UNIVERSITY OF ALBERTA

POPULATION DIFFUSION THROUGH AN ANNULAR PATCHY ENVIRONMENT

BY

GUIRONG CUI ©

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DEDICATED

TO MY PARENTS

AND MY WIFE HONGXIA

ABSTRACT

This thesis is concerned with populations diffusion through annular patchy environments. Firstly, perfect annular patchy environments are considered. The global stability of a unique positive radially symmetric steady state is obtained. Secondly, the models with periodically changing annular patchy environment are investigated, a threshold condition is obtained that population either goes to extinction or evolutes to a unique periodic solution. Thirdly, various annular patchy environments are analizied. Finally some numeric explanations are appended.

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INTRODUCTION

§ 1.1 What's a Patchy Environment?

Differential equations have been employed in the analysis of biological models for decades. At the early stages, mainly ordinary differential equations were used to analyze ecological models, e.g. the Volterra system describing fish ecology in the Mediteranean Sea, etc. ([F] and references therein). In these kinds of models, populations were assumed to be distributed homogeneously over the entire environment and the environment therefore was homogeneous. However, in a natural setting, neither the carrying capacity of the environment nor the spatial distribution of the population is constant. The spatial effects quite often need to be considered[AC], for example, in a model of various species growth and diffusion in a forest area with changing carrying capacities caused by changing soil conditions, elevations, and foliage cover, etc. In order to study the effect of environmental heterogeneity on the habitat, [PR] and [SR] suggested an efficient method to divide the habitat into homogeneous patches where the growth and diffusion coefficients for the several species may be different in different patches, but in each patch, the growth rate and diffusion coefficient is constant. This case can be modeled by a system of partial differential equations or ordinary differential equations with interface conditions between patches([AC],[CC1-9],[CCH],[CF] and [FK], etc). In some situations, the environment is viewed as a collection of discrete patches and within each patch the population is distributed homogeneously. This case is described by a system of ordinary differential equations where each equation describes the population in a single patch. These equations then are coupled through noflux interface conditions of population immigration between patches([FST], [FT], [FWu], [T1-3], [TL], [TOM] and references therein).

Here we mainly discuss population diffusion in a patchy environment such that within each patch the population diffuses at same rate. The main mathematical tools in this aspect are the theory of eigen-problems of PDEs, semigroups, and monotone dynamical systems. By skillfully exploiting the techniques of the eigenvalue problem of the elliptic equation, [AC] and [CC*] studied various aspects of models of populations in patchy environments, including persistence, permanence, effects of environmental size and asymptotic behavior, etc. The global stability of the steady state was also discussed by making use of semigroup techniques in [FK].

Some excellent applications of diffusion in patchy environments were conducted by [FF] and [LLC], where growths of organism tissues and cancers in different oxygen media can be controlled to a certain scale or to extinction by changing the carrying capacity of a particular part of the oxygen medium which forms a specific patch.

§ 1.2 Models and Results

The models we will consider are concerned with population diffusion through various kinds of annular patchy environments, such as perfect annular patchy environment and radially changing environments with holes or without holes, general annular patchy environments with holes or without holes. The environment may be constantly or periodically changing. We list the models and main results in the following.

(A) Population Diffusion Through a Perfect Annular Patchy Environment

In this case, the patchy environment consists of perfect annular patches with each except the last surrounded by an outer one.

We denote the patches by

$$\Omega_1 = \{x \in R^2 \mid |x| < r_1\},$$

$$\Omega_i = \{x \in R^2 \mid r_{i-1} < |x| < r_i\}, \quad i = 2, 3, \cdots, n;$$

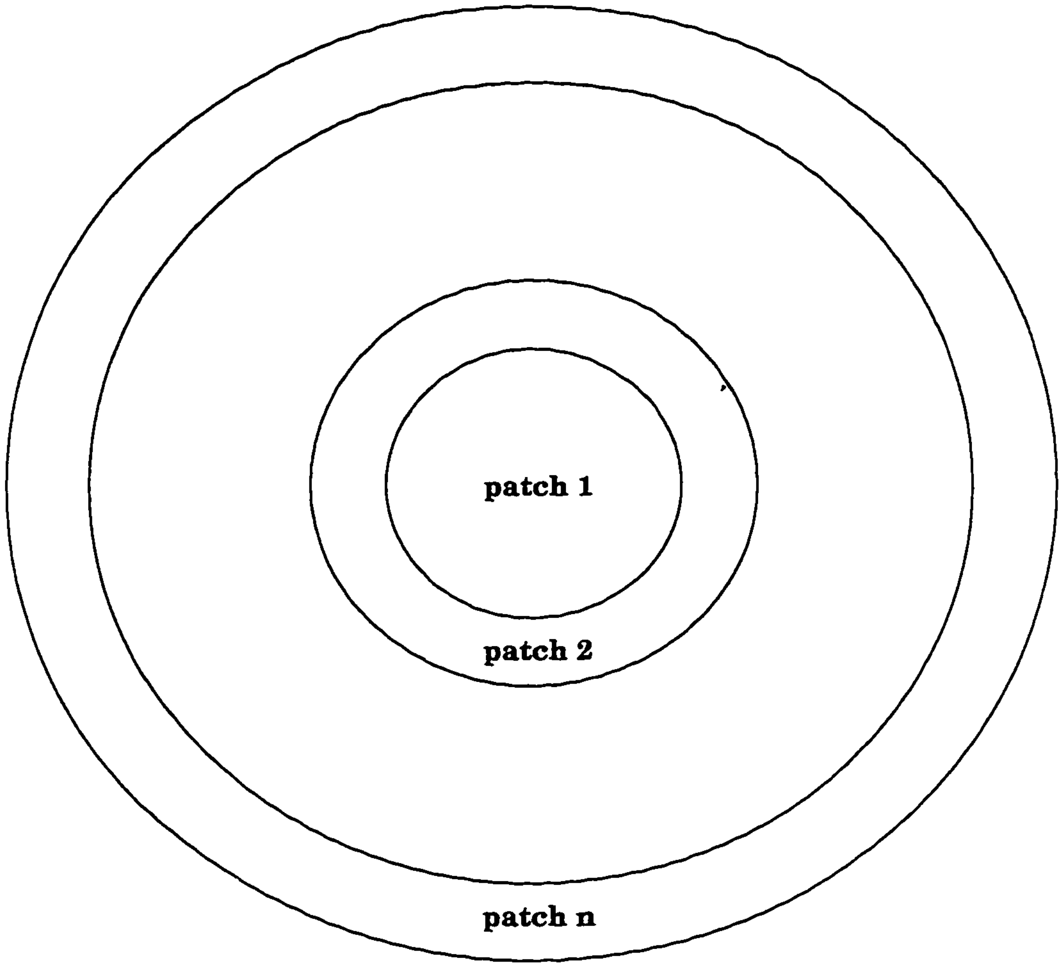
where Ω_i is the i-th patch respectively and $0 < r_1 < r_2 < \cdots < r_n$ are constants.

Population diffusion through patchy environment(A) is modeled by a system of parabolic equation,

$$\frac{\partial u_i}{\partial t} = d_i \Delta u_i + u_i g_i(u_i), \quad x \in \Omega_i, \ i = 1, 2, \cdots, n \tag{A.1}$$

with continuous flux matching conditions

$$\begin{aligned} u_i\big|_{r=r_i} &= u_{i+1}\big|_{r=r_i} \\ d_i \frac{\partial u_i}{\partial n_i}\bigg|_{r=r_i} &= d_{i+1} \frac{\partial u_{i+1}}{\partial n_i}\bigg|_{r=r_i} \end{aligned} \tag{A.2}$$



Patchy Environment (A)

Figure - 1

where $u_i = u_i(x,t)$, $t > 0$, $x \in \Omega_i$ denotes the population in patch Ω_i, $d_i > 0$ is the diffusive coefficient of population in patch Ω_i, $i = 12, \cdots, n$; n_i is the unit outer norm from Ω_i to Ω_{i+1}, $i = 1, 2, \cdots, n-1$, Δ is the Lapalacian, and g_i satisfies the Kolmogorov assumption[F],

$$g_i(0) > 0$$

$$g_i'(s) < 0 \quad for \quad s \geq 0$$

$$g_i(K_i) = 0 \quad for \quad some \quad K_i > 0. \qquad \text{(A.3)}$$

K_i is the capacity of the population in Ω_i, $i = 1, 2, \cdots, n$.

We assume that the boundary condition is of the form

$$u_n\big|_{r=r_n} = K_n \tag{A.4}$$

and the initial condition is given by

$$u_i(x, 0) = \eta_i(x) > 0, \quad x \in \Omega_i. \tag{A.5}$$

We write (A.1)-(A.2) in a concise form

$$\frac{\partial u}{\partial t} = d\Delta u + ug(u) \tag{A.6}$$

with $u(x, t) = u_i(x, t)$, $d(x, t) = d_i$, $g(u(x, t)) = g_i(u_i(x, t))$ for $x \in \Omega_i$ and $t > 0$, and u satisfies (A.2). Throughout this, whenever we consider system (A.6), the matching condition (A.2) is automatically implied. This is why we call (A.6) a patchy parabolic equation. For the same reason, we can have patchy elliptic equations. For convenience, we write $u = (u_1, u_2, \cdots, u_n)$, $\eta = (\eta_1, \eta_2, \cdots, \eta_n)$.

The main result for this case is that there exists a unique positive radially symmetric steady state of (A.1)-(A.2) and (A.4), which is globally attractive for those solutions of (A.1)-(A.2) and (A.4)-(A.5) with radially symmetric initial data η. Further, under certain conditions, it is globally attractive for all solutions with positive initial data η. Here a steady state solution of (A.1)-(A.2) and (A.4) is a solution of a system of elliptic equations

$$\Delta v_i + v_i g_i(v_i) = 0, \; x \in \Omega_i, \; t > 0, \; i = 1, 2, \cdots, n \tag{A.7}$$

with matching conditions

$$v_i\big|_{r=r_i} = v_{i+1}\big|_{r=r_i}$$
$$d_i \left.\frac{\partial v_i}{\partial n_i}\right|_{r=r_i} = d_{i+1} \left.\frac{\partial v_{i+1}}{\partial n_i}\right|_{r=r_i} \tag{A.8}$$

and boundary conditions

$$v_n\big|_{r=r_n} = K_n \tag{A.9}$$

We can also write (A.7)-(A.8) in a patchy elliptic form

$$\Delta v + vg(v) = 0 \tag{A.10}$$

where $v = (v_1, v_2, \cdots, v_n)$.

An example of patchy environment (A) is the polluted area surrounding an industrial chimney with the pollution changing dramatically with distance.

(A_0) Population Diffusion in a Radially Varying Environment

The model considered in this part is the continuous version of (A). We consider population diffusion in $\Omega = \{x \in R^2 \mid |x| < R_0\}$, $R_0 > 0$ is a constant. This model can be described by a parabolic equation

$$\frac{\partial u}{\partial t} = \nabla\left(d(r)\nabla u\right) + ug(u,r) \tag{A_0.1}$$

where $x \in \Omega$, $r = |x|$ and g satisfies the Kolmogorov assumptions[F],

$$g(0,r) > 0$$
$$g_u(s,r) < 0 \quad for \quad s \geq 0. \tag{A_0.2}$$

There exists a positive continuous function $K(r)$ such that

$$g(K(r), r) = 0$$

where $0 \leq r \leq R_0$.

The boundary condition is assumed to be of the form

$$u|_{r=R_0} = K(R_0), \tag{$A_0.3$}$$

and the initial condition is

$$u(x, 0) = \eta(x), \quad x \in \Omega. \tag{$A_0.4$}$$

The main result is similar to that in (A), i.e. there exists a unique positive radially symmetric steady state which is globally attractive for those solutions of (A_0.1), (A_0.3) and (A_0.4) with radially symmetric initial data, and under certain conditions, it is globally attractive for all solutions with positive initial data η.

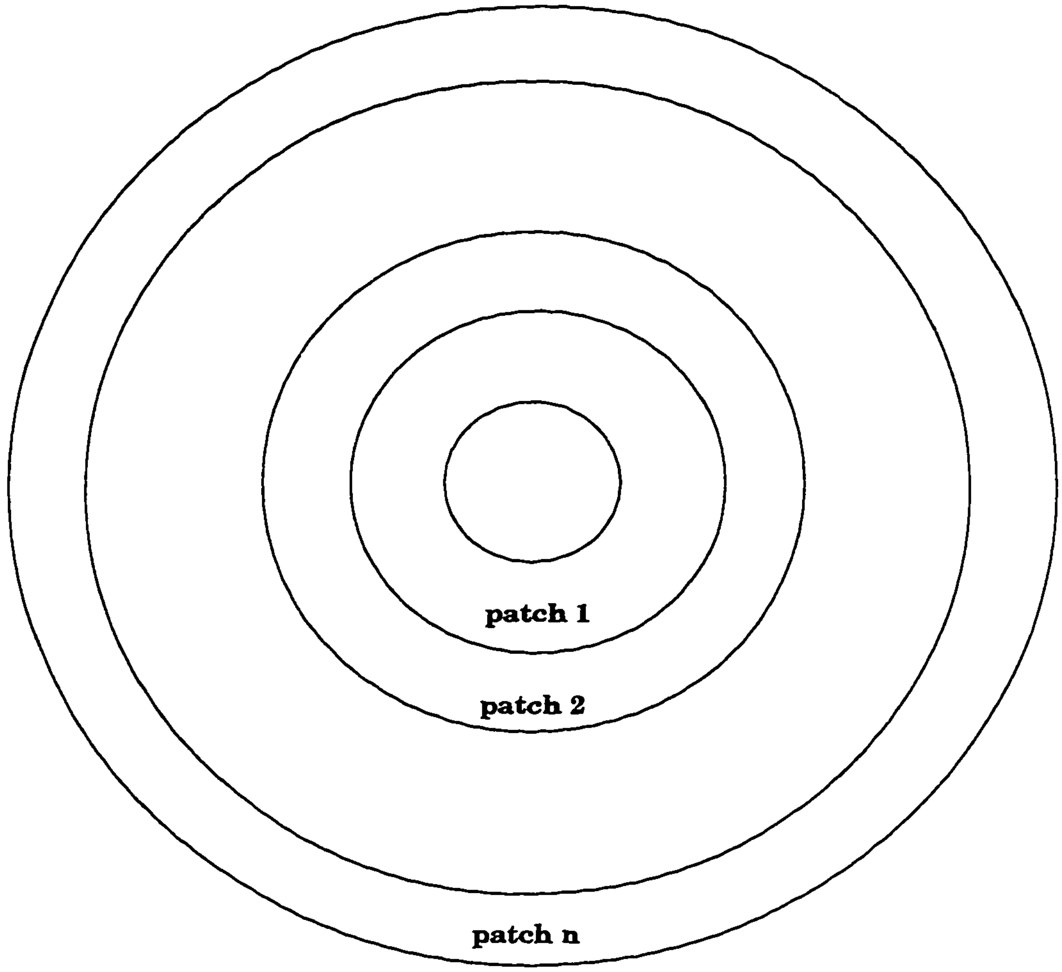
A steady state solution of (A_0.1) and (A_0.3) is a solution of the corresponding elliptic equation

$$\nabla (d(r)\nabla v) + vg(v, r) = 0 \tag{$A_0.5$}$$

with boundary condition

$$v|_{r=R_0} = K(R_0) \tag{$A_0.6$}$$

An example of a patchy environment (A_0) is a polluted lake surrounded by chemical factories.



Patchy Environment (B)

Figure - 2

(B) Population Diffusion Through a Perfectly Annular Patchy Environment With a Hole

This patchy environment is the patchy environment (A) with a hole inside.

where patch i is $\Omega_i = \{x \in R^2 \,|\, r_{i-1} < |x| < r_i\}$, $i = 1, 2, \cdots, n$; $0 < r_0 < r_1 < \cdots < r_n$ are constants.

Except for the boundary conditions, the model assumptions are the same as those in (A), i.e.

$$\frac{\partial u_i}{\partial t} = d_i \Delta u_i + u_i g_i(u_i), \quad x \in \Omega_i,\ i = 1, 2, \cdots, n \tag{B.1}$$

with continuous flux matching conditions

$$\begin{aligned} u_i\big|_{r=r_i} &= u_{i+1}\big|_{r=r_i} \\ d_i \left.\frac{\partial u_i}{\partial n_i}\right|_{r=r_i} &= d_{i+1} \left.\frac{\partial u_{i+1}}{\partial n_i}\right|_{r=r_i} \qquad i = 1, 2, \cdots, n-1 \end{aligned} \tag{B.2}$$

the initial condition is

$$u_i(x, 0) = \eta_i(x),\ x \in \Omega_i,\ i = 1, 2, \cdots, n \tag{B.3}$$

and the boundary conditions are

$$\begin{aligned} u_1\big|_{r=r_0} &= K_1 \\ u_n\big|_{r=r_n} &= K_n. \end{aligned} \tag{B.4}$$

The main result is same as that in (A), and so we don't repeat it here, but the precise statement will be given later. A steady state solution of (B.1)-(B.2) and (B.4) is a solution of the corresponding elliptic equations

$$d_i \Delta v_i + v_i g_i(v_i) = 0, \quad i = 1, 2, \cdots, n \tag{B.5}$$

with matching conditions

$$\begin{aligned} v_i\big|_{r=r_i} &= v_{i+1}\big|_{r=r_i} \\ d_i \left.\frac{\partial v_i}{\partial n_i}\right|_{r=r_i} &= d_{i+1} \left.\frac{\partial v_{i+1}}{\partial n_i}\right|_{r=r_i} \end{aligned} \tag{B.6}$$

and boundary conditions

$$v_1\big|_{r=r_0} = K_1$$

$$v_n\big|_{r=r_n} = K_n. \tag{B.7}$$

In (A), we wrote (A.1)-(A.2) in a concise form (A.6) by defining $u(x,t) = u_i(x,t)$, $d(x,t) = d_i$, $g(u(x,t)) = g_i(u_i(x,t))$ for $x \in \Omega_i$ and $t > 0$, where u satisfies (A.2). In the same manner, we can give the concise form of (B.1)-(B.4) and (B.5)-(B.7), they are a patchy parabolic equation and a patchy elliptic equation.

An example of a patchy environment (B) is a polluted area surrounding a huge industrial chimney.

(B_0) Population Diffusion in a Radially Varying Environment With Hole

(B_0) compared to (B) is the same as (A_0) compared to (A). We consider population diffusion in $\Omega = \{x \in R^2 \,|\, r_0 < |x| < R_0\}$, where $0 < r_0 < R_0$ are constants. This population can be described by

$$\frac{\partial u}{\partial t} = \nabla \cdot (d(r)\nabla u) + ug(u,r) \tag{B_0.1}$$

where $x \in \Omega$, $r = |x|$, g is the same as that in (A_0.2). The boundary conditions are

$$u\big|_{r=r_0} = K(r_0)$$

$$u\big|_{r=R_0} = K(R_0) \tag{B_0.2}$$

and the initial condition is

$$u(x,0) = \eta(x), \quad x \in \Omega. \tag{B_0.3}$$

A steady state solution of $(B_0.1) - (B_0.2)$ is a solution of the corresponding elliptic equation

$$d(r)\Delta v + vg(v,r) = 0 \tag{B_0.4}$$

$$v|_{r=r_0} = K(r_0)$$

$$v|_{r=R_0} = K(R_0). \tag{B_0.5}$$

The same conclusions as those in (A_0) can be obtained, and they will be precisely stated later.

