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Representations of affine truncations of representation involutive-semirings of Lie algebras and root systems of higher type

by

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**Abstract**

An important component of a rational conformal field theory is a representation of a certain involutive-semiring. In the case of Wess-Zumino-Witten models, the involutive-semiring is an affine truncation of the representation involutive-semiring of a finite-dimensional semisimple Lie algebra. We show how root systems naturally correspond to representations of an affine truncation of the representation involutive-semiring of $\mathfrak{sl}_2(\mathbb{C})$. By reversing this procedure, one can, to a representation of an affine truncation of the representation involutive-semiring of an arbitrary finite-dimensional semisimple Lie algebra, associate a root system and a Cartan matrix of higher type. We show how this same procedure applies to a special subclass of representations of the nontruncated representation involutive-semiring, leading to higher-type analogues of the affine Cartan matrices. Finally, we extend the known results for $\mathfrak{sl}_3(\mathbb{C})$ with a symmetric classification, a construction, and a list of computer-generated examples.

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# Chapter 1

# Introduction

Consider the root system of type $D_4$:

$$\Phi = \{\pm(e_i + e_j), \pm(e_i - e_j) : 1 \leq i < j \leq 4\}.$$

Let $W$ denote the Weyl group of $\Phi$. A Coxeter element of $W$ is a product of simple reflections for some choice of positive roots. For example, the orthogonal transformation

$$C = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

with respect to the ordered basis $(e_1, e_2, e_3, e_4)$ is a Coxeter element of $W$. One can verify that $C$ has order $h = 6$ with eigenvalues $q^1$, $q^3$, $q^3$, and $q^5$, where

$$q = \exp\left(\frac{2\pi i}{h}\right).$$

In general, all Coxeter elements of $W$ are conjugate with the same order and the same eigenvalues. Moreover, the eigenvalues of a Coxeter element are always roots of unity. Thus the quantity $h$ and the multiset $\{1, 3, 3, 5\}$ are invariants of $\Phi$. The quantity $h$ is known as the Coxeter number of $\Phi$. The elements of the multiset $\{1, 3, 3, 5\}$ are called exponents.

The action of the cyclic group $\langle C\rangle$ partitions $\Phi$ into four orbits:

$$\begin{aligned}
O_1 &= \langle C\rangle(e_1+e_3) = (e_1+e_3, -e_1+e_2, -e_2+e_3, -e_1-e_3, e_1-e_2, e_2-e_3)\\
O_2 &= \langle C\rangle(e_1+e_2) = (e_1+e_2, e_2+e_3, -e_1+e_3, -e_1-e_2, -e_2-e_3, e_1-e_3)\\
O_3 &= \langle C\rangle(e_2+e_4) = (e_2+e_4, e_3-e_4, -e_1+e_4, -e_2-e_4, -e_3+e_4, e_1-e_4)\\
O_4 &= \langle C\rangle(e_2-e_4) = (e_2-e_4, e_3+e_4, -e_1-e_4, -e_2+e_4, -e_3-e_4, e_1+e_4).
\end{aligned}$$

These orbits are written to exhibit their natural cyclic order which comes from the action of the cyclic group $\langle C\rangle$. For example, the first root in the orbit $O_1$ is $C^0(e_1+e_3)$, the second is $C^1(e_1+e_3)$, the third is $C^2(e_1+e_3)$, and so on. We say that two roots in the same orbit are adjacent if they are next to each other in the natural cyclic order of that orbit.

In column form, these four orbits of $\Phi$ have the preferred relative alignment

| $\lambda$ | $O_1$ | $O_2$ | $O_3$ | $O_4$ |
|---|---|---|---|---|
| 0 | | $e_1+e_2$ | | |
| 1 | $e_1+e_3$ | | $e_2+e_4$ | $e_2-e_4$ |
| 2 | | $e_2+e_3$ | | |
| 3 | $-e_1+e_2$ | | $e_3-e_4$ | $e_3+e_4$ |
| 4 | | $-e_1+e_3$ | | |
| 5 | $-e_2+e_3$ | | $-e_1+e_4$ | $-e_1-e_4$ |
| 6 | | $-e_1-e_2$ | | |
| 7 | $-e_1-e_3$ | | $-e_2-e_4$ | $-e_2+e_4$ |
| 8 | | $-e_2-e_3$ | | |
| 9 | $e_1-e_2$ | | $-e_3+e_4$ | $-e_3-e_4$ |
| 10 | | $e_1-e_3$ | | |
| 11 | $e_2-e_3$ | | $e_1-e_4$ | $e_1+e_4$ |
| 0 | | $e_1+e_2$ | | |

.

The last row in this table is a copy of the first: this is done to demonstrate the cyclic nature of the columns. Alternately, we can think of the index $\lambda$ as extending to $\pm\infty$ with the rows repeating cyclically as necessary.

Here is one possible way to arrive at this relative alignment of the orbits. The method we use constructs a graph (which happens to look a lot like the Dynkin diagram) using the orbits as nodes. More specifically, we connect orbit $O_i$ to orbit $O_j$ by $a_{ij} \in \mathbb{Z}_{\geq 0}$ arcs if the sum of any two adjacent roots $\alpha, \beta \in O_i$ can be written as

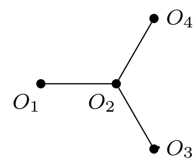
$$\alpha+\beta = \sum_{j=1}^{r} a_{ij}\gamma_j \text{ with } \gamma_j \in O_j.$$

We then align orbits $O_i$ and $O_j$ so that $\gamma_j$ appears in between $\alpha$ and $\beta$.

In our specific example, the sum of any two adjacent roots in each of the orbits $O_1$, $O_3$, and $O_4$ is always a root in $O_2$. For example, the roots $e_1 + e_3$ and $-e_1 + e_2$ are adjacent in $O_1$ and we have

$$(e_1 + e_3) + (-e_1 + e_2) = e_2 + e_3 \in O_2.$$

Thus $a_{11} = a_{13} = a_{14} = 0$ and $a_{12} = 1$. We slide columns $O_1$ and $O_2$ up and down relative to each other until the root $e_2 + e_3$ appears between $e_1 + e_3$ and $-e_1 + e_2$. Repeating this for $O_3$ and $O_4$ gives the complete relative alignment of the columns and we obtain the graph




