University of Alberta

MATHEMATICAL ANALYSIS OF NONLOCAL THERMISTOR PROBLEMS

by

Shuqing Ma ©

A thesis submitted to the Faculty of Graduate Studies and Research
in partial fulfillment of the requirements for the degree of
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in

Mathematics

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled **Mathematical Analysis of Nonlocal Thermistor Problems** submitted by **Shuqing Ma** in partial fulfillment of the requirements for the degree of **Doctor of Philosophy** in Mathematics.

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Dr. Weimin Han (University of Iowa)

August 1, 2003

To my wife and my parents

ABSTRACT

In this thesis, we consider some mathematical models for the simulation of thermistor behavior. We recall that the classical model consists of two partial differential equations, which govern the behavior of the temperature and the electrical potential respectively. Recently, the advent of micromachined microsensor devices has led to the introduction of a nonlocal term to one of the equations which represents heat losses to the surrounding gas. However, the presence of such a nonlocal term in the model could lead to negative temperatures at some points, a situation which has no physical meaning.

Several authors considered the stationary version of the nonlocal system and proved that for all sufficiently small gas pressures, the temperature is always positive. We are interested in extending this result to the time dependent case and prove that a similar result holds for all time periodic solutions. Moreover, we consider the long time behavior of the solutions of this nonlocal system. The existence of a uniform attractor is obtained and its Hausdorff dimension is estimated.

The previous results for the positivity of the temperature are valid only for small gas pressures. It is our next intention to develop new models which always ensure a nonnegative temperature under all gas pressures and an obstacle thermistor model is introduced. We show that all solutions with positive temperatures of this new obstacle model will also solve the previous nonlocal system and vice versa. Thus, the obstacle model is an extension to the previ-

ous nonlocal one. The existence of solutions of both its stationary and time dependent case is obtained and the long time behavior for the time dependent obstacle model is studied.

We also consider the effect of a current source on part of the boundary for the time dependent obstacle problem and obtain the existence of a unique Hölder continuous solution. Finally, a box discretization method is constructed for the obstacle problem and an optimal H^1-error estimate is derived.

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Chapter 1

Introduction

1.1 Preamble: thermistor problems

Thermistors are electrical devices whose resistance depends significantly on the operating temperature. The mathematical modeling of their behavior has a long history [11]. The classical mathematical model consists of two strongly coupled nonlinear partial differential equations. Specifically, let $u(x,t)$ be the distribution of temperature in the device and $\phi(x,t)$ the distribution of its electrical potential. Then u and ϕ satisfy a mathematical system as follows:

$$u_t - \nabla[k(u)\nabla u] = \sigma(u)|\nabla\phi|^2, \tag{1.1}$$

$$-\nabla[\sigma(u)\nabla\phi] = 0. \tag{1.2}$$

Here $k(u)$ is the thermal conductivity of the device, and $\sigma(u)$ is the temperature dependent electrical conductivity. The first equation in the above system describes the diffusion of heat in the presence of Joule heating, and the second equation describes current conservation. The system (1.1)-(1.2) and its stationary version, i.e., u and ϕ are independent of time, have received vast interests in the past decade. A great deal of research papers by mathematicians and engineers have been devoted to the analysis of these systems. We refer interested readers to [3, 4, 6, 8, 10, 12, 14, 15, 19, 20, 22] and the references therein. Moreover, the practical applications of thermistors can be found in

[16].

Recent interest in the problem of the modelling of certain micromachined microsensors has led to the addition of a nonlocal term to one of the equations, in order to account for the physically important effect of heat losses to the surrounding gas, [2, 17]. Specifically, the classical system (1.1)-(1.2) is replaced by the following nonlocal equations

$$u_t - \nabla[k(u)\nabla u] + \eta \int_\Omega G(x,y)u(y)dy + \gamma u^4 = \sigma(u)|\nabla\phi|^2, \quad (1.3)$$

$$-\nabla[\sigma(u)\nabla\phi] = 0. \quad (1.4)$$

Here the domain occupied by the device has been denoted by Ω. The nonlocal term (i.e., the integral term $\eta \int_\Omega G(x,y)u(y)dy$) represents the heat losses to the surrounding gas and is obtained by an ad hoc averaging technique first introduced in [17]. Here we have also considered the effect of energy loss by radiation which is incorporated in the equations by means of expressions derived from the Stefan-Boltzmann Law and is represented by the 4th power term γu^4.

However, the presence of a nonlocal term in a partial differential equation could render invalid maximum principle and lead to spurious results. In our case, the nonlocal term in (1.3) could lead to negative temperatures in some points of the device, [1], which makes no physical sense in our situation. It is necessary to investigate new mathematical models for these microsensors. We will introduce a novel obstacle problem as follows: find $u \geq 0$ and ϕ, such that

$$u(u_t - \nabla[k(u)\nabla u] + \eta \int_\Omega G(x,y)u(y)dy + \gamma u^4) \geq \sigma(u)|\nabla\phi|^2 u, \quad (1.5)$$

$$-\nabla[\sigma(u)\nabla\phi] = 0. \quad (1.6)$$

As we shall show, to some extent, this new obstacle problem can be viewed as an extension of the previous system of nonlocal equations.

In this thesis, we will concentrate on the study of the system (1.3)-(1.4) and the new obstacle problem (1.5)-(1.6). In Chapter 2, we will consider the system of nonlocal equations. The new obstacle problem will be introduced and carefully studied in Chapter 3. In Chapter 4, we will consider the case that there is a current driven source on part of the boundary and a box method will be constructed in Chapter 5 for numerically solving the obstacle problem. The main results of these chapters will be summarized in the following corresponding sections. Moreover, we note that all mathematical notations and assumptions on the initial and boundary data and coefficients will be given in each chapter in detail. We also note that

$$\sigma(u)|\nabla\phi|^2 = \nabla[\sigma(u)\phi\nabla\phi]$$

in the weak sense due to equation (1.2). Thus these two forms are exchangeable and the latter will be used most often in the rest chapters.

1.2 The time periodic system of nonlocal equations

In Chapter 2, we will consider the system of nonlocal equations (1.3)-(1.4). To simplify the discussion we assume that $k(u) \equiv 1$ and neglect the effect of energy loss by radiation, i.e., $\gamma = 0$. It is possible to extend the results in this chapter to the general case of $k(u)$. The effect of energy loss by radiation will be fully studied in Chapter 3. In other words, we consider the following system:

$$u_t - \Delta u + \eta \int_\Omega G(x,y)u(y)dy = \sigma(u)|\nabla\phi|^2, \tag{1.7}$$

$$-\nabla[\sigma(u)\nabla\phi] = 0. \tag{1.8}$$

As we mentioned before, the presence of a nonlocal term in the above system could render invalid the maximum principle and lead to negative temperatures at some points of the device. Indeed, the authors in [1] considered the stationary case of equations (1.7)-(1.8). They gave an example where the temperature will be negative for some points if the value of the parameter η is very big. But for small gas pressures, i.e., small values of η, and for a special class of functions of $G(x, y)$, the maximum principle still holds which ensures the positivity of the temperature of the solutions. This result was extended to the case of a more general class of functions of $G(x, y)$ later in [13]. All these results are for the stationary case only. To our best knowledge, there are no known results available for the time dependent case.

It is the purpose of Chapter 2 to extend the positivity result to the physically important case of time periodic input. Specifically, let $\partial\Omega$ be the boundary of Ω, the domain occupied by the thermistor. The boundary is decomposed into two parts Γ_D and Γ_N. We assume that Ω is smooth enough and Γ_D is nonempty. Specific descriptions about the domain can be found in [19]. The unknowns u and ϕ are associated with the following boundary conditions

$$u|_{\partial\Omega} = 0, \ \phi|_{\Gamma_D} = \phi_0(x, t), \ \frac{\partial \phi}{\partial \vec{n}}|_{\Gamma_N} = 0. \tag{1.9}$$

The electrical potential input $\phi_0(x, t)$ is time periodic with period T.

Suppose the temperature satisfies a periodic condition

$$u(x, \cdot) = u(x, \cdot + T). \tag{1.10}$$

We first establish the existence of a time periodic solution through the Faedo-Galerkin method and the Leray-Schauder degree theory. Next we are interested in the positivity of the time periodic solutions. We will show that, for all those potential sources ϕ_0 which satisfy a certain growth property, there exists an η_0 independent of the specific choice of ϕ_0 such that the temperature of all time periodic solutions is positive for all $0 \leq \eta < \eta_0$.

We next consider the case of an initial condition

$$u(x, 0) = u_0(x). \tag{1.11}$$

We are interested in the long time behavior of system (1.7)-(1.8) under the boundary and initial conditions (1.9) and (1.11). We show that there exists a uniform attractor in $L^2(\Omega)$ and that its Hausdorff dimension is finite. The main difficulty related to this problem is the lack of uniqueness of the weak solution for a given $u_0(x) \in L^2(\Omega)$. Specifically, to obtain uniqueness, a certain regularity of the gradient of ϕ, say,

$$|\nabla\phi| \in L^p(\Omega)\ for\ some\ p > 2, \tag{1.12}$$

is needed. Since in our case ϕ satisfies a mixed boundary condition, such a regularity (1.12) is not generally available. From a result in [19] we know that if $u(x,t)$ is Hölder continuous for each t, (1.12) will be true, but this will require $u_0(x)$ to be Hölder continuous as well. To overcome this difficulty, our idea is to show that for all positive time t, any weak solution $u(x,t)$ will be Hölder continuous. Moreover, for a given Hölder continuous initial value, the corresponding solution will also be Hölder continuous for all time t. Thus, the long time behavior of (1.7)-(1.8) in $L^2(\Omega)$ (i.e., for all $u_0(x) \in L^2(\Omega)$) is the same as that in $C^\alpha(\bar{\Omega})$ (i.e., for all $u_0(x) \in C^\alpha(\bar{\Omega})$).

To achieve the above results, we have assumed that $\sigma_0 \leq \sigma(u) \leq \sigma_1$ for some positive constants σ_0 and σ_1. The problem is still open for the degenerate case, $\lim_{s\to\infty}\sigma(s) \to 0$. The results obtained here are new. No previous long time results are known even for the classical system, and there are no positivity results for nonlocal parabolic equations.

1.3 A novel obstacle thermistor problem

We obtained positivity results in Chapter 2 for time periodic solutions of the nonlocal system (1.7)-(1.8). There are a lot of mathematical difficulties to overcome to extend the results to general initial conditions. Moreover, even if such results are extended, they only hold for the case of small gas pressures. It is always possible that a negative temperature will occur under the situation of large gas pressures. Therefore, instead of focusing on the positivity of the temperature of system (1.7)-(1.8), we will concentrate on the development of new mathematical models which guarantee the positivity of the temperature of all solutions under any gas pressure.

It is our purpose in Chapter 3 to introduce the obstacle thermistor system

(1.5)-(1.6). We first consider its stationary version: find $u \geq 0$ and ϕ such that

$$u(-\nabla[k(u)\nabla u] + \eta \int_\Omega G(x,y)u(y)dy + \gamma u^4) \geq \sigma(u)|\nabla \phi|^2 u, \tag{1.13}$$
$$-\nabla[\sigma(u)\nabla\phi] = 0. \tag{1.14}$$

It is convenient for what follows to introduce a family of related penalized problems given by:

$$-\nabla\left[k(x)\nabla u\right] + \left[\eta \int_\Omega G(x,y)u(y)dy + \gamma u^4\right] I_n(u) = \sigma(u)|\nabla\phi|^2, \tag{1.15}$$
$$-\nabla[\sigma(u)\nabla\phi] = 0, \tag{1.16}$$

with $I_n(s)$ a C^∞ function such that: $0 \leq I_n(s) \leq 1$; $I_n(s) = 0$ if $s \leq 0$; $I_n(s) \longrightarrow H(s)$ in $L^p(\Omega)$ for $1 \leq p < \infty$, where H denotes the Heaviside function. The existence of a solution (u_n, ϕ_n) of the above penalized problems for each n is derived from a truncation method [4, 19] and the Leray- Schauder Degree theory. Now by making use of the properties of I_n, we can show that the limit (u, ϕ) of a subsequence of (u_n, ϕ_n) will be a solution of the obstacle problem (1.13)-(1.14).

Next, we consider the time dependent obstacle problem (1.5)-(1.6). We recall that in our previous discussion of the stationary case, we have left the possibility open that both $k(u)$ and $\sigma(u)$ are degenerate, i.e., both $k(u)$ and $\sigma(u)$ will approach to zero if u approaches to infinity. However, it is quite challengeable for the time dependent problem even for the simpler case that only $\sigma(u)$ is degenerate, which we assume here, and there are positive constants k_0 and k_1 such that $k_0 \leq k(u) \leq k_1$. This degeneracy for the time dependent case was first considered in [20, 21] for the classical system (1.1)-(1.2) where the authors introduced the notion of "capacity solution" to overcome the difficulty caused by the term on the right hand side of (1.1). The authors also showed that, given that the temperature of the capacity solution is essentially bounded, the capacity solution will be also a weak solution. In Chapter 3, we will follow the capacity solution method as well as a similar penalized method as above to show that there exists a capacity solution of the time dependent obstacle problem.

We observe that for both the stationary and time dependent obstacle problems, their solutions with positive temperature will also solve the corresponding system of nonlocal equations, and vice versa. Thus, to this extent, the obstacle problem is an extension to the system of nonlocal equations.

Finally, we are interested in the long time behavior of the time dependent obstacle problem. The term simulating energy loss by radiation plays a significant role in the discussion. Precisely, this term enables us to obtain the existence of a uniform absorbing set $\mathcal{B}$, which means that there exists a positive constant t_0 such that for any initial value $u(x,0) = u_0(x)$ in $L^2(\Omega)$, its solution will enter into the set $\mathcal{B}$ after time t_0. This is a very strong result since typically the time t_0 will depend on the initial value $u_0(x)$. Besides the previous assumptions, if we further assume that $\sigma(u)$ is not degenerate and $\sigma_0 \leq \sigma(u) \leq \sigma_1$ as in Chapter 2, then there exists a global attractor of the system in $C^{\alpha}(\bar{\Omega})$ which is nonempty, compact and invariant.

1.4 A current driven source on part of the boundary

In Chapters 2 and 3, the thermistor devices are totally driven by an electrical potential source. We are concerned in Chapter 4, however, with a somewhat different situation which arises physically when the devices are also driven by a current source at the same time. In this case, the total current through part of the boundary of the device is known, but the applied potential on that part is not. Specifically, the boundary $\partial\Omega$ of the device consists of three parts Γ_0, Γ_1 and Γ_N. There is an electrical potential $\phi_0(x,t)$ applied on Γ_0 and Γ_N is electrically insulated. While on Γ_1, $\phi(x,t) = \xi(t)$. Here, $\xi(t)$ is an unknown constant for each t, but the total current $I(t)$ through Γ_1 is known for each time t. Thus, another nonlocal boundary condition for the problem is given by

$$I(t) = \int_{\Gamma_1} \sigma(u) \frac{\partial \phi}{\partial \vec{n}} ds. \tag{1.17}$$

We are particularly interested in the time dependent obstacle problem (1.5)-(1.6) associated with above boundary conditions. Before we proceed, we recall that a similar boundary condition to (1.17) has been studied by several authors for the stationary version of the classical system (1.1)-(1.2), see [5, 9, 14]. Various results related to the existence of solutions and their regularity have been achieved there. But all the results are obtained under the assumption that the potential ϕ satisfies a homogenous boundary condition on Γ_0, i.e., $\phi|_{\Gamma_0} = \phi_0(x) = 0$. In this thesis we will not impose such an assumption on ϕ, and therefore can't directly apply the methods in [5, 9, 14] even to this special version of our case. Moreover, their results are valid for the stationary problems only and there are no previous related results for the time dependent case. Finally, we refer the interested readers to [8, 12] for the detailed description of physical devices related to this kind of nonlocal boundary conditions and for their practical applications.

For simplicity, we assume in Chapter 4 that the thermal conductivity $k(u) \equiv 1$ and we shall not consider the situation where σ is degenerate. We will apply the penalized method to transfer the obstacle problem to a family of systems of equations. To overcome the difficulty caused by the nonlocal boundary condition, a decomposition of ϕ will be introduced. Roughly speaking, we decompose the elliptic equation (1.6) satisfied by ϕ into two other elliptic equations such that each of them is coupled with a usual boundary condition instead of a nonlocal one. Therefore, we are able to study these two equations by general methods for elliptic equations. This decomposition will play a significant role throughout Chapter 4 and details will be shown in Section 4.3. Finally, by arguments of Campanato spaces, we obtain the existence of a unique Hölder continuous solution.

1.5 A box method for the obstacle problem

In Chapter 5, we cite two thermistor devices. Each of them has a resistor in its center whose resistance varies with temperature. Possible loss of energy from the resistor occurs through the supporting arms, through the surrounding gas, and through radiation effects.

A possible application of such a structure as a gas pressure sensor is as follows: The electrical resistance of the structure is monitored and if the surrounding gas pressure were to drop - thereby decreasing the amount of heat lost by the device to the surrounding gas - the resistance would rise. It is therefore possible to determine the gas pressure by measuring the device resistance.

The simulation and modeling of such devices are now generally accepted as a very useful design tool. Accurate simulations offer the means to rapidly investigate the performance of proposed new devices, and to determine the effects on sensitivity of modifications of structures already constructed. These techniques avoid the lengthy cycle of iterating construction, device measurement, and reconstruction until - if ever - a suitable device is found.

The simulation begin with the formulation of a mathematical model which usually is a partial differential system. Then it is necessary to construct an appropriate numerical method to solve the corresponding system. Since we have to take into account heat losses to the surrounding gas and heat losses by radiation for above devices, the obstacle model (1.5)-(1.6) will be a good candidate for this simulation. It is our intention of Chapter 5 to introduce a box method, which is a technique commonly employed in practice, for this obstacle problem.

The box method, also so-called the finite volume element method, is a numerical method occupying an intermediate position between the finite difference and finite element methods. Usually it is characterized by a trial space consisting of continuous piecewise linear polynomials on the primary triangulation and by a test space consisting of piecewise constants on the dual box mesh. Nowadays, the box method has been extensively and successfully used not only for various differential equations but also for variational inequalities. For example, the author in [7] developed error estimates for a general self-adjoint elliptic boundary value problem and the author in [18] gave comparison results between the finite volume element and finite element methods for elliptic variational inequalities. However, there are few papers dealing with the box method for time dependent obstacle problems due to the increasing difficulties in analyzing its convergence.

The main result in this chapter is an optimal H^1 convergence theorem for the box method. To obtain such a result, we have assumed that $\Omega \subset R^2$, both u and ϕ satisfy Dirichlet boundary conditions, and both $\sigma(u)$ and $k(u)$ are not degenerate. Since the devices under consideration are very thin, the assumption $\Omega \subset R^2$ seems fairly reasonable. However, in realistic situations, ϕ usually satisfies a mixed Dirichlet/Neumann boundary condition and $\sigma(s)$, $k(s)$ may degenerate as we mentioned in the previous chapters. The study of these more general situations is presently under consideration.

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Chapter 2

On the Time Periodic Thermistor Problem

2.1 Introduction

The classical system of parabolic/elliptic equations modelling thermistor behavior is given as follows:

$$u_t - \Delta u = \nabla[\sigma(u)\phi\nabla\phi], \tag{2.1}$$

$$-\nabla[\sigma(u)\nabla\phi] = 0. \tag{2.2}$$

Here u and ϕ are the temperature and the electric potential in the thermistor respectively, and $\sigma(u)$ is the electric conductivity. The above system has a long history, [16], and has been the subject of a variety of mathematical investigations in the past decade. We refer in particular to the work of Cimatti, [10, 11, 12, 13], the results by various authors in [5, 6, 7, 17, 24, 32, 31, 35] and the references therein. The system under consideration often has two features which make analysis challenging: a degeneracy in the equations and mixed boundary conditions. An important step with regard to the first difficulty was taken by X. Xu who introduced the concept of a "capacity solution" in the time dependent case, see e.g. [34, 33].

Recent interest in the problem of the modelling of micromachined microsensors has led to the addition of a nonlocal term to one of the equations, in order to account for the physically important effect of heat losses to the surrounding gas, [4, 26]. Specifically, the classical system (2.1)-(2.2) is replaced by the following nonlocal equations

$$u_t - \Delta u + \eta \int_\Omega G(x,y)u(y)dy = \nabla[\sigma(u)\phi\nabla\phi], \tag{2.3}$$

$$-\nabla[\sigma(u)\nabla\phi] = 0. \tag{2.4}$$

In order to avoid spurious negative temperature points, this system has also been formulated as an "obstacle" problem, [3]. Indeed, we recall that the presence of the nonlocal term renders invalid maximum principle/order arguments. For the steady state (i.e. purely elliptic) case, results in [2] and [18] showed that some form of the maximum principle was indeed valid if the surrounding gas pressure was sufficiently small.

It is the purpose of this chapter to consider the long time behavior of the solutions to the nonlocal thermistor system (2.3)-(2.4) in the physically important case of periodic input. We show the existence of a uniform attractor and estimate its Hausdorff dimension. We also prove that the temperature of periodic solutions is positive if the gas pressure (i.e. η) is small. To the best of our knowledge, no previous long time results are known even for the classical system, and there are no positivity results for nonlocal parabolic equations.

Let Ω denote the three dimensional domain occupied by the microsensor. Its boundary is denoted by $\partial\Omega$ which is decomposed into two parts Γ_D and Γ_N. We assume that Ω is smooth enough and Γ_D is nonempty. Specific descriptions about the domain can be found in [32, 31]. The unknowns are associated with the following boundary conditions

$$u|_{\partial\Omega} = 0, \ \phi|_{\Gamma_D} = \phi_0(x,t), \ \frac{\partial\phi}{\partial\vec{n}}|_{\Gamma_N} = 0. \tag{2.5}$$

Here the potential source ϕ_0 is time periodic with period T. Moreover, u satisfies either a periodic condition

$$u(x,\cdot) = u(x,\cdot+T), \tag{2.6}$$

or an initial condition

$$u(x, 0) = u_0(x). \tag{2.7}$$

We first consider the periodic case (2.6). The existence of a time periodic solution is established through the Faedo-Galerkin method and the Leray-Schauder degree theory. Next we are interested in the positivity of the time periodic solutions. As we mentioned before, the temperature u could be somewhere negative if η is big. We will show that, for all those potential sources ϕ_0 which satisfy a certain growth property, there exists an η_0 independent of the specific choice of ϕ_0 such that the temperature of all time periodic solutions is positive for all $0 \leq \eta < \eta_0$.

Finally we study the initial value case (2.7) and consider the long time behavior of the non-autonomous system (2.3)-(2.5) and (2.7). We show that there exists a uniform attractor in $L^2(\Omega)$ and that its Hausdorff dimension is finite. The main difficulty related to this problem is the lack of uniqueness of the weak solution for a given $u_0(x) \in L^2(\Omega)$. Specifically, to obtain uniqueness, a certain regularity of the gradient of ϕ, say,

$$|\nabla\phi| \in L^p(\Omega) \; for \; some \; p > 2, \tag{2.8}$$

is needed. Since in our case ϕ satisfies a mixed boundary condition, such a regularity (2.8) is not generally available. From a result in [31] we know that if $u(x,t)$ is Hölder continuous for each t, (2.8) will be true, but this will require $u_0(x)$ to be Hölder continuous as well. To overcome this difficulty, our idea is to show that for all positive time t, any weak solution $u(x,t)$ will be Hölder continuous. Moreover for a given Hölder continuous initial value, the corresponding solution will also be Hölder continuous for all time t. Thus the long time behavior of (2.3)-(2.5) in $L^2(\Omega)$ (i.e., for all $u_0(x) \in L^2(\Omega)$) is the same as in $C^\alpha(\bar{\Omega})$ (i.e., for all $u_0(x) \in C^\alpha(\bar{\Omega})$).

We denote by $L^p(\Omega)$ the standard Lebesgue spaces with norm $\|w\|_{L^p(\Omega)} = \left(\int_\Omega |w|^p\right)^{1/p}$. The standard inner product and norm in $L^2(\Omega)$ are denoted by $(\cdot, \cdot)$ and $\|\cdot\|$ respectively. Let $H^1(\Omega)$ be the standard Sobolev space with norm $\|w\|_{H^1(\Omega)} = \left(\int_\Omega(|w|^2 + |\nabla w|^2)\right)^{1/2}$. Its dual space is denoted by $H^{-1}(\Omega)$. The closure of $C_0^\infty(\Omega)$ in $H^1(\Omega)$ is denoted by $H_0^1(\Omega)$. For simplicity we write $H^1_{\Gamma_D}(\Omega) = H_0^1(\Omega \cup \Gamma_N)$. Let X be a general normed space, the space $L^p(0,T;X)$ consists of functions from $(0,T)$ into X with $\|w\|_{L^p(0,T;X)} = \left(\int_0^T \|w\|_X^p dt\right)^{1/p} < \infty$. For $0 < \alpha < 1$, we denote by $C^{\alpha,\alpha/2}(Q_T)$ the collection of all Hölder continuous functions on $Q_T = [0,T] \times \bar{\Omega}$. Details of these spaces and norms can be found in [1] and [25]. Other notation will be given in the following.

For simplicity, we write either

$$W(T,\Omega) = \{u | u(x,\cdot) = u(x,\cdot + T), \;\; u \in L^2(0,T;H_0^1(\Omega)),$$
$$u_t \in L^2(0,T;H^{-1}(\Omega))\},$$

or

$$W(T,\Omega) = \{u | u \in L^2(0,T;H_0^1(\Omega)), \; u_t \in L^2(0,T;H^{-1}(\Omega))\},$$

corresponding to cases (2.6) or (2.7) respectively.

A tuple (u, ϕ) is called a weak solution of (2.3)-(2.4) if $u \in W(T,\Omega)$ and $\phi - \phi_0 \in L^\infty(0,T;H^1_{\Gamma_D}(\Omega))$ and satisfies that, for almost every t,

$$\int_\Omega \left[u_t v + \nabla u \nabla v + \eta \int_\Omega G(x,y)u(y)dyv\right] \tag{2.9}$$
$$= -\int_\Omega \sigma(u)\phi\nabla\phi\nabla v, \; \forall v \in H_0^1(\Omega),$$

$$\int_\Omega \sigma(u)\nabla\phi\nabla\psi = 0, \; \forall \psi \in H^1_{\Gamma_D}(\Omega). \tag{2.10}$$

Before we proceed we give some general assumptions on the given data and the coefficients.

A1. There are two positive finite numbers σ_0 and σ_1 such that $\sigma_0 \le \sigma(s) \le \sigma_1$. Moreover $\sigma(s)$ is Lipschitz continuous with Lipschitz constant l.

A2. $\sup_{x,y} |G(x,y)| \le G_0$ and $\int_\Omega \int_\Omega G(x,y)w(x)w(y)dxdy \ge 0$ for all $w(x) \in L^2(\Omega)$.

A3. $\sup_{x,t} |\phi_0| < \infty$. Moreover ϕ_0 can be extended to be a Lipschitz function over Q_T which satisfies the same boundary condition as ϕ. This extension will also be denoted by ϕ_0.

Observe that we have assumed that σ is nondegenerate in order to obtain the existence of a uniform attractor and its finite dimensionality. Thus the long time behavior for the "capacity solution" case remains open.

The chapter is structured as follows. The existence of time periodic solutions is presented in Section 2.2. The positivity of time periodic solutions is obtained in Section 2.3. In the last section, we first show the Hölder continuity of the weak solutions for positive time t. As a consequence, we obtain the existence of a uniform attractor and its finite dimensionality in $L^2(\Omega)$.

In the following, c, c_i will always denote some generic positive constants which may depend on the boundary conditions and the various bounds of the coefficients but are independent of the initial value and the time t except an explicit specification. Moreover their values may vary from one step to the next.

2.2 The existence of time periodic solutions

We will apply the Leray-Schauder degree theory and the Galerkin method to show the existence of time periodic solutions. Let w_n, $n = 1, 2, \cdots$ be a countable basis of $H_0^1(\Omega)$. Without loss of generality, we may assume $(w_n, w_m) = 1$, if $n = m$, and 0, otherwise. The n dimensional subspace spanned by $w_1, \cdots, w_n$ is denoted by H_n. We also write $X = \{d(t) | d(t) \in C([0,T]), d(\cdot) = d(\cdot + T)\}$ and X^n the n-power cartesian product of X. A function $u^n(x,t) = \sum_{j=1}^n d_j^n(t) w_j(x)$ is called a Galerkin approximation solution of

$u(x, t)$ if it satisfies that

$$\int_\Omega \left[u_t^n w_j + \nabla u^n \nabla w_j + \eta \int_\Omega G(x, y) u^n(y) dy w_j \right]$$

$$= - \int_\Omega \sigma(u^n) \phi^n \nabla \phi^n \nabla w_j, \quad j = 1, \cdots, n, \tag{2.11}$$

$$\int_\Omega \sigma(u^n) \nabla \phi^n \nabla \psi = 0, \ \forall \psi \in H^1_{\Gamma_D}(\Omega). \tag{2.12}$$

Here $d_j^n(t) \in X$. For simplicity, we write

$$F_j(z) = \eta \int_\Omega \int_\Omega G(x, y) z(y) dy w_j,$$

$$E_j(z) = - \int_\Omega \sigma(z) \varphi \nabla \varphi \nabla w_j,$$

$$a_{ij} = (\nabla w_i, \nabla w_j),$$

where φ denotes the unique solution of (2.10) with u replaced by z. By the assumption on w_n, we obtain that

$$d_t^n + A^n d^n = E^n(u^n) - F^n(u^n), \tag{2.13}$$

with

$$d^n = (d_1^n, \cdots, d_n^n)^*,$$

$$E^n(z) = (E_1(z), \cdots, E_n(z))^*,$$

$$F^n(z) = (F_1(z), \cdots, F_n(z))^*,$$

$$A^n = (a_{ij})_{n \times n}.$$

Here $*$ denotes the transpose of a vector. Let $b(t) = (b_1(t), \cdots, b_2(t))^* \in X^n$ and write $v = \sum_{j=1}^n b_j w_j$. For $0 \leq \lambda \leq 1$, we define a family of mappings T_λ : $b(t) \to d(t)$ from X^n into itself with $d(t) = (d_1(t), \cdots, d_n(t))^* \in X^n$ satisfying the following linear system

$$d_t + A^n d = \lambda [E^n(v) - F^n(v)]. \tag{2.14}$$

Since $v \in C(0, T; H_0^1(\Omega))$, there exists a unique solution φ of (2.10) with u replaced by v. By the weak maximum principle and the general estimates of linear elliptic equations we have

$$\sup_{x,t} |\varphi| \le \sup_{x,t} |\phi_0|, \quad \sup_t \|\nabla\varphi\| \le c \sup_t \|\nabla\phi_0\|. \tag{2.15}$$

Thus $\sup_t |E_j(v)| < \infty$. In view of the assumption of $G(x,y)$, we also have $\sup_t |F_j(v)| < \infty$. Moreover $E_j(v)$ and $F_j(v)$ are time periodic. Therefore, according to the general theory of linear ordinary differential equations, there exists a unique periodic solution of (2.14). Actually the solution is given by

$$d(t) = \lambda e^{-A^n t} \left[\int_0^t e^{A^n s}[E^n(v) - F^n(v)]ds \right.$$
$$\left. + (e^{A^n T} - I_{n\times n})^{-1} \int_0^T e^{A^n s}[E^n(v) - F^n(v)]ds \right]. \tag{2.16}$$

Here $I_{n\times n}$ is the $n \times n$ identity matrix. Thus for each λ, T_λ is well defined. We observe from (2.16) that $\{T_\lambda b(t) | \; \|b(t)\|_{X^n} \le 1\}$ is equi-continuous and equi-bounded. Consequently T_λ is compact for each λ.

Lemma 2.1. *T_λ is continuous for each λ.*

Proof. Let $b^{(m)}(t)$ be a convergent sequence of X^n with limit $b(t)$. Then it follows from (2.16) that

$$\|T_\lambda b^{(m)} - T_\lambda b\|_{X^n} \le \sup_t \|e^{A^n t}\| \left(\lambda + \|e^{A^n T} - I_{n\times n}\|^{-1}\right) \times \tag{2.17}$$
$$\int_0^T \left[|E^n(v^{(m)}) - E^n(v)| + |F^n(v^{(m)}) - F^n(v)| \right] ds.$$

Here $v^{(m)} = \sum_{j=1}^n b_j^{(m)} w_j$, similarly for v. By the definitions and the estimates (2.15) we have that

$$|E_j(v^{(m)}) - E_j(v)| \le c \int_\Omega \left[\, |\sigma(v^{(m)}) - \sigma(v)| \, |\nabla\varphi| \right. \tag{2.18}$$
$$\left. + |\varphi^{(m)} - \varphi| \, |\nabla\varphi| + |\nabla(\varphi^{(m)} - \varphi)| \, \right] |\nabla w_j|,$$

$$|F_j(v^{(m)}) - F_j(v)| \le \sum_{i=1}^n |d_i^{(m)} - d_i| \left| \int_\Omega \int_\Omega G(x,y) w_i(y) w_j(x) dx dy \right|. \tag{2.19}$$

Thus

$$\sup_t |F_j(v^{(m)}) - F_j(v)| \to 0. \tag{2.20}$$

Since $v^{(m)} \to v$ in $\Omega \times [0,T]$ and $\sigma(s)$ is continuous, the Lebesgue convergence theorem gives that

$$\int_{Q_T} |\sigma(v^{(m)}) - \sigma(v)||\nabla\varphi|\,|\nabla w_j| \to 0. \tag{2.21}$$

It follows from the equation (2.10) that

$$\int_\Omega |\nabla(\varphi^{(m)} - \varphi)|^2 \le c\int_\Omega |\sigma(v^{(m)}) - \sigma(v)|^2|\nabla\varphi|^2, \tag{2.22}$$

which implies that

$$\int_{Q_T} |\nabla(\varphi^{(m)} - \varphi)|^2 \to 0. \tag{2.23}$$

Therefore

$$\int_{Q_T} |\nabla(\varphi^{(m)} - \varphi)|\,|\nabla w_j| \to 0. \tag{2.24}$$

The Poincare Inequality and (2.23) give that (if necessary, after passing to a subsequence) $\varphi^{(m)} \to \varphi$ *a.e.* in $\Omega \times (0,T)$. We apply the Lebesgue convergence theorem again and obtain that

$$\int_{Q_T} |\varphi^{(m)} - \varphi|\,|\nabla\varphi|\,|\nabla w_j| \to 0. \tag{2.25}$$

Finally by combining (2.20), (2.21), (2.24) and (2.25) we obtain that

$$\int_0^T \left[|E^n(v^{(m)}) - E^n(v)| + |F^n(v^{(m)}) - F^n(v)|\right] \to 0, \tag{2.26}$$

which implies that $\|T_\lambda b_j^{(m)} - T_\lambda b_j\|_{X^n} \to 0$. Thus T_λ is continuous for each λ. □

Lemma 2.2. *There exists a constant β independent of λ such that, for all $d(t) \in X^n$ satisfying $T_\lambda d = d$, $\|d\|_{X^n} \le \beta$.*

Proof. We multiply both sides of equation (2.14) by d^* to obtain that

$$\frac{1}{2}\frac{d}{dt}|d|^2 + d^*A^n d = \lambda d^*[E^n(w) - F^n(w)]. \tag{2.27}$$

Here $w = \sum_{j=1}^{n} d_j w_j$. By the definition of A^n and the Poincare Inequality, there exists a positive constant ν such that

$$d^* A^n d = \|\nabla w\|^2 \geq \nu |d|^2.$$

Furthermore

$$d^* F^n(w) = \eta \int_\Omega \int_\Omega G(x,y) w(y) w(x) dy dx \geq 0,$$

$$|d^* E^n(w)| = \left| \int_\Omega \sigma(w) \varphi \nabla \varphi \nabla w \right| \leq c + \frac{1}{2} \|\nabla w\|^2,$$

where the assumptions on $G(x,y)$, the Schwarz Inequality and the estimates (2.15) have been used. Thus we obtain that

$$\frac{d}{dt} |d|^2 + \nu |d|^2 \leq c. \tag{2.28}$$

Integrating (2.28) from 0 to T gives $\int_0^T |d(s)|^2 ds \leq \frac{c}{\nu} T$. Thus there exists $t_0 \in [0, T]$ such that $|d(t_0)|^2 \leq \frac{c}{\nu}$. Now for any $t \in [t_0, t_0 + T]$, we integrate (2.28) from t_0 to t and obtain that

$$|d(t)|^2 \leq cT + \frac{c}{\nu}. \tag{2.29}$$

Thus Lemma 2.2 follows immediately from (2.29). □

Lemma 2.3. *There exists at least one time periodic solution to the approximation system (2.11)-(2.12) for each n.*

Proof. From previous discussion we conclude that the family of mappings T_λ satisfies all the conditions of the Leray-Schauder degree theory. Therefore the topological degrees of T_0 and T_1 are the same. Since 0 is the unique time periodic fixed point of T_0, the degree of T_0 is +1. Thus T_1 is of degree +1 and has at least one fixed point. By the definition of T_1, this fixed point is a time periodic solution of the system (2.11)-(2.12). □

Theorem 2.1. *There exists at least one time periodic solution to the system (2.3)-(2.4).*

Proof. Similarly to (2.27)-(2.29), we obtain that

$$\int_0^T \|\nabla u^n\|^2 ds, \quad \sup_t \|u^n\|^2 \le c(T). \tag{2.30}$$

Thus there exists a convergent subsequence of u^n (denoted by u^n itself for the sake of simplicity) and a function $u \in W(T, \Omega)$ such that

$$u^n \to u \textit{ weakly star in } L^\infty(0,T; L^2(\Omega)),$$
$$\textit{and weakly in } L^2(0,T; H_0^1(\Omega)).$$

By a similar discussion as in Cimatti [12], up to an another extracted subsequence, we conclude that

$$u^n \to u \textit{ strongly in } L^2(Q_T).$$

Furthermore ϕ^n satisfy a similar bound as in (2.15), thus there exists a function ϕ such that

$$\phi^n \to \phi \textit{ weakly star in } L^\infty(0,T; H^1_{\Gamma_D}) \textit{ and in } L^\infty(Q_T).$$

Now we fix an m and for any w_j in H_m we pass to the limit in (2.11)-(2.12) with respect to n to obtain (2.9)-(2.10) for all $v \in H_m$. Since $\cup_{m=1}^\infty H_m$ is dense in $H_0^1(\Omega)$, we conclude (2.9) is satisfied for all $v \in H_0^1(\Omega)$. Thus (u, ϕ) is a time periodic solution. □

2.3 The positivity of the time periodic solutions

The purpose of this section is to show that if η is small, the temperature u of periodic solutions of (2.9)-(2.10) is positive. We assume here that $G(x, y) \equiv 1$ (this may be weakened), and require a specific nature of the input $\phi_0(x, t)$ as given below. In particular, we ask that ϕ_0 satisfy the M-property stated below. We begin with some preliminary considerations.

