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AN INVESTIGATION OF A NONLINEAR MODIFICATION OF MAXWELL'S EQUATIONS

by

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled. An Investigation of a Nonlinear Modification of Maxwell's Equations

submitted by Richard Gary Nash

in partial fulfilment of the requirements for the degree of Doctor of Philosophy.

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TO MY

PARENTS

ABSTRACT

We use the equation $\partial_\alpha F^{\alpha\beta} = - A^\beta A_\alpha A^\alpha$ as the basis for our models of charged and neutral particles. A class of time dependent solutions of this equation is constructed using the fact that it is invariant under the special conformal group. We then investigate time dependent solutions of $\partial_\alpha F^{\alpha\beta} = - A^\beta(A_\alpha A^\alpha + b^2)$ where b is a parameter with the dimensions of inverse length which breaks the conformal invariance in the original system. Our results lead us to study a more sophisticated method of breaking the conformal symmetry based on the ansatz $A_\alpha = \overline{A_\alpha} + A'_\alpha$ where $\overline{A_\alpha}$ is a coherent excitation immersed in a randomly fluctuating field A'_α. We introduce a scalar field $f \equiv \overline{A'_\alpha A'^\alpha}$, where the bar represents a statistical average over the random fluctuations, that is considered as an observable in addition to $\overline{A_\alpha}$. We break the conformal invariance by using a particular expansion of $\overline{A_\alpha}$, f, and their derivatives. Various models of charged and neutral particles based on the nature of this expansion are studied. We show that it is possible to obtain some equations with a discrete set of charged solutions, all of them having the same charge (positive or negative) with a positive definite rest mass. A discrete set of neutral solutions which have a positive definite rest mass and the Yukawa asymptotic form are obtained (essentially) by setting $\phi = 0$ in the charged equations. Other neutral solutions are shown to exist but they necessarily have a degenerate mass spectrum.

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CHAPTER 1

INTRODUCTION

The problem of infinite self energies and singular potentials inherent in Maxwell's point charge theory of electrodynamics has led physicists for many years to search for charged particle models which are free from these difficulties. Most of these models are plagued with mathematical problems or inconsistencies with experimental observations, and the quantization of Maxwell's equation has not satisfactorily resolved the problem of infinities. Hence, the distinct possibility that we need a more complete classical theory of charged and neutral particles still remains. Dirac (1951) strongly believed this to be the case, and stated: "the troubles of the present quantum electrodynamics should be ascribed primarily, in my opinion, not to a fault in the general principles of quantization, but to our working from a wrong classical theory. To make progress one should therefore re-examine the classical theory of electrons and try to improve on it."

There are two different approaches that have been used to obtain a new theory of electrodynamics. The oldest and most familiar is the dualistic point of view. Here, the particles are the sources of the field, and are acted upon by the field, but are not part of it. The sources in the field equations must be completely specified before the field can be calculated.

The other approach is known as the unitary viewpoint, whereby the matter is constructed from a field. Particles appear as solutions to the equations governing the field which are finite everywhere, and exhibit an asymptotic form which is in accord with experimental observations.

Hence, the name "particle-like solutions". In what follows we have adopted this philosophy in our attempt to construct a new classical theory of charged and neutral particles.

We start with the equation

$$\partial_\alpha F^{\alpha\beta} = -A^\beta A_\alpha A^\alpha \qquad [1.1]$$

and use it as the basis for our models. The fact that it is not invariant under gauge transformations does not appear to lead to difficulties. What is more interesting is that [1.1] is the only equation with the current constructed entirely in terms of the fields A_α, that is invariant under the special conformal group. Since this is the largest continuous group of coordinate transformations in Minkowski space under which Maxwell's equations are invariant, it is reasonable to single out the equation which departs from Maxwell's theory but retains the same symmetries under coordinate transformations. As conformally invariant systems, however, have some undesirable properties, such as a continuous mass spectrum, (discussed in Chapter 3) this symmetry must therefore be broken so that we can obtain physically meaningful systems with non-zero rest masses. We achieve this goal by introducing a random background field into the classical field theory based on [1.1].

The use of fluctuating background fields have proved to be fruitful in extending the realm of classical field theories. Nelson (1966) has shown that nonrelativistic classical mechanics on which is superimposed a Brownian motion with diffusion coefficient $\frac{\hbar}{2m}$ is fully equivalent to nonrelativistic quantum theory. Boyer (1968, 1975) has been able to describe many well known phenomena by the introduction of "random electrodynamics", which is now briefly described.

In the Coulomb gauge, $\nabla\cdot\vec{A} = 0$, Maxwell's equations can be written as

[1.2] $$\nabla^2\phi = -4\pi\rho$$

[1.3] $$\Box\vec{A} = 4\pi\vec{J}_\perp$$

which have the general solutions

[1.4] $$\phi(\vec{r},t) = \int \frac{\rho(r',t)}{|r-r'|}\, d^3x'$$

[1.5] $$\vec{A}(\vec{r},t) = \vec{A}_0(\vec{r},t) + \int \vec{J}_\perp(\vec{r}',t')G(\vec{r},t;\vec{r}',t')d^4x'$$

where G is the familiar Green's function and $\vec{A}_0$ is a solution to the homogeneous vector wave equation. Traditionally, G is taken to be the retarded Green's function, and $\vec{A}_0$ is put to zero. These criteria require all radiation to come from somewhere at a finite time. The universe in the infinite past, as described by this model, would contain matter but not radiation. Boyer abandoned the boundary condition of $\vec{A}_0 = 0$ and introduced the notion of a random radiation associated with $\vec{A}_0$. Since the existence of this radiation was a fundamental hypothesis in his theory he reasoned that the random radiation "should possess the fundamental aspects of what is presently regarded as empty space"; it should be isotropic and homogeneous as no direction or position in space is preferred, and Lorentz invariant because no inertial frame is preferred.

He found that the requirement of Lorentz invariance implied a random radiation spectrum for $\vec{A}_0$ which was unique up to a multiplicative constant, which he denoted by $\frac{\hbar}{2\pi^2}$ in order to obtain a zero point energy of $\frac{1}{2}\hbar\omega$ per normal mode. As $\hbar \to 0$, $\vec{A}_0$ vanishes and he recovers the usual electrodynamics. Using the statistical model of Einstein and Hopf (1910) for a nonrelativistic particle with a dipole moment, Boyer was able to arrive at the Planck radiation law. He realized that if matter is present the zero-point radiation would interact with the matter and give rise to a different random radiation pattern, which in turn gives rise to forces between objects. He calculated these forces and showed that they agreed with the Van der Waals force calculations of quantum theory (Casimir 1948). Thus it is apparent, at least in these cases, that classical theories augmented with random background fields are quite capable of predicting fundamental results[1].

After discussing some solutions to [1.1] in Chapter 4, we look for time dependent solutions to the equation

$$\partial_\alpha F^{\alpha\beta} = -A^\beta (A_\alpha A^\alpha + b^2) \tag{1.6}$$

where b is a parameter with the dimensions of inverse length which breaks the conformal symmetry.

In Chapter 6, we introduce some equations which exhibit nicer properties than those associated with [1.6]. This is accomplished by assuming the existence of solutions to [1.1] which can be written as the sum of a coherent field and a fluctuating background field. The conformal invariance is then broken as a result of an ensemble averaging process.

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[1] Random background fields also prove useful in the study of quantum fluctuations (Dewitt 1967, Brown 1975).

CHAPTER 2

HISTORICAL BACKGROUND

The early history of a classical theory of a charged particle was directed primarily toward a theory of the electron. Abraham (1903) was the first to study in detail the model which depicted the electron to be a rigid sphere with a spherically symmetric charge distribution. He obtained the same expression as J.J. Thomson (1881) for the electromagnetic mass, namely

$$m_{e\ell m} = \frac{fe^2}{r_0 c^2} \tag{2.1}$$

where r_0 is the classical electron radius and f was a geometrical factor which was fixed when the manner in which the charge was distributed in the sphere was specified. Later, Abraham (1904) believed that his results were inconsistent with the transformations proposed by Lorentz in the same year[1]. Moreover, it was evident that the nonrelativistic theory of the electromagnetic electron was hampered by the fact that the various parts of the charged sphere must repel one another according to Coulomb's law, giving rise to an unstable electron.

Poincaré came to the rescue in 1905 by postulating some cohesive forces of non-electromagnetic origin which stabilizes the electron and also carries one third of the self energy so that the expressions for energy and momentum agreed with the special relativistic results.

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[1] It wasn't realized that this was due to an improper treatment of the transformation properties of the Coulomb field for a point charge (Rohrlich 1965).

Lorentz abandoned the idea of a rigid sphere and investigated in detail his "compressible" electron whose shape became that of an oblate spheroid under Lorentz transformations.

The facts that these theories must make geometrical assumptions about the shape and charge distribution of the electron, in addition to being compelled to introduce cohesive forces of some strange origin in order to achieve stability warrants them unsatisfactory.

It was not until 1938 that important progress with a dualistic model was made. Dirac constructed a relativistically invariant theory by treating the electron as a point charge. The difficulties with the infinite Coulomb energy were avoided by a procedure which allowed unwanted terms to cancel out. The equations obtained were not really new, but in their physical interpretation the finite size of the electron appeared in a new sense; the interior of the electron was a region of space through which signals could be transmitted faster than the speed of light.

The problem with causality and also the well known runaway solutions of the Lorentz-Dirac equation led physicists in the 1940's to search further for classical models of the electron. Bopp (1940) and Landé and Thomas (1941) proposed essentially the same ideas. They used a modified electrostatic potential $V = V'' - V'$ where V'' and V' satisfy Maxwell's equation

$$\nabla^2 V'' = -4\pi\rho \quad , \tag{2.2}$$

and Yukawa's equation

[2.3] $$\nabla^2 V' = -4\pi\rho + k^2 V'$$

respectively, so that

[2.4] $$V = \frac{e}{r} - \frac{e}{r} e^{-\frac{r}{r_0}} .$$

By introducing another field into the usual electrodynamic picture, with both fields having the same point charges as sources, they were able to construct a finite self energy $W = \frac{e^2}{2r_0}$. They were not able to understand from a physical point of view why V should be the difference rather than the sum of the two independent fields.

Podolsky (1942) solved this problem when he investigated a Lagrangian approach to deriving generalized linear electromagnetic field equations. He found the only non-trivial generalization of this kind leading to differential equations of order below 8 was obtained by taking the Lagrangian L_F to be

[2.5] $$L_F = \frac{1}{2} F_{\alpha\beta}{}^2 + a^2(\partial_\beta F_{\alpha\beta})^2 .$$

The resulting field equations contain the Landé-Thomas theory and account for the choice of sign required when one wishes to consider the total field to be constructed from the Maxwell and Yukawa fields. The higher order derivatives in the differential equations give an extra freedom of choice of solutions for a given problem. They appeared to be important for constructing finiteness conditions which serve to remove infinities inherent in the usual treatment of finite charges.

Another approach toward finding an extended charge model was proposed by McManus (1948) and Bohm et al. (1949). They essentially replaced the delta function in the usual charge density by some function f, which characterizes the shape of the charge, and has the same transformation and normalization properties of the delta function. The theory gave a stable electron with a finite self energy but contained a great deal of arbitrariness because the form factor f was not known from experiment.

I. Prigogine and F. Henin (1962) and others have made attempts to modify models like those discussed above, but the problems with the introduction of "form factors" or "cutoffs" have not been satisfactorily resolved.

Mie (1912) was the first person to dispense with the dualistic models. He adopted the unitary philosophy by demanding that matter in his theory was to be derivable from a field. Mie's ideas evolved around Maxwell's equations

[2.6] $$\partial_{[\alpha}F_{\beta\gamma]} = 0$$

[2.7] $$\partial_{\alpha}F^{\alpha\beta} = J^{\beta}$$

from which it follows that the conservation law

[2.8] $$\partial_{\beta}J^{\beta} = 0$$

holds. Mie noted that if J^{β} is derivable from an antisymmetric tensor $H^{\alpha\beta}$ according to

$$\partial_\alpha H^{\alpha\beta} = J^\beta \qquad [2.9]$$

the conservation law still holds, but $H^{\alpha\beta}$ could be a more general tensor than $F^{\alpha\beta}$; in fact, he set $H^{\alpha\beta}$ to be some function of $F^{\alpha\beta}$ and the fields A_α. He obtained some nonlinear field equations (Pauli (1958)) which he hoped would have solutions which differ from the Maxwell-Lorentz theory mainly inside the electron. However, no modification was found which gave satisfactory results. The main difficulty was that all trial solutions investigated had an arbitrary charge.

Mie's work, however, showed that the notion of a unitary theory was worthy of further investigation. Born and Infeld (1934) retained the unitary approach and attempted to remove the infinities in physical quantities according to "the principle of finiteness", which postulates that a satisfactory theory should be free of such infinities. They reasoned that if this principle is applied to the speed of a particle with mass, an upper bound c is required, and the Newtonian action function $\frac{1}{2} mv^2$ should be replaced by $mc^2(1 - \sqrt{1 - \frac{v^2}{c^2}})$. They carried this correspondence over to the electromagnetic field and constructed the gauge invariant Lagrangian density

$$\mathcal{L} = b^2\left(\sqrt{1 + \frac{1}{b^2} F_{\alpha\beta}^{\;2}} - 1\right) \qquad [2.10]$$

which they believed would represent a singularity free system. They were able to strengthen this argument by deriving $\mathcal{L}$ in a more rigorous manner. They constructed an action integral

$$I = \int \mathcal{L}\, d^4x$$

which was invariant under all space-time transformations by means of

Eddington's result which states that $\mathcal{L}$ must have the form

$$\mathcal{L} = \sqrt{|a_{\alpha\beta}|} \tag{2.11}$$

where $a_{\alpha\beta}$ is some asymmetric tensor. With an appropriate form of $a_{\alpha\beta}$ in terms of the electromagnetic fields and the space-time metric, the above Lagrangian density could be obtained. However, when they calculated the radial electric field $\vec{E}_r$, they obtained

$$\hat{r}E_r = \frac{e\ \hat{r}}{r_0^2\sqrt{1 + (\frac{r}{r_0})4}} \tag{2.12}$$

which does not vanish at r = 0 as a non-singular radial vector should.

Another attempt to construct a unitary theory of a charged particle was proposed by Dirac in 1951. He did not invoke gauge invariance in his model, believing that a more powerful theory could be developed without it, "capable of being transformed into a wide variety of different forms, and so providing better prospects for enabling one to introduce electric charges in a satisfactory way." He broke the gauge invariance by introducing the simplest relativistically invariant condition possible, namely

$$A_\mu A^\mu = - k^2 \tag{2.13}$$

which was used as a constraint in the usual Maxwell Lagrangian. The resulting field equations were

[2.14] $$\partial_\mu F^{\mu\nu} = \lambda A^\nu$$

where λ is some constant.

Dirac investigated the motion of charges in his theory in the following manner. With λ infinitesimal, let $F^{(o)}_{\mu\nu}$ be the field corresponding to no charges (i.e. $\partial_\mu F^{(o)\mu\nu} = 0$) constructed from A^*_μ. The new fields A_μ, which have charge associated with them, were connected to A^*_μ by a gauge transformation subject to the constraint that $A_\mu A^\mu = -k^2$, i.e.

[2.15] $$A_\mu = A^*_\mu + \partial_\mu S$$

means that

[2.16] $$(\partial_\mu S + A^*_\mu)(\partial^\mu S + A^{\mu *}) = -k^2.$$

Hence, if $k \equiv \frac{m}{e}$, he obtained the Hamilton-Jacobi equation for an electron moving according to Lorentz's equation in the field A^*_μ. The four velocity v_μ is then interpreted as

[2.17] $$v_\mu = k^{-1}(\partial_\mu S + A^*_\mu).$$

When λ is not small, the motion of the corresponding charge is obtained by making successive infinitesimal changes in the solution of the equations.

It was pointed out to Dirac by D. Gabor that his theory only allowed "electron streams" which are irrotational as follows from [2.17], and it is conceivable that vortical electron clouds can exist. In 1952 he modified his theory to include such effects, and in 1954 he extended it to interacting electron beams. However, his theory became increasingly complicated, and he abandoned it soon afterward.

H. Schiff (1962), (1969) introduced the field equations

[2.18] $$\partial_\alpha F^{\alpha\beta} = - A^\beta(A_\lambda A^\lambda + b^2)$$

where b^2 is a constant, and showed that particle-like solutions existed which were finite everywhere. With a given charge, a discrete mass spectrum was obtained. Neutral systems had a mutual interaction of the Yukawa type, and charges moved according to the Lorentz equations in weak external fields.

In the 1969 paper, he also proposed the equations

[2.19] $$\partial_\alpha F^{\alpha\beta} = - \frac{1}{e^2} A^\beta(A_\lambda A^\lambda) \text{ .}$$

He observed that the relation $F_{\alpha\beta}F^{\alpha\beta} = - \frac{2}{e^2} (A_\alpha A^\alpha)^2$ is satisfied identically in the Liénard-Wiechert gauge for a charge e in arbitrary accelerated motion. Using this as a constraint in the usual electromagnetic Lagrangian, he obtained [2.19]. He showed that nonsingular solutions to [2.19] exist which represent discrete neutral and charged states with zero energy.

Bisshopp (1972) also studied [2.19] and constructed a class of plane wave solutions. He chose the field vector A_α to be of the form $A_\alpha(x^\beta) = a_\alpha(\theta(x^\beta)) = a_\alpha(kz - \omega_0 t)$ with $\partial_\alpha \theta \equiv \omega_\alpha = (\vec{k}, \omega_0)$ which has constant components. This ansatz differs from that employed in Fourier analysis of a linear problem in so far as the functional dependence of $a_\alpha(\vec{k}\cdot\vec{x} - \omega_0 t)$ is not necessarily sinusoidal, but is determined from the field equations themselves.

CHAPTER 3

SOME PROPERTIES OF THE SPECIAL CONFORMAL GROUP

A general conformal coordinate transformation is defined by (Haantjes 1940)

$$[3.1] \qquad g^c_{\mu\nu}(x') = \lambda(x) \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(x) = \lambda(x) g'_{\mu\nu}(x')$$

where $g_{\mu\nu}(x)$ is the metric associated with the space-time coordinates x^α, $g^c_{\mu\nu}(x')$ is the conformally transformed metric in the new system of coordinates x'^α, and $\lambda(x)$ is an arbitrary differentiable function of x^α. From [3.1], it follows that the line element ds^2 transforms according to

$$[3.2] \qquad ds^{2c} = \lambda(x) ds^2$$

from which we can calculate the angle between two infinitesimal vectors dx^α and dy^β by means of the expression

$$[3.3] \qquad \cos\alpha = \frac{g_{\mu\nu} dx^\mu dy^\nu}{(g_{\mu\nu} dx^\mu dx^\nu)^{\frac{1}{2}} (g_{\mu\nu} dy^\mu dy^\nu)^{\frac{1}{2}}} .$$

It follows from [3.1] that [3.3] is a conformally invariant object and hence the name "conformal" transformations.

The conformal transformations which map a flat space into a flat space are known as the special conformal (abbreviated by S.C.) transformations. These transformations are defined by

$$[3.4] \qquad g^c_{\mu\nu}(x') \equiv \eta_{\mu\nu} = \lambda(x) \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} \eta_{\alpha\beta}$$

which is obtained from [3.1] by setting both $g_{\mu\nu}$ and $g^c_{\mu\nu}$ equal to the Minkowski metric

$$[3.5] \qquad \eta_{\mu\nu} = \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & -1 \end{pmatrix} .$$

As shown in Appendix 1, the 15 parameter S.C. group is generated from [3.4] and the infinitesimal transformations

$$[3.6] \qquad x'^\mu = x^\mu + \delta x^\mu(x)$$

where δx^μ is an infinitesimal differentiable real function. From [A1.14], the corresponding finite coordinate transformations are

$$[3.6a] \qquad x'^\mu = x^\nu a^\mu_\nu$$

$$[3.6b] \qquad x'^\mu = x^\mu + a^\mu$$

$$[3.6c] \qquad x'^\mu = \beta x^\mu$$

$$[3.6d] \qquad x'^\mu = \frac{x^\mu + c^\mu x^2}{\sigma}$$

where

$$[3.6e] \qquad \sigma = 1 + 2c^\mu x_\mu + c_\mu c^\mu x^2$$

and $x^2 \equiv x^\mu x_\mu$. Equations [3.6a] and [3.6b] together constitute the Poincaré group of coordinate transformations; eqn. [3.6c] defines the dilational transformations and [3.6d] are the S.C. or "acceleration" transformations.

The S.C. group is isomorphic (Dirac 1936) to the group SO(4,2) of orthogonal transformations in six dimensions. This group involves linear transformations in contrast to the nonlinear relations given by [3.6d] in Minkowski space, and is therefore useful in some calculations. However, the projection from six space to four space introduces non-linear quantities and makes most calculations as difficult as when [3.6d] are applied directly in four space. We therefore prefer to use equations [3.6] in Minkowski space for our purposes.

If, under a S.C. coordinate transformation, the covariant components of a vector $\vec{A}$ transform according to

[3.7] $$A'_{\alpha}(x') = A_{\beta}(x)\,\frac{\partial x^{\beta}}{\partial x'^{\alpha}}\ ,$$

then the contravariant components must transform like (Fulton et al. 1962)

[3.8] $$A'^{\alpha}(x') = \sigma^{2}(x)A^{\beta}(x)\,\frac{\partial x'^{\alpha}}{\partial x^{\beta}}\ .$$

The reason why a factor of σ^2 appears in [3.8] follows from [3.4] and [3.6d]:

[3.9] $$g'_{\mu\nu}(x') = \sigma^{2}(x)\eta_{\mu\nu} \equiv \frac{\eta_{\mu\nu}}{\lambda(x)}$$

with σ given by [3.6e]. The scalar product $A^{\alpha\prime}(x')A'_{\alpha}(x)$ has the form

[3.10] $$A'_{\alpha}(x')A'^{\alpha}(x') = \sigma^{2}A_{\alpha}(x)A^{\alpha}(x).$$

Conversely, if $A^{\alpha}(x)$ transforms like an ordinary contravariant vector, then we would have

$$A'_{\alpha}(x')A^{\alpha\prime}(x') = \frac{A_{\alpha}(x)A^{\alpha}(x)}{\sigma^2}$$

so we see that it is important to establish how the field variables should transform under a S.C. transformation.

The S.C. transformations [3.6d] are equivalent to an inversion $x'^{\mu} = \frac{k^2 x^{\mu}}{x_{\nu}x^{\nu}}$, followed by a translation $x''^{\mu} = x'^{\mu} + a^{\mu}$, followed by another inversion

$$x'''^{\mu} = \frac{x''^{\mu}k^2}{x''_{\nu}x''^{\nu}}, \qquad [3.11]$$

where k^2 and a^{μ} are some constants related to the parameters c^{μ} in [3.6e]. Hence, if an equation is invariant under an inversion like [3.11], then it is invariant under a S.C. transformation. Bateman (1910) and Cunningham (1909) used this fact to show that Maxwell's equations are invariant under the S.C. group of coordinate transformations - not just the Poincaré group.

Kastrup (1962) has shown that the dilations and S.C. transformations can be interpreted as a change in the units employed during a measurement of a given quantity. This observation can be established as follows. Let E be the interval between two events in space-time. E can be written as E = dse where e is the unit chosen for the measurement. If we had chosen some other unit e' which is related to e by means of the relation

[3.12] $$e' = \sigma(x)e$$

then, since E must remain unchanged,

$$E = ds e = ds' e' = ds' \sigma e$$

so we have

[3.13] $$ds'^2 = \frac{ds^2}{\sigma^2}$$

which is just [3.2] in flat space. We can therefore obtain the transformations [3.6d] as before. The S.C. transformations can thus be interpreted as a space-time dependent change in units. If we change the unit e by a scale factor β, i.e. $e' = \beta e$, then it can be shown in a similar fashion that the dilations are equivalent to a change in scale of the unit.

S.C. transformations are often referred to as acceleration transformations. This name comes from a study of uniform acceleration by Hill in 1947 which is now briefly summarized. A particle is said to be uniformly accelerated when it experiences a constant acceleration in its instantaneous rest frame. Let $\vec{r}$ and $v = \frac{d\vec{r}}{dt}$ be the position and velocity of a particle in a given frame of reference. Then, it can be shown that the equation

[3.14] $$(1-v^2)\ddot{\vec{v}} + 3\dot{\vec{v}}(\dot{\vec{v}}\cdot\vec{v}) = 0$$

must be satisfied by $\vec{v}$ if the particle is uniformly accelerated.

Hill showed that if $\vec{r}(t)$ is the trajectory of a uniformally accelerated particle, the point transformations [3.6d] map $\vec{r}(t)$ into $\vec{r}'(t')$ which satisfies [3.14]. He also proved that the S.C. group of transformations is the only one which leaves [3.14] invariant.