The gaps in the table are notational inconveniences that can be removed by inserting a disjoint copy of $\Phi$. This produces the new table

| $\lambda$ | $O_1$ | $O_2$ | $O_3$ | $O_4$ |
|---|---|---|---|---|
| 0 | $e_6 - e_7$ | $e_1 + e_2$ | $e_5 - e_8$ | $e_5 + e_8$ |
| 1 | $e_1 + e_3$ | $e_5 + e_6$ | $e_2 + e_4$ | $e_2 - e_4$ |
| 2 | $e_5 + e_7$ | $e_2 + e_3$ | $e_6 + e_8$ | $e_6 - e_8$ |
| 3 | $-e_1 + e_2$ | $e_6 + e_7$ | $e_3 - e_4$ | $e_3 + e_4$ |
| 4 | $-e_5 + e_6$ | $-e_1 + e_3$ | $e_7 - e_8$ | $e_7 + e_8$ |
| 5 | $-e_2 + e_3$ | $-e_5 + e_7$ | $-e_1 + e_4$ | $-e_1 - e_4$ |
| 6 | $-e_6 + e_7$ | $-e_1 - e_2$ | $-e_5 + e_8$ | $-e_5 - e_8$ |
| 7 | $-e_1 - e_3$ | $-e_5 - e_6$ | $-e_2 - e_4$ | $-e_2 + e_4$ |
| 8 | $-e_5 - e_7$ | $-e_2 - e_3$ | $-e_6 - e_8$ | $-e_6 + e_8$ |
| 9 | $e_1 - e_2$ | $-e_6 - e_7$ | $-e_3 + e_4$ | $-e_3 - e_4$ |
| 10 | $e_5 - e_6$ | $e_1 - e_3$ | $-e_7 + e_8$ | $-e_7 - e_8$ |
| 11 | $e_2 - e_3$ | $e_5 - e_7$ | $e_1 - e_4$ | $e_1 + e_4$ |
| 0 | $e_6 - e_7$ | $e_1 + e_2$ | $e_5 - e_8$ | $e_5 + e_8$ |

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Set $J = \{1, 2, 3, 4\}$ and define a map $\varphi : J \times \mathbb{Z} \to \Phi$ so that $\varphi(i, \lambda)$ is the root listed in column $i$ and row $\mu$ with $\lambda \in \mu + 12\mathbb{Z}$. Define the $4 \times 4$ matrices

$$[\widetilde{N}_\lambda]_{i,j} := \langle \varphi(i, \lambda), \varphi(j, 0) \rangle \text{ for all } \lambda \in \mathbb{Z}; i, j \in J.$$

Recursively define the $4 \times 4$ matrices

$$N_0 := 0$$
$$N_1 := I$$
$$N_\lambda := \widetilde{N}_{\lambda-1} + N_{\lambda-2} \text{ for all } \lambda \geq 2.$$

We work out the first six matrices below:

| $\lambda$ | $\widetilde{N}_\lambda$ | $N_\lambda$ |
|---|---|---|
| 0 | $\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$ | $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ |
| 1 | $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ | $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ |
| 2 | $\begin{bmatrix} -1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 1 \\ 1 & 0 & 1 & -1 \end{bmatrix}$ | $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ |
| 3 | $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ | $\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$ |
| 4 | $\begin{bmatrix} 1 & 0 & -1 & -1 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 1 & -1 \\ -1 & 0 & -1 & 1 \end{bmatrix}$ | $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ |
| 5 | $\begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & -1 & -1 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$ | $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ |
| 6 | $\begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$ | $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ |

Then the five matrices $\{N_1, N_2, \ldots, N_5\}$ form a representation of the level-6 affine truncation of the representation involutive-semiring of $\mathfrak{sl}_2(\mathbb{C})$. Essentially this means that these five matrices satisfy the product rule

$$N_2 \cdot N_\lambda = N_{\lambda-1} + N_{\lambda+1} \text{ for all } \lambda \in \{1, 2, 3, 4, 5\}$$

with $N_0 = 0 = N_6$. In mathematical physics, the five matrices $\{N_1, N_2, \ldots, N_5\}$ are also known as a nonnegative integer matrix representation of type $\mathfrak{sl}_2(\mathbb{C})$ at level 6.

The same procedure above works in reverse: starting with a representation of an affine truncation of the representation involutive-semiring of $\mathfrak{sl}_2(\mathbb{C})$, we can form the corresponding root system. The point of this procedure is that it also applies to arbitrary finite-dimensional semisimple Lie algebras.

## 1.1 Background

Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra; let $\mathrm{Rep}(\mathfrak{g})$ denote the involutive-semiring of representation isomorphism classes of $\mathfrak{g}$ with addition, multiplication, and involution derived, respectively, from the direct sum, the tensor product, and the dual of representations; and for any positive integer $k$, let $\mathrm{Rep}_k(\mathfrak{g})$ denote the affine truncation of $\mathrm{Rep}(\mathfrak{g})$ at level $k$. We give precise definitions of these concepts in Chapters 3 and 4.

The connection between root systems and representations of $\mathrm{Rep}_k(\mathfrak{sl}_2(\mathbb{C}))$ seems to be originally and solely due to Ocneanu [11]. In this paper, the author outlines a construction that associates a root system to a subgroup or module of a quantum Lie group at a root of unity. Part of the construction involves forming, in our terminology, a representation of $\mathrm{Rep}_k(\mathfrak{g})$ from the subgroup or module in question. Not all representations of $\mathrm{Rep}_k(\mathfrak{g})$ arise in this way (we provide examples of some in this thesis), but the root system construction still applies to arbitrary representations of $\mathrm{Rep}_k(\mathfrak{g})$. We know of no other references to this construction.

In mathematical physics, representations of $\mathrm{Rep}_k(\mathfrak{g})$ appear as a component of certain rational conformal field theories (RCFTs) known as Wess-Zumino-Witten (WZW) models. In this setting, representations of $\mathrm{Rep}_k(\mathfrak{g})$ are also known as nonnegative integer matrix representations (NIM-reps) or fusion representations. For good coverage of this topic, see [1, 8]. For the important connection between NIM-reps and modular invariants, see [7, 8, 9, 5, 6].

The essence of a representation of $\mathrm{Rep}_k(\mathfrak{sl}_n(\mathbb{C}))$ was first defined and studied just over twenty years ago by Di Francesco and Zuber [3]. In this paper, the authors define "graphs that generalize the A-D-E Dynkin diagrams" in terms of mainly spectral conditions on a set of nonnegative integer matrices. These graph have since been studied [1, 2, 4, 10, 12, 13, 14, 15, 16, 17, 18] (most notably for the case $n = 3$) under various names, such as "generalized

Dynkin diagrams" and "higher Coxeter graphs". The gradual trend has been to understand these graphs as representations of $\mathrm{Rep}_k(\mathfrak{sl}_n(\mathbb{C}))$, although there is still some discrepancy in the exact correspondence between the two.

