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TITLE OF THESIS HYPERQUANTIZATION OF SPIN 1/2 AND SPIN 3/2 ELECTROMAGNETIC INTERACTIONS

UNIVERSITY ALBERTA

DEGREE FOR WHICH THESIS WAS PRESENTED Ph.D.

YEAR THIS DEGREE GRANTED 1971

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HYPERQUANTIZATION OF SPIN 1/2 AND SPIN 3/2 ELECTROMAGNETIC INTERACTIONS

by

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A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES AND RESEARCH
IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE
OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF PHYSICS

EDMONTON, ALBERTA

FALL, 1971

UNIVERSITY OF ALBERTA

FACULTY OF GRADUATE STUDIES AND RESEARCH

The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled HYPERQUANTIZATION OF SPIN 1/2 AND SPIN 3/2 ELECTROMAGNETIC INTERACTIONS, submitted by Kenneth Wayne Dormuth, in partial fulfilment of the requirements for the degree of Doctor of Philosophy.

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ABSTRACT

An approach to quantum field theory known as hyperquantization is reviewed. This formalism is then applied to the quantum electrodynamics of a spin 1/2 field. The result is a gauge invariant theory containing an operator S whose matrix elements agree with those of the conventional S-matrix of quantum electrodynamics, but in which the use of an indefinite metric can be avoided. Furthermore, the approach is used to quantize the system of interacting electromagnetic and spin 3/2 fields, and it is found that the well-known difficulty of negative field anticommutators does not arise.

ACKNOWLEDGEMENTS

I wish to express my gratitude to Professor Yasushi Takahashi for suggesting the topic of this thesis, and for his constant guidance and supervision, without which this work would not have been possible.

To my wife, Alice, for her patience and encouragement during my studies goes my sincere appreciation.

I am indebted to Mr. W. John Heaney for his assistance in duplicating the thesis during my absence from campus.

My thanks to the National Research Council of Canada for awarding a scholarship to finance this work, and to the University of Alberta for providing part-time employment.

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1. INTRODUCTION

The theory of quantized fields involves two interrelated mathematical systems. First there is the general scheme of field operators satisfying equations of motion and commutation relations, and second, the linear vector space with its associated probability interpretation representing the quantum states.1) In the conventional approach to quantum field theory one begins with the equations of motion for the fields in the Heisenberg picture, and attempts to set up commutation relations in such a way that these equations of motion are compatible with the Heisenberg equation. The linear vector space is then constructed using the Fourier coefficients of the fields. While this procedure is satisfactory for comparatively simple systems, it leads to difficulties when dealing with more complicated interactions such as those involving higher spin fields2) or non-local interactions.3),4)

In this thesis we shall describe a somewhat different approach to quantum field theory. It begins with the introduction of creation and annihilation operators*), satisfying appropriate commutation relations, from which a general Fock space is constructed. Since the creation and annihilation operators do not satisfy equations of motion, the space so formed has, as yet, no connection with a physical system. The so called physical states are then formed from those vectors of the general space which satisfy certain supplementary conditions. It is these conditions which correspond to the equations of motion in the conventional theory. The energy-momentum operator for the system is defined in such a way that the defining commutation relations of the creation and annihilation operators lead identically to the Heisenberg equation. It turns out that this theory is the same as that proposed by Klein5) and Coester6), and is called hyperquantization.

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*) Technically, we should say raising and lowering operators since no particles are created or destroyed. However, to be consistent with other literature on the subject we shall continue to say creation and annihilation operators.

Chapter 2 reviews the general theory of hyperquantization.7) The physical states satisfying the supplementary conditions are explicitly constructed and shown to be eigenstates of the energy-momentum operator. Field operators are introduced such that their matrix elements between physical states yield the wave functions of the system. The matrix elements of simple products of these operators are found to satisfy equations analogous to the Matthews-Salam equations for time-ordered products of field operators in the conventional theory.8)

For interacting systems, the states are explicitly constructed by means of an operator S. It is this operator which is the analogue of the S-matrix in ordinary quantum field theory. It will be pointed out that the proof of the relativistic invariance of the S-matrix of hyperquantization is relatively straightforward, whereas the proof of conservation of probability (corresponding to unitarity in the usual theory) is complicated.9) In second quantization the opposite is true. Unitarity is easily proven whereas relativistic invariance can only be shown to given orders in the perturbation expansion.10) The reason for the simplification as regards the relativistic invariance in hyperquantization is that, as we shall see, it involves no non-relativistic operations such as chronological ordering, and terms depending on the normals to space-like surfaces do not arise as they do in the conventional theory when dealing with derivative couplings and non-local interactions.*)

It may be asked to what extent hyperquantization can reproduce the well-established results of ordinary

---

) In this sense hyperquantization may be said to correspond to the so called T-product formulation of the usual theory.11)

field theory. The fact is that for simple systems the results of this formalism can be shown to agree with those of Feynman-Dyson S-matrix theory.[9)] In particular, as will be shown in Chapter 3, the S-matrix elements of hyperquantization agree completely with those of the usual theory in the case of an electromagnetic-spin 1/2 interaction. In demonstrating the agreement between the two theories of quantum electrodynamics, we must take care on certain points. Since in hyperquantization there is no field equation, concepts such as current conservation and gauge invariance have to be reexamined. We shall see that, while the local current operator is not conserved, its expectation value between physical states is. Furthermore, we shall see that the expectation value of the electromagnetic potential is indeterminate to the extent that the four-divergence of an arbitrary c-number may be added to it. Corresponding to the gauge invariance of the usual theory, the S-matrix elements are not dependent on this arbitrary c-number. Conservation of probability (unitarity) can, in this case, be inferred from the exact agreement of the S-matrix elements with those of conventional quantum electrodynamics.

An important aspect of hyperquantization is the fact that the field operators satisfy an extremely simple algebra. In fact, the commutators or anticommutators of these operators are identically zero for arbitrary space-time separations.7) Now, Johnson and Sudarshan have shown that, in the case of an electromagnetic-spin 3/2 interaction, quantization using standard techniques leads to fermion anticommutators which are non-positive-definite.2) Indeed, by a suitable choice of Lorentz frames, they can always be made negative. Because of the simpler field algebra involved, we might expect that this problem would not arise at all if the same system were hyperquantized. That this is in fact the case will be shown in Chapter 4.

Unfortunately, the situation as regards the conservation of probability is not as simple as for the electromagnetic-spin 1/2 interaction, because no S-matrix elements have been found in the usual theory with which to compare our own. No general proof of this condition has as yet been found and we must rely on perturbation expansion arguments to demonstrate it.

2. A REVIEW OF HYPERQUANTIZATION

We shall here present the theory of hyperquantization as formulated by Y. Takahashi$^{7)}$ following Coester's original paper on the subject.$^{6)}$ The basic idea of hyperquantization is to introduce creation and annihilation operators from the outset, to construct a general Fock space using these operators, and from this space to select the physical subspace by imposing supplementary conditions. It is these supplementary conditions which replace the equations of motion of conventional field theory. As we shall see, in this formalism the field operators obey a much simpler algebra than do their counterparts in the usual theory, whereas the linear vector space involved becomes more complicated.

2.1 The Wave Equation

In this section we state some of the properties of the wave equation of the form

$$\Lambda(\partial)\, u_J^{(r)}(x) = 0 \tag{2.1.1}$$

where $u_J^{(r)}(x)$ is a wave function of the point x in four dimensional Minkowski space, J stands for a kinematical label such as momentum or angular momentum, r indicates a spin orientation, and $\Lambda(\partial)$ is a linear operator over the space of the wave functions. The material in this section is well known and is presented here only for convenience and clarity. Derivations and explanations may be found in reference 1.

We restrict ourselves to operators $\Lambda(\partial)$ which satisfy the following conditions:

(A) There exists a nonsingular matrix η such that

$$[\eta\Lambda(\partial)]^{\dagger} = \eta\Lambda(-\partial) \tag{2.1.2}$$

(B) $\Lambda(\partial)$ is of the form

$$\Lambda(\partial) = \sum_{\ell=0}^{\infty} \Lambda_{\mu_1 \cdots \mu_\ell}\, \partial_{\mu_1} \cdots \partial_{\mu_\ell} \tag{2.1.3}$$

(C) The Klein Gordon divisor $d(\partial)$ exists such that

$$\Lambda(\partial)d(\partial) = d(\partial)\Lambda(\partial) = \Box - m^2 \tag{2.1.4}$$

(D) The unitary, symmetric charge conjugation matrix C exists such that

$$[\eta\Lambda(\partial)]^{t} = \rho\, C^{-1}\, \eta\Lambda(-\partial)\, C \tag{2.1.5}$$

where

$$\begin{aligned} \rho &= 1 \text{ for fields with integer spin} \\ &= -1 \text{ for fields with half odd-integer spin .} \end{aligned} \tag{2.1.6}$$

The quantity defined by

$$\Gamma_\mu(\partial,-\overleftarrow{\partial}) = \Lambda_\mu + \Lambda_{\mu\nu}(\partial_\nu-\overleftarrow{\partial}_\nu) + \Lambda_{\mu\nu\lambda}(\partial_\nu\partial_\lambda-\partial_\nu\overleftarrow{\partial}_\lambda+\overleftarrow{\partial}_\nu\overleftarrow{\partial}_\lambda)+\dots \tag{2.1.7}$$

where $\Lambda_{\mu_1\dots\mu_\ell}$ have been symmetrized with respect to the indices, plays a role in the following discussion. It satisfies the identity

$$\Lambda(\partial) - \Lambda(-\overleftarrow{\partial}) = (\partial_\mu + \overleftarrow{\partial}_\mu)\Gamma_\mu(\partial,-\overleftarrow{\partial}) \ . \tag{2.1.8}$$

Equations (2.1.1), (2.1.4), and (2.1.8) may be rewritten

$$\bar{\Lambda}(\partial)u_J^{(r)}(x) = 0 \ , \tag{2.1.9}$$

$$\bar{\Lambda}(\partial)\bar{d}(\partial) = \bar{d}(\partial)\bar{\Lambda}(\partial) = \Box - m^2 \ , \tag{2.1.10}$$

$$\bar{\Lambda}(\partial) - \bar{\Lambda}(-\overleftarrow{\partial}) = (\partial_\mu + \overleftarrow{\partial}_\mu)\Gamma_\mu(\partial,-\overleftarrow{\partial}) \ , \tag{2.1.11}$$

respectively where

$$\bar{\Lambda}(\partial) = \eta\Lambda(\partial) \ , \tag{2.1.12}$$

$$\bar{d}(\partial) = d(\partial)\eta^{-1} \ , \tag{2.1.13}$$

$$\bar{\Gamma}_\mu(\partial,-\overleftarrow{\partial}) = \eta\Gamma_\mu(\partial,-\overleftarrow{\partial}) \ , \tag{2.1.14}$$

and it is this latter notation which is more convenient for use in the following discussion.

Now let us take $u_J^{(r)}(x)$ in (2.1.9) to be a positive frequency wave function and define its charge conjugate function

$$v_J^{(r)}(x) = C\, u_J^{(r)*}(x) \quad , \tag{2.1.15}$$

which satisfies

$$\bar{\Lambda}(\partial) v_J{}^{(r)}(x) = 0 \quad , \tag{2.1.16}$$

by virtue of (2.1.5) and (2.1.9). These functions satisfy the normalization and closure conditions

$$-i\int_\sigma d\sigma_\mu(x) u_J^{(r)^\dagger}(x) \bar{\Gamma}_\mu(\partial,-\overleftarrow{\partial}) u_{J'}^{(r')}(x) = \delta_{rr'}\delta_{JJ'} \quad , \tag{2.1.17}$$

$$-i\int_\sigma d\sigma_\mu(x) v_J^{(r)^\dagger}(x) \bar{\Gamma}_\mu(\partial,-\overleftarrow{\partial}) v_{J'}^{(r')}(x) = -\rho\delta_{rr'}\delta_{JJ'} \quad , \tag{2.1.18}$$

$$-i\int_\sigma d\sigma_\mu(x) v_J^{(r)^\dagger}(x) \bar{\Gamma}_\mu(\partial,-\overleftarrow{\partial}) u_{J'}^{(r')}(x) = 0 \quad , \tag{2.1.19}$$

$$\sum_{r,J} u_J^{(r)}(x) u_J^{(r)^\dagger}(y) = i\bar{d}(\partial)\Delta^{(+)}(x-y) \quad , \tag{2.1.20}$$

$$\sum_{r,J} v_J^{(r)}(x) v_J^{(r)^\dagger}(y) = -i\rho\bar{d}(\partial)\Delta^{(-)}(x-y) \quad , \tag{2.1.21}$$

and can also be shown to obey

$$u_J^{(r)^\dagger}(y) = \int_{x_o>y_o} d\sigma_\lambda(x) u_J^{(r)^\dagger}(x) \bar{\Gamma}_\lambda(\partial,-\overleftarrow{\partial}) \bar{d}(\partial)\Delta_c(x-y) \quad , \tag{2.1.22}$$

$$u_J^{(r)}(y) = \int_{x_o<y_o} d\sigma_\lambda(x) \bar{d}(-\partial)\Delta_c(y-x) \bar{\Gamma}_\lambda(\partial,-\overleftarrow{\partial}) u_J^{(r)}(x) \quad , \tag{2.1.23}$$

$$v_J^{(r)^\dagger}(y) = -\int_{x_o<y_o} d\sigma_\lambda(x) v_J^{(r)^\dagger}(x) \bar{\Gamma}_\lambda(\partial,-\overleftarrow{\partial}) \bar{d}(\partial)\Delta_c(x-y), \tag{2.1.24}$$

$$v_J^{(r)}(y) = -\int_{x_o>y_o} d\sigma_\lambda(x)\bar{d}(-\partial)\Delta_c(y-x)\bar{\Gamma}_\lambda(\partial,-\overleftarrow{\partial})v_J^{(r)}(x) \;, \tag{2.1.25}$$

where Δ_c is the causal Green's function of the Klein-Gordon equation and, therefore, obeys

$$\bar{\Lambda}(\partial)\bar{d}(\partial)\Delta_c(x-y) = \delta^{(4)}(x-y) \;. \tag{2.1.26}$$

2.2 Hyperquantization of Fields

Hyperquantization of a field begins not with an equation of motion, as would the conventional quantization of the same field, but with the introduction of creation and annihilation operators, depending on points in Minkowski space, and satisfying the relations

$$a_\alpha(x)a_\beta^\dagger(y) - \rho' a_\beta^\dagger(y)a_\alpha(x) = \delta_{\alpha\beta}\delta^{(4)}(x-y) \tag{2.2.1}$$

$$b_\alpha(x)b_\beta^\dagger(y) - \rho' b_\alpha^\dagger(y)b_\beta(x) = \delta_{\alpha\beta}\delta^{(4)}(x-y) \tag{2.2.2}$$

with similar commutators of other combinations of these operators vanishing. Here α and β are labels denoting different components of the same field as well as different fields. These labels will be suppressed in the following discussion. The constant ρ' is ± 1, and we shall fix it later according to (2.2.38).

The creation operators, $a^\dagger$ and $b^\dagger$, can now be used in conjunction with the vacuum Ω_o defined by

$$a(x)\Omega_o = b(x)\Omega_o = 0 \ , \tag{2.2.3}$$

to construct the orthonormal vectors of a Fock space in the usual manner. Since the operators in (2.2.1) and (2.2.2) do not satisfy any equations of motion, the space so constructed has no physical significance. The physics is brought in by selecting vectors Ω which satisfy the conditions

$$\bar{\Lambda}(\partial)a(x)\Omega = 0 \ , \tag{2.2.4}$$

and

$$b(x)\bar{\Lambda}(-\overleftarrow{\partial})\Omega = 0 \ . \tag{2.2.5}$$

These vectors Ω can be explicitly constructed by means of the operators

$$A_J^{(r)^\dagger} = \int d^4x \ a^\dagger(x)u_J^{(r)}(x) \ , \tag{2.2.6}$$

$$B_J^{(r)^\dagger} = \rho \int d^4x \ v_J^{(r)^\dagger}(x)b^\dagger(x) \ . \tag{2.2.7}$$

To see this we note that (2.2.1) and (2.2.2) give

$$a(x)A_J^{(r)^\dagger} - \rho' A_J^{(r)^\dagger} a(x) = u_J^{(r)}(x) \ , \tag{2.2.8}$$

$$b(x)B_J^{(r)^\dagger} - \rho' B_J^{(r)^\dagger} a(x) = \rho v_J^{(r)^\dagger}(x) \ , \tag{2.2.9}$$

and that consequently

$$\bar{\Lambda}(\partial)a(x)A_J^{(r)^\dagger} - \rho' A_J^{(r)^\dagger}\bar{\Lambda}(\partial)a(x) = \bar{\Lambda}(\partial)u_J^{(r)}(x) = 0, \tag{2.2.10}$$

$$b(x)\bar{\Lambda}(-\partial)B_J^{(r)^\dagger} - \rho' B_J^{(r)^\dagger} b(x)\bar{\Lambda}(-\overset{\dagger}{\partial}) = \rho v_J^{(r)^\dagger}(x)\bar{\Lambda}(-\overset{\dagger}{\partial}) = 0 \ , \tag{2.2.11}$$

by virtue of (2.1.9) and (2.1.16). Therefore, we have

$$\bar{\Lambda}(\partial)a(x)A_J^{(r)^\dagger} = \rho' A_J^{(r)^\dagger}\bar{\Lambda}(\partial)a(x) \ , \tag{2.2.12}$$

which proves that if a vector Ω satisfies (2.2.4) the vector $A_J^{(r)^\dagger}\Omega$ does also. Similarly, if Ω satisfies (2.2.5), so does $B_J^{(r)^\dagger}\Omega$. Since the vacuum Ω_o obviously satisfies (2.2.4) and (2.2.5) as a consequence of (2.2.3), we conclude that the vectors

$$\Omega(r_1J_1 \ldots r_mJ_m;\ s_1K_1 \ldots s_nK_n)$$

$$= \frac{1}{\sqrt{m!}}\ \frac{1}{\sqrt{n!}}\ \prod_{i=1}^{m} A_{J_i}^{(r_i)^\dagger}\ \prod_{j=1}^{n} B_{K_j}^{(s_j)^\dagger}\ \Omega_o \ , \tag{2.2.13}$$

and any linear combination of them will satisfy these conditions, and will be called physical state vectors.

