Cooking up Quantum Curves from Airy Structures

by

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Abstract

In the framework of topological recursion (TR), the quantum curve conjecture relates the initial spectral curve to a differential equation – the quantum curve – satisfied by the TR wave-function. Several admissibility conditions have been put forward to explain which input of spectral curve effectively produces a quantum curve.

In this thesis we propose a new way to prove the existence of quantum curves for genus zero spectral curves. This new approach is based on some early calculations relating the quantum curve and Virasoro constraints, and on the more recent Airy structure reformulation of TR in which Virasoro constraints play a central role.

The Airy structure approach gives a relation between the quantum curve wave function and the Airy structure partition function via the specialization map. We explain how the specialization map extended to differential operators could be the key to relate some generic combination of Virasoro constraints to the quantum curve differential equation. In this context, admissibility conditions arise when looking at which spectral curve produces a generic operator that can be specialized. Interestingly, when specialization works we always recover the expected quantum curve.

Although our method suggests a way to check the existence of any quantum curve, in practice it remains limited because of the pedestrian approach of specializing differential operators. For curves slightly more complicated than the easier examples, our expressions quickly get too complicated to manipulate. However we think that the connection which we outline here is interesting and could probably be extended further with a more technical treatment.

Contents

1 Introduction

1.1 *La Physique Mathématique expliquée à ma mère*

Quantum mechanics is weird. When the theory was first established a little over a century ago, it took the best physicists around three decades to come up with the right mathematical tools for it, and certainly much longer still to accept it conceptually. Yet as they slowly came to grip with it, they realized that it was not just some rickety fix to make the theory match to the experiments. It had a kind of necessity, a kind of inevitability to it. In his book *"Dreams of a final theory"* [44], Physics Nobel laureate Steven Weinberg argues that, if there's any piece of a physical theory which we know today which ought to survive in a final theory of nature, it would be quantum mechanics.

However there is still work ahead to find the appropriate language to talk about quantum mechanics. One of the biggest challenges of modern physics is to unify Einstein's general relativity with quantum mechanics: although we have a good description of both theories in their respective domains of application, they seem incompatible when brought together. *Mathematically* incompatible. What we are missing is the right *language*.

One of the ambitions of mathematical physics is to find that language, and so far the journey has been a fascinating and fruitful one. Except that the fruits have not always been the ones that we expected: Beautiful, unforeseen connections in various area of geometry, algebra and differential equations, sometimes bringing together abstract mathematical theories known for 50 years. All that, simply by taking different tools and trying to construct the same physical theory. And again with this kind of inevitability, *as if we could have discovered quantum mechanics with a pen and paper*, had we not been so unfortunate to stumble upon it first in the labs.

In this thesis we will see an example of just that: *Quantum curves*, a fundamentally geometric object that somehow magically exhibits quantum features, and that relates the most ancient of all fields of mathematics, that of "counting stuff", with some of the latest developments in algebraic geometry and infinite systems of differential equations.

1.2 Topological recursion and Quantum curves

A little more than a decade ago, Eynard and Orantin introduced the formula of topological recursion [32] as a universal formula for constructing different kinds of quantum geometric invariants. This generalized a formula known in the context of formal matrix models. In this formulation the input data is encoded in the spectral curve (Σ, x, y, B), where Σ is a compact Riemann surface of genus $\overline{g}$ with a choice of Bergman Kernel B, and x and y are two meromorphic functions defined on Σ, such that the zeroes of dx do not coincide with the zeroes of dy. The recursion produces a set of differential forms $\omega_{g,n}(z_1, \ldots, z_n)$ defined on Σ^n and whose asymptotic expansion encodes algebro-geometric invariants of some sort. Several enumerative geometry problems related to different areas of algebraic geometry have been shown to fall within the formalism of topological recursion, such as cohomological field theories[3], Hurwitz theory[26, 10, 21], Gromov-Witten theory for $\mathbb{C}P^1$[42] and toric Calabi-Yau threefolds [17, 34, 31], quantum knot invariants[8], Weil-Peterson volumes [33] and many more[1].

In the same spirit that topological recursion generalizes results known in matrix models, *quantum curves* are believed to exist as a generalization of determinental formulas [4]. The idea is the following. To the spectral curve we can associate an algebraic plane curve, that is the irreducible polynomial equation satisfied by x and y,

$$P(x, y) = 0. \tag{1.2.1}$$

The statement is that the wave function $\psi(x, \hbar)$,

$$\psi(x,\hbar) \propto \exp\left[\sum_{g\geq 0}\sum_{n\geq 1}\frac{\hbar^{2g-2+n}}{n!}\int_\alpha^z \cdots \int_\alpha^z \left(\omega_{g,n} - \delta_{g,0}\delta_{n,2}\frac{dx_1 dx_2}{(x_1-x_2)^2}\right)\right], \tag{1.2.2}$$

Produces the WKB asymptotic expansion of the solution of a quantization of the algebraic equation,

$$\hat{P}(\hat{x}, \hat{y}, \hbar)\, \psi(x, \hbar) = 0. \tag{1.2.3}$$

Here $\hat{x} = x$ and $\hat{y} = \hbar\frac{d}{dx}$ satisfy to the canonical commutation relations $[\hat{y}, \hat{x}] = \hbar$ and $\hat{P}$ is related to P in the classical limit $\hbar \to 0$ with the mapping $\hat{x} \to x$ and

[1] see [7] for an informal updated summary of results.

$\hat{y} \to y$. More precisely, one can write

$$\hat{P}(\hat{x}, \hat{y}, \hbar) = P(\hat{x}, \hat{y}) + \sum_{n \geq 1} \hbar^n P_n(\hat{x}, \hat{y}). \tag{1.2.4}$$

This statement has been verified for a large class of spectral curve of genus $\overline{g} = 0$ [12] under some admissibility, i.e. sufficient, condition. Moreover, a non-perturbative generalization of the conjecture was shown to hold up to a certain order for some specific examples of genus $\overline{g} = 1$ spectral curves [9, 8, 30, 11]. In this thesis we shall only be concerned with genus zero, and our main goal will be to study admissibility using a different approach.

1.3 Airy structures

More recently, Topological recursion has been reformulated in terms of Airy structures [38]. Airy structures are a generalization of the relations known as Virasoro constraints in the context of matrix models [27]. An Airy structure is a set of differential operators of the shape,

$$L_i = \hbar \partial_i - \frac{1}{2} \mathrm{A}_{iab} x^a x^b - \hbar \mathrm{B}_{ia}^b x^a \partial_b - \frac{\hbar^2}{2} \mathrm{C}_i^{ab} \partial_a \partial_b - \hbar \mathrm{D}_i, \tag{1.3.1}$$

where we use the repeated indices summation convention. Here $\hbar\partial_i$ and its dual x_i are basis elements of some $\mathbb{C}$-vector space V. We also impose that the L_i form a Lie algebra $[L_i, L_j] = f_{ij}^k L_k$. This condition guarantees existence and uniqueness of a solution to the infinite set of equations

$$L_i Z = 0. \tag{1.3.2}$$

In particular the solution is constructed recursively, in a very similar manner as the $\omega_{g,n}$ of the topological recursion. In fact, given a spectral curve we can construct a unique Airy structure that recovers the same enumerative information as the Eynard-Orantin topological recursion. Furthermore the partition function Z of the Airy structure then relates to the wave function ψ defined earlier.

This observation is the starting point of the investigation presented in this thesis: in the Airy structure formalism, we are by construction given a set of operators that annihilate the wave function, and the quantum curve conjecture is precisely a statement about the wave function being annihilated by some oper-

ator. The question is then **can we construct the quantum curve operator $\hat{P}$ directly, and in a general way, from the operators L_i of the Airy structure?**
We must stress that this hypothesis is not just a wild guess. Several authors [36, 23] have already related Virasoro constraints the quantum curve, however a general treatment using the new formulation of Airy structure has not yet been explored. This is what we hope to shed light on in this thesis.

1.4 Summary of results

Our method of proof consists in three main steps. First we write down the candidate Airy structure from the input data of the spectral curve, and we prove that it indeed forms a Lie algebra. The second step is to construct an evolution operator from the Airy structure, essentially a linear combination of the operators L_i, and using the Airy structure/topological recursion correspondence to translate it into a differential operator acting on the wave function. This translation is known as *specialization* in the literature, and amounts to sending the infinite number of variables x_i to set of meromorphic functions in the single variable z living on the spectral curve. While specializing we find new conditions for admissibility as the sufficient conditions for the specialization to work out nicely, and to recover a differential operator in the z variable. The third and last step is to check that the specialization of the evolution operator, after suitable conjugations, identifies with a quantization of the original spectral curve. We observe that all the curves which we considered that are passing step two are also passing step three, although we do not have a clear understanding of why this is true.

We first apply our method to genus zero spectral curves for which $x = \frac{1}{2}z^2$. This is a simple generalization of the calculation done in [23] on the Bessel curve. Moreover the existence of a corresponding Airy structure in those cases is a classical result. Therefore, all we do which is new is to proceed with the specialization step in all generality. Doing so we obtain a truncation condition on the series expansion of $\theta(z) = \frac{-2}{\omega_{0,1}(z)-\omega_{0,1}(-z)}$ (proposition 4.1.3), and we rederive the Bessel curve along with the Airy and Monotone Hurwitz curve which all satisfy to that condition. We also point out the existence of 3 corresponding families of quantizable spectral curves that somehow don't fall into the admissibility condition stated in [12] in terms of interior points in the

Newton polygon, but which are clear from our perspective. We find that these are the only spectral curves of this type that are quantizable according to our approach.

We then extend this to so called $n-$point functions which are relevant in the context of integrable systems. This is done by considering a multivariable specialization replacing $\psi(z)$ by $\psi(z_1, z_2, \ldots)$. We show that the multi-variable wave-function for the Airy curve satisfies to the Calogero-Moser Hamiltonian equation (proposition 4.3.7). We do the analog calculation for the Bessel spectral curve and find a similar result (proposition 4.3.12), which has yet to be compared with other sources.

Finally we look are several spectral curves with two branch points. Here we have to assume that we can set the branch points at 0 and ∞ while keeping the involution to be $\iota(z) = -z$. For these we have to define a new Airy structure that has not been mentioned before in the literature (proposition 4.4.1). We then apply successfully to reprove several results of [12]. This also gives us a new notion of admissibility for these cases, again in terms of a truncation condition (proposition 4.4.5).

1.5 Organization of the Thesis

The thesis is organized as follows:

In chapter 2 we summarize a classic application of topological recursion in the context of formal Hermitian matrix models. Our goal in doing so is twofold. Firstly historical, since topological recursion find its sources in random matrix theory, and to familiarize the reader with the structure of topological recursion in a concrete example instead of starting from the polished definition. Secondly, we show how the quantum curve arises in this setup, simply as a property of random matrix expectation values. This also helps us to explain the definition of the quantum curve. Later we will rederive the same quantum curve using Airy structures and only by looking at the corresponding algebraic curve.

In chapter 3 we introduce the new formulation of topological recursion in terms of Airy structures [38], and revisit the statement of the equivalence. Doing so we place some emphasis on the important points to make for our application.

Finally, in chapter 4 we focus our attention on a certain class of simpler spectral curves and we prove the last bit of lemma needed in our reasoning. Then we apply our construction and rederive some known quantum curves. In particular we revisit most of the examples done in [12]. In all these cases we find matching results in terms of correspondence between the choice of integration divisor and the choice of ordering in the canonical quantization of the spectral curve.

2 Quantum Curves in Random Matrices

2.1 Motivation

Like all good recipes, topological recursion is the product of a long history of experimentation, successes, and failures, rooted in an ancient culture that takes its root at the origin of civilization. But more concretely, without going back as far as the Babylonians [24], let us give a brief account of the story, and doing so describe in a bit more details one particular application of this universal formula. Topological recursion originated as a solution for matrix models. Random matrix theory is a very rich subject with applications in statistics, mathematical physics, engineering, finance and many more areas of both applied and pure mathematics. Despite having been studied for almost a century now [46], it continues to be an active field of research.

One typical problem in random matrices is the following: Consider a large matrix whose entries are given by random variables with a given distribution. What can we say about the distribution of its eigenvalues? This question first became relevant in the 1950s when physicists became interested in the excitation levels of heavy nuclei [45] . According to quantum mechanics, energy levels correspond to the eigenvalues of a self-adjoint operator, the Hamiltonian of the system. The inner structure of a large nuclei, although not well understood, is a system with many degrees of freedom and so it is natural to model its properties with some probabilistic law. More precisely we can represent its Hamiltonian as a matrix with random entries with some assumptions on the statistics, and the problem then becomes to compute the distribution of the eigenvalues of this matrix.

2.2 The Setup

Let us denote by H_N the set of $N \times N$ Hermitian matrices,

$$H_N = \{M \in \mathcal{M}_{N\times N}(\mathbb{C}) \mid M^\dagger = M\}. \tag{2.2.1}$$

We define a measure to integrate over this space,

$$d\mu(M) = e^{-\frac{N}{t}\operatorname{Tr} V(M)} dM, \tag{2.2.2}$$

for a choice of a least quadratic potential,

$$V(M) = \frac{M^2}{2} - \sum_{j=3}^{d} \frac{t_j}{j} M^j, \tag{2.2.3}$$

and dM the $U(N)$ invariant Lebesgue measure on H_N ,

$$dM = \frac{1}{Z_0} \prod_{i=1}^{N} dM_{ii} \prod_{i<j} d\operatorname{Re} M_{ij} d\operatorname{Im} M_{ij}, \tag{2.2.4}$$

where $Z_0 = 2^{N/2}(\pi t/N)^{N^2/2}$. We are interested in calculating the partition function

$$Z(t, t_3, \ldots, t_d; N) = \int_{H_N} d\mu(M), \tag{2.2.5}$$

(from now on we omit to write the dependence in the formal variables for tidiness) as well as expectation values,

$$\langle f \rangle = \frac{1}{Z} \int_{H_N} f(M)\, d\mu(M). \tag{2.2.6}$$

For example, the case of the characteristic polynomial $f(M) = \det(x - M)$[1], which contains information about the eigenvalues, will be of prime interest. We will refer to the special case $t_3, \ldots, t_d = 0$ as the *Gaussian* matrix model, as the corresponding partition function is simply a product of N^2 real Gaussian

[1] Here and from now on, one should read $x - M$ as $xI - M$ where I is the appropriately sized identity matrix.

integrals,

$$\operatorname{Tr} M^2 = \sum_{i,j} M_{ij} M_{ji} \tag{2.2.7}$$
$$= \sum_{i,j} M_{ij} M^*_{ij} \tag{2.2.8}$$
$$= \sum_{i,j} (\operatorname{Re} M_{ij})^2 + \sum_{i,j} (\operatorname{Im} M_{ij})^2 \tag{2.2.9}$$
$$= \sum_{i} M_{ii}^2 + 2\sum_{i<j} \left[(\operatorname{Re} M_{ij})^2 + (\operatorname{Im} M_{ij})^2 \right]. \tag{2.2.10}$$

Here we see the reason behind working with Hermitian matrices.

One can easily check, using the single variable case $\int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}}$, that the denominator in (2.2.4) is chosen precisely such that the Gaussian partition function is normalized to 1.

Formal vs convergent matrix models

Before we move on, there is an important point to clarify about what we mean by the integral sign in the expectation values and in the partition function.

When we expand the exponential in the measure we can write it as,

$$Z = \int_{H_N} e^{\frac{N}{t} \sum_{j=3}^{d} \frac{t_j}{j} \operatorname{Tr} M^j} e^{-\frac{N}{2t} \operatorname{Tr} M^2} dM, \tag{2.2.11}$$
$$= \int_{H_N} \prod_{k=3}^{d} \sum_{n_k \geq 0} \frac{1}{n_k!} (\frac{N}{t} \frac{t_k}{k} \operatorname{Tr} M^k)^{n_k} e^{-\frac{N}{2t} \operatorname{Tr} M^2} dM. \tag{2.2.12}$$

We must therefore ask whether order matters between the integration and the summation. The answer is *it does*, and this distinction draws the line between *convergent* and *formal* matrix models. In convergent matrix models, where we are essentially interested in numerical estimates of these expectation values, we are working with the above definition, that is, with the integration on the outside. The motivation given in the introduction, to calculate the distribution of the eigenvalues of a random matrix, falls into this framework [39].
In this thesis however, we will be working only with formal matrix integrals, where we first integrate each monomial before taking the sum,

$$Z := \prod_{k=3}^{d} \sum_{n_k \geq 0} \int_{H_N} \frac{1}{n_k!} (\frac{N}{t} \frac{t_k}{k} \operatorname{Tr} M^k)^{n_k} e^{-\frac{N}{2t} \operatorname{Tr} M^2} dM. \tag{2.2.13}$$

In general this yields a divergent series, but as we shall see it will be meaningful for us as a formal series, hence the name. For compactness we will keep the same notation as the convergent matrix models, but it should be understood that we ***first expand the integrand out, then commute integral and sum and integrate each monomials individually***.
It is important to distinguish between the two types of matrix models, as their domain of application are diametrically opposite to one another. For example, it is not hard to see that the convergent matrix model will only be valid for $t_j < 0$, while in the formal setting the interesting case is $t_j > 0$, as we shall see from combinatorial properties outlined below. For $t_j < 0$ the formal integral can be renormalized but even so it would not necessarily coincide with the convergent integral.
Formal matrix models were first described by t'Hooft [43] as a formalism for the strong interaction.

2.3 Resolvents

Another motivation for looking at matrix models is that they can be thought of as the simplest example of a quantum gauge theory in 0 dimensions.
Here the "field" is a unique[2] $N \times N$ matrix , as opposed to actual quantum field theories where fields are operators defined in a continuous space-time.
More concretely, the action given by the potential (2.2.3) is invariant under the gauge symmetry

$$M \to UMU^\dagger, \tag{2.3.1}$$

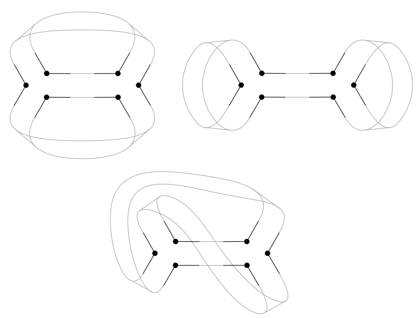
where U is a unitary matrix, $UU^\dagger = \mathrm{Id}$. We are therefore naturally looking at expectation values of $U(N)$ invariant functions of M. Such functions can always be decomposed as a linear combination of product of traces of some power of M, and so we focus our attention on integrals of the form

$$\mathcal{T}^*_{l_1,\ldots,l_n} = \int_{H_N} \operatorname{Tr} M^{l_1} \ldots \operatorname{Tr} M^{l_n} e^{-\frac{N}{t}V(M)} dM \tag{2.3.2}$$

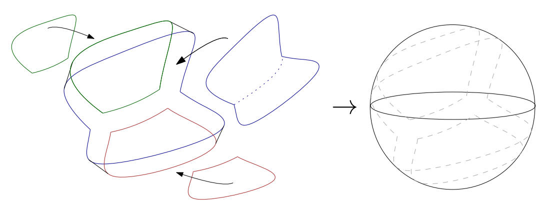
To calculate these we shall study their generating functions, so-called resolvents. To define them however we need the notion of *connected* expectation value, which makes sense thanks to the following result. Recall that a fatgraph (or

[2]There are also multi-matrix models, but in any case that is always a set, so in particular discrete, set.

ribbon graph) is a graph together with a choice of ordering of the edges at each vertex. Fixing this ordering is like giving edges a certain width: each side of the edge knows what side of another edge it continues to when meeting at a vertex. Here are the 3 inequivalent (orientable) fatgraphs that we can obtain from gluing two trivalent vertices:



We also have a notion of topology for a fatgraph: if we glue disks along each side of the ribbon edges, we obtain a closed surface (possibly disconnected and non-orientable), as in



(2.3.3)

Then we can talk about the Euler characteristic χ of a ribbon graph: simply look at the Euler characteristic of the corresponding surface.
We have the following property:

Proposition 2.3.1. *For the formal Hermitian matrix model, we have the following sum over graphs expansion,*

$$Z = \sum_{fatgraphs} \frac{1}{|Aut(\Gamma)|} \left(\frac{N}{t}\right)^{\chi} t^F \, t_3^{k_3} ... t_d^{k_d}, \tag{2.3.4}$$

where the sum is over all inequivalent fatgraphs, not necessarily connected, with

F faces, and k_j vertices of valence j, $j = 3, \dots, d$. Additionally we have,

$$\mathcal{T}^*_{l_1,\dots,l_n} = \sum_{\substack{fatgraphs\\ l_1,\dots,l_n}} \frac{1}{|Aut(\Gamma)|} \left(\frac{N}{t}\right)^{\chi-n} t^F\, t_3^{k_3} \dots t_d^{k_d}, \tag{2.3.5}$$

where the sum is over graphs, not necessarily connected, with n marked vertices of valences $l_1, \dots, l_n$.

Proof. This follows from Wick's theorem for Gaussian integrals and a brief analysis. Details can be found in [27]. ☕

This is where topology makes its entrée in our study. It is a prototypical example of how mathematical tools inspired from physics can be reinterpreted in terms of enumerative geometry, which opens a ramified and fruitful connection between both fields.

Remark 2.3.2. From the above result, we see that Hermitian matrix models are enumerating ways to draw a Ribbon graph on a Riemann surface, or, in a way, to discretize the surface. In particular this has applications in so called 2D-quantum gravity, and the famous work of Witten and Kontsevitch [47, 37].

We would like to define generating functions for these expectation values,

$$W_n(x_1, \dots, x_n) \stackrel{?}{=} \sum_{l_1,\dots,l_n \geq 0} \mathcal{T}^*_{l_1,\dots,l_n} \frac{1}{x_1^{l_1+1}} \cdots \frac{1}{x_n^{l_n+1}}. \tag{2.3.6}$$

This would define power series in the variables $t, t_3, \dots, t_d, N$ and the x_i 's, which we can then study analytically and hope to solve for its coefficients. However there is an obstacle here, apparent from the above result, that for fixed Euler characteristic χ and number of faces F, the number of *not necessarily connected* fatgraph is infinite, and infinite sums are harder to deal with analytically.
For this reason we are going to restrict our attention to *connected* maps. Compact connected orientable surfaces have their Euler characteristic given in terms of their genus g, $\chi = 2 - 2g$. One can easily check the following proposition.

Proposition 2.3.3. *The set of connected fatgraphs of genus g with unmarked vertices of degree at least* 3 *and at most d, k marked vertices, and F faces, is finite.*

Now it is a basic result in combinatorics that by defining the *free energy F*

as the log of the partition function

$$F = \log Z, \tag{2.3.7}$$

we are exactly reducing the sum in (2.3.4) to connected graphs,

$$F = \sum_{\substack{\text{fatgraphs} \\ connected}} \frac{1}{|\mathrm{Aut}(\Gamma)|} \left(\frac{N}{t}\right)^{\chi} t^F\, t_3^{k_3} \ldots t_d^{k_d}, \tag{2.3.8}$$

$$= \sum_{g\geq 0} \left(\frac{N}{t}\right)^{2-2g} \sum_{\substack{\text{fatgraphs} \\ connected \\ g}} \frac{1}{|\mathrm{Aut}(\Gamma)|} t^F\, t_3^{k_3} \ldots t_d^{k_d}, \tag{2.3.9}$$

where the second sum is over all connected fatgraphs with fixed genus g. Similarly, we can restrict expectation values to a sum over connected graphs by looking at cumulants. Consider the *connected expectation values* $\langle\cdot\rangle_c$,

$$\mathcal{T}_{l_1,\ldots,l_n} = \left\langle \mathrm{Tr}\, M^{l_1} \ldots \mathrm{Tr}\, M^{l_n} \right\rangle_c, \tag{2.3.10}$$

which we can define recursively as,

$$\mathcal{T}_{l_1} = \frac{1}{Z}\mathcal{T}^*_{l_1}, \tag{2.3.11}$$

$$\mathcal{T}_{l_1,\ldots,l_n} = \frac{1}{Z}\mathcal{T}^*_{l_1,\ldots,l_n} - \sum_{\mu\vdash\{l_1,\ldots,l_n\}}^{\circ} \prod_{j=1}^{|\mu|} \mathcal{T}_{\mu_j}, \tag{2.3.12}$$

where the sum is over all partitions of $\{l_1, \ldots, l_n\}$ except for $\mu = \{\{l_1, \ldots, l_n\}\}$. When $k = 1$, the factor $1/Z$ in the definition of the expectation value exactly removes all non connected fatgraphs, while the recursion ensures that we discard all ways to split marked faces among disconnected components. It follows immediately that,

$$\left\langle \mathrm{Tr}\, M^{l_1} \ldots \mathrm{Tr}\, M^{l_n} \right\rangle_c = \sum_{\substack{\text{fatgraphs} \\ connected \\ l_1,\ldots,l_n}} \frac{1}{|\mathrm{Aut}(\Gamma)|} \left(\frac{N}{t}\right)^{\chi - n} t^F\, t_3^{k_3} \ldots t_d^{k_d}, \tag{2.3.13}$$

$$= \sum_{g\geq 0} \left(\frac{N}{t}\right)^{2-2g-n} \sum_{\substack{\text{fatgraphs} \\ connected \\ g\,;\, l_1,\ldots,l_n}} \frac{1}{|\mathrm{Aut}(\Gamma)|} t^F\, t_3^{k_3} \ldots t_d^{k_d}, \tag{2.3.14}$$

$$= \sum_{g\geq 0} \left(\frac{N}{t}\right)^{2-2g-n} \mathcal{T}^{(g)}_{l_1,\ldots,l_n}, \tag{2.3.15}$$

where we have defined the generating function for connected fatgraphs with marked vertices of valence $l_1, \ldots, l_n$, at fixed genus g. We are finally ready to define the *resolvents* as the following generating functions

$$W_n(x_1, \ldots, x_n) = \sum_{l_1,\ldots,l_n \geq 0} \mathcal{T}_{l_1,\ldots,l_n} \frac{1}{x_1^{l_1+1}} \cdots \frac{1}{x_n^{l_n+1}}. \tag{2.3.16}$$

$$= \sum_{l_1,\ldots,l_n \geq 0} \left\langle \mathrm{Tr}\, \frac{M^{l_1}}{x_1^{l_1+1}} \ldots \mathrm{Tr}\, \frac{M^{l_n}}{x_n^{l_n+1}} \right\rangle_c \tag{2.3.17}$$

$$:= \left\langle \mathrm{Tr}\, \frac{1}{x_1 - M} \ldots \mathrm{Tr}\, \frac{1}{x_n - M} \right\rangle_c. \tag{2.3.18}$$

We dropped the dependence in t and $t_3, \ldots, t_d$ and N for tidiness. Here the last line (2.3.18) is **only a notation** for the series expansion (2.3.17). Note that the variables $x_1, \ldots, x_n$ are *not* formal variables, they are complex variables, and we are going to solve the matrix model by studying the W_k's analytical properties. In particular we can recover connected expectation values as residues at infinity,

$$\mathcal{T}_{l_1,\ldots,l_n} = (-1)^n \operatorname*{Res}_{x_1,\ldots,x_n \to \infty} x_1^{l_1} \ldots x_n^{l_n}\, W_n(x_1, \ldots, x_n) dx_1 \ldots dx_n. \tag{2.3.19}$$

It follows immediately from proposition 2.3.1 and by transitioning to connected maps that the F and W_n 's have a genus expansion.

Corollary 2.3.4. *The free energy has a genus expansion*

$$F = \sum_{g\geq 0} \left(\frac{N}{t}\right)^{2-2g} F_g, \tag{2.3.20}$$

where the F_g are power series in t, with each coefficient being a polynomial in *the $t_3, \ldots, t_d$. Similarly, the resolvents can be expanded as*

$$W_n(x_1, \ldots, x_n) = \sum_{g \geq 0} \left(\frac{N}{t}\right)^{2-2g-n} W_{g,n}(x_1, \ldots, x_n), \tag{2.3.21}$$

where the $W_{g,n}$ are powers series in t which, order by order, give polynomials in the variables $1/x_1, \ldots, 1/x_n$ and $t_3, \ldots, t_d$.

Together, the $W_{g,n}$ and F_g's contain all the information about the two questions asked earlier, that is, to calculate the partition function and expectation values of the formal matrix model. Leaving the F_g's aside for now – readers interested might refer to [27] – we will outline in the following section how analytical properties of the $W_{g,n}$ will allow us to derive a recursion relation to effectively calculate their expansion for all values of g and n, also known as the *topological recursion*[32].

2.4 Loop Equations

Divergence theorem. Our goal is now to find an effective way to calculate $W_{g,n}$ for all g and n. We start by deriving loop equations, which can be simply thought of as the divergence theorem applied to matrix integrals. In general, the divergence theorem tells us that the integral of a vector field in a region of space is the flux on the boundary,

$$\int_V (\nabla \cdot F)\, dV = \int_{\partial V} F \cdot dS. \tag{2.4.1}$$

In the case that the vector field vanishes on the boundary, the integral of the divergence must vanish as well. Applying this to our matrix model, we find, for any polynomial function $G(M)$,

$$\int_{H_N} \sum_{i \leq j} \frac{\partial}{\partial M_{ij}} \left(G(M)_{ij} e^{-\frac{N}{t} V(M)} \right) = 0. \tag{2.4.2}$$

Note that this equality holds both in the convergent and formal sense. In the convergent case it is because the factor $e^{-\frac{N}{t}V(M)}$, for $t_3, \ldots, t_d < 0$, would damp any polynomial when M becomes large; in the formal case it is because after expanding the exponential, the remaining Gaussian weight $e^{-\frac{N}{2t} \operatorname{Tr} M^2}$ would still damp any polynomial or product of traces when M becomes large.