Zuber [17] eventually distinguished between two types of generalized Dynkin diagrams: Class I and Class II. The Class I graphs were intended to be higher analogues of the simply-laced affine Dynkin diagrams while the Class II graphs were intended to be higher analogues of the simply-laced Dynkin diagrams. Both classes were defined using spectral conditions on a set of nonnegative integer matrices.

## 1.2 Outline

In this thesis, we outline a basic mathematical theory of representations of the involutive-semiring $\mathrm{Rep}_k(\mathfrak{g})$. The key original results are as follows.

- We show how Ocneanu's root system construction outlined in [11] applies to arbitrary representations of $\mathrm{Rep}_k(\mathfrak{g})$.
- We extend the notion of a representation of $\mathrm{Rep}_k(\mathfrak{g})$ slightly to include "involutionizable-representations" of $\mathrm{Rep}_k(\mathfrak{g})$. This allows root systems to have roots of varying length.
- We assign, to any involutionizable-representation of $\mathrm{Rep}_k(\mathfrak{g})$, a Cartan matrix of higher type. These Cartan matrices of higher type can be thought of as Dynkin diagrams, but they differ from the generalized Dynkin diagrams defined by Zuber [17] and others.
- We define "affine" involutionizable-representations of $\mathrm{Rep}(\mathfrak{g})$ and we show how these behave very much like involutionizable-representations of $\mathrm{Rep}_k(\mathfrak{g})$. In particular, they also define root systems and Cartan matrices of higher type. We argue that Class I generalized Dynkin diagrams should be understood as affine representations of $\mathrm{Rep}(\mathfrak{sl}_n(\mathbb{C}))$.
- We classify the symmetric representations of $\mathrm{Rep}_k(\mathfrak{sl}_3(\mathbb{C}))$ and the symmetric affine representations of $\mathrm{Rep}(\mathfrak{sl}_3(\mathbb{C}))$.
- We present an ad hoc construction, tentatively labelled "the automorphism construction", which generates large and unusual examples of representations of $\mathrm{Rep}_k(\mathfrak{sl}_3(\mathbb{C}))$ and affine representations of $\mathrm{Rep}(\mathfrak{sl}_3(\mathbb{C}))$.

- We provide a computer-generated list, up to some maximum number of vertices, of those representations of $\text{Rep}_k(\mathfrak{sl}_3(\mathbb{C}))$ and those affine representations of $\text{Rep}(\mathfrak{sl}_3(\mathbb{C}))$ which do not appear to arise from the automorphism construction.

The outline of this thesis is as follows. In Chapter 2, we provide a quick guide to involutive-semirings. This also serves to introduce some of our notation and terminology. In Chapter 3, we study the involutive-semiring structure of $\text{Rep}(\mathfrak{g})$. In Chapter 4, we study the involutive-semiring structure of $\text{Rep}_k(\mathfrak{g})$. In Chapter 5, we study representations of $\text{Rep}_k(\mathfrak{g})$. In Chapter 6, we study affine representations of $\text{Rep}(\mathfrak{g})$. In Chapter 7, we show how representations of $\text{Rep}(\mathfrak{g})$ and affine representations of $\text{Rep}_k(\mathfrak{g})$ correspond to root systems and Cartan matrices of higher type. In Chapter 8, we present our remaining original results. In the appendix, we provide a proof of our classification of the symmetric representations of $\text{Rep}_k(\mathfrak{sl}_3(\mathbb{C}))$ and the symmetric affine representations of $\text{Rep}(\mathfrak{sl}_3(\mathbb{C}))$.

# Chapter 2

# Abstract Algebra

The set of representation isomorphism classes of a finite-dimensional semisimple Lie algebra has a natural structure as an involutive-semiring, where addition comes from the direct sum, multiplication from the tensor product, and involution from the dual of representations. (There is also some additional structure which comes from things like exterior and symmetric powers, but we are not interested in those.) In this chapter, we provide a quick guide to the basic theory of involutive-semirings.

## 2.1 Monoids

A **monoid** is a triple $(M, \cdot, 1)$ such that $M$ is a set and $\cdot$ is an associative binary operation on $M$ with identity 1. The binary operation $\cdot$ is customarily written using infix notation or juxtaposition so that

$$\begin{aligned} \cdot\, &: M \times M \to M \\ \cdot\,&(m, n) \equiv m \cdot n \equiv mn \text{ for all } m, n \in M. \end{aligned}$$

The exact requirements on the binary operation $\cdot$ and identity 1 are

$$\begin{aligned} &m \cdot (n \cdot o) = (m \cdot n) \cdot o \text{ for all } m, n, o \in M, \\ &1 \cdot m = 1 = m \cdot 1 \text{ for all } m \in M. \end{aligned}$$

In general, the binary operation $\cdot$ and identity 1 are called multiplication and one, respectively. When $\cdot$ and 1 are unimportant or obvious, we will avoid cumbersome notation and write $M \equiv (M, \cdot, 1)$, in which case $M$ can refer to either the monoid or the set, depending on its context.

A monoid $(M, \cdot, 1)$ is **commutative** if the binary operation $\cdot$ is commutative:

$$m \cdot n = n \cdot m \text{ for all } m, n \in M.$$

In general, the binary operation and identity of a commutative monoid are called addition and zero, respectively, with the notations $\cdot \equiv +$ and $1 \equiv 0$.

Let $M \equiv (M, \cdot, 1)$ and $N \equiv (N, \cdot, 1)$ be two monoids. Here, as is often the case, we use the same symbols for the binary operations and identities of two different monoids; to which monoid the symbol $\cdot$ or 1 refers must be determined from context. A **monoid homomorphism** of $M$ to $N$ is a map $\pi : M \to N$ such that

$$\begin{aligned} &\pi(m \cdot n) = \pi(m) \cdot \pi(n) \text{ for all } m, n \in M, \\ &\pi(1) = 1. \end{aligned}$$

A homomorphism of a monoid to itself is called a **monoid endomorphism**. A bijective monoid homomorphism is called a **monoid isomorphism**. A bijective monoid endomorphism is called a **monoid automorphism**.

Let $M \equiv (M, \cdot, 1)$ be a monoid and let $N$ be a subset of $M$ that is closed under multiplication and contains one:

$$\begin{aligned} &m \cdot n \in N \text{ for all } m, n \in N, \\ &1 \in N. \end{aligned}$$

Then $N \equiv (N, \cdot|_{N \times N}, 1)$ is a monoid; more specifically, we say that $N$ is a **submonoid** of $M$.