We now define the momentum operator P_μ such that the Heisenberg equation is satisfied. That is

$$-i\partial_\mu a(x) = [a(x), P_\mu] \ , \tag{2.2.14}$$

$$-i\partial_\mu b(x) = [b(x), P_\mu] \ , \tag{2.2.15}$$

with similar relations for $a^\dagger(x)$ and $b^\dagger(x)$. It is easily seen that the operator

$$P_\mu = -\frac{1}{2}\, i\int d^4x\{a^\dagger(x)(\partial_\mu - \overleftarrow{\partial}_\mu)a(x) + b^\dagger(x)(\partial_\mu - \overleftarrow{\partial}_\mu)b(x)\} \qquad (2.2.16)$$

satisfies the above equations. This is not a unique construction, however, since the addition of a c-number to (2.2.16) will leave (2.2.14) and (2.2.15) unchanged. We can make it unique, in the absence of massless particles by requiring its vacuum expectation value to be zero. The problem in the presence of massless particles is as yet unresolved, and we shall not consider it further except to say that it is present in ordinary quantum field theory as well.*)

In the same way, we note that the operator

$$Q = \int d^4x\{a^\dagger(x)a(x) - b^\dagger(x)b(x)\} \ , \qquad (2.2.17)$$

satisfies

$$a(x) = [a(x),Q] \ , \qquad (2.2.18)$$

$$-b(x) = [b(x),Q] \ , \qquad (2.2.19)$$

and will, therefore, be identified with the charge operator.

It can be shown that the vectors (2.2.13) are eigenvectors of the momentum and charge operators. To prove, for instance, that they are eigenvectors of $H \equiv -iP_4$, we proceed as follows: Suppose a state Ω_1, say, is an eigenstate of H with eigenvalue E. That is,

---

*) The requirement that the four-momentum and angular momentum satisfy the Poincaré commutation relations does, however, result in a unique P_μ.

$$H\Omega_1 = E\Omega_1 \ . \tag{2.2.20}$$

Then it follows that

$$HA_J^{(r)^\dagger}\Omega_1 = [H,A_J^{(r)^\dagger}]\Omega_1 + E\,A_J^{(r)^\dagger}\Omega_1 \quad . \tag{2.2.21}$$

Now using (2.2.6) and (2.2.14) we find

$$\begin{aligned} H\,A_J^{(r)^\dagger} &= -i\int_{-\infty}^{\infty} d^4x\ u_J^{(r)}(x)\ \frac{\partial}{\partial t}\, a^\dagger(x)\Omega_1 + E\,A_J^{(r)^\dagger}\Omega_1 \\ &= -i\int_{-\infty}^{\infty} d^4x\ \frac{\partial}{\partial t}\,\{a^\dagger(x)u_J^{(r)}(x)\}\Omega_1 \\ &= +i\int_{-\infty}^{\infty} d^4x\ a^\dagger(x)\ \frac{\partial}{\partial t}\,u_J^{(r)}(x)\Omega_1 + E\,A_J^{(r)^\dagger}\Omega_1 . \end{aligned} \tag{2.2.22}$$

It can be shown7) that the expectation value of the first term in the final expression above vanishes, although we shall not give the proof here. Therefore we shall eliminate it. Then, noting that

$$i\dot{u}_J^{(r)}(x) = E_J u_J^{(r)}(x) \ , \tag{2.2.23}$$

where E_J is the frequency of the wave function $u_J^{(r)}$, we have

$$H\,A_J^{(r)^\dagger}\Omega_1 = (E_J + E)A_J^{(r)^\dagger}\Omega_1 \quad , \tag{2.2.24}$$

and similarly for $B_K^{(r)\dagger}\Omega_1$. Now, it is obvious from (2.2.3) and (2.2.16) that

$$H\Omega_o = 0 \quad . \tag{2.2.25}$$

Hence, (2.2.13) and (2.2.24) give

$$\begin{aligned} &H\Omega(r_1J_1,\ldots r_nJ_n;\ s_1K_1\ldots,s_mK_m) \\ &\quad=\{\sum_{i=1}^{n} E_{J_i} + \sum_{j=1}^{m} E_{K_j}\}\Omega(r_1J_1,\ldots,r_nJ_n;\ s_1K_1,\ldots s_mK_m) \end{aligned} \tag{2.2.26}$$

which completes the proof.

As is easily verified, the norm of the vectors (2.2.13) is infinite. To avoid this difficulty, we introduce the dual vectors

$$\begin{aligned} &\hat{\Omega}(r_1J_1,\ldots r_nJ_n;\ s_1K_1\ldots s_mK_m) \\ &\quad= \frac{1}{\sqrt{m!}}\ \frac{1}{\sqrt{n!}}\ \prod_{i=1}^{n} \hat{A}_{J_i}^{(r_i)\dagger} \prod_{j=1}^{m} \hat{B}_{K_j}^{(s_j)\dagger}\ \Omega_o\ , \end{aligned} \tag{2.2.27}$$

with

$$\hat{A}_J^{(r)\dagger} = \lim_{\sigma\to\infty} -i\int_\sigma d\sigma_\lambda(x)a^\dagger(x)\bar{\Gamma}_\lambda(\partial,-\overleftarrow{\partial})u_J^{(r)}(x)\ , \tag{2.2.28}$$

$$\hat{B}_K^{(s)\dagger} = \lim_{\sigma\to\infty}\ i\int_\sigma d\sigma_\lambda(x)v_K^{(s)\dagger}(x)\bar{\Gamma}_\lambda(\partial,-\overleftarrow{\partial})b^\dagger(x)\ . \tag{2.2.29}$$

We can now prove, for instance, that

$$(\hat{\Omega}(rJ;\ ),\ \Omega(r'J';\ )) = \delta_{rr'}\delta_{JJ'}\ , \tag{2.2.30}$$

$$(\hat{\Omega}(\ ;sK),\ \Omega(\ ;s'K') = \delta_{ss'}\delta_{KK'}\ , \tag{2.2.31}$$

but the details of this will be left for the next chapter. More generally, it can be shown that the orthonormalization condition holds between Ω and $\hat{\Omega}$. Accordingly, we define the matrix elements of operators by sandwiching them between $\hat{\Omega}$ and Ω. For example, consider the operators

$$\phi(x) = a(x) + i\int_{-\infty}^{\infty} d^4x'\ \bar{d}(\partial)\Delta_c(x-x')b^{\dagger}(x')\ , \tag{2.2.32}$$

$$\tilde{\phi}(x) = \rho' b(x) + i\int_{-\infty}^{\infty} d^4x'\ a^{\dagger}(x')\Delta_c(x'-x)\bar{d}(-\overleftarrow{\partial})\ . \tag{2.2.33}$$

The non-vanishing matrix elements of these operators are given by

$$(\Omega_o,\ \phi(x)\ \Omega(rJ;)) = u_J^{(r)}(x)\ , \tag{2.2.34}$$

$$\begin{aligned}(\hat{\Omega}(rJ;\ ),\ \tilde{\phi}(x)\ \Omega_o) &= \int_{\infty} d\sigma_\lambda(x')u_J^{(r)^{\dagger}}(x')\bar{\Gamma}_\lambda(\partial',-\overleftarrow{\partial}') \\ &\quad\times \Delta_c(x'-x)\bar{d}(-\overleftarrow{\partial}) \\ &= u_J^{(r)^{\dagger}}(x)\ ,\end{aligned} \tag{2.2.35}$$

where we have used (2.1.22). Similarly,

$$(\Omega_o,\ \tilde{\phi}(x)\Omega(\ ;sK)) = \rho\rho'\ v_K^{(s)^{\dagger}}(x)\ , \tag{2.2.36}$$

$$(\hat{\Omega}(\;;\,sK),\ \phi(x)\ \Omega_o) = v_K^{(s)}(x) \quad . \tag{2.2.37}$$

Equations (2.2.34)-(2.2.37) show that, if we assume the relation between spin and statistics

$$\rho = \rho' \quad , \tag{2.2.38}$$

we have

$$(\Omega_o,\ \phi(x)\ \Omega(rJ;\;)) = (\hat{\Omega}(rJ;\;),\ \tilde{\phi}(x)\ \Omega_o)^* \quad , \tag{2.2.39}$$

$$(\hat{\Omega}(\;;\,sK),\ \tilde{\phi}(x)\ \Omega_o) = (\Omega_o,\ \tilde{\phi}(x)\ \Omega(\;;\,sK))^* . \tag{2.2.40}$$

More generally, as will be shown in more detail for the specific case dealt with in the next chapter,

$$(\hat{\Omega}_2, :\phi(x_1)\ldots\phi(x_n)\ \tilde{\phi}(y_1)\ldots\tilde{\phi}(y_m):\Omega_1)$$

$$= (\hat{\Omega}_1, :\phi(y_m)\ldots\ \phi(y_1)\tilde{\phi}(x_n)\ldots\tilde{\phi}(x_1):\Omega_2)^* \ . \tag{2.2.41}$$

That is, $\tilde{\phi}(x)$ behaves like the Hermitian conjugate of $\phi(x)$ when sandwiched between physical states in a normal product. These operators, which we shall henceforth call the field operators, satisfy the remarkably simple commutation relations

$$\phi(x)\phi(y) - \rho'\phi(y)\phi(x) = \tilde{\phi}(x)\tilde{\phi}(y) - \rho'\tilde{\phi}(y)\tilde{\phi}(x) = 0, \tag{2.2.42}$$

$$\phi(x)\tilde{\phi}(y) - \rho'\tilde{\phi}(y)\phi(x) = 0 \quad , \tag{2.2.43}$$

as is readily verified using (2.2.1) and (2.2.2) along with the definitions (2.2.32) and (2.2.33). As mentioned previously, it is this simple operator algebra which is the basis for the usefulness of hyperquantization in discussing the electromagnetic-spin 3/2 interaction.

In the presence of an interaction we select state vectors which satisfy the conditions

$$\{\bar{\Lambda}(\partial)a(x) + J(x)\}\Psi = 0 , \qquad (2.2.44)$$

$$\{\rho' b(x)\bar{\Lambda}(-\overleftarrow{\partial}) + \tilde{J}(x)\}\Psi = 0 , \qquad (2.2.45)$$

where $J(x)$ and $\tilde{J}(x)$ are sources of the fields $\phi(x)$ and $\tilde{\phi}(x)$ which we shall assume are derivable from a functional $H(x)$ of $\phi(x)$, $\tilde{\phi}(x)$ and their derivatives according to the formal relation

$$\delta H(x) = J(x)\delta\tilde{\phi}(x) + \tilde{J}(x)\delta\phi(x) + (\text{divergence term}). \qquad (2.2.46)$$

As we shall demonstrate for the electromagnetic spin 1/2 interaction in the next chapter, the states Ψ can be explicitly constructed according to

$$\Psi = S\,\Omega \qquad (2.2.47)$$

with

$$S = \exp\{-i(\bar{H} + \bar{H}_o)\} \qquad (2.2.48)$$

where

$$\bar{H} = -i \int_{-\infty}^{\infty} d^4x \; H(x) \tag{2.2.49}$$

and $\bar{H}_o$ is a c-number constant defined such that

$$(\Omega_o, S\,\Omega_o) = 1 . \tag{2.2.50}$$

We shall see that the quantity

$$(\Omega_o, \phi(x_1)\ldots\phi(x_m)\;\tilde{\phi}(y_1)\ldots\tilde{\phi}(y_n)\Psi) , \tag{2.2.51}$$

satisfies an equation identical with that derived by Matthews and Salam for the chronological product of operators in ordinary field theory. For instance,[7)]

$$\bar{\Lambda}(\partial)(\Omega_o,\phi(x)\tilde{\phi}(y)\Psi) = i\delta^{(4)}(x-y)(\Omega_o,\Psi) - (\Omega_o,J(x)\tilde{\phi}(y)\Psi) , \tag{2.2.52}$$

while in ordinary theory, if $\psi(x)$ is a field operator and $|\rangle$ represents an arbitrary state with $|0\rangle$ the vacuum state, we have

$$\bar{\Lambda}(\partial)\langle 0|T(\psi(x),\psi^{\dagger}(y)|\rangle = i\delta^{(4)}(x-y)\langle 0|\rangle - \langle 0|T(J(x),\psi^{\dagger}(y)|\rangle . \tag{2.2.53}$$

Furthermore, we observe that the Wick expansion theorem holds for simple products of operators (2.2.32) and (2.2.33), i.e.[9)]

$$\phi(x)\tilde{\phi}(x') = :\phi(x)\tilde{\phi}(x'): + i\bar{d}(\partial)\Delta_c(x-x') , \tag{2.2.54}$$

and in general

$$\phi(x_1)\ldots\phi(x_m)\tilde{\phi}(y_1)\ldots\tilde{\phi}(y_n) = :\phi(x_1)\ldots\phi(x_m)\tilde{\phi}(y_1)\ldots\tilde{\phi}(y_n):$$

$$+ i(\rho')^{m-1}\,\bar{d}(\partial)\Delta_c(x_1-y_1):\phi(x_2)\ldots\phi(x_m)\tilde{\phi}(y_2)\ldots\tilde{\phi}(y_n):$$

$$+ \ldots \qquad (2.2.55)$$

So we see that the simple product of the operators ϕ and $\tilde{\phi}$ correspond to the chronological product of field operators in the interaction picture in the conventional theory.

The observations of the previous paragraph can be used to show the equivalence, to n'th order in the perturbation expansion, of the matrix elements of the operator S in (2.2.48) and those of the S-matrix in ordinary field theory.$^{9)}$ In the following chapter it is this equivalence which will be used to show that the conservation of probability condition

$$\sum_n (\hat{\Omega}_n, S\,\Omega_2)^*(\hat{\Omega}_n, S\,\Omega_1) = (\hat{\Omega}_2, \Omega_1)\ , \qquad (2.2.56)$$

holds. This condition corresponds to unitarity in the ordinary theory, and will be called the unitarity condition when no confusion is caused thereby. When no S-matrix expansion exists in the ordinary theory, as is the case for the electromagnetic-spin 3/2 interaction,$^{2)}$ the proof of (2.2.56) is not so straightforward. We shall return to this problem in Chapter 4.

It is notable that our "interaction Hamiltonian" H(x) does not contain the normal-dependent terms present in the ordinary theory when higher spin interactions or derivative couplings are present. Nor is there any chronological ordering of the field operators in S. This is because all the field operators either commute or anticommute making the algebra of these operators much simpler than in conventional theory. It is for this reason that hyperquantization might be expected to be useful in dealing with the spin 3/2 field discussed in Chapter 4. To pay for this simplification we must deal with a linear vector space whose structure has become more complicated.

Since our S-matrix elements contain no non-relativistic operations, such as chronological ordering, it might be expected that the relativistic invariance of the S-matrix can be proved more straightforwardly than in conventional field theory, where it can only be demonstrated to given orders in the perturbation expansion. That this is indeed the case has been shown by Y. Takahashi and R. Gourishankar. Since their proof is rather lengthy, and since the details of this proof are not necessary to an understanding of the remainder of this thesis, it is omitted here. The interested reader is referred to ref.9.

In the course of the proof, the above authors have derived the relations9)

$$[S, M_{\mu\nu}] = 0 \quad , \tag{2.2.57}$$

$$[S, P_{\mu}] = 0 \quad , \tag{2.2.58}$$

$$[S, Q] = 0 \quad , \tag{2.2.59}$$

which imply the conservation of angular momentum, energy-momentum, and charge respectively. Since hyperquantization is essentially an S-matrix theory, the conservation of local quantities at a point is not required and, in general, will not hold. What is required, rather, is the conservation of global quantities asymptotically which is insured by equations such as (2.2.57)-(2.2.59).

In the following chapter we shall apply the formalism presented here to hyperquantize the system of interacting electromagnetic and spin 1/2 fields. We shall show that our treatment can reproduce exactly the results of ordinary quantum electrodynamics and, in so doing, justify our use of hyperquantization in treating the problem of the electromagnetic-spin 3/2 interaction in Chapter 4.

3. THE ELECTROMAGNETIC-SPIN 1/2 INTERACTION

The purpose of this chapter is to demonstrate that quantum electrodynamics can be reproduced exactly using the method of hyperquantization.12) In particular, we shall show that our S-matrix elements agree completely with those of the ordinary theory. The theory presented below has the advantages, however, that the Gupta-Bleuler indefinite metric need not be introduced, and that the proof of covariance is greatly simplified as mentioned previously. Furthermore, since we can demonstrate the exact agreement of our S-matrix elements with those of the conventional theory, it follows that conservation of probability (unitarity) is satisfied.

Because in our formalism there is no field equation, the conservation of "current" and gauge invariance have to be reexamined. We shall investigate these problems as well as deriving Dysons equations for propagators and the vertex, and the Ward-Takahashi identity.

In order to establish the notation to be generalized to the electromagnetic-spin 3/2 interaction in Chapter 4, we shall repeat some of the arguments of Chapter 2.

