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INVARIANT NETS FOR AMENABLE GROUPS AND HYPERGROUPS

by

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Abstract

Let H be a hypergroup with left Haar measure. The amenability of H can be characterized by the existence of nets of positive, norm one functions in $L^1(H)$ which tend to left invariance in any of several ways. In this thesis we present a characterization of the amenability of H using configuration equations. Extending work of Rosenblatt and Willis we construct, for a certain class of hypergroups, nets in $L^1(H)$ which tend to left invariance weakly, but not in norm.

We define the semidirect product of H with a locally compact group. We show that the semidirect product of an amenable hypergroup and an amenable locally compact group is an amenable hypergroup and show how to construct Reiter nets for this semidirect product.

These results are generalized to Lau algebras providing a new characterization of left amenability of a Lau algebra and a notion of a semidirect product of a Lau algebra with a locally compact group. The semidirect product of a left amenable Lau algebra with an amenable locally compact group is shown to be a left amenable Lau algebra.

Some results towards the existence of a left Haar measure for amenable hypergroups are proven.

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Chapter 1

Introduction and Background

1.1 Introduction

A hypergroup is a locally compact space with a convolution product mapping each pair of points to a probability measure with compact support. Hypergroups are a generalization of locally compact groups wherein the convolution of two points corresponds to the point evaluation measure at their product. The abstract study of hypergroups began in the 1970s with Dunkl [8], Jewett [23], and Spector [52]. A detailed treatment can be found in the text of Bloom and Heyer [4]. Numerous authors continue to study various aspects of hypergroups including amenability properties [28–31, 51], Fourier transforms and spaces [39, 55], other function spaces [12, 13, 27, 56], and others [9, 15, 48–50, 54]. Within the literature there is some variation in the precise definition of a hypergroup. In this thesis we will use the definition of Jewett.

All locally compact groups are hypergroups. A hypergroup is a locally compact group precisely when the convolution product of every two point measures is again a point measure. Many hypergroups arise from semidirect products of locally compact groups. Consider a locally compact group G. If N is a compact, normal subgroup of G then the collection of (equivalently: left, right, or two-sided) cosets G/N is a group. If G/N is isomorphic to a subgroup of G with trivial intersection with N then G is isomorphic to the semidirect product $N \rtimes G/N$. If K is a compact, non-normal subgroup of G then the two-sided cosets no longer form a group, but instead inherit a hypergroup structure from G. That is, the double coset space $G//K = \{KgK : g \in G\}$ with the convolution inherited from the multiplication of G forms a hypergroup. Indeed, for any compact group $\{\tau_k : k \in K\}$ of automorphisms (inner or otherwise) of G we may be use the multiplication of G to induce a hypergroup structure on the orbit space $G^K = \{\{\tau_k(g) : k \in K\} : g \in G\}$. We see that these two ways of constructing hypergroups from groups are, in a sense, the same by observing that $G^K \equiv (G \rtimes K)//(\{e_G\} \times K)$ where $G \rtimes K$ is the semidirect product of K acting on G [23, 8.3B].

Let H be a hypergroup. Then the convolution extends to $M(H)$, the finite Borel measures on H, forming a Banach algebra. By considering the dual of this convolution, we define left translation on $C_C(H)$, the space of continuous and compactly supported functions, and on $C(H)$, the space of continuous bounded functions on H. A non-zero, positive, left invariant linear functional (possibly unbounded) on the former space corresponds to a left Haar measure, and a positive, norm one, left invariant linear functional on the latter space is a left invariant mean. The existence of a left Haar measure for every hypergroup remains an open question; however, it is known that such a measure exists if the hypergroup is commutative [52], compact [23], or discrete [23]. The existence of a left invariant mean on $C(H)$ characterizes the class of amenable hypergroups, studied in [51], which contains precisely those locally compact groups which are amenable as groups. If H admits a left Haar measure, then Skantharajah showed that the function spaces of $UCB_r(H)$, $UCB(H)$, $C(H)$, and $L^\infty(H)$ all either admit a left invariant mean (if H is amenable), or all do not. Using the same approach as Namioka [40] it is apparent that Day's famous result characterizing amenability of locally compact groups holds for hypergroups as well. Amenability can be characterized by the existence of a net of positive norm one elements of $L^1(H)$ which tend to left invariance with respect to either the weak or norm topologies. Also, Skantharajah and Lasser have shown that amenability of H can be characterized by any of several Reiter conditions [31, 51]. Particularly, that amenability is equivalent to the existence of a net of positive norm one functions in $L^1(H)$ tending to left invariance uniformly on compact sets. Nets satisfying these, or similar, properties are also interesting for semigroup algebras. They play an important role in the approximation of fixed points for semigroups of non-expansive mappings (see [33, 34]); in particular, Lau and Zhang have constructed such nets for the bicyclic semigroups in [36]. Characterizing those means which are limits of these nets in certain semigroup algebras is the focus of recent work of Hindman and Strauss [17].

In addition to these characterizations of amenability, there are many related topics of interest. In [51], Skantharajah points out that Johnson's theorem equating amenability of a locally compact group and algebra amenability of the group algebra does not hold for general hypergroups. In [28], Lasser investigates the amenability and weak amenability of a certain class of hypergroups.

A related notion, termed left amenability of F algebras by Lau [32] (often called Lau algebras), applied to the group algebra or the measure algebra does extend to hypergroups. A more general notion than left amenability, termed ϕ-amenability or α-amenability, has been studied for general algebras by Kaniuth, Lau and Pym [24] and for hypergroups in particular by Lasser [29] and Azimifard [2, 3]. Monfared [38] introduced the related concept of character amenability of a Banach algebra (see also [18]).

A net of positive norm one functions in $L^1(H)$ which tends to left invariance in norm must also tend to left invariance weakly, but the converse is not generally true. This raises the question of when nets exist which tend to left invariance weakly, but not in norm and how such nets might be constructed? Chapter 2 of the this thesis investigates these questions by building upon results of Rosenblatt and Willis. In [47] they constructed such nets for infinite locally compact groups. In so doing, they introduced the notion of configurations by considering colourings of the Cayley graph which encapsulates a finite amount of information from the multiplication table of G. For a hypergroup H we consider a finite partition and a finite subset of H. We define a system of linear equations associated to this partition and subset. A solution to these configuration equations corresponds to the existence of a mean which is left invariant in a restricted sense associated to the given partition and subset. We show that H is amenable if and only if for every choice of partition and subset, the configuration equations have a positive, normalized, inequality preserving solution. This generalizes proposition 3.2 of [47] from locally compact groups to all hypergroups. Due to the properties of translation being fundamentally different between groups and general hypergroups, the method of proof for the generalization is significantly different from that given by Rosenblatt and Willis and so the generalized result does not lend itself to constructing the nets of interest. It is interesting in other respects, however, since it provides a new characterization of amenability of hypergroups, and indeed can be extended to characterize the existence of left invariant means on other function spaces (Theorem 5.1.3). We conclude this section by using the result of Rosenblatt and Willis to construct, for a large class of double coset spaces of locally compact amenable groups, nets of positive, norm one L^1 functions which tend to left invariance in a weak sense, but not in norm.

In Chapter 3 of the thesis, we construct Reiter nets for semidirect products of locally compact groups. We then introduce the semidirect product of a hypergroup with a locally compact group of automorphisms of the hypergroup and show that the results also hold in this case. In particular, we show that the semidirect product of an amenable hypergroup with an amenable locally compact group is itself an amenable hypergroup. Additionally we give some examples of hypergroups which arise as semidirect products in this way.

In Chapter 4, we consider the more general class of Lau algebras (called F algebras in [32]) which contain the measure algebras of groups, hypergroups and semigroups. Left amenability of a Lau algebra has some known characterizations involving the existence of nets which tend to left invariance [32, 35]. We show that the constructions we have presented thus far – those of configurations and of semidirect products – have analogous concepts for Lau algebras. This construction of a semidirect product is somewhat similar to the θ-Lau algebra product of [32] and [37] and also to the crossed product of C^*-algebras with locally compact groups [43]. Using slight modifications of the proofs of earlier chapters, we provide a new characterization of left amenability of a Lau algebra. We show that the semidirect product of a Lau algebra and a locally compact group is again a Lau algebra. This is motivated by a result of Pier [43] that the crossed product of a C^* algebra and a locally compact group forms a C^* algebra. We further show that if the Lau algebra is left amenable and the group is amenable then the semidirect product is left amenable.

In Chapter 5, we first show that the approach of configurations can be applied to subalgebras of L^∞ which gives a characterization of amenability of a hypergroup without assuming the existence of a left Haar measure. We conclude by showing that the theory of locally compact quantum groups does not apply to hypergroups except in the case of locally compact groups.

In Chapter 6 we present two fixed point properties of hypergroups. The first is analogous to Rickert's fixed point theorem for semigroups. It equates the existence of a left invariant mean on the space of weakly right uniformly continuous functions to the existence of a fixed point for any action of the hypergroup. The second is a version of Ryll-Nardzewski's fixed point theorem applied to hypergroup actions. Using the former result, a certain class of amenable hypergroups are shown to have a left Haar measure.

1.2 Notation

Throughout this thesis, the following notation is used:

X	Topological space
$M(X)$	The vector space of complex bounded Radon measures on X
$M_C(X)$	The bounded Radon measures with compact support

δ_x	The point evaluation measure at $x \in X$
$C(X)$	The Banach space of continuous bounded functions on X
$C_C(X)$	The functions in $C(X)$ with compact support
$\mathrm{supp}(\mu)$	The support of the measure μ
χ_S	The characteristic function of the set S
E	Vector space (possibly ordered and/or normed)
E^+	The positive elements of E
E_1	The elements of E with norm 1

1.3 Definitions

Definition 1.3.1: A hypergroup, H, is a non-empty locally compact Hausdorff topological space which satisfies the following conditions:

1. There is a binary operation $*$, called convolution, on the vector space of bounded Radon measures turning it into an algebra.

2. For $x, y \in H$, the convolution of the two point measures is a probability measure, and $\mathrm{supp}(\delta_x * \delta_y)$ is compact.

3. The map $H \times H \ni (x, y) \mapsto \delta_x * \delta_y \in M_{1,C}(H)$ is continuous.

4. The map $H \times H \ni (x, y) \mapsto \mathrm{supp}(\delta_x * \delta_y)$ is continuous with respect to the Michael topology on the space of compact subsets of H.

5. There is a unique element $e \in H$ such that for every $x \in H$, $\delta_x * \delta_e = \delta_e * \delta_x = \delta_x$.

6. There exists a homeomorphism $\check{}: H \to H$ such that for all $x \in H$, $\check{\check{x}} = x$, which can be extended to $M(H)$ via $\check{\mu}(A) = \mu(\{x \in H : \check{x} \in A\})$, and

such that $(\mu * \nu)\check{} = \check{\nu} * \check{\mu}$.

7. For $x, y \in H$, $e \in \mathrm{supp}(\delta_x * \delta_y)$ if and only if $y = \check{x}$.

Remark 1.3.2: The Michael topology on $\mathfrak{C}(X)$ the space of compact subsets of X can be characterized in the following way; a net $C_\alpha \subset \mathfrak{C}(X)$ converges to a compact C if for every open set $U \subset X$ if $C \subset U$ then eventually $C_\alpha \subset U$ and if $U \cap C \neq \emptyset$ then eventually $C_\alpha \cap U \neq \emptyset$.

Theorem 1.3.3 (Jewett, [23]): *Let G be a hypergroup. G is a locally compact group if and only if the convolution product of every two elements $x, y \in G$ is a point measure.*

Definition 1.3.4: Let f be a Borel function on H and $\mu \in M(H)$. We define the left translation $\mu * f$ by $\mu * f(x) = \check{\mu} * \delta_x(f)$.

We say that H is *amenable* if there exists a positive linear functional of norm 1 on $C(H)$ which is invariant under left translation.

Definition 1.3.5: A left Haar measure for H is a non-zero regular Borel measure (with values in $[0, \infty]$), λ which is left-invariant in the sense that for any $f \in C_C(H)$, we have that $\lambda(\delta_x * f) = \lambda(f)$ for all $x \in H$.

Remark 1.3.6: It remains an open question whether every hypergroup admits a left Haar measure. If H does admit a left Haar measure λ, however, it is unique up to a scalar multiple [23]. For hypergroups with a left Haar measures we define the standard $L^p(H)$ function spaces.

Example 1.3.7: Let $H = \{e, a\}$ with the discrete topology. Then for any $0 < \gamma \leq 1$, we can make H into a hypergroup by defining the convolution via:

$$\delta_a * \delta_a = \gamma\delta_e + (1 - \gamma)\delta_a$$

(The convolution products involving e are forced).

As with every compact hypergroup, the normalized left Haar measure is a left invariant mean. In this case, it is

$$\lambda_H = \frac{\gamma}{\gamma+1}\delta_e + \frac{1}{\gamma+1}\delta_a.$$

Example 1.3.8: Let H be a commutative hypergroup, then $(L^1(H), L^\infty(H))$ form a commutative Lau algebra. By the Markov-Kakutani fixed point theorem, this Lau algebra is left amenable so there is a left invariant mean on $L^\infty(H)$.

Definition 1.3.9: We say that a continuous function $f \in C(H)$ is right uniformly continuous [weakly right uniformly continuous] if the map

$$H \ni x \mapsto \delta_x * f$$

is continuous in norm [weakly]. We denote the collection of right uniformly continuous functions [weakly right uniformly continuous] on H by $UCB_r(H)$ [$WUCB_r(H)$]. We similarly define the left versions of these spaces. Finally, we call $UCB(H) = UCB_r(H) \cap UCB_l(H)$ [$WUCB(H) = WUCB_r(H) \cap WUCB_l(H)$] be the space of uniformly continuous functions [weakly uniformly continuous functions] on H.

Remark 1.3.10: Skantharajah [51] showed that for hypergroups with left Haar measure, $UCB_r(H) = L^1(H) * L^\infty(H)$.

For a hypergroup H, with left Haar measure, we have the following characterizations of amenability.

This result is originally due to Reiter [44] for groups, and Skantharajah for hypergroups [51]

Theorem 1.3.11: *Let* X *be one of* $UCB(H), UCB_r(H), WUCB_r(H), C(H)$, *or* $L^\infty(H)$. *Then* H *is amenable if and only if there exists a left invariant mean on* X.

1.4 Invariant nets

Let H be a hypergroup with left Haar measure, λ. Then H is amenable if there is a positive, norm one linear functional on $L^\infty(H)$ which is left invariant. The positive unit sphere of $L^1(H)$ is weak-* dense in the positive unit sphere of $L^\infty(H)^*$ so the existence of a mean is equivalent to the existence of a net of positive, norm one functions $(f_\alpha)_\alpha \subset L^1(H)$ which, for every $\phi \in L^\infty(H)$ and $x \in H$

$$\langle \phi, f_\alpha - \delta_x * f_\alpha \rangle \to 0. \tag{1.4.1}$$

Such nets are said to *tend to left invariance weakly*.

The result below is originally due to Day when H is a group [7]. Namioka [40] showed it using a novel method for semigroups, and his method holds for hypergroups as well.

Theorem 1.4.1 (Day's Theorem): H *is amenable if and only if there is a net* $(f_\alpha)_\alpha \subset L^1(H)$ *such that for every* $x \in H$

$$\|f_\alpha - \delta_x * f_\alpha\| \to 0.$$

We say these nets tend to left invariance in norm.

The following result is originally due to Reiter in the group case and Skantharajah extended it to the hypergroup case. There are related results which hold for groups but not general hypergroups (see [31])

Theorem 1.4.2 (Reiter): *H is amenable if and only if it has Reiter's property (P1), that is, for any $\varepsilon > 0$ and compact $C \subset H$, there exists $f \in L^1(H)_1^+$ such that*

$$\|f - \delta_x * f\|_1 < \varepsilon \textit{ for all } x \in C.$$

By considering an ordered set consisting of values of ε and choices of compact C we see that the above result can be rephrased as the existence of a certain type of net of functions in $L^1(H)_1^+$. The ordering on $\mathfrak{C} \times \mathbb{R}^+$ is $(C, \varepsilon) \preceq (C', \varepsilon')$ if $C \subseteq C'$ and $\varepsilon \geq \varepsilon'$.

Definition 1.4.3: A net $(f_\alpha)_\alpha \subset L^1(H)_1^+$ is a Reiter net if for any compact $K \subset H$ and $\varepsilon > 0$ there exists α_0 such that for $\alpha \geq \alpha_0$

$$\|f_\alpha - \delta_x * f_\alpha\| < \varepsilon \quad \forall x \in K.$$

Remark 1.4.4: Hence H is amenable if and only if there is a Reiter net for H.
If H is σ-compact then by choosing a sequence of compact sets tending to all of H and $\varepsilon = 1/n$ we see that H is amenable if and only if there is Reiter sequence for H.

Convergence to left invariance weakly can be described in another way. By rearranging the term in 1.4.1, f_α tends to left invariance weakly if for every $\phi \in L^\infty(H)$ and $x \in H$

$$\langle \phi - \delta_{\check{x}} * \phi, f_\alpha \rangle \to 0.$$

We consider the subspace E of $L^\infty(H)$ which is generated by functions of the form $\phi - \delta_{\check{x}} * \phi$ for $\phi \in L^\infty(H)$ and $x \in H$. Then if H is not compact, E separates the elements of $L^1(H)$. (If H is compact, then E does not separate the constant functions)

Lemma 1.4.5: *Let H be a non-compact hypergroup with left Haar measure. Suppose that $(f_\alpha)_\alpha \subset L^1(H)_1^+$ is a net of positive norm one functions which satisfy*

$$\langle f_\alpha, \phi - \delta_x * \phi \rangle \to 0$$

for all $x \in H$ and $\phi \in UCB_r(H)$. Then

$$\langle f_\alpha, \phi - \mu * \phi \rangle \to 0$$

for all $\mu \in M_C(H)_1^+$ and $\phi \in UCB_r(H)$.

Proof. Let E be the vector subspace of $UCB_r(H)$ generated by functions of the form $\phi - \delta_x * \phi$ for $x \in H$ and $\phi \in UCB_r(H)$. Then E separates the functions of $L^1(H)$ so $L^1(H)$ and E form a dual pair. The net $(f_\alpha)_\alpha$ tends to zero in the $\sigma(L^1(H), E)$ topology and for each α, $\|f_\alpha\| = 1$. By [46][VI.1.2.3] the topologies $\sigma(L^1(H), E)$ and $\sigma(L^1(H), \overline{E}^{\|\cdot\|})$ coincide on closed balls in $L^1(H)$. Since $\phi \in UCB_r(H)$, and μ is the limit of affine combinations of point measures, it follows that $\phi - \mu * \phi$ is in the norm closure of E. □

The above lemma demonstrates that when applied to uniformly continuous functions, weak convergence to left invariance is equivalent to the slightly stronger convergence. However, this does not hold for functions which are not uniformly continuous. We shall see later that this lemma will allow us to construct, for a large class of hypergroups, nets which do not converge to left invariance in norm but do converge to 0 in the $\sigma(L^1(H), E)$ topology.

Remark 1.4.6: This result is intriguing when compared to the well-known result for locally compact groups that any left invariant mean on the uniformly continuous functions is automatically a topological left invariant mean (invariant under convolution by functions in $L^1(G)_1^+$). The proof given above could be modified to give a potentially new proof of that result.

Chapter 2

Configurations[1]

2.1 Invariant Nets from Configuration Equations

In [47], Rosenblatt and Willis introduced the notion of a configuration and the configuration equations corresponding to a locally compact group for the purpose of investigating certain properties of groups. In particular, they used configuration equations to provide a characterization of amenability. Using this characterization, they constructed a net which tends to left invariance weakly, but not in norm for any infinite, amenable, locally compact group. Configurations have also been used to study other group properties in [1].

