dx. \tag{4.24}$$

The first two boundary terms cancel by periodicity in the channel, where it then follows that

$$\iint\limits_{\Omega_H} \partial(f,g)\ dxdy = 0 \quad \text{if} \quad f_x = 0 \text{ on } y = 0, L, \tag{4.25}$$

since this forces the remaining terms in (4.24) to vanish individually. Since in our system both the state variables φ and h satisfy the boundary conditions mentioned above, this condition will be used repeatedly in the calculations to follow.

Any mean state that is a solution to the system of equations, must satisfy (4.19), (4.20) and (4.21). If we consider perturbations being superimposed on the mean flow, the sum of the two, must also satisfy these boundary conditions. Since each of the conditions are linear, this implies that the perturbations themselves satisfy the same equations

$$\delta\varphi_x = \delta h_x = 0 \quad \text{on} \quad y = 0, L, \tag{4.26}$$

$$\{\delta\varphi, \delta h\}_{x=x_R} = \{\delta\varphi, \delta h\}_{x=-x_R}\,, \tag{4.27}$$

in addition to the conservation of circulation along each channel wall.

In the discrete case the Hamiltonian function was nothing more than the total energy of the system. It has been found in Holm *et al.* (1985) that the Hamiltonian functional for the continuously stratified QG equations, is the integral of the total energy. This quantity must be contained in our Hamiltonian since the upper fluid in our model is QG. However, another term needs to be introduced because of the lower layer and the potential energy associated with it. This is done by introducing the square of the sum of the bottom topography and the lower layer height. Since we require that this potential energy term be zero if the height of the lower layer is zero, it is necessary to subtract the square of the bottom topography term (Swaters, 1993b). As well, we must include circulation terms for each channel wall, since this enables us to establish a variational principle.

The Hamiltonian is then chosen to be

$$H = \frac{\mu}{2} \iiint\limits_{\Omega} \nabla\varphi \cdot \nabla\varphi + (\varphi_z/N)^2 \, dxdydz$$

$$+\frac{\mu}{2} \iint\limits_{\Omega_H} (h + h_B/\mu)^2 - (h_B/\mu)^2 \, dxdy$$

$$+\mu \int_{-1}^{0} \left\{ \Gamma_0 \int_{-x_R}^{x_R} \varphi_y|_{y=0} \; dx - \Gamma_L \int_{-x_R}^{x_R} \varphi_y|_{y=L} \; dx \right\} dz, \tag{4.28}$$

with

$$q_1 = \Delta\varphi + \left(\varphi_z/N^2\right)_z, \qquad q_2 = \varphi_z|_{z=0},$$

$$q_3 = \varphi_z|_{z=-1} + N^2 (h + h_B/\mu), \qquad q_4 = h, \tag{4.29}$$

and

$$\Gamma_L = \varphi|_{y=L}, \qquad \Gamma_0 = \varphi|_{y=0}, \tag{4.30}$$

where Γ_L and Γ_0 are the functions of z that φ takes on $y = L$ and 0, respectively on account of (4.19). The Poisson bracket is defined to be

$$[F, G] = \iiint\limits_{\Omega} \frac{\delta F}{\delta q_1} \partial \left(\frac{\delta G}{\delta q_1}, q_1 \right) \; dxdydz - \iint\limits_{\Omega_H} \left[N^2 \frac{\delta F}{\delta q_2} \partial \left(\frac{\delta G}{\delta q_2}, q_2 \right) \right]_{z=0} \; dxdy$$

$$+ \iint\limits_{\Omega_H} \left[N^2 \frac{\delta F}{\delta q_3} \partial \left(\frac{\delta G}{\delta q_3}, q_3 \right) \right]_{z=-1} \; dxdy - \iint\limits_{\Omega_H} \frac{\delta F}{\delta q_4} \partial \left(\frac{\delta G}{\delta q_4}, q_4 \right) \; dxdy, \tag{4.31}$$

where F and G are arbitrary smooth functionals. This bracket requires that the cosymplectic form is

$$J = \begin{bmatrix} \partial(*, q_1) & 0 & 0 & 0 \\ 0 & -N^2(0)\,\partial(*, q_2) & 0 & 0 \\ 0 & 0 & N^2(-1)\,\partial(*, q_3) & 0 \\ 0 & 0 & 0 & -\partial(*, q_4) \end{bmatrix}, \tag{4.32}$$

where the inner product is the integral over the volume Ω for the first component and the other three integrals are over the horizontal region Ω_H. Equation (4.29) allows the governing equations to be written compactly as

$$q_{1_t} + \partial(\mu\varphi, q_1) = 0, \tag{4.33}$$

$$q_{2_t} + \partial\left(\mu\varphi|_{z=0}, q_2\right) = 0, \tag{4.34}$$

$$q_{3_t} + \partial\left(\mu\varphi|_{z=-1}, q_3\right) = 0, \tag{4.35}$$

$$q_{4_t} + \partial\left(\mu\varphi|_{z=-1} + h_B, q_4\right) = 0, \tag{4.36}$$

having used the self-commutation property of the Jacobian.

Using variational calculus, as explained in (Gelfand, 1963), we find the variation of the Hamiltonian to be

$$\begin{aligned} \delta H &= \mu \iiint_\Omega \nabla\varphi\cdot\nabla\delta\varphi + N^{-2}\varphi_z\delta\varphi_z \; dxdydz + \mu\iint_{\Omega_H}(h + h_B/\mu)\,\delta h\; dxdy \\ &\quad + \mu\int_{-1}^{0}\int_{-x_R}^{x_R}\left[\varphi\delta\varphi_y\right]_{y=0}\,dx - \int_{-x_R}^{x_R}\left[\varphi\delta\varphi_y\right]_{y=L}\,dxdz \\ &= \mu\iiint_\Omega \nabla\cdot(\varphi\nabla\delta\varphi) - \varphi\Delta\delta\varphi - \varphi\left(\delta\varphi_z/N^2\right)_z dxdydz \\ &\quad + \mu\iint_{\Omega_H}\left[N^{-2}\varphi\delta\varphi_z\right]_{z=-1}^{z=0} + (h + h_B/\mu)\,\delta h\; dxdy \\ &\quad + \mu\int_{-1}^{0}\Gamma_0\int_{-x_R}^{x_R}\delta\varphi_y|_{y=0}\,dx - \Gamma_L\int_{-x_R}^{x_R}\delta\varphi_y|_{y=L}\,dxdz \end{aligned} \tag{4.37}$$

where integration by parts and the identity (3.29) have been used. Rewriting the first term on the right-hand side by applying the divergence theorem and then cancelling terms due to periodicity gives,

$$\iiint\limits_{\Omega} \nabla\cdot(\varphi\nabla\delta\varphi)\, dxdydz = \int_{-1}^{0}\oint_{\partial\Omega_H} \varphi\nabla\delta\varphi\cdot\mathbf{n}\, dsdz$$

$$= \int_{-1}^{0}\Gamma_L \int_{-x_R}^{x_R} \delta\varphi_y\big|_{y=L}\, dx - \Gamma_0\int_{-x_R}^{x_R}\delta\varphi_y\big|_{y=0}\, dxdz, \tag{4.38}$$

by use of (4.30). Note that the no normal flow boundary condition prevents φ from depending on x on the channel walls which allows for Γ_L and Γ_0 to be brought in front of the x integral.

Substituting (4.38) into (4.37) and rewriting terms using (4.33) to (4.36) yields

$$\delta H = \iiint\limits_{\Omega} (-\mu\varphi)\,\delta q_1 dxdydz + \iint\limits_{\Omega_H} \left[\mu N^{-2}\varphi\delta q_2\right]_{z=0} dxdy$$

$$- \iint\limits_{\Omega_H} \left[\mu N^{-2}\varphi\delta q_3\right]_{z=-1} - (\mu\varphi + \mu h + h_B)\,\delta q_4 dxdy, \tag{4.39}$$

after cleverly adding zero in order that we can write δq_3. Hence, we conclude that the variational derivatives are

$$\frac{\delta H}{\delta q_1} = -\mu\varphi, \qquad \frac{\delta H}{\delta q_2} = \left[N^{-2}\mu\varphi\right]_{z=0},$$

$$\frac{\delta H}{\delta q_3} = -\left[N^{-2}\mu\varphi\right]_{z=-1}, \qquad \frac{\delta H}{\delta q_4} = \left(\mu\varphi|_{z=-1} + \mu h + h_B\right). \tag{4.40}$$

It is then possible to begin proving that this is indeed an appropriate Hamiltonian Structure.

4.3.1 H is a conserved quantity

To show that H is a conserved quantity, we take the time derivative of the Hamiltonian then follow the same steps that were taken when calculating the variational derivative,

$$\frac{dH}{dt} = \mu\iiint_{\Omega} \nabla\varphi\cdot\nabla\varphi_t + N^{-2}\varphi_z\varphi_{zt}\, dxdydz + \mu\iint_{\Omega_H}(h + h_B/\mu)\, h_t\, dxdy$$

$$
\begin{aligned}
&+\mu \int_{-1}^{0} \int_{-x_R}^{x_R} \left[\varphi\varphi_{yt}\right]_{y=0} \, dx - \int_{-x_R}^{x_R} \left[\varphi\varphi_{yt}\right]_{y=L} \, dxdz \\
=\; & -\mu \iiint_{\Omega} \varphi\Delta\varphi_t + \varphi\left(\varphi_{zt}/N^2\right)_z \, dxdydz + \mu \int_{-1}^{0} \oint_{\partial\Omega_H} \varphi\nabla\varphi_t \cdot \mathbf{n} \, dsdz \\
&+\mu \iint_{\Omega_H} \left[N^{-2}\varphi\varphi_{zt}\right]_{z=-1}^{z=0} + (h + h_B/\mu)\, h_t \, dxdy \\
&+\mu \int_{-1}^{0} \Gamma_0 \int_{-x_R}^{x_R} \varphi_{yt}\big|_{y=0} \, dx - \Gamma_L \int_{-x_R}^{x_R} \varphi_{yt}\big|_{y=L} \, dxdz. \qquad (4.41)
\end{aligned}
$$

Equation (4.21) enables us to eliminate the last two terms since each is a conserved quantity. The contour integral directly above is identically zero since it can be decomposed into four parts. The first two cancel each other out by periodicity, whereas the remaining two are each zero, as has just been stated,

$$
\begin{aligned}
\int_{-1}^{0} \oint_{\partial\Omega_H} \varphi\nabla\varphi_t \cdot \mathbf{n} \, dsdz \;=\; & \int_{-1}^{0} \left\{ \int_{0}^{L} \left[\varphi\varphi_{xt}\right]_{x=x_R} - \left[\varphi\varphi_{xt}\right]_{x=-x_R} \, dy \right. \\
& \left. - \int_{-x_R}^{x_R} \left[\varphi\varphi_{yt}\right]_{y=L} - \left[\varphi\varphi_{yt}\right]_{y=0} \, dx \right\} \\
=\; & 0. \qquad (4.42)
\end{aligned}
$$

Substituting (2.163) through to (2.165) and then rewriting terms using (4.29) produces

$$
\begin{aligned}
\frac{dH}{dt} =& -\iiint_{\Omega} \mu\varphi q_{1_t} dxdydz + \iint_{\Omega_H} \left[N^{-2}\mu\varphi q_{2_t}\right]_{z=0} dxdy \\
&- \iint_{\Omega_H} \left[N^{-2}\mu\varphi q_{2_t}\right]_{z=-1} - \left(\mu\, \varphi|_{z=-1} + \mu h + h_B\right) q_{4_t} dxdy. \qquad (4.43)
\end{aligned}
$$

Applying the governing equations (4.33) to (4.36), and grouping terms implies

$$
\begin{aligned}
\frac{dH}{dt} =& \frac{1}{2} \iiint_{\Omega} \partial\left((\mu\varphi)^2, q_1\right) dxdydz - \frac{1}{2} \iint_{\Omega_H} \partial\left((\mu\varphi/N)^2, q_2\right)_{z=0} dxdy \\
&+ \frac{1}{2} \iint_{\Omega_H} \partial\left((\mu\varphi/N)^2, q_3\right)_{z=-1} - \frac{1}{2} \iint_{\Omega_H} \partial\left((\mu\varphi + h_B)^2, q_4\right)_{z=-1} dxdy, \qquad (4.44)
\end{aligned}
$$

having used the fact that the Jacobian self-commutes. Clearly, each of these terms is equal to zero by (4.25) and (4.19), and therefore,

$$\frac{dH}{dt} = 0, \tag{4.45}$$

which ensures that H is an invariant of the dynamics with respect to time.

4.3.2 Governing equation derivation

An essential component of the Hamiltonian structure is that the equations of motion are contained within the framework. This is demonstrated by showing that (4.33) through to (4.36) is equivalent to (4.2). Substituting (4.40) and (4.32) into (4.2) yields this result easily. As has already been stated, this is equivalent to saying that the governing equations imply that (4.9) is true, which is what will be proven component wise. The first component upon expanding takes the form

$$[q_1, H] = \iiint\limits_{\Omega} \frac{\delta q_1}{\delta q_1} \partial \left(\frac{\delta H}{\delta q_1}, q_1 \right) \, dx dy dz. \tag{4.46}$$

Care must be taken in the interpretation of $\delta q_1 / \delta q_1$.

If we write q_i as the functional

$$q_i(\mathbf{x}) = \iiint\limits_{\bar{\Omega}} \delta(\mathbf{x} - \bar{\mathbf{x}}) \, q_i(\bar{\mathbf{x}}) \, d\bar{\mathbf{x}}, \tag{4.47}$$

where $\delta(\mathbf{x} - \bar{\mathbf{x}})$ is the Kronecker delta function centered at $\mathbf{x} = \bar{\mathbf{x}}$. It follows that

$$\delta q_i(\mathbf{x}) = \iiint\limits_{\bar{\Omega}} \delta(\mathbf{x} - \bar{\mathbf{x}}) \, \delta q_i(\bar{\mathbf{x}}) \, d\bar{\mathbf{x}}, \tag{4.48}$$

and by the definition of the variational derivative, we conclude that

$$\frac{\delta q_i}{\delta q_j} = \delta(\mathbf{x} - \bar{\mathbf{x}}). \tag{4.49}$$

For $i = 1$, the right-hand side of the above equation is the product of the delta functions of x, y and z, whereas for $i = 2, 3, 4$, it is only a product of delta functions of x and y.

Substituting (4.49) into (4.46), and denoting the variables of integration with overbars yields,

$$
\begin{aligned}
[q_1, H] &= -\iiint_\Omega \delta(x-\bar{x})\,\delta(y-\bar{y})\,\delta(z-\bar{z})\,\partial(\mu\varphi, q_1)\; d\bar{x}d\bar{y}d\bar{z} \\
&= -\partial(\mu\varphi, q_1) \\
&= q_{1_t},
\end{aligned} \tag{4.50}
$$

where (4.33) has been used. When the Jacobian appears under the integral sign it is a function of the overbared variables whereas when it does not, it is a function of the regular variables.

Proceeding analogously for the other three equations gives,

$$
\begin{aligned}
[q_2, H] &= -\iint_{\Omega_H} \delta(x-\bar{x})\,\delta(y-\bar{y}) \left[N^2\partial\left(\mu N^{-2}\varphi, q_2\right)\right]_{z=0} d\bar{x}d\bar{y} \\
&= -\partial\left(\mu\varphi|_{z=0}, q_2\right) \\
&= q_{2_t},
\end{aligned} \tag{4.51}
$$

as well

$$
\begin{aligned}
[q_3, H] &= -\iint_{\Omega_H} \delta(x-\bar{x})\,\delta(y-\bar{y}) \left[N^2\partial\left(\mu N^{-2}\varphi, q_3\right)\right]_{z=-1} d\bar{x}d\bar{y} \\
&= -\partial\left(\mu\varphi|_{z=-1}, q_3\right) \\
&= q_{3_t},
\end{aligned} \tag{4.52}
$$

and finally

$$
\begin{aligned}
[q_4, H] &= -\iint_{\Omega_H} \delta(x-\bar{x})\,\delta(y-\bar{y}) \left[\partial(\mu\varphi + h_B, q_4)\right]_{z=0} d\bar{x}d\bar{y} \\
&= -\partial\left(\mu\varphi|_{z=-1} + h_B, q_4\right) \\
&= q_{4_t}.
\end{aligned} \tag{4.53}
$$

Where again, the self-commutation of the Jacobian has been implemented along with (4.34) to (4.36). The second requirement of the Hamiltonian is then fulfilled.

Incidently, this also establishes (4.10), which then gives us another way of proving the conservation property of the Hamiltonian. Since it must be an admissible functional it must satisfy the equation

$$\frac{dH}{dt} = [H, H] = 0, \tag{4.54}$$

by the self-commutation law of the Poisson bracket.

4.3.3 Proof of the Algebraic properties

To prove the algebraic properties of the Poisson bracket requires making certain restrictions on the admissible functionals (McIntyre & Shepherd, 1987). The first is that they are smooth. The second is that each functional must be expressible in the following form

$$F = \iiint\limits_{\Omega} f(q_1)\ dxdydz + \iint\limits_{\Omega_H} g(q_2, q_3, q_4)\ dxdy, \tag{4.55}$$

for some functions f and g. The third is that the functionals are either Casimirs or satisfy the conditions that

$$\frac{\partial}{\partial x}\left(\frac{\delta F}{\delta q_i}\right) = 0 \quad \text{on} \quad y = 0, L, \tag{4.56}$$

$$\left\{\frac{\delta F}{\delta q_i}\right\}_{x=x_R} = \left\{\frac{\delta F}{\delta q_i}\right\}_{x=-x_R}, \tag{4.57}$$

as i ranges from 1 to 4. Both of these conditions are inspired by the fact that the functionals are really 'functionals of state' (McIntyre & Shepherd, 1987) in that they are completely determined by the state of the system. It has been shown that the entire system reduces to solving for two variables φ and h, each of which satisfy these two conditions listed because of the channel domain. Therefore, since the functionals depend only on variables that satisfy these conditions, they themselves, should satisfy the same conditions.

Before the algebraic properties are proven, an identity needs to be established,

$$\iint_{\Omega_H} A\partial(B, C)\ dxdy \;=\; \iint_{\Omega_H} \partial(B, AC) - \partial(B, A)\,C\ dxdy$$

$$= \iint_{\Omega_H} -\partial(B,A)\, C \; dxdy$$
$$= \iint_{\Omega_H} \partial(A,B)\, C \; dxdy, \tag{4.58}$$

where A, B and C are functions that satisfy (4.56) and (4.57). The first step used the property of the Jacobian being a derivation, the second (4.23) and (4.56) and the third the anti-symmetric property of the Jacobian. As well, certain weight functions will be defined in order that summation notation can be introduced,

$$w_1 = 1, \quad w_2 = -N^2(0), \quad w_3 = N^2(-1), \quad w_4 = -1. \tag{4.59}$$

To prove that our bracket is indeed a Poisson bracket relies on the fact that the Jacobian itself is a Poisson bracket, and the properties of the variational derivative. We begin by proving anticommutation. Applying the anti-symmetry property of the Jacobian, followed by (4.58), yields

$$[F,G] = -\iiint_{\Omega} \frac{\delta F}{\delta q_1}\partial\left(q_1, \frac{\delta G}{\delta q_1}\right)\; dxdydz - \sum_{i=2}^{4} w_i \iint_{\Omega_H} \frac{\delta F}{\delta q_i}\partial\left(q_i, \frac{\delta G}{\delta q_i}\right)\; dxdy,$$

$$= -\iiint_{\Omega} \frac{\delta G}{\delta q_1}\partial\left(\frac{\delta F}{\delta q_1}, q_1\right)\; dxdydz - \sum_{i=2}^{4} w_i \iint_{\Omega_H} \frac{\delta G}{\delta q_i}\partial\left(\frac{\delta F}{\delta q_i}, q_i\right)\; dxdy,$$

$$= -[G,F]. \tag{4.60}$$

To prove bilinearity, it is sufficient to prove linearity in one component because of the skew-symmetry property that has already been proven. Expanding the left-hand side of (4.5),

$$[\alpha F + \beta G, H] = \iiint_{\Omega} \frac{\delta(\alpha F + \beta G)}{\delta q_1}\partial\left(\frac{\delta H}{\delta q_1}, q_1\right)\; dxdydz$$
$$+ \sum_{i=2}^{4} w_i \iint_{\Omega_H} \frac{\delta(\alpha F + \beta G)}{\delta q_i}\partial\left(\frac{\delta H}{\delta q_i}, q_i\right)\; dxdy, \tag{4.61}$$

then using the fact that the variational derivative is linear

$$\frac{\delta\left(\alpha F+\beta G\right)}{\delta q_i}=\alpha\frac{\delta F}{\delta q_i}+\beta\frac{\delta G}{\delta q_i}, \tag{4.62}$$

allows each integral can be separated into two different terms. Finally, factoring the constants α and β in front of the integral, enables the above two equations to be written as

$$\left[\alpha F+\beta G,H\right]=\alpha\left[F,H\right]+\beta\left[G,H\right]. \tag{4.63}$$

The derivation property is what is more commonly referred to as the product rule. It insures that the differential operators have a nice enough structure to do algebraic manipulations. Writing out the left-hand side of (4.6)

$$\begin{aligned}\left[FG,H\right] &= \iiint_{\Omega}\frac{\delta\left(FG\right)}{\delta q_1}\partial\left(\frac{\delta H}{\delta q_1},q_1\right)\,dxdydz \\ &\quad+\sum_{i=2}^{4}w_i\iint_{\Omega_H}\frac{\delta\left(FG\right)}{\delta q_i}\partial\left(\frac{\delta H}{\delta q_i},q_i\right)\,dxdy,\end{aligned} \tag{4.64}$$

and expanding the variational derivative term, since it is a derivation, yields

$$\frac{\delta\left(FG\right)}{\delta q_i}=F\frac{\delta G}{\delta q_i}+\frac{\delta F}{\delta q_i}G. \tag{4.65}$$

Since every functional is in the form (4.55), they have no dependency on x, y or z, and hence can be factored our of the integral signs. Doing this enables us to conclude that

$$\left[FG,H\right]=F\left[G,H\right]+\left[F,H\right]G. \tag{4.66}$$

The final property is the Jacobi identity, which is the only one that is nontrivial to prove, as is usually the case. It can be written in the following form,

$$\left[\left[F,G\right],H\right]+cyc=0, \tag{4.67}$$

where *cyc* represents the cyclic terms, as in Scinocca & Shepherd (1992). The bracket

(4.31), can be rewritten using (4.58) and (4.59) as

$$[F,G]=\iiint\limits_{\Omega} q_1 J\left(\frac{\delta F}{\delta q_1},\frac{\delta G}{\delta q_1}\right)dxdydz+\sum_{i=2}^{4}\iint\limits_{\Omega_H} w_i q_i J\left(\frac{\delta F}{\delta q_i},\frac{\delta G}{\delta q_i}\right)dxdy. \tag{4.68}$$

The first step is to find the variation of $[F,G]$,

$$\delta[F,G]=\iiint\limits_{\Omega}\delta q_1 J\left(\frac{\delta F}{\delta q_1},\frac{\delta G}{\delta q_1}\right)dxdydz+\sum_{i=2}^{4}\iint\limits_{\Omega_H} w_i\delta q_i J\left(\frac{\delta F}{\delta q_i},\frac{\delta G}{\delta q_i}\right)dxdy$$

$$+\iiint\limits_{\Omega} q_1 J\left(\sum_{j=2}^{4}\frac{\delta^2 F}{\delta q_j\delta q_1}\delta q_j,\frac{\delta G}{\delta q_1}\right)+q_1 J\left(\frac{\delta F}{\delta q_1},\sum_{j=2}^{4}\frac{\delta^2 G}{\delta q_j\delta q_1}\delta q_j\right)dxdydz$$

$$+\sum_{i=2}^{4} w_i\iint\limits_{\Omega_H} q_i J\left(\sum_{j=2}^{4}\frac{\delta^2 F}{\delta q_j\delta q_i}\delta q_j,\frac{\delta G}{\delta q_i}\right)+q_i J\left(\frac{\delta F}{\delta q_i},\sum_{j=2}^{4}\frac{\delta^2 G}{\delta q_j\delta q_i}\delta q_j\right)dxdy, \tag{4.69}$$

which can be rewritten through (4.58) and the skew-symmetry of the Jacobian, as

$$\delta[F,G]=\iiint\limits_{\Omega}\delta q_1\left(J\left(\frac{\delta F}{\delta q_1},\frac{\delta G}{\delta q_1}\right)+\frac{\delta^2 F}{\delta q_1^2}J\left(\frac{\delta G}{\delta q_1},q_1\right)-\frac{\delta^2 G}{\delta q_1^2}J\left(\frac{\delta F}{\delta q_1},q_1\right)\right)dxdydz$$

$$+\sum_{i=2}^{4}\iint\limits_{\Omega_H} w_i\delta q_i J\left(\frac{\delta F}{\delta q_i},\frac{\delta G}{\delta q_i}\right)dxdy$$

$$+\sum_{i=2}^{4}\sum_{j=2}^{4}\iint\limits_{\Omega_H} w_j\delta q_i\left(\frac{\delta^2 F}{\delta q_j\delta q_i}J\left(\frac{\delta G}{\delta q_j},q_j\right)-\frac{\delta^2 G}{\delta q_j\delta q_i}J\left(\frac{\delta F}{\delta q_j},q_j\right)\right)dxdy, \tag{4.70}$$

where i and j have been interchanged in the line directly above. From this equation we can see what the variational derivatives of the bracket are, for $i=2,3,4$,

$$\frac{\delta[F,G]}{\delta q_1}=J\left(\frac{\delta F}{\delta q_1},\frac{\delta G}{\delta q_1}\right)+\frac{\delta^2 F}{\delta q_1^2}J\left(\frac{\delta G}{\delta q_1},q_1\right)-\frac{\delta^2 G}{\delta q_1^2}J\left(\frac{\delta F}{\delta q_1},q_1\right),$$

$$\frac{\delta[F,G]}{\delta q_i}=w_i J\left(\frac{\delta F}{\delta q_i},\frac{\delta G}{\delta q_i}\right),$$

$$+\sum_{j=2}^{4}\left[w_j\left(\frac{\delta^2 F}{\delta q_j\delta q_i}J\left(\frac{\delta G}{\delta q_j},q_j\right)-\frac{\delta^2 G}{\delta q_j\delta q_i}J\left(\frac{\delta F}{\delta q_j},q_j\right)\right)\right]. \tag{4.71}$$

Now that we have the variational derivatives of the bracket, we can begin to write out the left-hand side of the Jacobi identity using (4.31), then substituting in the variational derivatives,

$$[[F,G],H]+cyc$$

$$=\iiint\limits_{\Omega}\frac{\delta[F,G]}{\delta q_1}J\left(\frac{\delta H}{\delta q_1},q_1\right)dxdydz+\sum_{i=2}^{4}\iint\limits_{\Omega_H}w_i\frac{\delta[F,G]}{\delta q_i}J\left(\frac{\delta H}{\delta q_i},q_i\right)dxdy+cyc$$

$$=\iiint\limits_{\Omega}J\left(\frac{\delta F}{\delta q_1},\frac{\delta G}{\delta q_1}\right)J\left(\frac{\delta H}{\delta q_1},q_1\right)dxdydz+\sum_{i=2}^{4}\iint\limits_{\Omega_H}w_i^2J\left(\frac{\delta F}{\delta q_i},\frac{\delta G}{\delta q_i}\right)J\left(\frac{\delta H}{\delta q_i},q_i\right)dxdy$$

$$+\iiint\limits_{\Omega}\left[\frac{\delta^2F}{\delta q_1^2}J\left(\frac{\delta G}{\delta q_1},q_1\right)-\frac{\delta^2G}{\delta q_1^2}J\left(\frac{\delta F}{\delta q_1},q_1\right)\right]J\left(\frac{\delta H}{\delta q_1},q_1\right)dxdydz$$

$$+\sum_{i=2}^{4}\sum_{j=2}^{4}w_iw_j\iint\limits_{\Omega_H}\frac{\delta^2F}{\delta q_j\delta q_i}J\left(\frac{\delta G}{\delta q_j},q_j\right)J\left(\frac{\delta H}{\delta q_i},q_i\right)dxdy$$

$$-\sum_{i=2}^{4}\sum_{j=2}^{4}w_iw_j\iint\limits_{\Omega_H}\frac{\delta^2G}{\delta q_i\delta q_j}J\left(\frac{\delta F}{\delta q_i},q_i\right)J\left(\frac{\delta H}{\delta q_j},q_j\right)dxdy+cyc, \tag{4.72}$$

The i and j in the second last term in (4.72) have been interchanged. To see that the right-hand side of the equation above does vanish it is necessary to bring forth two identities. The first is that the Jacobian itself satisfies the Jacobi identity

$$J(A,J(B,C))+cyc=0. \tag{4.73}$$

This implies that the first four terms on the right-hand side of (4.72) vanish, since they are each being added to their cyclic terms, which together vanish. Secondly, because of the fact that variational derivatives commute, due to the functionals being smooth, another identity can be obtained,

$$
\begin{aligned}
&\frac{\delta^2 F}{\delta q_i \delta q_j} J\left(\frac{\delta G}{\delta q_i}, q_i\right) J\left(\frac{\delta H}{\delta q_j}, q_j\right) - \frac{\delta^2 G}{\delta q_j \delta q_i} J\left(\frac{\delta F}{\delta q_j}, q_j\right) J\left(\frac{\delta H}{\delta q_i}, q_i\right) + cyc \\
&= \frac{\delta^2 F}{\delta q_i \delta q_j} J\left(\frac{\delta G}{\delta q_i}, q_i\right) J\left(\frac{\delta H}{\delta q_j}, q_j\right) - \frac{\delta^2 G}{\delta q_j \delta q_i} J\left(\frac{\delta F}{\delta q_j}, q_j\right) J\left(\frac{\delta H}{\delta q_i}, q_i\right) \\
&+ \frac{\delta^2 G}{\delta q_i \delta q_j} J\left(\frac{\delta H}{\delta q_i}, q_i\right) J\left(\frac{\delta F}{\delta q_j}, q_j\right) - \frac{\delta^2 H}{\delta q_j \delta q_i} J\left(\frac{\delta G}{\delta q_j}, q_j\right) J\left(\frac{\delta F}{\delta q_i}, q_i\right) \\
&+ \frac{\delta^2 H}{\delta q_i \delta q_j} J\left(\frac{\delta F}{\delta q_i}, q_i\right) J\left(\frac{\delta G}{\delta q_j}, q_j\right) - \frac{\delta^2 F}{\delta q_j \delta q_i} J\left(\frac{\delta H}{\delta q_j}, q_j\right) J\left(\frac{\delta G}{\delta q_i}, q_i\right), \qquad (4.74)
\end{aligned}
$$

where each term on the right-hand sides cancels with another to give that

$$
\frac{\delta^2 F}{\delta q_i \delta q_j} J\left(\frac{\delta G}{\delta q_i}, q_i\right) J\left(\frac{\delta H}{\delta q_j}, q_j\right) - \frac{\delta^2 G}{\delta q_j \delta q_i} J\left(\frac{\delta F}{\delta q_j}, q_j\right) J\left(\frac{\delta H}{\delta q_i}, q_i\right) + cyc = 0. \qquad (4.75)
$$

Therefore, the second line on the right-hand side in (4.72) vanishes by itself, and the third and fourth together vanish. Hence we can conclude that the Jacobi identity is satisfied, and moreover that the proposed Hamiltonian and Poisson bracket do indeed form a Hamiltonian system.