Laue (1971) has calculated the orbits $r'(t')$ for a particle at rest at $\vec{r}(t) = \vec{r}_0$ = const., and shows that

$$[3.15] \qquad \vec{r}'(t') = \vec{r}_0' - \frac{\vec{\varepsilon}}{\vec{\varepsilon}^2} \pm \frac{\vec{\varepsilon}}{|\vec{\varepsilon}|} \sqrt{t'^2 + \frac{1}{\vec{\varepsilon}^2}}$$

with $c^\mu = (\vec{c},0)$, $\vec{r}_0' \equiv \dfrac{\vec{r}_0 + \vec{c}\,\vec{r}_0^2}{1 + 2\vec{c}\cdot\vec{r}_0 + \vec{c}^2\,\vec{r}_0^2}$, and the acceleration vector $\vec{\varepsilon}$ in the instantaneous rest frame

$$[3.16] \qquad \vec{\varepsilon} = 2\vec{c}^2\vec{r}_0 - 2\vec{c} - 4\vec{c}\cdot\vec{r}_0\vec{c} = \text{constant};$$

$$[3.17] \qquad \vec{r}'(t) = \frac{\vec{r}_0}{2(c_4)^2\vec{r}_0^2}\left[1 \pm \sqrt{1 + 4(c_4)^2\vec{r}_0^2(c_4t'+1)^2}\right]$$

with $c^\mu = (0,c^4)$ and $\vec{\varepsilon} = \pm\, 2(c_4)^2\vec{r}_0$. We see that the orbits are hyperbolas and that the acceleration depends on $\vec{r}_0$ and c^μ.

Milner (1921) demonstrated that when a charge q, initially at rest at x_0 = constant, is accelerated in a manner equivalent (Fulton and Rohrlich 1960) to a one dimensional special conformal point transformation, the resulting orbit is a hyperbola with branches consisting of equal and opposite charges. The effect of the acceleration in this example is therefore to "transform away" the original charge. The orbits given by [3.15] and [3.17] are geometrically similar to Milner's problem, but in three dimensions. This raises the following question - can we take a charge at a position $\vec{r}_0 \neq 0$ at some time t, perform a general S.C. point transformation on it, and obtain a configuration with zero total charge as a result of the acceleration? As shown in Appendix 2 this situation can be realized.

However, under a conformal coordinate transformation, the charge Q behaves in a different manner. It can be shown (Fulton et al. 1962) that Q defined by

[3.18] $$Q = \int I^{\nu} d\sigma_{\nu}$$

is a conformally invariant object:

[3.19] $$Q = \int I^{\nu} d\sigma_{\nu} = \int I'^{\nu} d\sigma'_{\nu} \quad .$$

Here, $I^{\nu} \equiv J^{\nu}\sqrt{|g|}$ is a vector density of weight + 1, and $d\sigma_{\nu}$ is a space-like surface of weight - 1.

Schouten and Haantjes (1936) observed that a S.C. or dilational transformation on a non-zero rest mass $(p_{\mu}p^{\mu})^{1/2}$ does not leave it invariant but rather allows it to take on any value in $(-\infty,\infty)$. This follows from the generators $x^{\nu}\partial_{\nu}$ and $k_{\mu} = 2x_{\mu}x^{\nu}\partial_{\nu}-x^2\partial_{\nu}$ for the dilational and S.C. transformations respectively. It can be shown (Barut and Haugen 1972) that

[3.20] $$e^{-\beta x^{\nu}\partial_{\nu}} p_{\mu}p^{\mu} e^{\beta x^{\nu}\partial_{\nu}} = e^{2\beta}p_{\mu}p^{\mu}$$

holds for the dilations, and

$$[3.21] \qquad e^{-c^{\lambda}k_{\lambda}} p_{\mu}p^{\mu} e^{c_{\lambda}k^{\lambda}} = \sigma^2 p_{\mu}p^{\mu}$$

is true for the S.C. transformations. We see that only zero mass particles will remain invariant under the special conformal group of transformations.

A S.C. transformation does not necessarily preserve the causal relation between a pair of events x_1, and x_2. From [3.6d], the length between x_1 and x_2 transforms according to

$$[3.22] \qquad [(x_1')_{\mu} - (x_2')_{\mu}][(x_1')^{\mu} - (x_2')^{\mu}] = \frac{[(x_1)_{\mu}-(x_2)_{\mu}][(x_1)^{\mu}-(x_2)^{\mu}]}{\sigma(x_1)\sigma(x_2)} .$$

If $(x_1-x_2)^2 \neq 0$, Rosen (1968) has shown that it is always possible to choose c^{μ} such that $\sigma(x_1)\sigma(x_2)$ can change its sign.

Using the fact that $\sigma(x) = 0$ defines a singular surface ($(x_1)_{\mu}$ and $(x_2)_{\mu}$ must lie on different sides of this surface in order to allow $\sigma(x_1)\sigma(x_2)$ to change its sign) Laue (1972) showed that the point transformations [3.6d] map the world line of a charged particle at rest into two branches of an hyperbola. The fields are given by the Liénard-Wiechert potentials for the charge q on one of the branches, the other corresponding to the advanced potentials of the opposite charge (the Liénard-Wiechert potentials are conformally invariant except for an unimportant gradient term (Haantjes 1940)).

This result comes from the fact that

$$A_{\alpha}^{adv}(-x^{\beta},-q) = A_{\alpha}^{ret}(x^{\beta},q).$$

We see that the conformal point transformations induce two symmetric

CHAPTER 4

THE EQUATION $\partial_\alpha F^{\alpha\beta} = - A^\beta A_\alpha A^\alpha$

4.1 Invariance of [1.1] under the S.C. group.

The equation[2]

$$\partial_\alpha F^{\alpha\beta} = - A^\beta A_\alpha A^\alpha \tag{1.1}$$

can be obtained from the Lagrangian

$$L = - \frac{1}{16\pi} \int [F_{\alpha\beta}F^{\alpha\beta} + (A_\alpha A^\alpha)^2] d^4x \ . \tag{4.1}$$

Under a S.C. transformation, the four dimensional volume element d^4x transforms in the following manner:

$$d^4x' = \frac{d^4x}{\sigma^4} \ . \tag{4.2}$$

If the transformation law [3.7] is employed, we see that $F'_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$ transforms like an ordinary tensor under a S.C. transformation, and

$$F'_{\alpha\beta}F'^{\alpha\beta} = \sigma^4 F_{\alpha\beta}F^{\alpha\beta} \ . \tag{4.3}$$

From [3.10], [4.2] and [4.3], it follows that

$$L(x) = L'(x')$$

which is a sufficient condition for the field equations obtained from L to be invariant under a S.C. transformation. Furthermore, we see that [4.1] is the only Lagrangian with an "interaction" term involving

[2]The coupling constant e in [2.19] has been removed by an amplitude transformation on A_α. We also take $c = \hbar = 1$ so $[A_\alpha] = L^{-1}$.

only the A_α's that is invariant under the group of S.C. coordinate transformations.

4.2 Solutions of [1.1] obtained by using the conformal symmetry of [1.1].

Dilational invariance of a given differential equation is useful in obtaining a certain class of solutions to such an equation. Morgan (1952) proved that if a partial differential equation exhibits dilational invariance, then solutions exist with the number of independent variables reduced by one. As shown by Schiff (1969), this theorem can be used to obtain radially symmetric solutions of [1.1] by setting

[4.4] $$A(r,t) = \frac{\chi_0(q)}{r} ,$$

[4.5] $$\phi(r,t) = \frac{\eta_0(q)}{r}$$

with $q \equiv \frac{r}{t}$. $\eta_0(q)$ satisfies the equation

[4.6] $$q^2(q^2-1)\ddot{\eta}_0 + 4q^3\dot{\eta}_0 + 2q^2\eta_0 + (q^2-1)\eta_0^3 = 0$$

where the dot represents $\frac{d}{dq}$. It can be shown that

[4.7] $$\chi_0(q) = q\eta_0 .$$

Equation [4.6] can be reduced to

[4.8] $$(s^2-1)^2 \frac{d^2y}{ds^2} = -y^3$$

by means of the substitutions

[4.9] $$s = \frac{3q-1}{q+1} \quad -1 \leq s \leq 3 \ ,$$

and

[4.10] $$y(s) = 2(q-1)\eta_o(q) \ .$$

Solutions of [4.8] are either antisymmetric or symmetric at $s = 0$ if $y = 0$ or $\frac{dy}{ds} = 0$ respectively. It can be shown (Pounder 1969) that symmetric solutions y_{2n} have the property

[4.11] $$y_{2n}(3) = 4\,\frac{dy_{2n}(3)}{ds}$$

and the antisymmetric ones y_{2n+1} satisfy

[4.12] $$y_{2n+1}(3) = 0 \ .$$

An expansion of $\eta(q)$ about $q = 0$ shows that

[4.13] $$\eta_o(q) = a_1 q + a_3 q^3 + a_5 q^5 + \ldots$$

and near $q = \infty$,

[4.14] $$\eta_o(q) = \frac{\alpha_1}{q} + \frac{\alpha_2}{q^2} + \ldots$$

with

[4.15] $$\alpha_2 = \alpha_4 , \ \alpha_3 = \frac{\alpha_1}{2}(2-\alpha_1{}^2), \ \alpha_3 - \alpha_5 = \alpha_1 \frac{2(3\alpha_3-\alpha_1)}{12} \ .$$

An expansion of y about $s = 3$ shows that $\alpha_2 = 0$ for symmetric solutions, and $\alpha_1 = 0$ for antisymmetric solutions.

A set of numerical solutions for [4.10] was obtained by choosing an appropriate eigenslope at s = 1. One algebraic solution was found to be

[4.16] $$\eta_0(q) = \frac{2\sqrt{6}\,q}{1+3\,q^2}$$

The electric field as a function of η_0 and $\dot{\eta}_0$ is

[4.17] $$E(r,t) = \frac{1}{r^2}\,[(q^2+1)\eta_0 + q(q^2-1)\dot{\eta}_0]$$

so the charge Q is given by

[4.18] $$Q = \lim_{r\to\infty} r^2 E = -\,\alpha_2\ .$$

We can obtain additional solutions to [1.1] by applying a S.C. transformation to the fields [4.4] and [4.5]. From [3.6d] and [3.7] with $c^\mu = (0,0,0,-k)$, we obtain

[4.19] $$A(r,t) = \frac{1-2kt}{t+k(r^2-t^2)}\ \eta_0\left(\frac{r}{t+k(r^2-t^2)}\right)\ ,$$

[4.20] $$\phi(r,t) = \frac{[t-k(t^2+r^2)]}{r[t+k(r^2-t^2)]}\ \eta_0\left(\frac{r}{t+k(r^2-t^2}\right)$$

where $\dfrac{r}{t+k(r^2-t^2)}$ is the argument of η_0.

We can write these potentials as

[4.21] $$A(r,t) = \frac{1}{r}\sum_{s=0}^{\infty} \chi_s^c(q)(kr)^s$$

[4.22] $$\phi(r,t) = \frac{1}{r}\sum_{0=0}^{\infty} \eta_s^c(q)(kr)^s$$

and determine χ_s^c and η_s^c from the Taylor expansions of [4.19] and [4.20]. The "c" on the fields signifies that they satisfy the conformally invariant equation [1.1]. The combinations

[4.23] $$f_s^c \equiv \chi_s^c - q\eta_s^c$$

and

[4.24] $$g_s^c \equiv \eta_s^c - q\chi_s^c$$

prove to be most convenient in various calculations, so we choose to work in terms of these functions.

When s = 0, $\eta_0^c \equiv \eta_0$, and

[4.25] $$f_0^c = 0$$

[4.26] $$g_0^c = \eta_0(1-q^2) \ .$$

With s = 1,

[4.27] $$f_1^c = \eta_0(q^2-1)$$

[4.28] $$g_1^c = (q^2-1)[q\eta_0 + (q^2-1)\dot{\eta}_0] \ ,$$

and s = 2 gives,

[4.29] $$f_2^c = -\frac{(q^2-1)^2}{q}[\eta_0+q\dot{\eta}_0] \ ,$$

[4.30] $$g_2^c = \frac{(q^2-1)^2}{2q^2}\ [-2q^2\eta_0 + 2q\dot{\eta}_0(1-2q^2)-q^2(q^2-1)\ddot{\eta}_0] \ .$$

These expressions are useful in the investigation of the equations to be discussed in the next chapter.

4.3 Calculation of the energy for the fields [4.19] and [4.20].

The energy ε is given by

$$[4.31]\quad \varepsilon = -\int T_{44}\, d^3x = \frac{1}{16\pi}\int d^3x \left[2E^2 + (A^2-\phi^2)(A^2+3\phi^2)\right] .$$

From [4.3], the new electric field[3] is

$$[4.32]\quad E(r,t) = \frac{1}{r^2}\left[\frac{(q^2+\tau^2)}{\tau^2}\, \eta_0(\tfrac{q}{\tau}) + \frac{q}{\tau}\,\frac{(q^2-\tau^2)}{\tau^2}\,\frac{d}{d(\frac{q}{\tau})}\,\eta_0(\tfrac{q}{\tau}) \right]$$

where $\tau \equiv 1 + kr(q - \frac{1}{q}) = 1 + kt(q^2-1)$. The energy is

$$[4.33]\quad \varepsilon = \frac{1}{16\pi}\int \frac{d^3x}{r^2\tau^4}\Big(2\eta_0^2(q^2+\tau^2)^2 + 2q^2\frac{(q^2-\tau^2)^2}{\tau^2}\,\dot{\eta}_0^2$$
$$+ 4(q^4-\tau^4)\,\frac{q}{\tau}\,\eta_0\dot{\eta}_0 + \eta_0^4(q^2-\tau^2)[q^2-\tau^2+4[1-kt(q^2+1)]^2] \Big)$$

where the dot denotes $\frac{d}{d(\frac{q}{\tau})}$. Using [4.16],

$$[4.34]\quad \varepsilon = \frac{24}{t}\,\frac{(\frac{\alpha}{\gamma})^{1/2}}{\alpha^4}\Big(L_1(\tfrac{\alpha}{\gamma})^{3/4}[16 + 16kt(kt-2)+9\alpha(1-\beta)-3v\alpha]$$
$$+ L_2(\tfrac{\alpha}{\gamma})^{5/4}[16\beta - 18\alpha\gamma + 3v(1-\beta)]$$
$$+ L_3(\tfrac{\alpha}{\gamma})^{7/4}[25 - 3v - 9\beta]\gamma$$
$$+ 9\alpha^2 L_0(\tfrac{\alpha}{\gamma})^{1/4} - 9L_4\gamma^2(\tfrac{\alpha}{\gamma})^{9/4} \Big)$$

---
[3] The corresponding charge is 0.

where $\alpha \equiv (1 - kt)^2$

$$\beta \equiv 2kt(1-kt)$$

$$\gamma \equiv k^2t^2$$

$$v \equiv 1 - 10kt + 10k^2t^2$$

$$L_n \equiv \int_0^\infty \frac{y^{n+\frac{1}{2}}dy}{[1+2\beta_1 y+y^2]^4}$$

$$2\beta_1 \equiv \frac{\beta + 3}{(\gamma\alpha)^{1/2}}$$

and $k \neq 0$. If $k = 0$, $\varepsilon = 0$.

In the intervals $\frac{3}{2} \leq kt < \infty$ and $-\infty < kt \leq -\frac{1}{2}$, $|\beta_1| \leq 1$ so we set $\cos w \equiv \beta_1$. L_n then has the form

$$[4.35] \quad L_n = 2^{7/2} \sin^{-7/2} w \, \Gamma(\tfrac{9}{2}) B^{(n)}(n + \tfrac{3}{2}, \tfrac{13}{2} - n) P_{n-3}^{-7/2} (\cos w)$$

where $B^{(n)}(x,y)$ is the beta function and P_ν^μ is the associated Legendre function. In the regions for which $|\beta_1| > 1$,

$$[4.36] \quad L_n = 2^{7/2} (\beta_1^2 - 1)^{-7/2} \Gamma(\tfrac{9}{2}) B^{(n)}(\tfrac{13}{2} - n, n + \tfrac{3}{2}) P_{n-3}^{-7/2}(\beta_1).$$

The recursion relations

$$[4.37] \quad (\beta_1^2 - 1)\frac{d}{d\beta_1} P_\nu^\mu = (\nu-\mu + 1)P_{\nu+1}^\mu - (\nu+1)\beta_1 P_\nu^\mu ,$$

$$[4.38] \quad (2\nu + 1)\beta_1 P_\nu^\mu = (\nu - \mu + 1)P_{\nu+1}^\mu + (\nu+\mu)P_{\nu-1}^\mu ,$$

and

$$P^{\mu}_{\nu} = P^{\mu}_{-\nu-1} \qquad [4.39]$$

hold for all β_1. It is clear that

$$B^{(4)} = B^{(1)} \text{ and } B^{(2)} = B^{(3)}$$

so we have

$$L_1 = L_4 \text{ and } L_2 = L_3. \qquad [4.40]$$

From [4.38] ,

$$L_0 = 3L_2 + 2\beta_1 L_1 \qquad [4.41]$$

and [4.37] gives

$$L_1 = \frac{L_2}{5}\,(7 + 2\beta_1) . \qquad [4.42]$$

We can easily evaluate L_3:

$$L_3 = \int_0^{\infty} \frac{x^{7/2}dx}{(1+2\beta_1 x+x^2)^4} = \frac{5\sqrt{2}\pi}{256(1+\beta_1)^{7/2}} \qquad [4.43]$$

for any $\beta_1 > -1$.

We must now determine the signs of the square roots of α and γ. When $-\infty < kt < 0$, $\alpha^{\frac{1}{2}} = 1 - kt$ and $\gamma^{\frac{1}{2}} = -kt$. We observe in [4.34] that any explicit appearance of α^m, where m is a fraction, always occurs with a product or quotient of γ^m. The energy for $kt < 0$ must therefore have the same form as when $kt > 1$, since $\alpha^{1/2} = kt - 1$ and $\gamma^{1/2} = kt$ in this region. When $0 \leq kt < 1$, we have $\alpha^{1/2} = 1 - kt$ and $\gamma^{1/2} = kt$.

From these values of $\alpha^{1/2}$ and $\gamma^{1/2}$, we have

$$[4.44] \qquad \varepsilon = \begin{cases} \dfrac{16k\pi}{3\sqrt{3}} & -\infty < kt < 0\ , \quad 1 < kt < \infty \\ 0 & 0 \leq kt < 1 \end{cases} .$$

At $kt = 1$, the energy diverges. Equation [4.44] shows how the conformal parameter k enters explicitly into ε and therefore demonstrates the continuous nature of the mass spectrum.

4.4 Propagating solutions of [1.1].

[illegible]ave solutions of [1.1] have been obtained by Bisshopp (19[illegible]r, the nature of propagating waves is more generally expl[illegible]oking at the characteristics of [1.1]. This was done by Sha[illegible]nd Capri (1974) who showed that the characteristic determinant [illegible] for [1.1] is

$$[4.45] \qquad D(n) = (n_\alpha n^\alpha)^3 \left[n_\alpha n^\alpha + 2 \frac{(A_\alpha n^\alpha)^2}{A_\beta A^\beta} \right] = 0$$

where n_μ is the normal to the characteristic surface. Hence, in addition to the light cones given by

$$[4.46] \qquad n_\alpha n^\alpha = 0\ ,$$

there exists another characteristic surface given by

$$n_\alpha n^\alpha = -2\,\frac{(A_\alpha n^\alpha)}{A_\beta A^\beta} \quad [4.47]$$

Solutions to [1.1] which allow [4.47] to be satisfied with real values of n_α guarantee the system to be hyperbolic.

CHAPTER 5

POINCARÉ INVARIANT SYSTEMS OBTAINED FROM [1.1]. I

5.1 Breaking the conformal symmetry.

As mentioned previously, conformally invariant systems have a continuous mass spectrum. We must therefore construct some means by which that symmetry can be broken in order to describe a physically meaningful system. One method to achieve this goal was constructed by Schiff (1969). He proposed that particle-like solutions of [1.1] consist of coherent excitations immersed in a randomly fluctuating background field. The observable states are obtained by some average over the random fluctuations. The coherent and incoherent fields were separated by introducing the complex fields

[5.1] $$\Gamma_\beta = A_\beta + iB_\beta$$

and

[5.2] $$\bar{\Gamma}_\beta = A_\beta - iB_\beta$$

where A_β and B_β represent the coherent and random fields respectively. It can then be shown that the observable field satisfies

[1.6] $$\partial_\alpha F^{\alpha\beta} = -A^\beta(b^2+A_\lambda A^\lambda)$$

where

[5.3] $$b^2 \equiv \frac{1}{T}\int_0^T B_\alpha B^\alpha dt$$

providing the random fields fluctuate over a period T much faster than the coherent fields. The parameter b has the dimensions of inverse length, so it is apparent that the random "background" field determines the length and time scales for this system.

5.2 Static solutions of [1.6]

Static solutions of [1.6] have been investigated by Schiff (1962) and Darewych (1969). The radially symmetric static solutions, as discussed by Schiff, are now briefly described.

By means of scale and amplitude transformations, the equations

[5.4] $$\nabla^2\phi = \phi(1 + \vec{A}^2 - \phi^2)$$

and

[5.5] $$\vec{A}(1 + A^2 - \phi^2) = 0$$

can be obtained from [1.6]. If $\vec{A}$ vanishes everywhere it can be shown that (Finkelstein et al. 1951)

[5.6] $$\nabla^2\phi = \phi - \phi^3$$

has spherically symmetric solutions with $\left.\frac{d\phi}{dr}\right|_{r=0} = 0$ which are finite everywhere. One type of solution forms an infinite discrete set asymptotic to zero as $\frac{e^{-r}}{r}$, and another continuous set exists which is asymptotic to ±1 as $\frac{\sin(\sqrt{2}r+\alpha)}{r}$ where α is arbitrary. The exponentially decaying solutions correspond to a discrete value of $\phi(0) > \sqrt{2}$, while the oscillatory set is characterized by any value of $\phi(0)$ in $(-\infty,\infty)$. The discrete solutions must represent neutral particles because ϕ decays faster than $\frac{1}{r}$. Furthermore, the mass of the neutral system M_N is

[5.7] $$M_N = \frac{1}{4\pi}\int \phi^2 d^3x$$

which is degenerate because both ϕ and $-\phi$ satisfy [5.6].

Other solutions to [5.4] and [5.5] can be obtained by setting

[5.8] $$\vec{A}(r) = 0 \qquad 0 \leq r \leq R_n$$

and

[5.9] $$\vec{A}^2(r) - \phi^2(r) = -1 \qquad r \geq R_n$$

where R_n is any zero of $1 - \phi^2$. The equations governing ϕ are then

[5.10] $$\nabla^2\phi = \phi - \phi^3 \qquad 0 \leq r \leq R_n$$

and

[5.11] $$\nabla^2\phi = 0 \qquad r \geq R_n.$$

The solution to [5.10] is that part of a solution to [5.6] which is asymptotic to ± 1; $\phi = a + \frac{b_1}{r}$ in the outer region, and ϕ and $\frac{d\phi}{dr}$ are required to be continuous at $r = R_n$. Since $A(r)$ is continuous at $r = R_n$, so is $\frac{d^2\phi}{dr^2}$. The parameters a, b_1, and R_n are related by the condition

$$a + \frac{b_1}{R_n} = \pm 1.$$

Equations [5.8 to 5.11] describe a localized charge distribution outside of which Maxwell's equations are satisfied. A discrete set of charged solutions are obtained for this system if a particular charge b_1 is chosen.

5.3 On time dependent solutions of [1.6] .

Although time dependent solutions to [1.6] in the form of plane waves are not difficult to find (Darewych 1966), we have not been able to construct any additional solutions in an unambiguous manner. At first sight there appears to be the possibility of formally writing down a time dependent series expansion, in the mass parameter b, of the form

$$A_\alpha = A^c_\alpha + b\, A^{(1)}_\alpha + b^2\, A^{(2)}_\alpha + \ldots$$

where A^c_α satisfies the conformal equation [1.1], which is just [1.6] with b = 0. We observe, however, that in [1.6] the magnitude of b can be made to have any value by a scale and amplitude transformation on x^α and A_α respectively. Consequently b cannot be regarded as a "small" parameter, and the validity of the above expansion is not clear.

Nevertheless, we thought it would be instructive to investigate the expansions

[5.12] $$A(r,t) \doteq \frac{1}{r} \sum_{s=0}^{\infty} (br)^s\, \chi_s(q) \cdot$$

and

[5.13] $$\phi(r,t) = \frac{1}{r} \sum_{s=0}^{\infty} (br)^s\, \eta_s(q)$$

where

$$q \equiv \frac{r}{t} \, .$$

Expressing the fields in this form allows us to use the conformal solutions [4.21] and [4.22] to simplify the problem at hand. Furthermore, using the variable q permits us to deal with two first order ordinary differential equations.