In the group setting, we begin with a finite partition, or colouring of G, a locally compact group into m measurable subsets, $\{E_1, \ldots, E_m\}$ and a selection of n group elements $\{g_1, \ldots, g_n\}$. A configuration $C = (C_0, C_1, \ldots, C_n)$ is an

[1]A version of this chapter has been submitted for publication.

ordered choice of $n+1$ (not necessarily distinct) colours (E_is). C is realized by $(x_0, x_1, \ldots, x_n) \in G^{n+1}$ if $x_j \in C_j$ for $j = 0, \ldots, n$ and $x_j = g_j x_0$ for $j = 1, \ldots, n$.

In [47] the notation $x_j(C)$ is used to denote the points which occur in the jth element of a realization of C.

This approach cannot be immediately extended to hypergroups primarily because in a hypergroup, the product of two points need not be another point, so the $g_j * x_0$ may not be contained in a single part of the partition. We define $\xi_0(C)$, a measurable function on H which in the group case is just the characteristic function on $x_0(C)$. With this approach we are able to give a characterization of amenability for hypergroups which is inspired by the result of Rosenblatt and Willis for locally compact groups.

Definition 2.1.1: Let H be a hypergroup with left Haar measure λ.

Let $\mathcal{E} = \{E_1, \ldots, E_m\}$ be a finite measurable partition of H and choose an n-tuple of elements of H, $\mathfrak{h} = \{h_1, \ldots, h_n\}$. A configuration is an $(n+1)$-tuple $C = (C_0, C_1, \ldots, C_n)$ where each $C_j \in \{1, \ldots, m\}$.

For a fixed configuration, C, we define $\xi_0(C)$ to be the real-valued function on H given by:

$$\xi_0(C)(x) := \prod_{j=0}^{n} \delta_{h_j} * \delta_x(E_{C_j})$$

where $h_0 = e$. In particular, if $x \in E_{C_0}$ and if for each $j \in \{1, \ldots, n\}$, $\text{supp}(\delta_{h_j} * \delta_x) \subset E_{C_j}$ then $\xi_0(C)(x) = 1$.

An alternate expression for $\xi_0(C)$ is:

$$\xi_0(C) = \prod_{j=0}^{n} \delta_{\check{h_j}} * \chi_{E_{C_j}}.$$

From this we see that $\xi_0(C)$ is the pointwise product of finitely many non-negative measurable functions bounded by 1 and so is itself in $L^\infty(H)^+$ and is norm bounded by 1.

For $f \in L^1(H)$, and a configuration, C, let f_C denote the integral

$$f_C := \int_H \xi_0(C)(t) f(t) d\lambda(t).$$

We denote by $\mathrm{Con}(\mathcal{E}, \mathfrak{h})$ the family of configurations associated to that particular choice of $\mathcal{E}$ and $\mathfrak{h}$.

Lemma 2.1.2: *Let H be a hypergroup with left Haar measure λ. Let $\mathcal{E}$ and $\mathfrak{h}$ be as above. For $f \in L^1(H)$, $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$ we have that*

$$\int_{E_i} f \, d\lambda = \sum_{\substack{C \in \mathrm{Con}(\mathcal{E}, \mathfrak{h}) \\ C_0 = i}} f_C$$

and

$$\int_{E_i} \delta_{h_j} * f \, d\lambda = \sum_{\substack{C \in \mathrm{Con}(\mathcal{E}, \mathfrak{h}) \\ C_j = i}} f_C.$$

Proof. First, notice that for $x \in H$

$$\begin{aligned}
\sum_{\substack{C \in \mathrm{Con}(\mathcal{E}, \mathfrak{h}) \\ C_0 = i}} \xi_0(C)(x) &= \sum_{\substack{C \in \mathrm{Con}(\mathcal{E}, \mathfrak{h}) \\ C_0 = i}} \left(\prod_{l=0}^{n} \delta_{h_l} * \delta_x(E_{C_l}) \right) \\
&= \chi_{E_i}(x) \prod_{l=1}^{n} \left(\sum_{k=1}^{m} \delta_{h_l} * \delta_x(E_k) \right) \\
&= \chi_{E_i}(x) \prod_{l=1}^{n} \delta_{h_l} * \delta_x(H) \\
&= \chi_{E_i}(x).
\end{aligned}$$

So, by integrating f multiplied by the above function, we get:

$$\begin{aligned}\int_{E_i} f\,d\lambda &= \int_H \chi_{E_i}(x)f(x)d\lambda(x)\\ &= \int_H \sum_{\substack{C\in\mathrm{Con}(\mathcal{E},\mathfrak{h})\\ C_0=i}} \xi_0(C)(x)f(x)d\lambda(x).\end{aligned}$$

By swapping the integration and summation, we get:

$$\begin{aligned}\int_{E_i} f\,d\lambda &= \sum_{\substack{C\in\mathrm{Con}(\mathcal{E},\mathfrak{h})\\ C_0=i}} \int_H \xi_0(C)(x)f(x)d\lambda(x)\\ &= \sum_{\substack{C\in\mathrm{Con}(\mathcal{E},\mathfrak{h})\\ C_0=i}} f_C.\end{aligned}$$

For the second equality, we again need to rearrange the sum of products to be the product of a sum. Indeed for $x \in H$

$$\begin{aligned}\sum_{\substack{C\in\mathrm{Con}(\mathcal{E},\mathfrak{h})\\ C_j=i}} \xi_0(C)(x) &= \sum_{\substack{C\in\mathrm{Con}(\mathcal{E},\mathfrak{h})\\ C_j=i}} \prod_{l=0}^{n} \delta_{h_l} * \delta_x(E_{C_l})\\ &= \delta_{h_j} * \delta_x(E_i) \prod_{\substack{l=0\\ l\neq j}}^{n} \sum_{k=1}^{m} \delta_{h_l} * \delta_x(E_k)\\ &= \chi_{E_i}(h_j * x). \qquad (2.1.1)\end{aligned}$$

It follows that:

$$\begin{aligned}\int_{E_i} \delta_{h_j} * f d\lambda &= \int_H \chi_{E_i}(h_j * t)f(t)d\lambda(t)\\ &= \int_H \sum_{\substack{C\in\mathrm{Con}(\mathcal{E},\mathfrak{h})\\ C_j=i}} \xi_0(C)(t)f(t)d\lambda(t).\end{aligned}$$

Again, by swapping the order of the integration and the summation, we get:

$$\int_{E_i} \delta_{h_j} * f d\lambda = \sum_{\substack{C\in\mathrm{Con}(\mathcal{E},\mathfrak{h})\\ C_j=i}} \int_H \xi_0(C)(t) f(t) d\lambda(t)$$
$$= \sum_{\substack{C\in\mathrm{Con}(\mathcal{E},\mathfrak{h})\\ C_j=i}} f_C.$$

□

Remark 2.1.3: We see from the above that summing over ALL configurations gives

$$\sum_{C\in\mathrm{Con}(\mathcal{E},\mathfrak{h})} \xi_0(C)(x) = 1 \ \ \forall x \in H.$$

Corollary 2.1.4: *Given $f \in L^1(H)$, we have for all $i = 1, \ldots, m$, $j = 1, \ldots, n$,*

$$\langle f - \delta_{h_j} * f, \chi_{E_i} \rangle = 0$$

if and only if for all $i = 1, \ldots, m$, $j = 1, \ldots, n$,

$$\sum_{\substack{C\in\mathrm{Con}(\mathcal{E},\mathfrak{h})\\ C_0=i}} f_C = \sum_{\substack{C\in\mathrm{Con}(\mathcal{E},\mathfrak{h})\\ C_j=i}} f_C.$$

Rather than start with some $f \in L^1(H)$ that generates the values f_C which satisfy the equations in the above corollary, we can consider those equations and solutions to them.

Definition 2.1.5: Fix $\mathcal{E}$ and $\mathfrak{h}$ as before. Let $\{z_C : C \in \mathrm{Con}(\mathcal{E},\mathfrak{h})\}$ be variables corresponding to the m^{n+1} configurations. Consider the $m \times n$ configuration equations

$$\sum_{\substack{C\in \mathrm{Con}(\mathcal{E},\mathfrak{h})\\ C_0=i}} z_C = \sum_{\substack{C\in \mathrm{Con}(\mathcal{E},\mathfrak{h})\\ C_j=i}} z_C$$

for each $i = 1, \ldots m$ and $j = 1, \ldots, n$.

We say that a solution to these configuration equations is

- *positive* if, for each $C \in \mathrm{Con}(\mathcal{E}, \mathfrak{h})$, $z_C \geq 0$;
- *normalized* if $\sum\limits_{C\in \mathrm{Con}(\mathcal{E},\mathfrak{h})} z_C = 1$; and
- *inequality preserving* if for every choice of m^{n+1} real numbers $\{a_C : C \in \mathrm{Con}(\mathcal{E}, \mathfrak{h})\}$

$$0 \leq \sum_{C\in \mathrm{Con}(\mathcal{E},\mathfrak{h})} a_C \xi_0(C) \text{ a.e. } \Rightarrow 0 \leq \sum_{C\in \mathrm{Con}(\mathcal{E},\mathfrak{h})} a_C z_C.$$

  i.e. any inequality which is satisfied by a linear combination of the functions $\{\xi_0(C) : C \in \mathrm{Con}(\mathcal{E}, \mathfrak{h})\}$ is also satisfied by the same linear combination of the values of the variables $\{z_C : C \in \mathrm{Con}(\mathcal{E}, \mathfrak{h})\}$.

Clearly, if there exists some $f \in L^1(H)_1^+$ for which $\langle f - \delta_{h_j} * f, \chi_{E_i}\rangle = 0$ for all i, j then $z_C = f_C$ is a positive, normalized, inequality preserving solution to these configuration equations. We will show in theorem 2.1.7 that H is amenable precisely when such solutions to the configuration equations exist for all choices of $m, n, \mathcal{E}$ and $\mathfrak{h}$.

Lemma 2.1.6: *Let (X, μ) be a measure space. Let $(f_\alpha)_{\alpha\in\Lambda}$ be a finite family of non-negative functions in $L^\infty(X, \mu)$ such that $\sum_\alpha f_\alpha = \chi_X$.*

Suppose that there are associated $(c_\alpha)_{\alpha\in\Lambda}$ non-negative real numbers such that *for any choice of real numbers* $(a_\alpha)_{\alpha\in\Lambda}$ *if*

$$0 \leq \sum_\alpha a_\alpha f_\alpha \text{ almost everywhere}$$

the associated inequality

$$0 \leq \sum_\alpha a_\alpha c_\alpha \tag{2.1.2}$$

also holds.

Then there exists $\hat{\Gamma} \in (L^\infty(X,\mu)^*)^+$ *such that* $\hat{\Gamma}(f_\alpha) = c_\alpha$ *for all* α.

Furthermore, $\|\hat{\Gamma}\| = \sum c_\alpha$.

Proof. Let $Y = \text{span}\{f_\alpha : \alpha \in \Lambda\}$. Then Y is a finite dimensional (hence closed) subspace of $L^\infty(X,\mu)$. Indeed, there is some subset Λ_0 of Λ such that $\{f_\alpha : \alpha \in \Lambda_0\}$ is a basis for Y.

Define $\Gamma : Y \to \mathbb{R}$ by letting $\Gamma(f_\alpha) = c_\alpha$ for $\alpha \in \Lambda_0$ and extending it linearly to all of Y.

Then for every $\alpha' \in \Lambda\backslash\Lambda_0$ there exist some real numbers $(a_\alpha)_{\alpha\in\Lambda_0}$ such that

$$f_{\alpha'} = \sum_{\alpha\in\Lambda_0} a_\alpha f_\alpha$$

So, by (2.1.2), the corresponding equality holds:

$$c_{\alpha'} = \sum_{\alpha\in\Lambda_0} a_\alpha c_\alpha$$

and we see that $\Gamma(f_\alpha) = c_\alpha$ for all $\alpha \in \Lambda$.

Define $\rho : L^\infty(X,\mu) \to \mathbb{R}_{\geq 0}$ via

$$\rho(f) := \inf\{\sum_{\alpha\in\Lambda} a_\alpha c_\alpha : a_\alpha \in \mathbb{R}_{\geq 0}, \sum_{\alpha\in\Lambda} a_\alpha f_\alpha \geq |f|\}.$$

Claim: ρ is a well-defined semi-norm on $L^\infty(X,\mu)$.

Since $\sum_{\alpha\in\Lambda} f_\alpha = \chi_X$, for any $f \in L^\infty(X,\mu)$, $\rho(f) \leq \sum_{\alpha\in\Lambda} c_\alpha \|f\|_\infty$.

Let $f, g \in L^\infty(X,\mu), a \in \mathbb{R}$.

$$\begin{aligned}
\rho(f) + \rho(g) = \inf&\left\{\sum_{\alpha\in\Lambda} a_\alpha c_\alpha : a_\alpha \in \mathbb{R}_{\geq 0}, \sum_{\alpha\in\Lambda} a_\alpha f_\alpha \geq |f|\right\} \\
&+ \inf\left\{\sum_{\alpha\in\Lambda} b_\alpha c_\alpha : b_\alpha \in \mathbb{R}_{\geq 0}, \sum_{\alpha\in\Lambda} b_\alpha f_\alpha \geq |g|\right\} \\
&\geq \inf\left\{\sum_{\alpha\in\Lambda}(a_\alpha + b_\alpha) c_\alpha : a_\alpha, b_\alpha \in \mathbb{R}_{\geq 0}, \sum_{\alpha\in\Lambda}(a_\alpha + b_\alpha) f_\alpha \geq |f| + |g|\right\} \\
&\geq \inf\left\{\sum_{\alpha\in\Lambda}(a_\alpha) c_\alpha : a_\alpha \in \mathbb{R}_{\geq 0}, \sum_{\alpha\in\Lambda} a_\alpha f_\alpha \geq |f+g|\right\} \\
&= \rho(f+g)
\end{aligned}$$

If a is zero then we clearly have $\rho(0) = 0$, otherwise, if a is non-zero then:

$$\begin{aligned}
\rho(af) &= \inf\left\{\sum_{\alpha\in\Lambda} a_\alpha c_\alpha : a_\alpha \in \mathbb{R}_{\geq 0}, \sum_{\alpha\in\Lambda} a_\alpha f_\alpha \geq |af|\right\} \\
&= |a| \inf\left\{\sum_{\alpha\in\Lambda} \frac{a_\alpha}{|a|} c_\alpha : a_\alpha \in \mathbb{R}_{\geq 0}, \sum_{\alpha\in\Lambda} \frac{a_\alpha}{|a|} f_\alpha \geq |f|\right\} \\
&= |a|\rho(f).
\end{aligned}$$

Hence ρ is a semi-norm.

Claim: For all $f \in Y$, $\Gamma(f) \leq \rho(f)$. Suppose $f \in Y$ and there are real numbers s_α for which $f = \sum_{\alpha \in \Lambda_0} s_\alpha f_\alpha$. Hence $\Gamma(f) = \sum_{\alpha \in \Lambda_0} s_\alpha c_\alpha$.

Suppose that for some $(a_\alpha)_\alpha \in \mathbb{R}_{\geq 0}$ we have $|f| \leq \sum_{\alpha \in \Lambda} a_\alpha f_\alpha$, then:

$$\begin{aligned} f &\leq \sum_{\alpha \in \Lambda} a_\alpha f_\alpha \\ \sum_{\alpha \in \Lambda_0} s_\alpha f_\alpha &\leq \sum_{\alpha \in \Lambda} a_\alpha f_\alpha \\ \sum_{\alpha \in \Lambda_0} s_\alpha c_\alpha &\leq \sum_{\alpha \in \Lambda} a_\alpha c_\alpha \\ \Gamma(f) &\leq \sum_{\alpha \in \Lambda} a_\alpha c_\alpha \end{aligned}$$

so by taking the infimum, $\Gamma(f) \leq \rho(f)$.

By the Hahn-Banach Theorem, there exists an extension, $\hat{\Gamma}$ to all of $L^\infty(X, \mu)$ which is bounded by ρ.

Claim: $\hat{\Gamma}$ is positive.

Suppose for contradiction that there exists some $f \in L^\infty(X, \mu)^+$ such that $\hat{\Gamma}(f) < 0$.

Let $a_\alpha \in \mathbb{R}_{\geq 0}$ such that $\sum_\alpha a_\alpha f_\alpha \geq f$. Then $\sum_\alpha a_\alpha f_\alpha \geq \sum_\alpha a_\alpha f_\alpha - f \geq 0$. So

$$\begin{aligned} \hat{\Gamma}\left(\sum_\alpha a_\alpha f_\alpha - f\right) &= \hat{\Gamma}\left(\sum_\alpha a_\alpha f_\alpha\right) - \hat{\Gamma}(f) \\ &= \sum_\alpha a_\alpha c_\alpha - \hat{\Gamma}(f) \\ &> \sum_\alpha a_\alpha c_\alpha. \end{aligned}$$

But,

$$\hat{\Gamma}\left(\sum_{\alpha} a_\alpha f_\alpha - f\right) \leq \rho\left(\sum_{\alpha} a_\alpha f_\alpha - f\right)$$
$$\leq \sum_{\alpha} a_\alpha c_\alpha$$

which is a contradiction, so $\hat{\Gamma}$ is positive.

Since $\sum_\alpha f_\alpha = \chi_X$, it follows that $\hat{\Gamma}$ has norm $\sum_\alpha c_\alpha$. □

We are now ready to prove our first main result.

Theorem 2.1.7: *Let H be a hypergroup with left Haar measure λ. H is amenable if and only if for all choices of $m, n, \mathfrak{h}$ and $\mathcal{E}$ the $m \times n$ configuration equations have a positive, normalized, inequality preserving solution.*

Proof. Assume that H is amenable. Then there is a left invariant mean m on $L^\infty(H)$. For a configuration C, let $z_C := m(\xi_0(C))$. By equation (2.1.1) and linearity of m, it follows that for any i, j

$$\sum_{\substack{C\in \mathrm{Con}(\mathcal{E},\mathfrak{h}) \\ C_0=E_i}} z_C = m(\chi_{E_i})$$

and

$$\sum_{\substack{C\in \mathrm{Con}(\mathcal{E},\mathfrak{h}) \\ C_j=E_i}} z_C = m(\delta_{\tilde{h}_j} * \chi_{E_i}).$$

Since m is left invariant, these are equal and so the configuration equations are satisfied by this choice of z_C. It is also apparent that because m is a mean, each z_C is non-negative and $\sum_C z_C = 1$. Since m is positive, it preserves inequalities so the solution to the configuration equations is inequality preserving.

For the converse, we apply lemma 2.1.6 to (H, λ), $\{\xi_0(C) : C \in \text{Con}(\mathcal{E}, \mathfrak{h})\}$, and $\{z_C : C \in \text{Con}(\mathcal{E}, \mathfrak{h})\}$ for each choice of $\mathcal{E}$ and $\mathfrak{h}$.

We consider an order on the family of choices of $(\mathcal{E}, \mathfrak{h})$ by saying that $(\mathcal{E}, \mathfrak{h}) \preceq (\mathcal{F}, \mathfrak{k})$ if $\mathcal{F}$ is a refinement of $\mathcal{E}$ and $\mathfrak{h} \subseteq \mathfrak{k}$. Under this order, the family becomes a directed set. By indexing with respect to this directed set and taking the means generated by Lemma 2.1.6 we get a net of means on $L^\infty(H)$ which converge in the weak* topology to left invariance. Since the set of means is weak* compact, there is an accumulation point of this net which must be a left invariant mean on $L^\infty(H)$ hence H is amenable. □

Remark 2.1.8: It is actually sufficient to use a collection of $\mathcal{E}$ and $\mathfrak{h}$ which is a directed set under the given ordering which contains each set of a basis of the topology of H in one of the partitions and each $h \in H$.