4.4 Invariances

There are two types of invariances that may arise in Hamiltonian systems, the first are Casimirs. To prove these are invariant, we calculate the time derivative of these functionals,

$$
\begin{aligned}
\frac{dC}{dt} &= [C, H] \\
&= -[H, C] \\
&= -\left\langle \frac{\delta H}{\delta \mathbf{q}}, J \frac{\delta C}{\delta \mathbf{q}} \right\rangle \\
&= 0, \qquad (4.76)
\end{aligned}
$$

by using (4.10), the skew-symmetry property of the bracket, (4.3), and finally the definition of a Casimir (4.8). Therefore, Casimirs are invariants with respect to time.

Substituting (4.32) into the definition of a Casimir (4.8) implies

$$\partial\left(\frac{\delta C}{\delta q_i}, q_i\right) = 0 \quad \text{for} \quad i = 1, ..., 4. \tag{4.77}$$

To solve this, we write the Jacobian as

$$\mathbf{e}_3 \cdot \left(\nabla \frac{\delta C}{\delta q_i} \times \nabla q_i\right) = 0. \tag{4.78}$$

If $i = 2, 3, 4$, it is known that each of the occurring functions, $\delta C/\delta q_i$ and q_i are only functions of x and y. Hence, the gradient of each function yields a vector that lies in the (x, y) plane, where taking the cross product of these two then results in a vector pointing in the z direction. Therefore, the only way that the quantity in (4.78) can be zero is if the bracket itself is zero. Assuming both vectors are non-zero requires that the normal vectors of the level curves of $\delta C/\delta q_i$ and q_i are everywhere parallel and therefore coincident, which allows us to deduce that the two variables have a direct functional relationship. The situation where $i = 1$, is slightly more complicated in that each function now has z dependency. For notational simplicity, this z dependency of F_1will be suppressed. Hence, the Casimirs must all be of the form

$$C = \iiint\limits_{\Omega} \Phi_1(q_1)\, dxdydz + \iint\limits_{\Omega_H} \Phi_2(q_2) + \Phi_3(q_3) + \Phi_4(q_4)\, dxdy, \tag{4.79}$$

where the Φ_i are the Casimir densities. Note that since there are nontrivial functions of this form, this proves that the symplectic matrix is noninvertible.

The second type of conserved quantity is obtainable from Noether's theorem. This theorem connects conserved quantities with a symmetry properties in the Hamiltonian structure. Taken directly form Shepherd (1990), it states the following.

Theorem 1 ***(Noether):*** *If H is invariant under translations in (some variable) ξ,* *and if the functional I satisfies*

$$J\frac{\delta I}{\delta \mathbf{q}} = -\mathbf{q}_{\xi}, \tag{4.80}$$

then I is an invariant in time.

For example, if ξ is chosen to be time, then it can be shown that the associated conserved functional is the Hamiltonian itself, since

$$\mathbf{q}_t = J\frac{\delta H}{\delta \mathbf{q}} = J\frac{\delta(-I)}{\delta \mathbf{q}}, \tag{4.81}$$

having combined (4.2) with (4.80), which lets us conclude that $H = -I$. Therefore, the invariance of H with time is connected to the conservation of H itself.

Invariances with respect to Cartesian coordinates gives rise to conservation of linear momentum, whereas those with respect to polar coordinates correspond to conservation of angular momentum. For our purposes it is only necessary to find the conserved quantity associated with the symmetry with respect to x, which is clearly present. The equations that need to be solved, written component wise, are

$$\partial\left(w_i\frac{\delta I}{\delta q_i}, q_i\right) = -q_{i_x} = \partial(y, q_i) \quad \text{for} \quad i = 1, ..., 4. \tag{4.82}$$

Choosing the variational derivatives as follows guarantees that the above equation is satisfied,

$$\frac{\delta I}{\delta q_i} = \frac{y}{w_i} \quad \text{for} \quad i = 1, ..., 4. \tag{4.83}$$

Therefore, the impulse functional is written as

$$I = \iiint\limits_{\Omega} y q_1 dx dy dz + \sum_{i=2}^{4} \frac{1}{w_i} \iint\limits_{\Omega_H} y q_i dx dy. \tag{4.84}$$

Re-writing this expression in terms of the dynamic pressure in the upper layer and the height of the lower layer and grouping terms results in

$$I = \iiint\limits_{\Omega} y\Delta\varphi + y\left(N^{-2}\varphi_z\right)_z dx dy dz + \iint\limits_{\Omega_H} y h_B/\mu - y\left[N^{-2}\varphi_z\right]_{z=-1}^{z=0} dx dy. \tag{4.85}$$

Using the Fundamental Theorem of calculus to rewrite the terms evaluated at $z = 0$ and $z = -1$, then cancelling these with terms that arise from the volume integral through integration by parts, yields

$$I = \iiint_{\Omega} y \Delta \varphi \; dx dy dz + \iint_{\Omega_H} y h_B / \mu \; dx dy, \tag{4.86}$$

where the second integral is a constant that does not depend on the dynamic variables.

4.5 Variational principle

Recall that in the discrete system, the Hamiltonian was invertible. To find the steady state equations, the left-hand side of (4.2) is set equal to zero. Then using the fact that the symplectic matrix is nonsingular, i.e. J^{-1} exists, the equation reduces to solving

$$\frac{\delta H}{\delta \mathbf{q}} = 0, \tag{4.87}$$

where it is understood that $\mathbf{q}$ is the vector that contains the q_i 's as components. The significance of this is that the steady state solutions satisfy the first-order conditions for an extrema of the Hamiltonian. It is for this reason that stability theorems can be established for this type of system, given that conditions can be found where the second variation of the Hamiltonian is sign definite.

However, in the noninvertible case the same conclusions cannot be made since the kernel space is no longer trivial. This is why it is necessary to consider, not the Hamiltonian, but a constrained Hamiltonian. This functional is defined to be the sum of the Hamiltonian and a Casimir, where the Casimirs are chosen to ensure that the steady state solutions satisfy the first-order conditions for an extrema of the Constrained Hamiltonian.

This can be interpreted in another way. It means that the steady state solutions are possible conditional extrema of H with the constraint that $C =$ constant (Shepherd, 1994). This is analogous to what is done in the method of Lagrange Multipliers, where a function needs to be optimized, with the added difficulty that it must lie within some constraining surface.

The method stated above can also be applied to steadily travelling solutions in exactly the same manner, where the steady state situation is a special case. Hence, instead of dealing with these two cases separately, it will only be done for the more general steadily travelling solutions.

To obtain equations of motion for the steadily travelling solutions along the channel at speed c, we assume solutions of the form

$$\varphi = \varphi_s (x - ct, y, z), \tag{4.88}$$

$$h = h_s (x - ct, y), \tag{4.89}$$

where the s subscript denotes that this is a steadily travelling solution. Setting $c = 0$ recovers the steady state situation.

Given this transformation, the time derivatives acting on φ and h take the form

$$\partial_t = -c\partial_x = \partial (cy, *), \tag{4.90}$$

which when substituted into (4.33) through to (4.36) yields

$$\partial (\mu\varphi_s + cy, q_{1s}) = 0, \tag{4.91}$$

$$\partial (\mu\varphi_s|_{z=0} + cy, q_{2s}) = 0, \tag{4.92}$$

$$\partial \left(\mu\varphi_s|_{z=-1} + cy, q_{3s}\right) = 0, \tag{4.93}$$

$$\partial \left(\mu\varphi_s|_{z=-1} + cy + h_B, q_{4s}\right) = 0, \tag{4.94}$$

where the s subscript denotes that the functions are being evaluated at φ_s and h_s.

Solving this system as was done in (4.77), introduces four steadily travelling functions F_i, for $i = 1, ..., 4$,

$$\mu\varphi_s + cy = F_1 (q_{1s}) = F_{1s}, \tag{4.95}$$

$$\mu\varphi_s|_{z=0} + cy = F_2 (q_{2s}) = F_{2s}, \tag{4.96}$$

$$\mu\varphi_s|_{z=-1} + cy = F_3(q_{3s}) = F_{3s}, \quad (4.97)$$

$$\mu\varphi_s|_{z=-1} + cy + h_B = F_2(q_{2s}) = F_{4s}. \quad (4.98)$$

Because the F_i functions specify the relation of the variables φ_s and h_s with the generalized coordinates, they completely describe the state of the system. This signifies that any steadily travelling solution is determined by these state functions.

As has been previously stated, for steady state solutions it is necessary to consider the constrained Hamiltonian which is the sum of the Hamiltonian and a Casimir. However, for steadily travelling solutions this is not sufficient (Shepherd, 1994). It is necessary to introduce the conservation of mass through using the Impulse functional as follows

$$\mathcal{H} = H - cI + C. \quad (4.99)$$

Computing the variational derivative of this functional using (4.39), (4.79) and (4.84) produces

$$\delta\mathcal{H} = \iiint\limits_{\Omega} \left(\Phi_1' - (\mu\varphi + cy)\right)\delta q_1 \; dxdydz + \iint\limits_{\Omega_H} \left(\Phi_2' + N^{-2}(\mu\varphi + cy)\right)_{z=0} \delta q_2 \; dxdy$$
$$+ \iint\limits_{\Omega_H} \left(\Phi_3' - N^{-2}(\mu\varphi + cy)\right)_{z=-1} \delta q_{34} \; dxdy$$
$$+ \iint\limits_{\Omega_H} \left(\Phi_4' + (\mu\varphi + \mu h + cy + h_B)\right)_{z=-1} \delta q_4 \; dxdy. \quad (4.100)$$

To make the steadily travelling solutions an extrema of $\mathcal{H}$ (i.e. $\delta\mathcal{H} = 0$ for all perturbations), it is necessary to choose the Casimir densities such that

$$\Phi_1'(q_{1s}) = \mu\varphi_s + cy = F_1(q_{1s}), \quad (4.101)$$

$$\Phi_2'(q_{2s}) = -\left[N^{-2}(\mu\varphi_s + cy)\right]_{z=0} = -N^{-2}(0)\,F_2(q_{2s}), \quad (4.102)$$

$$\Phi_3'(q_{3s}) = \left[N^{-2}(\mu\varphi_s + cy)\right]_{z=-1} = N^{-2}(-1)\,F_3(q_{3s}), \quad (4.103)$$

$$\Phi_4'(q_{4s}) = -\left[\mu\varphi_s + \mu h_s + cy + h_B\right]_{z=0} = -F_4(q_{4s}) - \mu q_{4s}, \quad (4.104)$$

having substituted in (4.95) through to (4.98).

To find explicit solutions we integrate each equation from 0 to q_i for $i = 1, 2, 4$. The $i = 3$ case needs to be integrated from $N^2 h_B/\mu$ to q_3. This is because when φ and h both approach 0, $q_3 \to N^2 h_B/\mu$, and hence for the integral to vanish as this limit is taken, we must have this quantity as being the lower bound of integration. The explicit expressions for the Casimir densities are then

$$\Phi_1 = \int_0^{q_1} F_1(\tau)\, d\tau, \tag{4.105}$$

$$\Phi_2 = -N^{-2}(0) \int_0^{q_2} F_2(\tau)\, d\tau, \tag{4.106}$$

$$\Phi_3 = N^{-2}(-1) \int_{N^2 h_B/\mu}^{q_3} F_3(\tau)\, d\tau, \tag{4.107}$$

$$\Phi_4 = -\int_0^{q_4} F_4(\tau)\, d\tau - \frac{\mu}{2} q_4^2, \tag{4.108}$$

which then makes the steadily travelling solutions an extrema of the functional $\mathcal{H}$. Therefore, we have established a variational principle for the steadily travelling solutions travelling in the x direction.

4.6 Formal stability

Formal stability is in reference to infinitesimal perturbations. It requires that the second variation of the constrained Hamiltonian, evaluated at the steadily travelling solution, is either positive or negative definite (Holm *et al.*, 1985). In the finite dimensional case with a system of particles, this also implies nonlinear stability in the sense of Liapunov (Drazin & Reid, 1981) However, this statement does not hold true for continuous systems.

When dealing with finite dimensional systems, it is well known that stability with respect to one norm implies stability with respect to all other norms. In the infinite dimensional case this equivalence is not present due to the loss of compactness (Gelfand, 1963). This is why when proving stability it is important what norm is chosen. We choose our norms so that they are connected with energy, since we associate instability with energy growth.

Andrews' Theorem (Andrews, 1984) implies that there are no steadily-travelling isolated solutions which are formally stable. Hence, in order to focus our attention on steady state solutions, in this section we set $c = 0$ and assume that the 0 subscript denotes the functions evaluated at that steady states.

To begin proving formal stability, it is necessary to find the second variation of the constrained Hamiltonian. Taking the variational derivative of (4.100) produces

$$\delta^2\mathcal{H} = \iiint\limits_{\Omega} (\Phi_1' - \mu\varphi)\,\delta^2 q_1 + \Phi_1''\,(\delta q_1)^2 - \mu\delta\varphi\delta q_1 \;dxdydz$$

$$+ \iint\limits_{\Omega_H} \left(\Phi_2' + N^{-2}\mu\varphi\right)_{z=0} \delta^2 q_2 + \Phi_2''\,(\delta q_2)^2 + \left(N^{-2}\mu\delta\varphi\right)_{z=0} \delta q_2 \;dxdy$$

$$+ \iint\limits_{\Omega_H} \left(\Phi_3' - N^{-2}\mu\varphi\right)_{z=-1} \delta^2 q_3 + \Phi_3''\,(\delta q_3)^2 - \left(N^{-2}\mu\delta\varphi\right)_{z=-1} \delta q_3 \;dxdy$$

$$+ \iint\limits_{\Omega_H} \left(\Phi_4' + (\mu\varphi + \mu h + h_B)\right)_{z=-1} \delta^2 q_4 \;dxdy$$

$$+ \iint\limits_{\Omega_H} \Phi_4''\,(\delta q_4)^2 + (\mu\delta\varphi + \mu\delta h)_{z=-1}\,\delta q_4 \;dxdy. \tag{4.109}$$

Evaluating this equation at the steady state implies that four of the terms vanish by (4.101) to (4.104). Integrating the resultant by parts with respect to z, applying Green's theorem, (4.25) along with the usual boundary conditions, yields

$$\delta^2\mathcal{H}(\mathbf{q}_0) = \iiint\limits_{\Omega} \mu\left(\nabla\delta\varphi\cdot\nabla\delta\varphi + (\delta\varphi_z/N)^2\right) + \Phi_{10}''\,(\delta q_1)^2 \;dxdydz$$

$$+ \iint\limits_{\Omega_H} \Phi_{20}''\,(\delta q_2)^2 + \Phi_{30}''\,(\delta q_3)^2 + (\Phi_{40}'' + \mu)\,(\delta q_4)^2 \;dxdy. \tag{4.110}$$

The quantity above is a sum of numerous terms, each of which have an interpretation. The first two terms form the total energy of the upper fluid, and is a sum of the disturbance kinetic energy and the baroclinic stretching term, which is due to the available potential energy contained in the slope-water. The next term is the disturbance enstrophy save for the presence of a Casimir density function. The final three terms are all disturbance available potential energies, each with a Casimir density present as well. The first two are due to the horizontal boundaries at the top and bottom respectively, whereas the final one is due to the existence of the lower layer and bottom topography.

The second variation of the Hamiltonian is an invariant of the linear dynamics. It is a good choice for a norm since firstly, each of its terms appears quadratically, and secondly, it is inherently associated with the energy of the system.

4.6.1 Invariance of the Second Variation

The equations of motion for the steady state are (4.91) through to (4.94) with $c = 0$. To find the equations that govern the perturbed state, it is necessary to substitute the following perturbed state,

$$\varphi = \varphi_0 + \delta\varphi, \tag{4.111}$$

$$h = h_0 + \delta h, \tag{4.112}$$

into the nonlinear equations (4.33) to (4.36). Doing so, while ignoring the nonlinear terms due to their small magnitudes, and utilizing the steady equations, yields

$$\delta q_{1_t} + \partial\left(\mu\delta\varphi, q_{10}\right) + \partial\left(\mu\varphi_0, \delta q_1\right) = 0, \tag{4.113}$$

$$\delta q_{2_t} + \partial\left(\mu\delta\varphi, q_{20}\right) + \partial\left(\mu\varphi_0, \delta q_2\right) = 0 \quad \text{on} \quad z = 0, \tag{4.114}$$

$$\delta q_{3_t} + \partial\left(\mu\delta\varphi, q_{30}\right) + \partial\left(\mu\varphi_0, \delta q_3\right) = 0 \quad \text{on} \quad z = -1, \tag{4.115}$$

$$\delta q_{4_t} + \partial\left(\mu\delta\varphi, q_{40}\right) + \partial\left(\mu\varphi_0 + h_B, \delta q_4\right) = 0 \quad \text{on} \quad z = -1. \tag{4.116}$$

The last term in the first equation is rewritten as

$$\begin{aligned} \partial\left(\mu\varphi_0, \delta q_1\right) &= \partial\left(F_{10}, \delta q_1\right) \\ &= \partial\left(F_{10}' q_{10}, \delta q_1\right) \\ &= \partial\left(-F_{10}' \delta q_1, q_{10}\right), \end{aligned} \tag{4.117}$$

by using (4.101) and skew-symmetry. The same simplification can be made on the other three equations, which enables us to rewrite the linearized equations as

$$\delta q_{1_t} + \partial\left(\mu\delta\varphi - F'_{10}\delta q_1, q_{10}\right) = 0, \tag{4.118}$$

$$\delta q_{2_t} + \partial\left(\mu\delta\varphi - F'_{20}\delta q_2, q_{20}\right) = 0 \quad \text{on} \quad z = 0, \tag{4.119}$$

$$\delta q_{3_t} + \partial\left(\mu\delta\varphi - F'_{30}\delta q_3, q_{30}\right) = 0 \quad \text{on} \quad z = -1, \tag{4.120}$$

$$\delta q_{4_t} + \partial\left(\mu\delta\varphi - F'_{40}\delta q_4, q_{40}\right) = 0 \quad \text{on} \quad z = -1. \tag{4.121}$$

It can now be shown that the second variation of the constrained Hamiltonian is an invariant with respect to the linear dynamics. We rewrite (4.110) in terms of the steady state functions rather than the Casimirs, then differentiate with respect to time to produce

$$\frac{d}{dt}\delta^2\mathcal{H}(\mathbf{q}_0) = 2\mu \iiint\limits_{\Omega} \nabla\delta\varphi\cdot\nabla\delta\varphi_t + N^{-2}\delta\varphi_z\delta\varphi_{zt} + 2F'_{10}\delta q_1\delta q_{1_t} + \left(F'_{10}\right)_t\left(\delta q_1\right)^2 \; dxdydz$$

$$- \iint\limits_{\Omega_H} 2N^{-2}(0)\, F'_{20}\delta q_2\delta q_{2_t} + \left(F'_{20}\right)_t\left(\delta q_2/N\right)^2_{z=0} dxdy$$

$$+ \iint\limits_{\Omega_H} 2N^{-2}(-1)\, F'_{30}\delta q_3\delta q_{3_t} + \left(F'_{30}\right)_t\left(\delta q_3/N\right)^2_{z=-1} dxdy$$

$$- \iint\limits_{\Omega_H} 2F'_{40}\delta q_4\delta q_{4_t} + \left(F'_{40}\right)_t\left(\delta q_4\right)^2 dxdy. \tag{4.122}$$

Applying integration by parts with respect to z, Green's theorem, (4.21) and recalling (4.90), changes the above equation to

$$\frac{d}{dt}\delta^2\mathcal{H}(\mathbf{q}_0) = -\iiint\limits_{\Omega} 2\left(\mu\delta\varphi - F'_{10}\delta q_1\right)\delta q_{1_t} \; dxdydz$$

$$+ \iint\limits_{\Omega_H} \left[2\left(N^{-2}\mu\delta\varphi + F'_{20}\delta q_2\right)\delta q_{2_t}/N^2\right]_{z=0} \; dxdy$$

$$-\iint_{\Omega_H} \left[2\left(N^{-2}\mu\delta\varphi - F'_{30}\delta q_3\right)\delta q_{3_t}/N^2\right]_{z=-1} dxdy$$

$$+\iint_{\Omega_H} 2\left(\mu\delta\varphi + F'_{40}\delta q_4\right)_{z=-1}\delta q_{4_t}\, dxdy. \quad (4.123)$$

Integrating by parts with respect to x and then using the periodicity of the functionals yields

$$\frac{d}{dt}\delta^2\mathcal{H}(\mathbf{q}_0) = -2\iiint_{\Omega}\left(\mu\delta\varphi - F'_{10}\delta q_1\right)\delta q_{1_t}\, dxdydz$$

$$+2\iint_{\Omega_H}\left[N^{-2}\left(\mu\delta\varphi - F'_{20}\delta q_2\right)\right]_{z=0}\delta q_{2_t}\, dxdy$$

$$-2\iint_{\Omega_H}\left[N^{-2}\left(\mu\delta\varphi - F'_{30}\delta q_3\right)\right]_{z=-1}\delta q_{3_t}\, dxdy$$

$$+2\iint_{\Omega_H}\left(\mu\delta\varphi - F'_{40}\delta q_4\right)_{z=-1}\delta q_{4_t} dxdy. \quad (4.124)$$

Finally, substituting in the linearized equations (4.118) through to (4.121) produces

$$\frac{d}{dt}\delta^2\mathcal{H}(\mathbf{q}_0) = \iiint_{\Omega}\partial\left(\left(\mu\delta\varphi - F'_{10}\delta q_1\right)^2, q_{10}\right) dxdydz$$

$$-\iint_{\Omega_H}\partial\left(N^{-2}\left(\mu\delta\varphi - F'_{20}\delta q_2\right)^2, q_{20}\right)_{z=0} dxdy$$

$$+\iint_{\Omega_H}\partial\left(N^{-2}\left(N^{-2}\mu\delta\varphi - F'_{30}\delta q_3\right)^2, q_{30}\right)_{z=-1} dxdy$$

$$-\iint_{\Omega_H}\partial\left(\left(\mu\delta\varphi - F'_{40}\delta q_4\right)^2, q_{40}\right)_{z=-1} dxdy = 0. \quad (4.125)$$

Each integral above vanishes because of the fact that each q_{i0} is independent of x on the channel walls because of the no normal flow boundary conditions and $h_B(y)$ combined with (4.25).

4.6.2 Arnol'd's First theorem

Arnol'd's first theorem finds sufficient criteria for linear stability by finding conditions that guarantee the second variation of the constrained Hamiltonian is positive definite. Notice that the kinetic energy and baroclinic stretching terms are each positive definite since they appear as squares with positive coefficients. The other four terms appear in quadrature as well, but their coefficients are variable depending on the type of steady state solution is under consideration. If the solution is such that the $(-1)^{i+1}F'_{i0} \geq 0$ for all i, this guarantees that $\delta^2\mathcal{H}$ is positive definite. This result is summarized in the following theorem.

Theorem 2 *The steady solutions $\varphi_0(x,y,z)$ and $h_0(x,y)$ as defined by (4.91) through to (4.94) with $c=0$, are linearly stable in the sense of Liapunov with respect to the perturbation norm $\|\delta\mathbf{q}\| = [\delta^2\mathcal{H}(\mathbf{q}_0)]^{\frac{1}{2}}$ if the Casimir functions are chosen to be (4.105) to (4.108) where the steady state functions, defined by (4.95) to (4.98) with $c=0$, must satisfy the conditions*

$$(-1)^{i+1} F'_{i0} \geq 0 \quad for \quad i = 1, ..., 4, \tag{4.126}$$

in the domain $(x,y,z) \in \Omega$.

It is desirable to translate these conditions, for the steady state case, in terms of the mean flow. Equation (4.126) becomes

$$\mu U_0 = F'_{10}\left(\Delta U_0 + \left(N^{-2}U_{0_z}\right)_z\right), \tag{4.127}$$

$$\mu U_0 = F'_{20}\left(U_{0_z}\right), \tag{4.128}$$

$$\mu U_0 = F'_{30}\left(U_{0_z} - N^2\left(h_{0_y} + h_{B_y}/\mu\right)\right), \tag{4.129}$$

$$\mu U_0 - h_{B_y} = F'_{40}\left(-h_{0_y}\right). \tag{4.130}$$

Therefore, we can use to rewrite (4.126) as follows

$$\frac{U_0}{(\Delta U_0 + (N^{-2}U_{0_z})_z)} \geq 0, \tag{4.131}$$

$$-\frac{U_0}{U_{0_z}} \geq 0, \tag{4.132}$$

$$\frac{U_0}{U_{0_z} - N^2 \left(h_{0_y} + h_{B_y}/\mu\right)} \geq 0, \tag{4.133}$$

$$\frac{\mu U_0 - h_{B_y}}{h_{0_y}} \geq 0. \tag{4.134}$$

For the special case where $U_0 = 0$, where the internal barotropic and baroclinic instabilities are filtered out, the conditions reduce to

$$h_{B_y} h_{0_y} \leq 0, \tag{4.135}$$

which is the same condition that was found in Chapter 3 for both the energetics and normal mode analysis.

4.6.3 Poincaré inequality

Arnol'd's second theorem establishes a different set of criteria for linear stability by finding conditions that imply $\delta^2\mathcal{H}(\mathbf{q}_s)$ is negative definite. This is more difficult because the total energy of the upper fluid in (4.110) is positive definite. Therefore, what is required, is to show that somehow the other terms are always more negative than this term is positive. This can be done by obtaining a Poincaré inequality. The one used here is the same as that for continuously stratified QG theory in Yongming *et al.* (1995). They obtained the inequality by considering two separate eigenvalue problems. This can more easily be done by considering one eigenvalue problem, which is what results from multiplying their two systems together.

The eigenvalue problem to be considered is

$$\left(\Delta\delta\varphi + \left(N^{-2}\delta\varphi_z\right)_z\right) + \kappa\delta\varphi = 0, \tag{4.136}$$

with boundary conditions

$$\alpha_2\delta\varphi_z - K\delta\varphi = 0 \quad \text{on} \quad z = 0, \tag{4.137}$$

$$\alpha_3\delta\varphi_z + K\delta\varphi = 0 \quad \text{on} \quad z = -1, \tag{4.138}$$

$$\delta\varphi = 0 \quad \text{on} \quad y = 0,\ L. \tag{4.139}$$

The positive constants α_2, α_3 and K are to be determined, and conditions (4.139) are special cases by which the no normal flow conditions are satisfied along the channel walls. Swaters (1993b) calls these 'natural' boundary conditions, since it is in the spirit of variational calculus where the only variations (or perturbations) considered are those that leave the boundary conditions unchanged. These specialized conditions are only used in derived the eigenvalue problem hence only apply to Arnol'd's second theorems.