Substituting [5.12] and [5.13] into the first order equation

$$\partial_\alpha [A^\alpha (A_\lambda A^\lambda + b^2)] = 0 \qquad [5.14]$$

gives

$$\left[\eta_0^2(q^2-1) + r^2b^2\right] \sum_{s=0}^{\infty} (br)^{1+s}(s\,\chi_{1+s} + q\,\dot{\chi}_{1+s} - q^2\dot{\eta}_{1+s}) \qquad [5.15]$$

$$+ 2\eta_0 \sum_{s=0}^{\infty} (br)^{2+s}\tilde{F}_{2+s} + \sum_{s=0}^{\infty} (br)^{3+s}\tilde{G}_{3+s}$$

$$+ 2r^2b^2(\chi_0 + \sum_{s=0}^{\infty} (br)^{1+s}\chi_{1+s})$$

$$+ 2q\eta_0 \sum_{s=0}^{\infty} (br)^{1+s} \left(\eta_0(1+s)(q\chi_{1+s} - \eta_{1+s}) + [\dot{\eta}_0(q^2-1)+q\dot{\eta}_0](\chi_{1+s} - q\eta_{1+s})\right)$$

$$+ \sum_{s=0} (br)^{2+s}[q\eta_0(\lambda_{2+s} - \beta_{2+s})(2+s) + 2A_{2+s} - 2q^2C_{2+s}]$$

$$+ \sum_{s=0}^{\infty} (br)^{3+s} [B_{3+s} - q^2D_{3+s}] = 0$$

where

$$\lambda_{2+s} \equiv \chi_1\chi_{1+s} + \chi_2\chi_s + \chi_3\chi_{s-1} + \ldots \qquad [5.16]$$

$$\beta_{2+s} \equiv \eta_1\eta_{1+s} + \eta_2\eta_s + \eta_3\eta_{s-1} + \ldots$$

$$A_{2+s} \equiv \chi_1[\eta_0(q\chi_{1+s} - \eta_{1+s})(1+s) + q(\dot{\chi}_0\chi_{1+s} + \chi_0\dot{\chi}_{1+s} - \dot{\eta}_0\eta_{1+s} - \eta_0\dot{\eta}_{1+s})]$$

$$+ \chi_2[\eta_0(q\chi_s - \eta_s)s + q(\dot{\chi}_0\chi_s + \chi_0\dot{\chi}_s - \dot{\eta}_0\eta_s - \eta_0\dot{\eta}_s)]$$

$$+ \ldots$$

$$B_{3+s} \equiv \chi_1[(\lambda_{2+s} - \beta_{2+s})(2+s) + q(\dot{\lambda}_{2+s} - \dot{\beta}_{2+s})] + \ldots$$

$$C_{2+s} \equiv \eta_1[\eta_0(q\chi_{1+s} - \dot{\eta}_{1+s}) + \dot{\chi}_0\chi_{1+s} - \dot{\eta}_0\eta_{1+s}] + \ldots$$

$$D_{3+s} \equiv \eta_1[\dot{\lambda}_{2+s} - \dot{\beta}_{2+s}] + \eta_2[\dot{\lambda}_{1+s} - \dot{\beta}_{1+s}] + \ldots$$

$$\tilde{F}_{2+s} \equiv (q\chi_1 - \eta_1)(s\chi_{1+s} + q\dot{\chi}_{1+s} - q^2\dot{\eta}_{1+s}) + \ldots$$

$$\tilde{G}_{3+s} \equiv (\lambda_2 - \beta_2)(s\chi_{1+s} + q\dot{\chi}_{1+s} - q^2\dot{\eta}_{1+s}) + \ldots$$

and the dot denotes $\frac{d}{dq}$.

From

[5.17] $$q\nabla\cdot\vec{E} + \frac{\partial E}{\partial t} = (A - q\phi)(A^2 - \phi^2 + b^2)$$

with the radial electric field given by

[5.18] $$E(r,t) = \frac{1}{r^2}(\eta_0 - q\dot{\eta}_0 + q^2\dot{\chi}_0) + \frac{1}{r^2}\sum_{s=0}^{\infty}(br)^{1+s}(-s\,\eta_{1+s} + q^2\dot{\chi}_{1+s} - q\dot{\eta}_{1+s})$$

we have another first order differential equation

[5.19] $$[r^2b^2 + \eta_0^2(q^2-1)]\sum_{s=0}^{\infty}(br)^{1+s}(\chi_{1+s} - q\,\eta_{1+s})$$

$$+ 2\,\eta_0\sum_{s=0}^{\infty}(br)^{2+s}[q(\lambda_{2+s} + \beta_{2+s} - q\tau_{2+s}) - \tau_{2+s}]$$

$$+ \sum_{s=0}^{\infty}(br)^{3+s}H_{3+s}$$

$$= -q\sum_{s=0}^{\infty}(br)^{1+s}[q(1+s)\dot{\eta}_{1+s} - q^2\dot{\chi}_{1+s}(1+s) + \eta_{1+s}\,s(1+s)]$$

where

[5.20] $$\tau_{2+s} \equiv \eta_1\chi_{1+s} + \eta_2\chi_s + \eta_3\chi_{s-1} + \ldots$$

and

$$H_{3+s} \equiv (\chi_{1+s} - q\eta_{1+s})(\lambda_2 - \beta_2) + \ldots \quad .$$

We observe that the parameter b does not explicitly appear in the field equations until we consider the $(br)^2$ and higher order coefficients. We therefore have

[5.21] $$\chi_1 = \chi_1{}^c, \; \eta_1 = \eta_1{}^c \quad .$$

The $(br)^2$ coefficients give the equations

[5.22] $$- 6q\eta_o{}^2 g_2 + \eta_o f_2(q^2-1)\left[2q\dot{\eta}_o + \eta_o\right] + qf_2\eta_o{}^2(q^2-1)$$

$$+ 2q\eta_o + \frac{2qf_1}{1-q^2}[f_1\eta_o + g_1\dot{\eta}_o(q^2-1) + q\eta_o g_1 - \eta_o\dot{\eta}_1(q^2-1)^2]$$

$$+ \frac{2\eta_o}{1-q^2}(-2qg_1{}^2 + qf_1{}^2 - f_1 g_1) = 0$$

and

[5.23] $$f_2[\eta_o{}^2(q^2-1)^2 - 4q^2] - 2q(q^2+1)g_2 - 2q^2(1-q^2)\dot{g}_2$$

$$- 2\,\eta_o f_1 g_1(q^2-1) = 0$$

from [5.15] and [5.19] respectively. We have used the definitions analogous to [4.23] and [4.24], namely

[5.24] $$f_s = \chi_s - q\eta_s \quad .$$

and

[5.25] $$g_s = \eta_s - q\chi_s \quad .$$

We know that [5.12] and [5.13] are equivalent to the conformal fields [4.21] and [4.22] when the parameter b vanishes in [1.6] . Furthermore, the equations obtained from the coefficients of $(br)^s$ in [5.15] and [5.19] are linear in f_s and g_s, so by setting

[5.26] $$f_s \equiv f_s^{\,c} + F_s$$

and

[5.27] $$g_s \equiv g_s^{\,c} + G_s$$

the equations for F_s and G_s are more easily solved. Hence, from [5.22] we obtain

[5.28] $$\dot{G}_2 + G_2 \frac{(q^2+1)}{q(1-q^2)} - F_2 \frac{\eta_o^{\,2}(q^2-1)^2-4q^2}{2q^2(1-q^2)} = 0$$

and [5.23] gives

[5.29] $$-6q\,\eta_o^{\,2} G_2 + \eta_o F_2 (q^2-1)(2q\dot{\eta}_o + \eta_o) + q\dot{F}_2 \eta_o^{\,2}(q^2-1) + 2q\eta_o = 0$$

which can be uncoupled to yield the equation

[5.30] $$\ddot{F}_2 + P(q)\dot{F}_2 + Q(q)F_2 = R(q)$$

where

[5.31] $$Q(q) \equiv \frac{\eta_o^{\,2}}{q^2} - 4\frac{(q^2+2)}{(q^2-1)^2} + \frac{2\dot{\eta}_o}{9\eta_o^{\,2}}\left[\eta_o \frac{(1+3q^2)}{1-q^2} - q\dot{\eta}_o\right]$$

$$P(q) \equiv 2\frac{(q\dot{\eta}_o + \eta_o)}{q\eta_o}$$

$$R(q) = \frac{2}{q\eta_0^2(1-q^2)^2}\left[\eta_0(1+q^2) + q\dot{\eta}_0(q^2-1)\right].$$

It is well known that any solution of the homogeneous equation

[5.32] $$\ddot{F} + P(q)\dot{F} + Q(q)F = 0$$

gives the general solution to [5.30]. Equation [5.32] with η_0 given by [4.16] corresponds to the Heun equation, as is shown in Appendix 3. This result, however, does not help us to obtain a closed form solution for F_2.

We now determine the asymptotic form of F_2 and G_2 as $q \to \infty$ so that we can numerically integrate

[5.33] $$\ddot{F}_2 + \frac{4}{q(1+3q^2)}\dot{F}_2 + 12\frac{(3-14q^2-5q^4)}{(q^2-1)^2(1+3q^2)^2}F_2 - \frac{8}{\sqrt{6}(q^2-1)^2} = 0$$

and

[5.34] $$6G_2 = -3\frac{(q^2-1)^2}{(1+3q^2)q^2}F_2 + (q^2-1)\dot{F}_2 + \frac{(1+3q^2)}{\sqrt{6}\,q}$$

which follow, respectively, from [5.30] and [5.29] with [4.16].

From [5.33], F_2 has the form

[5.35] $$F_2 = Aq + B + \frac{C}{q} + \ldots$$

and [5.34] requires

[5.36] $$G_2 = \tilde{A}q + \tilde{B} + \frac{\tilde{C}}{q} + \ldots .$$

We can eliminate some of the constants in these expressions by imposing the condition that the electric field vanishes at $t = 0$; i.e.

[5.37] $$E(r,0) \equiv E^C(r,0) + E^N(r,0) = 0$$

where E^C corresponds to the conformal electric field [4.32], and E^N is the non-conformal contribution. Equation [4.32] shows that the charge of the corresponding conformal system vanishes for any t. It is unreasonable to have a charge suddenly created at t = 0 by a contribution to $\bar{E}(r,0)$ from $E^N(r,0)$, so we require [5.37].

The conformal solutions we are concerned with correspond to the expansion of $\vec{A}$ and ϕ in the range $0 < br << 1$, so [4.14] with the argument $\frac{q}{\tau}$ gives

[5.38] $$\eta_o(\tfrac{q}{\tau})\Big|_{t=0} = \alpha_1 br + \alpha_2(br)^2 + \ldots$$

where k in the definition of τ is replaced by b. Hence,

[5.39] $$E^C(r,0) = \frac{1}{r^2}\,[2br(\alpha_1-\alpha_3) + 3(br)^2(\alpha_2-\alpha_4) + 4(br)^3(\alpha_3-\alpha_5) + 0(br)^4]$$

$$\equiv E_1^C + E_2^C + E_3^C + \ldots$$

with E_n^C proportional to $(br)^n$.

The relations in [4.15] require $E_2^C(r,0) = 0$ so

[5.40] $$E_2(r,0) = E_2^N(r,0) = b^2(-\eta_2^N - q\dot{\eta}_2^N + q^2\dot{\chi}_2^N)$$

where

[5.41] $$F_s \equiv \chi_s^N - q\eta_s^N$$

and

[5.42] $$G_s = \eta_s^N - q\chi_s^N .$$

It follows from [5.35] and [5.36] that

[5.43] $$E_2(r,0) = b^2[2A + \tilde{B} + \frac{2}{q}(B + \tilde{C}) + O(\frac{1}{q^2})]$$

so

[5.44] $$2A + \tilde{B} = 0.$$

From [5.35], [5.36], [5.33], and [5.34] we can obtain the relations

[5.45] $$\tilde{B} = \frac{2A - 3C}{9}$$

and

[5.46] $$\frac{8A}{3} = C$$

so from [5.44]

[5.47] $$A = C = \tilde{B} = 0.$$

Hence, the asymptotic forms of F_2 and G_2 become

[5.48] $$F_2 = B + \frac{D}{q^2} + \ldots$$

and

[5.49] $$G_2 = \tilde{A}q + \frac{\tilde{C}}{q} + \ldots$$

respectively.

Equation [5.48] and the requirement that F_2 should be finite everywhere were used in conjunction with the power series

$$F_2 = |q^2-1|^{\frac{1}{2}(1+\sqrt{13})} \sum_{n=0}^{\infty} a_n |q^2-1|^n + \sum_{n=0}^{\infty} |q^2-1|^n d_n$$

about $q^2 = 1$, where $a_0 = 1$, to numerically integrate [5.33].

The series involving $\frac{1}{2}(1 + \sqrt{13})$ is a solution to the homogeneous part of [5.33] which is discussed in Appendix 3. The integration was performed in two separate parts; the first covered the range $q \geq 1$ by matching F_2 with the asymptotic form [5.48] at $q = \infty$, and the second segment started at $q = 1$ and moved inwards to $q = 0$ with the condition that F_2 must be finite at $q = 0$. F_2 and its first two derivatives vanish at $q = 1$, but F_2 is not an analytic function. Hence, [5.34] implies that G_2 is not analytic either, and it follows that χ_2 and η_2 behave in a similar fashion.

Although it has been possible to construct a solution to second order in br which is physically acceptable - all physical quantities are continuous and finite at $q = 1$, because of the non-analytic behavior at $q = 1$ we expect that the higher order contributions would not be physically acceptable. Furthermore, because of the problem of defining the smallness of the mass parameter we cannot assign a radius of convergence for the series [5.12] and [5.13]. Whether or not the non-analytic property is a consequence of the lack of a "smallness" criterion is difficult to say, but would merit further study.

It is interesting to note that the time dependent systems based on [1.1] or [1.6] must be unstable. Schiff (1969) indicated that because the conformally invariant system has a traceless energy momentum tensor, localization cannot be achieved if the energy is non-zero. The system corresponding to [1.6] satisfies

$$\int d^4x \; T^{\alpha}_{\alpha} = 0$$

from which the same conclusion follows.

CHAPTER 6

[illegible] INVARIANT SYSTEMS OBTAINED FROM [1.1]. II

6.1 Brea[illegible]formal symmetry.

Hav[illegible]cussed one method of breaking the conformal symmetry as [illegible]ibe[illegible] Chapter 5, we now propose a different scheme. We assume that [illegible]re [illegible]st solutions to [1.1] which fluctuate rapidly and that an averag[illegible] these fluctuations yields a coherent (or mean) field which we cons[illegible] an observable.

[illegible] averaging process referred to above is to some extent arbitrary. [illegible] can imagine either an ensemble average over coordinate independen[illegible]andom fluctuations in a statistical sense, or a time average over the fluctuations. It follows that if we denote the averaging by a bar ($\bar{A}_\alpha$) then

[6.1] $$\bar{\bar{A}}_\alpha = \bar{A}_\alpha$$

and

[6.2] $$\overline{DA_\alpha} = D\,\bar{A}_\alpha$$

where D represents any differential or integral operation.

We define the fluctuating field A'_α by writing

[6.3] $$A_\alpha = \bar{A}_\alpha + A'_\alpha .$$

It follows from [6.1] that

[6.4] $$\bar{A}'_\alpha = 0 .$$

The "vacuum" state in this formulation is taken to be the one for which

[6.5] $$\overline{A}_{\alpha} = 0$$

and

[6.6] $$\overline{A'^{2}_{\alpha}} \equiv \overline{A^{(o)^{2}}_{\alpha}} = \text{constant}$$

corresponding to a randomly fluctuating "background" field, denoted by $A^{(o)}_{\alpha}$, which is homogeneous and isotropic in the sense of [6.6].

In addition to [6.6], the associated tensor

$$F^{(o)}_{\alpha\beta} \equiv \partial_{\alpha}A^{(o)}_{\beta} - \partial_{\beta}A^{(o)}_{\alpha}$$

has a special property directly observable from the equation for the "pure background" field

[6.7] $$\partial_{\beta}F^{\alpha\beta}_{(o)} = -A^{\alpha}_{(o)}A^{(o)}_{\beta}A^{\beta}_{(o)} \, .$$

Multiplying [6.7] by $A^{(o)}_{\alpha}$ gives

[6.8] $$\partial_{\beta}\left[F^{\alpha\beta}_{(o)}A^{(o)}_{\alpha}\right] + \frac{1}{2}\,F^{(o)}_{\alpha\beta}F^{\alpha\beta}_{(o)} = -\left[A^{(o)}_{\beta}A^{\beta}_{(o)}\right]^{2} .$$

Averaging both sides and noting that because of the homogeneity and isotropy in $\overline{F^{\alpha\beta}_{(o)}\,A^{(o)}_{\alpha}}$, we have

[6.9] $$-\frac{1}{2}\,\overline{F^{(o)}_{\alpha\beta}F^{\alpha\beta}_{(o)}} = \overline{E^{2(o)}} - \overline{H^{2(o)}} = \left[\overline{A^{(o)}_{\beta}A^{\beta}_{(o)}}\right]^{2} > 0$$

provided $A^{(o)}_{\beta}A^{\beta}_{(o)} \neq 0$.

This suggests[†] that if excited states of the "background" field exist they might exhibit electric properties, i.e. electric charge.

Inserting [6.3] into [1.1], and using an ensemble average with the properties [6.1], [6.2], and [6.5], gives the equations

$$[6.10] \quad \partial_\beta \overline{F^{\alpha\beta}} = -\bar{A}^\alpha(\bar{A}_\lambda\bar{A}^\lambda + \overline{A'^\lambda A'_\lambda}) - \overline{A'^\alpha(A'_\lambda A'^\lambda + 2\bar{A}_\lambda A'^\lambda)}$$

and

$$[6.11] \quad \partial_\beta F'^{\alpha\beta} = -\bar{A}^\alpha(A'_\lambda A'^\lambda - \overline{A'_\lambda A'^\lambda} + 2\bar{A}_\lambda A'^\lambda) - A'^\alpha(\bar{A}_\lambda\bar{A}^\lambda + A'_\lambda A'^\lambda + 2\bar{A}_\lambda A'^\lambda)$$

$$+ \overline{A'^\alpha(A'_\lambda A'^\lambda + 2\bar{A}_\lambda A'^\lambda)} \quad .$$

If we multiply [6.11] by A'_α and average the result, we obtain, after some algebraic manipulations,

$$[6.12] \quad \partial_\alpha\partial^\alpha f = 2\partial_\beta \overline{A'_\alpha\partial^\alpha A'^\beta} + 6\bar{A}^\alpha \overline{A'_\alpha A'_\lambda A'^\lambda} + 2\bar{A}_\alpha\bar{A}^\alpha f$$

$$+ \overline{4(\bar{A}^\alpha A'_\alpha)^2} + \overline{2(A'_\alpha A'^\alpha)^2} + \overline{F'_{\alpha\beta}F'^{\alpha\beta}}$$

where f is defined by

$$[6.13] \qquad f \equiv \overline{A'_\lambda A'^\lambda} \ .$$

We deal with [6.10] and [6.12] rather than [6.10] and [6.11] in order to avoid the details of the fluctuations.

Both [6.10] and [6.12] present us with a rather formidable mathematical problem (generally characteristic of nonlinear equations involving fluctuating fields). We consider f to be an observable

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[†]We also have $-4\overline{\vec{E}_{(o)}\cdot\vec{B}_{(o)}} = \overline{F^{\alpha\beta}_{(o)}F^{(o)}_{\alpha\beta}} = 2\partial_\alpha\overline{(\epsilon^{\alpha\beta\rho\sigma}A^{(o)}_\beta\partial_\rho A^{(o)}_\sigma)} = 0$. The symbol (o) is not a tensor index.

classical scalar field and for purposes of tractability, express the right hand sides of [6.10] and [6.12] as "functionals" of f, $\overline{A_\alpha}$, and derivatives of these quantities. This leads us to coupled equations of the form

$$[6.14] \quad \partial_\beta \overline{F^{\alpha\beta}} = -\bar{A}^\alpha(\bar{A}_\lambda \bar{A}^\lambda + f) - \Lambda_1^\alpha(\bar{A}^\beta, f, \partial_\beta f, \partial_\beta \bar{A}_\gamma, \text{etc.})$$

and

$$[6.15] \quad \partial_\alpha \partial^\alpha f = \Lambda_2(\bar{A}^\beta, f, \partial_\beta f, \partial_\beta \bar{A}_\gamma, \text{etc.})$$

where the functionals Λ_1^α and Λ_2 are given in terms of $\bar{A}^\beta$, f, and their derivatives. By a judicious choice of these functionals, it is possible to break the conformal invariance.

In principle, there is an infinite number of equations we could investigate, so we will look at the simplest ones which lead to reasonable results. We remove the complexity with the derivatives in Λ_1^α and Λ_2 by including derivatives only in Λ_2. Hence, $\Lambda_1^\alpha = \Lambda_1^\alpha(\bar{A}^\beta, f)$. This assumption is somewhat unwarranted but it leads to a system of equations which become much more tractable.

In addition, we require [6.14] and [6.15] to be derivable from a Lagrangian. This places further restrictions on Λ_1^α and Λ_2 as well as giving us an energy momentum tensor from which we can investigate the energy for various models we can construct.

It is well known that charged elementary particles occur in singlet pairs with opposite charges. Thus, for example, there is only one type of π^+ or π^- meson. Since we have an additional scalar field, namely f, which appears in our equations, we must therefore demand that -f is not a solution to the field equations if f satisfies them. Otherwise, we would necessarily have a description of doubly degenerate charged particles.

6.2 Systems without derivatives in Λ_2.

We consider static systems described by [6.14] and [6.15] with radial symmetry and $\vec{\bar{A}} = 0$.[1] When $\vec{\bar{A}} = 0$, it is reasonable to assume that the coupling between $\bar{\phi}$ and ϕ' is much stronger than that between $\bar{\phi}$ and $\vec{A}'$. This allows us to set $\vec{A}' \simeq \vec{A}_{(o)}$, so f is given by

$$f = \vec{A}'^2 - \phi'^2 \simeq \vec{A}^2_{(o)} - \phi'^2 \qquad [6.16]$$

which we take to vanish as $r \to \infty$. The case where $f \to$ constant as $r \to \infty$ is considered later.

In the rest frame we now investigate the equations derivable from a Lagrangian density of the form

$$\mathcal{L} = \frac{1}{4\pi}\left[\frac{(\nabla\phi)^2}{2} - \lambda^2\,\frac{(\nabla f)^2}{2} + G(f,\phi^2)\right] \qquad [6.17]$$

where we have dropped the bars on $\bar{\phi}$ for simplicity, and λ is a parameter with the dimensions of length. G is a scalar so it must contain even powers of ϕ. This ensures that $-\phi$ is a solution to the field equations derivable from [6.17] if ϕ is a solution. Hence, both positive and negative charges belong to the same solution ϕ. It is also clear that neutral solutions to the equations with $\phi \neq 0$ have a degenerate rest mass. We can, however, remove this degeneracy by setting $\phi = 0$ and taking the resulting equation for f to represent singlet neutral systems.

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[1] $\vec{A} = 0$ requires the intrinsic angular momentum (spin) to be zero.

In an arbitrary Lorentz frame, the Lagrangian density becomes

$$[6.18] \quad \mathcal{L} = \frac{1}{4\pi}\left[-\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} - \frac{1}{2}\lambda^2\,\partial_\alpha f\partial^\alpha f + G(f, A_\alpha A^\alpha)\right]$$

so that the pure electromagnetic term has the usual form and sign as in conventional electromagnetism.

Our main effort in this thesis will be to examine the consequences of such a Lagrangian density using different expressions for G.

From [6.18] we define the canonical energy-momentum tensor

$$[6.19] \quad T_{\alpha\beta} = \mathcal{L}\,\delta_{\alpha\beta} - \partial_\alpha A_\gamma \frac{\partial \mathcal{L}}{\partial(\partial_\beta A_\gamma)} - \partial_\alpha f \frac{\partial \mathcal{L}}{\partial(\partial_\beta f)}$$

The pure electromagnetic part can be symmetrized in the usual way.

The 4- momentum is

$$P^\beta = -\int T^{\alpha\beta} d\sigma_\alpha$$

where the integration is taken over an arbitrary space-like surface. If we choose the surface t = constant, we can write

$$P^\beta = -\int T^{\beta 4} d^3x$$

so the energy (mass) is

$$[6.20] \quad P^4 = M = -\int T^{44}\, d^3x\ .$$

For static fields, [6.19] gives

[6.21] $$M = -\int \mathcal{L}\, d^3x = -L \ .$$

M can be written more compactly if we use some integral relations which we now derive.

We can perform an amplitude variation, $f \to af$, and by requiring $\left(\frac{\partial L}{\partial a}\right)\Big|_{a=1} = 0$ for any f which satisfies the field equations and vanishes as $r \to \infty$, we obtain the integral relation

[6.22] $$0 = \int d^3x\, [\lambda^2(\nabla f)^2 - f\frac{\partial G}{\partial f}] \ .$$

Similarly, we can vary the amplitude of ϕ and obtain the relation

[6.23] $$0 = \int d^3x\, [(\nabla\phi)^2 + \phi\frac{\partial G}{\partial \phi}] \ .$$

If we change the scale in r by α, i.e. $r \to \alpha r$, and require $\frac{\partial L}{\partial \alpha}\Big|_{\alpha=1} = 0$, we have

[6.24] $$0 = \int d^3x\, [(\nabla\phi)^2 - \lambda^2(\nabla f)^2 + 6G] \ .$$

Hence, from [6.22], [6.23], and [6.24], M is given by

[6.25] $$M = \frac{1}{12\pi}\int d^3x\, [\lambda^2(\nabla f)^2 - (\nabla\phi)^2]$$

$$= \frac{1}{2\pi}\int d^3x\ G$$

$$= \frac{1}{12\pi}\int d^3x\ \ [f\frac{\partial G}{\partial f} + \phi\frac{\partial G}{\partial \phi}] \ .$$

This shows that M is generally indefinite in sign. It is possible in certain cases, however, to construct a positive definite rest mass. These situations are discussed later. It is apparent that neutral solutions with $\phi = 0$ always have a positive definite rest mass.

