SIAM J. SCI. COMPUT.
Vol. 34, No. 1, pp. B65–B85




# COMPUTATION OF TAIL PROBABILITIES VIA EXTRAPOLATION METHODS AND CONNECTION WITH RATIONAL AND PADÉ APPROXIMANTS*

PHILIPPE GAUDREAU†, RICHARD M. SLEVINSKY†, AND HASSAN SAFOUHI†

**Abstract.** We use the recently developed algorithm for the $G_n^{(1)}$ transformation to approximate tail probabilities of the normal distribution, the gamma distribution, the student's $t$-distribution, the inverse Gaussian distribution, and Fisher's $F$ distribution. Using this algorithm, which can be computed recursively when using symbolic programming languages, we are able to compute these integrals to high predetermined accuracies. Previous to this contribution, the evaluation of these tail probabilities using the $G_n^{(1)}$ transformation required symbolic computation of large determinants. With the use of our algorithm, the $G_n^{(1)}$ transformation can be performed relatively easily to produce explicit approximations. After a brief theoretical study, a connection between the $G_n^{(1)}$ transformation and rational and Padé approximants is established.

**Key words.** extrapolation methods, $G$ transformation, Slevinsky–Safouhi formulae, tails of probability distributions, rational and Padé approximants

**AMS subject classifications.** 65B05, 65D30, 41A20

**DOI.** 10.1137/100803778

**1. Introduction.** In numerical analysis, in applied mathematics and in physics, infinite series and infinite range integrals represent solutions of many problems. In practice, these series and integrals often have a very poor convergence, presenting severe numerical and computational difficulties. As a result, convergence accelerators and nonlinear transformation methods for accelerating the convergence of infinite series and integrals have been studied for many years and have been applied in various situations. They are based on the idea of extrapolation. Via sequence transformations, slowly convergent and divergent sequences and series can be transformed into sequences and series with better numerical properties. Thus, they are useful for accelerating convergence. In the case of nonlinear transformations the improvement of convergence can be remarkable. This approach forms the basis of many new methods for solving various problems which were unsolvable otherwise and have many applications as well [1, 2, 3, 4, 5, 6].

Previous work [7, 8] has shown that the $G_n^{(m)}$ transformation [9] can be exceptionally accurate in the computation of highly oscillatory integrals. The $G$ transformation was introduced in [10] and is extended to $G_n$ in [11] and to $G_n^{(m)}$ in [9]. The positive integer $m$ denotes the order of the linear homogeneous differential equation satisfied by the integrand and the positive integer $n$ stands for the truncation order of the asymptotic expansion used in the transformation. The $G_n^{(m)}$ transformation produces approximations to infinite range integrals by expressing the integral tails in asymptotic expansions. One of the main challenges facing the $G_n^{(m)}$ transformation is the

*Submitted to the journal's Computational Methods in Science and Engineering section July 27, 2010; accepted for publication (in revised form) November 4, 2011; published electronically February 9, 2012.
http://www.siam.org/journals/sisc/34-1/80377.html
†Mathematical Section, Campus Saint-Jean, University of Alberta, 8406 – 91 St, Edmonton, AB, Canada T6C 4G9 (pjgaudre@alberta.ca, mrslevinsky@hotmail.com, hsafouhi@ualberta.ca). The third author's research was supported by the Natural Sciences and Engineering Research Council of Canada (NSERC).

lack of efficient algorithms for its implementation. Brute force methods rely on the solution of large systems of linear equations involving successive (higher) derivatives of the integrands and are therefore algorithmically undesirable. Recent progress [12] has seen the introduction of a highly efficient algorithm for the implementation of the $G_n^{(1)}$ transformation for integrals whose integrands satisfy first order linear homogeneous differential equations. Its implementation has been greatly aided by the Slevinsky–Safouhi formula 1 (SSF 1) for higher order derivatives [13], which is given explicitly in section 2.2. The algorithm can also be computed recursively when using symbolic programming languages such as Maple or Mathematica.

The evaluation of tail integrals of probability distributions is a problem that arises in several fields, such as statistics, chemistry, and physics. For example, in certain types of clustering and reliability problems, it is necessary to compute extreme tail probabilities to a high accuracy [14]. Standard quadrature rules fail to provide sufficiently accurate computation of tail probabilities, leading to the need for approximation functions that yield adequate accuracy for probabilities in the range of interest. Unfortunately, as explained by Gray and Wang [15], there are few general methodologies for producing such functions.

Prior to this work, such integrals were hard to calculate with the $G_n^{(1)}$ transformation due to the increasingly cumbersome calculations of higher dimensional determinants for large $n$. McWilliams, Balusek, and Gray [16] were able to determine approximations to these tail probabilities for orders of $n = 10$ using the computer algebra program Mathematica. One deficiency of this method was the lack of control of the degree of accuracy because their algorithm cannot be computed recursively. In the present contribution, we use the algorithm introduced in [12] to compute the tail probabilities of five probability distributions: the normal distribution, the gamma distribution, the student's $t$-distribution, the inverse Gaussian distribution, and Fisher's $F$ distribution. The algorithm requires computation of derivatives of the form $(x^2 \frac{\mathrm{d}}{\mathrm{d}x})^n (x^{-\nu} f(x))$, where $\nu$ is some complex number and where $f(x)$ is either the probability density function (PDF) or its multiplicative inverse. To proceed efficiently, we use the SSF 1.

Tail probabilities of the aforementioned five probability distributions are computed and the numerical tables illustrate the high efficiency of the algorithm, which does not resort to any classical numerical integration scheme, such as a quadrature routine. In order to control the degree of accuracy, we develop a test which stops the computation once the relative error reaches a predetermined accuracy. In general, this test works very well. In certain cases, however, when the behavior of the sequence of approximations is erratic, or after a certain order of the transformation, the relative error grows. In such situations, we implement the same stopping criteria developed in [17] for another extrapolation method. This test stops the calculations and returns the most accurate approximation for the integral tails.

The numerical tables we produce replicate the values treated in [16] with an accuracy reaching as high as 15 correct digits in double precision arithmetic. In addition, some tables show computations resulting from different values of the parameters in question.

In section 5, we identify the similarities between the forms of the approximations $G_n^{(1)}$ to each of the tail probabilities, and we study the $G_n^{(1)}$ transformation on a theoretical level. We establish an analytical form for the approximations $G_n^{(1)}$ and demonstrate their connection with rational and Padé approximants [18].

## 2. Definitions and basic properties.

**2.1. Probability distributions.** In this section, we define the normal distribution, the gamma distribution, the student's $t$-distribution, the inverse Gaussian distribution, and Fisher's $F$ distribution. For more details on these distributions and their properties, see [19, 20].

The normal distribution (Gaussian distribution) has the PDF given by

$$f_N(x) = \frac{1}{\sqrt{2\,\pi\,\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\,\sigma^2}\right) \qquad \text{for} \qquad -\infty < x < +\infty, \tag{2.1}$$

where $\mu$ denotes the mean of the distribution and $\sigma^2$ represents the variance.

By making the change of variable $z = \frac{x-\mu}{\sigma}$ the normal distribution reduces to the standard normal distribution where $\mu = 0$ and $\sigma^2 = 1$. In this case, the PDF is given by

$$g_N(z) = \frac{1}{\sqrt{2\,\pi}} \exp\left(-\frac{z^2}{2}\right) \qquad \text{for} \qquad -\infty < z < +\infty. \tag{2.2}$$

The gamma distribution has the PDF

$$f_g(x) = \frac{x^{a-1}\, e^{-\frac{x}{b}}}{\Gamma(a)\, b^a} \qquad \text{for} \qquad 0 < x < +\infty, \tag{2.3}$$

for which $a > 0$ and $b > 0$ are two parameters and $\Gamma$ refers to the gamma function. The parameter $a$ is responsible for the shape of the distribution, whereas the parameter $b$ affects the scale. The mean of the gamma distribution is $\mu = a\,b$ and the variance is $\sigma^2 = a\,b^2$. The gamma distribution is transformed to the exponential distribution by setting $a = 1$. It is also related to the $\chi^2$ distribution by setting $a = v/2$ and $b = 2$, where $v$ is the number of degrees of freedom. Indeed, the gamma distribution also forms the heart of the noncentral gamma distribution studied in [21].

The student's $t$-distribution has the PDF

$$f_t(x) = \frac{\Gamma(\frac{v+1}{2})}{\sqrt{v\,\pi}\,\Gamma\left(\frac{v}{2}\right)} \left(1 + \frac{x^2}{v}\right)^{-(\frac{v+1}{2})} \qquad \text{for} \qquad -\infty < x < +\infty, \tag{2.4}$$

where the parameter $v > 0$ stands for the number of degrees of freedom. For a sample size of $n$ independent variables, the number of degrees of freedom is defined to be $v = n - 1$. The mean of the student's $t$-distribution is 0 and the variance is $\sigma^2 = \frac{v}{v-2}$ when $v > 2$, $\sigma^2 = \infty$ when $1 < v \leq 2$, and undefined otherwise. As $v$ tends to infinity, the student's $t$-distribution converges toward the standard normal distribution.

The inverse Gaussian distribution has the PDF

$$f_i(x) = \left(\frac{\lambda}{2\,\pi\,x^3}\right)^{\frac{1}{2}} \exp\left(-\frac{\lambda\,(x-\mu)^2}{2\,\mu^2\,x}\right) \qquad \text{for} \qquad 0 < x < +\infty, \tag{2.5}$$

where $\mu > 0$ is the mean and $\lambda > 0$ is the shape parameter. The variance is $\sigma^2 = \frac{\mu^3}{\lambda}$.

Fisher's $F$ distribution has the PDF

$$f_F(x) = \frac{\Gamma\left(\frac{a+b}{2}\right)}{\Gamma\left(\frac{a}{2}\right)\,\Gamma\left(\frac{b}{2}\right)} \left(\frac{a}{b}\right)^{a/2} \frac{x^{\frac{a-2}{2}}}{\left(1 + \left(\frac{a}{b}\right) x\right)^{\frac{a+b}{2}}} \qquad \text{for} \qquad 0 < x < +\infty, \tag{2.6}$$

which is characterized by two positive integral parameters $a$ and $b$. The mean of Fisher's $F$ distribution is $\mu = \frac{b}{b-2}$ for $b > 2$ and the variance is $\sigma^2 = \frac{2\,b^2\,(a+b-2)}{a\,(b-2)^2\,(b-4)}$ for $b > 4$.

**2.2. The SSF.** In [13], new formulae for higher order derivatives are shown to be widely applicable in numerical and computational settings. The recursive algorithm we detail in subsection 2.3 and implement in section 3 requires the computation of high order derivatives of the integrand or its multiplicative inverse. In order to facilitate the developments, we present the following theorem.