Let $0 < \sigma_0 < a(x) < \sigma_1$, and assume z solves

$$-\nabla[a(x)\nabla z] = 0, \tag{2.31}$$

$$z|_{\Gamma_0} = 0, \;\; z|_{\Gamma_1} = 1, \tag{2.32}$$

$$\frac{\partial z}{\partial \vec{n}}|_{\Gamma_N} = 0, \tag{2.33}$$

where $\Gamma_D = \Gamma_0 \cup \Gamma_1$, $\bar{\Gamma}_0 \cap \bar{\Gamma}_1$ is empty, and $\text{meas}(\Gamma_1) > 0$.

Lemma 2.4. *Let $w(x) > 0$ in Ω be a smooth function. Then there exists a positive constant m independent of the specific a, z such that $\int_\Omega a(x)|\nabla z|^2 w dx > m$.*

Proof. If not, there exist sequences $\{a_i(x)\}$, $\{z_i(x)\}$ with $\alpha < a_i(x) < \beta$ and $z_i(x)$ the corresponding solutions of (2.31)-(2.33) such that $\int_\Omega a(x)|\nabla z|^2 w dx \to 0$ as $i \to \infty$. Now $\{z_i(x)\}$ are bounded in $C^\alpha(\bar{\Omega})$. Thus without loss of generality, $z_i \to z$ in C^{α_1} for some $\alpha_1 < \alpha$. Clearly $z = 1$ on Γ_1, $z = 0$ on Γ_0. Now $\int_\Omega |\nabla z_i|^2$ is bounded. Without loss of generality $z_i \to z$ also in $L^2(\Omega)$. Let K be any compact subdomain of Ω, then $\int_K |\nabla z_i|^2 \to 0$. But $\int_K \varphi \nabla z_i = -\int_K \nabla\varphi z_i \to -\int_K \nabla \varphi z$ for all $\varphi \in C_0^\infty(K)$. Thus $\nabla z = 0$ in K which implies $z = constant$. Since K is arbitrary, $z = constant$ in $\bar{\Omega}$ which contradicts with $z = 1$ on Γ_1, $z = 0$ on Γ_0. □

Next we define the following:

Definition 2.1. For a given $M > 0$, a function $f \in L^2(0,T)$ has the M-property if and only if

$$||f||_{L^2(0,T)} \leq M||f||_{L^1(0,T)}.$$

Examples of collections of functions which satisfy this property are:

(a). There exist c_1, $c_2 > 0$ such that $c_1 \leq f \leq c_2$.

(b). There exist c_1, $c_2 > 0$ such that $\sup|f| \leq c_1$ and $\sup|f'| \leq c_2$.

(c). If t_0 is a point such that $\sup|f| = f(t_0) = c_1 > 0$ then there exists an interval I with $t_0 \in I$ and for any $t \in I$, $|f'(t)| \leq c_2 > 0$.

In each of these cases, $M = M(c_1, c_2)$ and does not depend on the specific f. We observe that physical limitations will force most possible practical inputs to satisfy the M-property for some M.

Now let u and ϕ solve the system (2.3)-(2.4) with $G(x,y) \equiv 1$, $\phi_0 = \phi_0(t)$, $\Gamma_D = \Gamma_0 \cup \Gamma_1$.

Theorem 2.2. *Suppose u, ϕ are smooth, periodic and $\phi_0^2(t)$ satisfies the M-property. There exists $\eta_0 > 0$, dependent on the data but independent of specific ϕ_0 except through M-property, such that if $0 \le \eta < \eta_0$ all periodic solutions (u, ϕ) satisfy $u > 0$.*

Proof. Through the transformation of $z = \frac{\phi}{\phi_0}$, we obtain that u and z satisfy

$$u_t - \Delta u + \eta \int_\Omega u = \phi_0^2(t) \nabla[\sigma(u) z \nabla z], \tag{2.34}$$

$$-\nabla[\sigma(u)\nabla z] = 0. \tag{2.35}$$

Here u satisfies the same initial and boundary conditions as before, but z satisfies

$$z|_{\Gamma_0} = 1,\ z|_{\Gamma_1} = 0,\ \frac{\partial z}{\partial \vec{n}}|_{\Gamma_N} = 0. \tag{2.36}$$

Let η be small enough, so that

$$-\Delta w_1 + \eta \int_\Omega w_1 = \lambda_1 w_1, \tag{2.37}$$

has a positive eigenvalue/eigenvector $\lambda_1 = \lambda_1(\eta)$, $w_1 = w_1(\eta)$ corresponding to the homogeneous Dirichlet condition. Note that $\lambda_1(\eta)$ is bounded away from zero as $\eta \to 0$. Then

$$\left(\int_\Omega w_1 u\right)_t + \lambda_1 \left(\int_\Omega w_1 u\right) = \phi_0^2(t) \int_\Omega \sigma(u) |\nabla z|^2 w_1.$$

Observe that $\int_\Omega \sigma(u)|\nabla z|^2 w_1$ is bounded above and below by Lemma 2.4. Write the value of this integral as $\tau(t)$ and obtain that

$$e^{\lambda_1 t} \int_\Omega w_1 u \Big|_t = \int_\Omega w_1 u \Big|_0 + \int_0^t e^{\lambda_1 s} \phi_0^2(s) \tau(s) ds.$$

Since u is periodic, we have

$$\int_\Omega w_1 u\Big|_t = \frac{e^{-\lambda_1 t}}{e^{\lambda_1 T}-1}\int_0^T e^{\lambda_1 s}\phi_0^2(s)\tau(s)ds$$
$$+e^{-\lambda_1 t}\int_0^t e^{\lambda_1 s}\phi_0^2(s)\tau(s)ds.$$

Thus replace u by $u/\int_0^T \phi_0^2(s)ds$ in (2.34) (then $\int_0^T \phi_0^2(s)ds = 1$ in (2.34) and $\int_0^T \phi_0^4(s)ds \le M^2$). We conclude that

$$\int_\Omega w_1 u\Big|_t > C, \tag{2.38}$$

for some positive constant C independent of u.

From equation (2.34) we obtain

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + \|\nabla u\|^2 + \eta\left(\int_\Omega u\right)^2 = \phi_0^2(t)\int_\Omega \sigma(u) z\nabla z\nabla u$$
$$\le \frac{1}{2}\|\nabla u\|^2 + c\phi_0^4(t)\int_\Omega \sigma^2(u)z^2|\nabla z|^2,$$

for some constant c, while (2.35)-(2.36) and the Poincare Inequality then yield for some c_1, c_2:

$$\frac{d}{dt}\|u\|^2 + c_1\|u\|^2 \le c_2\phi_0^4(t).$$

We obtain by periodicity:

$$\|u\|^2(t) \le c_2\left[1+\frac{1}{e^{c_1 T-1}}\right]\int_0^T e^{c_1 s}\phi_0^4(s)ds.$$

Thus $\|u\|^2(t)$ is bounded by the M-property, and we conclude that

$$\left|\int_\Omega u\right| < D,$$

for some constant D independent of the specific $\phi_0(t)$ but dependent on the M-property of ϕ_0, the domain and the bounds of the coefficients.

Finally we have

$$\int_\Omega u w_1 \ge C \;\; and \;\; \left|\int_\Omega u\right| \le D.$$

Now $u_t - \Delta u + \eta \int_\Omega u \geq 0$. Put $Lw = w_t - \Delta w$ with $w(x,0) = w(x,T)$. We obtain that for any $t \in [0,T]$:

$$\begin{aligned}
&\int_\Omega \left\{ u + \eta L^{-1} \left[\int_\Omega u \right] \right\} w_1 \\
&\geq \int_\Omega \left\{ u - \eta \sup_t \left| \int_\Omega u \right| L^{-1}(1) \right\} w_1 \\
&\geq \frac{C}{2},
\end{aligned}$$

if η is small enough as $\left|\int_\Omega u\right|$ and $L^{-1}(1)$ are bounded. On the other hand, by the same calculation,

$$\int_K \left\{ u + \eta L^{-1} \left[\int_\Omega u \right] \right\} w_1 \geq \frac{C}{4},$$

where K is a compact subset of Ω. Since $w_1 > 0$ in K and $u + \eta L^{-1} \left[\int_\Omega u\right] \geq 0$,

$$\int_K \left\{ u + \eta L^{-1} \left[\int_\Omega u \right] \right\} \geq c > 0.$$

But since

$$L \left[u + \eta L^{-1} \left(\int_\Omega u \right) \right] \geq 0,$$

we can apply Harnack's Inequality, [15, 25] and obtain for $t_1 > t_0$

$$\sup_K \left[u + \eta L^{-1} \left(\int_\Omega u \right) \right] \Bigg|_{t=t_0} \leq C \inf_K \left[u + \eta L^{-1} \left(\int_\Omega u \right) \right] \Bigg|_{t=t_1}.$$

Since the problem is periodic, $u + \eta L^{-1} \left(\int_\Omega u\right)$ at t_0 and $t_0 + T$ is the same. I.e.

$$\inf_K \left[u + \eta L^{-1} \left(\int_\Omega u \right) \right] \geq c > 0,$$

for any $t \in [0,T]$.

Let v be the solution of the following problem

$$\begin{aligned}
&-\Delta v = 0 \; in \; \Omega \backslash K, \\
&v = 1 \; on \; \partial K, \quad v = 0 \; on \; \partial \Omega.
\end{aligned}$$

Then $u + \eta L^{-1} \left(\int_\Omega u\right) \geq cv$ in $\Omega \backslash K$. We also have $u + \eta L^{-1} \left(\int_\Omega u\right) \geq c$ in K.

Thus if we choose η small enough, (independent of ϕ_0, except through the M-property), we obtain $u > 0$ in $\Omega \times [0,T]$. □

2.4 The existence of a uniform attractor and its dimension

2.4.1 The Hölder continuity of the weak solutions

In this part of the chapter we consider the Hölder continuity of the weak solutions for positive time t. For the case of a Hölder continuous initial value, we show that the solution belongs to $C^{\alpha,\alpha/2}(Q_T)$ and thus is unique. The results are obtained through Campanato space type arguments together with a cut-off function method.

Before we proceed, it will be convenient to recall some notations and results related to Campanato Spaces. For $0 \leq t_0 < t_1$, we denote $\Omega \times (t_0, t_1]$ by Q_{t_0,t_1}. For simplicity, if $t_0 = 0$, we write it as Q_{t_1}. A point $(x,t) \in Q_{t_0,t_1}$ is denoted by z. Let $B_r(x_0)$ be the ball centered at x_0 with radius r and $Q_r(z_0)$ be the cylinder $B_r(x_0) \times (t_0 - r^2, t_0]$. Then we define

$$\Omega[x_0, r] = B_r(x_0) \cap \Omega, \quad Q[z_0, r] = Q_r(z_0) \cap Q_{t_0,t_1}.$$

Moreover for $\mu \geq 0$, $\mathcal{L}^{2,\mu}(\Omega)$ and $\mathcal{L}^{2,\mu}(Q_{t_0,t_1})$ denote the Campanato spaces on Ω and Q_{t_0,t_1} associated with the standard norms $\|.\|_{2,\mu,\Omega}$ and $\|.\|_{2,\mu,Q_{t_0,t_1}}$ respectively. We refer interested readers to [23] , [28] and [35] for details on these spaces and norms.

Let δ_0 denote the Hölder exponent as stated in the De Giorgi - Nash theorem, see [28], [35]. In what follows, all α, α_i are in $(0, \delta_0)$ and μ_0, μ_1 are nonnegative numbers such that $\mu_0 < N - 2 + 2\delta_0$ and $\mu_1 < N + 2\delta_0$ where $N = 3$ is the dimension of Ω . They may differ from one step to the next. Furthermore $(\mu - 2)^+ = \max\{0, \mu - 2\}$.

Theorem 2.3. *Let $t_0 > 0$ and $h > 0$. There are generic constants $\rho_1 > 0$ and $\rho_2 > 0$ which only depend on h, the bounds of the coefficients, the boundary conditions and $|Q_{t_0,t_0+2h}|$ and are independent of t_0 and the initial value u_0 such that the weak solution u of (2.3)-(2.4) satisfies*

$$\|u\|_{C^{\alpha_0,\alpha_0/2}(Q_{t_0+h,t_0+2h})} \leq \rho_1 + \rho_2 e^{-\nu t}\|u_0\|, \qquad (2.39)$$

for all $0 < \alpha_0 < \delta_0$. *Here* ν *is a positive constant dependent on the domain* Ω *only.*

Proof. Let $0 \le \xi(t) \le 1$ be a smooth function such that $\xi(t) = 0$ for $t \le t_0$ and $\xi(t) = 1$ for $t \ge t_0 + h$. Furthermore assume $|\xi_t| \le \beta$ for some constant $\beta > 0$. Let (u, ϕ) be a weak solution of equations (2.3)-(2.4) and consider

$$\begin{aligned}(\xi u)_t - \Delta(\xi u) + \xi\eta \int_\Omega G(x,y)u(y,t)dy \\ = \xi\nabla[\sigma(u)\phi\nabla\phi] + \xi_t u, \\ \xi u(x,t_0) = 0, \quad \xi u|_{\partial\Omega} = 0.\end{aligned} \tag{2.40}$$

It follows from Theorem 3.5.1 of [31] that for all $0 \le \mu_0 < N - 2 + 2\delta_0$

$$\|\nabla\phi_n\|_{2,\mu_0,\Omega} \le c. \tag{2.41}$$

By Theorem 1.17 of [28], we have

$$r^{-\mu_0}\int_{t_0-r^2}^{t_0}\int_{B_r(x_0)} |\nabla\phi_n|^2 dx < cr^2. \tag{2.42}$$

Thus

$$\sup_{z_0\in Q_T, r>0} r^{-(\mu_0+2)}\int_{Q_r(z_0)} |\nabla\phi_n|^2 dz < c. \tag{2.43}$$

By using a result of [8] we obtain that for all $0 \le \mu_1 < N + 2\delta_0$

$$\|\nabla\phi_n\|_{2,\mu_1,Q_T} \le c. \tag{2.44}$$

Since $\sigma(u_n)\phi_n \in L^\infty(Q_T)$, we have $\sigma(u_n)\phi_n\nabla\phi_n \in \mathcal{L}^{2,\mu_1}(Q_T)$. It follows from Theorem 1 in [35] and (2.40) that

$$\begin{aligned}\|\xi\nabla u\|_{2,\mu_1,Q_{t_0,t_0+2h}} \le c[\|\xi\sigma(u)\phi\nabla\phi\|_{2,\mu_1,Q_{t_0,t_0+2h}} \\ +\|\xi_t u\|_{2,(\mu_1-2)^+,Q_{t_0,t_0+2h}} + \|\xi u\|_{L^2(t_0,t_0+2h;H^1(\Omega))}].\end{aligned} \tag{2.45}$$

First we apply the Energy Inequality and Gronwall lemma to (2.9) and obtain that

$$\|u\| \le c + e^{-\nu t}\|u_0\|, \tag{2.46}$$

and

$$\int_{t_0}^{t_0+2h} \|\nabla u\|^2 \leq c(1 + e^{-\nu t}\|u_0\|)^2. \tag{2.47}$$

Thus

$$\|\xi u\|_{L^2(t_0,t_0+2h;H^1(\Omega))} \leq c(1 + e^{-\nu t}\|u_0\|). \tag{2.48}$$

Applying (2.44) and (2.48) to (2.45) yields that

$$\|\xi \nabla u\|_{2,\mu_1,Q_{t_0,t_0+2h}} \leq c(1 + e^{-\nu t}\|u_0\|^2 + \|\xi_t u\|_{2,(\mu_1-2)^+,Q_{t_0,t_0+2h}}). \tag{2.49}$$

Now for $0 \leq \mu_2 \leq 2$, $(\mu_2 - 2)^+ = 0$. Since $\mathcal{L}^{2,0}(Q_{t_0,t_0+2h})$ is isomorphic to $L^2(Q_{t_0,t_0+2h})$, we obtain that

$$\|\xi \nabla u\|_{2,\mu_2,Q_{t_0,t_0+2h}} \leq c(1 + e^{-\nu t}\|u_0\|), \tag{2.50}$$

where (2.46) has been used. (2.50) implies that

$$\|\xi u\|_{2,\mu_2+2,Q_{t_0,t_0+2h}} \leq c(1 + e^{-\nu t}\|u_0\|), \tag{2.51}$$

Now for any $0 \leq \mu \leq N + 2\delta_0$, $(\mu - 2)^+ \leq 3$ since in our case $N = 3$. Thus we conclude from (2.49) and (2.51) that

$$\|\xi \nabla u\|_{2,\mu,Q_{t_0,t_0+2h}} \leq c(1 + e^{-\nu t}\|u_0\|), \tag{2.52}$$

for all $0 \leq \mu \leq N + 2\delta_0$. Consequently,

$$\|\xi u\|_{2,\mu+2,Q_{t_0,t_0+2h}} \leq c(1 + e^{-\nu t}\|u_0\|), \tag{2.53}$$

In particular, for $\mu = N + 2 + 2\alpha_0$ with $\alpha_0 < \delta_0$,

$$\|u\|_{2,N+2+2\alpha_0,Q_{t_0+h,t_0+2h}} \leq c(1 + e^{-\nu t}\|u_0\|), \tag{2.54}$$

where the definition of ξ and (2.53) have been used. Thus (2.39) follows from (2.54) immediately. □

Theorem 2.4. *If $u_0 \in C^\alpha(\bar{\Omega})$ and $u_0|_{\partial\Omega} = 0$ then the weak solution is Hölder continuous.*

Proof. We rewrite $u = w + z$. Here z is the unique solution of the simple equation

$$z_t - \Delta z = 0, \quad z(x,0) = u_0, \quad z|_{\partial\Omega} = 0, \tag{2.55}$$

and w satisfies

$$w_t - \Delta w = -\eta \int_\Omega G(x,y)u(y)dy + \nabla[\sigma(u)\phi\nabla\phi], \tag{2.56}$$

$$w(x,0) = 0, \quad w|_{\partial\Omega} = 0. \tag{2.57}$$

By a classic result in [25], $z \in C^{\alpha,\alpha/2}(Q_T)$. By a similar discussion as in Theorem 2.3 we also have $w \in C^{\alpha,\alpha/2}(Q_T)$. This completes the proof of the theorem. □

From Theorem 2.4, for a given initial value $u_0 \in C^\alpha(\bar{\Omega})$ with $u_0|_{\partial\Omega} = 0$, the function u is Hölder continuous. Thus the potential ϕ satisfies the regularity (2.8) with some $p > 3$ due to Lemma 5.3.2 in [31] which implies that the weak solution is unique. Moreover the following proposition holds, see [3].

Theorem 2.5. *Let (u_i, ϕ_i), $i = 1,\ 2$, be two $C^{\alpha,\alpha/2}(\bar{Q}_T)$ solutions to (2.3)-(2.5) corresponding to the initial data u_0^i, $i = 1,\ 2$ and the same ϕ_0. Write $w = u_1 - u_2$, $\varphi = \phi_1 - \phi_2$, $w_0 = u_0^1 - u_0^2$. Under the previous assumptions there exists a constant $c(t) > 0$ such that*

$$\|w\|^2 + \int_0^t \|\nabla w(s)\|^2 ds + \int_0^t \|\nabla\varphi(s)\|^2 ds \le c(t)\|w_0\|^2. \tag{2.58}$$

2.4.2 The uniform attractor and its dimension

Since the number h in Theorem 2.3 is arbitrary, we conclude that if the weak solution starts from an initial value in $L^2(\Omega)$ it will enter into the space $C^\alpha(\bar{\Omega})$ immediately. Since $C^\alpha(\bar{\Omega})$ is a subspace of $L^2(\Omega)$, we conclude that the uniform attractor in $C^\alpha(\bar{\Omega})$, if any, will be the same as in $L^2(\Omega)$. Thus the long time behavior of the system in $L^2(\Omega)$ is identical to that in $C^\alpha(\bar{\Omega})$. From Theorem 2.4, the weak solution corresponding to an initial value $u_0(x) \in C^\alpha(\bar{\Omega})$ is unique. Thus the solution operator of (2.3)-(2.4) is well defined. Therefore in the rest of the chapter we will focus on the case $u_0(x) \in C^\alpha(\bar{\Omega})$ only.

Following the notation of [21] and [29], we first briefly recall some definitions for the reader's convenience. Let E be a Banach space subject to the action of a two-parameter family of mappings $\{U(t,\tau)\} = \{U(t,\tau), t \geq \tau\}$, $U(t,\tau) : E \to E,\ t \geq \tau$.

Definition 2.2. A family of operators $\{U(t,\tau)\}$ is said to be a process in E if

1) $U(t,\tau) = U(t,s)U(s,\tau)\ \forall t \geq s \geq \tau$,

2) $U(\tau,\tau) = I$ is the identity operator $\forall \tau$.

Moreover a process $\{U(t,\tau)\}$ is said to be periodic with period T if

3) $U(t+T,\tau+T) = U(t,\tau)\ \forall t \geq \tau$.

Definition 2.3. A set $\mathcal{B} \in E$ is said to be uniformly attracting with respect to $\tau \in R$ for a process $\{U(t,\tau)\}$ if for all τ and for any set A that is bounded in E

$$\sup_{\tau} d(U(t+\tau,\tau)A, \mathcal{B}) \to 0 \ as \ t \to \infty.$$

Here $d(A_1, A_2)$ denotes the usual semi-distance of A_1 and A_2. Furthermore, a process is said to be uniformly asymptotically compact if $\mathcal{B}$ is also compact.

Definition 2.4. A closed subset of E is said to be a uniform attractor for a process $\{U(t,\tau)\}$ if it is the minimal closed uniformly attracting set for this process.

Definition 2.5. A curve $u(s) \in E,\ s \in R$, is called a complete trajectory of a process $\{U(t,\tau)\}$ if $U(t,\tau)u(\tau) = u(t)\ \ \forall t \geq \tau;\ \ t,\ \tau \in R$. The kernel $\mathcal{K}$ of a process $\{U(t,\tau)\}$ consists of all of its bounded complete trajectories. The set $\mathcal{K}(s) = \{u(s) : u(\cdot) \in \mathcal{K}\}$ of values of complete trajectories $u(s)$ with $t = s$ is called the kernel section of this process at the time $t = s$.

We refer the readers to [21] and [29] and the references therein for more specific descriptions of complete trajectories, kernels, kernel sections and so on.

We first show the existence of a uniform attractor for the initial boundary value problem (2.3)-(2.5) and (2.7). As in the previous sections, the driving source $\phi_0(x,t)$ is time-periodic with period T. We define the family of two-parameter operators $\{U(t,\tau)\}$: $C^\alpha(\bar{\Omega}) \to C^\alpha(\bar{\Omega})$, $U(t,\tau)u_\tau = u(t)$. Here $u(t)$ is the unique solution of the problem but with the initial value condition replaced by

$$u(x,\tau) = u_\tau(x), \quad u_\tau(x) \in C^\alpha(\bar{\Omega}), \quad u_\tau|_{\partial\Omega} = 0. \tag{2.59}$$

We observe that all conditions in Definition 2.2 are satisfied. Thus $\{U(t,\tau)\}$ defines a periodic process. Moreover, it follows from Theorem 2.5 that $U(t,\tau)$ is jointly continuous with respect to the potential $\phi_0(x,t)$ and the initial value $u_0(x)$. Let

$$\mathcal{B} = \{w | w \in C^\alpha(\bar{\Omega}), w|_{\partial\Omega} = 0, \ and \ \|w\|_{C^\alpha(\bar{\Omega})} \leq \rho_1 + 1\}, \tag{2.60}$$

where ρ_1 is the positive constant in Theorem 2.3. According to Theorem 2.3, $\mathcal{B}$ is a uniform attracting set of $U(t,\tau)$. Now write

$$\mathcal{B}_1 = \cup_{t\geq 0} \cup_{\tau\in[0,T)} U(t+\tau+2h,\tau)\mathcal{B}. \tag{2.61}$$

Here h is a positive constant as specified in Theorem 2.3. We observe that $\mathcal{B}_1$ is also a uniform attracting set. Since inequality (2.39) is satisfied for all $0 < \alpha_0 < \delta_0$, we obtain that $\mathcal{B}_1$ is a bounded subset of $C^{\alpha_0}(\bar{\Omega})$ for all $\alpha < \alpha_0 < \delta_0$. Thus by the compact imbedding theorem $\mathcal{B}_1$ is precompact in $C^\alpha(\bar{\Omega})$. This implies that the periodic process $U(t,\tau)$ is uniformly asymptotically compact. By a result in [29] (See Theorem 2.1) we conclude that

Theorem 2.6. *Under the previous assumptions, the system (2.3)-(2.5) has a uniform attractor $\mathcal{A}$ in $C^\alpha(\bar{\Omega})$ which is nonempty and compact. Furthermore,*

$$\mathcal{A} = \cup_{t\in[0,T)}\mathcal{K}(t). \tag{2.62}$$

Here $\mathcal{K}(t)$ are the kernel sections of $\{U(t,\tau)\}$ which have the following properties:

$$\mathcal{K}(t+T) = \mathcal{K}(t) \ \ \forall t \in R, \tag{2.63}$$

$$U(t,\tau)\mathcal{K}(\tau) = \mathcal{K}(t), \ \ t \geq \tau, \ t, \ \tau \in R. \tag{2.64}$$

In the rest of the chapter we deal with the finite dimensionality of the attractor $\mathcal{A}$ by considerations based on arguments related to these in [9, 20, 29]. We first find an upper bound of the dimensions of the kernel sections $\mathcal{K}(t)$. Then by an extension of a result in [29] we conclude that the dimension of the uniform attractor $\mathcal{A}$ is also bounded.

In this section, besides the assumptions (A1)-(A3), we further assume

A4. $\sigma(s)$ is continuously differentiable and there exist two positive constants σ_2 and θ_0 such that $|\sigma'(s)| \leq \sigma_2 < \infty$ and $|\sigma'(s_1) - \sigma'(s_2)| \leq \sigma_2 |s_1 - s_2|^{\theta_0}$ for all $s,\ s_1,\ s_2 \geq 0$.

We linearize equations (2.3)-(2.4) about (u, ϕ) and obtain

$$\frac{d\bar{u}}{dt} = L(u, \bar{u}), \tag{2.65}$$

$$\nabla[\sigma(u)\nabla\bar{\phi}] = -\nabla[\sigma'(u)\bar{u}\nabla\phi], \tag{2.66}$$

Here $u(t) = U(t,\tau)u_\tau$ with $u_\tau \in \mathcal{K}(\tau)$. The unknowns satisfy the following initial and boundary conditions:

$$\bar{u}|_{\partial\Omega} = 0, \quad \bar{u}(x,\tau) = \bar{u}_\tau, \tag{2.67}$$

$$\bar{\phi}|_{\Gamma_D} = 0, \quad \frac{\partial\bar{\phi}}{\partial\vec{n}}|_{\Gamma_N} = 0. \tag{2.68}$$

Moreover the operator $L(u, w)$ is given by

$$L(u, w) = \Delta w + L_1(w) + L_2(u, w), \tag{2.69}$$

with

$$L_1(w) = -\eta \int_\Omega G(x,y)w(y)dy, \tag{2.70}$$

$$L_2(u, w) = \nabla[\sigma(u)\bar{\phi}_w\nabla\phi + \sigma(u)\phi\nabla\bar{\phi}_w + \sigma'(u)\phi w\nabla\phi]. \tag{2.71}$$

Here $\bar{\phi}_w$ is the solution of (2.66) and (2.68) with $\bar{u}$ replaced by w. For the case of $w = \bar{u}$, it is simply denoted by $\bar{\phi}$ as before.

Theorem 2.7. *For any $w, \ v \in H_0^1(\Omega)$, the operator L satisfies*

$$|(L(u,w),v)| \leq c\|\nabla w\| \, \|\nabla v\|, \tag{2.72}$$
$$(-L(u,w),w) \geq \frac{1}{2}\|\nabla w\|^2 - c\|w\|^2. \tag{2.73}$$

Consequently the system (2.65)-(2.68) possesses a unique weak solution.

Proof. We recall that there exists a generic constant $c > 0$ such that

$$\|\phi\|_{L^\infty(\Omega)}(t), \ \|\nabla\phi\|_{L^p(\Omega)}(t) \leq c \quad for \ all \ t, \tag{2.74}$$

where $p > 3$ is some positive constant, see Lemma 5.3.2 of [31]. Now we estimate the terms of $L(u,w)$. We first have

$$|(L_1(w),v)| \leq c\|w\| \, \|v\|. \tag{2.75}$$

Next we replace $\bar{u}$ by w in (2.66), then multiply its both sides by $\bar{\phi}_w$ and integrate it over Ω to obtain

$$\begin{aligned}\int_\Omega \sigma(u)|\nabla\bar{\phi}_w|^2 &= -\int_\Omega \sigma'(u)w\nabla\phi\nabla\bar{\phi}_w \\ &\leq \sigma_2\|\nabla\phi\|_{L^p(\Omega)} \, \|w\|_{L^{2p/p-2}(\Omega)} \, \|\nabla\bar{\phi}_w\|,\end{aligned} \tag{2.76}$$

which gives that

$$\|\nabla\bar{\phi}_w\| \leq c\|w\|_{L^{2p/p-2}(\Omega)}. \tag{2.77}$$

Now we are ready to estimate $|(L_2(u,w),v)|$. In fact, it follows from (2.71) that

$$\begin{aligned}|(L_2(u,w),v)| &= |(\sigma(u)\bar{\phi}_w\nabla\phi + \sigma(u)\phi\nabla\bar{\phi}_w + \sigma'(u)\phi w\nabla\phi, \nabla v))| \\ &\leq [\sigma_1\|\nabla\phi\|_{L^p(\Omega)} \, \|\bar{\phi}_w\|_{L^{2p/p-2}(\Omega)} + \sigma_1\|\phi\|_{L^\infty(\Omega)} \, \|\nabla\bar{\phi}_w\| \\ &\quad +\sigma_2\|\phi\|_{L^\infty(\Omega)} \, \|\nabla\phi\|_{L^p(\Omega)} \, \|w\|_{L^{2p/p-2}(\Omega)}] \, \|\nabla v\|.\end{aligned} \tag{2.78}$$

By the Sobolev imbedding theorem, $\|\bar{\phi}_w\|_{L^{2p/p-2}(\Omega)} \leq c\|\nabla\bar{\phi}_w\|$, we conclude from (2.78) that

$$|(L_2(u,w),v)| \leq c\|w\|_{L^{2p/p-2}(\Omega)} \, \|\nabla v\|. \tag{2.79}$$

Finally, thanks to (2.75) and (2.79), we have

$$\begin{aligned}|(L(u,w),v)| &\leq \|\nabla w\|\ \|\nabla v\| + |(L_1(w),v)| + |(L_2(u,w),v)| \qquad (2.80)\\ &\leq \|\nabla w\|\ \|\nabla v\| + c(\|w\|\ \|v\| + \|w\|_{L^{2p/p-2}(\Omega)}\ \|\nabla v\|).\end{aligned}$$

Then (2.72) follows directly from (2.80), the Sobolev imbedding theorem and the Poincare Inequality.

On the other hand,

$$\begin{aligned}(-L(u,w),w) &= \|\nabla w\|^2 + (-L_1(w),w) + (-L_2(u,w),w) \qquad (2.81)\\ &\geq \|\nabla w\|^2 - c\|w\|_{L^{2p/p-2}(\Omega)}\ \|\nabla w\|,\end{aligned}$$

where the property $(-L_1(w),w) \geq 0$ and (2.79) are used. From the Sobolev interpolation inequality we have

$$\|w\|^2_{2p/(p-2)} \leq c\|w\|^{2-2N/p}\|\nabla w\|^{2N/p}. \qquad (2.82)$$

which together with Young inequality and (2.81) give (2.73).