If we then set $G(M) = M^{l_1} \prod_{j=2}^{n} \operatorname{Tr} M^{l_j}$, and with elementary calculations, we find,

$$\frac{\partial (M^{l_1})_{ij}}{\partial M_{ij}} = \prod_{j=0}^{l_1-1} (M^j)_{ii} (M^{l_1-1-j})_{jj}, \tag{2.4.3}$$

$$\frac{\partial \operatorname{Tr} M^{l_j}}{\partial M_{ij}} = l_j (M^{l_j-1})_{ji}, \tag{2.4.4}$$

$$\frac{\partial V(M)}{\partial M_{ij}} = V'(M)_{ji}. \tag{2.4.5}$$

Putting everything together, equation (2.4.2) yields,

$$\sum_{j=0}^{l_1-1} \left\langle \operatorname{Tr} M^j \operatorname{Tr} M^{l_1-1-j} \prod_{i=2}^{n} \operatorname{Tr} M^{l_i} \right\rangle + \sum_{j=2}^{n} l_j \left\langle \operatorname{Tr} M^{l_1+l_j-1} \prod_{\substack{i=2 \\ i\neq j}}^{n} \operatorname{Tr} M^{l_i} \right\rangle$$
$$= \frac{N}{t} \left\langle \operatorname{Tr} \left(M^{l_1} V'(M) \right) \prod_{i=2}^{n} \operatorname{Tr} M^{l_i} \right\rangle. \tag{2.4.6}$$

Or equivalently, denoting $L = \{l_2, \ldots, l_n\}$,

$$\sum_{j=0}^{l_1-1} \mathcal{T}^*_{j,l_1-1-j,L} + \sum_{j=2}^{n} l_j \mathcal{T}^*_{l_j+l_1-1,L\setminus\{l_j\}}$$
$$= \frac{N}{t} \left(\mathcal{T}^*_{l_1+1,L} - \sum_{j=3}^{d} t_j \mathcal{T}^*_{l_1+j-1,L} \right) \tag{2.4.7}$$

Upon passing to connected expectation values and plugging in the genus expansion, then collecting powers of N/t,

$$\sum_{j=0}^{l_1-1} \left[\sum_{h=0}^{g} \sum_{J\subset L} \mathcal{T}^{(h)}_{j,J} \mathcal{T}^{(g-h)}_{l_1-1-j,L\setminus J} + \mathcal{T}^{(g-1)}_{j,l_1-1-j,L} \right]$$
$$+ \sum_{j=2}^{n} l_j \mathcal{T}^{(g)}_{l_j+l_1-1,L\setminus\{l_j\}}$$
$$= \mathcal{T}^{(g)}_{l_1+1,L} - \sum_{j=3}^{d} t_j \mathcal{T}^{(g)}_{l_1+j-1,L}. \tag{2.4.8}$$

Re-expressing these relations in terms of the resolvents we finally obtain the following theorem:

Theorem 2.4.1 (Loop equations)**.** *For all g and n, and for $L = \{x_2, \dots, x_n\}$, the resolvents of the formal Hermitian matrix model satisfy:*

$$\sum_{h=0}^{g} \sum_{J \subset L} W_{h,1+|J|}(x_1, J) W_{g-h,n-|J|}(x_1, L \setminus J) + W_{g-1,n+1}(x_1, x_1, L)$$
$$+ \sum_{j=2}^{n} \frac{\partial}{\partial x_j} \frac{W_{g,n-1}(x_1, L \setminus \{x_j\}) - W_{g,n-1}(L)}{x_1 - x_j}$$
$$= V'(x_1) W_{g,n}(x_1, L) - P_{g,n}(x_1, L), \quad (2.4.9)$$

where $P_{g,n}(x_1, L)$ is a polynomial in x_1 of degree $d - 3 + \delta_{g,0}\delta_{n,1}$.

Proof. Take equation (2.4.8), multiply by $\prod_{i=1}^n 1/x_i^{l_i+1}$ then sum over all l_i's to obtain (2.4.9) line by line. In particular in the last line, the polynomial is

$$P_{g,n}(x_1, \dots, x_n) = -\sum_{j=2}^{d-1} t_{j+1} \sum_{i=0}^{j-1} x_1^i \sum_{l_2,\dots,l_n}^{\infty} \mathcal{T}_{j-1-i,L}^{(g)} + t\delta_{g,0}\delta_{n,1}, \quad (2.4.10)$$

and cancels exactly the positive part of $V'(x_1)W_{g,n}(x_1, L)$, while the negative part gives the third line in (2.4.8). ☕

Solving the loop equations: disk and cylinder amplitudes. The loop equation (2.4.9) is not useful yet in helping us solve for the resolvents. However we can work on it in order to make it a recursive relation on increasing values of the pair of indices (g, n), more exactly on increasing values of $2g - 2 + n$. To do so we need two things. One is to re-order terms to obtain a recursion. The other is to get rid of the polynomial $P_{g,n}$ in the relation so as to only get an expression of the $W_{g,n}$. Finally we have to solve separately for the base cases $(g, n) = (0, 1)$ and $(g, n) = (0, 2)$, corresponding to the topologies of a disk and cylinder respectively, which are the only ones with $2g - 2 + n \leq 0$. We will also refer to them as the *unstable* topologies, because the corresponding surfaces have an infinite automorphism group. Let us first look at the equation for $(g, n) = (0, 1)$, yielding

$$V'(x)W_{0,1}(x) = W_{0,1}(x)^2 + P_{0,1}(x), \quad (2.4.11)$$

which we can rewrite as

$$y(x)^2 = \frac{1}{4}V'(x)^2 - P_{0,1}(x). \quad (2.4.12)$$

where we've defined,

$$y(x) = W_{0,1}(x) - \frac{1}{2}V'(x). \tag{2.4.13}$$

We have the following lemma.

Lemma 2.4.2 (1-cut Brown's Lemma)**.** *[27, Lemma 3.1.1] The polynomial* $\frac{1}{4}V'(x) - P_{0,1}(x)$*, has only one pair of simple zeroes, and all the other zeroes are even. More precisely, there exists a polynomial* $M(x)$ *of degree* $d-2$ *whose roots are power series of* t*, and* a, b *power series in* $\sqrt{t}$ *such that* ab *and* $a+b$ *are power series of* t*, and such that*

$$\frac{1}{4}V'(x) - P_{0,1}(x) = M(x)^2(x-a)(x-b) \tag{2.4.14}$$

This lemma allows us to define a new variable z,

$$x(z) = \frac{a+b}{2} + \frac{a-b}{4}\left(z + \frac{1}{z}\right), \tag{2.4.15}$$

So that

$$\sqrt{(x-a)(x-b)} = \frac{a-b}{4}\left(z - \frac{1}{z}\right) \tag{2.4.16}$$

and when taking the square root of equation (2.4.12),

$$y(z) = M(x(z))\frac{a-b}{4}\left(z - \frac{1}{z}\right) \tag{2.4.17}$$

we find that $y(z)$ *must be a rational function of the* z *variables.* In particular one can show that

$$W_{0,1}(x(z)) = \sum_{j=1}^{d-1} u_j z^{-j} \tag{2.4.18}$$

where coefficients u_k are given by the expansion of the potential,

$$V'(x(z)) = \sum_{k=0}^{d-1} u_j (z^j - z^{-j}). \tag{2.4.19}$$

Moving on to the $(g,k) = (0,2)$ case, the loop equation yields,

$$2W_{0,1}(x_1)W_{0,2}(x_1,x_2) + \frac{\partial}{\partial x_2}\frac{W_{0,1}(x_1) - W_{0,1}(x_2)}{x_1 - x_2} \\ = V'(x_1)W_{0,2}(x_1,x_2) - P_{0,2}(x_1,x_2) \tag{2.4.20}$$

Grouping terms, then inserting the expression (2.4.13) for $W_{0,1}$,

$$W_{0,2}(x_1,x_2) = \frac{1}{2y(x_1)}\left[\frac{\partial}{\partial x_2}\frac{W_{0,1}(x_1)-W_{0,1}(x_2)}{x_1-x_2} - P_{0,2}(x_1,x_2)\right] \tag{2.4.21}$$

$$\begin{aligned} &= \frac{1}{2y(x_1)}\left[\frac{\partial}{\partial x_2}\frac{y(x_2)}{x_1-x_2} - \frac{1}{2}\frac{\partial}{\partial x_2}\frac{V'(x_1)-V'(x_2)}{x_1-x_2}\right. \\ &\qquad \left. - P_{0,2}(x_1,x_2)\right] - \frac{1}{2}\frac{1}{(x_1-x_2)^2}. \end{aligned} \tag{2.4.22}$$

It turns out that by looking at $W_{0,2}$ as a differential form, we can say the following.

Lemma 2.4.3 (1-cut Lemma for Cylinders)**.** *[27, Lemma 3.2.1]*
$W_{0,2}(x_1,x_2)dx_1dx_2$ is a rational differential form of z_1 and z_2 which behaves as $O(z_1^{-2})$ at large z_1 , and has only one pole at $z_1 = \frac{1}{z_2}$ which is a double pole with coefficients $-z_2^{-2}$. Furthermore, there is only one such rational bidifferential which is

$$W_{0,2}(x_1,x_2)dx_1dx_2 = \frac{-z_2^{-2}}{(z_1-z_2^{-1})^2}dz_1dz_2 = \frac{dz_1dz_2}{(z_1-z_2)^2} - \frac{dx_1dx_2}{(x_1-x_2)^2}. \tag{2.4.23}$$

We make one important observation from this lemma: if we define,

$$\omega_{0,2}(z_1,z_2) = W_2^{(0)}(x_1,x_2)dx_1dx_2 + \frac{dx_1dx_2}{(x_1-x_2)^2} = \frac{dz_1dz_2}{(z_1-z_2)^2}, \tag{2.4.24}$$

we get something which is completely independent of the input data of our matrix model, that is, the potential $V(x)$. We will come back to this shortly when generalizing the topological recursion.

Higher topologies. Finally, let us turn to the higher topology cases $2g-2+n>0$. The way we are going to get rid of the polynomial $P_{g,n}$ in (2.4.9) is by taking residues. To this end, let us define the differential forms $\omega_{g,n}$:

$$\omega_{g,n}(z_1,\dots,z_n) = W_{g,n}(x_1,\dots,x_n)\, dx_1\dots dx_n. \tag{2.4.25}$$

Remark 2.4.4. More precisely, in equation (2.4.25) we are defining differential forms in the z_i variables that recover the resolvents $W_{g,n}(x_1,\dots,x_n)$ in their expansion at infinity (large values for the x_i's). A careful analysis [27] shows that the resolvents are not well defined functions of x_i but are multi-valued, while they become well defined when seen as differentials that depends on the z_i variables. This is the meaning behind the different variables on both sides of equation (2.4.25): the claim is that we have well defined differentials $\omega_{g,n}$ that recover the resolvents in the limit $x_i \to \infty$.

Notice that doing so, we don't lose sight of our original problem, since the expectation values of the matrix model are related to the resolvents by taking residues at infinity (2.3.19). Let us multiply the loop equation (2.4.9) by $dx_1 \dots dx_n$ and re-arrange terms. We obtain

$$\begin{aligned}&\omega_{g,n}(z, L)\\ &= \frac{1}{2y(z)dx(1/z)} \sum_{h=0}^{g} \sum_{J\subset L}^{\circ} \omega_{h,|J|+1}(z, J)\omega_{g-h,n-|J|}(1/z, L \setminus J)\\ &\qquad + \frac{1}{2y(z)dx(1/z)}\omega_{g-1,n+1}(z, 1/z, L)\\ &\quad + \sum_{j=1}^{|L|} \frac{1}{2y(z)} \frac{\partial}{\partial z_j} \frac{\omega_{g,n}(z, L \setminus \{z_j\}) + \omega_{g,n}(L)dx(z)/dx(z_j)}{x(z) - x(z_j)}\\ &\qquad - \frac{dx(z)}{2y(z)} P_{g,n}(x(z), L)dx_1 \dots dx_n.\end{aligned} \tag{2.4.26}$$

Several properties can be shown about these differentials:

Lemma 2.4.5. *[27, Lemma 3.3.2, 3.3.3] For $2g - 2 + n$ we have the following,*

- *$\omega_{g,n}$ is antisymmetric under $z \to 1/z$,*

$$\omega_{g,n}(z_1, z_2, \dots, z_n) + \omega_{g,n}(1/z_1, z_2, \dots, z_n) = 0 \tag{2.4.27}$$

- *$\omega_{g,n}(z_1, \dots, z_n)$ is a rational function of the variables $z_1, \dots, z_n$, with poles only at $z_i = \pm 1$ and behaves as $O(z_i^{-2})$ at large z_i.*

And we finally obtain the topological recursion formula,

Theorem 2.4.6 (TR, Hermitian matrix models)**.** *[27, Theorem 3.3.1] The correlators of the formal Hermitian matrix model $\omega_{g,n}$, can be computed recursively for $2g - 2 + n > 1$ via*

$$\begin{aligned}\omega_{g,n}(z_1, L) &= \frac{1}{2} \operatorname*{Res}_{z\to\pm 1} \left(\frac{dz_1}{z_1 - z} - \frac{dz_1}{z_1 - 1/z}\right) \frac{1}{2y(z)dx(z)}\\ \times &\left[\sum_{h=0}^{g} \sum_{J\subset L}^{\circ} \omega_{h,1+|J|}(z, J)\omega_{g-h,n-|J|}(1/z, L \setminus J) + \omega_{n+1}^{(g-1)}(z, 1/z, L)\right]\end{aligned} \tag{2.4.28}$$

where the notation $\sum^{\circ}$ means that we excluded terms of the form $\omega_{0,1}\omega_{g,n}$.

Notice that all the $\omega_{g',n'}$ appearing on the right hand side now have $2g' - 2 + n' < 2g - 2 + n$, and so (2.4.28) is indeed a recursion relation on $2g - 2 + n$.

Proof. By Cauchy's formula we have

$$\omega_{g,n}(z_1, L) = -\operatorname*{Res}_{z \to z_1} \frac{dz_1}{z_1 - z}\omega_{g,n}(z, L). \tag{2.4.29}$$

By lemma 2.4.5, $\omega_{g,n}$ has poles only at $z = \pm 1$, and we can therefore move the contour such that

$$\omega_{g,n}(z_1, L) = \operatorname*{Res}_{z \to \pm 1} \frac{dz_1}{z_1 - z}\omega_{g,n}(z, L), \tag{2.4.30}$$

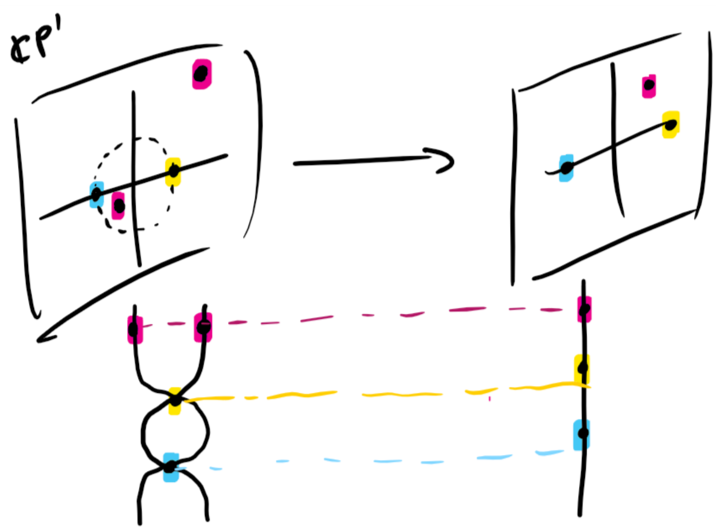
where $\operatorname{Res}_{z \to \pm 1}$ means we are summing the residues at ± 1. Here the minus sign is absorbed in the orientation of the contour. We can then symmetrize under $z \to 1/z$ (the residue of a differential form is independent of the parametrization) and use the second part of lemma 2.4.5 to obtain,

$$\omega_{g,n}(z_1, L) = \operatorname*{Res}_{z \to \pm 1} \frac{1}{2}\left(\frac{dz_1}{z_1 - z}\omega_{g,n}(z, L) + \frac{dz_1}{z_1 - 1/z}\omega_{g,n}(1/z, L)\right), \tag{2.4.31}$$

$$= \operatorname*{Res}_{z \to \pm 1} \frac{1}{2}\left(\frac{dz_1}{z_1 - z} - \frac{dz_1}{z_1 - 1/z}\right)\omega_{g,n}(z, L). \tag{2.4.32}$$

We can now insert the right hand side of (2.4.26) in (2.4.32). Since $y(z) = -y(1/z)$ as well as $x(z) = x(1/z)$, the terms $1/y(z)\omega_{g,n}(L \setminus \{z_j\})/(x(z) - x(z_j))$ in the third line of (2.4.26) are even under the involution, and they vanish when multiplied with the odd prefactor. Finally, all of $dx(z)/y(z)$, $P_{g,n}(x(z), L))$ and $\partial_{z_j}\omega_{g,n}(z, L \setminus z_j)/(dx(z_j)(x(z) - x(z_j)))$ have no pole at $z = \pm 1$ and therefore they drop out of the residue. ☕

This result theoretically solves the question posed in the introduction, to calculate expectation values for a general formal Hermitian matrix model. In practice, after solving for the initial case $W_{0,1}(x)$ and inserting the corresponding $y(z)$ and $x(z)$ into the recursion, the formula for topological recursion can easily be implemented in a computer to obtain expectation values at all order.
In the next section we will see how, after observing similar recursive structures in other problems of enumerative geometry, Eynard and Orantin [32] proposed what the general formulation of this recursion should be beyond Hermitian matrix models.



The double cover $x(z) = z + \frac{1}{z}$.

2.5 The Structure of TR

A first observation is that the initial data is equally given in terms of the potential $V(x)$ or the two meromorphic functions $x(z)$ and $y(z)$ together with their domain $\mathbb{C}P^1$. Furthermore, recall that

$$x(z) = \frac{a+b}{2} + \frac{a-b}{4}\left(z + \frac{1}{z}\right) \tag{2.5.1}$$

and that this is a double branched covering of $\mathbb{C}P^1$: it is a surjective map $\mathbb{C}P^1 \to \mathbb{C}P^1$, where each point has two preimages, $x(1/z) = x(z)$, except for the two branch points $z = \pm 1$. The involution which sends us between the two sheets is $\iota(z) = 1/z$.

The branched covering in homogeneous coordinates. To be more precise, $x(z)$ above represents the map on one of the charts in $\mathbb{C}P^1$. In homogeneous coordinates we can write $x(z_1, z_2) = (z_1^2 + z_2^2, z_1 z_2) \in \mathbb{C}P^1$. When z_1, z_2 are both non zero we recover the map above with $x(\frac{z_1}{z_2}, 1) = (\frac{z_1}{z_2} + \frac{z_2}{z_1}, 1)$, and in addition we see $x(0,1) = x(1,0) = (1,0) \sim \infty$. Notice that $(0,1)$ also has two preimages $(1,i)$ and $(1,-i)$. The involution is $\iota(z_1, z_2) = (z_2, z_1)$, which leaves invariant $(1,1) \sim 1$ and $(1,-1) \sim -1$.

The set of branch points corresponds to the zeroes of $dx(z)$ and we shall denote it by $\mathfrak{r}$. For formal Hermitian matrix models we have $\mathfrak{r} = \{-1, 1\}$. From x and y we can then define,

$$\omega_{0,1}(z) = \left(W_{0,1}(x(z)) - \frac{1}{2}V'(x(z)) \right) dx(z) = y(z)dx(z) \tag{2.5.2}$$

On the other hand, $\omega_{0,2}$, is defined as

$$\omega_{0,2}(z_1, z_2) = W_{0,2}(x_1, x_2)dx_1dx_2 + \frac{dx_1dx_2}{(x_1 - x_2)^2} = \frac{dz_1dz_2}{(z_1 - z_2)^2}, \tag{2.5.3}$$

Which is known as the fundamental second kind differential for the Riemann sphere $\mathbb{C}P^1$. On an arbitrary compact Riemann surface Σ, the fundamental second kind differential $B(z_1, z_2)$ is the unique bidifferential form which has a double poles at $z_1 = z_2$ and no other pole, and that behaves near the poles as

$$B(z_1, z_2) = \frac{dz_1dz_2}{(z_1 - z_2)^2} + O(1). \tag{2.5.4}$$

For uniqueness, we require it to satisfy some vanishing cycle integral condition: a genus $\overline{g}$ Riemann surface Σ can be equipped with a choice of symplectic basis of cycles $(A_1, \ldots, A_{\overline{g}}, B_1, \ldots, B_{\overline{g}}) \in H_1(\Sigma, \mathbb{Z})$ with $A_i \cap B_j = \delta_{ij}$, $A_i \cap A_j = 0$ and $B_i \cap B_j = 0$. We then require

$$\oint_{A_i} B = 0, \tag{2.5.5}$$

which fixes the holomorphic part of B. For the Riemann sphere all cycles are trivial and we get the unique (2.5.3).

The kernel B is a natural object in that it gives a map $f \to df$ for meromorphic functions defined on Σ:

$$df(p) = \operatorname*{Res}_{q \to p} B(p, q)f(q) \tag{2.5.6}$$

In this thesis we shall only consider the topological recursion for genus 0 spectral curves, i.e. on $\mathbb{C}P^1$, but it also has application when extended to higher genera curves, such as the Weiestrass spectral curve [11].

Observe that,

$$\frac{1}{2}\int_{\iota(z)}^{z} \omega_{0,2}(z_1, \cdot) = \frac{1}{2}\left(\frac{dz_1}{z_1 - z} - \frac{dz_1}{z_1 - 1/z} \right), \tag{2.5.7}$$

which was exactly the prefactor in (2.4.28). Wrapping up, with the above notation, we can summarize topological recursion as follows. The input data is given by a *spectral curve*.

Definition 2.5.1 (Spectral curve)**.** A spectral curve (Σ, x, y, B) is the data of choice of Riemann surface Σ together with a branched covering $x : \Sigma \to \Sigma' \subset \mathbb{C}P^1$, a meromorphic function y on Σ such that the zeroes of dy are distinct from the zeroes of dx, and a choice of fundamental second kind differential B on Σ.

In our thesis we will only be looking at spectral curves for which x has simple ramification points, but topological recursion has been generalized to higher branched coverings [13]. With simple branched coverings, we can define locally at each branch point a unique involution $p \to \iota(p)$ that sends us from one sheet to another. We define the topological recursion for such spectral curves as follows:

Definition 2.5.2 (Topological Recursion)**.** [32]
Given a spectral curve (Σ, x, y, B) with x a simple branched covering , define the base cases

$$\omega_{0,1}(z) = y(z)dx(z), \tag{2.5.8}$$

$$\omega_{0,2}(z) = B(z_1, z_2). \tag{2.5.9}$$

Then let $\mathfrak{r}$ be the set of branch points of x. Near each branch point we have a local involution ι. Define the recursion kernel as

$$K(z_0, z) = \frac{\frac{1}{2}\int_{\iota(z)}^{z} \omega_{0,2}(z_0, \cdot)}{\omega_{0,1}(z) - \omega_{0,1}(\iota(z))}, \tag{2.5.10}$$

and for $2g - 2 + n > 0$ let,

$$\begin{aligned}\omega_{g,n}(z_0, L) = \sum_{a \in \mathfrak{r}} \operatorname*{Res}_{z \to a} K(z_0, z) \Bigg[& \omega_{g-1,n+1}(z, \iota(z), L) \\ & + \sum_{h=0}^{g} \sum_{J \subset L}^{\circ} \omega_{h,1+|J|}(z, J)\omega_{g-h,n-|J|}(\iota(z), L \setminus J) \Bigg], \end{aligned} \tag{2.5.11}$$

where in the sum $\sum^{\circ}$ we omit terms with $\omega_{0,1}\omega_{g,n}$.

Notice that all $\omega_{g',n'}$ on the right hand side of (2.5.11) have $2g' - 2 + n' < 2g - 2 + n$, so that it effectively defines $\omega_{g,n}$ recursively for all g and n.

Remark 2.5.3. The assumptions on dy in definition 2.5.1 are required to generalize the property that the recursion defines differential forms that are totally symmetric [32]. Although we won't show why this is true at present, we will return to it later in section 3.3.2 after reformulating the recursion in terms of Airy structures.

Remark 2.5.4. The topological recursion defines differential forms in terms of their principal parts, i.e. in terms of their behaviours at their poles. Here we see the necessity to impose vanishing conditions of cycle integrals of B for higher genus curves, which is carried over to the $\omega_{g,n}$ and fixes the holomorphic part to give a well defined object. For genus zero there are no holomorphic forms globally defined on the Riemann sphere and so the prinpical parts determine the $\omega_{g,n}$'s uniquely.

Going back to our example, the correlators of the Hermitian matrix model are given by the topological recursion for the spectral curve $(\mathbb{C}P^1, x, y)$, where

$$\begin{cases} x(z) = \frac{a+b}{2} + \frac{a-b}{4}\left(z + \frac{1}{z}\right) \\ y(z) = -\frac{1}{2}\sum_{j=1}^{d-1} u_j(z^j - z^{-j}) \end{cases} \tag{2.5.12}$$

This might just seem like a complicated way to rewrite our original result, but the point is that this recursion was shown to apply to a larger class of problems in algebraic geometry. The table below gives a summary of some applications of topological recursion, from combinatorics, to the theory of integrable systems and Gromov-Witten theory, together with the corresponding spectral curve.

Remark 2.5.5. Let us very briefly recall the definition of Hurwitz numbers c) and monotone Hurwitz numbers d). For a more complete summary and further references we refer to [21]. Hurwitz numbers are one of the most classical enumerative results in algebraic-geometry. The Simple Hurwitz number $H_{g,n}(\mu_1, \dots, \mu_n)$ is the weighted count, up topological equivalence, of connected genus g branched covers $f : C \to \mathbb{C}P^1$ of the Riemann sphere with simple ramification except over ∞, where the ramification profile is given by the tuple $(\mu_1, \dots, \mu_n)$. One can then show that simple Hurwitz numbers can be computed by counting tuples of transpositions in the symmetric group $S_{|\mu|}$. Monotone Hurwitz numbers are then the restriction of this count to some monotonic condition on the tuples.

Remark 2.5.6. The cases e) and f) relate topological recursion to the theory of integrable systems and intersection theory on the moduli space of Riemann

APPLICATION	SPECTRAL CURVE
a) Enumeration of ribbon graphs [27]	see (2.5.12)
b) Enumeration of dessins d'enfants [26, 41]	$x = z + 1/z$ $y = z$
c) Simple/orbifold Hurwitz numbers [10, 26, 16, 22]	$x = z \exp(-z^a)$ $y = z^a$
d) Monotone Hurwitz numbers [21]	$x = (z-1)/z^2$ $y = -z$
e) Kontsevitch-Witten KdV $\tau-$function [32]	$x = z^2$ $y = z$
f) BGW KdV $\tau-$function [23]	$x = z^2$ $y = 1/z$
g) Weil-Peterson volumes of moduli spaces [33]	$x = z^2$ $y = \sin 2\pi z / 2\pi$
h) Stationary Gromov-Witten theory of $\mathbb{C}P^1$ [42]	$x = z + 1/z$ $y = \log(z)$
i) Gromov-Witten theory of toric CY3 [17, 34, 31]	mirror curves

Table 2.1: Applications of topological recursion.

surfaces $\overline{\mathcal{M}}_{g,n}$. This was initiated by Witten's conjecture [47] proved by Kontsevitch [37]. More generally it has been shown that any $\omega_{g,n}$ produced by topological recursion can be expanded in a basis where coefficients are given by some integral on $\overline{\mathcal{M}}_{g,n}$ [28]. In this context the "Airy" e) and "Bessel" f) curves capture the asymptotic behaviour of any such expansion close to the poles of $\omega_{g,n}$ [18]. The Airy and Bessel curves are the simplest spectral curves to deal with in the language of topological recursion and we shall get back to them later.

Remark 2.5.7. Application i) is in the context of topological string theory. Here topological recursion can be understood as a formalism to construct the B-model mirror to A-model topological string theory on toric Calabi-Yau threefolds. The mirror curve refers to the Hori-Vafa mirror [17].

Algebraic curve. Additionally when the spectral curve is the Riemann sphere $\mathbb{C}P^1$, we know that two meromorphic functions on $\mathbb{C}P^1$ always satisfy some algebraic equation. In the case of the Hermitian matrix model, we started with,

$$y^2 = M(x)(x-a)(x-b). \tag{2.5.13}$$

In particular, for the Gaussian case $V(x) = \frac{x^2}{2}$, [27] tells us $M(x) = 1/4$, $a = 2\sqrt{t}, b = -2\sqrt{t}$ and $u_j = \sqrt{t}\delta_{j,1}$. Then x and y reduce to

$$x(z) = \sqrt{t}\left(z + \frac{1}{z}\right) \tag{2.5.14}$$

$$y(z) = -\frac{1}{2}\sqrt{t}\left(z - \frac{1}{z}\right) \tag{2.5.15}$$

and the algebraic equation is

$$4y^2 - x^2 + 4t = 0. \tag{2.5.16}$$

Note that by setting all of $t_j = 0$ in the potential, we are losing the sum over ribbon graphs interpretation. However one can show that the Gaussian curve calculates Catalan numbers [27].

2.6 Quantum Curve

Let us now get back to the characteristic polynomial. First let us see how we can express it in terms of the resolvents. The usual relationship between traces and determinant $\det A = e^{\mathrm{Tr}\ln A}$ gives us

$$\begin{aligned}
&\langle \det(x - M)\rangle \\
&= \left\langle e^{\mathrm{Tr}\ln(x-M)}\right\rangle, && (2.6.1)\\
&= \exp\left(\mathrm{Tr}\int_\infty^x \left\langle \frac{dx'}{x'-M}\right\rangle\right), && (2.6.2)\\
&= \sum_{n=0}^{\infty}\frac{1}{k!}\int_\infty^x \cdots \int_\infty^x \left\langle \prod_{i=1}^{n}\mathrm{Tr}\,\frac{dx_i}{x_i - M}\right\rangle, && (2.6.3)\\
&= \exp\left(\sum_{n=1}^{\infty}\frac{1}{n!}\int_\infty^x \cdots \int_\infty^x \left\langle \prod_{i=1}^{n}\mathrm{Tr}\,\frac{dx_i}{x_i - M}\right\rangle_c\right), && (2.6.4)\\
&= \exp\left(\sum_{g=0}^{\infty}\sum_{n=1}^{\infty}\frac{(t/N)^{2g-2+n}}{n!}\int_\alpha^z \cdots \int_\alpha^z \omega_{g,n}\right.\\
&\qquad\left. + \frac{1}{(t/N)}\int_\alpha^z \frac{1}{2}V'(x_1)dx_1 - \frac{1}{2}\int_\alpha^z\int_\alpha^z \frac{dx_1 dx_2}{(x_1 - x_2)^2}\right), && (2.6.5)
\end{aligned}$$

where in (2.6.4) we are using the fact that the exponential of connected expectation values can be written in terms of the non-connected ones, and in (2.6.5) we inserted the genus expansion, and included extra terms according to definitions (2.5.2) and (2.5.3). Note that (2.6.3) is an equality specifically because $\frac{1}{x-M}$ is just a notation for the series expansion (2.3.17), for which integrating with base point at ∞ gives vanishing constant terms. Therefore, when passing to the differential forms $\omega_{g,n}$, we want to integrate with respect to some α being a pole of x. Finally, there will be some renormalization issues with the extra terms (2.6.5), which we shall discuss later in section 4.2.1 when doing actual calculations.

Now let us consider the special case of the Gaussian matrix model where $V(x) = \frac{1}{2}x^2$. In this case, it is known that the expectation value of the characteristic polynomial,

$$\langle \det(x-M) \rangle = \int_{H_N} \det(x-M) e^{-\frac{N}{2t}\operatorname{Tr} M^2} \tag{2.6.6}$$

is a monic orthogonal polynomial with respect to the measure $e^{-\frac{N}{2t}x^2}$ [39], and so it is a Hermite polynomial of degree N in the variable $x/\sqrt{t/N}$. Let us introduce the quantum parameter

$$\hbar = \frac{t}{N}. \tag{2.6.7}$$

Notice that, for fixed t, $N \to \infty$ corresponds to $\hbar \to 0$. We have,

$$\langle \det(x-M) \rangle = \hbar^{N/2} H_N(x/\sqrt{\hbar}) = \hbar^{N/2} H_N(\tilde{x}). \tag{2.6.8}$$

The Hermite polynomials satisfy the following differential equation,

$$H_N'' - \tilde{x} H_N' + N H_N = 0. \tag{2.6.9}$$

Now let us define the *wave function*

$$\psi(x) = e^{\frac{-x^2}{4\hbar}} \langle \det(x-M) \rangle, \tag{2.6.10}$$

where we have simply removed the factor associated to the potential in (2.6.5). After substituting in (2.6.9), we find

$$4\hbar^2 \psi(x)'' - (x^2 - 4t - 2\hbar)\psi(x) = 0. \tag{2.6.11}$$

Therefore we have shown that the wave function is annihilated by the canonical quantization of the spectral curve equation (2.5.16)

$$[4\hat{y}^2 - \hat{x}^2 + 4t + 2\hbar]\,\psi(x) = 0, \tag{2.6.12}$$

where $\hat{y} = \hbar\frac{d}{dx}$ and $\hat{x} = x$. In this case we obtain a "quantum correction term", $+2\hbar$, which vanishes in the classical limit $\hbar \to 0$.