THEOREM 2.1 (see [13]). *Let $G(x)$ be a function* k*th differentiable with the term* $(\frac{\mathrm{d}}{x^m \mathrm{d}x})^k (x^{-n}\, G(x))$ *well-defined. The SSF* 1 *is given by*

$$\left(\frac{\mathrm{d}}{x^{\mu}\mathrm{d}x}\right)^k (x^{-\nu}\, G(x)) = \sum_{i=0}^{k} A_k^i x^{n-\nu+i(m+1)-k(\mu+1)} \left(\frac{\mathrm{d}}{x^m \mathrm{d}x}\right)^i (x^{-n}\, G(x)) \tag{2.7}$$

*for any arbitrary $\mu$, $\nu$, $m$, $n \in \mathbb{C}$ and where the coefficients are given recursively by*

$$A_k^i = \begin{cases} 1 & \text{for } i = k, \\ (n-\nu-(k-1)(\mu+1))A_{k-1}^0 & \text{for } i = 0,\ k > 0, \\ (n-\nu+i(m+1)-(k-1)(\mu+1))A_{k-1}^i + A_{k-1}^{i-1} & \text{for } 0 < i < k. \end{cases} \tag{2.8}$$

In essence, while $\mu$, $\nu$, $m$, $n \in \mathbb{C}$, these constants are otherwise arbitrary and are typically chosen by the user to make derivations as easy as possible. As will be shown later in this manuscript, $\mu = -2$ in all cases. Applying the operator $(x^2 \frac{\mathrm{d}}{\mathrm{d}x})^k$ to the function $G(x) = e^{x^2/2}$ would certainly be cumbersome for large integer $k$. However, with the SSF 1, this problem is trivially solved as

$$\left(x^2 \frac{\mathrm{d}}{\mathrm{d}x}\right)^k e^{x^2/2} = \sum_{i=0}^{k} A_k^i\, x^{2i+k} \left(\frac{\mathrm{d}}{x\,\mathrm{d}x}\right)^i e^{x^2/2} = x^k e^{x^2/2} \sum_{i=0}^{k} A_k^i\, x^{2i}, \tag{2.9}$$

where the coefficients $A_k^i$ are those above with $(\mu, \nu, m, n) = (-2, 0, 1, 0)$.

In the SSF 1, one requires that the term $(\frac{\mathrm{d}}{x^m \mathrm{d}x})^k (x^{-n}\, G(x))$ be well-defined. To be used most efficiently, $m$ and $n$ should be chosen such that $(\frac{\mathrm{d}}{x^m \mathrm{d}x})^k (x^{-n}\, G(x))$ is easy to compute, even though the formula holds regardless.

The SSF 2 is developed as the specific case of above where $(\mu, \nu, m, n) = (0, 0, 1, 0)$ and for all intents and purposes is not relevant to the balance of this work. Note that the coefficients $A_k^i$ for both SSF 1 and SSF 2 also have explicit analytical expressions available in [13].

**2.3. The $G_n^{(1)}$ transformation.** For more details on the $G_n^{(m)}$ transformation, see [9, 5].

Let $f(x)$ be a function that satisfies a first order differential equation of the form

$$f(x) = p_1(x)\, f'(x), \tag{2.10}$$

where the coefficient $p_1(x)$ has an asymptotic expansion as $x \to +\infty$ of the form

$$p_1(x) \sim x^{\gamma} \sum_{i=0}^{\infty} \frac{\alpha_i}{x^i} \qquad \text{for some} \quad \gamma \in \mathbb{C}. \tag{2.11}$$

The approximation $G_n^{(1)}$ to $\int_0^\infty f(t)\,\mathrm{d}t$ is given as the solution of the system of $n+1$ linear equations [9],

$$\frac{\mathrm{d}^l}{\mathrm{d}x^l}\left(G_n^{(1)} - F(x) - x^{\xi_0} f(x) \sum_{i=0}^{n-1} \frac{\bar{\beta}_{0,i}}{x^i}\right) = 0 \quad \text{with} \quad l = 0, 1, \ldots, n, \tag{2.12}$$

where it is assumed that $\frac{\mathrm{d}^l}{\mathrm{d}x^l} G_n^{(1)} \equiv 0 \;\forall\, l > 0$. $F(x) = \int_0^x f(t)\,\mathrm{d}t$ and $\xi_0 = \min(s_0, 1)$, where $s_0$ is the largest of the integers $s$ such that $\lim_{x\to\infty} x^s f(x) = 0$ holds. Also, $G_n^{(1)}$ and the $\bar{\beta}_{0,i}$ with $0 \le i \le n-1$ are the respective set of $n+1$ unknowns.

Instead of solving the system of linear equations each time for each order $n$, however, it would be desirable to evaluate each approximation $G_n^{(1)}$ in a recursive manner. Here, we use the fact that $(x^2 \frac{\mathrm{d}}{\mathrm{d}x})^n$ annihilates polynomials of degree $n-1$ in inverse powers $\frac{1}{x}$.

By considering (2.12) for $l = 0$, by isolating for the summation, and by applying the $(x^2 \frac{\mathrm{d}}{\mathrm{d}x})$ annihilation operator $n$ times to the equation, we obtain [12]

$$G_n^{(1)} = \frac{\left(x^2 \dfrac{\mathrm{d}}{\mathrm{d}x}\right)^n \left(\dfrac{F(x)}{x^{\xi_0} f(x)}\right)}{\left(x^2 \dfrac{\mathrm{d}}{\mathrm{d}x}\right)^n \left(\dfrac{1}{x^{\xi_0} f(x)}\right)} = \frac{\mathcal{N}_n(x)}{\mathcal{D}_n(x)}, \tag{2.13}$$

which leads to the algorithm for computing the $G_n^{(1)}$ approximation [12].

For $n = 1, 2, \ldots,$ compute $\mathcal{N}_n(x)$ and $\mathcal{D}_n(x)$ recursively from

$$\mathcal{N}_n(x) = \left(x^2 \frac{\mathrm{d}}{\mathrm{d}x}\right) \mathcal{N}_{n-1}(x) \qquad \text{and} \qquad \mathcal{D}_n(x) = \left(x^2 \frac{\mathrm{d}}{\mathrm{d}x}\right) \mathcal{D}_{n-1}(x) \tag{2.14}$$

with

$$\mathcal{N}_0(x) = \frac{F(x)}{x^{\xi_0} f(x)} \qquad \text{and} \qquad \mathcal{D}_0(x) = \frac{1}{x^{\xi_0} f(x)}. \tag{2.15}$$

Since we are investigating integral tails $\int_x^\infty f(t)\,\mathrm{d}t$ rather than complete semi-infinite integrals, the remaining integrals $\int_0^x f(t)\,\mathrm{d}t$ appear on both sides of the above equation and we can then extract the approximation $\tilde{G}_n^{(1)}(x)$ to integral tails as follows:

$$\begin{aligned} \tilde{G}_n^{(1)}(x) &= G_n^{(1)}(x) - F(x) \\ &= \frac{\mathcal{N}_n(x) - F(x)\,\mathcal{D}_n(x)}{\mathcal{D}_n(x)} \\ &= \frac{\sum_{r=1}^n \binom{n}{r} \mathcal{D}_{n-r}(x) \left(x^2 \frac{\mathrm{d}}{\mathrm{d}x}\right)^{r-1} \left(x^2 f(x)\right)}{\mathcal{D}_n(x)} \\ &= \frac{\tilde{\mathcal{N}}_n(x)}{\mathcal{D}_n(x)}. \end{aligned} \tag{2.16}$$

The construction principle used above to derive the algorithmic form of the $G_n^{(1)}$ transformation is the same as that used in deriving algorithmic forms of several other Levin-type transformations. This construction principle occurs more widely in the discrete analogue, as discussed in [4, 22, 23]. In the discrete case, it is assumed that the elements of a model sequence $\{s_n\}_{n=0}^{\infty}$ with (generalized) limit $s$ can be expressed as follows:

$$s_n = s + \omega_n z_n, \qquad n \in \mathbb{N}_0. \tag{2.17}$$

Here, $\omega_n$ is a remainder estimate which is usually assumed to be known, and the correction terms $z_n$ should be chosen in such a way that the products $\omega_n z_n$ provide sufficiently accurate and rapidly convergent approximations to actual remainders. The model sequence (2.17) has one indisputable advantage: there exists a systematic approach for the construction of a sequence transformation which is exact for the model sequence (2.17). Let us assume that a linear operator $\hat{T}$ can be found which annihilates for all $n \in \mathbb{N}_0$ the correction term $z_n$ according to $\hat{T}(z_n) = 0$. Then, a sequence transformation, which is exact for the model sequence (2.17), can be obtained by applying $\hat{T}$ to $[s_n - s]/\omega_n = z_n$. Since $\hat{T}$ annihilates $z_n$ and is by assumption linear, the following sequence transformation $\mathcal{T}$ is exact for the model sequence (2.17):

$$\mathcal{T}(s_n, \omega_n) = \frac{\hat{T}(s_n/\omega_n)}{\hat{T}(1/\omega_n)} = s. \tag{2.18}$$

The above equation was introduced in [4] and applied for a rederivation of Levin's transformation [24] and for the derivation of related sequence transformations [4].

It is not very difficult to see the connection between the general transformation $\mathcal{T}$ in the discrete case and the approach used for the $G_n^{(1)}$ transformation in the continuous case. In (2.13), the linear operator $\hat{T}$ which annihilates an asymptotic expansion in inverse powers of $x$ of order $n-1$ is the operator $(x^2 \frac{\mathrm{d}}{\mathrm{d}x})^n$, while the partial sum $s_n$ is the integral $F(x)$ and the remainder estimate $\omega_n$ is the function $x^{\xi_0} f(x)$. In [23], Drummond's (discrete) process is extended to the continuous case. Indeed, it is similar to the $G_n^{(1)}$ transformation. Instead of an asymptotic expansion in inverse powers of $x$, however, the extended Drummond's process annihilates a polynomial in $x$. The differential operator $\frac{\mathrm{d}}{\mathrm{d}x}$ achieves this annihilation. So for the function $F(x)$ whose asymptotic expansion is given as

$$F(x) \sim F + \omega(x) P_n(x) \quad \text{as} \quad x \to 0, \tag{2.19}$$

where $P_n(x) \in \mathbb{R}[x]$ (i.e., $P_n(x)$ is in the ring of polynomials with coefficients in $\mathbb{R}$) and $\deg(P_n(x)) = n$, the extended Drummond's process is given by

$$D_n(x) = \frac{\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \left(\frac{F(x)}{\omega(x)}\right)}{\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \left(\frac{1}{\omega(x)}\right)}. \tag{2.20}$$

Evidently, there is a similarity in the developments.