From the properties (2.72) and (2.73), we conclude that the operator $-L$ is continuous and coercive. Thus the existence of a unique solution to the problem (2.65)-(2.68) is just a direct application of the results for abstract Cauchy problems presented in [30]. This completes the proof. □

Denote the process generated by the problem (2.65)-(2.68) by $\{U'(t,\tau)\}$, i.e., $\bar{u}(t) = U'(t,\tau)\bar{u}_\tau$. The following theorem holds.

Theorem 2.8. *The process $\{U(t,\tau)\}$ is uniformly quasi-differentiable on the kernel sections $\{\mathcal{K}(\tau)\}_{\tau\in[0,T)}$ in $L^2(\Omega)$ and $\{U'(t,\tau)\}$ is one of its differentials, i.e.,*

$$\lim_{\delta\to 0}\sup_{0<\|u^1_\tau-u^2_\tau\|\leq\delta}\frac{\|U(t,\tau)u^1_\tau - U(t,\tau)u^2_\tau - U'(t,\tau)(u^1_\tau - u^2_\tau)\|}{\|u^1_\tau - u^2_\tau\|} = 0, \qquad (2.83)$$

for all u^1_τ, $u^2_\tau \in \mathcal{K}(\tau)$, $\tau \in [0,T)$ and $t \geq \tau$.

The proof of this theorem is lengthy and similar to that of Theorem 2.5, thus we will leave it to interested readers.

Next we estimate the dimension of the kernel sections $\mathcal{K}(\tau)$, $\tau \in [0, T)$. Let us introduce the following quantities as in [9], [14] and [27],

$$q_m = \liminf_{t \to 0} \sup_{\tau \in [0,T)} \sup_{u_\tau \in \mathcal{K}(\tau)} \left(\frac{1}{t} \int_\tau^{\tau+t} Tr_m L ds \right). \tag{2.84}$$

Here L is the operator defined in (2.69), and $Tr_m L$ is the m-dimensional trace of L defined by $Tr_m L = \sup_Q TrLQ$ with the supremum taken over all the orthogonal projectors Q in $L^2(\Omega)$ on the space QL^2 of dimension m belonging to the domain of L (see [9]).

Before we estimate the bounds of the uniform attractors $\mathcal{A}$, we give an extension to the Proposition 3.2 of [9]. First let us recall some basic definitions. Let E be a metric space and $Y \subset E$ be a subset of E. Given two positive numbers d and ε, we write

$$\mu_H(Y, d, \varepsilon) = \inf \sum_i r_i^d, \tag{2.85}$$

$$\mu_H(Y, d) = \lim_{\varepsilon \to 0} \mu_H(Y, d, \varepsilon) = \sup_{\varepsilon > 0} \mu_H(Y, d, \varepsilon). \tag{2.86}$$

Here the infimum in (2.85) is for all coverings of Y by balls B_{r_i} of E with radius $r_i \leq \varepsilon$. Then the Hausdorff dimension of Y in E is defined by

$$\dim_H(Y) = \inf\{d : \ \mu_H(Y, d) = 0\}. \tag{2.87}$$

Similarly, let $J(\varepsilon, Y)$ be the minimum number of balls of E of radius ε which is necessary to cover Y, then the fractal dimension of Y in E is defined by

$$\dim_F(Y) = \inf\{d : \ \mu_F(Y, d) = 0\}, \tag{2.88}$$

with

$$\mu_F(Y, d) = \limsup_{\varepsilon \to 0} \varepsilon^d J(\varepsilon, Y). \tag{2.89}$$

Let K_0 be a compact subset in E and S be a mapping from $K_0 \times [0, T]$ to E such that $S(y, 0) = y$ for all $y \in K_0$. We assume that the mapping S is Hölder continuous with respect to both y and t with Hölder exponents ϵ_1 and ϵ_2 respectively, i.e., for all y_1, $y_2 \in K_0$ and t_1, $t_2 \in [0, T]$,

$$\|S(y_1, t_1) - S(y_2, t_2)\|_E \leq \nu(\|y_1 - y_2\|_E^{\epsilon_1} + |t_1 - t_2|^{\epsilon_2}). \tag{2.90}$$

The following theorem is an extension of Proposition 3.2 of [9] where the case $\epsilon_1 = \epsilon_2 = 1$ has been discussed.

Theorem 2.9. *Let K_0 and S be the compact set and the mapping described above, and also let $K_t = S(K_0, t)$, $t \in [0, T]$ and $Y = \cup_{t\in[0,T]} K_t$. Then*

$$\dim_F(Y) \le \frac{1}{\epsilon_1}\dim_F(K_0) + \frac{1}{\epsilon_2}, \tag{2.91}$$

$$\dim_H(Y) \le \frac{1}{\epsilon_1}\dim_H(K_0) + \frac{1}{\epsilon_2}. \tag{2.92}$$

Proof. We first consider the fractal dimension of Y. Given $\varepsilon > 0$, we cover K_0 by a family of balls $\{B_{\varepsilon'}(y_j)\}_{j=1}^J$ with $\varepsilon' = \left(\frac{\varepsilon}{2\nu}\right)^{\frac{1}{\epsilon_1}}$. Next we give a partition $\{t_i\}_{i=0}^{M+1}$ of $[0, T]$ with $t_0 = 0$ and $t_{M+1} = T$ such that

$$0 < t_{i+1} - t_i \le \left(\frac{\varepsilon}{2\nu}\right)^{\frac{1}{\epsilon_2}}, \quad M \le T\left(\frac{2\nu}{\varepsilon}\right)^{\frac{1}{\epsilon_2}}. \tag{2.93}$$

Now for any point (y, t) in $Y \times [0, T]$, we can choose some y_j and t_i such that $\|y - y_j\|_E \le \varepsilon'$ and $|t - t_i| \le \left(\frac{\varepsilon}{2\nu}\right)^{\frac{1}{\epsilon_2}}$. Consequently,

$$\|S(y, t) - S(y_j, t_i)\|_E \le \nu\left(\frac{\varepsilon}{2\nu} + \frac{\varepsilon}{2\nu}\right) = \varepsilon. \tag{2.94}$$

This implies that $B_\varepsilon(S(y_j, t_i))$, $j = 1, \cdots, J$, $i = 1, \cdots, M$ is a covering of Y. Thus by definition

$$\begin{aligned}\mu_F\left(Y, \frac{1}{\epsilon_1}d + \frac{1}{\epsilon_2}\right) &= \limsup_{\varepsilon\to 0} \varepsilon^{\frac{1}{\epsilon_1}d + \frac{1}{\epsilon_2}} JM \\ &\le T\,(2\nu)^{\frac{1}{\epsilon_1}d + \frac{1}{\epsilon_2}} \limsup_{\varepsilon\to 0} (\varepsilon')^d J \\ &= T\,(2\nu)^{\frac{1}{\epsilon_1}d + \frac{1}{\epsilon_2}} \mu_F(K_0, d),\end{aligned} \tag{2.95}$$

where the last inequality in (2.93) is used. Thus (2.91) follows immediately.

Similarly, to prove (2.92), we cover K_0 with balls $\{B_{\varepsilon_j}(y_j)\}_{j=1}^J$, where $\varepsilon_j \le \left(\frac{\varepsilon}{2\nu}\right)^{\frac{1}{\epsilon_1}}$. For each j, we give a partition $\{t_i^j\}_{i=0}^{M_j+1}$ such that

$$0 < t_{i+1}^j - t_i^j \le (\varepsilon_j)^{\frac{\epsilon_1}{\epsilon_2}}, \quad M \le T\,(\varepsilon_j)^{-\frac{\epsilon_1}{\epsilon_2}}. \tag{2.96}$$

Then for each point (y, t), we can choose y_j and t_i^j such that $\|S(y, t) - S(y_j, t_i^j)\|_E \le 2\nu\,(\varepsilon_j)^{\epsilon_1} \le \varepsilon$. Therefore, the family of balls

$$\{B_{2\nu(\varepsilon_j)^{\epsilon_1}}(S(y_j, t_i^j)) : \ j = 1, \cdots, J, \ i = 1, \cdots, M_j\}$$

is a covering of Y with radii less than ε. Hence

$$\mu_H(Y, \frac{1}{\epsilon_1}d + \frac{1}{\epsilon_2}) \leq \sum_{j=1}^{J}\sum_{i=1}^{M_j}\left[2\nu\left(\varepsilon_j\right)^{\epsilon_1}\right]^{\frac{1}{\epsilon_1}d+\frac{1}{\epsilon_2}} \tag{2.97}$$

$$= \sum_{j=1}^{J} M_j\left[2\nu\left(\varepsilon_j\right)^{\epsilon_1}\right]^{\frac{1}{\epsilon_1}d+\frac{1}{\epsilon_2}}$$

$$\leq T\left(2\nu\right)^{\frac{1}{\epsilon_1}d+\frac{1}{\epsilon_2}}\sum_{j=1}^{J}\left(\varepsilon_j\right)^d,$$

which implies (2.92). □

Theorem 2.10. *Under the assumptions (A1)-(A4), the Hausdorff dimensions in $L^2(\Omega)$ of the kernel sections of $U(t,\tau)$ are bounded and*

$$\dim_H(\mathcal{K}(\tau)) \leq m_0, \qquad \forall \tau \in [0,T), \tag{2.98}$$

where m_0 depends on the boundary conditions, the various bounds of the coefficients and the domain.

Proof. Let Q_m be an m-dimensional orthogonal projector in $L^2(\Omega)$, and $\{w_j\}_{j=1}^m$ be an orthonormal basis in $Q_mL^2(\Omega)$. We recall that

$$TrLQ_m = \sum_{j=1}^{m}(L(u, w_j), w_j). \tag{2.99}$$

Thus it follows from (2.73) that

$$TrLQ_m \leq -\frac{1}{2}\sum_{j=1}^{m}\|\nabla w_j\|^2 + c\sum_{j=1}^{m}\|w_j\|^2. \tag{2.100}$$

But by Lemma 2.1 (Page 390 of [27]), we have

$$\sum_{j=1}^{m}\|\nabla w_j\|^2 \geq cm^{1+2/N}. \tag{2.101}$$

Substituting (2.101) into (2.100) yields that

$$TrLQ_m \leq -\frac{1}{2}c_1m^{1+2/N} + c_2m \leq -c_3m^{1+2/N} + c_4, \tag{2.102}$$

where Young inequality is used. Then by the definition (2.84) of q_m we obtain

$$q_m \leq -c_3 m^{1+2/N} + c_4. \tag{2.103}$$

Thus if $m > \left(\frac{c_4}{c_3}\right)^{\frac{N}{N+2}}$, $q_m < 0$. By Theorem 4.1 of [9], we conclude that the Hausdorff dimensions of the kernel sections $\mathcal{K}(\tau)$ are bounded and (2.98) is satisfied with m_0 the minimal integer such that $m > \left(\frac{c_4}{c_3}\right)^{\frac{N}{N+2}}$. □

Finally the estimate of the Hausdorff dimension of the uniform attractor $\mathcal{A}$ is summarized in the following theorem

Theorem 2.11. *Under the assumptions (A1)-(A4), the Hausdorff dimension in $L^2(\Omega)$ of the uniform attractor $\mathcal{A}$ is bounded and satisfies*

$$\dim_H(\mathcal{A}) \leq m_0 + \frac{2}{\alpha}. \tag{2.104}$$

Proof. We only have to show that $U(t,0)$ satisfies a similar property to (2.90). In fact, let u_1, $u_2 \in \mathcal{K}(0)$ and t_1, $t_2 \geq 0$, then

$$\begin{aligned} &\|U(t_1,0)u_1 - U(t_2,0)u_2\| \\ &\leq \|U(t_1,0)u_1 - U(t_1,0)u_2\| + \|U(t_1,0)u_2 - U(t_2,0)u_2\|. \end{aligned} \tag{2.105}$$

In view of Theorem 2.5

$$\|U(t_1,0)u_1 - U(t_1,0)u_2\| \leq c\|u_1 - u_2\|. \tag{2.106}$$

Since $U(t,0)u_2 \in C^{\alpha,\frac{\alpha}{2}}(Q_T)$, we obtain

$$\|U(t_1,0)u_2 - U(t_2,0)u_2\| \leq c|t_1 - t_2|^{\frac{\alpha}{2}}. \tag{2.107}$$

Thus $U(t,0)$ satisfies the property (2.90) with $\epsilon_1 = 1$ and $\epsilon_2 = \frac{\alpha}{2}$. According to Theorems 2.9 and 2.10, (2.104) holds. □

2.5 Conclusions

In this chapter we have determined the positivity of periodic solutions to a nonlocal thermistor system if the surrounding gas pressure is small. We have also considered the long time behavior of initial value problem solutions and showed the existence of a uniform attractor. Finally, the Hausdorff dimension of the attractor was estimated. We believe these results to be new even for the classical thermistor system (i.e. equations (2.1)-(2.2)). The degenerate case involving "capacity solutions" remains open.

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Chapter 3

Existence and Long Time Behaviour of Solutions to Obstacle Thermistor Equations

3.1 Introduction

Equations that determine thermistor behaviour have been investigated for more than 100 years, [10] and the advent of micromachined microsensor devices has led to somewhat more general models, [3, 4, 5]. Specifically, if we include radiation effects as well as heat losses to the surrounding gas we obtain the system:

$$-\nabla[\sigma(u)\nabla\phi] = 0, \tag{3.1}$$

$$\frac{du}{dt} - \nabla[k(u)\nabla u] + \eta\int_\Omega G(x,y)u(y,t)dy + \gamma u^4 = \nabla[\sigma(u)\phi\nabla\phi], \tag{3.2}$$

in a smooth bounded domain $\Omega \subset R^N$, $N = 2$ or $N = 3$. The cases with $N > 3$ appear to be of primarily theoretical interest. Here $0 < \sigma(u)$, $k(u)$ are smooth functions and η, γ denote positive constants. We also assume $G(x,y) \geq 0$, and that G obeys the further properties given below. With (3.1)-(3.2) we associate suitable boundary/initial conditions: $u = u_0(x) \geq 0$ at $t = 0$; $u = 0$ on $\partial\Omega_D$, $\phi = \phi_0(x)$ on $\partial\Omega_D$, $\frac{\partial u}{\partial n} = \frac{\partial \phi}{\partial n} = 0$ on $\partial\Omega_N$ with: $\partial\Omega_D \cup \partial\Omega_N = \partial\Omega$ and $\partial\Omega_D$

closed in $\partial\Omega$, $\partial\Omega_N$ open in $\partial\Omega$, both nontrivial, smooth. Detailed regularity conditions needed for $\partial\Omega$ may be found in [24]. In practice we often have $u_0 \equiv 0$, and the boundary conditions on u, ϕ are piecewise constants. We assume that ϕ_0 is smooth, but it will be convenient to take $u_0 \in L^2(\Omega)$, and sometimes $u_0 \in C^\alpha(\Omega)$.

The presence of the nonlocal term in equation (3.2) leads for η big to a solution behavior that is at odds with what is physically expected, due to the failure of the maximum principle. More precisely and specifically, the equation

$$-\nabla[k(x)\nabla u] + \eta \int_\Omega G(x,y)u(y)dy + \gamma u^4 = f(x), \tag{3.3}$$

with $k(x) > 0$ and $u = 0$ on $\partial\Omega$, will have positive solutions u for any $f \geq 0$ iff $0 \leq \eta \leq \eta_0$ for some η_0 which depends on the data but not on the specific f. These results are explicitly shown for $\gamma = 0$ under various assumptions in e.g. [1, 11], where the parameter η_0 is also estimated in some special cases. If $\gamma \neq 0$, the lack of positivity will still follow by perturbation arguments.

Numerical simulations indicate that the same situation arises for system (3.1)-(3.2): we may have $u(x_0, t_0) < 0$ at some $x_0 \in \Omega$, $t_0 > 0$ for realistic values of $\eta > 0$, even though we expect $u \geq 0$ on physical arguments. The same situation arises in analogous problems from, for example, steady-state version of the non-cooperative system:

$$u_t - \Delta u = f(x) - \alpha v, \tag{3.4}$$

$$v_t - \Delta v = u, \tag{3.5}$$

with $u = v = 0$ on $\partial\Omega$, [17]. Indeed, putting $v = (-\Delta)^{-1}(u)$ reduces system (3.4)-(3.5) to a single equation of the general type (3.3).

It is our purpose to introduce an obstacle problem to replace equation (3.2), in such a way that (3.1)-(3.2) has solutions (ϕ, u) with $u \geq 0$ for any $\eta \geq 0$. Furthermore, solutions of (3.1)-(3.2) with $u \geq 0$, will also solve the new system so that the new model will extend (3.1)-(3.2), in the sense that physically meaningful solutions will be common to the two models. Some numerical methods for such extension was already studied in [2].

The chapter is structured as follows: we first consider the properties of equation (3.3) and consequently obtain results for the steady-state version of system (3.1)-(3.2). Next, under more restrictive conditions on σ and k, we consider the existence of solutions to (3.1)-(3.2) and also show the existence of an absorbing set. Finally, by still further assumptions, we show the existence of a compact, connected, maximal attractor. The presence of the fourth order nonlinear term will always be convenient in our analyses and, in some cases, essential to the proofs.

3.2 The new equation and results in steady-state

We consider, as a preliminary step, the obstacle problem

$$\begin{aligned}\int_\Omega k\nabla u\nabla(v-u)+\eta\int_\Omega\int_\Omega G(x,y)u(y)(v-u)(x)dydx \\ +\gamma\int_\Omega u^4(v-u)\geq\int_\Omega f(v-u),\end{aligned} \tag{3.6}$$

for $v,u \in V_D := H_0^1(\Omega\cup\partial\Omega_N)$. For the formal definition of $H_0^1(\Omega\cup\partial\Omega_N)$ we refer to [21, 24], and note that here we do allow the case: $\partial\Omega_N$ is empty. We assume that $0 < k_0 \leq k(x) \leq k_1$ for some constants k_0, k_1 and that $A(w) := \int_\Omega G(x,y)w(y)dy$ maps continuously $L^{p_0}(\Omega)$ to itself for some $p_0 > N$ such that $p_0 < \frac{2N}{N-2}$ if $N = 3$, and with $A(w) \geq 0$ if $w \geq 0$. The coefficient $k(x)$ is assumed smooth and, finally, we stipulate that the form associated with the left side of (3.6) is coercive over V_D which, in turn, is compactly embedded into L^p for $1 \leq p < \frac{2N}{N-2}$. The function f will always be assumed nonnegative, and of class L^2 unless otherwise specified.

We first note that if A is symmetric, (3.6) has a solution u obtained as the minimum of the functional J over the convex subset $K = \{w|w \geq 0,\ w \in V_D\}$, where:

$$J(w) = \frac{1}{2}\left[\int_\Omega k(x)|\nabla w|^2 + \eta \int_\Omega \int_\Omega G(x,y)w(x)w(y)dxdy\right]$$
$$+\gamma \int_\Omega \frac{w^5}{5}dx - \int_\Omega wf.$$

We recall that $N = 2$ or $N = 3$ so that by the Sobolev embedding theorem, the term $\int_\Omega w^5 dx$ is well defined if $w \in K$. We also observe that if A is positive definite, i.e. $(A(u-v),(u-v)) \geq 0$ for $u, v \in K$, then the solution u is unique. This will happen if, for example, $G(x,y)$ is a Green's function or if $G(x,y)$ is a positive constant.

It is convenient for what follows and to deal with cases where A is not symmetric, to introduce a family of related penalized problems given by:

$$-\nabla\left[k(x)\nabla u\right] + \left[\eta \int_\Omega G(x,y)u(y)dy + \gamma u^4\right] I_n(u) = f \qquad (3.7)$$

subject to $u \in V_D$, with $I_n(s)$ a C^∞ function such that: $0 \leq I_n(s) \leq 1$; $I_n(s) = 0$ if $s \leq 0$; $I_n(s) \to H(s)$ in $L^p(\Omega)$ for $1 \leq p < \infty$ where H denotes the Heaviside function.

We then observe:

Theorem 3.1. *(a) Equation (3.7) has a nonnegative solution u_n.*

(b) There exists a subsequence of u_n(also denoted by u_n) which converges to a solution of (3.6) strongly in L^2 and weakly in V_D.

(c) Equation (3.3) admits a positive solution for any $f \geq 0$ iff (3.6) admits a positive solution for any $f \geq 0$. Positive solutions of (3.3) also solve (3.6) and vice versa. By a positive solution u we mean that if $\Omega' \subset\subset \Omega \cup \partial\Omega_N$ then there exists $\epsilon > 0$ such that ess $\inf_{x\in\Omega'} u(x) \geq \epsilon$.

Proof. Choose a $p_0 > N$, as in the definition of $A(w)$.

(a) Put $Z(u) = [-\nabla(k\nabla)]^{-1}\left\{f - \left[\eta \int_\Omega G(x,y)u(y)dy + \gamma u^4\right] I_n(u)\right\}$. This is a continuous compact map $C^{\alpha_0} \to L^{p_0} \to C^{\alpha_0}$ for some $\alpha_0 > 0$, by e.g. [24]. If u solves $u = Z(u)$, then:

$$-\nabla[k(x)\nabla u] + \left[\eta \int_\Omega G(x,y)u(y)dy + \gamma u^4\right] I_n(u) = f.$$

Choosing u^- as a test function and recalling the definition of $I_n(u)$ and that $f \geq 0$, yield immediately $u^- \equiv 0$, i.e. $u \geq 0$. We also note that $-\nabla[k(x)\nabla u] \leq f$, and thus u is bounded in L^∞ for any η, γ, whence u is uniformly bounded in $C^{\alpha_0}(\bar{\Omega})$, and a homotopy argument gives:

$$deg_{LS}(I - Z, B_R, 0) = 1$$

for some $B_R \subset C^{\alpha_0}(\bar{\Omega})$, where deg_{LS} denotes the Leray- Schauder Degree and B_R the ball of radius R. The existence of a nonnegative solution follows.

(b) Let $n \to \infty$ in I_n, and let u_n denote the associated solution. Note that the arguments in (a) show that u_n is also bounded in V_D independently of n, η, γ. There is a subsequence of $\{u_n\}$ (also called u_n) and a function u such that $u_n \to u$ strongly in $L^2(\Omega)$ and weakly in V_D. Note that the u_n are also in C^{α_0} and bounded there and thus we assume $u_n \to u$ in C^{α_1} for some $\alpha_1 < \alpha_0$. Suppose at $x_0 \in \Omega$ we have $u(x_0) > 0$. Then $u_n(x_0) \to u(x_0)$ implies $I_n(u_n)(x_0) = 1$ for all large n. Note that $0 \leq I_n(u_n) \leq 1$, therefore, without loss of generality, $I_n(u_n) \to z$ weakly in L^2, and by the Theorem of Banach-Saks,

$$\frac{1}{n}\sum I_n(u_n) \to z \quad \textit{strongly in } L^2$$

and so $z \equiv 1$ where $u > 0$, while $0 \leq \frac{1}{n}\sum I_n(u_n) \leq 1$ and thus $0 \leq z(x) \leq 1$, for all $x \in \Omega$.

Passing to the limit, we have

$$-\nabla\left[k(x)\nabla u\right] + \left[\eta \int_\Omega G(x,y)u(y)dy + \gamma u^4\right] z = f,$$

$u \in V_D$, and (3.6) follows. We note that inside Ω, u is of class H^2 and so on the set $\{u|u = 0\} \cap \Omega$, we have $\nabla u = 0$ a.e. and thus $-\nabla[k(x)\nabla u] = 0$. We conclude $z = \frac{f}{\eta \int_\Omega G(x,y)u(y)dy}$ on this set.

(c) Follows immediately by the definitions.

□

Remark 3.1. Suppose $f = g_0 - \sum \frac{\partial}{\partial x_i}(g_i)$ in Theorem 3.1, with $\{g_j\}_{j=1}^n$ in $C^1(\Omega) \cap L^{p_0}(\Omega)$ or in suitable Campanato spaces – see Section 3.4, $g_0 \in L^2(\Omega)$. Then the result still holds.

Remark 3.2. We note that $0 \le z \le 1$, while $z = \frac{f}{\eta \int_\Omega G(x,y)u(y)dy}$ on the set $u = 0$. Thus if $G(x,y) \equiv 1$, $\gamma = 0$ and, consequently, $\int_\Omega G(x,y)u(y)dy = \int_\Omega u(y)dy = \frac{\int_\Omega gf}{(1+\eta \int_\Omega gz)}$ where $-\nabla[k\nabla g] = 1$ and g satisfies the given boundary conditions, then on the set $\left\{x | f(x) > \eta \frac{\int_\Omega gf}{(1+\eta\int_\Omega gz)}\right\}$ we have $u > 0$. In this case, it follows that: for any $\eta > 0$,

$$\{x | u(x) = 0\} \subset \left\{x | f(x) \le \eta \int_\Omega gf\right\}.$$

Remark 3.3. Let $w \in V_D$ solve $-\nabla[k\nabla w] \ge f$. Then we must have $0 \le u \le w$. Suppose $x_0 \in \Omega$, $\delta > 0$ and G smooth. Note that

$$\frac{f(x)}{\eta \int_\Omega G(x,y)u(y)} = \frac{f(x)}{\eta \int_{\Omega\cap\{u>0\}} G(x,y)u(y)} \ge \frac{f(x)}{\eta \int_{\Omega\cap\{u>0\}} G(x,y)w(y)}.$$

If $u \equiv 0$ in $\{y | |x_0 - y| < \delta\} \cap \Omega = B$, then $x \in B$ implies $\frac{f(x)}{\eta \int_{(\Omega\setminus B)} G(x,y)w(y)} \le 1$. Consequently, if $f(x) > \eta \int_{(\Omega\setminus B)} G(x,y)w(y)dy$ for some x in B, then $B \not\subset \{x | u(x) = 0\}$.

Remark 3.4. We observe that intuitively we expect the solutions of (3.6) to decrease as η increases. We can show that in one case this is true, namely if $G(x,y) = const > 0$. Indeed, suppose without loss of generality, that $G(x,y) \equiv 1$. Let u_1 (resp. u_2) solve (3.6) for η_1 (resp. η_2) with $0 < \eta_1 < \eta_2$. From (3.6) we obtain:

$$\begin{gathered}
\int_\Omega k(x)\nabla u_1 \nabla(u_2 - u_1) + \eta_1 \left(\int_\Omega u_1\right)\left(\int_\Omega (u_2 - u_1)\right) \\
+\gamma \int_\Omega u_1^4(u_2 - u_1) \ge \int_\Omega f(u_2 - u_1) \\
\int_\Omega k(x)\nabla u_2 \nabla(u_1 - u_2) + \eta_2 \left(\int_\Omega u_2\right)\left(\int_\Omega (u_1 - u_2)\right) \\
+\gamma \int_\Omega u_2^4(u_1 - u_2) \ge \int_\Omega f(u_1 - u_2)
\end{gathered}$$

Adding and putting $w = u_1 - u_2$, gives

$$\begin{gathered}
-\left[\int_\Omega k(x)|\nabla w|^2 + \gamma \int_\Omega (u_1^4 - u_2^4)(w)\right] \\
-\eta_2 \left(\int_\Omega w\right)^2 + [\eta_1 - \eta_2] \int_\Omega u_1 \int_\Omega (u_2 - u_1) \ge 0.
\end{gathered}$$

Since $u_1 \geq 0$ by construction, we conclude that $\int_\Omega u_2 \leq \int_\Omega u_1$. It follows that $\left[\eta_1 \int_\Omega u_1 - \eta_2 \int_\Omega u_2\right] \left[\int_\Omega (u_2 - u_1)\right] \geq 0$, i.e. $\eta_1 \int_\Omega u_1 \leq \eta_2 \int_\Omega u_2$. We can rewrite (3.6) in this case as

$$\int_\Omega k(x)\nabla u_1 \nabla(v - u_1) + \gamma \int_\Omega u_1^4(v - u_1) \geq \int_\Omega f_1(v - u_1),$$
$$\int_\Omega k(x)\nabla u_2 \nabla(v - u_2) + \gamma \int_\Omega u_2^4(v - u_2) \geq \int_\Omega f_2(v - u_2),$$

where $f - \eta_1 \int_\Omega u_1 = f_1 \geq f_2 = f - \eta_2 \int_\Omega u_2$. We observe that consequently $u_1 \geq u_2$(see [21]).

As an application of these results, we assume G smooth, $\sigma(s)$, $k(s) \to 0$ as $s \to \infty$ and replace (3.1)-(3.2) with the following extension in the steady-state:

$$-\nabla[\sigma(u)\nabla\phi] = 0, \tag{3.8}$$
$$\int_\Omega k(u)\nabla u \nabla(v - u) + \eta \int_\Omega \int_\Omega G(x, y)u(y)(v - u)(x)dydx \tag{3.9}$$
$$+\gamma \int_\Omega u^4(v - u) \geq - \int_\Omega \sigma(u)\phi\nabla\phi\nabla(v - u),$$

with $u \geq 0$, $v \geq 0$ and u, ϕ to be found subject to the earlier given mixed boundary conditions. An existence result for ϕ_0 of small variation (depending on $\int_0^\infty k/\sigma$), is now obtained by replacing (3.9) by the approximate equation:

$$-\nabla[k(u)\nabla u] + \left[\eta \int_\Omega G(x, y)u(y)dy + \gamma u^4\right] I_n(u) = \nabla[\sigma(u)\phi\nabla\phi], \tag{3.10}$$

and following the earlier arguments and those in [6, 24] which we sketch for convenience. Choose M by $\int_0^M \frac{k}{\sigma} = \frac{(\max\phi_0 - \min\phi_0)^2}{2}$, and truncate u in the coefficients $k(u)$, $\sigma(u)$ at M. We solve the problem with $k(\bar{u})$, $\sigma(\bar{u})$ in place of k, σ where $\bar{u} = \min(u, M)$. We observe that the solution u_n of (3.10) satisfies $0 \leq u_n \leq M$ even in the truncated case, and set up a map $C^{\alpha_0}(\bar{\Omega}) \to C^{\alpha_0}(\bar{\Omega})$, for some $\alpha_0 > 0$, along the lines given earlier and in [6, 24]. We conclude the existence of a solution pair (ϕ_n, u_n) to (3.8) and (3.10). We observe that (ϕ_n, u_n) are bounded in $C^{\alpha_0}(\bar{\Omega}) \times C^{\alpha_0}(\bar{\Omega})$, and passing to a limit – as in the earlier part of this section – we obtain the existence of a solution (ϕ, u) of class $C^\alpha(\bar{\Omega}) \times C^\alpha(\bar{\Omega})$, for some $\alpha > 0$, to (3.8)-(3.9) with associated (mixed) boundary conditions, and $u \geq 0$ for any value of $\eta > 0$.

$$\int_{Q_T} \sigma(u)\nabla\phi\nabla\psi = 0, \ \forall\psi \in L^2(0,T;V_D)$$

3.3 The time dependent case: existence and absorbing set

In this part of the chapter we begin considerations of the time dependent problem, show the existence of a capacity solution and of an absorbing set. For any $T > 0$, we denote by $Q_T = \Omega \times (0,T)$, the parabolic domain and by V_D' the dual space of V_D. Moreover we define the convex set

$$K = \{v \in L^2(0,T;\ V_D) \cap L^5(Q_T),\ v \geq 0 \ a.e.\ in\ Q_T\}.$$

Our time dependent obstacle problem extending (3.1)-(3.2) then becomes:

$$u \in K,\ u_t \in L^2(0,T;V_D') + L^{5/4}(Q_T)\ and\ \phi - \phi_0 \in L^2(0,T;V_D), \quad (3.11)$$

$$u \in K,\ u_t \in L^2(0,T;V_D') + L^{5/4}(Q_T)\ and\ \phi - \phi_0 \in L^2(0,T;V_D), \quad (3.11)$$

$$\int_\Omega \frac{du}{dt}(v-u) + \int_\Omega k(u)\nabla u\nabla(v-u) \quad (3.12)$$

$$+\eta\int_\Omega\int_\Omega G(x,y)u(y,t)(v-u)(x,t)dydx$$

$$+\gamma\int_\Omega u^4(v-u) \geq -\int_\Omega \sigma(u)\phi\nabla\phi\nabla(v-u),\ \forall v \in K,$$

$$\int_{Q_T} \sigma(u)\nabla\phi\nabla\psi = 0,\ \forall\psi \in L^2(0,T;V_D), \quad (3.13)$$

$$u(x,0) = u_0(x)\ in\ \Omega, \quad (3.14)$$

$$u|_{\partial\Omega_D} = 0,\ \frac{\partial u}{\partial n}|_{\partial\Omega_N} = 0, \quad (3.15)$$

$$\phi|_{\partial\Omega_D} = \phi_0(x),\ \frac{\partial\phi}{\partial n}|_{\partial\Omega_N} = 0, \quad (3.16)$$

where $u_0 \in L^2(\Omega)$ and $u_0 \geq 0$. The coefficients η, γ, $k(s)$, $\sigma(s)$ and $G(x,y)$ are the same as in the previous section but, for simplicity, we henceforth assume G is smooth. Furthermore we assume

$$0 < k_0 < k(s) < k_1,\ \ 0 < \sigma(s) \leq \sigma_1\ \ and\ \ \sigma(s) \to 0\ \ as\ \ s \to \infty \quad (3.17)$$

Since $\sigma(s) \to 0$, as $s \to \infty$, the system is degenerate and this leads to new mathematical difficulties. To deal with this problem, we adopt the notion of capacity solutions which were introduced by X. Xu in [26, 25, 27] to study the local thermistor problems.

We say a triplet (u, ϕ, g) is a capacity solution to (3.11)-(3.16) if it satisfies

$$u \in K, \quad u_t \in L^2(0,T;V_D') + L^{5/4}(Q_T), \tag{3.18}$$

$$\phi \in L^\infty(Q_T) \text{ and } g \in [L^2(Q_T)]^N,$$

$$\int_\Omega \frac{du}{dt}(v-u) + \int_\Omega k(u)\nabla u \nabla(v-u) \tag{3.19}$$

$$+\eta \int_\Omega \int_\Omega G(x,y)u(y,t)(v-u)(x,t)$$

$$+\gamma \int_\Omega u^4(v-u) \geq -\int_\Omega \phi g \nabla(v-u), \ \forall v \in K,$$

$$\int_{Q_T} g\nabla\psi = 0, \ \forall \psi \in L^2(0,T;V_D), \tag{3.20}$$

$$\text{for each } \rho \in C_0^1(R), \ \rho(u)\phi - \rho(0)\phi_0 \in L^2(0,T;V_D) \text{ and} \tag{3.21}$$

$$\rho(u)g = \sigma(u)[\nabla(\rho(u)\phi) - \phi\nabla(\rho(u))],$$

$$u(x,0) = u_0(x). \tag{3.22}$$

According to [26], we observe the following two remarks.

Remark 3.5. If u is bounded, then the capacity solution is equivalent to the solution of (3.11)-(3.16).

Remark 3.6. If there exists a capacity solution to the system, then $\nabla\phi$ is defined almost everywhere in Q_T, but may not belong to any space $[L^p(\Omega)]^N$ for $1 \leq p < \infty$. Moreover

$$g = \sigma(u)\nabla\phi, \ a.e. \ in \ Q_T. \tag{3.23}$$

Instead of the systems (3.11)-(3.16) and (3.18)-(3.22), we first consider their penalized version. The existence of a solution to this penalized system is obtained by the standard time discretization technique. Then by a series of boundedness estimates, a capacity solution of (3.18)-(3.22) is achieved as the limit of these solutions.