3.1 Formulation

We shall formulate the quantum electrodynamics of a spin 1/2 field.*) The first step is to introduce the creation and annihilation operators depending on points in four-dimensional Minkowski space and satisfying the relations

$$\{a_\alpha(x),\ a^\dagger_\beta(y)\} = \delta_{\alpha\beta}\delta^{(4)}(x-y) \tag{3.1.1}$$

$$\{b_\alpha(x),\ b^\dagger_\beta(y)\} = \delta_{\alpha\beta}\delta^{(4)}(x-y) \tag{3.1.2}$$

$$[c_\mu(x),\ c^\dagger_\nu(y)] = \delta_{\mu\nu}\delta^{(4)}(x-y) \tag{3.1.3}$$

with similar commutators or anticommutators vanishing. Here α and β are spinor indices and μ and ν are vector indices with all indices running from 1 to 4. We have assumed the relation between spin and statistics. The creation operators $a^\dagger_\alpha(x)$, $b^\dagger_\alpha(x)$ and $c^\dagger_\mu(x)$, together with the vacuum Ω_o defined by

$$a_\alpha(x)\Omega_o = 0 \ , \tag{3.1.4}$$

$$b_\alpha(x)\Omega_o = 0 \ , \tag{3.1.5}$$

$$c_\mu(x)\Omega_o = 0 \ , \tag{3.1.6}$$

can be used to establish a general Fock space. This Fock space is then given physical significance by restricting the states Ω to satisfy the conditions,

---

*) Note that in this formalism the Gupta-Bleuler indefinite metric can be avoided. This was conjectured by H. Umezawa.

$$(\gamma\partial + m)\ a(x)\Omega = 0\ , \tag{3.1.7}$$

$$b(x)(-\gamma\overleftarrow{\partial} + m)\Omega = 0\ , \tag{3.1.8}$$

$$\Box c_\mu(x)\Omega = 0\ , \tag{3.1.9}$$

$$\partial_\mu c_\mu(x)\Omega = 0\ . \tag{3.1.10}$$

These states can be explicitly constructed by using the spinor and vector wave functions satisfying

$$\left.\begin{aligned}(\gamma\partial + m)u_p^{(r)}(x) &= 0\\ (\gamma\partial + m)v_p^{(s)}(x) &= 0\end{aligned}\right\}\quad r,s = 1,2 \tag{3.1.11}$$

and

$$\left.\begin{aligned}\Box u_{k\mu}^{(t)}(x) &= 0\\ \partial_\mu\ u_{k\mu}^{(t)}(x) &= 0\end{aligned}\right\}\quad t = 1,2,3,4 \tag{3.1.12}$$

together with the normalization and closure conditions

$$i\int d\sigma_\lambda(x)\bar{u}_p^{(r)}(x)\gamma_\lambda u_{p'}^{(r')}(x) = 2\omega(\underset{\sim}{p})\delta_{rr'}\delta(\underset{\sim}{p}-\underset{\sim}{p}')\ , \tag{3.1.13}$$

$$i\int d\sigma_\lambda(x)\bar{v}_p^{(s)}(x)\gamma_\lambda v_{p'}^{(s')}(x) = 2\omega(\underset{\sim}{p})\delta_{ss'}\delta(\underset{\sim}{p}-\underset{\sim}{p}')\ , \tag{3.1.14}$$

$$-i\int d\sigma_\lambda(x)\bar{u}_{k\mu}^{(t)}(x)(\partial_\lambda-\overleftarrow{\partial}_\lambda)u_{k\mu}^{(t')}(x) = 2|\underset{\sim}{k}|g_{tt'}\delta(\underset{\sim}{k}-\underset{\sim}{k}'), \tag{3.1.15}$$

$$\sum_r\int\frac{d^3p}{2\omega(\underset{\sim}{p})}\ u_p^{(r)}(x)\bar{u}_p^{(r)}(x') = -i(\gamma\partial-m)\Delta^{(+)}(x-x') = iS^{(+)}(x-x'), \tag{3.1.16}$$

$$-\sum_{s}\int\frac{d^3q}{2\omega(\underset{\sim}{q})}\, v_q^{(s)}(x)\bar{v}_q^{(s)}(x') = i(\gamma\partial-m)\Delta^{(-)}(x-x') = iS^{(-)}(x-x'), \tag{3.1.17}$$

$$\sum_{t}\int\frac{d^3k}{2|\underset{\sim}{k}|}\, u_{k\mu}^{(t)}(x)\bar{u}_{k\sigma}^{(t)}(x') = i\,\delta_{\mu\sigma}\, D^{(+)}(x-x'), \tag{3.1.18}$$

where

$$\bar{u}_p^{(r)}(x) = u_p^{(r)\dagger}(x)\gamma_4 , \tag{3.1.19}$$

$$\bar{v}_q^{(s)}(x) = v_q^{(s)\dagger}(x)\gamma_4 , \tag{3.1.20}$$

$$\bar{u}_{k\mu}^{(t)}(x) = u_{k\nu}^{(t)\dagger}(x)g_{\nu\mu} . \tag{3.1.21}$$

For this purpose, we define

$$A_p^{(r)\dagger} = \int d^4x\; a^\dagger(x)u_p^{(r)}(x) , \tag{3.1.22}$$

$$B_q^{(s)\dagger} = -\int d^4x\; \bar{v}_q^{(s)}(x)b^\dagger(x) , \tag{3.1.23}$$

$$C_k^{(t)\dagger} = \int d^4x\; c_\mu^\dagger(x)u_{k\mu}^{(t)}(x) . \tag{3.1.24}$$

Equations (3.1.1) to (3.1.3) then imply

$$\{a(x),\, A_p^{(r)\dagger}\} = u_p^{(r)}(x) , \tag{3.1.25}$$

$$\{b(x),\, B_q^{(s)\dagger}\} = -\bar{v}_q^{(s)}(x) , \tag{3.1.26}$$

$$[c_\mu(x),\, C_k^{(t)\dagger}] = u_{k\mu}^{(t)}(x) . \tag{3.1.27}$$

Hence we find, with the aid of (3.1.11) and (3.1.12),

$$-(\gamma\partial+m)a(x)\ A_p^{(r)\dagger} = A_p^{(r)\dagger}(\gamma\partial+m)\ a(x)\quad , \tag{3.1.28}$$

$$-b(x)(-\gamma\overleftarrow{\partial}+m)\ B_q^{(s)\dagger} = B_q^{(s)\dagger}\ b(x)(-\gamma\overleftarrow{\partial}+m)\quad , \tag{3.1.29}$$

and

$$\Box c_\mu(x)\ C_k^{(t)\dagger} = C_k^{(t)\dagger}\ \Box c_\mu(x)\quad . \tag{3.1.30}$$

Since the vacuum Ω_o defined by (3.1.4)-(3.1.6) obviously satisfies the restrictions (3.1.7)-(3.1.10), the vectors

$$\begin{aligned}\Omega(m;n;\ell) &\equiv \Omega(p_1r_1,\ldots p_mr_m;\ q_1s_1\ldots q_ns_n;\ k_1t_1\ldots k_\ell t_\ell)\\ &= \frac{1}{\sqrt{\ell!}}\,\frac{1}{\sqrt{m!}}\,\frac{1}{\sqrt{n!}}\ \prod_{i=1}^{\ell} C_{k_i}^{(t_i)\dagger}\ \prod_{j=1}^{m} A_{p_j}^{(r_j)\dagger}\ \prod_{k=1}^{n} B_{q_k}^{(s_k)\dagger}\ \Omega_o\ ,\end{aligned} \tag{3.1.31}$$

will also satisfy those conditions and will be referred to as the physical states of the system. The proof involves only the repeated use of (3.1.28)-(3.1.30).

As was shown in Chapter 2, these vectors are eigenstates of the energy-momentum and charge operators

$$\begin{aligned}P_\mu = -\tfrac{i}{2}\int d^4x\ \{&a^\dagger(x)(\partial_\mu-\overleftarrow{\partial}_\mu)a(x) + b^\dagger(x)(\partial_\mu-\overleftarrow{\partial}_\mu)b(x)\\ &+ c_\rho^\dagger(x)(\partial_\mu-\overleftarrow{\partial}_\mu)c_\rho(x)\}\quad ,\end{aligned} \tag{3.1.32}$$

and

$$Q = e\int d^4x\ \{a^\dagger(x)a(x) - b^\dagger(x)b(x)\}\ , \tag{3.1.33}$$

which satisfy the relations

$$-i\ \partial_\mu a(x) = [a(x),\ P_\mu]\ , \tag{3.1.34}$$

$$-i\ \partial_\mu b(x) = [b(x),\ P_\mu]\ , \tag{3.1.35}$$

$$-i\ \partial_\mu c_\rho(x) = [c_\rho(x),\ P_\mu]\ , \tag{3.1.36}$$

and

$$e\ a(x) = [a(x),\ Q]\ , \tag{3.1.37}$$

$$-e\ b(x) = [b(x),\ Q]\ . \tag{3.1.38}$$

As indicated in Chapter 2, the norm of the vectors (3.1.31) is infinite. To render the states normalizable we define a set of dual vectors given by

$$\hat\Omega(m;n;\ell) = \frac{1}{\sqrt{\ell!}}\,\frac{1}{\sqrt{m!}}\,\frac{1}{\sqrt{n!}}\ \prod_{i=1}^{\ell} \hat C_{k_i}^{(t_i)\dagger}\ \prod_{j=1}^{m} \hat A_{p_j}^{(r_j)\dagger}\ \prod_{k=1}^{n} \hat B_{q_k}^{(s_k)\dagger}\ \Omega_o \tag{3.1.39}$$

with

$$\hat A_p^{(r)\dagger} = \lim_{\sigma\to\infty} i\int_\sigma d\sigma_\lambda(x)\,a^\dagger(x)\gamma_4\gamma_\lambda u_p^{(r)}(x)\ , \tag{3.1.40}$$

$$\hat B_q^{(s)\dagger} = \lim_{\sigma\to\infty} -i\int_\sigma d\sigma_\lambda(x)\,v_q^{(s)\dagger}(x)\gamma_\lambda b^\dagger(x)\ , \tag{3.1.41}$$

$$\hat{c}_k^{(t)\dagger} = \lim_{\sigma\to\infty} -i\int_\sigma d\sigma_\lambda(x)\, c_\mu^\dagger(x)\, g_{\mu\nu}(\partial_\lambda - \overleftarrow{\partial}_\lambda)\, u_{k\nu}^{(t)}(x) \ . \qquad (3.1.42)$$

By virtue of the normalization conditions (3.1.13)-(3.1.15), the above operators satisfy

$$\{\hat{A}_{p'}^{(r')}, A_p^{(r)\dagger}\} = \{\hat{B}_{p'}^{(r')}, B_p^{(r)\dagger}\} = 2\omega(p)\delta_{rr'}\delta(\underset{\sim}{p}-\underset{\sim}{p}') \ , \qquad (3.1.43)$$

$$[\hat{C}_{k'}^{(t')}, C_k^{(t)\dagger}] = 2|\underset{\sim}{k}|g_{tt'}\ \delta(\underset{\sim}{k}-\underset{\sim}{k}') \qquad (3.1.44)$$

from which it follows that the physical states satisfy, for instance, the orthonormalization conditions

$$(\hat{\Omega}(pr;\ ;),\ \Omega(p'r';\ ;)) = 2\omega(\underset{\sim}{p})\delta_{rr'}\delta(\underset{\sim}{p}-\underset{\sim}{p}') \ , \qquad (3.1.45)$$

$$(\hat{\Omega}(\ ;qs;),\ \Omega(;q's';)) = 2\omega(\underset{\sim}{q})\delta_{ss'}\delta(\underset{\sim}{q}-\underset{\sim}{q}') \ , \qquad (3.1.46)$$

$$(\hat{\Omega}(\ ;\ ;kt),\ \Omega(;\ ;k't')) = 2|\underset{\sim}{k}|g_{tt'}\delta(\underset{\sim}{k}-\underset{\sim}{k}') \ . \qquad (3.1.47)$$

The following remarks are in order. The wave functions satisfying the first of equations (3.1.12) with the normalization and closure conditions (3.1.15) and (3.1.18) may be written

$$u_{k\mu}^{(t)}(x) = (2\pi)^{-3/2}\ e_\mu^{(t)}(k)\ e^{ikx} \ , \qquad (3.1.48)$$

where $e_\mu^{(1)}$ and $e_\mu^{(2)}$ are two space-like vectors perpendicular to k_μ, and

$$e_\mu^{(3)}(k) = \frac{k_\mu + n_\mu(nk)}{(nk)}\ , \tag{3.1.49}$$

$$e_\mu^{(4)}(k) = n_\mu\ , \tag{3.1.50}$$

with n_μ a unit time-like vector. Equations (3.1.48)-(3.1.50) indicate that $u_{k\mu}^{(3)}(x)$ and $u_{k\mu}^{(4)}(x)$ will not satisfy the second of equations (3.1.12) separately. Consequently, the state constructed in (3.1.31) will not satisfy (3.1.10) unless we restrict ourselves to t, t' = 1,2, and take a certain linear combination of t = 3 and t = 4. In any case, we have

$$(\hat{\Omega}(;\ ;kt),\ \hat{\Omega}(;\ ;k't')) = 2|\underset{\sim}{k}|\delta_{tt'}\delta(\underset{\sim}{k}-\underset{\sim}{k}'),\qquad t = 1,2, \tag{3.1.51}$$

which is positive-definite.

Notice further that the definition of the dual vector which gives the orthonormalization conditions (3.1.45), (3.1.46), and (3.1.51) is not unique. For instance, we may add to the vector $\hat{\Omega}(pr;\ ;)$ the quantity

$$\hat{\Omega}^{(1)}(pr;\ ;) = \int d^4x\ a^\dagger(x)\gamma_4\gamma_\lambda u_p^{(r)}(x)\partial_\lambda\Lambda(x)\Omega_o, \tag{3.1.52}$$

where $\Lambda(x)$ is an arbitrary scalar which vanishes at $|x|\to\infty$. This will not affect the normalization (3.1.45) because

$$
\begin{aligned}
&(\hat{\Omega}^{(1)}(pr;\ ;),\ \Omega(p'r';\ ;)) \\
&= (\Omega_o, \int d^4x\partial_\lambda\Lambda(x)\bar{u}_p^{(r)}(x)\gamma_\lambda a(x)\int d^4x' a^\dagger(x')u_{p'}^{(r')}(x')\Omega_o) \\
&= -\int d^4x\Lambda(x)\ \partial_\lambda\bar{u}_p^{(r)}(x)\gamma_\lambda u_{p'}^{(r')}(x) \\
&= -\int d^4x\Lambda(x)\bar{u}_p^{(r)}(x)[(\gamma\partial+m) - (-\gamma\overleftarrow{\partial}+m)]u_{p'}^{(r')}(x) \\
&= 0\ . \qquad (3.1.53)
\end{aligned}
$$

We shall eliminate this arbitrariness by requiring the states $\hat{\Omega}$ to satisfy the conditions

$$a(x)\hat{\Omega} = b(x)\hat{\Omega} = 0 \quad \text{for} \quad x_o \neq \infty\ . \qquad (3.1.54)$$

The meaning of this condition will become clear in the discussion of gauge invariance. With this restriction the dual vectors are given uniquely by (3.1.39).

For the discussion of interacting fields we introduce the field operators

$$A_\mu(x) = c_\mu(x) + \int d^4y\ D_c(x-y)c_\mu^\dagger(y)\ , \qquad (3.1.55)$$

$$\psi(x) = a(x) - i\int d^4y\ S_c(x-y)b^\dagger(y)\ , \qquad (3.1.56)$$

$$\bar{\psi}(x) = -b(x) - i\int d^4y\ a^\dagger(y)S_c(y-x)\ , \qquad (3.1.57)$$

which obviously satisfy

$$[A_\mu(x),\ A_\nu(x)] = 0\ , \qquad (3.1.58)$$

$$\{\psi(x),\ \tilde{\psi}(y)\} = \{\psi(x),\ \psi(y)\} = \{\tilde{\psi}(x),\ \tilde{\psi}(y)\} = 0\ , \qquad (3.1.59)$$

and also

$$\Box A_\mu(x) = \Box c_\mu(x) + ic_\mu^\dagger(x) \quad , \qquad (3.1.60)$$

$$\partial_\mu A_\mu(x) = \partial_\mu c_\mu(x) + i\int d^4y\,\partial_\mu D_c(x-y)c_\mu^\dagger(y) \ , \qquad (3.1.61)$$

$$(\gamma\partial+m)\psi(x) = (\gamma\partial+m)a(x) - ib^\dagger(x) \ , \qquad (3.1.62)$$

$$\tilde{\psi}(x)(-\gamma\overleftarrow{\partial}+m) = -b(x)(-\gamma\overleftarrow{\partial}+m) - ia^\dagger(y) \ . \qquad (3.1.63)$$

As is readily verified, the non-vanishing matrix elements of $A_\mu(x)$ are given by

$$(\Omega_o,\ A_\mu(x)\Omega(;\ ;tk)) = u_{k\mu}^{(t)}(x) \ , \qquad (3.1.64)$$

$$(\hat{\Omega}(;\ ;tk),\ A_\mu(x)\Omega_o) = \bar{u}_{k\mu}^{(t)}(x) = u_{k\nu}^{(t)\dagger}(x)g_{\mu\nu} \ , \qquad (3.1.65)$$

so $A_i(x)\,(i=1,2,3)$ and $A_4(x)$ are hermitian and anti-hermitian respectively when sandwiched between Ω and $\hat{\Omega}$. In the same way we find that $\tilde{\psi}(x)\gamma_4$ is the hermitian conjugate of $\psi(x)$ when sandwiched between states.

The interaction between $\psi(x)$ and $A_\mu(x)$ can be dealt with as follows: Defining

$$I(x) = -ie\gamma_\lambda\psi(x)A_\lambda(x) \quad , \qquad (3.1.66)$$

$$\tilde{I}(x) = -ie\,\tilde{\psi}(x)\gamma_\lambda A_\lambda(x) \ , \qquad (3.1.67)$$

$$J_\lambda(x) = -ie\,\tilde{\psi}(x)\gamma_\lambda\psi(x) \quad , \qquad (3.1.68)$$

we impose the following field conditions on the state vectors

$$\{-(\gamma\partial+m)a(x) - I(x)\}\Psi = 0 \quad , \qquad (3.1.69)$$

$$\{b(x)(-\gamma\overleftarrow{\partial}+m) - \tilde{I}(x)\}\Psi = 0 \quad , \qquad (3.1.70)$$

$$\{\Box c_\mu(x) - J_\mu(x)\}\Psi = 0 \; , \qquad (3.1.71)$$

$$\{\partial_\mu c_\mu(x) - \int d^4y\,\partial_\mu D_C(x-y)J_\mu(x)\}\Psi = 0 \; , \qquad (3.1.72)$$

which may be rewritten

$$\{-(\gamma\partial+m)\psi(x) - I(x)\}\Psi = ib^\dagger(x)\Psi \; , \qquad (3.1.73)$$

$$\{-\tilde{\psi}(x)(-\gamma\partial+m) - \tilde{I}(x)\}\Psi = ia^\dagger(x)\Psi \; , \qquad (3.1.74)$$

$$\{\Box A_\mu(x) - J_\mu(x)\}\Psi = ic_\mu^\dagger(x)\Psi \; , \qquad (3.1.75)$$

$$\{\partial_\mu A_\mu(x) - \int d^4y\,\partial_\mu D_C(x-y)J_\mu(y)\}\Psi = i\int d^4y\,\partial_\mu D_C(x-y)c_\mu^\dagger(y)\Psi \; , \qquad (3.1.76)$$

as a consequence of (3.1.60)-(3.1.63).