## 3. Computing the probability distributions.

The objective of this manuscript is to compute integral tails $\int_x^\infty f(t)\,\mathrm{d}t$, where $f(t)$ is a PDF, using the algorithm presented above.

### 3.1. The normal distribution.

For simplicity, we first apply $G_n^{(1)}$ to the standard distribution. We derive the analytic expression of $\tilde{G}_n^{(1)}$ for the integral tail of the standard distribution, and then we make the change of variable $z = \frac{x-\mu}{\sigma}$ to obtain the analytic expression of $\tilde{G}_n^{(1)}$ for the integral tail of the normal distribution.

It is easy to show that the standard normal distribution PDF given by (2.2) satisfies a first order differential equation given by

$$g_N(z) = p_1(z)\, g'_N(z), \tag{3.1}$$

where the coefficient $p_1(z)$ is given by

$$p_1(z) \;=\; -\frac{1}{z} \;=\; -z^{-1} \quad \Rightarrow \quad \xi_0 = -1 \quad \text{(see (2.12) for the definition of } \xi_0\text{)}\,. \tag{3.2}$$

All the conditions required to apply the $G_n^{(1)}$ transformation to the standard distribution are satisfied. By using SSF 1 with $(\mu, \nu, m, n) = (-2, -1, 1, 0)$ we obtain

$$\begin{aligned} \mathcal{D}_n(z) &= \left(z^2 \frac{\mathrm{d}}{\mathrm{d}z}\right)^n z\sqrt{2\pi}e^{z^2/2} \\ &= \frac{z^{1+n}}{g_N(z)} \sum_{i=0}^{n} A_n^i\, z^{2i}, \end{aligned} \tag{3.3}$$

where the coefficients $A_k^i$ are calculated using the recurrence relations in (2.8) with $(\mu, \nu, m, n) = (-2, -1, 1, 0)$.

Since the elements $\mathcal{D}_n(z)$ are now available, to evaluate the numerator $\tilde{\mathcal{N}}_n(z)$, we only need to evaluate the following:

$$\left(z^2 \frac{\mathrm{d}}{\mathrm{d}z}\right)^{r-1} \left(z^2\, \frac{1}{\sqrt{2\,\pi}}\, e^{-\frac{z^2}{2}}\right). \tag{3.4}$$

We use the SSF 1 with $(\mu, \nu, m, n) = (-2, -2, 1, 0)$ to obtain

$$\left(z^2 \frac{\mathrm{d}}{\mathrm{d}z}\right)^{r-1} \left(z^2\, \frac{1}{\sqrt{2\,\pi}}\, e^{-\frac{z^2}{2}}\right) \;=\; z^{1+r}\, g_N(z) \sum_{i=0}^{r-1} B_{r-1}^i\, (-1)^i\, z^{2i}, \tag{3.5}$$

where the coefficients $B_k^i$ are calculated using the recurrence relations in (2.8) with $(\mu, \nu, m, n) = (-2, -2, 1, 0)$.

Inserting these results into (2.16), we obtain

$$\tilde{G}_n^{(1)}(z) \;=\; z\, g_N(z) \left[ \frac{\displaystyle\sum_{r=1}^{n} \binom{n}{r} \sum_{i=0}^{n-r} A_{n-r}^i\, z^{2i} \sum_{j=0}^{r-1} B_{r-1}^j\, (-1)^j\, z^{2j}}{\displaystyle\sum_{k=0}^{n} A_n^k\, z^{2k}} \right]. \tag{3.6}$$

Upon the change of variables $z = \frac{x-\mu}{\sigma}$ to return to the general normal distribution, we obtain

$$\tilde{G}_n^{(1)}(x) \;=\; (x-\mu)\, f_N(x) \left[ \frac{\displaystyle\sum_{r=1}^{n} \binom{n}{r} \sum_{i=0}^{n-r} A_{n-r}^i\, \omega^i \sum_{j=0}^{r-1} B_{r-1}^j\, (-\omega)^j}{\displaystyle\sum_{k=0}^{n} A_n^k\, \omega^k} \right], \tag{3.7}$$

where $\omega = (\frac{x-\mu}{\sigma})^2$.

Table 3.1
*Numerical evaluation of the tail integral of the normal distribution by* (3.7).

| $x$ | $\mu$ | $\sigma$ | $n$ | $\tilde{G}_n^{(1)}(x)$ | $\epsilon_n$ | Maple values |
|---|---|---|---|---|---|---|
| 1.2 | 0 | 1 | 48 | .115069670221707( 0) | .22(−15) | .115069670221708( 0) |
| 1.6 | 0 | 1 | 34 | .547992916995587(−1) | .36(−15) | .547992916995579(−1) |
| 2.0 | 0 | 1 | 28 | .227501319481791(−1) | .79(−15) | .2275013194817920(−1) |
| 3.0 | 0 | 1 | 19 | .134989803163009(−2) | .41(−15) | .134989803163009(−2) |
| 6.0 | 0 | 1 | 11 | .986587645037697(−9) | .00( 00) | .986587645037698(−9) |
| 10.0 | 0 | 1 | 7 | .761985302416049(−23) | .70(−15) | .7619853024160526(−23) |
| 12.0 | 0 | 1 | 7 | .177648211207767(−32) | .13(−15) | .1776482112077678(−32) |
| 45.0 | 18 | 6 | 12 | .339767312473006(−5) | .88(−15) | .339767312473006(−5) |
| 54.2 | 2 | 25 | 28 | .183989173418575(−1) | .28(−15) | .183989173418576(−1) |
| 0.3 | 0 | 1 | 162 | .382088577811118( 0) | .30(−13) | .382088577811047( 0) |

In Table 3.1, we list values of the normal distribution. This table reproduces the numerical results presented in [16] for the normal distribution. In Table 3.1 and subsequent tables, we use the following test based on an approximation to the relative error as a stopping criterion in the calculations:

$$\epsilon_n = \left| \frac{\tilde{G}_n^{(1)}(x) - \tilde{G}_{n-1}^{(1)}(x)}{\tilde{G}_n^{(1)}(x)} \right| \leq \epsilon, \tag{3.8}$$

where $\epsilon$ is chosen according to the desired degree of precision. In all our calculations, $\epsilon$ is set at $10^{-15}$.

We also used Maple to compute the tail probabilities with an accuracy of 15 correct digits and the values obtained are listed in the tables in the column Maple values. The evalf[15] command of Maple was used for a straightforward calculation of the integral tails, where the number 15 between brackets denotes the number of correct digits Maple will return in the computed expression. A straightforward calculation in Maple consists of entering the infinite-range integrals symbolically and recuperating a numerical value, whether the integral has a closed-form antiderivative or not.

Section 4 contains a more detailed numerical discussion.

**3.2. The gamma distribution.** The PDF $f_g(x)$ given by (2.3) satisfies the first order differential equation given by

$$f_g(x) = p_1(x) f_g'(x), \tag{3.9}$$

where the coefficient $p_1(x)$ is given by

$$p_1(x) \;=\; \frac{bx}{ab-b-x} \;=\; \sum_{n=0}^{\infty} \frac{-b^{n+1}(a-1)^n}{x^n} \quad \Rightarrow \quad \xi_0 = 0. \tag{3.10}$$

Using SSF 1 with $(\mu,\nu,m,n) = (-2, a-1, 0, 0)$, we obtain

$$\begin{aligned} \mathcal{D}_n(x) &= \left(x^2 \frac{\mathrm{d}}{\mathrm{d}x}\right)^n \Gamma(a)\, b^a\, x^{1-a} \, \exp\left(\frac{x}{b}\right) \\ &= \frac{x^n}{f_g(x)} \sum_{i=0}^{n} A_n^i \left(\frac{x}{b}\right)^i. \end{aligned} \tag{3.11}$$

For the numerator, we use $(\mu,\nu,m,n) = (-2, -a-1, 0, 0)$ to develop

$$\left(x^2 \frac{\mathrm{d}}{\mathrm{d}x}\right)^{r-1} (x^2 f_g(x)) \;=\; x^{r+1} f_g(x) \sum_{i=0}^{r-1} B_{r-1}^i \left(-\frac{x}{b}\right)^i. \tag{3.12}$$

TABLE 3.2
*Numerical evaluation of the tail integral of the gamma distribution by* (3.13).

| $x$ | $a$ | $b$ | $n$ | $\tilde{G}_n^{(1)}(x)$ | $\epsilon_n$ | Maple values |
|---|---|---|---|---|---|---|
| 13.0 | 7.000 | 2.0000 | 8 | .526523622518000( 0) | .00( 00) | .526523622517999( 0) |
| 15.0 | 7.000 | 2.0000 | 8 | .378154694323469( 0) | .00( 00) | .378154694323469( 0) |
| 20.0 | 7.000 | 2.0000 | 8 | .130141420882482( 0) | .00( 00) | .130141420882482( 0) |
| 35.0 | 7.000 | 2.0000 | 8 | .147001977487619(−2) | .00( 00) | .147001977487619(−2) |
| 40.0 | 7.000 | 2.0000 | 8 | .255122495856300(−3) | .00( 00) | .255122495856300(−3) |
| 45.0 | 7.000 | 2.0000 | 8 | .407935571774570(−4) | .00( 00) | .407935571774571(−4) |
| 50.0 | 7.000 | 2.0000 | 8 | .610629446192788(−5) | .00( 00) | .610629446192790(−5) |
| 60.0 | 7.000 | 2.0000 | 8 | .117319420023469(−6) | .00( 00) | .117319420023469(−6) |
| 120.0 | 7.000 | 2.0000 | 7 | .629224133230851(−18) | .38(−15) | .629224133230850(−18) |
| 12.0 | 2.000 | 3.0000 | 3 | .915781944436709(−1) | .00( 00) | .915781944436709(−1) |
| 25.5 | 4.430 | 2.0230 | 11 | .251747197371780(−2) | .15(−13) | .251747197371771(−2) |
| 45.0 | 5.432 | 4.5432 | 13 | .453930946920760(−1) | .24(−13) | .453930946920784(−1) |
| 14.0 | 1.111 | 9.0000 | 45 | .245873088348530( 0) | .33(−15) | .245873088348520( 0) |

Inserting these results into (2.16), we obtain

$$\tilde{G}_n^{(1)}(x) \;=\; x\, f_g(x) \left[ \frac{\displaystyle\sum_{r=1}^{n} \binom{n}{r} \sum_{i=0}^{n-r} A_{n-r}^{i} \left(\frac{x}{b}\right)^{i} \sum_{j=0}^{r-1} B_{r-1}^{j} \left(-\frac{x}{b}\right)^{j}}{\displaystyle\sum_{k=0}^{n} A_n^k \left(\frac{x}{b}\right)^{k}} \right]. \tag{3.13}$$

In Table 3.2, we list values of the gamma distribution. This table reproduces the numerical results presented in [16] for the gamma distribution.