This fact – the existence of a wave function constructed from the correlators and annihilated by a quantization of the spectral curve – is what is known as a *quantum curve* for topological recursion. For us, the quantum appellation will not have a rigorous physical meaning and we shall only use it as an analogy[3]. But there is something very interesting happening: We started with a problem related to quantum gravity, the enumeration of discrete graphs on a Riemann surface, and we find a natural construction that exhibits a behaviour similar to canonical quantization in quantum mechanics. Starting from a classical geometric object that is the spectral curve, we obtain a Schrödinger-like equation which, in the classical limit recovers the spectral curve.

This is perhaps simply a curiosity, or perhaps a hint that whatever we are manipulating here has in its mathematical core some ingredients of what a quantum theory should contain. It is therefore very exciting to try and understand why it is working in this way, and to see if for a general choice of spectral curve, the wave function constructed by topological recursion is also annihilated by a quantization of that spectral curve.

It has been observed in several cases, but the conditions for the existence of a quantum curve have yet to be precisely formulated.

2.7 Summary

Let us finish this section by briefly summarizing our observations and formulate the quantum curve conjecture.

The topological recursion produces a set of differential forms from the geometric data of a spectral curve (definition 2.5.1). These differential forms $\omega_{g,n}$, when expanded in a given basis of 1-forms, often produce the solution to some enumerative geometry problem. In the case of Hermitian matrix models, by construction we saw how we can obtain the enumeration of ribbon graphs

[3]Some authors have described quantum curves in a more physical language, in particular in relation to string theory [35, 20].

$\mathcal{T}^{(g)}_{l_1,\dots,l_n}$ by expanding the $\omega_{g,n}$ in the basis $\{dx_j/x_j^{l_j+1}\}$. However, there is more. Seen as defined on the base curve Σ (with corresponding z coordinate) it follows directly from the definition that such n-forms can only have poles are the zeroes of dx, and so the principal part of $\omega_{g,n}$ can be expanded in a basis of 1-forms with poles at each branch points $\{\xi_{k,r}\}_{k\geq 0, r\in\mathfrak{r}}$. We will return to that point in section 3.2. Different choices of basis can produce different enumerative invariants, for example there is a standard basis where the corresponding coefficients are given by integrals on the moduli space of curves [28].

On the other hand, in many cases the spectral curve is naturally associated to an algebraic equation

$$P(x,y) = 0. \tag{2.7.1}$$

This is true whenever Σ is a compact Riemann surface. From the correlators we can define a *wave function*,

$$\psi(x,\hbar) \propto \exp\left[\sum_{g\geq 0}\sum_{k\geq 1} \frac{\hbar^{2g-2+n}}{n!} \int_\alpha^z \cdots \int_\alpha^z \left(\omega_{g,n} - \delta_{g,0}\delta_{n,2}\frac{dx_1 dx_2}{(x_1-x_2)^2}\right)\right], \tag{2.7.2}$$

where $\propto$ indicates possible renormalization subtleties, and p is a point on the curve that is a pole of x. The quantum curve conjecture is then the statement that the wave function is annihilated by a quantization of the spectral curve

$$\hat{P}(\hat{x},\hat{y},\hbar)\,\psi(x,\hbar) = 0, \tag{2.7.3}$$

where $\hat{x} = x$, $\hat{y} = \hbar\frac{d}{dx}$, and $\hat{P}$ is such that in the classical limit, we have

$$\hat{P}(\hat{x},\hat{y},\hbar) \underset{\hbar\to 0}{\longrightarrow} P(x,y). \tag{2.7.4}$$

As we shall see, we will be able to obtain different quantum curves $(\hat{P}_\alpha, \psi_\alpha)$ depending on the choice of base point $\alpha \in \mathfrak{p}$.

There is a reason why the general statement of the quantum curve is still a conjecture: there are currently no algorithms to construct the quantum curve solely from the data of the spectral curve. As one can see in the baby example of the Gaussian curve, we used a property specific to that problem – that it was related to the Hermite differential equation – in order to find the quantization. What we propose to do next is to see whether we can exploit a recent reformulation of the topological recursion in terms of Airy structure, to produce such an algorithm.

3 Topological Recursion and Airy Structures

The goal of this section is to reformulate the topological recursion in the language of Airy structures, recently introduced by [38]. Airy structures are a generalization of the notion of Virasoro constraints, which have been known to exist for a long time in the context of matrix models, and can be thought of as a reformulation of the loop equations. See [27] or [15] for a summary. The difference now is that the notion of differential system which we need for quantum curves is no longer an emerging property but is taken as the starting point.

3.1 Airy Structures

About the name. Airy structures are named after George Airy who back in 1838 showed that the function,

$$Ai(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ipx} e^{i\frac{p^3}{3}} dp, \tag{3.1.1}$$

was giving the peaks of intensity of rainbows with respect to the angle between the sun and the observer [1, 25]. When Kontsevitch solved Witten's conjecture on the equivalence between two approaches of 2D quantum gravity [47, 37], he used the matrix Airy function, where variables x and p are replaced by Hermitian matrices. This yields a partition function for enumerating maps similar to the ones we outlined in section 2, which he managed to relate to the partition function of intersection theory on the moduli space of curves [19]. The Airy function satisfies the Airy differential equation,

$$\left\{ \frac{d^2}{dx^2} - x \right\} Ai(x) = 0, \tag{3.1.2}$$

which we will encounter again in section 4.2.

Definition. Let V be a $\mathbb{C}$-vector space. We denote its basis by $\hbar\partial_i$ and the dual basis by x_i, i.e. $[\hbar\partial_i, x_i] = \hbar$. Here $\hbar$ is a formal parameter. We denote by $\mathcal{W}_V$ the Weyl algebra of V, i.e. the algebra of differential operator in x_i with polynomial coefficients. We define a grading on $\mathcal{W}_V$ with $\deg(x_i) = \deg(\hbar\partial_i) = 1$ and $\deg(\hbar) = 2$. We will refer to operators being *quadratic* if they are of degree 2 according to this grading. We also denote by $\mathcal{W}_V(\leq 2)$ the sub-algebra of operators of degree at most 2 in x_i and $\hbar\partial_i$ (but arbitrary in $\hbar$). One can check that this forms a sub algebra of $\mathcal{W}_V$. A quantum Airy structure is given by a set of at most quadratic differential operators of the shape

$$L_i = \hbar\partial_i - \frac{1}{2}\mathrm{A}_{iab}x^a x^b - \hbar\mathrm{B}_{ia}^b x^a\partial_b - \frac{\hbar^2}{2}\mathrm{C}_i^{ab}\partial_a\partial_b - \hbar\mathrm{D}_i, \tag{3.1.3}$$

together with the constraint that they form a Lie algebra

$$[L_i, L_j] = \hbar f_{ij}^k L_k. \tag{3.1.4}$$

In particular it is a subalgebra of $\mathcal{W}_V(\leq 2)$. Here we use the convention that we are summing over repeated indices. The range of the indices $i, a, b \in \mathfrak{i}$ could be finite or infinite. In the infinite case and in the application that we will consider the sums will always be finite so that there is no issues of convergence. The condition (3.1.4) is equivalent to conditions on the coefficients ABCD [2]. In particular, that the coefficients A_{ijk} be fully symmetric

$$\mathrm{A}_{ijk} = \mathrm{A}_{jik}, \tag{3.1.5}$$

and that the following **BB-CA**, **BC**, **BA** and **D** relations hold

$$\mathrm{B}_{ij}^a\mathrm{B}_{ak}^l + \mathrm{B}_{ik}^a\mathrm{B}_{ja}^l + \mathrm{C}_i^{la}\mathrm{A}_{jak} = (i \leftrightarrow j), \tag{3.1.6}$$

$$\mathrm{B}_{ij}^a\mathrm{C}_a^{kl} + \mathrm{C}_i^{ka}\mathrm{B}_{ja}^l + \mathrm{C}_i^{la}\mathrm{B}_{ja}^k = (i \leftrightarrow j), \tag{3.1.7}$$

$$\mathrm{B}_{ij}^a\mathrm{A}_{akl} + \mathrm{B}_{ik}^a\mathrm{A}_{jal} + \mathrm{B}_{il}^a\mathrm{A}_{jak} = (i \leftrightarrow j), \tag{3.1.8}$$

$$\mathrm{B}_{ij}^a\mathrm{D}_a + \tfrac{1}{2}\mathrm{C}_i^{ab}\mathrm{A}_{jab} = (i \leftrightarrow j), \tag{3.1.9}$$

where by $(i \leftrightarrow j)$ we mean the same expression as the left hand side but with indices i and j exchanged. The structure constants are fully determined by the B_{ij}^k coefficients

$$f_{ij}^k = \mathrm{B}_{ij}^k - \mathrm{B}_{ji}^k. \tag{3.1.10}$$

Define the partition function to be the formal power series

$$Z = \exp\left(\sum_{g\geq 0} \sum_{n\geq 1}^{\circ\circ} \frac{\hbar^{g-1}}{n!} \sum_{i_1,\dots,i_n\in\mathfrak{i}} F_{g,n}[i_1,...,i_n]\, x_{i_1}...x_{i_n} \right), \tag{3.1.11}$$

where the sum $\sum^{\circ\circ}$ omits the terms $F_{0,1}$ and $F_{0,2}$, and the $F_{g,n}$ are assumed to be symmetric coefficients of the $i_1, \dots, i_n$. We have the following theorem:

Theorem 3.1.1. *[38] Given an quantum Airy structure, there exists a unique solution of the form* (3.1.11) *to the system of equations*

$$L_i \cdot Z = 0, \qquad \forall i \in \mathfrak{i}. \tag{3.1.12}$$

Proof. Uniqueness is based on the observation that the $F_{g,n}$ are uniquely determined by a recursion relation on $2g-2+n$. Applying the L_i to the right hand side of (3.1.11), collecting powers of $\hbar$, and then imposing the vanishing condition at all order yields the following base cases,

$$F_{0,3}[i,j,k] = \mathrm{A}_{ijk}, \qquad\qquad F_{1,1}[k] = \mathrm{D}_k, \tag{3.1.13}$$

and for $2g-2+n > 1$, the recursive relation

$$\begin{aligned} F_{g,n}[i_1, L] = \sum_{m=2}^{n} \mathrm{B}^a_{i_1 i_m} F_{g,n-1}[a, L\setminus\{i_m\}] + \frac{1}{2}\mathrm{C}^{ab}_{i_1} F_{g-1,n+1}[a,b,J] \\ + \frac{1}{2}\mathrm{C}^{ab}_{i_1} \sum_{h=0}^{g} \sum_{J\subset L}^{\circ\circ} F_{h,1+|J|}[a,J] F_{g-h,n-|J|}[b, L\setminus J], \end{aligned} \tag{3.1.14}$$

where sums over $a, b \in \mathfrak{i}$ are implied, and $\sum^{\circ\circ}$ excludes terms that would contain $F_{0,1}$ or $F_{0,2}$. Notice that each $F_{g',n'}$ on the right hand side has $2g'-2+n' < 2g-2+n$.

That would be it for existence as well if it weren't for the condition that the $F_{g,n}$ be symmetric in the indices $i_1, \dots, i_n$. This is a non trivial statement; when looking at the recursive formula (3.1.14) it seems that the first index i_1 is playing a special role. However we can work by induction and assume that all $F_{g',n'}$ on the right hand side are fully symmetric. The base cases $F_{1,1}$ and $F_{0,3}$ are certainly true, the first being trivial while the second is due to (3.1.5). After substituting formula (3.1.14) for all the $F_{g',n'}$ on the right hand side one finds that the $(i_1 \leftrightarrow i_2)$ symmetry for $F_{g,n}$ is exactly guaranteed by the 4 ABCD relations (3.1.6) - (3.1.9). Details are straightforward but tedious, readers may refer to [2, 6], or check them themselves ☺.

So far we have said nothing about the nature of the ABCD coefficients of the Airy structure or about the coefficients $F_{g,n}$ in the partition function, but we observed that from this simple setup we already obtain a recursion similar in structure to the topological recursion for spectral curves. In the next section we will make this correspondence explicit and we will say what the ABCD coefficients should be in order to recover the exact same recursion.

3.2 Relation with the Topological Recursion

We now want to rewrite the topological recursion for spectral curves (2.5.11) as a special case of the quantum Airy structure recursion relation. To do so we need to find a suitable vector space and basis. Following [2], suppose that there exists a basis of meromorphic 1-forms $d\xi_{k,r}$ indexed by some integer $k \in \mathbb{N}$, and branch point label $r \in \mathfrak{r}$, together with a set of germs of holomorphic 1-forms $d\xi^*_{k,r}$ defined locally at each branch point and such that,

$$\omega_{0,2}(p_0, p) \approx \sum_{\substack{r \in \mathfrak{r} \\ k \geq 0}} d\xi_{k,r}(p_0) d\xi^*_{k,r}(p). \tag{3.2.1}$$

Here the equality is only valid as a formal sum of the local behaviours for p around each $r \in \mathfrak{r}$. We also define germs of functions at each branch point,

$$\xi^*_{k,r}(p) = \int_r^p d\xi^*_{k,r}(p'). \tag{3.2.2}$$

We will show the following lemma:

Lemma 3.2.1 (Residue formula)**.** *There is a decomposition of the $\omega_{g,n}$ in the $d\xi_{k,r}$ basis with a finite number of non-zero terms,*

$$\omega_{g,n}(p_1, \ldots, p_n) = \sum_{\substack{r_1, \ldots, r_n \in \mathfrak{r} \\ k_1, \ldots, k_n \geq 0}} W_{g,n} \left[\begin{smallmatrix} r_1 & \cdots & r_n \\ k_1 & \cdots & k_n \end{smallmatrix} \right] \prod_{i=1}^{n} d\xi_{k_i, r_i}(p_i). \tag{3.2.3}$$

Furthermore the coefficients $W_{g,n}$ are given in terms of the recursion of a quan-

tum Airy structure (3.1.14) *from the data,*

$$\begin{aligned}
\mathcal{A}_{(k_1,r_1),(k_2,r_2),(k_3,r_3)} &= \operatorname*{Res}_{p\to r_1} \left[\xi^*_{k_1,r_1}(p) - \xi^*_{k_1,r_1}(\iota(p))\right] d\xi^*_{k_2,r_2}(p)\, d\xi^*_{k_3,r_3}(\iota(p))\, \tilde{\theta}(p),\\
\mathcal{B}^{(k_3,r_3)}_{(k_1,r_1),(k_2,r_2)} &= \operatorname*{Res}_{p\to r_1} \left[\xi^*_{k_1,r_1}(p) - \xi^*_{k_1,r_1}(\iota(p))\right] d\xi^*_{k_2,r_2}(p)\, d\xi_{k_3,r_3}(\iota(p))\, \tilde{\theta}(p),\\
\mathcal{C}^{(k_2,r_2),(k_3,r_3)}_{(k_1,r_1)} &= \operatorname*{Res}_{p\to r_1} \left[\xi^*_{k_1,r_1}(p) - \xi^*_{k_1,r_1}(\iota(p))\right] d\xi_{k_2,r_2}(p)\, d\xi_{k_3,r_3}(\iota(p))\, \tilde{\theta}(p),\\
\mathcal{D}_{(k,r)} &= \operatorname*{Res}_{p\to r_1} \left[\xi^*_{k,r}(p) - \xi^*_{k,r}(\iota(p))\right] \tfrac{1}{2}\, \omega_{0,2}(p,\iota(p))\, \tilde{\theta}(p).
\end{aligned} \tag{3.2.4}$$

Here, $\iota(p)$ is the involution that sends between the two sheets of x at the corresponding subscripted branch point, and we have defined for p in a neighbourhood of r_1,

$$\tilde{\theta}(p) = \frac{1}{\omega_{0,1}(p) - \omega_{0,1}(\iota(p))}. \tag{3.2.5}$$

Proof. Let us rewrite the recursion kernel in terms of the $d\xi$'s. Rewrite equation (3.2.1) as

$$\frac{1}{2}\sum_{\substack{r\in\mathfrak{r}\\ k\geq 0}} d\xi_{k,r}(p_0)\left[\xi^*_{k,r}(p) - \xi^*_{k,r}(\iota(p))\right] \approx \frac{1}{2}\left(\int_{\iota(p)}^{p} \omega_{0,2}(p_0,\cdot)\right). \tag{3.2.6}$$

Then we can write the kernel as

$$K(p_0,p) \approx \frac{1}{2}\sum_{\substack{k\geq 0\\ r\in\mathfrak{r}}} d\xi_{k,r}(p_0)\left[\xi^*_{k,r}(p) - \xi^*_{k_1,r_1}(\iota(p))\right] \tilde{\theta}(p). \tag{3.2.7}$$

This identity is, again, only true locally. However we can replace the recursion kernel with the above when taking residues with respect to $p \to r \in \mathfrak{r}$, in which case we drop the formal sum over $r \in \mathfrak{r}$.

All we have to do now is plug this expression in the formula for topological recursion (3.2.3). For base case $(g,n) = (0,3)$, we find

$$\begin{aligned}
\omega_{0,3}(p_1,p_2,p_3) = \sum_{r\in\mathfrak{r}} \operatorname*{Res}_{p\to r} K(p_1,p)\Big\{&\omega_{0,2}(p,p_2)\omega_{0,2}(\iota(p),p_3)\\
&+ \omega_{0,2}(p,p_3)\omega_{0,2}(\iota(p),p_2)\Big\}.
\end{aligned} \tag{3.2.8}$$

Since $K(p_1,p)$ is invariant under $p \to \iota(p)$ and taking the residue for $p \to r$ or

$\iota(p) \to r$ is the same thing, we get two terms with equal contributions.

$$\begin{aligned}
&\omega_{0,3}(p_1, p_2, p_3)\\
&= \sum_{r_1 \in \mathfrak{r}} \operatorname*{Res}_{p \to r_1} K(p_1, p) 2\omega_{0,2}(p, p_2)\omega_{0,2}(\iota(p), p_3), && (3.2.9)\\
&= \sum_{\substack{k_1,k_2,k_3 \geq 0\\ r_1,r_2,r_3 \in \mathfrak{r}}} \prod_{i=1}^{3} d\xi_{k_i,r_i}(p_i) && (3.2.10)\\
&\qquad \operatorname*{Res}_{p \to r_1} \left[\xi^*_{k_1,r_1}(p) - \xi^*_{k_1,r_1}(\iota(p))\right] d\xi^*_{k_2,r_2}(p) d\xi^*_{k_3,r_3}(\iota(p))\tilde{\theta}(p),
\end{aligned}$$

proving that (3.2.3) holds for $\omega_{0,3}$ with

$$W_{0,3}\left[\begin{smallmatrix} r_1 & r_2 & r_3 \\ k_1 & k_2 & k_3 \end{smallmatrix}\right] = \mathcal{A}_{(k_1,r_1),(k_2,r_2),(k_3,r_3)}. \tag{3.2.11}$$

Now turning to $\omega_{1,1}$,

$$\begin{aligned}
&\omega_{1,1}(p_1),\\
&= \sum_{r \in \mathfrak{r}} \operatorname*{Res}_{p \to r} K(p_1, p)\omega_{0,2}(p, \iota(p)), && (3.2.12)\\
&= \sum_{\substack{r \in \mathfrak{r}\\ k \geq 0}} \operatorname*{Res}_{p \to r} \left[\xi^*_{k,r}(p) - \xi^*_{k,r}(\iota(p))\right] \frac{1}{2}\omega_{0,2}(p, \iota(p))\tilde{\theta}(p) d\xi_{k,r}(p_1), && (3.2.13)
\end{aligned}$$

proving that (3.2.3) holds for $\omega_{1,1}$ with

$$W_{1,1}\left[\begin{smallmatrix} r \\ k \end{smallmatrix}\right] = \mathcal{D}_{(k,r)}. \tag{3.2.14}$$

For higher topologies we proceed by induction. Consider $2g - 2 + n > 1$ and assume that (3.2.3) holds for all (g', n') such that $2g' - 2 + n' < 2g - 2 + n$. We write $I = \{p_1, ..., p_n\}$. We first want to consider the terms from the sum $\sum^{\circ}$ that involve $\omega_{0,2}\omega_{g,n-1}$. Denote their contribution by $\omega^B_{g,n}$,

$$\begin{aligned}
\omega^B_{g,n}(p_1, I) = \sum_{j=2}^{n} \sum_{r \in \mathfrak{r}} \operatorname*{Res}_{p \to r} K(p_1, p) \Big[&\omega_{0,2}(p, p_j)\omega_{g,n-1}(\iota(p), I \setminus \{p_j\})\\
&+ \omega_{0,2}(\iota(p), p_j)\omega_{g,n-1}(p, I \setminus \{p_j\})\Big].
\end{aligned} \tag{3.2.15}$$

Again both terms give equal contributions, and we insert the decomposition

(3.2.3) for $\omega_{g,n-1}$

$$\begin{aligned}
&\omega^B_{g,n}(p_1, I) \\
&= \sum_{\substack{r_1,r_2,\dots,r_n\in\mathfrak{r}\\ k_1,k_2,\dots,k_n\geq 0}} \prod_{i=1}^{n} d\xi_{k_i,r_i}(p_i) \\
&\qquad \sum_{j=2}^{n}\sum_{\substack{r\in\mathfrak{r}\\ k\geq 0}} \operatorname*{Res}_{p\to r_1}\left[\xi^*_{k_1,r_1}(p) - \xi^*_{k_1,r_1}(\iota(p))\right] d\xi^*_{k_j,r_j}(p) d\xi_{k,r}(\iota(p))\tilde{\theta}(p) \\
&\qquad\qquad W_{g,n-1}\begin{bmatrix} r & r_2 & : & \hat{r}_j & : & r_n \\ k & k_2 & : & \hat{k}_j & : & k_n \end{bmatrix}.
\end{aligned} \tag{3.2.16}$$

Let us turn to the remaining terms,

$$\begin{aligned}
\omega^C_{g,n}(p_1, I) = \sum_{r_1\in\mathfrak{r}} \operatorname*{Res}_{p\to r_1} K(p_1, p)\Big[&\omega_{g-1,n+1}(p, \iota(p), I) \\
&+ \sum^{\circ\circ}_{\substack{g'+g''=g\\ J'\sqcup J''=I}} \omega_{g',1+|J'|}(p, J')\omega_{g'',1+|J''|}(\iota(p), J'')\Big].
\end{aligned} \tag{3.2.17}$$

This time the induction hypothesis applies to each $\omega_{g,n}$ on the right hand side, so we can insert the corresponding (3.2.3) expansion.

$$\begin{aligned}
&\omega^C_{g,n}(p_1, I) \\
&= \frac{1}{2} \sum_{\substack{r_1,r_2;r_n\in\mathfrak{r}\\ k_1,k_2;k_n\geq 0}} \prod_{i=1}^{n} d\xi_{k_i,r_i}(p_i) \\
&\sum_{\substack{k',k''\geq 0\\ r',r''\in\mathfrak{r}}} \operatorname*{Res}_{p\to r_1}\left[\xi^*_{k_1,r_1}(p) - \xi^*_{k_1,r_1}(\iota(p))\right] d\xi_{k',r'}(p) d\xi_{k'',r''}(\iota(p))\tilde{\theta}(p) \\
&\qquad \Bigg\{ W_{g-1,n+1}\begin{bmatrix} r' & r'' & r_2 & : & r_n \\ k' & k'' & k_2 & : & k_n \end{bmatrix} \\
&\qquad\qquad + \sum^{\circ\circ}_{\substack{g'+g''=g\\ J'\sqcup J''=I}} W_{g',1+|J'|}\begin{bmatrix} r' & (r_j)_{j\in J'} \\ k' & (k_j)_{j\in J'} \end{bmatrix} W_{g'',1+|J''|}\begin{bmatrix} r'' & (r_j)_{j\in J''} \\ k'' & (k_j)_{j\in J''} \end{bmatrix} \Bigg\}.
\end{aligned} \tag{3.2.18}$$

Summing both $\omega^B_{g,n}$ and $\omega^C_{g,n}$ contributions we get

$$
\begin{aligned}
& W_{g,n}\left[\begin{smallmatrix} r_1 & \cdots & r_n \\ k_1 & \cdots & k_n \end{smallmatrix}\right] \\
& = \sum_{i=2}^{n} \sum_{k',r'} \mathcal{B}^{(k',r')}_{(k_1,r_1),(k_i,r_i)} W_{g,n-1}\left[\begin{smallmatrix} r' & r_2 & \cdots & \hat{r}_i & \cdots & r_n \\ k' & k_2 & \cdots & \hat{k}_i & \cdots & k_n \end{smallmatrix}\right] \\
& \quad + \frac{1}{2} \sum_{\substack{k',k''\geq 0 \\ r',r''\in\mathfrak{r}}} \mathcal{C}^{(k',r'),(k'',r'')}_{(k_1,r_1)} \Bigg(W_{g-1,n+1}\left[\begin{smallmatrix} r' & r'' & r_2 & \cdots & r_n \\ k' & k'' & k_2 & \cdots & k_n \end{smallmatrix}\right] \\
& \quad + \sum_{\substack{g'+g''=g \\ J'\sqcup J''=I}}^{\infty} W_{g',1+|J'|}\left[\begin{smallmatrix} r' & (r_j)_{j\in J'} \\ k' & (k_j)_{j\in J'} \end{smallmatrix}\right] W_{g'',1+|J''|}\left[\begin{smallmatrix} r'' & (r_j)_{j\in J''} \\ k'' & (k_j)_{j\in J''} \end{smallmatrix}\right] \Bigg).
\end{aligned}
\tag{3.2.19}
$$

We see that we recover the same recursive structure as the $F_{g,n}$ from a quantum Airy structure (3.1.14).
Finally the fact that there are only finitely many non-zero terms is a direct consequence of the fact that the $\omega_{g,n}$'s are meromorphic, and that the poles must have finite order.

Wave function from the partition function

As a consequence of this lemma we can now relate the partition function of the quantum Airy structure with the wave function (2.7.2) of topological recursion. First we separate contributions of unstable topologies,

$$
\begin{aligned}
& \psi(x,\hbar) \\
& = \exp\left[\sum_{g\geq 0}\sum_{n\geq 1} \frac{\hbar^{2g-2+k}}{n!} \int_\alpha^z \cdots \int_\alpha^z \left(\omega_{g,n} - \delta_{g,0}\delta_{n,2}\frac{dx_1 dx_2}{(x_1-x_2)^2}\right)\right] \qquad (3.2.20) \\
& = \exp\left[\frac{1}{\hbar}\int_\alpha^z \omega_{0,1}\right] \exp\left[\frac{1}{2}\int_\alpha^z\int_\alpha^z \left(\omega_{0,2} - \frac{dx_1 dx_2}{(x_1-x_2)^2}\right)\right] \\
& \qquad \exp\left[\sum_{g\geq 0}\sum_{n\geq 1}^{\infty} \frac{\hbar^{2g-2+n}}{n!} \int_\alpha^z \cdots \int_\alpha^z \omega_{g,n}\right]. \qquad (3.2.21)
\end{aligned}
$$

Now insert the expansion (3.2.3) in the last factor and get

$$
\exp\left[\sum_{g\geq 0}\sum_{n\geq 1}^{\infty} \frac{\hbar^{2g-2}}{n!} \sum_{r\text{'s},k\text{'s}} W_{g,n}\left[\begin{smallmatrix} r_1 & \cdots & r_n \\ k_1 & \cdots & k_n \end{smallmatrix}\right] \prod_{i=1}^{n} \int_\alpha^z \hbar d\xi_{k_i,r_i}\right], \tag{3.2.22}
$$

which, knowing the recursion relation on the $W_{g,n}$, we recognize as the partition function Z of a quantum Airy structure (3.1.11), up to the change $\hbar \to \hbar^2$, and with the identification of variables $x_{k,r} = \int_\alpha^z \hbar d\xi_{k,r}$.

Now we would like to say that this partition function is annihilated by the operators $L_{k,r}$ to be defined from the coefficients ABCD (3.2.4) of the previous lemma; with this in hand we would have a very good base for our quantum curve equation soup. Before we make that step however we have to check that the coefficients (3.2.4) satisfy the ABCD relations; only then will we know that there exists a unique solution to $L_{k,r} \cdot Z = 0$, being precisely given by (3.2.22).

Therefore the next step in our reasoning consists in two points: First, given a spectral curve, can we show that the residue formulas (3.2.4) produces a quantum Airy structure? Second, can we relate the quantum curve operator to the $L_{k,r}$?

3.3 Choice of Polarization

We will now give an explicit basis of 1-forms which satisfies the hypotheses of the residue lemma. This is known as a choice of polarization. We will first do this in a special case of the Airy spectral curve. This is the spectral curve that relates to the Kontsevitch matrix model in 2D topological gravity [32]. It is also important because it captures the asymptotic behaviour of a large class of double sheeted spectral curves, which we will treat in section 3.3.3.