As in Chapter 2, we construct the states Ψ explicitly. They are given by

$$\Psi = S\Omega \quad , \qquad (3.1.77)$$

with

$$S = \exp\,[-i(\bar{H} + \bar{H}_0)] \qquad (3.1.78)$$

where

$$\bar{H} = \int d^4x\ J_\mu(x)A_\mu(x) \quad , \tag{3.1.79}$$

$\bar{H}_o$ being a c-number constant such that

$$(\Omega_o, S\Omega_o) = 1 \ . \tag{3.1.80}$$

To verify that Ψ given by (3.1.77) satisfies (3.1.72) for instance, we note that

$$\begin{aligned}[\partial_\mu c_\mu(x), \bar{H}] &= \int d^4x' J_\nu(x')\partial_\mu[c_\mu(x), A_\nu(x')] \\ &= i\int d^4x' J_\nu(x')\partial_\nu D_c(x'-x)\end{aligned} \tag{3.1.81}$$

from which we find

$$\begin{aligned}&\exp(i\bar{H})\partial_\mu c_\mu(x)\exp(-i\bar{H}) \\ &\quad = \partial_\mu c_\mu(x) - i\partial_\mu[c_\mu(x), \bar{H}] \\ &\quad = \partial_\mu c_\mu(x) + \int d^4x'\partial_\mu D_c(x-x')J_\mu(x') \ .\end{aligned} \tag{3.1.82}$$

Since $J_\mu(x)$ and $\bar{H}$ commute as a consequence of (3.1.58) and (3.1.59), we obtain with the help of (3.1.10)

$$\begin{aligned}&[\partial_\mu c_\mu(x)\exp(-i\bar{H}) - \int d^4y\partial_\mu D_c(x-y)J_\mu(y)\ \exp(-i\bar{H})]\Omega \\ &\qquad = \exp(-i\bar{H})\partial_\mu c_\mu(x)\Omega = 0 \quad ,\end{aligned} \tag{3.1.83}$$

and hence (3.1.72).

It is this quantity S whose matrix elements between Ω and $\hat{\Omega}$ agree with those of the S-matrix in the usual quantum electrodynamics. We shall postpone this argument until later.

3.2 Current Conservation and Gauge Invariance

The fact that there is no field equation for $\psi(x)$ and $A_\mu(x)$ casts some doubt upon the validity of current conservation and gauge invariance. We may expect that the current $J_\mu(x)$ would be conserved when operated on the state vector Ψ on account of the field conditions (3.1.73)-(3.1.76). However, the situation is not as simple as one can hope, since the field conditions (3.1.73)-(3.1.76) have non-vanishing terms on the right hand side which have no counter part in field equations in the Heisenberg picture in ordinary quantum electrodynamics. Indeed, if we calculate $\partial_\mu J_\mu(x)\Psi$, we obtain

$$\begin{aligned}
\partial_\mu J_\mu(x)\Psi &= -ie\,\tilde{\psi}(x)\gamma_\mu(\partial_\mu+\overleftarrow{\partial}_\mu)\psi(x)\Psi \\
&= ie\,\tilde{\psi}(x)\{-(\gamma\partial+m) + (-\gamma\overleftarrow{\partial}+m)\}\psi(x)\Psi \\
&= ie\,\psi(x)\{I(x) + ib^\dagger(x)\}\Psi \\
&\quad + ie\,\psi(x)\{\tilde{I}(x) + ia^\dagger(x)\}\Psi \\
&= e\,\{a^\dagger(x)a(x) - b^\dagger(x)b(x)\}\Psi \\
&\quad + ie\int d^4y\; a^\dagger(y)S_c(y-x)b^\dagger(x)\Psi \\
&\quad - ie\int d^4y\; a^\dagger(x)S_c(x-y)b^\dagger(y)\Psi \quad . \qquad (3.2.1)
\end{aligned}$$

Thus we see that the "current" is not conserved in the ordinary sense. However, if we multiply (3.2.1) from the left by $\hat{\Omega}$ and make use of the condition (3.1.54) we obtain

$$(\hat{\Omega},\partial_\mu J_\mu(x)\Psi) = 0, \quad x_0 \neq \infty , \tag{3.2.2}$$

which corresponds to the condition of current conservation in the usual theory. The exclusion of the point $x_0 = \infty$ causes no trouble as will be seen below.

Next let us investigate what analogue the gauge invariance of ordinary quantum electrodynamics has in our formalism. For this purpose it is convenient to use a different set of vectors $e_\mu^{(t)'}$ in place of $e_\mu^{(t)}$ in (3.1.48). The new vectors are given by

$$e_\mu^{(1)'} = e_\mu^{(1)} , \tag{3.2.3}$$

$$e_\mu^{(2)'} = e_\mu^{(2)} , \tag{3.2.4}$$

$$e_\mu^{(3)'} = \frac{1}{\sqrt{2}}(e_\mu^{(3)} - e_\mu^{(4)}) = \frac{k_\mu}{\sqrt{2}\,(nk)} , \tag{3.2.5}$$

$$e_\mu^{(4)'} = \frac{1}{\sqrt{2}}(e_\mu^{(3)} + e_\mu^{(4)}) = \frac{k_\mu + 2n_\mu(nk)}{\sqrt{2}\,(nk)} , \tag{3.2.6}$$

and the wave function is now expressed as

$$u_{k\mu}^{(t)'}(x) = (2\pi)^{-3/2}\, e_\mu^{(t)'}\, e^{ikx} . \tag{3.2.7}$$

In this case the second of equations (3.1.12) eliminates only the t = 4 component of the wave function so that Ω may now contain states with t = 3. Ω may be written as

$$\Omega = \Omega_T x \Omega^{(3)} \ , \qquad (3.2.8)$$

where Ω_T contains only particles described by the wave functions $u_{k\mu}^{(1,2)}(x)$, and $\Omega^{(3)}$ only those described by $u_{k\mu}^{(3)}(x)$. We expand $\Omega^{(3)}$ as

$$\Omega^{(3)} = \Omega_o + \int d^3k C^{(1)}(k)\phi(k) + \int d^3k \int d^3k' C^{(2)}(k,k')\phi(k,k') + \dots, \qquad (3.2.9)$$

with

$$\phi(k) = \int d^4x c_\mu^\dagger(x) u_{k\mu}^{(3)}(x)\Omega_o$$

$$\phi(k,k') = \int d^4x c_\mu^\dagger(x) u_{k\mu}^{(3)}(x) \int d^4x' c_\nu^\dagger(x') u_{k'\nu}^{(3)}(x')\Omega_o,$$

etc. (3.2.10)

As is easily verified, the states (3.2.10) have zero norm so that the expansion coefficients in (3.2.9) are arbitrary.

Taking Ω_T to contain no particles we obtain, using (3.1.55), (3.2.5) and (3.2.8)-(3.2.10)

$$\begin{aligned}
(\hat{\Omega}, A_\mu(x)\Omega) &= \int d^3k (\Omega_o, c_\mu(x) \int d^4x' c_\nu^\dagger(x') u_{k\nu}^{(3)}(x')\Omega_o) C^{(1)}(k) \\
&\quad + \int d^3k (\Omega_o, [-i \int d\sigma_\lambda (x') \bar{u}_{k\nu}^{(3)}(x') (\partial'_\lambda - \overleftarrow{\partial}'_\lambda) c_\nu(x')] \\
&\quad \times [i \int d^4y D_c(x-y) c_\mu^\dagger(y)]\Omega_o) C^{(1)*}(k) \\
&= \int d^3k [C^{(1)}(k) u_{k\mu}^{(3)}(x) + C^{(1)*}(k)\bar{u}_{k\mu}^{(3)}(x)] \\
&= (2\pi)^{-3/2} \int d^3k \frac{1}{\sqrt{2}\ (nk)} [C^{(1)}(k) k_\mu e^{ikx} + C^{(1)*}(k) k_\mu e^{-ikx}] \\
&= \partial_\mu \Lambda(x) \ , \qquad (3.2.11)
\end{aligned}$$

where

$$\Lambda(x) = -i(2\pi)^{-3/2}\int d^3k \frac{1}{\sqrt{2}\,(nk)} [C^{(1)}(k)e^{ikx} - C^{(1)*}(k)e^{-ikx}]. \tag{3.2.12}$$

Obviously

$$\Box\Lambda(x) = 0 .$$

From the above calculations it is seen that what corresponds to the arbitrariness of gauge of the electromagnetic potential in ordinary theory is the arbitrariness of the expansion coefficients $C^{(1)}(k)$ of (3.2.9) in hyper theory.

Denoting the second term in the expansion (3.2.9) by $\phi^{(1)}$ we may calculate the following two quantities:

$$\begin{aligned}(\hat{\Omega}'_T, S\phi^{(1)}\Omega_T) &= \int d^3kC^{(1)}(k)(\hat{\Omega}'_T, S\int d^4xc^{\dagger}_{\mu}(x)u^{(3)}_{k\mu}(x)\Omega_T)\\ &= -i\int d^3k\int d^4xC^{(1)}(k)(\Omega'_T, J_{\mu}(x)S\Omega_T)u^{(3)}_{k\mu}(x)\\ &= -\int d^4x'(\Omega'_T, J_{\mu}(x)S\Omega_T)\\ &\quad\times \partial_{\mu}[(2\pi)^{-3/2}\int d^3k \frac{1}{\sqrt{2}\,(nk)} C^{(1)}(k)e^{ikx}]\\ &= \int d^4x'(\Omega'_T, \partial_{\mu}J_{\mu}(x)S\Omega_T)\\ &\quad\times [(2\pi)^{-3/2}\int d^3k \frac{1}{\sqrt{2}\,(nk)} C^{(1)}(k)e^{ikx}] ,\end{aligned} \tag{3.2.14}$$

and

$$(\hat{\Omega}_T'\hat{\phi}^{(1)},\ S\Omega_T) = (\hat{\Omega}_T',\ [-i\int d\sigma_\lambda(x)\bar{u}_k^{(3)}(x)(\partial_\lambda-\overleftarrow{\partial}_\lambda)c_\mu(x)]S\Omega_T)$$

$$= -\ d^4x(\hat{\Omega}_T',\partial_\mu J_\mu(x)S\Omega_T)[(2\pi)^{-3/2}\int d^3k\ \frac{1}{\sqrt{2}\ (nk)}$$

$$\times\ C^{(1)*}(k)e^{-ikx}]\ , \qquad (3.2.15)$$

where use has been made of the commutation relations

$$[S,\ c_\mu^\dagger(x)] = -i\ S\ J_\mu(x)\ , \qquad (3.2.16)$$

$$[c_\mu(x),\ S] = S\int d^4x'J_\mu(x')D_c(x'-x)\ . \qquad (3.2.17)$$

Eqs. (3.2.14) and (3.2.15) together with (3.2.12) yield

$$(\hat{\Omega}_T',\ S\phi^{(1)}\Omega_T) + (\hat{\Omega}_T'\hat{\phi}^{(1)},\ S\Omega_T) = \int d^4x(\Omega_T',\partial_\mu J_\mu(x)S\Omega_T)\Lambda(x)\ . \qquad (3.2.18)$$

If we assume that $\Lambda(x)$ satisfies

$$\Lambda(x) = 0 \quad \text{for} \quad |x| \to \infty\ , \qquad (3.2.19)$$

then (3.2.18) and (3.2.2) give us

$$(\hat{\Omega}_T',\ S\phi^{(1)}\Omega_T) + (\hat{\Omega}_T'\hat{\phi}^{(1)},\ S\Omega_T) \equiv 0\ . \qquad (3.2.20)$$

Therefore, from (3.2.8) and (3.2.9), the S-matrix element between two arbitrary states Ω and $\hat{\Omega}'$ is given by

$$(\hat{\Omega}',\ S\Omega) = (\hat{\Omega}_T',\ S\Omega_T) + (\hat{\Omega}_T',\ S\phi^{(1)}\Omega_T) + (\hat{\Omega}_T'\hat{\phi}^{(1)},\ S\Omega_T)$$

$$= (\hat{\Omega}_T',\ S\Omega_T)\ , \qquad (3.2.21)$$

which shows that our S-matrix is "gauge invariant" in that it does not depend on the arbitrary function $\Lambda(x)$.

3.3 Dysons Equations and the Ward-Takahashi Identity

Having investigated gauge invariance and current conservation, we now derive Dyson's equations as well as the Ward-Takahashi identity. Hereby, the equivalence of our formalism to the ordinary theory is established.

We first note the equations

$$-(\gamma\partial+m)(\Omega_0,\psi(x)\tilde{\psi}(y)\Psi_0)=(\Omega_0,I(x)\tilde{\psi}(y)\Psi_0)+i\delta^{(4)}(x-y)\ , \tag{3.3.1}$$

$$\Box(\Omega_0, A_\mu(x)A_\nu(y)\Psi_0)=(\Omega_0,J_\mu(x)A_\nu(y)\Psi_0)+i\delta_{\mu\nu}\delta^{(4)}(x-y)\ , \tag{3.3.2}$$

$$\Box\partial_\mu(\Omega_0,A_\mu(x)A_\nu(y)\Psi_0)=i\partial_\nu\delta^{(4)}(x-y) \tag{3.3.3}$$

which follow from (3.1.69)-(3.1.72) and (3.2.1). If we define

$$-i\ S'_c(x-y)=(\Omega_0,\psi(x)\tilde{\psi}(y)\Psi_0)\ , \tag{3.3.4}$$

$$i\ D'_{c\mu\nu}=(\Omega_0,\ A_\mu(x)A_\nu(y)\Psi_0)\ , \tag{3.3.5}$$

$$ie\int d^4x'd^4y'd^4z'S'_c(y-y')\Gamma_\nu(y'-x';x'-z')S'_c(z'-z)D'_{c\nu\mu}(x'-x)$$

$$=(\Omega_0,\ A_\mu(x)\psi(y)\tilde{\psi}(z)\Psi_0)\ , \tag{3.3.6}$$

and substitute these into (3.3.1) and (3.3.2) we obtain

$$(\gamma\partial+m)S'_c(x-y) = \int d^4x'\Sigma^*(x-x')S'_c(x'-y) + \delta^{(4)}(x-y), \quad (3.3.7)$$

$$\Box D'_{c\mu\nu}(x-y) = \int d^4x'\Pi_{\mu\rho}(x-x')D'_{c\rho\nu}(x'-y) + \delta_{\mu\nu}\delta^{(4)}(x-y) , \quad (3.3.8)$$

where

$$\Sigma^*(x-x') = -ie^2\int d^4yd^4z'\gamma_\mu S'_c(x-x')\Gamma_\nu(y'-z';\ z'-x') \times D'_{c\nu\mu}(z'-x) , \quad (3.3.9)$$

$$\Pi_{\mu\nu}(x-x') = ie^2\int d^4y'd^4z'\ \mathrm{Tr}[\gamma_\mu S'_c(x-y')\Gamma_\nu(y'-x';\ x'-z') \times S'_c(z'-x)] . \quad (3.3.10)$$

If we transform (3.3.7) and (3.3.8) into momentum space and divide by $(i\gamma p+m)$ and k^2, respectively, we arrive at Dyson's equations for the propagators and the vertex.

In order to derive the Ward-Takahashi identity, we use (3.1.72) to obtain

$$\Box\partial_\mu(\Omega_0,\ A_\mu(x)\psi(y)\tilde{\psi}(z)\Psi_0) = (\Omega_0,\ \psi(y)\tilde{\psi}(z)\partial_\mu J_\mu(z)\Psi_0) . \quad (3.3.11)$$

If we now use (3.2.1) and shift the creation and annihilation operators to the left we arrive at

$$\Box\partial_\mu(\Omega_0,\ A_\mu(x)\psi(y)\tilde{\psi}(z)\Psi_0) = -e(\Omega_0,\ \psi(y)\tilde{\psi}(z)\Psi_0)\delta^{(4)}(z-x) +e(\Omega_0,\ \psi(x)\tilde{\psi}(z)\Psi_0)\delta^{(4)}(y-x) . \quad (3.3.12)$$

Then substituting (3.3.4) and (3.3.6) and transferring to momentum space we have finally

$$i\Gamma_\nu(p;\ q)(p-q)_\nu = S_c'^{-1}(p) - S_c'^{-1}(q) \ , \qquad (3.3.13)$$

which is just the Ward-Takahashi identity.$^{1)}$

From the above it is seen that our formalism is completely equivalent to Dyson's. Therefore the renormalization program can be carried out in exactly the same manner.