Multiplying (4.136) by the term in brackets, integrating over the volume Ω then applying integration by parts with respect to z and the divergence theorem along with the appropriate boundary conditions yields the following equation

$$\iiint\limits_{\Omega} \left(\Delta\delta\varphi + \left(N^{-2}\delta\varphi_z\right)_z\right)^2 dxdydz = \kappa \iiint\limits_{\Omega} \nabla\delta\varphi\cdot\nabla\delta\varphi + (\delta\varphi_z/N)^2\, dxdydz$$

$$-\kappa \iint\limits_{\Omega_H} \frac{\alpha_2}{K}\left(\delta\varphi_z^2/N^2\right)_{z=0} + \frac{\alpha_3}{K}\left(\delta\varphi_z^2/N^2\right)_{z=-1} dxdy. \tag{4.140}$$

Since the minimum eigenvalue is non-zero, we can divide through by κ to obtain the inequality

$$\iiint\limits_{\Omega} \nabla\delta\varphi\cdot\nabla\delta\varphi + (\delta\varphi_z/N)^2\, dxdydz \le \frac{\alpha_1}{K}\iiint\limits_{\Omega} q_1^2 dxdydz$$

$$+\iint\limits_{\Omega_H} \frac{\alpha_2}{K}(\delta\varphi_z/N)^2_{z=0} + \frac{\alpha_3}{K}(\delta\varphi_z/N)^2_{z=-1}\, dxdy, \tag{4.141}$$

where α_1 is a constant to be determined and K must satisfy the equation

$$\frac{\alpha_1}{K} = \frac{1}{\min\kappa(K)}. \tag{4.142}$$

The minimum eigenvalue is positive as will be shown in Appendix A. Moreover, it will be proven that, for the above equation to be true, K must be strictly positive.

4.6.4 Arnol'd's second theorem

In deriving Arnol'd's second theorem, it will be necessary to use the following algebraic identity as was done in Swaters (1993b),

$$ax^2 + b(x+y)^2 + cy^2 = (a+b)(x+\beta y)^2 + (a\beta + c)y^2, \tag{4.143}$$

where

$$\beta = \frac{b}{a+b}. \tag{4.144}$$

This is proven by expanding the left-hand side then completing the square

$$\begin{aligned} ax^2 + b(x+y)^2 + cy^2 &= (a+b)x^2 + 2bxy + (b+c)y^2 \\ &= (a+b)\left(x^2 + 2\beta xy\right) + (b+c)y^2 \\ &= (a+b)\left(x^2 + 2\beta xy\right) + \left(\beta^2(a+b) + (a\beta + c)\right)y^2 \\ &= (a+b)(x+\beta y)^2 + (a\beta + c)y^2. \end{aligned} \tag{4.145}$$

Applying the Poincaré inequality to (4.110) then using (4.143), gives an upper bound on the second variation of the constrained Hamiltonian to be

$$\begin{aligned} \delta^2\mathcal{H}(\mathbf{q}_0) \leq & \iiint\limits_{\Omega} \left(\frac{\alpha_1\mu}{K} + F_{10}'\right)(\delta q_1)^2 \; dxdydz + \iint\limits_{\Omega_H} \left(\frac{\alpha_2\mu}{K} - F_{20}'\right)(\delta q_2/N)^2_{z=0}\, dxdy \\ & + \iint\limits_{\Omega_H} \frac{\alpha_3\mu}{K}(\delta\varphi_z/N)^2_{z=-1} + F_{30}'(\delta q_3/N)^2_{z=-1} - F_{40}'(\delta q_4)^2 \; dxdy, \end{aligned}$$

$$\begin{aligned} \delta^2\mathcal{H}(\mathbf{q}_0) \leq & \iiint\limits_{\Omega} \left(\frac{\alpha_1\mu}{K} + F_{10}'\right)(\delta q_1)^2 \; dxdydz + \iint\limits_{\Omega_H} \left(\frac{\alpha_2\mu}{K} - F_{20}'\right)(\delta q_2/N)^2_{z=0}\, dxdy \\ & + \iint\limits_{\Omega_H} \left(\frac{\alpha_3\mu}{K} + F_{30}'\right)\left(\left(\delta\varphi_z + \gamma N^2\delta h\right)/N\right)^2_{z=-1} dxdy, \\ & + \iint\limits_{\Omega_H} \left(\frac{N^2(-1)\,\alpha_3\mu\gamma}{K} - F_{40}'\right)(\delta q_4)^2\, dxdy, \end{aligned} \tag{4.146}$$

where

$$\gamma = \frac{F'_{30}}{\frac{\alpha_3 \mu}{K} + F'_{30}}. \tag{4.147}$$

Conditions on the steady state functions can now be found to ensure that $\delta^2 \mathcal{H}(\mathbf{q}_0)$ is less than or equal to an expression that is negative definite, therefore it must be negative definite as well.

Theorem 3 *The steady solutions $\varphi_0(x,y,z)$ and $h_0(x,y)$ as defined by (4.91) through to (4.94) with $c = 0$, are linearly stable in the sense of Liapunov with respect to the perturbation norm $\|\delta \mathbf{q}\| = [-\delta^2 \mathcal{H}(\mathbf{q}_0)]^{\frac{1}{2}}$ if the Casimir functions are chosen to be (4.105) to (4.108) where the steady state functions, defined by (4.95) to (4.98) with $c = 0$, must satisfy the conditions*

$$(-1)^{i+1} F'_{i0} \leq -\frac{\alpha_i \mu}{K} \quad \textit{for} \quad i = 1, 2, 3,$$

$$-F'_{40} \leq -N^2 (-1) \frac{\alpha_3 \mu \gamma}{K}, \tag{4.148}$$

where γ is given by (4.147), in the domain $(x,y,z) \in \Omega$.

Translating these conditions in terms of the mean flow using (4.127) and (4.130) gives

$$\frac{\mu U_0}{(\Delta U_0 + (N^{-2} U_{0_z})_z)} \leq -\frac{\alpha_1 \mu}{K}, \tag{4.149}$$

$$-\frac{\mu U_0}{U_{0_z}} \leq -\frac{\alpha_3 \mu}{K}, \tag{4.150}$$

$$\frac{\mu U_0}{U_{0_z} - N^2 (h_{0_y} + h_{B_y}/\mu)} \leq -\frac{\alpha_3 \mu}{K}, \tag{4.151}$$

$$\frac{\mu U_0 - h_{B_y}}{h_{0_y}} \leq -N^2 (-1) \frac{\alpha_3 \mu \gamma}{K}. \tag{4.152}$$

Therefore Arnol'd's two sets of stability criteria for Formal stability are established. The usefulness in these theorems lies in the fact that given any steady solution that satisfies either set of criteria, since it is linearly stable, it would be conceivable that such a flow would persist in nature.

4.6.5 Andrews' theorem

Andrews (1984) showed, within the context of homogeneous QG fluids, that the class of flows which can be formally stable are restricted by the underlying symmetries of the Hamiltonian structure. For example, if the domains and boundary conditions are translationally invariant with respect to x, then only parallel shear flows, i.e. only depend on y, are formally stable. This is a general result that can be applied to many systems, ours of which is no exception. In our model parallel shear flows refer to solutions dependent on y and z, which is a motivation for why we considered steady state solutions in Chapter 3 that were translationally invariant.

To begin, it is necessary to differentiate (4.95) with respect to x, multiply by φ_{0_x} and integrate over the volume, to give

$$\mu \iiint_{\Omega} \varphi_{0_x} q_{10_x} dxdydz = \iiint_{\Omega} F'_{10}\varphi_{0_x}^2 dxdydz. \tag{4.153}$$

Integrating by parts with respect to z and using the divergence theorem produces

$$\iiint_{\Omega} \mu\left(\nabla\varphi_{0_x}\cdot\nabla\varphi_{0_x} + (\varphi_{0_{xz}}/N)^2\right) + F'_{10}\varphi_{0_x}^2 dxdydz$$

$$= \mu\int_{-1}^{0}\oint_{\partial\Omega_H} \varphi_{0_x}\mathbf{n}\cdot\nabla\varphi_{0_x} d0dz + \mu\iint_{\Omega_H}\left[N^{-2}\varphi_{0_x}\varphi_{0_{xz}}\right]_{z=-1}^{z=0} dxdy. \tag{4.154}$$

Expanding the circulation term

$$\begin{aligned}\oint_{\partial\Omega_H}\varphi_{0_x}\mathbf{n}\cdot\nabla\varphi_{0_x} d0 &= \int_0^L [\varphi_{0_x}\varphi_{0_{xx}}]_{x=x_R} - [\varphi_{0_x}\varphi_{0_{xx}}]_{x=-x_R}\, dy \\ &\quad + \int_{-x_R}^{x_R} [\varphi_{0_x}\varphi_{0_{xy}}]_{y=L} - [\varphi_{0_x}\varphi_{0_{xy}}]_{y=0}\, dx \\ &= 0,\end{aligned} \tag{4.155}$$

because the steady solution must satisfy the boundary conditions (4.19) and (4.20). To rewrite the term evaluated at $z = 0$ in (4.154) it is necessary to differentiate (4.96) with respect to x, multiply this by $N^{-2}\varphi_{0_x}|_{z=0}$, and integrate over the horizontal domain to give,

$$\mu \iint\limits_{\Omega_H} \left(N^{-2}\varphi_{0_x}\varphi_{0_{xz}}\right)_{z=0} dxdy = \iint\limits_{\Omega_H} F_{20}' \left(\varphi_{0_{xz}}/N\right)^2_{z=0} dxdy. \tag{4.156}$$

Following the same procedure for (4.97) yields

$$\mu\varphi_{0_x}|_{z=-1} = F_{30}'\left(\varphi_{0_{xz}} + N^2 h_{0_x}\right)_{z=-1} = F_{30}, \tag{4.157}$$

which when multiplied by $N^{-2}\varphi_{0_x}|_{z=-1}$, integrated over the horizontal region and then completing the square yields

$$\mu \iint\limits_{\Omega_H} \left(\varphi_{0_x}\varphi_{0_{xz}}/N^2\right)_{z=-1} dxdy = \iint\limits_{\Omega_H} F_{30}'\left(\varphi_{0_{xz}}^2/N^2 + \varphi_{0_{xz}}h_{0_x}\right)_{z=-1} dxdy$$

$$= \iint\limits_{\Omega_H} F_{30}'\left(q_{30_x}/N\right)^2_{z=-1} - F_{30}'\left(\varphi_{0_{xz}} + N^2 h_{0_x}\right)_{z=-1} h_{0_x}\ dxdy$$

$$= \iint\limits_{\Omega_H} F_{30}'\left(q_{30_x}/N\right)^2_{z=-1} - \left(\mu\varphi_{0_{xz}}h_{0_x}\right)_{z=-1} dxdy, \tag{4.158}$$

after substituting back in (4.157).

Finally, differentiating (4.98) with respect to x implies the following equation

$$\mu\varphi_{0_x} = F_{40}'h_{0_x}, \tag{4.159}$$

which when substituted into (4.158) results in

$$\mu \iint\limits_{\Omega_H} \left(N^{-2}\varphi_{0_x}\varphi_{0_{xz}}\right)_{z=-1} dxdy = \iint\limits_{\Omega_H} F_{30}'\left(q_{20_x}/N\right)^2_{z=-1} - F_{40}'h_{0_x}^2\ dxdy. \tag{4.160}$$

Substituting (4.155), (4.156) and (4.160) into (4.154) yields

$$\iiint\limits_{\Omega} \mu\left(\nabla\varphi_{0_x}\cdot\nabla\varphi_{0_x} + \left(\varphi_{0_{xz}}/N\right)^2\right) + F_{10}'\varphi_{0_x}^2 dxdydz$$

$$+ \iint\limits_{\Omega_H} \left(-F_{20}'\right)\left(\varphi_{0_{xz}}/N\right)^2_{z=0} + \iint\limits_{\Omega_H} F_{30}'\left(q_{20_x}/N\right)^2_{z=-1} - F_{40}'h_{0_x}^2\ dxdy$$

$$= 0. \tag{4.161}$$

In the context of Arnol'd's first theorem, the steady state functions are chosen such that the left-hand side of the above equation is positive definite. Since this quantity is also equal to zero, this means that each quadratic term is equal to zero, including the kinetic and potential energy-like terms. This implies that φ_{0_x} is a constant. Since it is equal to zero on the boundary by the no normal flow assumption, this constant must be identically zero, and hence

$$\varphi_0 = \varphi_0 (y, z) . \tag{4.162}$$

Therefore, the only steady solutions that satisfy the conditions of Arnol'd's first theorem, are parallel shear flows.

For Arnol'd's second theorem, we apply the Poincaré inequality and identity (4.143) to (4.161) as done before, to produce the equation

$$0 = \iiint\limits_{\Omega} \left(\frac{\alpha_1 \mu}{K} + F'_{10} \right) (q_{10_x})^2 \; dxdydz + \iint\limits_{\Omega_H} \left(\frac{\alpha_2 \mu}{K} - F'_{20} \right) (q_{20_x}/N)^2_{z=0} \, dxdy$$

$$+ \iint\limits_{\Omega_H} \left(\frac{\alpha_3 \mu}{K} + F'_{30} \right) (\varphi_{0_{zz}}/N + \gamma N h_{0_x})^2_{z=-1} \, dxdy$$

$$+ \iint\limits_{\Omega_H} \left(\frac{N^2 (-1) \, \alpha_3 \mu \gamma}{K} - F'_{40} \right) (q_{40_x})^2 \, dxdy. \tag{4.163}$$

Now, since the left-hand side is negative definite and equal to zero, this implies that

$$q_{10_x} = 0, \quad \varphi_{0_{xz}}|_{z=0} = 0,$$

$$\varphi_{0_{xz}}|_{z=-1} = 0, \quad h_{0_x} = 0. \tag{4.164}$$

Since the Poincaré inequality applies to φ_{0_x}, the above equations force the kinetic and potential energy-like terms to each vanish as before. Therefore, (4.162) also holds true for the second set of stability criteria.

4.7 Nonlinear stability

To find criteria for nonlinear stability it is insufficient to consider the constrained Hamiltonian as was done in the linear analysis, for two reasons. Firstly, the second variation is not an invariant of motion with respect to the nonlinear equations. Secondly, in finite dimensions $\delta^2\mathcal{H}(\mathbf{q}_0)$ being strictly positive implies that H is convex. However, this does not carry over to infinite dimensions because of the loss of compactness.

It is necessary to construct a new functional called the disturbance pseudo-energy functional,

$$\mathcal{A} = [H + C](\mathbf{q}_0 + \delta\mathbf{q}) - [H + C](\mathbf{q}_0) . \tag{4.165}$$

Recalling that the Hamiltonian, the Casimirs are each conserved quantities, we deduce that any sum of these will certainly be conserved, hence $\mathcal{A}$ is conserved.

Substituting (4.28), (4.79), (4.84) and the expressions for the Casimir densities (4.105) through to (4.108) gives rise to

$$\mathcal{A} = \iiint\limits_{\Omega} \frac{\mu}{2}\left(\nabla\delta\varphi\cdot\nabla\delta\varphi + (\delta\varphi_z/N)^2\right) + \mu\left(\nabla\varphi\cdot\nabla\delta\varphi + N^{-2}\varphi_{0_z}\delta\varphi_z\right)\, dxdydz$$

$$+ \iiint\limits_{\Omega} \int_{q_{10}}^{q_{10}+\delta q_1} F_1(\tau)\, d\tau\ dxdydz$$

$$- \iint\limits_{\Omega_H} N^{-2}(0) \int_{q_{20}}^{q_{20}+\delta q_2} F_2(\tau)\, d\tau dxdy$$

$$+ \iint\limits_{\Omega_H} N^{-2}(-1) \int_{q_{30}}^{q_{30}+\delta q_3} F_3(\tau)\, d\tau dxdy$$

$$- \iint\limits_{\Omega_H} \int_{q_{40}}^{q_{40}+\delta q_4} F_4(\tau)\, d\tau - h_B\delta q_4\ dxdy. \tag{4.166}$$

Using integration by parts and the divergence theorem, along with the boundary conditions as has been frequently done, and applying (4.95) to (4.98) reduces this expression to

$$\mathcal{A} = \iiint\limits_{\Omega} \frac{\mu}{2}\left(\nabla\delta\varphi\cdot\nabla\delta\varphi + (\delta\varphi_z/N)^2\right) + \int_{q_{10}}^{q_{10}+\delta q_1} (F_1(\tau) - F_{10})\, d\tau\ dxdydz$$

$$- \iint\limits_{\Omega_H} N^{-2}(0) \int_{q_{20}}^{q_{20}+\delta q_2} (F_2(\tau) - F_{20})\, d\tau\ dxdy$$

$$+ \iint\limits_{\Omega_H} N^{-2}(-1) \int_{q_{30}}^{q_{30}+\delta q_3} (F_3(\tau) - F_{30})\, d\tau\ dxdy$$

$$- \iint\limits_{\Omega_H} \int_{q_{40}}^{q_{40}+\delta q_4} (F_4(\tau)\, d\tau - F_{40})\ dxdy. \tag{4.167}$$

In order to find upper and lower bounds on the pseudo-energy, it is necessary to assume the convexity conditions

$$0 < \alpha_i < (-1)^{i+1} F_i' < \beta_i < \infty, \tag{4.168}$$

where α_i and β_i are constants for $i = 1, 2, 3, 4$, that appear in the Poincaré inequality. Integrating this equation with respect to q_i from q_{i0} to $q_{i0} + \delta q_i$ shows that

$$\alpha_i \delta q_i < (-1)^{i+1} (F_i(q_{i0} + \delta q_i) - F_{i0}) < \beta_i \delta q_i. \tag{4.169}$$

Then integrating with respect to δq_i from 0 to δq_i yields

$$\frac{\alpha_i \delta q_i^2}{2} < (-1)^{i+1} \int_{q_{i0}}^{q_{i0}+\delta q_{i0}} (F_i(\tau) - F_{i0})\, d\tau < \frac{\beta_i \delta q_i^2}{2}, \tag{4.170}$$

after having introduced a change of variable in the integration, $\tau = q_{i0} + \delta q_{i0}$.

Substituting (4.170) into (4.167) gives upper and lower bounds on $\mathcal{A}$,

$$\frac{1}{2} \iiint\limits_{\Omega} \mu\left(\nabla\delta\varphi\cdot\nabla\delta\varphi + (\delta\varphi_z/N)^2\right) + \alpha_1 (\delta q_1)^2\ dxdydz$$

$$+\frac{1}{2} \iint\limits_{\Omega_H} \alpha_2 (\delta q_2/N)^2_{z=0} + \alpha_3 (\delta q_3/N)^2_{z=-1} + \alpha_4 (\delta q_4)^2\, dxdy$$

$$\leq \mathcal{A} \leq$$

$$\frac{1}{2}\iiint\limits_{\Omega} \nabla\delta\varphi\cdot\nabla\delta\varphi + (\delta\varphi_z/N)^2 + \beta_1 (\delta q_1)^2 \;\, dxdydz$$

$$+\frac{1}{2}\iint\limits_{\Omega_H} \beta_2 (\delta q_2/N)^2_{z=0} + \beta_3 (\delta q_3/N)^2_{z=-1} + \beta_4 (\delta q_4)^2 \, dxdy, \qquad (4.171)$$

with which, it is possible to prove Arnol'd's first theorem for nonlinear stability. This is the analogous theorem of that found for a continuously stratified QG fluid in Swaters (1985).

Theorem 4 *If the steady state functions $F_i(\tau)$ for $i = 1, ..., 4$, determined from (4.95) to (4.98) satisfy (4.168) for some constants α_i and β_i, then the steady solutions φ_0 and h_0 are nonlinearly stable in the sense of Liapunov with respect to the disturbance norm $\|\delta\mathbf{q}\|$ given by*

$$\|\delta\mathbf{q}\|^2 = \iiint\limits_{\Omega} \nabla\delta\varphi\cdot\nabla\delta\varphi + (\delta\varphi_z/N)^2 + (\delta q_1)^2 \;\, dxdydz$$

$$+\iint\limits_{\Omega_H} (\delta q_2/N)^2_{z=0} + (\delta q_3/N)^2_{z=-1} + (\delta q_4)^2 \, dxdy. \qquad (4.172)$$

Proof. First notice that this norm is essentially the same as that used in establishing linear stability criteria, except that there are not any functions in front of the quadratures. Observe that if (4.168) holds, the pseudo-energy functional is positive definite by (4.171). What is then required to show is that it is possible to bound the disturbance norm by a scalar multiple of the initial value. The argument proceeds as follows

$$\begin{aligned} \|\delta\mathbf{q}\|^2 &\leq 2\tilde{\Gamma}\mathcal{A} \\ &= 2\tilde{\Gamma}\mathcal{A}\Big|_{t=0} \\ &\leq \tilde{\Gamma}\hat{\Gamma}\,\|\delta\mathbf{q}\|^2_{t=0}, \end{aligned} \qquad (4.173)$$

having used the invariance of $\mathcal{A}$ and the definitions

$$\tilde{\Gamma}^{-1} = \min_{i=1,\ldots,4}\{\mu, \alpha_i\},$$

$$\hat{\Gamma} = \max_{i=1,\dots,4} \{\mu, \beta_i\}. \quad (4.174)$$

Therefore, for all $\epsilon > 0$ there exists a $\delta = \left(\tilde{\Gamma}\hat{\Gamma}\right)^{-\frac{1}{2}} \epsilon > 0$ such that if $\|\delta \mathbf{q}\|_{t=0} < \delta$, then $\|\delta \mathbf{q}\| < \epsilon$ for all time. ∎

Arnol'd's second theorem for a QG fluid was proven in Yongming *et al.* (1996) without assuming that the perturbations were zero along the channel walls. Since we are assuming a stronger condition, that they are zero, this allows the work to be reduced a great deal. The method presented here uses the Poincaré inequality presented in this paper, but more closely resembles the work done in Swaters (1993b) in the two dimensional analogue.

To prove the theorem it is first necessary to assume the opposite convexity conditions

$$0 < \alpha_i < (-1)^i F_i' < \beta_i < \infty, \quad (4.175)$$

which by following the same procedure as above produces the following inequalities

$$-\frac{\beta_i \delta q_i^2}{2} < (-1)^{i+1} \int_{q_{i0}}^{q_{i0}+\delta q_{i0}} (F_i(\tau) - F_{i0})\, d\tau < -\frac{\alpha_i \delta q_i^2}{2}. \quad (4.176)$$

Substituting (4.176) into (4.167) yields these bounds on the pseudo-energy,

$$\begin{aligned}
&\frac{1}{2}\iiint_{\Omega} \mu\left(\nabla\delta\varphi\cdot\nabla\delta\varphi + (\delta\varphi_z/N)^2\right) - \beta_1(\delta q_1)^2 \; dxdydz \\
&\quad -\frac{1}{2}\iint_{\Omega_H} \beta_2(\delta q_2)^2 + \beta_3(\delta q_3)^2 + \beta_4(\delta q_4)^2\, dxdy \\
&\qquad\qquad \leq \mathcal{A} \leq \\
&\frac{1}{2}\iiint_{\Omega} \nabla\delta\varphi\cdot\nabla\delta\varphi + (\delta\varphi_z/N)^2 - \alpha_1(\delta q_1)^2 \; dxdydz \\
&\quad -\frac{1}{2}\iint_{\Omega_H} \alpha_2(\delta q_2)^2 + \alpha_3(\delta q_3)^2 + \alpha_4(\delta q_4)^2\, dxdy. \quad (4.177)
\end{aligned}$$

The Arnol'd's second theorem for nonlinear stability for our model, is as follows.

Theorem 5 *If the steady state functions $F_i(\tau)$ for $i = 1, \dots, 4$ determined from (4.95)* *to (4.98) satisfy (4.176) for the positive constants α_i and β_i and*

$$K > \mu, \tag{4.178}$$

$$\alpha_4 > N^2(-1)\frac{\alpha_3\mu\gamma}{K}, \tag{4.179}$$

with

$$\gamma = \frac{\beta_3}{\frac{\alpha_3}{K} + \beta_3}, \tag{4.180}$$

then the steady solutions φ_0 and h_0 are nonlinearly stable in the sense of Liapunov with respect to the disturbance norm $\|\delta\mathbf{q}\|$ given by

$$\|\delta\mathbf{q}\|^2 = \iiint\limits_{\Omega} (\delta q_1)^2 \ dxdydz + \iint\limits_{\Omega_H} (\delta q_2/N)^2_{z=0}\, dxdy$$

$$+ \iint\limits_{\Omega_H} (\delta\varphi_z/N)^2_{z=-1} + (\delta q_4)^2\, dxdy. \tag{4.181}$$

The condition $K > \mu$ will be translated in terms of the state in the system in Appendix A.

Proof. As with the linear case, to show that the pseudo-energy is negative definite will require the usage of the Poincaré inequality. Applying this inequality to the upper bound of $\mathcal{A}$, then identity (4.143) gives

$$\mathcal{A} \leq \frac{1}{2}\left(\frac{\mu}{K} - 1\right)\left(\iiint_{\Omega} \alpha_1 (\delta q_1)^2 \ dxdydz + \iint_{\Omega_H} \alpha_2 (\delta q_2/N)^2_{z=0}\, dxdy\right)$$

$$+\frac{1}{2}\iint\limits_{\Omega_H} \frac{\alpha_3\mu}{K}(\delta\varphi_z/N)^2_{z=-1} - \alpha_3(\delta q_3/N)^2_{z=-1} - \alpha_4(\delta q_4)^2\, dxdy,$$

$$= \frac{1}{2}\left(\frac{\mu}{K} - 1\right)\left(\iiint_{\Omega} \alpha_1 (\delta q_1)^2 \ dxdydz + \iint_{\Omega_H} \alpha_2 (\delta q_2/N)^2_{z=0}\, dxdy\right)$$

$$+\frac{1}{2}\left(\frac{\mu}{K} - 1\right)\iint\limits_{\Omega_H} \alpha_3 (\delta\varphi_z/N + \gamma N\delta h)^2_{z=-1}\, dxdy$$

$$+\frac{1}{2}\iint\limits_{\Omega_H} \left(N^2(-1)\frac{\alpha_3\mu\gamma}{K} + \beta_4\right)(\delta q_4)^2\, dxdy,$$

$$\leq \frac{1}{2}\tilde{\Upsilon}^{-1}\left\{\iiint_{\Omega}(\delta q_1)^2 \; dxdydz + \iint_{\Omega_H}(\delta q_2/N)^2_{z=0}\, dxdy \right.$$

$$\left. + \iint_{\Omega_H}(\delta\varphi_z/N + \gamma N\delta h)^2_{z=-1} + (\delta q_4)^2\, dxdy\right\}, \tag{4.182}$$

where the two new constants are γ, defined in (4.180), which is positive by assumption, and

$$\tilde{\Upsilon}^{-1} = \max_{i=1,2,3}\left\{\alpha_i\left(\frac{\mu}{K}-1\right), \left(N^2(-1)\frac{\alpha_3\mu\gamma}{K}+\beta_4\right)\right\} < 0, \tag{4.183}$$

which is negative because of (4.178) and (4.179).

Since $\tilde{\Upsilon}$ is negative, the last expression in (4.182) can be rewritten as

$$\iiint\limits_{\Omega}(\delta q_1)^2 \; dxdydz + \iint\limits_{\Omega_H}(\delta q_2/N)^2_{z=0}\, dxdy + \iint\limits_{\Omega_H}(\delta\varphi_z/N + \gamma N\delta h)^2_{z=-1} + (\delta q_4)^2\, dxdy$$

$$\leq 2\tilde{\Upsilon}\mathcal{A} = 2\tilde{\Upsilon}\mathcal{A}_{t=0}, \tag{4.184}$$

having used the invariance of the pseudo-energy functional.