3.3.1 Odd Basis of Differentials

In the residue formula of the previous section (3.2.4), we see that there is a simplification to do when the dual germs $\xi^*_{k,r}$ have some parity under the involution. On $\mathbb{C}P^1$, we can expand the Bergman kernel at 0 as follows:

$$\omega_{0,2}(z_1, z_2) = \frac{dz_1 dz_2}{(z_1 - z_2)^2} \tag{3.3.1}$$

$$\approx \sum_{k \geq 0} k z_1^{-k-1} dz_1 \, z_2^{k-1} dz_2, \qquad |z_2| < |z_1|, \tag{3.3.2}$$

and so we can pick the basis to be

$$d\xi_k(z) = k z^{-k-1} dz, \qquad\qquad d\xi^*_k(z) = z^{k-1} dz. \tag{3.3.3}$$

With this choice, and if the involution at 0 is given by $\iota(z) = -z$, we see that

$$\xi_k^*(p) - \xi_k^*(\iota(p)) = (1 - (-1)^k)\xi_k^*(p) \tag{3.3.4}$$

This means that whenever k_1 is even, the corresponding ABCD coefficients are all zero, and the Airy structure partition function is annihilated by the operators

$$L_{2i} = \hbar\partial_{2i} \tag{3.3.5}$$

$$L_{2i+1} = \hbar\partial_{2i+1} - \text{quadratic} \tag{3.3.6}$$

And so $\partial_{2i}Z = 0$, meaning that *the partition function does not depend on the even indexed variables* x_{2i}. So even though the ABCD coefficients of (3.2.4) with even k_2, k_3 indices may not vanish, we can safely ignore them and remove the ABCD coefficients with even indices from the remaining odd L_i's (3.3.6). Indeed, the even derivatives disappear and the remaining x_{2i}'s can be set to zero without changing Z and without changing $L_{2i+1}Z = 0$. Therefore, *it is enough to consider only the odd indices* in the basis, and redefine

$$d\xi_k(z) = \frac{(2k+1)}{z^{2k+2}}dz, \qquad d\xi_k^*(z) = z^{2k+1}dz, \tag{3.3.7}$$

as well as redefine the residue formula,

$$\begin{aligned}
A_{(k_1,r_1),(k_2,r_2),(k_3,r_3)} &= \operatorname*{Res}_{p\to r_1} \xi^*_{k_1,r_1}(p)\, d\xi^*_{k_2,r_2}(p)\, d\xi^*_{k_3,r_3}(p)\, \theta(p), \\
B^{(k_3,r_3)}_{(k_1,r_1),(k_2,r_2)} &= \operatorname*{Res}_{p\to r_1} \xi^*_{k_1,r_1}(p)\, d\xi^*_{k_2,r_2}(p)\, d\xi_{k_3,r_3}(p)\, \theta(p), \\
C^{(k_2,r_2),(k_3,r_3)}_{(k_1,r_1)} &= \operatorname*{Res}_{p\to r_1} \xi^*_{k_1,r_1}(p)\, d\xi_{k_2,r_2}(p)\, d\xi_{k_3,r_3}(p)\, \theta(p), \\
D_{(k,r)} &= \operatorname*{Res}_{p\to r_1} \xi^*_{k,r}(p)(-\tfrac{1}{2})\, \omega_{0,2}(p,\iota(p))\, \theta(p),
\end{aligned} \tag{3.3.8}$$

where we have absorbed a factor -2 in θ for convenience,

$$\theta(p) = \frac{-2}{\omega_{0,1}(p) - \omega_{0,1}(\iota(p))}. \tag{3.3.9}$$

In the rest of this thesis we will only be working with the coefficients (3.3.8), and leave as an open question to see if there exists other kind of involutions which one can work with effectively in this setup. See section 4.2.5 for some interesting curve with non "trivial" involution.

3.3.2 Airy curve

The Airy spectral curve is a genus zero curve given by the following data

$$\Sigma = \mathbb{C}P^1, \qquad x(z) = \frac{1}{2}z^2, \qquad y(z) = z. \tag{3.3.10}$$

The corresponding algebraic equation is,

$$P(x,y) = \frac{1}{2}y^2 - x = 0. \tag{3.3.11}$$

This curve has a simple ramification point at $z = 0$, with the involution locally given by $\iota(z) = -z$, and so we can use the $d\xi$ basis in (3.3.7).

The last thing we need is the expansion of $\theta(z)$. For the Airy curve we have $\omega_{0,1} = ydx = z^2dz$ and

$$\theta(z) = -\frac{1}{z^2}(dz)^{-1}. \tag{3.3.12}$$

However, we will be generalizing shortly and so we want to consider a full series expansion,

$$\theta(z) = \sum_{m\geq -1} \theta_m z^{2m}(dz)^{-1} \tag{3.3.13}$$

Remark 3.3.1. You can check that this particular expansion is the most general given the assumptions we made on dy in the definition of TR. Indeed, suppose that $x = \frac{1}{2}z^2$ and that $y(z)$ has a full series expansion. Since $y(z)$ is meromorphic the Laurent series is bounded from below,

$$y(z) = \sum_{k=d}^{\infty} A_k z^k \tag{3.3.14}$$

However if $y(z)$ has a pole of order greater than 1 at some ramification point, then the kernel $K(z_0, z)$ will have no pole there, the corresponding residue in the TR formula will vanish, and that branch point won't contribute to the recursion. Therefore we have

$$y(z) = \sum_{k=-1}^{\infty} A_k z^k, \tag{3.3.15}$$

But since $dy \neq 0$ at $z = 0$, either A_{-1} or A_1 is non zero, ensuring that the expansion of θ is bounded from below as in (3.3.13).
To resonate with remark 2.5.3, we shall see that the ABCD relations hold here precisely because the expansion is so bounded from below, so that the $W_{g,n}$'s, and the $\omega_{g,n}$'s, are fully symmetric.

Residue formula

All we have to do now is plug this data into the residue formula (3.3.8). Let us proceed,

$$\begin{aligned}
\mathrm{A}_{ijk} &= \operatorname*{Res}_{z=0} \xi_i^*(z) d\xi_j^*(z) d\xi_k^*(z)\theta(z) \\
&= \frac{1}{(2i+1)} \operatorname*{Res}_{z=0} \sum_{m\geq -1} \theta_m z^{2i+2j+2k+2m+1} dz \\
&= \frac{1}{(2i+1)} \delta_{i,j,k,0}\theta_{-1},
\end{aligned}$$

$$\begin{aligned}
\mathrm{B}_{ij}^k &= \operatorname*{Res}_{z=0} \xi_i^*(z) d\xi_j^*(z) d\xi_k(z)\theta(z) \\
&= \frac{(2k+1)}{(2i+1)} \operatorname*{Res}_{z=0} \sum_{m\geq -1} \theta_m z^{2i+2j-2k+2m-1} dz \\
&= \frac{(2k+1)}{(2i+1)} \theta_{k-i-j},
\end{aligned}$$

$$\begin{aligned}
\mathrm{C}_i^{jk} &= \operatorname*{Res}_{z=0} \xi_i^*(z) d\xi_j(z) d\xi_k(z)\theta(z) \\
&= \frac{(2j+1)(2k+1)}{(2i+1)} \operatorname*{Res}_{z=0} \sum_{m\geq -1} \theta_m z^{2i-2j-2k+2m-3} dz \\
&= \frac{(2j+1)(2k+1)}{(2i+1)} \theta_{k+j-1+1},
\end{aligned}$$

and,

$$\begin{aligned}
\mathrm{D}_i &= \operatorname*{Res}_{z=0} \frac{1}{2} \xi_{k,r}^*(p)\omega_{0,2}(p,\iota(p))\theta(p) \\
&= \frac{1}{8(2i+1)} \operatorname*{Res}_{z=0} \sum_{m\geq -1} \theta_m z^{2i+2m-1} dz \\
&= \delta_{i,0}\frac{\theta_0}{8} + \delta_{i,1}\frac{1}{24}\theta_{-1}.
\end{aligned}$$

We then show that these ABCD coefficients indeed satisfy the conditions for an Airy Structure.

Proposition 3.3.2. *The reduced ABCD coefficients obtained from the residue* *formula* (3.3.8) *with one branch point at* $r = 0$ *under the involution* $\iota(z) = -z$,

$$
\begin{aligned}
\mathrm{A}_{ijk} &= \delta_{i,j,k,0}\theta_{-1}, \\
\mathrm{B}_{ij}^{k} &= \frac{(2k+1)}{(2i+1)}\theta_{k-i-j}, \\
\mathrm{C}_{i}^{jk} &= \frac{(2j+1)(2k+1)}{(2i+1)}\theta_{1+j+k-i}, \\
\mathrm{D}_{k} &= \delta_{k,0}\frac{\theta_0}{8} + \delta_{k,1}\frac{\theta_{-1}}{24},
\end{aligned}
\tag{3.3.16}
$$

where $i, j, k \in \mathbb{N}$ *and* $\theta_m = 0$ *for* $m < -1$, *satisfy the* ***ABCD*** *relations* (3.1.5) – (3.1.9).

Proof. We shall check explicitly that each expression is symmetric under the exchange $i \leftrightarrow j$:

1 $\mathrm{A}_{ijk} \propto \delta_{i,j,k,0}$ is indeed fully symmetric.

2 The **BA** relation yields

$$
\begin{aligned}
&\sum_a \left(\mathrm{B}_{ij}^{a}\mathrm{A}_{akl} + \mathrm{B}_{ik}^{a}\mathrm{A}_{jal} + \mathrm{B}_{il}^{a}\mathrm{A}_{jak}\right) \\
&\qquad = \frac{\theta_{-1}}{(2i+1)}\left(\delta_{k,l,0}\theta_{-i-j} + \delta_{j,l,0}\theta_{-i-k} + \delta_{j,k,0}\theta_{-j-l}\right),
\end{aligned}
\tag{3.3.17}
$$

which vanishes for almost all values of (i, j, k, l) except

$$
\begin{aligned}
&(1,0,0,0) \qquad \left(\frac{1}{3} + \frac{1}{3} + \frac{1}{3}\right)(\theta_{-1})^2 = (\theta_{-1})^2 \\
&(0,1,0,0) \qquad (\theta_{-1})^2.
\end{aligned}
$$

Of course, there is nothing to check when $i = j$.

3 The **D** relation yields

$$
\begin{aligned}
&\sum_a \mathrm{B}_{ij}^{a}\mathrm{D}_a + \tfrac{1}{2}\sum_{a,b}\mathrm{C}_{i}^{ab}\mathrm{A}_{jab} \\
&= \frac{1}{8}\frac{1}{(2i+1)}\theta_{1-i-j}\theta_{-1} + \frac{1}{8}\frac{1}{(2i+1)}\theta_0\theta_{-i-j} + \frac{1}{2}\theta_{-1}\theta_{1-i}\delta_{j,0}\frac{1}{(2i+1)}.
\end{aligned}
\tag{3.3.18}
$$

This will vanish for most (i, j) pairs and we only need to check the following:

$$\begin{aligned}
&(1,0) \quad \left(\frac{1}{24}+\frac{1}{24}+\frac{1}{6}\right)\theta_0\theta_{-1}=\frac{1}{4}\theta_0\theta_{-1},\\
&(0,1) \quad \left(\frac{1}{8}+\frac{1}{8}\right)\theta_0\theta_{-1}=\frac{1}{4}\theta_0\theta_{-1},\\
&(2,0) \quad \left(\frac{1}{40}+\frac{1}{10}\right)(\theta_{-1})^2=\frac{1}{8}(\theta_{-1})^2,\\
&(0,2) \quad \frac{1}{8}(\theta_{-1})^2.
\end{aligned}$$

4 The **BB-AC** relation gives

$$\begin{aligned}
&\sum_a \left(\mathrm{B}^a_{ij}\mathrm{B}^l_{ak}+\mathrm{B}^a_{ik}\mathrm{B}^l_{ja}+\mathrm{C}^{la}_i\mathrm{A}_{jak}\right)\\
&=\frac{(2l+1)}{(2i+1)(2j+1)}\\
&\left(\sum_{a=0}\theta_{a-i-j}\theta_{l-a-k}(2j+1)+\sum_{a=0}\theta_{a-i-k}\theta_{l-j-a}(2a+1)\delta_{j,k,0}\theta_{-1}\theta_{l+1-i}\right).
\end{aligned} \tag{3.3.19}$$

First, we divide out by the symmetric factor $\frac{(2l+1)}{(2i+1)(2j+1)}$. Next we write down explicitly the bounds on the sums, knowing that $\theta_k = 0$ for $k < -1$. Doing so we also need extra terms to ensure the sums on the index a start at 0 and not -1.

$$\begin{aligned}
=&\sum_{a=i+j-1}^{l-k+1}\theta_{a-i-j}\theta_{l-a-k}(2j+1)+\sum_{a=i+k-1}^{l-j+1}\theta_{a-i-k}\theta_{l-j-a}(2a+1)\\
&+\delta_{j,k,0}\theta_{-1}\theta_{1+l-i}-\delta_{i,j,0}\theta_{-1}\theta_{l-k+1}+\delta_{i,k,0}\theta_{-1}\theta_{1+l-j}
\end{aligned} \tag{3.3.20}$$

The last three terms are symmetric, and after relabelling $a \to a - j + k$ in the second sum we get

$$= \sum_{a=i+j-1}^{l-k+1} \theta_{a-i-j}\theta_{l-a-k}(2j+1) + \sum_{a=i+j-1}^{l-k+1} \theta_{a-i-j}\theta_{l-a-k}(2a-2j+2k+1) \tag{3.3.21}$$

$$= \sum_{a=i+j-1}^{l-k+1} \theta_{a-i-j}\theta_{l-a-k}(2a+2k+2), \tag{3.3.22}$$

which is now explicitly symmetric under $(i \leftrightarrow j)$.

5 The **BC** relation rewrites as

$$\sum_a \left(\mathrm{B}_{ij}^a \mathrm{C}_a^{kl} + \mathrm{C}_i^{ka}\mathrm{B}_{ja}^l + \mathrm{C}_i^{la}\mathrm{B}_{ja}^k\right) \tag{3.3.23}$$

$$\begin{aligned} = \sum_{a=0} \Bigg(& \frac{(2a+1)}{(2i+1)}\theta_{a-i-j}\frac{(2k+1)(2l+1)}{(2a+1)}\theta_{1+k+l-a} \\ & + \frac{(2k+1)(2a+1)}{(2i+1)}\theta_{1+k+a-i}\frac{(2l+1)}{(2j+1)}\theta_{l-j-a} \\ & + \frac{(2l+1)(2a+1)}{(2i+1)}\theta_{1+l+a-i}\frac{(2k+1)}{(2j+1)}\theta_{k-j-a}\Bigg). \end{aligned} \tag{3.3.24}$$

Again our first step is to divide by the symmetric factor $\frac{(2k+1)(2l+1)}{(2i+1)(2j+1)}$, then write bounds on the sums,

$$= \sum_{a=i+j-1}^{k+l+2} (2j+1)\theta_{a-i-j}\theta_{1+k+l-a} - \delta_{i,j,0}\theta_{-1}\theta_{2+k+l} \tag{3.3.25}$$

$$+ \sum_{a=i-k-2}^{l-j+1} (2a+1)\theta_{1+k+a-i}\theta_{l-j-a} - \sum_{a=i-k-2}^{-1} (2a+1)\theta_{1+k+a-i}\theta_{l-j-a} \tag{3.3.26}$$

$$+ \sum_{a=i-l-2}^{k-j+1} (2a+1)\theta_{1+l+a-i}\theta_{k-j-a} - \sum_{a=i-l-2}^{-1} (2a+1)\theta_{1+l+a-i}\theta_{k-j-a}. \tag{3.3.27}$$

The second term is symmetric. Focusing our attention on the three left terms, relabelling $a \to a - k - j - 1$ in (3.3.26) and $a \to i + k - a$ in (3.3.27), we find,

$$\begin{aligned}
&\sum_{a=i+j-1}^{k+l+2} (2j+1)\theta_{a-i-j}\theta_{1+k+l-a} \\
&+ \sum_{a=i+j-1}^{k+l+2} (2a-2k-2j-1)\theta_{1+k+a-i}\theta_{1+k+l-a} \qquad (3.3.28) \\
&+ \sum_{a=i+j-1}^{k+l+2} (2i+2k-2a+1)\theta_{1+l+a-i}\theta_{1+k+l-a} \\
&= \sum_{a=i+j-1}^{k+l+2} (2i+1)\theta_{a-i-j}\theta_{1+k+l-a}, \qquad (3.3.29)
\end{aligned}$$

which is $(i \leftrightarrow j)$ of the first left term in (3.3.25). Now turning to the remaining two bottom right terms in (3.3.26) and (3.3.27), relabelling $a \to -a - 1$ in both while subtracting the original sums where $(i \leftrightarrow j)$,

$$\begin{aligned}
&\sum_{a=0}^{k-i+1} (2a+1)\theta_{k-i-a}\theta_{1+a+l-j} \sum_{a=0}^{l-i+1} (2a+1)\theta_{l-i-a}\theta_{1+a+k-j} \\
&+ \sum_{a=j-k-2}^{-1} (2a+1)\theta_{1+k+a-j}\theta_{l-i-a} + \sum_{a=j-l-2}^{-1} (2a+1)\theta_{1+l+a-j}\theta_{k-i-a} \qquad (3.3.30) \\
&= \sum_{a=j-k-2}^{l-i+1} (2a+1)\theta_{1+k+a-j}\theta_{l-i-a} + \sum_{a=j-l-2}^{k-i+1} (2a+1)\theta_{1+l+a-j}\theta_{k-i-a}, \qquad (3.3.31)
\end{aligned}$$

which is $(i \leftrightarrow j)$ of the two bottom left terms in (3.3.26) and (3.3.27). More concisely we have showed

$$\begin{aligned}
\mathrm{X}_{i,j} + \mathrm{Y}_{i,j} &= \mathrm{X}_{j,i}, \\
\mathrm{Z}_{i,j} - \mathrm{Z}_{j,i} &= \mathrm{Y}_{j,i}, \\
\Rightarrow \mathrm{X}_{i,j} + \mathrm{Y}_{i,j} + \mathrm{Z}_{i,j} &= \mathrm{X}_{j,i} + \mathrm{Y}_{j,i} + \mathrm{Z}_{j,i}.
\end{aligned}$$

In the next section we will apply this result to quantize the Airy spectral curve. First let us see how we can generalize it to an arbitrary spectral curve.

3.3.3 Local basis

For completeness, in this section we shall outline an possible approach to dealing with any spectral curve with simple ramification point, as a generalization of the Airy curve.
While we show that we can construct an Airy structure for any spectral curve, out of which we could also potentially build a quantum curve, in practice the expression we would get are too complicated to use the same techniques, more precisely the coefficients of the corresponding Airy structure, as well as the specialization map, both introduce series expansions in the expressions we are dealing with. We expect that these series should somehow compensate each other, but it remains unclear how to proceed.

The starting point is the following: for an arbitrary spectral curve with simple ramification points, we can always find local coordinates ζ such that in a neighbourhood of the ramification points we can write,

$$x(\zeta) = x(r) + \frac{1}{2}\zeta^2. \tag{3.3.32}$$

With the involution again locally given as $\iota(\zeta) = -\zeta$. By introducing this local coordinate however we change the Bergman kernel to potentially have a full series expansion

$$\omega_{0,2}(p_1, p_2) = \delta_{r_1,r_2} \frac{d\zeta_1 d\zeta_2}{(\zeta_1 - \zeta_2)^2} + \sum_{l_1,l_2 \geq 0} \varphi \left[\begin{smallmatrix} r_1 & r_2 \\ l_1 & l_2 \end{smallmatrix} \right] \zeta_1^{l_1} \zeta_2^{l_2} d\zeta_1 d\zeta_2. \tag{3.3.33}$$

Following the same lines as before, we find the $d\xi_{k,r}$ basis in local coordinate to be,

$$d\xi_{k,r}(p_0) = \left(\frac{(2k+1)}{\zeta_0^{2k+2}} \delta_{r,r_0} + \sum_{l \geq 0} \varphi \left[\begin{smallmatrix} r & r_0 \\ 2k & l \end{smallmatrix} \right] \zeta_0^l \right) d\zeta_0, \tag{3.3.34}$$

$$\xi^*_{k,r}(p_0) = \delta_{r,r_0} \frac{\zeta_0^{2k+1}}{2k+1}. \tag{3.3.35}$$

Here δ_{r,r_0} is equal to 1 if p_0 is in the neighbourhood of r and 0 otherwise. Finally, after inserting this into the residue formula we find the ABCD coefficients,

$$\begin{aligned}
A_{(k_1,r_1),(k_2,r_2),(k_3,r_3)} &= \delta_{r_1,r_2,r_3}\delta_{k_1,k_2,k_3,0}\,\theta_{-1,r},\\
B^{(k_3,r_3)}_{(k_1,r_1),(k_2,r_2)} &= \tfrac{(2k_3+1)}{(2k_1+1)}\,\delta_{r_1,r_2,r_3}\theta_{k_3-k_2-k_1,r_1}\\
&\quad + \delta_{r_1,r_2}\tfrac{1}{(2k_1+1)}\delta_{k_1,k_2,0}\,\varphi\left[\begin{smallmatrix} r_3 & r_1\\ 2k_3 & 0\end{smallmatrix}\right]\theta_{-1,r_1},\\
C^{(k_2,r_2),(k_3,r_3)}_{(k_1,r_1)} &= \tfrac{(2k_3+1)(2k_2+1)}{(2k_1+1)}\delta_{r_1,r_2,r_3}\,\theta_{1+k_2+k_3-k_1,r_1}\\
&\quad + \tfrac{(2k_3+1)}{(2k_1+1)}\sum_{m=0}^{1+k_3-k_1}\delta_{r_1,r_3}\,\varphi\left[\begin{smallmatrix} r_2 & r_1\\ 2k_2 & 2m\end{smallmatrix}\right]\theta_{k_3-k_1-m,r_1} \qquad (3.3.36)\\
&\quad + \tfrac{(2k_2+1)}{(2k_1+1)}\sum_{m=0}^{1+k_2-k_1}\delta_{r_1,r_2}\,\varphi\left[\begin{smallmatrix} r_3 & r_1\\ 2k_3 & 2m\end{smallmatrix}\right]\theta_{k_2-k_1-m,r_1}\\
&\quad + \delta_{k_1,0}\,\varphi\left[\begin{smallmatrix} r_2 & r_1\\ 2k_2 & 0\end{smallmatrix}\right]\varphi\left[\begin{smallmatrix} r_3 & r_1\\ 2k_3 & 0\end{smallmatrix}\right]\theta_{-1,r_1},\\
D_{(k,r)} &= \delta_{k,0}\left(\frac{\theta_{-1,r}}{2}\varphi\left[\begin{smallmatrix} r & r\\ 0 & 0\end{smallmatrix}\right] + \frac{\theta_{0,r}}{8}\right) + \delta_{k,1}\frac{\theta_{-1,r}}{24}.
\end{aligned}$$

Do the ABCD relations still hold for these coefficients? It would be a pain to check them explicitly, as we already did in the previous section. Instead let us appeal to some transformation properties of Airy structures.

Operations on Airy structures

Given an Airy structure which is a sub-algebra of the Lie algebra $\mathcal{W}_V(\leq 2)$, we can act on it by conjugation with some element $U \in \exp(\mathcal{W}_V(\leq 2))$.

$$\tilde{L}_i = UL_iU^{-1} \quad \tilde{Z} = UZ \qquad (3.3.37)$$

then the structure constants stay the same, and the $\tilde{L}_i$ annihilate the new partition function $\tilde{Z}$. However only a few carefully chosen U will conserve the shape (3.1.3) of an Airy structure. One such U is given by

$$U = \exp(\frac{\hbar}{2}u^{cd}\partial_c\partial_d) \qquad (3.3.38)$$

where u is a symmetric matrix. Upon conjugating we find,

$$\begin{aligned}\tilde{L}_i &= \hbar\partial_i - \frac{1}{2}A_{iab}x^a x^b - \hbar\left(B_{ia}{}^b + A_{iac}u^{cb}\right)x^a\partial_b \\ &\quad - \frac{\hbar^2}{2}\left(C_i{}^{ab} + A_{icd}u^{ac}u^{bd} + B_{ic}{}^a u^{cb} + B_{ic}{}^b u^{ca}\right)\partial_a\partial_b - \hbar\left(D_i + A_{icd}u^{cd}\right).\end{aligned} \tag{3.3.39}$$

In particular we preserve the single derivative in the linear term, and the rest is still at most quadratic. In other words we get a new Airy structure with,

$$\begin{aligned}\tilde{A}_{ijk} &= A_{ijk}, \\ \tilde{B}^k_{ij} &= B^k_{ij} + A_{ijc}u^{ck}, \\ \tilde{C}^{jk}_i &= C^{jk}_i + B^j_{ic}u^{ck} + B^k_{ic}u^{cj} + A_{icd}u^{jc}u^{kd}, \\ \tilde{D}_i &= D_i + A_{icd}u^{cd}.\end{aligned} \tag{3.3.40}$$

Since we know that the Lie algebra structure is conserved and hence that the ABCD relations still hold, $\tilde{Z} = UZ$ should be precisely the partition function given by the topological recursion of this new Airy structure.

If we look back at ABCD coefficients for the local coordinates (3.3.36), we see that for each ramification point, they are related to those of the single branch point (3.3.16) by conjugation with

$$U = e^{-\frac{\hbar}{2}\varphi\left[\begin{smallmatrix} r_1 & r_2 \\ k_1 & k_2\end{smallmatrix}\right]\partial_{k_1,r_1}\partial_{k_2,r_2}}. \tag{3.3.41}$$

More precisely, we can first introduce the extra indices (r_1, r_2, r_3) to the coefficients in (3.3.16) by multiplying A, B and C by δ_{r_1,r_2,r_3} – adding a symmetric factor will not change the $(i \leftrightarrow j)$ symmetry in the ABCD relations – and then use the conjugation by U. This proves that the ABCD coefficients of (3.3.36) satisfy the ABCD relations.

3.4 Summary

In this section we have seen how to reformulate the topological recursion of a spectral curve in terms of Airy structures.

We also saw that we have a way to define, for any spectral curve, differential operators operators that annihilate a partition function $Z(x_1, x_2, ...)$ that is related to the quantum curve via

$$\psi(x,\hbar) = \exp\left[\frac{1}{\hbar}\int_\alpha^z \omega_{0,1}\right] \exp\left[\frac{1}{2}\int_\alpha^z\int_\alpha^z \omega_{0,2} - \frac{dx_1 dx_2}{(x_1-x_2)^2}\right] \times Z(x_1,x_2,...)\big|_{x_i=\int_\alpha^z \hbar d\xi_i}. \quad (3.4.1)$$

We will refer to the process of substituting $x_i \to \int_\alpha^z \hbar d\xi_i$ as *specialization*, and denote it by

$$\mathcal{S}[f(x_i)](z) = f(x_i)\big|_{x_i=\int_\alpha^z \hbar d\xi_i}. \quad (3.4.2)$$

Let us call A the contribution of unstable maps,

$$\mathrm{A} = \exp\left[\frac{1}{\hbar}\int_\alpha^z \omega_{0,1}\right] \exp\left[\frac{1}{2}\int_\alpha^z\int_\alpha^z \omega_{0,2} - \frac{dx_1 dx_2}{(x_1-x_2)^2}\right], \quad (3.4.3)$$

We have that,

$$\mathcal{S}[Z] = \mathrm{A}^{-1}\psi, \quad (3.4.4)$$

and also

$$L_i \cdot Z = 0. \quad (3.4.5)$$

We want to define the specialization of an element $M \in \mathcal{W}_V$. Suppose we can find B an operator in the variable z such that *for any monomial* $x_{\mu_1}\dots x_{\mu_n}$,

$$B\,\mathcal{S}[x_{\mu_1}\dots x_{\mu_n}] = \mathcal{S}[M\,x_{\mu_1}\dots x_{\mu_n}]. \quad (3.4.6)$$

Then we say it is the specialization of M,

$$B = \mathcal{S}[M]. \quad (3.4.7)$$

If B and B' are two specializations of M then $B - B'$ annihilates the specialization of any monomial, and in particular it annihilates any function of the shape $\frac{1}{z^{2k+1}}$, therefore we expect $B - B' = 0$ and the specialization is well defined.
Suppose we can find $\mathcal{S}[M]$ for M that is some combination of the operators L_i so that $MZ = 0$. Then we would have a ready made operator which annihilates

the wave function,

$$\begin{aligned}
&\left(\mathrm{A}\,\mathcal{S}[M]\mathrm{A}^{-1}\right)\,\psi \\
&= \mathrm{A}\,\mathcal{S}[M]\mathcal{S}[Z] \\
&= \mathrm{A}\,\mathcal{S}[\underbrace{M\,Z}_{0}] \\
&= 0.
\end{aligned}$$

Where we used that Z is a formal sum of monomials and order by order in $\hbar$ there are only finitely many non zero $W_{g,n}$ coefficients in its expansion (3.2.22). The idea being that this could give us a way to find a quantization of the original spectral curve associated with ψ:

$$\hat{P} = \mathrm{A}\,\mathcal{S}[M]\mathrm{A}^{-1}. \tag{3.4.8}$$

In the next section we will apply this strategy to the simplest cases of the Airy and Bessel curves.

4 Quantum Curves from Airy Structures

We now have all collected all the ingredients which we needed for our new quantum curve recipe. The last piece of information we require is what should we use as the operator M to be specialization. Kazarian and Zograf [36] and Norbury and Do [23] noticed that by defining the following evolution operator from the Virasoro constraints L_i,

$$M = \sum_i x_i L_i, \tag{4.0.1}$$

one could recover the quantum curve after specializing and conjugating with the contribution of unstable topologies. We can therefore wrap up our method in the following.

Recipe 4.0.1. Here are our suggested steps to construct quantum curves:

1. Start with a spectral curve on $\mathbb{C}P^1$ given by the tuple $(x(z), y(z))$
2. Grind x and y into the inverse 1-form $\theta(z)$.
3. Find a choice of polarization $\{d\xi_{k,r}\}$, such that the residue formula (3.2.4) are easy enough to work with.
4. **Check** that the ABCD coefficients obtained in this way satisfy to the ABCD relations (3.1.5)-(3.1.9).
5. Construct the evolution operator $M \sim \sum_{k,r} x_{k,r} L_{k,r}$. Here some adjustment might be needed (add spices of your choice for the desired result).
6. **Specialize** the evolution operator to obtain a global differential operator on the spectral curve expressed in the z variable. Doing so we might find out that not all inputs of $\theta(z)$ will work.
7. Conjugate the evolution operator with the left overs unstable topologies of the A factor (3.4.3).

8. (a) Substitute $x(z) = \hat{x}$ and $\hbar \left(\frac{dx}{dz}\right)^{-1} \frac{d}{dz} = \hat{y}$ and check that you indeed obtain a quantization of the original spectral curve.

   (b) Alternatively, start with the canonical quantization of the spectral curve, substitute $\hat{x} = x(z)$ and $\hat{y} = \hbar \left(\frac{dx}{dz}\right)^{-1} \frac{d}{dz}$ and take the difference with what you obtained at step 7): what you get should be only composed of quantum correction terms.

In this last chapter we will apply these steps to one and two branch point spectral curves. We will also explore a multi-variable variant of the specialization step.

4.1 One Branch Point

Let us apply our strategy first to the generalized Airy spectral curve given by the ABCD coefficients in (3.3.16) which we substitute in equation (3.1.3). In light of (3.2.22), we also do the change $\hbar \to \hbar^2$. We get,

$$\begin{aligned} L_i = \hbar^2 \partial_i - \frac{1}{2}\theta_{-1}\delta_{i,0}(x_0)^2 \\ - \hbar^2 \sum_{a,b} \frac{(2b+1)}{(2i+1)} \theta_{b-i-a} x_a \partial_b \\ - \frac{\hbar^4}{2} \sum_{a,b} \frac{(2a+1)(2b+1)}{(2i+1)} \theta_{a+b+1-i} \partial_a \partial_b \\ - \hbar^2 \delta_{i,0} \frac{\theta_0}{8} - \hbar^2 \delta_{i,1} \frac{\theta_{-1}}{24}. \end{aligned} \tag{4.1.1}$$

Our strategy to specializing will be pretty pedestrian, we shall look at how a differential operator acts on a monomial of x_i's, then translate this ourselves as an operator of the single z variable.