The preceding result gives a new characterization of amenability for hypergroups. In chapters 4 and 5 we show that similar characterizations of amenability can be found for other algebras. In chapter 4 we give a characterization of the existence of a topological left invariant mean on a Lau algebra using a generalized notion of configurations. In chapter 5 we show that by using a different notion of partitioning the identity, we can use configurations to characterize the existence of a left invariant mean on function algebras on H other than $L^\infty(H)$.

In [47], Rosenblatt and Willis proved a version of theorem 2.1.7 for locally compact groups using a more constructive approach and because of this are able to construct a net $(f_\alpha)_\alpha \in L^1(G)^+_1$ which tends to left invariance weakly, but which, for any $x \in G\backslash\{e\}$, eventually the supports of f_α and $\delta_x * f_\alpha$ are disjoint. Such a result is impossible in general for hypergroups (see example

2.1.14). However, the approach of [47] is helpful for constructing nets for hypergroups which tend to left invariance in a weak sense but not in norm as demonstrated below in theorem 2.1.12.

Corollary 2.1.9 (Rosenblatt and Willis [47]): *Let G be a locally compact group. There is a positive, normalized solution of every possible instance of the essential configuration equations if and only if G is amenable.*

Proof. Since G is a group, $\delta_x * \chi_{E_i} = \chi_{xE_i}$ so $\xi_0(C) = \chi_{x_0(C)}$ for some set $x_0(C)$. For two configurations $C \neq C'$ the sets $x_0(C)$ and $x_0(C')$ are disjoint. Rosenblatt and Willis call a configuration C essential if $\lambda_G(x_0(C)) > 0$. Subsequently the condition that a solution be inequality preserving is equivalent to $z_C = 0$ for each non-essential configuration. □

For an infinite locally compact amenable group G, Rosenblatt and Willis [47] used their constructive proof of 2.1.9 to construct nets of positive, norm one functions $\{f_\alpha\}_\alpha$ in $L^1(G)$ for which $\delta_x * f_\alpha - f_\alpha$ tends weakly to 0 but $\|\delta_x * f_\alpha - f_\alpha\| = 2$ eventually for every $x \in G\backslash\{e\}$. The key to their proof lies in being able to choose a function which, when integrated against a $\xi_0(C)$, yields the corresponding z_C yet is supported on a small enough set so that the supports of f and $\delta_x * f$ are disjoint. Being able to accomplish this for general hypergroups is impossible because of the 'spreading' behaviour of translation as we see in example 2.1.14. However, we are able to modify their result for a certain class of hypergroups.

Theorem 2.1.10 (Rosenblatt and Willis [47]): *If G is an infinite amenable locally compact group, then there exists a net (f_α) in $P(G)$ converging weakly to invariance, such that for every $x \in G\backslash\{e\}$ eventually $\|f_\alpha - \delta_x * f_\alpha\| = 2$.*

Lemma 2.1.11: *Let G be a locally compact group and K be a compact subgroup of G. If $(f_\alpha)_\alpha$ is a net converging weakly to left invariance in $L^1(G)_1^+$ then $(\mathring{f}_\alpha)_\alpha$ is a net in $L^1(G//K)_1^+$ satisfying $\langle \mathring{f}_\alpha - \delta_{KxK} * \mathring{f}_\alpha, \phi\rangle \to 0$ for all $\phi \in UCB_r(G//K)$ where $\mathring{f}(KyK) = \int_K \int_K f(syt) d\lambda_K(s) d\lambda_K(t)$.*

Proof. Let $\phi \in UCB_r(G//K)$. By [51, 2.2] there exist $\gamma \in L^1(G//K)$ and $\psi \in L^\infty(G//K)$ such that $\phi = \gamma * \psi$. Let $\gamma_\circ \in L^1(G)$ and $\psi_\circ \in L^\infty(G)$ be given by $\gamma_\circ(y) = \gamma(KyK)$ and $\psi_\circ(y) = \psi(KyK)$. Define $\phi_\circ = \gamma_\circ * \psi_\circ$ in $UCB_r(G)$ so $\phi_\circ(y) = \int_G \gamma(KyzK)\psi(Kz^{-1}K) d\lambda_G(z)$. For $y \in G$ we have

$$\begin{aligned}\phi(KyK) &= \int_G \gamma(KyK * KzK)\psi(Kz^{-1}K) d\lambda_G(z)\\ &= \int_K \int_G \gamma(K(yt)zK)\psi(Kz^{-1}K) d\lambda_G(z) d\lambda_K(t)\\ &= \int_K \phi_\circ(yt) d\lambda_K(t)\\ &= \phi_\circ * \check{\lambda}_K(y).\end{aligned}$$

Similarly, for $x, y \in G$ we have $\phi(KxK * KyK) = \int_K \phi_\circ * \check{\lambda}_K(xsy) d\lambda_K(s)$.

Then for $x \in G$ and $f \in L^1(G)$, we have the following:

$$\begin{aligned}\langle \mathring{f} - \delta_{KxK} * \mathring{f}, \phi\rangle &= \int_G \left(\mathring{f} - \delta_{KxK} * \mathring{f}\right)(KyK)\phi(KyK) d\lambda_G(y)\\ &= \int_G \mathring{f}(KyK)\left(\phi - \check{\delta}_{KxK} * \phi\right)(KyK) d\lambda_G(y)\\ &= \iiint\limits_{G\,K\,K} f(syt) d\lambda_K(s) d\lambda_K(t)\left(\phi - \check{\delta}_{KxK} * \phi\right)(KyK) d\lambda_G(y)\\ &= \iiint\limits_{K\,K\,G} f(y)\left(\phi - \check{\delta}_{KxK} * \phi\right)(Ks^{-1}yt^{-1}K) d\lambda_G(y) d\lambda_K(s) d\lambda_K(t).\end{aligned}$$

Since K is compact, the modular function of G for any element of K is 1.

Therefore the integral with respect to y can be translated on both the left and right by elements of K and remain unchanged. This yields:

$$\begin{aligned}\langle \mathring{f} - \delta_{KxK} * \mathring{f}, \phi\rangle &= \int_G f(y)\left(\phi(KyK) - \phi(KxK * KyK)\right) d\lambda_G(y) \\ &= \int_G f(y)\left(\phi_\circ * \check{\lambda}_K(y) - \int_K \phi_\circ * \check{\lambda}_K(xsy) d\lambda_K(s)\right) d\lambda_G(y) \\ &= \left\langle f, \phi_\circ * \check{\lambda}_K - \int_K \delta_{s^{-1}x^{-1}} * \phi_\circ * \check{\lambda}_K d\lambda_K(s)\right\rangle.\end{aligned}$$

Since $\phi_\circ$ is in $UCB_r(G)$, so is $\phi_\circ * \check{\lambda}_K$. Hence the map from K to $C(G)$, $s \mapsto \delta_{s^{-1}x^{-1}} * \phi_\circ * \check{\lambda}_K$ is continuous so $\int_K \delta_{s^{-1}x^{-1}} * \phi_\circ * \check{\lambda}_K d\lambda_K(s)$ is in the norm closure of the convex hull of the functions $\left\{\delta_{s^{-1}x^{-1}} * \phi_\circ * \check{\lambda}_K : s \in K\right\}$. Since $\langle f_\alpha, \phi_\circ * \check{\lambda}_K - \delta_{s^{-1}x^{-1}} * \phi_\circ * \check{\lambda}_K\rangle$ tends to zero for any $s \in K$ and (f_α) is norm bounded , it follows that $\langle f_\alpha, \phi_\circ * \check{\lambda}_K - \int_K \delta_{s^{-1}x^{-1}} * \phi_\circ * \check{\lambda}_K d\lambda_K(s)\rangle$ also tends to zero as in lemma 1.4.5. So $\langle \mathring{f}_\alpha - \delta_{KxK} * \mathring{f}_\alpha, \phi\rangle \to 0$ for all $\phi \in UCB_r(G//K)$ and all $x \in G$. □

Theorem 2.1.12: *Let G be an amenable, non-compact, locally compact group. Let K be a compact subgroup of G. Suppose that for any $\varepsilon > 0$, finite $F \subset G\backslash K$ and subset X of G which does not have zero measure outside of a compact set, we can find a relatively compact $X' \subset X$ such that*

$$\lambda_G(KFKX'K \cap KX'K) < \frac{\varepsilon}{2}\lambda_G(KX'K). \tag{2.1.3}$$

Then there exists a net $(\mathring{f}_\alpha)_\alpha \subset L^1(G//K)_1^+$ such that $\langle \mathring{f}_\alpha - \delta_{KxK} * \mathring{f}_\alpha, \phi\rangle$ tends to 0 for all $\phi \in UCB_r(G//K)$ and $x \in G$ but for which $\|\mathring{f}_\alpha - \delta_{KxK} * \mathring{f}_\alpha\| \to 2$ whenever $x \notin K$.

Proof. Fix $\varepsilon > 0$ and a finite subset $F \subset G\backslash K$ and take $\mathcal{E}$ and $\mathfrak{g}$ as before for the group G.

Since G is amenable there is a positive, normalized solution to the configurations corresponding to $\mathrm{Con}(\mathcal{E}, \mathfrak{g})$.

Since G is a group, for each $C \in \mathrm{Con}(\mathcal{E}, \mathfrak{g})$ the function $\xi_0(C)$ is the characteristic function of the set $X_0(C)$. If the value of z_C is non-zero, then because G is non-compact $X_0(C)$ does not have measure zero outside any compact set.

Choose an order, $\{C_a\}_{a=1}^{N}$ for the $C \in \mathrm{Con}(\mathcal{E}, \mathfrak{g})$ with non-zero z_C and then iteratively select relatively compact $X_0^C \subset X_0(C)$ satisfying inequality (2.1.3) such that

$$FKX_0^{C_a} \cap KX_0^{C_b}K = \emptyset \quad \text{whenever} \quad a \neq b. \tag{2.1.4}$$

This is possible since each $X_0(C_a)$ has non-zero measure outside the compact set

$$\bigcup_{b=1}^{a-1} (KFKX_0^{C_b}K \cup KF^{-1}KX_0^{C_b}K).$$

As in [47, 3.2], let

$$f = \sum_{s=1}^{N} \frac{z_{C_s}}{\lambda_G(X_0^{C_s})} \chi_{X_0^{C_s}}.$$

Then $f \in L^1(G)_1^+$ and $\langle f - \delta_g * f, \chi_E \rangle = 0$ for each $g \in \mathfrak{g}$ and $E \in \mathcal{E}$. Let $\mathring{f} \in L^1(G//K)$ be as in lemma 2.1.11.

Observe that for $y \in G$,

$$\mathring{f}(KyK) = \sum_{C \in \mathrm{Con}(\mathcal{E}, \mathfrak{g})} \frac{z_C}{\lambda_G(X_0^C)} \int_K \int_K \chi_{X_0^C}(syt) d\lambda_K(s) d\lambda_K(t)$$

and if $\mathring{f}(KyK) > 0$ then $y \in KX_0^C K$ for some C.

Similarly, for $x \in F$,

$$\mathring{f}(Kx^{-1}K * KyK) = \sum_{C\in \mathrm{Con}(\mathcal{E},\mathfrak{g})} \frac{z_C}{\lambda_G(X_0^C)} \iiint\limits_{K\,K\,K} \chi_{X_0^C}(rx^{-1}syt)d\lambda_K(r)d\lambda_K(s)d\lambda_K(t)$$

and if $\mathring{f}(Kx^{-1}K * KyK) > 0$ then $y \in KFKX_0^CK$ for some C.

By condition (2.1.4), if both $\mathring{f}(KyK) > 0$ and $\mathring{f}(Kx^{-1}K * KyK) > 0$ for some $x \in F$ then there is a unique $C \in \mathrm{Con}(\mathcal{E}, \mathfrak{g})$ with $y \in KX_0^CK \cap KFKX_0^CK$.

For each $C \in \mathrm{Con}(\mathcal{E}, \mathfrak{g})$, let

$$A_C = \{y \in KX_0^CK \cap KFKX_0^CK : \mathring{f}(KyK) > 0, \mathring{f}(KxK * KyK) > 0\}.$$

This yields

$$\begin{aligned}
&\int\limits_{A_C} |\mathring{f}(KyK) - \mathring{f}(Kx^{-1}K * KyK)| d\lambda_G(y) \\
&\quad \le \int\limits_{A_C} \frac{z_C}{\lambda_G(X_0^C)} \left| \int\limits_{K^3} f(syt) - f(rx^{-1}syt)d\lambda_K(r)d\lambda_K(s)d\lambda_K(t)\right| d\lambda_G(y) \\
&\quad \le \frac{z_C}{\lambda_G(X_0^C)} \int\limits_{K^3}\int\limits_{A_C} \left|\chi_{X_0^C}(syt) - \chi_{X_0^C}(rxsyt)\right| d\lambda_G(y)d\lambda_K(r)d\lambda_K(s)d\lambda_K(t) \\
&\quad \le \frac{z_C}{\lambda_G(X_0^C)} \int\limits_{K^3}\int\limits_{A_C} (2)d\lambda_G(y)d\lambda_K(r)d\lambda_K(s)d\lambda_K(t) \\
&\quad \le \frac{2z_C}{\lambda_G(X_0^C)} \lambda_G(KX_0^CK \cap KFKX_0^CK) \\
&\quad < z_C\varepsilon.
\end{aligned}$$

By inequality (2.1.3),

$$\int_G |\mathring{f} - \delta_{KxK} * \mathring{f}| d\lambda_G \geq \|\mathring{f}\| + \|\delta_{KxK} * \mathring{f}\| - 2 \int_{\bigcup_C A_C} |\mathring{f} - \delta_{KxK} * \mathring{f}| d\lambda_G,$$

and so

$$\begin{aligned}\int_G |\mathring{f} - \delta_{KxK} * \mathring{f}| d\lambda_G &\geq 2 - 2\sum_C z_C \varepsilon \\ &= 2 - 2\varepsilon.\end{aligned}$$

Since our choices of $\varepsilon, F, \mathcal{E}$, and $\mathfrak{g}$ were arbitrary, we can find an f as above for each such choice. Now consider the order on $\{(\varepsilon, F, \mathcal{E}, \mathfrak{g})\}$ where

$$(\varepsilon, F, \mathcal{E}, \mathfrak{g}) \preceq (\varepsilon', F', \mathcal{E}', \mathfrak{g}')$$

if $\varepsilon \geq \varepsilon'$, $F \subseteq F'$, $\mathcal{E}'$ is a refinement of $\mathcal{E}$, and $\mathfrak{g} \subseteq \mathfrak{g}'$.

Using this order the net $(f_{(\varepsilon,F,\mathcal{E},\mathfrak{g})})$ converges weakly to left invariance on $L^\infty(G)$. Therefore by lemma 2.1.11 $\langle \mathring{f}_{(\varepsilon,F,\mathcal{E},\mathfrak{g})} - \delta_{KxK} * \mathring{f}_{(\varepsilon,F,\mathcal{E},\mathfrak{g})}, \phi\rangle \to 0$ for all $x \in G$ and $\phi \in UCB_r(G//K)$.

On the other hand, $\|\mathring{f}_{(\varepsilon,F,\mathcal{E},\mathfrak{g})} - \delta_{KxK} * \mathring{f}_{(\varepsilon,F,\mathcal{E},\mathfrak{g})}\| \geq 2 - 2\varepsilon$ for all $x \in F$. Hence, for any $x \in G\backslash K$, $\|\mathring{f}_{(\varepsilon,F,\mathcal{E},\mathfrak{g})} - \delta_{KxK} * \mathring{f}_{(\varepsilon,F,\mathcal{E},\mathfrak{g})}\| \to 2$. □

If K is finite, then the net we construct does tend to left invariance weakly.

Example 2.1.13: Let H be the hypergroup $(\mathbb{Z}[i] \rtimes \langle i\rangle)//\langle i\rangle$ where the action of $\langle i\rangle$ on $\mathbb{Z}[i]$ is multiplication. The coset of $a + ib \in \mathbb{Z}[i]$ is the four points $\{a+ib, -b+ia, -a-ib, b-ia\}$. For any finite set F, if $\|a+ib\|$ is sufficiently large,

$X' = \{a + ib\}$ will satisfy inequality (2.1.3) of theorem 2.1.12. Hence, using the method of 2.1.12, we can construct a net which tends to left invariance weakly, but not in norm for $(\mathbb{Z}[i] \rtimes \langle i \rangle)//\langle i \rangle$.

Example 2.1.14: Let H be the hypergroup $(\mathbb{R}^2)^{\mathbb{T}}$ (equivalently $(\mathbb{R}^2 \rtimes \mathbb{T})//\mathbb{T}$) where the action of the torus on $\mathbb{R}^2$ is rotation about the origin. More details on this example can be found in [23] or [4, 1.1.18]. The underlying space of H is $\mathbb{R}_{\geq 0}$ and for any $f \in L^1(H)$ and $x \in H$, the support of the translation of f by x is given by:

$$\operatorname{supp}(\delta_x * f) = \{x\} * \operatorname{supp} f = \bigcup_{y \in \operatorname{supp} f} [|x - y|, x + y].$$

From this we see that as long as $\operatorname{supp} f$ is not contained in the interval $[0, x/2)$ the two supports are not disjoint. Since the support of f_α must eventually not be contained in such an interval, if f_α tends weakly to left invariance, then the supports of f_α and $\delta_x * f_\alpha$ are not eventually disjoint.

However, this hypergroup does satisfy the condition of theorem 2.1.12 so we can construct a net which tends weakly to left invariance against right uniformly continuous functions but does not tend to left invariance in norm.

Chapter 3

Reiter nets for semidirect products[1]

In this chapter, we present two methods for combining Reiter nets of two locally compact groups to form a Reiter net for their semidirect product. We apply this result to Reiter nets where each function is a normalized characteristic function of some compact subset of G to achieve similar results for Følner nets of subsets of amenable groups. We then define the semidirect product hypergroup of a hypergroup and a locally compact group. We extend the results for combining Reiter nets to this case and show that the semidirect product of an amenable locally compact group and amenable hypergroup is an amenable hypergroup. We also prove analogous results for discrete semigroups.

[1] A version of this chapter has been published. Willson 2009. Proceedings of the American Mathematical Society. 137:3823-3832

3.1 Semidirect products of groups

There are two methods for defining the semidirect product of groups. Definition 3.1.1 uses the *external* method for defining semidirect products.

Definition 3.1.1: Let N and H be locally compact groups with H acting on N, i.e. there exists τ, a group homomorphism from H to $\mathrm{Aut}(N)$ such that $(n,h) \to \tau_h(n)$ is continuous with respect to the product topology on $N \times H$ where $\mathrm{Aut}(N)$ is the group of continuous group automorphisms of N. We say that $G := N \rtimes_\tau H$ is the *semidirect product* of N and H with respect to τ if G is the group consisting of elements of the form (n,h) where $n \in N$ and $h \in H$ equipped with multiplication given by:

$$(n_1, h_1) * (n_2, h_2) = (n_1 \tau_{h_1}(n_2), h_1 h_2).$$

If G is equipped with the product topology then G is a locally compact group. We will refer to N and H as the factor groups of $N \rtimes_\tau H$.

The second method used to define a semidirect product is called the internal method. The internal method considers a group with two subgroups satisfying certain conditions and uses conjugation by one of the subgroups on the other as the τ given in 3.1.1. For this method, we begin with a group action which determines τ rather than constructing the group action from an arbitrary τ.

If $f : N \to \mathbb{C}$ and $g : H \to \mathbb{C}$, then we define $f \times g : N \rtimes_\tau H \to \mathbb{C}$ via $f \times g(n,h) = f(n)g(h)$.

It is well known (cf. [16, 15.29]) that the left Haar measure of a semidirect product depends on the left Haar measures of the factor groups. It can be defined in the following way:

$$d\lambda_{N \rtimes_\tau H}(n,h) = \delta(h) d\lambda_N(n) d\lambda_H(h)$$

where $\delta(h) = \frac{\lambda_N(A)}{\lambda_N(\tau_h(A))}$ for each measurable $A \subset N$. Then δ is a continuous group homomorphism from H to $\mathbb{R}^+$.