Let $M \equiv (M, \cdot, 1)$ be a monoid, let $X$ be any set, let $M^X$ denote the set of all maps of $X$ to $M$, let 1 denote the constant map of $X$ to $M$ with image 1, and let $\cdot$ denote the binary operation on $M^X$ defined so that

$$\begin{aligned} &\pi \cdot \sigma : M \to X \text{ for all } \pi, \sigma \in M^X \\ &(\pi \cdot \sigma)(m) := \pi(m) \cdot \sigma(m) \text{ for all } m \in M. \end{aligned}$$

Then $M^X \equiv (M^X, \cdot, 1)$ is a monoid. We refer to the binary operation $\cdot$ on $M^X$ as pointwise multiplication. If $M$ is commutative, then so is $M^X$. If we let $M[X]$ denote the set of all maps of $X$ to $M$ with finite support, then $M[X]$ is a submonoid of $M^X$. If we let $\text{End}(M)$ denote the set of all monoid endomorphisms of $M$, then $\text{End}(M)$ is a submonoid of $M^M$.

Let $M \equiv (M, \cdot, 1)$ and $N \equiv (N, \cdot, 1)$ be two monoids, and let $\cdot$ denote the binary operation on the set $M \times N$ defined so that

$$(m, n) \cdot (o, p) := (m \cdot o, n \cdot p) \text{ for all } (m, n), (o, p) \in M \times N.$$

Then $M \oplus N \equiv (M \times N, \cdot, (1, 1))$ is a monoid; we refer to it as the **direct sum monoid** of $M$ and $N$.

Let $M \equiv (M, +, 0)$ be a commutative monoid. A **basis** of $M$ is a subset $B$ of $M$ such that every element of $M$ can be uniquely written (up to reordering) as a finite sum of elements in $B$. We say that a commutative monoid is **free** if it has a basis. The basis of a free commutative monoid is unique. The **rank** of a free commutative monoid is the cardinality of its basis. For any cardinal number $r$, there is a monoid isomorphism of any free commutative monoid of rank $r$ to the monoid direct sum of $r$ copies of $\mathbb{Z}_{\geq 0} \equiv (\mathbb{Z}_{\geq 0}, +, 0)$.

## 2.2 Involutive-Monoids

An **involutive-monoid** is a quadruple $(M, \cdot, 1, *)$ such that $(M, \cdot, 1)$ is a monoid and $*$ is a self-inverse unary operation on $M$ that reverses multiplication and fixes one. The unary operation $*$ is usually written using superscript notation so that

$$\begin{aligned} * &: M \to M \\ *(m) &\equiv m^* \text{ for all } m \in M. \end{aligned}$$

The exact requirements on the unary operation $*$ are

$$\begin{aligned} &(m \cdot n)^* = (m^*) \cdot (n^*) \text{ for all } m, n \in M, \\ &(m^*)^* = m \text{ for all } m \in M, \\ &1^* = 1. \end{aligned}$$

In general, the unary operation of an involutive-monoid is called involution. When $\cdot$, 1, and $*$ are known implicitly or from context, we will avoid cumbersome notation and write $M \equiv (M, \cdot, 1, *)$, in which case $M$ can refer to either the involutive-monoid, the monoid, or the set, depending on its context.

An involutive-monoid $(M, \cdot, 1, *)$ is **commutative** if the monoid $(M, \cdot, 1)$ is commutative. For any commutative monoid $(M, +, 0)$, the quadruple $(M, +, 0, \mathrm{id})$ is always a commutative involutive-monoid, where id denotes the identity map on $M$. Thus every commutative monoid is always a commutative involutive-monoid in a way that offers no additional structure. This allows us to consider only commutative involutive-monoids without losing any generality.

Let $M \equiv (M, \cdot, 1, *)$ and $N \equiv (N, \cdot, 1, *)$ be two involutive-monoids. An **involutive-monoid homomorphism** of $M$ to $N$ is a monoid homomorphism $\pi$ of $(M, \cdot, 1)$ to $(N, \cdot, 1)$ such that

$$\pi(m^*) = \pi(m)^* \text{ for all } m \in M.$$

A bijective involutive-monoid homomorphism is called an **involutive-monoid isomorphism**.

Let $M \equiv (M, \cdot, 1, *)$ be an involutive monoid and let $N$ be a submonoid of $M$ that is closed under involution:

$$n^* \in N \text{ for all } n \in N.$$

Then $N \equiv (N, \cdot|_{N \times N}, 1, *|_N)$ is an involutive-monoid; more specifically, we say that $N$ is a **subinvolutive-monoid** of $M$.

A commutative involutive-monoid $(M, +, 0, *)$ is **free** if the commutative monoid $(M, +, 0)$ is free. Note that the requirements on the unary operation $*$ are such that $*$ is an automorphism of $M$. This implies that $*$ acts as a permutation on the basis of $M$.

## 2.3 Semirings

A **semiring** is a quintuple $(S, +, 0, \cdot, 1)$ such that

$S$ is a set,
$(S, +, 0)$ is a commutative monoid,
$(S, \cdot, 1)$ is a monoid,
$a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ for all $a, b, c \in S$,
$(a + b) \cdot c = (a \cdot c) + (b \cdot c)$ for all $a, b, c \in S$,
$0 \cdot a = 0 = a \cdot 0$ for all $a \in S$.

In general, the binary operation $+$ and identity 0 of a semiring are called addition and zero, respectively, while the binary operation $\cdot$ and identity 1 are called multiplication and one, respectively. When $+$, 0, $\cdot$, and 1 are unimportant or obvious, we will allow the symbol $S$ to refer to either the semiring or the set, depending on its context.

A semiring $(S, +, 0, \cdot, 1)$ is **commutative** if the monoid $(S, \cdot, 1)$ is commutative, and **free** if the commutative monoid $(S, +, 0)$ is free.

Let $S \equiv (S, +, 0, \cdot, 1)$ and $T \equiv (T, +, 0, \cdot, 1)$ be two semirings. A **semiring homomorphism** of $S$ to $T$ is a map $\pi : S \to T$ such that

$$\pi \text{ is a monoid homomorphism of } (S, +, 0) \text{ to } (T, +, 0),$$
$$\pi \text{ is a monoid homomorphism of } (S, \cdot, 1) \text{ to } (T, \cdot, 1).$$

A bijective semiring homomorphism is called a **semiring isomorphism**.

Let $S \equiv (S, +, 0, \cdot, 1)$ be a semiring and let $T$ be a subset of $S$ that contains one and is closed under addition and multiplication:

$$s + t \in T \text{ for all } s, t \in T,$$
$$s \cdot t \in T \text{ for all } s, t \in T,$$
$$1 \in T.$$

Then $T \equiv (T, +|_{T\times T}, 0, \cdot|_{T\times T}, 1)$ is a semiring; more specifically, we say that $T$ is a **subsemiring** of $S$.