(C) Population Diffusion Through a General Annular Patchy Environment

In this situation, the patchy environment consists of annular patches which can have holes inside and which may not be perfectly annular.

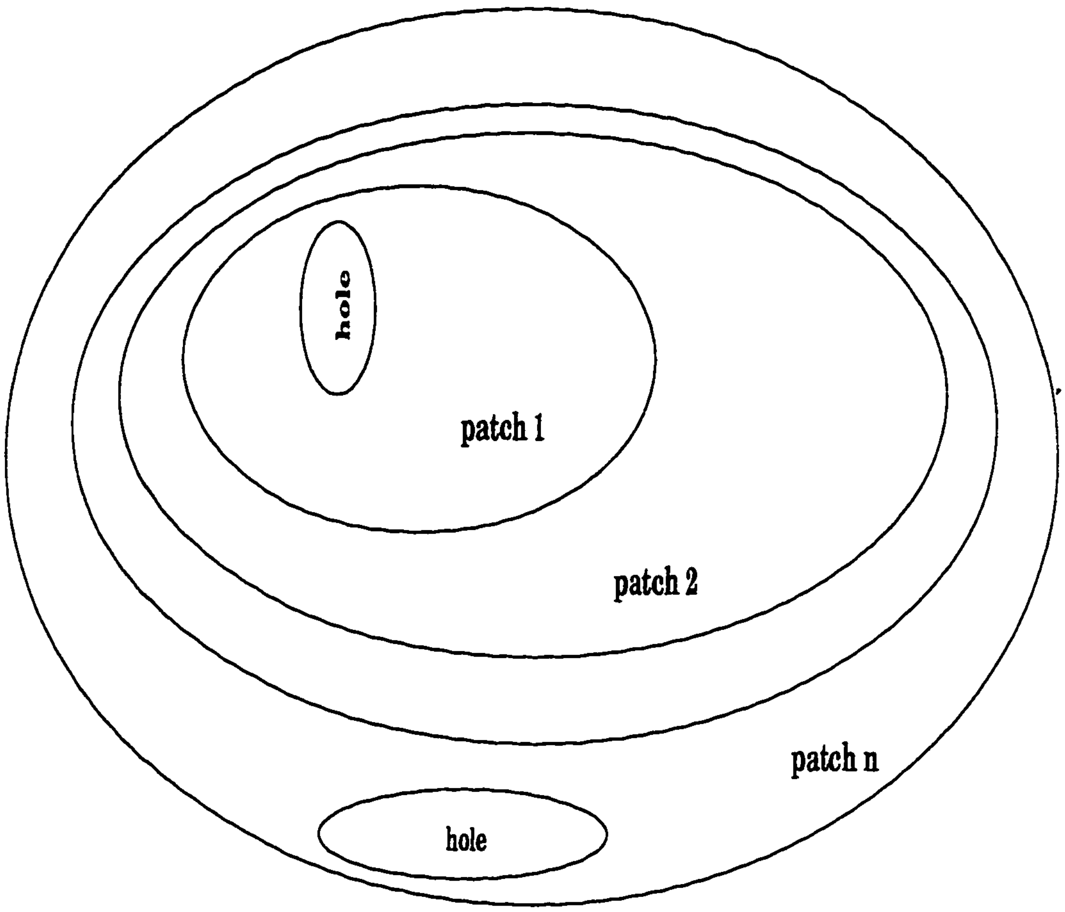
Patch i, denoted by Ω_i, $i = 1, 2, \cdots, n$, is an annular patch which can have holes inside of it. Except for the boundary conditions, we adopt the equations of model (A), i.e.

$$\frac{\partial u_i}{\partial t} = d_i \Delta u_i + u_i g_i(u_i),\ x \in \Omega_i,\ i = 1, 2, \cdots, n \tag{C.1}$$

with matching conditions

$$u_i|_{\overline{\Omega}_i \cap \overline{\Omega}_{i+1}} = u_{i+1}|_{\overline{\Omega}_i \cap \overline{\Omega}_{i+1}}$$

$$d_i \left.\frac{\partial u_i}{\partial n_i}\right|_{\overline{\Omega}_i \cap \overline{\Omega}_{i+1}} = d_{i+1} \left.\frac{\partial u_{i+1}}{\partial n_i}\right|_{\overline{\Omega}_i \cap \overline{\Omega}_{i+1}} \qquad i = 1, 2, \cdots, n-1 \tag{C.2}$$



Patchy Environment (C)

Figure - 3

and initial conditions

$$u_i(x,0) = \eta_i(x) > 0, \quad x \in \Omega,\ i = 1, 2, \cdots, n \tag{C.3}$$

The boundary condition is assumed to be of Dirichlet type, i.e.

$$u|_{\partial\Omega} = 0 \tag{C.4}$$

where $\Omega = \cup_{i=1}^{n} \Omega_i$, $u = (u_1, u_2, \cdots, u_n)$ is the concise form rather than a vector.

The main result is that there exists a switch condition giving that either 0 is globally stable or there exists a positive steady state which is globally asymptotic stable. A steady state solution of (C.1)-(C.2) and (C.4) is a solution of the corresponding patchy elliptic equation,

$$d_i \Delta v_i + v_i g_i(v_i) = 0, \quad x \in \Omega, \, i = 1, 2, \cdots, n \tag{C.5}$$

with matching conditions

$$\begin{aligned} v_i|_{\overline{\Omega}_i \cap \overline{\Omega}_{i+1}} &= v_{i+1}|_{\overline{\Omega}_i \cap \overline{\Omega}_{i+1}} \\ d_i \left. \frac{\partial v_i}{\partial n_i} \right|_{\overline{\Omega}_i \cap \overline{\Omega}_{i+1}} &= d_{i+1} \left. \frac{\partial v_{i+1}}{\partial n_i} \right|_{\overline{\Omega}_i \cap \overline{\Omega}_{i+1}} \qquad i = 1, 2, \cdots, n-1 \end{aligned} \tag{C.6}$$

and boundary condition

$$v|_{\partial\Omega} = 0, \tag{C.7}$$

where $v = (v_1, v_2, \cdots, v_n)$ is a concise form.

An example of patchy environment (C) is a polluted lake containing many islands inside.

(C_0) population Diffusion Through General Periodically Changing Annular Patchy Environments

We asume the patchy environment (C) is valid here, i.e. the model is basically generalized from model (C),

$$\frac{\partial u_i}{\partial t} = d_i \Delta u_i + u_i g_i(u_i, t), \, x \in \Omega_i, \, i = 1, 2, \cdots, n \tag{C_0.1}$$

with matching conditions

$$\begin{aligned} u_i|_{\overline{\Omega}_i \cap \overline{\Omega}_{i+1}} &= u_{i+1}|_{\overline{\Omega}_i \cap \overline{\Omega}_{i+1}} \\ d_i \left. \frac{\partial u_i}{\partial n_i} \right|_{\overline{\Omega}_i \cap \overline{\Omega}_{i+1}} &= d_{i+1} \left. \frac{\partial u_{i+1}}{\partial n_i} \right|_{\overline{\Omega}_i \cap \overline{\Omega}_{i+1}} \qquad i = 1, 2, \cdots, n-1 \end{aligned} \tag{C_0.2}$$

and boundary condition

$$u|_{\partial\Omega} = 0. \tag{C_0.3}$$

the initial condition is

$$u_i(x,0) = \eta_i(x) > 0, \quad x \in \Omega_i,\ i = 1, 2, \cdots, n \tag{C_0.4}$$

where g_i is a T-periodic function for some positive constant T, and $g_i(\cdot, t)$ satisfies the Kolmogorov assumption, i.e. there exists a positive continuous T-periodic function $K_i(t)$ on $[0, T]$, which is the carrying capacity in Ω_i at time t, $t \in [0, t]$, $i = 1, 2, \cdots, n$, such that

$$\begin{aligned}
&g_i(\cdot, t+T) = g_i(\cdot, t), \quad \forall t > 0 \\
&g_i(0, t) > 0 \\
&\frac{\partial g_i}{\partial u_i}(\xi, t) < 0, \quad \forall \xi \geq 0 \\
&g_i(K_i(t), t) = 0, \quad t \in [0, T]; \qquad i = 1, 2, \cdots, n.
\end{aligned} \tag{C_0.5}$$

The main result is that there exists a switch condition such that either 0 is globally stable or there exists a T-periodic solution which is globally asymptotic stable.

Chapter 2

Maximum Principle in Patchy PDEs

Maximum Principles play central roles in the theory of parabolic partial differential equations. They provide the fundamental tools for establishing the existence of positively invariant sets, comparisons between solutions of different parabolic equations and monotonicity of the solution operator. The classic reference is [PW].

The aim of this chapter is to develop various maximum principles for patchy partial differential equations. These maximum principles will be useful for obtaining results of the kind just mensioned.We will show that the classical maximum principles still hold in patchy PDEs. Actually the maximum principles in patchy PDEs have already been widely used without being proved, such as (4.8) in [FK] where the boundedness of the solution of a patchy parabolic equation is obtained.

§2.1 Elliptic Cases

Before developing the maximum principles in patchy elliptic equations, we introduce the classical maximum principles in elliptic equations[PW].

Theorem 2.1.1. *Let $\Omega \subset R^N$ be a domain, $h(x) \leq 0$ be bounded in $\overline{\Omega}$, $d > 0$ be a constant and $d\Delta u + h(x)u \geq 0$ in Ω. If u attains its nonnegative maximum M at a point of Ω, then $u \equiv M$ in Ω. Furthermore if $M = 0$, the restriction on $h(x)$ can be removed.*

The strong version of the maximum principle in elliptic equations is:

Theorem 2.1.2. *Under the conditions of Theorem 2.1.1, if u attains its nonnegative maximum M at a boundary point P, assuming P lies on the boundary of a ball in Ω, and u is continuous in $\Omega \cup P$, then $\frac{\partial u}{\partial \nu} > 0$ unless $u \equiv M$ in Ω. Here ν is the unit outer normal at P. Furthermore, if $M = 0$, the nonnegative restriction on $h(x)$ can be removed.*

Now we consider the patchy elliptic inequality system,

$$d_i \Delta u_i + h_i(x) u_i \geq 0 \quad \text{for} \quad x \in \Omega_i, i = 1, 2, ..., n$$

where $(\Omega_1, \Omega_2, ..., \Omega_n)$ could be a patchy environment of type (A), (B) or (C). $d_i > 0 \quad (i = 1, 2, ..., n)$ is constant, $h_i(x) \leq 0 \quad (i = 1, 2, ..., n)$ is bounded in Ω_i, and $u_i \quad (i = 1, 2, ..., n)$ satisfies the matching conditions in (A), (B) or (C). We can write the system as

$$d\Delta u + h(x) u \geq 0, \quad x \in \Omega$$

where $d(x) = d_i(x)$, $h(x) = h_i(x)$ and $u(x) = u_i(x)$ for $x \in \Omega_i$, $i = 1, 2, ..., n$, $\Omega = \cup_{i=1}^{n} \Omega_i$.

Theorem 2.1.3. *If u attains its nonnegative maximum M at a point of Ω, then $u \equiv M$ in Ω. Furthermore if $M = 0$, the nonpositive restriction on $h(x)$ can be removed.*

Proof. If u attains M in Ω_i, then by Theorem 2.1.2, $u_i \equiv M$ in Ω_i. Considering u_{i+1} in Ω_{i+1} and u_{i-1} in Ω_{i-1}, we have that u_{i+1} and u_{i-1} attain their maximums on the boundaries of Ω_{i+1} and Ω_{i-1}, and the outer norm derivatives of u_{i+1} and u_{i-1} will be zero by the matching conditions and $u_i \equiv M_i$ in Ω_i. Thus by Theorem 2.1.2, $u_{i+1} \equiv M$, $u_{i-1} \equiv M$. Hence $u \equiv M$ in Ω.

If u attains M on $\overline{\Omega}_i \cap \overline{\Omega}_{i+1}$, $i = 1, 2, ..., n-1$, the outward directional derivatives of u_i in Ω_i and u_{i+1} in Ω_{i+1} must be nonnegative. By the matching conditions, both of them must be zero. Hence by Theorem 2.1.2, $u_i \equiv M$, $u_{i+1} \equiv M$. Thus $u \equiv M$.

The strong version of the maximum principle for patchy elliptic PDEs is a direct corollary of the classical strong maximum principle.

Theorem 2.1.4. *If u attains its nonnegative maximum M at a boundary point P, assume P lies on the boundary of a ball in Ω and u is continuous in $\Omega \cup P$. Then $\frac{\partial u}{\partial \nu} > 0$ unless $u \equiv M$ in Ω. Here ν is the outward unit normal vector at P. Furthermore if $M = 0$, the nonpositive restriction on $h(x)$ can be removed.*

§2.2 Parabolic Cases

We can develop the maximum principle for patchy parabolic equations in a manner similar to patchy elliptic equations. Similarly as in the previous section, we introduce the classical version first in order to compare them [PW].

Theorem 2.2.1. *Let $\Omega \subset R^N$ be a domain, $h(x,t) \leq 0$ be bounded in $\overline{\Omega} \times [0,T]$ for some $T > 0$, $d > 0$ is a constant, and $-u_t + d\Delta u + h(x,t)u \geq 0$ in $\Omega \times$*

$[0,T]$. *Suppose* u *attains a nonnegative maximum* M *at an interior point* $(\bar{x},\bar{t})$, *i.e.* $(\bar{x},\bar{t}) \in \Omega \times (0,T]$. *Denote by* $E(\bar{t})$ *the connected component of the intersection of the hyperplane* $t = \bar{t}$ *with* $\Omega \times [0,T]$ *which contains* $(\bar{x},\bar{t})$. *Then* $u \equiv M$ *in* $E(\bar{t})$. *Furthermore, if* Q *is a point of* E *which can be connected to* P *by a path in* $\Omega \times [0,T]$ *consisting only of horizontal segments and upward vertical segments, then* $u = M$ *at* Q. *Finally, if M=0, the nonpositive restriction on* $h(x,t)$ *can be removed.*

The strong classical maximum principle for parabolic equations is:

Theorem 2.2.2. *Under the assumptions of Theorem 2.2.1, suppose the nonnegative maximum* M *is attained at a point* P *on the boundary* $\partial\Omega \times [0,T]$. *Assume that a sphere through* P *can be constructed whose interior lies entirely in* $\Omega \times [0,T]$ *and in which* $u < M$. *Also suppose that the radial direction from the center of the sphere to* P *is not parallel to the t-axis. Then* $\frac{\partial u}{\partial \nu} > 0$, *where* ν *is any outward norm. Furthermore, if* $M = 0$, *then nonpositive restriction on* $h(x,t)$ *can be removed.*

The patchy parabolic system is

$$-\frac{\partial u_i}{\partial t} + d_i \Delta u_i + h_i(x,t)u_i \geq 0, \quad (x,t) \in \Omega_i \times [0,T], \quad i = 1,2,...,n$$

$d_i > 0$ is a constant, $h_i(x,t) \leq 0$ is bounded, $(\Omega_1, \Omega_2, ..., \Omega_n)$ could be a patchy environment of type (A), (B) or (C), and u_i, $i = 1,2,...,n$ satisfies the corresponding matching conditions. Similarly to the treatment of the patchy elliptic system, we write the patchy parabolic system in a compact form

$$-u_t + d\Delta u + h(x,t)u \geq 0, \quad (x,t) \in \Omega \times [0,T],$$

where $d(x) = d_i(x)$, $h(x,t) = h_i(x,t)$, $u(x,t) = u_i(x,t)$, for $(x,t) \in \Omega_i \times [0,T]$, $i = 1, 2, ..., n$, and $\Omega = \cup_{i=1}^{n} \Omega_i$. The following theorems are the maximum principle and its strong version in patchy parabolic systems. The arguments basically follow analogously to those in patchy elliptic systems, and so we omit them here.

Theorem 2.2.3. *The conclusions of Theorem 2.2.1 hold for the patchy parabolic system above in* Ω.

Theorem 2.2.4. *The conclusions of Theorem 2.2.2 hold for the patchy parabolic system above in* Ω.

Chapter 3

Population Diffusion Through Perfect Annular Patches

Models (A), (A_0), (B) and (B_0) are analized in this chapter. Section 3.1 - 3.4 are concerned with (A), whereas Section 3.5, 3.6, and 3.7 deal with (A_0), (B) and (B_0) respectively.

§3.1 Notations

In Section 3.1 - 3.4, we discuss population diffusion through perfect annular patches. Model (A), i.e. $(A.1)-(A.10)$ is utilizied. For convenience, we repeat them here and relabel them as following:

$$\frac{\partial u_i}{\partial t} = d_i \Delta u_i + u_i g_i(u_i), \quad i = 1, 2, ..., n \tag{3.1.1}$$

matching conditions

$$\begin{aligned} u_i\big|_{r=r_i} &= u_{i+1}\big|_{r=r_i} \\ d_i \left.\frac{\partial u_i}{\partial n_i}\right|_{r=r_i} &= d_{i+1} \left.\frac{\partial u_{i+1}}{\partial n_i}\right|_{r=r_i} \end{aligned} \tag{3.1.2}$$

and boundary condition

$$u_n\big|_{r=r_n} = K_n \tag{3.1.3}$$

and initial condition

$$u(x, 0) = \eta(x). \tag{3.1.4}$$

The concise form of (3.1.1) is

$$u_t = d\Delta u + ug(u). \tag{3.1.5}$$

A solution of (3.1.1) – (3.1.4) in the classical sense is a function $u \in C(\Omega^{(n)} \times R^+, R^+)$ such that

(i) the partial derivatives $\frac{\partial}{\partial t}u_i$, $\frac{\partial^2}{\partial {x_k}^2}u_i$, $i = 1, 2, ..., n$, $k = 1, 2$, of the restriction u_i of u are continuous on $\Omega_i \times R^+$,

(ii) the restriction u_i, $i = 1, 2, ..., n$ satisfies (3.1.1) – (3.1.4).

The steady state equations of (3.1.1) – (3.1.3) is given by the corresponding elliptic equations

$$d_i \Delta v_i + v_i g_i(v_i) = 0, \quad i = 1, 2, ..., n \tag{3.1.6}$$

with matching conditions

$$\begin{aligned} v_i\big|_{r=r_i} &= v_{i+1}\big|_{r=r_i} \\ d_i \left.\frac{\partial v_i}{\partial n_i}\right|_{r=r_i} &= d_{i+1} \left.\frac{\partial v_{i+1}}{\partial n_i}\right|_{r=r_i} \end{aligned} \tag{3.1.7}$$

and boundary condition

$$v_n\big|_{r=r_n} = K_n. \tag{3.1.8}$$

In Section 3.2, the existence and uniqueness of a positive radially symmetric steady state solution will be obtained. In Section 3.3, the Cauchy problem for (3.1.1) – (3.1.4) will be investigated. The global stability of the radially symmetric steady state is considered in Section 3.4.