From the action, τ, of H on N we define

$$\begin{aligned} T :& H \to B(L^1(N)) \\ & h \mapsto T_h \end{aligned}$$

via $(T_h f)(n) = f(\tau_{h^{-1}}(n))\delta(h)$ for $f \in L^1(N)$, $n \in N$, $h \in H$. It can be easily verified that each T_h is a linear isometry and preserves convolution.

3.2 Reiter nets in $L^1(N \rtimes_\tau H)_1^+$

When investigating Reiter nets for semidirect products, it is natural to wonder how Reiter nets for N and H may be combined to produce a Reiter net for $N \rtimes_\tau H$. We present two methods for doing so.

Throughout this section, N and H will be amenable locally compact groups, and τ a continuous group homomorphism from H to $\mathrm{Aut}(N)$. Also, $(f_\alpha)_\alpha$ and $(g_\beta)_\beta$ will be Reiter nets in $L^1(N)_1^+$ and $L^1(H)_1^+$, respectively.

Theorem 3.2.1 (Method 1): *Let $E_{\alpha,\beta} := f_\alpha \times g_\beta \delta^{-1}$. The net $(E_{\alpha,\beta})_{\alpha,\beta}$ is a Reiter net for $N \rtimes_\tau H$ if and only if the following condition holds:*

$$\|T_y f_\alpha - f_\alpha\|_{L^1(N)} \to 0 \tag{3.2.1}$$

uniformly in y on compact subsets of H.

Proof. We begin by assuming that (3.2.1) holds. Then:

$$\begin{aligned}
&\|l^*_{(x,y)}E_{\alpha,\beta} - E_{\alpha,\beta}\|_{L^1(N\rtimes_\tau H)} \\
&\qquad = \iint \left|l^*_x T_y f_\alpha(n) l^*_y g_\beta(h) - f_\alpha(n) g_\beta(h)\right| d\lambda_H(h) d\lambda_N(n) \\
&\qquad \leq \|(l^*_x T_y f_\alpha) - f_\alpha\|_{L^1(N)} \|l^*_y g_\beta\|_{L^1(H)} \\
&\qquad\qquad + \|f_\alpha\|_{L^1(N)} \|l^*_y g_\beta - g_\beta\|_{L^1(H)} \\
&\qquad \leq \|l^*_x(T_y f_\alpha - f_\alpha)\|_{L^1(N)} + \|l^*_x f_\alpha - f_\alpha\|_{L^1(N)} \\
&\qquad\qquad + \|l^*_y g_\beta - g_\beta\|_{L^1(H)}.
\end{aligned}$$

Each of these terms tends to zero uniformly on compact subsets of $N \rtimes_\tau H$. Therefore $(E_{\alpha,\beta})_{\alpha,\beta}$ is a Reiter net.

For the converse we will apply similar triangle inequalities to get, for $y \in H$:

$$\begin{aligned}
\|T_y f_\alpha - f_\alpha\|_{L^1(N)} &= \int_N \int_H |T_y f_\alpha(n) g_\beta(h) - f_\alpha(n) g_\beta(h)| d\lambda_H(h) d\lambda_N(n) \\
&\leq \|T_y f_\alpha\|_{L^1(N)} \|g_\beta - l^*_y g_\beta\|_{L^1(H)} \\
&\qquad + \|l^*_{(e_N,y)} E_{\alpha,\beta} - E_{\alpha,\beta}\|_{L^1(N\rtimes_\tau H)}.
\end{aligned}$$

Since each of the terms on the right tend to zero uniformly in y on compact subsets of H, so does the left side of the inequality. □

Proposition 3.2.2: *There exists a Reiter net $(f_\alpha)_\alpha \subset L^1(N)^+_1$ that satisfies (3.2.1).*

Proof. Since N and H are amenable, so is the semidirect product $N \rtimes_\tau H$.

Therefore there is a Reiter net $(F_\alpha)_\alpha \subset L^1(N \rtimes_\tau H)^+_1$.

For each α, define $f_\alpha \in L^1(N)$ via

$$f_\alpha(n) := \int_{h \in H} F_\alpha(n,h)\delta(h)d\lambda_H(h).$$

Then it is easily verified that $(f_\alpha)_\alpha$ is a net in $L^1(N)^+_1$. Furthermore, for $y \in H$

$$\begin{aligned}
\|T_y f_\alpha - f_\alpha\|_{L^1(N)} &= \int_{n \in N} \left| \int_{h \in H} F_\alpha(\tau_{y^{-1}}(n), h)\delta(y)\delta(h)d\lambda_H(h) \right. \\
&\qquad \left. - \int_{h \in H} F_\alpha(n,h)\delta(h)d\lambda_H(h) \right| d\lambda_N(n) \\
&\leq \int_{n \in N} \int_{h \in H} \left| F_\alpha(\tau_{y^{-1}}(n), y^{-1}h)\delta(y)\delta(y^{-1}h) \right. \\
&\qquad \left. - F_\alpha(n,h)\delta(h) \right| d\lambda_H(h)d\lambda_N(n) \\
&= \int_{N \rtimes_\tau H} |F_\alpha(\tau_{y^{-1}}(n), y^{-1}h) - F_\alpha(n,h)| d\lambda_{N \rtimes_\tau H}(n,h) \\
&= \|l^*_{(e_N, y)} F_\alpha - F_\alpha\|_{L^1(N \rtimes_\tau H)}.
\end{aligned}$$

Since $(F_\alpha)_\alpha$ is a Reiter net, $\|l^*_{(e_N,y)}F_\alpha - F_\alpha\|_{L^1(N \rtimes_\tau H)} \to 0$ uniformly in y on compact subsets of H. It remains to show that $(f_\alpha)_\alpha$ is a Reiter net for N. For $x \in N$

$$\begin{aligned}
\|l^*_x f_\alpha - f_\alpha\|_{L^1(N)} &= \int_{n \in N} \left| \int_{h \in H} F_\alpha(x^{-1}n, h)\delta(h) - F_\alpha(n,h)\delta(h)d\lambda_H(h) \right| d\lambda_N(n) \\
&\leq \int_{n \in N} \int_{h \in H} \left| F_\alpha(x^{-1}n, h) - F_\alpha(n,h) \right| \delta(h)d\lambda_H(h)d\lambda_N(n) \\
&= \|l^*_{(x, e_H)} F_\alpha - F_\alpha\|_{L^1(N \rtimes_\tau H)}.
\end{aligned}$$

Again, since $(F_\alpha)_\alpha$ is a Reiter net, so is $(f_\alpha)_\alpha$. □

Remark 3.2.3: Janzen [22] provides a similar result for the existence of a Følner net which satisfies a condition analogous to (3.2.1) in the case where δ is constantly one.

Remark 3.2.4: If $(E_{\alpha,\beta})_{\alpha,\beta}$ is a Reiter net, then so is any subnet. In particular, if $(f_k)_{k=1}^{\infty}$ and $(g_k)_{k=1}^{\infty}$ are Reiter sequences for N and H respectively, which satisfy the conditions of the theorem, then the diagonal sequence $(f_k \times g_k\delta^{-1})_{k=1}^{\infty}$ is a Reiter sequence for $N \rtimes_\tau H$.

Theorem 3.2.5 (Method 2): *For $n \in N, h \in H$, let $D_{\alpha,\beta} \in L^1(N \rtimes_\tau H)$ be given by $D_{\alpha,\beta}(n,h) := T_h f_\alpha(n) g_\beta(h)\delta(h^{-1})$. If each g_β is compactly supported, then there exists a subnet of $(D_{\alpha,\beta})_{\alpha,\beta}$ which is a Reiter net for $N \rtimes_\tau H$.*

Proof. Observe that

$$\begin{aligned}
&\|l^*_{(x,y)} D_{\alpha,\beta} - D_{\alpha,\beta}\|_{L^1(N\rtimes_\tau H)} \\
&\qquad = \int \left| l^*_x T_h f_\alpha(n) l^*_y g_\beta(h) - T_h f_\alpha(n) g_\beta(h) \right| \delta(h^{-1}) d\lambda_G(n,h) \\
&\qquad = \iint |T_h l^*_{\tau_{h^{-1}}(x)} f_\alpha(n) l^*_y g(h) - T_h f_\alpha(n) g_\beta(h)| d\lambda_N(n) d\lambda_H(h) \\
&\qquad \le \iint |T_h(l^*_{\tau_{h^{-1}}(x)} f_\alpha - f_\alpha)(n)||l^*_y g_\beta(h)| d\lambda_N(n) d\lambda_H(h) \\
&\qquad\qquad + \iint |T_h f_\alpha(n)||l^*_y g_\beta(h) - g_\beta(h)| d\lambda_N(n) d\lambda_H(h) \\
&\qquad = \int \|l^*_{\tau_{(yh)^{-1}}(x)} f_\alpha - f_\alpha\|_{L^1(N)} |g_\beta(h)| d\lambda_H(h) + \|l^*_y g_\beta - g_\beta\|.
\end{aligned}$$

For $K \subset N \rtimes_\tau H$ compact and $\varepsilon > 0$ there exists $\beta_{K,\varepsilon}$ such that for $\beta \ge \beta_{K,\varepsilon}$,

$$\|l^*_y g_\beta - g_\beta\|_{L^1(H)} < \varepsilon/2 \quad \forall y \in K_H = \{y : (x,y) \in K \text{ for some } x \in N\}.$$

For each such β there is an $\alpha_{\beta,K,\varepsilon}$ such that for $\alpha \ge \alpha_{\beta,K,\varepsilon}$ for all $h \in \operatorname{supp}(g_\beta)$

and all $(x,y) \in K$,

$$\|l^*_{\tau_{(yh)^{-1}}(x)} f_\alpha - f_\alpha\|_{L^1(N)} < \varepsilon/2.$$

Consider the directed set Λ, where each element of Λ is a quadruple consisting of a compact subset $K \subset G$, an $\varepsilon > 0$, a $\beta \geq \beta_{K,\varepsilon}$, and an $\alpha \geq \alpha_{\beta,K,\varepsilon}$. The order we put on Λ is $\preceq$, where $(K_1, \varepsilon_1, \beta_1, \alpha_1) \preceq (K_2, \varepsilon_2, \beta_2, \alpha_2)$ if $K_1 \subset K_2$, $\varepsilon_1 \geq \varepsilon_2$, $\beta_1 \leq \beta_2$, and $\alpha_1 \leq \alpha_2$. From the above observations, it is apparent that for any ε and K, there exists α and β such that $\|l^*_{(x,y)} D_{\alpha,\beta} - D_{\alpha,\beta}\| < \varepsilon$ for all $(x,y) \in K$.

Then $(E_{\alpha,\beta})_{K,\varepsilon,\beta,\alpha}$ is a subnet of $(E_{\alpha,\beta})_{\alpha,\beta}$ which is a Reiter net for $N \rtimes_\tau H$. $\square$

Remark 3.2.6: If $N \rtimes_\tau H$ is σ-compact, then we can find a sequence of compact sets $(F_n)_n$ such that $F_n \subset F_{n+1}$ and $N \rtimes_\tau H = \bigcup F_n$. In this case, we can find a sequence of elements of Λ such that $(E_{\alpha_n,\beta_n})_{n=1}^\infty$ is a Reiter sequence.

Example 3.2.7 ('ax+b' Group): Let $G = \mathbb{R} \rtimes_\tau \mathbb{R}^+$ where $\tau_a(b) = ab$ for $a \in \mathbb{R}^+$ and $b \in \mathbb{R}$. Let $f_n = \frac{1}{2n}\chi_{[-n,n]}$ and $g_n = \frac{1}{2\ln(n)}\chi_{[\frac{1}{n},n]}$. Using the method of 3.2.5, $D_{n,m} = \frac{1}{4n\ln(n)}\chi_{\{(ab,a)|b\in[-n,n],a\in[\frac{1}{m},m]\}}$. It follows that $(D_{n,n})_n$ is a Reiter sequence for G. Further details of this example can be found in [42, Example 0.5] or [14].

By applying the method of Proposition 3.2.2, we get the Reiter sequence for N, $(h_n)_n$ given by:

$$h_n(b) = \begin{cases} \frac{n^2-1}{4n^2\ln(n)} & \text{if } |b| < 1, \\ \frac{n^2-|b|}{4n^2|b|\ln(n)} & \text{if } 1 \leq |b| < n^2, \\ 0 & \text{if } n^2 \leq |b|. \end{cases}$$

Combining this with the sequence $(g_n)_n$ using Method 1 gives a Reiter sequence for the 'ax+b' group which is different from the standard example found in the literature:

$$E_n(b,a) = h_n(b)g_n(a)a.$$

3.3 Følner Nets

As in Example 3.2.7 it is often possible to have Reiter nets $(f_\alpha)_\alpha$ of the form $f_\alpha = \frac{\chi_{A_\alpha}}{\lambda(A_\alpha)}$. The sets $(A_\alpha)_\alpha$ form a Følner net, thus termed because of the condition introduced in [10]. Namioka investigated numerous Følner type conditions in [40]. For further reference, the reader is directed to [14, Section 3.6] and [42, Chapter 4].

Definition 3.3.1: Let G be a locally compact group. A net [resp. sequence] $(A_\alpha)_\alpha$ of measurable subsets of G such that $0 < \lambda_G(A_\alpha) < \infty$ is called a *Følner net [resp. Følner sequence]* if for any $\varepsilon > 0$, and any compact $F \subset G$, there exists β such that for $\alpha > \beta$

$$\frac{\lambda_G(xA_\alpha \triangle A_\alpha)}{\lambda_G(A_\alpha)} < \varepsilon \quad \forall x \in F.$$

Remark 3.3.2: A net $(A_\alpha)_\alpha$ of measurable subsets of G with $0 < \lambda_G(A_\alpha) < \infty$ is a Følner net if and only if for any $\varepsilon > 0$, and any compact $F \subset G$, there exists β such that, for $\alpha > \beta$

$$\frac{\lambda_G(xA_\alpha \backslash A_\alpha)}{\lambda_G(A_\alpha)} < \varepsilon \quad \forall x \in F$$

This is because $\lambda_G(A \backslash xA) = \lambda_G(x^{-1}A \backslash A)$ since λ_G is a left Haar measure.

Remark 3.3.3: Suppose that $(A_\alpha)_\alpha$ is a Følner net for G. For each α, let $f_\alpha(x) := \frac{1}{\lambda_G(A_\alpha)}\chi_A(x)$. Then $(f_\alpha)_\alpha$ is a Reiter net in $L^1(G)^+_1$.

Janzen [22] studied Følner nets in semidirect products where δ is constantly 1. As a consequence of the previous section, we now present one of Janzen's results as a corollary to Theorem 3.2.1.

Corollary 3.3.4 (Janzen): *Let $N \rtimes_\tau H$ be a semidirect product of locally compact groups such that $\delta \equiv 1$. Suppose $(A_\alpha)_\alpha$, $(B_\beta)_\beta$ are Følner nets for N and H respectively. Then $(A_\alpha \times B_\beta)_{\alpha,\beta}$ is a Følner net for $N \rtimes_\tau H$ if and only if*

$$\frac{\lambda_N(\tau_y(A_\alpha)\triangle A_\alpha)}{\lambda_N(A_\alpha)} \to 0$$

uniformly in y on compact subsets of H.

Proof. Let $f_\alpha = \frac{\chi_{A_\alpha}}{\lambda_N(A_\alpha)}$ and let $g_\beta = \frac{\chi_{B_\beta}}{\lambda_H(B_\beta)}$. By Theorem 3.2.1 $(f_\alpha \times g_\beta)_{\alpha,\beta}$ is a Reiter net if and only if $\|T_y f_\alpha - f_\alpha\| \to 0$. But $f_\alpha \times g_\beta = \frac{\chi_{A_\alpha \times B_\beta}}{\lambda_{N\rtimes_\tau H}(A_\alpha \times B_\beta)}$ and $\|T_y f_\alpha - f_\alpha\| = \frac{\lambda_N(\tau_y(A_\alpha)\triangle A_\alpha)}{\lambda_N(A_\alpha)}$. □

We cannot use Theorem 3.2.1 to generalize this result to the case where δ is not constantly 1 since in this case, $\chi_A \times \chi_B \delta^{-1}$ is not equal to $\chi_{A\times B}$. In fact, we will now show that if the product of two Følner nets is a Følner net for the semidirect product, then δ must be constantly 1.

Lemma 3.3.5: *Let $G = N \rtimes_\tau H$, for N and H locally compact amenable groups. Let $(A_\alpha)_\alpha$ be a Følner net for N and $(B_\beta)_\beta$ be a Følner net for H. If $(A_\alpha \times B_\beta)_{\alpha,\beta}$ is a Følner net for G, then*

$$\frac{\lambda_N(\tau_y(A_\alpha)\backslash A_\alpha)}{\lambda_N(A_\alpha)} \to 0$$

uniformly in y on compact sets of H.

Proof. Observe that for $x \in N, y \in H$ we have that

$$(x\tau_y(A_\alpha)\backslash A_\alpha) \times yB_\beta \subset ((x,y) * (A_\alpha \times B_\beta))\backslash(A_\alpha \times B_\beta).$$

Since $(A_\alpha \times B_\beta)_{\alpha,\beta}$ is a Følner net for G

$$\frac{\lambda_G\left(((x,y) * (A_\alpha \times B_\beta))\backslash(A_\alpha \times B_\beta)\right)}{\lambda_G(A_\alpha \times B_\beta)} \to 0$$

uniformly in (x,y) on compact subsets of G. Therefore

$$\frac{\lambda_G((x\tau_y(A_\alpha)\backslash A_\alpha) \times yB_\beta)}{\lambda_G(A_\alpha \times B_\beta)} \to 0$$

uniformly in (x,y) on compact subsets of G. Observe that

$$\begin{aligned}\frac{\lambda_G((x\tau_y(A_\alpha)\backslash A_\alpha) \times yB_\beta)}{\lambda_G(A_\alpha \times B_\beta)} &= \frac{\lambda_N(x\tau_y(A_\alpha)\backslash A_\alpha)}{\lambda_N(A_\alpha)}\frac{\int_{yB_\beta}\delta(h)d\lambda_H(h)}{\int_{B_\beta}\delta(h)d\lambda_H(h)}\\ &= \frac{\lambda_N(x\tau_y(A_\alpha)\backslash A_\alpha)}{\lambda_N(A_\alpha)}\frac{\int_{B_\beta}\delta(y)\delta(h)d\lambda_H(h)}{\int_{B_\beta}\delta(h)d\lambda_H(h)}\\ &= \frac{\lambda_N(x\tau_y(A_\alpha)\backslash A_\alpha)}{\lambda_N(A_\alpha)}\delta(y).\end{aligned}$$

Choose a compact $F \subset H$. The set $\{e_N\} \times F$ is compact in G, so

$$\frac{\lambda_N(\tau_y(A_\alpha)\backslash A_\alpha)}{\lambda_N(A_\alpha)}\delta(y) \to 0$$

uniformly on F. Then since δ is continuous, it is bounded below on F by some positive value. Therefore

$$\frac{\lambda_N(\tau_y(A_\alpha)\backslash A_\alpha)}{\lambda_N(A_\alpha)} \to 0$$

uniformly on F. □

Theorem 3.3.6: *If $(A_\alpha \times B_\beta)_{\alpha,\beta}$ is a Følner net for G then $\delta \equiv 1$.*

Proof. Observe that for $y \in H$:

$$\begin{aligned}\lambda_N(\tau_y(A_\alpha)\backslash A_\alpha) &\geq \lambda_N(\tau_y(A_\alpha)) - \lambda_N(A_\alpha)\\ &= \lambda_N(A_\alpha)(\delta(y^{-1}) - 1).\end{aligned}$$

By the lemma, we know that $\frac{\lambda_N(\tau_y(A_\alpha)\backslash A_\alpha)}{\lambda_N(A_\alpha)} \to 0$ for each $y \in H$. If there exists a $y_0 \in H$ such that $\delta(y_0^{-1}) > 1$ then for all α, $\frac{\lambda_N(\tau_{y_0}(A_\alpha)\backslash A_\alpha)}{\lambda_N(A_\alpha)} \geq \delta(y_0^{-1}) - 1 > 0$ which is a contradiction, so $\delta \equiv 1$. □

As indicated by Example 3.2.7, the method of Theorem 3.2.5 does apply even if δ is not constantly 1.