3.4 The S-matrix

We shall now show that the matrix elements of the quantity S defined by (3.1.78) with (3.1.79) agree with the S-matrix elements in the conventional approach to quantum electrodynamics. Using the commutation relations

$$[c_\mu(x),\ c_k^{(t)\dagger}] = u_{k\mu}^{(t)}(x) \ , \qquad (3.4.1)$$

$$[\hat{c}_k^{(t)},\ \{i\int d^4y D_c(x-y) c_\mu^\dagger(y)\}] = \bar{u}_{k\mu}^{(t)}(x) \ , \qquad (3.4.2)$$

we find that,

$$(\hat{\Omega}'(;\ ;q_1'\cdots q_m'),\ :A_\mu(x_1)A_\nu(x_2):\ \Omega(;\ ;q_1\cdots q_n))$$

$$= \sum_{i=1}^{n}\ \sum_{j=1}^{n}\ (1-\delta_{ij})(\hat{\Omega}'(;\ ;q_1'\cdots q_m'),\ \Omega(;\ ;q_1\cdots q_{j-1},$$

$$q_{j+1}\cdots q_{i-1},\ q_{i+1}\cdots q_n))u_{k_j\mu}^{(t_j)}(x_1)u_{k_i\nu}^{(t_i)}(x_2)$$

$$+ \sum_{i=1}^{n} \sum_{j=1}^{m} (\hat{\Omega}'(q'_1 \ldots q'_{j-1}, q'_{j+1} \ldots q'_m, \Omega(; ;q_1 \ldots q_{i-1},$$

$$q_{i+1} \ldots q_n)) \bar{u}^{(t'_j)}_{k'_j\nu}(x_2) u^{(t_i)}_{k_i\mu}(x_1)$$

$$+ \sum_{i=1}^{n} \sum_{j=1}^{m} (\hat{\Omega}'(; ;q'_1 \ldots q'_{j-1}, q'_{j+1} \ldots q'_m), \Omega(; ;q_1 \ldots q_{i-1},$$

$$q_{i+1} \ldots q_n)) \bar{u}^{(t'_j)}_{k'_j\mu}(x_1) u^{(t_i)}_{k_i\nu}(x_2)$$

$$+ \sum_{i=1}^{m} \sum_{j=1}^{m} (1-\delta_{ij}) (\hat{\Omega}'(; ;q'_1 \ldots q'_{j-1}, q'_{j+1} \ldots q'_{i-1},$$

$$q'_{i+1} \ldots q_m), \Omega(; ;q_1 \ldots q_n)) \bar{u}^{(t'_i)}_{k'_i\mu}(x_1) \bar{u}^{(t'_j)}_{k'_j\nu}(x_2),$$

where the q's indicate specifications of both k and t. Comparison with the similar equation for $(\hat{\Omega}(; ;q_1 \ldots q_n), : A_\nu(x_2)\ A_\mu(x_1)\ :\ \Omega'(; ;q'_1 \ldots q'_m))$ yields

$$(\hat{\Omega}'(; ;q'_1 \ldots q'_m), : A_\mu(x_1) A_\nu(x_2)\ :\ \Omega(; ;q_1 \ldots q_n))^*$$

$$= (\hat{\Omega}(; ;q_1 \ldots q_n), : A_\sigma(x_2) A_\rho(x_1) : \Omega'(; ;q'_1 \ldots q'_m)) g_{\sigma\nu} g_{\rho\mu}$$

(3.4.3)

In general we have

$$(\hat{\Omega}', : A_{\mu_1}(x_1) A_{\mu_2}(x_2) \ldots A_{\mu_n}(x_n)\ :\ \Omega)^*$$

$$= (\hat{\Omega}, : A_{\rho_n}(x_n) \ldots A_{\rho_2}(x_2) A_{\rho_1}(x_1) : \Omega') g_{\rho_n\mu_n} \ldots g_{\rho_1\mu_1} .$$

(3.4.4)

Moreover, it follows from (3.1.55) that

$$A_\mu(x_1)A_\nu(x_2) = :A_\mu(x_1)A_\nu(x_2): + iD_c(x_2-x_1)\delta_{\mu\nu}\ , \qquad (3.4.5)$$

and in general

$$A_{\mu_1}(x_1)A_{\mu_2}(x_2)\ldots A_{\mu_n}(x_n) = :A_{\mu_1}(x_1)\ldots A_{\mu_n}(x_n): +$$

$$iD_c(x_2-x_1)\ :A_{\mu_3}(x_3)\ldots A_{\mu_n}(x_n): +\ldots \qquad (3.4.6)$$

Therefore, the product of operators on the left side of (3.4.6) corresponds to the chronological product of the interaction picture operators in ordinary field theory.

In the same way it can be shown that $\tilde{\psi}(x)\gamma_4$ behaves like the hermitian conjugate of $\psi(x)$ when it occurs in a normal product sandwiched between Ω and $\hat{\Omega}$, and that the Wick expansion in normal products holds for products of these operators.

From the foregoing discussion it follows that the matrix elements of the operator S in (3.1.78) agree with the S-matrix elements in the usual theory. We shall demonstrate this agreement to second order.

The second order term in the expansion of our S-matrix is given by

$$S^{(2)} = \frac{(-e)^2}{2!}\int d^4x_1 d^4x_2 :\tilde{\psi}(x_1)\gamma_\mu\psi(x_1)A_\mu(x_1)::\tilde{\psi}(x_2)\gamma_\nu\psi(x_2)A_\nu(x_2):$$

$$= \frac{(-e)^2}{2!} \int d^4x_1 d^4x_2 \{ : A_\mu(x_1) A_\nu(x_2) : + iD_c(x_2-x_1)\delta_{\mu\nu} \}$$

$$\times \{ -b(x_1)\gamma_\mu a(x_1) - i\int d^4y_1 b^\dagger(y_1)\gamma_\mu S_c(x_1-y_1) b(x_1)$$

$$- i\int d^4z_1 a^\dagger(z_1)\gamma_\mu S_c(z_1-x_1) a(x_1)$$

$$- \int d^4y_1 d^4z_1 a^\dagger(z_1)\gamma_\mu S_c(z_1-x_1) S_c(x_1-y_1) b^\dagger(y_1) \}$$

$$\times \{ -b(x_2)\gamma_\nu a(x_2) - i\int d^4y_2 b^\dagger(y_2)\gamma_\nu S_c(x_2-y_2) b(x_2)$$

$$- i\int d^4z_2 a^\dagger(z_2)\gamma_\nu S_c(z_2-x_2) a(x_2)$$

$$- d^4y_2 d^4z_2 a^\dagger(z_2)\gamma_\nu S_c(z_2-x_2) S_c(x_2-y_2) b^\dagger(y_2) \}$$

$$= \frac{(-e)^2}{2!} \int d^4x_1 d^4x_2 \{ : A_\mu(x_1) A_\nu(x_2) : + i\delta_{\mu\nu} D_c(x_2-x_1) \}$$

$$\times \{ : \tilde{\psi}(x_1)\gamma_\mu \psi(x_1)\tilde{\psi}(x_2)\gamma_\nu\psi(x_2) : -iS_c(x_1-x_2)$$

$$\times \; : \tilde{\psi}(x_1)\gamma_\mu\gamma_\nu\psi(x_2) :$$

$$+ iS_c(x_2-x_1) : \gamma_\mu\psi(x_1)\tilde{\psi}(x_2)\gamma_\nu : + S_c(x_1-x_2) S_c(x_2-x_1)\gamma_\mu\gamma_\nu \},$$

(3.4.7)

where we have used equations (3.1.55)-(3.1.57) and (3.4.5) along with the easily derived relations

$$\tilde{\psi}(x_1)\psi(x_2) = \; : \tilde{\psi}(x_1)\psi(x_2) : + iS_c(x_2-x_1) \;, \qquad (3.4.8)$$

and

$$\psi(x_1)\tilde{\psi}(x_2) = :\psi(x_1)\tilde{\psi}(x_2): - iS_c(x_1-x_2) \ . \qquad (3.4.9)$$

From (3.4.5), (3.4.8) and (3.4.9) we see that the contractions of the various field operators are given by

$$\underbrace{\tilde{\psi}(x_1)\psi}_{}(x_2) = iS_c(x_2-x_1) \ , \qquad (3.4.10)$$

$$\underbrace{\psi(x_1)\tilde{\psi}}_{}(x_2) = -iS_c(x_1-x_2) \ , \qquad (3.4.11)$$

$$\underbrace{A_\mu(x_1)A_\nu}_{}(x_2) = i\delta_{\mu\nu}D_c(x_2-x_1) \ , \qquad (3.4.12)$$

and in the same way we find

$$\underbrace{\tilde{\psi}(x_1)\underbrace{\psi(x_1)\tilde{\psi}}_{}(x_2)\psi}_{}(x_2) = S_c(x_2-x_1)S_c(x_1-x_2) \qquad (3.4.13)$$

putting (3.4.10)-(3.4.13) in (3.4.7) we see that to second order our S-matrix elements agree with those in ordinary quantum electrodynamics.

From this agreement of the matrix elements it follows that our S-matrix satisfies conservation of probability; i.e.,

$$\sum_n (\Omega_n, S\Omega_2)^*(\hat{\Omega}_n, S\Omega_1) = (\hat{\Omega}_2, \Omega_1) \ , \qquad (3.4.14)$$

and

$$\sum_n (\hat{\Omega}_2, S\Omega_n)(\hat{\Omega}_1, S\Omega_n)^* = (\hat{\Omega}_2, \Omega_1) \ , \qquad (3.4.15)$$

corresponding to the unitarity of the S-matrix in the usual theory.

We have applied the general theory developed in Chapter 2 to the quantum electrodynamics of a spin 1/2 field and have shown that the matrix elements of our S agree with those of the conventional S-matrix. We have also shown that the formalism is gauge invariant. Dyson's equations for propagators and the vertex, and also the Ward-Takahashi identity have been derived. One advantage of our formalism is that the indefinite metric does not have to be referred to. We note, however, that it is very difficult in our formalism to prove the spectral representation of propagators. This is because the causal propagator appears directly in our formalism without referring to the positive and negative frequency propagators. A proof of the spectral representation is usually carried out by asserting that the positive and negative frequency propagators separately satisfy the spectral representation and then combining these two to form the causal propagator. We cannot use this argument here.

Now that we have demonstrated that the electromagnetic field can be hyperquantized consistently in a gauge invariant manner, and indeed that the results of quantum electrodynamics can be exactly reproduced using this formalism, we shall proceed to apply this technique to the electromagnetic-spin 3/2 interaction where ordinary quantization procedures fail.

4. THE ELECTROMAGNETIC-SPIN 3/2 INTERACTION

In this chapter we shall apply the technique of hyperquantization to the system of an electromagnetic field coupled to a spin 3/2 field.[13)] This system presents problems when handled using conventional quantization procedures as will be explained shortly. We shall see that the difficulties inherent in the use of the ordinary theory will not arise in our formalism. However, the proof of probability conservation (unitarity) is not as straightforward as it was for the electromagnetic-spin 1/2 coupling. This is because we cannot compare our S-matrix elements with those of the conventional theory since the latter have not been found. No general proof of unitarity has been forthcoming. In this chapter we demonstrate it to lowest order in the perturbation expansion.

4.1 The Johnson-Sudarshan Inconsistency

The system of a spin 3/2 field interacting with an electromagnetic field cannot be quantized by conventional methods. The reason for this was pointed out by Johnson and Sudarshan[2)] and is explained below. A knowledge of the explicit calculations involved is not necessary for an understanding of this thesis and will not be given here.

The reader wishing a detailed knowledge of these calculations is referred to the original paper.$^{2)}$

According to the Schwinger's action principle,$^{14)}$ the variation of the action integral over a volume bounded by two space-like surfaces may be expressed as

$$\delta \int_{\sigma_1}^{\sigma_2} d^4x\, \mathcal{L}(x) = G(\sigma_2) - G(\sigma_1) \qquad (4.1.1)$$

where σ_1 and σ_2 are space-like surfaces bounding the volume of integration, $\mathcal{L}(x)$ is the Lagrangian for the system, and G is the generator of the transformation giving rise to the variation. Thus, the variation in the field operators undergoing this transformation is given by

$$\delta\psi(x) = [\psi(x), G] \quad . \qquad (4.1.2)$$

Equations (4.1.1) and (4.1.2) will determine both the equations of motion and the field commutation or anticommutation relations.

Under Lorentz transformations the Rarita-Schwinger field has components which transform as spin 1/2 and spin 3/2 fields. It turns out that these components are not independent, but are related by equations of constraint. Therefore, the variations of these components cannot be done independently in (4.1.1). Taking the constraints between these variations into account, Johnson and Sudarshan have shown using (4.1.1) and (4.1.2) that, when the Lagrangian $\mathcal{L}(x)$ contains coupled electromagnetic and spin 3/2 fields, the requirement that the anticommutators of the fermion field operators be positive definite demands that

$$m^2 > \frac{2}{3} |e \underset{\sim}{H}| \qquad (4.1.3)$$

everywhere. Here m and e are, respectively, the mass and charge of the fermion field and $\underset{\sim}{H}$ is the magnetic field intensity. However, we can always find a Lorentz frame in which (4.1.3) is violated. So we have an inconsistency when ordinary quantization procedures are applied to this system.

Now, as we have seen previously, the field commutators or anticommutators are fixed at zero in the formalism of hyperquantization. In the case of the electromagnetic-spin 3/2 interaction, therefore, the aforementioned difficulty of negative anticommutators will not arise. In the following sections we shall investigate the hyperquantization of this interaction to determine whether other inconsistencies are present.

4.2 Formulation

Let us consider the interaction of the Rarita-Schwinger and electromagnetic fields. The electromagnetic field can be hyperquantized in a gauge invariant manner as was shown in Chapter 3, and no further discussion on this point will be given here.

As usual, we begin by introducing the creation and annihilation operators obeying

$$\{a_\sigma(x), a_\rho^\dagger(y)\} = \delta_{\sigma\rho}\delta^{(4)}(x-y) \ , \tag{4.2.1}$$

$$\{b_\sigma(x), b_\rho^\dagger(y)\} = \delta_{\sigma\rho}\delta^{(4)}(x-y) \ , \tag{4.2.2}$$

$$[c_\mu(x), c_\nu^\dagger(y)] = \delta_{\mu\nu}\delta^{(4)}(x-y) \ , \tag{4.2.3}$$

with similar commutators or anticommutators of other operator combinations vanishing. Here μ, ν, σ, and ρ are vector indices running from 1 to 4, and spinor indices have been suppressed. In constructing the above relations we have assumed the relation between spin and statistics. The vacuum state Ω_o is defined such that

$$a_\sigma(x)\Omega_o = b_\sigma(x)\Omega_o = 0 \ , \tag{4.2.4}$$

$$c_\mu(x)\Omega_o = 0 \ , \tag{4.2.5}$$

and this state, together with the creation operators, may be used to construct a Fock space which is as yet not connected with physical reality. From the states of this space, we now select the physical states which satisfy the restrictions

$$\Lambda_{\sigma\rho}(\partial)a_\rho(x)\Omega = 0 \ , \tag{4.2.6}$$

$$b_\rho(x)\Lambda_{\rho\sigma}(-\overleftarrow{\partial})\Omega = 0 \ , \tag{4.2.7}$$

$$\left.\begin{aligned} \Box c_\mu(x)\Omega &= 0 \ , \\ \partial_\mu c_\mu(x)\Omega &= 0 \ , \end{aligned}\right\} \tag{4.2.8}$$

with1)

$$\Lambda_{\sigma\rho}(\partial) = -(\gamma\partial+m)\delta_{\sigma\rho} + \frac{1}{3}(\gamma_\sigma\partial_\rho + \gamma_\rho\partial_\sigma) - \frac{1}{3}\gamma_\sigma(\gamma\partial-m)\gamma_\rho \ . \tag{4.2.9}$$

The operator (4.2.9) satisfies the identity

$$\Lambda(\partial)d(\partial) = d(\partial)\Lambda(\partial) = \Box - m^2 \ , \tag{4.2.10}$$

with the Klein-Gordon divisor $d(\partial)$ given by1)

$$\begin{aligned} d_{\sigma\rho}(\partial) = -(\gamma\partial-m)[\delta_{\sigma\rho} - \frac{1}{3}\gamma_\sigma\gamma_\rho + \frac{1}{3m}(\gamma_\sigma\partial_\rho - \gamma_\rho\partial_\sigma \\ - \frac{2}{3m^2}\partial_\sigma\partial_\rho] - \frac{2}{3m^2}(\Box - m^2)[(\gamma_\sigma\partial_\rho - \gamma_\rho\partial_\sigma) \\ + (\gamma\partial-m)\gamma_\sigma\gamma_\rho] \ . \end{aligned} \tag{4.2.11}$$

The two operators1)

$$\Gamma_{\mu,\sigma\rho} = -\gamma_\mu\delta_{\sigma\rho} + \frac{1}{3}(\gamma_\rho\delta_{\sigma\mu} + \gamma_\rho\delta_{\rho\mu}) - \frac{1}{3}\gamma_\sigma\gamma_\mu\gamma_\rho , \tag{4.2.12}$$

and

$$\eta_{\sigma\rho} = g_{\sigma\rho}\gamma_4 \ , \tag{4.2.13}$$

can be shown to satisfy the relations

$$\Lambda(\partial) - \Lambda(-\overleftarrow{\partial}) = (\partial_\mu + \overleftarrow{\partial}_\mu)\Gamma_\mu \quad , \tag{4.2.14}$$

and

$$[\eta\Lambda(\partial)]^\dagger = \eta\Lambda(-\partial) \quad . \tag{4.2.15}$$

Moreover, the energy-momentum and charge operators,

$$P_\mu = -\tfrac{i}{2}\int d^4x\{a_\sigma^\dagger(x)(\partial_\mu-\overleftarrow{\partial}_\mu)a_\sigma(x) + b_\sigma^\dagger(x)(\partial_\mu-\overleftarrow{\partial}_\mu)b_\sigma(x)$$

$$+ c_\lambda^\dagger(x)(\partial_\mu-\overleftarrow{\partial}_\mu)c_\lambda(x)\} \quad , \tag{4.2.16}$$

and

$$Q = e\int d^4x \ \{a_\sigma^\dagger(x)a_\sigma(x) - b_\sigma^\dagger(x)b_\sigma(x)\} \quad , \tag{4.2.17}$$

are easily seen to obey

$$-i\partial_\mu a_\sigma(x) = [a_\sigma(x), P_\mu] \quad , \tag{4.2.18}$$

$$-i\partial_\mu b_\sigma(x) = [b_\sigma(x), P_\mu] \quad , \tag{4.2.19}$$

$$-i\partial_\mu c_\lambda(x) = [c_\lambda(x), P_\mu] \quad , \tag{4.2.20}$$

$$e a_\sigma(x) = [a_\sigma(x), Q] \quad , \tag{4.2.21}$$

$$-e b_\sigma(x) = [b_\sigma(x), Q] \quad , \tag{4.2.22}$$

$$0 = [c_\lambda(x), Q] \quad , \tag{4.2.23}$$

with similar equations for $a_\sigma^\dagger(x)$, $b_\sigma^\dagger(x)$, and $c_\lambda^\dagger(x)$. For example, taking the Hermitian conjugate of (4.2.21) gives

$$ea_\sigma^\dagger(x) = [a_\sigma^\dagger(x), Q] \; . \tag{4.2.24}$$

The states Ω satisfying (4.2.6)-(4.2.8) and the dual states $\hat{\Omega}$ are constructed from products of the operators

$$A_p^{(r)\dagger} = \int d^4x a_\sigma^\dagger(x) u_{p\sigma}^{(r)}(x) \; , \tag{4.2.25}$$

$$B_q^{(s)\dagger} = -\int d^4x v_{q\sigma}^{(s)}(x) b_\sigma^\dagger(x) \; , \tag{4.2.26}$$

$$C_k^{(t)\dagger} = \int d^4x c_\mu^\dagger(x) u_{k\mu}^{(t)}(x) \; , \tag{4.2.27}$$

and

$$\hat{A}_p^{(r)\dagger} = -i\int_\infty d\sigma_\lambda(x) a_\sigma^\dagger(x) \bar{\Gamma}_{\lambda,\sigma\rho} u_{p\rho}^{(r)}(x) \; , \tag{4.2.28}$$

$$\hat{B}_p^{(r)\dagger} = i\int_\infty d\sigma_\lambda(x) v_{q\sigma}^\dagger(x) \bar{\Gamma}_{\lambda,\sigma\rho} b_\rho^\dagger(x) \; , \tag{4.2.29}$$

$$\hat{C}_k^{(t)\dagger} = -i\int_\infty d\sigma_\lambda(x) c_\mu^\dagger(x) g_{\mu\nu}(\partial_\lambda - \overleftarrow{\partial}_\lambda) u_{k\nu}^{(t)}(x) \; , \tag{4.2.30}$$

respectively acting on the vacuum Ω_o in the same way as in Chapter 2. The wave functions under the above integrals obey

$$\Lambda_{\sigma\rho}(\partial) u_{p\rho}^{(r)}(x) = \Lambda_{\sigma\rho}(\partial) v_{q\rho}^{(s)}(x) = 0 \; , \tag{4.2.31}$$

$$\Box u_{k\mu}^{(t)}(x) = \partial_\mu u_{k\mu}^{(t)}(x) = 0 \; , \tag{4.2.32}$$

together with normalization and closure conditions similar to (3.1.13)-(3.1.18). For example

$$-i\int d\sigma_{\lambda}(x)\bar{u}^{(r)}_{p\sigma}(x)\Gamma_{\lambda,\sigma\rho}u^{(r')}_{p'}(x) = 2\omega(\underset{\sim}{p})\delta_{rr'}\delta(\underset{\sim}{p}-\underset{\sim}{p}') , \quad (4.2.33)$$

$$\sum_{r} \frac{d^3p}{2\omega(\underset{\sim}{p})} u^{(r)}_{p\sigma}(x)\bar{u}^{(r)}_{p\rho}(x') = id_{\sigma\rho}(\partial)\Delta^{(+)}(x-x') . \quad (4.2.34)$$