Recall that for a fixed choice of α a pole of x, the specialization map $\mathcal{S}$ acts on functions of x_i as

$$x_i \to \mathcal{S}[x_i] = \int_\alpha^z \hbar d\xi_i. \tag{4.1.2}$$

Here we are working with the odd basis in (3.3.7) where $\xi_i = \frac{(2k+1)}{z^{2k+2}} dz$ and the pole is at $\alpha = \infty$, therefore the map becomes:

$$\mathcal{S}[x_i] = -\hbar z^{-2i-1}. \tag{4.1.3}$$

By extension, for an arbitrary monomial of x_i's,

$$\mathcal{S}\left[x_{\mu_1}\dots x_{\mu_n}\right] = (-1)^n \hbar^n z^{-\sum_{i=0}^{n}(2\mu_i+1)}. \tag{4.1.4}$$

We must now combine the L_i's of (4.1.1) such that they can be specialized.

Let us illustrate with the simplest term, the linear term in L_i (4.1.11). Note that we can extend the basis ξ_i naturally to half integers using the same definition (3.3.7): these just correspond to basis of 1-form with odd order poles which we didn't need when decomposing the $\omega_{g,n}$. Multiplying by $(2i+1)x_{i+\frac{1}{2}}$ and summing over i, its specialization is given by the following lemma.[1]

Lemma 4.1.1.

$$\mathcal{S}\left[\hbar\sum_{i=0}^{\infty}(2i+1)x_{i+\frac{1}{2}}\partial_i(x_{\mu_1}\cdots x_{\mu_n})\right] = -\hbar\frac{d}{dz}\mathcal{S}[x_{\mu_1}\cdots x_{\mu_n}]. \tag{4.1.5}$$

Proof.

$$\begin{aligned}
&\mathcal{S}\left[\sum_{i=0}^{\infty}(2i+1)x_{i+\frac{1}{2}}\partial_i(x_{\mu_1}\cdots x_{\mu_n})\right]\\
&= \mathcal{S}\left[\sum_{i=0}^{\infty}(2i+1)\sum_{\rho=1}^{n}x_{i+\frac{1}{2}}\delta_{i,\mu_\rho}\frac{1}{x_{\mu_\rho}}(x_{\mu_1}\cdots x_{\mu_n})\right] && (4.1.6)\\
&= \mathcal{S}\left[\sum_{\rho=1}^{n}(2\mu_\rho+1)\frac{x_{\mu_\rho+\frac{1}{2}}}{x_{\mu_\rho}}x_{\mu_1}\cdots x_{\mu_n}\right] && (4.1.7)\\
&= -\sum_{\rho=1}^{n}(2\mu_\rho+1)\hbar z^{2\mu_\rho-2}\frac{\mathcal{S}\left[x_{\mu_1}\cdots x_{\mu_n}\right]}{\mathcal{S}\left[x_{\mu_\rho}\right]} && (4.1.8)\\
&= -\frac{d}{dz}\mathcal{S}[x_{\mu_1}\cdots x_{\mu_n}]. && (4.1.9)
\end{aligned}$$

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[1]The reason for this shift is only for tidiness here: this way we won't have to multiply by extra factors to recover the right quantum curve at the end. However in the multivariable specialization in the next section it does seem to be necessary. More observations/examples would be necessary to get a better understanding of why this is the right combination to consider.

Evolution operator

This motivates us to define the evolution operator M as the following linear combination,

$$M = -\frac{1}{\hbar}\sum_{i=0}^{\infty}(2i+1)x_{i+\frac{1}{2}}L_i. \tag{4.1.10}$$

Inserting the above expression of L_i,

$$M = -\hbar\sum_{i}(2i+1)x_{i+\frac{1}{2}}\partial_i \tag{4.1.11}$$
$$+\frac{\hbar^{-1}}{2}\theta_{-1}x_{\frac{1}{2}}(x_0)^2 \tag{4.1.12}$$
$$+\hbar\sum_{a,b,i}(2b+1)\theta_{b-i-a}x_{i+\frac{1}{2}}x_a\partial_b \tag{4.1.13}$$
$$+\frac{\hbar^3}{2}\sum_{a,b,i}(2a+1)(2b+1)\theta_{a+b+1-i}x_{i+\frac{1}{2}}\partial_a\partial_b \tag{4.1.14}$$
$$+\hbar x_{\frac{1}{2}}\frac{\theta_0}{8}+\hbar x_{\frac{3}{2}}\frac{\theta_{-1}}{8}. \tag{4.1.15}$$

We will refer to (4.1.12), (4.1.13), (4.1.14) and (4.1.15) respectively as the A,B,C and D terms. Continuing on , and trying to apply specialization to the B and C terms, we observe that not all choices of θ will work out nicely.
We introduce some notation for convenience:

$$\vartheta(z) = \theta(z)dz, \tag{4.1.16}$$

So $\vartheta(z)$ has the same expansion as $\theta(z)$ but removing the 1-form inverse factor $(dz)^{-1}$. We have the following lemma:

Lemma 4.1.2. *If θ_{-1} and θ_0 are the only non zero coefficients in the expansion of θ, then the B and C terms specialize as,*

$$\mathcal{S}\Bigg[\hbar\sum_{a,b,i}(2b+1)\theta_{b-i-a}x_{i+\frac{1}{2}}x_a\partial_b + \frac{\hbar^3}{2}\sum_{a,b,i}(2a+1)(2b+1)\theta_{a+b+1-i}x_{i+\frac{1}{2}}\partial_a\partial_b\Bigg] = -\frac{\hbar^2}{2}\vartheta(z)'\frac{d}{dz}-\frac{\hbar^2}{2}\vartheta(z)\frac{d^2}{dz^2}. \tag{4.1.17}$$

Proof. Multiply the left hand side by $-2\hbar^{-2}$ for convenience, we have,

$$\mathcal{S}\left[-2\left(\hbar^{-1}\sum_{a,b,i}(2b+1)\theta_{b-i-a}x_{i+\frac{1}{2}}x_a\partial_b + \frac{\hbar}{2}\sum_{a,b,i}(2a+1)(2b+1)\theta_{a+b+1-i}x_{i+\frac{1}{2}}\partial_a\partial_b\right)(x_{\mu_1}\dots x_{\mu_n})\right] \tag{4.1.18}$$

$$= \mathcal{S}\left[-2\left(\hbar^{-1}\sum_{\rho=1}^{n}\sum_{a,b,i}(2b+1)\theta_{b-i-a}\delta_{b,\mu_\rho}\frac{x_{i+\frac{1}{2}}x_a}{x_b} + \frac{\hbar}{2}\sum_{\substack{\rho,\eta=1\\ \rho\neq\eta}}^{n}\sum_{a,b,i}(2a+1)(2b+1)\theta_{a+b+1-i}\delta_{a,\mu_\eta}\delta_{b,\mu_\rho}\frac{x_{i+\frac{1}{2}}}{x_a x_b}\right)(x_{\mu_1}\dots x_{\mu_n})\right] \tag{4.1.19}$$

$$= \left(2\sum_{\rho=1}^{n}\sum_{a,b,i}(2b+1)\theta_{b-i-a}\delta_{b,\mu_\rho}z^{-2i-2a+2b-2} + \sum_{\substack{\rho,\eta=1\\ \rho\neq\eta}}^{n}\sum_{a,b,i}(2a+1)(2b+1)\theta_{a+b+1-i}\delta_{a,\mu_\eta}\delta_{b,\mu_\rho}z^{-2i+2a+2b}\right)\mathcal{S}[x_{\mu_1}\dots x_{\mu_n}]. \tag{4.1.20}$$

Let us do a bit of rewriting on the operator. Let $j = b - i - a$ in the first sum, which will range from -1 to b. In the second sum let $j = a + b - i + 1$, which will range from -1 to $a+b+1$.

$$\begin{aligned} &2z^{-2}\sum_{\rho=1}^{n}\sum_{b}(2b+1)\sum_{j=-1}^{b}\sum_{a=0}^{b-j}\theta_j\delta_{b,\mu_\rho}z^{2j} \\ &\qquad + z^{-2}\sum_{\substack{\rho,\eta=1\\ \rho\neq\eta}}^{n}\sum_{a,b}(2a+1)(2b+1)\delta_{a,\mu_\eta}\delta_{b,\mu_\rho}\sum_{j=-1}^{a+b+1}\theta_j z^{2j} \end{aligned} \tag{4.1.21}$$

$$\begin{aligned} &= z^{-2}\sum_{\rho=1}^{n}\sum_{b}(2b+1)\delta_{b,\mu_\rho}\sum_{j=-1}^{b}(2b-2j+2)\theta_j z^{2j} \\ &\qquad + z^{-2}\sum_{\substack{\rho,\eta=1\\ \rho\neq\eta}}^{n}\sum_{a,b}(2a+1)(2b+1)\delta_{a,\mu_\eta}\delta_{b,\mu_\rho}\sum_{j=-1}^{a+b+1}\theta_j z^{2j} \end{aligned} \tag{4.1.22}$$

$$\begin{aligned} &= z^{-2}\sum_{\rho=1}^{n}(2\mu_\rho+1)\sum_{j=-1}^{\mu_\rho}(2\mu_\rho-2j+2)\theta_j z^{2j} \\ &\qquad + z^{-2}\sum_{\substack{\rho,\eta=1\\ \rho\neq\eta}}^{n}(2\mu_\eta+1)(2\mu_\rho+1)\sum_{j=-1}^{\mu_\eta+\mu_\rho+1}\theta_j z^{2j}\ . \end{aligned} \tag{4.1.23}$$

Here we see that the bounds on the sums will depend on the monomial which we consider, since $\mu_i \geq 0$. Therefore if we want to reconstruct the series of θ independently of the monomial considered, we have to assume $\theta_k = 0$ for $k > 0$. Doing so, we can rewrite (4.1.23) as,

$$\begin{aligned} &z^{-2}\sum_{\rho=1}^{n}(2\mu_\rho+1)\sum_{j\geq -1}(2\mu_\rho-2j+2)\theta_j z^{2j} \\ &\qquad + z^{-2}\sum_{\substack{\rho,\eta=1\\ \rho\neq\eta}}^{n}(2\mu_\eta+1)(2\mu_\rho+1)\sum_{j\geq -1}\theta_j z^{2j} \end{aligned} \tag{4.1.24}$$

$$\begin{aligned} &= -z^{-2}\sum_{\rho=1}^{n}(2\mu_\rho+1)\sum_{j\geq -1}(2j-1)\theta_j z^{2j} \\ &\qquad + z^{-2}\sum_{\rho,\eta=1}^{n}(2\mu_\eta+1)(2\mu_\rho+1)\sum_{j\geq -1}\theta_j z^{2j}\ . \end{aligned} \tag{4.1.25}$$

Since $\sum_{\rho=1}^{n}(2\mu_\rho+1) \sim -z\frac{d}{dz}$ on the specialized monomials, we can finally rewrite (4.1.20) as,

$$= \left(\sum_{j=-1}^{\infty} (2j-1)\theta_j z^{2j-1} \frac{d}{dz} + z^{-1} \sum_{j=-1}^{\infty} \theta_j z^{2j} \frac{d}{dz} z \frac{d}{dz} \right) \mathcal{S}[x_{\mu_1} \ldots x_{\mu_n}] \quad (4.1.26)$$

$$= \left(\sum_{j=-1}^{\infty} 2j\theta_j z^{2j-1} \frac{d}{dz} + \sum_{j=-1}^{\infty} \theta_j z^{2j} \frac{d^2}{dz^2} \right) \mathcal{S}[x_{\mu_1} \ldots x_{\mu_n}] \quad (4.1.27)$$

$$= \left(\vartheta'(z) \frac{d}{dz} + \vartheta(z) \frac{d^2}{dz^2} \right) \mathcal{S}[x_{\mu_1} \ldots x_{\mu_n}]. \quad (4.1.28)$$

Multiply back by $-\frac{1}{2}\hbar^2$ to finish the proof.

The A and D terms don't contain partial derivatives and specialize easily. We can conclude with the following proposition.

Proposition 4.1.3. *Let a spectral curve (x, y) on $\mathbb{C}P^1$ be such that $x(z) = a + b\, z^2$, and such that θ_{-1}, θ_0 are the only non zero coefficients in the expansion of $\theta(z)$ at $z = 0$. Then the specialization operator $\mathcal{S}[x_i] = \int_{\infty}^{z} \hbar d\xi_i$ gives*

$$\mathcal{S}\left[-\frac{1}{\hbar} \sum_{i \geq 0} (2i+1) x_{i+\frac{1}{2}} L_i \right]$$
$$= \hbar \frac{d}{dz} - \frac{1}{2}\hbar^2 \vartheta'(z) \frac{d}{dz} - \frac{1}{2}\hbar^2 \vartheta(z) \frac{d^2}{dz^2} - \frac{5}{8}\hbar^2 \theta_{-1} z^{-4} - \frac{1}{8}\hbar^2 \theta_0 z^{-2}. \quad (4.1.29)$$

Remark 4.1.4. The fact that the specialization only works with the base point α chosen as the pole at infinity is only implicit here: the proof worked in part because there are no extra constant terms to deal with. Why we need a pole of x will be clearer when we extend the calculation to two branch points and the poles will be at ± 1.

As we can see, the admissibility condition in terms of the expansion of θ greatly restricts the curves which we can quantize. However this proposition gives us a fairly general result for curves with one branch point and a double-cover that can be parametrized with the coordinate z in that way. In the next section we review all such quantizable curves.

4.2 Applications

It is difficult to proceed further with the same level of generality ; while we can straightforwardly compute $\theta(z)$ from a general input of (x, y), we do not know in general what algebraic equation these will correspond to and what quantum curve operator $\hat{P}$ to expect. However for specific cases where we know x and y, it will be an effective technique to show that a quantum curve operator does exist.

There are 3 interesting spectral curves which are of the form treated in the previous section. One is the Airy curve, which we mentioned a few times already and which is related to intersection theory on the moduli space of curves and the Kontsevitch-Witten τ-function [32]. Another one is the Bessel curve that governs asymptotic behaviour of so called irregular spectral curves, when y has a simple pole at zeroes of dx, and which is also related to the BGW τ-function for the KdV hierarchy [23]. Finally the monotone Hurwitz curve calculates monotone Hurwitz numbers, that counts branched coverings of the Riemann sphere with given ramification profile, modulo some monotonicity condition. See remark 2.5.5 and [21] for more details. They are given by the following (x, y) pairs:

$$x = \frac{1}{2}z^2 \qquad y = z, \qquad \text{(Airy)}$$
$$x = \frac{1}{2}z^2 \qquad y = \frac{1}{z}, \qquad \text{(Bessel)}$$
$$x = \frac{1}{4} - \frac{1}{4}z^2 \qquad y = -\frac{2}{1-z}. \qquad \text{(Monot. Hurwitz)}$$

The corresponding algebraic curves are

$$\frac{1}{2}y^2 - x = 0, \qquad \text{(Airy)}$$
$$2xy^2 - 1 = 0, \qquad \text{(Bessel)}$$
$$xy^2 + y + 1 = 0, \qquad \text{(Monot. Hurwitz)}$$

and we get for $\theta(z)$,

$$\vartheta(z) = z^{-2}, \qquad \text{(Airy)}$$
$$\vartheta(z) = 1, \qquad \text{(Bessel)}$$
$$\vartheta(z) = -z^{-2} + 1. \qquad \text{(Monot. Hurwitz)}$$

Note that they all satisfy $\theta_k = 0, \quad k > 0$. We will present the "algorithm" in the case of the Airy curve. Extending these calculations to the other special is better done with a computer algebra system, and we will simply state the results.

Now that we have the specialization of the evolution operator, we need to conjugate it with the A factor. Afterwards we can compare this to a guess of the quantization of the algebraic curve equation via the canonical choice $x \to \hat{x} = x(z)$ and $y \to \hat{y} = \hbar \frac{d}{dx} = \hbar \left(\frac{dx}{dz}\right)^{-1} \frac{d}{dz}$.

4.2.1 Airy curve

Computation of A. We need to conjugate the specialization operator by A, which contains the contributions of $\omega_{0,1}$ and $\omega_{0,2}$. Recall that

$$\mathrm{A} = \exp\left[\frac{1}{\hbar}\int_\alpha^z \omega_{0,1}\right] \exp\left[\frac{1}{2}\int_\alpha^z \int_\alpha^z \omega_{0,2} - \frac{dx_1 dx_2}{(x_1 - x_2)^2}\right], \tag{4.2.1}$$

where α is a pole of x. For $x = \frac{1}{2}z^2$ we have $\alpha = \infty$, and the second factor is

$$\begin{aligned} &\exp\left[\frac{1}{2}\left(\log\frac{z_1 - z_2}{x_1 - x_2}\right)\Big|_{z_1=\infty}^{z}\Big|_{z_2=\infty}^{z}\right] \\ &\qquad = \sqrt{\frac{z_1 - z_2}{x_1 - x_2}}\Big|_{\substack{z_1=z\\z_2=z}} \left(\sqrt{\frac{z_1 - z_2}{x_1 - x_2}}\Big|_{\substack{z_1=z\\z_2=\infty}}\right)^{-2} \sqrt{\frac{z_1 - z_2}{x_1 - x_2}}\Big|_{\substack{z_1=\infty\\z_2=\infty}}. \end{aligned} \tag{4.2.2}$$

Furthermore,

$$\sqrt{\frac{z_1 - z_2}{x_1 - x_2}} = \sqrt{\frac{2}{z_1 + z_2}}. \tag{4.2.3}$$

As one can see, substituting ∞ in (4.2.2) can make some infinities or zero factors appear in A. We can deal with this by realizing that when we write the quantum curve equation $\hat{P}\,\psi = 0$ what we mean strictly speaking is that $\hat{P}$ acting on the argument of the exponential in ψ, which produces an element in $\mathbb{C}[x][[\hbar]]$, vanishes. Concisely, if $\psi = \exp[\phi]$ then what we want to vanish is

$$\hat{P}\,\psi = \underbrace{\left(\hat{P}\,\phi\right)}_{=0}\psi \tag{4.2.4}$$

In other words what we ask is

$$\psi^{-1}\hat{P}\psi = \tilde{\psi}^{-1}\mathrm{A}^{-1}\hat{P}\mathrm{A}\tilde{\psi} = 0, \tag{4.2.5}$$

where $\tilde{\psi}(z)$ is the specialization of the Airy structure partition function $\mathcal{S}[Z]$ as in (3.2.22),

$$\tilde{\psi}(z) = \exp\left[\sum_{g\geq 0}\sum_{k\geq 1}^{\circ\circ} \frac{\hbar^{2g-2}}{k!} \sum_{\mu_1,\dots,\mu_n} W_{g,n}\left[\,\mu_1 \cdots \mu_n\,\right] \prod_{i=1}^{n} \int_{\infty}^{z} \hbar d\xi_{\mu_i}\right]. \tag{4.2.6}$$

So to deal with the infinities in A, we can integrate with respect to some different base point α' which cuts off infinities, conjugate to get rid of these constant terms, and then take the limit $\alpha' \to \alpha$. In practice, this means that we can just ignore any diverging constant factor when calculating A, and really only keep the part which is a function of z. We indicate that with the symbol $\propto$. We have,

$$\exp\left[\frac{1}{2}\int_{\infty}^{z}\int_{\infty}^{z}\left(\frac{dz_1 dz_2}{(z_1-z_2)^2} - \frac{dx_1 dx_2}{(x_1-x_2)^2}\right)\right] \propto z^{-1/2}. \tag{4.2.7}$$

For the Airy curve, $\omega_{0,1} = ydx = z^2 dz$, one finds

$$\exp\left[\frac{1}{\hbar}\int_{\infty}^{z}\omega_{0,1}\right] \propto \exp\left(\frac{1}{3\hbar}z^3\right). \tag{4.2.8}$$

So that

$$\mathrm{A} \propto \exp\left(\frac{1}{3\hbar}z^3\right) z^{-1/2}. \tag{4.2.9}$$

Constructing the Quantum curve. All we have left to do is input the specific values of $\theta(z)$ into equation (4.1.29) to get the operator that annihilates $\tilde{\psi}(z) = \mathcal{S}\left[Z\right]$. For $\theta(z) = z^{-2}(dz)^{-1}$, we have

$$\left(\hbar\frac{d}{dz} + \frac{\hbar^2}{2z^2}\frac{d^2}{dz^2} - \frac{\hbar^2}{z^3}\frac{d}{dz} + \frac{5\hbar^2}{8z^4}\right)\tilde{\psi}(z) = 0. \tag{4.2.10}$$

We then conjugate with the A factor (4.2.9) to get the operator which annihilates the wave-function $\psi = A\mathcal{S}\left[Z\right]$:

$$\begin{aligned}\exp\left(\frac{1}{3\hbar}z^3\right) z^{-1/2}\left(\hbar\frac{d}{dz} + \frac{\hbar^2}{2z^2}\frac{d^2}{dz^2} - \frac{\hbar^2}{z^3}\frac{d}{dz} + \frac{5\hbar^2}{8z^4}\right)\exp\left(-\frac{1}{3\hbar}z^3\right) z^{1/2} \\ = \frac{\hbar^2}{2z^2}\frac{d^2}{dz^2} - \frac{\hbar^2}{2z^3}\frac{d}{dz} - \frac{z^2}{2},\end{aligned} \tag{4.2.11}$$

and finally by identifying $\hat{x} = \frac{1}{2}z^2$ and $\hat{y} = \hbar\frac{d}{dx} = \frac{\hbar}{z}\frac{d}{dz}$,

$$\left(\frac{\hbar^2}{2z^2}\frac{d^2}{dz^2} - \frac{\hbar^2}{2z^3}\frac{d}{dz} - \frac{1}{2}z^2\right)\psi(z) = 0, \tag{4.2.12}$$

$$\iff \left(\frac{1}{2}\frac{\hbar}{z}\frac{d}{dz}\frac{\hbar}{z}\frac{d}{dz} - \frac{1}{2}z^2\right)\psi(z) = 0, \tag{4.2.13}$$

$$\iff \left(\frac{1}{2}\hat{y}^2 - \hat{x}\right)\psi(z) = 0. \tag{4.2.14}$$

So we have proved the quantum curve equation for the Airy spectral curve. In other words, we showed that the wave-function $\psi(z)$, constructed as in (3.4.1) from the specialization of the Airy structure partition function, is annihilated by the differential operator $\hat{P} = \frac{1}{2}\hbar^2\frac{d^2}{dx^2} - x$, which is the canonical quantization of the Airy spectral curve $\frac{1}{2}y^2 - x = 0$.

4.2.2 Bessel curve

Let us now repeat this process with the Bessel spectral curve. Feed the input data of the Bessel curve $\theta(z) = (dz)^{-1}$ into the specialization of the evolution operator (4.1.29), which gives

$$\left(\hbar\frac{d}{dz} + \frac{\hbar^2}{2}\frac{d^2}{dz^2} + \frac{\hbar^2}{8z^2}\right)\tilde{\psi} = 0. \tag{4.2.15}$$

Then conjugate by the A factor (4.2.1) corresponding to that curve:

$$\exp\left(\frac{1}{\hbar}z\right)z^{-1/2}\left(\hbar\frac{d}{dz} + \frac{\hbar^2}{2}\frac{d^2}{dz^2} + \frac{\hbar^2}{8z^2}\right)\exp\left(-\frac{1}{\hbar}z\right)z^{1/2} = \frac{\hbar^2}{2}\frac{d^2}{dz^2} + \frac{\hbar^2}{2z}\frac{d}{dz} - \frac{1}{2}. \tag{4.2.16}$$

Doing the substitution $\hat{x} = \frac{1}{2}z^2$, $\hat{y} = \frac{\hbar}{z}\frac{d}{dz}$, we get:

$$\left(\frac{\hbar^2}{2}\frac{d^2}{dz^2} + \frac{\hbar^2}{2z}\frac{d}{dz} - \frac{1}{2}\right)\psi(z) = 0, \tag{4.2.17}$$

$$\iff \left(\frac{\hbar}{z}\frac{d}{dz}\frac{1}{2}z^2\frac{\hbar}{z}\frac{d}{dz} - \frac{1}{2}\right)\psi(z) = 0, \tag{4.2.18}$$

$$\iff \left(\hat{y}\hat{x}\hat{y} - \frac{1}{2}\right)\psi(z) = 0. \tag{4.2.19}$$

Therefore, the wave function (3.4.1) constructed from the Bessel spectral curve is annihilated by the operator $\hat{P} = \hat{y}\hat{x}\hat{y} - \frac{1}{2}$, which is a quantization of the spectral curve $xy^2 - \frac{1}{2} = 0$. Here we have some non trivial choice of ordering to do, and we see that what comes out of this construction is the ordering $xy^2 \to \hat{y}\hat{x}\hat{y}$.

4.2.3 Monotone Hurwitz curve

Continue the exercise with the Monotone Hurwitz curve, inserting $\theta(z)dz = -z^{-2} + 1$ into (4.1.29)

$$\left(\hbar\frac{d}{dz} - \frac{\hbar^2}{z^3}\frac{d}{dz} - \frac{\hbar^2}{2}(1 - \frac{1}{z^2})\frac{d^2}{dz^2} + \frac{5\hbar^2}{8z^4} - \frac{\hbar^2}{8z^2}\right)\tilde{\psi} = 0. \tag{4.2.20}$$

Conjugate by the A factor (4.2.1) (omitting some constant factors in the $\omega_{0,2}$ contribution),

$$\begin{aligned}
&\exp\left(-\frac{1}{\hbar}\left(z + \log(z-1)\right)\right)z^{-1/2} \\
&\qquad\left(\hbar\frac{d}{dz} - \frac{\hbar^2}{z^3}\frac{d}{dz} - \frac{\hbar^2}{2}(1 - \frac{1}{z^2})\frac{d^2}{dz^2} + \frac{5\hbar^2}{8z^4} - \frac{\hbar^2}{8z^2}\right) \\
&\qquad\qquad\exp\left(\frac{1}{\hbar}\left(z + \log\left(z-1\right)\right)\right)z^{1/2} \\
&= \frac{\hbar^2}{2z^2}\frac{d^2}{dz^2} - \frac{\hbar}{2z^3}\frac{d}{dz} - \frac{\hbar^2}{2}\frac{d^2}{dz^2} - \frac{\hbar^2}{2z}\frac{d}{dz} - \frac{\hbar}{z}\frac{d}{dz} + \frac{1}{2}.
\end{aligned} \tag{4.2.21}$$

This time $\hat{x} = \frac{1}{4} - \frac{1}{4}z^2$ and $\hat{y} = -\frac{2\hbar}{z}\frac{d}{dz}$, so we can conclude,

$$\left(\frac{\hbar^2}{2z^2}\frac{d^2}{dz^2} - \frac{\hbar}{2z^3}\frac{d}{dz} - \frac{\hbar^2}{2}\frac{d^2}{dz^2} - \frac{\hbar^2}{2z}\frac{d}{dz} - \frac{\hbar}{z}\frac{d}{dz} + \frac{1}{2}\right)\psi(z) = 0 \tag{4.2.22}$$

$$\iff \left(2\frac{\hbar}{z}\frac{d}{dz}\left(\frac{1}{4} - \frac{1}{4}z^2\right)\frac{\hbar}{z}\frac{d}{dz} - \frac{\hbar}{z}\frac{d}{dz} + \frac{1}{2}\right)\psi(z) = 0 \tag{4.2.23}$$

$$\iff \left(\frac{1}{2}\hat{y}\hat{x}\hat{y} + \frac{1}{2}\hat{y} + \frac{1}{2}\right)\psi(z) = 0. \tag{4.2.24}$$

The operator $\hat{P} = \frac{1}{2}\hat{y}\hat{x}\hat{y} + \frac{1}{2}\hat{y} + \frac{1}{2}$ is again a quantization of the spectral curve $xy^2 + y + 1$ with the ordering $xy^2 \to yxy$ for the first term, multiplied by an overall factor of $\frac{1}{2}$.

4.2.4 More single branch point curves

In the previous section we treated the 3 standard cases that are allowed by our lemma, when $\theta_k = 0$ for $k > 0$. However there is some information about the curve that is not carried over to $\theta(z)$. Recall that, from the definition,

$$\theta(z) = \frac{-2}{(y(z) - y(\sigma(z)))\, dx(z)}, \tag{4.2.25}$$

so any contribution to y that would be even under the involution is ignored in the topological recursion. In particular, we can modify our curves to be

$$x = \frac{1}{2}z^2 \qquad y = z + f(x) \tag{4.2.26}$$

$$x = \frac{1}{2}z^2 \qquad y = \frac{1}{z} + f(x) \tag{4.2.27}$$

$$x = \frac{1}{4} - \frac{1}{4}z^2 \qquad y = -\frac{2}{1-z} + f(x) \tag{4.2.28}$$

corresponding to some algebraic curves, respectively,

$$\frac{1}{2}(y - f(x))^2 - x = 0, \tag{4.2.29}$$

$$2x(y - f(x))^2 - 1 = 0, \tag{4.2.30}$$

$$x(y - f(x))^2 + y - f(x) + 1 = 0. \tag{4.2.31}$$

The operators obtained from proposition 4.1.3 will be the same as the Airy, Bessel and Monotone Hurwitz curves respectively. However the factor A changes to,

$$\mathrm{A} \to \mathrm{A} \exp\left(-\frac{1}{\hbar} \int f(x)dx\right), \tag{4.2.32}$$

So we have to conjugate the quantum curve by the factor $\exp\left(-\frac{1}{\hbar}\int f(x)dx\right)$, but this is exactly the same as shifting $\hat{y} \to \hat{y} - f(\hat{x})$, and therefore we obtain quantum curves for this whole family of spectral curves as well.

This gives rise to many spectral curves that are not in the class studied in our paper of reference [12] since their Newton polygons may have interior points. While this is nothing specific to our approach (this is already clear from the formula of topological recursion (2.5.11) and the definition of the wave function (2.7.2)), it provides a straightforward extension of the class of curves for which quantum curves exist studied in [12].

4.2.5 Shortcomings

Minimal models

Unfortunately, there are some known spectral curves which would be of a similar shape as the ones treated above but that do not satisfy the conditions of the Lemma to be quantized in this way. Let us mention the case of the $(p, q) = (3, 2)$ minimal model (for an introduction, see [27, 5]), also known as pure gravity. The spectral curve is

$$\Sigma = \mathbb{C}P^1, \qquad x = z^2 - 2, \qquad y = z^3 - 3z, \tag{4.2.33}$$

corresponding to the algebraic curve

$$y^2 = x^3 - 3x + 2. \tag{4.2.34}$$

While this indeed has a single branch point curve with a branched covering of the shape $x = a + bx^2$, it has a θ with higher degree terms and does not satisfy to the hypothesis of proposition 4.1.3. We do not know if a quantum curve is proved to exist by other techniques in this case, however the most general result known for genus zero curves [12] also does not apply, since the corresponding Newton polytope has interior points.