3.4 Semidirect products with hypergroup factors

The examples (2.1.13, 2.1.14) concluding the previous chapter are double coset hypergroups. These hypergroups arise from taking semidirect products of locally compact groups and quotienting by the non-normal factor. Subsequently we are motivated to investigate further notions of semidirect products as related to hypergroups.

Jewett [23] showed that the direct product of two hypergroups is again a hypergroup and much of the theory that applies for the direct product of groups applies verbatim for hypergroups. However, when it comes to invesigating semidirect products, we cannot simply replace groups by hypergroups in the definition since one of the groups needs to act as automorphisms on the other.

In this section we will present the definition of a semidirect product with a hypergroup factor and several related results. We will conclude the section with several results on amenability of semidirect products.

The semidirect product of two hypergroups only makes sense if there is a homomorphism from one hypergroup to a subgroup of the automorphisms of the other (in the case of a direct product, this is the trivial group).Because of this, we will consider semidirect products where one factor is a hypergroup and the other factor is a locally compact group acting as automorphisms on the hypergroup. Hypergroup automorphisms and, more generally, homomorphisms are interesting and have been mentioned in [5] and [25].

This definition does not appear as part of the published literature, but does appear in a technical report of Rösler [48] where she applies this construction to Bessel-Kingman hypergroups.

Definition 3.4.1: Let H be a hypergroup. A homeomorphism $\phi : H \to H$ is a (hypergroup) *automorphism* if $\phi(e_H) = e_H$ and for $x, y \in H$ and $A \subset H$ a Borel subset we have that $\delta_x * \delta_y(A) = \delta_{\phi(x)} * \delta_{\phi(y)}(\{\phi(a) : a \in A\})$. The collection of automorphisms of H (equipped with the topology of pointwise convergence) form a topological group denoted $\mathrm{Aut}(H)$.

Let G be a locally compact group. Suppose that there exists a continuous group homomorphism $\tau : G \to \mathrm{Aut}(H)$. We then define the semidirect product of G and H (with respect to τ) as the topological space $H \times G$ with a convolution defined by:

$$\delta_{(h_1,g_1)} * \delta_{(h_2,g_2)} = \delta_{h_1} * \delta_{\tau_{g_1}(h_2)} \otimes \delta_{g_1 g_2}$$

where we embed the tensor product $M(H) \otimes M(G)$ into $M(H \times G)$.

With this convolution, $H \rtimes_\tau G$ becomes a hypergroup. The identity of $H \rtimes_\tau G$ is (e_H, e_G) and the involution is given by: $(h, g)\check{} = (\tau_{g^{-1}}(\check{h}), g^{-1})$.

If we further suppose that H has a left Haar measure λ_H then for each $g \in G$, the measure on H, $\lambda_H \circ \tau_g$, is a positive multiple of λ_H. Letting $\delta(g) = \frac{\lambda_H}{\lambda_H \circ \tau}$ we get the following left Haar measure on $H \rtimes G$:

$$d\lambda_{H \rtimes G}(h, g) = \delta(g) d\lambda_H(h) d\lambda_G(g).$$

Proposition 3.4.2: *Let G, G' be locally compact groups and K a compact subgroup of G. Suppose that G' acts on G and that for each $g' \in G'$, $g'(K) = K$. Then $(G \rtimes G')//(K \times e_{G'})$ is isomorphic to $(G//K) \rtimes G'$.*

Proof. The elements of $(G \rtimes G')//(K \times e_{G'})$ are of the form

$$\begin{aligned} [(g, g')] &= \{(k_1, e_{G'})(g, g')(k_2, e_{G'}) : k_1, k_2 \in K\} \\ &= \{(k_1 g \tau_{g'}(k_2), g') : k_1, k_2 \in K\} \\ &= \{(k_1 g k_2, g') : k_1, k_2 \in K\} \end{aligned}$$

because the action of G' restricts to K.

Similarly, the elements of $G//K \rtimes G'$ are of the form

$$([g], g') = (\{k_1 g k_2 : k_1, k_2 \in K\}, g')$$

hence there is a natural identification between the two hypergroups.

The multiplication in the former is given by:

$$\begin{aligned}[(g_1, g_1')] * [(g_2, g_2')] &= \int_{K \times e_{G'}} \delta_{[(g_1, g_1')(s, e_{G'})(g_2, g_2')]} d\lambda_K(s) \\ &= \int_{K \times e_{G'}} \delta_{[(g_1 \tau_{g_1'}(s g_2), g_1' g_2')]} d\lambda_K(s) \\ &= \int_{K \times e_{G'}} \delta_{[(g_1 s \tau_{g_1'}(g_2), g_1' g_2')]} d\lambda_K(s)\end{aligned}$$

We note that λ_K is invariant under the action of g_1' because the action on K is 'unimodular' since K is compact.

On the latter, the convolution is:

$$\begin{aligned}([g_1], g_1') * ([g_2], g_2') &= ([g_1] * \tau_{g_1'}([g_2])) \otimes g_1' g_2' \\ &= \left(\int_K \delta_{[g_1 s \tau_{g_1'}(g_2)]} d\lambda_K(s) \right) \otimes g_1' g_2'\end{aligned}$$

and so we see that they coincide. □

Remark 3.4.3: The first semidirect product in the above proposition is a semidirect product of groups while the latter has a hypergroup factor.

Example 3.4.4: Here we provide a non-trivial example of a semidirect product with a hypergroup factor. The hypergroup is not a group and the action of G on H is not the trivial action.

Let $\mathbb{Z}_5$ be the additive group of integers modulo 5.

Let $\{e_\tau, \tau\}$ be the two element group acting on $\mathbb{Z}_5$ via $\tau(z) = -z$.

Then $\mathbb{Z}_5//\langle\tau\rangle$ is a hypergroup of three elements with the following multiplication table:

$*$	$[0]$	$[1]$	$[2]$
$[0]$	$\delta_{[0]}$	$\delta_{[1]}$	$\delta_{[2]}$
$[1]$	$\delta_{[1]}$	$\frac{1}{2}(\delta_{[0]} + \delta_{[2]})$	$\frac{1}{2}(\delta_{[1]} + \delta_{[2]})$
$[2]$	$\delta_{[2]}$	$\frac{1}{2}(\delta_{[1]} + \delta_{[2]})$	$\frac{1}{2}(\delta_{[0]} + \delta_{[1]})$

Let $\{e_\sigma, \sigma\}$ be the two element group acting on $\mathbb{Z}_5//\langle\tau\rangle$ which swaps [1] and [2]. It is straightforward to verify that σ is a hypergroup automorphism using the convolution table above.

Forming the semidirect product $(\mathbb{Z}_5//\langle\tau\rangle) \rtimes \langle\sigma\rangle$ we get the following six element hypergroup:

$*$	$([0], e_\sigma)$	$([0], \sigma)$	$([1], e_\sigma)$	$([1], \sigma)$	$([2], e_\sigma)$	$([2], \sigma)$
$([0], e_\sigma)$	$(\delta_{[0]}, e_\sigma)$	$(\delta_{[0]}, \sigma)$	$(\delta_{[1]}, e_\sigma)$	$(\delta_{[1]}, \sigma)$	$(\delta_{[2]}, e_\sigma)$	$(\delta_{[2]}, \sigma)$
$([0], \sigma)$	$(\delta_{[0]}, \sigma)$	$(\delta_{[0]}, e_\sigma)$	$(\delta_{[2]}, \sigma)$	$(\delta_{[2]}, e_\sigma)$	$(\delta_{[1]}, \sigma)$	$(\delta_{[1]}, e_\sigma)$
$([1], e_\sigma)$	$(\delta_{[1]}, e_\sigma)$	$(\delta_{[1]}, \sigma)$	$(\frac{\delta_{[0]}+\delta_{[2]}}{2}, e_\sigma)$	$(\frac{\delta_{[0]}+\delta_{[2]}}{2}, \sigma)$	$(\frac{\delta_{[1]}+\delta_{[2]}}{2}, e_\sigma)$	$(\frac{\delta_{[1]}+\delta_{[2]}}{2}, \sigma)$
$([1], \sigma)$	$(\delta_{[1]}, \sigma)$	$(\delta_{[1]}, e_\sigma)$	$(\frac{\delta_{[1]}+\delta_{[2]}}{2}, \sigma)$	$(\frac{\delta_{[1]}+\delta_{[2]}}{2}, e_\sigma)$	$(\frac{\delta_{[0]}+\delta_{[2]}}{2}, \sigma)$	$(\frac{\delta_{[0]}+\delta_{[2]}}{2}, e_\sigma)$
$([2], e_\sigma)$	$(\delta_{[2]}, e_\sigma)$	$(\delta_{[2]}, \sigma)$	$(\frac{\delta_{[1]}+\delta_{[2]}}{2}, e_\sigma)$	$(\frac{\delta_{[1]}+\delta_{[2]}}{2}, \sigma)$	$(\frac{\delta_{[0]}+\delta_{[1]}}{2}, e_\sigma)$	$(\frac{\delta_{[0]}+\delta_{[1]}}{2}, \sigma)$
$([2], \sigma)$	$(\delta_{[2]}, \sigma)$	$(\delta_{[2]}, e_\sigma)$	$(\frac{\delta_{[0]}+\delta_{[1]}}{2}, \sigma)$	$(\frac{\delta_{[0]}+\delta_{[1]}}{2}, e_\sigma)$	$(\frac{\delta_{[1]}+\delta_{[2]}}{2}, \sigma)$	$(\frac{\delta_{[1]}+\delta_{[2]}}{2}, e_\sigma)$

.

We will now address amenability of semidirect products. As mentioned in the introduction, the amenability of a hypergroup can be characterized by the existence of a Reiter net of approximate means in $L^1(H)_1^+$. Using the approach of the previous section, we show that the semidirect product of an amenable hypergroup and an amenable locally compact group is amenable.

Theorem 3.4.5: *Let H be an amenable hypergroup with left Haar measure and G be an amenable locally compact group. Then $H \rtimes_\tau G$ is an amenable hypergroup.*

Proof. Let (f_α) be a Reiter net for H and (d_β) be a Reiter net for G with each d_β supported on a compact subset of G.

For $h \in H, g \in G$ let $F_{\alpha,\beta} \in L^1(H \rtimes_\tau G)$ be given by

$$F_{\alpha,\beta}(h,g) := f_\alpha(\tau_{g^{-1}}(h))d_\beta(g).$$

Observe that

$$\begin{aligned}
&\|\delta_{(x,y)} * F_{\alpha,\beta} - F_{\alpha,\beta}\|_{L^1(H\rtimes_\tau G)} \\
&\qquad = \int_G\int_H \left| f_\alpha\left(\tau_{(yg)^{-1}}(\check{x} * h)\right) d_\beta(y^{-1}g) - f_\alpha(\tau_{g^{-1}}(h))d_\beta(g)\right| \delta(g)d\lambda_H(h)d\lambda_G(g) \\
&\qquad \leq \int_G \|\delta_{\tau_{(yg)^{-1}}(x)} * f_\alpha - f_\alpha\|_{L^1(H)}|d_\beta(g)|d\lambda_G(g) + \|\delta_y * d_\beta - d_\beta\|_{L^1(G)}.
\end{aligned}$$

For $K \subset H \rtimes_\tau G$ compact and $\varepsilon > 0$ there exists $\beta_{K,\varepsilon}$ such that

$$\|\delta_y * d_\beta - d_\beta\|_{L^1(G)} < \varepsilon/2 \quad \forall y \text{ such that } \exists x \text{ with } (x,y) \in K$$

and there exists $\alpha_{K,\varepsilon}$ such that

$$\|\delta_{\tau(yg)^{-1}(x)} * f_{\alpha_{K,\varepsilon}} - f_{\alpha_{K,\varepsilon}}\| < \frac{\varepsilon}{2}$$

for all $(x, y) \in K$ and $g \in \operatorname{supp}(d_{\beta_{K,\varepsilon}})$.

So the semidirect product satisfies the Reiter condition and hence is amenable.

□

3.5 Semigroups

Analogous results to those presented in previous sections can be formulated for semigroups. This section will deal with discrete semigroups. Rather than using a left Haar measure we consider the counting measure. The concept of a semidirect product is suitably modified. It is well known that a semigroup S is left amenable if and only if there is a net $(f_\alpha)_\alpha$ of elements in $\ell^1(S)_1^+$ for which $\|l_x^* f_\alpha - f_\alpha\| \to 0$ for all $x \in S$ ([7], [40]). Since S has the discrete topology, the collection of such nets is precisely the Reiter nets for S.

Definition 3.5.1: Let U and V be semigroups. Assume that V acts on U on the left, i.e., assume that there is a semigroup homomorphism τ from V to $\operatorname{End}(U)$ such that for each $v \in V$ there exists $\tau_v : U \to U$ such that $\tau_{v_1}(\tau_{v_2}(u)) = \tau_{v_1 v_2}(u)$ for all $u \in U$, $v_1, v_2 \in V$.

We say that $S = U \rtimes_\tau V$ is the *semidirect product* of U and V with respect to τ if S is the semigroup consisting of elements of the form (u, v) where $u \in U$

and $v \in V$ equipped with multiplication given by:

$$(u_1, v_1)(u_2, v_2) = (u_1 \tau_{v_1}(u_2), v_1 v_2).$$

Using τ, we can define a right action of V on $\ell^\infty(U)$ by:

$$\begin{aligned} T_v : &\ell^\infty(U) \to \ell^\infty(U) \\ &\phi \mapsto T_v \phi \end{aligned}$$

where $(T_v \phi)(u) = \phi(\tau_v(u))$. So we get a left action on $\ell^\infty(U)^*$ by considering T_v^* for each $v \in V$. Unlike in the group case each T_v^* is not necessarily isometric but $\|T_v^*\| \leq 1$. If $U = V$ and $\tau_v(u) = vu$ then we denote T_v by l_v.

We say that a net, $(f_\alpha)_\alpha$, in $\ell^1(U)_1^+$ is a *Reiter net* if

$$\|l_u^* f_\alpha - f_\alpha\|_{\ell^1(U)} \to 0$$

uniformly in u on compact subsets of U.

Remark 3.5.2: Since U has the discrete topology, the uniform convergence on compact (i.e. finite) subsets of U is equivalent to convergence for all $u \in U$. This makes our definition of Reiter nets equivalent to that of strongly regular nets found in [33], [34], and [36].

Left amenability of a semigroup U is defined in terms of a left invariant mean (see [7]). For simplicity, we remark that U is left amenable if and only if there is a Reiter net in $\ell^1(U)_1^+$. The counting measure on the semidirect product is precisely the product of the counting measures on N and H. For the remainder of this section U and V will be semigroups and τ a semigroup homomorphism from V to $\mathrm{End}(U)$. As well, $(f_\alpha)_\alpha$ and $(g_\beta)_\beta$ will be Reiter nets for U and V respectively.

Theorem 3.5.3: *Assume that* $\|T_y^* f_\alpha - f_\alpha\| \to 0$ *uniformly in* y *on compact subsets of* V*. Then* $(E_{\alpha,\beta})_{\alpha,\beta}$ *is a Reiter net for* $U \rtimes_\tau V$ *where* $E_{\alpha,\beta}(u,v) := f_\alpha(u)g_\beta(v)$*.*

Proof. The proof proceeds in a similar fashion to that of Theorem 3.2.1. We safely suppress the details. □

Remark 3.5.4: Unlike in the group case, there is not always a Reiter net which satisfies the condition of the theorem. Indeed, there are examples semidirect products of left-amenable semigroups which are not left amenable (see [26, 3.6]).

Theorem 3.5.5: *Define* $D_{\alpha,\beta} \in \ell^1(U \rtimes_\tau V)$ *via* $D_{\alpha,\beta}(u,v) := T_v^* f_\alpha(u) g_\beta(v)$*. Assume that each* g_β *is finitely supported. Further suppose that for each* $v \in V$ *the net* $(T_v^* f_\alpha)_\alpha$ *is a Reiter net. Then there is a subnet of* $(D_{\alpha,\beta})_{\alpha,\beta}$ *which is a Reiter net for* $U \rtimes_\tau V$*.*

Proof. It is straightforward to verify that $l_{(x,y)}^* D_{\alpha,\beta}(u,v) = l_x^* T_v^* f_\alpha(u) l_y^* g_\beta(v)$. It is then apparent that

$$\|l_{(x,y)}^* D_{\alpha,\beta} - D_{\alpha,\beta}\| \leq \sum_{v \in V} \|l_x^* T_v^* f_\alpha - T_v^* f_\alpha\| g_\beta(v) + \|l_y^* g_\beta - g_\beta\|.$$

The remainder of this proof mimics that of Theorem 3.4.5. We again suppress the details. □

Remark 3.5.6: For $x \in U$ and $v \in V$ $l_{\tau_v(x)}^* T_v^* f_\alpha = T_v^* l_x^* f_\alpha$ so if $\tau : V \to \mathrm{Sur}(U)$

then $U \rtimes_\tau V$ is left amenable. This provides a new proof of a result of Klawe [26, 3.4].

Remark 3.5.7: Explicit constructions of Reiter nets for bicyclic semigroups can be found in [36].

Chapter 4

Lau Algebras

4.1 Definition of Lau algebras

In [32], Lau introduced a type of Banach algebra (called F algebras in [32]) and defined left amenability of these algebras to correspond to left amenability of the measure algebra of a semigroup. The $L^1(H)$ algebra of a hypergroup (with left Haar measure) is a Lau algebra [51] and is left amenable precisely when the hypergroup is amenable. Other examples of Lau algebras include the Fourier and Fourier-Stieljes algebras of locally compact groups. In this chapter, the constructions of chapters 2 and 3 are adapted to the more general setting of Lau algebras. We present a characterization of left amenability using Lau algebra configuration equations. We define the semidirect product of a Lau algebra with a locally compact group and show that this semidirect product is again a Lau algebra. Furthermore, if the Lau algebra factor is left amenable and the group factor is amenable then the semidirect product is also left amenable.

Definition 4.1.1: [32] A *Lau Algebra* is a pair (A, M) such that A is a complex Banach algebra and M is a W^*-algebra such that $A = M_*$ and e, the identity of M, is a multiplicative linear functional on A.

Lau, [32] gives several equivalent characterizations of left amenability of (A, M). In particular, the following are equivalent:

1. The Lau algebra (A, M) is *left amenable*.

2. A^* has a topological left invariant mean. That is, there exists an $m \in (A^{**})_1^+$ such that

$$m(x \cdot \phi) = m(x) \quad \forall \phi \in A_1^+, x \in A^*.$$

3. There exists a net $\phi_\alpha \in A_1^+$ such that $\|\phi \cdot \phi_\alpha - \phi_\alpha\| \to 0$ for each $\phi \in A_1^+$.

4.2 Lau algebra configurations

Definition 4.2.1: Let (A, M) be a Lau algebra. Let $(\phi_1, \ldots, \phi_n) \in (A_1^+)^n$ and $\{f_1, \ldots, f_m\} \subset M$ such that each $f_i \geq 0$ and $\sum_{i=1}^m f_i = e_M$. We define an (A, M)-configuration as an ordered choice $C = (C_0, C_1, \ldots, C_n)$ with each $C_j \in \{1, \ldots, m\}$ and define $\xi_0(C)$ as before via:

$$\xi_0(C) = \prod_{j=0}^{n} C_j \cdot \phi_j.$$

Here the $\cdot$ represents the dual module action of A on M. In case the multiplication in M is non-commutative, we need only fix a convention for the ordering and keep to it throughout. For convenience, we'll assume that the multiplication is done left to right as j goes from 0 to n.