**3.3. The student's *t*-distribution.** The PDF $f_t(x)$ given by (2.4) satisfies the first order differential equation given by

$$f_t(x) = p_1(x) f_t'(x), \tag{3.14}$$

where the coefficient $p_1(x)$ is given by

$$p_1(x) \;=\; -\frac{x^2+v}{x(v+1)} \;=\; x\left(-\frac{1}{v+1} - \frac{v}{(v+1)x^2}\right) \quad \Rightarrow \quad \xi_0 = 1. \tag{3.15}$$

To calculate $D_n(x)$, we use the SSF 1 with $(\mu, \nu, m, n) = (-2, 1, 1, 0)$ to obtain

$$\begin{aligned} \mathcal{D}_n(x) &= \left(x^2 \frac{\mathrm{d}}{\mathrm{d}x}\right)^n x^{-1} \frac{\sqrt{v\pi}\,\Gamma(\frac{v}{2})}{\Gamma(\frac{v+1}{2})} \left(1 + \frac{x^2}{v}\right)^{(\frac{v+1}{2})} \\ &= \frac{x^{n-1}}{f_t(x)} \sum_{i=0}^{n} A_n^i \prod_{k=0}^{i-1} (v+1-2k) \left(\frac{x^2}{v+x^2}\right)^{i}. \end{aligned} \tag{3.16}$$

For the numerator, we use $(\mu, \nu, m, n) = (-2, -2, 1, 0)$ to develop

$$\left(x^2 \frac{\mathrm{d}}{\mathrm{d}x}\right)^{r-1} (x^2 f_t(x)) \;=\; x^{r+1} f_t(x) \sum_{i=0}^{r-1} B_{r-1}^{i} \prod_{k=0}^{i-1} (v+1+2k) \left(-\frac{x^2}{v+x^2}\right)^{i}. \tag{3.17}$$

TABLE 3.3
*Numerical evaluation of the tail integral of the student's t-distribution by* (3.18).

| $x$ | $v$ | $n$ | $\tilde{G}_n^{(1)}(x)$ | $\epsilon_n$ | Maple values |
|---|---|---|---|---|---|
| 1.812 | 10 | 28 | .500376310329719(−1) | .17(−11) | .500376310329236(−1) |
| 2.228 | 10 | 26 | .250058859084132(−1) | .16(−11) | .250058859085556(−1) |
| 3.169 | 10 | 19 | .500231668217826(−2) | .57(−12) | .500231668219242(−2) |
| 4.587 | 10 | 21 | .499918645935931(−3) | .62(−13) | .499918645938171(−3) |
| 6.927 | 20 | 12 | .500032563506471(−6) | .46(−13) | .500032563506499(−6) |
| 5.449 | 60 | 11 | .499901999489751(−6) | .85(−14) | .499901999489723(−6) |
| 3.373 | 120 | 21 | .500752580750085(−3) | .45(−13) | .500752580749990(−3) |
| 20.0 | 120 | 11 | .255269495907817(−39) | .16(−15) | .255269495907814(−39) |

Inserting these results into (2.16), we obtain

$$\tilde{G}_n^{(1)}(x) \;=\; x f_t(x) \left[ \frac{\displaystyle\sum_{r=1}^{n} \binom{n}{r} \sum_{i=0}^{n-r} A_{n-r}^{i}\,(\omega)_i\, z^i \sum_{j=0}^{r-1} B_{r-1}^{j}\,(-\omega)_j\, z^j}{\displaystyle\sum_{k=0}^{n} A_n^k\,(\omega)_k\, z^k} \right], \tag{3.18}$$

where $\omega = -\frac{v+1}{2}$, $z = -\frac{2\,x^2}{v+x^2}$ and $(x)_n = x(x+1)\cdots(x+n-1)$ is a Pochhammer symbol.

In Table 3.3, we list values of the student's $t$-distribution. This table reproduces the numerical results presented in [16] for the student's $t$-distribution.

**3.4. The inverse Gaussian distribution.** The PDF $f_i(x)$ given by (2.5) satisfies the first order differential equation given by

$$f_i(x) \;=\; p_1(x) f_i'(x), \tag{3.19}$$

where the coefficient $p_1(x)$ is given by

$$p_1(x) \;=\; \frac{-2\mu^2x^2}{\lambda x^2 + 3\mu^2 x - \lambda\mu^2} = \left(-\frac{2\mu^2}{\lambda} + \frac{6\mu^2}{\lambda^2 x} - \cdots\right) \quad\Rightarrow\quad \xi_0 = 0. \tag{3.20}$$

Using SSF 1 with $(\mu,\nu,m,n) = (-2,-\frac{3}{2},0,0)$, we obtain

$$\begin{aligned} \mathcal{D}_n(x) &= \left(x^2\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \left(\frac{\lambda}{2\pi x^3}\right)^{-\frac{1}{2}} \exp\left(\frac{\lambda(x-\mu)^2}{2\mu^2 x}\right) \\ &= \frac{1}{f_i(x)} \sum_{k=0}^{n} \binom{n}{k} x^k \sum_{i=0}^{k} A_k^i \left(\frac{\lambda\, x}{2\mu^2}\right)^i \left(-\frac{\lambda}{2}\right)^{n-k}. \end{aligned} \tag{3.21}$$

For the numerator, we use $(\mu,\nu,m,n) = (-2,-\frac{1}{2},0,0)$ to develop

$$\left(x^2\frac{\mathrm{d}}{\mathrm{d}x}\right)^{r-1} (x^2 f_i(x)) \;=\; x^2 f_i(x) \sum_{q=0}^{r-1} \binom{r-1}{q} x^q \sum_{i=0}^{q} B_q^i \left(-\frac{\lambda\, x}{2\mu^2}\right)^i \left(\frac{\lambda}{2}\right)^{r-1-q}. \tag{3.22}$$

TABLE 3.4
*Numerical evaluation of the tail integral of the inverse Gaussian distribution by* (3.23).

| $x$ | $\mu$ | $\lambda$ | $n$ | $\tilde{G}_n^{(1)}(x)$ | $\epsilon_n$ | Maple values |
|---|---|---|---|---|---|---|
| 1.50 | 1.00 | 1.00 | 107 | .189232007000019( 0) | .92(−15) | .189232007000020( 0) |
| 2.00 | 1.00 | 1.00 | 82 | .114524574013992( 0) | .43(−15) | .114524574013993( 0) |
| 3.00 | 1.00 | 1.00 | 61 | .468120792572114(−1) | .63(−15) | .468120792572116(−1) |
| 4.50 | 1.00 | 1.00 | 42 | .143011829460930(−1) | .00( 00) | .143011829460931(−1) |
| 6.00 | 1.00 | 1.00 | 35 | .484988213370218(−2) | .00( 00) | .484988213370217(−2) |
| 10.00 | 1.00 | 1.00 | 36 | .350414537208826(−3) | .65(−15) | .350414537208819(−3) |
| 16.00 | 1.00 | 1.00 | 24 | .943916863494728(−5) | .39(−15) | .943916863494723(−5) |
| 32.00 | 1.00 | 1.00 | 16 | .122006566375975(−8) | .72(−15) | .122006566375975(−8) |
| 24.00 | 2.00 | 4.00 | 27 | .510429100438049(−6) | .14(−15) | .510429100438016(−6) |
| 33.46 | 4.54 | 2.78 | 40 | .621975008388134(−2) | .72(−15) | .621975008388144(−2) |
| 23.00 | 6.54 | 6.00 | 55 | .333636164607366(−1) | .71(−15) | .333636164607370(−1) |
| 0.50 | 1.00 | 1.00 | 165 | .635024451788276( 0) | .49(−11) | .635024451827040( 0) |

Inserting these results into (2.16), we obtain

$$\tilde{G}_n^{(1)}(x) = \frac{2\,x^2\,f_i(x)}{\lambda} \times \left[ \frac{\sum_{r=1}^{n} \binom{n}{r} (-1)^r \sum_{k=0}^{n-r} \binom{n-r}{k} (-\omega)^k \sum_{i=0}^{k} A_k^i\, z^i \sum_{q=0}^{r-1} \binom{r-1}{q} \omega^q \sum_{l=0}^{q} B_q^l\,(-z)^l}{\sum_{m=0}^{n} \binom{n}{m} (-\omega)^m \sum_{p=0}^{m} A_m^p\, z^p} \right], \tag{3.23}$$

where $\omega = \frac{2\,x}{\lambda}$ and $z = \frac{\lambda\,x}{2\,\mu^2}$.

In Table 3.4, we list values of the inverse Gaussian distribution. This table reproduces the numerical results presented in [16] for the inverse Gaussian distribution.

**3.5. Fisher's *F* distribution.** The PDF $f_F(x)$ given by (2.6) satisfies the first order differential equation given by

$$f_F(x) \;=\; p_1(x) f_F'(x), \tag{3.24}$$

where the coefficient $p_1(x)$ is given by:

$$p_1(x) \;=\; -\frac{2ax^2+2bx}{(2a+ab)x+2b-ab} = x\left(-\frac{2}{2+b} - \frac{2b(b+a)}{(2+b)^2ax} - \cdots\right) \quad\Rightarrow\quad \xi_0 = 1. \tag{3.25}$$

Using SSF 1 with $(\mu,\nu,m,n) = (-2,\frac{a}{2},0,0)$ and $\beta = \frac{\Gamma(\frac{a+b}{2})}{\Gamma(\frac{a}{2})\Gamma(\frac{b}{2})}(\frac{a}{b})^{\frac{a}{2}}$, we obtain

$$\begin{aligned} \mathcal{D}_n(x) &= \left(x^2\frac{\mathrm{d}}{\mathrm{d}x}\right)^n x^{-1}\beta^{-1}x^{-\frac{a-2}{2}}\left(1+\left(\frac{a}{b}\right)x\right)^{\frac{a+b}{2}} \\ &= \frac{x^{n-1}}{f_F(x)}\sum_{i=0}^{n} A_n^i \prod_{k=0}^{i-1}\left(\frac{a+b}{2}-k\right)\left(\frac{ax}{ax+b}\right)^i. \end{aligned} \tag{3.26}$$