§3.2 Positive Radially Symmetric Steady State Solution

Because of the symmetry of the environment, we try for a radially symmetric steady state solution which is the solution of the radially symmetric form of

(3.1.6) – (3.1.8),

$$d_i v_i''(r) + \frac{d_i}{r} v_i'(r) + v_i(r) g_i(v_i(r)) = 0, r \in (r_{i-1},\ r_i),\ i = 1, 2, ..., n$$

$$v_n(r_n) = K_n$$

$$v_i(r_i) = v_{i+1}(r_i)$$

$$d_i v_i'(r_i) = d_{i+1} v_{i+1}'(r_i),\ i = 1, 2, ..., n-1 \tag{3.2.1}$$

Before discussing the radially symmetric steady state solution of (3.2.1), two preliminary lemmas are given concerning the radially symmetric steady state solution of a parabolic equation in two typical domains Ω_1 and Ω_i, $i > 1$. The equation is of the form

$$u_t = d\Delta u + h(u) \tag{3.2.2}$$

where $d > 0$ is a constant,

$$h(u) = \begin{cases} ug(u) & u \geq 0 \\ 0 & otherwise \end{cases}$$

$$g(0) > 0$$

$$g'(0) < 0 \quad for \quad u > 0$$

$$g(K) = 0 \quad for \quad some \quad K > 0. \tag{3.2.3}$$

Here d, K, $h(u)$ and $g(u)$ in (3.2.2) – (3.2.3) have slightly different meanings from those in (3.1.5). In (3.1.5), $g(u)$ is a concise form of all $g_i, i = 1, 2, \cdots, n$, more precisely $g(u) = g_i(u)$ for $x \in \Omega_i$. In these two lemmas, we may treat $g(u)$ as one of the $g_i(u), i = 1, 2, \cdots, n$, depending on which patch Ω_i is considered. We also may treat d and K as certain d_i and K_i in the same way. In the rest of this section, d, $h(u)$ and $g(u)$ follow the sense given in (3.1.5).

Lemma 3.2.1. *In Ω_i, $i > 1$, there exists a radial symmetric steady state solution $v(r)$ of (3.2.2) for the boundary value problem*

$$v|_{r=r_{i-1}} = \widetilde{K}, \quad \left.\frac{\partial v}{\partial n_{i-1}}\right|_{r=r_{i-1}} = \alpha$$

where $\widetilde{K}$ is a constant, and n_{i-1} is the outward normal unit vector from Ω_{i-1} to Ω_i. Furthermore, if $\widetilde{K} > K$ and $\alpha > 0$, then v will be strictly increasing; if $\widetilde{K} < K$ and $\alpha < 0$, then v will be strictly decreasing.

Proof. The existence of a radially symmetric steady state $v(r)$ of (3.2.2) is implied by the radially symmetric form of (3.2.2) which is a second order ODE

$$dv''(r) + \frac{d}{r}v'(r) + h(v(r)) = 0, \ r \in (r_{i-1}, \ r_i). \tag{3.2.4}$$

The monotonicity of v can be easily derived from the integral form of (3.2.4)

$$\begin{aligned} v'(r) &= \frac{r_{i-1}}{r}v'(r_{i-1}) - \frac{1}{rd}\int_{r_{i-1}}^{r} s \cdot h(v(s))ds \\ &= \frac{r_{i-1}\alpha}{r} - \frac{1}{rd}\int_{r_{i-1}}^{r} s \cdot h(v(s))ds \end{aligned} \tag{3.2.5}$$

and the property of g and h, i.e. $h(v) > 0$ while $v < K$ and $h(v) < 0$ while $v < K$.

Lemma 3.2.2. *In Ω_1, there exists a radially symmetric steady state solution $v(r)$ of (3.2.2) satisfying the central condition $v(0) = \widetilde{K}$, where $\widetilde{K} > 0$ is a constant. Furthermore if $\widetilde{K} > K$, v is strictly increasing; if $\widetilde{K} < K$, v is strictly decreasing.*

Proof. It is not difficult to see that the precondition $v'(0) = 0$ must hold for the existence of a radially symmetric steady state solution $v(r)$ from the radially symmetric form of (3.2.2)

$$dv''(r) + \frac{d}{r}v'(r) + h(v(r)) = 0, \; 0 \le r < r_1. \tag{3.2.6}$$

The monotonicity of $v(r)$ is obvious by the integral form of (3.2.6) in Ω_1

$$v'(r) = -\frac{1}{rd}\int_0^r s \cdot h(v(s))ds \tag{3.2.7}$$

and the properties of h.

For the existence, we will prove that the radially symmetric steady state solution is a limit of $\{v^{(m)}\}$ by the Ascoli-Arzela lemma, where $v^{(m)}$ is the radially symmetric steady state of

$$\begin{aligned} d\Delta v^{(m)} + h(v^{(m)}) &= 0 \; in \; \Omega_{r_{\epsilon_m} r_1} \\ v|_{\partial\Omega_{r_{\epsilon_m}}} &= \widetilde{K} \\ \left.\frac{\partial v}{\partial n}\right|_{\partial\Omega_{r_{\epsilon_m}}} &= 0 \end{aligned} \tag{3.2.8}$$

where

$$\Omega_{R_{\epsilon_m} r_1} = \{ (x,y) | \quad r_{\epsilon_m}^2 < x^2 + y^2 < r_1^2 \}$$

$$\Omega_{r_{\epsilon_m}} = \{ (x,y) | \quad x^2 + y^2 < r_{\epsilon_m}^2 \}$$

$$r_1 > r_{\epsilon_1} > r_{\epsilon_2} > \cdots \to 0$$

The radially symmetric integral form of (3.2.8) is

$$v^{(m)\prime}(r) = -\frac{1}{rd}\int_{r_{\epsilon_m}}^r s \cdot h(v^{(m)}(s))ds, \; r_{\epsilon_m} < r < r_1 \tag{3.2.9}$$

In order to use the Ascoli-Arzela Lemma, we only need to check the uniform boundedness of $\{v^{(m)}\}$, since then its equi-continuity is implied by the uniform boundedness which results from (3.2.9).

If $\widetilde{K} < K$, then $v^{(m)}(r) < K$ by Lemma 3.2.1. By (3.2.9),

$$\left|v^{(m)\prime}(r)\right| \leq \frac{1}{rd}\frac{r^2 - r_{\epsilon_m}^2}{2}M_1 < \frac{M_1 r_1}{2d}, \; r_{\epsilon_m} < r < r_1$$

where $M_1 = \sup_{v \leq K} |h(v)|$ exists by the properties of h. Hence

$$\begin{aligned} \left|v^{(m)}(r)\right| &\leq \left|v^{(m)}(r_{\epsilon_m})\right| + \left|v^{(m)}(r) - v^{(m)}(r_{\epsilon_m})\right| \\ &\leq \widetilde{K} + \frac{M_1 r_1}{2d}r_1, \qquad r_{\epsilon_m} < r < r_1. \end{aligned}$$

If $\widetilde{K} > K$, then $v^{(m)}(r)$ is strictly increasing by Lemma 3.2.1. Since $h'(0) < 0$ and $g(v) < 0$ for $v > K$, we have from (3.2.9),

$$\begin{aligned} v^{(m)\prime}(r) &= -\frac{1}{rd}\int_{r_{\epsilon_m}}^{r} s \cdot v^{(m)}(s) g(v^{(m)}) ds \\ &< -\frac{r_1}{d} v^{(m)}(r) g(v^{(m)}(r)), \qquad r_{\epsilon_m} < r < r_1 \end{aligned} \tag{3.2.10}$$

Let $\widetilde{v}(r)$ be the solution of

$$\begin{aligned} v'(r) &= -\frac{r_1}{d} v(r) g(v(r)) \\ v(0) &= \widetilde{K} \qquad 0 \leq r \leq r_1. \end{aligned} \tag{3.2.11}$$

It is not difficult to compare (3.2.10) to (3.2.11), and we get

$$v^{(m)}(r) < \widetilde{v}(r), \qquad r_{\epsilon_m} \leq r \leq r_1.$$

Thus in both cases, $\{v^{(m)}(r)\}$ is uniformly bounded. By the Ascoli-Arzela Lemma, there exists a uniformly convergent subsequence of $\{v^{(m)}(r)\}$ in $[r_{\epsilon_1}, r_1]$, say $\{v^{(1,m)}(r)\}_{m=1}^{\infty}$. Similarly there exists a uniformly convergent subsequence of $\{v^{(1,m)}(r)\}_{m=1}^{\infty}$ in $[r_{\epsilon_2}, r_1]$, say $\{v^{(2,m)}(r)\}_{m=1}^{\infty}$. Extending this procedure, we get a uniformly convergent subsequence of $\{v^{(l-1,m)}(r)\}_{m=1}^{\infty}$ in $[r_{\epsilon_l}, r_1]$, say $\{v^{(l,m)}(r)\}_{m=1}^{\infty}$, $l = 1, 2, \cdots$. It is obvious that $\{v^{(m,m)}(r)\}_{m=1}^{\infty}$ is uniformly convergent in any interval $[\alpha, \beta] \subset (0, r_1]$. Without loss of generality, we assume $\{v^{(m)}(r)\}$ is uniformly convergent to $v^*(r)$ in $(0, r_1]$. It is easy to see that $v^*(r)$ is continuous and bounded in $(0, r_1]$, and so we can extend the definition of v^* to $r = 0$. Actually from the construction of $\{v^{(m)}(r)\}$, we have $v^*(r) = \widetilde{K}$.

An immediate result of (3.2.9) is that $\{v^{(m)\prime}(r)\}$ is uniformly convergent in $[\alpha, \beta] \subset (0, r_1]$. Actually (3.2.9) can be written in two parts

$$v^{(m)\prime}(r) = -\frac{1}{rd}\int_{r_{\epsilon_m}}^{max\{r_{\epsilon_m}, r^*\}} s \cdot h(v^{(m)}(s))ds - \frac{1}{rd}\int_{max\{r_{\epsilon_m}, r^*\}}^{r} s \cdot h(v^{(m)}(s))ds \tag{3.2.12}$$

where $r^* < \alpha$ doesn't depend on m and is small enough to make the first part as uniformly small as we want by the uniform boundedness of $\{v^{(m)}(r)\}$. The second part is uniformly convergent to

$$-\frac{1}{rd}\int_{r^*}^{r} s \cdot h(v^*(s))ds$$

in $[\alpha, \beta]$ as $m \to \infty$ by the uniformly convergence of $\{v^{(m)}(r)\}$ in $[r^*, r_1]$. Furthermore by this argument and that

$$-\frac{1}{rd}\int_{0}^{r^*} s \cdot h(v^*(s))ds$$

could be as small as we want only if r^* is small enough, we obtain that the right side of (3.2.9) uniformly converges to

$$-\frac{1}{rd}\int_0^r s\cdot h(v^*(s))ds$$

in $[\alpha,\beta]$ as $m\to\infty$. Therefore $\{v^{(m)'}(r)\}$ uniformly converges in $[\alpha,\beta]$, and hence $v^*(r)$ is differentiable and $v^{(m)'}(r)\to v^{*'}(r)$ uniformly in $[\alpha,\beta]$, and (3.2.9) becomes

$$v^{*'}(r) = -\frac{1}{rd}\int_0^r s\cdot h(v^*(s))ds$$

i.e. $v^*(r)$ satisfies (3.2.7) and thus is a radially symmetric steady state solution of (3.2.2). This finishes the proof of Lemma 3.2.2. □

The next two theorems are concerned with the existence and uniqueness of the positive radially symmetric steady state solution of (3.2.1).

Theorem3.2.1. (Existence) *There exists a positive radially symmetric steady state solution of* (3.2.1) *which is located in* $[\underline{K},\overline{K}]$, *where*

$$\overline{K} = \max_{1\le i\le n}\{K_i\},\ \underline{K} = \min_{1\le i\le n}\{K_i\}$$

Proof. We assume that the growth function is $h_i(u_i)$ instead of $u_i g_i(u_i)$ by defining

$$h_i(u_i) = \begin{cases} u_i g_i(u_i) & u_i\ge 0 \\ 0 & otherwise \end{cases} \tag{3.2.13}$$

This is in case that negative solutions appear. Now consider two special radially symmetric steady state solutions, one being $\bar{v}=(\bar{v}_1,\bar{v}_2,\cdots,\bar{v}_n)$ with component

$\bar{v}_1(r)$ to be determined by the initial condition $\bar{v}_1(0) = \bar{K}$ and Lemma 3.2.1, and component $\bar{v}_i(r)$ to be determined consecutively by $\bar{v}_{i-1}(r)$, $i = 1, 2, \cdots, n$ and Lemma 3.2.2. The other one is $\underline{v} = (\underline{v}_1, \underline{v}_2, \cdots, \underline{v}_n)$ with initial condition $\underline{v}_1(0) = \underline{K}$, and its components can be determined similarly. It is easy to see that $\underline{v}_n(r_n) \leq K_n \leq \overline{v}_n(r_n)$ by the monotonicity shown in Lemmas 3.2.1 and 3.2.2. By the continuous dependence of solution on initial data (which is valid since each solution continously depends on the data on the boundary of each patch consectively), we will get a radially symmetric steady state solution $(v_1, v_2, \cdots, v_n)$ with $v_n(r_n) = K_n$ and $\underline{K} \leq v_1(0) \leq \overline{K}$. This solution must be located in $[\underline{K}, \overline{K}]$, since once it leaves this interval, it will never return by the monotonicity in Lemmas 3.2.1 and 3.2.2.

Theorem 3.2.2. (**Uniqueness**) *The positive radially symmetric steady state solution of* (3.2.1) *is unique.*

Proof. Suppose $\widetilde{v} = (\widetilde{v}_1, \widetilde{v}_2, \cdots, \widetilde{v}_n)$ and $\widetilde{\widetilde{v}} = (\widetilde{\widetilde{v}}_1, \widetilde{\widetilde{v}}_2, \cdots, \widetilde{\widetilde{v}}_n)$ are two positive radially symmetric steady state solutions. We basically adopt the technique in [FK], defining the function $w = (w_1, w_2, \cdots, w_n)$ on $[0,\ r_n]$ by

$$w_i(r) = \frac{d_i \widetilde{\widetilde{v}}_i'(r)}{\widetilde{\widetilde{v}}_i(r)} - \frac{d_i \widetilde{v}_i'(r)}{\widetilde{v}_i(r)},\ r \in [r_{i-1},\ r_i], \quad i = 1, 2, \cdots, n.$$

If $w(r_n) = 0$, then $\widetilde{\widetilde{v}}_n'(r_n) = \widetilde{v}_n'(r_n)$ since $\widetilde{\widetilde{v}}_n(r_n) = \widetilde{v}_n(r_n) = K_n$. Hence $\widetilde{\widetilde{v}} = \widetilde{v}$ in $[0,\ r_n]$ by considering the uniqueness of the solution of (3.2.1) in $[\epsilon,\ r_n]$ for arbitrary $\epsilon > 0$. If $w(r_n) \neq 0$, then without loss of generality, we assume $w(r_n) < 0$. Let $r^* = \sup_{0 \leq r \leq r_n} \{w(r) = 0\}$ which exists since $w(0) = 0$. It is easy to see that

$\widetilde{\widetilde{v}}(r) > \widetilde{v}(r)$ for $r \in (r^*, r_n)$. We calculate by (3.2.1)

$$w_i'(r) = g_i(\widetilde{v}_i) - g_i(\widetilde{\widetilde{v}}_i) - \left(\frac{1}{r} + \frac{\widetilde{\widetilde{v}}_i'}{\widetilde{\widetilde{v}}_i} + \frac{\widetilde{v}_i'}{\widetilde{v}_i}\right) w_i(r).$$

Since $w(r^*) = 0$ and $\widetilde{v} < \widetilde{\widetilde{v}}$ in (r^*, r_n) and $g_i' < 0$, we have $w(r) > 0$ in (r^*, r_n). This contradicts $w(r_n) < 0$. This complete the proof of Theorem 3.2.2. □

When the carrying capacities are monotone, the radially symmetric steady state solution is also monotone. This is the content of the next theorem.

Theorem 3.2.3.. *When the carrying capacities are decreasing, the positive radially symmetric steady state solution is decreasing, and vice versa.*

Proof. Suppose $K_1 > K_2 > \cdots > K_n$, and that $(v_1, v_2, \cdots, v_n)$ is the positive radially symmetric steady state solution. Because of $v'(0) = 0$, we can pick

$$r_* = \max\{r \leq r_n | \quad v' \leq 0 \quad in \quad [0, r]\}.$$

If $r_* < r_n$, say $r_* \in [r_{i-1}, r_i)$, then $v'(r_*) = 0$. Furthermore, by the monotonicity in Lemmas 3.2.1 and 3.2.2, $v(r_*) > K_i$ will result in $v(r) > K_i$ for $r \geq r_*$ which contradicts $v(r_n) = K_n < K_i$; $v(r_*) \leq K_i$ and $v'(r_*) = 0$ will contradict the definition of r_*. Therefore $r_* = r_n$, i.e. v must be decreasing. The case $K_1 < K_2 < \cdots < K_n$ can be treated similarly. □

§3.3 Cauchy Problem

The existence of a unique, globally defined solution of (3.1.1) – (3.1.4) is proved in this section. The regularity and boundedness of the solution are also given for the stability analysis in the next section. For convenience, we need to introduce a suitable Sobolev space in which we can discuss our model by employing well developed tools in partial differential equations and ordinary differential equations in Banach spaces. We define

$$\Omega = \cup_{i=1}^{n} \Omega_i$$

and the Hilbert space

$$X = \prod_{i=1}^{n} L^2(\Omega_i)$$

or more precisely

$$X = \{\varphi = (\varphi_1, \varphi_2, \cdots, \varphi_n) | \quad \varphi_i = \varphi|_{\Omega_i} \in L^2(\Omega_i), \quad i = 1, 2, \cdots, n\}$$

with inner product

$$< \varphi, \psi >= \sum_{i=1}^{n} < \varphi_i, \psi_i >_{L^2(\Omega_i)} .$$

Another Hilbert space Y is defined by

$$Y = \left\{ \varphi = (\varphi_1, \varphi_2, \cdots, \varphi_n) \left| \begin{array}{l} \varphi_i = \varphi|_{\Omega_i} \in H^1(\Omega_i),\ i = 1, 2, \cdots, n \\ \\ \varphi_i|_{r=r_i} = \varphi_{i-1}|_{r=r_i},\ i = 1, 2, \cdots, n-1 \\ \\ \varphi|_{r=r_n} = 0 \end{array} \right. \right\}$$

with inner product

$$< \varphi, \psi >= \sum_{i=1}^{n} < \nabla\varphi_i, \nabla\psi_i >_{L^2(\Omega_i)} .$$

Here $\varphi = (\varphi_1, \varphi_2, \cdots, \varphi_n)$, $\psi = (\psi_1, \psi_2, \cdots, \psi_n)$ are not vector forms, but rather they are piecewise forms. H^1 is the standard Sobolev space [J].