As well, an upper bound on the norm can be established

$$\begin{aligned}
\|\delta\mathbf{q}\|^2 &= \iiint_{\Omega}(\delta q_1)^2 \; dxdydz + \iint_{\Omega_H}(\delta q_2/N)^2_{z=0}\, dxdy \\
&\quad + \iint_{\Omega_H}(\delta\varphi_z/N + \gamma N\delta q_4 - \gamma N\delta q_4)^2_{z=-1} + (\delta q_4)^2\, dxdy, \\
&\leq \iiint_{\Omega}(\delta q_1)^2 \; dxdydz + \iint_{\Omega_H}(\delta q_2/N)^2_{z=0}\, dxdy \\
&\quad + \iint_{\Omega_H} 2\left(\delta\varphi_z/N + \gamma N\delta q_4\right)^2_{z=-1} + \left(1 + 2\gamma^2 N^2(-1)\right)(\delta q_4)^2\, dxdy, \\
&\leq \tilde{\Upsilon}\left\{\iiint_{\Omega}(\delta q_1)^2 \; dxdydz + \iint_{\Omega_H}(\delta q_2/N)^2_{z=0}\, dxdy \right. \\
&\quad \left. + \iint_{\Omega_H}(\delta\varphi_z/N + \gamma N\delta q_4)^2_{z=-1} + (\delta q_4)^2\, dxdy\right\},
\end{aligned} \tag{4.185}$$

having added zero, applying the simple algebraic identity

$$(x+y)^2 \leq 2\left(x^2 + y^2\right), \tag{4.186}$$

and finally introducing the constant

$$\bar{\Upsilon} = \max\left\{2, 1 + 2\gamma^2 N^2 (-1)\right\} > 0. \tag{4.187}$$

Now substituting (4.184) into (4.185) yields, upon noting that $\bar{\Upsilon}\tilde{\Upsilon} < 0$,

$$\|\delta\mathbf{q}\|^2 \leq 2\bar{\Upsilon}\tilde{\Upsilon}\mathcal{A}_{t=0}$$

$$\begin{aligned}
\leq \;& \bar{\Upsilon}\tilde{\Upsilon}\left\{\iiint_\Omega \mu\left(\nabla\delta\varphi\cdot\nabla\delta\varphi + (\delta\varphi_z/N)^2\right) - \beta_1(\delta q_1)^2 \; dxdydz \right. \\
& \left. - \iint_{\Omega_H} \beta_2 (\delta q_2/N)^2_{z=0} + \beta_3(\delta q_3/N)^2_{z=-1} + \beta_4(\delta q_4)^2 \, dxdy \right\}_{t=0}, \\
\leq & -\bar{\Upsilon}\tilde{\Upsilon}\left\{\iiint_\Omega \beta_1(\delta q_1)^2 \; dxdydz + \iint_{\Omega_H} \beta_2(\delta q_2/N)^2_{z=0} \; dxdy \right. \\
& \left. + \iint_{\Omega_H} 2\beta_3(\delta\varphi_z/N)^2_{z=-1} + \left(2N^2(-1)\beta_3 + \beta_4\right)(\delta q_4)^2 \, dxdy \right\}_{t=0}, \\
\leq \;& \Upsilon \|\delta\mathbf{q}\|^2_{t=0},
\end{aligned} \tag{4.188}$$

having again used (4.186), and the fact that

$$\bar{\Upsilon}\tilde{\Upsilon}\iiint_\Omega \mu\left(\nabla\delta\varphi\cdot\nabla\delta\varphi + (\delta\varphi_z/N)^2\right) dxdydz \leq 0, \tag{4.189}$$

since $\bar{\Upsilon}\tilde{\Upsilon} < 0$ and finally the definition

$$\Upsilon = \bar{\Upsilon}\tilde{\Upsilon}\max_{i=1,2}\left\{\beta_i, 2\beta_3, 2N^2(-1)\beta_3 + \beta_4\right\} > 0. \tag{4.190}$$

Therefore, for all $\epsilon > 0$ there exists a $\delta = \Upsilon^{-\frac{1}{2}}\epsilon > 0$ such that if $\|\delta\mathbf{q}\|_{t=0} < \delta$, then $\|\delta\mathbf{q}\| < \epsilon$ for all time. ■

It might seem at first glance that the conditions for linear and nonlinear stability are identical because of the fact that the upper and lower bounds are the same. However, the bounds in the linear stability case are on the steady state functions evaluated at the steady solutions. The bounds for nonlinear stability, are on the same functions, but must hold true on their entire domain not just at the single state under consideration. This is due to the fact that when doing a linear stability analysis we assume that the perturbations are infinitesimal, hence the F_i functions will not vary very much from the steady solution, if it is stable that is.

Chapter 5

Density driven eddy solutions

Chapters 3 and 4 studied the stability characteristics of a mesoscale gravity current within a continuously stratified fluid. When instability does occur, it has been demonstrated numerically (Swaters, 1997), that the current may break up into isolated cold-pools. This chapter will study the propagation of these cold-pools along the shelf, as well as the effect they have on the upper fluid. Two solutions are to be obtained. The first is an exact solution to the nonlinear equations which assumes that both fluids are radially symmetric and that wave drag forces are negligible. The second, is an a weakly radiating solution that assumes there is only weak interaction between the two fluids, which allows for the existence of an upper layer topographic Rossby wave field.

5.1 Isolated steadily travelling eddies

Before we begin, it is necessary to explain exactly what is meant by isolated. In our context, it means that the volume/area integrated energy, enstrophy and cold-pool mass are all finite quantities.

The geometry is such that the horizontal region is unbounded,

$$\Omega_H = \{(x, y) \mid -\infty < x, y < \infty\}. \tag{5.1}$$

The bottom topography is chosen to idealize a gently sloping continental shelf where the slope-water gets deeper in the positive y direction, i.e. $h_B = -y$.

We assume that the cold dome is travelling at constant speeds c and $\tilde{c}$, in the x and y directions, respectively. To change into a coordinate system that has the cold-pool as being stationary, requires the following transformation

$$\xi = x - ct, \quad \zeta = y - \tilde{c}t. \tag{5.2}$$

It follows that ξ and ζ both range from $-\infty$ to $+\infty$. The symbol Ω_H is used to denote both the (x, y) and (ξ, ζ) coordinate systems depending on the context.

In the new coordinate system, the time derivative takes the form

$$\begin{aligned} \partial_t &= \frac{\partial x}{\partial t}\partial_x + \frac{\partial y}{\partial t}\partial_y \\ &= -c\partial_x - \tilde{c}\partial_y, \end{aligned} \tag{5.3}$$

which when substituted into the nonlinear governing equations yields the following system

$$\partial\left(\mu\varphi + c\zeta - \tilde{c}\xi, \Delta\varphi + \left(N^{-2}\varphi_z\right)_z\right) = 0, \tag{5.4}$$

$$\partial\left(\mu\varphi + c\zeta - \tilde{c}\xi, \varphi_z\right) = 0 \quad \text{at} \quad z = 0, \tag{5.5}$$

$$\partial\left(\mu\varphi + c\zeta - \tilde{c}\xi, \varphi_z + N^2\left(h - \frac{\zeta}{\mu}\right)\right) = \frac{N^2\tilde{c}}{\mu} \quad \text{at} \quad z = -1, \tag{5.6}$$

$$\partial\left(\mu\varphi + (c-1)\zeta - \tilde{c}\xi, h\right) = 0 \quad \text{at} \quad z = -1. \tag{5.7}$$

Note that the bottom topography has been written as $h_B = -\zeta$ since the Jacobian is now in terms of (ξ, ζ). The Jacobian in (5.7), simplifies to,

$$\left(\left(\mu\varphi_\xi|_{z=-1} - \tilde{c}\right)h\right)_\zeta - \left(\left(\mu\varphi_\zeta|_{z=-1} + c - 1\right)h\right)_\xi = 0, \tag{5.8}$$

after having regrouped and cancelled terms.

To find the first moment of the cold-pool it is necessary to multiply (5.8) by ξ then integrate over the horizontal region

$$\iint_{\Omega_H} \left(\mu\varphi_\zeta|_{z=-1} + c - 1\right) h \; d\xi d\zeta = 0, \tag{5.9}$$

upon integrating by parts twice and using the fact that h vanishes at the boundary. Solving this expression for c gives

$$c = 1 - \mu \frac{\iint_{\Omega_H} h\varphi_\zeta|_{z=-1} \, d\xi d\zeta}{\iint_{\Omega_H} h \, d\xi d\zeta}, \tag{5.10}$$

The second moment is found through multiplying (5.8) by ζ and then following the same steps as above, which results in

$$\tilde{c} = \mu \frac{\iint_{\Omega_H} h\varphi_\xi|_{z=-1} \, d\xi d\zeta}{\iint_{\Omega_H} h \, d\xi d\zeta}. \tag{5.11}$$

If $\mu = 0$, the moment equations simplify to $(c, \tilde{c}) = (1, 0)$. Upon considering the scalings, this is seen to be the same result found in Nof (1983) where the motion is strictly along the shelf. The means by which this occurs is that gravity acts on the cold-pool, trying to pull it down the slope since this is a state of less potential energy. However, the Coriolis force in the northern hemisphere, deflects it to the right. If these two forces are evenly balanced there will be no down-slope component of the velocity. This is what should be expected since the *interaction parameter* being zero reduces our model to the reduced gravity case, which is what was studied in Nof (1983). If $\mu \neq 0$, then the along-shelf speed will generally not be the Nof speed, and a cross-shelf component can develop.

It is known that a flat steadily travelling object moving inviscidly through an homogeneous fluid within a strongly rotating system with a flat bottom, will create a Taylor column directly above it (Pedlosky, 1987). This is a means by which the fluid above the object can be transported along with the moving object. Because our slope-water is continuously stratified and the cold dome is not generally flat, we cannot expect the same result to be applied, but it is of interest to determine whether the fluid above the cold-pool can be transported.

To find conditions under which this can occur, it is necessary to understand streaklines. In the reference frame of the moving eddy, the streaklines are the paths the fluid parcels follow. Because the channel walls are impermeable, the streaklines cannot originate on, or end at, these walls. Therefore, the only remaining possibilities are for the paths to begin upstream and end down stream, or for them to have closed orbits. If closed streaklines exist, then the interior fluid is transported along with the cold-pool. Upon considering (5.4) to (5.6), it is seen that the streaklines are denoted by the level curves of the following function

$$\psi_{streakline} = \mu\varphi + c\zeta - \tilde{c}\xi. \tag{5.12}$$

When the interaction parameter is small, closed orbits do not occur. However, for large μ, we can expect closed streaklines since the streakfunction is dominated by the streamfunction which does contain closed orbits. Therefore, there must be a critical value of μ, call it μ_c where for $\mu > \mu_c$ closed streaklines will certainly occur. Conversely, if $\mu < \mu_c$, no streaklines will arise. When we establish a weakly radiating solution, we will consider different values of the *interaction parameter* to demonstrate this graphically.

Assuming that a closed streakline does exist, we denote the area within it and the curve itself by A and ∂A, respectively. Integrating (5.6) over A gives that

$$\int_{\partial A} \left(\varphi_z + N^2\left(h - \frac{\zeta}{\mu}\right)\right)\left(\mathrm{e}_3 \times \nabla\left(\mu\varphi + c\zeta - \tilde{c}\xi\right)\right)\cdot \mathbf{n}\; ds = \frac{N^2\tilde{c}}{\mu}\left\|A\right\|, \tag{5.13}$$

using $\|A\|$ to denote the area contained in A, as well as (4.23) and the divergence theorem. An identity that can easily be proven is

$$(\mathrm{e}_3 \times \nabla B)\cdot \mathbf{n} \;= \nabla B \times \mathbf{n}, \tag{5.14}$$

which when applied to (5.13) yields

$$\int_{\partial A} \left(\varphi_z + N^2\left(h - \frac{\zeta}{\mu}\right)\right)\nabla\left(\mu\varphi + c\zeta - \tilde{c}\xi\right) \times \mathbf{n}\; ds = \frac{N^2\tilde{c}}{\mu}\left\|A\right\|. \tag{5.15}$$

Taking the gradient of $\psi_{streakline}$ in the (ξ, ζ) coordinate system results in a vector that is normal to the boundary of a streakline, since it is a level curve of the function. Hence, the cross product of this gradient term with the normal vector of the boundary is zero because these vectors are parallel. This allows us to conclude that for general stratification,

$$\tilde{c} = 0, \tag{5.16}$$

if a closed streakline exists. This is equivalent to saying that there cannot be any cross-slope component of the velocity of the steadily travelling cold-pool when fluid transport occurs.

Substituting (5.16) into (5.4) through to (5.7), yields the same set of steadily travelling equations that were considered in Chapter 4, (4.91) to (4.94), with the sole difference of the ζ's replacing the y's. To this system, we can introduce steadily travelling functions as in (4.95) to (4.98). It is possible to determine the form of F_3 by evaluating the analogue of (4.97) in the far field. This involves taking the limit as $\xi^2 + \zeta^2 \rightarrow \infty$, and using the fact that φ, h and all of their derivatives must vanish in this region, since the disturbance is assumed to be isolated.

Evaluating this equation in the far field gives

$$c\zeta = F_3\left(-\frac{N^2(-1)\zeta}{\mu}\right), \tag{5.17}$$

which implies that

$$F_3(*) = -\frac{c\mu}{N^2(-1)}*, \tag{5.18}$$

for all closed streaklines that extend to infinity. Therefore (4.97) can be replaced by

$$\varphi_z + \frac{N^2}{c}\varphi = -N^2 h \quad \text{at} \quad z = -1, \tag{5.19}$$

upon doing some algebraic manipulations. Therefore, the governing equations for steadily travelling isolated cold-pools within a continuously stratified fluid are (4.91), (4.92), (5.19) and (4.94).

In the derivation of this equation we have assumed that the streaklines extend to infinity. However, this clearly does not hold true for those streaklines that are closed and is a difficulty since these are precisely the streaklines of most interest.

Indeed, there is no known method of determining $F_3(*)$ for a region of closed streaklines. Here, we simply adopt the ansatz, following the arguments of Hogg (1980) and

Swaters & Mysak (1985), that (5.18) is assumed to hold everywhere. This choice will ensure that φ and its derivatives will be continuous across the boundary separating the regions of open and closed streaklines.

5.2 Radially symmetric Solutions

In general, three-dimensional cold-pools are difficult to solve for analytically. It is for this reason that we make the radially symmetric assumption which reduces the problem to that of a two-dimensional one, through the implementation of cylindrical coordinates. Solutions are sought of the following form,

$$h = h(r), \quad \varphi = \varphi(r, z), \tag{5.20}$$

where

$$r = \sqrt{\xi^2 + \zeta^2}, \quad \theta = \arctan(\zeta/\xi),$$
$$\xi = r\cos\theta, \quad \zeta = r\sin\theta. \tag{5.21}$$

Note that because the bottom topography is gently sloping, to leading order, it is perpendicular to the z axis, which is why there need not be θ dependency in our solution.

Changing the Jacobian to polar variables, by use of the definition form Chapter 2, results in

$$\begin{aligned} \partial(A, B) &\equiv \frac{\partial(A, B)}{\partial(x, y)} \\ &= \frac{\partial(r, \theta)}{\partial(x, y)} \frac{\partial(A, B)}{\partial(r, \theta)} \\ &= \frac{1}{r} \frac{\partial(A, B)}{\partial(r, \theta)} \\ &= \frac{1}{r}(A_r B_\theta - A_\theta B_r), \end{aligned} \tag{5.22}$$

since the determinate of the Jacobian from polar to Cartesian coordinates is r. Therefore, since φ and h are both assumed to be independent of θ, the Jacobian terms of φ and h, or any of their derivatives, vanish since each term involves a derivative with respect to the variable θ. This implies that solutions of the form (5.20) are fully nonlinear solutions of the model equations.

The implications of (5.20) on the velocity of the cold-pool can be found by evaluating the integrals that appear in the numerators of the non-Nof speed components of (5.10) and (5.11). The first becomes, by converting to polar coordinates

$$\begin{aligned} \iint_{\Omega_H} h\varphi_\zeta|_{z=-1}\ d\xi d\zeta &= \int_0^\infty h\varphi_r|_{z=-1} \int_0^{2\pi} \sin\theta d\theta dr \\ &= 0, \end{aligned} \tag{5.23}$$

having used (5.21) to transform the partial derivative, calculating the azimuthal integral and using the periodicity of sine. Similarly,

$$\begin{aligned} \iint_{\Omega_H} h\varphi_\xi|_{z=-1}\ d\xi d\zeta &= \int_0^\infty h\varphi_r|_{z=-1} \int_0^{2\pi} \cos\theta d\theta dr \\ &= 0, \end{aligned} \tag{5.24}$$

which signifies that all steadily travelling, radially symmetric solutions propagate at the Nof speed along the continental shelf, with no cross-slope component.

Substituting the above two equations into the steadily travelling equations reduces (5.4) and (5.5) to

$$\left(\Delta\varphi + \left(N^{-2}\varphi_z\right)_z\right)_\xi = 0, \tag{5.25}$$

$$(\varphi_z)_\xi = 0 \quad \text{at} \quad z = 0, \tag{5.26}$$

since the nonlinear terms vanish due to radial symmetry. Notice that (5.7) reduces to the statement that $0 = 0$, which contains no information and hence is ignored. It is the degeneracy of this equation that changes the state of h from a variable to be solved for, to that which must be predetermined. This is a problem in the theory in that we are unable to determine criteria for which cold-pool profiles are allowable, as in Swaters & Flierl (1991).

In looking for wave-like solutions travelling in the along-shelf direction we can integrate (5.25) and (5.26) while setting the functions of integration to zero since the solutions are isolated. These two equations, combined with (5.19), yield the following linear system

$$\Delta\varphi + \left(N^{-2}\varphi_z\right)_z = 0, \tag{5.27}$$

$$\varphi_z = 0 \quad \text{at} \quad z = 0, \tag{5.28}$$

$$\varphi_z + N^2\varphi = -N^2 h \quad \text{at} \quad z = -1. \tag{5.29}$$

After choosing a lower layer profile, we then solve for the dynamic pressure of the upper layer to see how the upper fluid reacts to this given profile. This system of equations above is the same as that in Johnson (1978). This paper considered the situation of continuously stratified QG fluid moving over topographic disturbances. If we take the perspective of the cold-pool then the situation is similar to that in the aforementioned paper, due to the fact that the lower layer is coherent and travelling along the shelf at a constant speed. By coherent, we mean that the lower layer height field does not deform to leading order in time. Deformations will take place, but on a much slower scale than what we are considering.

Since we are in cylindrical coordinates the Laplacian must also be written in this coordinate system,

$$\Delta\varphi = \frac{1}{r}\left(r\varphi_r\right)_r. \tag{5.30}$$

No θ derivatives occur because of the radial symmetry of φ.

System (5.27) through to (5.29) is difficult to solve for several reasons. Firstly, there is not a unique solution and hence some other condition need be imposed. Secondly, since in (5.29) $N^2 > 0$, boundary condition (5.29) makes the system not Sturm-Liouville. Thirdly, this boundary condition contains r dependency, since h is present, which means that the system is not separable.

The system not being Sturm-Liouville suggests that we cannot expect all the eigenvalues associated with the horizontal problem to be positive, which indeed is true for the special case where the buoyancy frequency is constant. Positive eigenvalues yield ordinary Bessel functions which decay like $O\left(r^{-\frac{1}{2}}\right)$ as $r \to \infty$. These modes can have an upstream wave-field which is problematic since it is known that this does not occur in nature. To rectify this difficulty it is necessary to impose a radiation condition. This added constraint solves the problem of the lack of uniqueness in the solution. The radiation condition that will be applied is the zero-wave drag condition, as was done in Swaters & Flierl (1991). This condition is very strong and has the net effect of eliminating any upstream as well as downstream waves. Hence no wave field is generated outside the support of the lower layer. The negative eigenvalues that arise, result in modified Bessel functions that are known to decay exponentially as $r \rightarrow \infty$, and therefore do not give rise to any wave-field.

Integrating (5.27) over the entire volume, using the divergence theorem along with the fact that the gradient of φ vanishes in the far field, and then integrating by parts with respect to z along with boundary conditions (5.28) and (5.29), yields

$$\iiint_{\Omega} \Delta\varphi + \left(N^{-2}\varphi_z\right)_z \; dxdydz = 0,$$

$$\iint_{\Omega_H} \left[N^{-2}\varphi_z\right]_{z=-1}^{z=0} dxdy = 0, \tag{5.31}$$

which implies

$$\iint_{\Omega_H} \varphi|_{z=-1} + h \; dxdy = 0. \tag{5.32}$$

This is the *Stern Isolation Constraint* and is a statement of conservation of angular momentum (Mory, 1985). Because we are considering cold-pool anomalies $h \geq 0$, if $\varphi|_{z=-1}$ is only of one sign, as turns out to be true, it must then be negative. Therefore, above the cold-pool there exists a negative pressure anomaly which corresponds to a low pressure system. Since low pressure systems correspond to cyclonic motion, the induced eddy in the upper fluid will be a cyclone.

This same conclusion can be obtained through a more heuristic argument. It is known with surface waves that concave and convex deformations correspond to low and high pressures, respectively. Since the lower layer height is positive, this implies that there must be a low pressure system directly above it. Because low pressure systems are associated with cyclonic motion, the induced eddy above the cold-pool must be a cyclone.

Converting (5.32) in terms of polar coordinates, integrating by parts with respect to r and using the far field condition results in the following, after having multiplied by 2,

$$\iint\limits_{\Omega_H} r^2 \left(\varphi|_{z=-1} + h\right)_r drd\theta = 0. \tag{5.33}$$

By a similar argument as above, we conclude that for the integral to be zero, there must be regions where $(\varphi|_{z=-1} + h)_r$ is positive and others where it is negative.

In Chapter 2 it was determined that the horizontal velocity fields in the upper and lower layers were determined geostrophically. The upper layer depends on φ, whereas the lower layer on φ, h and h_B. The geostrophic relation (2.130) can be separated into two components, the first being

$$\mathbf{e}_3 \times \nabla h_B = \mathbf{e}_1, \tag{5.34}$$

for the topography under consideration, where $\mathbf{e}_1$ is the unit vector in the positive x direction. This shows that it is the bottom topography that causes the cold-pool to move along the shelf because of the balance of the buoyancy and Coriolis forces. The other component is what causes the rotation of the eddy, and is called the swirl velocity. This velocity is denoted with $\mathbf{u}_{swirl}$ and is represented by

$$\begin{aligned} \mathbf{u}_{swirl} &= \mathbf{e}_3 \times \nabla \mu (\varphi + h), \\ &= \mu (\varphi + h)_r \mathbf{e}_r, \end{aligned} \tag{5.35}$$

where $\mathbf{e}_r$ denotes the unit vector directed along the polar axis. Similarly, the swirl velocity for the upper layer $\mathbf{u}_s$, is

$$\mathbf{u}_s = \varphi_r \mathbf{e}_r. \tag{5.36}$$

Equations (5.33) and (5.35) imply that there are regions of cyclonic and anticyclonic vorticity. This is counter-intuitive in that we expect the eddy to spin in only one direction, but clearly this is not the case as was also found in Swaters & Flierl (1991). Assuming the height of the lower layer has its peak at $r = 0$ and gradually decreases to zero means that $h_r < 0$ on the support of h. If φ_r is of only one sign, (5.33) implies that it must be positive. Because $h_r(0) = 0$ for smooth profiles, the core of the cold-pool is cyclonic, which means that it is the peripheral region that must be anticyclonic, if there is only one point where the vorticity changes sign. Also, upon considering the equation directly above, we see through a different means, that the vorticity in the upper layer eddy must be cyclonic.

Going back to the system (5.27) to (5.29), exact solutions can be found if we assume that the buoyancy frequency is constant, or equivalently, that the stratification of the background state is linear. To solve the system we use the method of Finite Fourier Transforms (Zauderer, 1989). This requires that we first consider the following eigenvalue problem with the associated homogeneous boundary conditions from above,

$$N^{-2}\Phi_{zz} = \lambda\Phi, \tag{5.37}$$

$$\Phi_z = 0 \quad \text{on} \quad z = 0, \tag{5.38}$$

$$\Phi_z + N^2\Phi = 0 \quad \text{on} \quad z = -1, \tag{5.39}$$

where $\Phi = \Phi(z)$. There will be two sets of solutions, those that correspond to positive and negative eigenvalues respectively.

First assuming the eigenvalues are positive, which we denote by λ_0, allows the solutions to be written as

$$\Phi = A_0 \cosh\left(N\sqrt{\lambda_0}z\right) + B_0 \sinh\left(N\sqrt{\lambda_0}z\right), \tag{5.40}$$

where (5.38) requires that $B_0 = 0$, therefore

$$\Phi = A_0 \cosh\left(N\sqrt{\lambda_0}z\right). \tag{5.41}$$

Substituting the new solution into the other boundary condition (5.39), yields the dispersion relation

$$\tanh\left(N\sqrt{\lambda_0}\right) = \frac{N}{\sqrt{\lambda_0}}. \tag{5.42}$$

Observe that the right-hand side of the above equation is a function of λ_0, that originates at $+\infty$ and decreases to 0. The left-hand side, on the other hand, is increasing from 0 to 1. These two properties when combined, is enough for us to conclude that there is exactly on point of intersection, ergo one solution. The graph for the special case where $N = 1$ is shown in Figure 5.1

If $\lambda < 0$, we denote the eigenvalues with λ_n. The general solution to (5.37) is

$$\Phi = A_n \cos\left(N\sqrt{-\lambda_n} z\right) + B_n \sin\left(N\sqrt{-\lambda_n} z\right), \tag{5.43}$$

where a subscript n has been used in anticipation of the fact that there will be many solutions. Equation (5.38) implies that $B_n = 0$,

$$\Phi = A_n \cos\left(N\sqrt{-\lambda_n} z\right). \tag{5.44}$$

Similarly, substituting this into (5.39) gives the dispersion relation for the negative eigenvalues

$$\tan\left(N\sqrt{-\lambda_n}\right) = -\frac{N}{\sqrt{-\lambda_n}}. \tag{5.45}$$

Unlike the previous case, this equation has an infinite number of solutions because tangent is an oscillatory function. We denote the solutions with $n = 1, 2, \ldots$ The graph for the special case where $N = 1$ is shown in Figure 5.2, where it is seen that the solutions are not periodically spaced.

Equations (5.41) and (5.44) give the set of vertical eigenfunctions for the solution space. The system can be orthonormalized by calculating the constants A_n for $n = 0, 1, 2, \ldots$, such that the integral of each of the functions multiplied by itself is one. Squaring (5.41) and integrating the result from -1 to 0, which then set equal to one, results in

$$A_0^2 \int_{-1}^{0} \cosh^2\left(N\sqrt{\lambda_0} z\right) dz = 1,$$

$$\frac{1}{2} A_0^2 \int_{-1}^{0} 1 + \cosh\left(2N\sqrt{\lambda_0} z\right) dz = 1,$$

$$A_0^2\left[z+\frac{\sinh\left(2N\sqrt{\lambda_0}z\right)}{2N\sqrt{\lambda_0}}\right]_{z=-1}^{z=0}=2,$$

$$A_0=\left[\frac{4N\sqrt{\lambda_0}}{2N\sqrt{\lambda_0}+\sinh\left(2N\sqrt{\lambda_0}\right)}\right]^{\frac{1}{2}}, \tag{5.46}$$

upon using a trigonometric identity and computing the integration. Doing a similar calculation for the other eigenfunctions requires that

$$A_n=\left[\frac{4N\sqrt{-\lambda_n}}{2N\sqrt{-\lambda_n}+\sin\left(2N\sqrt{-\lambda_n}z\right)}\right]^{\frac{1}{2}}. \tag{5.47}$$

Therefore, the orthonormal set eigenfunctions that solve the system are $\{\lambda_n, \Phi_n(z)\}_{n=0}^{\infty}$ with,

$$\Phi_0=\left[\frac{4N\sqrt{\lambda_0}}{2N\sqrt{\lambda_0}+\sinh\left(2N\sqrt{\lambda_0}\right)}\right]^{\frac{1}{2}}\cosh\left(\sqrt{\lambda_0}Nz\right), \tag{5.48}$$

$$\Phi_n=\left[\frac{4N\sqrt{-\lambda_n}}{2N\sqrt{-\lambda_n}+\sin\left(2N\sqrt{-\lambda_n}\right)}\right]^{\frac{1}{2}}\cos\left(\sqrt{-\lambda_n}Nz\right), \tag{5.49}$$

for $n = 1, 2, ...$, where the eigenvalues are determined from (5.42) and (5.45) and the following relation is true

$$\int_{-1}^{0}\Phi_m(z)\,\Phi_n(z)\,dz=\delta_{mn}. \tag{5.50}$$

The gravest mode λ_0, is the only positive eigenvalue, and is the source of the eddy generated above the cold dome. The remaining modes λ_n for $n = 1, 2, ...$, that correspond to negative eigenvalues, are the internal baroclinic modes for the continuously stratified fluid (Cushman-Roisin, 1994).