For $\phi \in A$ we define

$$\phi_C = \langle \xi_0(C), \phi \rangle.$$

We define the configuration equations as before as the mn equations in the m^{n+1} variables (z_C corresponding to the configuration C) as

$$\sum_{C, C_0 = i} z_C = \sum_{C, C_j = i} z_C.$$

A solution to the configuration equations is again said to be positive if each $z_C \geq 0$, normalized if $\sum_C z_C = 1$ and inequality preserving if for any choice of real numbers $\{a_C\}$

$$0 \leq \sum_C a_C \xi_0(C) \Rightarrow 0 \leq \sum_C a_C z_C.$$

Theorem 4.2.2: *A Lau algebra (A, M) is left amenable if and only if for all choices of $(\phi_1, \ldots, \phi_n) \in (A_1^+)^n$ and $\{f_1, \ldots, f_m\} \subset M$ such that each $f_i \geq 0$ and $\sum_{i=1}^m f_i = e_M$ the associated (A, M)-configuration equations have a positive, normalized, inequality preserving solution.*

Proof. We apply the method of lemma 2.1.2 to get:

$$\begin{aligned}
\sum_{C, C_j = i} \xi_0(C) &= \sum_C \prod_{l=0}^{n} f_{C_l} \cdot \phi_l \\
&= (\prod_{l=0}^{j-1} \sum_{k=1}^{m} f_k \cdot \phi_l)(f_i \cdot \phi_j)(\prod_{l=j}^{n} \sum_{k=1}^{m} f_k \cdot \phi_l) \\
&= (\prod_l e \cdot \phi_l) f_i \cdot \phi_j (\prod_l e \cdot \phi_l) \\
&= f_i \cdot \phi_j
\end{aligned}$$

so

$$\langle f_i, \phi_j \cdot \phi \rangle = \sum_{C, C_j = i} \phi_C$$

for any i and j and $\phi \in A$, noting that any rearrangment is only of a sum, and that the order of the multiplication in any term is unchanged.

If (A, M) is left amenable, there exists a topological left invariant mean, m, on M [32]. By letting $z_C = m(\xi_0(C))$, we gain a positive, inequality preserving, normalized solution to the configuration equations since

$$\begin{aligned}
\sum_{C, C_0 = i} z_C &= \sum_{C, C_0 = i} m(\xi_0(C)) \\
&= m(f_i) = m(f_i \cdot \phi_j) \\
&= \sum_{C, C_j = f_i} m(\xi_0(C)) \\
&= \sum_{C, C_j = f_i} z_C.
\end{aligned}$$

For the converse, Lemma 2.1.6 holds with $L^\infty(X, \mu)$ replaced by M (with the partial order of M replacing the a.e. ordering of $L^\infty(X)$) and we apply the same net construction as we do in theorem 2.1.7 to gain a net of means in $(A^{**})_1^+$ which tends weakly to topological left invariance which must have some accumulation point which is then a topological left invariant mean on M. □

4.3 Semidirect products of Lau algebras

We now define the notion of a semidirect product of a locally compact group with a Lau algebra. We remark that if the Lau algebra in question is the group algebra of a locally compact group, then the resulting semidirect product corresponds to the group algebra of the semidirect product.

Definition 4.3.1: Let G be a locally compact group and (A, M) be a Lau algebra. We say that T is an *action* of G on (A, M) if:

1. For each $g \in G$ there is an isometric isomorphism $T_g : A \to A$.

2. The map $g \in G \mapsto T_g \in \mathrm{Aut}(A)$ is a continuous group homomorphism.

3. For each $g \in G$, the dual map T_g^* is an isometric *-isomorphism of M onto itself.

If G acts on (A, M) the we define the *semidirect product* of G with (A, M) as the Lau algebra $(L^1_T(G, A), L^\infty(G)\bar{\otimes}M)$. Here, $L^1_T(G, A)$ is the Banach space $L^1(G, A)$ of integrable A-valued functions on G with a twisted multiplication. That is, for $F_1, F_2 \in L^1_T(G, A)$ we define the function $F_1 * F_2$ from G to A by:

$$F_1 * F_2(g) = \int_{h \in G} F_1(h) T_h(F_2(h^{-1}g))\, dh.$$

It is well-known (eg [53]) that $L^\infty(G)\bar{\otimes}M$ is indeed a von Neumann algebra and the dual of (the Banach space) $L^1_T(G, A)$.

Remark 4.3.2: The multiplication defined above is well defined and with it, the norm of $L^1_T(G, A)$ is submultiplicative. To see this, first consider simple tensors $f_1 \otimes a_1, f_2 \otimes a_2 \in K(G) \otimes A$. Then for $g \in G$

$$f_1 \otimes a_1 * f_2 \otimes a_2(g) = \int_G f_1(h) f_2(h^{-1}g) a_1 T_h(a_2)\, dh.$$

So $\operatorname{supp}(f_1 \otimes a_1 * f_2 \otimes a_2) \subset \operatorname{supp}(f_1)\operatorname{supp}(f_2)$ which is compact. Furthermore, the range of $f_1 \otimes a_1 * f_2 \otimes a_2$ is contained in $a_1 T_{\operatorname{supp}(f_1)\operatorname{supp}(f_2)}(a_2)$. Since the map $h \mapsto T_h$ is continuous the range of $f_1 \otimes a_1 * f_2 \otimes a_2$ is relatively compact in A, so $f_1 \otimes a_1 * f_2 \otimes a_2 \in L^1_T(G, A)$.

By linearity, we can extend this argument to all functions in $K(G) \otimes A$.

For $F_1, F_2 \in K(G) \otimes A$,

$$\begin{aligned} \|F_1 * F_2\| &= \int_{g \in G} \int_{h \in G} \|F_1(h) T_h(F_2(h^{-1}g))\|\, dhdg \\ &\leq \int_{h \in G} \|F_1(h)\| dh \int_{g \in G} \|F_2(g)\| dg \\ &= \|F_1\|\, \|F_2\| \end{aligned}$$

and so by density of $K(G) \otimes A$ in $L^1_T(G, A)$, we conclude that $F_1 * F_2 \in L^1_T(G, A)$ for any $F_1, F_2 \in L^1_T(G, A)$ and that with this multiplication, $L^1_T(G, A)$ is a Banach algebra.

Proposition 4.3.3: $(L^1_T(G, A), L^\infty \bar{\otimes} A^*)$ *is a Lau algebra.*

Proof. It is apparent that $L^1_T(G, A)$ is a Banach algebra and $L^\infty(G) \bar{\otimes} A^*$ is its dual. Since $L^\infty(G)$ and A^* are both W^* algebras the tensor product $L^\infty(G) \bar{\otimes} A^*$ is also a W^* algebra with identity $1 \otimes E_{A^*}$.

All that remains to show is that $1 \otimes E_{A^*}$ is a multiplicative linear functional on $L^1_T(G, A)$. Let $F_1, F_2 \in L^1_T(G, A)$.

Then

$$
\begin{aligned}
\langle F_1 * F_2, 1 \otimes e_{A^*} \rangle &= \int \langle F_1 * F_2(g), e_{A^*} \rangle dg \\
&= \int \int \langle F_1(h) T_h F_2(h^{-1}g), e_{A^*} \rangle dh dg \\
&= \int \int \langle F_1(h), e_{A^*} \rangle \langle T_h F_2(h^{-1}g), e_{A^*} \rangle dh dg \\
&= \int \int \langle F_1(h), e_{A^*} \rangle \langle F_2(g), T_h^* e_{A^*} \rangle dg dh.
\end{aligned}
$$

The image of e_{A^*} is always e_{A^*} and therefore these integrals are separable. Thus

$$
\begin{aligned}
\langle F_1 * F_2, 1 \otimes e_{A^*} \rangle &= \int \langle F_1(h), e_{A^*} \rangle dh \int \langle F_2(g), e_{A^*} \rangle dg \\
&= \langle F_1, 1 \otimes e_{A^*} \rangle \langle F_2, 1 \otimes e_{A^*} \rangle
\end{aligned}
$$

□

Lemma 4.3.4: *The positive elements of* $L^1_T(G, A)$ *can be characterized by:*

$$
L^1(G, A)^+_1 = \{F \in L^1_T(G, A) : F(g) \in A^+_1 \textit{ for almost every } g \in G\}.
$$

Proof. It is clear that "$\supseteq$" holds.

To see the converse, suppose that $F \in L^1_T(G, A)$ such that $\{g \in G : F(g) \notin A^+_1\}$ has positive measure. Then we can find $\varepsilon > 0$ such that $\{g \in G : \exists m \in A^*, \|m\| = 1, \inf_{\alpha \in \mathbb{R}_{\geq 0}} |\langle F(g), m^* m \rangle - \alpha| > \varepsilon\}$ has positive measure. Let K be a compact subset of this set with $L := \lambda(K) > 0$. Then we can find a compact $K_0 \subset K$ such that F is continuous on K_0 and $\lambda(K \backslash K_0) \leq L/2$. Pick $x_0 \in K_0$ such that $C := \{g \in K_0 : \|F(g) - F(x_0)\| < \varepsilon/2\}$ has positive measure. Let

$m_0 \in A^*$ with $\|m_0\| = 1$ and $\inf_{\alpha \in \mathbb{R}_{\geq 0}} |\langle F(x_0), m_0^* m_0 \rangle - \alpha| > \varepsilon$. Then consider

$$\langle F, (\chi_C \otimes m_0)^*(\chi_C \otimes m_0) \rangle = \int_C \langle F(g), m_0^* m_0 \rangle dg$$
$$\int_C \langle F(x_0), m_0^* m_0 \rangle dg = \lambda(C) \langle F(x_0), m_0^* m_0 \rangle$$
$$\left| \int_C \langle F(x_0) - F(g), m_0^* m_0 \rangle dg \right| \leq \lambda(C)\varepsilon/2$$

So for $\alpha \in \mathbb{R}_{\geq 0}$ we have

$$\begin{aligned} &|\langle F, (\chi_C \otimes m_0)^*(\chi_C \otimes m_0) \rangle - \alpha| \\ &\qquad = |\int \langle F(g) - f(x_0), m_0^* m_0 \rangle dg + \int \langle f(x_0), m_0^* m_0 \rangle dg - \alpha| \\ &\qquad \geq |\lambda(C) \langle f(x_0), m_0^* m_0 \rangle - \alpha| - \lambda(C)\varepsilon/2 \end{aligned}$$

Hence $\langle F, (\chi_C \otimes m_0)^*(\chi_C \otimes m_0) \rangle$ is not in $\mathbb{R}_{\geq 0}$. □

Theorem 4.3.5: *Let G be an amenable locally compact group which acts continuously on a left-amenable Lau algebra (A, M). Suppose that $(f_\beta)_\beta \subset L^1(G)_1^+$ is a Reiter net for G and that $(\phi_\alpha)_\alpha \subset A_1^+$ is a net satisfying condition 3 of definition 4.1.1. Suppose also that*

$$\|T_g \phi_\alpha - \phi_\alpha\|_A \to 0$$

uniformly in g on compact subsets of G. For each α and β, let $F_{\alpha,\beta} \in L_T^1(G, A)_1^+$ be given by $F_{\alpha,\beta}(g) = f_\beta(g)\phi_\alpha$. Then the net $(F_{\alpha,\beta})_{\alpha,\beta}$ satisfies condition 3 of definition 4.1.1 for $L_T^1(G, A)$.

Proof. Fix $\varepsilon > 0$. Consider $F \in L^1(G, A)_1^+$.

Then for any α and β:

$$\begin{aligned}\|F * F_{\alpha,\beta} - F_{\alpha,\beta}\|_1 &= \int_G \|F * F_{\alpha,\beta}(g) - F_{\alpha,\beta}(g)\| \, dg \\ &= \int_G \left\| \int_{\text{supp}(F)} F(h) T_h \phi_\alpha f_\beta(h^{-1}g) dh - \phi_\alpha f_\beta(g) \right\| dg \\ &= \int_G \left\| \int_{\text{supp}(F)} F(h) T_h(\phi_\alpha) f_\beta(h^{-1}g) - \|F(h)\| \phi_\alpha f_\beta(g) dh \right\| dg\end{aligned}$$

By taking the norm inside the integral and changing the order of integration we get:

$$\begin{aligned}\|F * F_{\alpha,\beta} - F_{\alpha,\beta}\|_1 &\le \int\limits_{\text{supp}(F)} \|F(h)\| \left(\int_G \left\| \frac{F(h)}{\|F(h)\|} T_h(\phi_\alpha) f_\beta(h^{-1}g) - f_\beta(h^{-1}g)\phi_\alpha \right\| \right. \\ &\qquad \left. + \left\| f_\beta(h^{-1}g)\phi_\alpha - \phi_\alpha f_\beta(g) \right\| dg \right) dh \\ &= \int\limits_{\text{supp}(F)} \|F(h)\| \left\| \frac{F(h)}{\|F(h)\|} T_h(\phi_\alpha) - \phi_\alpha \right\| dh \\ &\qquad + \int \|F(h)\| \|\delta_h * f_\beta - f_\beta\| dh\end{aligned}$$

So if we chose a compact $K \subset G$ such that $\int_{G \backslash K} \|F(h)\| dh < \varepsilon$ then we can find β_0 so that if $\beta \ge \beta_0$, $\|\delta_h * f_\beta - f_\beta\| < \varepsilon$ for all $h \in K$. Hence

$$\begin{aligned}\int_G \|F(h)\| \|\delta_h * f_\beta - f_\beta\| dh &\le \int_K \|F(h)\| \|\delta_h * f_\beta - f_\beta\| dh \\ &\qquad + \int_{G\backslash K} \|F(h)\| \|\delta_h * f_\beta - f_\beta\| dh \\ &< \varepsilon + 2\varepsilon\end{aligned}$$

Since the map $h \mapsto T_{h^{-1}} \left(\frac{F(h)}{\|F(h)\|} \right)$ is measurable, we can find a compact $K_0 \subset K$ with $\lambda(K \backslash K_0) < \varepsilon$ and $T_{h^{-1}} \left(\frac{F(h)}{\|F(h)\|} \right)$ continuous on K_0. Hence we can find an

α_0 such that for all $h \in K_0$ and all $\alpha \geq \alpha_0$

$$\left\| T_{h^{-1}} \left(\frac{F(h)}{\|F(h)\|} \right) \phi_\alpha - \phi_\alpha \right\| < \varepsilon$$

Also, by the assumption, we can find $\alpha_1 \geq \alpha_0$ such that for $\alpha \geq \alpha_1$ it follows that $\|T_h\phi_\alpha - \phi_\alpha\| < \varepsilon$ for all $h \in K_0$. So for $\alpha \geq \alpha_1$

$$\begin{aligned}
&\int_{\mathrm{supp}(F)} \|F(h)\| \left\| \frac{F(h)}{\|F(h)\|} T_h(\phi_\alpha) - \phi_\alpha \right\| dh \\
&\quad \leq \int_{\mathrm{supp}(F)} \|F(h)\| \left(\left\| \frac{F(h)}{\|F(h)\|} T_h(\phi_\alpha) - T_h\phi_\alpha \right\| + \|T_h\phi_\alpha - \phi_\alpha\| \right) dh \\
&\quad = \int_{K_0} \|F(h)\| \left(\|T_h\| \left\| T_{h^{-1}} \left(\frac{F(h)}{\|F(h)\|} \right) \phi_\alpha - \phi_\alpha \right\| + \|T_h\phi_\alpha - \phi_\alpha\| \right) dh \\
&\qquad + \int_{G \setminus K_0} \|F(h)\| \left(\left\| \frac{F(h)}{\|F(h)\|} T_h(\phi_\alpha) - T_h\phi_\alpha \right\| + \|T_h\phi_\alpha - \phi_\alpha\| \right) dh \\
&\quad < \varepsilon + \varepsilon + 4\varepsilon.
\end{aligned}$$

Then, for $\alpha \geq \alpha_1$ and $\beta \geq \beta_0$ we have $\|F * F_{\alpha,\beta} - F_{\alpha,\beta}\|_1 < 9\varepsilon$. □

Theorem 4.3.6: *Let G be an amenable locally compact group which acts continuously on a left-amenable Lau algebra (A, M). Suppose that $(f_\beta)_\beta$ is a Reiter net for G such that* $\mathrm{supp}(f_\beta)$ *is compact for each β and that $(\phi_\alpha)_\alpha$ is a net in A_1^+ satisfying condition 3 of definition 4.1.1. For each α and β, let $F_{\alpha,\beta} \in L^1_T(G, A)^+_1$ be given by $F_{\alpha,\beta}(g) = f_\beta(g)T_g(\phi_\alpha)$. Then there exists a subnet of $(F_{\alpha,\beta})_{\alpha,\beta}$ which satisfies condition 3 of definition 4.1.1 for $L^1_T(G, A)$.*

Proof. Fix $\varepsilon > 0$ and $F \in L^1_T(G, A)^+_1$. Then for any α and β:

$$\begin{aligned}
\|F * F_{\alpha,\beta} - F_{\alpha,\beta}\|_1 &= \int_G \left\| \int_G F(h)T_h(T_{h^{-1}g}\phi_\alpha f_\beta(h^{-1}g))dh - T_g\phi_\alpha f_\beta(g) \right\| dg \\
&\leq \|T_g\| \int_G \int_G \left\| T_{g^{-1}}(F(h))\phi_\alpha f_\beta(h^{-1}g) - \|F(h)\|\phi_\alpha f_\beta(g) \right\| dhdg.
\end{aligned}$$

By applying the triangle inequality we get

$$\begin{aligned}\|F * F_{\alpha,\beta} - F_{\alpha,\beta}\|_1 \le \int_G \int_G & \left\| T_{g^{-1}}(F(h))\phi_\alpha f_\beta(h^{-1}g) - \|F(h)\|\phi_\alpha f_\beta(h^{-1}g)\right\| \\ & + \left\| \|F(h)\|\phi_\alpha f_\beta(h^{-1}g) - \|F(h)\|\phi_\alpha f_\beta(g)\right\| dhdg \\ \le & \int_{\mathrm{supp}(F)} \|F(h)\| \int_{\mathrm{supp}(g_\beta)} \left\| T_{(hg)^{-1}}\left(\frac{F(h)}{\|F(h)\|}\right)\phi_\alpha - \phi_\alpha \right\| f_\beta(g) dg dh \\ & + \int_{\mathrm{supp}(F)} \|F(h)\| \, \|l_h^* f_\beta - f_\beta\| \, dh.\end{aligned}$$

Since the compactly supported functions are dense in $L^1_T(G, A)$, we may assume that F is compactly supported. Since $(f_\beta)_\beta$ is a Reiter net, there is a $\beta_{F,\varepsilon}$ such that for $h \in \mathrm{supp}(F)$, $\|l_h^* f_{\beta_{F,\varepsilon}} - f_{\beta_{F,\varepsilon}}\| < \varepsilon$.

We can also find an $\alpha_{F,\varepsilon,\beta_{F,\varepsilon}}$ such that for all $g \in \mathrm{supp}(g_{\beta_{F,\varepsilon}})$ and $h \in \mathrm{supp}(F)$, $\left\| T_{(hg)^{-1}}\left(\frac{F(h)}{\|F(h)\|}\right)\phi_\alpha - \phi_\alpha\right\| < \varepsilon$.