Let $v^*(r) = (v_1^*(r), v_2^*(r), \cdots, v_n^*(r))$ be the positive radially symmetric steady state solution of (3.1.6) – (3.1.8). Define $w = (w_1, w_2, \cdots, w_n)$ by

$$w(x,t) = u(x,t) - v^*(r), \quad t \geq 0, \, x \in \Omega, \, r = |x|, \, 0 \leq r \leq r_n.$$

For any solution $u(x,t)$ of (3.1.1) – (3.1.4), $w(x,t)$ satisfies

$$\frac{\partial}{\partial t} w_i = d_i \Delta w_i + h_i(w_i + v_i^*) - h_i(v_i^*), \quad t > 0, \, x \in \Omega, \, r = |x|, \tag{3.3.1}$$

the homogeneous boundary condition

$$w_n\big|_{r=r_n} = 0, \tag{3.3.2}$$

the matching conditions

$$w_i\big|_{r=r_i} = w_{i+1}\big|_{r=r_i} \tag{3.3.3}$$

$$d_i \left.\frac{\partial w_i}{\partial n_i}\right|_{r=r_i} = d_{i+1} \left.\frac{\partial w_{i+1}}{\partial n_i}\right|_{r=r_i} \tag{3.3.4}$$

and the initial condition

$$w(x,0) = \gamma(x) \in Y. \tag{3.3.5}$$

Here h_i is defined in (3.2.13). It is time to indicate that the precise requirement on $\eta(x)$ in (3.1.4) is such that $\eta(x) - v^*(r) \in Y$. Checking with the definition of H_0^1 [J], we see that Y is a subspace of $H_0^1(\Omega)$. The gradient norm on $H_0^1(\Omega)$ is eqivalent to the L^2–norm([J], 4.(5.19)). Therefore Y is complete and there exists c_1, $c_2 > 0$ such that

$$c_1 |\varphi|_X \leq |\varphi|_Y \leq c_2 |\varphi|_X, \quad \forall \varphi \in Y, \tag{3.3.6}$$

where $|\varphi|_X$, $|\varphi|_Y$ are the norms on X and Y induced by their inner products respectively.

Now we interpret (3.3.1)-(3.3.5) in Hilbert space X in the distributional sense, and employ abstract semigroup theory of the operator[H,P,M] in order to obtain the existence of a distributional solution. We then utilize the regularity of the solution with an embedding theorem[H] in order to prove this distributional solution is also a classical solution. Here by a classical solution of (3.3.1)-(3.3.5) we mean $w(x,t) + v^*(r)$ is a solution of (3.1.1)-(3.1.4) in the classical sense.

We define a linear operator $A : X \to X$ by

$$A(\varphi_1, \varphi_2, \cdots, \varphi_n) = (d_1 \Delta\varphi_1, d_2 \Delta\varphi_2, \cdots, d_n \Delta\varphi_n)$$

where $\Delta\varphi_i$ denotes the second distributional Laplacian derivative of φ_i, and with domain

$$D(A) = \left\{ \varphi \in Y \cap \prod_{i=1}^{n} H^2(\Omega_i) \;\middle|\; d_i \left.\frac{\partial\varphi_i}{\partial n_i}\right|_{r=r_i} = d_{i+1} \left.\frac{\partial\varphi_{i+1}}{\partial n_i}\right|_{r=r_i}, i = 1, 2, \cdots, n-1 \right\}.$$

Lemma 3.3.1. *A will generate a compact, analytic, uniformly bounded semigroup $T(t)$, $|T(t)| \le M_0$ and $0 \in \rho(A)$.*

Proof. $\prod_{i=1}^{n} C^\infty(\Omega_i) \cap D(A)$ is dense in X, and therefore $D(A)$ is dense in X, i.e. A is densely defined in X.

We prove that $R(A) = X$ and $0 \in \rho(A)$ by considering the weak form of $A\varphi = \psi$ for any $\psi \in X$. i.e. $\forall \tilde{\varphi} \in C_0^\infty(\Omega)$

$$\int_\Omega A\varphi\tilde{\varphi}\, d\Omega = \int_\Omega \psi\tilde{\varphi}\, d\Omega.$$

By the matching condition in $D(A)$ and Green's formula, this is equivalent to

$$\sum_{i=1}^{n}\int_{\Omega_i} d_i\nabla\varphi_i\cdot\nabla\widetilde{\varphi}_i d\Omega=\sum_{i=1}^{n}\int_{\Omega_i}\psi_i\widetilde{\varphi}_i d\Omega.$$

By the theory of weak solutions of elliptic equations[A], $A\varphi=\psi$ has a unique solution $A^{-1}\psi$ in $D(A)$ and A^{-1} is compact, giving $0\in\rho(A)$.

For $\varphi,\ \psi\in D(A)$, $<\varphi,A\psi>_X=<A\varphi,\ \psi>_X$ by integrating by parts, and thus A is symmetric. $R(A)=X$ and the symmetry of A imply that A is selfadjiont, and hence A is closed. A is also negative definite, because of (3.3.6) and $\forall\varphi\in D(A)$

$$\begin{aligned}<A\varphi,\ \varphi>_X &=\sum_{i=1}^{n}\int_{\Omega_i} d_i\Delta\varphi_i\cdot\varphi_i d\Omega\\ &=-\sum_{i=1}^{n} d_i\int_{\Omega-i}|\nabla\varphi_i|^2 d\Omega\\ &\le-\min_{1\le i\le n} d_i\cdot c_1|\varphi|_X^2. \end{aligned}\tag{3.3.7}$$

Now A is selfadjoint, closed, densely defined and bounded above, and therefore A will generate an analytic semigroup $T(t)$, $t\geq 0$[H, page 19]. Inequality (3.3.7) implies $|T(t)|\leq M_0$ for some $M_0>0$. The compactness of $T(t)$ comes from the analiticity of $T(t)$ and the compactness of A^{-1} ([P], Corollary2.3.4). This completes the proof of Lemma 3.3.1. □

Considering the fractional power $(-A)^{\frac{1}{2}}$([Fr], [H],[P]), $X^{\frac{1}{2}}$ is the domain of $(-A)^{\frac{1}{2}}$, which is a Banach space with norm $|\varphi|_{X^{\frac{1}{2}}}=\left|(-A)^{\frac{1}{2}}\varphi\right|_X$. Since $-A$ is positive definite, $<-A\varphi,\ \varphi>_X=\left|(-A)^{\frac{1}{2}}\varphi\right|_X^2$ for $\varphi\in D(A)$, and by (3.3.7),

$$\min_{1\le i\le n} d_i\cdot|\varphi|_Y^2\le<-A\varphi,\varphi>\le\max_{1\le i\le n} d_i\cdot|\varphi|_Y^2.$$

The denseness of $D(A)$ in both Y and $X^{\frac{1}{2}}$ implies $Y = X^{\frac{1}{2}}$.

Let $f : X \to X$ be the substitution operator induced by

$$h_i(\varphi(x) - v^*(r)) - h_i(v^*(r)), \quad x \in \Omega_i,\ |x| = r,\ i = 1, 2, \cdots, n.$$

f is locally Lipschitzian by the definition of h_i, and therefore (3.1.1)-(3.1.4) can be written in the abstract form

$$\begin{aligned} &w'(t) = Aw(t) + f(w(t)), \quad t > 0 \\ &w(0) = w_0 \end{aligned} \tag{3.3.8}$$

where $w_0 \in Y$. A function $w : [0, T) \to X$ is a solution of (3.3.8) on $[0, T)$ if $w \in C([0, T), X) \cap C^1((0, T), X)$ and (3.3.8) holds for $t \in [0, T)$. We write the local existence, uniqueness, continuity and compactness results in next theorem. The proof is routine ([P], Theorem 6.3.1; [H], Theorems 3.3.4, 3.3.6, 3.4.1).

Theorem 3.3.1. *For any $w_0 \in Y$, the Cauchy problem of (3.3.8) has a unique solution on $[0, T)$ for some $T > 0$ such that:*

(i) $w \in C([0, T), Y)$.

(ii) If $T < \infty$, then there exists a sequence $\{t_n\}$ such that $t_n \to T - 0$ and $|w(t_n)|_Y \to \infty$ as $n \to \infty$.

(iii) If $T = \infty$ and $\{w(t) \mid t \geq 0\}$ is bounded in Y, then $\{w(t) \mid t \geq 0\}$ is precompact in Y.

(iv) The mapping $w_0 \mapsto w(t)$ is continuous from Y to Y uniformly on compact subintervals of $[0, T)$.

By a standard way in ([H], Section3.5 & 3.6), we can prove that $w(t)$ is actually a classical solution. We write this result in the next lemma for later purposes.

Lemma 3.3.2. *Let w be the unique solution of (3.3.8) on $[0, T)$. Then $\frac{\partial}{\partial t}w_i$, $\frac{\partial}{\partial x_k}w_i$, $\frac{\partial^2}{\partial x_k^2}w_i$, $\frac{\partial^2}{\partial x_k \partial t}w_i$, $\frac{\partial^2}{\partial t \partial x_k}w_i$, $i = 1, 2, \cdots, n$; $k = 1, 2$, $0 < t < T$, exist and are continuous in the corresponding patch Ω_i.*

Actually elements of X are equivalence classes of functions. A more exact statement of Lemma 3.3.2 is that there is a function with the stated smoothness properties in the equivalence class of the solution. Therefore we have the existence and uniqueness of the solution $u(x,t)$ of (3.1.1)-(3.1.4) in some interval $t \in [0, T)$.

According to the maximum principle in patchy parabolic equations, we claim that for $x \in \Omega$ and $t \in [0, T)$,

$$\min_{1 \le i \le n} \left\{ K_i, \min_{x \in \Omega_i} \eta_i(x) \right\} \le u(x,t) \le \max_{1 \le i \le n} \left\{ K_i, \max_{x \in \Omega_i} \eta_i(x) \right\}. \tag{3.3.9}$$

Actually if $u(\overline{x}, \overline{t})$ reaches $M^* = \max_{1 \le i \le n} \{K_i, \max_{x \in \Omega_i} \eta_i(x)\}$ at some $\overline{x} \in \Omega$, $\overline{t} > 0$, then by $[u_i g_i(u_i)]' = g_i(u_i) + u_i g_i'(u_i) < 0 \, for \, u_i \ge K_i$, there exists a neighborhood $U(\overline{x})$ of $\overline{x}$ and $(\overline{t} - \delta, \overline{t})$ of $\overline{t}$ such that $[ug(u)]' < 0$ in $U(\overline{x}) \times (\overline{t} - \delta, \overline{t})$. Therefore applying the mean value theorem and from $-\frac{\partial u}{\partial t} + d\Delta u + ug(u) = 0$ and $M^* g(M^*) \le 0$, we have

$$-\frac{\partial \overline{u}}{\partial t} + d\Delta \overline{u} + h(x,t)\overline{u} \ge 0$$

for some $h(x,t) \le 0$, where $\overline{u} = u - M^*$. Now utilizing Theorem 2.2.3, we get $\overline{u} \equiv 0$ in $U(\overline{x}) \times (\overline{t} - \delta, \overline{t})$ i.e. $u \equiv M^*$ in $U(\overline{x}) \times (\overline{t} - \delta, t)$. Furthermore, this argument means $u \equiv M^*$ in $\Omega \times [0, \bar{t}]$. therefore $u \leq M^*$. The other direction of (3.3.8) can be obtained in the same way. Hence $|f(w(t)|_X$ is bounded on $[0, T)$. By the variation of parameters formula

$$w(t) = T(t)w_0 + \int_0^t T(t-s)f(w(s))ds$$

and some results on fractional powers([P], Theorem 2.6.13), i.e. $(-A)^{\frac{1}{2}}w_0 \in X$, $|(-A)^{\frac{1}{2}}T(t)| \leq M_{\frac{1}{2}}t^{-\frac{1}{2}}$, $t > 0$ and $|T(t)| \leq M_0$, we have

$$\begin{aligned}
|w(t)|_Y = |w(t)|_{X^{\frac{1}{2}}} &\leq |T(t)w_0|_{X^{\frac{1}{2}}} + \int_0^t |T(t-s)f(w(s))|_{X^{\frac{1}{2}}}ds \\
&\leq |(-A)^{\frac{1}{2}}T(t)w_0|_X + \int_0^t |(-A)^{\frac{1}{2}}T(t-s)f(w(s))|_X ds \\
&\leq M_0|(-A)^{\frac{1}{2}}w_0|_X + M_{\frac{1}{2}} \sup_{s\in[0,T)} |f(w(s))|_X \times \int_0^t (t-s)^{-\frac{1}{2}}ds \\
&\leq M < \infty
\end{aligned}$$

for some $M > 0$ and all $t \in [0, T)$. Hence $T = \infty$ by Theorem 3.3.1(ii), and consequently we get

Theorem 3.3.2. *System (3.1.1)-(3.1.4) has a unique classical solution defined on $\Omega \times R^+$ satisfying (3.3.9).*

§3.4 Global Stability

In this section, we treat system (3.1.1)-(3.1.4) as a dynamical system $S(t)$ in a complete metric space

$$Z = \{\varphi \geq 0 \mid \varphi - v^* \in Y\}$$

with metric $d_Z(\varphi, \psi) = |\varphi - \psi|_Y$, and then obtain global stability by the Liapunov invariance principle. $S(t)$ is defined by $S(t)\varphi = w(\varphi - v^*)(t) + v^*$, where v^* is the positive radially symmetric steady state solution of (3.2.1), $w(\varphi - v^*)(t)$ is the solution of (3.3.8) with initial condition $w_0 = \varphi - v^*$. Theorems 3.3.1 and 3.3.2 guarantee that $S(t)$ is a dynamical system on Z([H], Chapter 4).

Define the functional $V: Z \to R$ by

$$V(\varphi) = \sum_{i=1}^{n} \left(\int_{\Omega_i} \frac{1}{2} d_i |\nabla \varphi_i|^2 d\Omega - \int_{\Omega_i} \int_0^{\varphi_i} h_i(s) ds d\Omega \right).$$

V is continous by the continuity of the inclusion $Y \to C(\Omega)$. We will show that V is a Liapunov functional, i.e. its derivative is nonpositive,

$$\dot{V}(\varphi) = \limsup_{t \to 0+} \frac{1}{t} [V(S(t)\varphi) - V(\varphi)] \leq 0, \ \varphi \in Z. \tag{3.4.1}$$

For $\varphi \in Z$, $u(x,t) = (S(t)\varphi)(x)$ satisfies (3.1.1)-(3.1.4) with $\eta = \varphi$ and $u(x,t) = w(x,t) + v^*(r)$, $w(x,t)$ is the solution of (3.3.8) with $w_0 = \varphi - v^*$. By Lemma 3.3.2, the mixed partial derivatives of u with respect to t and x are continous. Thus the order of differentiation can be exchanged. So by the matching and boundary conditions

$$\begin{aligned}
\frac{d}{dt} V(u(\cdot,t)) &= \sum_{i=1}^{n} \left(\int_{\Omega_i} d_i \nabla u_i \cdot \nabla u_{it} d\Omega - \int_{\Omega_i} h_i(u_i) u_{it} d\Omega \right) \\
&= \sum_{i=1}^{n} \left(\int_{\partial\Omega_i} d_i u_{it} \frac{\partial u_i}{\partial n} dS - \int_{\Omega_i} d_i u_{it} \Delta u_i d\Omega - \int_{\Omega_i} h_i(u_i) u_{it} d\Omega \right) \\
&= - \sum_{i=1}^{n} \int_{\Omega_i} u_{it} (d_i \Delta u_i + h_i(u_i)) d\Omega \\
&= - \sum_{i=1}^{n} \int_{\Omega_i} (d_i \Delta u_i + h_i(u_i))^2 d\Omega.
\end{aligned} \tag{3.4.2}$$

Hence (3.4.1) holds because of

$$\frac{1}{t}[V(S(t)\varphi - V(\varphi)] = \frac{d}{dt} V(u(\cdot, t^*))$$

for some $t^* \in (0, t)$ by the mean value theorem.

We define the ω-limit set $\omega(\psi)$ for $\psi \in Z$ by

$$\omega(\psi) = \{\varphi \in Z \mid \exists t_n \to \infty \, such \, that \, S(t_n)\psi \to \varphi\}$$

By Theorem 3.3.1, $\{S(t)\psi;\ t \geq 0\}$ is precompact in Z, and therefore $\omega(\psi)$ is nonempty, compact, invariant, connected and $dist(S(t)\psi, \omega(\psi)) \to 0$ as $t \to \infty$([H], Theorem 4.3.3), see also ([H], Theorem 4.3.4)

$$\omega(\psi) \subset E = \{\varphi \in Z \mid \dot{V}(\varphi) = 0\}.$$

We are going to prove that E consists of a positive steady state solution of (3.1.6)-(3.1.8). By (3.4.2), $\dot{V}(\varphi) = 0$ implies that there exists a sequence $t_n \to 0+$ such that $\sum_{i=1}^{n} \int_{\Omega_i} (d_i \Delta u_i + h_i(u_i))^2 d\Omega \to 0$, where $u = S(t)\varphi$. This implies that $Aw(t_n) + f(w(t_n)) \to 0$ in X as $n \to \infty$, where $w(\cdot, t) = u(\cdot, t) - v^*(\cdot)$. By Theorem 3.3.1, $w(t_n) \to w(0)$ in Y as $n \to \infty$, thus $w(t_n) \to w(0)$ also in X, and furthermore $f(w(t_n)) \to f(w(0))$ in X by the continuity of $f : Y \to X$. Therefore $Aw(t_n) \to -f(w(0))$. Because A is closed, $w(0) \in D(A)$ and $Aw(0) = -f(w(0))$, i.e. $Aw(0) + f(w(0)) = 0$. This means that $w(0)$ is a steady state solution of (3.1.6)-(3.1.8), and is positive by (3.3.9), i.e. E consists of a positive steady state solution of (3.1.6)- (3.1.8).

By Theorems 3.2.1 and 3.2.2, we have:

Theoren 3.4.1. $\|u(\cdot, t) - v^*\|_Y \to 0$ *as* $t \to \infty$ *for all the solutions* u *of (3.1.1)-(3.1.4) with positive radially symmetric initial data.*

The next theorem is concerned with the uniqueness of the positive radially symmetric steady state solution under certain conditions. Therefore E consists of this unique positive steady state solution, i.e. it is globally stable. This is the content of the second theorem below.

Theorem 3.4.2. *If $K^* \leq \underline{K}$, then the positive steady state solution of (3.1.6)-(3.1.8) is unique. Here $K^* = \max_{1\leq i\leq n}\{K_i^*\}$, $K_i^* = \sup\{\xi \geq 0 \mid [\xi g_i(\xi)]' > 0\}$, $i = 1, 2, \cdots, n$, $\underline{K} = \min_{1\leq i\leq n}\{K_i\}$ is defined in Theorem 3.2.1.*

Proof. Suppose $\overline{v} = (\overline{v}_1, \overline{v}_2, \cdots, \overline{v}_n)$ and $\overline{\overline{v}} = (\overline{\overline{v}}_1, \overline{\overline{v}}_2, \cdots, \overline{\overline{v}}_n)$ are two positive steady state solutions of

$$d_i\Delta\overline{v}_i + \overline{v}_i g_i(\overline{v}_i) = 0$$

$$d_i\Delta\overline{\overline{v}}_i + \overline{\overline{v}}_i g_i(\overline{\overline{v}}_i) = 0 \quad i = 1, 2, \cdots, n. \tag{3.4.3}$$

We claim that $\overline{v} \geq \underline{K}$. For otherwise let

$$\overline{m} = \min_{\Omega^{(n)}} \overline{v}(x,y) = \overline{v}_i(x_0, y_0), \quad (x_0, y_0) \in \overline{\Omega}_i$$

$$\overline{H}(x,y) = -g_i(\overline{v}_i(x,y))$$

$$\overline{V}_i = \overline{m} - \overline{v}_i(x,y), \quad (x,y) \in \Omega_i.$$

Then

$$d_i\Delta\overline{V}_i + \overline{H}(x,y)\overline{V}_i = \overline{m} g_i(\overline{v}_i). \tag{3.4.4}$$

Since $\overline{m} < \underline{K}$, there exists a neighborhood $U(x_0, y_0)$ of (x_0, y_0) in Ω_i, in which

$$\overline{H}(x,y) \leq 0$$

$$\overline{m} g_i(\overline{v}_i) > 0.$$

Therefore (3.4.4) satisfies the condition of Theorem 2.1.3. By boundary condition $(A.4)$, we know that $(x_0, y_0) \in int\Omega_i$, and thus by Theorem 2.1.3. $\overline{v} \equiv \overline{m}$ in $U(x_0, y_0)$. We can extend this conclusion to the entire $\Omega^{(n)}$. This contradicts the boundary condition $(A.4)$ and hence $\overline{v} \geq \underline{K}$. Similarly $\overline{\overline{v}} \geq \underline{K}$.