As in Chapter 3, the orthonormalization conditions for the physical states can now be shown to be

$$(\hat{\Omega}(pr; ;), \Omega(p'r'; ;)) = 2\omega(\underset{\sim}{p})\delta_{rr'}\delta(\underset{\sim}{p}-\underset{\sim}{p}') , \quad (4.2.35)$$

$$(\hat{\Omega}(;qs;), \Omega(;q's';)) = 2\omega(\underset{\sim}{q})\delta_{ss'}\delta(\underset{\sim}{q}-\underset{\sim}{q}') , \quad (4.2.36)$$

$$(\hat{\Omega}(; ;kt), \Omega(; ;k't')) = 2|\underset{\sim}{k}|\delta_{tt'}\delta(\underset{\sim}{k}-\underset{\sim}{k}') , \quad (4.2.37)$$

where

$$\left.\begin{aligned} r,r',s,s' &= 1,2,3,4, \\ t,t' &= 1,2 . \end{aligned}\right\} \quad (4.2.38)$$

We now introduce the field operators

$$\psi_{\sigma}(x) = a_{\sigma}(x) + i\int d^4x'd_{\sigma\rho}(\partial)\Delta_c(x-x')b^{\dagger}_{\rho}(x') , \quad (4.2.39)$$

$$\tilde{\psi}_{\sigma}(x) = -b_{\sigma}(x) + i\int d^4x'a^{\dagger}_{\rho}(x')\Delta_c(x-x')d_{\rho\sigma}(-\overleftarrow{\partial}'), \quad (4.2.40)$$

$$A_{\mu}(x) = c_{\mu}(x) + i\int d^4x' \, D_c(x-x')c^{\dagger}_{\mu}(x') , \quad (4.2.41)$$

for which we have

$$\{\psi_\sigma(x), \psi_\rho(x')\} = \{\tilde{\psi}_\sigma(x), \tilde{\psi}_\rho(x')\} = \{\psi_\sigma(x), \tilde{\psi}_\rho(x')\} = 0 , \tag{4.2.42}$$

$$[A_\mu(x), A_\rho(x')] = [A_\mu(x), \psi_\rho(x')] = [A_\mu(x), \tilde{\psi}_\rho(x')] = 0 , \tag{4.2.43}$$

To obtain the interacting states of the system, we make the substitution $\partial \to \partial - ieA$ in (4.2.6) and (4.2.7) with the result

$$[\Lambda_{\sigma\rho}(\partial)\psi_\rho(x) + J_\sigma(x) - ib_\sigma^\dagger(x)]\Psi = 0 , \tag{4.2.44}$$

$$[\psi_\rho(x)\Lambda_{\rho\sigma}(-\overset{\dagger}{\partial}) + \tilde{J}_\sigma(x) - ia_\sigma^\dagger(x)]\Psi = 0 , \tag{4.2.45}$$

where

$$J_\sigma(x) = -ie\, \Gamma_{\lambda,\sigma\rho}\psi_\rho(x)A_\lambda(x) , \tag{4.2.46}$$

$$\tilde{J}_\sigma(x) = -ie\, \tilde{\psi}_\rho(x)\Gamma_{\lambda,\rho\sigma}A_\lambda(x) , \tag{4.2.47}$$

use having been made of (4.2.39)-(4.2.41). The states Ψ satisfying (4.2.44) and (4.2.45) are given explicitly by

$$\Psi = S\Omega , \tag{4.2.48}$$

with

$$S = \exp[-i(\bar{H} + \bar{H}_0)] , \tag{4.2.49}$$

$$\bar{H} = -\int d^4y\, \tilde{\psi}_\sigma(y)J_\sigma(y) , \tag{4.2.50}$$

and H_0 a c-number constant such that

$$(\Omega_0, S\Omega_0) = 1 . \tag{4.2.51}$$

It is the operator S which corresponds to the S-matrix in the conventional theory, and which will be the object of discussion in the next section.

4.3 The S-matrix

From the foregoing formulation it can be seen that the Johnson-Sudarshan difficulty, which besets the conventional approach to the quantization of the Rarita-Schwinger field, does not arise in the hyperquantized formalism. This is because we fixed our field operator commutation relations (4.2.42) and (4.2.43) at the beginning, and we do not find Lorentz frames in which they become negative. However, we are, as usual, faced with the task of showing that the operator S, defined by (4.2.49), satisfies

$$\sum_n (\hat{\Omega}_n, S\Omega_2)^* (\hat{\Omega}_n, S\Omega_1) = (\hat{\Omega}_2, \Omega_1) , \qquad (4.3.1)$$

corresponding to S-matrix unitarity in the usual theory. We have thus far been unable to prove this condition in general, and we content ourselves here with an explicit demonstration to second order in the perturbation expansion.

Equation (4.3.1) is satisfied to second order provided that

$$(\hat{\Omega}_1, S^{(2)}\Omega_2)^* + (\hat{\Omega}_2, S^{(2)}\Omega_1) = 0 , \qquad (4.3.2)$$

where, according to (4.2.49), (4.2.50) and (4.2.46),

$$S^{(2)} = \frac{e^2}{2}\int d^4x d^4y \ : \tilde{\psi}(x)\Gamma_\lambda\psi(x)A_\lambda(x)::\tilde{\psi}(y)\Gamma_\kappa\psi(y)A_\kappa(y) \ : \ . \tag{4.3.3}$$

In the usual way, we can expand the matrix elements of this operator in normal products to give

$$\begin{aligned}(\hat{\Omega}_2,\ S^{(2)}\Omega_1) = \frac{e^2}{2}\int d^4x d^4y (\hat{\Omega}_2, :\tilde{\psi}(x)\eta^{-1}\bar{\Gamma}_\lambda\psi(x)\tilde{\psi}(y)\eta^{-1}\bar{\Gamma}_\kappa\psi(y):\\ + i:\tilde{\psi}(x)\eta^{-1}\bar{\Gamma}_\lambda\bar{d}(\partial)\Delta_c(x-y)\bar{\Gamma}_\kappa\psi(y):\\ + i:\tilde{\psi}(y)\eta^{-1}\bar{\Gamma}_\kappa\bar{d}(-\partial)\Delta_c(x-y)\bar{\Gamma}_\lambda\psi(x):\\ + \mathrm{Tr}\{\bar{d}(-\partial)\Delta_c(x-y)\bar{\Gamma}_\lambda\bar{d}(\partial)\Delta_c(x-y)\bar{\Gamma}_\kappa\}\\ \times\ [:A_\lambda(x)A_\kappa(y): + i\delta_{\kappa\lambda}D_c(x-y)]\Omega_1) \quad .\end{aligned} \tag{4.3.4}$$

In writing (4.3.4) we have used the contraction formulas

$$(\Omega_o,\ \psi_\sigma(x)\tilde{\psi}_\rho(y)\eta^{-1}\Omega_o) = i\bar{d}_{\sigma\rho}(\partial)\Delta_c(x-y) \quad , \tag{4.3.5}$$

$$(\Omega_o,\ \tilde{\psi}_\sigma(x)\eta^{-1}\psi_\rho(y)\Omega_o) = -i\bar{d}_{\rho\sigma}(-\partial)\Delta_c(x-y) \quad , \tag{4.3.6}$$

$$(\Omega_o,\ A_\lambda(x)A_\kappa(y)\Omega_o) \quad = i\delta_{\kappa\lambda}D_c(x-y) \quad . \tag{4.3.7}$$

Now, $(\hat{\Omega}_1,\ S^{(2)}\Omega_2)^*$ is found by taking the complex conjugate of the above expansion and making the substitution

$1 \longleftrightarrow 2$. To do this we note that, as shown in Chapter 2, $\tilde{\psi}(x)\eta^{-1}$ behaves as the Hermitian conjugate of $\psi(x)$ when sandwiched between states in a normal product. Furthermore, using the relation

$$[\bar{d}(\partial)]^{\dagger} = [d(\partial)\eta^{-1}]^{\dagger} = \bar{d}(-\partial) \quad , \qquad (4.3.8)$$

which follows from (2.1.2) and (2.1.4), we find

$$(\Omega_o, \psi_\sigma(x)\tilde{\psi}_\rho(y)\eta^{-1}\Omega_o)^* = -i\bar{d}_{\rho\sigma}(-\partial)\Delta_c^*(x-y) \quad , \qquad (4.3.9)$$

$$(\Omega_o, \tilde{\psi}_\sigma(x)\eta^{-1}\psi_\rho(y)\Omega_o)^* = i\bar{d}_{\sigma\rho}(\partial)\Delta_c^*(x-y) \quad ; \qquad (4.3.10)$$

and from (3.4.3) we have

$$(\hat{\Omega}_2, :A_\lambda(x)A_\kappa(y):\Omega_1)^* = g_{\lambda\mu}g_{\kappa\nu}(\hat{\Omega}_1, :A_\nu(y)A_\mu(x):\Omega_1) . \qquad (4.3.11)$$

Using the above formulas together with[1)]

$$\bar{\Gamma}_\lambda^{\dagger} = [\eta\Gamma_\lambda]^{\dagger} = -g_{\lambda\rho}\eta\Gamma_\rho = -g_{\lambda\rho}\bar{\Gamma}_\rho \quad , \qquad (4.3.12)$$

we can generate the matrix element

$$\begin{aligned}(\hat{\Omega}_1, S^{(2)}\Omega_2)^* = \frac{e^2}{2}\int d^4x d^4y (\hat{\Omega}_2, &:\tilde{\psi}(y)\eta^{-1}\bar{\Gamma}_\kappa\psi(y)\tilde{\psi}(x)\eta^{-1}\bar{\Gamma}_\lambda\psi(x): \\ &- i:\tilde{\psi}(y)\eta^{-1}\bar{\Gamma}_\kappa\bar{d}(-\partial)\Delta_c^*(x-y)\bar{\Gamma}_\lambda\psi(x): \\ &- i:\tilde{\psi}(x)\eta^{-1}\bar{\Gamma}_\lambda\bar{d}(\partial)\Delta_c^*(x-y)\bar{\Gamma}_\kappa\psi(x): \\ &+ \mathrm{Tr}\{\bar{\Gamma}_\kappa\bar{d}(-\partial)\Delta_c(x-y)\bar{\Gamma}_\lambda\bar{d}(\partial)\Delta_c(x-y)\} \\ &\times [:A_\kappa(y)A_\lambda(x): - i\delta_{\kappa\lambda}D_c^*(x-y)]\Omega_1)\end{aligned} \qquad (4.3.13)$$

The sum on the left hand side of (4.3.2) is obtained directly from (4.3.4) and (4.3.13). The term in each of these equations with no contractions is zero by energy-momentum conservation, and will not be considered further.

We shall now investigate the terms in the sum (4.3.2) and show them each to be zero. The first such term we consider is

$$I_1 = \frac{e^2}{2}\int d^4x d^4y : \tilde{\psi}(x)\eta^{-1}\bar{\Gamma}_\lambda\psi(x)\tilde{\psi}(y)\eta^{-1}\bar{\Gamma}_\kappa\psi(y):$$

$$\times \{i\delta_{\kappa\lambda}D_c(x-y) - i\delta_{\kappa\lambda}D_c^*(x-y)\} \qquad (4.3.14)$$

where we have made the interchanges $x \longleftrightarrow y$ and $x \longleftrightarrow \lambda$ in (4.3.14), and sandwiching between $\hat{\Omega}_2$ and Ω_1 is implicit. The quantity in curly brackets in (4.3.14) may be rewritten

$$i\delta_{\kappa\lambda}D_c(x-y) - i\delta_{\kappa\lambda}D_c^*(x-y)$$

$$= i\delta_{\kappa\lambda}[-\tfrac{i}{2}D^{(1)}(x-y) - \bar{D}(x-y) - \tfrac{i}{2}D^{(1)}(x-y) + \bar{D}(x-y)]$$

$$= \delta_{\kappa\lambda}D^{(1)}(x-y) \qquad (4.3.15)$$

where use has been made of the identity$^{1)}$

$$D_c(x) = -\tfrac{i}{2}D^{(1)}(x) - \bar{D}(x) \quad , \qquad (4.3.16)$$

with the real functions $D^{(1)}$ and $\bar{D}$ given by

$$D^{(1)}(x) = \frac{1}{(2\pi)^3} \int d^4k \ \delta(k^2) e^{ikx} \quad , \qquad (4.3.17)$$

$$\bar{D}(x) = \frac{1}{(2\pi)^4} P \int_{-\infty}^{\infty} d^4k \ \frac{1}{k^2} \ e^{ikx} \quad , \qquad (4.3.18)$$

P denoting the principal part of the integral. Putting (4.3.15) in (4.3.14) and noting the delta function in (4.3.17), we see that I_1 contains both fermions and the photon on-shell and is therefore zero by energy-momentum conservation.

Another term in the sum (4.3.2) is

$$I_2 = \frac{e^2}{2} \int d^4x d^4y : \tilde{\psi}(x) \eta^{-1} \bar{\Gamma}_\lambda \{ i\bar{d}(\partial) \Delta_c(x-y)$$

$$- i\bar{d}(\partial) \Delta_c^*(x-y) \} \bar{\Gamma}_\kappa \psi(y) : : A_\lambda(x) A_\kappa(y) : \quad . \qquad (4.3.19)$$

For the factor in curly brackets above we have

$$i\bar{d}(\partial)\Delta_c(x-y) - i\bar{d}(\partial)\Delta_c^*(x-y)$$

$$= i\bar{d}(\partial) [- \tfrac{i}{2} \Delta^{(1)}(x-y) - \bar{\Delta}(x-y) - \tfrac{i}{2} \Delta^{(1)}(x-y) + \bar{\Delta}(x-y)]$$

$$= \bar{d}(\partial) \ \Delta^{(1)}(x-y) \qquad (4.3.20)$$

use having been made of the equations$^{1)}$

$$\Delta_c(x) = - \tfrac{i}{2} \Delta^{(1)}(x) - \bar{\Delta}(x) \quad , \qquad (4.3.21)$$

$$\Delta^{(1)}(x) = \frac{1}{(2\pi)^3} \int d^4p\,\delta(p^2+m^2)e^{ipx} \quad , \tag{4.3.22}$$

$$\bar{\Delta}(x) = \frac{1}{(2\pi)^4} P \int d^4p \frac{1}{p^2+m^2} e^{ipx} \quad . \tag{4.3.23}$$

Equation (4.3.20) substituted into (4.3.19) indicates that I_2 again contains three on-shell particles, and is therefore zero. An identical argument shows that the term

$$\begin{aligned} I_3 = \frac{e^2}{2} \int d^4x d^4y :\tilde{\psi}(y)\eta^{-1}\bar{\Gamma}_\kappa\{ i\bar{d}(-\partial)\Delta_c(x-y) \\ - i\bar{d}(-\partial)\Delta_c^*(x-y)\}\bar{\Gamma}_\lambda\psi(x)::A_\lambda(x)A_\kappa(y): \end{aligned} \tag{4.3.24}$$

of (4.3.2) also vanishes.