We thus define $\sigma_n(s) = \frac{1}{n} + \sigma(s)$. Then by (3.17) we have

$$0 < \frac{1}{n} \leq \sigma_n(s) \leq \sigma_1 + 1, \ \text{for all } s \in R. \tag{3.24}$$

Our new penalized system is given by the following

$$u_n \in K, \ \frac{du_n}{dt} \in L^2(0,T;V_D') + L^{5/4}(Q_T), \tag{3.25}$$

$$\begin{aligned} &\frac{du_n}{dt} - \nabla(k(u_n)\nabla u_n) \\ &\qquad + \left[\eta \int_\Omega G(x,y)u_n(y,t)dy + \gamma u_n^4\right] I_n(u_n) \\ &= \nabla[\sigma_n(u_n)\phi_n\nabla\phi_n], \ \ in \ \ L^2(0,T;V_D') + L^{5/4}(Q_T), \end{aligned} \tag{3.26}$$

$$\nabla[\sigma_n(u_n)\nabla\phi_n] = 0, \ \ in \ \ L^2(0,T;V_D'), \tag{3.27}$$

coupled with the same initial and boundary conditions.

Before we proceed with our main theorems we state a lemma which is a slight modification of the one given for the stationary problem. See in particular Theorem 3.1, Remark 3.1 and the arguments at the end of Section 2.

Lemma 3.1. *For each $0 \le H(x) \in L^2(\Omega)$, there exists a weak solution (u,ϕ) with $0 \le u \in V_D$, $\phi - \phi_0 \in V_D \cap L^\infty(\Omega)$ and $h > 0$ a constant to the system:*

$$\begin{aligned} &-\nabla(k(u)\nabla u) + \frac{1}{h}u + \left[\eta \int_\Omega G(x,y)u(y)dy + \gamma u^4\right] I_n(u) \\ &\quad = \nabla[\sigma_n(u)\phi\nabla\phi] + H(x), \end{aligned} \tag{3.28}$$

$$\nabla(\sigma_n(u)\nabla\phi) = 0, \tag{3.29}$$

$$u|_{\partial\Omega_D} = 0, \ \frac{\partial u}{\partial n}|_{\partial\Omega_N} = 0, \tag{3.30}$$

$$\phi|_{\partial\Omega_D} = \phi_0(x), \ \frac{\partial \phi}{\partial n}|_{\partial\Omega_N} = 0, \tag{3.31}$$

Moreover,

$$ess \sup_{x\in\Omega} |\phi(x)| \le \sup_{x\in\Gamma_D} |\phi_0(x)|, \tag{3.32}$$

$$\|\nabla\phi\| \le n(\sigma_1 + 1)\|\nabla\phi_0\|. \tag{3.33}$$

Next we obtain the following existence result by Rothe's Method. Hereafter for the sake of simplicity, we understand that a sequence is convergent if it has a convergent subsequence and we identify the subsequence with the sequence itself. Moreover C always stands for a positive constant which depends only on N and Ω, the coefficients η, γ and $G(x, y)$, the bounds on $k(s)$ and $\sigma(s)$ and the initial/boundary data except otherwise specified. It also may differ from one line to another.

Theorem 3.2. *For each n there exists a solution (u_n, ϕ_n) to the problem (3.25)-(3.27) which satisfies uniformly with respect to n that*

$$\text{ess } \sup_{Q_T} |\phi_n(x,t)| \leq C, \tag{3.34}$$

$$\text{ess } \sup_{0\leq t\leq T} \int_\Omega u_n^2 + \int_{Q_T} |\nabla u_n|^2 + \int_{Q_T} u_n^5 \leq C, \tag{3.35}$$

$$\int_{Q_T} \sigma_n(u_n)|\nabla \phi_n|^2 \leq C. \tag{3.36}$$

Moreover $\frac{du_n}{dt}$ is also uniformly bounded in $L^2(0,T;V_D') + L^{5/4}(Q_T)$ and thus in $L^{5/4}(0,T;V_D')$.

Proof. Let m be a positive integer. We decompose the interval $[0, T]$ evenly into m sub-intervals. The corresponding uniform partition is denoted by $\{t_j\}_{j=0}^m$ with $t_j = jh$, $h = T/m$. Discretizing equation (3.26) with respect to t and combining with (3.27) give the following system of $2m$ equations:

$$\begin{aligned} &-\nabla[k(u_n^j)\nabla u_n^j] + \frac{1}{h}(u_n^j - u_n^{j-1}) \\ &\quad + \left[\eta \int_\Omega G(x,y)u_n^j(y)dy + \gamma(u_n^j)^4\right] I_n(u_n^j) \\ &\quad = \nabla[\sigma_n(u_n^j)\phi_n^j \nabla\phi_n^j], \ j = 1, \ \cdots, \ m, \end{aligned} \tag{3.37}$$

$$\nabla[\sigma_n(u_n^j)\nabla\phi_n^j] = 0, \ j = 1, \ \cdots, \ m. \tag{3.38}$$

Since u_0 is in $L^2(\Omega)$, thanks to Lemma 3.1, we can solve the above system successively. First we derive some a priori estimates of the solutions. To do so, we choose u_n^j as a test function in (3.37) and obtain

$$\begin{aligned} &\int_\Omega k(u_n^j)|\nabla u_n^j|^2 + \frac{1}{h}\int_\Omega (u_n^j - u_n^{j-1})u_n^j \\ &+ \int_\Omega \left[\eta \int_\Omega G(x,y)u_n^j(y)dy + \gamma(u_n^j)^4\right] I_n(u_n^j)u_n^j \\ &= -\int_\Omega \sigma_n(u_n^j)\phi_n^j \nabla\phi_n^j \nabla u_n^j, \ j = 1, \ \cdots, \ m. \end{aligned} \tag{3.39}$$

Simple calculations show that for each $j = 1, \cdots, m$, the following property then holds:

$$\frac{1}{h}\left[\|u_n^j\|^2 - \|u_n^{j-1}\|^2 + \|u_n^j - u_n^{j-1}\|^2\right] \tag{3.40}$$
$$+k_0\|\nabla u_n^j\|^2 + 2\int_\Omega I_n(u_n^j)(u_n^j)^5 \le C.$$

We next sum inequality (3.40) from 1 to j and drop the other positive terms to obtain

$$\frac{1}{h}(\|u_n^j\|^2 - \|u_0\|^2) \le jC. \tag{3.41}$$

Thus $\|u_n^j\|^2 \le C \;\; for \;\; j = 1, \;\; \cdots, \;\; m$. Summing inequality (3.40) from 1 to m, we find

$$\frac{1}{h}\left[\|u_n^m\|^2 - \|u_0\|^2 + \sum_{j=1}^{m}\|u_n^j - u_n^{j-1}\|^2\right] \tag{3.42}$$
$$+k_0\sum_{j=1}^{m}\|\nabla u_n^j\|^2 + 2\sum_{j=1}^{m}\int_\Omega I_n(u_n^j)(u_n^j)^5 \le mC,$$

which gives:

$$h\sum_{j=1}^{m}\|\nabla u_n^j\|^2 \le C, \tag{3.43}$$

$$h\sum_{j=1}^{m}\int_\Omega (u_n^j)^5 \le C, \tag{3.44}$$

$$\sum_{j=1}^{m}\|u_n^j - u_n^{j-1}\|^2 \le C, \tag{3.45}$$

by the definition of I_n.

Now we define the Rothe's functions $u_n^h(x,t)$, $\phi_n^h(x,t)$ and $w_n^h(x,t)$ by the following:

$$u_n^h(x,t) = u_n^j \;\; for \;\; t \in ((j-1)h, jh], \tag{3.46}$$
$$j = 1, \;\cdots, \; m \;\; and \;\; u_n^h(x,0) = u_0,$$
$$\phi_n^h(x,t) = \phi_n^j \;\; for \;\; t \in ((j-1)h, jh], \tag{3.47}$$
$$j = 1, \;\cdots, \; m,$$
$$w_n^h(x,t) = u_n^{j-1} + \frac{(t - t_{j-1})}{h}(u_n^j - u_n^{j-1}) \;\; for \;\; t \in [(j-1)h, jh], \tag{3.48}$$
$$j = 1, \;\cdots, \; m.$$

Thus we may write the system (3.37)-(3.38) as

$$\frac{dw_n^h}{dt} - \nabla[k(u_n^h)\nabla u_n^h] + \left[\eta \int_\Omega G(x,y)u_n^h(y,t)dy + \gamma(u_n^h)^4\right] I_n(u_n^h) \tag{3.49}$$
$$= \nabla[\sigma_n(u_n^h)\phi_n^h \nabla \phi_n^h],$$
$$\nabla[\sigma_n(u_n^h)\nabla\phi_n^h] = 0. \tag{3.50}$$

The boundedness of u_n^h and w_n^h in $L^2(0,T;V_D) \cap L^\infty(0,T;L^2(\Omega)) \cap L^5(Q_T)$ is then a direct application of the earlier estimates, while to show the boundedness of $\frac{dw_n^h}{dt}$ in $L^2(0,T;V_D') + L^{5/4}(Q_T)$, we observe that

$$\|I_n(u_n^h)(u_n^h)^4\|_{L^{5/4}(Q_T)} \leq \|u_n^h\|^4_{L^5(Q_T)} \leq C, \tag{3.51}$$

and all other terms in (3.49) (except $\frac{dw_n^h}{dt}$) are uniformly bounded in $L^2(0,T;V_D')$. Thus $\frac{dw_n^h}{dt}$ is in a bounded set of $L^2(0,T;V_D') + L^{5/4}(Q_T)$. By the definitions of u_n^h and w_n^h,

$$w_n^h(x,t) - u_n^h(x,t) = \frac{t - jh}{h}(u_n^j - u_n^{j-1}) \;\; for \;\; (j-1)h < t \leq jh \tag{3.52}$$

Thus,

$$\|w_n^h - u_n^h\|^2_{L^2(Q_T)} = \sum_{j=1}^m \int_{(j-1)h}^{jh} \|w_n^h - u_n^h\|^2(t)dt$$
$$= \frac{h}{3}\sum_{j=1}^m \|u_n^j - u_n^{j-1}\|^2. \tag{3.53}$$

Therefore by (3.45) and (3.53), $w_n^h - u_n^h \to 0$ *strongly in* $L^2(Q_T)$ *as* $h \to 0$.

We conclude that there exists a common function $u_n(x,t)$ such that as $h \to 0$,

$$w_n^h \to u_n, \ \ u_n^h \to u_n \ \ weakly \ \ in \ \ L^2(0,T;V_D), \tag{3.54}$$
$$weak - star \ \ in \ \ L^\infty(0,T;L^2(\Omega)),$$
$$\frac{dw_n^h}{dt} \to \frac{du_n}{dt} \ \ weakly \ \ in \ \ L^2(0,T;V_D') + L^{5/4}(Q_T). \tag{3.55}$$

Because $\frac{dw_n^h}{dt}$ is uniformly bounded in $L^2(0,T;V_D')+L^{5/4}(Q_T)$, it is also bounded in $L^{5/4}(0,T;V_D')$. So by a compactness result (see [19], page 271), we also have

$$w_n^h \to u_n, \ \ u_n^h \to u_n \ \ strongly \ \ in \ \ L^2(Q_T). \tag{3.56}$$

Due to (3.32) and (3.33), ϕ_n^h is uniformly bounded in $L^\infty(0,T;V_D)$ and $L^\infty(Q_T)$. Therefore

$$\phi_n^h \to \phi_n \ \ weak - star \ \ in \ \ L^\infty(0,T;V_D) \ \ and \ \ in \ \ L^\infty(Q_T), \tag{3.57}$$

and as a consequence, $\phi_n^h \to \phi_n$ strongly in $L^2(Q_T)$ (see Lemma 4.10 in [18]).

Since $I_n(s)$, $\sigma_n(s)$ and $k(s)$ are bounded continuous functions, we have

$$I_n(u_n^h) \to I_n(u_n), \sigma_n(u_n^h) \to \sigma_n(u_n), k(u_n^h) \to k(u_n) \tag{3.58}$$
$$strongly \ \ in \ \ L^p(Q_T) \ \ for \ \ any \ \ p \geq 1.$$

Now we pass to the limit in (3.50) and obtain (3.27). To obtain (3.26), we note that $I_n(u_n^h)(u_n^h)^4$ converges to $I_n(u_n)(u_n)^4$ weakly in $L^{5/4}(Q_T)$. Indeed, without loss of generality we may assume $u_n^h \to u_n$ pointwise in Q_T. Since $(u_n^h)^4 v$ is uniformly bounded in $L^1(Q_T)$, and $(u_n^h)^4 v$ converges to $(u_n)^4 v$ pointwise and thus almost uniformly. In view of (3.58) we obtain the desired result.

We now pass to the limit in (3.49). We note that the dual space of $L^2(0,T;V_D') + L^{5/4}(Q_T)$ is $L^2(0,T;V_D) \cap L^5(Q_T)$. Thus if we take the duality product in (3.49) with $v \in L^2(0,T;V_D) \cap L^5(Q_T)$, then by passing to the limit we obtain

$$\begin{aligned} &< \frac{du_n}{dt}, v > - < \nabla(k(u_n)\nabla u_n), v > \\ &\quad + < \left[\eta \int_\Omega G(x,y)u_n(y,t)dy + \gamma(u_n)^4\right] I_n(u_n), v > \\ &=< \nabla[\sigma_n(u_n)\phi_n \nabla\phi_n], v > . \end{aligned} \tag{3.59}$$

Thus equation (3.26) is satisfied. It remains to show that $u_n(x,0) = u_0$. In fact, from (3.54) and (3.55),

$$(w_n^h - u_n, \psi) \to 0, \quad for \ \psi \in V_D' \ and \ for \ every \ t \in [0,T]. \tag{3.60}$$

Since $w_n^h(x,0) = u_0$, we have $u_n(x,0) = u_0$.

Equation (3.34) follows from (3.32). Estimate (3.35) and the boundedness of $\frac{du_n}{dt}$ follow from the given estimates and (3.54)-(3.55). We have (3.36) by using $\phi_n - \phi_0$ as a test function in (3.27). This completes the proof of Theorem 3.2. □

It follows from Theorem 3.2 that there are $u \in L^2(0,T;V_D) \cap L^\infty(0,T;L^2(\Omega)) \cap L^5(Q_T)$, $\phi \in L^\infty(Q_T)$, $g \in [L^2(Q_T)]^N$, $z \in L^\infty(Q_T)$ such that

$$u_n \to u \ weakly \ in \ L^2(0,T;V_D) \ and \ strongly \ in \ L^2(Q_T), \tag{3.61}$$

$$\frac{du_n}{dt} \to \frac{du}{dt} \ weakly \ in \ L^2(0,T;V_D') + L^{5/4}(Q_T), \tag{3.62}$$

$$\phi_n \to \phi \ weak-star \ in \ L^\infty(Q_T), \tag{3.63}$$

$$\sigma_n(u_n)\nabla\phi_n \to g \ weakly \ in \ [L^2(Q_T)]^N, \tag{3.64}$$

$$I_n(u_n) \to z \ weak-star \ in \ L^\infty(Q_T). \tag{3.65}$$

We recall that in [26] (see Claim 1 and Claim 3) it is shown that for each $\rho \in C_0^1(R)$, $\rho(u_n)\phi_n \to \rho(u)\phi$ weakly in $L^2(0,T;V_D)$, and that

$$\lim_{n\to\infty} \int_{Q_T} |\phi_n - \phi| = 0. \tag{3.66}$$

From (3.63) and (3.66) we then conclude that

$$\sigma_n(u_n)\phi_n \to \sigma(u)\phi \ strongly \ in \ L^p(Q_T) \ for \ each \ p \geq 1. \tag{3.67}$$

Theorem 3.3. *(3.18)-(3.22) are satisfied for (u, ϕ, g) with (3.19) replaced by*

$$\begin{aligned} u_t - \nabla[k(u)\nabla u] + z\left(\eta \int_\Omega G(x,y)u(y,t)dy + \gamma u^4\right) \\ = \nabla(\phi g) \quad in \ L^2(0,T;V_D') + L^{5/4}(Q_T). \end{aligned} \tag{3.68}$$

Proof. Obviously (3.18), (3.20) and (3.22) hold. For each $\rho \in C_0^1(R)$,

$$\rho(u_n)\sigma_n(u_n)\nabla\phi_n = \sigma_n(u_n)(\nabla(\phi_n\rho(u_n)) - \phi_n\nabla\rho(u_n)). \tag{3.69}$$

Equation (3.21) follows by letting $n \to \infty$ in (3.69). For each $v \in L^5(Q_T)$, we have

$$\int_{Q_T} u_n^4 I_n(u_n)v \to \int_{Q_T} u^4 zv. \tag{3.70}$$

To obtain (3.68), we take the duality product with $v \in L^2(0,T;V_D) \cap L^5(Q_T)$ in equation (3.26) and get

$$\begin{aligned} &< \frac{du_n}{dt}, v > + \int_{Q_T} k(u_n)\nabla u_n \nabla v \\ &+ \int_{Q_T} \left(\eta \int_\Omega G(x,y)u_n(y,t)dy + \gamma u_n^4\right) I_n(u_n)v \\ &= - \int_{Q_T} \sigma_n(u_n)\phi_n\nabla\phi_n\nabla v. \end{aligned} \tag{3.71}$$

We can pass to the limit in (3.71) to obtain

$$\begin{aligned} &< \frac{du}{dt}, v > + \int_{Q_T} k(u)\nabla u \nabla v \\ &+ \int_{Q_T} \left(\eta \int_\Omega G(x,y)u(y,t)dy + \gamma u^4\right) zv \\ &= - \int_{Q_T} g\nabla v, \end{aligned} \tag{3.72}$$

and (3.68) follows. □

Theorem 3.4. *There exists a capacity solution to (3.11)-(3.16) which satisfies (3.18)-(3.22).*

Proof. Actually we only need to show the solution (u, ϕ, g) in Theorem 3.3 also satisfies (3.19). Similarly to the steady state problem, z also has the following property

$$0 \le z \le 1, \quad and \quad z = 1 \quad if \quad u > 0. \tag{3.73}$$

Thus for all $v \ge 0$, $v \in L^2(0,T;V_D) \cap L^5(Q_T)$,

$$z(v-u) \le v - u. \tag{3.74}$$

Hence, from (3.68) we have

$$\int_\Omega u_t(v-u)+\int_\Omega k(u)\nabla u\nabla(v-u)+\eta\int_\Omega\int_\Omega G(x,y)u(y)(v-u)(x,t)dydx$$
$$+\gamma\int_\Omega u^4(v-u)\geq-\int_\Omega \phi g\nabla(v-u).$$

This means that (3.19) holds. So (u,ϕ,g) is a capacity solution to (3.11)-(3.16). □

The remaining part of this section is devoted to the existence of an absorbing set for the obstacle problem. We first give a generalized Gronwall type inequality. Related results are given in [20], but for completeness and the reader's convenience, we give it here.

Lemma 3.2. *Assume α, γ, δ, p are positive constants with $p>1$. If $y(t)$, a positive function, satisfies*

$$\frac{dy}{dt}+\alpha y+\gamma y^p\leq\delta,\quad t\in(0,\infty), \tag{3.75}$$

then $y(t)$ satisfies the uniform estimate

$$y(t)\leq\mu+\left(\frac{\alpha}{\gamma}\right)^{\frac{1}{p-1}}\frac{e^{-\alpha t}}{[1-e^{-\alpha(p-1)t}]^{\frac{1}{p-1}}},\quad \textit{for all } t>0, \tag{3.76}$$

where μ is the unique solution of $\alpha y+\gamma y^p=\delta$.

Proof. First we observe that if $y(0)\leq\mu$, then $y(t)\leq\mu$ for all $t\in[0,\infty)$ and if $y(0)>\mu$, there exists a $0<t_0\leq\infty$ such that $y(t)\leq\mu$ for $t\geq t_0$ and $y(t)>\mu$ for $t\in[0,t_0)$. Thus we only need consider the case that $y(0)>\mu$ and $t\in[0,t_0)$.

Let $z(t)=y(t)-\mu$ for $t\in[0,t_0)$. Clearly the inequality $z^p+\mu^p\leq(z+\mu)^p=y^p$ implies

$$\frac{dz}{dt}+\alpha z+\gamma z^p\leq\frac{dy}{dt}+\alpha y+\gamma y^p-\alpha\mu-\gamma\mu^p \tag{3.77}$$
$$=\frac{dy}{dt}+\alpha y+\gamma y^p-\delta\leq 0.$$

Since $z(t)$ is positive by the assumption, we may define

$$v(t) = \frac{z^{1-p}}{1-p}. \tag{3.78}$$

Then a simple computation yields that

$$\frac{dv}{dt} = z^{-p}\frac{dz}{dt} \le \alpha(p-1)v - \gamma. \tag{3.79}$$

It follows that by integrating above formula from 0 to t,

$$v(t) \le \frac{\gamma}{\alpha(p-1)}[1 - \exp\{\alpha(p-1)t\}] + v(0)\exp\{\alpha(p-1)t\}. \tag{3.80}$$

By the definition of $v(t)$ we have

$$z(t) \le \left\{ \frac{\exp\{-\alpha(p-1)t\}}{z_0^{1-p} + \frac{\gamma}{\alpha}[1 - \exp\{-\alpha(p-1)t\}]} \right\}^{\frac{1}{p-1}} \quad \textit{for all } t \in [0, t_0). \tag{3.81}$$

In view of $z_0 > 0$,

$$z(t) \le \left(\frac{\alpha}{\gamma}\right)^{\frac{1}{p-1}} \frac{e^{-\alpha t}}{[1 - e^{-\alpha(p-1)t}]^{\frac{1}{p-1}}}. \tag{3.82}$$

This completes the proof by the definition of $z(t)$. □

Denote

$$\|\phi_0(x)\|_\infty := ess\,\sup_{x\in\partial\Omega_D} |\phi_0(x)|.$$

By the weak maximal principle, we actually have

$$ess\,\sup_{x\in\Omega}|\phi_n(x,t)| \le \|\phi_0\|_\infty \quad a.e.\ 0 < t < \infty. \tag{3.83}$$

Furthermore, in view of Remark 3.6 and (3.20) we have

$$\int_\Omega g\nabla(\phi - \phi_0) = \int_\Omega \sigma(u)\nabla\phi\nabla(\phi - \phi_0) = 0 \tag{3.84}$$

by using $\phi - \phi_0$ as a test function. Thus,

$$\begin{aligned} \int_\Omega \sigma(u)|\nabla\phi|^2 &= \int_\Omega \sigma(u)\nabla\phi\nabla\phi_0 \\ &\le \left(\int_\Omega \sigma(u)|\nabla\phi|^2\right)^{1/2}\left(\int_\Omega \sigma(u)|\nabla\phi_0|^2\right)^{1/2}, \end{aligned}$$

or

$$\left(\int_\Omega \sigma(u)|\nabla\phi|^2\right)^{1/2} \le \left(\int_\Omega \sigma(u)|\nabla\phi_0|^2\right)^{1/2}.$$

Write $(\int_\Omega |\nabla\phi_0|^2)^{1/2} = \|\nabla\phi_0\|$. Thanks to (3.17) and (3.23),

$$\int_\Omega |g|^2 \le \sigma_1^2\|\nabla\phi_0\|^2 \quad \textit{for all} \quad 0 < t < \infty. \tag{3.85}$$

Theorem 3.5. *There exists an absorbing set to the obstacle problem which is a ball $\mathcal{B}$ in $K_0 = \{u \in L^2(\Omega)|\ u \ge 0\}$ centered at 0 with radius ρ_0'. This ball absorbs the elements of K_0 uniformly, i.e., there exists a fixed $t_0(\rho_0')$ such that for any $u_0 \in K_0$, $S(t)u_0$ will enter into $\mathcal{B}$ after time $t_0(\rho_0')$. Here ρ_0' and $t_0(\rho_0')$ are determined in the proof.*

Proof. One can easily verify that the solution (u, ϕ, g) satisfies

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2 + \int_\Omega k(u)|\nabla u|^2 + \gamma\int_\Omega u^5 \le -\int_\Omega \phi g \nabla u. \tag{3.86}$$

Thanks to (3.83), (3.85) and the Schwarz Inequality,

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2 + \int_\Omega k(u)|\nabla u|^2 + \gamma\int_\Omega u^5 \le \|\phi_0\|_\infty \left(\int_\Omega |g|^2\right)^{1/2}\left(\int_\Omega |\nabla u|^2\right)^{1/2}. \tag{3.87}$$

By the inequality $ab \le \frac{1}{2\epsilon}a^2 + \frac{\epsilon}{2}b^2$ and the Hölder Inequality, (3.87) can be written as

$$\frac{d}{dt}\|u\|^2 + k_0\|\nabla u\|^2 + \frac{2\gamma}{|\Omega|^{\frac{3}{2}}}\|u\|^5 \le \frac{\sigma_1^2}{k_0}\|\phi_0\|_\infty^2\ \|\nabla\phi_0\|^2. \tag{3.88}$$

Finally, by the Poincaré Inequality $\|\nabla u\|^2 \ge P_0\|u\|^2$ we obtain

$$\frac{d}{dt}\|u\|^2 + b_1\|u\|^2 + b_2\|u\|^5 \le b_3, \tag{3.89}$$

where $b_1 = k_0 P_0$, $b_2 = \frac{2\gamma}{|\Omega|^{\frac{3}{2}}}$ and $b_3 = \frac{\sigma_1^2}{k_0}\|\phi_0\|_\infty^2\ \|\nabla\phi_0\|^2$. Then by the above Gronwall type inequality with $\alpha = b_1$, $\gamma = b_2$, $\delta = b_3$ and $p = 2.5$ we have

$$\|u\|^2 \le \rho_0^2 + \left(\frac{b_1}{b_2}\right)^{\frac{2}{3}} \frac{e^{-b_1 t}}{\left[1 - e^{-\frac{3}{2}b_1 t}\right]^{\frac{2}{3}}}, \quad \textit{for all} \quad t > 0, \tag{3.90}$$

with ρ_0 the unique solution of $b_1\rho + b_2\rho^{5/2} = b_3$. Now we can choose any $\rho_0' > \rho_0$ to obtain our absorbing ball $\mathcal{B}$. Then, from (3.90) we easily obtain

$$t_0(\rho_0') = \frac{2}{3b_1}\ln\left(1 + \frac{b_1}{b_2\left(\rho_0'^2 - \rho_0^2\right)^{3/2}}\right).$$

□

Since the time $t_0(\rho_0')$ is independent of the initial value u_0, the set $\mathcal{B}$ absorbs the elements of K_0 uniformly. This property is due to the 4-th order nonlinear term which leads to bounds on u in (3.90) which don't depend on the initial value u_0.

3.4 The time dependent case: global attractor

In this final part of the chapter we show the existence of a global attractor. In addition to the previous assumptions, we further assume $k(x) \equiv 1$, $\Omega \subset R^3$, u satisfies a pure Dirichlet boundary condition, i.e. $u|_{\partial\Omega} = 0$, but keep the mixed conditions on ϕ, and

$$0 < \sigma_1 < \sigma(s) < \sigma_2 < \infty, \quad \textit{for all } s \geq 0, \tag{3.91}$$

$$\textit{there exists a positive constant } L \textit{ such that} \tag{3.92}$$
$$|\sigma(s) - \sigma(s')| \leq L|s - s'| \textit{ for all } s,\ s' \geq 0.$$

Under these assumptions the capacity solutions are also weak solutions since the gradient of the potential ϕ is L^2−integrable.

Let us denote the solution operators of (3.11)-(3.16) and (3.25)-(3.27) by $S(t)$ and $S_n(t)$ respectively. The main difficulty here is that $S(t)$ ($S_n(t)$) may not define a semigroup in $L^2(\Omega)$ since the weak solutions could not be unique. To circumvent this difficulty, we first show that if the initial data belong to $C^\alpha(\bar{\Omega})$, the solution is unique. Thus we can follow the procedure in [20] to prove the existence of global attractors. Systematically, we replace $L^2(\Omega)$ by $C^\alpha(\Omega)$ for some $\alpha > 0$ and show that $S(t)$ ($S_n(t)$) does define a semigroup in this Banach space. By making use of the term describing radiation heat losses, we derive a local a priori estimate for the solutions of (3.11)-(3.16) and (3.25)-(3.27) which says that any solution will be of class $C^{\alpha_0}(\bar{\Omega})$ for some $\alpha_0 > \alpha$ after a certain time independent of the initial data. Thus there exists a uniform absorbing set and the semigroups are uniformly compact for t large. The existence of global attractors follows immediately.

It will be convenient to recall some notation. For $0 \leq t_0 < t_1$, we denote $\Omega \times (t_0, t_1]$ by Q_{t_0,t_1}. For simplicity, if $t_0 = 0$, we write it as Q_{t_1}. A point $(x,t) \in Q_{t_0,t_1}$ is denoted by z. Let $B_r(x_0)$ be the ball centered at x_0 with radius r and $Q_r(z_0)$ be the cylinder $B_r(x_0) \times (t_0 - r^2, t_0]$. Then we define

$$\Omega[x_0, r] = B_r(x_0) \cap \Omega, \quad Q[z_0, r] = Q_r(z_0) \cap Q_{t_0,t_1}.$$

Moreover for $\mu \geq 0$, $\mathcal{L}^{2,\mu}(\Omega)$ and $\mathcal{L}^{2,\mu}(Q_{t_0,t_1})$ denote the Campanato spaces on Ω and Q_{t_0,t_1} associated with the standard norms $\|.\|_{2,\mu,\Omega}$ and $\|.\|_{2,\mu,Q_{t_0,t_1}}$ respectively. We refer to interested readers to [28], [21] and [14] for details on these spaces and norms. The following proposition can be found in [21], Theorem 1.17.

Lemma 3.3. *(i) If $0 \leq \mu < N$, the mapping*

$$u \longrightarrow \left(\sup_{x_0 \in \bar{\Omega}, r>0} r^{-\mu} \int_{\Omega[x_0,r]} u^2 dx \right)^{1/2}$$

defines a norm on $\mathcal{L}^{2,\mu}(\Omega)$ which is equivalent to $\|\cdot\|_{2,\mu,\Omega}$.

(ii) $\mathcal{L}^{2,N+2\mu}(\Omega)$ is isomorphic to $C^{\mu}(\bar{\Omega})$ for $\mu \in (0,1)$.

Similar results for $\mathcal{L}^{2,\mu}(Q_{t_0,t_1})$, established in [8], are summarized in the following lemma.

Lemma 3.4. *(i) If $0 \leq \mu < N+2$, the mapping*

$$u \longrightarrow \left(\sup_{z_0 \in \bar{Q}_{t_0,t_1}, r>0} r^{-\mu} \int_{Q[z_0,r]} u^2 dz \right)^{1/2}$$

defines a norm on $\mathcal{L}^{2,\mu}(Q_{t_0,t_1})$ which is equivalent to $\|\cdot\|_{2,\mu,Q_{t_0,t_1}}$.

(ii) $\mathcal{L}^{2,N+2+2\mu}(Q_{t_0,t_1})$ is isomorphic to $C^{\mu,\mu/2}(\bar{Q}_{t_0,t_1})$ for $\mu \in (0,1)$.

We choose and fix a p, $3 < p < 4$ and recall the following lemma from [6].

Lemma 3.5. *Let $a_{ij} \in C^{\alpha}(\bar{\Omega})$ and satisfy uniform elliptic conditions. Consider the operator $\mathcal{L}$ given by the formal expression $-\sum D_i(a_{ij}D_j v)$ and boundary conditions:*

$$(\sum a_{ij}\frac{\partial v}{\partial x_i}) \cdot \vec{n} + \beta v = \beta v_0 \quad on \quad \partial\Omega_N, \tag{3.93}$$

$$v = v_0 \qquad on \quad \partial\Omega_D. \tag{3.94}$$

Then $\mathcal{L}$ maps $v_0 + H^{1,p}(\Omega \cup \partial\Omega_N)$ onto $H^{-1,p}(\Omega)$, the dual space of $H^{1,p}(\Omega \cup \partial\Omega_N)$. Furthermore, if $\{f_i\}_{i=0}^{3}$ in L^p denotes a representation of a member of $H^{-1,p}$ and $\mathcal{L}v = f_0 + \sum D_i(f_i)$, then

$$\|v\|_{H^{1,p}} \leq C[\sum_{i=0}^{3} \|f_i\|_{L^p(\Omega)} + \|v_0\|_{C^1}], \tag{3.95}$$

where C is independent of v.

The Gagliardo-Nirenberg interpolation inequality, where $G_0 > 0$, constant,

$$\|w\|^2_{L^{2p/(p-2)}(\Omega)} \leq G_0\|w\|^{2-2n/p}\|\nabla w\|^{2n/p} \tag{3.96}$$

yields:

Lemma 3.6. *For all $w \in V_D$, the estimate:*

$$\|w\|^2_{L^{2p/(p-2)}(\Omega)} \leq \epsilon\|\nabla w\|^2 + C_\epsilon\|w\|^2 \tag{3.97}$$

holds, where

$$C_\epsilon = \frac{G_0^{p/(p-n)}}{\frac{p}{p-n}(\frac{\epsilon p}{n})^{n/(p-n)}}. \tag{3.98}$$

We say a solution (u, ϕ) (respectively (u_n, ϕ_n)) is a $C^{\alpha,\alpha/2}(\bar{Q}_T)$ solution of (3.11)-(3.16) (respectively (3.25)-(3.27)) iff it is a weak solution and is in $C^{\alpha,\alpha/2}(\bar{Q}_T)$.

Let δ_0 denote the Hölder exponent as stated in the De Giorgi - Nash theorem, see [21], Theorem 2.14, page 115 (also see [28]). In what follows, all α, α_i's are in $(0, \delta_0)$ and may differ from one step to the next. Now we are ready to claim the following theorem (which could also be used to show existence of $C^{\alpha,\alpha/2}(\bar{Q}_T)$ solutions).