From (3.4.3), we have

$$d_i\Delta(\overline{\overline{v}}_i - \overline{v}_i) + \int_0^1 [v_i g_i(v_i)]'\Big|_{v_i=\overline{v}_i+\xi(\overline{\overline{v}}_i-\overline{v}_i)} d\xi \cdot (\overline{\overline{v}}_i - \overline{v}_i) = 0. \tag{4.3.5}$$

Since $\overline{v}, \overline{\overline{v}} \geq \underline{K}$, then $\overline{v}$, $\overline{\overline{v}} \geq \underline{K}$, giving $\overline{v}$, $\overline{\overline{v}} \geq K^*$, and hence

$$\int_0^1 [v_i g_i(v_i)]'\Big|_{v_i=\overline{v}_i+\xi(\overline{\overline{v}}_i-\overline{v}_i)} d\xi \leq 0.$$

By applying the boundary condition and Theorem 2.1.3 to (4.3.5) we get that $\overline{v} \equiv \overline{\overline{v}}$.

From the above statement and Theorem 3.4.2, we have:

Theorem 3.4.3.. *If $K^* \leq \underline{K}$, then the positive radially symmetric steady state is globally asymptotically stable with $\|u(\cdot,t) - v^*\|_Y \to 0$ as $t \to \infty$ for any solution of (3.1.1)-(3.1.4) with positive initial data.*

Corollary3.4.1. (Equi-capacities). *If $K_1 = K_2 = \cdots = K_n$, then $u \equiv K_1$ is globally asymptotic stable, i.e. $\|u(\cdot,t) - K_1\|_Y \to 0$ as $t \to \infty$.*

§3.5 Radially Continuously Varying Patchy Environments

This section concerns model (A_0), i.e. $(A_0.1) - (A_0.4)$. We briefly repeat them here and relabel them

$$\frac{\partial u}{\partial t} = \nabla \cdot (d(r)\nabla u) + ug(u,r),\ u = u(x,t),\ x \in \Omega,\ t > 0,\ r = |x|, \tag{3.5.1}$$

with boundary condition

$$u|_{\partial\Omega} = K(R_0). \tag{3.5.2}$$

The steady state equation is the corresponding elliptic equation

$$\nabla \cdot (d(r)\nabla v) + vg(v, r) = 0 \tag{3.5.3}$$

$$v(R_0) = K(R_0). \tag{3.5.4}$$

Theorem 3.5.1. *There exists a unique positive radially symmetric steady state of (3.5.3)-(3.5.4). It is located in* $[\underline{K}, \overline{K}]$*, where* $\underline{K} = \min_{r\in[0,R_0]} K(r)$ *and* $\overline{K} = \max_{r\in[0,R_0]} K(r)$.

Proof. For each positive integer n, we devide the entire environment Ω into n annular patches

$$\Omega_i^{(n)} = \{x \,|\, r_{i-1}^{(n)} < |x| < r_i^{(n)}\},\ i = 1, 2, \cdots, n,$$

where $r_i^{(n)} = \frac{i}{n}R_0$. In each patch $\Omega_i^{(n)}$, $i = 1, 2, \cdots, n$, we discretize (3.5.3)- (3.5.4) into

$$\nabla \cdot \left(d(r_i^{(n)}\nabla v_i^{(n)}\right) + v_i^{(n)} g_i^{(n)}(v_i^{(n)}) = 0 \tag{3.5.5}$$

with matching and boundary conditions

$$\begin{aligned} v_i^{(n)}\Big|_{r=r_i^{(n)}} &= v_{i+1}^{(n)}\Big|_{r=r_i^{(n)}} \\ d(r_i^{(n)})\,\frac{\partial v_i^{(n)}}{\partial n_i}\Bigg|_{r=r_i^{(n)}} &= d(r_{i+1}^{(n)})\,\frac{\partial v_{i+1}^{(n)}}{\partial n_i}\Bigg|_{r=r_i^{(n)}} \\ v_n^{(n)}\Big|_{\partial\Omega} &= K(R_0), \quad i = 1, 2, \cdots, n-1. \end{aligned} \tag{3.5.6}$$

Here

$$v^{(n)} = (v_1^{(n)}, v_2^{(n)}, \cdots, v_n^{(n)})$$

$$g_i^{(n)}\left(v_i^{(n)}\right) = g\left(v_i^{(n)}, r_i^{(n)}\right), \quad i = 1, 2, \cdots, n.$$

By Theorems 3.2.1 and 3.2.2, there exists a unique positive radially symmetric steady state solution $\{v^{(n)}(r)\}$ of (3.5.5)- (3.5.6) satisfying

$$\underline{K} \le v^{(n)} \le \overline{K} \quad and \quad v^{(n)'}(0) = 0. \tag{3.5.7}$$

The radially symmetric form of (3.5.5) is

$$d(r_i^{(n)})v_i^{(n)''}(r) + \frac{d(r_i^{(n)})}{r}v_i^{(n)'}(r) + v_i^{(n)}(r)g_i^{(n)}(v_i^{(n)}(r)) = 0, \quad r \in [r_{i-1}^{(n)}, r_i^{(n)}),$$

$$i = 1, 2, \cdots, n. \tag{3.5.8}$$

Thereafter we assume that the derivatives at partition points refer to the right side derivatives where they exist. The matching and boundary conditions become

$$v_i^{(n)}(r_i^{(n)}) = v_{i+1}^{(n)}(r_i^{(n)})$$

$$d(r_i^{(n)})v_i^{(n)'}(r_i^{(n)}) = d(r_{i+1}^{(n)})v_{i+1}^{(n)\,'}(r_i^{(n)})$$

$$v_n^{(n)}(r_n^{(n)}) = K(R_0), \quad i = 1, 2, \cdots, n-1. \tag{3.5.9}$$

The integral form of (3.5.8) is

$$v_i^{(n)'}(r) = \frac{r_{i-1}^{(n)}}{r}v_i^{(n)'}(r_{i-1}^{(n)}) - \frac{1}{rd(r_i^{(n)})}\int_{r_{i-1}}^{r} sv_i^{(n)}(s)g_i^{(n)}(v_i^{(n)}(s))ds, \quad r \in [r_{i-1}^{(n)}, r_i^{(n)}),$$

$$i = 1, 2, \cdots, n. \tag{3.5.10}$$

It is easy to get the concise form of (3.5.10) from (3.5.9) and (3.5.10), namely

$$v^{(n)'}(r) = -\frac{1}{rd^{(n)}(r)} \int_0^r sv^{(n)}(s) g^{(n)}(v^{(n)}(s)) ds \tag{3.5.11}$$

where

$$d^{(n)}(r) = d(r_i^{(n)})$$

$$v^{(n)}(s) = v_i^{(n)}(s)$$

$$g^{(n)}(v^{(n)}(s)) = g_i^{(n)}(v_i^{(n)}(s))$$

for $s, r \in [r_{i-1}^{(n)}, r_i^{(n)})$, $\quad i = 1, 2, \cdots, n$. Therefore

$$\left| v^{(n)'} \right| \leq \frac{r_1 M_1}{2d_*} \tag{3.5.12}$$

where

$$M1 = \max_{\substack{v \in [\underline{K}, \overline{K}] \\ r \in [0, R_0]}} |vg(v, r)|$$

$$d_* = \min_{r \in [0, R_0]} d(r).$$

There exists a uniformly convergent subsequence of $\{v^{(n)}\}$ and without loss of generality, assume $\{v^{(n)}\}$ convergences to $v(r)$ by applying the Ascoli-Arzela Lemma, (3.5.7) and (3.5.12). Because of (3.5.11) and the Lemma 3.5.1 to follow, $\{v^{(n)'}(r)\}$ must be uniformly convergent to $v'(r)$. Similarly by (3.5.8) and Lemma 3.5.1, $v^{(n)''}(r)$ uniformly convergences to $v''(r)$. Hence $v(r)$ is a positive radially symmetric steady state solution of (3.5.1)-(3.5.2). By (3.5.7), it is located in $[\underline{K}, \overline{K}]$. The uniqueness can be obtained by considering

$$w(r) = d(r)\frac{v'(r)}{v(r)} - d(r)\frac{\overline{v}'(r)}{\overline{v}(r)}$$

for two positive radially symmetric steady state solutions $v(r)$ and $4\overline{v}(r)$ in the same way as that in the proof of Theorem 3.2.2.

Lemma 3.5.1.. *Suppose* $\{f_n(\xi)\}$, $n = 1, 2, \cdots\cdots$ *are absolutely continuous on* $[a, b]$, $\{f_n(\xi)\}$ *converges to* $f(\xi)$, *their right derivatives* $\{\frac{d^+}{d\xi} f_n(\xi)\}$ *uniformly converges to a continuous function* $g(\xi)$. *Then* $f(\xi)$ *is differentiable in* $[a, b]$ *and* $f'(\xi) = g(\xi)$.

Proof. Because $\{\frac{d^+}{dx} f_n(\xi)\}$ uniformly converges to $g(\xi)$, $\int_a^\xi \frac{d^+}{d\xi} f_n(\xi) d\xi$ uniformly converges to $\int_a^\xi g(\xi) d\xi$. From the absolute continuity of $\{f_n(\xi)\}$, we have

$$f_n(\xi) - f_n(a) = \int_a^\xi \frac{d^+}{d\xi} f_n(\xi) d\xi.$$

Thus by the convergence of $\{f_n(\xi)\}$ to $f(\xi)$, we get

$$f(\xi) - f(a) = \int_a^\xi g(\xi) d\xi.$$

Hence $f(\xi)$ is differentiable and $f'(\xi) = g(\xi)$. □

We needn't specially investigate the Cauchy problem of (3.5.1)-(3.5.2), since it is a classical boundary value problem. For the global stability of the radially symmetric steady state solution $v^*(r)$ of (3.5.3)-(3.5.4), we introduce the Liapunov functional

$$V(\varphi) = \int_\Omega \left(\frac{1}{2} d(r) |\nabla \varphi|^2 - \int_0^\varphi h(s, r) ds \right) d\Omega,$$

where

$$h(s, r) = \begin{cases} sg(s, r) & s \geq 0 \\ 0 & otherwise. \end{cases}$$

Similar to the analysis in Section 3.4, results comparable to Theorems 3.4.1, 3.4.2 and 3.4.3 can be obtained.

Theorem 3.5.2. *$\|u(\cdot,t)-v^*\|_{H_0^1(\Omega)} \to 0$ as $t \to \infty$ for all solutions u of (3.5.1)-(3.5.2) with positive radially symmetric initial data.*

Theorem 3.5.3. *If $K^* \le \underline{K}$, then the positive steady state of (3.5.3)-(3.5.4) is unique, i.e. it is $v^*(r)$ which is globally asymtotic stable with $\|u(\cdot,t)-v^*\|_{H_0^1(\Omega)} \to 0$ as $t \to \infty$ for any solution u of (3.5.1)-(3.5.2) with positive initial data. Here*

$$K^* = \sup_{r\in[0,R_0]} \left\{ v \ge 0 \;\middle|\; \frac{\partial}{\partial v}[vg(v,r)] \ge 0 \right\}.$$

§3.6 Perfect Annular Patchy Environment With a Hole

This section is devoted to model (B) or (B.1)-(B.7) which is

$$\frac{\partial u_i}{\partial t} = d_i \Delta u_i + u_i g_i(u_i), \quad i = 1, 2, \cdots, n \tag{3.6.1}$$

with matching conditions

$$\begin{aligned} u_i\big|_{r=r_i} &= u_{i+1}\big|_{r=r_i} \\ d_i \left.\frac{\partial u_i}{\partial n_i}\right|_{r=r_i} &= d_{i+1} \left.\frac{\partial u_{i+1}}{\partial n_i}\right|_{r=r_i} . \end{aligned} \tag{3.6.2}$$

The boundary conditions are

$$\begin{aligned} u_n\big|_{r=r_n} &= K_n \\ u_1\big|_{r=r_0} &= K_1 \end{aligned} \tag{3.6.3}$$

and the initial condition is

$$u(x,t) = \eta(x) > 0, \quad x \in \Omega \tag{3.6.4}$$

where $\Omega = \cap_{i=1}^{n} \Omega_i$, and η is such that $\eta - v^*(r) \in Y$, where Y is slightly different from that in Section 3.3,

$$Y = \left\{ \varphi \;\middle|\; \begin{array}{ll} \varphi_i = \varphi|_{\Omega_i} \in H^1(\Omega_i), & i = 1, 2, \cdots, n \\ \varphi_i|_{r=r_i} = \varphi_{i+1}|_{r=r_i}, & i = 1, 2, \cdots, n-1 \\ \varphi|_{r=r_0} = \varphi|_{r=r_n} = 0 & \end{array} \right\}.$$

and $v^*(r)$ is the positive radially symmetric steady state solution of the corresponding elliptic equations (B.5)-(B.7) whose existence is guaranteed by the following theorem.

Theorem 3.6.1. *There exists a unique positive radially symmetric steady state solution of (3.6.1)-(3.6.3), which is located in $[\underline{K}, \overline{K}]$. Here $\underline{K}, \overline{K}$ are same as those in Theorem 3.2.1.*

We don't give the proof, since it is analogous to that in Section 3.2. The following theorems are the corresponding versions of those of the Cauchy problem and stability problem in Section 3.3 and 3.4. We omit their proofs, since their proofs are basically the same as those in Section 3.3 and 3.4.

Theorem 3.6.2. *(3.6.1)-(3.6.4) has a unique solution defined on $\Omega \times R^+$.*

Theorem 3.6.3. *$\|u(\cdot,t) - v^*\|_Y \to 0$ as $t \to \infty$ for all solution u of (3.6.10-(3.6.4) with radially symmetric initial data.*

Theorem 3.6.4. *If $K^* \leq \underline{K}$, then the positive steady state is unique, and $\|u(\cdot,t) - v^*\|_Y \to 0$ as $t \to \infty$ for all solutions of (3.6.1)-(3.6.4) with positive initial data, where $\underline{K}$ and K^* are defined in Theorems 3.2.1 and 3.4.2.*

Corollary 3.6.1. *If $K_1 = K_2 = \cdots = K_n$, then $u \equiv K_1$ is globally asymptotically stable.*

§3.7 Radially Continuously Varying Environment With a Hole

This section deals with model (B_0) or $(B_0.1) - (B_0.5)$, that is

$$\frac{\partial u}{\partial t} = \nabla \cdot (d(r)\nabla u) + ug(u,r) \tag{3.7.1}$$

with boundary conditions

$$u|_{r=r_0} = K(r_0)$$

$$u|_{r=R_0} = K(R_0). \tag{3.7.2}$$

By the same argument as those in Section 3.5, we have:

Theorem 3.7.1. *There exists a unique positive radially symmetric steady state of (3.7.1)-(3.7.2), which globally attracts all the solutions with positive radially symmetric initial data.*

Theorem 3.7.2. *If $K^* \leq \underline{K}$, the positive steady state is unique and radially symmetric, and it is globally asymptotic stable, where K^* and $\underline{K}$ are defined in Theorems 3.5.3 and 3.5.1.*

Chapter 4

Population Diffusion Through a Periodically Changing Annular Patchy Environment

Models (C) and (C_0) are considered in this chapter. Note that Section (C) is a special case of (C_0) by viewing (C) as periodic of any positive period. Section 4.1 - 4.3 consider the Cauchy problem, and uniqueness and global stability of the periodic solution of model (C_0). In Section 4.4 the result for (C) are stated.

§4.1 Cauchy Problem for (C_0)

We repeat model (C_0) here for convenience:

$$\frac{\partial u_i}{\partial t} = d_i \Delta u_i + u_i g_i(u_i, t), \quad x \in \Omega_i, \quad i = 1, 2, \cdots, n \tag{4.1.1}$$

with matching conditions

$$\begin{aligned} u_i|_{\overline{\Omega}_i \cap \overline{\Omega}_{i+1}} &= u_{i+1}|_{\overline{\Omega}_i \cap \overline{\Omega}_{i+1}} \\ d_i \left.\frac{\partial u_i}{\partial n_i}\right|_{\overline{\Omega}_i \cap \overline{\Omega}_{i+1}} &= d_{i+1} \left.\frac{\partial u_{i+1}}{\partial n_i}\right|_{\overline{\Omega}_i \cap \overline{\Omega}_{i+1}}, \quad i = 1, 2, \cdots, n-1, \end{aligned} \tag{4.1.2}$$

boundary condition

$$u|_{\partial\Omega} = 0 \tag{4.1.3}$$

and initial condition

$$u_i(x, 0) = \eta_i(x) > 0, \quad x \in \Omega_i, \quad i = 1, 2, \cdots, n \tag{4.1.4}$$

In order to parallel the analysis in Section 3.3, we define

$$h(u, t) = h_i(u_i, t) = \begin{cases} u_i g_i(u_i, t) & u_i \geq 0 \\ 0 & otherwise \end{cases} \tag{4.1.5}$$

and introduce two Hilbert spaces

$$X = \prod_{i=1}^{n} L^2(\Omega_i) \tag{4.1.6}$$

with inner product

$$< \varphi, \psi >_X = \sum_{i=1}^{n} < \varphi_i, \psi_i >_{L^2(\Omega_i)}$$

and

$$Y = \left\{ \varphi = (\varphi_1, \varphi_2, \cdots, \varphi_n) \;\middle|\; \begin{array}{ll} \varphi_i = \varphi|_{\Omega_i} \in H^1(\Omega_i), & i = 1, 2, \cdots, n \\ \varphi_i|_{\overline{\Omega}_i \cap \overline{\Omega}_{i+1}} = \varphi_{i+1}|_{\overline{\Omega}_i \cap \overline{\Omega}_{i+1}} & i = 1, 2, \cdots, n-1 \\ \varphi|_{\partial\Omega} = 0 & \end{array} \right\} \tag{4.1.7}$$

with inner product

$$< \varphi, \psi >_Y = \sum_{i=1}^{n} < \nabla\varphi_i, \nabla\psi_i >_{L^2(\Omega_i)} .$$

We can define the operator A similarly to that in Section 3.3, and write (4.1.1)-(4.1.4) in abstract concise form

$$\begin{cases} u'(t) = Au(t) + f(u(t), t) \\ u(0) = u_0. \end{cases} \tag{4.1.8}$$

Here f is the substitution operator induced by $h(u(t), t)$. We can repeat the argument in the same way as that in Section 3.3 to obtain the same conclusions as those in Theorems 3.3.1 and 3.3.2 and in (3.3.9). we write them in the following manner

$$0 \leq u(x,t) \leq \max_{\substack{1 \leq i \leq n \\ 0 \leq t \leq T}} \left\{ K_i(t), \sup_{x \in \Omega_i} \eta_i(x) \right\}. \tag{4.1.9}$$

Theorem 4.1.1. *System* (4.1.1) – (4.1.4) *has a unique classical solution defined on* $\Omega \times R^+$ *satisfying* (4.1.9).