We wish to work with equations which involve dimensionless quantities, so we perform the scale and amplitude transformations

[6.26] $$\phi = \frac{\hat{\phi}}{\lambda}, \quad f = \frac{\hat{f}}{\lambda^2}, \quad r = \hat{r}\lambda$$

where $\hat{\phi}$, $\hat{f}$, and $\hat{r}$ are dimensionless. The energy calculated from [6.26], $\hat{M}$, is related to M according to

[6.27] $$M = \frac{\hat{M}}{\lambda} .$$

We now drop the hats on ϕ,f, and r for the sake of simplicity, and investigate various models of charged and neutral particles based on the Lagrangian density

[6.28] $$\mathcal{L} = \frac{1}{4\pi}\left[\frac{(\nabla\phi)^2}{2} - \frac{(\nabla f)^2}{2} + G(f,\phi^2)\right] .$$

The simplest equation we can obtain from [6.14] is

[6.29] $$\nabla^2\phi = \phi(f - \phi^2) .$$

Charged particle solutions to [6.29] therefore require $f \to \phi^2$ as $r \to \infty$. Furthermore, the Lagrangian density for [6.29] must contain the terms $-\frac{\phi^4}{4} + \frac{f\phi^2}{2}$, so that the f equation involves the term $\frac{\phi^2}{2}$. Therefore, G is of the form

$$G = -\frac{1}{4}(f - \phi^2)^2 + N(f)$$

where N(f) is some polynomial in f. For simplicity we choose

$$N(f) = -\frac{\beta}{3}f^3$$

where β is a dimensionless parameter. The equation for f is then

$$\nabla^2 f = \frac{1}{2}(f-\phi^2) + \beta f^2. \tag{6.30}$$

It is easy to verify that

$$\phi = \frac{b_1}{\sqrt{r^2+a^2}} \tag{6.31}$$

and

$$f = \frac{b_1{}^2(r^2+a^2)-3a^2}{(r^2+a^2)^2} \tag{6.32}$$

satisfy [6.29] and [6.30] with

$$\beta = 8,\ b_1{}^2 = \frac{9}{10},\ a^2 = \frac{78}{25}.$$

We thus have a non-singular solution for the potential ϕ which is asymptotic to the Coulomb[2] solution.

Neutral solutions which have an asymptotic expansion of the form

$$\phi = \sum_{n=2}^{\infty} \frac{b_n}{r^n} \tag{6.33}$$

require f to have the series expansion

$$f = \sum_{n=2}^{\infty} \frac{a_n}{r^n} \tag{6.34}$$

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[2] The Coulomb potential $\phi = \frac{b_1}{r}$ is also an exact solution to these equations (singular at $r = 0$).

with $a_2 \neq 0$ for any ϕ of the type [6.33]; i.e. $\phi = \frac{b_2}{r^2} + \ldots$, $\phi = \frac{b_3}{r^3} + \ldots$, etc. Hence, these expansions require the f equation to be of the form

[6.35] $$\nabla^2 f = -\frac{\phi^2}{2} + \beta_N f^2$$

where the subscript N here, and in what follows, refers to neutral systems with $\phi \neq 0$.

One exact neutral solution of [6.29] and [6.35] is

[6.36] $$\phi_\circ = \frac{b_N}{r^2 + a_N^2}$$

with

[6.37] $$f = \frac{2r^2 + b_N^2 - 6a_N^2}{(r^2 + a_N^2)^2}$$

and

$$\frac{78}{5} = a_N^2, \; b_N^2 = \frac{104}{5}, \; \beta_N = \frac{18}{5}.$$

The rest mass for charged particles, M_c, is given by

$$M_c = \frac{1}{24\pi} \int d^3x (f^2 - \phi^4).$$

For the exact solution presented above,

$$M_c = -\frac{3\pi}{320a}$$

which is negative.

Other charged particle solutions were sought for by numerically integrating [6.29] and [6.30], but none were found. Furthermore, we see from [6.12] that a term involving $f\phi^2$ appears explicitly in the f equation. This leads to our next model.

We consider the equation

[6.38] $$\nabla^2 f = \frac{1}{2}(f-\phi^2) + \beta f^2 - \gamma f^3 + df\phi^2$$

which forces us to write the ϕ equation as

[6.39] $$\nabla^2\phi = \phi(f-\phi^2) - d\phi f^2$$

for a charged system. Neutral solutions with a degenerate mass follow from the equations

[6.40] $$\nabla^2\phi = \phi(f-\phi^2) - d_N\phi f^2$$

and

[6.41] $$\nabla^2 f = -\frac{\phi^2}{2} + \beta_N f^2 - \gamma_N f^3 + d_N f\phi^2 ,$$

and singlet neutral states are obtained by solving

[6.42] $$\nabla^2 f = \frac{f}{2} + \beta_n f^2 - \gamma_n f^3 .$$

where the subscript n is used for neutral systems with $\phi = 0$.

It is interesting to note that the charged and neutral equations with $\phi \neq 0$ are the simplest ones we can construct which are equivalent to equation [1.6] with $\vec{A} = 0$,

$$\nabla^2\phi = \phi(b^2 - \phi^2)$$

when f = constant everywhere. In fact, we have $b^2 \equiv \frac{f}{2}$.

An exact solution of [6.38] and [6.39] with ϕ of the form [6.31] is obtained by setting $f = \phi^2$. The parameters in these equations satisfy the relations

[6.43] $$3a^2 = b_1{}^4 d,\ 2 = b_1{}^2(\beta+d),\ 8a^2 = \gamma b_1{}^4 .$$

A neutral solution of the form [6.36] with

[6.44] $$f = \frac{2}{r^2 + a_N^2}$$

satisfies [6.40] and [6.41] provided

[6.45] $$4 = -\frac{b_N^2}{2} + 4\beta_N,\ -8a_N^2 = b_N^2 d_N - 4\gamma_N,\ 8a_N^2 = 4d_N + b_N^2.$$

The rest mass of the exact charged solution is

[6.46] $$M_c = \frac{\pi}{96a^3}(4 - 6a^2),$$

and that for the exact neutral solution is

[6.47] $$M_N = \frac{\pi}{24a_N^3}(4 - b_N^2)$$

so it is clear that the parameters may be chosen to ensure both masses are positive.

Other solutions for these charged and neutral equations were obtained numerically. The results are presented in tables 6.1 and 6.2. Figures 6.1 and 6.2 represent the nodeless solution for ϕ and the corresponding solution with one node in f. The solutions were obtained by integrating inwards from $r = \infty$ with the asymptotic expansions $\phi = \frac{b_1}{r} + \frac{b_3}{r^3} + \ldots$, and $f = \frac{a_2}{r^2} + \frac{a_3}{r^3} + \ldots$, with ϕr and fr both zero at $r = 0$. Two free parameters which become eigenvalues were generally required to satisfy these boundary conditions.

We now present a phase plane analysis of the neutral equation [6.42]. This is the equation for which it is practical to carry out such an analysis because when ϕ is not zero, we must deal with derivative terms in the Lagrangian with an indefinite relative sign.

We can perform scale and amplitude transformations in [6.42] to cast it into the form

[6.48] $$\nabla^2 f = f + \beta_1 f^2 - \mu f^3 .$$

where $\mu \equiv 2\gamma_n$, and $2\beta_n \equiv \beta_1$.

The critical points of [6.48] are situated at

$$f = A_+ , \quad f = A_- , \quad \text{and } f = 0$$

where

[6.49] $$A_\pm \equiv \frac{\beta_1 \pm \sqrt{\beta_1^2 + 4\mu}}{2\mu}$$

because [6.48] can be expressed as

[6.50] $$\nabla^2 f = -\mu f(f-A_+)(f-A_-) .$$

The solutions about the critical points are obtained by setting $f = c + w$, where w is small and c is put equal to zero or $A_\pm$. Inserting this form for f into [6.50] and retaining the linear terms in w gives

[6.51] $$\nabla^2 w = -w[c(c^2 + A_+A_- - cA_- - cA_+) + w(3c^2 - 2cA_- - 2cA_+ + A_+A_-)] .$$

When $c = 0$,

$$\nabla^2 w = w$$

which has the bounded solutions for large r,

[6.52] $$w = \alpha_1 \frac{e^{-r}}{r}$$

where α_1 is some constant.

At $c = A_\pm$, we set $w \equiv \frac{v}{r}$, and see that v satisfies

$$\frac{d^2v}{dr^2} = -\sqrt{\frac{\beta_1^2 + 4\mu}{2\mu}}\left[\pm \beta_1 + \sqrt{\beta_1^2 + 4\mu}\right]v$$

so the asymptotically bounded solution is

[6.53] $$w = \frac{\sin \alpha_2 r}{r}$$

where α_2 is the square root of the coefficient of $-v$.

As the name suggests, phase plane analysis draws some useful notions over from classical mechanics (Finkelstein et al., 1951). We set

$$K \equiv T + V \equiv \frac{1}{2}\dot{f}^2 + V$$

where

[6.54] $$V = -\beta_1\frac{f^3}{3} + \frac{\mu}{4}f^4 - \frac{f^2}{2}$$

is the "potential" energy function, and K plays the role of the energy for a conservative system.

The critical or equilibrium points follow from $\frac{dV}{dr} = 0$, and the equilibrium curves are obtained when $\frac{dK}{dr} = 0$. If we multiply [6.48] by $\dot{f}$, we have

[6.55] $$\frac{dK}{dr} = \frac{d}{dr}\left(\frac{\dot{f}^2}{2} + V\right) = -\frac{2}{r}\dot{f}^2 < 0$$

so the orbits in the phase plane must move inwards across the lines of constant K.

When K = 0,

[6.56] $$\frac{1}{2}\dot{f}^2 = f^2\left(\frac{1}{2} + \frac{\beta_1}{3}f - \frac{\mu}{4}f^2\right) > 0 .$$

so f must lie in the range $f_1 < f < f_0$ where f_0 and f_1 are the greater and smaller of

$$\frac{2\beta_1}{3\mu} \pm \frac{2}{\mu}\sqrt{\frac{\beta_1^{\,2}}{g} + \frac{\mu}{2}}$$

respectively.

The curve for $K = 0$ must be symmetrical about the f axis in the phase plane, but not about the $\dot{f}$ axis if $\beta_1 \neq 0$. The situation is plotted in Figure 6.3.

When K = constant = $-A$ with $A > 0$, we have

$$\frac{\dot{f}^2}{2} = -\mu\frac{f^4}{4} + \frac{\beta_1}{3}f^3 + \frac{f^2}{2} - A\,. \qquad [6.57]$$

Applying Descarte's rule of signs shows us that the right hand side of [6.57] has two or no real positive roots. Setting $f \to -f$ in this equation also gives two variations in sign so there are two or no negative real roots. Furthermore, f cannot vanish in [6.57] because this would imply $\frac{\dot{f}^2}{2} = -A$, which is impossible. Hence the curves with $K < 0$ enclose only one of A_+ or A_- , and must lie inside the curve defined by $K = 0$. This situation is depicted in Figure 6.4.

When $K > 0$, we have

$$\frac{\dot{f}^2}{2} = -\mu\frac{f^4}{4} + \frac{\beta_1}{3}f^3 + \frac{f^2}{2} + A \qquad [6.58]$$

so there is one positive real root, and one negative real root with $\beta_1 \gtrless 0$. These curves lie outside of those with $K = 0$, and $f = 0$ allows $\frac{\dot{f}^2}{2} = A$, so both A_+ and A_- can be encompassed by these curves as is shown in Figure 6.5.

From [6.55] we see that any orbit which passes inside one of the $K = 0$ lobes must terminate in that lobe. Also, any initial value of f which keeps the orbit outside of the $K = 0$ lobes forces the orbit to terminate at $f = 0$. These solutions are the exponentially decaying eig[illegible]this system. The eigenvalues of f must clearly be greater than f_0 or less than f_1. Note that $f = 0$ is an unstable critical point while $A_\pm$ are both stable. Near $r = 0$, a power series of the form $f = \sum^{\infty} \bar{a}_n r^n$ satisfies [6.48] with $a_{2n+1} = 0$, so we must have

$$\dot{f}(0) = 0 .$$

We thus have an infinite discrete set of eigensolutions asymptotic to zero as $\frac{e^{-r}}{r}$, and a continuous set asymptotic to $A_\pm$ as $\sin \frac{\alpha_2 r}{r}$. Both types of solutions are finite everywhere, and the rest energy for this neutral system is given by the positive definite expression

[6.59] $$M_n = \frac{1}{3} \int (\nabla f)^2 \, d^3x .$$

6.3 Systems with derivative terms in Λ_2.

By looking at the power series expansion of ϕ and f for large r, we can ascertain whether or not all solutions to the equations have the same charge. This must be the case if we hope to construct a theory of charged scalar elementary particles. The models presented above do not generally exhibit this property, so we now depart from them and construct some equations which give us fixed charge solutions.

We can achieve this goal by including derivative terms in Λ_2. We consider a Lagrangian density of the form

[6.60] $$\mathcal{L} = k(f,\phi) \frac{(\nabla f)^2}{2} + \nu \frac{(\nabla\phi)^2}{2} + h(f,\phi)\nabla\phi\cdot\nabla f + G(f,\phi)$$

where h and k and G are polynomials in f and ϕ, and ν is a parameter; h is actually the fourth component of a 4-vector h_β since the covariant generalization of [6.60] is

$$\mathcal{L} = \frac{k}{2} \partial_\alpha f \partial^\alpha f - \frac{\nu}{4} F_{\alpha\beta}F^{\alpha\beta} + \partial_\alpha f F^{\alpha\beta} h_\beta + G .$$

We obtain the field equations

[6.61] $$\frac{\partial G}{\partial f} = \frac{1}{2}\frac{\partial k}{\partial f}(\nabla f)^2 + \frac{\partial k}{\partial \phi}\nabla\phi\cdot\nabla f + k\nabla^2 f + h\nabla^2\phi + \frac{\partial h}{\partial \phi}(\nabla\phi)^2 .$$

and

[6.62] $$\frac{\partial G}{\partial \phi} + \frac{\partial k}{\partial \phi}\frac{(\nabla f)^2}{2} = \nu\nabla^2\phi + h\nabla^2 f + \frac{\partial h}{\partial f}(\nabla f)^2$$

which give

[6.63] $$\nabla^2\phi\left(\nu - \frac{h^2}{k}\right) = \frac{\partial G}{\partial \phi} + (\nabla f)^2\left(\frac{1}{2}\frac{\partial k}{\partial \phi} - \frac{\partial h}{\partial f} + \frac{h}{2k}\frac{\partial k}{\partial f}\right)$$

$$- \frac{h}{k}\frac{\partial G}{\partial f} + \frac{h}{k}\frac{\partial k}{\partial \phi}\nabla\phi\cdot\nabla f + \frac{h}{k}\frac{\partial h}{\partial \phi}(\nabla\phi)^2 .$$

The only way we can avoid the derivative terms in the right hand side of [6.63] with $h \neq 0$ is to have

[6.64] $$k = ah^2 = k(f)$$

where "a" is a constant. Our equations then become

[6.65] $$\nabla^2\phi = \frac{a}{\nu a-1}\left[\frac{\partial G}{\partial \phi} - \frac{1}{ah}\frac{\partial G}{\partial f}\right]$$

and

[6.66] $$\nabla^2 f = \frac{\nu}{h^2(\nu a-1)}\frac{\partial G}{\partial f} - \frac{1}{h}\frac{dh}{df}(\nabla f)^2 - \frac{1}{h(\nu a-1)}\frac{\partial G}{\partial \phi} .$$

If we require a $\phi(f - \phi^2)$ term in [6.57] and the condition that $f \to 0$ as $r \to \infty$, the asymptotic form of f and ϕ are given by

[6.67] $$\phi = \frac{b_1}{r} + \ldots, \quad f = \frac{a_2}{r^2} + \ldots$$

with $b_1^2 = a_2$. The charge b_1 is therefore fixed if a_2 is fixed, which leads us to investigate the $\frac{1}{r^4}$ coefficients in [6.66].

From [6.65],

[6.68] $$\frac{\partial G}{\partial \phi} - \frac{1}{ah}\frac{\partial G}{\partial f} \sim \phi(f-\phi^2) + \text{higher order asymptotic terms odd in } \phi$$

so

[6.69] $$\frac{\nu}{h^2}\frac{\partial G}{\partial f} - \frac{1}{h}\frac{\partial G}{\partial \phi} \sim -\frac{1}{h}\left[\phi(f-\phi^2) + \frac{1}{h}\frac{\partial G}{\partial f}\left(\frac{1}{a} - \nu\right) + \text{higher order asymptotic terms odd in } \phi\right].$$

Now G is a scalar, and integrals involving G, such as the rest mass, are finite only if $G \sim O(r^{-4})$. Hence $G \sim \phi^4, f\phi^2, f^2, \ldots +$ higher order terms so $\frac{\partial G}{\partial f} \sim \phi^2, f \ldots, +$ higher order terms. If $h \sim \frac{1}{f^n}$ where n is a positive integer, [6.69] drops off too fast to contain a $\frac{1}{r^4}$ contribution. If $h \sim f^m$ with m a positive integer, then the term $\frac{\phi f}{f^m}$ would force b_1 to vanish from [6.66], and this term, because of the manner in which the odd and even quantities involving ϕ enter in [6.69], can never be cancelled by another term. The only possibility we have to fix the charge in a system with G given by

$$G \sim -\frac{\phi^4}{4} + f\frac{\phi^2}{2} + O(r^{-4})$$

with the boundary condition $f \to 0$ as $r \to \infty$, is therefore with $h = $ constant, which we now consider. We take $k = -1$ and split $G(f,\phi)$ into the even and odd powers of ϕ, $\tilde{G}(f,\phi^2)$ and $H(f,\phi)$ respectively, according to the prescription

[6.70] $$G = \tilde{G} + hH .$$

The equation

[6.71] $$\nabla^2\phi(\nu + h^2) - h\left(\frac{\partial G}{\partial f} + h\frac{\partial H}{\partial f}\right) = \frac{\partial \tilde{G}}{\partial \phi} + h\frac{\partial H}{\partial \phi}$$

is obtained from [6.65]. Equating the odd terms in ϕ gives

[6.72] $$h^2 \frac{\partial H}{\partial f} + \frac{\partial \tilde{G}}{\partial \phi} = Z\,\phi(f - \phi^2) + Z\,S(f,\phi)$$

where S contains odd powers in ϕ, and $Z \equiv \nu + h^2$. The even powers in ϕ require

[6.73] $$-\frac{\partial \tilde{G}}{\partial f} - \frac{\partial H}{\partial \phi} = 0.$$

Hence [6.66] becomes

[6.74] $$\nabla^2 f = h\phi(f-\phi^2) + h\,S - \frac{\partial \tilde{G}}{\partial f} - h\,\frac{\partial H}{\partial f}\,.$$

Differentiating [6.72] with respect to f, [6.73] with respect to ϕ, and adding the results gives

[6.75] $$-\frac{\partial^2 H}{\partial \phi^2} + h^2\,\frac{\partial^2 H}{\partial f^2} = Z\phi + Z\,\frac{\partial S}{\partial f}\,.$$

We now consider the simplest equation we can obtain from [6.71], namely that when S = 0:

[6.76] $$\nabla^2\phi = \phi(f - \phi^2)\,.$$

Charged solutions require $\phi \sim \frac{b_1}{r}$ and $f \sim \frac{a_2}{r^2}$ and we keep all odd terms in H which drop off as r^{-11}. Hence,

[6.77] $$\begin{aligned} H = {} & a\,\phi^5 + bf\,\phi^3 + cf^2\phi + df\phi^5 + \ell f^2\phi^3 + vf^3\phi + pf^4\phi \\ & + q\phi^7 + nf^3\phi^3 + mf^2\phi^5 + sf\phi^7 + t\phi^9 + A\phi^{11} + B\phi^9 f \\ & + Df^2\phi^7 + Ef^3\phi^5 + Rf^4\phi^3 + Pf^5\phi\,. \end{aligned}$$

Inserting [6.77] into [6.75] gives the conditions

[6.78] $P = \frac{1}{h^4} d, \; n = \frac{10}{3}\frac{d}{h^2}, \; v = \frac{b}{h^2}, \; \ell = 2ph^2, \; 10a = \ell h^2, \; 2ch^2 = Z$

with all other coefficients except b and c equal to zero. Equation [6.74] becomes

[6.79] $$\nabla^2 f = h\phi(f - \phi^2) + 5a\phi^4 + 3bf\phi^2 + cf^2 + 5df\phi^4 + 3\ell f^2\phi^2$$
$$+ vf^3 + pf^4 + 3nf^3\phi^2 + Pf^5$$
$$- h[b\phi^3 + 2cf\phi + d\phi^5 + 2\ell f\phi^3 + 3vf^2\phi + 4p\phi f^3$$
$$+ 3nf^2\phi^3 + 5Pf^4\phi]$$

which, with [6.78] and the scale and amplitude transformation,

[6.80] $$f \to \tilde{f}\,\alpha, \; \phi \to \tilde{\phi}\,\alpha^{\frac{1}{2}}, \; x \to \tilde{x}\;\alpha^{-\frac{1}{2}}$$

where $\alpha^{\frac{1}{2}} \equiv h$, can be transformed into the equation

[6.81] $$\nabla^2 f = b(f+\phi)^3 - (2b+1)\phi^3 + (1-2c)f\phi + cf^2 + 5a(f+\phi)^4$$
$$+ dh^2(f+\phi)^5$$

after dropping the twiddle on f, ϕ, and r for the sake of simplicity. Here, $c = \frac{1}{2}$ and $b = -1$ for charged solutions; "a", h, and d are parameters. Equation [6.76] retains the same form under [6.80]. The $\frac{1}{r^4}$ coefficients in [6.81] give

[6.82] $$b_1^{\,2} = a_2 = \frac{4}{-5+10a}$$

so the charge is fixed for a given value of "a". Equations [6.81] and

[6.76] can be more easily studied by letting $f + \phi$ be a new variable.

From [6.78] the relation

$$2ch^2 = h^2 = Z = \nu + \bar{h}^2 \qquad [6.83]$$

can only be satisfied provided $\nu = 0$. Consequently the pure electromagnetic contribution to the Lagrangian [6.60] disappears. Thus even though we can show that the rest mass is given by

$$M = \frac{1}{4\pi} \int d^3x \; (\nabla f)^2 \qquad [6.84]$$

which is positive definite, such a Lagrangian would have to be rejected (this object would not be able to radiate electromagnetic energy). Hence, we continue our search for a better model which contains fixed charge solutions. If we put $h = 0$ in [6.61], we have

$$\nabla^2 f + \frac{1}{2k}\frac{dk}{df}(\nabla f)^2 = \frac{1}{k}\frac{\partial G}{\partial f} \qquad [6.85]$$

where $k = k(f)$. If $k \sim \frac{1}{f^n}$, then a $\frac{1}{r^4}$ term in $\frac{1}{k}\frac{\partial G}{\partial f}$ can occur only when $n = 1$. However, it is easy to verify that the corresponding $\frac{1}{r^4}$ contribution to the left hand side of [6.85] vanishes, and we cannot fix the charge. Furthermore, [6.60] shows that L diverges if $k \sim f^n$ with $n > 1$, and [6.85] does not allow $n = 1$ to lead to a fixed charge either. The only other possible form for k is when it is a constant but this case has already been investigated in the first two models we discussed.

One final alternative approach to construct equations with a fixed charge from [6.60] with $f \to 0$ as $r \to \infty$ is to introduce singular terms into $\int G(f,\phi)d^3x$. For example, consider the Lagrangian density

[6.86] $\mathcal{L} = \frac{(\nabla\phi)^2}{2} + \frac{\varepsilon}{2}\left[\frac{(\nabla f)^2}{f^2} - 4\alpha f\right] - \frac{\phi^2}{2}\left(\frac{\phi^2}{2} - f\right) + \tilde{N}(f) + \ell\,\frac{\phi^6}{6}$

where $\varepsilon = \pm 1$. We must have

[6.87] $$\alpha = \frac{1}{a_2}$$

where a_2 is again defined by $f \doteq \frac{a_2}{r^2} + \ldots$, in order to obtain a finite rest energy $M = - L$.

The field equations obtained from [6.86] are

[6.88] $$\nabla^2\phi = \phi(f - \phi^2) + \ell\,\phi^5$$

and

[6.89] $$\nabla^2 f - \frac{(\nabla f)^2}{f} = - 2\alpha f^2 + \varepsilon f^2\,\frac{d\tilde{N}}{df} + \frac{f^2\phi^2\varepsilon}{2} \quad .$$

From [6.88] we need ${b_1}^2 = a_2$, as before, and the $\frac{1}{r^4}$ term in [6.89] is consistent with [6.87]. Hence the charge is fixed for a given value of α.

With $\tilde{N}(f) = - \frac{\gamma f^2}{2}$, we introduce the scale and amplitude transformations,

[6.90] $$f = s^2\tilde{f}(sr), \quad \phi = \tilde{\phi}(sr)^{\dagger}$$

and calculate $\left.\frac{d\tilde{L}}{ds}\right|_{s=1} = 0$ to obtain the integral relation

[6.91] $$0 = \int d^3x \left[- \frac{(\nabla\phi)^2}{2} - \varepsilon\frac{(\nabla f)^2}{2f^2} + 2\alpha f\varepsilon - \frac{f\phi^2}{2} - \frac{\gamma f^2}{2} + \frac{3\phi^4}{4} - \frac{\ell}{2}\,\phi^6\right]$$

which allows the rest mass to be obtained from

[6.92] $$M = \frac{1}{4\pi}\int d^3x \left[- \frac{\phi^4}{2} + \gamma f^2 + \ell\,\frac{\phi^6}{3}\right] .$$

An exact charged solution to [6.88] and [6.89] is given by

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†This choice ensures that the ε term in the Lagrangian gives a finite contribution.

$f=\phi^2$ and

[6.93] $$\phi = \frac{b_1}{\sqrt{r^2 + a^2}}$$

with $-3a^2 = \ell b_1{}^4$, $\alpha b_1{}^2 = 1$, $8a^2 = \varepsilon b_1{}^4(2\gamma - 1)$. The corresponding rest mass is

[6.94] $$M_c = \frac{\pi}{16a}(16a^2\varepsilon - b_1{}^2)$$

which is positive for $a^2 > \frac{b_1{}^2\varepsilon}{16}$.