Let $M \equiv (M, +, 0)$ be a commutative monoid. Let $\mathrm{End}(M)$ denote the set of all monoid endomorphisms of $M$. Let $+$ denote the binary operation on $\mathrm{End}(M)$ defined so that

$$\pi + \sigma : M \to M \text{ for all } \pi, \sigma \in \mathrm{End}(M)$$
$$(\pi + \sigma)(m) = \pi(m) + \sigma(m) \text{ for all } m \in M.$$

Let 0 denote the endomorphism of $M$ that maps every element to zero. Let $\circ$ denote the binary operation on $\mathrm{End}(M)$ defined so that

$$\pi \circ \sigma : M \to M \text{ for all } \pi, \sigma \in \mathrm{End}(M)$$
$$(\pi \circ \sigma)(m) = \pi(\sigma(m)) \text{ for all } m \in M.$$

Let id denote the endomorphism of $M$ that maps every element to itself. Then one can verify that $\mathrm{End}(M) \equiv (\mathrm{End}(M), +, 0, \circ, \mathrm{id})$ is a semiring; we refer to it as the **monoid endomorphism semiring** of $M$.

## 2.4 Modules

Let $S$ be an semiring. A **module** of $S$ is a quadruple $(M, +, 0, \pi)$ such that $M \equiv (M, +, 0)$ is a commutative module and $\pi$ is a semiring homomorphism of $S$ to $\text{End}(M)$. We will often write the map $\pi$ as a left action of $S$ on $M$ so that

$$sa := (\pi(s))(a) \text{ for all } s \in S, a \in M.$$

When the map $\pi$ is unimportant or obvious, we will avoid cumbersome notation and write $M \equiv (M, +, 0, \pi)$, in which case $M$ can refer to either the module, the monoid, or the set, depending on its context.

Let $S$ be a semiring and let $M$ and $N$ be two modules of $S$. A **module homomorphism** of $M$ to $N$ is a monoid homomorphism $\tau : M \to N$ such that

$$\tau(sa) = s\tau(a) \text{ for all } s \in S, a \in M.$$

A bijective module homomorphism is called a **module isomorphism**.

Let $S$ be a semiring, let $M$ be a module of $S$, and let $N$ be a submonoid of $M$ that is closed under the action of $S$:

$$sn \in N \text{ for all } s \in S, n \in N.$$

Then $N$ is a module of $S$ with this action; more specifically, we say that $N$ is a **submodule** of $M$.

Let $S$ be a semiring and let $M$ and $N$ be two modules of $S$. Define a left action of $S$ on the monoid direct sum $M \oplus N$ so that

$$s(m, n) := (sm, sn) \text{ for all } s \in S, (m, n) \in M \times N.$$

Then one can verify that $M \oplus N$ is a module of $S$ with this action; more specifically, we refer to $M \oplus N$ as the **module direct sum** of $M$ and $N$.

Let $M \equiv (M, +, 0)$ be a commutative monoid and let $\mathbb{Z}_{\geq 0}$ denote the semiring of nonnegative integers. Define an action of $\mathbb{Z}_{\geq 0}$ on $M$ so that

$$sm := \underbrace{m + \cdots + m}_{s} \text{ for all } s \in \mathbb{Z}_{\geq 0}, m \in M.$$

Then $M$ is a module of $\mathbb{Z}_{\geq 0}$ with this action. In this way, all commutative monoids have natural interpretations as modules of $\mathbb{Z}_{\geq 0}$.

We can think of an abelian group as a module of $\mathbb{Z}$.

## 2.5 Involutive-Semirings

An **involutive-semiring** is a sextuple $(S, +, 0, \cdot, 1, *)$ such that

$$(S, +, 0, \cdot, 1) \text{ is a semiring,}$$
$$(S, +, 0, *) \text{ is an involutive-monoid,}$$
$$(S, \cdot, 1, *) \text{ is an involutive-monoid.}$$

In general, the unary operation of an involutive-semiring is called involution. When $+$, $0$, $\cdot$, $1$, and $*$ are unimportant or obvious, we will avoid cumbersome notation and write $S \equiv (S, +, 0, \cdot, 1, *)$, in which case $S$ can refer to either the involutive-semiring, the semiring, or the set, depending on its context.

Let $S \equiv (S, +, 0, \cdot, 1, *)$ and $T \equiv (T, +, 0, \cdot, 1, *)$ be two involutive-semirings. An **involutive-semiring homomorphism** of $S$ to $T$ is a map $\pi : S \to T$ such that

$$\pi \text{ is a involutive-monoid homomorphism of } (S, +, 0, *) \text{ to } (T, +, 0, *),$$
$$\pi \text{ is an involutive-monoid homomorphism of } (S, \cdot, 1, *) \text{ to } (T, \cdot, 1, *).$$

A bijective involutive-semiring homomorphism is called an **involutive-semiring isomorphism**.

Let $S \equiv (S, +, 0, \cdot, 1, *)$ be an involutive-semiring and let $T$ be a subsemiring of $S$ that is closed under involution:

$$t^* \in T \text{ for all } t \in T.$$

Then $T \equiv (T, +|_{T \times T}, 0, \cdot|_{T \times T}, 1, *|_T)$ is an involutive semiring; more specifically, we say that $T$ is a **subinvolutive-semiring** of $S$.

An involutive-semiring is **commutative** if multiplication is commutative. For any commutative semiring $(S, +, 0, \cdot, 1)$, the sextuple $(S, +, 0, \cdot, 1, \mathrm{id})$ is always a commutative involutive-semiring, where id denotes the identity map on $S$. Thus every commutative semiring is always a commutative involutive-semiring in a way that offers no additional structure. This allows us to consider only commutative involutive-semirings without losing any generality.

The most important example of an involutive-semiring is the set of endomorphisms on a free commutative monoid of finite rank. Let $M \equiv (M, +, 0)$ be a free commutative monoid of finite rank. Fix an ordering $B = (b_1, \ldots, b_r)$ of the basis of $M$. Let $\pi$ denote the map of $\mathrm{M}_r(\mathbb{Z}_{\geq 0})$ to $\mathrm{End}(M)$ defined so that

$$\begin{aligned}
&\pi : \mathrm{M}_r(\mathbb{Z}_{\geq 0}) \to \mathrm{End}(M) \\
&\pi(A) : M \to M \\
&(\pi(A))(b_i) := \sum_{j=1}^{r} [A]_{ij} b_j \text{ for all } i \in \{1, \ldots, r\}.
\end{aligned}$$

Then one can verify that $\pi$ is a semiring isomorphism. Thus we can think of monoid endomorphisms of free commutative monoids as nonnegative integer matrices acting on nonnegative integer column vectors, and vice versa.