Since Φ satisfies (5.37) and (5.38), any linear superposition will also satisfy the same equations. Therefore, the solution of (5.27) through to (5.29) is constructed by assuming the following series solution

$$\varphi(r,z)=\sum_{n=0}^{\infty}A_n(r)\,\Phi_n(z), \tag{5.51}$$

Figure 5.1

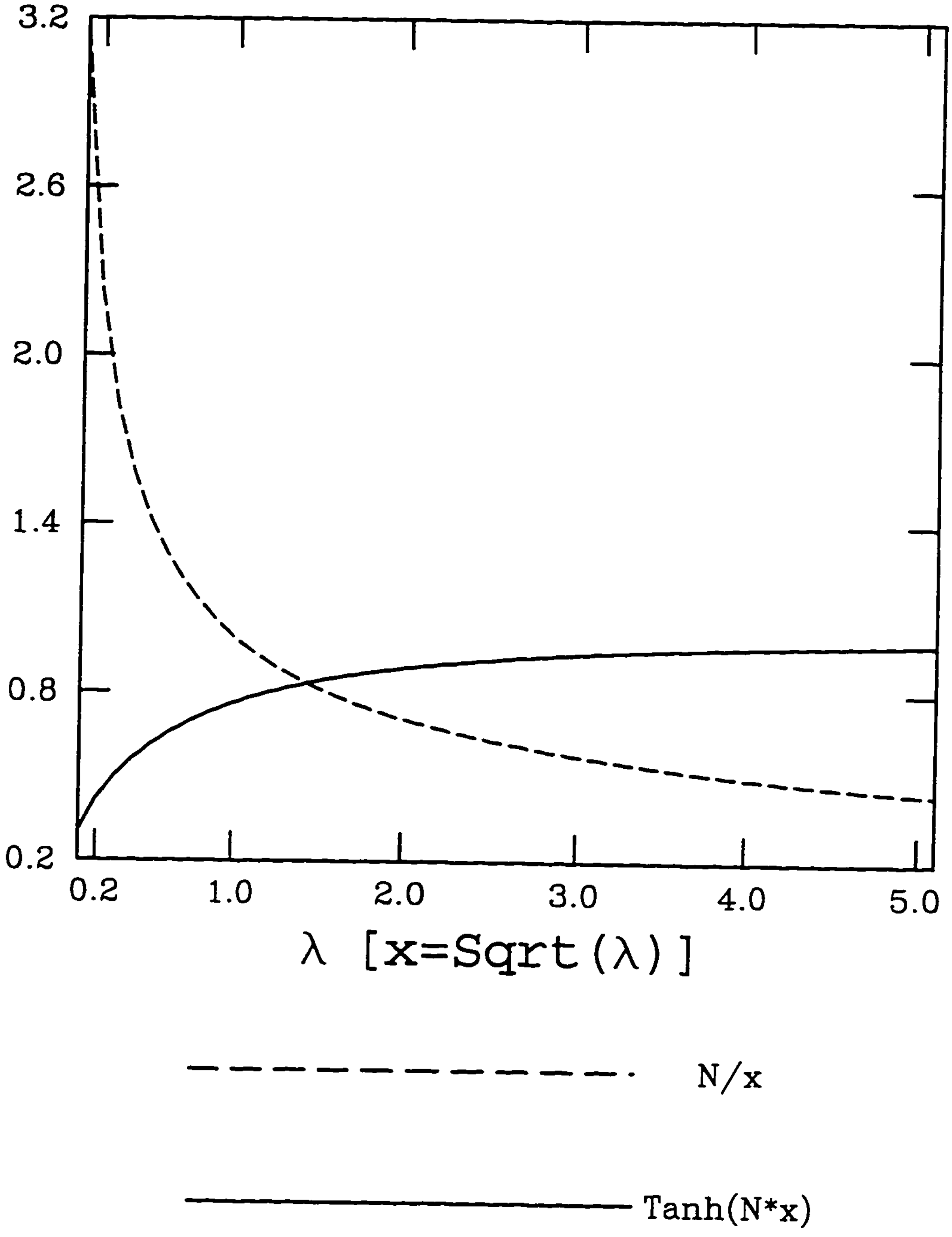
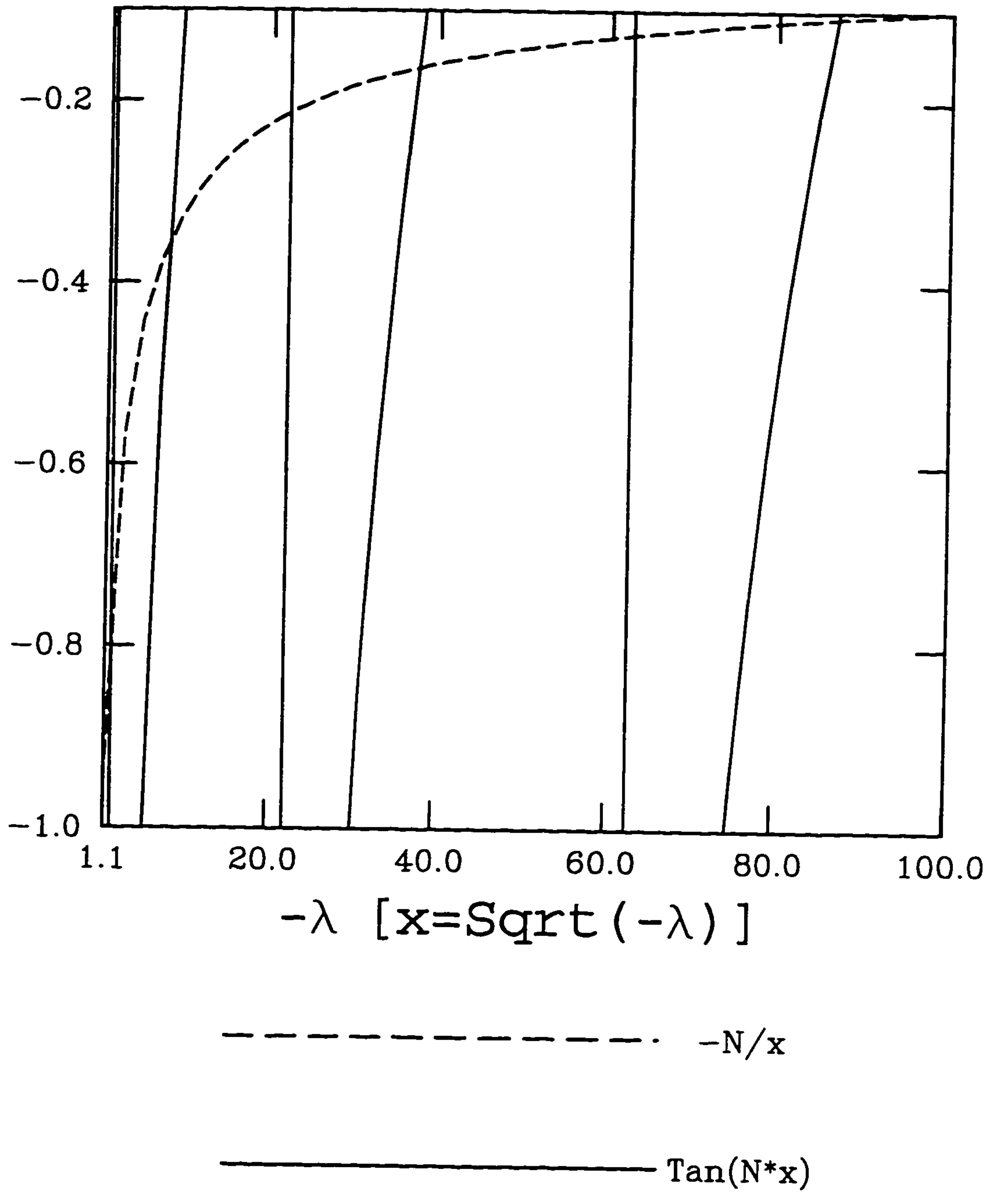


Figure 5.2



since (5.27) and (5.29) are the same as (5.37) and (5.38). We assume that this solution satisfies (5.29) and then obtain a differential equation in A_n, which must then be solved.

To obtain this differential equation, we multiply (5.28) by $\Phi_m(z)$, and integrate with respect to z from -1 to 0, then substitute (5.51) to yield

$$\sum_{n=0}^{\infty} \Delta A_n \int_{-1}^{0} \Phi_m \Phi_n dz + \int_{-1}^{0} \Phi_m \left(N^{-2}\varphi_z\right)_z dz = 0. \tag{5.52}$$

The first term can be rewritten because of (5.50). Integrating by parts twice, applying (5.51) and the orthonormal relation produces

$$\Delta A_n + \lambda_n \int_{-1}^{0} \Phi_n \varphi \, dz = -\left[N^{-2}\Phi_n \varphi_z - N^{-2}\Phi_{n_z}\varphi\right]_{z=-1}^{z=0},$$

$$\Delta A_n + \lambda_n A_n = -\left[N^{-2}\left(\Phi_n \varphi_z - \Phi_{n_z}\varphi\right)\right]_{z=-1}^{z=0}. \tag{5.53}$$

Substituting in the boundary conditions (5.28), (5.29), (5.38) and (5.39) reduces the above expression to

$$\Delta A_n + \lambda_n A_n = -\Phi_n(-1)\, h. \tag{5.54}$$

This differential equation needs to be solved for $n = 0, 1, 2, ...$ The gravest mode equation is the same equation that which arose in Flierl (1984) and Swaters & Flierl (1991), this being the inhomogeneous Helmholtz equation. Note that a solution to (5.54) for a n, would yield an exact solution to the nonlinear governing equations since no approximations have been made, only simplifications based on the given assumptions.

Assuming the cold-pool profile is

$$h = \begin{cases} h(r) & \text{for} \quad 0 \leq r < a \\ 0 & \text{for} \quad r \geq a \end{cases}, \tag{5.55}$$

with $h(a) = 0$, so that h is continuous, allows us to solve (5.54). We begin by solving for the gravest mode.

If the eigenvalue is positive, the left-hand side of (5.54) is the zeroth order Bessel equation. For $r \geq a$, the equation is homogeneous because $h = 0$ by assumption, and the solution is

$$A_0 = c_0 J_0\left(\sqrt{\lambda_0} r\right) + d_0 Y_0\left(\sqrt{\lambda_0} r\right), \tag{5.56}$$

for some constants c_0 and d_0, where J_0 and Y_0 are the zeroth order Bessel functions of the first and second kind respectively (see Watson (1962)). Certain properties that can be derived, but instead of doing so, they will simply be stated in Appendix B and cited when need be.

For $r \leq a$, the equation is inhomogeneous and hence the solution is a sum of the solution to the associated homogeneous equation and a particular solution. To obtain a particular solution we use the method of variation of parameters. This replaces the constants in (5.56) with functions of r, say $\alpha_0(r)$ and $\beta_0(r)$, which must be solved for. It can be found in Zauderer (1989), that these functions must satisfy the following system of equations

$$\begin{bmatrix} J_0\left(\sqrt{\lambda_0} r\right) & Y_0\left(\sqrt{\lambda_0} r\right) \\ J_{0_r}\left(\sqrt{\lambda_0} r\right) & Y_{0_r}\left(\sqrt{\lambda_0} r\right) \end{bmatrix} \begin{bmatrix} \alpha_0' \\ \beta_0' \end{bmatrix} = \begin{bmatrix} 0 \\ -\Phi_0(-1)\, h \end{bmatrix}. \tag{5.57}$$

The solution is obtained by multiplying this equation by the inverse of the matrix that appears on the left-hand side. Doing this, and noting that its determinate is the Wronskian which is $W(J_0, Y_0) = \frac{2}{\pi r}$, brings about

$$\begin{bmatrix} \alpha_0' \\ \beta_0' \end{bmatrix} = \frac{\pi r}{2} \begin{bmatrix} Y_0\left(\sqrt{\lambda_0} r\right) \Phi_0(-1)\, h \\ -J_0\left(\sqrt{\lambda_0} r\right) \Phi_0(-1)\, h \end{bmatrix}. \tag{5.58}$$

Integrating this system yields, while ignoring the constants of integration since we are looking for a particular solution, gives

$$\alpha_0 = -\frac{\pi \Phi_0(-1)}{2} \int_r^a \tau Y_0\left(\sqrt{\lambda_0} \tau\right) h(\tau)\ d\tau, \tag{5.59}$$

$$\beta_0 = -\frac{\pi \Phi_0(-1)}{2} \int_0^r \tau J_0\left(\sqrt{\lambda_0} \tau\right) h(\tau)\ d\tau. \tag{5.60}$$

Therefore, a particular solution is

$$A_{0p} = -\frac{\pi\Phi_0(-1)}{2}\left\{J_0\left(\sqrt{\lambda_0}r\right)\int_r^a \tau Y_0\left(\sqrt{\lambda_0}\tau\right)h(\tau)\ d\tau\right.$$

$$\left.+Y_0\left(\sqrt{\lambda_0}r\right)\int_0^r \tau J_0\left(\sqrt{\lambda_0}\tau\right)h(\tau)\ d\tau\right\}, \tag{5.61}$$

for the region $0 \leq r \leq a$. Evaluating this expression at $r = a$, shows that the solution of the particular solution for that outer region that matches to this particular solution must be

$$A_{0p} = -\frac{\pi\Phi_0(-1)}{2}Y_0\left(\sqrt{\lambda_0}r\right)\int_0^a \tau J_0\left(\sqrt{\lambda_0}\tau\right)h(\tau)\ d\tau, \tag{5.62}$$

for $r \geq a$.

In order to eliminate the wave-field ahead of the eddy, and therefore everywhere, we must impose an added constraint of

$$\int_0^a \tau J_0\left(\sqrt{\lambda_0}\tau\right)h(\tau)\ d\tau = 0. \tag{5.63}$$

This is a zero-wave drag condition and is the same as what was used in Swaters & Flierl (1991). It implies that the outer solution reduces to $A_{0p} = 0$. The derivative of (5.61) with respect to r is,

$$\frac{dA_{0p}}{dt} = -\frac{\pi\Phi_0(-1)}{2}\left\{J_{0_r}\left(\sqrt{\lambda_0}r\right)\int_r^a \tau Y_0\left(\sqrt{\lambda_0}\tau\right)h(\tau)\ d\tau - J_0\left(\sqrt{\lambda_0}r\right)rY_0\left(\sqrt{\lambda_0}r\right)h\right.$$

$$\left.+Y_{0_r}\left(\sqrt{\lambda_0}r\right)\int_0^r \tau J_0\left(\sqrt{\lambda_0}\tau\right)h(\tau)\ d\tau + Y_0\left(\sqrt{\lambda_0}r\right)rJ_0\left(\sqrt{\lambda_0}r\right)h\right\}, \tag{5.64}$$

where, evaluating at $r = a$ gives,

$$\frac{dA_{0p}}{dt}(a) = -\frac{\pi\Phi_0(-1)}{2}Y_{0_r}\left(\sqrt{\lambda_0}a\right)\int_0^a \tau J_0\left(\sqrt{\lambda_0}\tau\right)h(\tau)\ d\tau = 0, \tag{5.65}$$

after applying (5.63). Therefore, (5.63) is enough to make the particular solution continuously differentiable over the entire domain for the case of no wave-field in the outer region.

When the eigenvalues are negative, the left-hand side of (5.54) is the homogeneous zeroth order modified Bessel equation. In the region $r \geq a$, the solution is

$$A_n = c_n I_0\left(\sqrt{-\lambda_n} r\right) + d_n K_0\left(\sqrt{-\lambda_n} r\right), \tag{5.66}$$

where c_n and d_n are constants and I_0 and K_0 are the zeroth order Bessel functions of the third and fourth kind, respectively. To find a particular solution for $r \leq a$ we replace these constants with functions of r, $\alpha_n(r)$ and $\beta_n(r)$ and solve the following system

$$\begin{bmatrix} I_0\left(\sqrt{-\lambda_n} r\right) & K_0\left(\sqrt{-\lambda_n} r\right) \\ I_{0_r}\left(\sqrt{-\lambda_n} r\right) & K_{0_r}\left(\sqrt{-\lambda_n} r\right) \end{bmatrix} \begin{bmatrix} \alpha_n' \\ \beta_n' \end{bmatrix} = \begin{bmatrix} 0 \\ -\Phi_n(-1)\, h \end{bmatrix}. \tag{5.67}$$

Multiply by the inverse of the matrix results in

$$\begin{bmatrix} \alpha_n' \\ \beta_n' \end{bmatrix} = -r \begin{bmatrix} K_0\left(\sqrt{-\lambda_n} r\right) \Phi_n(-1)\, h \\ -I_0\left(\sqrt{-\lambda_n} r\right) \Phi_n(-1)\, h \end{bmatrix}, \tag{5.68}$$

after having used the fact $W(I_0, K_0) = -\frac{1}{r}$. Integrating, and ignoring the constants of integration, then yields

$$\alpha_n = \Phi_n(-1) \int_r^a \tau K_0\left(\sqrt{-\lambda_n} \tau\right) h(\tau)\, d\tau, \tag{5.69}$$

$$\beta_n = \Phi_n(-1) \int_0^r \tau I_0\left(\sqrt{-\lambda_n} \tau\right) h(\tau)\, d\tau. \tag{5.70}$$

Hence a particular solution on the interval $r \leq a$ is

$$A_{np} = \Phi_n(-1) \left\{ I_0\left(\sqrt{-\lambda_n} r\right) \int_r^a \tau K_0\left(\sqrt{-\lambda_n} \tau\right) h(\tau)\, d\tau \right.$$

$$\left. + K_0\left(\sqrt{-\lambda_n} r\right) \int_0^r \tau I_0\left(\sqrt{-\lambda_n} \tau\right) h(\tau)\, d\tau \right\}. \tag{5.71}$$

Evaluating this at $r = a$ shows that this solution is matched smoothly to the following particular outer solution,

$$A_{np} = \Phi_n(-1)\, K_0\left(\sqrt{-\lambda_n} r\right) \int_0^a \tau I_0\left(\sqrt{-\lambda_n} \tau\right) h(\tau)\, d\tau. \tag{5.72}$$

A radiation condition need not be imposed on the eigenfunctions that correspond to the negative eigenvalues because these modes decay much quicker then do waves. Waves in the far field are larger than $o(r^{-\frac{1}{2}})$ as $r \to \infty$, and since K_0 decays exponentially, none of these eigenfunctions contribute to the wave field.

5.3 Parabolic eddy solution

The previous section found exact solutions for an isolated radially symmetric lower layer of arbitrary profile. This section reduces the solution to the special case where the cold-pool is parabolic,

$$h = \begin{cases} 1 - (\frac{r}{a})^2 & \text{for} \quad 0 \leq r < a \\ 0 \quad \text{for} \quad \mathrm{r} \geq a \end{cases} . \tag{5.73}$$

For the gravest mode, there are three integrals that must be evaluated. The first two appear in A_{0p}, for $r < a$. The first set of calculations shall be done explicitly, whereas since the second is simply stated because it is done through an analogous process. Using the Bessel identity (B.1), in Appendix B, and integration by parts twice, along with the fact that $J_2(0) = 0$, allows the following to be deduced

$$\begin{aligned} \int_0^r \tau J_0\left(\sqrt{\lambda_0}\tau\right)\left(1-\left(\frac{\tau}{a}\right)^2\right)\, d\tau &= \frac{1}{\sqrt{\lambda_0}}\int_0^r \left(1-\left(\frac{\tau}{a}\right)^2\right)\frac{d}{d\tau}\left(\tau J_1\left(\sqrt{\lambda_0}\tau\right)\right)\, d\tau, \\ &= \left[\frac{(\tau-\tau^3/a^2)}{\sqrt{\lambda_0}}J_1\left(\sqrt{\lambda_0}\tau\right)\right]_0^r + \frac{2}{\sqrt{\lambda_0}a^2}\int_0^r \tau^2 J_1\left(\sqrt{\lambda_0}\tau\right)\, d\tau, \\ &= \frac{(r-r^3/a^2)}{\sqrt{\lambda_0}}J_1\left(\sqrt{\lambda_0}r\right) + \frac{2}{\lambda_0 a^2}\int_0^r \frac{d}{d\tau}\left(\tau^2 J_2\left(\sqrt{\lambda_0}\tau\right)\right)\, d\tau, \\ &= \frac{(r-r^3/a^2)}{\sqrt{\lambda_0}}J_1\left(\sqrt{\lambda_0}r\right) + \frac{2r^2}{\lambda_0 a^2}J_2\left(\sqrt{\lambda_0}r\right). \end{aligned} \tag{5.74}$$

The second integral, using identity (B.5), becomes

$$\int_r^a \tau Y_0\left(\sqrt{\lambda_0}\tau\right)\left(1-\left(\frac{\tau}{a}\right)^2\right)\, d\tau = -\frac{(r-r^3/a^2)}{\sqrt{\lambda_0}}Y_1\left(\sqrt{\lambda_0}r\right)$$

$$-\frac{2r^2}{\lambda_0 a^2} Y_2\left(\sqrt{\lambda_0} r\right) + \frac{2}{\lambda_0} Y_2\left(\sqrt{\lambda_0} a\right). \tag{5.75}$$

which enables us to write A_{0p} for $r \leq a$ as

$$A_{0p} = -\frac{\pi\Phi_0(-1)}{2}\left\{\frac{(r - r^3/a^2)}{\sqrt{\lambda_0}}\left(Y_0\left(\sqrt{\lambda_0} r\right) J_1\left(\sqrt{\lambda_0} r\right) - J_0\left(\sqrt{\lambda_0} r\right) Y_1\left(\sqrt{\lambda_0} r\right)\right)\right.$$

$$\frac{2r^2}{\lambda_0 a^2}\left(Y_0\left(\sqrt{\lambda_0} r\right) J_2\left(\sqrt{\lambda_0} r\right) - J_0\left(\sqrt{\lambda_0} r\right) Y_2\left(\sqrt{\lambda_0} r\right)\right)$$

$$\left. + \frac{2}{\lambda_0} Y_2\left(\sqrt{\lambda_0} a\right) J_0\left(\sqrt{\lambda_0} r\right)\right\}. \tag{5.76}$$

Identities (B.15) and (B.16), transform this equation to

$$A_{0p} = -\frac{\Phi_0(-1)}{\lambda_0}\left(\left(1 - \left(\frac{r}{a}\right)^2\right) + \frac{4}{\lambda_0 a^2} + \pi Y_2\left(\sqrt{\lambda_0} a\right) J_0\left(\sqrt{\lambda_0} r\right)\right). \tag{5.77}$$

The solution (5.62), for $r \geq a$, simplifies to

$$A_{0p} = -\frac{\pi\Phi_0(-1)}{\lambda_0} Y_0\left(\sqrt{\lambda_0} r\right) J_2\left(\sqrt{\lambda_0} a\right), \tag{5.78}$$

upon rewriting the integral using (5.74) evaluated at $r = a$, and the zero-wave drag condition reduces to

$$J_2\left(\sqrt{\lambda_0} a\right) = 0. \tag{5.79}$$

This restricts what is allowable for the radius of the eddy, this being a discrete set of values. The first zero of $J_2(x)$ is $x = 0$, which is the trivial case and hence ignored. Any other root of this equation can be considered. When plotting graphs the first non-zero root will be utilized.

The remaining modes also have integrals that must be evaluated. Again, going through the same calculations as above, using Bessel identities (B.5) and (B.7), allows the following equations to be derived,

$$\int_0^r \tau I_0\left(\sqrt{-\lambda_n}\tau\right)\left(1 - \left(\frac{\tau}{a}\right)^2\right) d\tau = \frac{1}{\sqrt{-\lambda_n}}\left(r - \frac{r^3}{a^2}\right) I_1\left(\sqrt{-\lambda_n} r\right)$$

$$-\frac{2r^2}{\lambda_n a^2} I_2\left(\sqrt{-\lambda_n} r\right), \tag{5.80}$$

$$\int_r^a \tau K_0\left(\sqrt{-\lambda_n}\tau\right)\left(1-\left(\frac{\tau}{a}\right)^2\right)d\tau = \frac{1}{\sqrt{-\lambda_n}}\left(r-\frac{r^3}{a^2}\right)K_1\left(\sqrt{-\lambda_n}r\right)$$

$$+\frac{2r^2}{\lambda_n a^2}K_2\left(\sqrt{-\lambda_n}r\right)-\frac{2}{\lambda_n}K_2\left(\sqrt{-\lambda_n}a\right), \tag{5.81}$$

where substituting $r = a$ into (5.80) gives

$$\int_0^a \tau I_0\left(\sqrt{-\lambda_n}\tau\right)\left(1-\left(\frac{\tau}{a}\right)^2\right)d\tau = -\frac{2}{\lambda_n}I_2\left(\sqrt{-\lambda_n}a\right), \tag{5.82}$$

and enables (5.71) to be written as

$$A_{np} = \Phi_n(-1)\left\{\frac{(r-r^3/a^2)}{\sqrt{-\lambda_n}}\left(K_0\left(\sqrt{-\lambda_n}r\right)I_1\left(\sqrt{-\lambda_n}r\right)+I_0\left(\sqrt{-\lambda_n}r\right)K_1\left(\sqrt{-\lambda_n}r\right)\right)\right.$$

$$-\frac{2r^2}{\lambda_n a^2}\left(K_0\left(\sqrt{-\lambda_n}r\right)I_2\left(\sqrt{-\lambda_n}r\right)-I_0\left(\sqrt{-\lambda_n}r\right)K_2\left(\sqrt{-\lambda_n}r\right)\right)$$

$$\left.-\frac{2}{\lambda_n}K_2\left(\sqrt{-\lambda_n}a\right)\right\}. \tag{5.83}$$

Substituting identities (B.17) and (B.18), then yields

$$A_{np} = -\frac{\Phi_n(-1)}{\lambda_n}\left[\left(1-\left(\frac{r}{a}\right)^2\right)+\frac{4}{\lambda_n a^2}+2K_2\left(\sqrt{-\lambda_n}a\right)I_0\left(\sqrt{-\lambda_n}r\right)\right]. \tag{5.84}$$

For the region $r \geq a$, substituting in (5.82) into (5.72) results in

$$A_{np} = -\frac{2\Phi_n(-1)}{\lambda_n}I_2\left(\sqrt{-\lambda_n}a\right)K_0\left(\sqrt{-\lambda_n}r\right). \tag{5.85}$$

The complete solutions are chosen to be the particular solutions.

Asymptotic expansion for K_0 in (B.24), shows that the solution above decays exponentially. Since waves that occur in the far field must be larger than $o(r^{-\frac{1}{2}})$, which is larger than any exponential, we have that the non-grave modes do not contribute to the wave-field, but modify it. It should be noted that the functions $\{A_n(r)\}_{n=1}^{\infty}$ do satisfy the property of having finite volume/area integrated energy and enstrophy as demanded and are continuously differentiable everywhere.

The exact solution is then given to be the infinite sum of all the modes. The gravest mode is what gives the eddy its truncated cone shape, whereas the subsequent modes modify the form only slightly. Therefore, to get the qualitatively behavior of the eddy, it is enough to plot the gravest mode along with several subsequent modes. In Figure 5.3 we plot the first four modes, with the radius of the cold-pool chosen to be the smallest positive value that solves (5.79), the result being $a = 4.28$. The buoyancy frequency set equal to one, and the spacing between each contour is 0.2.

The center of the induced eddy shows that the solution is tapered, as was hypothesized. The contour lines are negative, a low pressure system, with greater strength towards the center and bottom. The contour lines being closer together at the bottom shows that this eddy is bottom intensified. This is due to the stratification present in the upper layer, which acts to cushion disturbances. Observe that φ increases with r as was predicted, hence the induced eddy is proven to be cyclonic.

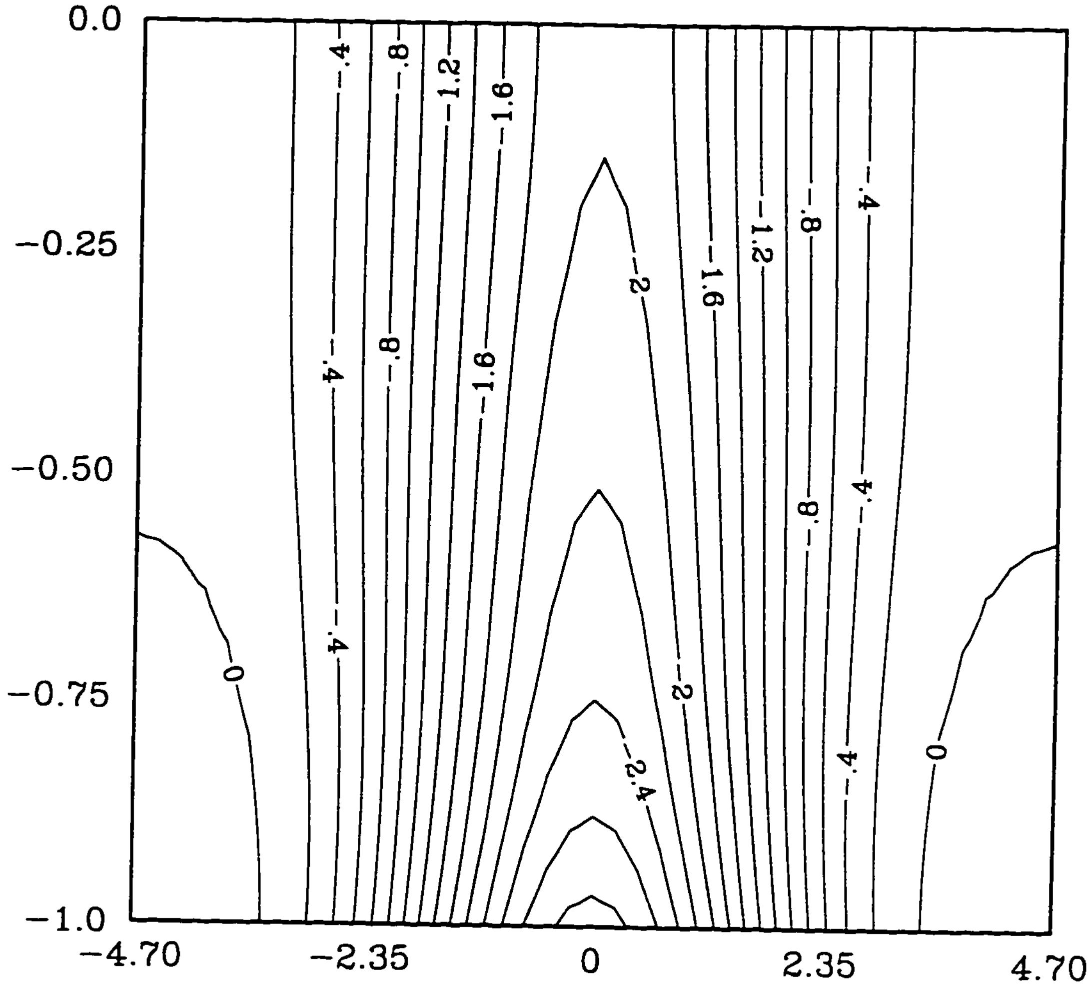


Figure 5.3 The vertical cross-section of the pressure field in the Radially Symmetric Solution with N=1 and a=4.28.