Theorem 3.6. *Assume that $u_0(x) \in C^{\alpha}(\bar{\Omega})$, $u_0 = 0$ on $\partial\Omega$. Then any weak solution (u_n, ϕ_n) of the system (3.25)-(3.27) is in $C^{\alpha,\alpha/2}(\bar{Q}_T)$. Consequently there is a $C^{\alpha,\alpha/2}(\bar{Q}_T)$ solution to the system (3.11)-(3.16).*

Proof. Let (u_n, ϕ_n) be a weak solution of (3.25) - (3.27). Consider the following equation:

$$w_t - \Delta w = \nabla[\sigma(u_n)\phi_n \nabla \phi_n]. \tag{3.99}$$

Here w satisfies the same initial and boundary conditions as u_n. By the comparison principles we have $0 \le u_n \le w$. On the other hand, it follows from the results of [23] that for all $0 \le \mu < N - 2 + 2\alpha$,

$$\|\nabla \phi_n\|_{2,\mu,\Omega} \le C. \tag{3.100}$$

By Lemma 3.3, we have

$$r^{-\mu} \int_{t_0 - r^2}^{t_0} \int_{B_r(x_0)} |\nabla \phi_n|^2 dx < Cr^2. \tag{3.101}$$

Thus,

$$\sup_{z_0 \in Q_T, r > 0} r^{-(\mu+2)} \int_{Q_r(z_0)} |\nabla \phi_n|^2 dz < C. \tag{3.102}$$

By using Lemma 3.4 we obtain that for all $0 \le \mu < N + 2\alpha$,

$$\|\nabla \phi_n\|_{2,\mu,Q_T} \le C. \tag{3.103}$$

Since $\sigma(u_n)\phi_n \in L^{\infty}(Q_T)$, we have $\sigma(u_n)\phi_n \nabla \phi_n \in \mathcal{L}^{2,\mu}(Q_T)$. Thus by the results of [28], it follows from equation (3.99) that for all $0 \le \mu \le N + 2\alpha$,

$$\|\nabla w\|_{2,\mu,Q_T} \le C. \tag{3.104}$$

Consequently,

$$\|w\|_{L^{\infty}(Q_T)} \le C. \tag{3.105}$$

In view of $0 \le u_n \le w$, we have

$$\|u_n\|_{L^{\infty}(Q_T)} \le C. \tag{3.106}$$

We now decompose u_n into two parts as $u_n = v + (u_n - v)$. Here v is the solution of the following simple equation:

$$v_t - \Delta v = 0, \tag{3.107}$$

$$v(x,0) = u_0(x). \tag{3.108}$$

Since $u_0 \in C^{\alpha}(\bar{\Omega})$ and $u_0 = 0$ on $\partial\Omega$, the above equation has a unique $C^{\alpha,\alpha/2}(\bar{Q}_T)$ solution by a classic result in [15]. Now we apply the results of [28] to $u_n - v$ and obtain for all $0 \le \mu \le N + 2\alpha$ that

$$\begin{aligned} \|\nabla(u_n - v)\|_{2,\mu,Q_T} &\le C\{\|\sigma(u_n)\phi_n\nabla\phi_n\|_{2,\mu,Q_T} \\ &+ \left\| I_n(u_n)\left[\eta\int_\Omega G(x,y)u_n(y,t)dy + ku_n^4\right]\right\|_{2,(\mu-2)^+,Q_T} \\ &+ \|u_n - v\|_{L^2(0,T;H^1(\Omega))}\}. \end{aligned} \tag{3.109}$$

By the imbedding theorems and inequality (3.106), we have

$$\begin{aligned} &\left\| I_n(u_n)\left[\eta\int_\Omega G(x,y)u_n(y,t)dy + ku_n^4\right]\right\|_{2,(\mu-2)^+,Q_T} \\ &\le C\left\| I_n(u_n)\left[\eta\int_\Omega G(x,y)u_n(y,t)dy + ku_n^4\right]\right\|_{L^\infty(Q_T)} \le C. \end{aligned} \tag{3.110}$$

Thus, for $0 \le \mu \le N + 2\alpha$,

$$\|\nabla(u_n - v)\|_{2,\mu,Q_T} \le C. \tag{3.111}$$

Therefore for each $0 < \mu \le N + 2 + 2\alpha$, $u_n - v$ is bounded in $\mathcal{L}^{2,\mu}(Q_T)$. But $v \in C^{\alpha,\alpha/2}(\bar{Q}_T)$ and hence in $\mathcal{L}^{2,\mu}(Q_T)$ by Lemma 3.4, we conclude that $u_n \in \mathcal{L}^{2,\mu}(Q_T)$. Moreover, from the previous section, the solution u is attained as the limit of a subsequence of u_n, we also have $u \in \mathcal{L}^{2,\mu}(Q_T)$. Thus if we set $\mu = N + 2 + 2\alpha$, due to Lemma 3.4, we finally have $u,\ u_n \in C^{\alpha,\alpha/2}(\bar{Q}_T)$. This completes the proof. □

Theorem 3.7. *Let (u_i, ϕ_i) (respectively (u_{ni}, ϕ_{ni})), $i = 1,\ 2$, be two $C^{\alpha,\alpha/2}(\bar{Q}_T)$ solutions to (3.11)-(3.16) (respectively (3.25)-(3.27)) corresponding to the initial data u_0^i, $i = 1,\ 2$, and the same ϕ_0. Write $w = u_1 - u_2$,*

$\varphi = \phi_1 - \phi_2$, $w_0 = u_0^1 - u_0^2$ *(respectively* $w_n = u_{n1} - u_{n2}$, $\varphi_n = \phi_{n1} - \phi_{n2}$, $w_{n0} = u_0^1 - u_0^2$*). Under the previous assumptions there exist constants* $C_1(t)$, $C_2(t) > 0$ *such that*

$$\|w\|^2 + \int_0^t \|\nabla w(s)\|^2 ds + \int_0^t \|\nabla \varphi(s)\|^2 ds \leq C_1(t)\|w_0\|^2, \tag{3.112}$$

$$\|w_n\|^2 + \int_0^t \|\nabla w_n(s)\|^2 ds + \int_0^t \|\nabla \varphi_n(s)\|^2 ds \leq C_2(t)\|w_{n0}\|^2. \tag{3.113}$$

Proof. We first prove (3.112). It follows from (3.12) that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|w\|^2 + \|\nabla w\|^2 + \eta \int_\Omega \int_\Omega G(x,y) w(y,t) w(x,t) dy dx \\ + \gamma \int_\Omega (u_1^2 + u_2^2)(u_1 + u_2) w^2 \\ \leq - \int_\Omega \left[\sigma(u_1)\phi_1 \nabla \phi_1 - \sigma(u_2)\phi_2 \nabla \phi_2\right] \nabla w dx \end{aligned} \tag{3.114}$$

and from (3.13) that

$$\nabla[\sigma(u_1)\nabla\phi_1] - \nabla[\sigma(u_2)\nabla\phi_2] = 0. \tag{3.115}$$

We observe that

$$\eta \int_\Omega \int_\Omega G(x,y) w(y,t) w(x,t) dy dx \leq \eta \|G\| \, \|w\|^2, \tag{3.116}$$

$$\gamma \int_\Omega (u_1^2 + u_2^2)(u_1 + u_2) w^2 dx \geq 0. \tag{3.117}$$

Thus we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|w\|^2 + \|\nabla w\|^2 \leq - \Bigg[\int_\Omega (\sigma(u_1) - \sigma(u_2))\phi_1 \nabla\phi_1 \nabla w dx \\ + \int_\Omega \sigma(u_2)\varphi \nabla\phi_1 \nabla w dx + \int_\Omega \sigma(u_2)\phi_2 \nabla\varphi \nabla w dx\Bigg] + \eta\|G\| \, \|w\|^2. \end{aligned} \tag{3.118}$$

In view of Lemma 3.5, we can easily see that $\|\nabla\phi_1\|_{L^p}$ is uniformly bounded in t. Indeed, $\phi_1 - \phi_0$ satisfies

$$\nabla[\sigma(u_1)\nabla(\phi_1 - \phi_0)] = -\nabla(\sigma(u_1)\nabla\phi_0),$$

$$(\phi_1 - \phi_0)|_{\partial\Omega_D} = 0, \quad \frac{\partial\phi_1}{\partial n}|_{\partial\Omega_N} = 0,$$

which is just a special case of Lemma 3.5. Now we apply the Hölder's Inequality and the inequality (3.83) to estimate the right hand side of (3.118).

$$\begin{aligned}&\left|\int_\Omega (\sigma(u_1)-\sigma(u_2))\phi_1\nabla\phi_1\nabla w dx\right| \\ &\le C\|\nabla\phi_1\|^2_{L^p(\Omega)}\ \|w\|^2_{L^{2p/(p-2)}(\Omega)}+\frac{1}{8}\|\nabla w\|^2.\end{aligned} \tag{3.119}$$

$$\left|\int_\Omega \sigma(u_2)\varphi\nabla\phi_1\nabla w dx\right| \le C\|\nabla\phi_1\|^2_{L^p(\Omega)}\ \|\nabla\varphi\|^2+\frac{1}{8}\|\nabla w\|^2. \tag{3.120}$$

$$\left|\int_\Omega \sigma(u_2)\phi_2\nabla\varphi\nabla w dx\right| \le C\|\nabla\varphi\|^2+\frac{1}{8}\|\nabla w\|^2. \tag{3.121}$$

Moreover from (3.115) we have

$$\begin{aligned}\int_\Omega \sigma(u_2)|\nabla\varphi|^2 dx &= -\int_\Omega (\sigma(u_1)-\sigma(u_2))\nabla\phi_1\nabla\varphi dx \\ &\le L\int_\Omega |\nabla\phi_1|\ |\nabla\varphi|\ |w|dx \\ &\le L\|\nabla\varphi\|\ \|\nabla\phi_1\|_{L^p(\Omega)}\ \|w\|_{L^{2p/(p-2)}(\Omega)}.\end{aligned} \tag{3.122}$$

Hence

$$\|\nabla\varphi\| \le \frac{L}{\sigma_1}\|\nabla\phi_1\|_{L^p(\Omega)}\ \|w\|_{L^{2p/(p-2)}(\Omega)}. \tag{3.123}$$

Finally by combining (3.118)-(3.123) it follows that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|w\|^2+\|\nabla w\|^2 \\ &\le C\|\nabla\phi_1\|^2_{L^p(\Omega)}\left(1+\|\nabla\phi_1\|^2_{L^p(\Omega)}\right)\|w\|^2_{L^{2p/(p-2)}(\Omega)} \\ &+\frac{3}{8}\|\nabla w\|^2+\eta\|G\|\ \|w\|^2.\end{aligned} \tag{3.124}$$

Thus it follows that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|w\|^2+\|\nabla w\|^2 \\ &\le C\|w\|^2_{L^{2p/(p-2)}(\Omega)}+\frac{3}{8}\|\nabla w\|^2+\eta\|G\|\ \|w\|^2.\end{aligned} \tag{3.125}$$

Applying Lemma 3.6 with $\epsilon=\frac{1}{8}$ yields that

$$\frac{d}{dt}\|w\|^2+\|\nabla w\|^2 \le C\|w\|^2. \tag{3.126}$$

In (3.126) we first drop the gradient term and integrate from 0 to t. Then

$$\|w\|^2 \leq \exp{(Ct)}\|w_0\|^2. \tag{3.127}$$

Integrating (3.126) with respect to time again yields

$$\int_0^t \|\nabla w(s)\|^2 ds \leq \exp(Ct)\|w_0\|^2. \tag{3.128}$$

Recalling (3.123), applying Lemma 3.6 again and using (3.127), (3.128) we obtain

$$\int_0^t \|\nabla\varphi(s)\|^2 ds \leq C' \exp(Ct)\|w_0\|^2. \tag{3.129}$$

These complete the proof of (3.112). The essential part of the proof of (3.113) is similar and left to interested readers. □

By Theorem 3.6 and Theorem 3.7, there exists a unique $C^{\alpha,\alpha/2}(\bar{Q}_T)$ solution to (3.11)-(3.16) and (3.25)-(3.27) respectively. Since u and $u_n \in C^{\alpha,\alpha/2}(\bar{Q}_T)$ for any $T > 0$, for each $t > 0$, we conclude that u and $u_n \in C^{\alpha}(\bar{\Omega})$. Thus the solution operators $S(t)$ and $S_n(t)$ define two semigroups from $C^{\alpha}(\bar{\Omega})$ into itself, and we can show the existence of global attractors in $C^{\alpha}(\bar{\Omega})$. To establish this result we need first the following local a priori estimate.

Theorem 3.8. *There is a generic constant $\rho_1 > 0$ which only depends on the bounds of the coefficients, the boundary conditions and $|Q_{t_0,t_0+3}|$ and is independent of t_0, n and the initial value u_0 such that the $C^{\alpha,\alpha/2}(\bar{Q}_T)$ solutions satisfy*

$$\|u_n\|_{C^{\alpha_0,\alpha_0/2}(Q_{t_0+2,t_0+3})} \leq \rho_1, \tag{3.130}$$

$$\|u\|_{C^{\alpha_0,\alpha_0/2}(Q_{t_0+2,t_0+3})} \leq \rho_1, \tag{3.131}$$

for all $t_0 \geq 1$ and all $0 < \alpha_0 < \delta_0$.

Proof. In the proof of this theorem, the positive constant C is also a generic constant which has the same dependence on the data as ρ_1. It may be different from line to line. Let $\xi(t)$ be a smooth function such that $\xi(t) = 0$ for $t \leq t_0$

and $\xi(t) = 1$ for $t \geq t_0 + 1$. Furthermore, assume $|\xi_t| \leq \beta$ for some constant $\beta > 0$. Let (u_n, ϕ_n) be a weak solution of equations (3.25)-(3.27) and consider

$$\begin{aligned}(\xi u_n)_t - \Delta(\xi u_n) + \xi I_n(u_n)\left[\eta \int_\Omega G(x,y)u_n(y,t)dy + ku_n^4\right] \quad (3.132)\\ = \xi\nabla[\sigma(u_n)\phi_n\nabla\phi_n] + \xi_t u_n,\\ \xi u_n(x,t_0) = 0, \quad \xi u_n|_{\partial\Omega} = 0.\end{aligned}$$

Similarly to the previous results, we first consider

$$\begin{aligned}w_t - \Delta w = \xi\nabla[\sigma(u_n)\phi_n\nabla\phi_n] + \xi_t u_n, \quad (3.133)\\ w(x,t_0) = 0, \quad w|_{\partial\Omega} = 0,\end{aligned}$$

and obtain $0 \leq \xi u_n \leq w$. Moreover,

$$\begin{aligned}&\|\nabla w\|_{2,\mu,Q_{t_0,t_0+3}} \quad (3.134)\\ &\leq C[\|\xi\sigma(u_n)\phi_n\nabla\phi_n\|_{2,\mu,Q_{t_0,t_0+3}} + \|\xi_t u_n\|_{2,(\mu-2)^+,Q_{t_0,t_0+3}}\\ &\quad + \|w\|_{L^2(t_0,t_0+3,H^1(\Omega))}].\end{aligned}$$

By simple calculations we conclude just as before that

$$\|\xi\sigma(u_n)\phi_n\nabla\phi_n\|_{2,\mu,Q_{t_0,t_0+3}} \leq C. \quad (3.135)$$

Since u_n satisfies equation (3.26), by using u_n as a test function in (3.26) we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|u_n\|^2 + \|\nabla u_n\|^2 \quad (3.136)\\ &\quad +\eta\int_\Omega\int_\Omega G(x,y)I_n(u_n)(x)u_n(x,t)u_n(y,t)dydx\\ &\quad +\gamma\int_\Omega I_n(u_n)u_n^5\\ &\leq -\int_\Omega \sigma(u_n)\phi_n\nabla\phi_n\nabla u_n.\end{aligned}$$

Since the third term of the left hand side in the above equation is nonnegative, we have

$$\frac{d}{dt}\|u_n\|^2 + \|\nabla u_n\|^2 + 2\gamma\int_\Omega I_n(u_n)u_n^5 \leq C, \quad (3.137)$$

where the Schwarz Inequality is used. Applying the Poincaré's Inequality, we find that

$$\frac{d}{dt}\|u_n\|^2 + d_1\|u_n\|^2 + 2\gamma\int_\Omega I_n(u_n)u_n^5 \le C. \tag{3.138}$$

Then we can rewrite (3.138) as

$$\frac{d}{dt}\|u_n\|^2 + d_1\|u_n\|^2 + 2\gamma\int_\Omega u_n^5 \le C \tag{3.139}$$

and obtain

$$\frac{d}{dt}\|u_n\|^2 + d_1\|u_n\|^2 + d_2\|u_n\|^5 \le C, \tag{3.140}$$

which is similar to (3.89). Now we apply Lemma 3.2 to (3.140) with $p = \frac{5}{2}$ and ρ the unique solution of $d_1\rho + d_2\rho^{5/2} = C$ and obtain

$$\|u\|^2 \le \rho^2 + (\frac{d_1}{d_2})^{\frac{2}{3}}\frac{e^{-d_1 t}}{\left[1 - e^{-\frac{3}{2}d_1 t}\right]^{\frac{2}{3}}}, \quad for\ \ all\ \ t > 0. \tag{3.141}$$

Write

$$\rho' = \rho^2 + (\frac{d_1}{d_2})^{\frac{2}{3}}\frac{e^{-d_1}}{\left[1 - e^{-\frac{3}{2}d_1}\right]^{\frac{2}{3}}}.$$

Then for all $t \ge 1$, we have

$$\|u_n\|^2(t) \le \rho'. \tag{3.142}$$

Now integrating (3.139) from t_0 to $t_0 + 3$ yields that

$$\int_{t_0}^{t_0+3}\int_\Omega u_n^5 dxdt \le C, \ for\ all\ t_0 \ge 1, \tag{3.143}$$

where the inequality (3.142) is used. Thus by the Hölder Inequality, we have

$$\|u_n\|_{2,3,Q_{t_0,t_0+3}} \le C.$$

Finally it follows from $(\mu - 2)^+ < 3$ that

$$\|u_n\|_{2,(\mu-2)^+,Q_{t_0,t_0+3}} \le C\|u_n\|_{2,3,Q_{t_0,t_0+3}} \le C. \tag{3.144}$$

Combining the inequalities (3.134), (3.135), (3.142) and (3.144), yields that

$$\|\nabla w\|_{2,\mu,Q_{t_0,t_0+3}} \le C, \tag{3.145}$$

for all $0 \le \mu < N+2\delta_0$. Hence for $0 < \alpha_0 < \delta_0$, w is of class $C^{\alpha_0,\alpha_0/2}(\bar{Q}_{t_0,t_0+3})$. Consequently

$$\|w\|_{L^\infty(Q_{t_0,t_0+3})} \le C. \tag{3.146}$$

Since $0 \le \xi u_n \le w$, we have

$$\|\xi u_n\|_{L^\infty(Q_{t_0,t_0+3})} \le C. \tag{3.147}$$

In view of the definition of ξ, we have

$$\|u_n\|_{L^\infty(Q_{t_0+1,t_0+3})} \le C, \tag{3.148}$$

for all $t_0 \ge 1$. Now if we shift the graph of $\xi(t)$ to the right hand side by one unit, then it follows from (3.132) and [28] that

$$\begin{aligned}
&\|\nabla(\xi u_n)\|_{2,\mu,Q_{t_0+1,t_0+3}} \\
&\le C\{\|\xi I_n(u_n)[\eta \int_\Omega G(x,y)u_n(y,t)dy + ku_n^4]\|_{2,(\mu-2)^+,Q_{t_0+1,t_0+3}} \\
&\quad +\|\xi\sigma(u_n)\phi_n\nabla\phi_n\|_{2,\mu,Q_{t_0+1,t_0+3}} + \|\xi_t u_n\|_{2,(\mu-2)^+,Q_{t_0+1,t_0+3}} \\
&\quad +\|\xi u_n\|_{L^2(t_0+1,t_0+3;H^1(\Omega))}\}.
\end{aligned} \tag{3.149}$$

We estimate the right hand side of the above inequality term by term.

$$\begin{aligned}
&\left\|\xi I_n(u_n)\left[\eta \int_\Omega G(x,y)u_n(y,t)dy + ku_n^4\right]\right\|_{2,(\mu-2)^+,Q_{t_0+1,t_0+3}} \\
&\le C\left\|\xi I_n(u_n)\left[\eta \int_\Omega G(x,y)u_n(y,t)dy + ku_n^4\right]\right\|_{L^\infty(Q_{t_0+1,t_0+3})} \le C,
\end{aligned} \tag{3.150}$$

$$\|\xi_t u_n\|_{2,(\mu-2)^+,Q_{t_0+1,t_0+3}} \le C, \tag{3.151}$$

where (3.148) is used. Integrating (3.137) from t_0+1 to t_0+3 and using (3.142) yield that

$$\int_{t_0+1}^{t_0+3} \|\nabla u_n\|^2 dt \le C. \tag{3.152}$$

It follows from (3.142) and (3.152) that

$$\|\xi u_n\|_{L^2(t_0+1,t_0+3;H^1(\Omega))} \le C. \tag{3.153}$$

The boundedness for the second term is obvious. Thus we finally obtain for $\mu < N + 2\delta_0$,

$$\|\nabla(\xi u_n)\|_{2,\mu,Q_{t_0+1,t_0+3}} \leq C. \tag{3.154}$$

Therefore, for $\mu < N + 2 + 2\delta_0$,

$$\|\xi u_n\|_{2,\mu,Q_{t_0+1,t_0+3}} \leq C. \tag{3.155}$$

Since u_n converges to u strongly in $L^2(Q_{t_0+1,t_0+3})$, we also have

$$\|\xi u\|_{2,\mu,Q_{t_0+1,t_0+3}} \leq C. \tag{3.156}$$

Finally, by the definition of $\xi(t)$ and Lemma 3.4, the desired results (3.130) and (3.131) are obtained. □

From the above theorem we observe that

$$\|u_n\|_{C^\alpha(\bar{\Omega})}(t) \leq \rho_1 \;\; and \;\; \|u\|_{C^\alpha(\bar{\Omega})}(t) \leq \rho_1$$

for all $t \geq 3$. Since the constant ρ_1 is independent of t, n and u_0, we easily obtain the following theorem.

Theorem 3.9. *Let $K_1 = \{u \geq 0 - u = 0$ on $\partial\Omega$, $u \in C^\alpha(\bar{\Omega})\}$. Then the set $\mathcal{B}_1 = B(0, \rho_1) \cap K_1$ is a common absorbing set for $S(t)$ and $S_n(t)$, where $B(0, \rho_1)$ is the ball in $C^\alpha(\bar{\Omega})$ centered at 0 with radius ρ_1. The set $\mathcal{B}_1$ absorbs the elements of K_1 uniformly and all solutions u and u_n will enter into this absorbing set after time $t \geq 3$.*

Also in Theorem 3.8, the inequalities (3.130) and (3.131) are satisfied for all $0 < \alpha_0 < \delta_0$. In particular, picking $\alpha_0 > \alpha$ yields that $S(t)$ and $S_n(t)$ are uniformly compact for all $t \geq 3$, since the imbedding $C^{\alpha_0}(\bar{\Omega}) \hookrightarrow C^\alpha(\bar{\Omega})$ is compact.

Now the existence of global attractors for $S(t)$ and $S_n(t)$ is just a direct consequence of Theorem 1.1 in [20]. We summarize the results in the following theorem.

Theorem 3.10. *The dynamical system (3.11)-(3.16) (respectively, (3.25)-(3.27)), under the previous assumptions, possesses an attractor* $\mathcal{A}$ *(respectively,* $\mathcal{A}_n$*) which is compact, connected, and maximal in* K_1*.* $\mathcal{A}$ *(respectively,* $\mathcal{A}_n$*) attracts every element of* K_1*. Furthermore,* $\mathcal{A}$ *(respectively,* $\mathcal{A}_n$*) is contained in* $C^{\alpha_0}(\bar{\Omega})$ *for all* $\alpha < \alpha_0 < \delta_0$*.*

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Chapter 4

Hölder Continuous Solutions of an Obstacle Thermistor Problem

4.1 Introduction

Recently the authors in [2] introduced the following obstacle problem which models the behavior of certain micromachined microsensor devices:

$$(\frac{\partial u}{\partial t} - \nabla[k(u)\nabla u] + \eta \int_{\Omega} G(x,y)u(y,t)dy + \gamma u^4)u \geq \nabla[\sigma(u)\phi\nabla\phi]u, \tag{4.1}$$
$$-\nabla[\sigma(u)\nabla\phi] = 0. \tag{4.2}$$

Here the unknown functions u and ϕ denote the distributions of the temperature and the electrical potential in the device. The coefficient $\sigma(u)$ represents the temperature dependent electrical conductivity and $k(u)$ the thermal conductivity. The parameters η and γ are positive constants. The integral term in the first equation describes heat losses to the surrounding gas and the 4th-order nonlinear term models the radiation effects. We refer to [2] and the references therein for more background information on this obstacle problem. In [2], the usual boundary conditions are considered. Specifically, let Ω be a domain in R^N with boundary $\partial\Omega$ divided into two parts: Γ_0 and Γ_N. On the boundary, the temperature u satisfies either a homogenous Dirichlet or a homogenous mixed boundary condition and the potential ϕ satisfies $\phi|_{\Gamma_0} = \phi_0(x)$ and $\frac{\partial \phi}{\partial n}|_{\Gamma_N} = 0$. Here $\phi_0(x)$ is a known function. The authors in [2] discussed the existence of solutions to (4.1)-(4.2), and the long time behavior of the solutions was described by consideration of global attractors.

In this chapter, we are interested in an analysis of the situation where the microsensor is driven by a current source. We are first interested in the nonautonomous case, i.e., the source $\phi_0(x,t)$ on the boundary is time dependent. Secondly we will need to consider a nonlocal boundary condition case. This time the boundary $\partial\Omega$ is decomposed into three parts, besides Γ_0 and Γ_N, there is another piece Γ_1, i.e., $\partial\Omega = \Gamma_0 \cup \Gamma_1 \cup \Gamma_N$. The boundary condition for u is the same as before while ϕ satisfies $\phi|_{\Gamma_1} = \xi(t)$ and the previous boundary conditions on the other two parts of the boundary. Here $\xi(t)$ is an unknown constant for each t, but the total current $I(t)$ through Γ_1 is known for each time t. Thus another nonlocal boundary condition for the problem is given by

$$I(t) = \int_{\Gamma_1} \sigma(u) \frac{\partial \phi}{\partial n} ds. \tag{4.3}$$

We recall that for microsensor devices operating under conditions which imply that radiation effects and heat losses are irrelevant, the following well known elliptic system is widely used to model their steady state behavior:

$$-\nabla[k(u)\nabla u] = \nabla[\sigma(u)\phi\nabla\phi], \tag{4.4}$$

$$\nabla[\sigma(u)\nabla\phi] = 0. \tag{4.5}$$

Several authors have studied this elliptic system with a nonlocal boundary condition similar to (4.3), see [3, 7, 9], and various results related to the existence of solutions are given in these papers. But all results are obtained under the assumption that the potential ϕ satisfies a homogenous boundary condition on Γ_0, i.e., $\phi|_{\Gamma_0} = \phi_0(x) = 0$. In this chapter we will not impose this assumption on ϕ, and therefore can't directly apply the methods in [3, 7, 9] even to this special version of our case. To overcome this difficulty a decomposition of ϕ will be introduced. It will play a significant role throughout the chapter and details will be shown in Section 4.3. We also refer to [6, 8] for the description of physical devices related to this kind of nonlocal boundary conditions. Finally we mention here that the time dependent version of (4.4)-(4.5) associated with the usual (i.e., ϕ known on $\partial\Omega$) boundary conditions is also well studied. Related results can be found in [4, 17, 16] and the references therein.

In this chapter we will show the unique solvability of the initial-boundary value problem (4.1)-(4.2). Since all results will still hold if Γ_1 is empty, our theorems are also extensions of those in [2] where only the autonomous case was studied.

For simplicity we will assume that the thermal conductivity $k(s) \equiv 1$ and we shall not consider the situation where σ, k degenerate. Mathematically the domain Ω could be a connected domain in R^N for any N, but in practice Ω is a bounded three dimensional domain. Thus we will restrict the explicit presentation to the case $N = 3$. Furthermore, in view of the physically meaningful situation and of the presence of the fourth power term on the left hand side of (4.1), we shall consider $C^{\alpha,\alpha/2}$ solutions only. As stated before $\partial\Omega = \Gamma_0 \cup \Gamma_1 \cup \Gamma_N$, and we will assume that both Γ_0 and Γ_1 are closed and nonempty. Moreover $\Omega \cup \Gamma_N$ is Lipschitz. More specific descriptions about the domain can be found in [15]. In particular, we need Poincare's Inequality for functions certain vanishing on Γ_0. Then the associated initial and boundary conditions to (4.1)-(4.2) are given as

$$u(x,0) = u_0(x) \; in \; \Omega, \quad and \quad u|_{\partial\Omega} = 0, \tag{4.6}$$

$$\phi|_{\Gamma_0} = \phi_0(x,t), \; \phi|_{\Gamma_1} = \xi(t), \; \frac{\partial\phi}{\partial n}|_{\Gamma_N} = 0, \tag{4.7}$$

$$I(t) = \int_{\Gamma_1} \sigma(u) \frac{\partial\phi}{\partial n} ds, \tag{4.8}$$

with (u, ϕ, ξ) denoting the unknowns. Before we proceed, we summarize more formal assumptions in the following:

A1. $0 \leq u_0(x) \in C^\alpha(\bar{\Omega})$ and $u_0|_{\partial\Omega} = 0$. The current source $I(t)$ belongs to

$C^{\alpha/2}([0,T])$.

A2. There exist two Lipschitz functions $\Phi_0(x,t)$ and $\Phi_1(x)$ such that $\Phi_0|_{\Gamma_0} = \phi_0$, $\Phi_0|_{\Gamma_1} = 0$, $\Phi_1|_{\Gamma_0} = 0$, $\Phi_1|_{\Gamma_1} = 1$, $\frac{\partial \Phi_0}{\partial n}|_{\Gamma_N} = \frac{\partial \Phi_1}{\partial n}|_{\Gamma_N} = 0$. Moreover for some constant $\kappa > 0$, $\|\nabla(\Phi_0(x,t_1) - \Phi_0(x,t_2))\| \leq \kappa|t_1 - t_2|^{\alpha/2}$.

A3. There exist three positive constants σ_0, σ_1 and l such that $\sigma_0 \leq \sigma(s) \leq \sigma_1$ for all $s \geq 0$ and $|\sigma(s_1) - \sigma(s_2)| \leq l|s_1 - s_2|$ for all $s_1, s_2 \geq 0$.

A4. $G(x,y) \geq 0$ and $\sup_{x,y\in\Omega}|G(x,y)| < \infty$. The parameters η and γ are positive constants.

For any $T > 0$, we denote the parabolic domain by $Q_T = \Omega \times (0,T)$. For simplicity, we write $V = H_0^1(\Omega)$, $V_D = H_0^1(\Omega \cup \Gamma_N)$ and V', V_D' the corresponding dual spaces of V, V_D respectively. Moreover, we define the convex set

$$K = \{v \in L^2(0,T;\, V)|\ \ v \geq 0\ \ a.e.\ \ in\ \ Q_T\}.$$

We take advantage of the system structure and say a triplet (u,ϕ,ξ) is a weak solution of (4.1)-(4.2) if it satisfies the following conditions:

$$\begin{aligned}
& u \in K,\ u_t \in L^2(0,T;V'), \\
& \int_{Q_T} \frac{du}{dt}(v-u) + \int_{Q_T} \nabla u \nabla(v-u) \qquad\qquad (4.9)\\
& \quad +\eta \int_{Q_T}\int_\Omega G(x,y)u(y,t)(v-u)(x,t)dydxdt \\
& \quad +\gamma \int_{Q_T} u^4(v-u) \geq -\int_{Q_T} \sigma(u)\phi\nabla\phi\nabla(v-u),\ \forall v \in K; \\
& \phi - \Phi_\xi \in L^2(0,T;V_D), \\
& \int_{Q_T} \sigma(u)\nabla\phi\nabla v = 0,\ \forall v \in L^2(0,T;V_D), \qquad\qquad (4.10)\\
& I(t) = \int_\Omega \sigma(u)\nabla\phi\nabla g,\ \forall g \in S. \qquad\qquad (4.11)
\end{aligned}$$

Here $\Phi_\xi = \Phi_0 + \xi\Phi_1$ and $S = \{v|v \in H^1(\Omega),\ v|_{\Gamma_0} = 0,\ v|_{\Gamma_1} = 1\}$. Moreover we say that (u,ϕ,ξ) is a $C^{\alpha,\alpha/2}(\bar{Q}_T)$ solution if u and ϕ are in $C^{\alpha,\alpha/2}(\bar{Q}_T)$ and $\xi(t) \in C^{\alpha/2}([0,T])$.

We observe that, if $\phi \in H^2(\Omega)$, for all $g \in S$,

$$\int_{\Gamma_1} \sigma(u)\frac{\partial \phi}{\partial n} ds = \int_{\Omega} \sigma(u)\nabla\phi\nabla g, \tag{4.12}$$

where the divergence theorem and equation (4.2) are used. Thus, equation (4.11) is a weak form of the expression (4.3) which is actually not formally defined for $\phi \in H^1(\Omega)$.

We will follow the penalized method introduced in [2] to solve this problem. Let $0 \leq I_n(s) \leq 1$ be a sequence of smooth functions which converges to the Heaviside function. Then the related penalized system is given by:

$$\begin{aligned} &\frac{du_n}{dt} - \Delta u_n \qquad (4.13)\\ &\qquad + \left[\eta \int_{\Omega} G(x,y)u_n(y,t)dy + \gamma u_n^4\right] I_n(u_n) \\ &= \nabla[\sigma(u_n)\phi_n\nabla\phi_n], \\ &\nabla[\sigma(u_n)\nabla\phi_n] = 0, \qquad (4.14) \end{aligned}$$

coupled with the same initial and boundary conditions. Similar definitions of weak solutions and $C^{\alpha,\alpha/2}(\bar{Q}_T)$ solutions hold for these penalized systems. We first establish the existence of solutions to the above penalized systems for each n. The method used here involves Leray-Schauder degree theory together with Campanato type arguments. We thus obtain a sequence of solutions, $\{u_n\}$, and through a series of a priori estimates we show that a subsequence converges to a solution of the original obstacle problem. The unique solvability of the problems is then obtained through the property of the continuous dependency of the solutions on the given data.

The rest of the chapter is structured as the following. In Section 4.2, we recall some preliminary results related to Campanato spaces. In Section 4.3, a related linear elliptic equation with a nonlocal boundary condition is considered. The existence and uniqueness of solutions are given in Section 4.4.

In the entire chapter, C and C_i always stand for positive generic constants which only depend on the various norms of the boundary conditions, the various bounds of the coefficients and the domain Ω. Dependence on other quantities will be explicitly specified. These constants may differ from one step to another.

4.2 Notations and known results

It will be convenient to recall some notations and results related to Campanato spaces. For $0 \le t_0 < t_1$, we denote $\Omega \times (t_0, t_1]$ by Q_{t_0,t_1}. For simplicity, if $t_0 = 0$, we write it as Q_{t_1}. A point $(x, t) \in Q_{t_0,t_1}$ is denoted by z. Let $B_r(x_0)$ be the ball centered at x_0 with radius r and $Q_r(z_0)$ be the cylinder $B_r(x_0) \times (t_0 - r^2, t_0]$. Then we define

$$\Omega[x_0, r] = B_r(x_0) \cap \Omega, \quad Q[z_0, r] = Q_r(z_0) \cap Q_{t_0,t_1}.$$

Moreover, for $\mu \ge 0$, $\mathcal{L}^{2,\mu}(\Omega)$ and $\mathcal{L}^{2,\mu}(Q_{t_0,t_1})$ denote the Campanato spaces on Ω and Q_{t_0,t_1} associated with the standard norms, $\|.\|_{2,\mu,\Omega}$ and $\|.\|_{2,\mu,Q_{t_0,t_1}}$ respectively. We refer interested readers to [10], [12] and [18] for details on these spaces and norms.

Let δ_0 denote the Hölder exponent as stated in the De Giorgi - Nash theorem, see [12], [18]. In what follows, all α, α_i are in $(0, \delta_0)$, and μ_0, μ_1 are nonnegative numbers such that $\mu_0 < N - 2 + 2\delta_0$ and $\mu_1 < N + 2\delta_0$. They may differ from one step to the next. Furthermore $(\mu - 2)^+ = \max\{0, \mu - 2\}$.

We also denote the standard L^2 inner product and norm by $(\cdot, \cdot)$ and $\|\cdot\|$, respectively. For a general normed space E, we denote its norm by $\|\cdot\|_E$. For instance, $\|\cdot\|_{H^1(\Omega)}$ denotes the standard norm of the Sobolev space $H^1(\Omega)$ (see [1]).

The following proposition can be found in [12], Theorem 1.17.