Define a unary operation $\mathsf{T}$ on $\mathrm{End}(M)$ to be matrix transpose. Then

$$\mathrm{End}(M) \equiv (\mathrm{End}(M), +, 0, \circ, \mathrm{id}, \mathsf{T})$$

is an involutive-semiring, which we refer to as the **monoid endomorphism involutive-semiring** of $M$.

An involutive-semiring $(S, +, 0, \cdot, 1, *)$ is **free** if the commutative monoid $(S, +, 0)$ is free. Multiplication in a free involutive-semiring has a particularly nice description in terms of structure coefficients. Let $B = (b_i)_{i \in I}$ denote the basis of $S$ for some index set $I$. Then there are nonnegative integer coefficients ${\mathcal{C}_{ij}}^k$ such that

$$b_i \cdot b_j = \sum_{k \in I} {\mathcal{C}_{ij}}^k b_k \text{ for all } i, j \in I.$$

Since every element of $S$ can be uniquely written (up to reordering) as a sum of the basis elements, this completely determines multiplication in $S$. We refer to the nonnegative integer ${\mathcal{C}_{ij}}^k$ as the **structure coefficients** of $S$.

## 2.6 Involutive-Algebras

Let $S$ be a commutative involutive-semiring. An **involutive-algebra** of $S$ is a septuple $(A, +, 0, \cdot, 1, *, \pi)$ such that $A \equiv (A, +, 0, \cdot, 1, *)$ is an involutive-semiring and $\pi$ is an involutive-semiring homomorphism of $S$ to $A$. We will often write the map $\pi$ as a left action of $S$ on $A$ so that

$$sa := \pi(s) \cdot a \text{ for all } s \in S, a \in A.$$

When the map $\pi$ is unimportant or obvious, we will avoid cumbersome notation and write $A \equiv (A, +, 0, \cdot, 1, *, \pi)$, in which case $A$ can refer to either the involutive-algebra, the involutive-semiring, or the set, depending on its context. Note that all involutive-algebras of $S$ are also modules of $S$, but not vice versa.

If involution in both $S$ and $A$ is the identity, then we simply say that $A$ is an **algebra** of $S$.

Let $S$ be a commutative involutive-semiring and let $A$ and $B$ be two involutive-algebras of $S$. An **involutive-algebra homomorphism** of $A$ to $B$ is an involutive-semiring homomorphism $\tau : A \to B$ such that

$$\tau(sb) = s\tau(b) \text{ for all } s \in S, b \in B.$$

A bijective involutive-algebra homomorphism is called an **involutive-algebra isomorphism**.

Let $S$ be a commutative involutive-semiring, let $A$ be an involutive-algebra of $S$, and let $B$ be a subinvolutive-semiring of $A$ that is closed under the action of $S$:

$$sb \in B \text{ for all } s \in S, b \in B.$$

Then $B$ is an involutive-algebra of $S$ with this action; more specifically, we say that $B$ is a **subinvolutive-algebra** of $A$.

Let $A \equiv (A, +, 0, \cdot, 1, *)$ be an involutive-semiring and let $\mathbb{Z}_{\geq 0}$ denote the commutative involutive-semiring of nonnegative integers. Define the map

$$\begin{aligned} &\pi : \mathbb{Z}_{\geq 0} \to A \\ &\pi(s) := \underbrace{1 + \cdots + 1}_{s} \text{ for all } s \in \mathbb{Z}_{\geq 0}. \end{aligned}$$

Then $A \equiv (A, +, 0, \cdot, 1, *, \pi)$ is an involutive-algebra of $\mathbb{Z}_{\geq 0}$. In this way, all involutive-semirings have natural interpretations as involutive-algebras of $\mathbb{Z}_{\geq 0}$.

We can think of an involutive-ring as an involutive-algebra of $\mathbb{Z}$.

Let $S$ be an involutive-semiring. We will often find it convenient to extend $S$ into an involutive-algebra of $\mathbb{Z}$, $\mathbb{R}$, or $\mathbb{C}$. (Involution of complex numbers is understood to be complex conjugation.) Without going into details, we formally define

$$\begin{aligned} S^{\mathbb{Z}} &:= \mathbb{Z} \otimes_{\mathbb{Z}_{\geq 0}} S \\ S^{\mathbb{R}} &:= \mathbb{R} \otimes_{\mathbb{Z}_{\geq 0}} S \\ S^{\mathbb{C}} &:= \mathbb{C} \otimes_{\mathbb{Z}_{\geq 0}} S. \end{aligned}$$

We refer to $S^{\mathbb{Z}}$, $S^{\mathbb{R}}$, and $S^{\mathbb{C}}$ as the integerization, real-form, and complexification of $S$, respectively. We have the obvious inclusions

$$S \hookrightarrow S^{\mathbb{Z}} \hookrightarrow S^{\mathbb{R}} \hookrightarrow S^{\mathbb{C}}.$$

We will not provide a precise definition of the tensor product of involutive-algebras in this thesis; the idea is the same as for rings.

## 2.7 Representations of Involutive-Semirings

Let $S$ be an involutive-semiring and let $M$ be a free commutative monoid of finite rank. A **representation** of $S$ on $M$ is an involutive-semiring homomorphism

$$\pi : S \to \mathrm{End}(M).$$

We will often make the map $\pi$ implicit by writing

$$sa := (\pi(s))(a) \text{ for all } s \in S, a \in M.$$

The commutative monoid $M$ with this action is a module of $S$. The notion of module isomorphism defines an equivalence relation on the set of all representations of $S$. For any semiring representation $M$ of $S$, we use $[M]$ to denote the equivalence class of $M$ under this equivalence relation. We refer to $[M]$ as the **isomorphism class** of $M$.

Let $S$ be a semiring. The module direct sum of two representations of $S$ is also a representation. Let $\mathrm{Rep}(S)$ denote the set of all isomorphism classes of representations of $S$. Let $+$ denote the binary operation on $\mathrm{Rep}(S)$ defined so that

$$\begin{aligned}&+ : \mathrm{Rep}(S) \times \mathrm{Rep}(S) \to \mathrm{Rep}(S)\\ &[M] + [N] := [M \oplus N] \text{ for all } [M], [N] \in \mathrm{Rep}(S).\end{aligned}$$

Let 0 denote the representation isomorphism class of $S$ that contains a representation of rank 0. Then $\mathrm{Rep}(S) \equiv (\mathrm{Rep}(S), +, 0)$ is a well-defined commutative monoid.