For the numerator, we use $(\mu,\nu,m,n) = (-2,-\frac{a+2}{2},0,0)$ to develop

$$\left(x^2\frac{\mathrm{d}}{\mathrm{d}x}\right)^{r-1}(x^2 f_F(x)) \;=\; x^{r+1}f_F(x)\sum_{j=0}^{r-1}B_{r-1}^j\prod_{q=0}^{j-1}\left(\frac{a+b}{2}+q\right)\left(-\frac{ax}{ax+b}\right)^j. \tag{3.27}$$

TABLE 3.5
*Numerical evaluation of the tail integral of Fisher's F distribution by* (3.28).

| $x$ | $a$ | $b$ | $n$ | $\tilde{G}_n^{(1)}(x)$ | $\epsilon_n$ | Maple values |
|---|---|---|---|---|---|---|
| 4.190 | 3 | 4 | 14 | .100029643896889( 0) | .68(−13) | .100029643896895( 0) |
| 6.590 | 3 | 4 | 12 | .500168891790474(−1) | .11(−12) | .500168891790405(−1) |
| 9.980 | 3 | 4 | 11 | .249965339234578(−1) | .31(−13) | .249965339234568(−1) |
| 16.70 | 3 | 4 | 9 | .999383733001448(−2) | .17(−13) | .999383733001462(−2) |
| 5.750 | 5 | 1 | 10 | .306042577763992( 0) | .15(−12) | .306042577763857( 0) |
| 3.340 | 1 | 1 | 18 | .318737836141563( 0) | .43(−12) | .318737836141636( 0) |
| 23.23 | 10 | 5 | 7 | .142310351602081(−2) | .17(−13) | .142310351602084(−2) |
| 12.05 | 8 | 3 | 6 | .325796489130341(−1) | .53(−14) | .325796489130337(−1) |

Inserting these results into (2.16), we obtain

$$\tilde{G}_n^{(1)}(x) \;=\; x f_F(x) \left[ \frac{\displaystyle\sum_{r=1}^{n} \binom{n}{r} \sum_{i=0}^{n-r} A_{n-r}^{i}\,(\omega)_i\, z^i \sum_{j=0}^{r-1} B_{r-1}^{j}\,(-\omega)_j\, z^j}{\displaystyle\sum_{k=0}^{n} A_n^k\,(\omega)_k\, z^k} \right], \tag{3.28}$$

where $\omega = -\frac{a+b}{2}$ and $z = -\frac{a\,x}{a\,x+b}$.

In Table 3.5, we list values of Fisher's $F$ distribution. This table reproduces the numerical results presented in [16] for Fisher's $F$ distribution.

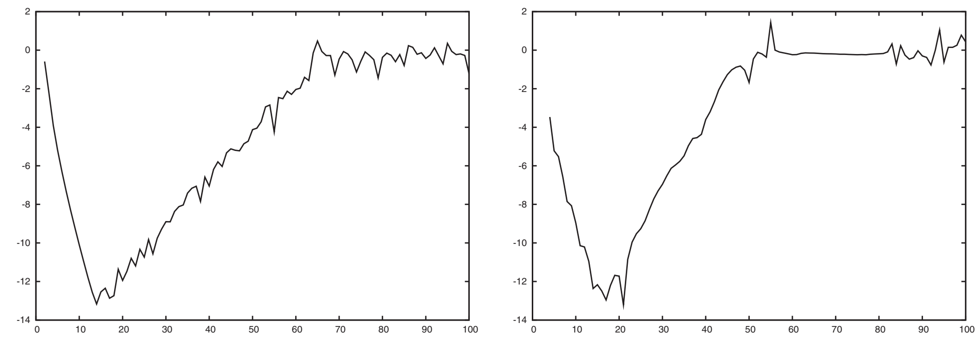
**4. Numerical discussion.** The SSF 1 greatly simplifies the computation of the explicit expressions when using a Fortran compiler due to the recurrence relations (2.8) satisfied by the coefficients $A_k^i$. This internal recursion leads to a considerable gain in the calculation times compared to the use of a method to solve the linear system (2.12). In addition, the recurrence relations (2.8) allow us to stop the calculation as soon as the desired precision is attained. That is, we calculate the approximation $\tilde{G}_{n+1}^{(1)}$ only if the precision attained by $\tilde{G}_n^{(1)}$ is insufficient. We use the test given by (3.8) as a stopping criterion in the calculations.

In general, the test of accuracy (3.8) works well. However, in certain instances, when the behavior of the sequence of approximations $\{\tilde{G}_n^{(1)}(x)\}$ is unstable and after an optimal order of approximation, the error will only grow larger (see Figures 4.1(a) and 4.1(b) and their corresponding Tables 4.1 and 4.2). In such a situation, we must therefore stop the calculation, knowing that the error will only grow larger. The stopping criterion in such an instance is determined by the following test:

$$\varrho_i = \left| \frac{\tilde{G}_{n-i}^{(1)}(x) - \tilde{G}_{n-1-i}^{(1)}(x)}{\tilde{G}_{n-1-i}^{(1)}(x) - \tilde{G}_{n-2-i}^{(1)}(x)} \right| > 1 \qquad \text{for} \qquad i = 0, 1, 2, \ldots. \tag{4.1}$$

We found that $\varrho_0 > 1$ was a sufficient stopping criterion for Fisher's $F$ and the gamma distributions. For the student's $t$-distribution, $\varrho_i > 1$ for $i = 0, 1, 2$ are required in order to achieve the best numerical result. For Fisher's $F$ distribution, when $\varrho_0 > 1$ is used, we found that the $\tilde{G}_{n-1}^{(1)}(x)$ term gives the best result. In Table 4.1, we have $\varrho_0 > 1$ at $n = 15$ and the best approximation is given by $\tilde{G}_{14}^{(1)}(x)$.

For the gamma distribution, we found that the $\tilde{G}_n^{(1)}(x)$ term gives the best result. Using $\varrho_i > 1$ for $i = 0, 1, 2$ for the student's $t$-distribution, we found that the $\tilde{G}_{n-3}^{(1)}(x)$ term gives the best result. In Table 4.2, we have $\varrho_i > 1$ for $i = 0, 1, 2$ at $n = 24$ and the best approximation is given by $\tilde{G}_{21}^{(1)}(x)$.




(a) (b)

FIG. 4.1. *Plot of* $\log_{10}(\epsilon_n)$ *given by* (3.8) *as a function of the order* $n$ *of the approximations* $\tilde{G}_n^{(1)}(x)$ *for* (a) *Fisher's* $F$ *distribution with* $x = 4.19$, $a = 3$, *and* $b = 4$ *(corresponds to Table* 4.1*) and for* (b) *the student's* $t$*-distribution with* $x = 4.587$ *and* $v = 10$ *(corresponds to Table* 4.2*).*

TABLE 4.1
*Error table for Fisher's* $F$ *distribution for* $x = 4.19$ *and* $a = 3$ *and* $b = 4$ *by* (3.28). $\tilde{G}_{14}^{(1)}$ *is the approximation obtained from our algorithm.*

| $n$ | $\tilde{G}_n^{(1)}$ | $\epsilon_n$ | $\varrho_0$ |
|---|---|---|---|
| ⋮ | ⋮ | ⋮ | ⋮ |
| 7 | .100 029 643 323 462( 0) | .46(−07) | .09(0) |
| 8 | .100 029 643 826 732( 0) | .50(−08) | .10(0) |
| 9 | .100 029 643 887 614( 0) | .60(−09) | .12(0) |
| 10 | .100 029 643 895 591( 0) | .79(−10) | .13(0) |
| 11 | .100 029 643 896 704( 0) | .11(−10) | .13(0) |
| 12 | .100 029 643 896 869( 0) | .16(−11) | .14(0) |
| 13 | .100 029 643 896 896( 0) | .27(−12) | .16(0) |
| 14 | .100 029 643 896 889( 0) | .68(−13) | .24(0) |
| 15 | .100 029 643 896 918( 0) | .28(−12) | .42(1) |
| 16 | .100 029 643 896 873( 0) | .44(−12) | .15(1) |
| 17 | .100 029 643 896 887( 0) | .13(−12) | .30(0) |
| ⋮ | ⋮ | ⋮ | ⋮ |
| 59 | .978 730 933 819 339(-1) | .50(−02) | .68(0) |
| 60 | .987 747 825 057 981(-1) | .91(−02) | .18(1) |
| 61 | .977 349 542 823 236(-1) | .10(−01) | .11(1) |
| 62 | .940 746 788 876 968(-1) | .38(−01) | .35(1) |
| ⋮ | ⋮ | ⋮ | ⋮ |

Figure 4.1 shows representative plots of $\log_{10}(\epsilon_n)$ for Fisher's $F$ distribution and for the student's $t$-distribution. Figure 4.1(a) shows a typical error curve which allows the test for $\varrho_0 > 1$ to achieve the optimal approximation. In contrast, Figure 4.1(b) shows a typical error curve which prevents the test for $\varrho_0 > 1$ from achieving the optimal approximation. Heuristically speaking, the irregularities in the sequence of relative errors present in Figure 4.1(b) but absent in Figure 4.1(a) cause the test for $\varrho_0 > 1$ to stop the algorithm prematurely before reaching the optimal approximation. After analysis of numerous similar plots, we find that only the student's $t$-distribution requires additional test.

Generally speaking, the accuracy improves as the order $n$ in the $\tilde{G}_n^{(1)}$ transformation increases. However, after a certain value of iteration depending on the arguments

TABLE 4.2
*Error table for the student's t-distribution for $x = 4.587$ and $v = 10$ by (3.18). $\tilde{G}_{21}^{(1)}$ is the approximation obtained from our algorithm.*

| $n$ | $\tilde{G}_n^{(1)}$ | $\epsilon_n$ | $\varrho_0$ | $\varrho_1$ | $\varrho_2$ |
|---|---|---|---|---|---|
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| 16 | .499 918 645 938 139(−3) | .32(−12) | .48(0) | .16(1) | .03(0) |
| 17 | .499 918 645 938 192(−3) | .11(−12) | .33(0) | .48(0) | .16(1) |
| 18 | .499 918 645 937 877(−3) | .63(−12) | .59(1) | .33(0) | .48(0) |
| 19 | .499 918 645 936 838(−3) | .21(−11) | .33(1) | .59(1) | .33(0) |
| 20 | .499 918 645 935 900(−3) | .19(−11) | .90(0) | .33(1) | .59(1) |
| 21 | .499 918 645 935 931(−3) | .62(−13) | .03(0) | .90(0) | .33(1) |
| 22 | .499 918 645 928 744(−3) | .14(−10) | .23(3) | .03(0) | .90(0) |
| 23 | .499 918 645 876 004(−3) | .11(−09) | .73(1) | .23(3) | .03(0) |
| 24 | .499 918 645 725 138(−3) | .30(−09) | .29(1) | .73(1) | .23(3) |
| 25 | .499 918 645 449 595(−3) | .55(−09) | .18(1) | .29(1) | .73(1) |
| 26 | .499 918 644 750 484(−3) | .14(−08) | .25(1) | .18(1) | .29(1) |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| 49 | .906 689 643 668 337(−3) | .91(−01) | .65(0) | .14(1) | .16(1) |
| 50 | .925 745 751 045 273(−3) | .21(−01) | .23(0) | .65(0) | .14(1) |
| 51 | .143 490 605 510 311(−2) | .35( 00) | .27(2) | .23(0) | .65(0) |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |

provided, overflow occurs. The program returns the message NaN (not a number) due to the divergent nature of the coefficients in the SSF 1. In such a situation, we used the following stopping criteria in our algorithm:

$$\tilde{\mathcal{N}}_n(x) > \text{Huge} \qquad \text{or} \qquad \mathcal{D}_n(x) > \text{Huge}. \tag{4.2}$$

The value of Huge is chosen close but not equal to the largest real number that can be stored by the machine. We chose the value $10^{300}$ in our program. In this case, we pick the value $\tilde{G}_{n-1}^{(1)}(x)$ as the best result.