Simple Hurwitz numbers

Another limitation in our approach is the dependency on the existence of a "nice" parametrization of the spectral curve: if the involution is anything different than the simple change of sign $z \to -z$, we do not know how to deal with the coefficients arising in our expressions. One interesting such example is the simple Hurwitz numbers spectral curve. Simple Hurwitz numbers have been shown to be generated by topological recursion [14, 26], on the spectral curve

$$e^x - ye^{-y} = 0. \tag{4.2.35}$$

It is also a genus zero curve which can be parametrized by

$$x = \ln z - z, \qquad y = z. \tag{Simple Hurwitz}$$

Here, dx has a single zero at $z = 1$, and the involution is defined by the equation

$$ze^{-z} = T(z)e^{-T(z)}. \tag{4.2.36}$$

It is related to the Lambert W function which gives the solution to

$$z = W(z)e^{W(z)}, \tag{4.2.37}$$

via $T(z) = -W(-ze^{-z})$. Hence the branch point at $z = 1$ coincides with the branch point of W at $-\frac{1}{e}$, where it is locally a degree 2 cover. Here the quantum curve is known to exist [48] and to be given by

$$\hat{P} = \hat{y} - e^{\hat{x}}e^{\hat{y}} \tag{4.2.38}$$

with the usual $\hat{y} = \hbar\frac{d}{dx}$ and $\hat{x} = x$. How could we deal with this curve in our setting? One way would be to use the Airy structure associated to local coordinate in (3.3.36), working out the specialization lemma including the extra coefficients $\varphi\left[\begin{smallmatrix} r_1 & r_2 \\ l_1 & l_2 \end{smallmatrix}\right]$ and using some relations with the coefficients θ_m to eventually simplify stuff out. Unfortunately we weren't successful in this way.

Another possible choice of basis could be to move the branch point to 0 and then use the simpler basis of (3.3.3) and the standard form of $\omega_{0,2}$, but then one has to deal with coefficients of the expansion of the involution (4.2.36). We were also unable to make this approach work.

One apparent difficulty is that from the specialization of a linear combination of the L_i we can only expect a quadratric differential operator. Hence a quantum curve that is quadratic in $\hat{y}$ and not a full series like (4.2.38).

Despite these difficulties, and because the existence of the quantum curve has been proven already, we feel like one should be able to tackle the simple Hurwitz case using the Airy structure approach, and we leave this problem for future work.

4.3 Multivariable Case

In the previous section, we considered the specialization operator $\mathcal{S}$ which sends the infinite basis $\{x_i\}$ of V to polynomials of z. This process is of course not reversible and one can ask if there exists a similar process that would instead be bijective.

A natural way to address this question is to consider the specialization

$$\mathcal{P}[x_i] = \hbar\left(\int_\infty^{p_1} + \int_\infty^{p_2} + \dots\right) d\xi_i, \tag{4.3.1}$$

where the integration divisor now includes an infinite set of points. We will write $\int_D$ has a short hand notation. In the special case that we have been considering so far where $d\xi_i = \frac{(2i+1)}{z^{2i+2}}dz$, this amounts to

$$\mathcal{P}[x_i] = -\hbar\sum_{\alpha=1}^{N} z_\alpha^{-(2i+1)} \tag{4.3.2}$$

and then letting $N \to \infty$. We could write $\mathcal{P}_N$ but we will drop the N to avoid cluttered notation. Besides, the calculations are independent of N. The right hand side here is a variant of power sum symmetric polynomials, and we shall refer to $\mathcal{P}$ as the power-sum specialization.

The wave-function also changes to a multi-variable variant,

$$\begin{aligned}&\psi(x_1, x_2, \dots; \hbar)\\ &\quad\propto \exp\left[\sum_{g\geq 0}\sum_{k\geq 1}\frac{\hbar^{2g-2+n}}{n!}\int_D \cdots \int_D \left(\omega_{g,n} - \delta_{g,0}\delta_{n,2}\frac{dx_1' dx_2'}{(x_1'-x_2')^2}\right)\right],\end{aligned} \tag{4.3.3}$$

where $\int_D = \int_\infty^{z_1} + \int_\infty^{z_2} + \dots$. This in turns has connections to the theory of integrable systems [29].

If we are doing things correctly, what we get at the end should also allow us to recover the wave function specialization of M,

$$\mathcal{S}(M)(z) = \mathcal{P}(M)(z_\alpha)\Big|_{N=1}.$$

4.3.1 Airy curve

Let us apply this to the Airy curve. Taking the evolution operator with the input $\theta_k = \delta_{k,-1}$ yields

$$\begin{aligned} M = &-\hbar\sum_{i=0}^{\infty}(2i+1)x_{i+\frac{1}{2}}\partial_i - \frac{\hbar^{-1}}{2}x_{\frac{1}{2}}x_0^2 - \frac{\hbar}{8}x_{\frac{3}{2}} \\ &-\hbar\sum_{s=0}^{\infty}\sum_{m=0}^{s+1}(2s+1)x_{s-m+\frac{3}{2}}x_m\partial_s \\ &-\frac{\hbar^3}{2}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}(2m+1)(2k+1)x_{m+k+\frac{5}{2}}\partial_m\partial_k, \end{aligned} \tag{4.3.4}$$

and we want to compute the power sum specialization $\mathcal{P}(M)$.

Starting with the linear term, we have a generalization of lemma 4.1.1.

Lemma 4.3.1.

$$\mathcal{P}\left[\sum_{i=0}^{\infty}(2i+1)x_{i+\frac{1}{2}}\partial_i(x_{\mu_1}\cdots x_{\mu_n})\right] = -\sum_{\alpha}\frac{\partial}{\partial z_\alpha}\mathcal{P}[x_{\mu_1}\cdots x_{\mu_n}]. \tag{4.3.5}$$

Proof. As in the proof of lemma 4.1.1, we have:

$$\mathcal{P}\left[\sum_{i=0}^{\infty}(2i+1)x_{i+\frac{1}{2}}\partial_i(x_{\mu_1}\cdots x_{\mu_n})\right] = \sum_{\rho=1}^{n}(2\mu_\rho+1)\frac{\mathcal{P}[x_{\mu_\rho+\frac{1}{2}}]}{\mathcal{P}[x_{\mu_\rho}]}\mathcal{P}\left[x_{\mu_1}\cdots x_{\mu_n}\right]. \tag{4.3.6}$$

On the other hand,

$$\sum_{\alpha}\frac{\partial}{\partial z_\alpha}\mathcal{P}[x_{\mu_1}\cdots x_{\mu_n}] = \hbar\sum_{\alpha}\sum_{\rho=1}^{n}(2\mu_\rho+1)z_\alpha^{-2\mu_\rho-2}\frac{1}{\mathcal{P}[x_{\mu_\rho}]}\mathcal{P}[x_{\mu_1}\cdots x_{\mu_n}] \tag{4.3.7}$$

$$= -\sum_{\rho=1}^{n}(2\mu_\rho+1)\frac{\mathcal{P}[x_{\mu_\rho+\frac{1}{2}}]}{\mathcal{P}[x_{\mu_\rho}]}\mathcal{P}\left[x_{\mu_1}\cdots x_{\mu_n}\right]. \tag{4.3.8}$$

☕

The next lemma is for the B and C terms:

Lemma 4.3.2.

$$\mathcal{P}\left[\sum_{s=0}^{\infty}\sum_{m=0}^{s+1}(2s+1)x_{s-m+\frac{3}{2}}x_m\partial_s+ \frac{\hbar^2}{2}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}(2m+1)(2k+1)x_{m+k+\frac{5}{2}}\partial_m\partial_k\right] \tag{4.3.9}$$

$$= \left(\hbar\sum_{\alpha}z_\alpha^{-3}\frac{\partial}{\partial z_\alpha}-\frac{\hbar}{2}\sum_{\alpha}z_\alpha^{-2}\frac{\partial}{\partial z_\alpha^2}\right) + \sum_{\alpha\neq\beta}\sum_{\rho=1}^{n}\sum_{m=0}^{\mu_\rho+1}(2\mu_\rho+1)\frac{(-\hbar z_\alpha^{-2\mu_\rho+2m-4})(-\hbar z_\beta^{-2m-1})}{\mathcal{P}\left[x_{\mu_\rho}\right]}. \tag{4.3.10}$$

Proof. First, we calculate:

$$\mathcal{P}\left[\sum_{s=0}^{\infty}\sum_{m=0}^{s+1}(2s+1)x_{s-m+\frac{3}{2}}x_m\partial_s(x_{\mu_1}\cdots x_{\mu_n})\right]$$

$$=\mathcal{P}\left[\sum_{\rho=1}^{n}\sum_{m=0}^{\mu_\rho+1}(2\mu_\rho+1)x_{\mu_\rho-m+\frac{3}{2}}x_m\frac{1}{x_{\mu_\rho}}(x_{\mu_1}\cdots x_{\mu_n})\right] \tag{4.3.11}$$

$$=\sum_{\rho=1}^{n}\sum_{m=0}^{\mu_\rho+1}(2\mu_\rho+1)\frac{\mathcal{P}[x_{\mu_\rho-m+\frac{3}{2}}]\mathcal{P}[x_m]}{\mathcal{P}[x_{\mu_\rho}]}\mathcal{P}\left[x_{\mu_1}\cdots x_{\mu_n}\right] \tag{4.3.12}$$

$$=\sum_{\alpha,\beta}\sum_{\rho=1}^{n}\sum_{m=0}^{\mu_\rho+1}(2\mu_\rho+1)(-\hbar z_\alpha^{-2\mu_\rho+2m-4})(-\hbar z_\beta^{-2m-1})\frac{\mathcal{P}\left[x_{\mu_1}...x_{\mu_n}\right]}{\mathcal{P}\left[x_{\mu_\rho}\right]}. \tag{4.3.13}$$

Now turn to the second term,

$$\mathcal{P}\left[\frac{\hbar^2}{2}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}(2m+1)(2k+1)x_{m+k+\frac{5}{2}}\partial_m\partial_k(x_{\mu_1}\cdots x_{\mu_n})\right]$$

$$=\mathcal{P}\left[\frac{\hbar^2}{2}\sum_{\substack{\rho,\eta=1\\ \rho\neq\eta}}^{n}(2\mu_\rho+1)(2\mu_\eta+1)x_{\mu_\rho+\mu_\eta+\frac{5}{2}}\frac{1}{x_{\mu_\rho}x_{\mu_\eta}}(x_{\mu_1}\cdots x_{\mu_n})\right] \tag{4.3.14}$$

$$=\frac{\hbar^2}{2}\sum_{\substack{\rho,\eta=1\\ \rho\neq\eta}}^{n}(2\mu_\rho+1)(2\mu_\eta+1)\frac{\mathcal{P}[x_{\mu_\rho+\mu_\eta+\frac{5}{2}}]}{\mathcal{P}[x_{\mu_\rho}]\mathcal{P}[x_{\mu_\eta}]}\mathcal{P}[x_{\mu_1}\cdots x_{\mu_n}]. \tag{4.3.15}$$

For this second term notice that

$$
\begin{aligned}
&-\frac{\hbar}{2}\sum_\alpha z_\alpha^{-2}\frac{\partial^2}{\partial z_\alpha^2}\mathcal{P}[x_{\mu_1}...x_{\mu_n}] \\
&=-\frac{\hbar}{2}\sum_\alpha z_\alpha^{-2}\frac{\partial}{\partial z_\alpha}\sum_{\rho=1}^{n}(2\mu_\rho+1)\hbar z_\alpha^{-2\mu_\rho-2}\frac{\mathcal{P}[x_{\mu_1}...x_{\mu_n}]}{\mathcal{P}[x_{\mu_\rho}]} \qquad (4.3.16)\\
&=\frac{\hbar}{2}\sum_\alpha z_\alpha^{-2}\sum_{\rho=1}^{n}(2\mu_\rho+1)(2\mu_\rho+2)\hbar z^{-2\mu_\rho-3}\frac{\mathcal{P}[x_{\mu_1}...x_{\mu_n}]}{\mathcal{P}[x_{\mu_\rho}]} \\
&\quad-\frac{\hbar}{2}\sum_\alpha z_\alpha^{-2}\sum_{\substack{\rho,\eta=1\\ \rho\neq\eta}}^{n}\hbar^2 z^{-2\mu_\rho-2\mu_\eta-4}(2\mu_\rho+1)(2\mu_\eta+1)\frac{\mathcal{P}[x_{\mu_1}...x_{\mu_n}]}{\mathcal{P}[x_{\mu_\rho}]\mathcal{P}[x_{\mu_\eta}]} \\
&\qquad (4.3.17)\\
&=\sum_\alpha\sum_{\rho=1}^{n}\sum_{m=0}^{\mu_\rho}(2\mu_\rho+1)(-\hbar z_\alpha^{-2\mu_\rho+2m-4})(-\hbar z_\alpha^{-2m-1})\frac{\mathcal{P}[x_{\mu_1}...x_{\mu_n}]}{\mathcal{P}[x_{\mu_\rho}]} \\
&\quad+\frac{\hbar^2}{2}\sum_{\substack{\rho,\eta=1\\ \rho\neq\eta}}^{n}(2\mu_\rho+1)(2\mu_\eta+1)\frac{\mathcal{P}[x_{\mu_\rho+\mu_\eta+\frac{5}{2}}]}{\mathcal{P}[x_{\mu_\rho}]\mathcal{P}[x_{\mu_\eta}]}\mathcal{P}[x_{\mu_1}...x_{\mu_n}] \qquad (4.3.18)\\
&=\sum_\alpha\sum_{\rho=1}^{n}\sum_{m=0}^{\mu_\rho+1}(2\mu_\rho+1)(-\hbar z_\alpha^{-2\mu_\rho+2m-4})(-\hbar z_\alpha^{-2m-1})\frac{\mathcal{P}[x_{\mu_1}...x_{\mu_n}]}{\mathcal{P}[x_{\mu_\rho}]} \\
&\quad-\hbar\sum_\alpha z_\alpha^{-3}\frac{\partial}{\partial z_\alpha}\mathcal{P}[x_{\mu_1}...x_{\mu_n}] \qquad (4.3.19)\\
&\quad+\mathcal{P}\left[\frac{\hbar^2}{2}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}(2m+1)(2k+1)x_{m+k+\frac{5}{2}}\partial_m\partial_k(x_{\mu_1}\cdots x_{\mu_n})\right],
\end{aligned}
$$

where in (4.3.18) we introduced a sum with $\mu_\rho+1$ identical terms and in (4.3.19) we used

$$
\begin{aligned}
&\hbar\sum_\alpha z_\alpha^{-3}\frac{\partial}{\partial z_\alpha}\mathcal{P}[x_{\mu_1}...x_{\mu_n}] \\
&=\sum_\alpha\sum_{\rho=1}^{n}(2\mu_\rho+1)(\hbar z_\alpha^{-2\mu_\rho-5})\frac{\mathcal{P}[x_{\mu_1}...x_{\mu_n}]}{\mathcal{P}[x_{\mu_\rho}]} \qquad (4.3.20)\\
&=\sum_\alpha\sum_{\rho=1}^{n}(2\mu_\rho+1)(-\hbar z_\alpha^{-2\mu_\rho+2m-4})(-\hbar z_\alpha^{-2m-1})\frac{\mathcal{P}[x_{\mu_1}...x_{\mu_n}]}{\mathcal{P}[x_{\mu_\rho}]}. \qquad (4.3.21)
\end{aligned}
$$

Rearranging terms,

$$\begin{aligned}
&\mathcal{P}\left[\frac{\hbar^2}{2}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}(2m+1)(2k+1)x_{m+k+\frac{5}{2}}\partial_m\partial_k(x_{\mu_1}\cdots x_{\mu_n})\right] \\
&= -\frac{\hbar}{2}\sum_{\alpha} z_\alpha^{-2}\frac{\partial^2}{\partial z_\alpha^2}\mathcal{P}[x_{\mu_1}...x_{\mu_n}] + \hbar\sum_{\alpha} z_\alpha^{-3}\frac{\partial}{\partial z_\alpha}\mathcal{P}[x_{\mu_1}...x_{\mu_n}] \\
&\qquad - \sum_{\alpha=\beta}\sum_{\rho=1}^{n}\sum_{m=0}^{\mu_\rho+1}(2\mu_\rho+1)(-\hbar z_\alpha^{-2\mu_\rho+2m-4})(-\hbar z_\beta^{-2m-1})\frac{\mathcal{P}[x_{\mu_1}...x_{\mu_n}]}{\mathcal{P}[x_{\mu_\rho}]}.
\end{aligned} \tag{4.3.22}$$

Summing both contributions completes the proof. ☕

Notice that we get a result similar to the one variable case , except for that extra term. Now let's rewrite it in terms of partials $\frac{\partial}{\partial z_\alpha}$.

Lemma 4.3.3.

$$\begin{aligned}
&\sum_{\alpha\neq\beta}\sum_{\rho=1}^{n}\sum_{m=0}^{\mu_\rho+1}(2\mu_\rho+1)(-\hbar z_\alpha^{-2\mu_\rho+2m-4})(-\hbar z_\beta^{-2m-1})\frac{\mathcal{P}\left[x_{\mu_1}\cdots x_{\mu_n}\right]}{\mathcal{P}\left[x_{\mu_\rho}\right]} \\
&= \hbar\sum_{\alpha\neq\beta}\sum_{m=0}^{\infty}\left(\frac{z_\alpha}{z_\beta}\right)^{2m}\left(z_\alpha^{-2}z_\beta^{-1}\frac{\partial}{\partial z_\alpha} - z_\beta^{-3}\frac{\partial}{\partial z_\beta}\right)\mathcal{P}\left[x_{\mu_1}\cdots x_{\mu_n}\right].
\end{aligned} \tag{4.3.23}$$

Proof. We may write

$$\begin{aligned}
&\hbar\sum_{m=0}^{\infty} z_\alpha^{2m-2}z_\beta^{-2m-1}\frac{\partial}{\partial z_\alpha}\mathcal{P}\left[x_{\mu_1}...x_{\mu_n}\right] \\
&= \sum_{\rho=1}^{n}\sum_{m=0}^{\infty}(2\mu_\rho+1)(-\hbar z_\alpha^{-2\mu_\rho+2m-4})(-\hbar z_\beta^{-2m-1})\frac{\mathcal{P}\left[x_{\mu_1}...x_{\mu_n}\right]}{\mathcal{P}\left[x_{\mu_\rho}\right]}
\end{aligned} \tag{4.3.24}$$

$$\begin{aligned}
&= \sum_{\rho=1}^{n}\sum_{m=0}^{\mu_\rho+1}(2\mu_\rho+1)(-\hbar z_\alpha^{-2\mu_\rho+2m-4})(-\hbar z_\beta^{-2m-1})\frac{\mathcal{P}\left[x_{\mu_1}...x_{\mu_n}\right]}{\mathcal{P}\left[x_{\mu_\rho}\right]} \\
&\quad + \sum_{\rho=1}^{n}\sum_{m=\mu_\rho+2}^{\infty}(2\mu_\rho+1)(-\hbar z_\alpha^{-2\mu_\rho+2m-4})(-\hbar z_\beta^{-2m-1})\frac{\mathcal{P}\left[x_{\mu_1}...x_{\mu_n}\right]}{\mathcal{P}\left[x_{\mu_\rho}\right]}.
\end{aligned} \tag{4.3.25}$$

Let's do a little bit of rewriting on the second term, by shifting the indices $m \to m + \mu_\rho + 2$ we get

$$\sum_{\rho=1}^{n} \sum_{m=\mu_\rho+2}^{\infty} (2\mu_\rho + 1)(-\hbar z_\alpha^{-2\mu_\rho+2m-4})(-\hbar z_\beta^{-2m-1}) \frac{\mathcal{P}\left[x_{\mu_1} ... x_{\mu_n}\right]}{\mathcal{P}\left[x_{\mu_\rho}\right]}$$

$$= \sum_{\rho=1}^{n} \sum_{m=0}^{\infty} (2\mu_\rho + 1)(-\hbar z_\alpha^{2m})(-\hbar z_\beta^{-2m-2\mu_\rho-5}) \frac{\mathcal{P}\left[x_{\mu_1} ... x_{\mu_n}\right]}{\mathcal{P}\left[x_{\mu_\rho}\right]} \tag{4.3.26}$$

$$= \hbar \sum_{m=0}^{\infty} z_\alpha^{2m} z_\beta^{-2m-3} \frac{\partial}{\partial z_\beta} \mathcal{P}\left[x_{\mu_1} ... x_{\mu_n}\right]. \tag{4.3.27}$$

Sending that term to the other side, factoring out $\left(\frac{z_\alpha}{z_\beta}\right)^{2m}$, and summing over $\alpha \neq \beta$ completes the proof. ☕

Wrapping up, we managed to specialize the evolution operator according to the following lemma.

Lemma 4.3.4. *The power sum specialization of the evolution operator for the Airy spectral curve* (4.3.4) *is*

$$\begin{aligned} \mathcal{P}[M] = \hbar \sum_\alpha \frac{\partial}{\partial z_\alpha} + \frac{\hbar^2}{2} (\sum_\alpha z_\alpha^{-2})(\sum_\alpha z_\alpha^{-1})^2 + \frac{\hbar^2}{8} \sum_\alpha z_\alpha^{-4} \\ + \frac{\hbar^2}{2} \sum_\alpha z_\alpha^{-2} \frac{\partial}{\partial z_\alpha^2} - \hbar^2 \sum_\alpha z_\alpha^{-3} \frac{\partial}{\partial z_\alpha} \\ - \hbar^2 \sum_{\alpha \neq \beta} \sum_{m=0}^{\infty} \left(\frac{z_\alpha}{z_\beta}\right)^{2m} \left(z_\alpha^{-2} z_\beta^{-1} \frac{\partial}{\partial z_\alpha} - z_\beta^{-3} \frac{\partial}{\partial z_\beta} \right). \end{aligned} \tag{4.3.28}$$

A factor

As in the one variable case, the next thing to do is to conjugate with the Afactor (4.2.1) to obtain an operator that acts on the multi-variable analog of the wave function (3.4.1). A straightforward calculation gives,

$$\mathrm{A} = \exp\left(\frac{1}{\hbar} \int_D \omega_{0,1}\right) \exp\left(\frac{1}{2} \iint_D \left(\omega_{0,2} - \frac{dx_1 dx_2}{(x_1 - x_2)^2}\right)\right), \tag{4.3.29}$$

$$\propto \exp\left(\frac{1}{3\hbar} \sum_\mu z_\mu^3\right) \prod_{\mu,\eta} \left(\frac{z_\mu + z_\eta}{2}\right)^{-1/2}. \tag{4.3.30}$$

We are going to need the following conjugation relation:

Lemma 4.3.5.

$$
\begin{aligned}
A\partial_\alpha A^{-1} &= \partial_\alpha - \hbar^{-1} z_\alpha^2 + \sum_\gamma (z_\alpha + z_\gamma)^{-1}, \\
A\partial_\alpha^2 A^{-1} &= \partial_\alpha^2 + \hbar^{-2} z_\alpha^4 - 2\hbar^{-1} z_\alpha - 2\hbar^{-1} z_\alpha^2 \partial_\alpha \\
&\quad + \sum_{\gamma\neq\delta} (z_\alpha + z_\gamma)^{-1} (z_\alpha + z_\delta)^{-1} - \frac{1}{4} z_\alpha^{-2} \\
&\quad + 2\sum_\gamma (z_\gamma + z_\alpha)^{-1} \partial_\alpha \\
&\quad - 2\hbar^{-1} z_\alpha^2 \sum_\gamma (z_\alpha + z_\gamma)^{-1}.
\end{aligned}
$$

We omit the proof, since it is a simple and straightforward calculation. With these we compute $\mathrm{A}\left(\mathcal{P}[M]\right)\mathrm{A}^{-1}$. We get a lot of cancellations happening and we end up with:

$$\mathrm{A}\left(\mathcal{P}[M]\right)\mathrm{A}^{-1} \tag{4.3.31}$$

$$= \frac{\hbar^2}{2} \sum_\alpha z_\alpha^{-2} \frac{\partial^2}{\partial z_\alpha^2} - \frac{\hbar^2}{2} \sum_\alpha z_\alpha^{-3} \frac{\partial}{\partial z_\alpha} \tag{4.3.32}$$

$$+ \hbar^2 \sum_{\alpha\neq\beta} \frac{1}{z_\alpha^2 - z_\beta^2} \left(z_\alpha^{-2} z_\beta \frac{\partial}{\partial z_\alpha} - z_\beta^{-1} \frac{\partial}{\partial z_\beta} \right) \tag{4.3.33}$$

$$+\hbar^2 \sum_{\alpha\neq\gamma} z_\alpha^{-2} (z_\gamma + z_\alpha)^{-1} \frac{\partial}{\partial z_\alpha} \tag{4.3.34}$$

$$-\frac{1}{2} \sum_\alpha z_\alpha^2 \tag{4.3.35}$$

$$+\hbar^2 \sum_{\alpha\neq\beta} z_\alpha^{-3} z_\beta^{-1} + \frac{\hbar^2}{2} \sum_{\alpha\neq\beta} z_\alpha^{-2} z_\beta^{-2} + \frac{\hbar^2}{2} \sum_{\alpha\neq\gamma\neq\beta} z_\alpha^{-2} z_\beta^{-1} z_\gamma^{-1} \tag{4.3.36}$$

$$-\hbar^2 \sum_{\alpha\neq\beta} z_\alpha^{-3} (z_\alpha + z_\beta)^{-1} \tag{4.3.37}$$

$$+ \frac{\hbar^2}{2} \sum_\alpha \sum_{\gamma\neq\beta} z_\alpha^{-2} (z_\alpha + z_\gamma)^{-1} (z_\alpha + z_\beta)^{-1} \tag{4.3.38}$$

$$+ \hbar^2 \sum_\gamma \sum_{\alpha\neq\beta} \frac{1}{z_\alpha^2 - z_\beta^2} \left(z_\alpha^{-2} z_\beta (z_\alpha + z_\gamma)^{-1} + z_\beta^{-1} (z_\beta + z_\gamma)^{-1} \right). \tag{4.3.39}$$

Here we used the formal representation of the geometric series.

$$\sum_{m=0}^{\infty}\left(\frac{z_\alpha}{z_\beta}\right)^{2m} = -\frac{z_\beta^2}{z_\alpha^2 - z_\beta^2}. \tag{4.3.40}$$

We now show that all the last four lines (4.3.36), (4.3.37), (4.3.38) and (4.3.39) cancel out:

Lemma 4.3.6.

$$\begin{aligned}
&\sum_{\alpha\neq\beta} z_\alpha^{-3} z_\beta^{-1} + \frac{1}{2}\sum_{\alpha\neq\beta} z_\alpha^{-2} z_\beta^{-2} + \frac{1}{2}\sum_{\alpha\neq\gamma\neq\beta} z_\alpha^{-2} z_\beta^{-1} z_\gamma^{-1} \\
&-\sum_{\alpha\neq\beta} z_\alpha^{-3}(z_\alpha + z_\beta)^{-1} \\
&+\frac{1}{2}\sum_{\alpha}\sum_{\gamma\neq\beta} z_\alpha^{-2}(z_\alpha + z_\gamma)^{-1}(z_\alpha + z_\beta)^{-1} \\
&+\sum_{\gamma}\sum_{\alpha\neq\beta}\frac{1}{z_\alpha^2 - z_\beta^2}\left(z_\alpha^{-2} z_\beta (z_\alpha + z_\gamma)^{-1} + z_\beta^{-1}(z_\beta + z_\gamma)^{-1}\right) \\
&= 0.
\end{aligned}$$

Proof. The first step is to express all the summations in a symmetric way. For example, split sums $\sum_\gamma \sum_{\alpha\neq\beta}$ into $\gamma = \alpha$, $\gamma = \beta$ and $\gamma \neq \alpha \neq \beta$,

$$\begin{aligned}
&\sum_{\alpha\neq\beta} z_\alpha^{-3} z_\beta^{-1} + \frac{1}{2}\sum_{\alpha\neq\beta} z_\alpha^{-2} z_\beta^{-2} + \frac{1}{2}\sum_{\alpha\neq\gamma\neq\beta} z_\alpha^{-2} z_\beta^{-1} z_\gamma^{-1} \\
&-\sum_{\alpha\neq\beta} z_\alpha^{-3}(z_\alpha + z_\beta)^{-1} \\
&+\frac{1}{2}\sum_{\alpha\neq\beta} z_\alpha^{-3}(z_\beta + z_\alpha)^{-1} \\
&+\frac{1}{2}\sum_{\alpha\neq\beta\neq\gamma} z_\alpha^{-2}(z_\alpha + z_\gamma)^{-1}(z_\alpha + z_\beta)^{-1} \\
&+\frac{1}{2}\sum_{\alpha\neq\beta}\frac{1}{z_\alpha - z_\beta} z_\alpha^{-3} \\
&+\sum_{\alpha\neq\beta}\frac{1}{z_\alpha - z_\beta} z_\alpha^{-2}(z_\alpha + z_\beta)^{-1} \\
&+\sum_{\alpha\neq\beta\neq\gamma}\frac{1}{z_\alpha - z_\beta} z_\alpha^{-2}(z_\alpha + z_\gamma)^{-1}.
\end{aligned}$$

Now look at all the double sums

$$\begin{aligned}
&\sum_{\alpha\neq\beta} z_\alpha^{-3} z_\beta^{-1} + \frac{1}{2}\sum_{\alpha\neq\beta} z_\alpha^{-2} z_\beta^{-2} \\
&- \sum_{\alpha\neq\beta} z_\alpha^{-3}(z_\alpha + z_\beta)^{-1} \\
&+ \frac{1}{2}\sum_{\alpha\neq\beta} z_\alpha^{-3}(z_\beta + z_\alpha)^{-1} \\
&+ \frac{1}{2}\sum_{\alpha\neq\beta} \frac{1}{z_\alpha - z_\beta} z_\alpha^{-3} \\
&+ \sum_{\alpha\neq\beta} \frac{1}{z_\alpha - z_\beta} z_\alpha^{-2}(z_\alpha + z_\beta)^{-1} \\
&= \frac{1}{2}\sum_{\alpha\neq\beta} z_\alpha^{-2}\left(\frac{2}{z_\alpha z_\beta} + \frac{1}{z_\beta^2} - \frac{1}{z_\alpha(z_\alpha + z_\beta)}\right. \\
&\qquad\qquad \left. + \frac{1}{z_\alpha(z_\alpha - z_\beta)} + \frac{2}{(z_\alpha - z_\beta)(z_\alpha + z_\beta)}\right) \\
&= \frac{1}{2}\sum_{\alpha\neq\beta} \frac{z_\alpha^{-2} z_\beta^{-2}}{(z_\alpha^2 - z_\beta^2)}(z_\alpha + z_\beta)^2 \\
&= 0,
\end{aligned}$$

which vanishes as the summand and antisymmetric in $\alpha \leftrightarrow \beta$ while the summation is symmetric. Next look at the triple sums

$$\begin{aligned}
&-\frac{\hbar}{2}\sum_{\alpha\neq\beta\neq\gamma} z_\alpha^{-2} z_\beta^{-1} z_\gamma^{-1} - \frac{\hbar}{2}\sum_{\alpha\neq\beta\neq\gamma} z_\alpha^{-2}(z_\alpha + z_\gamma)^{-1}(z_\alpha + z_\beta)^{-1} \\
&\qquad - \hbar \sum_{\alpha\neq\beta\neq\gamma} z_\alpha^{-2}(z_\alpha + z_\gamma)^{-1}(z_\alpha - z_\beta)^{-1} \\
&= -\frac{\hbar}{2}\sum_{\alpha\neq\beta\neq\gamma} z_\alpha^{-2}\left(\frac{1}{z_\beta z_\gamma} + \frac{1}{(z_\alpha + z_\gamma)(z_\alpha + z_\beta)} + \frac{2}{(z_\alpha + z_\gamma)(z_\alpha - z_\beta)}\right) \\
&= -\frac{\hbar}{2}\sum_{\alpha\neq\beta\neq\gamma} \frac{z_\alpha^{-1} z_\beta^{-1} z_\gamma^{-1}}{(z_\alpha^2 - z_\beta^2)(z_\alpha + z_\gamma)}\left(z_\alpha^2 + z_\alpha z_\gamma + 3 z_\gamma z_\beta - z_\beta^2\right).
\end{aligned}$$

To see that this sum vanishes, the most straight forward way is to complete the denominator to

$$\frac{1}{(z_\alpha^2 - z_\beta^2)(z_\beta^2 - z_\gamma^2)(z_\gamma^2 - z_\alpha^2)}.$$

This factor is totally anti-symmetric under permutations of (α, β, γ). What we get is

$$\begin{aligned}\frac{\hbar}{2}\sum_{\alpha\neq\beta\neq\gamma}&\frac{z_\alpha^{-1}z_\beta^{-1}z_\gamma^{-1}}{(z_\alpha^2-z_\beta^2)(z_\beta^2-z_\gamma^2)(z_\gamma^2-z_\alpha^2)}\\ &\times\Big(z_\alpha^3z_\beta^2-z_\alpha^3z_\gamma^2-z_\beta^2z_\gamma^3-3z_\beta^3z_\gamma^2\\ &\quad+3z_\alpha z_\beta^3z_\gamma-3z_\alpha z_\beta z_\gamma^3\\ &\quad+z_\beta^4z_\gamma-z_\alpha z_\beta^4+z_\alpha z_\gamma^4+3z_\beta z_\gamma^4\Big)=0,\end{aligned}$$

where each line cancels with the appropriate relabelling.