§4.2 Uniqueness of the Positive Periodic Solution

In this section, we define a suitable solution space and introduce order in this space to show that the Poincare map is strongly monotone and strictly sub-linear([H], [T]), and then apply Lemma 1 in [Zhao] to get the uniqueness of the positive periodic solution.

According to Theorem 4.1.1, the solution of (4.1.1)-(4.1.4) is classical. Rather than Y, a natural choice for the solution space is the subspace of continuous functions

$$Z = \left\{ \varphi \in C(\overline{\Omega}) \quad \middle| \quad \varphi|_{\Omega_i} \in C^1(\Omega_i), \quad \varphi|_{\partial\Omega} = 0, \quad i = 1, 2, \cdots, n \right\}.$$

Let

$$Z^+ = \{\varphi \in Z \quad | \quad \varphi(x) \geq 0, \quad x \in \Omega\}$$

and

$$intZ^+ = \left\{ \varphi \in Z^+ \quad \middle| \quad \begin{array}{l} u(x) > 0, \quad x \in \Omega \\ \dfrac{\partial u(x)}{\partial n} < 0, x \in \partial\Omega \end{array} \right\}$$

We can define an ordering on Z for $\varphi, \psi \in Z$ by

$$\varphi < \psi \quad iff \quad \psi - \varphi \in Z^+ \quad \text{but} \quad \varphi \neq \psi,$$

$$\varphi << \psi \quad iff \quad \psi - \varphi \in intZ^+,$$

$$\varphi \leq \psi \quad iff \quad \psi - \varphi \in Z^+,$$

Let S_t denote the solution map of (4.1.8), i.e. $S_t\varphi = u(t,\varphi)$, where $u(t,\varphi)$ is the solution of (4.1.8) with $u_0 = \varphi$. Then we have:

Theorem 4.2.1. *Z^+ is positively invariant in S_t. S_t is strongly monotone increasing, i.e. for any $\varphi, \psi \in Z$ with $\varphi < \psi$, $S_t(\varphi) << S_t(\psi)$, $\forall t > 0$.*

Proof. Z^+ is positively invariant for the abstract ordinary differential equation

$$u'(t) = f(u(t), t)$$

because it satisfies the Nagumo condition

$$\lim_{\lambda\to 0+} \lambda^{-1} dist(Z^+, \varphi + \lambda f(\varphi, t)) = \lim_{\lambda\to 0+} \lambda^{-1} dist(Z^+, \varphi + \lambda\varphi g(\varphi, t)) = 0$$

for all $\varphi \in Z^+$ and $t > 0$. Z^+ is also positively invariant under the semigroup $T(t)$ which is generated by A, since $T(t)\varphi$ is the solution of $u'(t) = Au(t)$ with $\varphi \in Z^+$, i.e. the solution $u(t) \in Z^+$ of $\frac{\partial u_i}{\partial t} = d_i \Delta u_i, \quad i = 1, 2, \cdots, n$ with initial condition $u(0) = \varphi \geq 0$, and by the maximum principle in the patchy parabolic equation of Theorem 2.2.3, $T(t)\varphi \in Z^+$. A conclusion of Theorem 7.3.1 of [Smith] is that Z^+ is positively invariant for $u'(t) = Au(t) + f(u(t), t)$. Therefore Z^+ is a positively invariant set of S_t.

Let

$$Q(x,t) = u(x,t,\varphi) - u(x,t,\psi)$$

where $u(x,t,\varphi)$ and $u(x,t,\psi)$ are two solutions of (4.1.8) with initial data φ and

ψ, respectively. Thus

$$\frac{\partial Q}{\partial t} = d\Delta Q + h\big(u(x,t,\varphi),t\big) - h\big(u(x,t,\psi),t\big)$$
$$= d\Delta Q + \left[\int_0^1 \frac{\partial h}{\partial u}\big(su(x,t,\varphi) + (1-s)u(x,t,\psi),t\big)\,ds\right] Q.$$

Since $Q|_{t=0} < 0$, $Q|_{\partial\Omega} = 0$, and by the maximum principle for patchy partial differential equations of Theorems 2.2.3 and 2.2.4, we have $Q(x,t) < 0$, $x \in \Omega$, $t > 0$ and $\left.\frac{\partial Q}{\partial n}\right|_{\partial\Omega} > 0$. This says that S_t is strongly monotone increasing.

Theorem 4.2.2. *S_t is strictly sublinear, i.e. $S_t(\alpha\varphi) > \alpha S_t(\varphi)$ for all $\alpha \in (0,1)$, $t > 0$ and $\varphi >> 0$. (see [H] [T] and [Zhao] for the definition).*

Proof. Let

$$V(\cdot,t) = \alpha u(t,\varphi)(\cdot) - u(t,\alpha\varphi)(\cdot).$$

Then

$$\frac{\partial V}{\partial t} - d\Delta V = \alpha u(t,\varphi) g\big(u(t,\varphi),t\big) - u(t,\alpha\varphi) g\big(u(t,\alpha\varphi),t\big)$$
$$< \alpha u(t,\varphi) g\big(\alpha u(t,\varphi),t\big) - u(t,\alpha\varphi) g\big(u(t,\alpha\varphi),t\big)$$
$$= \int h'\big(s\alpha u(t,\varphi) + (1-s)u(t,\alpha\varphi),t\big)\,ds \cdot V$$

Applying the maximum principle to patchy parabolic equations of Theorem 2.2.3 and being aware of the strict inequality above, we get for $t > 0$,

$$V < 0.$$

This finishes the proof of Theorem 4.2.2.

Now define the Poincare map $S = S_T$. Then S is strongly monotone increasing and strictly sublinear. It is evident that a fixed point of S is a T-periodic solution of (4.1.1)-(4.1.3). From Remark 5.2[H] and Lemma 1[Zhao], we have:

Theorem 4.2.3. *S admits at most one positive fixed point in Z^+, i.e. there exists at most one positive T-periodic solution of* $(4.1.1) - (4.1.3)$.

§4.3 Global Stability of the Positive Periodic Solution

In this section, the techniques developed in [DH1] [DH2] [H] [ST][T] and [Zhao] will be employed in order to obtain the switching conditions needed for the existence and global stability of a positive periodic solution. The Krein-Rutman Theorem plays a central role in these techniques, and so we write them out(Theorems 7.2 and 7.3 [H]) for an ordered Banach space $(Z, intZ^+)$:

Theorem 4.3.1 (Krein-Rutman). *Let K be a linear, compact, strongly positive functional on Z. Then $r = spr(K) > 0$ and r is the unique eigenvalue of K having a positive eigenfunction φ. Further $\varphi >> 0$, and r is an algebraically simple eigenvalue. Furthermore, $abs(\lambda) < r$ for all $\lambda \in \sigma(K)$, $\lambda \neq r$. K is called strongly positive if $K(Z^+ \backslash \{0\}) \subset intZ^+$, $\sigma(K)$ is the spectrum of K, and $spr(K)$ is the spectral radius of K.*

Theorem 4.3.2 (Krein-Rutman). *Consider the inhomogeneous equation*

$$\lambda u - Ku = \psi \quad in \quad Z, \qquad \psi > 0 \tag{4.3.1}$$

under the assumptions of Theorem 4.3.1. We have:

(i) (4.3.1) *has a unique solution* u *if* $\lambda > r$*, and* $u >> 0$*; equation* (4.3.1) *has no positive solution if* $\lambda \leq r$*,*

(ii) for $\lambda = r$*, there exists no solution of equation* (4.3.1).

Lemma 1.1 in [H] will be used here frequently, and so we write it and related terminology out for convenience:

Definition 4.3.1. *K is called strictly order preserving in Z, if $K(\varphi) >> K(\psi)$ whenever $\varphi > \psi$, $\varphi, \psi \in Z$. φ is called a subequilibrium (superequilibrium) of the equation $Ku = u$, if $\varphi \leq K\varphi$ ($\varphi \geq K\varphi$).*

Theorem 4.3.3. *Suppose K is strictly order preserving , K is compact, $\varphi < \psi$ are a subequilibrium and a superequilibrium of $Ku = u$ respectively , and*

$$\varphi_{n+1} = K\varphi_n, \quad \varphi_0 = \varphi$$

$$\psi_{n+1} = K\psi_n, \quad \psi_0 = \psi.$$

Then $\{\varphi_n\}$ is an increasing sequence converging to the minimal fixed point $\underline{u}$ in $[\varphi, \psi] = \{u : \quad \varphi \leq u \leq \psi\}$, and $\{\psi_n\}$ is a decreasing sequence converging to the maximal fixed point $\overline{u}$ in $[\varphi, \psi]$.

The next step is to set up a Poincare map and delineate some properties of it which will be used to obtain the existence and global stability of a positive periodic solution.

S_T is called a Poincare map and denoted by K. We will show that K has the same properties as that in Theorem 4.3.1.

From Theorems 4.2.1 and 4.2.2 or a similar argument to that in (Prop.21.2 [H]) or [DH2], it follows that K is strongly order-preserving, compact and strictly sublinear. The same argument as that in (Prop.23.1 [H]) implies that K is Frechet differentiable and $DK(0) = \overline{U}(T,0)$, where $\overline{U}(T,0)$ is the evolution operator of the linear variational equation of (4.1.1)-(4.1.3), or

$$\begin{cases} u_t = Au + g(0,t)u \\ u|_{\partial\Omega} = 0. \end{cases} \tag{4.3.2}$$

According to (Chapter II, [H]), the evolution operator $\overline{U}(t,\tau)$, $\quad 0 \le \tau \le t \le T$ is compact and strongly positive, and thus $DK(0)$ is strongly positive and compact. Let $r = spr(DK(0))$ be the spectral radius of $DK(0)$. By Theorem 4.3.1, r is the principal eigenvalue of K. From (Prop.14.4, [H]), μ is the eigenvalue with positive eigenfunction of

$$\begin{cases} u_t = Au + g(0,t)u + \mu u \\ u|_{\partial\Omega} = 0 \\ u \quad \text{is} \quad T-periodic \end{cases} \tag{4.3.3}$$

where

$$\mu = -\frac{1}{T} ln r. \tag{4.3.4}$$

Now we are ready to state main result of this chapter.

Theorem 4.3.4.

(i) If $\mu \ge 0$, then $u = 0$ is globally asymptotically stable with respect to those solutions with nonnegative initial data in Z^+.

(ii) If $\mu < 0$, and there exists a positive supersolution Ψ of (4.1.8) (i.e. a T-periodic function $\Psi(t)$ satisfying (4.1.2) and (4.1.3) but $\Psi'(t) \ge A\Psi(t) + f(\Psi(t),t)$), then there exists a unique positive T-periodic solution which is

globally asymptotically stable with respect to those solutions with positive initial data in Z^+.

Proof.

(i) Because $DK(0)$ is strongly positive and compact, by Theorem 4.3.1, $DK(0)$ has a principal eigenfunction $e >> 0$ with a principal eignvalue r, such that $r \leq 1$ by (4.3.4).

We first show that for any $u >> 0$,

$$K(u) < DK(0)u. \tag{4.3.5}$$

In fact by the strict sublinearity of K and $K(0) = 0$, we have for any $\alpha \in (0, 1)$,

$$\begin{aligned} K(u) &< \frac{K(\alpha u)}{\alpha} \\ &= \frac{K(0) + DK(0)(\alpha u) + o(\|\alpha u\|)}{\alpha} \\ &= DK(0)u + \frac{o(\|\alpha u\|)}{\|\alpha u\|} \cdot \|u\| \end{aligned}$$

Let $\alpha \to 0$. Then we obtain that $K(u) \leq DK(0)u$. If for some $u_0 >> 0$, $K(u_0) = DK(0)u_0$, then for any $\alpha \in (0,1)$,

$$\alpha K(u_0) < K(\alpha u_0) \leq DK(0)(\alpha u_0) = \alpha K(u_0),$$

which is a contradiction. Thus

$$K(u) < DK(0)u, \qquad u >> 0$$

Now we show that there exists no periodic solution of (4.1.1)-(4.1.3). If there is, call it u, then

$$u = K(u) < DK(0)$$

or

$$1 \cdot (-u) - DK(0)(-u) > 0.$$

Letting $\psi = DK(0)u - u, \quad \lambda = 1$, we get a contradiction by Theorem 4.3.1(i). Hence there exists no periodic solution of (4.3.1)-(4.3.3).

Now for any initial data $\eta > 0$, there exists a constant $\theta > 0$ such that

$$0 < \eta < \theta e.$$

Because

$$K(\theta e) < DK(0)(\theta e) = \theta DK(0)e = \theta \cdot re \leq \theta e,$$

i.e. θe is a strict superequilibrium, by Theorem 4.3.3, $K^n(\theta e)$ will approach a fixed point of K as $n \to \infty$, which is zero by the nonexistence of positive periodic solutions. Furthermore by the monotonicity of K, we have

$$0 = K^n(0) < K^n(\eta) < K^n(\theta e)$$

and thus $u = 0$ is stable with respect to all the solutions with nonnegative initial data in Z^+.

(ii) If $\mu < 0$, then $r > 1$ by (4.3.4). We still use $e >> 0$ to denote the principal eigenfunction of $DK(0)$. $\forall \epsilon > 0$,

$$K(\epsilon e) = K(0) + DK(0)\epsilon e + o(\epsilon)$$
$$= \epsilon r e + o(\epsilon)$$

and therefore there exists $\epsilon_0 > 0$ such that for $\epsilon \in (0, \epsilon_0]$,

$$K(\epsilon e) > \epsilon e$$

i.e. ϵe is a strict subequilibrium.

Because Ψ is a positive supersolution of (4.1.8), it is easy to see for any $m > 1$, $m\Psi(t)$ is also a positive supersolution of (4.1.8). Comparing the solution of (4.1.1)-(4.1.4) with initial data $m\Psi(0)$ to the supersolution $m\Psi(t)$ at time T, we get

$$Km\Psi(0) \leq m\Psi(0)$$

and therefore $m\Psi(0)$ is a strict superequilibrium.

Now for any initial data $\eta > 0$, we can find $\theta, m > 0$ such that

$$\theta e < \eta < m\Psi(\theta).$$

By Theorem 4.3.3 and the uniqueness of the positive periodic solution, we get (ii). This completes the proof of Theorem 4.3.4. □

§4.4 Population Diffusion Through a General Annular Patchy Environment

We repeat (C.1)-(C.4) here for convenience,

$$\frac{\partial u_i}{\partial t} = d_i \Delta u_i + u_i g_i(u_i), \quad x \in \Omega_i, \quad i = 1, 2, \cdots, n \tag{4.4.1}$$

with matching conditions

$$\begin{aligned} u_i|_{\overline{\Omega}_i \cap \overline{\Omega}_{i+1}} &= u_{i+1}|_{\overline{\Omega}_i \cap \overline{\Omega}_{i+1}} \\ d_i \left.\frac{\partial u_i}{\partial n_i}\right|_{\overline{\Omega}_i \cap \overline{\Omega}_{i+1}} &= d_{i+1} \left.\frac{\partial u_{i+1}}{\partial n_i}\right|_{\overline{\Omega}_i \cap \overline{\Omega}_{i+1}}, \quad i = 1, 2, \cdots, n-1 \end{aligned} \tag{4.4.2}$$

boundary condition

$$u|_{\partial\Omega} = 0 \tag{4.4.3}$$

and initial conditions

$$u_i(x, 0) = \eta_i(x) > 0, \quad x \in \Omega_i, \quad i = 1, 2, \cdots, n. \tag{4.4.4}$$

Here the patchy environment is assumed to be of type (C). Model (C) may be thought as of a kind of model (C_0) with arbitrary period. For the main result, Theorem 4.3.4 in Section 4.1-4.3 can be similarly adopted here, and the positive periodic solution in Theorem 4.3.4 should become the positive steady state solution of (C.5)-(C.7). The main results on model (C) can be stated as follows.

Theorem4.4.1.

(i) If $\mu \geq 0$, then $u = 0$ is globally asymptotically stable with respect to those solutions of system $(4.4.1) - (4.4.4)$ *with nonnegative initial data in Z^+.*

(ii) If $\mu < 0$, and there exists a positive supersolution Ψ of (4.4.1) *(i.e. a Ψ satisfying* (4.4.2) *and* (4.4.4) *but $\Psi' \geq A\Psi + f(\Psi)$), then there exists a unique*

positive steady state solution of $(C.5)-(C.7)$ *which is globally asymptotically stable with respect to those solutions of* $(4.4.1)-(4.4.4)$ *with positive initial data in* Z^+.

where A *is the same as in* $(4.1.8)$ *and* f *is the substitution operator induced by*

$$h(u) = \begin{cases} ug(u) & u \geq 0 \\ 0 & otherwise, \end{cases}$$

and μ *is the eigenvalue with a positive eigenfunction of*

$$\begin{cases} u_t = Au + g(0)u + \mu u \\ u|_{\partial\Omega} = 0. \end{cases}$$

Chapter 5

NUMERICAL ANALYSIS

In the previous chapters, the unique existence and global stability of the radially symmetric steady state solution has been theoretically investigated. Here we proceed to some numerical analysis in order to illustrate what the radially symmetric steady state solution looks like in some typical biological models. By comparing the contours of the radially symmetric steady state solutions, we can discuss the effects on the radially symmetric steady state solution by varying the sizes of patches, the diffusion rates, and other parameters. In Section 5.1, a numerical algorithm is developed with easily extendable codes in programming language C for the models from [F] and the references therein. In Section 5.2, by comparing the contours of the radially symmetric steady state solutions, we can discuss the effects on the radially symmetric steady state solution by varying the sizes of patches, diffusion rates, carrying capacities, etc.

§ 5.1 Algorithm

The system for the radially symmetric steady state is indicated in (3.2.1), namely

$$d_i v_i''(r) + \frac{d_i}{r} v_i'(r) + v_i(r) g_i(v_i(r)) = 0, r \in (r_{i-1},\ r_i),\ i = 1, 2, \ldots, n$$

$$v_i(r_i) = v_{i+1}(r_i)$$

$$d_i v_i'(r_i) = d_{i+1} v_{i+1}'(r_i),\ i = 1, 2, \ldots, n-1 \tag{5.1.1}$$

with boundary conditions

$$v_n(r_n) = K_n$$
$$v_1'(0) = 0. \tag{5.1.2}$$

We solve the system above by considering (5.1.1) with initial conditions

$$v_1(0) = \alpha$$
$$v_1'(0) = 0. \tag{5.1.3}$$

We want to determine α such that the solution of the Cauchy problem (5.1.1) with (5.1.3) satisfies the boundary condition (5.1.2), and therefore this solution is the radially symmetric steady state solution by the uniqueness in Theorem 3.2.2.

In the numerical computation to determine α, we use a binary partition. We first consider interval $[\min K_i, \max K_i]$. Let α be the midpoint of the interval $[\min K_i, \max K_i]$. If the solution v of (5.1.1) with (5.1.3) obtained in this way is close to K_n at $r = r_n$ with some accuracy, then α adopts this midpoint. Otherwise, if $v(r_n) > K_n$ or $v(r_n) < K_n$, we will consider the right half interval or left half interval of $[\min K_i, \max K_i]$. and continue this procedure until the required accuracy is attained.