Lemma 4.1. *(i) If $0 \le \mu < N$ the mapping*

$$u \to \left(\sup_{x_0 \in \bar{\Omega}, r > 0} r^{-\mu} \int_{\Omega[x_0,r]} u^2 dx \right)^{1/2}$$

defines a norm on $\mathcal{L}^{2,\mu}(\Omega)$ which is equivalent to $\|\cdot\|_{2,\mu,\Omega}$.

(ii) $\mathcal{L}^{2,N+2\mu}(\Omega)$ is isomorphic to $C^\mu(\bar{\Omega})$ for $\mu \in (0, 1)$.

Similar results for $\mathcal{L}^{2,\mu}(Q_{t_0,t_1})$, established in [5], are summarized in the following lemma.

Lemma 4.2. *(i) If $0 \le \mu < N+2$, the mapping*

$$u \longrightarrow \left(\sup_{z_0 \in \bar{Q}_{t_0,t_1}, r>0} r^{-\mu} \int_{Q[z_0,r]} u^2 dz \right)^{1/2}$$

defines a norm on $\mathcal{L}^{2,\mu}(Q_{t_0,t_1})$, which is equivalent to $\|\cdot\|_{2,\mu,Q_{t_0,t_1}}$.

(ii) $\mathcal{L}^{2,N+2+2\mu}(Q_{t_0,t_1})$ is isomorphic to $C^{\mu,\mu/2}(\bar{Q}_{t_0,t_1})$ for $\mu \in (0,1)$.

The next two lemmas are special cases of some results in [15, 14].

Lemma 4.3. *(Lemma 5.3.2, [14]) Let $a(x) \in C^{\alpha}(\bar{\Omega})$ and satisfy $0 < a_0 \le a(x) \le a_1 < \infty$. Assume w solves the boundary value problem:*

$$\nabla[a(x)\nabla w] = f_0 + \sum_{i=1}^{3} D_i f_i \ \ in\ \ \Omega \tag{4.15}$$

$$\frac{\partial w}{\partial n} = 0 \ \ on\ \ \partial\Omega_N, \qquad w = w_D \ \ on\ \ \partial\Omega_D. \tag{4.16}$$

Then there exists a positive number p, $3 < p < 4$,

$$\|w\|_{1,p} \le C[\sum_{i=0}^{3} \|f_i\|_{L^p(\Omega)} + \|w_D\|_{C^1(\Omega)}],$$

whenever the norms on the right hand side are bounded. Here C is a positive constant independent of w. We denote by w_D also the extension of w_D to Ω.

Lemma 4.4. *(Theorem 3.5.1, [14]) Let $a(x) \in L^\infty(\Omega)$ and $0 < a_0 \le a(x) \le a_1 < \infty$. If w solves (4.15)-(4.16), then for $0 \le \mu_0 < N - 2 + 2\delta_0$,*

$$\|\nabla w\|_{2,\mu_0,\Omega} \le C(\|f_0\|_{2,(\mu_0-2)^+,\Omega} + \|\nabla w_D\|_{2,(\mu_0-2)^+,\Omega} + \sum_{i=1}^{3} \|f_i\|_{2,\mu_0,\Omega} + \|w_D\|_{2,\mu_0,\Omega} + \|w\|_{H^1(\Omega)}),$$

whenever the norms of the right hand side are bounded. In particular, $w \in C^{\alpha_1}(\Omega)$ with $\alpha_1 = (\mu_0 - N + 2)/2$ for $\mu_0 > N - 2$.

The following lemma is an analogy of Lemma 4.4 for the time dependent case which is a special case of the results in [18].

Lemma 4.5. *(Theorem 1, [18]) Let $a(x) \in L^\infty(\Omega)$ and $0 < a_0 \le a(x) \le a_1 < \infty$. If w is a weak solution of the initial and boundary value problem*

$$\frac{dw}{dt} - \nabla[a(x)\nabla w] = f_0 + \sum_{i=1}^{3} D_i f_i \ \ in \ \ \Omega, \tag{4.17}$$

$$w = 0 \ \ on \ \ \partial\Omega, \ \ w(x,0) = 0, \tag{4.18}$$

then for $0 \le \mu_1 < N + 2\delta_0$,

$$\|\nabla w\|_{2,\mu_1,Q_T} \le C(\|f_0\|_{2,(\mu_1-2)^+,Q_T} + \sum_{i=1}^{3} \|f_i\|_{2,\mu_1,Q_T} + \|w\|_{L^2(0,T;H^1(\Omega))}),$$

whenever the norms of the right hand side are bounded. In particular, for $\mu_1 > N$, $w \in C^{\alpha_2,\frac{\alpha_2}{2}}(Q_T)$ with $\alpha_2 = (\mu_1 - N)/2$.

4.3 A linear elliptic equation with a nonlocal boundary condition

Let u be a known measurable function. We first consider the following linear nonlocal elliptic problem and establish results that will be useful in what follows: Find (ϕ, ξ) such that $\phi - \Phi_\xi \in V_D$ and

$$\int_\Omega \sigma(u)\nabla\phi\nabla v = 0, \ \forall v \in V_D, \tag{4.19}$$

$$I = \int_\Omega \sigma(u)\nabla\phi\nabla g, \ \forall g \in S. \tag{4.20}$$

Here σ, ϕ_0, I, ξ and Φ_ξ are the same as before. As in the rest of the chapter, the functions $u(x,t)$, $\phi(x,t)$, $\xi(t)$, $I(t)$ and so on may be also time dependent, but for convenience, we will suppress the variable t in this section, and the results presented will hold for every t.

Lemma 4.6. *There exists a unique solution (ϕ, ξ) to the equations (4.19)-(4.20) and it is given by $\phi = \psi + \xi\varphi$. Here, ψ is the unique solution of the following elliptic boundary value problem*

$$\psi - \Phi_0 \in V_D, \quad \int_\Omega \sigma(u)\nabla\psi\nabla v = 0, \ \forall v \in V_D, \tag{4.21}$$

and φ satisfies

$$\varphi - \Phi_1 \in V_D, \quad \int_\Omega \sigma(u)\nabla\varphi\nabla v = 0, \ \forall v \in V_D. \tag{4.22}$$

The constant ξ is given by

$$\xi = (I - \int_\Omega \sigma(u)\nabla\psi\nabla g) / \int_\Omega \sigma(u)\nabla\varphi\nabla g, \ \forall g \in S. \tag{4.23}$$

Moreover, there exists a positive constant ξ^* which depends only on the data, I, ϕ_0 and the bounds σ_0, σ_1 of σ such that

$$|\xi| < \xi^*.$$

The explicit form of ξ^* is given in the proof. Furthermore, $\phi \in L^\infty(\Omega)$.

Proof. By standard arguments about linear elliptic equations, we see that the two systems (4.21) and (4.22) are both uniquely solvable. If we let $\phi = \psi + \xi\varphi$, then clearly ϕ satisfies (4.19). Substituting this ϕ into the nonlocal boundary condition (4.20) yields that

$$\xi \int_\Omega \sigma(u)\nabla\varphi\nabla g = I - \int_\Omega \sigma(u)\nabla\psi\nabla g. \tag{4.24}$$

In view of (4.22), the value of $\int_\Omega \sigma(u)\nabla\varphi\nabla g$ will not depend on the specific choice of g. Thus $\int_\Omega \sigma(u)\nabla\varphi\nabla g = \int_\Omega \sigma(u)|\nabla\varphi|^2 > 0$ and we may divide both sides of equation (4.24) by $\int_\Omega \sigma(u)\nabla\varphi\nabla g$ to obtain (4.23). By observation, if there exist two different solutions ϕ^1 and ϕ^2 to the equations (4.19)-(4.20), then $\xi^1 - \xi^2 = \phi^1|_{\Gamma_1} - \phi^2|_{\Gamma_1} \neq 0$. Since both ϕ^1 and ϕ^2 satisfy the nonlocal boundary condition (4.20), it follows that

$$0 = \frac{1}{\xi^1 - \xi^2} \int_\Omega \sigma(u)|\nabla(\phi^1 - \phi^2)|^2,$$

by choosing $g = (\phi^1 - \phi^2)/(\xi^1 - \xi^2)$ as a test function. Thus $\int_\Omega |\nabla(\phi^1 - \phi^2)|^2 dx = 0$. Consequently, $\phi^1 = \phi^2$ due to Poincare's Inequality. This contradicts with $\phi^1 \neq \phi^2$, and the uniqueness follows.

The rest of the proof deals with the existence of an upper bound ξ^* of $|\xi|$. By standard estimates of linear elliptic equations we have

$$||\nabla\psi|| \leq \frac{\sigma_1}{\sigma_0}||\nabla\Phi_0||, \quad ||\nabla\varphi|| \leq \frac{\sigma_1}{\sigma_0}||\nabla\Phi_1||. \tag{4.25}$$

Due to equations (4.21) and (4.22), the value of ξ doesn't depend on any specific g. Thus we set $g = \varphi$ and obtain

$$\xi = (I - \int_\Omega \sigma(u)\nabla\psi\nabla\varphi)/\int_\Omega \sigma(u)\nabla\varphi\nabla\varphi. \tag{4.26}$$

Moreover, by standard variational arguments $\inf_{v\in S}||\nabla v||^2$ exists and is a positive number. Define

$$m_* = \inf_{v\in S}||\nabla v||^2. \tag{4.27}$$

Therefore, we obtain that $|\xi| \leq \xi^*$ with

$$\xi^* = \frac{|I| + \frac{\sigma_1^3}{\sigma_0^2}||\nabla\Phi_0||\ ||\nabla\Phi_1||}{\sigma_0 m_*}, \tag{4.28}$$

where (4.25), (4.26), (4.27) and Schwarz Inequality are used. Finally, that $\phi \in L^\infty(\Omega)$ follows from the boundedness of Φ_0 and the weak maximum principle. This completes the proof. □

Now we give some estimates of ψ, φ and ϕ in the Campanato spaces.

Lemma 4.7. *The following holds for all* $0 \leq \mu_0 < N - 2 + 2\delta_0$*:*

$$||\nabla\varphi||_{2,\mu_0,\Omega} \leq C, \tag{4.29}$$

$$||\nabla\psi||_{2,\mu_0,\Omega} \leq C(||\Phi_0||_{2,\mu_0,\Omega} + ||\nabla\Phi_0||_{2,(\mu_0-2)^+,\Omega}), \tag{4.30}$$

$$||\nabla\phi||_{2,\mu_0,\Omega} \leq C(|I| + ||\Phi_0||_{2,\mu_0,\Omega} + ||\nabla\Phi_0||_{2,(\mu_0-2)^+,\Omega}). \tag{4.31}$$

Proof. Due to Lemma 4.4, ψ and φ satisfy the following estimates:

$$||\nabla\psi||_{2,\mu_0,\Omega} \leq C(||\Phi_0||_{2,\mu_0,\Omega} + ||\nabla\Phi_0||_{2,(\mu_0-2)^+,\Omega} + ||\psi||_{H^1(\Omega)}), \tag{4.32}$$

$$||\nabla\varphi||_{2,\mu_0,\Omega} \leq C(||\Phi_1||_{2,\mu_0,\Omega} + ||\nabla\Phi_1||_{2,(\mu_0-2)^+,\Omega} + ||\varphi||_{H^1(\Omega)}). \tag{4.33}$$

On the other hand, ψ and φ also satisfy

$$\|\nabla\psi\| \le C\|\nabla\Phi_0\|_{2,(\mu_0-2)^+,\Omega}, \quad \|\nabla\varphi\| \le C\|\nabla\Phi_1\|_{2,(\mu_0-2)^+,\Omega}, \tag{4.34}$$

where (4.25) and $\mathcal{L}^{2,(\mu_0-2)^+}(\Omega) \hookrightarrow L^2(\Omega)$ are used. Thus (4.29) and (4.30) follow immediately. Moreover, by definition (4.28),

$$\xi^* = \frac{|I| + \frac{\sigma_1^3}{\sigma_0^2}\|\nabla\Phi_0\|\;\|\nabla\Phi_1\|}{\sigma_0 m_*} \le C(|I| + \|\nabla\Phi_0\|_{2,(\mu_0-2)^+,\Omega}). \tag{4.35}$$

Since $\phi = \psi + \xi\varphi$, we have

$$\|\nabla\phi\|_{2,\mu_0,\Omega} \le C(|I| + \|\Phi_0\|_{2,\mu_0,\Omega} + \|\nabla\Phi_0\|_{2,(\mu_0-2)^+,\Omega}), \tag{4.36}$$

where (4.32)-(4.35) and Poincare's Inequality are used as well as the fact that $L^\infty(\Omega)$ is a multiplier for $\mathcal{L}^{2,\mu_0}(\Omega)$ if $\mu_0 < N$. □

In the remaining part of this section, we show the continuous dependence of the solutions of (4.19)-(4.20) on the given data. Let (ϕ^1,ξ^1) and (ϕ^2,ξ^2) be the solutions of (4.19)-(4.20) corresponding to the data (u^1, ϕ_0^1, I^1) and (u^2, ϕ_0^2, I^2) respectively. Due to Lemma 4.6, we may write ϕ^1 and ϕ^2 as

$$\phi^1 = \psi^1 + \xi^1\varphi^1, \quad \phi^2 = \psi^2 + \xi^2\varphi^2. \tag{4.37}$$

Here, ψ^i and φ^i with $i = 1,\ 2$ are the solutions of (4.21) and (4.22), respectively, associated with the given data. Then the following lemma holds.

Lemma 4.8. *Assume $u^i \in L^{2p/(p-2)}(\Omega)$ and $\nabla\psi^i,\ \nabla\varphi^i \in L^p(\Omega)$ for some $3 < p < 4$ with $i = 1,\ 2$, then*

$$\|\phi^1 - \phi^2\|_{H^1(\Omega)} \le C(|I^1 - I^2| + \|\nabla(\Phi_0^1 - \Phi_0^2)\| + \|u^1 - u^2\|_{L^{2p/(p-2)}(\Omega)}) \tag{4.38}$$

Proof. Due to (4.37),

$$\begin{aligned} &\|\nabla(\phi^1 - \phi^2)\| = \|\nabla(\psi^1 + \xi^1\varphi^1 - \psi^2 - \xi^2\varphi^2)\| \\ &\le \|\nabla(\psi^1 - \psi^2)\| + |\xi^1|\;\|\nabla(\varphi^1 - \varphi^2)\| + |\xi^1 - \xi^2|\;\|\nabla\varphi^2\|. \end{aligned} \tag{4.39}$$

We will estimate the right hand side of (4.39) term by term. According to (4.21), $\psi^1 - \psi^2$ satisfies

$$\int_\Omega \sigma(u^1)|\nabla(\psi^1 - \psi^2)|^2 = \int_\Omega \sigma(u^1)\nabla(\psi^1 - \psi^2)\nabla(\Phi_0^1 - \Phi_0^2) \quad (4.40)$$
$$+ \int_\Omega [\sigma(u^1) - \sigma(u^2)]\nabla\psi^2\nabla(\Phi_0^1 - \Phi_0^2)$$
$$- \int_\Omega [\sigma(u^1) - \sigma(u^2)]\nabla\psi^2\nabla(\psi^1 - \psi^2).$$

By Schwarz Inequality we have

$$\int_\Omega \sigma(u^1)\nabla(\psi^1 - \psi^2)\nabla(\Phi_0^1 - \Phi_0^2) \leq \sigma_1\|\nabla(\psi^1 - \psi^2)\| \, \|\nabla(\Phi_0^1 - \Phi_0^2)\| \quad (4.41)$$
$$\leq \frac{\sigma_0}{4}\|\nabla(\psi^1 - \psi^2)\|^2 + C\|\nabla(\Phi_0^1 - \Phi_0^2)\|^2.$$

Moreover by Hölder Inequality and the assumptions on $\sigma(s)$ we have

$$\int_\Omega [\sigma(u^1) - \sigma(u^2)]\nabla\psi^2\nabla(\phi_0^1 - \phi_0^2) \quad (4.42)$$
$$\leq l\|\nabla(\Phi_0^1 - \Phi_0^2)\| \, \|\nabla\psi^2\|_{L^p(\Omega)} \, \|u^1 - u^2\|_{L^{2p/(p-2)}(\Omega)}$$
$$\leq \frac{l}{2}\|\nabla(\Phi_0^1 - \Phi_0^2)\|^2 + C\|u^1 - u^2\|^2_{L^{2p/(p-2)}(\Omega)}.$$

Here l is the Lipschitz constant in A3. Similarly to (4.42), we have

$$\int_\Omega [\sigma(u^1) - \sigma(u^2)]\nabla\psi^2\nabla(\psi^1 - \psi^2) \quad (4.43)$$
$$\leq \frac{\sigma_0}{4}\|\nabla(\psi^1 - \psi^2)\|^2 + C\|u^1 - u^2\|^2_{L^{2p/(p-2)}(\Omega)}.$$

Hence it follows from (4.40)-(4.43) that

$$\|\nabla(\psi^1 - \psi^2)\|^2 \leq C(\|\nabla(\Phi_0^1 - \Phi_0^2)\|^2 + \|u^1 - u^2\|^2_{L^{2p/(p-2)}(\Omega)}). \quad (4.44)$$

Similarly to the estimate of (4.40), $\varphi^1 - \varphi^2$ satisfies

$$\|\nabla(\varphi^1 - \varphi^2)\|^2 \leq C\|u^1 - u^2\|^2_{L^{2p/(p-2)}(\Omega)}. \quad (4.45)$$

We next estimate $|\xi^1 - \xi^2|$. From the proof of Lemma 4.6, we have by setting $g = \varphi^2$ that

$$\xi^i \int_\Omega \sigma(u^i)|\nabla\varphi^i\nabla\varphi^2 = I^i - \int_\Omega \sigma(u^i)\nabla\psi^i\nabla\varphi^2, \;\; i = 1,\ 2. \quad (4.46)$$

Consequently

$$(\xi^1 - \xi^2)\int_\Omega \sigma(u^2)|\nabla\varphi^2|^2 = I^1 - I^2 \tag{4.47}$$
$$+\int_\Omega [\sigma(u^2)\nabla(\psi^2-\psi^1) + (\sigma(u^2)-\sigma(u^1))\nabla\psi^1]\nabla\varphi^2$$
$$+\xi^1\int_\Omega [\sigma(u^2)\nabla(\varphi^2-\varphi^1) + (\sigma(u^2)-\sigma(u^1))\nabla\varphi^1]\nabla\varphi^2.$$

Thus, it follows from the Hölder Inequality, (4.27), (4.44) and (4.45) that

$$|\xi^1 - \xi^2| \le C(|I^1 - I^2| + \|\Phi_0^1 - \Phi_0^2\| + \|u^1 - u^2\|_{L^{2p/(p-2)}(\Omega)}). \tag{4.48}$$

Then it follows immediately from the combination of (4.44), (4.45) and (4.48) that

$$\|\nabla(\phi^1 - \phi^2)\| \le C(|I^1 - I^2| + \|\nabla(\Phi_0^1 - \Phi_0^2)\| + \|u^1 - u^2\|_{L^{2p/(p-2)}(\Omega)}). \tag{4.49}$$

Note that exactly the same estimate holds for $\|\nabla(\phi^1-\phi^2-\Phi_0^1+\Phi_0^2)\|$, and since $\phi^1 - \phi^2 - \Phi_0^1 + \Phi_0^2 = 0$ on Γ_0, we obtain (4.38) by Poincare's Inequality. □

As we shall see, the assumptions in Lemma 4.8 are reasonable. Indeed based on physical considerations, we are only interested in the $C^{\alpha,\alpha/2}(\bar{Q}_T)$ solutions. Thus, in our case, $u \in C^\alpha(\bar{\Omega})$ for each t. Then by Lemma 4.3, ψ and φ both belong to $H^{1,p}(\Omega)$ for some $3 < p < 4$.

4.4 The existence and uniqueness of $C^{\alpha,\alpha/2}(\bar{Q}_T)$ solutions

In this part we first will show the existence of solutions to the penalized equations for each n by applying Leray-Schauder degree theory. Then the solution of the original obstacle system will be obtained by passing to the limit of some subsequence of the previous penalized solutions. For notational convenience we write

$$E(w) = I_n(w)\left[\eta\int_\Omega G(x,y)w(y)dy + \gamma w^4\right], \tag{4.50}$$
$$F(w,\phi) = \sigma(w)\phi\nabla\phi. \tag{4.51}$$

We now decompose the solution u_n of (4.13) into two parts as $u_n = v + z_n$. Here v is the solution of the following simple initial-boundary value problem:

$$v_t - \Delta v = 0, \tag{4.52}$$

$$v(x,0) = u_0(x), \;\; v(x,t) = 0 \;\; if \;\; x \in \partial\Omega. \tag{4.53}$$

Since $u_0 \in C^{\alpha}(\bar{\Omega})$ and $u_0 = 0$ on $\partial\Omega$, the above equation has a unique $C^{\alpha,\alpha/2}(\bar{Q}_T)$ solution by a classic result in [11]. In view of the equation satisfied by u_n, we note that z_n and ϕ_n satisfy the following equations

$$z_{nt} - \Delta z_n + E(z_n + v) = \nabla F(z_n + v, \phi_n), \tag{4.54}$$

$$\nabla[\sigma(z_n + v)\nabla\phi_n] = 0, \tag{4.55}$$

with $z_n(x,0) = 0$ and the same boundary conditions. Since $v \in C^{\alpha,\alpha/2}(\bar{Q}_T)$, it is sufficient for us to show $z_n \in C^{\alpha,\alpha/2}(\bar{Q}_T)$ as well. Define the family of operators

$$L(\lambda): \; Z_n \to w_n, \;\; w_n = L(\lambda)Z_n, \tag{4.56}$$

with $0 \leq \lambda \leq 1$. Here $Z_n \in C^{\alpha,\alpha/2}(\bar{Q}_T)$ and w_n is the solution of the following system:

$$w_{nt} - \Delta w_n = -\lambda(E(Z_n + v) - \nabla F(Z_n + v, \phi_n)), \tag{4.57}$$

$$\nabla[\sigma(Z_n + v)\nabla\phi_n] = 0, \tag{4.58}$$

with the same initial and boundary conditions as for system (4.54)-(4.55).

Our results are summarized in the following several lemmas and theorems.

Lemma 4.9. *For each $0 \leq \lambda \leq 1$, $L(\lambda)$ is a well-defined compact operator from $C^{\alpha,\alpha/2}(\bar{Q}_T)$ into itself.*

Proof. In view of Lemma 4.6, the equation (4.58) is uniquely solvable and its solution $\phi_n \in H^1(\Omega) \cap L^\infty(\Omega)$ for each t, uniformly bounded and continuous in t by Lemma 4.3 and Lemma 4.8. Thus, by classic results of linear parabolic equations [11], there exists a unique weak solution w_n to the equation (4.57).

It remains to show that $w_n \in C^{\alpha,\alpha/2}(\bar{Q}_T)$. According to Lemma 4.7, for almost every t,

$$\|\nabla\phi_n\|_{2,\mu_0,\Omega}(t) \leq C. \tag{4.59}$$

By Lemma 4.1, we have

$$\sup_{x_0\in\bar{\Omega},r>0} r^{-\mu_0}\int_{\Omega[x_0,r]} |\nabla\phi_n|^2 \leq \|\nabla\phi_n\|^2_{2,\mu_0,\Omega}(t) \leq C, \tag{4.60}$$

which implies that

$$r^{-(\mu_0+2)}\int_{Q[z_0,r]} |\nabla\phi_n|^2 dz \leq C, \tag{4.61}$$

for all $z_0 \in Q_T$ and $r > 0$. Therefore, applying Lemma 4.2 together with (4.61) yields that, for all $0 \leq \mu_1 \leq N + 2\delta_0$,

$$\|\nabla\phi_n\|^2_{2,\mu_1,Q_T} \leq C \sup_{z_0\in\bar{Q}_T,r>0} r^{-\mu_1}\int_{Q[z_0,r]} |\nabla\phi_n|^2 dz \leq C. \tag{4.62}$$

Moreover, by the weak maximum principle $\phi_n \in L^\infty(Q_T)$. Thus, $\sigma(Z_n+v)\phi_n \in L^\infty(Q_T)$. Since $L^\infty(Q_T)$ is a multiplier of $\mathcal{L}^{2,\mu_1}(Q_T)$, we have $\sigma(Z_n+v)\phi_n\nabla\phi_n \in \mathcal{L}^{2,\mu_1}(Q_T)$. Now applying Lemma 4.5 to equation (4.57) yields that

$$\begin{aligned}\|\nabla w_n\|_{2,\mu_1,Q_T} \leq C[&\|\nabla\phi_n\|_{2,\mu_1,Q_T} \\ &+\|E(Z_n+v)\|_{2,(\mu_1-2)^+,Q_T} + \|w_n\|_{L^2(0,T;H^1(\Omega))}].\end{aligned} \tag{4.63}$$

According to (4.62), the first term of the right hand side of (4.63) is bounded. We notice that $E(Z_n+v) \in L^\infty(Q_T)$ and $F(Z_n+v,\phi_n) \in L^2(Q_T)$. Thus, by standard a priori estimates for linear parabolic equations [11], the third term of the right hand side of (4.63) is bounded too. Since in our case $0 \leq \mu_1 < N+2\delta_0$, we have $(\mu_1-2)^+ < N$. By the imbedding theorem $L^\infty(Q_T) \hookrightarrow \mathcal{L}^{2,\mu}(Q_T)$ for $0 \leq \mu < N$, we have that the second term is also bounded. Therefore, there exists a positive constant C such that $\|\nabla w_n\|_{2,\mu_1,Q_T} \leq C$ and $\|w_n\|_{2,\mu_1+2,Q_T} \leq C$. This implies that $w_n \in C^{\alpha_0,\alpha_0/2}(\bar{Q}_T)$ for all $0 < \alpha_0 < \delta_0$. Therefore, the operator $L(\lambda)$ is well defined for each parameter $0 \leq \lambda \leq 1$. Now if we choose $\alpha_0 > \alpha$, $L(\lambda)$ maps $C^{\alpha,\alpha/2}(\bar{Q}_T)$ into $C^{\alpha_0,\alpha_0/2}(\bar{Q}_T)$ and thus it is a compact operator for each λ. □

Lemma 4.10. *For each $0 \le \lambda \le 1$, $L(\lambda)$ is continuous.*

Proof. Let $Z_n^m \to Z_n$ in $C^{\alpha,\alpha/2}(\bar{Q}_T)$ as $m \to \infty$. We have to show $w_n^m \to w_n$ in $C^{\alpha,\alpha/2}(\bar{Q}_T)$.

We first give an estimate for $\phi_n^m - \phi_n$. By following a procedure similar to the proof of Lemma 4.8, we write $\phi_n^m - \phi_n = \psi_n^m - \psi_n + \xi_n^m \varphi_n^m - \xi_n \varphi_n$ and obtain

$$\begin{aligned} \|\nabla(\phi_n^m - \phi_n)\|_{2,\mu_0,\Omega} \le \|\nabla(\psi_n^m - \psi_n)\|_{2,\mu_0,\Omega} \\ + |\xi_n^m| \, \|\nabla(\varphi_n^m - \varphi_n)\|_{2,\mu_0,\Omega} + |\xi_n^m - \xi_n| \, \|\nabla \varphi_n\|_{2,\mu_0,\Omega}. \end{aligned} \tag{4.64}$$

Here ψ_n^m (ψ_n) and φ_n^m (φ_n) satisfy (4.21) and (4.22) respectively with u replaced by $Z_n^m + v$ ($Z_n + v$). Therefore, $\psi_n^m - \psi_n$ and $\varphi_n^m - \varphi_n$ satisfy

$$\begin{aligned} &\int_\Omega \sigma(Z_n + v)\nabla(\psi_n^m - \psi_n)\nabla v \\ &= -\int_\Omega [\sigma(Z_n^m + v) - \sigma(Z_n + v)]\nabla \psi_n^m \nabla v, \end{aligned} \tag{4.65}$$

$$\begin{aligned} &\int_\Omega \sigma(Z_n + v)\nabla(\varphi_n^m - \varphi_n)\nabla v \\ &= -\int_\Omega [\sigma(Z_n^m + v) - \sigma(Z_n + v)]\nabla \varphi_n^m \nabla v, \end{aligned} \tag{4.66}$$

for all $v \in V_D$. Applying Lemma 4.4 to (4.65) gives

$$\begin{aligned} \|\nabla(\psi_n^m - \psi_n)\|_{2,\mu_0,\Omega} \le C \, [\|\psi_n^m - \psi_n\|_{H^1(\Omega)} \\ + \|(\sigma(Z_n^m + v) - \sigma(Z_n + v))\nabla \psi_n^m\|_{2,\mu_0,\Omega}]. \end{aligned} \tag{4.67}$$

Similarly to the estimates of (4.32), (4.34) and (4.44),

$$\|\nabla \psi_n^m\|_{2,\mu_0,\Omega} \le C(\|\Phi_0\|_{2,\mu_0,\Omega} + \|\nabla \Phi_0\|_{2,(\mu_0-2)^+,\Omega}), \tag{4.68}$$

$$\|\nabla(\psi_n^m - \psi_n)\| \le C \, \|Z_n^m - Z_n\|_{L^{2p/(p-2)}(\Omega)} \le C\|Z_n^m - Z_n\|_{L^\infty(Q_T)}. \tag{4.69}$$

Therefore, it follows from (4.67), (4.68) and (4.69) that for all $0 \le \mu_0 < N - 2 + 2\delta_0$,

$$\|\nabla(\psi_n^m - \psi_n)\|_{2,\mu_0,\Omega} \le C\|Z_n^m - Z_n\|_{L^\infty(Q_T)}. \tag{4.70}$$

Similarly,

$$\|\nabla(\varphi_n^m - \varphi_n)\|_{2,\mu_0,\Omega} \leq C\|Z_n^m - Z_n\|_{L^\infty(Q_T)}. \tag{4.71}$$

According to (4.48) we also have

$$|\xi_n^m - \xi_n| \leq C\|Z_n^m - Z_n\|_{L^\infty(Q_T)}. \tag{4.72}$$

Now we substitute (4.70)-(4.72) into (4.64) and keep in mind that ξ_n^m is uniformly bounded by $\sup_t \xi^*(t)$ to obtain

$$\|\nabla(\phi_n^m - \phi_n)\|_{2,\mu_0,\Omega} \leq C\|Z_n^m - Z_n\|_{L^\infty(Q_T)}. \tag{4.73}$$

Thus, due to Lemma 4.1 and Lemma 4.2, by a procedure similar to (4.60)-(4.62), we obtain

$$\|\nabla(\phi_n^m - \phi_n)\|_{2,\mu_1,Q_T} \leq C\|Z_n^m - Z_n\|_{L^\infty(Q_T)}. \tag{4.74}$$

It also follows from (4.73) that, for $\mu_0 > N-2$,

$$\|\phi_n^m - \phi_n\|_{L^\infty(\Omega)} \leq C\|\nabla(\phi_n^m - \phi_n)\|_{2,\mu_0,\Omega} \leq C\|Z_n^m - Z_n\|_{L^\infty(Q_T)}. \tag{4.75}$$

Next we derive an estimate for $w_n^m - w_n$. Actually $w_n^m - w_n$ satisfies the following equation

$$\begin{aligned}&(w_n^m - w_n)_t - \Delta(w_n^m - w_n) \\ &= -\lambda\{E(Z_n^m + v) - E(Z_n + v) - \nabla[F(Z_n^m + v, \phi_n^m) - F(Z_n + v, \phi_n)]\}.\end{aligned} \tag{4.76}$$

Applying Lemma 4.5 to equation (4.76) yields

$$\begin{aligned}\|\nabla(w_n^m - w_n)\|_{2,\mu_1,Q_T} &\leq C[\|E(Z_n^m + v) - E(Z_n + v)\|_{2,(\mu_1-2)^+,Q_T} \\ &+\|F(Z_n^m + v, \phi_n^m) - F(Z_n + v, \phi_n^m)\|_{2,\mu_1,Q_T} + \|w_n^m - w_n\|_{L^2(0,T;H^1(\Omega))}].\end{aligned} \tag{4.77}$$

From (4.74), (4.75) we obtain

$$\|F(Z_n^m + v, \phi_n^m) - F(Z_n + v, \phi_n)\|_{2,\mu_1,Q_T} \leq C\|Z_n^m - Z_n\|_{L^\infty(Q_T)}. \tag{4.78}$$

It follows from the assumptions on G, I_n and the fact $Z_n + v \in C^{\alpha,\alpha/2}(Q_T)$ that

$$\|E(Z_n^m + v) - E(Z_n + v)\|_{2,(\mu_1-2)^+,Q_T} \le C\|Z_n^m - Z_n\|_{L^\infty(Q_T)}. \quad (4.79)$$

To estimate the last term in the right hand side of (4.77), we multiply equation (4.76) by $w_n^m - w_n$ and integrate over Q_t to obtain

$$\begin{aligned}\|\nabla(w_n^m - w_n)\|^2_{L^2(Q_T)} &\le C[\|E(Z_n^m + v) - E(Z_n + v)\|_{L^2(Q_T)}] \quad (4.80)\\ &+\|F(Z_n^m + v, \phi_n^m) - F(Z_n + v, \phi_n)\|_{L^2(Q_T)} \\ &\le C\|Z_n^m - Z_n\|_{L^\infty(Q_T)}.\end{aligned}$$

Substituting inequalities (4.78)-(4.80) into equation (4.77) gives

$$\|\nabla(w_n^m - w_n)\|_{2,\mu_1,Q_T} \le C\|Z_n^m - Z_n\|_{L^\infty(Q_T)}.$$

Thus, $\|\nabla(w_n^m - w_n)\|_{2,\mu_1,Q_T} \to 0$. In particular, let $\mu_1 = N + 2\alpha$ and obtain $w_n^m - w_n \to 0$ when $Z_n^m - Z_n \to 0$ in $C^{\alpha,\alpha/2}(\bar{Q}_T)$. We conclude that $L(\lambda)$ is continuous in $C^{\alpha,\alpha/2}(\bar{Q}_T)$ for each λ. □

Lemma 4.11. *There exists a positive constant M which is independent of λ such that for all $0 \le \lambda \le 1$ if $Z_n = L(\lambda)Z_n$, then $\|Z_n\|_{C^{\alpha,\alpha/2}(\bar{Q}_T)} \le M$.*

Proof. We write $Z_n = \mathcal{X} + (Z_n - \mathcal{X})$, where $\mathcal{X}$ is the solution of the following equation:

$$\mathcal{X}_t - \Delta\mathcal{X} = \lambda\nabla F(Z_n + v, \phi_n), \quad (4.81)$$

with zero initial and boundary conditions. Due to the weak maximum principle, Lemma 4.1, Lemma 4.2 and Lemma 4.7,

$$\|\phi_n\|_{L^\infty(Q_T)}, \quad \|\nabla\phi_n\|_{2,\mu_1,Q_T} \le C,$$

which implies that

$$\|F(Z_n + v, \phi_n)\|_{2,\mu_1,Q_T} \le \sigma_1\|\phi_n\|_{L^\infty(Q_T)}\,\|\nabla\phi_n\|_{2,\mu_1,Q_T} \le C. \quad (4.82)$$

Then by the earlier procedures we find that

$$\|\nabla \mathcal{X}\|_{2,\mu_1,Q_T} \leq C. \tag{4.83}$$

Thus, $\mathcal{X}$ is in $C^{\alpha,\alpha/2}(\bar{Q}_T)$. Consequently, $\mathcal{X}$ is bounded in $L^\infty(Q_T)$. Moreover, $Z_n - \mathcal{X}$ satisfies

$$(\mathcal{X} - Z_n)_t - \Delta(\mathcal{X} - Z_n) = \lambda E(Z_n + v), \tag{4.84}$$

again with zero initial and boundary conditions. Since $Z_n + v$ is a solution of a parameterized version of (4.13)-(4.14), we conclude $Z_n + v \geq 0$ by using $(Z_n+v)^-$ as a test function in (4.13)-(4.14). Thus, the right hand side of (4.84) is nonnegative. By the maximum principle, we have $\mathcal{X} - Z_n \geq 0$. Equivalently, $0 \leq Z_n + v \leq \mathcal{X} + v$, and both $\mathcal{X}$ and v are bounded in $L^\infty(Q_T)$ with a bound independent of λ and Z_n. By summarizing the previous results and applying the energy inequality to Z_n we obtain

$$\begin{aligned} \|\nabla Z_n\|_{2,\mu_1,Q_T} &\leq C[\|E(Z_n+v)\|_{2,(\mu_1-2)^+,Q_T} \\ &+\|F(Z_n+v,\phi_n)\|_{2,\mu_1,Q_T} + \|Z_n\|_{L^2(0,T;H^1(\Omega))}] \leq C. \end{aligned} \tag{4.85}$$

Finally we conclude that there is a positive constant M which is independent of λ and Z_n such that

$$\|Z_n\|_{C^{\alpha,\alpha/2}(\bar{Q}_T)} \leq M. \tag{4.86}$$

□

Lemma 4.12. *There exists a positive constant C such that, for all n*

$$\begin{aligned} &\|\psi_n\|_{C^{\alpha,\alpha/2}(\bar{Q}_T)}, \ \|\varphi_n\|_{C^{\alpha,\alpha/2}(\bar{Q}_T)}, \\ &\qquad \|\xi_n\|_{C^{\alpha/2}([0,T])}, \ \|\phi_n\|_{C^{\alpha,\alpha/2}(\bar{Q}_T)} \leq C. \end{aligned} \tag{4.87}$$

Proof. Given $t_1,\ t_2 \in [0,T]$, we consider $\Psi(x) = \psi_n(x,t_1) - \psi_n(x,t_2)$ which satisfies

$$\nabla[\sigma(w(x,t_1))\nabla\Psi] = -\nabla[(\sigma(w(x,t_1)) - \sigma(w(x,t_2)))\nabla\psi_n(x,t_2)], \tag{4.88}$$

$$\Psi|_{\Gamma_0} = \Phi_0(x,t_1) - \Phi_0(x,t_2) := \Theta(x), \ \Psi|_{\Gamma_1} = 0, \ \frac{\partial\Psi}{\partial n}|_{\Gamma_n} = 0, \tag{4.89}$$

where $w = Z_n + v$. Again from Lemma 4.4,

$$\|\nabla\Psi\|_{2,\mu_0,\Omega} \leq C(\|w(x,t_1) - w(x,t_2)\|_{L^\infty(\Omega)} \, \|\nabla\psi_n(x,t_2)\|_{2,\mu_0,\Omega} \quad (4.90)$$
$$+\|\Theta\|_{2,\mu_0,\Omega} + \|\nabla\Theta\|_{2,(\mu_0-2)^+,\Omega} + \|\Psi\|_{H^1(\Omega)}).$$

We choose $\Psi - \Theta$ as a test function for (4.88) and integrate it over Ω. By direct calculations we find that

$$\|\nabla\Psi\| \leq C(\|w(x,t_1) - w(x,t_2)\|_{L^\infty(\Omega)} + \|\nabla\Theta\|). \qquad (4.91)$$

Now we choose a μ_0 such that $N - 2 < \mu_0 < 2$. Then it follows from (4.90), (4.91) and the Poincaré Inequality that

$$\|\nabla\Psi\|_{2,\mu_0,\Omega} \leq C(\|w(x,t_1) - w(x,t_2)\|_{L^\infty(\Omega)} \qquad (4.92)$$
$$+\|\Theta\|_{2,\mu_0,\Omega} + \|\nabla\Theta\|).$$

Due to $w \in C^{\alpha,\alpha/2}(\bar{Q}_T)$ and assumption A2

$$\|w(x,t_1) - w(x,t_2)\|_{L^\infty(\Omega)}, \ \|\Theta\|_{2,\mu_0,\Omega}, \ \|\nabla\Theta\| \leq C|t_1 - t_2|^{\alpha/2}. \ (4.93)$$

Consequently

$$\|\nabla\Psi\|_{2,\mu_0,\Omega} \leq C|t_1 - t_2|^{\alpha/2}. \qquad (4.94)$$

Since $\mu_0 > N - 2$, we conclude that $\Psi \in C^{\frac{N-\mu_0}{2}}(\Omega)$ and $\|\Psi\|_{C^{\frac{N-\mu_0}{2}}(\Omega)} \leq C|t_1 - t_2|^{\alpha/2}$ which implies that

$$\|\Psi\|_{L^\infty(\Omega)} \leq C|t_1 - t_2|^{\alpha/2}. \qquad (4.95)$$

Now $\psi_n(x,t) \in C^\alpha(\Omega)$ for each t and (4.95) implies $\psi_n(x,t) \in C^{\alpha,\alpha/2}(Q_T)$ and is uniformly bounded with respect to n.