We also do a bit of rewriting on the terms (4.3.33) and (4.3.34) with

$$\sum_{\alpha\neq\beta}\frac{1}{z_\alpha^2-z_\beta^2}z_\beta\frac{\partial}{\partial z_\alpha}+\sum_{\alpha\neq\beta}\frac{1}{z_\alpha+z_\beta}\frac{\partial}{\partial z_\beta}=\sum_{\alpha\neq\beta}\frac{1}{z_\alpha^2-z_\beta^2}z_\alpha\frac{\partial}{\partial z_\alpha}. \tag{4.3.41}$$

To conclude, we just showed that:

Proposition 4.3.7. *The multi-variable wave-function* (4.3.3) *for the Airy spectral curve is annihilated by the following operator*

$$\begin{aligned}&A(\mathcal{P}[M])A^{-1}\\ &\quad=\frac{\hbar^2}{2}\sum_\alpha z_\alpha^{-2}\frac{\partial^2}{\partial {z_\alpha}^2}-\frac{\hbar^2}{2}\sum_\alpha z_\alpha^{-3}\frac{\partial}{\partial z_\alpha}\\ &\qquad+\hbar^2\sum_{\alpha\neq\beta}(z_\alpha^2-z_\beta^2)^{-1}\left(z_\alpha^{-1}\frac{\partial}{\partial z_\alpha}-z_\beta^{-1}\frac{\partial}{\partial z_\beta}\right)-\frac{1}{2}\sum_\alpha z_\alpha^2.\end{aligned} \tag{4.3.42}$$

Equivalently, in terms of $x_\alpha=\frac{1}{2}z_\alpha^2$,

$$\begin{aligned}&A(\mathcal{P}[M])A^{-1}\\ &\quad=\hbar^2\sum_\alpha\frac{\partial^2}{\partial x_\alpha^2}+\sum_{\alpha\neq\beta}(x_\alpha-x_\beta)^{-1}\left(\frac{\partial}{\partial x_\alpha}-\frac{\partial}{\partial x_\beta}\right)-\sum_\alpha x_\alpha.\end{aligned} \tag{4.3.43}$$

We can recover $\mathcal{S}$ by setting $N=1$. The sum over $\alpha\neq\beta$ disappears, and what is left is

$$\mathrm{A}(\mathcal{P}[M])\mathrm{A}^{-1}\Big|_{N=1}=\hbar^2\frac{d^2}{dx^2}-x, \tag{4.3.44}$$

which is the same as the Airy quantum curve (4.2.14).

This is the expected result; the operator is given by

$$(H - \sum_{\alpha} x_\alpha),$$

where

$$H = \sum_{\alpha} \frac{\partial^2}{\partial x_\alpha^2} + \sum_{\alpha \neq \beta} (x_\alpha - x_\beta)^{-1} \left(\frac{\partial}{\partial x_\alpha} - \frac{\partial}{\partial x_\beta} \right),$$

is the Calogero-Moser Hamiltonian. For more details on this and the connection to integrable systems, we refer the reader to [29].

4.3.2 Bessel curve

We can repeat this process with the Bessel curve. Start from the evolution operator with the input $\theta_k = \delta_{k,0}$

$$\begin{aligned} M = & -\hbar \sum_{i=0}^{\infty} (2i+1) x_{i+\frac{1}{2}} \partial_i - \frac{\hbar}{8} x_{\frac{1}{2}} \\ & - \hbar \sum_{s=0}^{\infty} \sum_{m=0}^{s} (2s+1) x_{s-m+\frac{1}{2}} x_m \partial_s \\ & - \frac{\hbar^3}{2} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} (2m+1)(2k+1) x_{m+k+\frac{3}{2}} \partial_m \partial_k . \end{aligned} \tag{4.3.45}$$

We need to update lemma 4.3.2.

Lemma 4.3.8.

$$\begin{aligned} \mathcal{P} \Bigg[& \sum_{s=0}^{\infty} \sum_{m=0}^{s} (2s+1) x_{s-m+\frac{1}{2}} x_m \partial_s \\ & + \frac{\hbar^2}{2} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} (2m+1)(2k+1) x_{m+k+\frac{3}{2}} \partial_m \partial_k \Bigg] \end{aligned} \tag{4.3.46}$$

$$\begin{aligned} = & -\frac{\hbar}{2} \sum_{\alpha} \frac{\partial}{\partial z_\alpha^2} \\ & + \sum_{\alpha \neq \beta} \sum_{\rho=1}^{n} \sum_{m=0}^{\mu_\rho} (2\mu_\rho + 1) \frac{(-\hbar z_\alpha^{-2\mu_\rho + 2m - 2})(-\hbar z_\beta^{-2m-1})}{\mathcal{P}\left[x_{\mu_\rho}\right]} . \end{aligned} \tag{4.3.47}$$

Proof. Follow the same steps as lemma 4.3.2, except that everything is multiplied by z_α^2, and the upper bound in the sum is s instead of $s+1$. Consequently there is no need for the second term that was needed in the other lemma. ☕

Then we can easily specialize the rest of the terms in M. We obtain the following.

Lemma 4.3.9.

$$\begin{aligned}\mathcal{P}[M] = \hbar \sum_{\alpha} \frac{\partial}{\partial z_\alpha} + \frac{\hbar^2}{8} \sum_{\alpha} z_\alpha^{-2} \\ + \frac{\hbar^2}{2} \sum_{\alpha} \frac{\partial}{\partial z_\alpha^2} \\ - \hbar^2 \sum_{\alpha \neq \beta} \sum_{m=0}^{\infty} \left(\frac{z_\alpha}{z_\beta}\right)^{2m} \left(z_\beta^{-1} \frac{\partial}{\partial z_\alpha} - z_\beta^{-1} \frac{\partial}{\partial z_\beta}\right).\end{aligned} \tag{4.3.48}$$

We need to update the conjugation relations, now with

$$\mathrm{A} \propto \exp\left(\frac{1}{\hbar}\sum_{\mu} z_\mu\right) \prod_{\mu,\eta} \left(\frac{z_\mu + z_\eta}{2}\right)^{-1/2}. \tag{4.3.49}$$

And lemma 4.3.5 becomes:

Lemma 4.3.10.

$$\begin{aligned}A\partial_\alpha A^{-1} &= \partial_\alpha - \hbar^{-1} + \sum_{\gamma} (z_\alpha + z_\gamma)^{-1}, \\ A\partial_\alpha^2 A^{-1} &= \partial_\alpha^2 + \hbar^{-2} - 2\hbar^{-1}\partial_\alpha \\ &\quad + \sum_{\gamma \neq \delta} (z_\alpha + z_\gamma)^{-1} (z_\alpha + z_\delta)^{-1} - \frac{1}{4} z_\alpha^{-2} \\ &\quad + 2\sum_{\gamma} (z_\gamma + z_\alpha)^{-1} \partial_\alpha \\ &\quad -2\hbar^{-1} \sum_{\gamma} (z_\alpha + z_\gamma)^{-1}.\end{aligned}$$

Now let us apply these formulas and compute $\mathrm{A}\left(\mathcal{P}[M]\right)\mathrm{A}^{-1}$.

$$\mathrm{A}\left(\mathcal{P}[M]\right)\mathrm{A}^{-1} = \frac{\hbar^2}{2}\sum_\alpha \frac{\partial^2}{\partial z_\alpha^2} + \frac{\hbar^2}{2}\sum_\alpha z_\alpha^{-1}\frac{\partial}{\partial z_\alpha} \tag{4.3.50}$$

$$+\hbar^2 \sum_{\alpha\neq\beta} \frac{1}{z_\alpha^2 - z_\beta^2}\left(z_\beta \frac{\partial}{\partial z_\alpha} - z_\beta \frac{\partial}{\partial z_\beta}\right) \tag{4.3.51}$$

$$+\hbar^2 \sum_{\alpha\neq\gamma} (z_\gamma + z_\alpha)^{-1}\frac{\partial}{\partial z_\alpha} \tag{4.3.52}$$

$$-\frac{1}{2}\sum_\alpha 1 \tag{4.3.53}$$

$$+\frac{\hbar^2}{2}\sum_\alpha \sum_{\gamma\neq\beta} (z_\alpha + z_\gamma)^{-1}(z_\alpha + z_\beta)^{-1} \tag{4.3.54}$$

$$+\hbar^2 \sum_\gamma \sum_{\alpha\neq\beta} \frac{1}{z_\alpha^2 - z_\beta^2}\left(z_\beta(z_\alpha + z_\gamma)^{-1} - z_\beta(z_\beta + z_\gamma)^{-1}\right). \tag{4.3.55}$$

Once again the last two lines are vanishing,

Lemma 4.3.11.

$$\frac{1}{2}\sum_\alpha \sum_{\gamma\neq\beta} (z_\alpha + z_\gamma)^{-1}(z_\alpha + z_\beta)^{-1}$$
$$+\sum_\gamma \sum_{\alpha\neq\beta} \frac{1}{z_\alpha^2 - z_\beta^2}\left(z_\beta(z_\alpha + z_\gamma)^{-1} - z_\beta(z_\beta + z_\gamma)^{-1}\right)$$
$$= 0.$$

Proof. First let's rewrite

$$\sum_\gamma \sum_{\alpha\neq\beta} \frac{1}{z_\alpha^2 - z_\beta^2}\left(z_\beta(z_\alpha + z_\gamma)^{-1} - z_\beta(z_\beta + z_\gamma)^{-1}\right)$$
$$= \sum_\gamma \sum_{\alpha\neq\beta} \frac{1}{z_\alpha^2 - z_\beta^2}\left(z_\beta(z_\alpha + z_\gamma)^{-1} + z_\alpha(z_\alpha + z_\gamma)^{-1}\right)$$
$$= \sum_\gamma \sum_{\alpha\neq\beta} \frac{1}{z_\alpha - z_\beta}(z_\alpha + z_\gamma)^{-1},$$

then combine with the second term.

$$\begin{aligned}
&\frac{1}{2}\sum_{\alpha}\sum_{\gamma\neq\beta}\frac{1}{z_\alpha+z_\beta}(z_\alpha+z_\gamma)^{-1}+\sum_{\gamma}\sum_{\alpha\neq\beta}\frac{1}{z_\alpha-z_\beta}(z_\alpha+z_\gamma)^{-1}\\
&=\frac{1}{2}\sum_{\gamma}\sum_{\alpha\neq\beta}\frac{1}{z_\gamma+z_\beta}\frac{1}{z_\alpha+z_\gamma}+\sum_{\gamma}\sum_{\alpha\neq\beta}\frac{1}{z_\alpha-z_\beta}\frac{1}{z_\alpha+z_\gamma}\\
&=\frac{1}{2}\sum_{\gamma}\sum_{\alpha\neq\beta}\frac{z_\alpha+z_\beta+2z_\gamma}{(z_\alpha+z_\gamma)(z_\alpha-z_\beta)(z_\beta+z_\gamma)}\\
&=0
\end{aligned}$$

Because the summation is symmetric under $\alpha\leftrightarrow\beta$ while the summand is anti-symmetric.

We can also combine lines (4.3.51) and (4.3.52) along the same lines as (4.3.41). To conclude we have shown the following proposition,

Proposition 4.3.12. *The multi-variable wave-function* (4.3.3) *for the Bessel spectral curve is annihilated by the following operator,*

$$\begin{aligned}
&A(\mathcal{P}[M])A^{-1}\\
&=\frac{\hbar^2}{2}\sum_{\alpha}\frac{\partial^2}{\partial {z_\alpha}^2}+\frac{\hbar^2}{2}\sum_{\alpha}z_\alpha^{-1}\frac{\partial}{\partial z_\alpha}\\
&\quad+\hbar^2\sum_{\alpha\neq\beta}(z_\alpha^2-z_\beta^2)^{-1}\left(z_\alpha\frac{\partial}{\partial z_\alpha}-z_\beta\frac{\partial}{\partial z_\beta}\right)-\frac{1}{2}\sum_{\alpha}1.
\end{aligned}\tag{4.3.56}$$

Equivalently, in terms of $x_\alpha=\frac{1}{2}z_\alpha^2$*,*

$$\begin{aligned}
&A(\mathcal{P}[M])A^{-1}\\
&=\hbar^2\sum_{\alpha}\frac{\partial}{\partial x_\alpha}x_\alpha\frac{\partial}{\partial x_\alpha}\\
&\quad+\hbar^2\sum_{\alpha\neq\beta}(x_\alpha-x_\beta)^{-1}\left(x_\alpha\frac{\partial}{\partial x_\alpha}-x_\beta\frac{\partial}{\partial x_\beta}\right)-\sum_{\alpha}\frac{1}{2}.
\end{aligned}$$

Again, we recover the Bessel curve by setting $N=1$.

$$\mathrm{A}(\mathcal{P}[\sum_i x_{i+\frac{1}{2}}\Lambda_i])\mathrm{A}^{-1}\Big|_{N=1}=\hat{y}\hat{x}\hat{y}-\frac{1}{2}.\tag{4.3.57}$$

It would be interesting to compare this result with what is expected from the theory of integrable systems, as in [29] for the Airy curve. We leave this for future work.

4.4 Two Branch Points

The examples considered so far all had a unique branch point and a unique basis of differential forms. The obvious next step is to apply this method to spectral curves with more branch points, such as the spectral curve of Hermitian matrix models which we studied in section 2.

4.4.1 A usable choice of basis

As we have described earlier in section 3.3.3 , we have a way of constructing an Airy structure for an arbitrary spectral curve from a choice of local coordinates. However it is hard to extend our pedestrian methods of proof to the complicated ABCD coefficients of (3.3.36). So instead of doing it with the local basis, we now define a new basis of differentials that naturally generalizes our method to the case of two branch points.

Recall that our choice of polarization for $\omega_{0,2}$ was odd under the corresponding local involution, and satisfied,

$$\sum_{\substack{r\in\mathfrak{r}\\ k\geq 0}} d\xi_{k,r}(p_0)\xi^*_{k,r}(p) = \frac{1}{2}\left(\int_{\iota(p)}^{p} \omega_{0,2}(p_0,\cdot)\right). \tag{4.4.1}$$

In the two branch point case at genus zero, we can always use a Möbius transformation to map any three points to $0, 1$ and ∞. Recall that a Möbius transformation is defined as

$$f(z) = \frac{a\,z+b}{c\,z+d}, \tag{4.4.2}$$

for $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$. They satisfy to the nice property that they leave the Bergman Kernel unchanged,

$$\omega_{0,2}(f(z_1), f(z_2)) = \omega_{0,2}(z_1, z_2). \tag{4.4.3}$$

We can use a Möbius transformation to map our two branch points to 0 and ∞. Doing so we get two patches, one with coordinate z at 0 and one with coordinate w at ∞, related via $w = \frac{1}{z}$. If the involution of x is locally given by $\iota_0(z) = -z$

and $\iota_\infty(w) = -w$, we can define a $d\xi$ basis in the following way. For $k \geq 0$ let

$$d\xi_{0,k}(z) = \frac{(2k+1)}{z^{2k+2}}dz, \qquad \xi^*_{0,k}(z) = \delta_{r,0}\frac{z^{2k+1}}{(2k+1)}, \tag{4.4.4}$$

$$d\xi_{\infty,k}(w) = \frac{(2k+1)}{w^{2k+2}}dw, \quad \xi^*_{\infty,k}(w) = \delta_{r,\infty}\frac{w^{2k+1}}{(2k+1)}. \tag{4.4.5}$$

Note that the Kronecker delta's in $\xi^*_{0,r}$ and $\xi^*_{\infty,r}$ are there to recall that these are germs of functions, i.e. they are zero outside of a neighbourhood of 0 and ∞ respectively. It is clear that (4.4.1) still holds since $\omega_{0,2}(w_1, w_2) = dw_1 dw_2/(w_1 - w_2)^2$ in the other patch. Furthermore we can merge the two basis to a single basis indexed by $k \in \mathbb{Z}$ by mapping indices $(0, k)$ to k, (∞, k) to $(-1-k)$.

$$d\xi_k(w) = d\xi_{\infty,-k-1}(w) = -(2k+1)w^{2k}dw, \qquad k \leq -1. \tag{4.4.6}$$

$$\xi^*_k(w) = \xi^*_{\infty,-k-1}(w) = -\frac{w^{-2k-1}}{(2k+1)}, \qquad k \leq -1. \tag{4.4.7}$$

So we get a single basis of differentials ξ_k defined as

$$d\xi_k = \frac{(2k+1)}{z^{2k+2}}dz = -(2k+1)w^{2k}dw, \quad k \in \mathbb{Z}. \tag{4.4.8a}$$

$$\xi^*_k = \begin{cases} \frac{z^{2k+1}}{(2k+1)}, & k \geq 0, \\ -\frac{w^{-2k-1}}{(2k+1)}, & k < 0. \end{cases} \tag{4.4.8b}$$

It is the sign of the first index in the ABCD coefficients which determines where we take the residue in the residue formula.

Finally, recall that the conditions on x and y from the definition of a spectral curve implied that $\theta(p)$ have an expansion of the shape

$$\theta(z) = \sum_{m \geq -1} \theta_{0,m} z^{2m}(dz)^{-1} \tag{4.4.9}$$

$$\theta(w) = \sum_{m \geq -1} \theta_{\infty,m} w^{2m}(dw)^{-1} \tag{4.4.10}$$

at 0 and ∞ respectively. We expect some kind of relation between these two expansions since they are related to the same globally defined object θ. This can be very specific to the curve. However when θ has an infinite radius of convergence at both points so that both expansions are equally valid at any given point, and since $w = 1/z$, we have that $\theta_{0,m} = -\theta_{\infty,1-m}$. In particular $\theta_{0,m} = 0$ for $m > 2$. We will return to this constraint later. For now, we will write $\theta_m = \theta_{0,m}$ and $\overline{\theta}_m = \theta_{\infty,m}$.

4.4.2 ABCD relations

We now need to check whether the ABCD coefficients constructed from this basis form an Airy structure. Remark that this Airy structure not the same as the local basis, and hence we have to verify that the ABCD relations also hold in this basis.

It is straightforward to compute the ABCD coefficients; if the first index is positive, express everything in terms of z and take the coefficient of $z^{-1}dz$, if the first index is negative, express everything in terms of w and take the coefficient of $w^{-1}dw$. We can then verify that the resulting coefficients satisfy the ABCD relations.

Proposition 4.4.1. *The ABCD coefficients obtained from the residue formula* (3.3.8) *with the $d\xi$ basis given in* (4.4.8)*,*

$$\mathrm{A}_{ijk} = \delta_{i,j,k,0}\theta_{-1} + \delta_{i,j,k,-1}\overline{\theta}_{-1}, \tag{4.4.11a}$$

$$\mathrm{B}^k_{ij} = \frac{(2k+1)}{(2i+1)}\begin{cases}\theta_{k-i-j} & i,j \geq 0\\ \overline{\theta}_{1+i+j-k} & i,j<0\\ 0 & otherwise\end{cases}, \tag{4.4.11b}$$

$$\mathrm{C}^{jk}_i = \frac{(2k+1)(2j+1)}{(2i+1)}\begin{cases}\theta_{k+j+1-i} & i\geq 0\\ -\overline{\theta}_{i-j-k} & i<0\end{cases}, \tag{4.4.11c}$$

$$\mathrm{D}_i = \frac{1}{8(2i+1)}\left[\theta_0\delta_{i,0} - \overline{\theta}_0\delta_{i,-1} + \theta_{-1}\delta_{i,1} - \overline{\theta}_{-1}\delta_{i,-2}\right], \tag{4.4.11d}$$

where $i,j,k \in \mathbb{Z}$ and $\theta_m, \overline{\theta}_m = 0$ for $m < -1$,satisfy to the ABCD relations (3.1.5) *–* (3.1.9)*.*

Proof. Note that we recover the Airy structure from proposition 3.3.2 when coefficients are either all positive or negative. Therefore we only need to check that the ABCD symmetries still hold for the additional mixed coefficients. Also note that the situation is symmetric between 0 and ∞, so if the relation holds for $i \geq 0, j < 0$, it is also true for $i < 0, j \geq 0$, and similarly for $i,j \geq 0$ and $i,j < 0$. Let us proceed.

1 A_{ijk} is still fully symmetric.

2 The **BA** relation yields

$$\begin{aligned}&\sum_a \left(\mathrm{B}^a_{ij}\mathrm{A}_{akl} + \mathrm{B}^a_{ik}\mathrm{A}_{jal} + \mathrm{B}^a_{il}\mathrm{A}_{jak}\right)\\ &= \theta_{-1}\left(\mathrm{B}^0_{ij}\delta_{k,l,0} + \mathrm{B}^0_{ik}\delta_{j,l,0} + \mathrm{B}^0_{il}\delta_{j,k,0}\right)\\ &+ \overline{\theta}_{-1}\left(\mathrm{B}^{-1}_{ij}\delta_{k,l,-1} + \mathrm{B}^{-1}_{ik}\delta_{j,l,-1} + \mathrm{B}^{-1}_{il}\delta_{j,k,-1}\right).\end{aligned} \tag{4.4.12}$$

If $i, j \geq 0$, the first brackets only have positive indices contributing, and the second bracket only has $\mathrm{B}^{-1}_{0,0}\delta_{i,j,0}$ contributing, which is symmetric.
If $i \geq 0, j < 0$, the first bracket vanishes and the second bracket yields

$$-\overline{\theta}_{-1}\theta_{-1}\delta_{j,l,-1}\delta_{k,i,0} - \overline{\theta}_{-1}\theta_{-1}\delta_{j,k,-1}\delta_{l,i,0}. \tag{4.4.13}$$

If one now swaps $(i, j) \to (j, i)$, then the second bracket vanishes, and the first bracket will give the same as above.

3 Consider the **D** relation,

$$\sum_a \mathrm{B}^a_{ij}\mathrm{D}_a + \tfrac{1}{2}\sum_{a,b}\mathrm{C}^{ab}_i\mathrm{A}_{jab}. \tag{4.4.14}$$

If $i, j \geq 0$, all the coefficients in the sums are positive except for $B^{-1}_{0,0}D_{-1}$ which is symmetric.
If $i \geq 0, j < 0$, the first term vanishes and the second term is

$$\frac{1}{2}C^{-1,-1}_i\delta_{j,-1}\overline{\theta}_{-1} = \frac{1}{2}\overline{\theta}_{-1}\theta_{-1}\delta_{i,0}\delta_{j,-1}. \tag{4.4.15}$$

If we swap $(i, j) \to (j, i)$ we get

$$\frac{1}{2}C^{0,0}_j\delta_{i,0}\theta_{-1} = \frac{1}{2}\overline{\theta}_{-1}\theta_{-1}\delta_{i,0}\delta_{j,-1}, \tag{4.4.16}$$

which is the same.

4 For the **BB-AC** condition,

$$\sum_a \left(\mathrm{B}^a_{ij}\mathrm{B}^l_{ak} + \mathrm{B}^a_{ik}\mathrm{B}^l_{ja} + \mathrm{C}^{la}_i\mathrm{A}_{jak}\right), \tag{4.4.17}$$

for $i \geq 0, j < 0$, the first term vanishes and we have

$$\begin{aligned}&\sum_{a\in\mathbb{Z}} \frac{(2a+1)}{(2i+1)}\delta_{k\geq 0}\theta_{a-i-k}\frac{(2l+1)}{(2j+1)}\delta_{a<0}\overline{\theta}_{1+j+a-l} - \delta_{j,k,-1}\frac{(2l+1)}{(2i+1)}\theta_{l-i}\overline{\theta}_{-1}\\ &= -\theta_{-1}\delta_{i,k,0}\frac{(2l+1)}{(2j+1)}\overline{\theta}_{j-l} - \delta_{j,k,-1}\overline{\theta}_{-1}\theta_{l-i}\frac{(2l+1)}{(2j+1)}.\end{aligned} \tag{4.4.18}$$

If we swap $(i, j) \to (j, i)$, we instead have

$$\begin{aligned}&\sum_{a\in\mathbb{Z}} \frac{(2a+1)}{(2i+1)}\delta_{k<0}\overline{\theta}_{1+j+k-a}\frac{(2l+1)}{(2j+1)}\delta_{a\geq 0}\theta_{l-i-a} - \delta_{i,k,0}\frac{(2l+1)}{(2j+1)}\overline{\theta}_{j-l}\theta_{-1}\\ &= -\delta_{j,k,-1}\overline{\theta}_{-1}\theta_{l-i}\frac{(2l+1)}{(2j+1)} - \theta_{-1}\delta_{i,k,0}\frac{(2l+1)}{(2j+1)}\overline{\theta}_{j-l},\end{aligned} \tag{4.4.19}$$

hence it is symmetric. For $i, j \geq 0$, it yields,

$$\begin{aligned}&\sum_{a\geq 0} \frac{(2a+1)}{(2i+1)}\theta_{a-i-j}\frac{(2l+1)}{(2a+1)}\theta_{l-a-k}\delta_{k\geq 0}\\ &+\sum_{a<0} \frac{(2a+1)}{(2i+1)}\theta_{a-i-j}\frac{(2l+1)}{(2a+1)}\overline{\theta}_{1+a+k-l}\delta_{k<0}\\ &+\sum_{a\geq 0} \frac{(2a+1)}{(2i+1)}\theta_{a-i-k}\frac{(2l+1)}{(2j+1)}\theta_{l-j-a}\delta_{k\geq 0} + \delta_{j,k,0}\frac{(2l+1)}{(2i+1)}\theta_{1+l-i}\theta_{-1}.\end{aligned} \tag{4.4.20}$$

If $k \geq 0$, this is the sum we did before (which works regardless if $l \geq 0$ or $l < 0$). If $k < 0$ the remaining sum only contributes with the symmetric term $\propto \delta_{i,j,0}$.

5 Finally let's look at the **BC** condition,

$$\sum_a \left(\mathrm{B}_{ij}^a \mathrm{C}_a^{kl} + \mathrm{C}_i^{ka}\mathrm{B}_{ja}^l + \mathrm{C}_i^{la}\mathrm{B}_{ja}^k\right). \tag{4.4.21}$$

We see that if $i, j \geq 0$, all first coefficients have a positive first index except in $B_{0,0}^{-1}C_{-1}^{k,l}$ in the first sum, which is symmetric. Hence the result from the previous lemma applies with the same calculation.

If $i \geq 0, j < 0$ we have

$$= \sum_{a\in\mathbb{Z}} \frac{(2k+1)(2a+1)}{(2i+1)} \theta_{1+k+a-i} \frac{(2l+1)}{(2j+1)} \overline{\theta}_{1+j+a-l} \delta_{a<0} + (k \leftrightarrow l). \tag{4.4.22}$$

If we swap $i \leftrightarrow j$ we obtain

$$= -\sum_{a\in\mathbb{Z}} \frac{(2k+1)(2a+1)}{(2i+1)} \overline{\theta}_{j-k-a} \frac{(2l+1)}{(2j+1)} \theta_{l-i-a} \delta_{a\geq 0} + (k \leftrightarrow l). \tag{4.4.23}$$

These two sums are the same as can be seen after re-indexing $a \to -a-1$. ☕

With this proof, proceeding as in the previous sections, we know that the resulting Airy structure L_i with $i \in \mathbb{Z}$ annihilates the partition function Z, and we also know that the specialization $x_i = \hbar \int_\alpha^z d\xi_i$ recovers the wave function modulo $\omega_{0,1}$ and $\omega_{0,2}$ factors. The specialization naturally gives a function of a single variable z globally defined on the Riemann sphere.