With $\phi = 0$, an exact neutral solution was found to be

[6.95] $$f = \frac{B}{a_N{}^2 + r^2}$$

with $A = \frac{1}{\alpha}$, $4a^2 = \gamma B^2\varepsilon$,

so ε must be + 1 if $\gamma > 0$, which is necessary for M_n to be > 0.

Another charged solution was found numerically for $\varepsilon = +1$ with the property that $f \to 0$ as $r \to 0$ in such a way that $\frac{(\nabla f)}{f}$ vanished at the origin. No other solutions with charge in addition to the ones mentioned here were obtained.

However, this model has the drawback that it describes a system with infinite static self-stresses. This is evident from [6.86] by calculating

[6.96] $$T_\alpha{}^\alpha = \int d^3x\left[(\nabla\phi)^2 - \frac{\varepsilon}{f^2}(\nabla f)^2 + 8\varepsilon\alpha f + 4\tilde{N}(f) - \phi^4 + 2f\phi^2 + \frac{2}{3}\ell\phi^6\right].$$

Such an unstable static system seems physically unrealistic, so we have given little weight to this type of construction.

6.4 The boundary condition f $\to$ constant as $r \to \infty$.

All of the systems we have discussed so far had the property that f vanished at spatial infinity. This condition is now relaxed, and we will see that having f going to a constant as $r \to \infty$ leads to the most fruitful equations of any we have constructed.

We set

$$[6.16] \quad f = \overline{A_{(o)}^{\;2}} - \overline{\phi'^{2}} = \overline{A_{(o)}^{\;2}} - \overline{\phi_{(o)}^{\;2}} + \phi_{(o)}^{\;2} - \phi'^{2} \equiv (\varepsilon + g(r))\frac{1}{\lambda^2}.$$

where the length scale λ is taken to be

$$\lambda = (|\overline{A_{(o)}^{\;2}} - \overline{\phi_{(o)}^{\;2}}|)^{-\frac{1}{2}}$$

and g(r) vanishes as $r \to \infty$; $\varepsilon = \pm 1$ since the relative sign of $\overline{A_{(o)}^{\;2}} - \overline{\phi_{(o)}^{\;2}}$ is indeterminate. We must remember that λ has been removed from the field equations to obtain a dimensionless system, so in what follows,

$$f = \varepsilon + g.$$

Our previous models dealt with terms in Λ_1^4 which dropped off faster than the quantity $\phi(f - \phi^2)$ as $r \to \infty$. Now, any higher order terms in f will have a nonzero contribution at spatial infinity, so there is no a priori reason why we should not include quantities in Λ_1^4 of the same or of similar type as ϕf and ϕ^3. Hence we write G as

$$[6.97] \quad G = \frac{\phi^2}{2}\left[R + (T+1)f + Af^2 + Bf^3\right] + \frac{\phi^4}{4}\left[S-1 + Cf + Df^2\right]$$

$$- \frac{\alpha}{2} g^2 + \frac{\beta}{3} g^3 + \frac{\gamma}{4} g^4 + \frac{\sigma}{5} g^5 + \frac{\omega}{6} g^6$$

which, from [6.28], gives the field equations

[6.98] $$\nabla^2\phi = \phi[R + (T+1)f + Af^2 + Bf^3] + \phi^3[S-1 + Cf + Df^2]$$

and

[6.99] $$\nabla^2 f = -\frac{\phi^2}{2}[T + 1 + 2Af + 3Bf^2] - \frac{\phi^4}{4}(C + 2Df) + \alpha g - \beta g^2 - \gamma g^3 - \sigma g^4 - \omega g^5 \; .$$

It should be pointed out that for $S < 1$ and $T > -1$, we can make the scale and amplitude transformations

[6.100] $$\phi = \tilde{\phi}(1+T)(1-S)^{-1}, \; f = f(1+T)(1-S)^{-1}, \; r = \tilde{r}(1+T)^{-1}(1-S)^{\frac{1}{2}}$$

and deal with equations in which S and T are removed. We do not wish to do this because a system with $S > 1$ can be constructed with a positive definite rest mass.

Inserting the power series

[6.101] $$\phi = \sum_{n=1}^{\infty} \frac{b_n}{r^n}$$

and

[6.102] $$g = \sum_{n=1}^{\infty} \frac{a_n}{r^n} \; .$$

into [6.98] and [6.99] allows us to investigate the charge b_1. The ϕ equation gives

[6.103] $$0 = T_1 \varepsilon + R + A + B\varepsilon$$

[6.104] $$0 = a_1(T_1 + 2A\varepsilon + 3B)$$

[6.105] $$0 = a_2(T_1 + 2A\varepsilon + 3B) + a_1^2(A+3B\varepsilon) + b_1^2(\varepsilon C+D+T_2)$$

where

[6.106] $$T_1 \equiv T + 1 \,, \quad T_2 \equiv S - 1 \,.$$

From [6.99] we have

[6.107] $$\alpha\, a_1 = 0 \,,$$

[6.108] $$0 = \frac{b_1^{\,2}}{2}\,(T_1 + 2A\varepsilon + 3B) + \beta a_1^{\,2} + \alpha\, a_2$$

[6.109] $$0 = b_1^{\,2}(A + 3B\varepsilon)a_1 + 2a_1\, a_2\beta + \alpha a_3 + \gamma a_1^{\,3} + b_1 b_2(T_1 + 2A\varepsilon + 3B)$$

[6.110] $$-2a_2 = (2b_1b_2 + b_2^{\,2})\left(\frac{T_1}{2} + A\varepsilon + \frac{3B}{2}\right) + 2b_1b_2a_1(A + 3B\varepsilon) + \alpha a_4 + b_1^{\,2}\left[(A + 3B\varepsilon)a_2 + \frac{3B}{2}\,a_1^{\,2}\right] + \frac{b_1^{\,4}}{4}\,(C + 2D\varepsilon) + \beta(2a_1a_3 + a_2^{\,2}) + 3\gamma a_1^{\,2}a_2 + \sigma a_1^{\,4} \,.$$

Suppose $a_1 \neq 0$. Then we must have

[6.111] $$\alpha = 0,\ T_1 + 2A\varepsilon + 3B = 0,\ \beta = 0$$

so

[6.112] $$b_1^{\,2} = - a_1^{\,2}\,\frac{(A + 3B\varepsilon)}{C\varepsilon + D + T_2}$$

which shows that the charge is arbitrary.

If $a_1 = 0$ and $\alpha \neq 0$, we obtain the relations, [6.113] $$b_1^2 = -a_2 \frac{(T_1 + 2A\varepsilon + 3B)}{\varepsilon C + D + T_2} = -\frac{2\alpha a_2}{T_1 + 2A\varepsilon + 3B}$$

and from [6.98] we have

[6.114] $$2b_2 = b_1 a_3(T_1 + 2A\varepsilon + 3B) + b_2 a_2(T_1 + 2A\varepsilon + 3B) + 3b_1^2 b_2(T_2 + \varepsilon C + D).$$

If $b_2 \neq 0$, we can obtain another relation between b_1 and a_2. However, it is easy to check from [6.113] and [6.114] that this leads to a contradiction, and we cannot obtain a fixed charge with $\alpha \neq 0$.

Our goal can be achieved with $\alpha = a_1 = 0$. This gives

[6.115] $$0 = T_1 + 2A\varepsilon + 3B$$

[6.116] $$0 = \varepsilon C + D + T_2$$

which forces b_2 to vanish and requires

[6.117] $$2a_2 = -b_1^2 a^2(A + 3B\varepsilon) - \frac{b_1^4}{4}(C + 2D\varepsilon) - \beta a_2^2.$$

with

[6.118] $$A = -\frac{\varepsilon}{2}(3B + T_1), \quad R = \varepsilon\frac{(B - T_1)}{2}.$$

There are two ways we can arrive at a fixed charge, depending on whether or not a_3 vanishes.

Suppose $a_3 \neq 0$. Then, from [6.99],

$$6 = -b_1^2(A + 3B\varepsilon) - 2\beta a_2,$$

which, from [6.118], is equivalent to

[6.119] $$a_2 = \frac{-b_1^2(R+B\varepsilon) - C}{2\beta}$$

Inserting this into [6.117] gives

[6.120] $$0 = b_1^4[\beta(C+2D\varepsilon) - (R+B\varepsilon)^2] - 4b_1^2(R+B\varepsilon) + 12$$

from which we can fix the charge.

We looked at the case when $\omega = \sigma = R = T = S = 0$ which, from [6.118], means $A = -2\varepsilon$, $B = 1$, and the equations become

[6.121] $$\nabla^2\phi = g^2\phi(\varepsilon + g) + \phi^3[(1+D)g\varepsilon + Dg^2]$$

[6.122] $$\nabla^2 g = -\frac{\phi^2 g}{2}(2\varepsilon + 3g) - \frac{\phi^4}{4}(\varepsilon + D\varepsilon + 2Dg) - \beta g^2 - \gamma g^3.$$

An exact solution to these equations was found to be given by

[6.123] $$\phi = \frac{b_1}{\sqrt{r^2 + a^2}}$$

and

[6.124] $$g = \frac{\lambda_c}{r^2 + a^2}$$

with

[6.125] $$D = \frac{\lambda_c}{b_1^2}, \quad -3a^2 = b_1^2\lambda_c\varepsilon, \quad 8a^2 = \lambda_c(b_1^2 + \gamma\lambda_c),$$

$$-2 = \left(3 + \frac{b_1^2}{\lambda_c}\right)\frac{b_1^2\varepsilon}{4} + \beta\lambda_c .$$

The rest mass for this solution is

[6.126] $$M_c = \frac{\pi}{48a^3}(2\lambda_c^3 - 3a^2b_1^2)$$

which has the values

[6.127] $$M_c = \begin{cases} \dfrac{\pi}{16\sqrt{11\gamma}}\left(\dfrac{22}{b_1^2} - 3\gamma\right) & \text{for } \varepsilon = 1 \\ \dfrac{\pi}{16\sqrt{5\gamma}}\left(\dfrac{10}{b_1^2} - 3\gamma\right) & \text{for } \varepsilon = -1. \end{cases}$$

Other numerical solutions were found with ϕ and g having no nodes. The results are presented in Table 6.3.

With $\phi = 0$, a neutral solution of

[6.128] $$\nabla^2 g = -\beta_n g^2 - \gamma_n g^3$$

is

[6.129] $$g = \frac{\lambda_n}{a_n^2 + r^2}$$

where

[6.130] $$2 = -\beta_n \lambda_n \ , \quad 8a_n^2 = \gamma_n \lambda_n^2 .$$

The corresponding rest mass has the value

[6.131] $$M_n = \frac{1}{12\pi}\int d^3x(\nabla g)^2 = \frac{\pi|\beta_n|\sqrt{2}}{3\,\gamma_n^{3/2}} .$$

If we allow the parameter α to vanish only for these charged and neutral systems, we can recover an equation analogous to [6.42] which has the discrete set of solutions asymptotic to the Yukawa form, and a non-degenerate mass spectrum which is positive definite:

[6.132] $$\nabla^2 g = \alpha_n g - \beta_n g^2 - \gamma_n g^3 .$$

The exact solutions [6.123] and [6.124] must have $a_3 = 0$, so they do not have the same "symmetry" as the numerical solutions presented in Table 6.3. It is possible, however, to fix the charge with $a_3 = 0$.

We arrive at the relation

[6.133] $$2 = - b_1^2(R + B\varepsilon)$$

from [6.117] by setting

$$\beta = 0$$

and

$$C = -2D\varepsilon .$$

If $\beta = 0$, then [6.133] forces a_3 to vanish. The field equations now take the form

[6.134] $$\nabla^2\phi = - \frac{2\phi g^2}{b_1^2} + Bg^3\phi + D\phi^3 g^2$$

and

[6.135] $$\nabla^2 g = \frac{\phi^2}{2}\left(\frac{4g}{b_1^2} - 3Bg^2\right) - \frac{D}{2}\phi^4 g - \gamma g^3 - \sigma g^4 - \omega g^5 ,$$

and equation [6.116] with $C = - 2D\varepsilon$ requires

[6.136] $$D = T_2 .$$

An exact solution of the form [6.123] with [6.124] when $\sigma = \omega = 0$ was found. The parameters satisfy the relations

[6.137] $$3a^2 b_1^2 = 2\lambda_c^2$$

and

[6.138] $$\lambda_c = - b_1^2 \frac{D}{B} , \quad 2\frac{b_1^4 D}{a^2} + 16 = 3b_1^2 \gamma .$$

The rest mass for this solution vanishes, which is evident from [6.126] and [6.137]. One numerical solution with no nodes in ϕ and g was found with $B = 1$, $D = -1$, and $\sigma = \omega = 0$. The result is shown in Table 6.4 along with the corresponding parameters for the exact solution.

6.5 Charged systems with a positive definite mass spectrum.

All of the charged systems considered thus far allowed both positive and negative values for the rest mass. We now construct two expressions for the rest energy corresponding to equations [6.134] and [6.135] which can be made positive definite.

From the integral relations

[6.139] $$0 = \int d^3x \, [g \frac{\partial G}{\partial g} - (\nabla g)^2] \quad ,$$

[6.140] $$0 = \int d^3x [(\nabla\phi)^2 + \phi\frac{\partial G}{\partial \phi}] \quad ,$$

and

[6.141] $$0 = \int d^3x [\frac{(\nabla\phi)^2}{2} - \frac{(\nabla g)^2}{2} + 3G] \quad ,$$

the rest mass can be expressed as

[6.142a] $$M = \frac{1}{12\pi}\int d^3x [(\nabla g)^2 - (\nabla\phi)^2] \quad ,$$

[6.142b] $$M = \frac{1}{2\pi}\int d^3x \, G \quad ,$$

or

[6.142c] $$M = \frac{1}{12\pi}\int d^3x [g \frac{\partial G}{\partial g} + \phi\frac{\partial G}{\partial \phi}]$$

where G is given by

[6.143] $$G = -\frac{g^2\phi^2}{b_1^2} + \frac{B}{2}\phi^2 g^3 + \frac{D}{4}\phi^4 g^2 + \frac{\gamma}{4} g^4 + \frac{\sigma}{5} g^5 + \frac{\omega}{6} g^6 \; .$$

We take the linear combination

$$\frac{1}{4\pi(p+c)} \int d^3x\,[2pG + \frac{c}{3}\,(g\,\frac{\partial G}{\partial g} + \phi\,\frac{\partial G}{\partial \phi})]$$

to construct

[6.144] $$M_c = \frac{1}{4\pi(p+c)} \int d^3x \left[-\frac{g^{2\;2}}{3b_1^2}\,(4c+6p) + B\,\frac{\phi^2 g^3}{6}\,(5c+6p) + \frac{D}{2}\,\phi^4 g^2 (c+p) + \frac{\gamma g^4}{6}\,(3p+2c) + \frac{\sigma}{15}\,g^5 (6p+5c) + \frac{\omega}{3}\,g^6 (p+c) \right].$$

We can set 5c = -6p and obtain the result

[6.145] $$M_c = \frac{1}{4\pi} \int d^3x \left(\frac{2g^2\phi^2}{b_1^2} + D\,\frac{\phi^4 g^2}{2} - \frac{\gamma g^4}{2} + \frac{\omega}{3}\,g^6 \right),$$

or we can choose 3p = -2c to get

[6.146] $$M_c = \frac{1}{4\pi} \int d^3x \left(\frac{D}{2}\,\phi^4 g^2 + \frac{\sigma}{5}\,g^5 + \frac{\omega}{3}\,g^6 + \frac{B}{2}\,\phi^2 g^3 \right).$$

If we set D = 1 and $\omega = \sigma = 0$, then,

[6.147] $$\nabla^2\phi = (-\frac{2g^2}{b_1^2} + Bg^3)\phi + g^2\phi^3$$

and

[6.148] $$\nabla^2\phi = g\phi^2(\frac{2}{b_1^2} - \frac{3}{2}\,Bg) - \frac{g\phi^4}{2} - \gamma g^3.$$

A thorough numerical study of these equations for values of γ and B over a wide range (including B and γ both zero or vanishing separately) did not reveal any solutions.

It is apparent that the sign of D in [6.134] plays an important role concerning the existence of solutions and the nature of M_c; solutions do exist when D is negative, but this does not guarantee that the corresponding rest mass is positive. We can avoid this predicament by demanding that D = 0.

The next set of equations we consider are thus

[6.149] $$\nabla^2\phi = -2\phi g^2 + B\phi g^3$$

and

[6.150] $$\nabla^2 g = g\phi^2\left(2 - \frac{3B}{2}g\right) - \gamma g^3 - \sigma g^4$$

where we have set $b_1 = 1$ for convenience. An expression of the form

[6.151] $$\phi = \frac{(r^2 - a^2)}{(r^2 + a^2)^{3/2}}$$

with

[6.152] $$g = \frac{\lambda_c}{r^2 + a^2}$$

satisfies these equations provided

[6.153] $$\sigma = \frac{8a^4}{\lambda_c^3}, \; \lambda_c^2 = \frac{15}{2}a^2, \; \gamma = 0, \; B = 0 .$$

Incidentally, [6.151] and [6.152] force D and ω to vanish. The rest mass for this solution is

[6.154] $$M_c = \frac{\sigma}{20\pi}\int d^3x\, g^5 = \frac{1}{2\pi}\int d^3x\, g^2\phi^2 = \frac{15\pi}{64a} .$$

We now show that B must always vanish; otherwise, we will have a continuous set of nodeless solutions which contradicts the requirements of having a discrete set. We put

[6.155] $$g = \lambda_c \phi^2$$

into [6.149] and [6.150] , which uncouples these equations to give

[6.156] $$\left(\frac{d\phi}{dr}\right)^2 = \lambda_c^2 \phi^6 \left[2 - \frac{3B}{4\lambda_c} - \frac{\gamma}{2} - \lambda_c \phi^2 (B + \frac{\sigma}{2})\right] + \phi^4 .$$

We then use [6.156] to calculate $\nabla^2 \phi$ and equate that result to the right hand side of [6.149] to obtain the expression

[6.157] $$r^2\phi^2\left[2 + (3N + 2)\lambda_c^2\phi^2 - \lambda_c^3\phi^4(B + 4P)\right]^2 = 4\left[\lambda_c^2 N\phi^2 - \lambda_c^3 P\phi^4 + 1\right]$$

where

[6.158] $$N \equiv 2 - 3\frac{3B}{4\lambda_c} - \frac{\gamma}{2} \quad \text{and} \quad P \equiv B + \frac{\sigma}{2} .$$

When λ_c is infinitesimally small and ϕ bounded, we have

$$r^2\phi^2 = 1 - 3\frac{B\lambda_c}{4}\phi^2 + 0(\lambda_c^2)$$

so that

[6.159] $$\phi^2 = \frac{1}{r^2 + \frac{3B\lambda_c}{4}}$$

which shows that a continuum of ϕ's must exist unless[3] B = 0. With B = 0, ϕ is finite at the origin if $\sigma \neq 0$ and it approaches 1 as $r \to \infty$.

[3]A similar calculation with all of the parameters in [6.134] and [6.135] included requires only that B = 0.

We are therefore obliged to study the equations

[6.160] $$\nabla^2\phi = -2\phi g^2$$

and

[6.161] $$\nabla^2 g = 2g\phi^2 - \gamma g^3 - \sigma g^4 .$$

With $\gamma = 0$, no solutions in addition to [6.151] with [6.152] could be found. However, with $\gamma < 0$, we have a positive definite rest mass,

[6.162] $$M_c = \frac{1}{4\pi}\int d^3x(2g^2\phi^2 - \frac{\gamma g^4}{2})$$

which follows from [6.145], and the numerical solutions presented in Table 6.5, were obtained. We have the unusual result that the energies decrease as the number of nodes in ϕ increases. The function g has no nodes for any of the ϕ's. The results are plotted in Figures 6.6 to 6.11.

We also studied the equations

[6.160] $$\nabla^2\phi = -2g^2\phi$$

and

[6.163] $$\nabla^2 g = 2g\phi^2 - \gamma g^3 - \omega g^5$$

which has the corresponding rest mass

[6.164] $$M_c = \frac{\omega}{12\pi}\int g^6\, d^3x .$$

Although these equations have the undesirable property that both g and -g are solutions with the same ϕ (leading to a mass degeneracy), numerical solutions were obtained with the energies increasing as the number of nodes in ϕ increases. The ground state is therefore more easily defined than that of the preceding model. The results are presented in Table 6.6 with Figures 6.12 to 6.15 depicting the solutions with one and two nodes in ϕ; g is nodeless in all cases.

The obvious system to study now is that with both of the g^4 and g^5 terms included in the g equation, i.e.

[6.165] $$\nabla^2 g = 2g\phi^2 - \gamma g^3 - \sigma g^4 - \omega g^5$$

with

[6.160] $$\nabla^2 \phi = - 2g^2 \phi .$$

The numerical solutions presented in Table 6.7 were obtained. As expected, g is nodeless for all of the ϕ's. Note that M_c increases and then decreases as the number of nodes in ϕ increases.

By adding even higher order terms in g to the g equation in an appropriate manner, we can always obtain a positive definite rest mass with no $g \to - g$ symmetry. For example, consider

[6.166] $$\nabla^2 g = 2g\phi^2 - \gamma g^3 - \sigma g^4 - \omega g^5 - \frac{7}{6}\omega g^6 - \frac{4}{9}\omega g^7$$

with

[6.160] $$\nabla^2 \phi = -2g^2 \phi .$$

The rest energy is

[6.167] $$M_c = \frac{1}{4\pi}\int d^3x\left[2g^2\phi^2 - \frac{\gamma}{2} g^4 + \frac{\omega}{3} g^6(1+g)^2\right]$$

which is guaranteed to be positive definite for $\gamma < 0$.

It is interesting to note that spherically symmetric charged particles corresponding to a Lagrangian density

$$\mathcal{L} = \frac{1}{4\pi}\left[\frac{(\nabla\phi)^2}{2} - \frac{(\nabla g)^2}{2} + G(g,\phi)\right] \qquad [6.168]$$

that are far apart and instantaneously at rest (Rosen and Rosenstock 1952) interact according to the Coulomb force (Schiff 1962) plus some higher order corrections. The Coulomb contribution is calculated in Appendix 4.

Finally, we comment on the uniqueness of the asymptotic expansions which we have used to obtain fixed charge solutions. In all of the models constructed above we employed the series

$$\phi = \frac{b_1}{r} + \frac{b_2}{r^2} + \dots \qquad [6.169]$$

and

$$g = \frac{a_2}{r^2} + \frac{a_3}{r^3} + \dots \quad . \qquad [6.170]$$

Other possible expansions we could write down are

$$\phi = \frac{b_1}{r} + \frac{1}{r}\sum_{n=0}^{\infty} \frac{d_n e^{-c_n r}}{r^n} , \qquad [6.171]$$

and

$$g = \frac{1}{r}\sum_{n=0}^{\infty} \frac{a_n e^{-p_n r}}{r^n} \quad . \qquad [6.172]$$

However, when we put these expressions into [6.98] and [6.99] we find that $c_0 = 0$ and d_0 is arbitrary. We cannot therefore fix the charge with such expansions of ϕ and g.

It is possible to construct other types of power series which may or may not lead to fixed charges. This point requires more study to answer correctly.

CHAPTER 7

DISCUSSION

We have shown that it is possible to obtain a set of equations from [6.14] and [6.15] such that all charged solutions had the same charge, a non-degenerate positive definite rest mass, and were finite everywhere. Numerical integration of these equations indicated that the solutions form a discrete set characterized by the number of nodes in ϕ. Two free parameters which become eigenvalues were required to obtain these solutions.

Furthermore, it was possible to obtain a discrete set of neutral solutions with a non-degenerate rest mass which were asymptotic to the Yukawa form, by taking ϕ to vanish everywhere and allowing the parameter α in [6.97] to vanish only for charged solutions. There were also "long range" neutral particles given by [6.129], the physical significance of which (if any) is not understood, by allowing α_n to vanish but not β_n and γ_n.

Although these equations were obtained by requiring $f \to$ constant as $r \to \infty$, we could replace g in these equations by f with the boundary condition $f \to 0$ as $r \to \infty$ and deal with essentially the same system. In either case, the important physical properties were a consequence of allowing Λ_1^4 to contain terms proportional to ϕf and ϕ^3; in fact with $f \equiv g$

$$\Lambda_1^4 = -\phi(f - \phi^2) + \text{higher order terms.}$$

We also saw that including derivatives in Λ_2 helped us to obtain fixed charge solutions but complicated the situation regarding stability and rest masses.

To summarize then, we have shown that it is possible to devise a vector-scalar system which has the desired behavior to represent discrete bosons with fixed charge and positive mass. If we adopt a unitary philosophy it is possible to conceive of the vector and scalar fields as having a common origin - a conformally invariant 4-vector. However, the transition from the latter requires a heuristic statement regarding the functional properties of various averaged quantities. The great complexity of the system is such that we have not been able to establish the validity of the expansions used. Presumably a concrete model for the fluctuations themselves would lead to a better understanding of the possibilities of our approach.

The essential problem we faced was in determining a satisfactory expression for the non-derivative terms in the Lagrangian. To a large extent our approach was an ad hoc one. Needless to say that if a finite classical field theory can be shown to provide a satisfactory pre-quantum model for particles, the form of the Lagrangian would have to exhibit special properties particularly with regard to some yet to be defined symmetry operations. This is clearly crucial to any acceptable theory and could represent the point of departure for further investigation.

Table 6.1 Numerical Integration of [6.38] and [6.39].