5.4 Radiating eddies

This section obtains a weakly radiating solution to the nonlinear equations where there is a Rossby wave-wake but no upstream waves. The cold-pool profile is still assumed to be radially symmetric but the reduced pressure is not. Therefore it is necessary to introduce θ dependency.

The mathematical difficulties in this are that the nonlinear Jacobian terms no longer vanish, as in the radially symmetric case. We are unaware of how to solve these equations exactly, and hence have found it necessary to make an added simplification. Taking advantage of the fact that all nonlinear terms are preceded with a μ, we assume this parameter to be small,

$$\mu \ll 1, \tag{5.86}$$

perform an asymptotic expansion and then solve the leading order system. Physically, this assumption translates to saying that the two fluids are experiencing only weak interactions. We hope that the solution we obtain to this simplified system, yields the same qualitative behavior as for cases where μ is larger.

Since the cross-slope velocity is zero to leading order, we rescale it in the following manner

$$\tilde{c} \rightarrow \mu \tilde{c}. \tag{5.87}$$

Now, an asymptotic solution can be found to the nonlinear system, in the form

$$(\varphi, h) \simeq (\varphi, h)^{(0)} + \mu (\varphi, h)^{(1)} + ..., \tag{5.88}$$

$$(c, \tilde{c}) \simeq (c, \tilde{c})^{(0)} + \mu (c, \tilde{c})^{(1)} + ... \tag{5.89}$$

The substitution of these two expressions, along with (5.87), into (5.10) and (5.11) yields,

$$c^{(0)} = 1, \tag{5.90}$$

$$\left(c^{(1)}, \bar{c}^{(0)}\right) = \left(-\frac{\iint_{\Omega_H} h^{(0)} \varphi_\zeta^{(0)}\Big|_{z=-1} d\xi d\zeta}{\iint_{\Omega_H} h^{(0)}\, d\xi d\zeta}, \frac{\iint_{\Omega_H} h^{(0)} \varphi_\xi^{(0)}\Big|_{z=-1} d\xi d\zeta}{\iint_{\Omega_H} h^{(0)}\, d\xi d\zeta} \right). \tag{5.91}$$

Substituting (5.87) through to (5.91) into (5.4), (5.5) and (5.7), implies that $(\varphi^{(0)}, h^{(0)})$ are governed by, after having dropped the superscripts,

$$\left(\Delta\varphi + \left(N^{-2}\varphi_z\right)_z\right)_\xi = 0, \tag{5.92}$$

$$(\varphi_z)_\xi = 0 \quad \text{at} \quad z = 0. \tag{5.93}$$

Equation (5.7), reduces to the statement, to leading order $0 = 0$. This signifies a type of degeneracy, as occurred in the radially symmetric case, since h is no longer a variable to be solved for, but is to be chosen a priori.

Before substituting into (5.6), it is advantageous to simplify this expression. This is done by substituting in (5.87), expanding, using the antisymmetry property of the Jacobian and then cancelling terms, to give

$$\partial\left(\mu\varphi + c\zeta - \mu\bar{c}\xi, \varphi_z + N^2 h\right) - N^2\varphi_\xi = 0 \quad \text{at} \quad z = -1. \tag{5.94}$$

Now, substituting in (5.88) to (5.91), yields the leading order equation, again after dropping the superscripts,

$$\left(\varphi_z + N^2(\varphi + h)\right)_\xi = 0 \quad \text{at} \quad z = -1. \tag{5.95}$$

Integrating (5.92), (5.93) and (5.95) and setting the functions of integration to zero, results in the same equations that arose in Section 5.2, (5.27) to (5.29), with several differences. Since the dynamic pressure field now has θ dependency, the Laplacian is no longer given by (5.30), but is

$$\Delta\varphi = \frac{1}{r}(r\varphi_r)_r + \frac{1}{r^2}\varphi_{\theta\theta}. \tag{5.96}$$

As well, solving this system does not yield an exact solution as before, but a leading order solution in terms of the *interaction parameter*.

It is worth pointing out that for our derivation of (5.29) for the radially symmetric solution, we needed to implement a far field condition and hence, assume that the streaklines extended to infinity. The fact that here, for a weakly radiating solution, we found the same equation without making such an assumption suggests there was merit in doing so even though it seemed problematic at the time.

As before, we use Finite Fourier Transforms which assumes a particular solution is of the form (5.51), where the eigenfunctions Φ_n are given by (5.48) and (5.49). This reduces the problem to solving the ordinary differential equation (5.54), where the Laplacian is given by (5.96). Even though our new equations have an additional term, the particular solutions from before, (5.61), (5.62), (5.71) and (5.72), still form a solution to this new system. The weakly radiating solution is then a sum of this particular solution with a homogeneous one which must be chosen to eliminate any upstream waves. This is done by imposing the Sommerfield radiation condition, which requires that the waves satisfy the far field condition

$$r^{\frac{1}{2}}\varphi \rightarrow 0 \quad \text{as} \quad r \rightarrow \infty \quad \text{for} \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}. \tag{5.97}$$

Because the $n = 1, 2, \ldots$ modes all satisfy this property, this condition need only be imposed on the gravest mode. Therefore, the particular solution for the non-grave modes are chosen to be (5.71) and (5.72).

Note that the zero-wave drag condition has not been applied because it is too strong, in that it kills the wave-field everywhere outside the radius of the eddy. The constant

$$\gamma \equiv \int_0^a \tau J_0\left(\sqrt{\lambda_0}\tau\right) h(\tau)\, d\tau, \tag{5.98}$$

must be non-zero, which is the constraint on the radius of the cold-pool. In the limit as this parameter goes to zero, the nonradiating limit must be obtained.

The method to be used will follow closely that of Flierl (1984). Observe that the left-hand side of (5.54), for the radiating situation, is the Helmholtz equation, which in polar coordinates is the n th order Bessel equation. The solution of which is obtained using separation of variables. The azimuthal part of the solution is

$$B_n \cos(n\theta) + C_n \sin(n\theta) \quad \text{for} \quad n = 0, 1, 2, ... \tag{5.99}$$

where it is the periodic boundary conditions that imply n is an integer. The solution to the radial part that is regular at the origin, and hence everywhere, is

$$J_n\left(\sqrt{\lambda_0} r\right) \quad \text{for} \quad n = 0, 1, 2, ... \tag{5.100}$$

Therefore, the regular solution to the homogeneous n-th order Bessel equation is

$$A_{0h} = \sum_{n=0}^{\infty} \left(B_n \cos(n\theta) + C_n \sin(n\theta)\right) J_n\left(\sqrt{\lambda_0} r\right). \tag{5.101}$$

In anticipation of the fact that the particular and homogeneous solutions will be evaluated in the far field, it is necessary to find asymptotic expansions for the Bessel functions as $r \to \infty$. Simplifying the asymptotic expression for J_n using (B.21), brings about two different relations for the even and odd integers respectively, after having applied a trigonometric identity

$$J_{2n}(x) \sim \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} \cos\left(x - \frac{\pi}{4}\right)(-1)^n \quad \text{as} \quad r \to \infty, \tag{5.102}$$

$$J_{2n+1}(x) \sim \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} \sin\left(x - \frac{\pi}{4}\right)(-1)^n \quad \text{as} \quad r \to \infty. \tag{5.103}$$

Also, the asymptotic expansion for Y_0 is similarly simplified by considering (B.22),

$$Y_0(x) \sim \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} \sin\left(x - \frac{\pi}{4}\right) \quad \text{as} \quad r \to \infty. \tag{5.104}$$

The general solution for A_0 is therefore

$$A_0 = A_{0p} + A_{0h}. \tag{5.105}$$

Substituting (5.62) and (5.101) into this relation, taking the limit as $r \to \infty$ by substituting in (5.102) to (5.104) and imposing the radiation condition (5.97) produces, after ignoring the term $[2/(\pi\sqrt{\lambda_0}r)]^{1/2}$ since it appears in all the terms,

$$\frac{\pi\Phi_0\,(-1)\,\gamma}{2}\sin\left(\sqrt{\lambda_0}r-\frac{\pi}{4}\right)$$

$$=\sum_{n=0}^{\infty}\left(B_{2n+1}\cos\left((2n+1)\,\theta\right)+C_{2n+1}\sin\left((2n+1)\,\theta\right)\right)\sin\left(\sqrt{\lambda_0}r-\frac{\pi}{4}\right)(-1)^n$$

$$+\sum_{n=0}^{\infty}\left(B_{2n}\cos\left(2n\theta\right)+C_{2n}\sin\left(2n\theta\right)\right)\cos\left(\sqrt{\lambda_0}r-\frac{\pi}{4}\right)(-1)^n. \qquad (5.106)$$

First observe that the left-hand side has a $\sin(\sqrt{\lambda_0}r-\frac{\pi}{4})$, whereas the right-hand side has both sines and cosines of the same argument. We can conclude that the Fourier coefficients corresponding to the cosines must all be zero,

$$B_{2n}=C_{2n}=0 \quad \text{for} \quad n=0,1,2,... \qquad (5.107)$$

Also the left-hand side of (5.106) has no θ dependency, which means that it is even with respect to that variable. Since even functions are written in terms of cosines,

$$C_{2n+1}=0 \quad \text{for} \quad n=0,1,2,... \qquad (5.108)$$

Equations (5.106) and (5.107) state that the wave-field must be representable in terms of the product of odd cosines with odd Bessel functions of the first kind, which is exactly what was done in Flierl (1984). This method stems from work done in Miles (1968), which, using sines instead of cosines, exploited the completeness of this basis in order to obtain a wave-field expression.

Substituting (5.107) and (5.108) into (5.106) and considering only the coefficients of the radial sine term, gives

$$\frac{\pi\Phi_0\,(-1)\,\gamma}{2}=\sum_{m=0}^{\infty}(-1)^m\,B_{2m+1}\cos\left((2m+1)\,\theta\right). \qquad (5.109)$$

It is necessary to compute the Fourier series of the left-hand side in terms of the odd cosine functions. To do so, note the following orthogonality relation from elementary calculus

$$\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \cos\left(\left(2n+1\right)\theta\right)\cos\left(\left(2m+1\right)\theta\right)d\theta = \frac{\pi}{2}\delta_{nm}. \tag{5.110}$$

Multiplying (5.109) by $\cos\left(\left(2n+1\right)\theta\right)$ and integrating from $-\frac{\pi}{2}$ to $+\frac{\pi}{2}$ with respect to θ yields

$$B_{2m+1} = \frac{2\Phi_0\left(-1\right)}{2m+1}\gamma, \tag{5.111}$$

having used the orthogonality relation above. Therefore, the solution is

$$\varphi = \sum_{n=0}^{\infty}\left[A_{np}\left(r\right)\Phi_n\left(z\right) + 2\Phi_0\left(z\right)\Phi_0\left(-1\right)\gamma\frac{\cos\left(\left(2n+1\right)\theta\right)}{2n+1}J_{2n+1}\left(\sqrt{\lambda_0}r\right)\right], \tag{5.112}$$

with A_{np} and Φ_n are given by (5.48), (5.49), (5.61), (5.62), (5.71) and (5.72), and the radius of the eddy is chosen so that γ is non-zero.

In the limit as $\gamma \to 0$, we recover the non-radiating solution as in the previous section. This is significant since at no point in obtaining this solution, have we assumed that the streaklines extend to infinity. This shows that using this particular far field condition, which at the time seemed dubious, is justifiable since it yields the same result as obtained by not applying this condition.

The solution in (5.112) is for general cold dome profiles. For the special case where the lower layer is parabolic, as was considered in Section 5.3, the solution is still represented by (5.112), but now the radial functions are defined by (5.77), (5.78), (5.84) and (5.85). Figures 5.4, 5.5 and 5.6 plot this solution with the first four modes of the radially symmetric part, as before, and the first eleven terms in the radiating sum. We choose $a = 3.85$, this number being 0.9 times the radius of the cold-pool in the previous case, and $N = 1$. The plots are of the pressure field at the bottom, middle and top of the upper layer respectively. The contour separation in the first case is 0.2 and in the second and third cases it is 0.125.

The first structure to observe is that there is still an induced eddy as before, where the cold-pool lies directly beneath. This eddy possess the same properties as before, i.e. cyclonic vorticity, tapering and bottom intensification, but the eddy itself is no longer radially symmetric due to the effect of the wave-drag forces.

In these plots the bottom axis corresponds to the x axis, and since we know that the cold-pool moves in the positive x direction, in the wake of the induced eddy we see a wave field. This field is entitled a Rossby wave-wake (Johnson, 1978) since it emphasizes the point that we are considering a subinertial regime. The first wave behind the eddy is a high, the next a low, and the subsequent waves alternate in the same order. The amplitude of these highs and lows get progressively smaller radially away from the eddy core, since the effect of the cold-pool is weaker further away. Also, towards the rigid-lid, the amplitude of the waves are seen to decrease as well. This is due to the cushioning of the ever present stratification.

Given the weakly radiating solution (5.112) for general lower layer profiles, it is possible to go back and calculate the correction terms to the steadily travelling velocity. The second integral in (5.91) that need be evaluated is,

$$\begin{aligned} \iint_{\Omega_H} h\varphi_\xi|_{z=-1} \; d\xi d\zeta &= -\iint_{\Omega_H} h_\xi \varphi|_{z=-1} \; d\xi d\zeta \\ &= \iint_{\Omega_H} \cos\theta h_r \varphi|_{z=-1} \, r dr d\theta, \qquad (5.113) \end{aligned}$$

having used integration by parts, along with the fact that the variables vanish in the far field and introduced polar coordinates. The integral is over $0 < \theta < 2\pi$ and $0 < r < \infty$. However, since $h = h_r = 0$ outside of the radius eddy, we can restrict this to $0 < r < a$. Observing, as has been previously proven, that the radially symmetric part of φ integrates to zero, signifies that we only need consider the radiating part of the solution. Substituting this into (5.113), using the orthogonal relationship of cosine terms, integration by parts and applying identity (B.1),

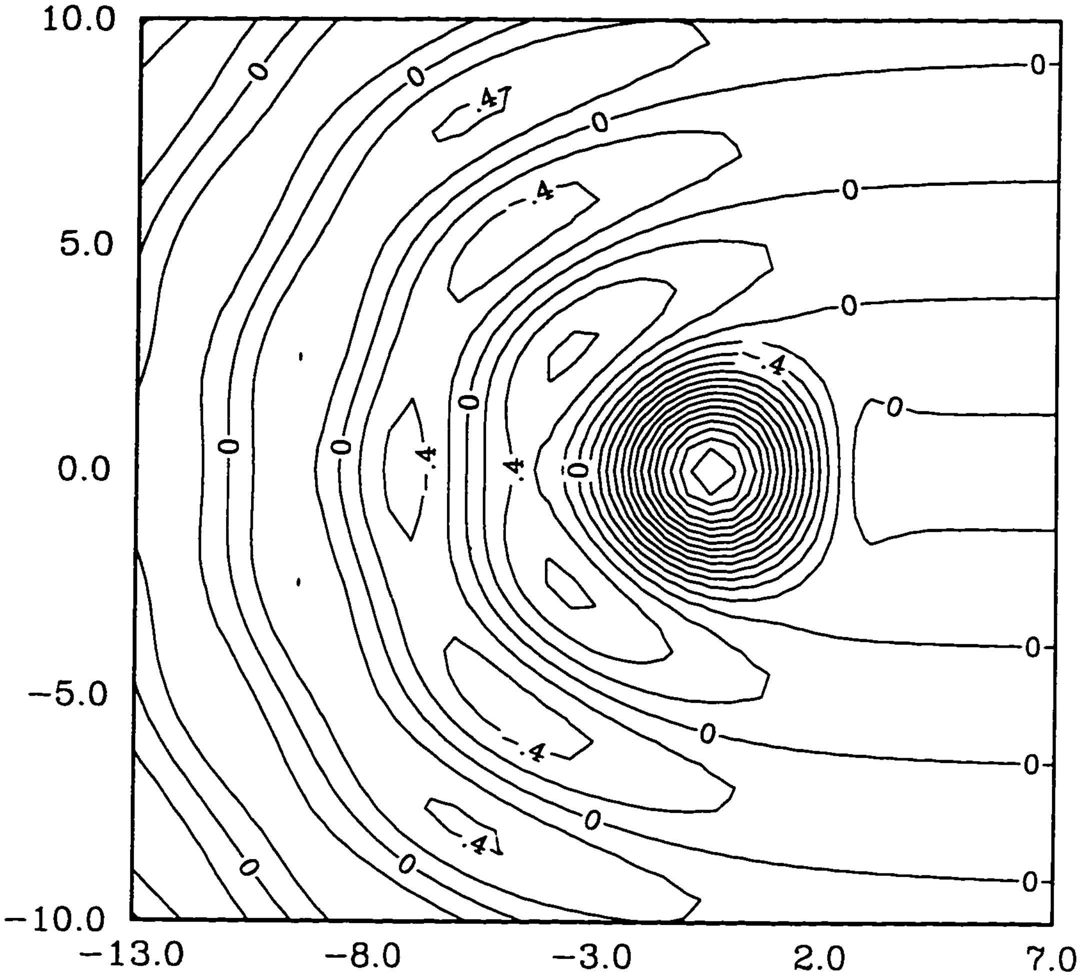


Figure 5.4 The horizontal cross-section of the Radiating Eddy Solution at z=-1.0, with N=1 and a=3.85.

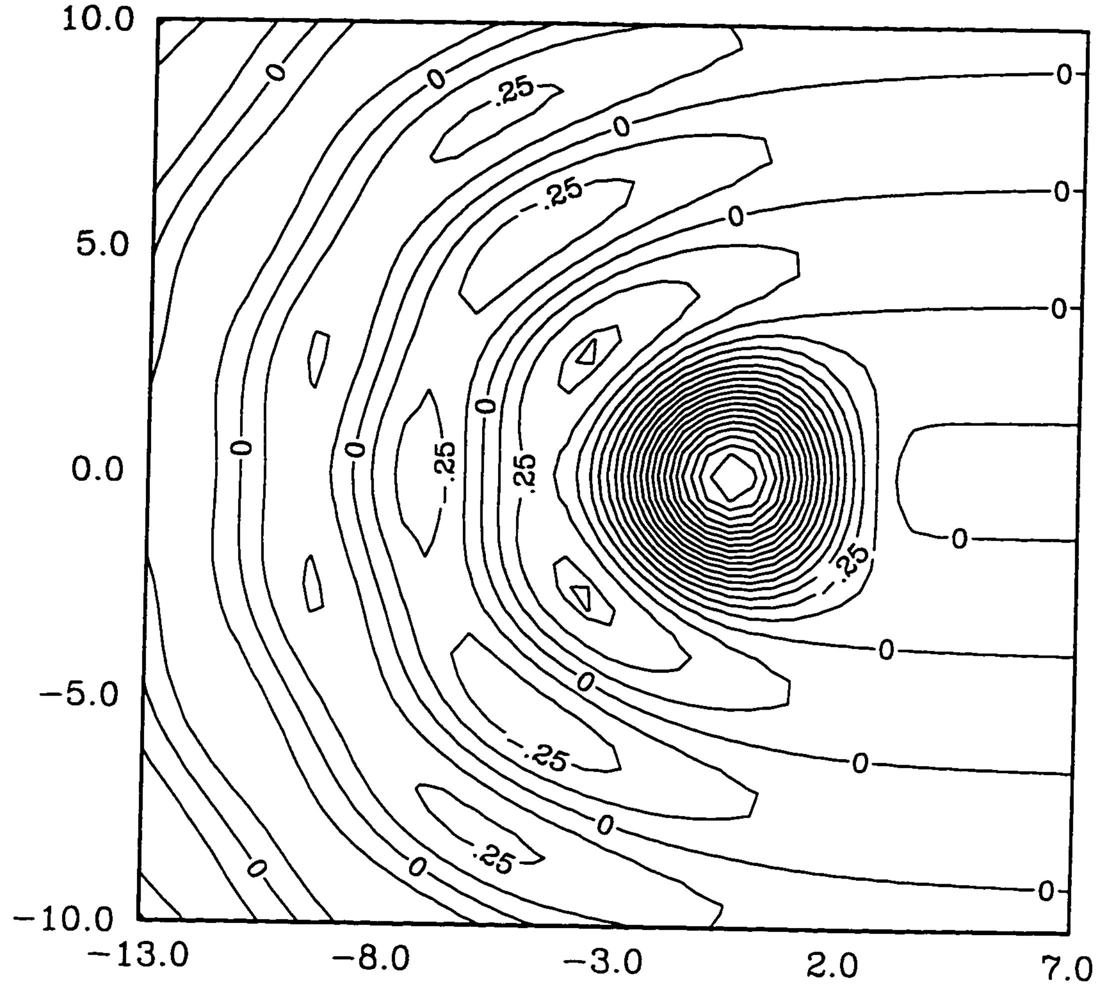


Figure 5.5 The horizontal cross-section of the Radiating Eddy Solution at z=-0.5, with N=1 and a=3.85.

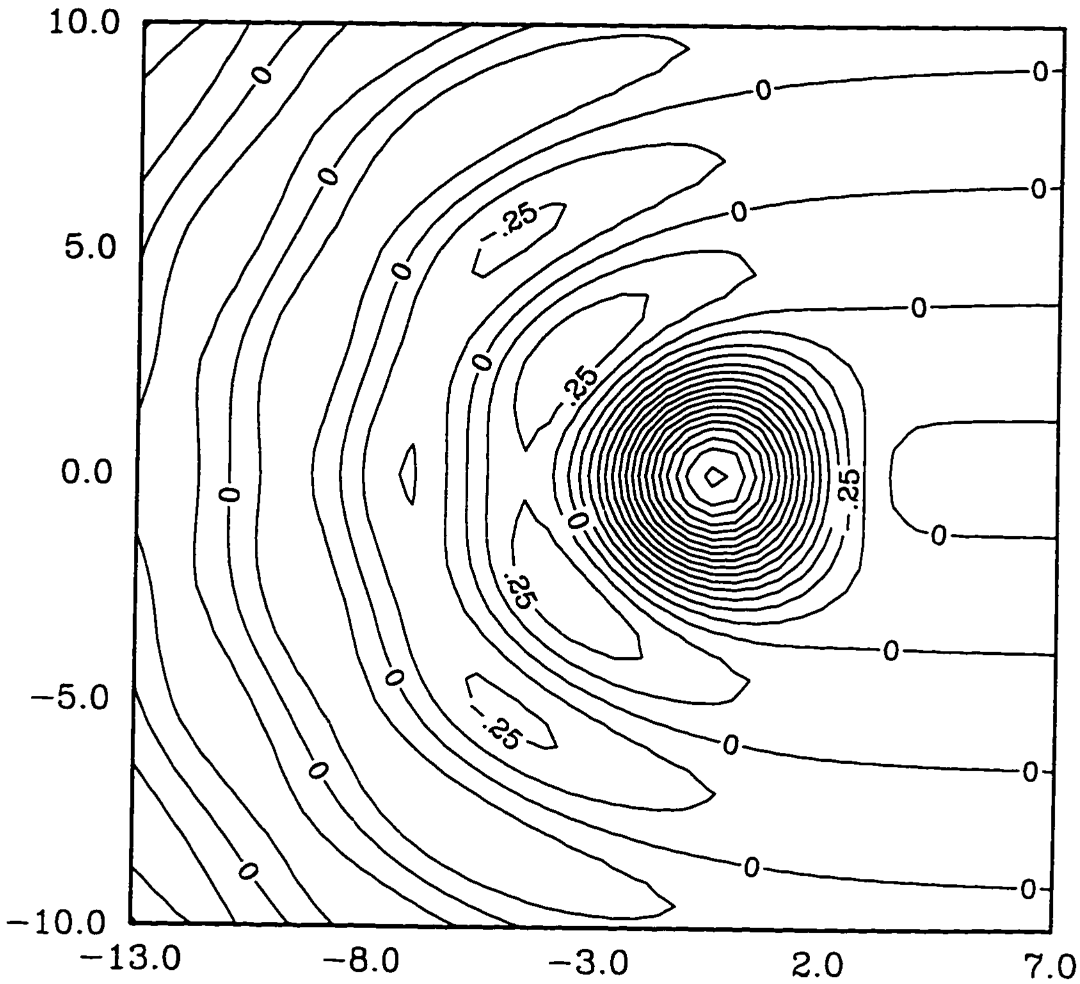


Figure 5.6 The horizontal cross-section of the Radiating Eddy Solution at z=0.0, with N=1 and a=3.85.

$$\iint_{\Omega_H} h\varphi_\xi|_{z=-1}\ d\xi d\zeta$$

$$= 2\Phi_0^2(-1)\gamma\sum_{m=0}^{\infty}\int_0^a J_{2m+1}\left(\sqrt{\lambda_0}r\right)h_r r\int_0^{2\pi}\frac{\cos\theta\cos((2m+1)\theta)}{2m+1}d\theta dr,$$

$$= 2\pi\Phi_0^2(-1)\gamma\int_0^a J_1\left(\sqrt{\lambda_0}r\right)h_r r dr,$$

$$= 2\pi\Phi_0^2(-1)\gamma\int_0^a \frac{d}{dr}\left(rJ_1\left(\sqrt{\lambda_0}r\right)\right)h dr,$$

$$= 2\pi\Phi_0^2(-1)\gamma\sqrt{\lambda_0}\int_0^a \tau J_0\left(\sqrt{\lambda_0}\tau\right)h(\tau)\,d\tau, \tag{5.114}$$

after changing the dummy variable of integration.

A similar calculation, as what has been done above, can be done for (5.10). However, because the associated version of (5.113) contains a sine which is orthogonal to all the cosines from the radiation term, the corresponding integral vanishes. Therefore, the along-shelf speed for the first two orders, is precisely the Nof speed.

Substituting these results into (5.91), yield

$$c^{(1)} = 0, \tag{5.115}$$

$$\tilde{c}^{(0)} = \mu\sqrt{\lambda_0}\left(\frac{2\gamma\Phi_0(-1)}{a}\right)^2. \tag{5.116}$$

This result is for general cold-pool profiles.

Since the right-hand side of (5.116) is positive, we see that the next order approximation to the cross-channel velocity is positive, and ergo the cold-pool has a down-slope drift. This is because the cold-pool generates the Rossby wave-wake behind it, which radiates energy away from the cold-pool. This radiation of energy must come from the release of available potential energy. Since we are assuming that the cold-pool maintains it shape, potential energy is released in the only other possible way. This being by the cold-pool moving down the slope since this corresponds to a state of less available potential energy for the cold-pool.

In a different light, this down-slope drift can be interpreted as being due to the presence of drag forces. Drag acts against the Coriolis force and hence disrupts the balance that it strives to achieve with buoyancy forces, which is what allows for the cold-pool to slide down the slope. A similar result was found in Flierl (1984) and Swaters & Flierl (1991).

The Sommerfield condition is not the only radiation condition that can be used. Another possibility is that explained in Lighthill (1965). This method demands that the solution which is physically relevant is obtainable from the limiting case of an initial value problem. Johnson (1978) used this radiation condition to determine the behavior of QG flow over an isolated topographic disturbance. Our situation can be viewed as being the same except that in our model the QG fluid is at rest, and the disturbance beneath it is in motion. This paper yielded essentially the same result as what we have just obtained.

To plot the streakfunction (5.12), we substitute (5.112) in the parabolic case, and the leading order expressions for the cold-pool speed $(c, \tilde{c}) = (1, 0)$. We include the same modes as in the streamfunction plot of the weakly radiating solution, as well as $a = 3.85$ and $N = 1$. The contour spacings for all the plots is 1.1.