4.4.3 Specialization

Let us go ahead and specialize but now with the Airy structure from (4.4.11), namely

$$L_i = \hbar^2 \partial_i - \frac{1}{2}\theta_{-1}\delta_{i,0}(x_0)^2 - \frac{1}{2}\overline{\theta}_{-1}\delta_{i,-1}(x_{-1})^2 \tag{4.4.24}$$

$$- \delta_{i\geq 0}\hbar^2 \sum_{\substack{a\geq 0\\ b\in\mathbb{Z}}} \frac{(2b+1)}{(2i+1)} \theta_{b-i-a} x_a \partial_b \tag{4.4.25}$$

$$- \delta_{i<0}\hbar^2 \sum_{\substack{a<0\\ b\in\mathbb{Z}}} \frac{(2b+1)}{(2i+1)} \overline{\theta}_{1+i+a-b} x_a \partial_b \tag{4.4.26}$$

$$- \delta_{i\geq 0}\frac{\hbar^4}{2} \sum_{a,b\in\mathbb{Z}} \frac{(2a+1)(2b+1)}{(2i+1)} \theta_{a+b+1-i} \partial_a \partial_b \tag{4.4.27}$$

$$- \delta_{i<0}\frac{\hbar^4}{2} \sum_{a,b\in\mathbb{Z}} \frac{(2a+1)(2b+1)}{(2i+1)} (-\overline{\theta}_{i-a-b}) \partial_a \partial_b \tag{4.4.28}$$

$$- \frac{\hbar^2}{8(2i+1)} \left[\theta_0\delta_{i,0} - \overline{\theta}_0\delta_{i,-1} + \theta_{-1}\delta_{i,1} - \overline{\theta}_{-1}\delta_{i,-2}\right] \tag{4.4.29}$$

Let us consider a slightly different definition of the evolution operator (4.1.10). Instead let,

$$M = -\frac{1}{\hbar}\sum_{i\in\mathbb{Z}}(2i+1)w_i L_i, \tag{4.4.30}$$

with the short hand notation $w_i = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}$. The reason for this choice is the following: previously we were integrating with base point at ∞ which resulted in no terms associated to the base point while specializing. However here ∞ is already assumed to be a branch point, so we have to integrate with respect to some $\alpha \in \mathbb{C}$ instead. Upon specializing we will have

$$x_i \to -\hbar(z^{-2i-1} - \alpha^{-2i-1}), \tag{4.4.31}$$

$$w_i \to -\hbar(1-z^2)z^{-2i-2} + \hbar(1-\alpha^2)\alpha^{-2i-2}. \tag{4.4.32}$$

The constant term in x_i would be problematic when specializing, but by setting α to be at ± 1, which we can do with the remaining freedom in the Möbius transformation, the constant term in w_i vanishes and the specialization will work out nicely. Furthermore we observed in practice that the extra factor $(1-z^2)$ was necessary to recover the expected quantum curve. Ultimately it would be nice to get a better conceptual understanding of why (4.4.30) is the right evolution operator to consider. After plugging in L_i in the evolution operator we have,

$$M = -\sum_{i\in\mathbb{Z}}\hbar(2i+1)w_i\partial_i + \frac{\hbar^{-1}}{2}\theta_{-1}w_0(x_0)^2 - \frac{\hbar^{-1}}{2}\overline{\theta}_{-1}w_{-1}(x_{-1})^2 \tag{4.4.33}$$

$$+\hbar\sum_{\substack{i,a\geq 0\\ b\in\mathbb{Z}}}(2b+1)\theta_{b-i-a}w_i x_a\partial_b \tag{4.4.34}$$

$$+\hbar\sum_{\substack{i,a<0\\ b\in\mathbb{Z}}}(2b+1)\overline{\theta}_{1+i+a-b}w_i x_a\partial_b \tag{4.4.35}$$

$$+\frac{\hbar^3}{2}\sum_{\substack{i\geq 0\\ a,b\in\mathbb{Z}}}(2a+1)(2b+1)\theta_{a+b+1-i}w_i\partial_a\partial_b \tag{4.4.36}$$

$$+\frac{\hbar^3}{2}\sum_{\substack{i<0\\ a,b\in\mathbb{Z}}}(2a+1)(2b+1)(-\overline{\theta}_{i-a-b})w_i\partial_a\partial_b \tag{4.4.37}$$

$$+\frac{\hbar}{8}\left[\theta_0 w_0 - \overline{\theta}_0 w_{-1} + \theta_{-1}w_1 - \overline{\theta}_{-1}w_{-2}\right]. \tag{4.4.38}$$

Let us apply this operator to the monomials and specialize:

$$x_{\mu_1} \cdots x_{\mu_n} \xrightarrow{\mathcal{S}} (-1)^n \hbar^n \prod_{\gamma=1}^{n} (z^{-2\mu_\gamma - 1} - \alpha^{-2\mu_\gamma - 1}). \tag{4.4.39}$$

We start with the linear term.

Lemma 4.4.2.

$$\mathcal{S}\left[-\hbar \sum_{i\in\mathbb{Z}} (2i+1) w_i \partial_i\right] = (1-z^2)\hbar \frac{d}{dz}. \tag{4.4.40}$$

Proof.

$$\mathcal{S}\left[\sum_{i\in\mathbb{Z}} (2i+1) w_i \partial_i (x_{\mu_1} \cdots x_{\mu_n})\right]$$

$$= \mathcal{S}\left[\sum_{\rho=1}^{n} \sum_{i\in\mathbb{Z}} (2i+1) w_i \delta_{i,\mu_\rho} \frac{1}{x_{\mu_\rho}} (x_{\mu_1} \cdots x_{\mu_n})\right] \tag{4.4.41}$$

$$= \mathcal{S}\left[\sum_{\rho=1}^{n} (2\mu_\rho + 1) \frac{w_{\mu_\rho}}{x_{\mu_\rho}} (x_{\mu_1} \cdots x_{\mu_n})\right] \tag{4.4.42}$$

$$= (1-z^2) \sum_{\rho=1}^{n} (2\mu_\rho + 1) \frac{z^{2\mu_\rho - 2}}{z^{-2\mu_\rho - 1} - \alpha^{-2\mu_\rho - 1}} \mathcal{S}\left[x_{\mu_1} \cdots x_{\mu_n}\right] \tag{4.4.43}$$

$$= -(1-z^2) \frac{d}{dz} \mathcal{S}[x_{\mu_1} \cdots x_{\mu_n}]. \tag{4.4.44}$$

We also have the following lemma.

Lemma 4.4.3. *under $\overline{\theta}_{1-k} = -\theta_k$, the C terms specialize as*

$$\mathcal{S}\left[\frac{\hbar^3}{2}\sum_{\substack{i\geq 0\\ a,b\in\mathbb{Z}}}(2a+1)(2b+1)\theta_{a+b+1-i}w_i\partial_a\partial_b\right.$$

$$\left.\frac{\hbar^3}{2}\sum_{\substack{i<0\\ a,b\in\mathbb{Z}}}(2a+1)(2b+1)(-\overline{\theta}_{i-a-b})w_i\partial_a\partial_b\right]$$

$$= -\frac{\hbar^2}{2}(1-z^2)\theta(z)\frac{d^2}{dz^2} + \frac{\hbar^2}{2}(1-z^2)\theta(z)\sum_{\rho=1}^{n}(2\mu_\rho+1)(2\mu_\rho+2)\frac{z^{-2\mu_\rho-3}}{z^{-2\mu_\rho-1}-\alpha^{-2\mu_\rho-1}}. \tag{4.4.45}$$

Proof. Multiply by an overall $-2\hbar^{-2}$. Looking at the first term, after applying the derivatives and specializing, we have

$$\sum_{\substack{\rho,\eta=1\\ \rho\neq\eta}}^{n}\sum_{\substack{i\geq 0\\ a,b\in\mathbb{Z}}}(2a+1)(2b+1)\theta_{1+a+b-i}\frac{(1-z^2)z^{-2i-2}\,\delta_{a,\mu_\rho}\delta_{b,\mu_\eta}}{(z^{-2a-1}-\alpha^{-2a-1})(z^{-2b-1}-\alpha^{-2b-1})} \tag{4.4.46}$$

$$= \sum_{\substack{\rho,\eta=1\\ \rho\neq\eta}}^{n}\sum_{j\geq -1}^{\mu_\rho+\mu_\eta+1}(2\mu_\rho+1)(2\mu_\eta+1)\frac{(1-z^2)\theta_j z^{-2(\mu_\rho+\mu_\eta-j)-4}}{(z^{-2\mu_\rho-1}-\alpha^{-2\mu_\rho-1})(z^{-2\mu_\eta-1}-\alpha^{-2\mu_\eta-1})}, \tag{4.4.47}$$

where in the second line we set $j = 1+a+b-i$ so for fixed a, b, j will run from -1 to $1+a+b$. Now consider the second term. Using that $-\overline{\theta}_{i-a-b} = \theta_{1+a+b-i}$, and with the same relabelling of summation index, it gives us the exact same thing but with j running from $2+a+b$ to ∞, so the sum becomes

$$\begin{aligned} &= (1-z^2) \\ &\quad \times \sum_{\substack{\rho,\eta=1 \\ \rho\neq\eta}}^{n} \sum_{j\geq -1} (2\mu_\rho+1)(2\mu_\eta+1) \frac{\theta_j z^{-2(\mu_\rho+\mu_\eta-j)-4}}{\left(z^{-2\mu_\rho-1}-\alpha^{-2\mu_\rho-1}\right)\left(z^{-2\mu_\eta-1}-\alpha^{-2\mu_\eta-1}\right)} \end{aligned} \tag{4.4.48}$$

$$\begin{aligned} &= (1-z^2)\theta(z) \\ &\quad \times \sum_{\substack{\rho,\eta=1 \\ \rho\neq\eta}}^{n} (2\mu_\rho+1)(2\mu_\eta+1) \frac{z^{-2\mu_\rho-2}}{\left(z^{-2\mu_\rho-1}-\alpha^{-2\mu_\rho-1}\right)} \frac{z^{-2\mu_\eta-2}}{\left(z^{-2\mu_\eta-1}-\alpha^{-2\mu_\eta-1}\right)}, \end{aligned} \tag{4.4.49}$$

where we recognize a double derivative minus diagonal terms.

For the B terms we have the next lemma.

Lemma 4.4.4. *Under $\overline{\theta}_{1-k} = -\theta_k$ and $\theta(\alpha)=0$,the B terms specialize as*

$$\begin{aligned} &\mathcal{S}\left[\hbar \sum_{\substack{i,a\geq 0 \\ b\in\mathbb{Z}}} (2b+1)\theta_{b-i-a} w_i x_a \partial_b + \hbar \sum_{\substack{i,a<0 \\ b\in\mathbb{Z}}} (2b+1)\overline{\theta}_{1+i+a-b} w_i x_a \partial_b\right] \\ &\quad = -\frac{\hbar^2}{2}(1-z^2)\theta(z)\sum_{\rho=1}^{n}(2\mu_\rho+1)(2\mu_\rho+2)\frac{z^{-2\mu_\rho-3}}{z^{-2\mu_\rho-1}-\alpha} \\ &\qquad\qquad + \frac{\hbar^2}{2}((1-z^2)\left(\theta'(z)+2\frac{\alpha}{\alpha^2-z^2}\theta(z)\right)\frac{d}{dz}. \end{aligned} \tag{4.4.50}$$

Proof. Multiply by an overall $-2\hbar^{-2}$. Let us look at the first term,

$$\begin{aligned} &2\sum_{\rho=1}^{n} \sum_{\substack{i,a\geq 0 \\ b\in\mathbb{Z}}} (2b+1)\theta_{b-i-a}(1-z^2)\frac{z^{-2i-2}\left(z^{-2a-1}-\alpha^{-2a-1}\right)}{\left(z^{-2b-1}-\alpha^{-2b-1}\right)}\delta_{b,\mu_\rho} \\ &\quad = 2(1-z^2)\sum_{\rho=1}^{n}\sum_{j=-1}^{\mu_\rho}\sum_{a=0}^{\mu_\rho-j}(2\mu_\rho+1)\theta_j\frac{z^{-2(\mu_\rho-j-a)-2}\left(z^{-2a-1}-\alpha^{-2a-1}\right)}{\left(z^{-2\mu_\rho-1}-\alpha^{-2\mu_\rho-1}\right)}, \end{aligned} \tag{4.4.51}$$

where we set $j=b-i-a$ which runs from -1 to b, and for fixed b and j we have a running from 0 to $b-j$. Upon evaluating the sum on a, we get two terms, one sum independent on a that yields $\mu_\rho - j + 1$ identical terms, and one truncated geometric series $\sum_{a=0}^{\mu_\rho-j}(z/\alpha)^{2a} = \frac{1-(z/\alpha)^{2(\mu_\rho-j+1)}}{1-(z/\alpha)^2}$,

$$= (1-z^2)\sum_{\rho=1}^{n}\sum_{j=-1}^{\mu_\rho}(2\mu_\rho+1)(2\mu_\rho-2j+2)\theta_j z^{2j}\frac{z^{-2\mu_\rho-3}}{\left(z^{-2\mu_\rho-1}-\alpha^{-2\mu_\rho-1}\right)}$$
$$-2\alpha\frac{1-z^2}{\alpha^2-z^2}\sum_{\rho=1}^{n}\sum_{j=-1}^{\mu_\rho}(2\mu_\rho+1)\theta_j\frac{z^{-2(\mu_\rho-j)-2}-\alpha^{-2(\mu_\rho-j)-2}}{\left(z^{-2\mu_\rho-1}-\alpha^{-2\mu_\rho-1}\right)}. \quad (4.4.52)$$

We follow the same procedure with the second term, except now the sum on $i, a < 0$ gives $j = b - i - a$ running from $b+2$ to ∞, and for fixed b and j, a is going from $b-j+1$ to -1. We work under $\overline{\theta}_{1+i+a-b} = -\theta_{b-i-a}$. The constant sum yields $j-\mu_\rho-1$ identical terms, and the geometric series here is $\sum_{a=b-j+a}^{-1}(z/\alpha)^{2a} = \sum_{a=0}^{j-b-2}(z/\alpha)^{-2a-2} = -\frac{1-(z/\alpha)^{2(b-j+1)}}{1-(z/\alpha)^2}$, therefore equation (4.4.52) becomes,

$$2(1-z^2)\sum_{\rho=1}^{n}\sum_{j=\mu_\rho+2}^{\infty}\sum_{a=\mu_\rho-j+1}^{-1}(2\mu_\rho+1)\theta_j\frac{z^{-2(\mu_\rho-j-a)-2}\left(z^{-2a-1}-\alpha^{-2a-1}\right)}{\left(z^{-2\mu_\rho-1}-\alpha^{-2\mu_\rho-1}\right)} \quad (4.4.53)$$

$$= (1-z^2)\sum_{\rho=1}^{n}\sum_{j=\mu_\rho+2}^{\infty}(2\mu_\rho+1)(2\mu_\rho-2j+2)\theta_j z^{2j}\frac{z^{-2\mu_\rho-3}}{\left(z^{-2\mu_\rho-1}-\alpha^{-2\mu_\rho-1}\right)}$$
$$-2\alpha\frac{(1-z^2)}{\alpha^2-z^2}\sum_{\rho=1}^{n}\sum_{j=\mu_\rho+2}^{\infty}(2\mu_\rho+1)\theta_j\frac{z^{-2(\mu_\rho-j)-2}-\alpha^{-2(\mu_\rho-j)-2}}{\left(z^{-2\mu_\rho-1}-\alpha^{-2\mu_\rho-1}\right)}. \quad (4.4.54)$$

Notice how the missing $j = \mu_\rho + 1$ in the sum is identically zero, so the sums nicely complete each other in

$$(1-z^2)\sum_{\rho=1}^{n}\sum_{j\geq -1}(2\mu_\rho+1)(2\mu_\rho+2)\theta_j z^{2j}\frac{z^{-2\mu_\rho-3}}{\left(z^{-2\mu_\rho-1}-\alpha^{-2\mu_\rho-1}\right)}$$
$$-(1-z^2)\sum_{\rho=1}^{n}\sum_{j\geq -1}(2\mu_\rho+1)2j\theta_j z^{2j-1}\frac{z^{-2\mu_\rho-2}}{\left(z^{-2\mu_\rho-1}-\alpha^{-2\mu_\rho-1}\right)}$$
$$-2\alpha\frac{(1-z^2)}{(\alpha^2-z^2)}\sum_{\rho=1}^{n}\sum_{j\geq -1}(2\mu_\rho+1)\theta_j\frac{z^{-2\mu_\rho+2j-2}-\cancel{\alpha^{-2\mu_\rho+2j-2}}}{\left(z^{-2\mu_\rho-1}-\alpha^{-2\mu_\rho-1}\right)} \quad (4.4.55)$$

$$= (1-z^2)\vartheta(z)\sum_{\rho=1}^{n}(2\mu_\rho+1)(2\mu_\rho+2)\frac{z^{-2\mu_\rho-3}}{(z^{-2\mu_\rho-1}-\alpha^{-2\mu_\rho-1})}$$

$$+(1-z^2)\vartheta(z)'\frac{d}{dz}+2\alpha\frac{1-z^2}{\alpha^2-z^2}\vartheta(z)\frac{d}{dz}. \quad (4.4.56)$$

Where we again used the notation $\vartheta(z)=\theta(z)dz$. Here we assumed $\theta(\alpha)=0$ while cancelling. ☕

Hence we just showed that,

$$\mathcal{S}\,[\text{B term} + \text{C term}]$$
$$= -\frac{\hbar^2}{2}(1-z^2)\vartheta(z)\frac{d^2}{dz^2}-\frac{\hbar^2}{2}(1-z^2)\left(\vartheta'(z)+2\frac{\alpha}{\alpha^2-z^2}\vartheta(z)\right)\frac{d}{dz}. \quad (4.4.57)$$

Continuing, the A term in (4.4.33) yields

$$\mathcal{S}\left[\frac{\hbar^{-1}}{2}\theta_{-1}w_0(x_0)^2-\frac{\hbar^{-1}}{2}\overline{\theta}_{-1}w_{-1}(x_{-1})^2\right]$$
$$= -\frac{\hbar^2}{2}(1-z^2)(\theta_{-1}z^{-2}(z^{-1}-\alpha^{-1})^2+\theta_2(z-\alpha)^2). \quad (4.4.58)$$

Finally, the D term in (4.4.38), again under the assumption $\overline{\theta}_{1-k}=-\theta_k$ and $\theta_k=0\,,k>2$, gives

$$\mathcal{S}\left[\frac{\hbar}{8}\sum_{i\geq-1}\theta_i w_{-i}\right]=-\frac{\hbar^2}{8}(1-z^2)\sum_{i\geq-1}\theta_i z^{2i-2} \quad (4.4.59)$$

$$=-\frac{\hbar^2}{8}(1-z^2)z^{-2}\vartheta(z). \quad (4.4.60)$$

The result is summarized in the next proposition.

Proposition 4.4.5. *Let (x,y) be a spectral curve on $\mathbb{C}P^1$ with two branch points set at 0 and ∞ and the poles of x set at $\alpha=\pm1$. Furthermore assume that the coefficients θ_k and $\overline{\theta}_k$ of the local expansions of θ at 0 and ∞ respectively, satisfy to $\overline{\theta}_{1-k}=-\theta_k$ and in particular $\theta_k=0$ for $k>2$. Then the specialization of* *the evolution operator* (4.4.30) *is:*

$$\begin{aligned}
&\mathcal{S}[M] \\
&= (1-z^2)\left[\hbar\frac{d}{dz} - \frac{\hbar^2}{2}\vartheta(z)\frac{d^2}{dz^2} - \frac{\hbar^2}{2}\left(\vartheta'(z) + \frac{2}{\alpha^2 - z^2}\vartheta(z)\right)\frac{d}{dz}\right. \\
&\qquad \left. - \frac{\hbar^2}{2}\theta_{-1}z^{-2}(\frac{1}{z} - \frac{1}{\alpha})^2 - \frac{\hbar^2}{2}\theta_2(z-\alpha)^2 - \frac{\hbar^2}{8}z^{-2}\vartheta(z)\right],
\end{aligned} \tag{4.4.61}$$

where $\vartheta(z) = \theta(z)dz$.

4.4.4 Applications

Here are 3 spectral curves that can be treated with the above method and which are found in [12]: the Gaussian curve which we talked about in the introduction, the curve related to the enumeration of Grothendieck's dessins d'enfants, which is mentioned in table 2.1, and what we will call the generalized Catalan number curve, which is related to ribbon graphs in [12], but differs to the latter by some weights, as defined in [40][2].

We recall here the corresponding algebraic curves:

$$4y^2 - x^2 + 4 = 0, \qquad \text{(Gaussian Curve)}$$
$$y^2 - xy + 1 = 0, \qquad \text{(Gen. Catalan)}$$
$$xy^2 - xy + 1 = 0. \qquad \text{(Dessins d'enfants)}$$

We shall use the parametrizations:

$$x = 2\frac{z^2+1}{z^2-1}, \qquad y = 2\frac{z}{1-z^2}, \qquad \text{(Gaussian Curve)}$$
$$x = 2\frac{z^2+1}{z^2-1}, \qquad y = \frac{1+z}{1-z}, \qquad \text{(Gen. Catalan)}$$
$$x = 4\frac{z^2}{z^2-1}, \qquad y = \frac{z-1}{2z}, \qquad \text{(Dessins d'enfants)}$$

These parametrizations all have the two branch points at 0 and ∞, as well as the two poles of x at ± 1. Therefore we have two choices for the value of α in proposition 4.4.5, and two possible quantizations.

[2] In particular, the generalized Catalan numbers $C_{g,n}$ are defined as $C_{g,n}(\mu_1, \ldots, \mu_n) = \mu_1 \ldots \mu_n \mathcal{T}_n^{(g)}(\mu_1, \ldots, \mu_n)$ where $\mathcal{T}_{g,n}$ is the generating function for connected fatgraphs described in chapter 2.

The corresponding $\theta(z)$ are globally defined, and read:

$$\vartheta(z) = -\frac{1}{16z^2} + \frac{3}{16} - \frac{3}{16}z^2 + \frac{1}{16}z^4 \qquad \text{(Gaussian Curve)}$$

$$\vartheta(z) = -\frac{1}{16z^2} + \frac{3}{16} - \frac{3}{16}z^2 + \frac{1}{16}z^4 \qquad \text{(Gen. Catalan)}$$

$$\vartheta(z) = -\frac{1}{4} + \frac{1}{2}z^2 - \frac{1}{4}z^4 \qquad \text{(Dessins d'enfants)}$$

Notice that they all satisfy to the truncation condition in proposition 4.4.5, that $\theta_k = 0$ for $k > 2$, and $\overline{\theta}_{1-k} = -\theta_k$. Moreover, for $\alpha = \pm 1$, $\theta(\alpha) = 0$.

Computation of A

We now want to conjugate with the A factor as usual. Recall that A contains the contribution of $\omega_{0,1}$ and $\omega_{0,2}$ in the wave-function (2.7.2),

$$\mathrm{A} = \exp\left[\frac{1}{\hbar}\int_\alpha^z \omega_{0,1}\right]\exp\left[\frac{1}{2}\int_\alpha^z\int_\alpha^z \omega_{0,2} - \frac{dx_1 dx_2}{(x_1-x_2)^2}\right], \tag{4.4.62}$$

where α is a pole of x. Furthermore, after integration the $\omega_{0,2}$ contribution is equal to,

$$\exp\left[\frac{1}{2}\left(\log\frac{z_1-z_2}{x_1-x_2}\right)\Big|_{z_1=\alpha}^{z}\Big|_{z_2=\alpha}^{z}\right] = \sqrt{\frac{z_1-z_2}{x_1-x_2}}\Big|_{\substack{z_1=z\\z_2=z}}\left(\sqrt{\frac{z_1-z_2}{x_1-x_2}}\Big|_{\substack{z_1=z\\z_2=\alpha}}\right)^{-2}\sqrt{\frac{z_1-z_2}{x_1-x_2}}\Big|_{\substack{z_1=\alpha\\z_2=\alpha}}. \tag{4.4.63}$$

Let us return to the problem of the normalization of this factor which we introduced back the one branch point case 4.2.1. Recall that we can throw away any infinite or zero constant factor, but we must keep the stuff that depends on z. We are integrating a new function $x(z)$ and the choices of poles are now $\alpha = \pm 1$. In all the cases above one can check,

$$\sqrt{\frac{z_1-z_2}{x_1-x_2}} \propto \sqrt{\frac{(z_1-1)(z_1+1)(z_2-1)(z_2+1)}{z_1+z_2}}. \tag{4.4.64}$$

Plugging this into the right hand side in (4.4.63) and substituting $\alpha = \pm 1$, the third factor yields 0 which we discard, and we are left with,

$$\exp\left[\frac{1}{2}\left(\log\frac{z_1-z_2}{x_1-x_2}\right)\Big|_{z_1=\alpha}^{z}\Big|_{z_2=\alpha}^{z}\right] \propto \sqrt{\frac{z_1-z_2}{x_1-x_2}}\Big|_{\substack{z_1=z\\z_2=z}} \times \frac{1}{(z-\alpha)}. \tag{4.4.65}$$

The $\omega_{0,1}$ factor is easily computed case by case.

Results

All we have to do is to plug the corresponding $\theta(z)$ in proposition 4.4.5, conjugate with the A factor, and check whether it agrees with some quantization of $P(x,y)$ after substituting $\hat{x} = x(z)$ and $\hat{y} = \hbar \left(\frac{dx}{dz}\right)^{-1} \frac{d}{dz}$. This is the kind of thing that's easier done with a computer algebra system.

Gaussian curve. When $\alpha = 1$, the output is

$$\begin{aligned} A(\mathcal{S}[M])A^{-1} = &\frac{1}{64z^3\left(z^4-2z^2+1\right)} \\ &\times\big[\hbar^2 z\left(z^{12}-6z^{10}+15z^8-20z^6+15z^4-6z^2+1\right)\frac{d^2}{dz^2} \\ &+\hbar^2\left(3z^{12}-14z^{10}+25z^8-20z^6+5z^4+2z^2-1\right)\frac{d}{dz} \\ &-32z^3\left(\hbar z^4-2\hbar z^2+\hbar+8z^2\right)\big]. \end{aligned} \tag{4.4.66}$$

By comparison, we find that this is the same as

$$\hat{P}(\hat{x},\hat{y}) = 4\hat{y}^2 - \hat{x}^2 + 4 - 2\hbar \tag{4.4.67}$$

but multiplied by an overall $\frac{1}{4}$. This is a quantization of the spectral curve $4y^2 - x^2 + 4 = 0$ with a quantum correction term $-2\hbar$.
When $\alpha = -1$, the output of the lemma is

$$\begin{aligned} A(\mathcal{S}[M])A^{-1} = &\frac{1}{64z^3\left(z^4-2z^2+1\right)} \\ &\times\big[\hbar^2 z\left(z^{12}-6z^{10}+15z^8-20z^6+15z^4-6z^2+1\right)\frac{d^2}{dz^2} \\ &+\hbar^2\left(3z^{12}-14z^{10}+25z^8-20z^6+5z^4+2z^2-1\right)\frac{d}{dz} \\ &-32z^3\left(-\hbar z^4+2\hbar z^2-\hbar+8z^2\right)\big] \end{aligned} \tag{4.4.68}$$

Which corresponds to,

$$\hat{P}(\hat{x},\hat{y}) = 4\hat{y}^2 - \hat{x}^2 + 4 + 2\hbar \tag{4.4.69}$$

which is, again, multiplied by an overall $\frac{1}{4}$. This is again a quantization of the spectral curve, this time with a quantum correction term $+2\hbar$. This is in agreement with the results in [12, 5].

Generalized Catalan numbers. When $\alpha = 1$, the output is

$$\begin{aligned} A(\mathcal{S}[M])A^{-1} = & \left(\frac{\hbar^2 z^6}{64} - \frac{\hbar^2 z^4}{16} + \frac{3\hbar^2 z^2}{32} - \frac{\hbar^2}{16} + \frac{\hbar^2}{64z^2}\right)\frac{d^2}{dz^2} \\ & + \left(\frac{3\hbar^2 z^5}{64} - \frac{\hbar^2 z^3}{8} + \frac{3\hbar^2 z}{32} - \frac{\hbar^2}{64z^3} + \frac{\hbar z^3}{4} - \frac{\hbar}{4z}\right)\frac{d}{dz} \\ & + 1 - \hbar. \end{aligned} \tag{4.4.70}$$

By comparison, we find that this is the same as

$$\hat{P}(\hat{x},\hat{y}) = \hat{y}^2 - \hat{x}\hat{y} + 1, \tag{4.4.71}$$

This is the canonical quantization of the spectral curve $y^2 - xy + 1 = 0$, with the ordering $xy \to \hat{x}\hat{y}$.

When $\alpha = -1$, the output of the lemma is

$$\begin{aligned} A(\mathcal{S}[M])A^{-1} = & \left(\frac{\hbar^2 z^6}{64} - \frac{\hbar^2 z^4}{16} + \frac{3\hbar^2 z^2}{32} - \frac{\hbar^2}{16} + \frac{\hbar^2}{64z^2}\right)\frac{d^2}{dz^2} \\ & + \left(\frac{3\hbar^2 z^5}{64} - \frac{\hbar^2 z^3}{8} + \frac{3\hbar^2 z}{32} - \frac{\hbar^2}{64z^3} + \frac{\hbar z^3}{4} - \frac{\hbar}{4z}\right)\frac{d}{dz} \\ & + 1, \end{aligned} \tag{4.4.72}$$

which corresponds to,

$$\hat{P}(\hat{x},\hat{y}) = \hat{y}^2 - \hat{y}\hat{x} + 1, \tag{4.4.73}$$

We now obtain a quantization of the same spectral curve but with a different ordering $xy \to \hat{y}\hat{x}$.

Dessins d'enfants. When $\alpha = 1$, the output is

$$\begin{aligned} A(\mathcal{S}[M])A^{-1} = & -\left(\frac{\hbar^2 z^6}{16} - \frac{3\hbar^2 z^4}{16} + \frac{3\hbar^2 z^2}{16} - \frac{\hbar^2}{16}\right)\frac{d^2}{dz^2} \\ & - \left(\frac{3\hbar^2 z^5}{16} - \frac{7\hbar^2 z^3}{16} + \frac{5\hbar^2 z}{16} - \frac{\hbar^2}{16z} + \frac{\hbar z^3}{2} - \frac{\hbar z}{2}\right)\frac{d}{dz} \\ & - 1. \end{aligned} \tag{4.4.74}$$

By comparison, we find that this is the same as

$$\hat{P}(\hat{x},\hat{y}) = \hat{y}\hat{x}\hat{y} - \hat{x}\hat{y} + 1 \tag{4.4.75}$$

multiplied by an overall $-$ sign, Which is a quantization of the spectral curve

$xy^2 + xy + 1 = 0$.
When $\alpha = -1$, the output of the lemma is

$$
\begin{aligned}
A(\mathcal{S}[M])A^{-1} = &-\left(\frac{\hbar^2 z^6}{16} - \frac{3\hbar^2 z^4}{16} + \frac{3\hbar^2 z^2}{16} - \frac{\hbar^2}{16}\right)\frac{d^2}{dz^2} \\
&-\left(\frac{3\hbar^2 z^5}{16} - \frac{7\hbar^2 z^3}{16} + \frac{5\hbar^2 z}{16} - \frac{\hbar^2}{16z} + \frac{\hbar z^3}{2} - \frac{\hbar z}{2}\right)\frac{d}{dz} \\
&-(1-\hbar),
\end{aligned}
\tag{4.4.76}
$$

which corresponds to,

$$\hat{P}(\hat{x},\hat{y}) = \hat{y}\hat{x}\hat{y} - \hat{y}\hat{x} + 1, \tag{4.4.77}$$

again multiplied by an overall $-$ sign, a quantization of the spectral curve but with a different ordering.

Comparison. All the cases above where already proved in [12]. The quantum curves obtained agree with the known results for each choice of base point.

Remark 4.4.6. One notices that the curves for the generalized Catalan curve and the Gaussian curve are in fact related by the transformation

$$y_{\text{Gauss}}(z) = y_{\text{Cat}}(z) - \frac{1}{2}x(z) \tag{4.4.78}$$

We already commented on curves related in this way in section 4.2.4. In particular it is not a surprise that they have the same $\theta(z)$. Moreover, their quantization are mutually related: if we take the quantum curve of the Gaussian curve for the base point $\alpha = 1$ and shift it with $\hat{y} \to \hat{y} - \frac{1}{2}\hat{x}$, we obtain:

$$4\left(\hat{y} - \frac{1}{2}\hat{x}\right)^2 - \hat{x}^2 + 4 + 2\hbar = 4\hat{y}^2 - 4\hat{x}\hat{y} + 4 \tag{4.4.79}$$

which is 4 times the canonical quantization associated to the base point $\alpha = 1$ for the Catalan curve, $\hat{y} - \hat{x}\hat{y} + 1$. The same correspondence holds for $\alpha = -1$. It is interesting to see that the perhaps unexpected quantum corrections $\pm 2\hbar$ in the Gaussian curve are related to the fact that the Catalan curve has the simpler looking quantization where we just substitute $x \to \hat{x}$ and $y \to \hat{y}$ modulo choice of ordering.

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