Nodes in (ϕ,f)	β	γ	d	M_c
(0,1)	7.50282	0.023265	-0.457609	-0.048981
(0,2)	0.924144	156.899	8.43991	0.045280
(1,1)	2.03910	1.88183	7.14245	0.330544
(1,2)	2.64277	16.2527	6.44739	-0.406639

Table 6.2 Numerical Integration of [6.40] and [6.41].

Nodes in (ϕ,f)	β_N	γ_N	d_N	M_N
(0,1)	3.61890	0.1	0.742758	-0.012786
(0,1)	3.62390	0.2	1.15088	-0.010141
(1,1)	2.11053	0.5	0.209529	129.834
(1,2)	3.14663	1.93155	0	40.8391

Table 6.3 Numerical Integration of [6.121] and [6.122] with $\varepsilon=-1$.

b_1^2	β	γ	D	a_2	a_3	M_c
5	$\frac{11}{50}$	0.655989	5	-25	9.33536	0.700409
5	$\frac{11}{50}$	2.243058	5	-25	-75.9834	-0.261806

Table 6.4 Numerical Integration of [6.134] and [6.135] with $\omega = \sigma = 0$.

b_1^2	β	γ	D	B	a_2	M_c
2	0	$\frac{5}{3}$	-1	1	2	0
2	0	2.21364	-1	1	1.80625	-0.266970

Table 6.5. Numerical Integration of [6.160] and [6.161].

Nodes in (ϕ,g)	γ	σ	a_2	M_c
(1,0)	0	1	7.03125	0.860361
(2,0)	-0.109856	1	34.0377	0.841526
(3,0)	-0.097999	1	120.252	0.751935

Table 6.6 Numerical Integration of [6.160] and [6.163].

Nodes in (ϕ,g)	γ	ω	a_2	M_c
(1,0)	0.278677	1	8.02775	0.751951
(2,0)	0.0592413	1	32.8343	0.921674
(3,0)	0.0179429	1	108.894	0.979628
(4,0)	0.00633783	1	328.409	1.00286

Table 6.7 Numerical Integration of [6.160] and [6.165].

Nodes in (ϕ,g)	γ	σ	ω	a_2	M_c
(1,0)	0.114634	1	1	12.1634	0.496645
(2,0)	-0.0550019	1	1	52.3550	0.554541
(1,0)	0.678249	1	1	5.64892	1.08111
(3,0)	-0.0679895	1	1	172.062	0.541872

$\phi(r)$ vs. r with (0,1) nodes in (ϕ,f)

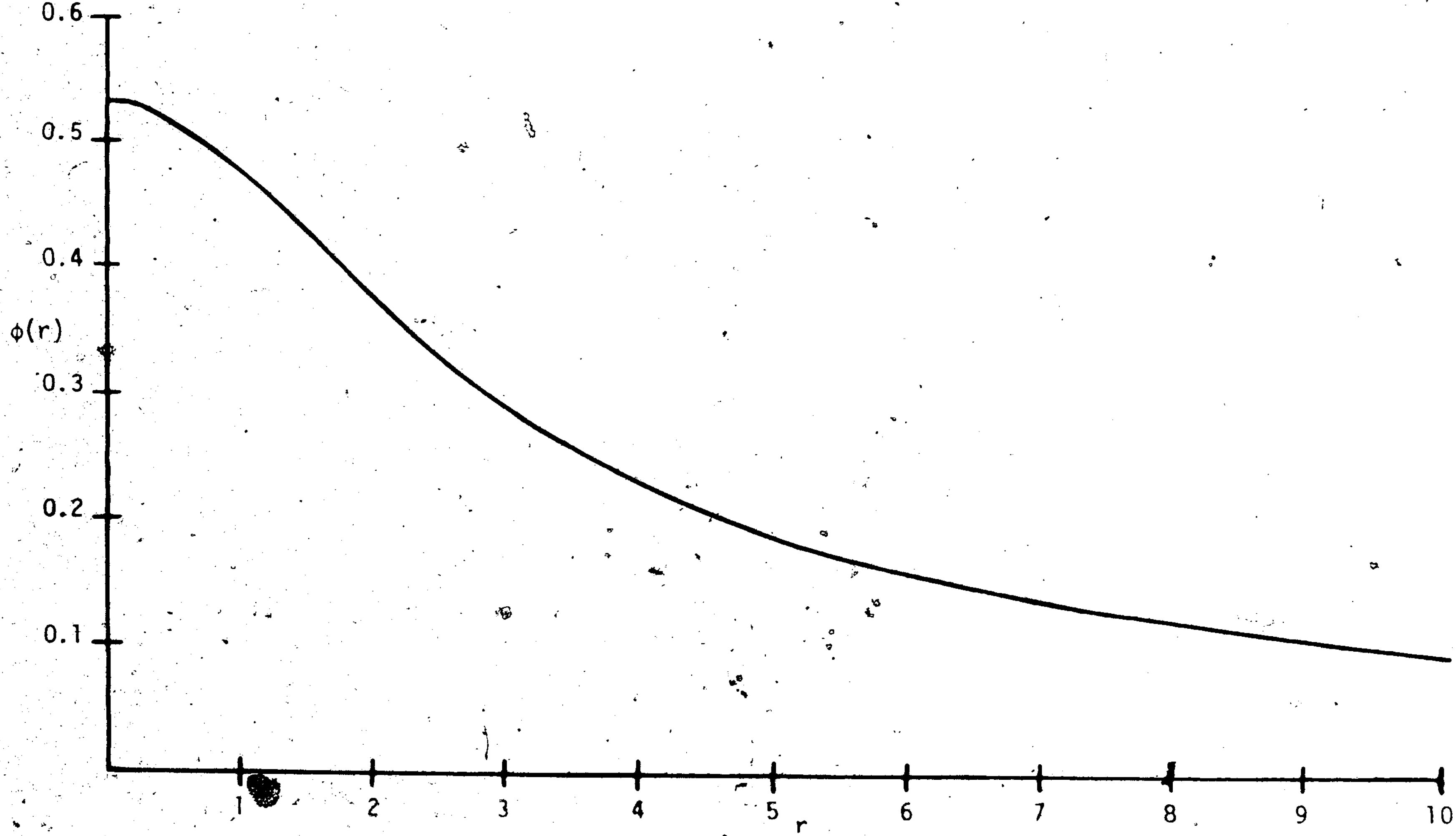


Figure 6.1

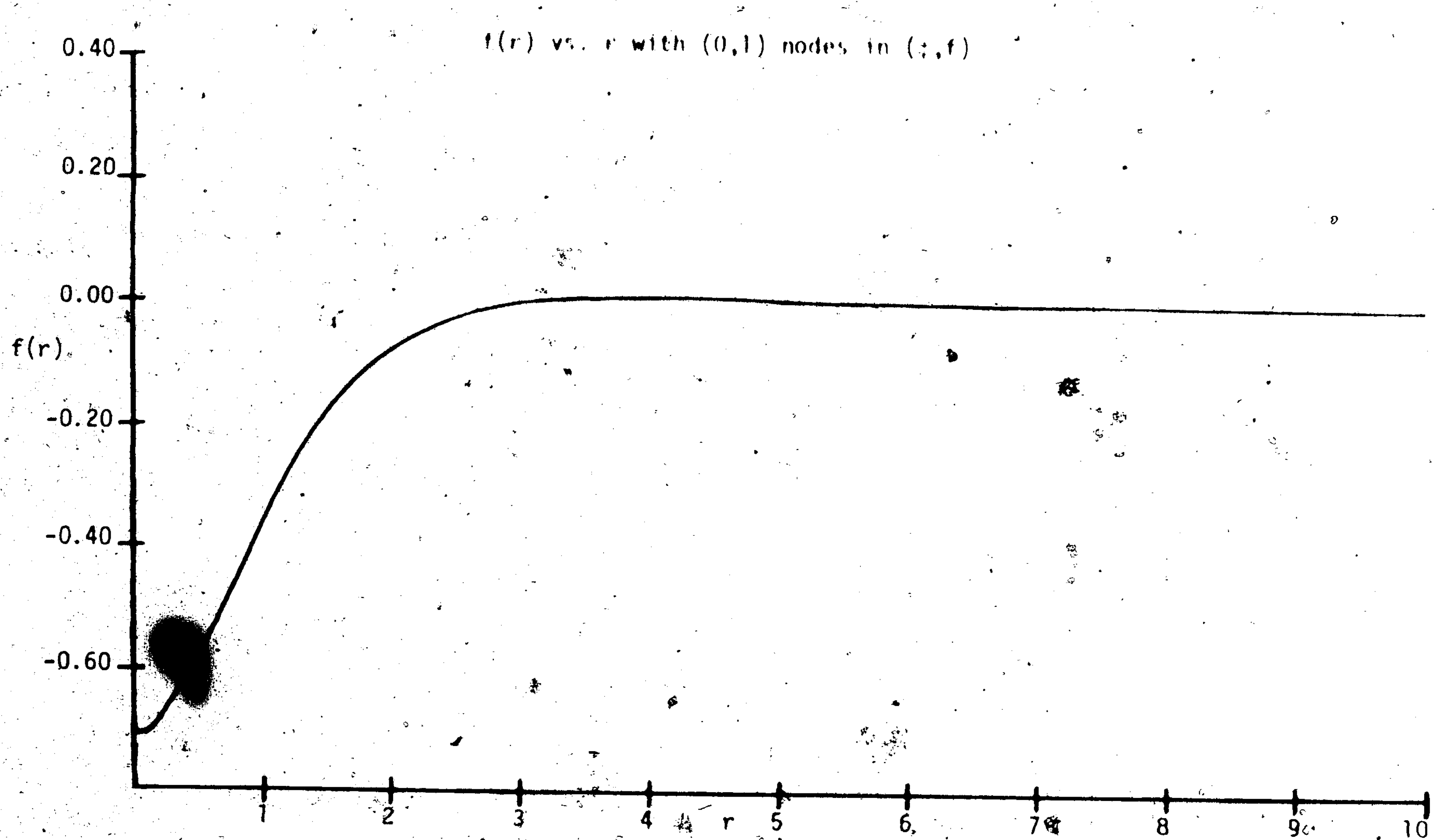

f(r) vs. r with (0,1) nodes in (:,f)
0.40
0.20
0.00
f(r)
-0.20
-0.40
-0.60
1
2
3
4
r
5
6
7
8
9
10

Figure 6.2

PHASE PLANE DIAGRAMS

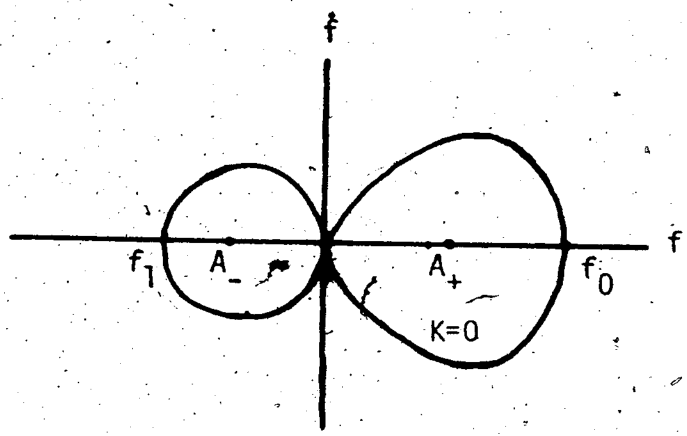


Figure 6.3: K=0

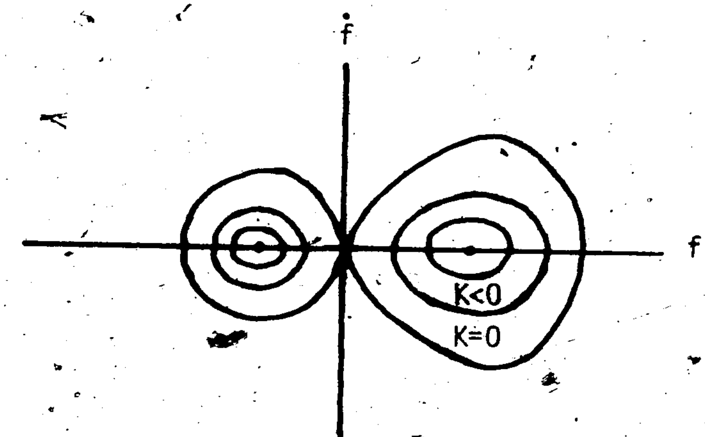


Figure 6.4: K<0

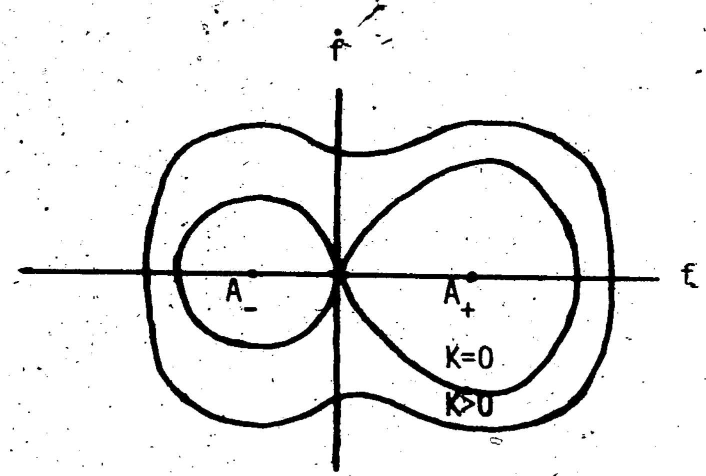


Figure 6.5: K>0

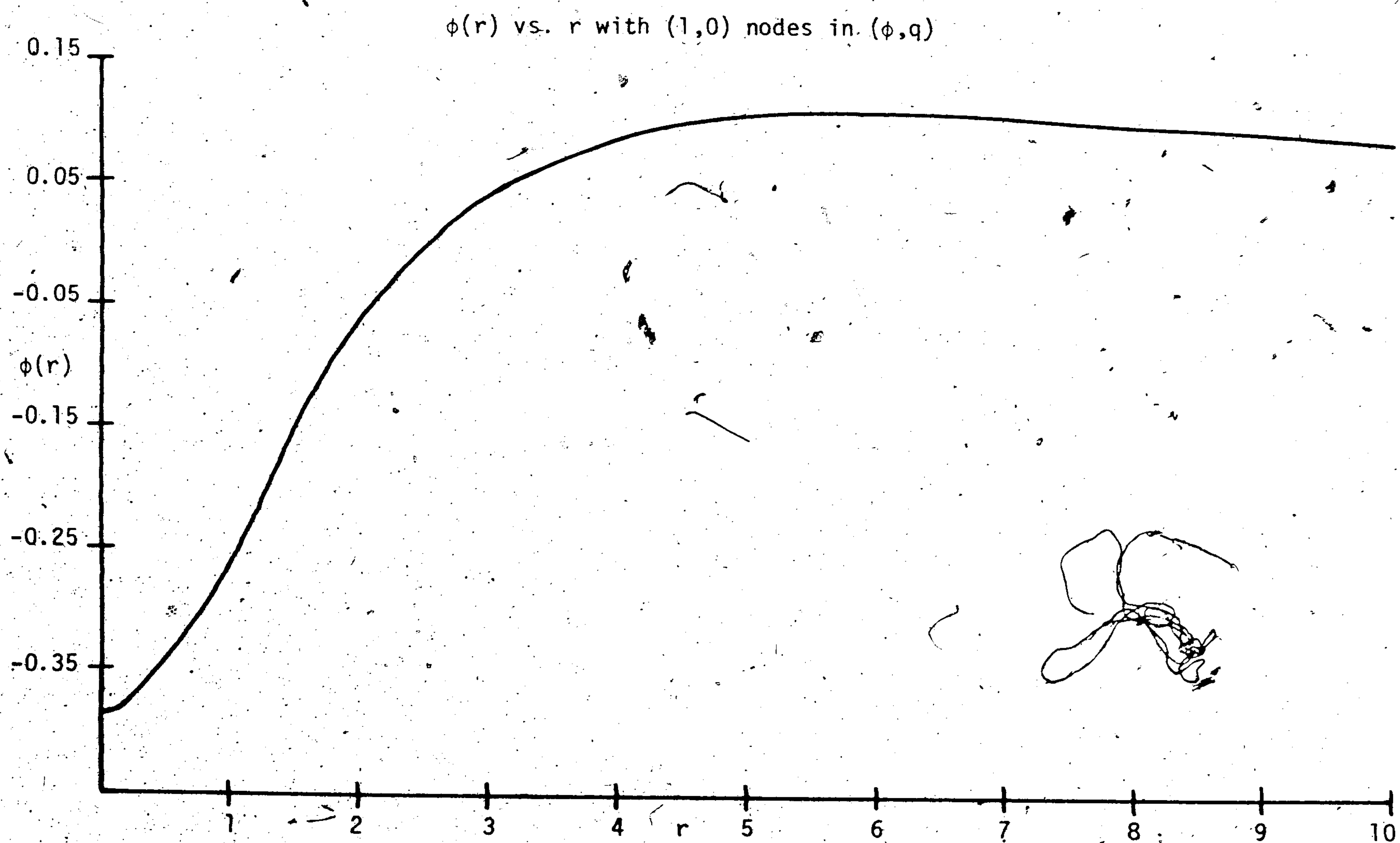

ϕ(r) vs. r with (1,0) nodes in (ϕ,q)
0.15
0.05
-0.05
ϕ(r)
-0.15
-0.25
-0.35
1
2
3
4
r
5
6
7
8
9
10

Figure 6.6

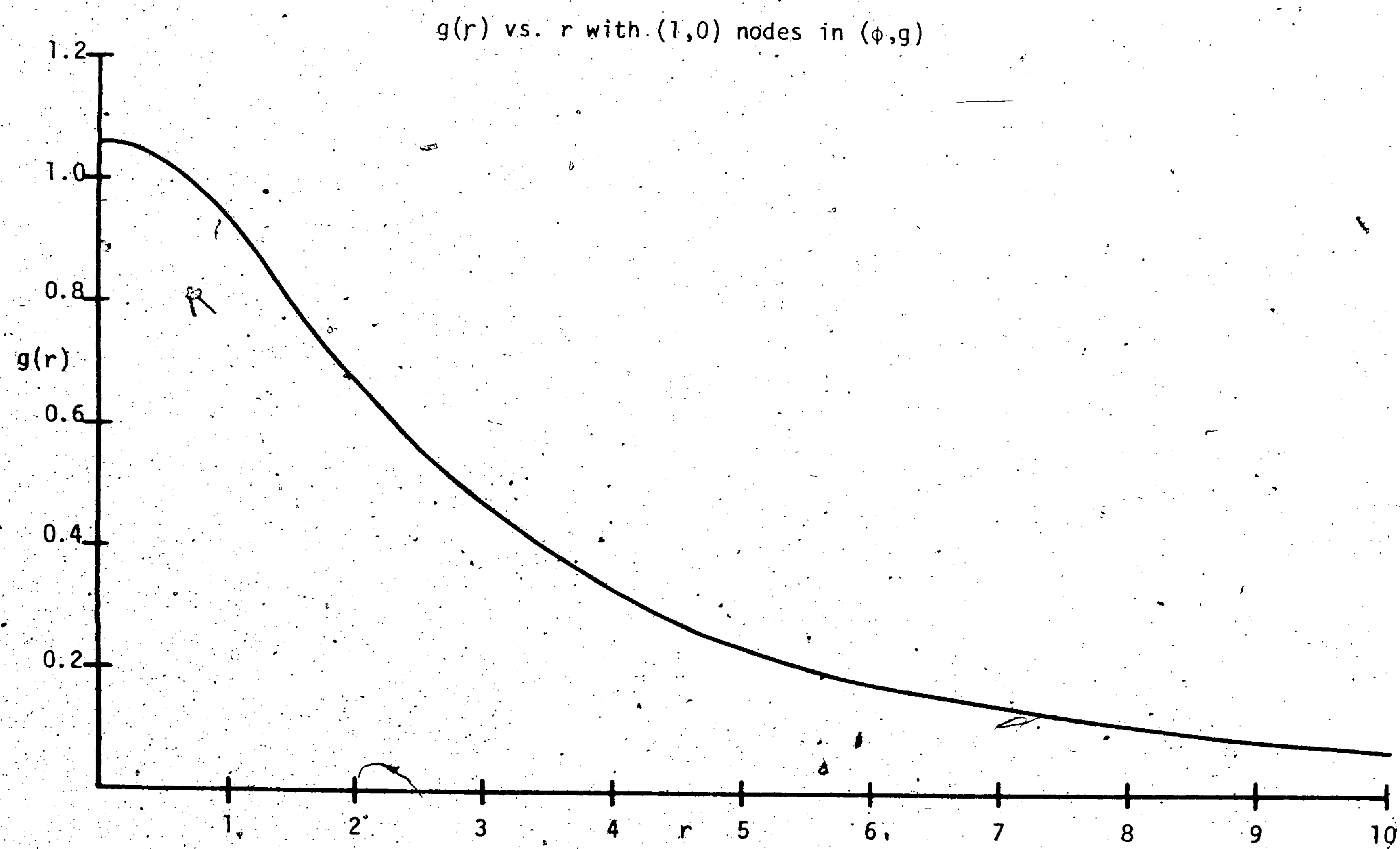


Figure 6.7

$\phi(r)$ vs. r with (2,0) nodes in (ϕ, g)

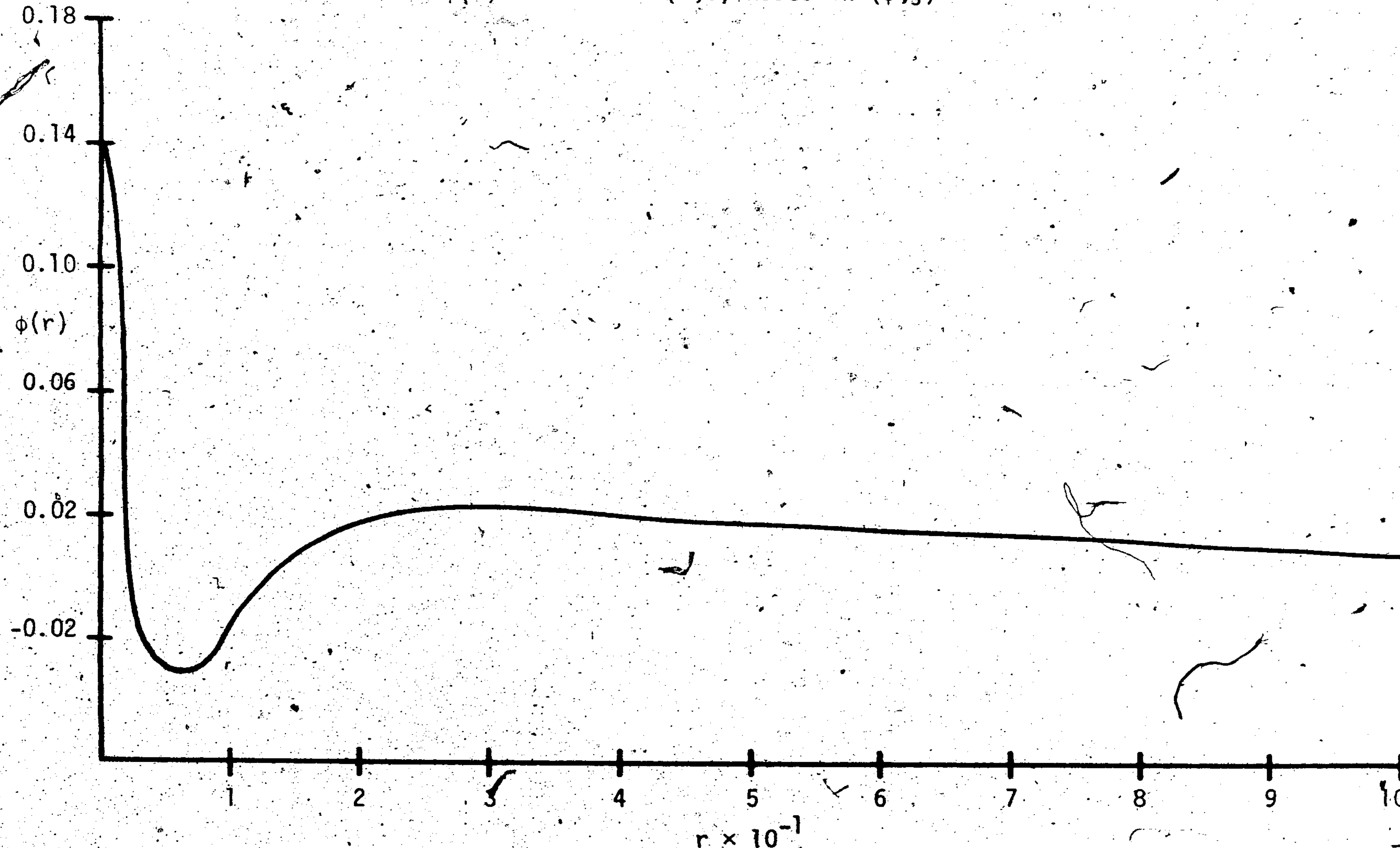


Figure 6.8

g(r) vs. r with (2,0) nodes in (ϕ,g)