Let $S$ be a semiring and let $M$ be a representation of $S$. Then $M$ is **decomposable** if it can be written as a direct sum of two nonzero representations of $S$. Otherwise $M$ is **indecomposable**. A basis of $\mathrm{Rep}(S)$ is provided by the isomorphism classes of indecomposable representations.

If $M$ has rank $r$, then using the isomorphism of $\mathrm{M}_r(\mathbb{Z}_{\geq 0})$ to $\mathrm{End}(M)$, we can think of a representation of $S$ as an involutive-semiring homomorphism

$$\pi : S \to \mathrm{M}_r(\mathbb{Z}_{\geq 0}).$$

Most of this thesis is devoted to studying representations of an affine truncation of the representation involutive-semiring of a finite-dimensional semisimple Lie algebra.

# Chapter 3

# The Involutive-Semiring $\mathrm{Rep}(\mathfrak{g})$

Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra and let $\mathrm{Rep}(\mathfrak{g})$ denote the set of all isomorphism classes of representations of $\mathfrak{g}$. Define addition, multiplication, and involution in $\mathrm{Rep}(\mathfrak{g})$ so that

$$\begin{aligned}[U] + [V] &:= [U \oplus V] \text{ for all } [U], [V] \in \mathrm{Rep}(\mathfrak{g}) \\ [U] \cdot [V] &:= [U \otimes V] \text{ for all } [U], [V] \in \mathrm{Rep}(\mathfrak{g}) \\ [U]^* &:= [U^*] \text{ for all } [U] \in \mathrm{Rep}(\mathfrak{g}).\end{aligned}$$

Let 0 denote the isomorphism class of the zero-dimensional representation of $\mathfrak{g}$ and let $R_\rho \equiv [V_\rho]$ denote the isomorphism class of the one-dimensional representation of $\mathfrak{g}$ on which $\mathfrak{g}$ acts as zero. Then

$$\mathrm{Rep}(\mathfrak{g}) \equiv (\mathrm{Rep}(\mathfrak{g}), +, 0, \cdot, R_\rho, *)$$

is a commutative involutive-semiring which we refer to as the **representation involutive-semiring** of $\mathfrak{g}$. By complete reducibility, the isomorphism classes of irreducible representations of $\mathfrak{g}$ form a basis of $\mathrm{Rep}(\mathfrak{g})$. Thus $\mathrm{Rep}(\mathfrak{g})$ is a free commutative involutive-semiring.

## 3.1 The Weyl Group

Much of the involutive-semiring structure of $\mathrm{Rep}(\mathfrak{g})$ comes from the action of the Weyl group on the weight lattice. Let $W$ denote the Weyl group of $\mathfrak{g}$ and let $P$ denote the weight lattice of $\mathfrak{g}$. Then $W$ acts freely and transitively on the Weyl chambers of the weight space. Choose one of the Weyl chambers to be dominant. Let $P^+$ (respectively $P^{++}$) denote the intersection of $P$ with the dominant Weyl chamber union its boundary (respectively the dominant Weyl chamber). Then $P^+$ is a set of representatives for the orbits of $P$ under the action of the Weyl group and $P^{++}$ is the subset of $P^+$ consisting of all weights with trivial stabilizer.

We can describe the sets $P^+$ and $P^{++}$ more explicitly in terms of fundamental weights. Let $\Lambda_1, \ldots, \Lambda_r$ denote the fundamental weights of $\mathfrak{g}$ with respect to the dominant Weyl chamber. As an abelian group over addition, we have

$$P \cong \bigoplus_{i=1}^{r} \mathbb{Z}\Lambda_i.$$

In terms of this direct sum decomposition, we have

$$\begin{aligned} P^{++} &= \left\{ \sum_{i=1}^{r} \lambda^i \Lambda_i \in P : \lambda^1 \geq 1, \ldots, \lambda^r \geq 1 \right\} \\ P^{+} &= \left\{ \sum_{i=1}^{r} \lambda^i \Lambda_i \in P : \lambda^1 \geq 0, \ldots, \lambda^r \geq 0 \right\}. \end{aligned}$$

Define the Weyl vector

$$\rho := \sum_{i=1}^{r} \Lambda_i \in P^{++}.$$

It is well-known that the isomorphism classes of irreducible representations of $\mathfrak{g}$ are in a one-to-one correspondence with the set $P^{++}$. More specifically, all isomorphism classes of irreducible representations of $\mathfrak{g}$ are of the form

$$R_\lambda \equiv [V_\lambda] \text{ for all } \lambda \in P^{++}$$

where $V_\lambda$ is an irreducible representation of $\mathfrak{g}$ with highest weight $\lambda - \rho$. As a commutative monoid over addition, we have

$$\mathrm{Rep}(\mathfrak{g}) \cong \bigoplus_{\lambda \in P^{++}} \mathbb{Z}_{\geq 0} R_\lambda.$$

We should point out here that $V_\lambda$ customarily denotes an irreducible representation of $\mathfrak{g}$ with highest weight $\lambda$, so that the irreducible representations of $\mathfrak{g}$ are in a one-to-one correspondence with the set $P^+$. However, the more natural correspondence, and the one that we use, is with the set $P^{++}$.

The Weyl group is often too large for practical computations. For example, the Weyl group of $E_8$ has order 696 729 600. Fortunately, the action of the

Weyl group on the weight lattice can be understood in terms of two maps that do have efficient computer implementations. We use the notations

$$\zeta : P \to P^{++}$$
$$\eta : P \to \{-1, 0, 1\}$$

and define these maps as follows. Let $\lambda \in P$. Recall that $P^+$ is a set of representatives for the orbits of $P$ under the action of $W$. Thus there is exactly one weight in the intersection of $P^+$ and the orbit of $\lambda$. Define $\zeta(\lambda)$ to be this weight. If there is a unique $w \in W$ such that $w\lambda = \zeta(\lambda)$, then set $\eta(\lambda) = \mathrm{sgn}(w)$. Otherwise, set $\eta(\lambda) = 0$.

We refer to the map $\zeta$ (respectively $\eta$) as the **zeta map** (respectively **eta map**) of $\mathfrak{g}$. Some of properties of the zeta and eta maps of $\mathfrak{g}$ are listed below.

**Proposition 3.1**: The zeta and eta maps of $\mathfrak{g}$ satisfy

(i) $\zeta(w\lambda) = \zeta(\lambda)$ for all $w \in W, \lambda \in P$,

(ii) $\eta(w\lambda) = \mathrm{sgn}(w)\eta(\lambda)$ for all $w \in W, \lambda \in P$,

(iii) $\eta(\lambda) = 0 \Leftrightarrow \zeta(\lambda) \in P^+ \setminus P^{++}$ for all $\lambda \in P$.