We have yet to consider the terms which have two internal lines. Let us first examine the one corresponding to the fermion self-energy, namely

$$\begin{aligned} I_4 &= -\frac{e^2}{2} \int d^4x d^4y \;(\hat{\Omega}_2, :\tilde{\psi}(x)\eta^{-1}\bar{\Gamma}_\lambda\bar{d}(\partial)\Delta_c(x-y)\bar{\Gamma}_\kappa\psi(y): \\ &\qquad \times \delta_{\kappa\lambda}D_c(x-y)\Omega_1) \\ &= -\frac{e^2}{2} \int d^4x d^4y \; u_{p_1}^{(r)}(x)\bar{\Gamma}_\lambda\bar{d}(\partial)\Delta_c(x-y)\bar{\Gamma}_\lambda u_{p_2}^{(r)}(y)D_c(x-y) \quad . \end{aligned} \tag{4.3.25}$$

Making the substitutions

$$u_p^{(r)}(x) = (2\pi)^{-3/2} u^{(r)}(p)e^{ipx} \quad , \tag{4.3.26}$$

$$\Delta_c(x) = -(2\pi)^{-4}\int d^4p \,\frac{1}{p^2+m^2-i\varepsilon}\, e^{ipx} \ , \qquad (4.3.27)$$

$$D_c(x) = -(2\pi)^{-4}\int d^4k \,\frac{1}{k^2-\,i\varepsilon}\, e^{ikx} \ , \qquad (4.3.28)$$

and performing the integrations over x and y, we obtain

$$I_4 = -\frac{e^2}{2}\,\frac{1}{(2\pi)^3}\int d^4p\,d^4k\,\delta^{(4)}(k+p-p_1)\,\delta^{(4)}(p_2-k-p)$$

$$\times\, u^{(r)\dagger}(\underset{\sim}{p}_1)\bar{\Gamma}_\lambda\,\frac{\bar{d}(ip)}{p^2+m^2-i\varepsilon}\,\bar{\Gamma}_\lambda\,\frac{1}{k^2-i\varepsilon}\,u^{(r)}(\underset{\sim}{p}_2)$$

$$= -\frac{e^2}{2}\,\frac{1}{(2\pi)^3}\int d^4k\,\delta^{(4)}(p_1-p_2)$$

$$\times\, u^{(r)\dagger}(\underset{\sim}{p}_1)\bar{\Gamma}_\lambda\,\frac{\bar{d}(i(k-p_1))}{(k-p_1)^2+m^2-i\varepsilon}\,\bar{\Gamma}_\lambda u^{(r)}(\underset{\sim}{p}_1)\,\frac{1}{k^2-i\varepsilon} \ .$$

$$(4.3.29)$$

We can use (4.3.8) and (4.3.12) to show that the numerator of the above integral is real as follows:

$$[u^{(r)\dagger}(\underset{\sim}{p}_1)\bar{\Gamma}_\lambda\bar{d}(ip)\bar{\Gamma}_\lambda u^{(r)}(\underset{\sim}{p}_1)]^\dagger$$

$$= u^{(r)\dagger}(\underset{\sim}{p}_1)\bar{\Gamma}_\kappa g_{\kappa\lambda}\bar{d}(ip)\bar{\Gamma}_\mu g_{\mu\lambda}u^{(r)}(\underset{\sim}{p}_1)$$

$$= u^{(r)\dagger}(\underset{\sim}{p}_1)\bar{\Gamma}_\kappa\bar{d}(ip)\bar{\Gamma}_\kappa u^{(r)}(\underset{\sim}{p}_1)] \ . \qquad (4.3.30)$$

For simplicity we shall examine the integral (4.3.29) in the particular frame $\underset{\sim}{p}_1 = 0$ and define the real quantity

$$D(ik) = -[\frac{e^2}{2}\frac{1}{(2\pi)^3}\delta^{(4)}(p_1-p_2)u^{(r)\dagger}(\underset{\sim}{p}_1)\bar{\Gamma}_\lambda$$

$$\times\ \bar{d}(i(k-p_1))\bar{\Gamma}_\lambda u^{(r)}(\underset{\sim}{p}_1)]_{p_1=0}\ . \tag{4.3.31}$$

Equation (4.3.29) then takes the form

$$I_4 = \int d^3k \int dk_o \frac{D(ik)}{-k_o^2+2mk_o+\underset{\sim}{k}^2+m^2-i\varepsilon}\cdot\frac{1}{-k_o^2+\underset{\sim}{k}^2-i\varepsilon}\ . \tag{4.3.32}$$

The denominator in the above integral can be split into partial fractions in the form

$$\frac{(\underset{\sim}{k}^2+i\varepsilon)k_o}{2m(\underset{\sim}{k}^4+\varepsilon^2)}\cdot\frac{1}{-k_o^2+\underset{\sim}{k}^2-i\varepsilon}+\frac{\underset{\sim}{k}^2+i\varepsilon}{2m(k^4+\varepsilon^2)}[2m-k_o]\cdot\frac{1}{-k_o^2+2mk_o+\underset{\sim}{k}^2-i\varepsilon} \tag{4.3.33}$$

so that

$$I_4 = \int d^3k\ G(\underset{\sim}{k}^2)\{\frac{D(ik)k_o}{-k_o^2+\underset{\sim}{k}^2-i\varepsilon}+\frac{D(ik)[2m-k_o]}{-k_o^2+2mk_o+\underset{\sim}{k}^2-i\varepsilon}\}\ dk_o \tag{4.3.34}$$

where

$$G(\underset{\sim}{k}^2) = \frac{1}{2m}\frac{k^2+i\varepsilon}{k^4+\varepsilon^2}\ . \tag{4.3.35}$$

The integrals in (4.3.34) are not convergent. We may, however, put in a real cutoff which will not affect the reality properties which are the objects of interest here.

With this in mind, we shall perform the k_o integration indicated as though the cutoff factor were already absorbed in the quantity D(ik). The first and second integrals have poles at

$$k_o = \pm(|\underset{\sim}{k}| - i\delta), \qquad \delta = \frac{\varepsilon}{2|\underset{\sim}{k}|} \quad , \tag{4.3.36}$$

and

$$k_o = m \pm (\sqrt{k^2+m^2} - i\delta')$$

$$= m \pm (E - i\delta'), \qquad \delta' = \frac{\varepsilon}{2E} \quad , \tag{4.3.37}$$

respectively. Therefore, choosing to close the contours in the lower half plane, we have

$$\int dk_o \frac{D(ik)k_o}{-k_o^2+\underset{\sim}{k}^2-i\varepsilon} = -\pi i[D(ik)]_{k_o = |\underset{\sim}{k}| - i\delta} \; , \tag{4.3.38}$$

and

$$\int dk_o \frac{D(ik)[2m-k_o]}{-k_o^2+2mk_o+\underset{\sim}{k}^2-i\varepsilon} = -\pi i[D(ik)]_{k_o=m+E-i\delta'} \frac{m-E+i\delta'}{E-i\delta'} \; . \tag{4.3.39}$$

Putting (4.3.38) and (4.3.39) in (4.3.34) and taking the limit $\varepsilon \to 0$, we have finally

$$I_4 = -\frac{\pi i}{2m} \int \frac{d^3k}{|\underset{\sim}{k}|^2} \{[D(ik)]_{k_o=|\underset{\sim}{k}|} + \frac{m-E}{E}[D(ik)]_{k_o=m+E}\} \; , \tag{4.3.40}$$

which is pure imaginary. Consequently, the fermion self-energy term in (4.3.4) cancels the corresponding term in (4.3.13) when they occur in the sum (4.3.2).

The photon self-energy term in (4.3.4) is given by

$$I_5 = \frac{e^2}{2} \int d^4x d^4y (\hat{\Omega}_2, \mathrm{Tr}\{\bar{d}(-\partial)\Delta_c(x-y)\bar{\Gamma}_\lambda \bar{d}(\partial)\Delta_c(x-y)\bar{\Gamma}_\kappa\}$$

$$\times :A_\lambda(x)A_\kappa(y):\Omega_1)$$

$$= \frac{e^2}{2} \int d^4x d^4y \ \mathrm{Tr}\{\bar{d}(-\partial)\Delta_c(x-y)\bar{\Gamma}_\lambda \bar{d}(\partial)\Delta_c(x-y)\bar{\Gamma}_\kappa\}\bar{u}^{(t)}_{k\lambda}(x)u_{k'\kappa}(y).$$

$$(4.3.41)$$

Substituting (4.3.27) and

$$u^{(t)}_{k\mu}(x) = (2\pi)^{-3/2} \ e^{(t)}_\mu(k)e^{ikx} \qquad (4.3.42)$$

in (4.3.41) and integrating over x and y we obtain

$$I_5 = \frac{e^2}{2(2\pi)^3} \frac{\bar{e}^{(t)}_\lambda(k)e^{(t)}_\kappa(k)}{[2\omega(k)]^2} \delta^4(k-k')$$

$$\times \int d^4p \ \frac{\mathrm{Tr}\{\bar{d}(-ip)\bar{\Gamma}_\lambda \bar{d}(i(k-p))\bar{\Gamma}_\kappa\}}{(p^2+m^2-i\varepsilon)[(k-p)^2+m^2-i\varepsilon]} \ . \qquad (4.3.43)$$

Now,

$$[\bar{e}^{(t)}_\lambda(k)e^{(t)}_\kappa(k)\mathrm{Tr}\{\bar{d}(-ip)\bar{\Gamma}_\lambda \bar{d}(i(k-p))\bar{\Gamma}_\kappa\}]^\dagger$$

$$= e^{(t)}_\kappa(k)e^{(t)}_\mu(k)g_{\mu\lambda} \ \mathrm{Tr}\{\bar{\Gamma}_\alpha g_{\alpha\kappa}\bar{d}(i(k-p))\bar{\Gamma}_\beta g_{\beta\lambda}\bar{d}(-ip)\}$$

$$= \bar{e}^{(t)}_\alpha(k)e^{(t)}_\beta(k) \ \mathrm{Tr}\{\bar{d}(-ip)\bar{\Gamma}_\alpha d(i(k-p))\bar{\Gamma}_\beta\}, \qquad (4.3.44)$$

where we have used the cyclic property of the trace. Therefore, we may define the real quantity

$$R(ip) = \frac{e^2}{2(2\pi)^3} \frac{\bar{e}_\lambda^{(t)}(k) e_\kappa^{(t)}(k)}{[2\omega(k)]^2} \delta^4(k-k')\mathrm{Tr}\{\bar{d}(-ip)\bar{\Gamma}_\lambda \bar{d}(i(k-p))\bar{\Gamma}_\kappa\}, \tag{4.3.45}$$

and rewrite (4.3.43)

$$I_5 = \int d^4p \frac{R(ip)}{(p^2+m^2-i\varepsilon)[(k-p)^2+m^2-i\varepsilon]}$$

$$= \int d^3p \, \{\int dp_o \frac{[c_1(\underset{\sim}{p})p_o + c_2(\underset{\sim}{p})]R(ip)}{-p_o^2 + W^2 - i\varepsilon}$$

$$+ \int dp_o \frac{[c_1(\underset{\sim}{p})p_o + c_4(\underset{\sim}{p})]R(ip)}{-p_o^2 + 2\omega p_o + W^2 - 2\underset{\sim}{k}.\underset{\sim}{p} - i\varepsilon} \quad , \tag{4.3.46}$$

where the final form was obtained by splitting the denominator into partial fractions, and where

$$\omega = k_o = \sqrt{\underset{\sim}{k}^2} \quad , \tag{4.3.47}$$

$$W = \sqrt{\underset{\sim}{p}^2 + m^2} \quad , \tag{4.3.48}$$

$$c_1 = -c_3 = \frac{1}{2\Delta} \{\omega[\omega^2 W^2 - (k.p)^2] + i\omega^2\varepsilon\} \ , \tag{4.3.49}$$

$$c_2 = \frac{1}{2\Delta} \{(k.p)[\omega^2 W^2 - (k.p)^2] + i\omega^2(\underset{\sim}{k}.\underset{\sim}{p})\varepsilon\} \ , \tag{4.3.50}$$

$$c_4 = \frac{1}{2\Delta} [2\omega^2 - (\underset{\sim}{k}.\underset{\sim}{p})]\{\omega^2 W^2 - (\underset{\sim}{k}.\underset{\sim}{p})^2 + i\omega^2[2\omega^2 - \underset{\sim}{k}.\underset{\sim}{p}]\varepsilon\}, \tag{4.3.51}$$

with

$$\Delta = \omega^4(W^4 + \varepsilon^2) - (\underset{\sim}{k}.\underset{\sim}{p})^4 \quad . \qquad (4.3.52)$$

The first and second integrals in curly brackets in (4.3.46) have poles respectively at

$$p_o = \pm(W - i\delta) \quad , \qquad \delta = \frac{\varepsilon}{2W} \quad ,$$

and

$$p_o = \omega \pm (\sqrt{W^2 - 2(\underset{\sim}{k}.\underset{\sim}{p}) + \omega^2} - i\delta' \quad , \; \delta' = \frac{\varepsilon}{2\sqrt{E^2 - 2(\underset{\sim}{k}.\underset{\sim}{p}) + \omega^2}} \quad .$$

Consequently, if we note that the quantities defined by equations (4.3.47)-(4.3.51) are real in the limit as $\varepsilon \to 0$, it follows that I_5 is pure imaginary by the same procedure as was used to obtain (4.3.40). This means again that in the sum (4.3.2) this term will cancel the corresponding term from (4.3.13).

As a final point we note that the terms in (4.3.4) and (4.3.13) with no external lines need not be considered since they will be eliminated by an appropriate choice of H_o in (4.2.49) so that (4.2.51) is satisfied.

This completes the demonstration that to second order in the perturbation expansion conservation of probability is satisfied. No complete proof of untarity has been forthcoming and it remains an interesting problem for further investigation. It is an interesting point that the demonstration given depends on the spin-statistics relation (2.2.38) being satisfied. Otherwise, for example, we could not show that $\tilde{\psi}(x)\eta^{-1}$ behaves as the Hermitian conjugate of $\psi(x)$ when sandwiched between states in a normal product.

4.4 Spin 3/2 Compton Scattering

It is desirable to give an example of a practical calculation using the foregoing formalism. For this purpose we shall consider the simplest interaction between the electromagnetic and spin 3/2 fields, namely the scattering of a photon from a spin 3/2 particle with a spin 3/2 intermediate particle. From (4.3.4) the matrix element for this interaction is

$$<\Omega'(p'r';\ ;k't')|S|\Omega(pr;\ ;kt)>$$

$$= \int d^4x d^4y <\hat{\Omega}'(p'r';\ ;k't')|ie^2 :\tilde{\psi}(x)\eta^{-1}\bar{\Gamma}_\lambda \bar{d}(\partial)\Delta_c(x-y)$$

$$\times\ \bar{\Gamma}_\kappa \psi(y) : :A_\lambda(x) A_\kappa(y) : |\Omega(pr;\ ;kt)> \qquad (4.4.1)$$

where p,p',r,r',k,k',t, and t' are the initial and final fermion four-momenta, the initial and final fermion spins, the initial and final photon four-momenta, and the initial and final photon spins respectively.

Now from (4.2.25)-(4.2.30) we have

$$<\hat{\Omega}'| \equiv <\hat{\Omega}(p'r';\ ;k't')| = <\Omega_o|\hat{A}_{p'}^{(r')}\hat{C}_{k'}^{(t')}$$

$$= <\Omega_o|[-i\int_\infty d\sigma_\mu(z)u_{p'}^{(r')^\dagger}(z)\bar{\Gamma}_{\mu,\sigma\rho}a_\rho(z)]$$

$$\times\ [-i\int_\infty d\sigma_\nu(x)u_{k'\lambda}^{(t')^\dagger}(x)(\partial_\nu-\overleftarrow{\partial}_\nu)g_{\lambda\kappa}c_\kappa(x)]\ , \qquad (4.4.2)$$

$$|\Omega> \equiv |\Omega(pr;\ ;kt)> = A_p^{(r)^\dagger}C_k^{(t)^\dagger}|\Omega_o>$$

$$= [\int d^4za_\nu^\dagger(z)u_p^{(r)}(z)][\int d^4xc_\mu^\dagger(x)u_{k\mu}^{(t)}(x)]|\Omega_o>. \qquad (4.4.3)$$

Substituting (4.4.2), (4.4.3) and (4.2.39)-(4.2.41) in (4.4.1), and making use of the commutation relations (4.2.1)-(4.2.3) as well as the conditions (4.2.4) and (4.2.5), we obtain

$$<\hat{\Omega}'|S|\Omega> = ie^2\int d^4xd^4y$$

$$\times\ \int_\infty d\sigma_\mu(z)u_{p'}^{(r')^\dagger}(z)\bar{\Gamma}_\mu\Delta_c(z-x)\bar{d}(-\partial)\bar{\Gamma}_\lambda\bar{d}(\partial)\Delta_c(x-y)\bar{\Gamma}_\kappa u_p^{(r)}(y)$$

$$\times\ [u_{k\lambda}^{(t)}(x)\int_\infty d\sigma_\nu(x')u_{k'\rho}^{(t')^\dagger}(x')(\partial_\nu'-\overleftarrow{\partial}_\nu')D_c(y-x')g_{\rho\kappa}$$

$$+\ u_{k\kappa}^{(t)}(y)\int_\infty d\sigma_\nu(x')u_{k'\rho}^{(t')^\dagger}(x')(\partial_\nu'-\overleftarrow{\partial}_\nu')D_c(y-x')g_{\rho\lambda}]$$

$$= \int d^4xd^4yu_{p'}^{\dagger(r')}(x)\bar{\Gamma}_\lambda\bar{d}(\partial)\Delta_c(x-y)\bar{\Gamma}_\kappa u_p^{(r)}(y)$$

$$\times\ [u_{k\lambda}^{(t)}(x)u_{k'\kappa}^{(t')^\dagger}(y)\ +\ u_{k\kappa}^{(t)}(y)u_{k'\lambda}^{(t')^\dagger}(x)], \qquad (4.4.4)$$

where use has been made of equation (2.1.22). In (4.4.4) we now make the replacements$^{1)}$

$$u_p^{(r)}(y) = \frac{1}{\sqrt{V}}\, u^{(r)}(\underset{\sim}{p})\; e^{ipy} \quad , \qquad (4.4.5)$$

$$u_{p'}^{(r')^\dagger}(x) = \frac{1}{\sqrt{V}}\, u^{(r')^\dagger}(\underset{\sim}{p}')\; e^{-ip'x} \; , \qquad (4.4.6)$$

$$u_{k\lambda}^{(t)}(x) = \frac{1}{\sqrt{2\omega V}}\; e_\lambda^{(t)}\; e^{ikx} \quad , \qquad (4.4.7)$$

$$u_{k'\kappa}^{(t')^\dagger}(y) = \frac{1}{\sqrt{2\omega' V}}\; e_\kappa^{(t')}\; e^{-ik'y} \quad , \qquad (4.4.8)$$

$$\bar{d}(\partial)\Delta_c(x-y) = -\frac{1}{(2\pi)^4}\int d^4p\; e^{ip(x-y)}\; \frac{\bar{d}(ip)}{p^2+m^2-i\varepsilon} \; , \qquad (4.4.9)$$

where V is the volume of quantization and ω and ω' are the initial and final photon energies respectively. On performing the integrations over x and y, we have

$$\langle\hat{\Omega}'|S|\Omega\rangle = -ie^2\,\frac{1}{V^2}\,\frac{1}{\sqrt{4\omega\omega'}}\,(2\pi)^4\int d^4p$$

$$\times\,[\delta^{(4)}(q+k-p')\,\delta^{(4)}(q+k'-p)\,u^{(r')^\dagger}(\underset{\sim}{p}')$$

$$\times\,\bar{\Gamma}.e\,\frac{\bar{d}(i\underset{\sim}{p})}{p^2+m^2-i\varepsilon}\,\bar{\Gamma}\,.\,e'u^{(r)}(\underset{\sim}{p})$$

$$+\,\delta^{(4)}(q-k'-p')\,\delta^{(4)}(q-k-p)\,u^{(r')^\dagger}(\underset{\sim}{p}')$$

$$\times\,\bar{\Gamma}\,.\,e'\,\frac{\bar{d}(ip)}{p^2+m^2-i\varepsilon}\,\bar{\Gamma}\,.\,e\;u^{(r)}(\underset{\sim}{p})]\;. \qquad (4.4.10)$$