Consider the directed set Λ, where each element of Λ is a quadruple consisting of a compactly supported $F \in L^1_T(G, A)^+_1$, an $\varepsilon > 0$, a $\beta \ge \beta_{F,\varepsilon}$, and an $\alpha \ge \alpha_{F,\varepsilon,\beta}$. The order we put on Λ is $\preceq$ where $(F_1, \varepsilon_1, \beta_1, \alpha_1) \preceq (F_2, \varepsilon_2, \beta_2, \alpha_2)$ if $\mathrm{supp}(K_1) \subset \mathrm{supp}(K_2)$, $\varepsilon_1 \ge \varepsilon_2$, $\beta_1 \le \beta_2$, and $\alpha_1 \le \alpha_2$. From the above observations, it is apparent that for any ε and F, there exists α and β such that $\|F * F_{\alpha,\beta} - F_{\alpha,\beta}\|_1 < \varepsilon$.

Then $(F_{\alpha,\beta})_{F,\varepsilon,\beta,\alpha}$ is a subnet of $(F_{\alpha,\beta})_{\alpha,\beta}$ which is an appropriate net. □

Corollary 4.3.7: *The semidirect product of an amenable locally compact group with a left-amenable Lau algebra is again a left-amenable Lau algebra.*

Chapter 5

Other Generalizations

5.1 Configurations on subspaces of $L^\infty(H)$

Remark 5.1.1: The approach of configuration equations can be extended from considering partitions of H into subsets to considering partitions of the identity into continuous functions and dealing with the existence of a left invariant mean on a space of continuous functions rather than $L^\infty(H)$. One motivation for this approach is to characterize amenability without assuming the existence of a left Haar measure.

Let H be a hypergroup and $\mathcal{A}$ be a norm closed subalgebra of $C(H)$ which is closed under left translation, pointwise multiplication, lattice operations (min and max) and contains the identity. Examples of such algebras include the continuous and bounded functions on H, the uniformly continuous functions on H and if H is a locally compact group, the almost periodic functions.

Lemma 2.1.6 applies verbatim to such algebras with $\mathcal{A}$ in place of $L^\infty(X, \mu)$.

Definition 5.1.2: Let $(h_1, \ldots, h_n) \in H^n$ and $\{f_1, \ldots, f_m\} \subset \mathcal{A}$ such that each $f_i \geq 0$ and $\sum_{i=1}^n f_i = \chi_H$. We define a $\mathcal{A}$-configuration as an ordered choice of the f_is, $C = (c_0, c_1, \ldots, c_n)$ and define $\xi_0(C)$ as before via:

$$\xi_0(C) = \prod_{j=0}^{n} \delta_{\check{h}_j} * c_j.$$

Because $\mathcal{A}$ is translation invariant and closed under multiplication, it follows that $\xi_0(C)$ is an element of $\mathcal{A}$.

For $\mu \in M(H)$ we define $\mu_C = \int_H \xi_0(C)(t) d\mu(t)$.

Lemma 2.1.2 can be applied verbatim and we see that

$$\int f_i d\mu = \sum_{C, c_0 = f_i} \mu_C$$

and

$$\int \delta_{\check{h}_j} * f_i d\mu = \sum_{C, c_j = f_i} \mu_C$$

We define the $\mathcal{A}$-configuration equations similar to before as the mn equations in the m^{n+1} variables (z_C corresponding to the $\mathcal{A}$-configuration C) as

$$\sum_{C, C_0 = i} z_C = \sum_{C, C_j = i} z_C.$$

A solution to the $\mathcal{A}$-configuration equations is again said to be positive if each $z_C \geq 0$, normalized if $\sum_C z_C = 1$ and inequality preserving if for any choice of real numbers $\{a_C\}$

$$0 \leq \sum_C a_C \xi_0(C) \;\Rightarrow 0 \leq \sum_C a_C z_C.$$

Indeed, even further, we can show by using the method of proof of Theorem 2.1.7 that the following result holds for generalized configurations.

Theorem 5.1.3: *There exists a left invariant mean on $\mathcal{A}$ iff for all choices of $(h_1, \ldots, h_n) \in H^n$ and partitions of χ_H, $\{f_1, \ldots, f_m\} \subset \mathcal{A}$ the associated $\mathcal{A}$-configuration equations have a positive, normalized, inequality preserving solution.*

5.2 When is $L^\infty(H)$ a Hopf von Neumann algebra

Most of the results of this thesis were motivated by considering hypergroups as a generalization of locally compact groups. Chapter 4 is devoted to considering Lau algebras as a generalization of hypergroup algebras. Subsequently it is natural to ask whether there are other interesting classes of algebras which generalize hypergroup algebras and might retain more structure.

Locally compact quantum groups and Hopf-von Neumann algebras have recently been investigated as interesting generalizations of locally compact groups (eg. [19, 20]). Indeed, recently Daws and Runde [6] have generalized the Reiter's property characterization of amenability to the locally compact quantum group case. Unfortunately, as we show below, the overlap between locally compact quantum groups and hypergroups is limited to just the group case.

Definition 5.2.1: A *Hopf-von Neumann algebra* is a pair $(\mathcal{A}, \Gamma)$ where $\mathcal{A}$ is a von Neumann algebra and Γ is a *co-multiplication*, ie. $\Gamma : \mathcal{A} \to \mathcal{A}\bar{\otimes}\mathcal{A}$ is a normal, unital, injective *-homomorphism satisfying $(I \otimes \Gamma) \circ \Gamma = (\Gamma \otimes I) \circ \Gamma$.

Let G be a locally compact group. Then $L^\infty(G)$ is a Hopf-von Neumann algebra with co-multiplication induced from convolution on its predual, $L^1(G)$.

Theorem 5.2.2: *Let H be a hypergroup with left Haar measure λ. Then $L^\infty(H)$ is a Hopf-von Neumann algebra with the co-multiplication induced by convolution on $L^1(H)$ if and only if H is a locally compact group.*

Proof. It is well known that $L^\infty(G)$ is a Hopf-von Neumann algebra for any locally compact group G (see, for example, [6] for details).

For the converse suppose that $L^\infty(H)$ is indeed a Hopf-von Neumann algebra and that $\langle \Gamma(F), f \otimes g\rangle = \langle F, f * g\rangle$ for $F \in L^\infty(H)$ and $f, g \in L^1(H)$.

Claim: For any $F \in L^\infty(H)$

$$\Gamma(F)(x,y) = F(x * y) \text{ for almost every } (x,y) \in H \times H.$$

For any $f, g \in L^1(H)$:

$$\begin{aligned}
\langle \Gamma(F), f \otimes g\rangle &= \langle F, f * g\rangle \\
&= \iint F(y) f(y * x) g(\check{x})\, dx\, dy \\
&= \iint F(y) f(x) \check{g}(\check{y} * x)\, dx\, dy \\
&= \iint F(y) f(x) g(\check{x} * y)\, dx\, dy \\
&= \iint F(x * y) f(x) g(y)\, dx\, dy
\end{aligned}$$

hence the claim holds.

Now we will show that because Γ is a homomorphism, for each $x, y \in H$ the product of the point measures $\delta_x * \delta_y$ is again a point measure.

Suppose for contradiction that there exist $x, y \in H$ such that $t_1 \neq t_2 \in \operatorname{supp}(\delta_x * \delta_y)$. Then we can find open sets $\mathcal{O}_1, \mathcal{O}_2$ such that $t_i \in \mathcal{O}_i$ for $i = 1, 2$ and $\mathcal{O}_1 \cap \mathcal{O}_2 = \emptyset$.

Choose $k_1, k_2 \in C_C^+(H)$ such that $\operatorname{supp}(k_i) \subset \mathcal{O}_i$ and $k_i(x * y) = 1$. Since the map $(u, v) \mapsto \delta_u * \delta_v$ is continuous (from the product topology to the cone topology), there exist compact neighbourhoods A of x and B of y such that for any $(u, v) \in A \times B$, we have

$$\frac{1}{2} \leq k_i(u * v) \leq \frac{3}{2}.$$

Let $F_2 \in C_C^+(H) \subset L^\infty(H)$ such that F_2 is zero on $\operatorname{supp}(k_1)$ and one on $\operatorname{supp}(k_2)$ and $F_1 := \alpha k_1 + k_2 (\in L^\infty(H))$ where $\alpha = \frac{3}{\inf\{F_2(u*v):(u,v)\in A\times B\}}$.

Since Γ is a homomorphism, $\langle \Gamma(F_1 F_2), f \otimes g \rangle = \langle \Gamma(F_1)\Gamma(F_2), f \otimes g \rangle$ for any $f, g \in L^1(H)$. Consider $f = \chi_A$ and $g = \chi_B$. Then:

$$\begin{aligned} \langle \Gamma(F_1 F_2), \chi_A \otimes \chi_B \rangle &= \iint (F_1 F_2)(u * v) \chi_A(u) \chi_B(v) \, du \, dv \\ &= \iint k_2(u * v) \chi_A(u) \chi_B(v) \, du \, dv \end{aligned}$$

So

$$\frac{\lambda(A)\lambda(B)}{2} \leq \langle \Gamma(F_1 F_2), \chi_A \otimes \chi_B \rangle \leq \frac{3\lambda(A)\lambda(B)}{2}.$$

But,

$$\begin{aligned} \langle \Gamma(F_1)\Gamma(F_2), \chi_A \otimes \chi_B \rangle &= \iint F_1(u * v) F_2(u * v) \chi_A(u) \chi_B(v) \, du \, dv \\ &= \iint (\alpha k_1(u * v) + k_2(u * v)) F_2(u * v) \chi_A(u) \chi_B(v) \, du \, dv \end{aligned}$$

So

$$\frac{(\alpha+1)\inf\{F_2(u*v) : (u,v)\in A\times B\}\lambda(A)\lambda(B)}{2} \leq \langle \Gamma(F_1F_2), \chi_A\otimes\chi_B\rangle.$$

Then

$$\langle \Gamma(F_1F_2), \chi_A\otimes\chi_B\rangle < \langle \Gamma(F_1F_2), \chi_A\otimes\chi_B\rangle$$

which contradicts Γ being a homomorphism. Therefore the support of any measure $\delta_x * \delta_y$ must be a singleton. By a result of Jewett [23, 4.1] this implies that H is a locally compact group. □

Chapter 6

Fixed point properties and left Haar measures

In this chapter, we present two generalizations of fixed point properties which apply to hypergroups. Using a fixed point property motivated by a similar property for groups proved by Rickert [45], we prove the existence of a left Haar measure for a class of amenable hypergroups satisfying a certain uniformity of convolution property.

6.1 Fixed point properties

Definition 6.1.1: Let E be a Hausdorff locally convex vector space and let $K \subset E$ be a compact, convex subset. Suppose that there is a separately continuous mapping $\cdot : H \times K \to K$. Then for $x, y \in H$ and $\xi \in K$ the weak integral

$$\int_H (t \cdot \xi) d(\delta_x * \delta_y)(t)$$

exists uniquely in K.

An *action* of H on K is a separately continuous mapping $\cdot : H \times K \to K$ such that

- $e \cdot \xi = \xi$ for all $\xi \in K$;
- $x \cdot (y \cdot \xi) = \int_H (t \cdot \xi) d(\delta_x * \delta_y)(t)$.

Furthermore, the action is called affine if, for each $x \in H$ $\xi \mapsto x \cdot \xi$ is affine.

The following theorem is a modification of a result of Rickert as presented in [42].

Theorem 6.1.2: *Suppose that $UCB_r(H)$ has a left invariant mean m. Then for each jointly continuous affine action of H on some K, a compact convex subset of a Hausdorff locally convex vector space, there is a point $\xi_0 \in K$ such that $x \cdot \xi_0 = \xi_0$ for all $x \in H$. Furthermore, the result holds if $UCB_r(H)$ is replaced by $WUCB_r(H)$ and jointly continuous is replaced by separately continuous.*

Proof. Suppose that there is a hypergroup action of H on K. We denote the set of affine functions on K by $Aff(K)$. It is clear that for each point $\xi \in K$, evaluation at ξ is a mean on $Aff(K)$. Indeed, K can be identified with the collection of **all** means on $Aff(K)$ and this identification is an affine homeomorphism. (See for instance [42] 2.20)

Given this identification, we see that the existence of a fixed point in K is equivalent to the existence of a mean on $Aff(K)$ which is invariant under the action of H.

Suppose that $\phi \in Aff(K)$ and $\xi_0 \in K$. Let $\hat{\xi}_0(\phi) \in C(H)$ be defined by $\hat{\xi}_1(\phi)(x) = \phi(x \cdot \xi_1)$. It is clear that $\hat{\xi}_1(\phi)$ is a continuous function provided that that action of H is (separately) continuous. We further claim that if the action is separately continuous then $\hat{\xi}_1(\phi)$ is in $WUCB_r(H)$ and if it is jointly continuous then $\hat{\xi}_1(\phi)$ is in $UCB_r(H)$.

To show the first claim, consider some mean F on $C(H)$ and some net $x_\alpha \to x$ in H. Then $F \circ \hat{\xi}_1 : Aff(K) \to \mathbb{C}$ is a mean on $Aff(K)$. It follows then, that there is some $\xi_2 \in K$ such that $F \circ \hat{\xi}_1(\phi) = \phi(\xi_2)$. Therefore,

$$\begin{aligned} \langle F, \delta \check{x_\alpha} * \hat{\xi}_1(\phi) \rangle &= \langle F \circ \hat{\xi}_1, \delta \check{x_\alpha} * \phi \rangle \\ &= \delta \check{x_\alpha} * \phi(\xi_2) \\ &= \phi(x_\alpha \cdot \xi_2) \\ &\to \phi(x \cdot \xi_2) \\ &= \langle F, \delta \check{x} * \hat{\xi}_1(\phi) \rangle \end{aligned}$$

From this we conclude that for any $F \in C(H)^*$ the same holds. That is, that $\hat{\xi}_0(\phi)$ is in $WUCB_r(H)$. Now since there is a left invariant mean M on $WUCB_r(H)$ it follows that $M \circ \hat{\xi}_1$ is a mean on $Aff(K)$ which corresponds to evaluation at some point $\xi_0 \in K$. It is apparent that ξ_0 is a fixed point of the action of H.

For the second claim, observe that $\|\delta_{\check{x_\alpha}} * \phi - \delta_{\check{x}} * \phi\| \to 0$ as $x_\alpha \to x$ since the action is jointly continuous. Furthermore, since $\hat{\xi}$ is contractive for each $\xi \in K$, $\|\delta_{\check{x_\alpha}} * \hat{\xi}_1(\phi) - \delta_{\check{x}} * \hat{\xi}_1(\phi)\| \to 0$ as $x_\alpha \to x$. But this shows that $\hat{\xi}_1(\phi)$ is in $UCB_r(H)$ as required. By a similar argument as above, the mean on

$UCB_r(H)$ generates a fixed point in K.

$\square$

The Ryll-Nardzewski fixed point theorem states that if a group G acts affinely on a compact convex subset K of a locally convex space in such a way that for some continuous seminorm ρ, for any two distinct points $x, y \in K$ there exists an ε such that $\rho(g.x - g.y) > \varepsilon$ for all $g \in G$, then there is a common fixed point. A key idea of the proof of given by [41] is that given two points in K we can 'dent' K by a sufficiently small amount so that the orbit of the midpoint of the two points remains in the 'dented' compact convex subset of K. Then, using Zorn's Lemma, it follows that the minimal such subset must be a singleton - a common fixed point.

We now present a variation of the Ryll-Nardzewski theorem by considering the action of a hypergroup on a locally convex lattice. We borrow many ideas from the proof by [41] but we use a strong order unit rather than a seminorm to describe the 'small dent'.

Definition 6.1.3: A *vector lattice* or Riesz space is a vector space V over the field $\mathbb{R}$ along with a partial order $\leq$ such that for any $x, y, z \in V$ and $\alpha \in \mathbb{R}$:

- if $x \leq y$ then $x + z \leq y + z$;
- if $x \geq 0$ and $\alpha \geq 0$ then $\alpha x \geq 0$;
- $\sup\{x, y\}$ and $\inf\{x, y\}$ exist.

The collection of vectors $\{x \in V : x \geq 0\}$ is called the positive cone of V.

Definition 6.1.4: A linear functional f from V to $\mathbb{R}$ is *increasing* if $f(x) \geq 0$ whenever $x \geq 0$. The span of increasing linear functionals forms the order dual space V^*. V^* is an order complete vector lattice [11][31A].

Let X be a locally compact Hausdorff topological space. With the pointwise ordering, $C_C(X)$ is a vector lattice ([11][A2D]). By the Riesz representation theorem the regular Borel measures on X can be identified with the increasing linear functionals on $C_C(X)$.(See [16] §11 or [11][73D])

Definition 6.1.5: A seminorm ρ on a vector lattice X is a *lattice seminorm* if for any $x, y \in X$ with $|x| \leq |y|$ it follows that $\rho(x) \leq \rho(y)$.

A *locally convex vector lattice* is a locally convex vector space $(X, \mathfrak{T})$ with a lattice order $\leq$ such that $\mathfrak{T}$ is generated by lattice seminorms.

Suppose that H acts on K. Let K' be a non-empty compact convex subset of K. We say that K' is fixed under the action of H if for each $x \in H$ we have that $T_x(K') \subset K'$. Order the non-empty compact convex subsets of K which are fixed under the action of H by inclusion. For any chain $\mathfrak{C}$ of such subsets, it is apparent that the intersection $\cap\{C \in \mathfrak{C}\}$ is again convex and fixed under the action of H and is compact and non-empty by the finite intersection property of $\mathfrak{C}$ and the compactness of K. Hence $\mathfrak{C}$ has a upper bound. By Zorn's Lemma we can then conclude that there is a minimal non-empty compact convex $K_0 \subset K$ which is fixed under the action of H.

Theorem 6.1.6: *Let H be a hypergroup, V a locally convex vector lattice, $K \subset V$ be a non-empty compact convex subset of the positive cone of V. Suppose that H acts affinely on K such that for any $v, w \in K$, $v \neq w$, there exists an element $u \in V$ such that $|x \cdot (v) - x \cdot (w)| > 2u$ for each $x \in H$ and* *such that $cu \leq K$ and $(c+1)u \nleq K$ for some $c \in \mathbb{R}^+$. Then there is a common fixed point in K for the action of H.*

Proof. Let K_0 be a minimal non-empty compact convex subset of K which is invariant under the action of H.

Assume for contradiction that K_0 is not a singleton. Let $v \neq w \in K_0$. Let $u \in \text{span}(K_0)$ and $c \in \mathbb{R}^+$ be such that $|x \cdot (v) - x \cdot (w)| > 2u$ for each $x \in H$ and such that $cu \leq K_0$ and $(c+1)u \nleq K_0$.

Let $K_1 = \{z \in K_0 : (c+1)u \leq z\}$. It is clear that K_1 is proper non-empty convex subset of K_0. Since the positive cone V^+ is closed, and $K_1 = K_0 \cap ((c+1)u + V^+)$ it follows that K_1 is compact.

If $x \cdot (\frac{1}{2}(v+w)) \notin K_1$ for some $x \in H$ then $(c+1)u \nleq c \cdot (\frac{1}{2}(v+w))$. Since $|x \cdot v - x \cdot (\frac{1}{2}(v+w))| = \frac{1}{2}|x \cdot (v) - x \cdot (w)| \nleq u$ and $x \cdot (\frac{1}{2}(v+w)) \in K_0 \backslash K_1$ it follows from the consideration that $K_0 \backslash K_1$ is contained in a band of width u that $x \cdot v \notin K_0 \backslash K_1$ so $x \cdot v \in K_1$. Similarly, $x \cdot w \in K_1$. This cannot be since K_1 is convex and $x \cdot (\frac{1}{2}(v+w))$ would then necessarily be in K_1. Hence $\overline{\{x \cdot (\frac{1}{2}(v+w)) : x \in H\}} \subset K_1$ is a proper non-empty compact convex subset of K_0 which is invariant under the action of H. This contradicts the minimality of K_0 so $K_0 = \{x_0\}$ is a singleton and x_0 is a common fixed point of the action of H. □

6.2 Towards a left Haar measure

The existence of a Haar measure on an arbitrary hypergroup is still an open question. In this section we present an approach motivated by Izzo [21]. Izzo used the Markov-Kakutani fixed point theorem to prove the existence of a Haar measure on abelian groups. We use a more general fixed point theorem (proven by Rickert for topological groups [45]) to show that amenable hypergroups which satisfy an additional property have a left Haar measure.