Tables 4.1 and 4.2 show sequences of values obtained at each iteration of the $\tilde{G}_n^{(1)}$ transformation for the $F$ distribution and the $t$-distribution, respectively. From these tables, we can clearly see how the use of the $\varrho_i > 1$ test helped us reach the most accurate numerical results.

Tables 3.1 to 3.5 reproduce all the numerical results presented in [16].

In the calculation presented in Table 3.2, it is important to note that for integer values of $a$, the integrand has a closed-form antiderivative; thus $\tilde{G}_a^{(1)}(x)$ is equal to the true value. For the numerical calculation of the gamma function we use the subroutine Mgamma.for [25]. In this table, we also add additional values with noninteger parameters to show how our method performs when no closed-form anti-derivatives are known.

When we compare our numerical results with those obtained in [16], all values agree except for the seventh entry in Table 1 in [16]. For this entry, we find .1776482112077677(−32), which is in agreement with the value that we obtained using the symbolic programming language Maple .177648211207767(−32). The value given in [16] is 0.367097(−50) and we suspect this is a typographical error.

All the computations were performed in Fortran 77 using double precision arithmetic. We used the Lahey ED compiler (15 significant decimals in double precision). Calculations were performed on a PC with an AMD Athlon 64 processor 4000+2.40 GHz.

We also used Maple to compute the tail probabilities with an accuracy of 15 correct digits, and the values obtained are listed in Tables 3.1–3.5 in the column Maple values. The evalf[15] command of Maple was used for a straightforward calculation of the integral tails, where the [15] denotes the number of digits Maple will return in the computed expression. A straightforward calculation in Maple consists of entering the infinite-range integrals symbolically and recuperating a numerical value, whether the integral has a closed-form antiderivative or not.

**5. Connection with rational and Padé approximants.** As can be seen from (3.7), (3.13), (3.18), (3.23), and (3.28), the approximations $\tilde{G}_n^{(1)}(x)$ to the tail probabilities all have similar forms. To study the forms more closely, including their connection with other approximation methods, we require the following lemma.

LEMMA 5.1. *Let $f(x)$ have the form*

$$f(x) = A\,x^{\mu} e^{r(x)}, \tag{5.1}$$

*where $A \in \mathbb{R} \setminus 0$, $\mu \in \mathbb{R}$, and $r(x) \in \mathbb{R}[x]$ with $\deg(r(x)) = r_0 \geq 0$. Then, for $i = 0, 1, \ldots,$ and for $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$, it follows that*

$$\left(x^{\alpha+1}\frac{\mathrm{d}}{\mathrm{d}x}\right)^i (x^{\beta} f(x)) = x^{\alpha\, i+\beta} f(x) s_i(x), \tag{5.2a}$$

$$\left(x^{\alpha+1}\frac{\mathrm{d}}{\mathrm{d}x}\right)^i \left(\frac{x^{\beta}}{f(x)}\right) = \frac{x^{\alpha\, i+\beta}}{f(x)} t_i(x), \tag{5.2b}$$

*where $s_i(x) \in \mathbb{R}[x]$ with $\deg(s_i(x)) = i\, r_0$ and $t_i(x) \in \mathbb{R}[x]$ with $\deg(t_i(x)) = i\, r_0$.*

*Proof.* The demonstration of either (5.2a) or (5.2b) is sufficient, as $\frac{1}{f(x)} = A^{-1}x^{-\mu}e^{-r(x)}$ is still in the form of (5.1) if, for example, we take $B = A^{-1}$, $\nu = -\mu$, and $s(x) = -r(x)$.

To prove (5.2a), we begin with $i = 0$. In this case $s_0(x) = 1$.

For $i = 1$,

$$\begin{aligned} x^{\alpha+1}\frac{\mathrm{d}}{\mathrm{d}x}(x^{\beta} f(x)) &= x^{\alpha+1}\left[A(\beta+\mu)x^{\beta+\mu-1} + Ax^{\beta+\mu}r'(x)\right]e^{r(x)} \\ &= x^{\alpha+\beta} A\,x^{\mu} e^{r(x)}\left[(\beta+\mu) + x\,r'(x)\right] \\ &= x^{\alpha+\beta} f(x) s_1(x), \end{aligned} \tag{5.3}$$

where $s_1(x) \in \mathbb{R}[x]$ with $\deg(s_1(x)) = r_0$.

For $i > 1$, the proof follows by induction:

$$\begin{aligned} \left(x^{\alpha+1}\frac{\mathrm{d}}{\mathrm{d}x}\right)(x^{\beta+\alpha(i-1)} f(x) s_{i-1}(x)) &= x^{\alpha(i-1)} s_{i-1}(x)\left(x^{\alpha+1}\frac{\mathrm{d}}{\mathrm{d}x}\right)(x^{\beta} f(x)) \\ &\quad + x^{\beta} f(x)\left(x^{\alpha+1}\frac{\mathrm{d}}{\mathrm{d}x}\right)(x^{\alpha(i-1)} s_{i-1}(x)) \\ &= x^{\alpha(i-1)} s_{i-1}(x) x^{\alpha+\beta} f(x) s_1(x) \\ &\quad + x^{\beta} f(x)(\alpha(i-1)x^{\alpha\, i} s_{i-1}(x) + x^{\alpha\, i+1} s'_{i-1}(x)) \\ &= x^{\alpha\, i+\beta} f(x) s_{i-1}(x) s_1(x) \\ &\quad + x^{\alpha\, i+\beta} f(x)(\alpha(i-1) s_{i-1}(x) + x\, s'_{i-1}(x)) \\ &= x^{\alpha\, i+\beta} f(x) s_i(x), \end{aligned} \tag{5.4}$$

where $s_i(x) \in \mathbb{R}[x]$ with $\deg(s_i(x)) = i\, r_0$. ∎

LEMMA 5.2. *Let $\omega(x)$ be such that*

$$\omega(x) \sim A\,x^{\mu} e^{r(x)} \quad \text{as} \quad x \to \infty, \tag{5.5}$$

*where $A \in \mathbb{R}/0$, $\mu \in \mathbb{R}$, and $r(x) \in \mathbb{R}[x]$ with $\deg(r(x)) = r_0 \geq 0$. Then, for $n = 0, 1, \ldots,$ and for $\alpha \in \mathbb{R}$, it follows that*

$$\frac{\left(x^{\alpha+1}\dfrac{\mathrm{d}}{\mathrm{d}x}\right)^{n-1}\left(\dfrac{1}{\omega(x)}\right)}{\left(x^{\alpha+1}\dfrac{\mathrm{d}}{\mathrm{d}x}\right)^{n}\left(\dfrac{1}{\omega(x)}\right)} \sim \frac{\omega_n}{x^{\alpha+r_0}} \quad \text{as} \quad x \to \infty, \tag{5.6}$$

*where $\omega_n \in \mathbb{R}$ is a constant.*

*Proof.* By Lemma 5.1, since $\omega(x) \sim f(x)$, where $f(x)$ is given in (5.1), the left-hand side of (5.6) is then (as $x \to \infty$)

$$\frac{\left(x^{\alpha+1}\dfrac{\mathrm{d}}{\mathrm{d}x}\right)^{n-1}\left(\dfrac{1}{\omega(x)}\right)}{\left(x^{\alpha+1}\dfrac{\mathrm{d}}{\mathrm{d}x}\right)^{n}\left(\dfrac{1}{\omega(x)}\right)} \sim \frac{\left(x^{\alpha+1}\dfrac{\mathrm{d}}{\mathrm{d}x}\right)^{n-1}\left(\dfrac{1}{f(x)}\right)}{\left(x^{\alpha+1}\dfrac{\mathrm{d}}{\mathrm{d}x}\right)^{n}\left(\dfrac{1}{f(x)}\right)} \tag{5.7}$$

$$= \frac{\dfrac{x^{\alpha(n-1)}}{f(x)} t_{n-1}(x)}{\dfrac{x^{\alpha\, n}}{f(x)} t_n(x)} \tag{5.8}$$

$$= \frac{t_{n-1}(x)}{x^{\alpha}\, t_n(x)} \tag{5.9}$$

$$\sim \frac{\omega_n}{x^{\alpha+r_0}}, \tag{5.10}$$

where $\omega_n \in \mathbb{R}$ is a constant whose dependence on $n$ is highlighted. ∎

We now have the tools to derive asymptotic error estimates for the $G_n^{(1)}$ transformation. However, we do this with a generalized notation.

Let $F(x) = \int_0^x f(t)\,\mathrm{d}t$ and let $I[f] = \lim_{x\to\infty} F(x)$. Let also

$$I[f] - F(x) \sim \omega(x) \sum_{i=0}^{\infty} \frac{\beta_i}{x^{\alpha\, i}} \qquad \text{as} \qquad x \to \infty, \tag{5.11}$$

where $\alpha \in \mathbb{R}^+$ and $\omega : \mathbb{R} \to \mathbb{R}$.

Then, using the analogous annihilation operator to the one used in (2.16), we obtain the approximations

$$G_n^{(1,\alpha)}(x) = \frac{\left(x^{\alpha+1}\dfrac{\mathrm{d}}{\mathrm{d}x}\right)^{n}\left(\dfrac{F(x)}{\omega(x)}\right)}{\left(x^{\alpha+1}\dfrac{\mathrm{d}}{\mathrm{d}x}\right)^{n}\left(\dfrac{1}{\omega(x)}\right)}. \tag{5.12}$$

When $\omega(x) = x^{\xi_0} f(x)$, we have $G_n^{(1,1)}(x) = G_n^{(1)}(x)$.