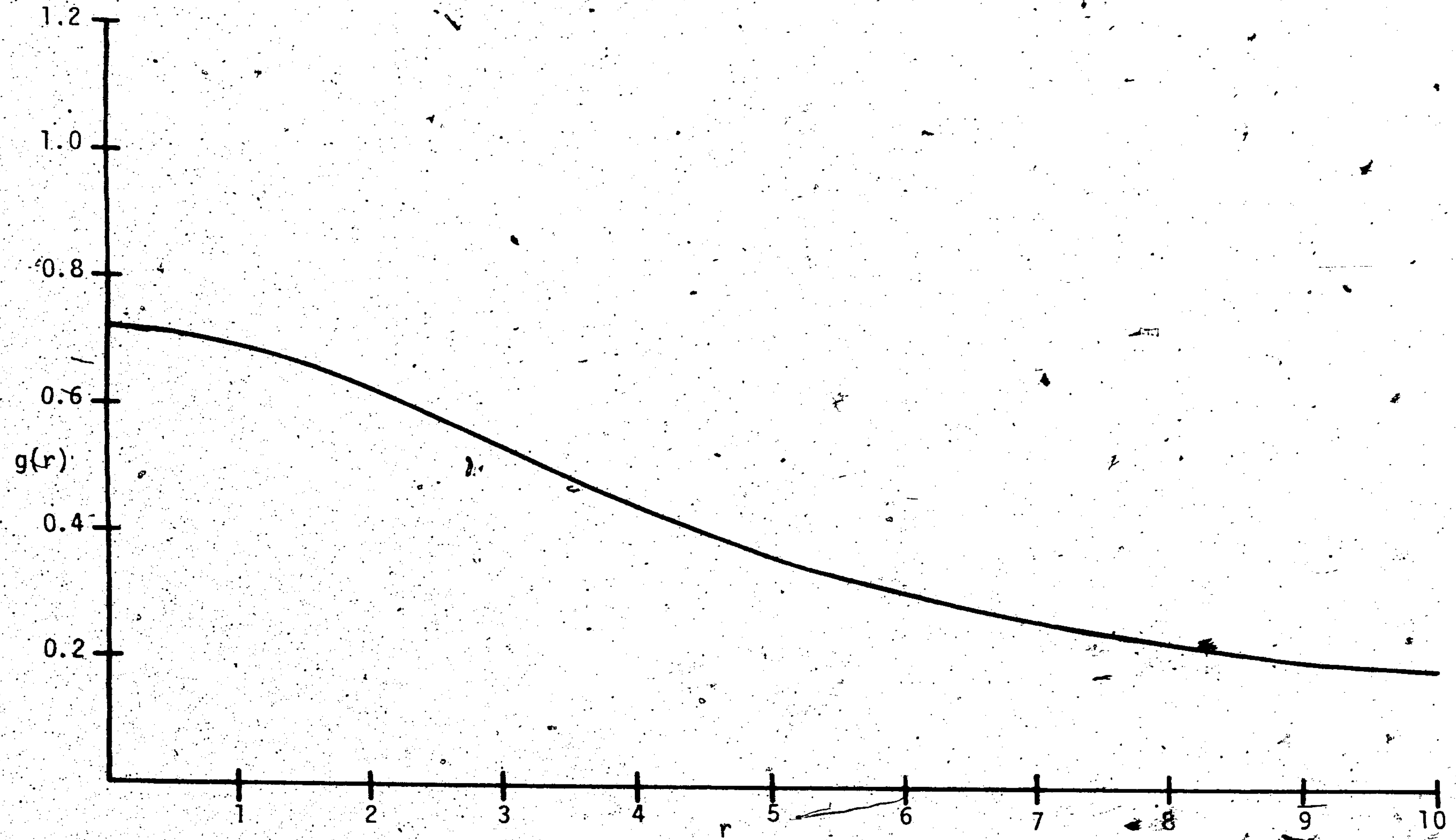


Figure 6.9

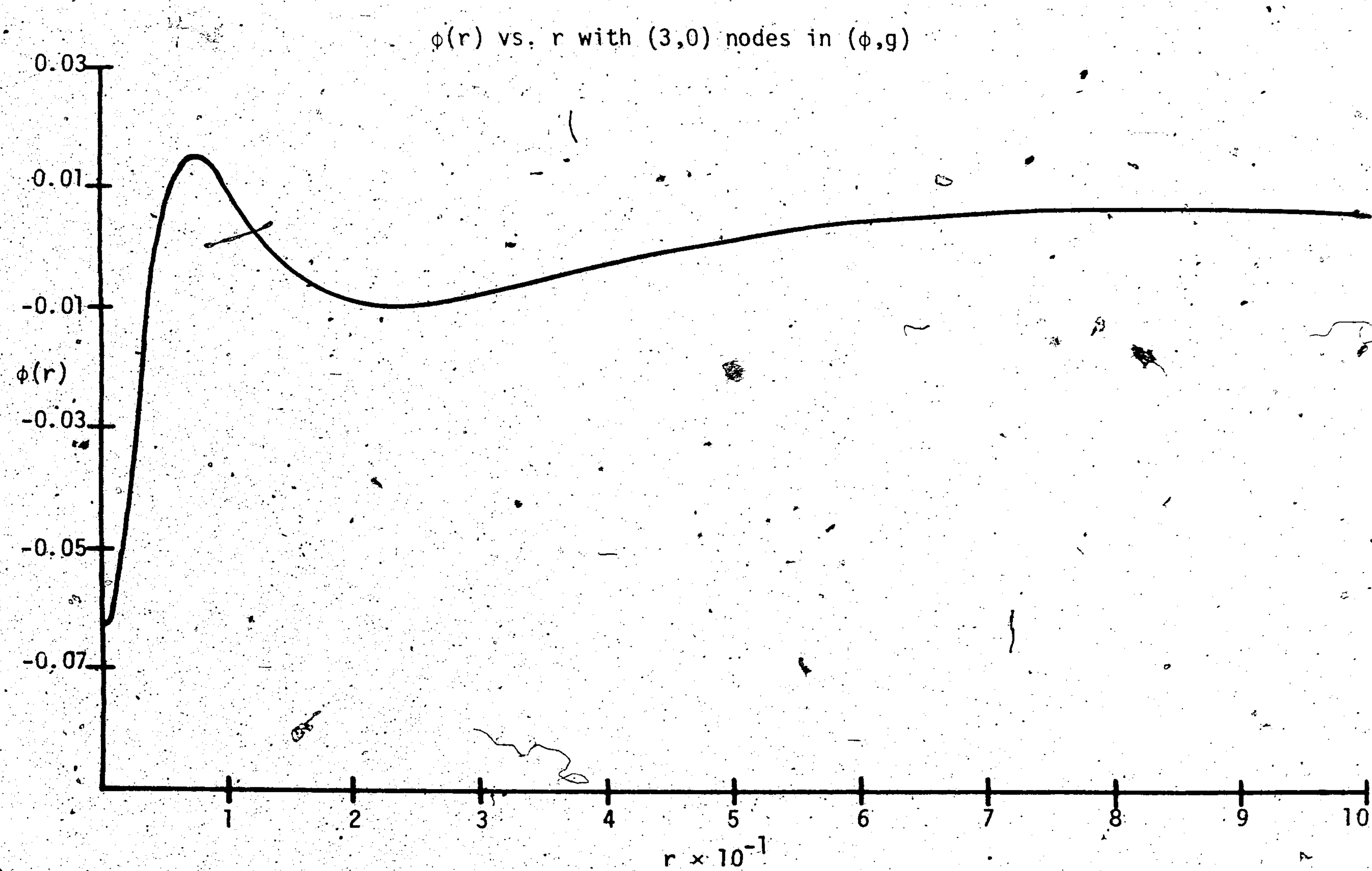

φ(r) vs. r with (3,0) nodes in (φ,g)
0.03
0.01
-0.01
φ(r)
-0.03
-0.05
-0.07
1
2
3
4
5
6
7
8
9
10
r × 10^-1

Figure 6.10

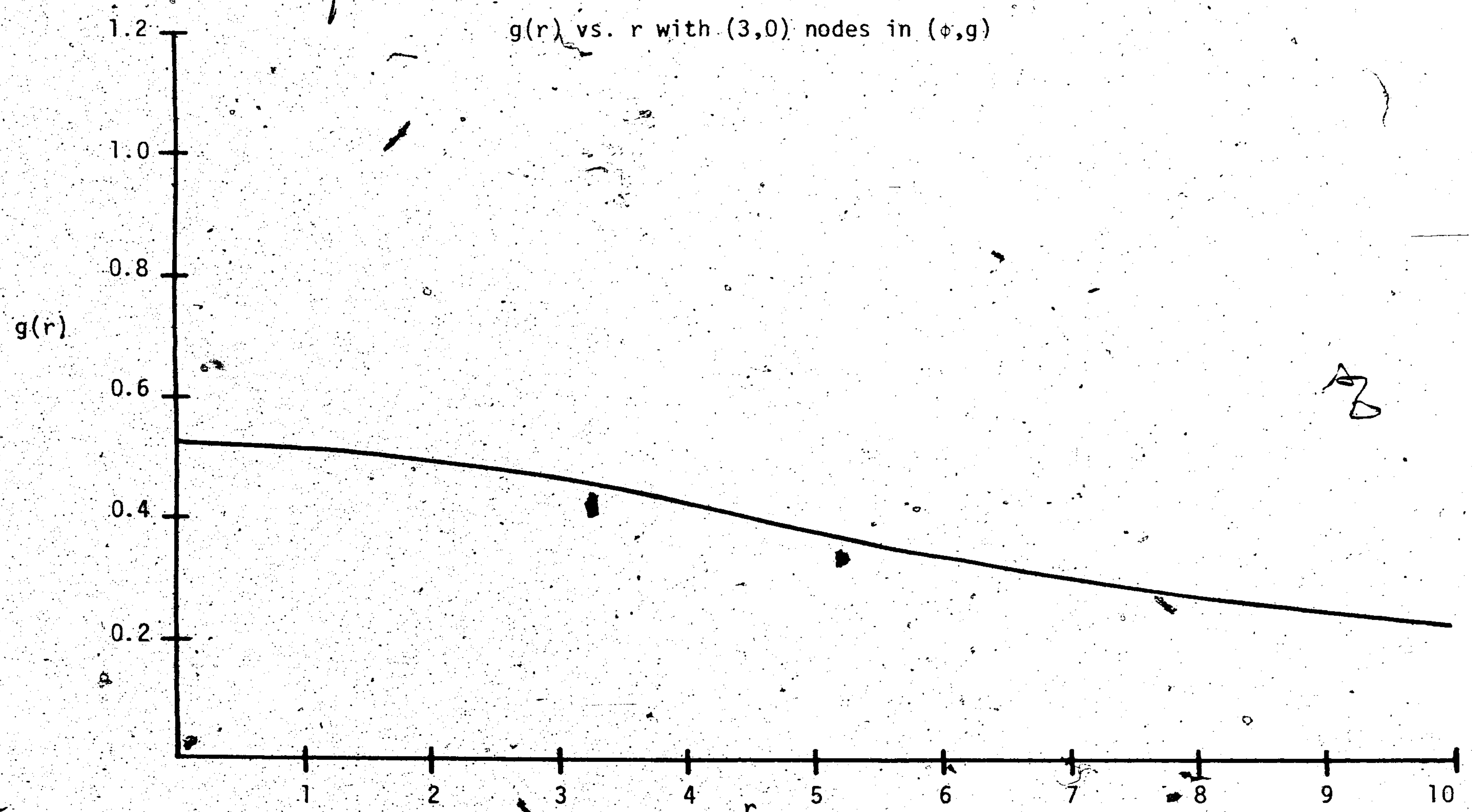


Figure 6.11

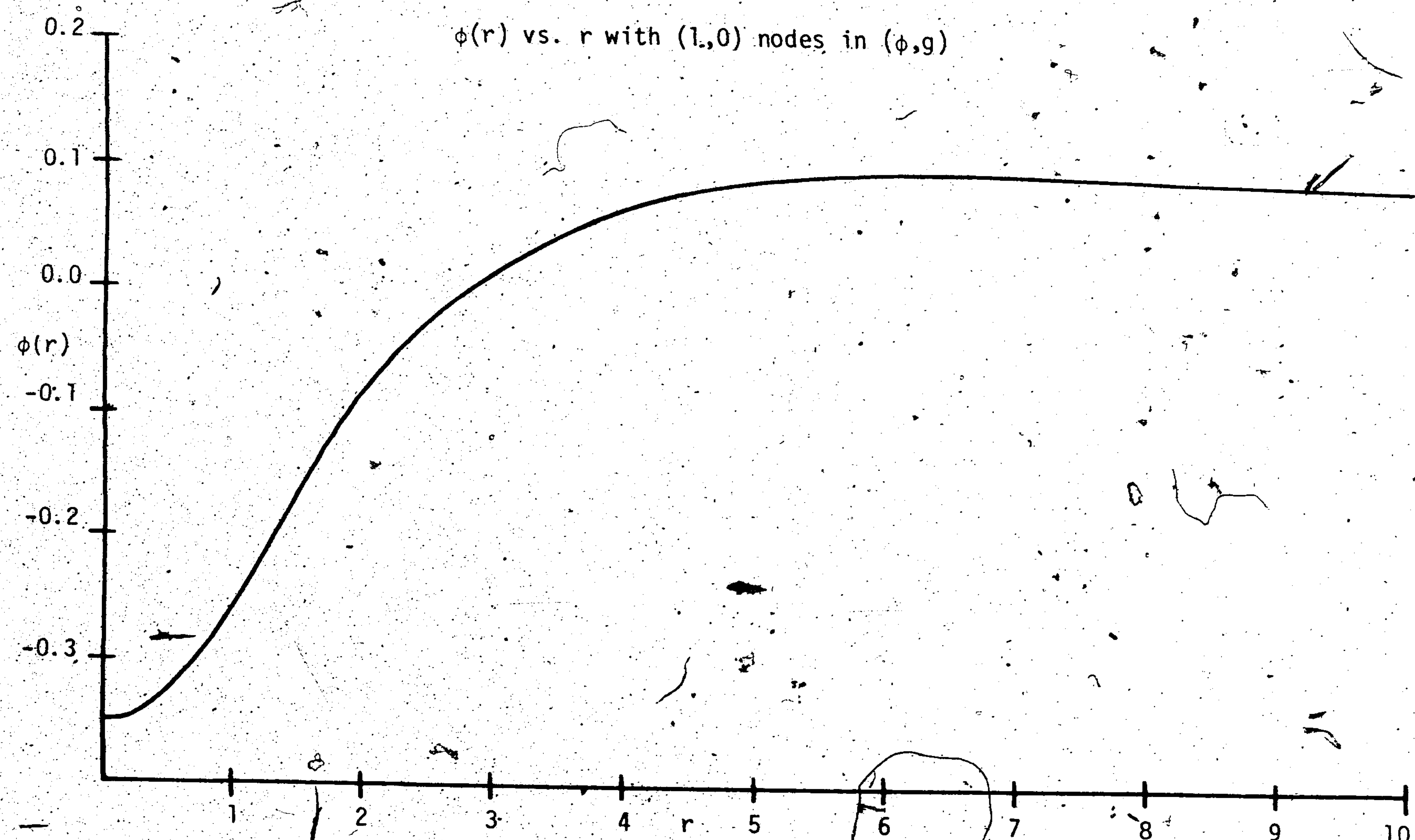


Figure 6.12

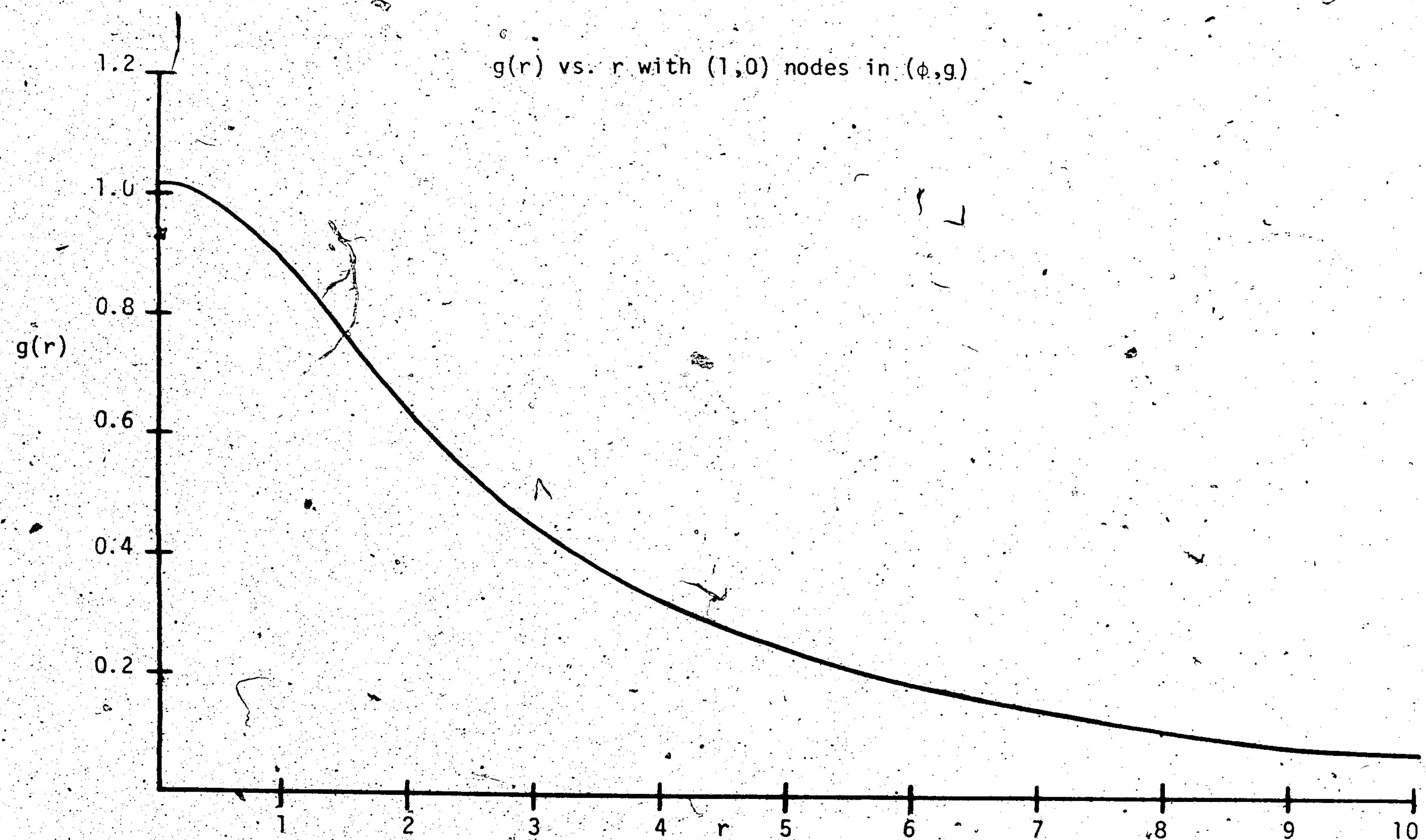
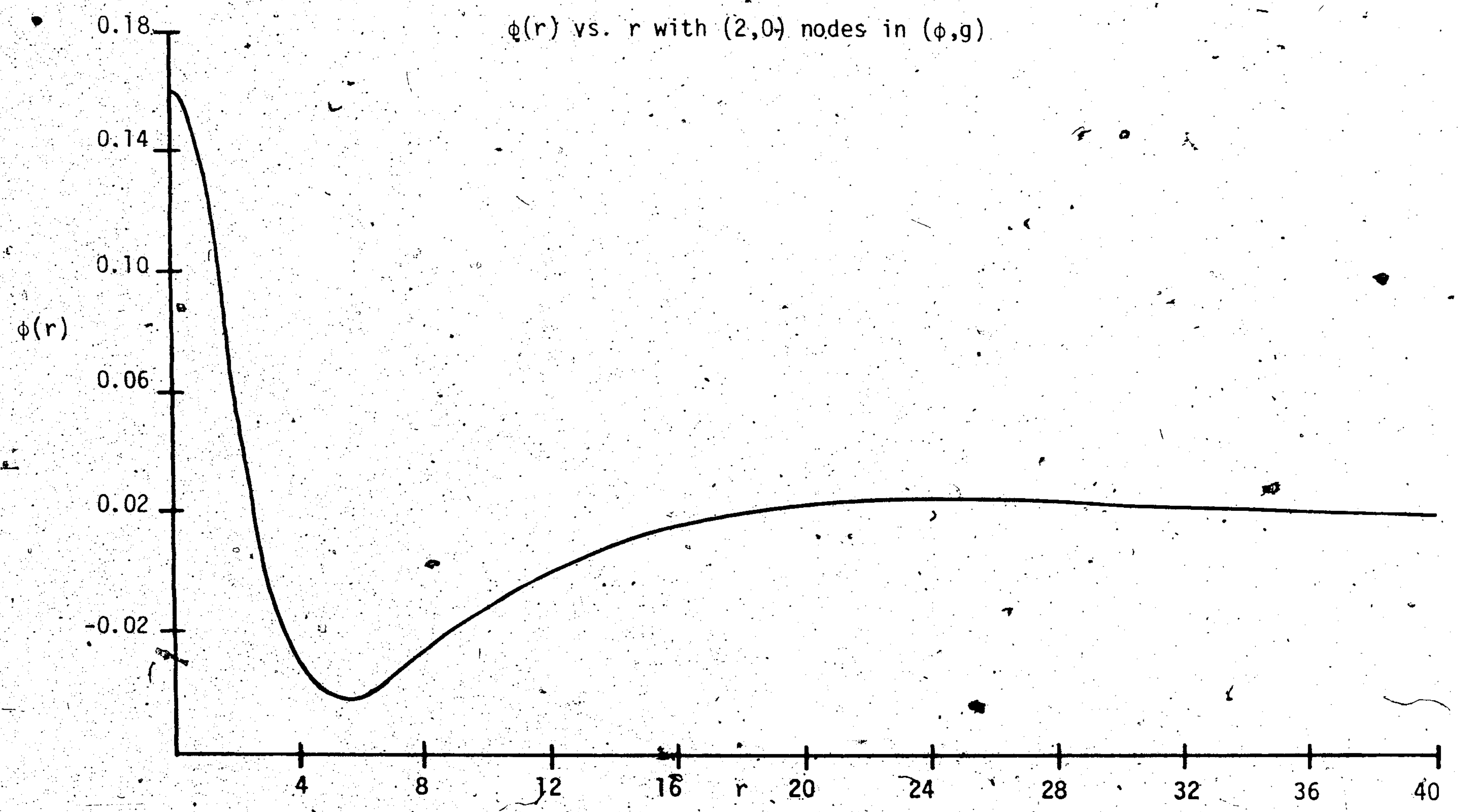


Figure 6.13



ϕ(r) vs. r with (2,0) nodes in (ϕ,g)
ϕ(r)
0.18
0.14
0.10
0.06
0.02
-0.02
4
8
12
16
r
20
24
28
32
36
40

Figure 6.14

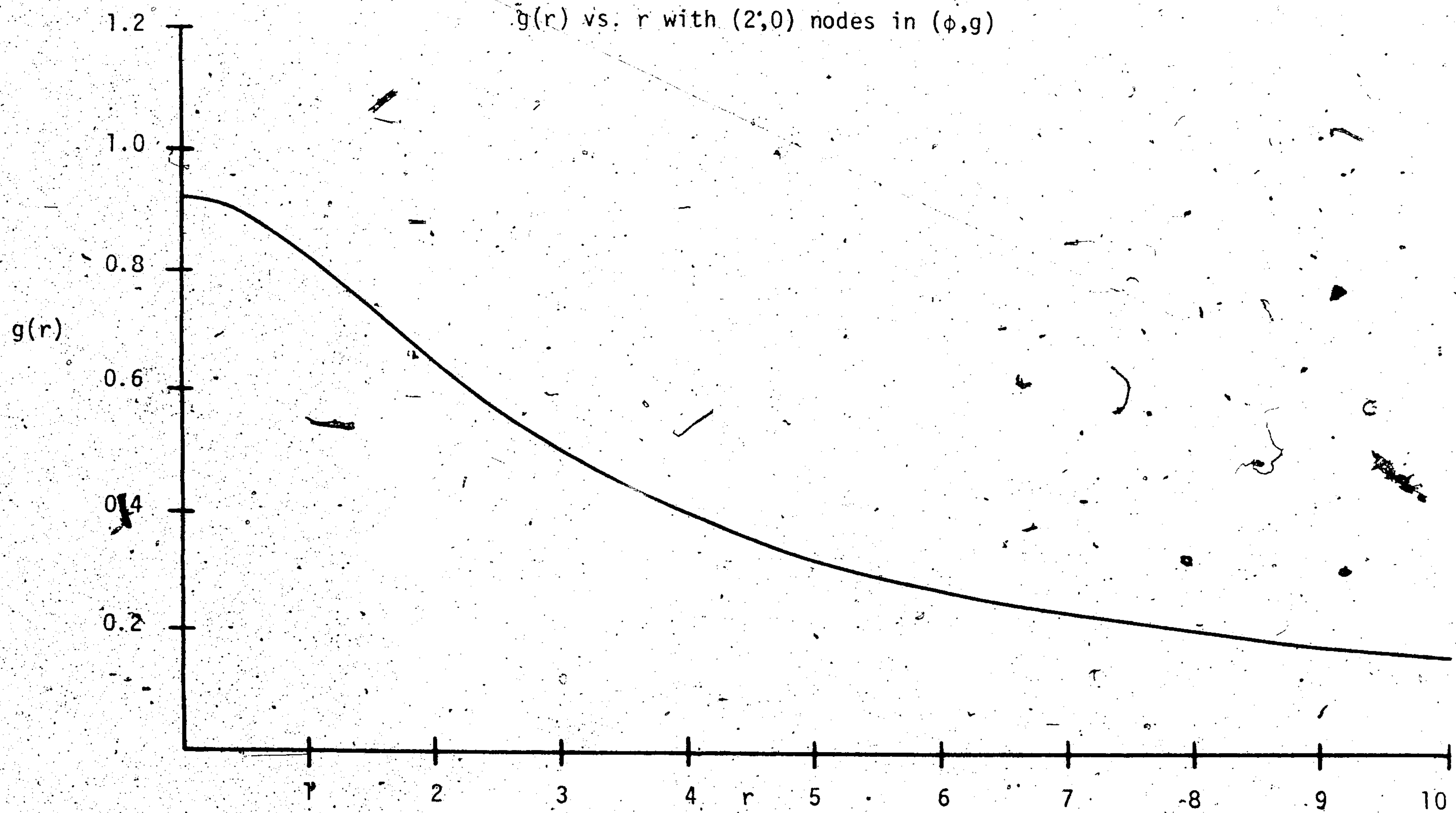


Figure 6.15

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APPENDIX 1: Derivation of the Generators for the S.C. Group

The calculation is similar to that presented by Rosen (1972) but is presented here for the sake of completeness.