Similar results hold for φ_n. Through the relations between ξ_n, ϕ_n and ψ_n, φ_n, the remaining two bounds can be obtained. □

Theorem 4.1. *There exists a $C^{\alpha,\alpha/2}(\bar{Q}_T)$ solution to the penalized equations (4.13)-(4.14) for each n as well as to the original obstacle problem (4.9)-(4.11).*

Proof. When $\lambda = 0$, the system (4.57)-(4.58) is uniquely solvable and the degree of $I - L(\lambda)$ in a large $C^{\alpha,\alpha/2}(\bar{Q}_T)$ ball at $\lambda = 0$ is 1. Thus by Lemmas 4.9-4.11 and Leray-Schauder degree theory, the degree of $I - L(1)$ is also 1. Therefore, there is a fixed point z_n of $L(1)$ such that $u_n = z_n + v \in C^{\alpha,\alpha/2}(\bar{Q}_T)$ is a solution of the penalized equations (4.13)-(4.14).

From Lemma 4.12 and the proof of Lemma 4.11, we can choose a constant $M_1 > 0$ which only depends on σ_0, σ_1, Φ_0, Φ_1 and I such that

$$\begin{aligned} &\|\phi_n\|_{C^{\alpha,\alpha/2}(\bar{Q}_T)},\ \|u_n\|_{C^{\alpha,\alpha/2}(\bar{Q}_T)},\ \|\xi_n\|_{C^{\alpha/2}([0,T])}, \\ &\qquad \|\nabla\phi_n\|_{2,\mu_0,Q_T},\ \|\nabla u_n\|_{2,\mu_0,Q_T} \le M_1. \end{aligned} \tag{4.96}$$

Consequently, both $E(u_n)$ and $F(u_n,\phi_n)$ are uniformly bounded in $L^2(Q_T)$. Thus, it follows from equation (4.13) that u_{nt} is uniformly bounded in $L^2(0,T;V')$, and passing to subsequences there exist functions $u(x,t)$, $\phi(x,t)$ and $\xi(t)$ such that for some α smaller than the one in (4.96),

$$u_n \to u \quad \textit{weakly in } L^2(0,T;V) \textit{ and strongly in } C^{\alpha,\alpha/2}(Q_T), \tag{4.97}$$

$$u_{nt} \to u_t \quad \textit{weakly in } L^2(0,T;V'), \tag{4.98}$$

$$\phi_n \to \phi \quad \textit{weakly in } L^2(0,T;V_D) \textit{ and strongly in } C^{\alpha,\alpha/2}(Q_T), \tag{4.99}$$

$$\xi_n(t) \to \xi(t) \quad \textit{strongly in } C^{\alpha/2}([0,T]). \tag{4.100}$$

Since $0 \le I_n(u_n) \le 1$, there is a function $0 \le \varrho \le 1$ such that

$$I_n(u_n) \to \varrho \quad \textit{weak} - \textit{star in } L^\infty(Q_T). \tag{4.101}$$

We now multiply both sides of (4.14) by a test function $w \in L^2(0,T;V_D)$ and integrate it over Q_T, then pass to the limit with respect to n to obtain (4.10). Next we multiply both sides of (4.13) by $w \in L^2(0,T;V)$ and integrate it over Q_T, then passing to the limit gives the following

$$\begin{aligned} &\int_{Q_T} \frac{du}{dt} w + \int_{Q_T} \nabla u \nabla w \\ &\quad + \eta \int_{Q_T}\int_\Omega G(x,y)u(y,t)\varrho(x,t)w(x,t)dydxdt \\ &\quad + \gamma \int_{Q_T} u^4 \varrho w = -\int_{Q_T} \sigma(u)\phi\nabla\phi\nabla w. \end{aligned} \tag{4.102}$$

Since u_n is nonnegative by the weak maximum principle, so is u. Moreover, we observe that $\varrho(x_0, t_0) = 1$ if $u(x_0, t_0) > 0$. Thus $\varrho(x,t)(w-u) \le w - u$ for all $w \ge 0$. Then if we replace w by $w-u$ in equation (4.102) with $w \in K$, (4.9) is obtained. Since u_n satisfies conditions (4.6), so does u. It remains to show that ϕ satisfies conditions (4.7) and (4.8). We recall that $\phi_n = \psi_n + \xi_n \varphi_n$, and that $\psi_n \to \psi$ and $\varphi_n \to \varphi$ where $\phi = \psi + \xi\varphi$. Thus ϕ satisfies the boundary condition (4.7). Furthermore,

$$\xi_n = \frac{I - \int_\Omega \sigma(u_n)\nabla\psi_n\nabla g}{\int_\Omega \sigma(u_n)\nabla\varphi_n\nabla g} \to \frac{I - \int_\Omega \sigma(u)\nabla\psi\nabla g}{\int_\Omega \sigma(u)\nabla\varphi\nabla g} = \xi.$$

Therefore, the nonlocal condition (4.8) holds. □

In view of Lemma 4.8, we also have the following uniqueness result whose proof is by energy inequality arguments (see also [2]).

Theorem 4.2. *Let (u^i, ϕ^i, ξ^i), $i = 1, 2$, be two $C^{\alpha,\alpha/2}(\bar{Q}_T)$ solutions to (4.9)-(4.11) corresponding to the data (u_0^i, ϕ_0^i, I^i) $i = 1, 2$. Then there exists a positive constant $C(t)$ such that*

$$\begin{aligned} &\|u^1 - u^2\|^2 + \int_0^t \|\nabla(u^1 - u^2)\|^2 ds + \int_0^t \|\nabla(\phi^1 - \phi^2)\|^2 ds \\ &\le C(t)\left[\|u_0^1 - u_0^2\|^2 + \int_0^t |I^1 - I^2| ds + \int_0^t \|\nabla(\Phi_0^1 - \Phi_0^2)\| ds\right]. \end{aligned} \tag{4.103}$$

Similar result holds for the penalized system (4.13)-(4.14). Consequently, both the obstacle problem and the penalized system for each n have a unique $C^{\alpha,\alpha/2}(\bar{Q}_T)$ solution.

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Chapter 5

On the Box Method for a Non-local Parabolic Variational Inequality

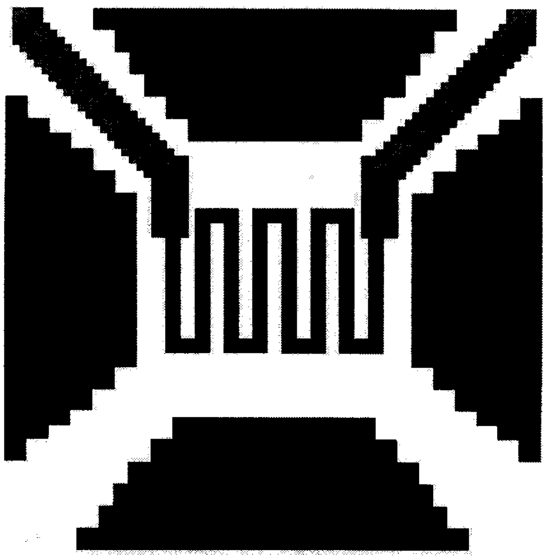
5.1 Introduction

Micromachined structures fabricated in standard technologies have been proposed and investigated in recent years for microsensor applications, see, e.g. [5, 6] and the references therein. These devices consist of very thin structures ($\simeq$ 5-10μm) suspended over a deep trench ($\simeq$ 80μm). Typical examples are Devices 1 and 2 shown in Figure 5-1. The length of Device 1 is approximately $1000\mu m$, while the serpentine resistor in the center of Device 2 has a width of $6\mu m$. The other dimensions may be estimated from the two figures, and further practical device details may be found in [5, 6, 7].

An applied current flows through the central resistor in Device 1, and through the zigzag resistor in the center of Device 2, which consist of a polycrystalline silicon layer whose resistance varies with temperature. Loss of energy from the resistor occurs through the supporting arms, through the surrounding gas and through radiation effects.

A possible application of such a structure as a gas pressure sensor is as follows: The electrical resistance of the structure is monitored and if the surrounding gas pressure were to drop - thereby decreasing the amount of heat loss by the device to the surrounding gas - the resistance would rise. It is therefore possible to determine the gas pressure by measuring the device resistance.

Device 1



Device 2

Figure 5-1. Device 1 and Device 2 ([7])

The simulation and modeling of such devices is now generally accepted as a very useful design tool. Accurate simulations offer the means to rapidly investigate the performance of proposed new devices, and to determine the effects on sensitivity of modifications of structures already constructed. These techniques avoid the lengthy cycle of iterating construction, device measurement, and reconstruction until - if ever - a suitable device is found. The simulation begins with the formulation and analysis of a system of partial differential equations, which in its most classic form has been studied for more than 100 years, [12]. It consists of the two equations:

$$u_t - \nabla[k(u)\nabla u] = \sigma(u)|\nabla\phi|^2, \tag{5.1}$$

$$-\nabla[\sigma(u)\nabla\phi] = 0, \tag{5.2}$$

and does not include terms to account for radiation losses nor for heat losses to the surrounding gas.

However, these terms are of paramount importance in the realistic simulation of the structures presently considered as we show below by actual example: their omission leads to very large errors between simulation and experiment. Radiation terms can be incorporated in the equations by means of expressions derived from the Stefan-Boltzmann Law, while gas losses are simulated in practice by means of an ad hoc averaging technique first introduced by Mastrangelo in 1991, [15]. In this approach, the heat loss is described by a nonlocal (i.e. integral) term, and we obtain the system consisting of (5.2) together with:

$$u_t - \nabla[k(u)\nabla u] + \eta \int_\Omega G(x,y)u(y)dy + \alpha u^4 = \sigma(u)|\nabla\phi|^2 \tag{5.3}$$

There have been many papers dealing with (5.1)-(5.2) in recent years. We refer the interested readers to [2] for theoretical results dealing with (5.2)-(5.3) in steady state (with $\alpha = 0$) and for further theoretical references, and to [4] for numerical results for (5.1)-(5.2) and further numerical references. These modifications of the classical thermistor equations lead to good agreement between simulation and experiment if global quantities such as total resistance changes versus applied current at various pressures are calculated. A practical illustration of the above comments can be seen in Figure 5-2 and Figure 5-3. These figures involve a comparison - at steady-state - of the percent changes of resistance ($\frac{\Delta R}{R}\%$) for Device 1 in a vacuum and at one atmosphere, respectively, as a function of the applied current (in milliamps).

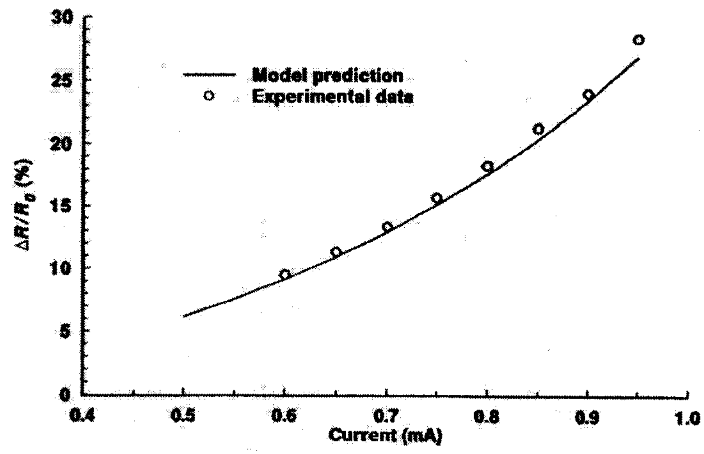


Figure 5-2. % change in resistance as a function of current for Device 1 in a vacuum ([7]).

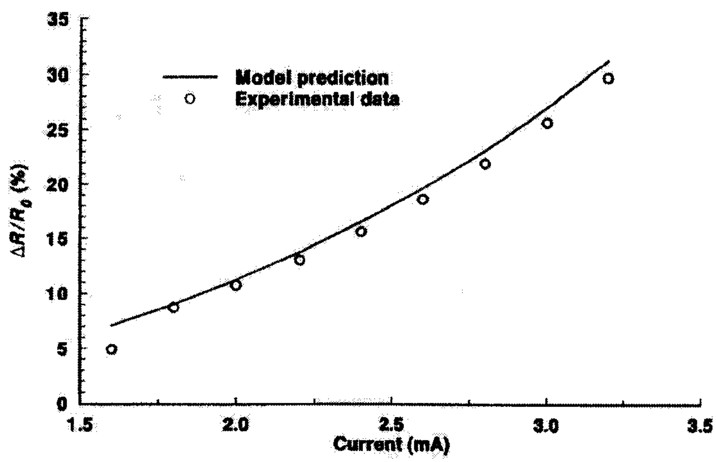


Figure 5-3. % change in resistance as a function of current for Device 1 at one atmosphere ([7]).

We observe the agreement between simulation and experiment, and the fact that the current needed to obtain a given $\frac{\Delta R}{R}\%$ is approximately four times the current required in a vacuum. However, if a more detailed analysis is required, such as an estimate of the heat loss at the base of one of the supporting arms, then this model can lead to unrealistic results for large gas pressures , i.e. large η. Indeed, it is possible, as a consequence of (5.2)-(5.3), to have negative temperatures in some parts of the device in cases where the surrounding temperature is assumed to be zero. To avoid this discrepancy, a new model has been suggested, [3], which involves a differential inequality in place of (5.3) with zero as the obstacle, thus ensuring that the temperature is always nonnegative. Furthermore and most importantly, the new model extends the old, i.e. solutions with positive temperature of the old model also are solutions for the new.

It is our intention to present results for the numerical discretization of the new model using the box method, which is the technique commonly employed in practice. This method (also so-called the finite volume element method) is a numerical method occupying an intermediate position between the finite difference and finite element methods. Usually, it is characterized by a trial space consisting of continuous piecewise linear polynomials on the primary triangulation and by a test space consisting of piecewise constants on the dual box mesh. Nowadays, the box method has been extensively and successfully used not only for various differential equations but also for variational inequalities. For example, Z. Cai developed the error estimates for a general self-adjoint elliptic boundary value problem in [10] and J. Steinbach gave comparison results between the finite volume element and finite element methods in [17] for elliptic variational inequalities. One of the important reasons for the box method being popular is that most of the time it is derived from local balance equations directly and so it conserves important physical properties. Another important reason is that it is easy to implement and provides effective discretization processes for multilevel adaptive methods (see [16]).

In summary, in this chapter we apply the box method to the following nonlocal obstacle thermistor problem:

$$\textit{Find } u \in K = \{v \in H_0^1(\Omega) | v \geq 0\} \textit{ such that}$$

$$(u_t, v-u) + (k(u)\nabla u, \nabla(v-u)) \tag{5.4}$$
$$+\eta(\int_\Omega G(x,y)u(y)dy, v-u) + \alpha(u^4, v-u)$$
$$\geq (\sigma(u)|\nabla\phi|^2, v-u), \qquad \forall\ v \in K,$$

$$(\sigma(u)\nabla\phi, \nabla v) = 0, \qquad \forall\ v \in H_0^1(\Omega), \tag{5.5}$$

where $(\cdot,\cdot)$ denotes the standard $L^2(\Omega)$ inner product. Here, u denotes the temperature in the thermistor and φ the potential. We shall construct and analyze a box scheme for (5.4)-(5.5), subject to the initial/boundary conditions given below. Our main result is an optimal H^1 error estimate (Theorem 5.2) for the scheme.

We refer the reader to [1] and [11] for the standard definition of $L^2(\Omega)$, $H^s(\Omega)$, $W^{s,p}(\Omega)$ and their associated norms $\|\cdot\|$, $\|\cdot\|_s$, $\|\cdot\|_{s,p}$. We refer to [18] for the definition of the space $L^p(0,T;X)$ with X a Banach space and its associated norm.

For simplicity, we assume that $\Omega \subset R^2$ is a polygonal domain. Furthermore, we associate with the system (5.4)-(5.5) the initial and boundary conditions: $u(x,0) = u_0(x)$, $u|_{\partial\Omega} = 0$, $\phi|_{\partial\Omega} = g(x)$.

Throughout this chapter we assume that

(A1) $u_0(x) \in H_0^1(\Omega)$, $g(x) \in C^\infty(\bar{\Omega})$ and $\eta, \alpha > 0$.

(A2) $G(x,y)$ is positive definite and

$\int_\Omega G(x,y)u(y)dy \geq 0$ if $u \geq 0$, $\int_\Omega \int_\Omega G^2(x,y) < \infty$.

(A3) $0 < m \leq \sigma(s), k(s) \leq M < \infty$ and there exists $m_0 > 0$ such that

$|\sigma(s) - \sigma(s')| + |k(s) - k(s')| \leq m_0|s - s'|$, s, $s' \in R$.

Given these conditions on the data, we assume the solution of (5.4)-(5.5) exists and is unique, and also satisfies the following regularity:

(A4) $(u,\phi) \in L^\infty(0,T;H_0^1(\Omega) \cap H^2(\Omega)) \times L^\infty(0,T;H^2(\Omega) \cap W^{1,\infty}(\Omega))$,

$u_t \in L^2(0,T;H^1(\Omega))$.

For theoretical results for (5.4)-(5.5) we refer the interested reader to [3].

Finally, we comment on our assumptions: Equations (5.4)-(5.5) and assumptions (A1)-(A4) represent a simplification of the equations actually employed to simulate structures such as Devices 1 and 2 described earlier, and we are not aware of theoretical or numerical results for the full model used in practice. We observe, in particular, that since the devices under consideration are very thin, the assumption $\Omega \subset R^2$ seems fairly reasonable. However, in realistic situations, ϕ usually satisfies a mixed Dirichlet/Neumann boundary condition and $\sigma(s)$, $k(s)$ may degenerate as $s \to \infty$, as is described in some of the cited references. The study of these more general situations is presently under consideration.

5.2 Basics of finite volume methods and known results

Let T^h be a regular and quasi-uniform triangulation (see [4]) of Ω and T^h_v the set of vertices of T^h. For each $p \in T^h_v$, we associate the box $b_p \in B^h$ which consists of the union of the subregions which have p as a corner. Here B^h denotes the dual mesh based upon T^h. In this chapter we only discuss the case that B^h is a so-called circumcenter dual mesh (See Figure 5-4). We refer to [4] for detailed information on constructing such a dual mesh.

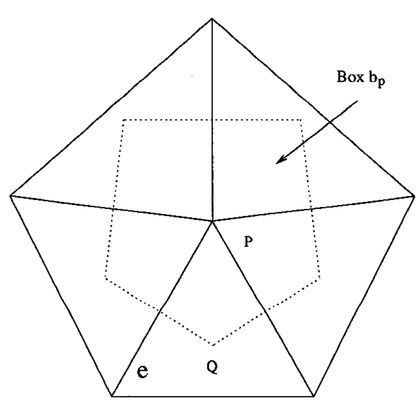


Figure 5-4. A box b_p centered at P with Q as the circumcenter

of the element $e \in T^h$.

Our piece-wise linear finite element subspace $S^h \subset H^1(\Omega)$ corresponding to the triangulation T^h is given by

$$S^h = \{v \in H^1(\Omega) : \; v|_e \; is \; a \; linear \; function \; for \; all \; e \in T^h\}$$

and

$$S_0^h = S^h \cap H_0^1(\Omega).$$

Then the convex trial subset $K^h \subset K$ is defined by

$$K^h = \{v \in S_0^h | v(p) \geq 0 \;\; for \;\; all \;\; p \in T_v^h\}.$$

It follows from [11, Chapter 3], [9] that the following inequalities hold:

Lemma 5.1. *There exists a positive constant C independent of S^h such that*

$$\|v\|_{\beta,q} \leq C h^{\mu-\beta-2\max\{0,1/p-1/q\}} \|v\|_{\mu,p}, \tag{5.6}$$

$$0 \leq \mu \leq \beta \leq 1, \;\; 1 \leq p, \; q \leq \infty, \;\; \forall v \in S^h,$$

$$\|v\|_{0,\infty} \leq C |\ln h|^{1/2} \|v\|_1, \qquad \forall v \in S^h, \tag{5.7}$$

$$\|w - P_h w\| + h\|w - P_h w\|_1 \leq C h^2 \|w\|_2, \quad \forall w \in H^2(\Omega), \tag{5.8}$$

$$\|w - P_h w\|_{0,\infty} \leq C h \|w\|_2, \tag{5.9}$$

$$\|P_h w\|_{1,\infty} \leq C \|w\|_{1,\infty}, \; \forall w \in H^2(\Omega),$$

where $P_h : L^2(\Omega) \to S^h$ is the standard L^2-projection.

Let $N^h(p)$ denote the set of the neighboring vertices of $p \in T_v^h$ and ∂b_p denote the boundary of b_p, whence $\partial b = \cup_{p \in T_v^h} \partial b_p$, $\partial b_p = \cup_{p^* \in N^h(p)} \{\Gamma_{pp^*}\}$ where $\Gamma_{pp^*} = \partial b_p \cap \partial b_{p^*}$, and let $l_{\partial b} : \partial b \to R^+$ be defined as follows: For $p \in T_v^h$ and $b_p \in B_h$,

$$l_{\partial b}|_{\Gamma_{pp^*}} = |p - p^*| \qquad for \quad p^* \in N^h(p). \tag{5.10}$$

Then it is easy to see that there exists a constant $C > 0$ such that $C^{-1}h \leq l_{\partial b}|_{\Gamma_{pp^*}} \leq Ch$, $\forall b \in B^h$.

For $b \in B^h$ and $x \in \partial b$, we denote the jump in w across ∂b at x by $[w]_{\partial b}(x) = w(x+0) - w(x-0)$, where $w(x \pm 0)$ are the two (outside and inside) limit values of $w(x)$ along the normal directions to ∂b.

Moreover, we define the piece-wise constant interpolation operator $I_h : C(\Omega) \rightarrow L^2(\Omega)$ by

$$I_h u = u(p), \quad on \;\; b_p \in B^h, \quad \forall p \in T_v^h \tag{5.11}$$

and the corresponding discrete norms by

$$\|v\|_{1,h} \equiv (\sum_{l \in \partial b} |[I_h v]_l|^2)^{1/2} \;\; and \;\; \|v\|_{0,h} \equiv \|I_h v\|. \tag{5.12}$$

The subscript "h" will be employed for a norm notation only in these cases.

Let us recall the following two lemmas from [4, 8, 10].

Lemma 5.2. *There exists a constant $C > 0$ such that*

$$C^{-1}\|\nabla v\| \leq \|v\|_{1,h} \leq C\|\nabla v\|, \quad \forall v \in S^h, \tag{5.13}$$

$$C^{-1}\|v\| \leq \|v\|_{0,h} \leq C\|v\|, \quad \forall v \in S^h, \tag{5.14}$$

$$\|v - I_h v\| \leq Ch\|v\|_1 \quad \forall v \in S^h. \tag{5.15}$$

Lemma 5.3. *For any $a \in C(\bar{\Omega})$, there exists a constant $C > 0$ such that*

$$|-\sum_{b \in B^h} \int_{\partial b} a \frac{\partial u}{\partial \nu} I_h v| \leq C\|u\|_1 \|v\|_1, \quad \forall u, v \in S^h. \tag{5.16}$$

Moreover, if there exists a constant $a_0 > 0$ such that $a \geq a_0$ in Ω, then

$$-\sum_{b \in B^h} \int_{\partial b} a \frac{\partial v}{\partial \nu} I_h v \geq C^{-1}\|v\|_1^2, \quad \forall v \in S_0^h, \tag{5.17}$$

where ν is the outward normal.

As in [8] and [10], let $Q_h : H^2(\Omega) \to S^h$ be defined by $Q_h w - i_h w \in S_0^h$ and

$$-\sum_{b\in B^h}\int_{\partial b} a\frac{\partial(w-Q_h w)}{\partial\nu} I_h v = 0, \quad \forall v \in S_0^h, \tag{5.18}$$

where $i_h : C(\Omega) \to S^h$ is the Lagrangian interpolation operator with $w \in H^2(\Omega)$.

It also follows from [4, 8, 10] that the following two lemmas hold.

Lemma 5.4. *Assume that $a \in C(\bar{\Omega})$ with $a \geq a_0 > 0$. Then there exists $C > 0$ such that for $w \in H^2(\Omega)$,*

$$\|w - Q_h w\|_1 \leq Ch\|w\|_2. \tag{5.19}$$

Furthermore, if $w \in H^2(\Omega) \cap W^{1,\infty}(\Omega)$, then

$$\|Q_h w\|_{1,\infty} \leq C(\|w\|_{1,\infty} + \|w\|_2). \tag{5.20}$$

Lemma 5.5. *(also see [11]) There holds for each $b \in B^h$*

$$h^{1/2}\|w\|_{L^2(\partial b)} \leq C(\|w\|_{L^2(b)} + h\|w\|_{H^1(b)}), \quad \forall w \in H^1(b). \tag{5.21}$$

5.3 The box approximation of the obstacle problem

5.3.1 The box scheme

We construct the box scheme for (5.4)-(5.5) as follows:

$$\begin{aligned}
&\textit{Find} \quad (u^h, \phi^h) \in K^h \times S^h \quad \textit{such that} \\
&(I_h u_t^h, I_h(v^h - u^h)) - \sum_{b\in B^h}\int_{\partial b} k(u^h)\frac{\partial u^h}{\partial\nu} I_h(v^h - u^h) \qquad (5.22)\\
&\quad +\eta(\int_\Omega G(x,y)u^h(y)dy, I_h(v^h - u^h)) + \alpha((u^h)^4, I_h(v^h - u^h)) \\
&\geq (\sigma(u^h)|\nabla\phi^h|^2, I_h(v^h - u^h)), \quad \forall v^h \in K^h, \\
&-\sum_{b\in B^h}\int_{\partial b} \sigma(u^h)\frac{\partial\phi^h}{\partial\nu} I_h v^h = 0, \quad \forall v^h \in S_0^h, \qquad (5.23)
\end{aligned}$$

and

$$u^h(0) = P_h u_0, \quad \phi^h|_{\partial\Omega} = i_h g, \tag{5.24}$$

where P_h is the L^2 projection and i_h is the Lagrangian interpolation operator as stated in the previous section.

We comment that by $u^h \in K^h$ we mean that this is true for all t, i.e. $u^h(t,\cdot) \in K^h$. To avoid complicating the notation, we shall not explicitly make reference to "t" in what follows. Similar remarks apply to ϕ^h, etc.

Under the assumptions (A.1)-(A.4) in Section 5.1 we derive an a priori estimate for the solutions of (5.22)-(5.24) which will be used later.

Lemma 5.6. *Assume that (5.22)-(5.24) has a solution $(u^h, \phi^h) \in K^h \times S^h$. Then there exists a constant $C > 0$, independent of t and S^h, such that*

$$\|\phi^h(t)\|_1 \leq C\|\phi(t)\|_{1,p}, \quad p \geq 2, \quad 0 \leq t \leq T. \tag{5.25}$$

Also there exists a constant $C_0 = C_0(h) > 0$ such that

$$\begin{aligned} &\|u^h\|_{L^\infty(0,T;L^2)} + \|u^h\|_{L^2(0,T;H^1)} \\ &\leq C\|u_0\| + C_0(h)T^{1/2}\|\phi\|^2_{L^\infty(0,T;W^{1,p})}, \quad p \geq 2. \end{aligned} \tag{5.26}$$

Proof. By choosing $v^h = \phi^h - i_h\phi$ as the test function in (5.23) we obtain

$$\begin{aligned} &-\sum_{b\in B^h}\int_{\partial b}\sigma(u^h)\frac{\partial(\phi^h - i_h\phi)}{\partial\nu}I_h(\phi^h - i_h\phi) \\ &= \sum_{b\in B^h}\int_{\partial b}\sigma(u^h)\frac{\partial(i_h\phi)}{\partial\nu}I_h(\phi^h - i_h\phi). \end{aligned} \tag{5.27}$$

In view of Lemma 5.3 we have

$$C^{-1}\|\phi^h - i_h\phi\|_1^2 \leq C\|i_h\phi\|_1\|\phi^h - i_h\phi\|_1. \tag{5.28}$$

On the other hand,

$$\|\phi^h\|_1 - \|i_h\phi\|_1 \leq \|\phi^h - i_h\phi\|_1, \tag{5.29}$$

$$\|i_h\phi\|_1 \leq C\|\phi\|_{1,p}, \quad p \geq 2. \tag{5.30}$$

Thus, (5.25) follows from (5.28)-(5.30) immediately.

By assumption (A.2) together with $u^h \geq 0$ on Ω, we have

$$\eta(\int_\Omega G(x,y)u^h(y)dy, I_h u^h) + \alpha((u^h)^4, I_h u^h) \geq 0. \tag{5.31}$$

Therefore, if we take $v^h = 0$ as the test function in (5.22), then it follows from (5.17), (5.31) that

$$\frac{1}{2}\frac{d}{dt}\|I_h u^h\|^2 + C^{-1}\|u^h\|_1^2 \leq (\sigma(u^h)|\nabla\phi^h|^2, I_h u^h). \tag{5.32}$$

But from (5.25) and (5.7) we also have

$$\begin{aligned}(\sigma(u^h)|\nabla\phi^h|^2, I_h u^h) &\leq C\|\phi^h\|_1^2\|I_h u^h\|_{0,\infty} \leq C\|\phi^h\|_1^2\|u^h\|_{0,\infty} \\ &\leq C|\ln h|^{1/2}\|\phi\|_{1,p}^2\|u^h\|_1 \leq C(h)\|\phi\|_{1,p}^4 + \frac{1}{2C}\|u^h\|_1^2.\end{aligned} \tag{5.33}$$

Hence,

$$\frac{d}{dt}\|I_h u^h\|^2 + \|u^h\|_1^2 \leq C(h)\|\phi\|_{1,p}^4. \tag{5.34}$$

Integrating both sides of (5.34) from 0 to t gives

$$\|I_h u^h\|^2 + \int_0^t \|u^h\|_1^2 \leq \|I_h P_h u_0\|^2 + C(h)\int_0^t \|\phi\|_{1,p}^4. \tag{5.35}$$

Therefore,

$$\|u^h\|^2 + \int_0^t \|u^h\|_1^2 \leq C\|u_0\|^2 + C(h)T\|\phi\|_{L^\infty(0,T;W^{1,p})}^4. \tag{5.36}$$

Finally, (5.26) follows immediately from (5.36). □

5.3.2 Existence and uniqueness of the semi-discrete system

In this section we will show that the semi-discrete system (5.22)-(5.24) has a unique solution.