Figures 5.7 to 5.9, 5.10 to 5.12 and 5.13 to 5.15 are for $\mu = 0.5$, 1.5 and 2.5 respectively. The first, second and third in each set are at the bottom, middle and top respectively. It is true that our solution is only valid for small μ, so we cannot expect our plots for $\mu = 1.5$ and 2.5 to be accurate. These are extrapolations and are believed to yield the same qualitative behavior.

In all of these plots, the cold dome is positioned in the center. For $\mu = 0.5$, the effect of the lower layer is to deflect the fluid parcels passing directly above it slightly towards deeper water, where then it returns to its original path after h begins to decrease. For $\mu = 1.5$ and 2.5, this path deflection is stronger in that it extends further outwards and enables some fluids directly over it, to travel in closed orbits. As well, to the left of the cold-pool, the Rossby wave-wake gets more apparent with increasing values of μ.

It is seen that with increasing values of μ, closed streaklines begin to form and then in greater numbers. The fact that more closed paths occur, hence more fluid is transported with increasing μ signifies that the transport of fluid in the upper layer is due to baroclinic processes.

In addition, closed contours form more frequently near the bottom. The greater number of closed streaklines at the bottom signifies that baroclinicity is stronger at the bottom due to the existence of stratification in the upper layer which cushions the effect of the lower layer throughout the upper layer depth.

In Section 5.1 it was determined that for steadily travelling solutions, if there exists a closed streakline, the cross-slope velocity must be zero. The conclusion for weakly radiating solutions, that there is a down-slope drift, may seem to conflict with this, but indeed it does not. This is because the radiating solution is only an approximate one, in that we have only obtained the leading order behavior. Substituting the radiation term into the full nonlinear equations, yields a term which is not necessarily zero and must be compensated for by time evolution (as in Flierl, 1984 and Swaters & Flierl, 1991). Therefore, this system is no longer steadily travelling and the zero cross-slope velocity conclusion is not applicable.

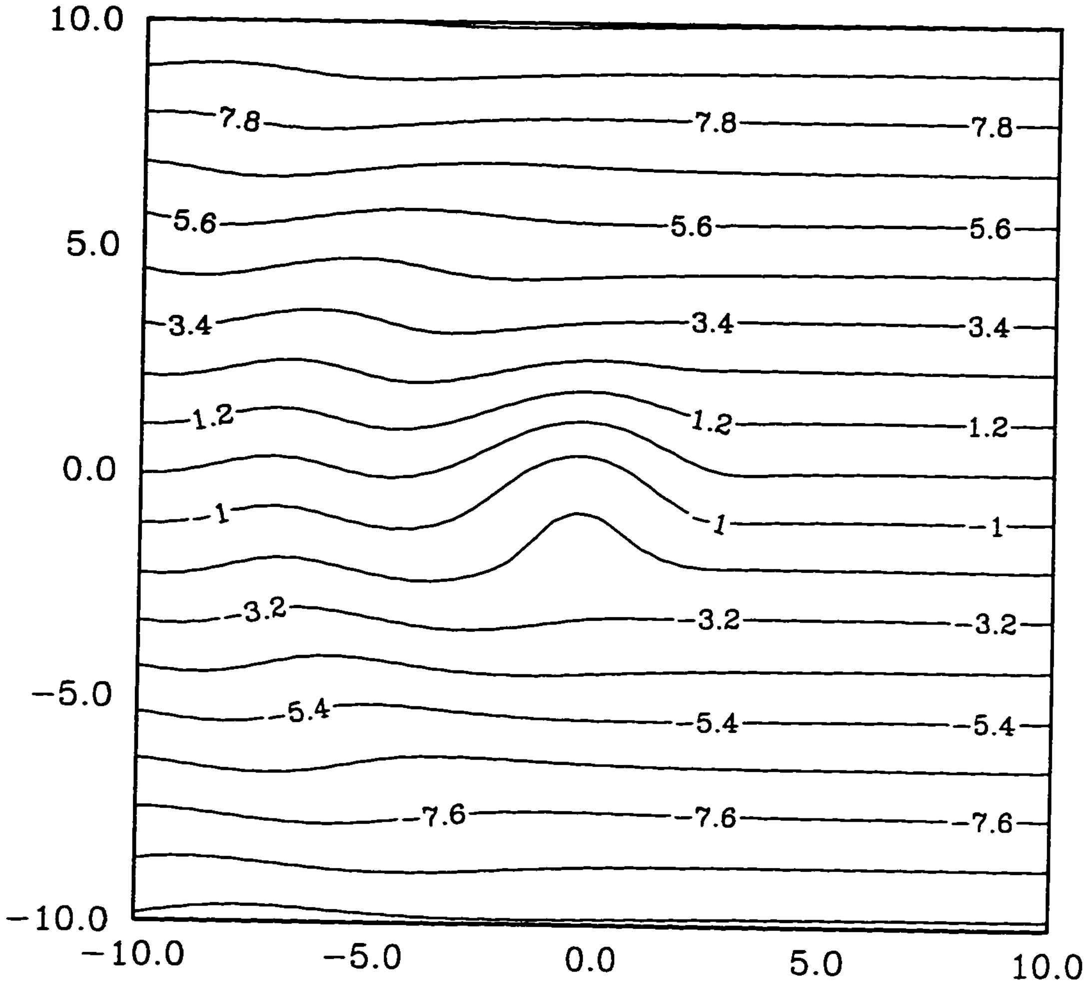


Figure 5.7 The streaklines of the Radiating Eddy Solution at z=-1.0, with mu=0.5, N=1 and a=3.85.

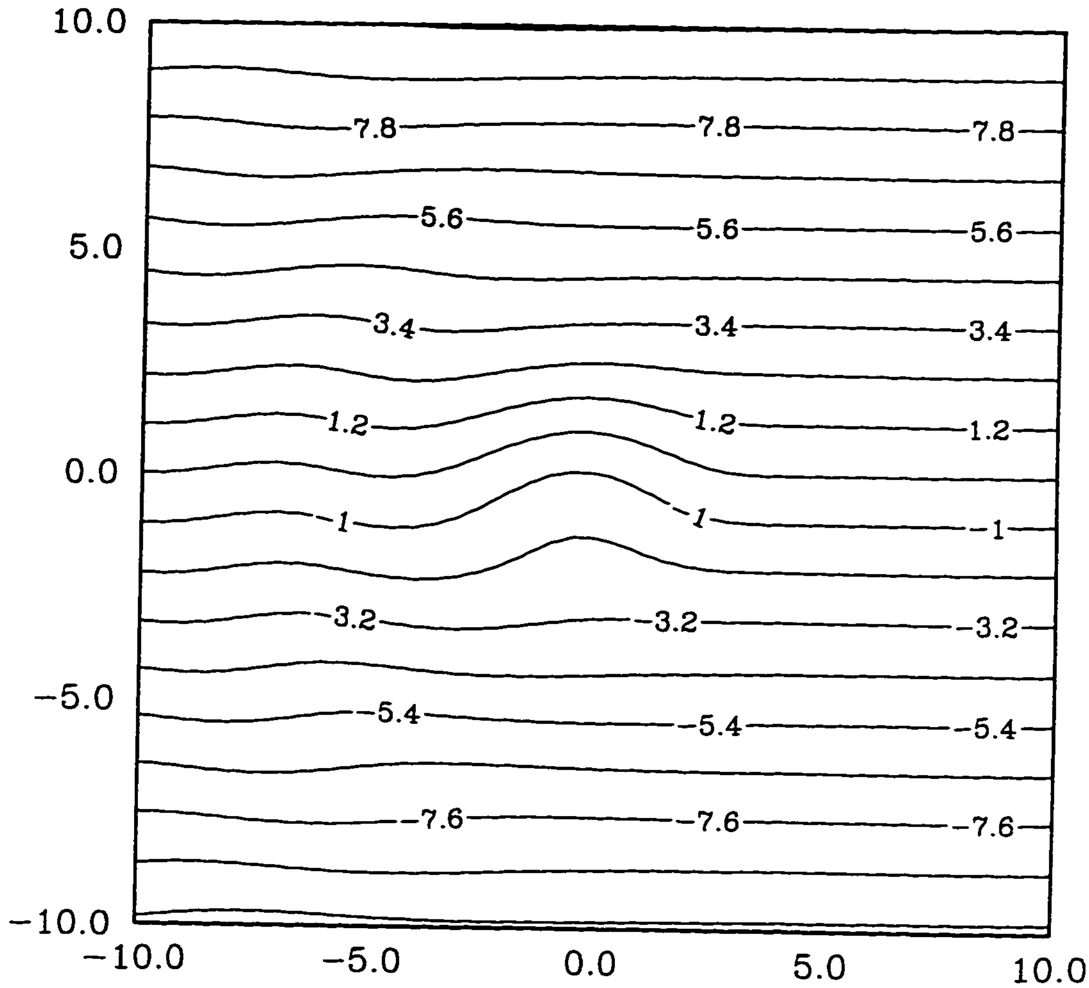


Figure 5.8 The streaklines of the Radiating Eddy Solution at z=-0.5, with mu=0.5, N=1 and a=3.85.

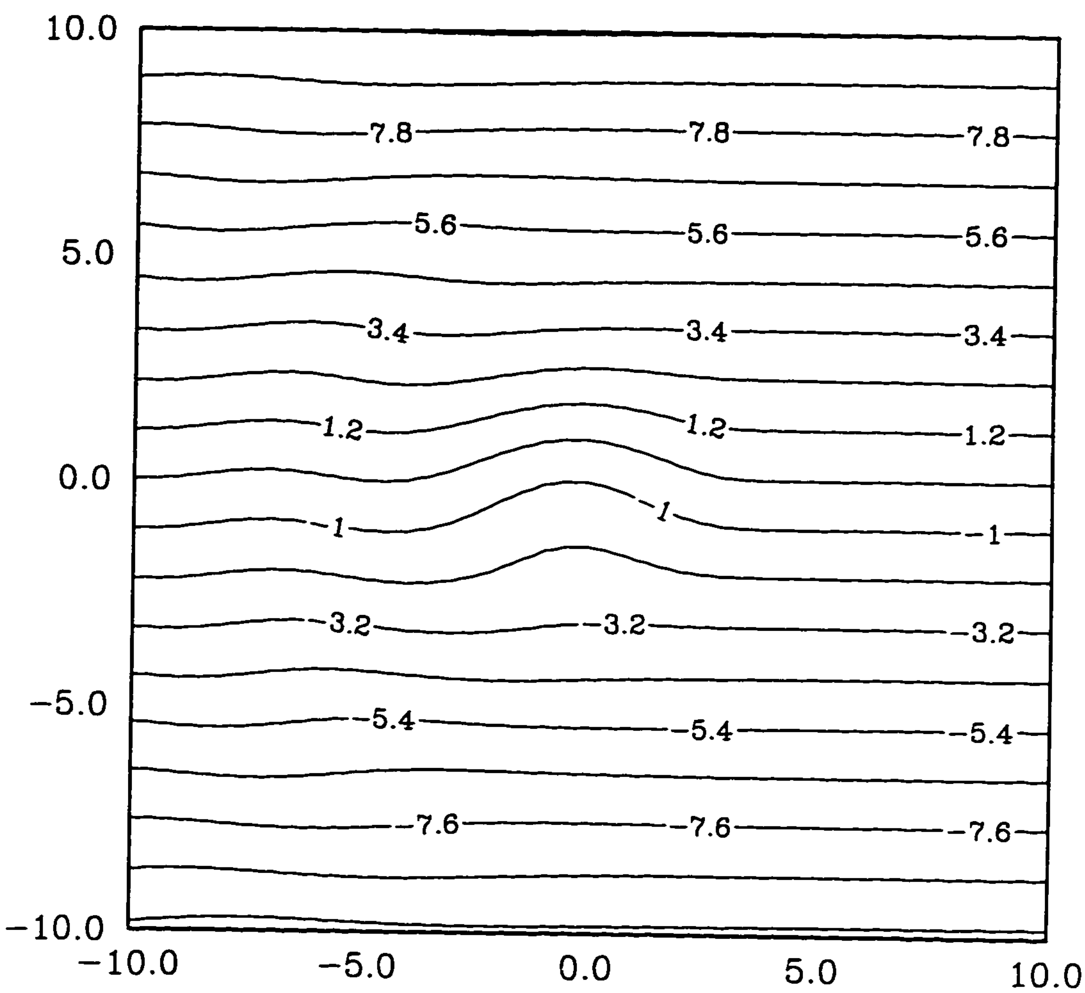


Figure 5.9 The streaklines of the Radiating Eddy Solution at z=0.0, with mu=0.5, N=1 and a=3.85.

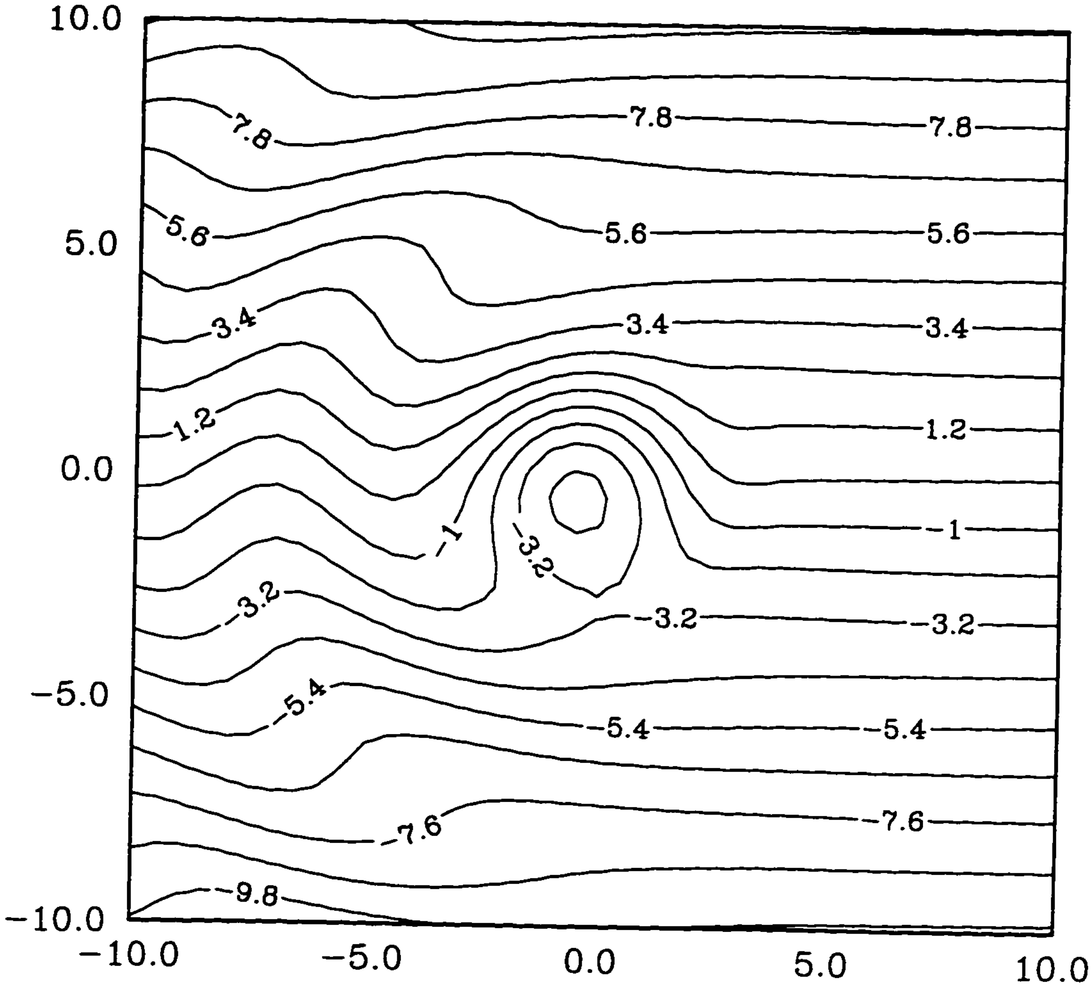


Figure 5.10 The streaklines of the Radiating Eddy Solution
at z=-1.0, with mu=1.5, N=1 and a=3.85.

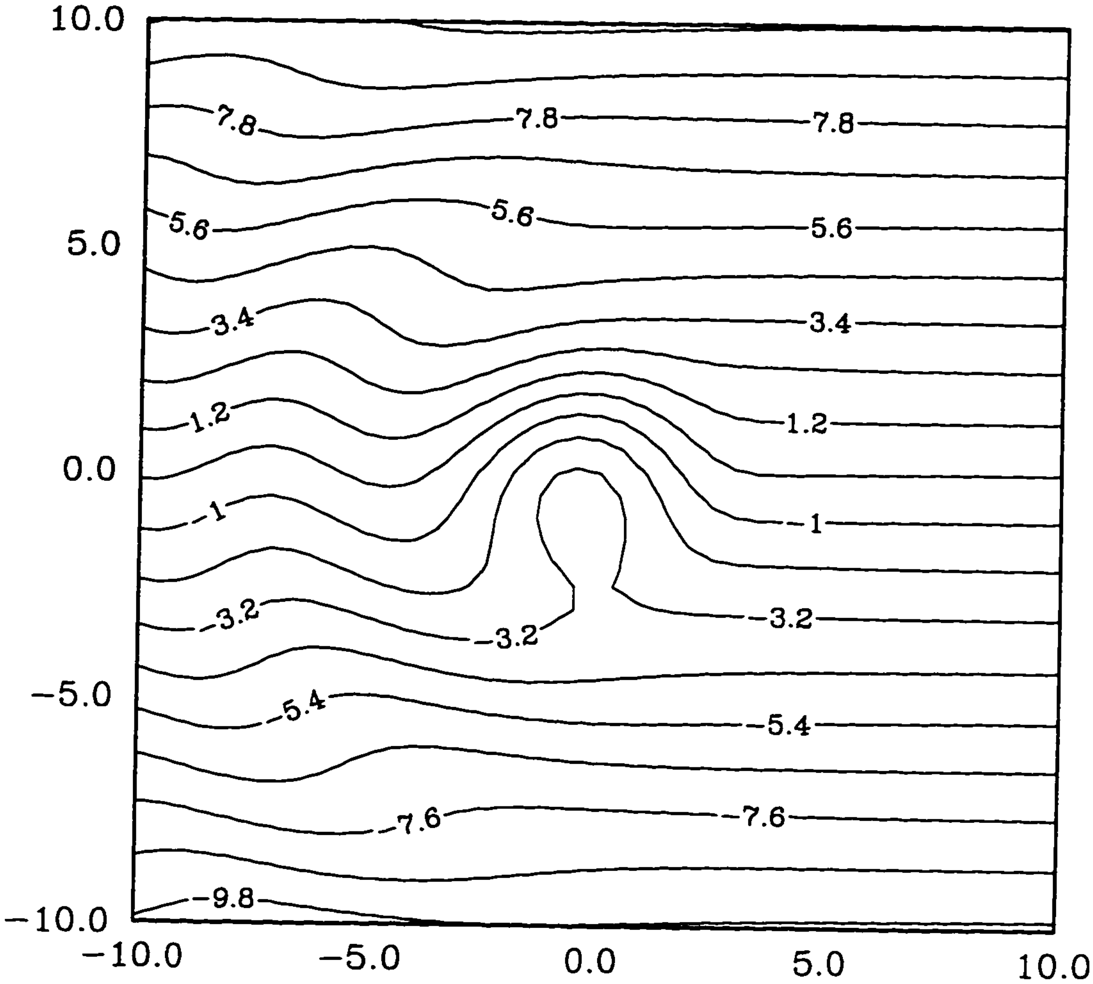


Figure 5.11 The streaklines of the Radiating Eddy Solution at z=-0.5, with mu=1.5, N=1 and a=3.85.

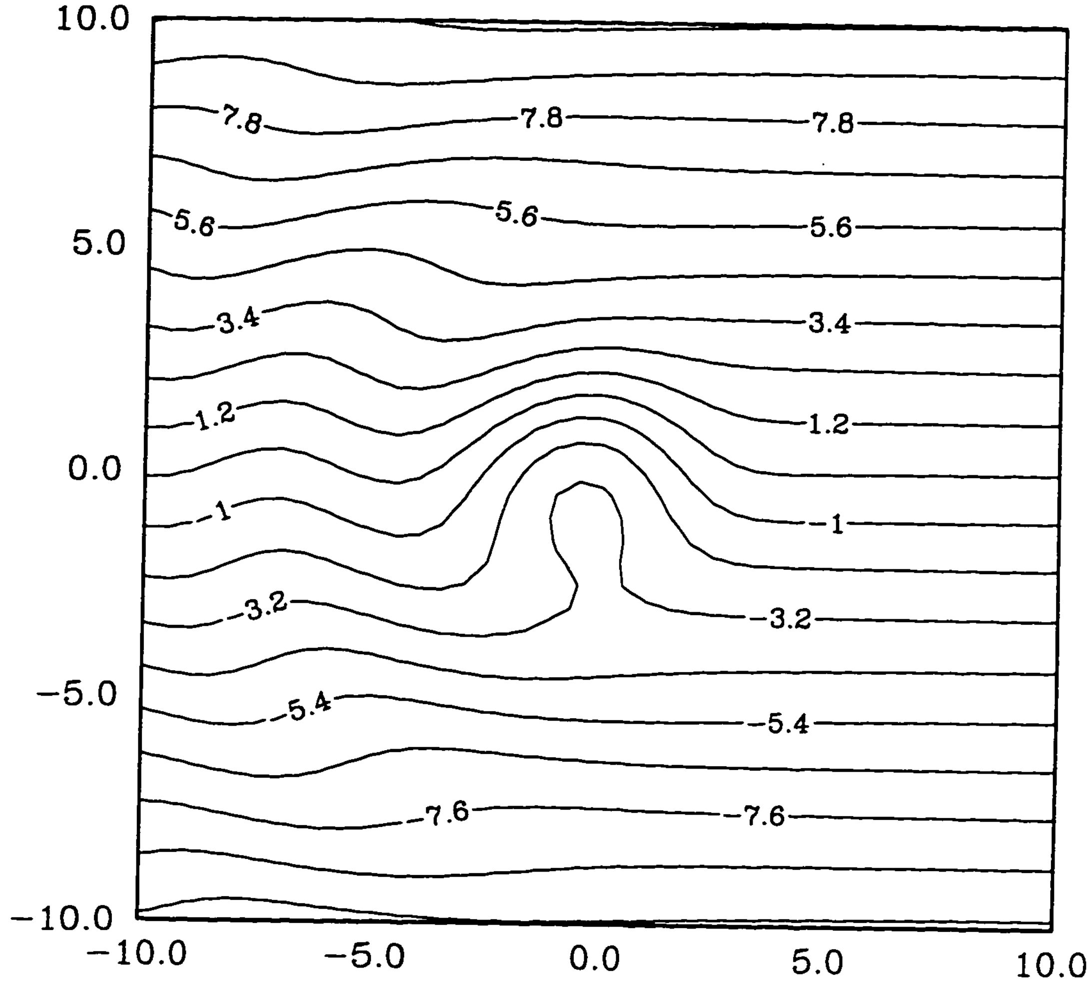


Figure 5.12 The streaklines of the Radiating Eddy Solution at z=0.0, with mu=1.5, N=1 and a=3.85.

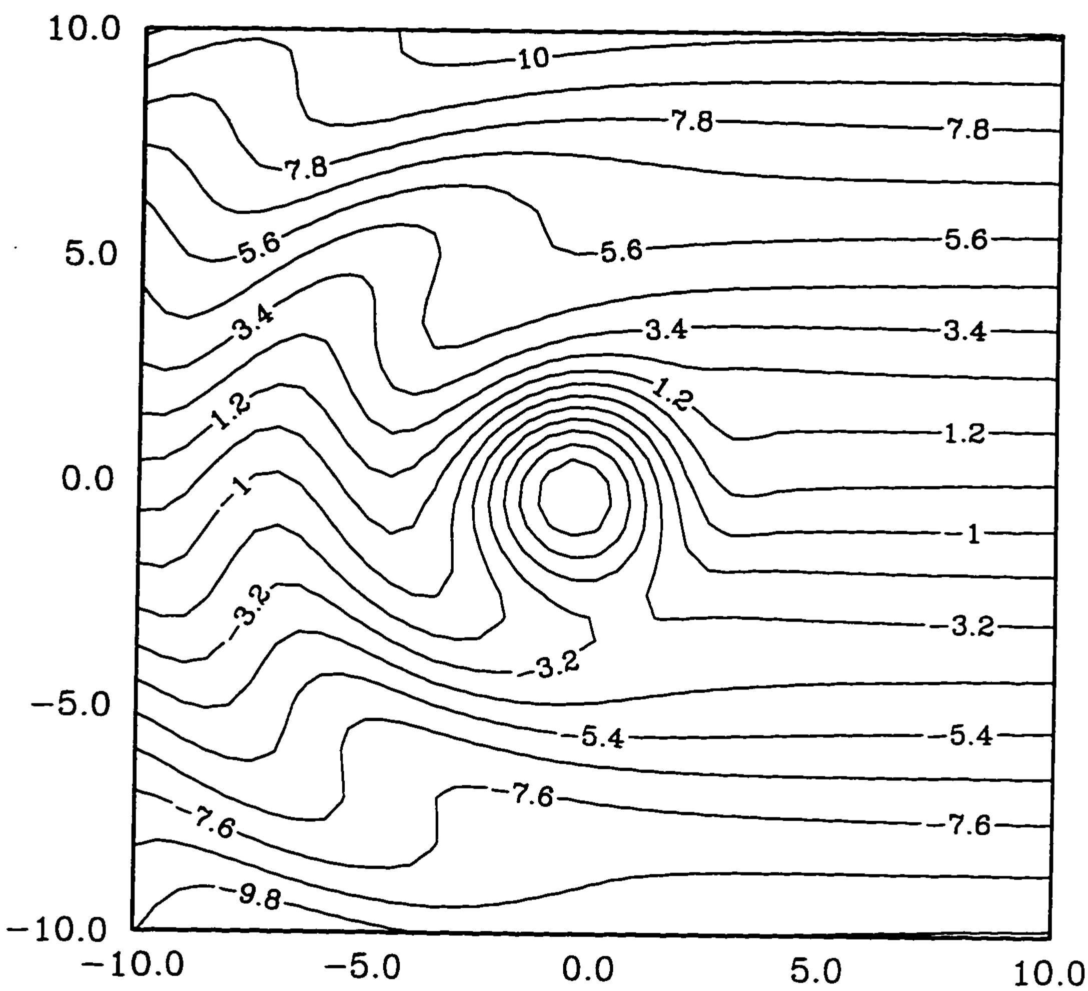


Figure 5.13 The streaklines of the Radiating Eddy Solution at z=-1.0, with mu=2.5, N=1 and a=3.85.

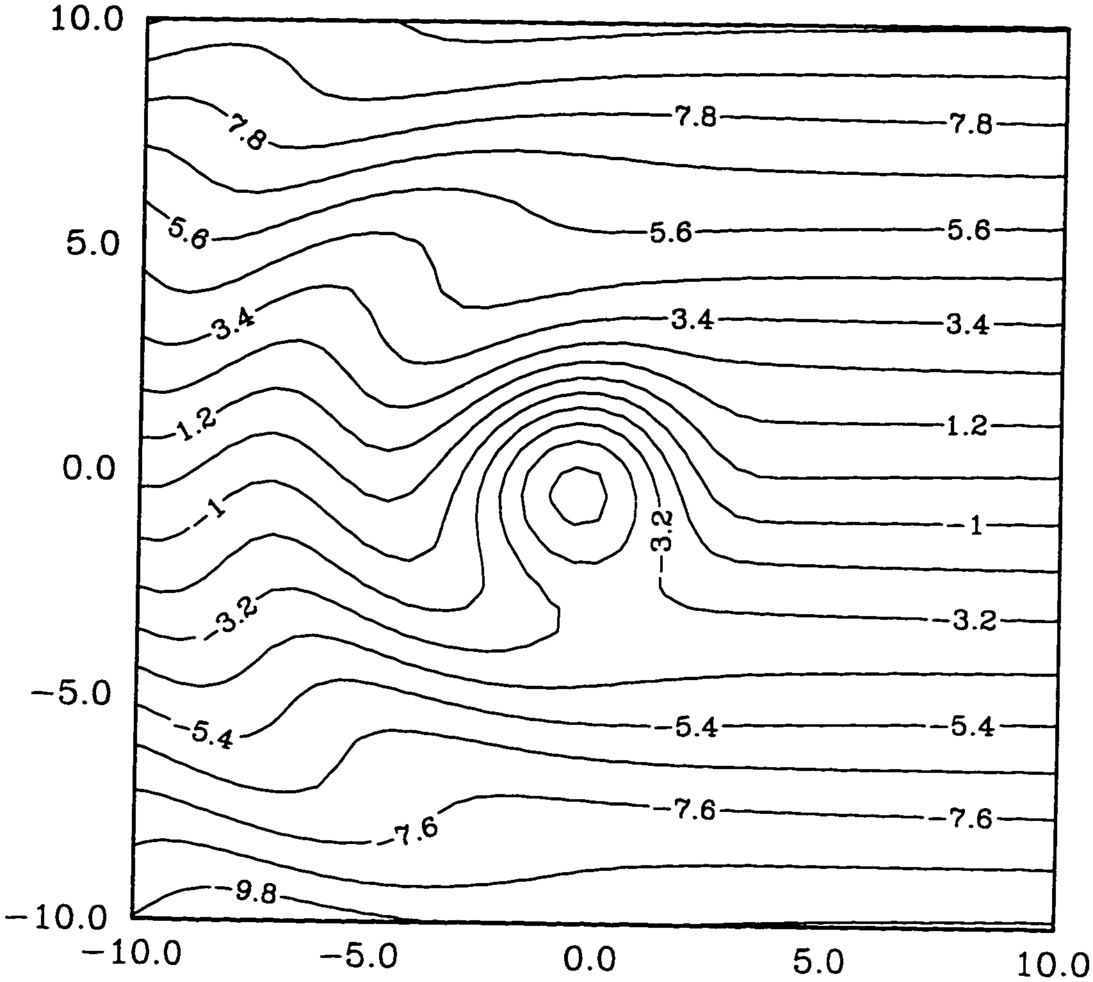


Figure 5.14 The streaklines of the Radiating Eddy Solution at z=-0.5, with mu=2.5, N=1 and a=3.85.