We now develop the asymptotic error estimate for this generalized $G_n^{(1,\alpha)}$ transformation.

THEOREM 5.3. *Let* $\lim_{x\to\infty} F(x) = I[f]$. *Let* $I[f]-F(x)$ *have the asymptotic expansion given by* (5.11), *where* $\omega(x)$ *is such that* (5.5) *holds. Then the approximations* $G_n^{(1,\alpha)}(x)$ *given in* (5.12) *satisfy*

$$\frac{I[f]-G_n^{(1,\alpha)}(x)}{I[f]-G_{n-1}^{(1,\alpha)}(x)} = \mathcal{O}\left(\frac{1}{x^{\alpha+r_0}}\right) \quad \text{as} \quad x\to\infty. \tag{5.13}$$

*Proof.* Using (5.12), the ratio (5.13) is given by

$$\frac{I[f]-G_n^{(1,\alpha)}(x)}{I[f]-G_{n-1}^{(1,\alpha)}(x)} = \frac{\left(x^{\alpha+1}\frac{\mathrm{d}}{\mathrm{d}x}\right)^n\left(\frac{I[f]-F(x)}{\omega(x)}\right)}{\left(x^{\alpha+1}\frac{\mathrm{d}}{\mathrm{d}x}\right)^{n-1}\left(\frac{I[f]-F(x)}{\omega(x)}\right)} \tag{5.14}$$

$$\times\frac{\left(x^{\alpha+1}\frac{\mathrm{d}}{\mathrm{d}x}\right)^{n-1}\left(\frac{1}{\omega(x)}\right)}{\left(x^{\alpha+1}\frac{\mathrm{d}}{\mathrm{d}x}\right)^{n}\left(\frac{1}{\omega(x)}\right)}, \tag{5.15}$$

since $I[f]$ is a constant and since $(x^{\alpha+1}\frac{\mathrm{d}}{\mathrm{d}x})$ is a linear operator.

Note that the discrete analogue of (5.15) is typical of Levin's transformation [24].

We investigate both ratios separately. First, from the asymptotic condition (5.11), the first ratio is asymptotic to a constant (as $x\to\infty$):

$$\frac{\left(x^{\alpha+1}\frac{\mathrm{d}}{\mathrm{d}x}\right)^n\left(\frac{I[f]-F(x)}{\omega(x)}\right)}{\left(x^{\alpha+1}\frac{\mathrm{d}}{\mathrm{d}x}\right)^{n-1}\left(\frac{I[f]-F(x)}{\omega(x)}\right)} \sim \frac{\left(x^{\alpha+1}\frac{\mathrm{d}}{\mathrm{d}x}\right)^n\sum_{i=0}^{\infty}\frac{\beta_i}{x^{\alpha\, i}}}{\left(x^{\alpha+1}\frac{\mathrm{d}}{\mathrm{d}x}\right)^{n-1}\sum_{i=0}^{\infty}\frac{\beta_i}{x^{\alpha\, i}}} \tag{5.16}$$

$$= \frac{\sum_{i=n}^{\infty}\frac{(-\alpha)^n i!\beta_i}{(i-n)!x^{\alpha(i-n)}}}{\sum_{i=n-1}^{\infty}\frac{(-\alpha)^{n-1} i!\beta_i}{(i-n+1)!x^{\alpha(i-n+1)}}} \tag{5.17}$$

$$\sim \frac{(-\alpha)^n n!\beta_n}{(-\alpha)^{n-1}(n-1)!\beta_{n-1}} \tag{5.18}$$

$$= -\frac{\alpha\, n\, \beta_n}{\beta_{n-1}}. \tag{5.19}$$

And from Lemma 5.2, the second ratio is asymptotic to

$$\frac{\left(x^{\alpha+1}\frac{\mathrm{d}}{\mathrm{d}x}\right)^{n-1}\left(\frac{1}{\omega(x)}\right)}{\left(x^{\alpha+1}\frac{\mathrm{d}}{\mathrm{d}x}\right)^{n}\left(\frac{1}{\omega(x)}\right)} \sim \frac{\omega_n}{x^{\alpha+r_0}} \quad \text{as} \quad x\to\infty. \tag{5.20}$$

Combining these ratios, it is trivial to obtain the asymptotic condition (5.13). ∎

By induction on the order $n$ of the transformation, it is equivalent to state

$$\frac{I[f]}{\omega(x)} - \frac{G_n^{(1,\alpha)}(x)}{\omega(x)} = \mathcal{O}\left(\frac{1}{x^{(\alpha+r_0)n}}\right) \quad \text{as} \quad x\to\infty \tag{5.21}$$

or for the $\tilde{G}_n^{(1,\alpha)}(x)$ transformation defined by $\tilde{G}_n^{(1,\alpha)}(x) = G_n^{(1,\alpha)}(x) - F(x)$

$$\frac{\tilde{G}_n^{(1,\alpha)}(x)}{\omega(x)} - \frac{\int_x^\infty f(t)\,\mathrm{d}t}{\omega(x)} = \mathcal{O}\left(\frac{1}{x^{(\alpha+r_0)n}}\right) \quad \text{as} \quad x \to \infty. \tag{5.22}$$

With these tools, we are able to describe the general form of the approximations $\tilde{G}_n^{(1)}(x)$ to $\int_x^\infty f(t)\,\mathrm{d}t$ for integrals whose integrands are of the form (5.1).

THEOREM 5.4. *Let $f(x)$ be integrable at infinity, i.e., $|\int_x^\infty f(t)\,\mathrm{d}t| < \infty$ for some $x \in \mathbb{R}$, and have the general form prescribed by* (5.1). *The approximations $\tilde{G}_n^{(1,\alpha)}(x)$ to $\int_x^\infty f(t)\,\mathrm{d}t$ take the form*

$$\tilde{G}_n^{(1,\alpha)}(x) = x\, f(x) \frac{a_n(x)}{b_n(x)}, \tag{5.23}$$

*where $a_n(x) \in \mathbb{R}[x]$ with $\deg(a_n(x)) \le (n-1)\, r_0$ and $b_n(x) \in \mathbb{R}[x]$ with $\deg(b_n(x)) = n\, r_0$.*

*Proof.* The function (5.1) satisfies

$$f(x) = p_1(x) f'(x), \tag{5.24}$$

where

$$p_1(x) = \frac{x}{\mu + x\, r'(x)} \sim x^{1-r_0} \sum_{i=0}^{\infty} \frac{\alpha_i}{x^i} \qquad \text{as} \qquad x \to \infty. \tag{5.25}$$

Therefore, the approximations $\tilde{G}_n^{(1,\alpha)}(x)$ can be constructed as in subsection 2.3.

Applying Lemma 5.1 to (2.16) or its generalization (5.12), we obtain

$$\begin{aligned}
\tilde{G}_n^{(1,\alpha)}(x) &= \frac{\displaystyle\sum_{r=1}^{n} \binom{n}{r} \mathcal{D}_{n-r}(x) \left(x^{\alpha+1} \frac{\mathrm{d}}{\mathrm{d}x}\right)^{r-1} \left(x^{\alpha+1} f(x)\right)}{\mathcal{D}_n(x)} \\
&= \frac{\displaystyle\sum_{r=1}^{n} \binom{n}{r} \frac{x^{-\xi_0+\alpha(n-r)}}{f(x)} t_{n-r}(x)\, x^{\alpha+1+\alpha(r-1)} f(x) s_{r-1}(x)}{\frac{x^{-\xi_0+\alpha\, n}}{f(x)} t_n(x)} \\
&= x\, f(x) \frac{\displaystyle\sum_{r=1}^{n} \binom{n}{r} t_{n-r}(x)\, s_{r-1}(x)}{t_n(x)} \\
&= x\, f(x) \frac{a_n(x)}{b_n(x)},
\end{aligned} \tag{5.26}$$

where the polynomials $a_n(x)$ and $b_n(x)$ and the bounds on their degrees are as prescribed above. ∎

Before applying these tools to the five examples above, we would like to discuss a special class of rational approximants, the Padé approximants [18]. Consider the (formal) power series $\mathcal{F}(x) = \sum_{i=0}^{\infty} f_i x^i$ as $x \to 0$. Then the Padé approximants $[l/m]_f(x)$ to $\mathcal{F}(x)$ are the rational approximants:

$$\frac{P^{[l/m]}(x)}{Q^{[l/m]}(x)} = \frac{p_0 + p_1 x + \cdots + p_l x^l}{q_0 + q_1 x + \cdots + q_m x^m} \qquad \text{with} \qquad l, m \in \mathbb{N}_0, \tag{5.27}$$

which satisfy the maximal accuracy-through-order condition:

$$\mathcal{F}(x) - \frac{P^{[l/m]}(x)}{Q^{[l/m]}(x)} = \mathcal{O}(x^{l+m+1}) \quad \text{as} \quad x \to 0. \tag{5.28}$$

This formalism for the construction of Padé approximants also works if we start from an inverse power series. The only difference is that we then obtain Padé approximants in $1/x$ instead of $x$. This modified asymptotic condition is then

$$\mathcal{F}(1/x) - \frac{P^{[l/m]}(1/x)}{Q^{[l/m]}(1/x)} = \mathcal{O}\left(\frac{1}{x^{l+m+1}}\right) \qquad \text{as} \qquad x \to \infty. \tag{5.29}$$

**5.1. The distributions.** In applying Theorems 5.3 and 5.4, we first consider the standard normal distribution, where

$$g_N(x) = A\, x^{\mu} e^{r(x)}, \tag{5.30}$$

where $A^{-1} = \sqrt{2\,\pi}$, $\mu = 0$, and $r(x) = -x^2/2$. The approximations take the form

$$\tilde{G}_n^{(1)}(x) = x\, g_N(x) \frac{a_n(x)}{b_n(x)}, \tag{5.31}$$

where $\deg(a_n(x)) \leq 2n - 2$ and $\deg(b_n(x)) = 2n$, which is in agreement with (3.6). The approximations $\tilde{G}_n^{(1)}(x) = \tilde{G}_n^{(1,1)}(x)$ satisfy the asymptotic error estimate:

$$\frac{\tilde{G}_n^{(1)}(x)}{g_N(x)/x} - \frac{\int_x^{\infty} g_N(t)\,\mathrm{d}t}{g_N(x)/x} = \mathcal{O}\left(\frac{1}{x^{3n}}\right) \quad \text{as} \quad x \to \infty. \tag{5.32}$$

The case for the general normal distribution $f_N(x)$ may be developed similarly.