Consider the infinitesimal transformations

$$x'^{\mu} = x^{\mu} + \delta x^{\mu}(x) \qquad [3.6]$$

where $\delta x^{\mu}(x)$ is an infinitesimal differentiable function of x^{α}. From [3.4] we can write

$$\partial_{\nu}\delta x_{\mu} + \partial_{\mu}\delta x_{\nu} = \delta\lambda\eta_{\mu\nu} \qquad [A1.1]$$

where

$$\delta\lambda = 1/2\ \partial_{\mu}\delta x^{\mu}$$

is obtained by contracting μ and λ. Equation [A1.1] is differentiated with respect to x^{λ} to give

$$\partial_{\lambda}\partial_{\nu}\delta x_{\mu} + \partial_{\lambda}\partial_{\mu}\delta x_{\nu} = \partial_{\lambda}(\delta\lambda)\eta_{\mu\nu} \quad , \qquad [A1.2]$$

and ν and λ are interchanged to obtain

$$\partial_{\nu}\partial_{\lambda}\delta x_{\mu} + \partial_{\nu}\partial_{\mu}\delta x_{\lambda} = \partial_{\nu}(\delta\lambda)\eta_{\mu\lambda} \quad . \qquad [A1.3]$$

Subtracting [A1.2] and [A1.3] yields

$$\partial_{\lambda}\partial_{\mu}\delta x_{\nu} - \partial_{\mu}\partial_{\nu}\delta x_{\lambda} = \partial_{\lambda}(\delta\lambda)\eta_{\mu\nu} - \partial_{\nu}(\delta\lambda)\eta_{\mu\lambda} \quad , \qquad [A1.4]$$

which is differentiated with respect to x^ρ to get

[A1.5] $\partial_\rho \partial_\mu (\partial_\lambda \delta x_\nu - \partial_\nu \delta x_\lambda) = \partial_\rho \partial_\lambda (\delta\lambda) \eta_{\mu\nu} - \partial_\rho \partial_\nu (\delta\lambda) \eta_{\mu\lambda}$.

Using the symmetry in μ and ρ gives the integrability conditions

[A1.6] $\partial_\rho \partial_\lambda (\delta\lambda) \eta_{\mu\nu} - \partial_\rho \partial_\nu (\delta\lambda) \eta_{\mu\lambda} - \partial_\mu \partial_\lambda (\delta\lambda) \eta_{\rho\nu} + \partial_\mu \partial_\nu (\delta\lambda) \eta_{\rho\lambda} = 0.$

Multiplying [A1.6] by $\eta^{\mu\alpha}$ and contracting α and ν gives

[A1.7] $$2\, \partial_\rho \partial_\lambda (\delta\lambda) + \partial^\alpha \partial_\alpha (\delta\lambda) \eta_\rho{}^\lambda = 0 ,$$

from which we also have

[A1.8] $$\partial^\alpha \partial_\alpha (\delta\lambda) = 0$$

so it follows that $\delta\lambda$ is a linear function of x^α:

[A1.9] $$\delta\lambda \equiv 4\delta\beta_\alpha x^\alpha + 2\delta\beta \ .$$

Inserting [A1.9] into [A1.4] gives

[A1.10] $$\partial_\lambda \partial_\mu \delta x_\nu - \partial_\mu \partial_\nu \delta x_\lambda = 4\delta\beta_\lambda \eta_{\mu\nu} - 4\delta\beta_\nu \eta_{\mu\lambda} \ ,$$

and integrating [A1.10] with respect to dx^μ yields

[A1.11] $$\partial_\lambda (\delta x_\nu) - \partial_\nu (\delta x_\lambda) = 4\delta\beta_\lambda x_\nu - 4\delta\beta_\nu x_\lambda + 2\delta\alpha_{\nu\lambda}$$

where $\delta\alpha_{\nu\lambda} = -\delta\alpha_{\lambda\nu}$ are six constants of integration.

From [A1.1] and [A1.9], we have

$$\partial_\nu \delta x_\lambda + \partial_\lambda \delta x_\nu = \eta_{\lambda\nu}(4\delta\beta_\alpha x^\alpha + 4\delta\beta) \qquad \text{[A1.12]}$$

which, when added to [A1.11], means

$$\partial_\lambda(\delta x_\nu) = 2[\eta_{\lambda\nu}\delta\beta_\alpha x^\alpha + \delta\beta_\lambda x_\nu - \delta\beta_\nu x_\lambda] + \delta\beta\eta_{\lambda\nu} + \delta\alpha_{\nu\lambda} . \qquad \text{[A1.13]}$$

Integrating [A1.13] with respect to x^λ gives

$$\begin{aligned} \delta x_\nu &= 2\int [\eta_{\lambda\nu}\delta\beta^\alpha x_\alpha + \delta\beta_\lambda x_\nu] dx^\lambda - \delta\beta_\nu x^\alpha x_\alpha \\ &\quad + \delta\beta x_\nu + \delta\alpha_{\nu\alpha} x^\alpha + \delta\alpha_\nu \\ &= 2\,\eta_{\nu\beta} x^\beta x_\alpha \delta\beta^\alpha - \delta\beta_\nu x^\alpha x_\alpha + \delta\beta x_\nu + \delta\alpha_{\nu\alpha} x^\alpha + \delta\alpha_\nu \\ &= -2\, x_\nu \delta c_\alpha x^\alpha + \delta c_\nu x_\alpha x^\alpha + \delta\beta x_\nu + \delta\alpha_{\nu\alpha} x^\alpha + \delta\alpha_\nu \end{aligned} \qquad \text{[A1.14]}$$

where $\delta\alpha_\nu$ are four more integration constants and we have set $\delta c_\nu \equiv -\delta\beta_\nu$ to bring [A1.14] into the usual notation. The parameters δa_ν and $\delta a_{\mu\nu}$ describe the translations and homogeneous Lorentz transformations of the Poincaré group; $\delta\beta$ corresponds to the dilations and δc_ν are the four special conformal parameters. It is now clear how to write the generators for the S.C. group.

APPENDIX 2: Calculation of the Charge Under a General S.C. Point Transformation.

The S.C. point transformations written in spherical coordinates are

[A2.1] $$r \sin\theta \cos\alpha = \frac{r' \sin\theta' \cos\alpha' - c_x(r'^2 - t'^2)}{\sigma'}$$

[A2.2] $$r \sin\theta \sin\alpha = \frac{r' \sin\theta' \sin\alpha' - c_y(r'^2 - t'^2)}{\sigma'}$$

[A2.3] $$r \cos\theta = \frac{r' \cos\theta' - c_z(r'^2 - t'^2)}{\sigma'}$$

[A2.4] $$t = \frac{t' + k(r'^2 - t'^2)}{\sigma'}$$

where $\sigma' = 1 - 2\, cr' \cos\gamma - 2kt' + (c^2-k^2)(r'^2-t'^2)$, $c^\mu = (\vec{c},-k)$, and γ is the angle between $\vec{c}$ and $\vec{r}'$. We denote the electric field obtained through the point transformation as E'. The charge Q is then

[A2.5] $$Q = \frac{1}{4\pi}\int d^3x'\nabla'\cdot\vec{E}' = \frac{1}{4\pi}\int_{S'}\vec{E}'\cdot\hat{n}dS' = \frac{1}{4\pi}\lim_{r'\to\infty}\int d\Omega' E_r' r'^2$$

since S is the surface over the sphere with normal $\hat{n}$ in the radial direction. Here, $\vec{r}'$ and t' denote the new space-time points obtained from $\vec{r}$,t in the same coordinate system. We place a charge at some non-zero point in space $\vec{r} = \vec{r}_0$ at some time t, perform the point transformations [A2.1] to [A2.4], and calculate Q from [A2.5].

From the transformed fields

[A2.6] $$A_{r'} = A_r \frac{\partial r}{\partial r'} + A_\theta \frac{\partial\theta}{\partial r'} + A_\alpha \frac{\partial\alpha}{\partial r'} - Q\frac{\partial t}{\partial r'}$$

[A2.7] $$\phi' = -A_r \frac{\partial r}{\partial t'} - A_\theta \frac{\partial \theta}{\partial t'} - A_\alpha \frac{\partial \alpha}{\partial t'} + \phi \frac{\partial t}{\partial t'}$$

we have

[A2.8] $$-E_r' = \frac{\partial A_r'}{\partial t'} + \frac{\partial \phi'}{\partial r'}$$

$$= -E_r\left(\frac{\partial r}{\partial r'}\frac{\partial t}{\partial t'} - \frac{\partial r}{\partial t'}\frac{\partial t}{\partial r'}\right) + \left(\frac{\partial A_\theta}{\partial t} + \frac{\partial \phi}{\partial \theta}\right)\left(\frac{\partial \theta}{\partial r'}\frac{\partial t}{\partial t'} - \frac{\partial \theta}{\partial t'}\frac{\partial t}{\partial r}\right)$$

$$+ \left(\frac{\partial A_\alpha}{\partial t} + \frac{\partial \phi}{\partial \alpha}\right)\left(\frac{\partial \alpha}{\partial r'}\frac{\partial t}{\partial t'} - \frac{\partial \alpha}{\partial t'}\frac{\partial t}{\partial t'}\right) + \left(\frac{\partial A_r}{\partial \alpha} - \frac{\partial A_\alpha}{\partial r}\right)\left(\frac{\partial r}{\partial r'}\frac{\partial \alpha}{\partial t'} - \frac{\partial r}{\partial t'}\frac{\partial \alpha}{\partial r'}\right)$$

$$+ \left(\frac{\partial A_\theta}{\partial \alpha} - \frac{\partial A_\alpha}{\partial \sigma}\right)\left(\frac{\partial \theta}{\partial r'}\frac{\partial \alpha}{\partial t'} - \frac{\partial \theta}{\partial t'}\frac{\partial \alpha}{\partial r'}\right) + \left(\frac{\partial A_r}{\partial \theta} - \frac{\partial A_\theta}{\partial r}\right)\left(\frac{\partial r}{\partial r'}\frac{\partial \theta}{\partial t'} - \frac{\partial r}{\partial t'}\frac{\partial \theta}{\partial r'}\right)$$

From equations [A2.1] to [A2.4], we see that $\vec{r}$ and t tend to some combination of $\vec{c}$,k as $r' \to \infty$ so that the asymptotic form of E_r' is determined by the derivatives of r,θ,α,t with respect to r' and t'.

We write

[A2.9] $$\sigma' = r'^2\left(a_2 + \frac{a_1}{r'} + \frac{\bar{a}_0}{r'^2}\right)$$

where

$$a_1 \equiv -2c \cos\gamma,\ a_2 \equiv c^2 - k^2,\ \bar{a}_0 \equiv 1 - 2kt' - t'^2 a_2 .$$

Since

[A2.10] $$r^2 = \frac{r'^2 - 2(r'^2 - t'^2)r'F + c^2(r'^2 - t'^2)^2}{\sigma'^2}$$

where

$$F \equiv c_z \cos\theta' + \sin\theta'(c_x \cos\alpha' + c_y \sin\alpha')$$

we can show that for large r',

[A2.11] $$r\frac{\partial r}{\partial r'} = \frac{1}{r'^2}\left[\frac{F}{a_2^2} - 2c^3\frac{\cos\gamma}{c_2^3}\right] + O(r'^{-3})$$

and

[A2.12] $$r\frac{\partial r}{\partial t'} = \frac{2c^2k}{r'^2 a_2^3} + O(r'^{-3})$$

[A2.13] $$\frac{\partial t}{\partial r'} = -\frac{2kc\cos\gamma}{a_2^2 r'^2} + O(r'^{-3})$$

[A2.14] $$\frac{\partial t}{\partial t'} = \frac{c^2 + k^2}{r'^2 a_2^2} + O(r'^{-3})$$

providing $a_2 \neq 0$. Because

[A2.15] $$\sec^2\theta \rightarrow \frac{c^2}{c_z^2} + O(r'^{-1})$$

with $c_z \neq 0$, it follows that

[A2.16] $$\frac{\partial\theta}{\partial t'} \rightarrow O(r'^{-3})$$

and

[A2.17] $$\frac{\partial\theta}{\partial r'} \rightarrow O(r'^{-2}) \text{ at least.}$$

If $c_z = 0$,

[A2.18] $$\frac{\partial\theta}{\partial t'} \to \pm \frac{2\cos\theta' t'}{r'^3\sqrt{c_x^2 + c_y^2}} + O(r'^{-4})$$

[A2.19] $$\frac{\partial\theta}{\partial r'} \to \pm \frac{\cos\theta'}{\sqrt{c_x^2 + c_y^2}} \frac{1}{r'^2} + O(r'^{-3})$$

If $c_x = c_y = c_z = 0$, then $\theta = \theta'$ and $\alpha = \alpha'$ so

$$\frac{\partial\theta}{\partial t'} = \frac{\partial\theta}{\partial r'} = \frac{\partial\alpha}{\partial t'} = \frac{\partial\alpha}{\partial r'} = 0 .$$

For the angle α, we have the expression

[A2.20] $$\tan\alpha = \frac{r'\sin\theta'\sin\alpha' - c_y(r'^2 - t'^2)}{r'\sin\theta'\cos\alpha' - c_x(r'^2 - t'^2)}$$

If $c_x \neq 0$,

[A2.21] $$\frac{\partial\alpha}{\partial t'} \to O(r'^{-3})$$

[A2.22] $$\frac{\partial\alpha}{\partial r'} \to O(r'^{-2}) .$$

If $c_x = 0$,

[A2.23] $$\frac{\partial\alpha}{\partial t'} \to \frac{2t'\sin\theta'\cos\alpha'}{r'^3 c_y} + O(r'^{-4})$$

[A2.24] $$\frac{\partial\alpha}{\partial r'} \to -\frac{\sin\theta'\cos\alpha'}{c_y\, r'^2}\left(1 + O(r'^{-1})\right) .$$

If $c_x = c_y = 0$, then $\alpha = \alpha'$.

From [A2.1] to [A2.4], we see that all functions of θ', α', γ are multiplied by r', so in our asymptotic expansions of the derivatives as given above, no functions of the angles can appear in the denominators. Furthermore, integration over the angles does not change the r' dependence in E_r'. We therefore see that

$$\lim_{r'\to\infty} r'^2 \int E_r'\, d\Omega' \to 0$$

if $a_2 \neq 0$. If $a_2 = 0$, then [A2.10] shows that $r \to \infty$ if $r' \to \infty$. We want r to be situated at a finite point in space after the charge is accelerated so we must have $c^2 \neq k^2$. We also require $\vec{r}_0 \neq 0$ because r' must vanish with $kt \neq 1$ and $\vec{c} = 0$ if r is zero. We wish to have the freedom to put $\vec{c} = 0$, so we ensure that r' will not vanish by demanding $r \neq 0$.

We have therefore shown that applying a general S.C. point transformation to a charge, situated at a point $\vec{r}_0 \neq \hat{0}$ in space at a time $t \neq \frac{1}{k}$, results in a configuration with zero total charge.

APPENDIX 3: Solution of the Homogeneous Equation [5.32].

If we put the power series expansions of η_0 about $q = 0, 1,$ and ∞ into [5.32], we see that these points are regular singularities. A solution of the form

[A3.1] $$F = \sum_{n=o}^{\infty} a_n(q-q_o)^{n+c_1}$$

with $a_o = 1$ therefore exists provided the roots c_1 and c_2 of the indicial equation (called the exponents at the singular point q_o)

[A3.2] $$c^2 + (P_o - 1)c + Q_o = 0$$

are such that the Re $c_1 \geq$ Re c_2 (Rainville 1964). The Laurent expansion of P and Q about q_o defines P_o and Q_o:

[A3.3] $$P = \sum_{n=o}^{\infty} P_n(q-q_o)^{n-1}$$

[A3.4] $$Q = \sum_{n=o}^{\infty} Q_n(q-q_o)^{n-2} .$$

At $q = 0$, $c_1 = 1$ and $c_2 = -3$; at $q = \pm 1$, $c_1 = \frac{1}{2}(1 + \sqrt{13})$ and $c_2 = \frac{1}{2}(1 - \sqrt{13})$.

A complete examination of [5.32] is feasible if we have a closed form for η_0, so we now employ [4.16]. Setting $x \equiv q^2$ and inserting [4.16] into [5.32] gives

[A3.5] $$F'' + \frac{3(x+3)(1-5x)F}{x(x-1)^2(1+3x)^2} + F' \frac{(5+3x)}{2x(1+3x)} = 0$$

where the prime denotes $\frac{d}{dx}$. This equation has regular singularities at $x = 0, 1, -\frac{1}{3}$, and ∞, and is therefore of the Fuchsian type. A well known theorm for Fuchsian equations with singular points at $x = x_1, x_2, x_3$, and ∞ allows [A3.5] to be written in the form

[A3.6]
$$F'' + \left[\frac{1-\alpha_{11}-\alpha_{21}}{x-x_1} + \frac{1-\alpha_{12}-\alpha_{22}}{x-x_2} + \frac{1-\alpha_{13}-\alpha_{23}}{x-x_3}\right]F'$$
$$+ \left[\frac{\alpha_{11}\alpha_{21}}{(x-x_1)^2} + \frac{\alpha_{12}\alpha_{22}}{(x-x_2)^2} + \frac{\alpha_{13}\alpha_{23}}{(x-x_3)^2} + \right.$$
$$\left. + \frac{\alpha_{1\infty}\alpha_2 - \alpha_{11}\alpha_{21} - \alpha_{12}\alpha_{22} - \alpha_{13}\alpha_{23}}{(x-x_1)(x-x_2)} + \frac{C_3(x_1-x_3(x_2-x_3)}{(x-x_1)(x-x_2)(x-x_3)}\right]F = 0$$

where α_{1k} and α_{2k} are the exponents at $x = x_k$; $\alpha_{1\infty}$ and $\alpha_{2\infty}$ are those at $x = \infty$; C_3 is a constant equal to -12 here, and is the coefficient of the $\frac{1}{x-x_3}$ factor when $Q(x)$ is broken into partial fractions. The exponents at the singular points of [A3.5] are the following:

[A3.7]

$x_1 = 0$	$x_2 = 1$	$x_3 = -\frac{1}{3}$	∞
0	$\frac{1}{2}(1+\sqrt{13})$	-1	0
$-\frac{3}{2}$	$\frac{1}{2}(1-\sqrt{13})$	4	$-\frac{1}{2}$

It is convenient to represent [A3.6] in terms of the Riemann P symbol[1]

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[1]This is not a unique representation if more than three regular singularities are present.

$$[A3.8] \qquad F = P\begin{Bmatrix} x_1 & x_2 & x_3 & \infty & \\ \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{1\infty} & x \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{2\infty} & \end{Bmatrix}$$

We can remove the poles of order two in [A3.6] by using transformations of the form

$$[A3.9] \qquad F = (x-x_2)^{\alpha_{12}} w.$$

Hence, the transformation

$$[A3.10] \qquad F = (x-x_2)^{\alpha_{12}}(x-x_3)^{\alpha_{13}} v$$

allows v to be determined from an equation of the symbolic form

$$[A3.11] \qquad v = P\begin{Bmatrix} x_1 & x_2 & x_3 & \infty \\ 0 & 0 & 0 & \alpha_{1\infty}+\alpha_{12}+\alpha_{13} \\ \alpha_{21} & \alpha_{22}-\alpha_{12} & \alpha_{23}-\alpha_{13} & \alpha_{2\infty}+\alpha_{12}+\alpha_{13} \end{Bmatrix},$$

which, from [A3.10], [A3.5], and [A3.6], represents the equation

$$[A3.12] \quad v'' + \left[\frac{5}{2x} + \frac{1+\sqrt{13}}{x-1} - \frac{4}{x+\frac{1}{3}}\right]v' + \left[\frac{\sqrt{13}-1}{12x(x-1)(x+\frac{1}{3})} + \frac{3(5-\sqrt{13})}{4(x-1)(x+\frac{1}{3})}\right]v = 0.$$

We can now identify [A3.12] with the Heun equation

$$[A3.13] \quad x(x-1)(x+\tfrac{1}{3})v'' + \left[-x\{\alpha+\beta-\delta+1-\frac{(\gamma+\delta)}{3}\} + \alpha\gamma + x^2(\alpha+\beta+1)\right]v'$$
$$+ \alpha\beta(x-k)\,v = 0$$

where the exponents at 0, 1, $-\frac{1}{3}$, and ∞ are, respectively, $(0,1-\gamma)$, $(0,1-\delta)$, $(0,1-\varepsilon)$, and (α,β); k is a parameter which is fixed for a given equation. The relation

$$\alpha + \beta + 1 = \gamma + \delta + \varepsilon$$

holds from Fuch's theorm. The Heun equation is a generalization of the hypergeometric equation; the former involves four regular singularities while the latter is characterized uniquely by three. A solution to [A3.13] is denoted by

[A3.14] $$v = v(-\frac{1}{3}, k; \alpha, \beta, \gamma, \delta : x) = \sum_{n=0}^{\infty} v_n x^n$$

and converges absolutely for $|x| < \frac{1}{3}$; v has the following form (Murphy 1960):

[A3.15] $$v(a,k;\alpha,\beta,\gamma,\delta:x) = 1 + \alpha\beta(\gamma_1 z + \gamma_2 z^2 + \ldots)$$

where $a = -\frac{1}{3}$, $z \equiv \frac{x}{a}$, and

[A3.16] $$\gamma_n = \delta_n(k)/[n!\gamma(\gamma+1)\ldots(\gamma+n)] ,$$

$$\delta_1(k) = k ,$$

$$\delta_2(k) = \alpha\beta k^2 + [1 + \alpha + \beta - \delta + a(\gamma+\delta)]k - \alpha\gamma ,$$

$$\delta_{n+1}(k) = [n\{\alpha+\beta-\delta+n+a(\gamma+\delta+n-1)\} + \alpha\beta k]\delta_n(k)$$

$$- (\alpha+n-1)(\beta+n-1)(\gamma+n-1)na\delta_{n-1}(k).$$

There has not been a great deal of work on the Heun equation, so it is of little value in obtaining closed form or numerical solutions to [A3.5].

APPENDIX 4: Asymptotic Static Interaction Between Charged Particles

We now demonstrate that two similar spherically symmetric particles which are far apart and instantaneously at rest interact according to the Coulomb force.

Suppose that at a certain moment charges q_1 and q_2 are located on the z axis with a large separation R. The force between the particles is

[A4.1]
$$F_3 = -\int_S T_{33}\, dS$$

where S is a two dimensional surface which encloses one of the particles, and has the geometry of a plane (normal to the z axis) located at a distance z from particle 1 (closed with an infinite hemisphere which gives no contribution to F_3). T_{33} is the zz component of the symmetrized stress tensor and is given by

[A4.2]
$$T_{33} = \frac{1}{4\pi}\left[-\frac{(\nabla\phi)^2}{2} + \left(\frac{\partial\phi}{\partial x}\right)^2 + O(r^{-5})\right]$$

since we are interested only in the leading contributions to T_{33}. Higher order corrections are difficult to calculate. We must specify the total field on S because both particles will contribute.

We take each particle to satisfy

[A4.3]
$$\nabla^2\phi \simeq = 0$$

and

[A4.4]
$$g \simeq 0 .$$

Hence,

[A4.5] $$\phi \simeq \frac{q_1}{r} + \frac{q_2}{|\vec{r}-\vec{R}|}$$

is the total potential, and T_{33} becomes

[A4.6] $$T_{33} = -\frac{1}{4\pi}\left[\frac{q_1^{\,2}}{2r^4} - \frac{z^2 q_1^{\,2}}{r^6} + \frac{q_2^{\,2}}{(R^2+r^2-2zR)^3}\left(-z^2-zR-\frac{R^2}{z}+\frac{r^2}{2}\right)\right.$$

$$\left. + \frac{q_1 q_2}{r(R^2+r^2-2zR)^{3/2}}\left(1+\frac{2z^2}{r^2}+\frac{zR}{r^2}\right)\right] + O(r^{-5}) \; .$$

With the help of cylindrical coordinates ρ, and θ we find that the $q_1^{\,2}$ terms integrate to zero, as do the $q_2^{\,2}$ terms. The problem then simplifies to integrations over the $q_1 q_2$ contributions which give

[A4.7] $$F_3 = -\int_S dS T_{33}$$

$$= \frac{q_1 q_2}{R^2} + O(R^{-3}) \; .$$

Here, q_1 and q_2 are equivalent to $\pm b_1$.

Charged particles therefore interact according to the Coulomb law plus some higher order corrections. The structure of G is unimportant until these corrections are calculated.