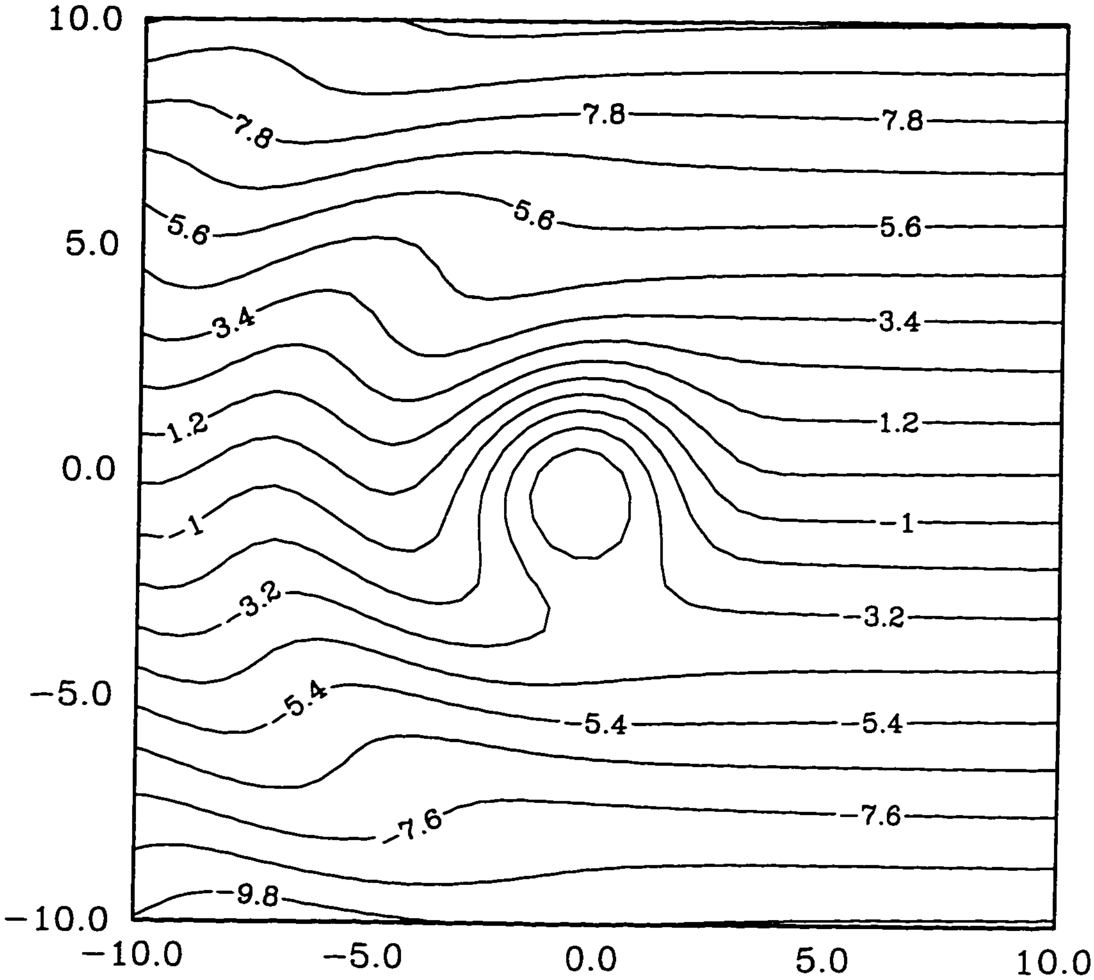


Figure 5.15 The streaklines of the Radiating Eddy Solution at z=0.0, with mu=2.5, N=1 and a=3.85.

Chapter 6

Summary and Conclusions

The occurrence of mesoscale gravity currents over gently sloping continental shelves and their destabilization into cold-pools inspires the need for an analytical model in order to better understand these phenomena. The Swaters (1991) model did this, and demonstrated the importance of baroclinicity along with other achievements. However, experimental observations and numerical simulations suggested the need for modifying this model so that the overlying fluid is continuously stratified. The model described in this thesis, has done exactly this, where the upper layer is QG and the lower layer is an 'intermediate length scale' model that is in the form of either a gravity current or cold-pool.

The model considered simultaneously the destabilizing effects of vortex tube stretching versus the stabilizing effects of topography. The ratio of these two quantities is the ***interaction parameter*** μ, which determines the stability characteristics of the system. It is the down-sloping side of the gravity current where the instability most easily develops, as in the original Swaters' model. A semicircle theorem was established which gives a bound on the growth rate of the unstable modes.

By considering a simple wedge, we have established a high-wavenumber cutoff which eliminates the possibility of ultra-violet catastrophe. As well, it has been determined that the presence of stratification tends to increase the growth rate of the most unstable mode, in addition to decreasing its wavelength. This lower layer profile is highly idealized, yet this qualitative behavior is believed to carry through to more complicated situations.

The governing equations for each layer in their primitive form, are known to have a noncanonical Hamiltonian structure. This is advantageous since symmetries in the Hamiltonian each imply the existence of a conserved quantity. As well, criteria can be obtained to guarantee linear or nonlinear stability. Putting these two fluids together in one model and doing an asymptotic expansion, reveals that a noncanonical Hamiltonian structure is still present, which we have exploited to obtain stability conditions on the steady state solutions that are invariant with respect to the along-shelf coordinate. Such conditions could not be found for steadily travelling solutions because of Andrews' theorem.

By considering steadily travelling, isolated and coherent cold-core eddy solutions a degeneracy arose. This being that conditions could not be found to determine what cold-pool shapes are allowable. For general lower layer profiles, two different types of solutions were obtained. The first is an exact solution where the entire system is radially symmetric and has a zero-wave drag condition imposed, which eliminated any waves outside the support of the cold-pool. The induced cyclone directly above the cold-pool is tapered in shape, which correlates with numerical predictions.

The second solution is weakly radiating in that it assumes the *interaction parameter* is small which allows for wave drag forces to be present. Solving for the leading order behavior results in an induced eddy as before, but the drag on the cold-pool creates a Rossby wave-wake behind it. This wake interacts with the cold-pool to create a down-slope drift. The stratification present in the upper layer cushions the interaction with the lower layer which is what causes the induced eddy to be tapered and the wake to decrease in amplitude radially away from the cold-pool and with height.

By plotting the streaklines of the weakly radiating solution where the lower layer is parabolic, it was found that for small values of μ, no closed streaklines occurred and hence no fluid was transported along in the upper layer. Extrapolating this solution to larger values of μ, it was found that closed streaklines did occur. Moreover, the larger the *interaction parameter* the more closed streaklines, therefore the greater the fluid transport.

This thesis has derived a new model and has analyzed certain aspects of it, but there are still more analysis that could be done. For instance, it is possible to do a weakly nonlinear analysis in order to obtain amplitude equations for the slightly linearly unstable modes. Numerical simulations could be performed that would explicitly show the destabilization of gravity currents into cold-pools and also how the cold-pools propagate with either zero or non-zero wave drags. Ventilation processes could be introduced, as was done in Swaters & Flierl (1991), to study the thermodynamic interaction of the two layers. Given that the conservative form of this model has shown merit, it is justifiable in proceeding with the introduction of friction, to study the effects that it has on the system.

Since the inspiration for this work was the Strait of Georgia, it would be desirable to apply this model directly to this body of water. It is believed that this model should be more accurate then the original Swaters' model, since it is known that for gravity currents to form it is necessary that the upper fluid is stratified (LeBlond *et al.*, 1991).

A further extension that could be made is introducing a third layer above the stratified one that is homogeneous and less dense then both the other layers. This is because during gravity current formation in the Strait of Georgia, a freshwater layer is present at the surface due to runoff from melted snow. This third layer would have an impact on the dynamics of the system and hence should be considered. Benlov (1996) did this adjustment for the original Swaters' model, but we believe that adjusting our model in this manner would be an even better approximation since it would be closer to physical observations.

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Appendix A

Technical Proofs

A.1 The Eigenvalues are Positive

In deriving the Poincaré inequality the fact that the eigenvalues are positive has been used, and is essential. Therefore, it is necessary to prove this fact or else the foundation of our analysis collapses. This is done in the same manner as what was done in Yongming *et al.* (1996). We begin by separating the eigenvalue problem into two separate eigenvalue problems. The method of separation of variables, suggests assuming $\varphi(x,y,z) = u(x,y)\,v(z)$, which when substituted into (4.136) through to (4.139) yields the following equations with their respective boundary conditions

$$\left(N^{-2}v_z\right)_z + \tau v = 0, \tag{A.1}$$

$$\alpha_2 v_z - Kv = 0 \quad \text{on} \quad z = 0, \tag{A.2}$$

$$\alpha_3 v_z + Kv = 0 \quad \text{on} \quad z = -1, \tag{A.3}$$

and

$$\Delta u + \sigma u = 0, \tag{A.4}$$

$$u = 0 \quad \text{on} \quad y = 0, L. \tag{A.5}$$

The parameter τ and σ are the eigenvalues for the vertical and horizontal problems respectively, which sum to the total eigenvalue. These are the two eigenvalue problems considered in Yongming *et al.* (1996). As shown in the aforementioned paper, the eigenvalues of (A.4) and (A.5) are non-negative, hence we need only concern ourselves with the vertical problem.

Equations (A.1) to (A.3) are Sturm-Liouville because all the constants that appear in the boundary conditions are positive. To gain an expression for the eigenvalue we multiply (A.1) by v and integrate over z from -1 to 0, which becomes after substituting in the boundary conditions

$$\begin{aligned}\tau\int_{-1}^{0} v^2 dv &= -\int_{-1}^{0} v\left(N^{-2}v_z\right)_z dz \\ &= \int_{-1}^{0} N^{-2}v_z^2 dz - \left[N^{-2}vv_z\right]_{z=-1}^{z=0} \\ &= \int_{-1}^{0} N^{-2}v_z^2 dz - \frac{K}{\alpha_2}\left[N^{-2}v^2\right]_{z=0} - \frac{K}{\alpha_3}\left[N^{-2}v^2\right]_{z=-1},\end{aligned} \tag{A.6}$$

which enables us to solve for τ,

$$\tau = \frac{\int_{-1}^{0} N^{-2}v_z^2 dz - \frac{K}{\alpha_2}\left[N^{-2}v^2\right]_{z=0} - \frac{K}{\alpha_3}\left[N^{-2}v^2\right]_{z=-1}}{\int_{-1}^{0} v^2 dv}. \tag{A.7}$$

The minimum of all these eigenvalues is defined to be τ_0,

$$\tau_0 = \min\left\{\frac{\int_{-1}^{0} N^{-2}v_z^2 dz - \frac{K}{\alpha_2}\left[N^{-2}v^2\right]_{z=0} - \frac{K}{\alpha_3}\left[N^{-2}v^2\right]_{z=-1}}{\int_{-1}^{0} v^2 dv}\right\}, \tag{A.8}$$

where the minimum operator is over all non-zero smooth functions of z.

Without loss of generality, we can normalize v such that

$$\int_{-1}^{0} v^2 dz = 1. \tag{A.9}$$

The function v depends on the variable z but also varies with K since it is present in the boundary conditions. Therefore, it makes sense to differentiate v with respect to K if we assume there is a smooth dependency. Differentiating (A.9) with respect to K

gives the following orthogonality relation,

$$\int_{-1}^{0} vv_K dz = 0. \tag{A.10}$$

Recall that K was chosen to be that parameter that solved the expression

$$\tau_0(K) + \sigma_0 = \frac{K}{\alpha_1}, \tag{A.11}$$

where $\tau_0(K) = \min \tau(K)$ and $\sigma_0 = \min \sigma$. The right-hand side of this expression begins at zero and is monotonically increasing. To first show that there is a unique solution it is necessary to show that the left-hand side is monotonically decreasing. We begin by differentiating (A.7) with respect to K, after substituting in (A.9)

$$\begin{aligned} \frac{d\tau}{dK} &= 2\int_{-1}^{0} N^{-2} v_z v_{zK} dz - \frac{2K}{\alpha_2}\left[N^{-2} vv_K\right]_{z=0} - \frac{2K}{\alpha_3}\left[N^{-2} vv_K\right]_{z=-1} \\ &\quad -\frac{1}{\alpha_2}\left[N^{-2}v^2\right]_{z=0} - \frac{1}{\alpha_3}\left[N^{-2}v^2\right]_{z=-1}. \end{aligned} \tag{A.12}$$

The first three terms can be rewritten using conditions (A.2) and (A.3)

$$\begin{aligned} & 2\left(\int_{-1}^{0} N^{-2} v_z v_{zK} dz - \frac{K}{\alpha_2}\left[N^{-2} vv_K\right]_{z=0} - \frac{K}{\alpha_3}\left[N^{-2} vv_K\right]_{z=-1}\right) \\ &= 2\left(\int_{-1}^{0} N^{-2} v_z v_{zK} dz - \left[N^{-2} v_z v_K\right]_{z=-1}^{z=0}\right) \\ &= -2\int_{-1}^{0} v_K \left(N^{-2} v_z\right)_z \\ &= 2\tau \int_{-1}^{0} vv_K dz \\ &= 0, \end{aligned} \tag{A.13}$$

implementing the product rule to eliminate terms, substituting in (A.1) and finally applying (A.10). Substituting this result into (A.12) yields

$$\frac{d\tau}{dK} = -\frac{1}{\alpha_2}\left[N^{-2}v^2\right]_{z=0} - \frac{1}{\alpha_3}\left[N^{-2}v^2\right]_{z=-1}. \tag{A.14}$$

The functions and constants that appear in the right-hand side are all positive, and each occurs with a negative sign in front, to make this expression negative definite. This proves that the left-hand side of (A.11) is decreasing, and hence τ_0 is also decreasing. The right-hand side of (A.11) increases from zero to infinity with K, whereas the left-hand side decreases starting from $\tau(0) + \sigma$, which is strictly positive. Therefore, the intersection must occur at a positive value, which need be for a positive K. This then proves the positivity of the eigenvalues.

A.2 Translation of $K > \mu$

Arnol'd's second theorem for nonlinear stability required that $K > \mu$. This condition as it stands is not very insightful, and should be translated in terms of the boundary conditions. This is done by first deducing that since (A.1) is a linear second order differential equation, there must exist two linearly independent solutions, each of which depend on the eigenvalue τ. We call these two solutions $U(z;\tau)$ and $V(z;\tau)$, and choose them such that they satisfy

$$\begin{aligned} U(-1;\tau) &= 1, \quad V(-1,\tau) = 0, \\ U_z(-1;\tau) &= 0, \quad V_z(-1,\tau) = 1. \end{aligned} \tag{A.15}$$

The general solution is

$$v = c_1 U(z;\tau) + c_2 V(z;\tau), \tag{A.16}$$

which when substituted into (A.3) yields the requirement

$$B = -\frac{K}{\alpha_3} A, \tag{A.17}$$

Hence, if we choose $A = \alpha_3$, (A.16) reduces to

$$v = \alpha_3 U(z;\tau) - KV(z;\tau). \tag{A.18}$$

Since this solution satisfies (A.1) and (A.3), we need only impose the final boundary condition. Substituting (A.18) into (A.2) yields, after simplifying,

$$\alpha_3 \left(\alpha_2 U_z (0;\tau) - KU(0;\tau)\right) = K \left(\alpha_2 V_z (0;\tau) - V(0;\tau)\right), \tag{A.19}$$

which then represents the eigenvalue problem under consideration. For each K, there are an infinite number of τ solutions, where τ_0 is the minimum of those values. For any $\tau < \tau_0 (0)$, there must exist at least one solution with $K > 0$, which we call K_0, that satisfies the equation $\tau_0 (K_0) = \tau$, since τ_0 decreases from $\tau_0(0)$ to $-\infty$ as K increases from 0 to ∞. This K_0 is the minimum value that satisfies this equation.

Define a new function τ_1 to be,

$$\tau_1 = \min \frac{\int_{-1}^{0} N^{-2} v_z^2 dz}{\int_{-1}^{0} v^2 dv} \quad \text{with} \quad v(-1) = 0 = v(0), \tag{A.20}$$

where the minimum operator is taken over the smooth functions on the interval -1 to 0. To show that $V(0;\tau) > 0$ for $\tau < \tau_1$, we begin by proving that it is non-zero, which is done by contradiction. Substituting V into (A.7) along with the boundary condition (A.15), and the assumption that $V(0;\tau) = 0$, yields

$$\tau = \frac{\int_{-1}^{0} N^{-2} V_z^2 dz}{\int_{-1}^{0} V^2 dv}, \tag{A.21}$$

which contradicts the assumption that $\tau < \tau_1$, upon realizing that this is one of the functions being summed over in (A.20) and therefore cannot be strictly less than τ_1. Therefore, we conclude that $V(0;\tau) \neq 0$ for $\tau < \tau_1$.

Before we can determine the sign of V, it is necessary to determine the sign of VV_z on $-1 \leq z \leq 0$ and for $\tau > \tau_1$. This is done by evaluating (A.1) at V, then multiplying this expression by V and integrating over the z interval. Using integration by parts and (A.15) then simplifies the expression to,

$$N^{-2} V V_z = \int_{-1}^{z} N^{-2} V_z^2 - \tau V^2 dz. \tag{A.22}$$

Substituting in for τ using (A.7) evaluated at V, with the associated boundary conditions, brings the above equation into the following form,

$$N^{-2}VV_z = \int_{-1}^{z} N^{-2}V_z^2 dz - \left(\frac{\int_{-1}^{z} V^2 dz}{\int_{-1}^{0} V^2 dv}\right) \int_{-1}^{0} N^{-2}V_z^2 dz$$

$$+\frac{\int_{-1}^{z} V^2 dz}{\int_{-1}^{0} V^2 dv}\frac{K}{\alpha_2}\left[N^{-2}V^2\right]_{z=0}. \qquad \text{(A.23)}$$

Since the quantity in brackets is at most one, this means that the first row on the right-hand side is bigger than equal to zero. Whereas, the second row, using the fact that $V(0;\tau) \neq 0$ for $\tau < \tau_1$, is strictly positive. Therefore, we deduce that $VV_z \neq 0$, where by continuity and the fact that $V_z(-1;\tau) > 0$, that $V_z > 0$ for this entire regime, which then allows us to conclude that $V > 0$ for $z > -1$ and $\tau < \tau_1$, since these two quantities must be positively correlated.

Equation (A.19) is a quadratic equation in K, which can be rewritten as

$$VK^2 - (\alpha_3 U + \alpha_2 V_z) K + \alpha_2\alpha_3 U_z = 0 \quad \text{on} \quad z = 0, \qquad \text{(A.24)}$$

which allows us to solve for K using the quadratic equation

$$K = \frac{(\alpha_3 U + \alpha_2 V_z)}{2V} \pm \frac{\left[(\alpha_3 U + \alpha_2 V_z)^2 - 4\alpha_2\alpha_3 V U_z\right]^{\frac{1}{2}}}{2V} \quad \text{on} \quad z = 0,$$

$$= \frac{(\alpha_3 U + \alpha_2 V_z)}{2V} \pm \frac{\left[(\alpha_3 U + \alpha_2 V_z)^2 - 4\alpha_2\alpha_3 (VU_z + UV_z - UV_z)\right]^{\frac{1}{2}}}{2V} \quad \text{on} \quad z = 0,$$

$$= \frac{(\alpha_3 U + \alpha_2 V_z)}{2V} \pm \frac{\left[(\alpha_3 U - \alpha_2 V_z)^2 - 4\alpha_2\alpha_3 (VU_z + UV_z)\right]^{\frac{1}{2}}}{2V} \quad \text{on} \quad z = 0, \qquad \text{(A.25)}$$

upon adding zero then re-completing the square.

To get an expression for $VU_z + UV_z$, consider the differential equation (A.1) which governs both U and V. Taking the U equation multiplying it by V then subtracting this from the V equation multiplied by U, results in after integrating, cancelling

$$\int_{-1}^{0} U\left(N^{-2}V_z\right)_z - V\left(N^{-2}U_z\right)_z dz = 0,$$

$$(UV_z - VU_z)_{z=0} = \frac{N^2(0)}{N^2(-1)}, \tag{A.26}$$

and using (A.15). Substituting this result into (A.25) produces the following expression determining K for $\tau < \tau_1$,

$$\begin{aligned} K &= \frac{(\alpha_3 U + \alpha_2 V_z)}{2V} \pm \frac{\left[(\alpha_3 U - \alpha_2 V_z)^2 + 4\alpha_2\alpha_3 \frac{N^2(0)}{N^2(-1)}\right]^{\frac{1}{2}}}{2V} \quad \text{on} \quad z = 0 \\ &\equiv M_{\pm}(\tau). \end{aligned} \tag{A.27}$$

Note that $\tau_0(0)$ and τ_1 are the same in appearance, but they differ in that the functions that are being summed over in the first quantity contain those of the second and more, which implies that

$$\tau_0(0) \leq \tau_1. \tag{A.28}$$

Therefore, $\tau_0(K)$ for $K > 0$, must be the inverse of one of the two solutions of (A.27). By definition, $\tau_0(K)$ matches to the smallest value of K for a particular τ, and hence corresponds with $M_-(\tau)$. This means that these two functions must satisfy the following relation, $\tau_0(M_-(\tau)) = \tau$ for all $\tau < \tau_0(K)$, or

$$M_-(\tau_0(K)) = K, \tag{A.29}$$

for $K > 0$. Substituting (A.11) into (A.29) yields

$$M_-\left(\frac{K}{\alpha_1} - \sigma_0\right) = K \quad \text{and} \quad \tau < \tau_0(0). \tag{A.30}$$

Clearly the right-hand side of (A.29) is an increasing function of K. The left-hand side is a decreasing function of K because $\tau_0(K)$ is decreasing and these functions are inverses of each other. Hence we see that a solution having $K > \mu$, as what arose in the second nonlinear stability theorem, corresponds to requiring

$$M_{-}\left(\frac{K}{\alpha_1}-\sigma_0\right) > \mu \quad \text{and} \quad \frac{K}{\alpha_1}-\sigma_0 < \tau_0(0)\,. \tag{A.31}$$

Appendix B

Bessel identities

The Bessel identities listed here can all be found in standard texts on Bessel functions, such as Watson (1962). The first set of identities are differentiation formulas,

$$\frac{d}{dx}\left(x^{\alpha} J_{\alpha}(x)\right) = x^{\alpha} J_{\alpha-1}(x), \tag{B.1}$$

$$\frac{d}{dx}\left(x^{-\alpha} J_{\alpha}(x)\right) = -x^{-\alpha} J_{\alpha+1}(x), \tag{B.2}$$

$$\frac{d}{dx}\left(x^{\alpha} Y_{\alpha}(x)\right) = x^{\alpha} Y_{\alpha-1}(x), \tag{B.3}$$

$$\frac{d}{dx}\left(x^{-\alpha} Y_{\alpha}(x)\right) = -x^{-\alpha} Y_{\alpha+1}(x), \tag{B.4}$$

$$\frac{d}{dx}\left(x^{\alpha} I_{\alpha}(x)\right) = x^{\alpha} I_{\alpha-1}(x), \tag{B.5}$$

$$\frac{d}{dx}\left(x^{-\alpha} I_{\alpha}(x)\right) = x^{-\alpha} I_{\alpha+1}(x), \tag{B.6}$$

$$\frac{d}{dx}\left(x^{\alpha} K_{\alpha}(x)\right) = -x^{\alpha} K_{\alpha-1}(x), \tag{B.7}$$

$$\frac{d}{dx}\left(x^{-\alpha} K_{\alpha}(x)\right) = -x^{-\alpha} K_{\alpha+1}(x), \tag{B.8}$$

where α is an integer. The second set are recurrence relations,

$$J_{\alpha+1}(x) = \frac{2\alpha}{x} J_{\alpha}(x) - J_{\alpha-1}(x), \tag{B.9}$$

$$Y_{\alpha+1}(x) = \frac{2\alpha}{x} Y_\alpha(x) - Y_{\alpha-1}(x), \tag{B.10}$$

$$I_{\alpha+1}(x) = -\frac{2\alpha}{x} I_\alpha(x) + I_{\alpha-1}(x), \tag{B.11}$$

$$K_{\alpha+1}(x) = \frac{2\alpha}{x} K_\alpha(x) + K_{\alpha-1}(x). \tag{B.12}$$

The Wronskian for each set of Bessel functions are

$$W(x; J_\alpha, Y_\alpha) = J_\alpha(x) \frac{d}{dx}(Y_\alpha(x)) - Y_\alpha(x) \frac{d}{dx}(J_\alpha(x)) = \frac{2}{\pi x}, \tag{B.13}$$

$$W(x; I_\alpha, K_\alpha) = I_\alpha(x) \frac{d}{dx}(K_\alpha(x)) - K_\alpha(x) \frac{d}{dx}(I_\alpha(x)) = -\frac{1}{x}. \tag{B.14}$$

Four other formulas, that are not so standard, that can be derived from the above are

$$Y_0(x) J_1(x) - J_0(x) Y_1(x) = \frac{2}{\pi x}, \tag{B.15}$$

$$x^2 (Y_0(x) J_2(x) - J_0(x) Y_2(x)) = \frac{4}{\pi}, \tag{B.16}$$

$$K_0(x) I_1(x) + I_0(x) K_1(x) = \frac{1}{x}, \tag{B.17}$$

$$x^2 (K_0(x) I_2(x) - I_0(x) K_1(x)) = -2. \tag{B.18}$$

The first identity above is established by substituting (B.2) and (B.4) for $\alpha = 1$ into (B.13) evaluated at $\alpha = 0$. Equation (B.16) is proven by substituting (B.9) and (B.10) for $\alpha = 2$ into the left-hand side of (B.16), then cancelling off terms and using (B.15),

$$\begin{aligned} x^2 (Y_0(x) J_2(x) - J_0(x) Y_2(x)) &= x^2 Y_0(x) \left(\frac{2}{x} J_1(x) - J_0(x) \right) \\ &\quad - x^2 J_0(x) \left(\frac{2}{x} Y_1(x) - Y_0(x) \right), \\ &= 2x (Y_0(x) J_1(x) - J_0(x) Y_1(x)), \\ &= \frac{4}{\pi}. \end{aligned} \tag{B.19}$$

Substituting (B.6) and (B.8) into (B.14) for $\alpha = 0$ yields (B.17). The final equality is proven through a similar procedure as above,

$$\begin{aligned} x^2\left(K_0(x)\,I_2(x) - I_0(x)\,K_1(x)\right) &= x^2K_0(x)\left(-\frac{2}{x}I_1(x) + I_0(x)\right) \\ &\quad -x^2I_0(x)\left(\frac{2}{x}K_1(x) + K_0(x)\right), \\ &= -2x\left(K_0(x)\,I_1(x) + I_0(x)\,K_1(x)\right), \\ &= -2. \end{aligned} \tag{B.20}$$

Finally, the asymptotic expansions for the regular and modified Bessel functions are

$$J_\alpha(x) \sim \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} \cos\left(x - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right) \quad \text{as} \quad x \to \infty, \tag{B.21}$$

$$Y_\alpha(x) \sim \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} \sin\left(x - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right) \quad \text{as} \quad x \to \infty, \tag{B.22}$$

$$I_\alpha(x) \sim \left(\frac{1}{2\pi x}\right)^{\frac{1}{2}} e^{x} \quad \text{as} \quad x \to \infty, \tag{B.23}$$

$$K_\alpha(x) \sim \left(\frac{\pi}{2x}\right)^{\frac{1}{2}} e^{-x} \quad \text{as} \quad x \to \infty. \tag{B.24}$$