We next consider the gamma distribution, where

$$f_g(x) = A\, x^{\mu} e^{r(x)}, \tag{5.33}$$

where $A^{-1} = \Gamma(a)\, b^a$, $\mu = a - 1$, and $r(x) = -x/b$. The approximations, then, take the form

$$\tilde{G}_n^{(1)}(x) = x\, f_g(x) \frac{a_n(x)}{b_n(x)}, \tag{5.34}$$

where $\deg(a_n(x)) \leq n - 1$ and $\deg(b_n(x)) = n$, which is in agreement with (3.13). The approximations $\tilde{G}_n^{(1)}(x) = \tilde{G}_n^{(1,1)}(x)$ satisfy the asymptotic error estimate:

$$\frac{\tilde{G}_n^{(1)}(x)}{f_g(x)} - \frac{\int_x^{\infty} f_g(t)\,\mathrm{d}t}{f_g(x)} = \mathcal{O}\left(\frac{1}{x^{2n}}\right) \quad \text{as} \quad x \to \infty. \tag{5.35}$$

In fact, given the accuracy-through-order condition derived above, the bound on the degree of $a_n(x)$ can be made more precise. Indeed, $\deg(a_n(x)) = n - 1$. Furthermore, the rational approximants of (3.13) are Padé approximants in inverse powers of $x$ as $x \to \infty$.

We consider the student's $t$-distribution. In this case, only the asymptotic error estimate for the approximations $\tilde{G}_n^{(1)}(x) = \tilde{G}_n^{(1,1)}(x)$ can be deduced, as Theorem 5.4 does not apply. Since $\omega(x) = x\, f_t(x)$, the approximations satisfy

$$\frac{\tilde{G}_n^{(1)}(x)}{x\, f_t(x)} - \frac{\int_x^{\infty} f_t(t)\,\mathrm{d}t}{x\, f_t(x)} = \mathcal{O}\left(\frac{1}{x^{n}}\right) \quad \text{as} \quad x \to \infty. \tag{5.36}$$

We consider the inverse Gaussian distribution. In this case, only the asymptotic error estimate for the approximations $\tilde{G}_n^{(1)}(x) = \tilde{G}_n^{(1,1)}(x)$ can be deduced, as Theorem 5.4 does not apply. Since $\omega(x) = f_i(x)$, the approximations satisfy

$$\frac{\tilde{G}_n^{(1)}(x)}{f_i(x)} - \frac{\int_x^\infty f_i(t)\,\mathrm{d}t}{f_i(x)} = \mathcal{O}\left(\frac{1}{x^{2n}}\right) \quad \text{as} \quad x \to \infty. \tag{5.37}$$

Finally, we consider Fisher's $F$ distribution. In this case, only the asymptotic error estimate for the approximations $\tilde{G}_n^{(1)}(x) = \tilde{G}_n^{(1,1)}(x)$ can be deduced, as Theorem 5.4 does not apply. Since $\omega(x) = x\, f_F(x)$, the approximations satisfy

$$\frac{\tilde{G}_n^{(1)}(x)}{x\, f_F(x)} - \frac{\int_x^\infty f_F(t)\,\mathrm{d}t}{x\, f_F(x)} = \mathcal{O}\left(\frac{1}{x^{n}}\right) \quad \text{as} \quad x \to \infty. \tag{5.38}$$

*Remark* 5.5.

1. The asymptotic expansion suggested by (2.11) and (2.12) is the asymptotic expansion in (5.11) with $\omega(x) = x^{\xi_0} f(x)$ and $\alpha = 1$. It is known that for $f(x)$ satisfying a first order linear homogeneous differential equation of the form (2.10), this asymptotic expansion is certainly true and valid. However, it may be that different values of $\alpha$ exist which may lead to different approximations $G_n^{(1,\alpha)}$. For example, it is known that the asymptotic expansion of the tail integral of the standard normal distribution is more concisely given in inverse powers of $x^2$ as in

$$\int_x^\infty \frac{e^{-t^2/2}}{\sqrt{2\,\pi}}\,\mathrm{d}t \sim \frac{e^{-x^2/2}}{x\sqrt{2\,\pi}} \sum_{i=0}^{\infty} \frac{(-2)^i (1/2)_i}{x^{2\,i}} \quad \text{as} \quad x \to \infty \tag{5.39}$$

than just in inverse powers of $x$. In cases like these, where such an $\alpha \neq 1$ can be legitimately found, the $G_n^{(1,\alpha)}$ transformation may be applied to obtain a higher accuracy-through-order condition than that obtained through $G_n^{(1)}$. However, in approximating the value of a tail integral as $x \to 0$, or even for a medium-valued $x$, a higher accuracy-through-order condition may not help, as there are other factors which govern convergence.
2. Theorem 5.4 does not apply to the approximations of the student's $t$, Fisher's $F$, and inverse Gaussian distributions. Other general forms for integrands such as $f(x) = x^\mu \frac{p(x)}{q(x)}$ or even $f(x) = x^\mu \frac{p(x)}{q(x)} \exp(x^\nu r(x))$, where $\mu \in \mathbb{R}$, $\nu \in \mathbb{Z}$, $p(x) \in \mathbb{R}[x]$, $q(x) \in \mathbb{R}[x]$, and $r(x) \in \mathbb{R}[x]$, may be able to provide further insight in the forms of rational approximants derived from the $G$ transformation; however, the introduction of a rational function dramatically increases the upper bounds to the degrees in the polynomials in the analogous Lemma 5.1 (by the quotient rule for derivatives) such that they are neither accurate nor helpful in characterizing the overall solution.

**6. Conclusion.** In this paper, we develop explicit approximations to tail probabilities of five probability distributions from the recursive algorithm for the $\tilde{G}_n^{(1)}(x)$ transformation. The implementation of the recursive algorithm for $\tilde{G}_n^{(1)}(x)$ is justified since all the probability distributions discussed in this work satisfy first order linear homogeneous differential equations of the form required to apply the $G$ transformation. This algorithm has numerous advantages over previous methods for calculating tail probabilities. First, it can be used to compute the integrands to any degree of accuracy that is required. Furthermore, due to the recursive nature of the coefficients involved in the SSF 1, the algorithm is relatively easy to compute using standard computer software. Finally, this method does not require any numerical integration.

## REFERENCES


[1] D. C. Joyce, *Survey of extrapolation processes in numerical analysis*, SIAM Rev., 13 (1971), pp. 435–490.

[2] C. Brezinski and M. Redivo-Zaglia, *Extrapolation Methods: Theory and Practice*, North-Holland, Amsterdam, 1991.

[3] C. Brezinski, M. Redivo-Zaglia, and E. J. Weniger, eds., *Special issue: Approximation and extrapolation of convergent and divergent sequences and series*, Appl. Numer. Math., 60 (2010), pp. 1183–1464.

[4] E. J. Weniger, *Nonlinear sequence transformations for the acceleration of convergence and the summation of divergent series*, Comput. Phys. Rep., 10 (1989), pp. 189–371.

[5] A. Sidi, *Practical Extrapolation Methods: Theory and Applications*, Cambridge University Press, Cambridge, UK, 2003.

[6] P. Rabinowitz, *Extrapolation methods in numerical integration*, Numer. Algorithms, 3 (1992), pp. 17–28.

[7] R. M. Slevinsky and H. Safouhi, *Numerical treatment of a twisted tail using extrapolation methods*, Numer. Algorithms, 48 (2008), pp. 301–316.

[8] R. M. Slevinsky and H. Safouhi, *The $S$ and $G$ transformations for computing three-center nuclear attraction integrals*, Int. J. Quantum Chem., 109 (2009), pp. 1741–1747.

[9] H. L. Gray and S. Wang, *A new method for approximating improper integrals*, SIAM J. Numer. Anal., 29 (1992), pp. 271–283.

[10] H. L. Gray and T. A. Atchison, *Nonlinear transformation related to the evaluation of improper integrals*. I, SIAM J. Numer. Anal., 4 (1967), pp. 363–371.

[11] H. L. Gray, T. A. Atchison, and G. V. McWilliams, *Higher order $G$-transformations*, SIAM J. Numer. Anal., 8 (1971), pp. 365–381.

[12] R. M. Slevinsky and H. Safouhi, *A recursive algorithm for the $G$ transformation and accurate computation of incomplete Bessel functions*, Appl. Numer. Math., 60 (2010), pp. 1411–1417.

[13] R. M. Slevinsky and H. Safouhi, *New formulae for higher order derivatives and applications*, J. Comput. Appl. Math., 233 (2009), pp. 405–419.

[14] I. J. Good, *Very small tails of the $t$ distribution and significance tests for clustering*, J. Statist. Comput. Simulation, 23 (1986), pp. 243–248.

[15] H. L. Gray and S. Wang, *A general method for approximating tail probabilities*, J. Amer. Statist. Assoc., 86 (1991), pp. 159–166.

[16] J. McWilliams, J. Balusek, and H. L. Gray, *Approximation of tail probabilities using the $G_n$-transform*, Commun. Statist. Simu. Comput., 33 (2004), pp. 1–17.

[17] S. Duret and H. Safouhi, *The $W$ algorithm and the $\bar{D}$ transformation for the numerical evaluation of three-center nuclear attraction integrals*, Int. J. Quantum Chem., 107 (2007), pp. 1060–1066.

[18] G. A. Baker, Jr., and P. Graves-Morris, *Padé Approximants*, 2nd ed., Cambridge University Press, New York, 1996.

[19] D. Wackerly, W. Mendenhall, and R. Scheaffer, *Mathematical Statistics with Applications*, 7th ed., Thompson Brooks/Cole, Belmont, CA, 2008.

[20] T. P. Ryan, *Modern Engineering Statistics*, Wiley Interscience, Hoboken, NJ, 2007.

[21] L. Knüsel and B. Bablok, *Computation of the noncentral gamma distribution*, SIAM J. Sci. Comput., 17 (1996), pp. 1224–1231.

[22] C. Brezinski and M. Redivo-Zaglia, *Extended procedures for extrapolation to the limit*, J. Comput. Appl. Math., 235 (2010), pp. 631–645.

[23] C. Brezinski and M. Redivo-Zaglia, *Extensions of Drummond's process for convergence acceleration*, J. Comput. Appl. Math., 60 (2010), pp. 1231–1241.

[24] D. Levin, *Developement of non-linear transformations for improving convergence of sequences*, Int. J. Comput. Math., B3 (1973), pp. 371–388.

[25] J. Jin, *FORTRAN Routines for Computation of Special Functions*, http://jin.ece.uiuc.edu/routines/routines.html.