Remark 6.2.1: Let G be a locally compact group. G satisfies the property that for any neighbourhood U of the identity and any $x \in G$

$$\delta_{x^{-1}} * \delta_x(U) \geq \delta_t * \delta_x(U) \ \ \forall\ t \in G.$$

However, this property is not satisfied by general hypergroups. In particular, it is not satisfied by the double coset hypergroup $(\mathbb{R}^2)^{\mathbb{T}}$.

Definition 6.2.2: We say that a hypergroup H satisfies condition (C1) if there exist neighbourhoods U and V of the identity such that for every $x \in H$

$$\delta_{\check{x}} * \delta_x(U) \geq \delta_t * \delta_x(V)\ \forall\ t \in H.$$

We say H satisfies (C2) if for every neighbourhood U, there exists a neighbourhood V of the identity such that for every $x \in H$

$$\delta_{\check{x}} * \delta_x(U) \geq \delta_t * \delta_x(V)\ \forall\ t \in H.$$

Lemma 6.2.3: *Suppose that H has (C1) and U and V are given as above. Then the collection K of all positive linear functionals on $C_C(H)$ satisfying:*

- $\Lambda(f) \leq 1$ *for any* $f \in C_C(H)$ *such that* $f \leq \nu * \chi_V$ *for some* $\nu \in M(H)_1^+$.

- $\Lambda(f) \geq 1$ *for any* $f \in C_C(H)$ *such that* $f \geq \nu * \chi_U$ *for some* $\nu \in M(H)_1^+$.

is non-empty.

Proof. By Zorn's Lemma there exists a maximal $S \subset M^+(H)$ subset of positive measures satisfying

$$\sum_{\mu \in S} \chi_V * \mu(x) \leq 1 \quad \forall x \in G.$$

We claim that $\sum_{\mu \in S} \chi_U * \mu(x) \geq 1$ for all $x \in G$.

Suppose for contradiction that there exists $y \in G$ such that $1 - \sum_{\mu \in S} \chi_U * \mu(y) =: \varepsilon > 0$. Then for $x \in Ny$ we have:

$$\begin{aligned}
\sum_{\mu \in S} \chi_V * \mu(x) + \varepsilon \chi_V * \delta_y(x) &\leq \sum_{\mu \in S} \chi_V * \mu(x) + \varepsilon \\
&\leq \sum_{\mu \in S} \delta_{x^{-1}} * \chi_V * \mu(e) + \varepsilon \\
&\leq \sum_{\mu \in S} \delta_{y^{-1}} * \chi_U * \mu(e) + \varepsilon \\
&= \sum_{\mu \in S} \chi_U * \mu(y) + \varepsilon \\
&= 1
\end{aligned}$$

This contradicts the maximality of S, so it follows that

$$\sum_{\mu \in S} \chi_U * \mu(x) \geq 1 \quad \forall x \in H.$$

Let $\Lambda(f) := \sum_{\mu \in S} \mu(\check{f})$ where $\check{f}(x) = f(x^{-1})$. Suppose $f \in C_C(H)$ such that

$f \leq \nu * \chi_V$ for some compactly supported $\nu \in M(H)_1^+$. Then

$$\begin{aligned}\Lambda(f) &\leq \Lambda(\nu * \chi_V) \\ &= \sum_{\mu \in S} \check{\mu}(\nu * \chi_V) \\ &= \sum_{\mu \in S} \check{\nu} * \check{\mu}(\chi_V) \\ &= \check{\nu}(\sum_{\mu \in S} \chi_V * \mu) \\ &\leq 1\end{aligned}$$

Similarly, if $f \in C_C(H)$ such that $f \geq \nu * \chi_U$ for some compactly supported $\nu \in M(H)_1^+$ then $\Lambda(f) \geq 1$.

□

Theorem 6.2.4: *Suppose H has property (C1). Suppose also that $WUCB_r(H)$ has a left invariant mean. Then H has a left Haar measure.*

Proof. Let K be the collection of all positive linear functionals on $C_C(H)$ satisfying:

- $\Lambda(f) \leq 1$ for any $f \in C_C(H)$ such that $f \leq \nu * \chi_V$ for some $\nu \in M(H)_1^+$.
- $\Lambda(f) \geq 1$ for any $f \in C_C(H)$ such that $f \geq \nu * \chi_U$ for some $\nu \in M(H)_1^+$.

We equip K with the weak* topology induced by $C_C(H)$. If $\Lambda_\alpha \to \Lambda$ and each $\Lambda_\alpha \in K$ then since non-strict inequalities are preserved when taking limits $\Lambda \in K$ so K is closed. For $f \in C_C(H)^+$ there is a $\mu \in M(H)^+$ such that $f \leq \mu * \chi_V$. Subsequently for any $\Lambda \in K$, $\Lambda(f) \leq \Lambda(\mu * \chi_V) \leq \|\mu\|$. Hence K is compact.

To see that K is convex, suppose that $\Lambda_1, \Lambda_2 \in K$ and $0 \leq c \leq 1$. If $\Lambda_1(f) \leq 1$ and $\Lambda_2(f) \leq 1$ then $(c\Lambda_1 + (1-c)\Lambda_2)(f) \leq c + 1 - c = 1$ so $(c\Lambda_1 + (1-c)\Lambda_2)$ satisfies the first condition for being an element of K and a similar argument shows that it satisfies the second condition.

It now suffices to show that left translation is an action of H on K and that this action is separately continuous. From this we can apply the previous theorem to find a point in K which is fixed under left translation. It is clear that the zero functional is not in K so this fixed point must be a left Haar measure.

For $x \in H$ and $\Lambda \in K$ we define $x \cdot \Lambda(f) = \Lambda(\delta_{\check{x}} * f)$. Then $e \cdot \Lambda = \Lambda$ and

$$\begin{aligned} x \cdot (y \cdot \Lambda)(f) &= y \cdot \Lambda(\delta_{\check{x}} * f) \\ &= \Lambda(\delta_{\check{y}} * \delta_{\check{x}} * f) \\ &= \int \Lambda(\delta_{\check{t}} * f) d(\delta_x * \delta_y)(t)) \\ &= \int t \cdot \Lambda(f) d(\delta_x * \delta_y)(t). \end{aligned}$$

To see that this action is separately continuous, consider a net $x_\alpha \to x \in H$. Eventually x_α will stay within a compact neighbourhood N of x. Then for $\Lambda \in K$ and $f \in C_C(H)$

$$\begin{aligned} x_\alpha \cdot \Lambda(f) &= \Lambda(\delta_{\check{x_\alpha}} * f) \\ &= \Lambda|_{\check{N} * \mathrm{supp}(f)}(\delta_{\check{x_\alpha}} * f) \\ &= \delta_{x_\alpha}(f * \check{\Lambda}|_{\check{N} * \mathrm{supp}(f)}) \\ &\to \delta_x(f * \check{\Lambda}|_{\check{N} * \mathrm{supp}(f)}) \\ &= x \cdot \Lambda(f). \end{aligned}$$

Now suppose there is a net $\Lambda_\beta \to \Lambda \in K$. So for $x \in H$ and $f \in C_C(H)$

$$\begin{aligned} x \cdot \Lambda_\beta(f) &= \Lambda_\beta(\delta_{\check{x}} * f) \\ &\to \Lambda(\delta_{\check{x}} * f) \\ &= x \cdot \Lambda(f). \end{aligned}$$

Therefore the action is separately continuous. $\square$

Chapter 7

Conclusion and open problems

7.1 Introduction

This chapter is a collection of questions which arose during the preparation of this thesis. Some are questions posed by other authors that are related to, but beyond the scope of this work. Others are natural continuations of some of the results presented herein.

7.2 Open Questions

Question 1: *Can we find a constructive proof for the results of chapter 2?*

The motivating problem for those results was to try to find a net of positive, norm one, integrable functions on a hypergroup which tend to left invariance weakly, but not in norm. Is it possible to construct such a net for general hypergroups using configurations as Rosenblatt and Willis [47] did for groups.

Question 2: *Are there other interesting types of convergence to left invariance for nets of functions in* $L^1(H)^+_1$*?*

We have discussed weak convergence, convergence in norm, and Reiter nets. For locally compact groups, we have discussed Følner nets (although these are more accurately nets of subsets of G). In proving a result in chapter 2, we touch upon weak convergence against uniformly continuous functions as well as weak convergence to left invariance of translation by compactly supported probability functions.

Question 3: *It is clear that left invariant means are accumulation points of nets tending to left invariance weakly. Are topological left invariant means precisely the weak accumulation points of Reiter nets? If so, what can we conclude from characterizations or enumerations of the set of topological left invariant means about the collection of Reiter nets?*

Question 4: *Can we extend the results of chapter 3 to other types of nets which tend to left invariance?*

In Chapter 3, we talk only about Reiter nets for semidirect products. Certainly it is true that nets tending to left invariance weakly or in norm exist for semidirect products. Can they be constructed from nets for the factors of the semidirect product in the same way as Reiter nets?

Question 5: *For other specific examples of Lau algebras, can we devise a notion of configurations that allow us to characterize left amenability usefully?*

We defined configurations for Lau algebras. These configurations are perhaps too general of a notion as they rely upon considering translations by elements of the predual. In the case of hypergroup or group algebras, we need only consider translation by the group elements to get a type of convergence to left invariance (weak). Perhaps other Lau algebras have similar sufficient sets. We saw that $L^\infty(H)$ is almost never a Hopf-von Neumann algebra (unless H is a group), but perhaps Hopf-von Neumann algebras, or locally compact quantum groups have such sets.

Question 6: *Does every hypergroup admit a left Haar measure?*

This is a long-standing open problem in the theory of hypergroups. In chapter 5, we presented some new results in this direction. There are other questions that can be asked in this direction.

Question 7: *Does property (C1) hold for every hypergroup?*

Question 8: *For an arbitrary hypergroup, can we find a non-negative, Borel measure which, when convoluted with a compactly supported, continuous function yields an almost periodic function?*

The property (C1) allows us to find a measure which yields a uniformly continuous function. If H admits a left Haar measure, then that yields a constant function. The space of almost periodic, or weakly almost periodic functions lie between the constants and the uniformly continuous functions. Additionally, for a locally compact group, the almost periodic functions have a unique left invariant mean.

Bibliography

[1] Abdollahi, A., Rejali, A., and Willis, G. A. Group properties characterised by configurations. *Illinois J. Math. 48*, 3 (2004), 861–873.

[2] Azimifard, A. Hypergroups with unique α-means. *C. R. Math. Acad. Sci. Soc. R. Can. 30*, 2 (2008), 33–39.

[3] Azimifard, A. On the α-amenability of hypergroups. *Monatsh. Math. 155*, 1 (2008), 1–13.

[4] Bloom, W. R., and Heyer, H. *Harmonic analysis of probability measures on hypergroups*, vol. 20 of *de Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, 1995.

[5] Bloom, W. R., and Walter, M. E. Isomorphisms of hypergroups. *J. Austral. Math. Soc. Ser. A 52*, 3 (1992), 383–400.

[6] Daws, M., and Runde, V. Reiter's properties (p1) and (p2) for locally compact quantum groups. *Journal of Mathematical Analysis and Applications 364*, 2 (2010), 352 – 365.

[7] Day, M. M. Amenable semigroups. *Illinois J. Math. 1* (1957), 509–544.

[8] Dunkl, C. F. The measure algebra of a locally compact hypergroup. *Trans. Amer. Math. Soc. 179* (1973), 331–348.

[9] Dunkl, C. F., and Ramirez, D. E. A family of countably compact P_*-hypergroups. *Trans. Amer. Math. Soc. 202* (1975), 339–356.

[10] Følner, E. On groups with full Banach mean value. *Math. Scand. 3* (1955), 243–254.

[11] Fremlin, D. H. *Topological Riesz spaces and measure theory.* Cambridge University Press, London, 1974.

[12] Ghahramani, F., and Medgalchi, A. R. Compact multipliers on weighted hypergroup algebras. *Math. Proc. Cambridge Philos. Soc. 98*, 3 (1985), 493–500.

[13] Ghahramani, F., and Medgalchi, A. R. Compact multipliers on weighted hypergroup algebras. II. *Math. Proc. Cambridge Philos. Soc. 100*, 1 (1986), 145–149.

[14] Greenleaf, F. P. *Invariant means on topological groups and their applications.* Van Nostrand Mathematical Studies, No. 16. Van Nostrand Reinhold Co., New York, 1969.

[15] Hartmann, K., Henrichs, R. W., and Lasser, R. Duals of orbit spaces in groups with relatively compact inner automorphism groups are hypergroups. *Monatsh. Math. 88*, 3 (1979), 229–238.

[16] Hewitt, E., and Ross, K. A. *Abstract harmonic analysis. Vol. I*, second ed., vol. 115 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences].* Springer-Verlag, Berlin, 1979. Structure of topological groups, integration theory, group representations.

[17] Hindman, N., and Strauss, D. Density and invariant means in left amenable semigroups. *Topology Appl. 156*, 16 (2009), 2614–2628.

[18] Hu, Z., Monfared, M. S., and Traynor, T. On character amenable Banach algebras. *Studia Math. 193*, 1 (2009), 53–78.

[19] Hu, Z., Neufang, M., and Ruan, Z.-J. On topological centre problems and SIN quantum groups. *J. Funct. Anal. 257*, 2 (2009), 610–640.

[20] Hu, Z., Neufang, M., and Ruan, Z.-J. Multipliers on a new class of Banach algebras, locally compact quantum groups, and topological centres. *Proc. Lond. Math. Soc. (3) 100*, 2 (2010), 429–458.

[21] Izzo, A. J. A functional analysis proof of the existence of Haar measure on locally compact abelian groups. *Proc. Amer. Math. Soc. 115*, 2 (1992), 581–583.

[22] Janzen, D. Følner nets for semidirect products of amenable groups. *Canad. Math. Bull. 51*, 1 (2008), 60–66.

[23] Jewett, R. I. Spaces with an abstract convolution of measures. *Advances in Math. 18*, 1 (1975), 1–101.

[24] Kaniuth, E., Lau, A. T.-M., and Pym, J. On ϕ-amenability of Banach algebras. *Math. Proc. Cambridge Philos. Soc. 144*, 1 (2008), 85–96.

[25] Kaniuth, E., Lau, A. T.-M., and Ülger, A. Homomorphisms of commutative Banach algebras and extensions to multiplier algebras with applications to Fourier algebras. *Studia Math. 183*, 1 (2007), 35–62.

[26] Klawe, M. Semidirect product of semigroups in relation to amenability, cancellation properties, and strong Følner conditions. *Pacific J. Math. 73*, 1 (1977), 91–106.

[27] Lasser, R. Almost periodic functions on hypergroups. *Math. Ann. 252*, 3 (1980), 183–196.

[28] Lasser, R. Amenability and weak amenability of l^1-algebras of polynomial hypergroups. *Studia Math. 182*, 2 (2007), 183–196.

[29] Lasser, R. On α-amenability of commutative hypergroups. In *Infinite dimensional harmonic analysis IV*. World Sci. Publ., Hackensack, NJ, 2009, pp. 184–195.

[30] Lasser, R. Various amenability properties of the L^1-algebra of polynomial hypergroups and applications. *J. Comput. Appl. Math. 233*, 3 (2009), 786–792.

[31] Lasser, R., and Skantharajah, M. Reiter's condition for amenable hypergroups. *Monatsh Math* (2010), 1–12.

[32] Lau, A. T.-M. Analysis on a class of Banach algebras with applications to harmonic analysis on locally compact groups and semigroups. *Fund. Math. 118*, 3 (1983), 161–175.

[33] Lau, A. T.-M., Miyake, H., and Takahashi, W. Approximation of fixed points for amenable semigroups of nonexpansive mappings in Banach spaces. *Nonlinear Anal. 67*, 4 (2007), 1211–1225.

[34] Lau, A. T.-M., Shioji, N., and Takahashi, W. Existence of nonexpansive retractions for amenable semigroups of nonexpansive mappings and nonlinear ergodic theorems in Banach spaces. *J. Funct. Anal. 161*, 1 (1999), 62–75.

[35] Lau, A. T.-M., and Wong, J. C. S. Invariant subspaces for algebras of linear operators and amenable locally compact groups. *Proceedings of the American Mathematical Society 102*, 3 (1988), pp. 581–586.

[36] Lau, A. T.-M., and Zhang, Y. Fixed point properties of semigroups of non-expansive mappings. *J. Funct. Anal. 254*, 10 (2008), 2534–2554.

[37] Monfared, M. S. On certain products of Banach algebras with applications to harmonic analysis. *Studia Math. 178*, 3 (2007), 277–294.

[38] MONFARED, M. S. Character amenability of Banach algebras. *Math. Proc. Cambridge Philos. Soc. 144*, 3 (2008), 697–706.

[39] MURUGANANDAM, V. Fourier algebra of a hypergroup. I. *J. Aust. Math. Soc. 82*, 1 (2007), 59–83.

[40] NAMIOKA, I. Følner's conditions for amenable semi-groups. *Math. Scand. 15* (1964), 18–28.

[41] NAMIOKA, I., AND ASPLUND, E. A geometric proof of Ryll-Nardzewski's fixed point theorem. *Bull. Amer. Math. Soc. 73* (1967), 443–445.

[42] PATERSON, A. L. T. *Amenability*, vol. 29 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1988.

[43] PIER, J.-P. *Amenable Banach algebras*, vol. 172 of *Pitman Research Notes in Mathematics Series*. Longman Scientific & Technical, Harlow, 1988.

[44] REITER, H. On some properties of locally compact groups. *Nederl. Akad. Wetensch. Proc. Ser. A 68=Indag. Math. 27* (1965), 697–701.

[45] RICKERT, N. W. Amenable groups and groups with the fixed point property. *Trans. Amer. Math. Soc. 127* (1967), 221–232.

[46] ROBERTSON, A. P., AND ROBERTSON, W. *Topological vector spaces*, second ed. Cambridge University Press, London, 1973. Cambridge Tracts in Mathematics and Mathematical Physics, No. 53.

[47] ROSENBLATT, J. M., AND WILLIS, G. A. Weak convergence is not strong convergence for amenable groups. *Canad. Math. Bull. 44*, 2 (2001), 231–241.

[48] RÖSLER, M. Radial wavelets and bessel-kingman hypergroups. Tech. rep., 1997.

[49] ROSS, K. A. Hypergroups and centers of measure algebras. In *Symposia Mathematica, Vol. XXII (Convegno sull'Analisi Armonica e Spazi di Funzioni su Gruppi Localmente Compatti, INDAM, Rome, 1976)*. Academic Press, London, 1977, pp. 189–203.

[50] ROSS, K. A. Centers of hypergroups. *Trans. Amer. Math. Soc. 243* (1978), 251–269.

[51] SKANTHARAJAH, M. Amenable hypergroups. *Illinois J. Math. 36*, 1 (1992), 15–46.

[52] Spector, R. Mesures invariantes sur les hypergroupes. *Trans. Amer. Math. Soc. 239* (1978), 147–165.

[53] Takesaki, M. *Theory of operator algebras. I.* Springer-Verlag, New York, 1979.

[54] Voit, M. Projective and inductive limits of hypergroups. *Proc. London Math. Soc. (3) 67*, 3 (1993), 617–648.

[55] Vrem, R. C. Harmonic analysis on compact hypergroups. *Pacific J. Math. 85*, 1 (1979), 239–251.

[56] Wolfenstetter, S. Weakly almost periodic functions on hypergroups. *Monatsh. Math. 96*, 1 (1983), 67–79.