For convenience, in order to solve numerically for the radially symmetric steady state solution of (5.1.1) with (5.1.3), we write it as the system

$$\frac{dv}{dr} = V$$
$$\frac{dV}{dr} = G(r, v, V) \tag{5.1.4}$$

where

$$G(r, v, V) = -\frac{V}{r} - \frac{1}{d}vg(v). \tag{5.1.5}$$

Following the above idea, we try initial condition $v(0) = \alpha$ and $V(0) = 0$, where α is the midpoint of a subinterval obtained by a series of binary partitions of $[\min K_i, \max K_i]$. We make use of the Runge-Kutta fourth order iterative method [PL] in each patch, and convert the corresponding values of v and V at the interfaces between patches by the interface condition, and then check if the solution obtained in this way satisfies the accuracy requirement. This procedure may be repeated until the accuracy is met. We may want to include the case $r = 0$ in the definition of $G(r, v, V)$ by redefining

$$G(r, v, V) = \begin{cases} -\frac{1}{d}vg(v) & r = 0 \\ -\frac{V}{r} - \frac{1}{d}vg(v) & r > 0. \end{cases}$$

This binary partition algorithm is feasible according to the inequalities $\min K_i \leq v \leq \max K_i$ in Theorem 3.2.1 and the monotonicity of v about α in the following Lemma 5.1.1.

Lemma 5.1.1. *The solution v of (5.1.4) with initial condition (5.1.3) is increasing with respect to α, i.e. if $v(r, \alpha_1)$ and $v(r, \alpha_2)$ are solutions of (5.1.4) with initial conditions $v(0, \alpha_1) = \alpha_1$, $V((0, \alpha_1) = 0$ and $v(0, \alpha_2) = \alpha_2$, $V((0, \alpha_2) = 0$, respectively, and $\alpha_1 \leq \alpha_2$, then $v(r, \alpha_1) \leq v(r, \alpha_2)$.*

Lemma 5.1.1 can be proved in the same way as was done for Theorem 3.2.2.

The algorithm developed above is realized in C code. The program is composed of functional modules. We particularly add a module for the Volterra model, since it is the most typical classical biological model. The main algorithm is realized through a pointer to virtual functions. It is independent of the particular model considered and feasible for general models. In the case of other types of models, we only need to add a functional module for each of these models. This C code is well documented and stands on its own for both usage and extension. This program named "patch.c" is listed in the appendix.

§ 5.2 Analysis

In this section, we investigate the numerical solution of the radially symmetric steady state of a Volterra population diffusion model in a patchy environment. Particularly, we observe the monotonicity of the solution. We also discuss the effect of patch sizes, carrying capacities and diffusion rates.

§ 5.2.1 Monotonicity

In this subsection, we demonstrate the monotonicity of the radially symmetric steady state solution in Theorem 3.2.3. The environment consists of three patches with radii $r_1 = 10$, $r_2 = 20$, and $r_3 = 30$. The diffusion rates in these patches are $D_1 = 10$, $D_2 = 20$, $D_3 = 30$. The population models in these patches are Volterra types with

$$g_1(v) = 4(1 - \frac{v}{K_1})$$

$$g_2(v) = 4(1 - \frac{v}{K_2})$$

$$g_3(v) = 4(1 - \frac{v}{K_3}).$$

In Figure-4, the diamond line is a increasing radially symmeric steady state solution of the above model with increasing capacities $K_1 = 5$, $K_2 = 15$ and $K_3 = 30$. The crossed line is a decreasing one with decreasing capacities $K_1 = 30$, $K_2 = 20$ and $K_3 = 15$. The squared line is neither increasing nor decreasing with non monotone capacities $K_1 = 10$, $K_2 = 30$ and $K_3 = 15$.

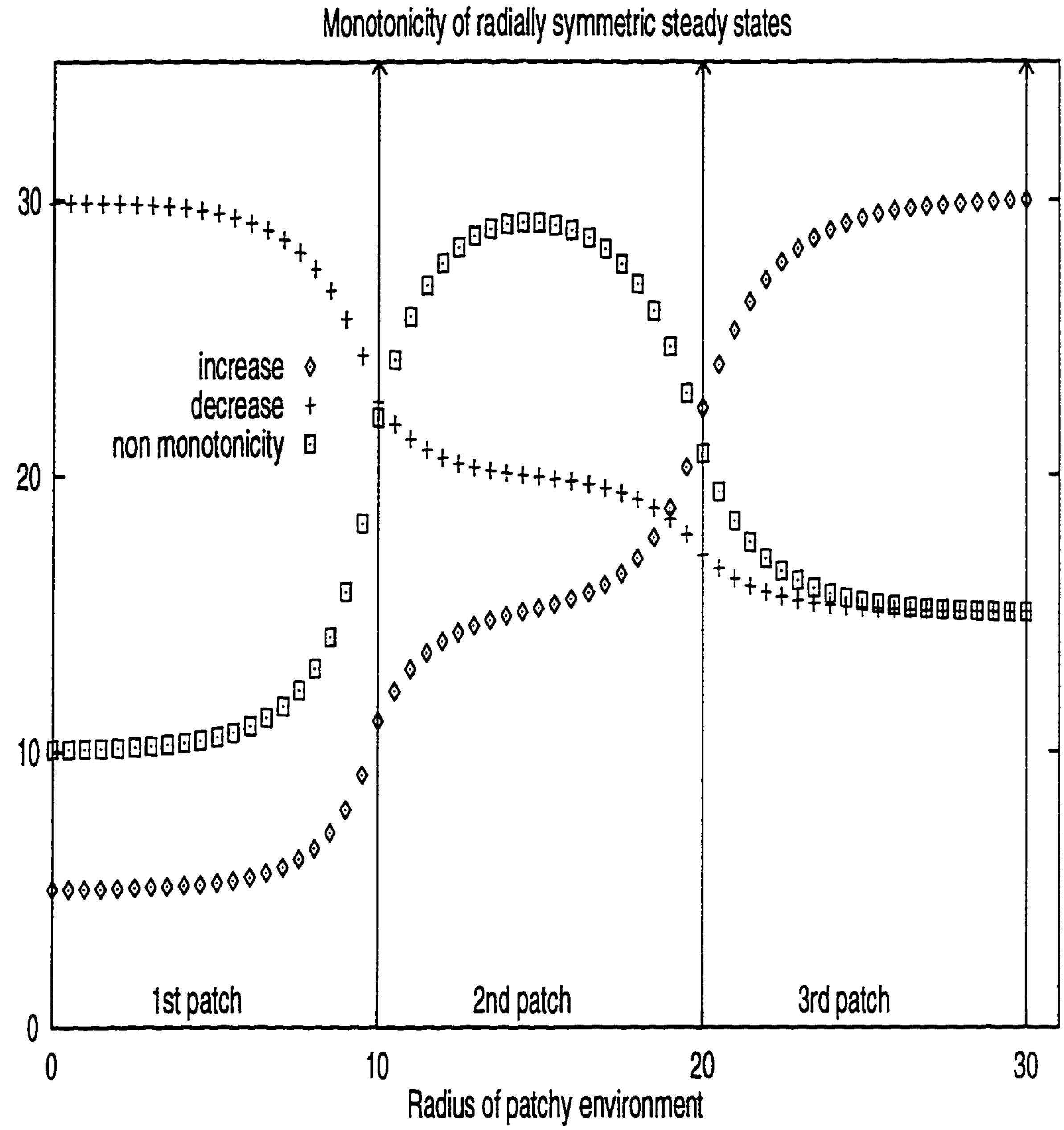

Monotonicity of radially symmetric steady states
increase
decrease
non monotonicity
1st patch
2nd patch
3rd patch
0
10
20
30
Radius of patchy environment

Figure - 4

§ 5.2.2 Effect of Diffusion

In this subsection, we discuss the effect of diffusion on the radially symmetric steady state solution in (3.2.1). The patchy environment consists of three patches with radii $r_1 = 10$, $r_2 = 20$, and $r_3 = 30$. The population models in the three patches are Volterra types with

$$g_1(v) = 0.4(1 - \frac{v}{10})$$
$$g_2(v) = 0.8(1 - \frac{v}{20})$$
$$g_3(v) = 0.7(1 - \frac{v}{30}).$$

In Figure-5, we compare two radially symmetric steady state solutions of the same models above but different diffusion rates. The crossed line has higher diffusion with $D_1 = 10$, $D_2 = 20$, and $D_3 = 30$, the diamond line has lower diffusion with $D_1 = 1$, $D_2 = 2$, and $D_3 = 3$. We can see the one with lower diffusion rates retains local properties such as remaining in their carrying capacities in some sense. The other one with higher diffusion rates shows more of an immigration trend than staying in their capacity levels.

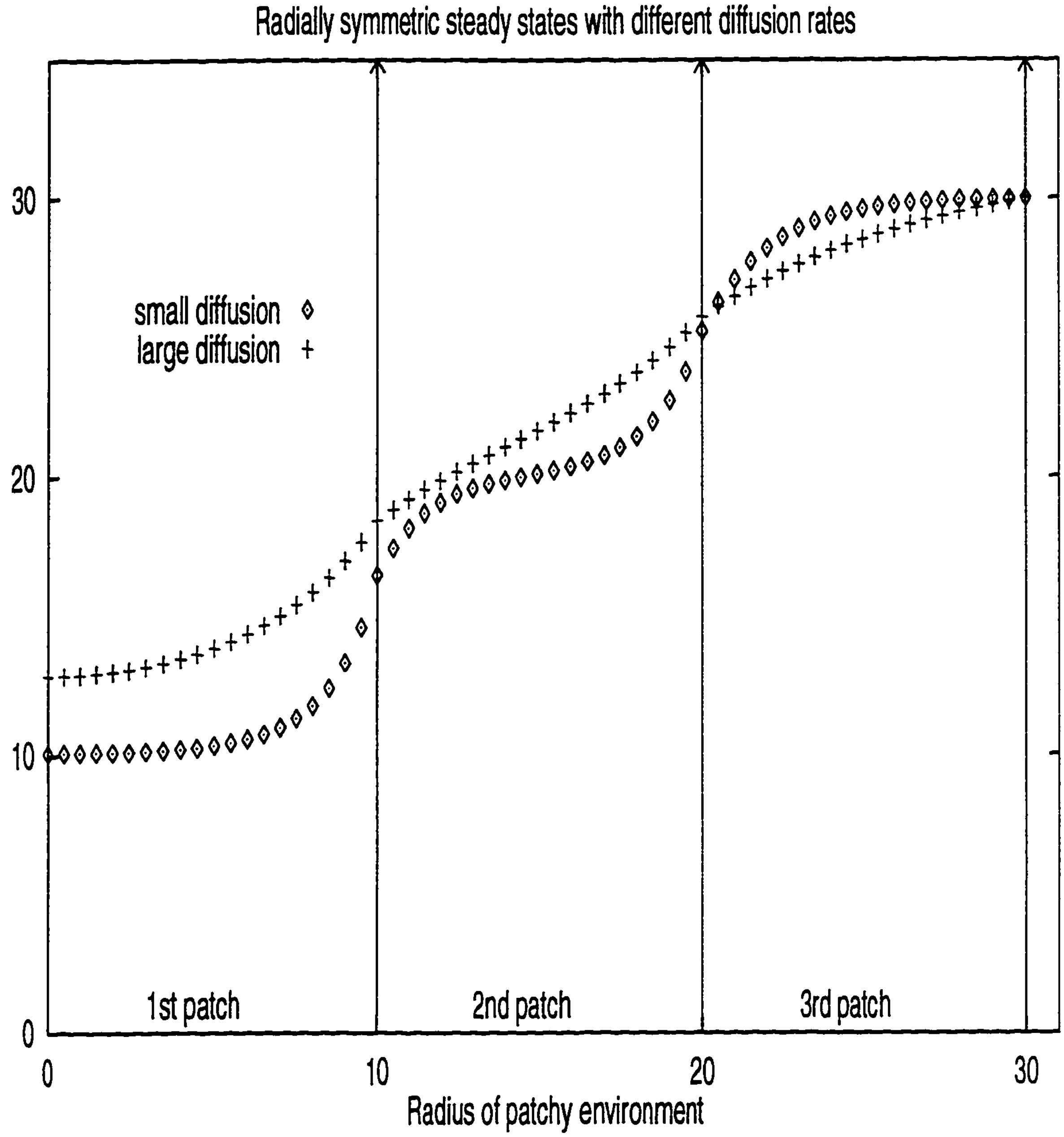

Radially symmetric steady states with different diffusion rates
small diffusion
large diffusion
30
20
10
0
1st patch
2nd patch
3rd patch
0
10
20
30
Radius of patchy environment

Figure - 5

§ 5.2.3 Effect of Patch Sizes

Varying the sizes of patches can significantly affect the population dynamics [AC]. In this subsection, we discuss population diffusion models in patchy environments with

$$g_1(v) = 5(1 - \frac{v}{25})$$

$$g_2(v) = 5(1 - \frac{v}{5})$$

and diffusion rates

$$D_1 = D_2 = 8.$$

In Figure-6, the diamond line is the numerical solution of the radially symmetric steady state of this model in a patchy environment with smaller inner patch ($r_1 = 5$) and larger outer patch ($r_2 = 30$). The crossed line is for an environment with larger inner patch ($r_1 = 25$) and smaller outer patch ($r_2 = 30$). We can see the population behavior is relatively stable in larger patches.

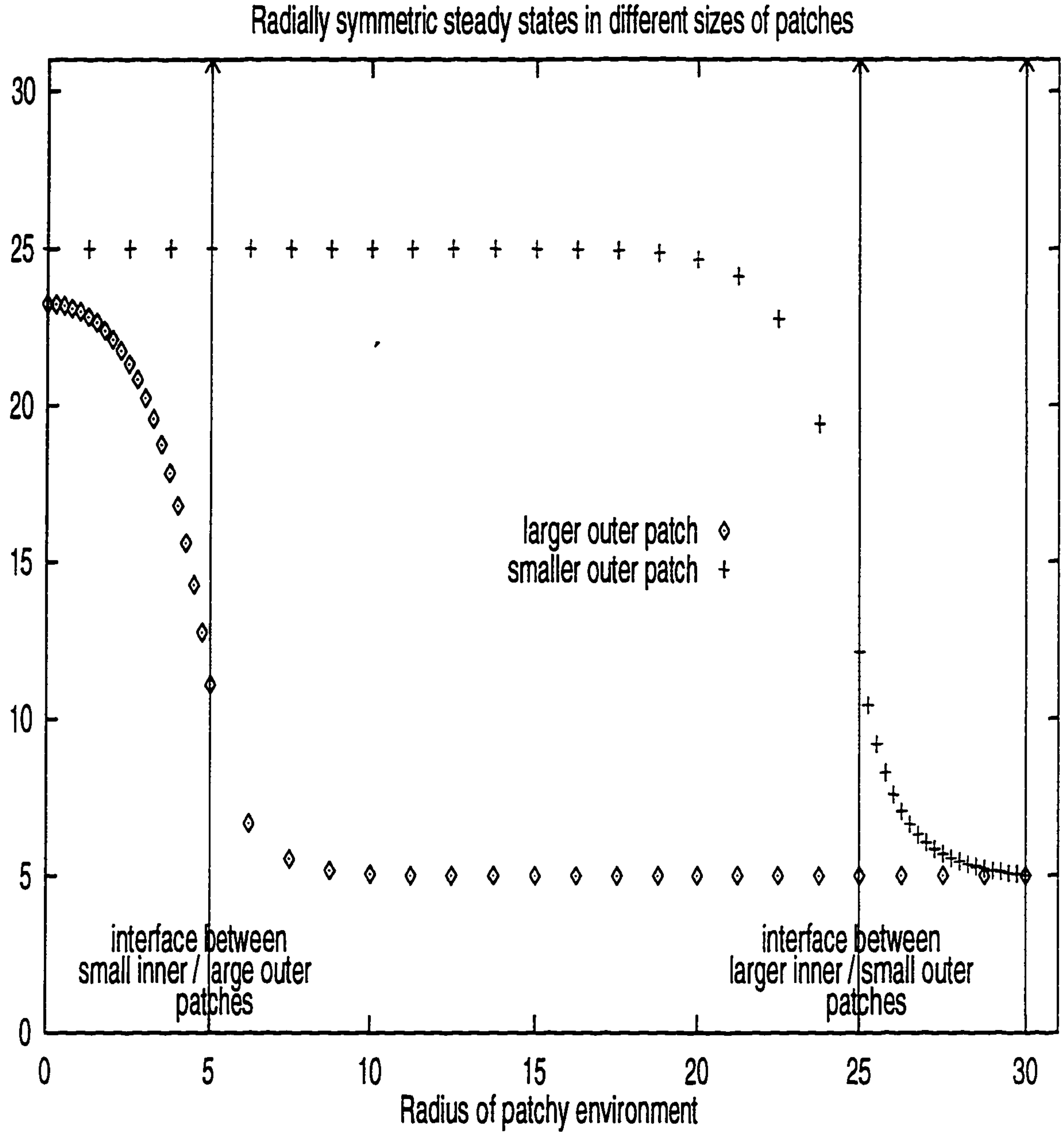

Radially symmetric steady states in different sizes of patches
larger outer patch
smaller outer patch
interface between small inner / large outer patches
interface between larger inner / small outer patches
Radius of patchy environment
0
5
10
15
20
25
30

Figure - 6

§ 5.2.4 Effect of Boundary Carrying Capacities

In this subsection, we investigate the effects of boundary carrying capacities on the radially symmetric steady state solution of the population model given by

$$g_1(v) = 8(1 - \frac{v}{15})$$

$$g_2(v) = 2(1 - \frac{v}{K_2})$$

with diffusion rates

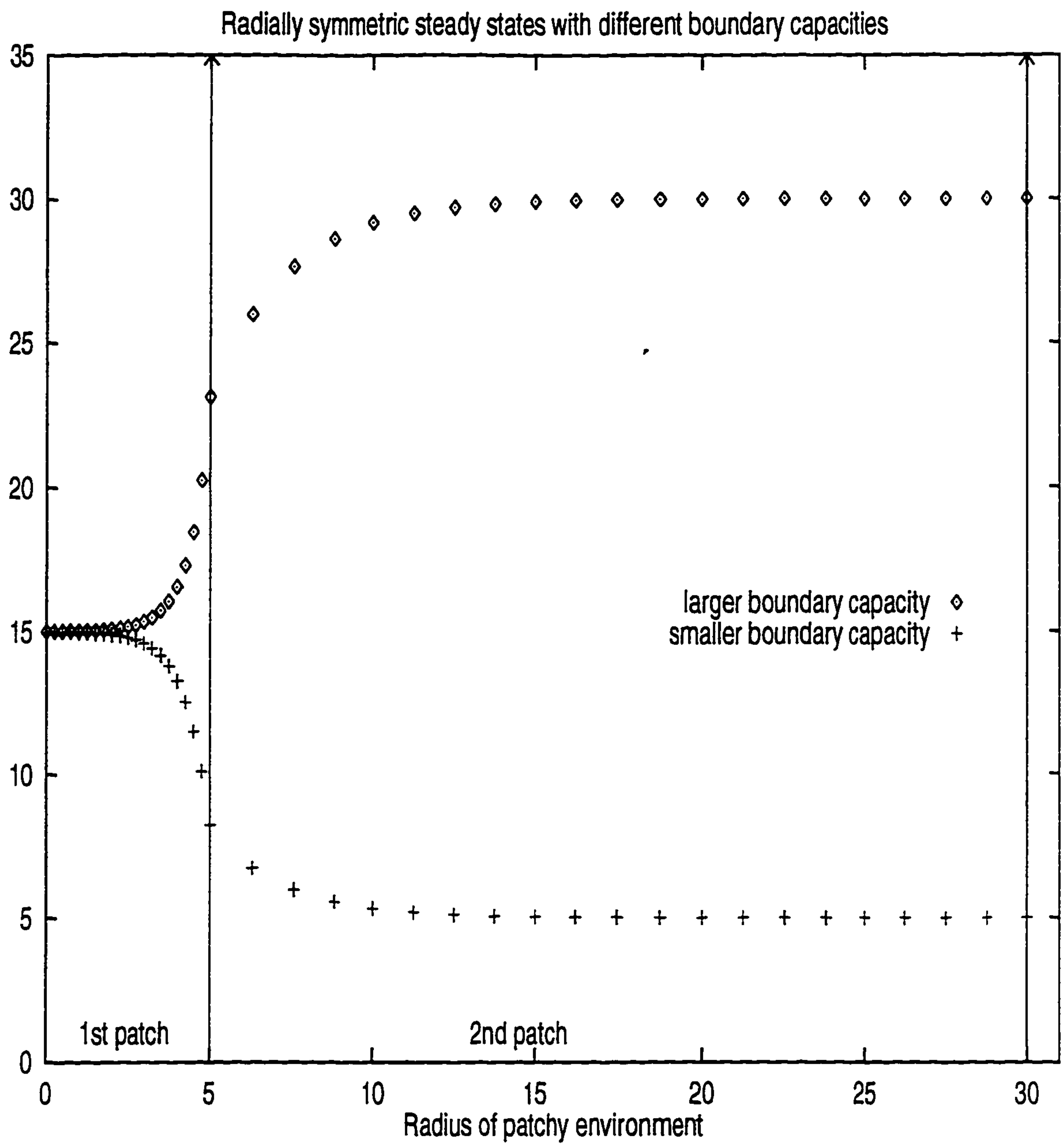
$$D_1 = 3$$

$$D_2 = 15$$

in a patchy environment with radii $r_1 = 5$, $r_2 = 30$.