First of all, we define an operator $u^h = Fw^h$: $S^h \to S^h$ in the following way: Given w^h,

$$\begin{aligned} & \textit{find } u^h \in K^h,\ \phi^h \in S^h,\ \textit{such that } u^h(0) = P_h u_0,\ \phi^h|_{\partial\Omega} = i_h g \textit{ and} \\ & (I_h u_t^h, I_h(v^h - u^h)) + a_{w^h}(u^h, v^h - u^h) \qquad (5.37) \\ & \geq (\sigma(w^h)|\nabla\phi^h|^2, I_h(v^h - u^h)), \quad \forall v^h \in K^h, \\ & -\sum_{b\in B^h} \int_{\partial b} \sigma(w^h)\frac{\partial\phi^h}{\partial\nu} I_h v^h = 0, \quad \forall v^h \in S_0^h. \qquad (5.38) \end{aligned}$$

Here, the bilinear form $a_{w^h}(\cdot,\cdot)$ is defined by

$$\begin{aligned} a_{w^h}(u^h, v^h) = & -\sum_{b\in B^h} \int_{\partial b} k(w^h)\frac{\partial u^h}{\partial\nu} I_h v^h \qquad (5.39) \\ & +\eta(\int_\Omega G(x,y)u^h(y)dy, I_h v^h(x)) \\ & +\alpha((w_+^h)^3 u^h, I_h v^h), \quad \forall u^h \in K^h, \quad v^h \in S_0^h. \end{aligned}$$

with

$$w_+^h = w^h, \quad \textit{if} \quad w^h \geq 0; \quad 0, \quad \textit{otherwise}. \qquad (5.40)$$

Obviously, $a_{w^h}(\cdot,\cdot)$ satisfies

$$a_{w^h}(u^h, u^h) \geq C\|u^h\|_1^2, \qquad \forall u^h \in K^h, \qquad (5.41)$$

$$a_{w^h}(u^h, v^h) \leq C\|u^h\|_1\|v^h\|_1, \qquad \forall u^h \in K^h, \quad v^h \in S_0^h. \qquad (5.42)$$

These two properties (i.e., (5.41), (5.42)) ensure that the linear system (5.37), (5.38) has a unique solution in $K^h \times S^h$ (see [14]).

If there exists a fixed point for the map F, i.e., for some u^h, $Fu^h = u^h$, then u^h is a solution of the system (5.22)-(5.24), since $u^h \geq 0$ and $u_+^h = u^h$ by the definition. We now show that F is a contraction map and hence it has a fixed point.

Similar to Lemma 5.6, the solution (u^h, ϕ^h) of (5.37) and (5.38) also satisfies (5.26). If we define the subset $E \subset S_0^h$ by

$$E = \{w^h \in S_0^h : \|w^h\|_{L^\infty(0,T;L^2)} \leq C\|u_0\| + C_0(h)T^{1/2}\|\phi\|^2_{L^\infty(0,T;W^{1,p})}\}, \quad (5.43)$$

then clearly $F : E \to E$ for $0 \le t \le T$.

Let w_1^h, $w_2^h \in E$. The corresponding solutions of (5.37) and (5.38) are defined by (u_1^h, ϕ_1^h) and (u_2^h, ϕ_2^h). Then we have the following lemma.

Lemma 5.7. *The solutions* (u_1^h, ϕ_1^h) *and* (u_2^h, ϕ_2^h) *satisfy*

$$\|\phi_1^h - \phi_2^h\| \le C(h)\|w_1^h - w_2^h\|, \tag{5.44}$$

$$\|u_1^h - u_2^h\|^2 \le C_2(h)\exp(C_1(h)t)\int_0^t \|w_1^h - w_2^h\|^2 ds. \tag{5.45}$$

Proof. It follows from (5.38) by taking $v^h = \phi_1^h - \phi_2^h$ that

$$\begin{aligned} &-\sum_{b\in B^h}\int_{\partial b}\sigma(w_2^h)\frac{\partial(\phi_1^h - \phi_2^h)}{\partial\nu} I_h(\phi_1^h - \phi_2^h) \\ &= \sum_{b\in B^h}\int_{\partial b}(\sigma(w_1^h) - \sigma(w_2^h))\frac{\partial\phi_1^h}{\partial\nu} I_h(\phi_1^h - \phi_2^h). \end{aligned} \tag{5.46}$$

In view of (5.17) and (A.3) we have

$$\|\phi_1^h - \phi_2^h\|_1 \le C\|\phi_1^h\|_{1,\infty}\sum_{b\in B^h}\|w_1^h - w_2^h\|_{L^2(\partial b)}. \tag{5.47}$$

The inverse inequality (5.6) and Lemma 5.5 show that

$$\|\phi_1^h\|_{1,\infty} \le Ch^{-1}\|\phi_1^h\|_1, \tag{5.48}$$

$$\begin{aligned} &\|w_1^h - w_2^h\|_{L^2(\partial b)} \\ &\le Ch^{-1/2}(\|w_1^h - w_2^h\|_{L^2(b)} + h\|w_1^h - w_2^h\|_{H^1(b)}) \\ &\le Ch^{-1/2}\|w_1^h - w_2^h\|_{L^2(b)}. \end{aligned} \tag{5.49}$$

Substituting (5.48) and (5.49) into (5.47) yields (5.44).

To show (5.45), we choose the test function $v^h = u_2^h$ with respect to w_1^h and $v^h = u_1^h$ with respect to w_2^h in (5.37) and add the resulting inequalities together to obtain:

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|I_h(u_1^h - u_2^h)\|^2 + a_{w_1^h}(u_1^h, u_1^h - u_2^h) - a_{w_2^h}(u_2^h, u_1^h - u_2^h) \\ &\le (\sigma(w_1^h)|\nabla\phi_1^h|^2 - \sigma(w_2^h)|\nabla\phi_2^h|^2, I_h(u_1^h - u_2^h)). \end{aligned} \tag{5.50}$$

We first estimate the left hand side of (5.50) term by term.

$$
\begin{aligned}
&-\sum_{b\in B^h}\int_{\partial b}\sigma(w_1^h)\frac{\partial u_1^h}{\partial \nu}I_h(u_1^h-u_2^h)+\sum_{b\in B^h}\int_{\partial b}\sigma(w_2^h)\frac{\partial u_2^h}{\partial \nu}I_h(u_1^h-u_2^h)\\
&=-\sum_{b\in B^h}\int_{\partial b}\sigma(w_1^h)\frac{\partial (u_1^h-u_2^h)}{\partial \nu}I_h(u_1^h-u_2^h) \qquad (5.51)\\
&+\sum_{b\in B^h}\int_{\partial b}(\sigma(w_1^h)-\sigma(w_2^h))\frac{\partial u_2^h}{\partial \nu}I_h(u_1^h-u_2^h)\\
&\geq C^{-1}\|u_1^h-u_2^h\|_1^2-C\|u_2^h\|_{1,\infty}\,\|w_1^h-w_2^h\|\;\|u_1^h-u_2^h\|_1,\\
&\geq C^{-1}\|u_1^h-u_2^h\|_1^2-C(h)\|w_1^h-w_2^h\|\;\|u_1^h-u_2^h\|_1,
\end{aligned}
$$

where (5.6), (5.17), (5.21) and (A.3) have been used. Furthermore, thanks to (5.6) and (5.43), we have

$$
\begin{aligned}
&\eta(\int_\Omega G(x,y)(u_1^h-u_2^h)(y)dy, I_h(u_1^h-u_2^h)) \qquad (5.52)\\
&=\eta\int_\Omega\int_\Omega G(x,y)(u_1^h-u_2^h)(y)I_h(u_1^h-u_2^h)dydx\\
&\leq \eta\|u_1^h-u_2^h\|_{0,\infty}\;\|G(x,y)\|_{L^2(\Omega\times\Omega)}\;\|u_1^h-u_2^h\|_1\\
&\leq C(h)\|u_1^h-u_2^h\|^2+\frac{1}{3}C^{-1}\|u_1^h-u_2^h\|_1^2,
\end{aligned}
$$

$$
\begin{aligned}
&\alpha((w_{1+}^h)^3u_1^h-(w_{2+}^h)^3u_2^h, I_h(u_1^h-u_2^h)) \qquad (5.53)\\
&=\alpha((w_{1+}^h)^3(u_1^h-u_2^h)[(w_{1+}^h)^3-(w_{2+}^h)^3]u_2^h, I_h(u_1^h-u_2^h))\\
&\leq C(h)\|u_1^h-u_2^h\|^2+C(h)\|w_{1+}^h-w_{2+}^h\|\;\|u_1^h-u_2^h\|\\
&\leq C_1(h)\|u_1^h-u_2^h\|^2+C_2(h)\|w_1^h-w_2^h\|^2.
\end{aligned}
$$

Thus, the combination of (5.51) - (5.53) gives that

$$
\begin{aligned}
&a_{w_1^h}(u_1^h,u_1^h-u_2^h)-a_{w_2^h}(u_2^h,u_1^h-u_2^h) \qquad (5.54)\\
&\geq\frac{1}{3}C^{-1}\|u_1^h-u_2^h\|_1^2-C_1(h)\|u_1^h-u_2^h\|^2-C_2(h)\|w_1^h-w_2^h\|^2.
\end{aligned}
$$

Now we estimate the right hand side of (5.50). In view of (5.6) and (5.44), we find

$$
\begin{aligned}
&(\sigma(w_1^h)|\nabla\phi_1^h|^2 - \sigma(w_2^h)|\nabla\phi_2^h|^2, I_h(u_1^h - u_2^h)) \qquad (5.55)\\
&= (\sigma(w_1^h)\nabla(\phi_1^h + \phi_2^h)\nabla(\phi_1^h - \phi_2^h), I_h(u_1^h - u_2^h))\\
&\quad +((\sigma(w_1^h) - \sigma(w_2^h))|\nabla\phi_2^h|^2, I_h(u_1^h - u_2^h))\\
&\le C(h)\|w_1^h - w_2^h\|\, \|u_1^h - u_2^h\|.
\end{aligned}
$$

Substituting (5.54) and (5.55) into (5.50) together with (5.6) and (5.14), yields

$$
\frac{d}{dt}\|I_h(u_1^h - u_2^h)\|^2 - C_1(h)\|I_h(u_1^h - u_2^h)\|^2 \le C_2(h)\|w_1^h - w_2^h\|^2. \qquad (5.56)
$$

Applying the integrating factor $\exp(-C_1(h)t)$ to both sides of (5.56) and integrating from 0 to t gives

$$
\|I_h(u_1^h - u_2^h)\|^2 \le C_2(h)\exp(C_1(h)t)\int_0^t \exp(-C_1(h)s)\|w_1^h - w_2^h\|^2 ds. \quad (5.57)
$$

By the equivalence $\|I_h \cdot\|$ and $\|\cdot\|$, estimate (5.45) is obtained. This completes the proof. □

Now we are ready to state our main result of this section.

Theorem 5.1. *There exists a unique solution to (5.22)-(5.24) for all* $0 \le t \le T$.

Proof. By estimate (5.45) in Lemma 5.7, the map F earlier defined is a contraction under the norm $\|\cdot\|_{L^\infty(0,T;L^2(\Omega))}$ for t sufficiently small, say $0 \le t \le t_0$. Thus it has a fixed point in E for $t \in (0, t_0]$. Therefore, (5.22)-(5.24) has this fixed point as a solution in $(0, t_0]$. Moreover, it is easy to show that the solution is unique. Otherwise, let (u_1^h, ϕ_1^h) and (u_2^h, ϕ_2^h) be two different solutions. A procedure similar to that used in the proof of Lemma 5.7 shows that

$$
\|\phi_1^h - \phi_2^h\|^2 \le C(h)\|u_1^h - u_2^h\|^2 \le C_2(h)\exp\{C_1(h)t_0\}\int_0^t \|u_1^h - u_2^h\|^2 ds. \quad (5.58)
$$

Then Gronwall's Lemma implies that $u_1^h = u_2^h$, so $\phi_1^h = \phi_2^h$. Due to the boundedness of the solutions, as a consequence of Lemma 5.6, a standard bootstrapping argument shows that one can extend this solution to a unique global solution. □

5.3.3 An H^1 error estimate

In what follows, we will analyze the proposed box scheme (5.22)-(5.24) for the obstacle problem (5.4)-(5.5). An optimal H^1 error estimate will be derived. Our main results are summarized in the following theorem.

Theorem 5.2. *Assume that (A.1)-(A.4) hold. For sufficiently small h, there exists a positive constant $C > 0$, independent of h, such that*

$$\|\phi - \phi^h\|_{L^\infty(0,T;H^1)} + \|u - u^h\|_{L^\infty(0,T;L^2)} + \|u - u^h\|_{L^2(0,T;H^1)} \leq Ch. \quad (5.59)$$

Proof. We simplify the long calculations by decomposing the proof into several lemmas.

Lemma 5.8. $\nabla[k(u)\nabla u] \in L^2(\Omega)$ *for a.e.* $t \in [0, T]$.

Proof. (A3) implies that $k(s)$ is differentiable a.e. in R. Moreover, if the derivatives $k'(s)$ exists, it is uniformly bounded by m_0. Thus,

$$\nabla[k(u)\nabla u] = k'(u)|\nabla u|^2 + k(u)\Delta u. \quad (5.60)$$

Obviously, the Sobolev imbedding theorem guarantees that the right hand sides of (5.60) is in $L^2(\Omega)$. So the lemma holds by (A.4). □

Hereafter, we denote I as the identity.

Lemma 5.9. *Let (u, ϕ) be the exact solution of (5.4)-(5.5) and (u^h, ϕ^h) be the solution of the box scheme (5.22)-(5.24), then*

$$\|\phi - \phi^h\|_1 \leq C(h + \|u^h - P_h u\|). \quad (5.61)$$

Proof. Let Q_h be defined as in (5.18) with a and w replaced by $\sigma(u)$ and ϕ, respectively. Letting $v^h = \phi^h - Q_h\phi$ in (5.23) , we obtain

$$
\begin{aligned}
0 = &- \sum_{b\in B^h} \int_{\partial b} \sigma(u^h)\frac{\partial \phi^h}{\partial \nu} I_h(\phi^h - Q_h\phi) \qquad (5.62)\\
= &- \sum_{b\in B^h} \int_{\partial b} \sigma(u^h)\frac{\partial (\phi^h - Q_h\phi)}{\partial \nu} I_h(\phi^h - Q_h\phi)\\
&- \sum_{b\in B^h} \int_{\partial b} (\sigma(u^h) - \sigma(u))\frac{\partial (Q_h\phi)}{\partial \nu} I_h(\phi^h - Q_h\phi)\\
&- \sum_{b\in B^h} \int_{\partial b} \sigma(u)\frac{\partial (Q_h\phi - \phi)}{\partial \nu} I_h(\phi^h - Q_h\phi)\\
&- \sum_{b\in B^h} \int_{\partial b} \sigma(u)\frac{\partial \phi}{\partial \nu} (I_h - I)(\phi^h - Q_h\phi)\\
&- \sum_{b\in B^h} \int_{\partial b} \sigma(u)\frac{\partial \phi}{\partial \nu} (\phi^h - Q_h\phi)\\
= &\ I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned}
$$

We estimate (5.62) term by term. First, by (5.17) we obtain

$$I_1 \geq C^{-1}\|\phi^h - Q_h\phi\|_1^2. \qquad (5.63)$$

Since σ is Lipschitz continuous and $I_h(\phi^h - Q_h\phi)$ is a constant on any box $b \in B^h$, it follows from (5.14), (5.20) and the Schwarz inequality that

$$
\begin{aligned}
I_2 &\leq C\|\phi^h - Q_h\phi\|_1 (\sum_{b\in B^h} (\int_{\partial b} |u^h - u|)^2 (\int_{\partial b} |\frac{\partial Q_h\phi}{\partial \nu}|^2))^{1/2} \qquad (5.64)\\
&\leq C\|\phi^h - Q_h\phi\|_1 (\sum_{b\in B^h} \int_{\partial b} ds \int_{\partial b} (u^h - u)^2)^{1/2}\\
&\leq Ch^{1/2}\|\phi^h - Q_h\phi\|_1 (\sum_{b\in B^h} \|u^h - u\|^2_{L^2(\partial b)})^{1/2}.
\end{aligned}
$$

Using the trace inequality (5.21), inverse inequality (5.6) and the property (5.8) of P_h, we obtain

$$
\begin{aligned}
\sum_{b\in B^h} h\|u - u^h\|^2_{L^2(\partial b)} &\leq C \sum_{b\in B^h} (\|u^h - u\|^2_{L^2(b)} + h^2\|u^h - u\|^2_{H^1(b)}) \qquad (5.65)\\
&\leq C \sum_{b\in B^h} (\|u - P_h u\|^2_{L^2(b)} + h^2\|u - P_h u\|^2_{H^1(b)}\\
&\quad + \|u^h - P_h u\|^2_{L^2(b)} + h^2\|u^h - P_h u\|^2_{H^1(b)})\\
&\leq C(h^4 + \|u^h - P_h u\|^2).
\end{aligned}
$$

Substituting (5.65) into (5.64) yields

$$I_2 \le C\|\phi^h - Q_h\phi\|_1(h^2 + \|u^h - P_h u\|). \tag{5.66}$$

By the definition (5.18) of Q_h, $I_3 = 0$. Applying Green's theorem in I_4 together with (5.5), (5.15) and (5.8) give that

$$\begin{aligned} I_4 &= \int_\Omega \nabla[\sigma(u)\nabla\phi][(I - I_h)(\phi^h - Q_h\phi)] \\ &\quad + \int_\Omega \sigma(u)\nabla\phi\nabla(\phi^h - Q_h\phi) \\ &= \int_\Omega \nabla[\sigma(u)\nabla\phi][(I - I_h)(\phi^h - Q_h\phi)] \\ &= 0. \end{aligned} \tag{5.67}$$

We observe

$$I_5 = \int_{\partial\Omega} \sigma(u)\frac{\partial\phi}{\partial\nu}(\phi^h - Q_h\phi) = 0. \tag{5.68}$$

Finally, (5.9) follows by substituting the estimates of I_1 to I_5 into (5.62) and (5.19). This completes the proof. □

Lemma 5.10. *The following inequality holds:*

$$\begin{aligned} &(\sigma(u^h)|\nabla\phi^h|^2, I_h(u^h - P_h u)) + (\sigma(u)|\nabla\phi|^2, u - u^h) \\ &\le C_1 h^2 + C_2(h + \|u^h - P_h u\|)\|u^h - P_h u\|_1 \\ &\quad + C_3|\ln h|^{1/2}\|u^h - P_h u\|\ \|u^h - P_h u\|_1^2. \end{aligned} \tag{5.69}$$

Proof. A simple computation gives that

$$\begin{aligned} &(\sigma(u^h)|\nabla\phi^h|^2, I_h(u^h - P_h u)) + (\sigma(u)|\nabla\phi|^2, u - u^h) \\ &= (\sigma(u^h)(|\nabla\phi^h|^2 - |\nabla\phi|^2), I_h(u^h - P_h u)) \\ &\quad + ((\sigma(u^h) - \sigma(u))|\nabla\phi|^2, I_h(u^h - P_h u)) \\ &\quad + (\sigma(u)|\nabla\phi|^2, (I_h - I)(u^h - P_h u)) \\ &\quad + (\sigma(u)|\nabla\phi|^2, u - P_h u) \\ &= J_1 + J_2 + J_3 + J_4. \end{aligned} \tag{5.70}$$

By (5.14), Lemma 5.9 and the inverse inequality (5.7), we have

$$\begin{aligned} J_1 &= (\sigma(u^h)|\nabla\phi^h - \nabla\phi|^2, I_h(u_h - P_h u)) \\ &\quad +2(\sigma(u^h)\nabla\phi(\nabla\phi^h - \nabla\phi), I_h(u^h - P_h u)) \\ &\le C\|\phi^h - \phi\|_1^2 \, \|u^h - P_h u\|_{0,\infty} + C\|\phi^h - \phi\|_1 \, \|u^h - P_h u\| \\ &\le C(h^2 + \|u^h - P_h u\|^2)|\ln h|^{1/2}\|u^h - P_h u\|_1 \\ &\quad +C(h + \|u^h - P_h u\|)\|u^h - P_h u\|. \end{aligned} \tag{5.71}$$

If h is small enough, say $h \le \frac{1}{\sqrt{e}}$ and $h|\ln h|^{1/2} \le 1$, then

$$\begin{aligned} J_1 &\le C(h + \|u^h - P_h u\|)\|u^h - P_h u\|_1 \\ &\quad +C|\ln h|^{1/2}\|u^h - P_h u\| \, \|u^h - P_h u\|_1^2. \end{aligned} \tag{5.72}$$

By the triangle inequality, (5.8) and (5.15) the following inequalities hold:

$$\begin{aligned} J_2 &\le C\|\phi\|_{1,\infty}^2 \|u^h - u\| \, \|u^h - P_h u\| \\ &\le C(\|u - P_h u\| + \|u^h - P_h u\|)\|u^h - P_h u\| \\ &\le C(h^2 + \|u^h - P_h u\|)\|u^h - P_h u\|_1, \end{aligned} \tag{5.73}$$

$$J_3 \le C\|\phi\|_{1,4}^2 \, \|(I - I_h)(u^h - P_h u)\| \le Ch\|u^h - P_h u\|_1, \tag{5.74}$$

$$J_4 \le C\|\phi\|_{1,4}^2 \, \|u - P_h u\| \le Ch^2. \tag{5.75}$$

The proof is completed by substituting (5.71)-(5.75) into (5.70). □

Lemma 5.11. *The following inequality holds:*

$$\begin{aligned} &-\sum_{b \in B^h} \int_{\partial b} k(u^h)\frac{\partial u^h}{\partial \nu} I_h(u^h - P_h u) + (k(u)\nabla u, \nabla(u - u^h)) \\ &\ge C_1^{-1}\|u^h - P_h u\|_1^2 - C_2(h + \|u^h - P_h u\|)\|u^h - P_h u\|_1 - C_3 h^2. \end{aligned} \tag{5.76}$$

Proof. Actually we have

$$-\sum_{b\in B^h}\int_{\partial b} k(u^h)\frac{\partial u^h}{\partial \nu} I_h(u^h - P_h u) + (k(u)\nabla u, \nabla(u-u^h)) \tag{5.77}$$

$$= -\sum_{b\in B^h}\int_{\partial b} k(u^h)\frac{\partial (u^h - P_h u)}{\partial \nu} I_h(u^h - P_h u)$$

$$-\sum_{b\in B^h}\int_{\partial b} k(u^h)\frac{\partial (P_h u - u)}{\partial \nu} I_h(u^h - P_h u)$$

$$-\sum_{b\in B^h}\int_{\partial b} (k(u^h) - k(u))\frac{\partial u}{\partial \nu} I_h(u^h - P_h u)$$

$$[-\sum_{b\in B^h}\int_{\partial b} k(u)\frac{\partial u}{\partial \nu} (I_h - I)(u^h - P_h u) + (k(u)\nabla u, \nabla(u-u^h))]$$

$$-\sum_{b\in B^h}\int_{\partial b} k(u)\frac{\partial u}{\partial \nu} (u^h - P_h u).$$

$$= K_1 + K_2 + K_3 + K_4 + K_5.$$

By the coerciveness condition (5.17),

$$K_1 \geq C^{-1}\|u^h - P_h u\|_1^2. \tag{5.78}$$

Let $u \in C(\bar{\Omega})$ and denote by u_I its piecewise linear interpolation in S_0^h. Then by Lemma 4.2 in [10] together with (5.17) and (5.8), we obtain

$$\begin{aligned} K_2 &\leq C\|P_h(u-u_I)\|_1\|u^h - P_h u\|_1 \\ &\quad + \sum_{b_p\in B^h} \left|\int_{\partial b_p} k(u^h)\frac{\partial (u_I - u)}{\partial \nu} I_h(u^h - P_h u)\right| \\ &\leq Ch\|u^h - P_h u\|_1 + Ch\|u\|_2\,\|u^h - P_h u\|_1 \\ &\leq Ch\|u^h - P_h u\|_1. \end{aligned} \tag{5.79}$$

We also have:

$$K_3 \;\leq\; C(h + \|u^h - P_h u\|)\|u^h - P_h u\|_1. \tag{5.80}$$

To estimate K_4, we observe that

$$-\sum_{b\in B^h}\int_{\partial b} k(u)\frac{\partial u}{\partial \nu}(I_h - I)(u^h - P_h u) \tag{5.81}$$

$$= -\int_\Omega \nabla(k(u)\nabla u)(I_h - I)(u^h - P_h u) + \int_\Omega k(u)\nabla u \nabla(u^h - P_h u).$$

Thus, by Lemmas 5.1, 5.2 and 5.8, we have

$$
\begin{aligned}
K_4 = & -\int_\Omega \nabla(k(u)\nabla u)(I_h - I)(u^h - P_h u) \\
& + \int_\Omega k(u)\nabla u \nabla(u - P_h u) \\
\leq & \; Ch\|u^h - P_h u\|_1 + |\int_\Omega \nabla(k(u)\nabla u)(u - P_h u)| \\
\leq & \; Ch^2 + Ch\|u^h - P_h u\|_1.
\end{aligned} \tag{5.82}
$$

Obviously,

$$
K_5 = -\int_{\partial\Omega} k(u)\frac{\partial u}{\partial \nu}(u^h - P_h u) = 0. \tag{5.83}
$$

Finally, (5.76) is obtained by substituting (5.78)-(5.83) into (5.77). □

Lemma 5.12. *The following inequalities hold:*

$$
\begin{aligned}
& \eta(\int_\Omega G(x,y)u^h(y)dy, I_h(u^h - P_h u)) \\
& \quad + \eta(\int_\Omega G(x,y)u(y)dy, u - u^h) \\
& \leq Ch^2 + C(h + \|u^h - P_h u\|)\|u^h - P_h u\|_1,
\end{aligned} \tag{5.84}
$$

$$
\begin{aligned}
& \alpha((u^h)^4, I_h(u^h - P_h u)) + \alpha(u^4, u - u^h) \\
& \leq C_1(h^{-1}\|u^h - P_h u\|^3 + \|u^h - P_h u\|^2 + \|u^h - P_h u\|)\|u^h - P_h u\|_1^2 \\
& \quad + C_2 h\|u^h - P_h u\|_1 + C_3\|u^h - P_h u\|^2 + C_4 h^2.
\end{aligned} \tag{5.85}
$$

Proof. First to prove (5.84), we write it in the following form and apply (5.8), (A.2) and (5.15) to obtain

$$\begin{aligned}
&\eta(\int_\Omega G(x,y)u^h(y)dy, I_h(u^h - P_hu)) \qquad (5.86)\\
&\quad +\eta(\int_\Omega G(x,y)u(y)dy, u - u^h)\\
&= \eta(\int_\Omega G(x,y)(u^h - P_hu)(y)dy, I_h(u^h - P_hu))\\
&\quad +\eta(\int_\Omega G(x,y)(P_hu - u)(y)dy, I_h(u^h - P_hu))\\
&\quad +\eta(\int_\Omega G(x,y)u(y)dy, (I_h - I)(u^h - P_hu))\\
&\quad +\eta(\int_\Omega G(x,y)u(y)dy, u - P_hu)\\
&\le C\|u^h - P_hu\|^2 + Ch\|u^h - P_hu\| + Ch\|u^h - P_hu\|_1 + Ch^2\\
&\le Ch^2 + C(h + \|u^h - P_hu\|)\|u^h - P_hu\|_1.
\end{aligned}$$

Before we prove (5.85), we recall the Gagliardo-Nirenberg interpolation inequality. For $w \in H_0^1(\Omega)$ and $p > \max(2, n)$, there exists a constant $C > 0$ such that

$$\|w\|_{0,2p/(p-2)} \le C\|w\|^{1-n/p}\|w\|_1^{n/p}. \qquad (5.87)$$

Similarly to (5.84), we decompose (5.85) into four terms,

$$\begin{aligned}
&\alpha((u^h)^4, I_h(u^h - P_hu)) + \alpha(u^4, u - u^h) \qquad (5.88)\\
&= \alpha(((u^h)^4 - (P_hu)^4), I_h(u^h - P_hu))\\
&\quad +\alpha(((P_hu)^4 - u^4), I_h(u^h - P_hu))\\
&\quad +\alpha(u^4, (I_h - I)(u^h - P_hu)) + \alpha(u^4, u - P_hu)\\
&= L_1 + L_2 + L_3 + L_4.
\end{aligned}$$

We estimate (5.88) term by term. Simple computations give

$$\begin{aligned}
L_1 &= \alpha((u^h - P_hu)^4 + 4P_hu(u^h - P_hu)^3 + 6(P_hu)^2(u^h - P_hu)^2\\
&\quad +4(P_hu)^3(u^h - P_hu), I_h(u^h - P_hu)) \qquad (5.89)\\
&\le \alpha\|u^h - P_hu\|_{0,\infty}\, \|u^h - P_hu\|_{0,6}^3\, \|u^h - P_hu\|\\
&\quad +C\|u^h - P_hu\|_{0,6}^3\, \|u^h - P_hu\|\\
&\quad +C\|u^h - P_hu\|_{0,4}^2\, \|u^h - P_hu\| + C\|u^h - P_hu\|^2.
\end{aligned}$$

By the above interpolation inequality (5.87) and the inverse inequalities (5.6), it follows that

$$\|u^h - P_h u\|_{0,\infty} \le Ch^{-1}\|u^h - P_h u\|, \tag{5.90}$$

$$\|u^h - P_h u\|_{0,6}^3 \le C\|u^h - P_h u\| \, \|u^h - P_h u\|_1^2, \tag{5.91}$$

$$\begin{aligned} \|u^h - P_h u\|_{0,4}^2 &\le C\|u^h - P_h u\| \, \|u^h - P_h u\|_1 \\ &\le C\|u^h - P_h u\|_1^2. \end{aligned} \tag{5.92}$$

Therefore, L_1 can be estimated as

$$\begin{aligned} L_1 \le{} & C(h^{-1}\|u^h - P_h u\|^2 + \|u^h - P_h u\| + 1)\|u^h - P_h u\| \, \|u^h - P_h u\|_1^2 \\ & + C\|u^h - P_h u\|^2. \end{aligned} \tag{5.93}$$

It follows from (5.6)-(5.7) together with (5.8), (5.9), and (5.15) that

$$\begin{aligned} L_2 &= \alpha(((P_h u)^2 + u^2)(P_h u + u)(P_h u - u), I_h(u^h - P_h u)) \\ &\le Ch^2\|u^h - P_h u\|. \end{aligned} \tag{5.94}$$

$$\begin{aligned} L_3 &\le \alpha\|u\|_{0,\infty}^4 \, \|(I_h - I)(u^h - P_h u)\| \\ &\le Ch\|u^h - P_h u\|_1. \end{aligned} \tag{5.95}$$

$$\begin{aligned} L_4 &\le \alpha\|u\|_{0,\infty}^4 \, \|u - P_h u\| \\ &\le Ch^2. \end{aligned} \tag{5.96}$$

Now substituting the above inequalities into (5.88) gives (5.85). □

The last lemma deals with the error resulted from the time derivative part.

Lemma 5.13. *The following inequality holds:*

$$\begin{aligned} & (I_h u_t^h, I_h(u^h - P_h u)) + (u_t, u - u^h) \\ & \ge \frac{1}{2}\frac{d}{dt}\|u^h - P_h u\|^2 - Ch\|u_t\|_1\|u^h - P_h u\| \\ & \quad - Ch\|u_t\| \, \|u^h - P_h u\|_1 + Ch^2\|u_t\|. \end{aligned} \tag{5.97}$$

Proof. Write the left hand side of (5.97) in the form

$$\begin{aligned} (I_h u_t^h, I_h(u^h - P_h u)) + (u_t, u - u^h) ={} & (I_h(u_t^h - P_h u_t), I_h(u^h - P_h u)) \\ & + ((I_h - I)P_h u_t, I_h(u^h - P_h u)) + (P_h u_t - u_t, I_h(u^h - P_h u)) \\ & + (u_t, (I_h - I)(u^h - P_h u)) + (u_t, u - P_h u). \end{aligned}$$

Then, by using Lemma 5.1 and Lemma 5.2, we easily obtain the desired result.

□

To complete the proof of the main Theorem 5.2, we choose $v = u^h$ in (5.4) and $v^h = P_h u$ in (5.22) and add them together. By applying Lemma 5.9-Lemma 5.13 and simplifying the resulting inequality, we obtain

$$\begin{aligned}
&\frac{d}{dt}\|I_h(u^h - P_h u)\|^2 + \|u^h - P_h u\|_1^2 \\
&\le C_1 h^2(1 + \|u_t\|_1^2) + C_2\|u^h - P_h u\|^2 + C_3(h^{-1}\|u^h - P_h u\|^2 \\
&\quad + \|u^h - P_h u\| + |\ln h|^{1/2} + 1)\|u^h - P_h u\|\ \|u^h - P_h u\|_1^2.
\end{aligned} \tag{5.98}$$

To estimate the right hand side of (5.98), we apply the induction method discussed in [13]. First we assume that

$$C_3(h^{-1}\|u^h - P_h u\|^2 + \|u^h - P_h u\| + |\ln h|^{1/2} + 1)\|u^h - P_h u\| \le \frac{1}{2} \tag{5.99}$$

for $t \in (0, T)$. Then (5.98) can be written as

$$\begin{aligned}
&\frac{d}{dt}\|I_h(u^h - P_h u)\|^2 + \|u^h - P_h u\|_1^2 \\
&\le Ch^2(1 + \|u_t\|_1^2) + C\|u^h - P_h u\|^2.
\end{aligned} \tag{5.100}$$

Integrating (5.100) from 0 to t and keeping in mind the equivalence of $\|I_h \cdot\|$ and $\|\cdot\|$ we obtain

$$\|u^h - P_h u\|^2 + \int_0^t \|u^h - P_h u\|_1^2 \le Ch^2 + \int_0^t \|u^h - P_h u\|^2. \tag{5.101}$$

Thus Gronwall's inequality leads to

$$\|u^h - P_h u\|_{L^\infty(0,T;L^2)}^2 + \|u^h - P_h u\|_{L^2(0,T;H^1)}^2 \le Ch^2, \tag{5.102}$$

which implies (5.59).

Now we show that for h small enough, (5.99) holds. By definition $u^h(0) = P_h u_0$, thus (5.99) holds for $t = 0$. Assume that (5.99) is not true for some $t \in (0, T]$. Then there exists a $\tau \in (0, T]$ such that

$$\begin{aligned}
\tau := \inf\{0 < t \le T : C_3(h^{-1}\|u^h - P_h u\|^2 + \|u^h - P_h u\| \\
+ |\ln h|^{1/2} + 1)\|u^h - P_h u\| \ge \frac{1}{2}\} > 0.
\end{aligned} \tag{5.103}$$

So

$$C_3(h^{-1}\|(u^h - P_h u)(\tau)\|^2 + \|(u^h - P_h u)(\tau)\| \\ +|\ln h|^{1/2} + 1)\|(u^h - P_h u)(\tau)\| \geq \frac{1}{2}. \tag{5.104}$$

Thus (5.99) holds for $t \in (0, \tau)$ and similar to (5.102) we have

$$\|u^h - P_h u\|^2_{L^\infty(0,\tau;L^2)} + \|u^h - P_h u\|^2_{L^2(0,\tau;H^1)} \leq Ch^2, \tag{5.105}$$

Therefore,

$$\|(u^h - P_h u)(\tau)\| \leq Ch. \tag{5.106}$$

This contradicts (5.104) for sufficiently small h and completes the proof. □

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