The integration over p then gives

$$<\hat{\Omega}'|S|\Omega> = -i\delta^{(4)}(p+k-p'-k')\ t_{fi} \quad , \qquad (4.4.11)$$

where

$$t_{fi} = \frac{e^2}{2V^2}\ \frac{(2\pi)^4}{\sqrt{\omega\omega'}}\ \bar{u}^{(r')}(\underset{\sim}{p}')\ [\Gamma.e'\ \frac{d[i(p+k)]}{(p+k)^2+m^2}\ \Gamma.e$$

$$+ \Gamma.e\ \frac{d[i(p-k')]}{(p-k')^2+m^2}\ \Gamma.e']\ u^{(r)}(\underset{\sim}{p}) \quad . \qquad (4.4.12)$$

The total cross-section is then give by[10)]

$$\sigma = \frac{V^2}{(2\pi)^4\ v_r} \cdot \frac{V}{(2\pi)^3} \int d^3p'\ \frac{V}{(2\pi)^3} \int d^3k'\delta^{(4)}(p+k-p'-k')|t_{fi}|^2 , \qquad (4.4.13)$$

where v_r is the relative velocity of the incident particles and in this case

$$v_r = c = 1 \ .$$

As usual we shall average over initial spins and sum over final spins. For this purpose we calculate the quantity

$$\frac{1}{4}\sum_{r=1}^{4}\ \sum_{r'=1}^{4}|t_{fi}|^2 = \frac{1}{4}\sum_{r=1}^{4}\ \sum_{r'=1}^{4}\ \bar{u}^{(r')}(\underset{\sim}{p}')[\Gamma.e'\ \frac{d[i(p+k)]}{(p+k)^2+m^2}\ \Gamma.e$$

$$+ \Gamma.e\ \frac{d[i(p-k')]}{(p-k')^2+m^2}\ \Gamma.e']u^{(r)}(\underset{\sim}{p})\bar{u}^{(r)}(\underset{\sim}{p})$$

$$\times\ [\Gamma.e\ \frac{d[i(p+k)]}{(p+k)^2+m^2}\ \Gamma.e' + \Gamma.e'\ \frac{d[i(p-k')]}{(p-k')^2+m^2}\ \Gamma.e]u^{(r')}(p')[\frac{e^4.(2\pi)^8}{4V^4\omega\omega'}] . \qquad (4.4.14)$$

We now make use of the relation1)

$$\sum_r u^{(r)}(\underset{\sim}{p})\bar{u}^{(r)}(\underset{\sim}{p}) = \frac{1}{2p_o}\, d(ip) \quad , \qquad (4.4.15)$$

to obtain

$$\frac{1}{4}\sum_{r=1}^{4}\ \sum_{r'=1}^{4}|t_{fi}|^2 = \frac{1}{4}\ \mathrm{Tr}\{[\Gamma.e'\ \frac{d[i(p+k)]}{(p+k)^2+m^2}\ \Gamma.e$$

$$+\ \Gamma.e\ \frac{d[i(p-k')]}{(p-k')^2+m^2}\ \Gamma.e']\ \frac{d(ip)}{2p_o}$$

$$\times\ [\Gamma.e\ \frac{d[i(p+k)]}{(p+k)^2+m^2}\ \Gamma.e' + \Gamma.e'\ \frac{d[i(p-k')]}{(p-k')^2+m^2}\ \Gamma.e]\}$$

$$\times\ [\frac{e^4}{2V^4}\ \frac{(2\pi)^8}{\omega\omega'}] \quad . \qquad (4.4.16)$$

In the rest frame of the target ($\underset{\sim}{p} = 0$) this becomes

$$\frac{1}{4}\sum_{r=1}^{4}\ \sum_{r'=1}^{4}|t_{fi}|^2 = \frac{1}{64m^2}\ \mathrm{Tr}\{[\Gamma.e'\ \frac{d[i(p+k)]}{\omega}\ \Gamma.e$$

$$-\ \Gamma.e\ \frac{d[i(p-k')]}{\omega'}\ \Gamma.e']\ d(ip)$$

$$\times\ [e \longleftrightarrow e']\ \frac{d(ip')}{p_o'}\}\ [\frac{e^4}{2V^4}\ \frac{(2\pi)^8}{\omega\omega'}] \quad . \qquad (4.4.17)$$

Therefore, averaging over initial spins and summing over final spins in (4.4.13) we find through the use of (4.4.17)

$$\frac{d\sigma}{d\Omega} = \frac{e^4}{1024\pi^2}\ \frac{1}{m^3\omega}\int \frac{d^3\underset{\sim}{p}'}{p_o'}\int \omega' d\omega'\ \mathrm{Tr}\{\ \}\,\delta^{(3)}(\underset{\sim}{k}-\underset{\sim}{p}'-\underset{\sim}{k}')\,\delta(m+\omega-p_o'-\omega') \ ,$$

$$(4.4.18)$$

where { } denotes the quantity in curly brackets in (4.4.17) and

$$d\Omega = \sin\theta \; d\theta \; d\phi$$

where θ is the angle between the initial and final photon directions and ϕ is the azimuthal angle about the direction of the incident photon. Use of the Compton formula[10)]

$$\omega' = \frac{\omega m}{m+\omega(1 - \cos\theta)} \quad , \qquad (4.4.19)$$

allows us to perform the integrations in (4.4.18) with the result

$$\frac{d\sigma}{d\Omega} = \frac{e^4}{2048\pi^2 m^2} \left(\frac{\omega'}{\omega}\right)^2 \sum_{t,t'=1}^{2} \mathrm{Tr}\{-\Gamma.e^{(t')} \frac{d[i(p+k)]}{\omega} \Gamma.e^{(t)}$$

$$+ \Gamma.e^{(t)} \frac{d[i(p-k')]}{\omega'} \Gamma.e^{(t')}] \; d(ip)$$

$$\times \; [-\Gamma.e^{(t)} \frac{d[i(p+k)]}{\omega} \Gamma.e^{(t')}$$

$$+ \Gamma.e^{(t')} \frac{d[i(p-k')]}{\omega'} \Gamma.e^{(t)}] \; d(ip')\} \; , \qquad (4.4.20)$$

where we have averaged over initial and summed over final photon polarizations, and energy-momentum conservation holds so that

$$p' = p - k + k' \; . \qquad (4.4.21)$$

Note that if $1/\omega$ is factored out of the trace in (4.4.20), and the factors ω'/ω which occur are replaced according to (4.4.19), there will be no explicit variable denominator under the trace. Moreover, if the trace is expressed in terms of $\gamma=\omega/m$, the only quantities which will contribute to the denominator are powers of ω'. Thus, it can be seen from (4.4.19) that the denominator will be some power of $[1 + \gamma(1 - \cos\theta)]$. Then, if multiply by the appropriate power of $[1 + \gamma(1 - \cos\theta)]$, what remains as the trace term will be a rational polynomial in γ and $\cos\theta$. With these facts in mind a numerical analysis of (4.4.20) was done as explained in what follows.14)

The parameters in (4.4.20) may be explicitly constructed as explained in reference 1. We find that we may take

$$k = \begin{pmatrix} 0 \\ 0 \\ \omega \\ i\omega \end{pmatrix} , \qquad k' = \begin{pmatrix} 0 \\ \omega' \sin\theta \\ \omega' \cos\theta \\ i\omega' \end{pmatrix} , \qquad (4.4.22)$$

$$e^{(1)} = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad e^{(2)} = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \end{pmatrix} , \qquad (4.4.23)$$

$$e^{(1')} = \begin{pmatrix} 0 \\ -\cos\theta \\ \sin\theta \\ 0 \end{pmatrix} , \quad e^{(2')} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} , \qquad (4.4.24)$$

$$P = (0, 0, 0, im) \quad . \qquad (4.4.25)$$

A $1/\omega$ was factored from the trace in (4.4.20). Using (4.4.21)-(4.4.25) the trace was then calculated for various values of θ and γ on the I.B.M. 360/67 computer in double precision (16 significant figures). The resulting data points were multiplied by successive powers of $[1 + \gamma(1 - \cos\theta)]$ until a fit was obtained to a rational polynomial in γ and $\cos\theta$. The fitted expression was put into (4.4.20) to give the result

$$\frac{d\sigma}{d\Omega} = \frac{R_o^2}{162} \sum_{j=1}^{7} \sum_{k=1}^{6} \frac{C_{jk}\, \gamma^{j-1} (\cos\theta)^{k-1}}{[1 + \gamma(1 - \cos\theta)]^5} , \qquad (4.4.26)$$

where

$$R_o = \frac{e^2}{4\pi m^2} , \qquad (4.4.27)$$

$$\gamma = \omega/m , \qquad (4.4.28)$$

and

$$C = \begin{bmatrix} 81 & 0 & 81 & 0 & 0 & 0 \\ 243 & -243 & 243 & -243 & 0 & 0 \\ 564 & -696 & 487 & -486 & 243 & 0 \\ 723 & -1305 & 796 & -326 & 243 & -81 \\ 527 & -1284 & 1010 & -244 & 7 & 0 \\ 206 & -638 & 700 & -300 & 38 & -6 \\ 50 & -172 & 224 & -132 & 30 & 0 \end{bmatrix} \qquad (4.4.29)$$

Results calculated from (4.4.26) agreed with those computed from (4.4.20) to a minimum of 12 significant figures which is well within error limits due to round-off by the computer.

A straightforward but tedious integration of (4.4.26) gives the total cross-section

$$\sigma = \frac{2\pi R_o^2}{162} \{ \frac{2}{3\gamma^2(1+2\gamma)^4} [486 + 3,888\gamma + 12,219\gamma^2 + 18,927\gamma^3$$
$$+ 15,510\gamma^4 + 8182\gamma^5 + 4344\gamma^6$$
$$+ 1,444\gamma^7 + 232\gamma^8 + 144\gamma^9]$$
$$- \frac{\log(1+2\gamma)}{\gamma^3} (162 + 162\gamma + 23\gamma^2 - 8\gamma^3 - 30\gamma^4) \} \qquad (4.4.30)$$

Note that for $\gamma = 0$, (4.4.26) is identical to the differential cross-section for spin 1/2 Compton scattering,10) and is symmetrical about $\theta = \pi/2$. The total cross-section in this case is therefore equal to the Thompson cross-section. However, putting $\theta = 0$ in (4.4.26) we find

$$\frac{d\sigma}{d\Omega} = \frac{R_o^2}{27} \sum_{j=1}^{7} C_{jk}\,\gamma^{j-1} \quad , \tag{4.4.31}$$

which shows that at zero scattering angle the differential cross-section increases with increasing incident energy, whereas in the spin 1/2 case it is energy independent.10)

This completes our study of the electromagnetic-spin 3/2 interaction. A discussion of what we have accomplished and of the results we have obtained follows in the next chapter.

5. DISCUSSION

We have hyperquantized the system of a Rarita-Schwinger field interacting with the electromagnetic field, and have shown that the Johnson-Sudarshan inconsistency does not arise. The reason for this may be stated as follows: In conventional quantum field theory the field operator commutation relations are determined in such a way that they are consistent with the equations of motion. On the other hand, in hyperquantization the commutation relations are given at the outset and the physical state vectors are determined so as to satisfy the supplementary conditions. Thus the way in which kinematics and dynamics are separated is completely different in the two theories.

As we have seen, the presence of negative fermion anticommutators, when the Schwinger action principle is used to quantize the electromagnetic-spin 3/2 system, is due to the dependence of these anticommutators on the dynamics of the interaction. Johnson and Sudarshan2) have shown that this dynamical dependence arises for any field with half-odd-integer spin greater than 1/2. However, because the field anticommutators are fixed in hyperquantization, they can never depend on the dynamics of an interaction.

The major difficulty in hyperquantization is that no general proof has been found that the S-matrix satisfies conservation of probability which corresponds to unitarity in the usual theory. In the case of the electromagnetic-spin 1/2 interaction we were able to prove it by showing the exact agreement of our S-matrix elements with those of conventional quantum electrodynamics. The situation is more complicated for the electromagnetic-spin 3/2 interaction. No S-matrix elements have been found using conventional methods with which to compare our own, and we have so far only been able to verify conservation of probability using the perturbation expansion.

In other respects the S-matrix of hyperquantization is a much simpler object than that of ordinary quantum field theory. Its relativistic invariance can be proven directly without recourse to the perturbation expansion.9) This is because non-relativistic operations such as chronological ordering do not enter. Nor does the interaction Hamiltonian contain terms depending on the normals to space-like surfaces. One might expect this simplification to prove very useful since the S-matrix is a most powerful tool for dealing with strong interactions.

Indeed, we have seen that hyperquantization is essentially an S-matrix theory. Conservation laws, such as that of the current in quantum electrodynamics, are not required to hold at a point but only asymptotically when bracketted between states.

As a final point it is worth noting that, since normal dependent terms are absent from the theory, hyperquantization may prove very useful for dealing with non-local interactions provided a more general proof can be found that unitarity is satisfied.

APPENDIX

NOTATION AND CONVENTIONS

The following notation and conventions are used throughout this thesis.

1) Natural units are employed in which $\hbar=c=1$.

2) There is an implicit summation over repeated indices.

3) The fourth component of a four-vector is imaginary. For example

$$x_\mu = (\underset{\sim}{x}, ix_o) \quad , \qquad \text{(A.1)}$$

and no distinction is made between contravariant and covariant four-vectors.

4) The volume elements which occur in integrals are denoted by

$$d^4x = dx_o\, dx_1\, dx_2\, dx_3 \quad , \qquad \text{(A.2)}$$

$$d^3p = dp_1\, dp_2\, dp_3 \quad , \qquad \text{(A.3)}$$

and similarly for other vectors.

5) For the four-gradient differential operator operating to the right we use

$$\partial_\mu = \frac{\partial}{\partial x_\mu} = (\underset{\sim}{\nabla}, - i \frac{\partial}{\partial x_o}) \qquad \text{(A.4)}$$

$$\partial'_\mu = \frac{\partial}{\partial x'_\mu} = (\underset{\sim}{\nabla}', -i \frac{\partial}{\partial x'_o}) \qquad \text{(A.5)}$$

etc.

6) The differential operator $\overleftarrow{\partial}_\mu$ operates on functions standing to its left. For example, if f(x) and g(x) are arbitrary functions

$$f(x)\overleftarrow{\partial}_\mu g(x) = [\partial_\mu f(x)]\, g(x) \quad , \tag{A.6}$$

$$f(x)(\partial_\mu + \overleftarrow{\partial}_\mu)g(x) = \partial_\mu [f(x)g(x)] \quad . \tag{A.7}$$

7) The D'Alembertian operator is defined by

$$\Box = \partial_\mu \partial_\mu = \nabla^2 - \frac{\partial^2}{\partial x_o^2} \quad . \tag{A.8}$$

8) We define the matrix $g_{\mu\nu}$ such that

$$\begin{aligned} g_{\mu\nu} &= 1 && \text{for } \mu=\nu=1,2,3 \\ &= -1 && \text{for } \mu=\nu=4 \\ &= 0 && \text{for } \mu\neq\nu \quad . \end{aligned} \tag{A.9}$$

9) A space-like surface in four dimensional Minkowski space is denoted by σ, or, if we wish to specify that the surface passes through a point x, by $\sigma(x)$. Moreover, we define a four-vector differential surface area at the point x by

$$d\sigma_\mu(x) = (dx_2dx_3dx_o,\ dx_1dx_3dx_o,\ dx_1dx_2dx_o, -idx_1dx_2dx_3) . \tag{A.10}$$

10) We employ the standard conventions

$$[A, B] = AB - BA \quad , \tag{A.11}$$

$$\{A, B\} = AB + BA \quad , \tag{A.12}$$

where A and B are operators.

11) The symbol † denotes hermitian conjugation while * denotes complex conjugation.

LIST OF REFERENCES

1) Y. Takahashi, An Introduction to Field Quantization (Pergamon Press, Oxford, 1969).

2) K. Johnson and E.C.G. Sudarshan, Ann. of Phys. 13 (1961), 126.

3) C. Hayashi, Prog. Theor. Phys. 11 (1954), 226.

4) E. Arnous, W. Heitler and Y. Takahashi, Nuovo Cimento 16 (1960), 671.

5) O. Klein, Niels Bohr and the Development of Physics (McGraw-Hill, New York, 1955).

6) F. Coester, Phys. Rev. 95 (1954), 1318.
See also J.M. Jauch, Helv. Phys. Acta 29 (1956) 287.

7) Y. Takahashi, Phys. Rev. 166 (1968), 1645.

8) P.T. Matthews and A. Salam, Proc. Roy. Soc. (London) A221 (1953), 128.

9) Y. Takahashi and R. Gourishankar, Nucl. Phys. B12 (1969), 301.

10) K. Nishijima, Fields and Particles (Benjamin, New York, 1969).

11) H. Umezawa, Quantum Field Theory (North Holland, Amsterdam, 1956).

12) K. Dormuth and Y. Takahashi, Prog. Theor. Phys. 43 (1970), 1386.

13) K. Dormuth and Y. Takahashi, Prog. Theor. Phys. 44 (1970), 1077.

14) K. Dormuth and R. Teshima, Lettere al Nuovo Cimento 4 (1970), 796.