In Figure-7, the diamond line is the numerical solution of the steady state with larger boundary carrying capacity $K_2 = 30$, the crossed line is that with smaller boundary carrying capacity $K_2 = 5$. The higher boundary carrying capacity raised the distribution of the steady state in the inner patch, and vise versa, the lower boundary carrying capacity caused a substantial drop.

An interesting example of the effects of the boundary carrying capacity can be found in [FF]. [FF] considered a model of diffusion of oxygen to spheroids growing in a stationary medium. The spheroids are used as tumor models. Actually we can treat spheroids and stationary media as a two-patch environment for the oxygen diffusion. The lower oxygen concerntration in the vicinity of the medium can cause a substantial depletion of the oxygen in the spheroid to recess the growth of the tumors.



Radially symmetric steady states with different boundary capacities
35
30
25
20
15
10
5
0
larger boundary capacity
smaller boundary capacity
1st patch
2nd patch
0
5
10
15
20
25
30
Radius of patchy environment

Figure - 7

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```
/*********************************************************************
 *                                                                   *
 *      FILE:           patch.c                                      *
 *                                                                   *
 *      PURPOSE:        Generates numerical solution data for the    *
 *                      radially symmetric steady state solution     *
 *                      of (3.2.1) or (5.1.1) - (5.1.2).             *
 *                                                                   *
 *      EXPLANATION:    patch.c consists of many modules.            *
 *                      It is easily extendable to include           *
 *                      any type of model by adding global           *
 *                      variables for model parameters and           *
 *                      designing functions for the model            *
 *                      functions. There is a global function        *
 *                      pointer which may point any models.          *
 *                                                                   *
 *********************************************************************/

#include        <stdio.h>
#include        <stdlib.h>
#include        <ctype.h>
#include        <malloc.h>
#include        <string.h>
#include        <math.h>

/*** error function ***/
#define ERROR(x)        { perror(x); exit(1); }

/*** print function ***/
#define PRINT(x)        { fprintf(stdout, x); }


/*** in case of iterative overflow ***/
typedef enum{ TooSml, Normal, TooBig } STATUS;

/*** type of the function in population models ***/
typedef double  MODEL(int, double);

extern double fabs( double  );


/*************  parameters of patch environment  ********************
 *                                                                   *
 *      PatNum   -  number of patches                                *
 *      R        -  array of the outer radius of each patch          *
 *      D        -  array of the diffusion rates in each patch       *
 *      Capa     -  array of the capacities in each patch            *
 *      MinCapa  -  minimum of carrying capacities                   *
 *      MaxCapa  -  maximum of carrying capacities                   *
 *      g        -  pointer to the function in population models     *
 *      index    -  index of the selected model                      *
 *                                                                   *
 *********************************************************************/

int     PatNum;
double  *R;
double  *D;
double  *Capa;
double  MinCapa;
double  MaxCapa;
```

```
MODEL   *g;
int     index;


/****************  parameters of algorithm   ***************************
 *                                                                     *
 *      Accuracy - accuracy of iteration                               *
 *      Steps    - steps of iteration                                  *
 *      H        - array of length of iterative steps in each patches  *
 *      BigNum   - used to control overflow                            *
 *      status   - indicate overflow or normal of iteration            *
 *      MAXPARTITION                                                   *
 *               - max steps of binary partition                       *
 *                                                                     *
 ***********************************************************************/

double  Accuracy       =      0.01;
double  Steps          =      10000;
double  *H;
int     BigNum         =      1000;
int     MAXPARTITION   =      1000;
STATUS  status;


/************************   proto types   ******************************/

/*** input patch parameters ***/
void input_patch_parameters();

/*** input model parameters ***/
void input_model_parameters();

/*** output parameters ***/
void output_parameters(char *parameterfile);

/*** output data ***/
void output_data(char *datafile);


/***  function in steady state equation  ***/
double G(int pat, double r, double v, double V);

/***    Runge-Kutta iterative   ***/
void RungeKutta(int pat, double *r, double h, double *v, double *V);

/***   Determine boundary value by central value  ***/
void RungeKuttaBoundValue(double CVal, double *BVal);

/***   determine the central value   ***/
double  CenterValue( );

/*************   parameters in Volterra model    ************************
 *                                                                     *
 *      Volterra model:                                                *
 *                        x' = rx(1-x/k)                               *
 *      volterra_r  -  array of parameter r                            *
 *      volterra_k  -  array of parameter k                            *
 *                                                                     *
 ***********************************************************************/
```

```
double  *volterra_r;
double  *volterra_k;

/***  function in Volterra model ***/
double volterra(int pat, double v);

/*** input parameters in Volterra model ***/
void input_volterra_parameter();


void main( )
{

        char    parameterfile[30];        /* to store parameters  */
        char    datafile[30];             /* to store data        */

        /***   input parameters of patches   ***/
        input_patch_parameters();

        /***   input parameters of models    ***/
        input_model_parameters();

        /*** select file name to store parameters ***/
        PRINT(" file name to store parameters, e.g. volterral.para?  ");
        scanf("%s", parameterfile);

        /*** select file name to store data ***/
        PRINT("file name to store data, e.g. volterral.dat?   ");
        scanf("%s", datafile);

        /*** output parameters ***/
        output_parameters( parameterfile );

        /*** output data ***/
        output_data( datafile );

}


/*****************    input_patch_parameters   ***************************
 *                                                                       *
 *      input number of patches                                          *
 *      input outer radius of each patch                                 *
 *      input diffusion rate in each patch                               *
 *      calculate the iterative length in each patch i:                  *
 *              H[i] = ( R[i] - R[i-1] ) / Steps                         *
 *                                                                       *
 ************************************************************************/

 void input_patch_parameters( )
 {

        int     pat;

        PRINT( "\nNumber of patches? ");
        scanf("%d", &PatNum);

        if(PatNum <= 0)
                ERROR( "Invalid PatNum!" );
```

```
	/***   allocate spaces for global variables ***/
	R	= (double *) calloc( PatNum + 1, sizeof(double) );
	D	= (double *) calloc( PatNum + 1, sizeof(double) );
	H	= (double *) calloc( PatNum + 1, sizeof(double) );
	Capa	= (double *) calloc( PatNum + 1, sizeof(double) );

	/*** input outer radius of each patch  ***/
	for( pat = 1; pat <= PatNum; pat++)
	{
		fprintf(stdout, "Outer radius of %d-th patch?  ", pat);
		scanf( "%lf", R + pat );
	}

	/*** postulate R[0] = 0 ***/
	R[0] = 0;

	/***  input diffusion rate in each patch  ***/
	for( pat = 1; pat <= PatNum; pat++)
	{
		fprintf(stdout, "Diffusion rate in %d-th patch?  ", pat);
		scanf( "%lf", D + pat );
	}

}


/*****************    input_model_parameters   ***************************
 *                                                                         *
 *      input index of model                                               *
 *      input the parameters for the selected model                        *
 *                                                                         *
 ***************************************************************************/

void input_model_parameters( )
{

	/***  select model  ***/
	PRINT( "Select model:\n ");
	PRINT( "---> 1 for Volterra model?  ");
	scanf("%d", &index);

	/*** case Volterra ***/
	if( index == 1 )
	{
		input_volterra_parameter();
		g = volterra;
	}

	/*** case not considered yet ***/
	else
		ERROR("Invalid index of model ");

}


/*********************  output_parameters    ***************************
 *                                                                         *
 *      open parameterfile                                                 *
```

```
 *      output number of patches                                         *
 *      output radius and diffusion rate in each  patch                  *
 *      output index of the selected model                               *
 *      output parameters of the model in each patch                     *
 *      for volterra model:                                              *
 *               output growth rate and capacity in each patch           *
 *                                                                       *
 ************************************************************************/

void output_parameters( char *parameterfile )
{
       int      pat;     /* index of patches      */
       FILE     *fptr;   /* file pointer          */

       /*** store parameters ***/
       fptr = fopen( parameterfile, "w");

       /*** output number of patches ***/
       fprintf( fptr, "%d\n", PatNum);

       /*** output radius and diffusion rate in each patch ***/
       for( pat = 1; pat <= PatNum; pat++ )
               fprintf( fptr, " %f  %f\n", R[pat], D[pat] );

       /*** output index of the selected model ***/
       fprintf( fptr, "%d\n", index);

       /*** case Volterra ***/
       if( index == 1)
       {
         for( pat = 1; pat <= PatNum; pat++ )
          fprintf( fptr, " %f  %f\n", volterra_r[pat], volterra_k[pat] );
       }

       fclose( fptr );
}


/************************      output_data      ***************************
 *                                                                       *
 *      open datafile to store iterative data                            *
 *      determine the central value: v(0)                                *
 *      iterate by Runge-Kutta method                                    *
 *                                                                       *
 ************************************************************************/

void output_data( char *datafile )
{

       double  CVal;                    /* central value          */
       double  r;                       /* radius variable        */
       double  v;                       /* steady state variable*/
       double  V;                       /* derivative of v        */
       int     pat;                     /* index of patches       */
       FILE    *fptr;                   /* file pointer           */
       int     i;

       /*** determine the Center Value ***/
       CVal = CenterValue();

       /*** output the numerical solution ***/
```

```
	r = R[0];
	v = CVal;
	V = 0;

	/*** store data ***/
	fptr = fopen(datafile, "w");
	fprintf( fptr, "%f  %f\n", r, v);

	for( pat = 1; pat <= PatNum; pat++)
	{

		for(i = 1; i <= Steps; i++)
		{
		,	RungeKutta(pat, &r, H[pat], &v, &V);

			/*** output data ***/
			fprintf( fptr, "%f  %f\n", r, v);
		}

		/*** at interface between patches ***/
		if( pat < PatNum )
		{
			/*** go to next patch ***/
			r = R[pat];
			V = V * D[pat] / D[pat+1];
		}

	}
	fclose( fptr );


}

/**************     function in steady state equation     ****************
 *                                                                        *
 *      G(r, v, V) = - V / r  -  1 / d * v * g(v)                         *
 *                                                                        *
 *************************************************************************/

double G(int pat, double r, double v, double V)
{
	if( r == 0 )
		return   - 1/D[pat] * v * g(pat, v);

	return     - V/r - 1/D[pat] * v * g(pat, v);
}


/**********************     RungeKutta    **** **************************
 *                                                                        *
 *      4th order Runge-Kutta iterative method to solve system:           *
 *              { dv/dr = V                                               *
 *              { dV/dr = G(r, v, V)                                      *
 *                                                                        *
 *      PreCondition:                                                     *
 *              pat     -  patch index                                    *
 *              r       -  radius variable                                *
 *              h       -  length of iteration                            *
 *              v       -  value of current v                             *
 *              V       -  value of current V                             *
```

```
 *                                                                              *
 *      PostCondition:                                                          *
 *              r       -  value of r after iteration                           *
 *              v       -  value of v after iteration                           *
 *              V       -  value of V after iteration                           *
 *                                                                              *
 ********************************************************************************/

void RungeKutta(int pat, double *r, double h, double *v, double *V)
{

        /*** auxilary variables ***/
        double k1,k2,k3,k4,l1,l2,l3,l4;

        /*** iterative ***/
        k1 = h * (*V);
        l1 = h * G( pat, *r, *v, *V );

        k2 = h * ( (*V) + l1/2 );
        l2 = h * G( pat, (*r) + h/2, (*v) + k1/2, (*V) + l1/2 );

        k3 = h * ( (*V) + l2/2 );
        l3 = h * G( pat, (*r) + h/2, (*v) + k2/2, (*V) + l2/2 );

        k4 = h * ((*V) + l3);
        l4 = h * G( pat, (*r) + h, (*v) + k3, (*V) + l3 );

        /*** next step ***/
        *r += h;
        *v += ( k1 + 2*k2 + 2*k3 + k4 )/6;
        *V += ( l1 + 2*l2 + 2*l3 + l4 )/6;

        return;

}


/******************* RungeKuttaBoundValue     **************************
 *                                                                              *
 *      PreCondition:                                                           *
 *                                                                              *
 *              dv/dr = V                                                       *
 *              dV/dr = G(r, v, V)                                              *
 *                                                                              *
 *              with central condition  v(0) = CVal                             *
 *                                                                              *
 *      PostCondition:                                                          *
 *                                                                              *
 *              indicate whether underflow, overflow or normal.                 *
 *              if iteration is normal, the boundary value                      *
 *              of the steady state solution is stored in BVal.                 *
 *                                                                              *
 ********************************************************************************/

void RungeKuttaBoundValue(double CVal, double *BVal)
{

        /*** initial values of the iteration ***/
        double  v=CVal,
                V=0,
                r=R[0];
```

```
        /*** index of patches ***/
        int     pat;

        /*** index of iteration ***/
        int     i;


        /*** iterative ***/
        for(pat = 1; pat <= PatNum; pat++)
        {
                /*** iterative in patch with index pat ***/
                for(i = 1; i <= Steps; i++)
                {
                        /*** values in next step ***/
                        RungeKutta(pat, &r, H[pat], &v, &V);

                        /*** case underflow ***/
                        if( v <= 4)
                        {
                                status = TooSml;
                                return;
                        }

                        /*** case overflow ***/
                        if(v > BigNum)
                        {
                                status = TooBig;
                                return;
                        }

                }

                /*** interface between patches ***/
                if(pat < PatNum)
                {
                        /*** go to next patch ***/
                        r = R[pat];
                        V = V * D[pat]/D[pat+1];
                }
        }

        /*** case normal ***/
        *BVal = v;
        status = Normal;

}


/*************************   CenterValue    ****************************
 *                                                                    *
 *      determine the central value by binary partition               *
 *      the interval [ MinCapa, Maxcapa ].                            *
 *                                                                    *
 *      detemine length of iteration in each patch.                   *
 *      determine MinCapa and MaxCapa.                                *
 *                                                                    *
 *      iteration is terminated by the boundary condition,            *
 *      or optimal case of the iteration, or maximum                  *
 *      partition step.                                               *
```

```
 *                                                                              *
 ********************************************************************************/

double CenterValue( )
{

        /*** index of patch ***/
        int     pat;

        /***  used in binary partition  ***/
        double  CVal1;
        double  CVal2;
        double  CVal;
        double  BVal;

        /*** initial iterative error ***/
        double  olderr = BigNum;
        double  newerr;

        int i = 0;

        /*** calculate step length of iteration in each patch ***/
        for( pat = 1; pat <= PatNum; pat++)
        {
                H[pat] = ( R[pat] - R[pat - 1] ) / Steps;
        }

        /*** decide the Mincapa and Maxcapa ***/
        MinCapa = Capa[1];
        MaxCapa = Capa[1];
        for( pat = 2; pat <= PatNum; pat++)
        {
                if( Capa[pat] > MaxCapa )
                        MaxCapa = Capa[pat];

                if( Capa[pat] < MinCapa)
                        MinCapa = Capa[pat];
        }

        /***  binary partition to determine Center Value ***/
        CVal1 = MinCapa;
        CVal2 = MaxCapa;
        do{

                CVal = ( CVal1 + CVal2 ) / 2;

                RungeKuttaBoundValue( CVal, &BVal );

                if( status == TooBig)
                {
                        CVal2 = CVal;
                        continue;
                }
                else if( status == TooSml)
                {
                        CVal1 = CVal;
                        continue;
                }
                else if( status == Normal)
                {
                        if( BVal > Capa[PatNum])
```

```
                        {
                                CVal2 = CVal;
                        }
                        else
                        {
                                CVal1 = CVal;
                        }
                }

                /*** Error of Boundary Condition ***/
                newerr = fabs( BVal - Capa[PatNum] );

                /*** meet boundary condition ***/
                if( newerr < Accuracy )
                {
                        break;
                }

                /*** almost reach the iterative optimal case ***/
/*
                if ( fabs( olderr - newerr ) < Accuracy )
                {
                        break;
                }
*/
                /*** reach the maximum partition ***/
                if( ( i++ ) >= MAXPARTITION  )
                {
                        break;
                }

                /*** record the iterative error ***/
                olderr = newerr;

        } while( 1 );

        return CVal;

}

/**************     function in Volterra model    ************************
 *                                                                      *
 *      g(x) = r ( 1 - x / k )  in certain patch                        *
 *                                                                      *
 ************************************************************************/

 double volterra(int pat, double v)
 {

        return volterra_r[pat] * ( 1 - v / volterra_k[pat] );

 }


/*****************     input_volterra_parameter   ************************
 *                                                                      *
 *      input parameters of Volterra model:                             *
 *              x' = r x ( 1 - x / k )                                  *
 *      volterra_r ...... r                                             *
 *      volterra_k ...... k                                             *
```

```
*       Calculate capacities of each patch i:                                  *
*                                 Capa[i] = volterra_k[i]                      *
*                                                                              *
*******************************************************************************/

void input_volterra_parameter()
{

        int     pat;

        /*** allocate spaces ***/
        volterra_r = (double *) calloc( PatNum + 1, sizeof(double) );
        volterra_k = (double *) calloc( PatNum + 1, sizeof(double) );

        for( pat = 1; pat <= PatNum; pat++ )
        {
           fprintf(stdout, "volterra_r for %d-th patch?  ", pat);
           scanf( "%lf", volterra_r + pat);

           fprintf(stdout, "volterra_k for %d-th patch?  ", pat);
           scanf( "%lf", volterra_k + pat);

           /*** capacities ***/
           Capa[pat] = volterra_k[pat];
        }

}
```