We present here a brief description of one algorithm for computing the output of the zeta and eta maps of $\mathfrak{g}$. Let $s_1, \ldots s_r$ denote the simple reflections in $W$ with respect to the dominant Weyl chamber. For any $\lambda \in P$, write

$$\lambda = \sum_{i=1}^{r} \lambda^i \Lambda_i.$$

If $\lambda^i \geq 0$ for all $i \in \{1, \ldots, r\}$, then $\zeta(\lambda) = \lambda$. Otherwise, let $j \in \{1, \ldots, r\}$ be the first index with $\lambda^j < 0$, apply $s_j$ to $\lambda$, and repeat the test. One can show that this procedure eventually terminates after a finite number of steps. The output of $\eta$ may be computed by keeping track of the sign changes that occur with each application of $s_j$.

The zeta map gives a convenient description of involution in $\mathrm{Rep}(\mathfrak{g})$. Since $\mathrm{Rep}(\mathfrak{g})$ is free, we know that its basis is closed under involution. In particular, there is a self-inverse map

$$* : P^{++} \to P^{++}$$
$$*(\lambda) \equiv \lambda^* \text{ for all } \lambda \in P^{++}$$

called involution which satisfies

$$(R_\lambda)^* = R_{\lambda^*} \text{ for all } \lambda \in P^{++}.$$

In terms of the zeta map, it turns out that we have

$$\lambda^* = \zeta(-\lambda) \text{ for all } \lambda \in P^{++}.$$

For a simple example, consider the Lie algebra $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$:

$$\begin{aligned}
W &= \{\pm 1\} \\
P &= \mathbb{Z}\Lambda_1 \cong \mathbb{Z} \\
\rho &= 1 \\
P^+ &= \{0, 1, \ldots\} \\
P^{++} &= \{1, 2, \ldots\} \\
\zeta(\lambda) &= \left\{\begin{matrix} \lambda & \text{if } \lambda \in P^{++} \\ 0 & \text{if } \lambda = 0 \\ -\lambda & \text{if } -\lambda \in P^{++} \end{matrix}\right\} \text{ for all } \lambda \in P \\
\eta(\lambda) &= \left\{\begin{matrix} 1 & \text{if } \lambda \in P^{++} \\ 0 & \text{if } \lambda = 0 \\ -1 & \text{if } -\lambda \in P^{++} \end{matrix}\right\} \text{ for all } \lambda \in P.
\end{aligned}$$

Using the zeta map above, we can easily see that the involution map on $P$ is just the identity.

For a slightly more complex example, consider the Lie algebra $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$:

$$\begin{aligned}
W &= \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} \right\} \\
P &= \mathbb{Z}\Lambda_1 \oplus \mathbb{Z}\Lambda_2 \cong \mathbb{Z}^2 \\
\rho &= (1, 1) \\
P^+ &= \{\lambda = (\lambda^1, \lambda^2) \in \mathbb{Z}^2 : \lambda^1 \geq 0, \lambda^2 \geq 0\} \\
P^{++} &= \{\lambda = (\lambda^1, \lambda^2) \in \mathbb{Z}^2 : \lambda^1 \geq 1, \lambda^2 \geq 1\}.
\end{aligned}$$

The zeta and eta maps in this case do not have nice case-by-case descriptions. With some work, one can show that the involution map on $P$ swaps the first and second components.

## 3.2 Tensor Characters

There are three types of characters that we assign to elements of the involutive-semiring $\text{Rep}(\mathfrak{g})$; we refer to them as tensor characters, multiplicity characters, and exponent characters, respectively. Characters are, more or less, a convenient way to model the involutive-semiring structure of $\text{Rep}(\mathfrak{g})$. In this section, we look at tensor characters.

Let $R \in \text{Rep}(\mathfrak{g})$ and let $\lambda \in P^{++}$. Since $\text{Rep}(\mathfrak{g})$ is a free involutive-semiring, we know that the product $R \cdot R_\lambda$ can be uniquely written (up to reordering) as a finite sum of basis elements. In other words, there is a map

$$\mathcal{T}_R : P^{++} \times P^{++} \to \mathbb{Z}_{\geq 0}$$

with finite support in the second component so that

$$R \cdot R_\lambda := \sum_{\mu \in P^{++}} \mathcal{T}_R(\lambda, \mu) R_\mu \text{ for all } \lambda \in P^{++}.$$

We refer to the map $\mathcal{T}_R$ as the **tensor character** of $R$. We say that $\mathcal{T}_R$ is irreducible if $R$ is irreducible and we use the shorthand notation

$$\mathcal{T}_\lambda := \mathcal{T}_{R_\lambda} \text{ for all } \lambda \in P^{++}.$$

Some properties of the tensor characters of $\mathfrak{g}$ are listed below.

**Proposition 3.2**: The tensor characters of $\mathfrak{g}$ satisfy

(i) $\mathcal{T}_{R^*}(\lambda, \mu) = \mathcal{T}_R(\lambda^*, \mu^*) = \mathcal{T}_R(\mu, \lambda)$ for all $R \in \text{Rep}(\mathfrak{g}); \lambda, \mu \in P^{++}$,

(ii) $\mathcal{T}_\lambda(\mu, \nu) = \mathcal{T}_\mu(\lambda, \nu)$ for all $\lambda, \mu, \nu \in P^{++}$,

(iii) $\mathcal{T}_\rho(\lambda, \mu) = \left\{ \begin{matrix} 1 & \text{if } \lambda = \mu \\ 0 & \text{otherwise} \end{matrix} \right\}$ for all $\lambda, \mu \in P^{++}$.

Part (ii) of the above proposition shows that tensor characters also have finite support in the first component. Let $\text{TCh}(\mathfrak{g})$ denote the set of all tensor characters of $\mathfrak{g}$. It turns out that we can define addition, multiplication, and involution of tensor characters so that the set $\text{TCh}(\mathfrak{g})$ has the structure of an involutive-semiring. We refer to $\text{TCh}(\mathfrak{g})$ with this structure as the **tensor character involutive-semiring** of $\mathfrak{g}$.

We derive the involutive-semiring structure of $\text{TCh}(\mathfrak{g})$ from the following larger involutive-semiring. Let $\mathbb{Z}_{\geq 0}[P^{++} \times P^{++}]$ denote the set of all maps of $P^{++} \times$

$P^{++}$ to $\mathbb{Z}_{\geq 0}$ with finite support. If we think of maps in this set as infinite nonnegative integer matrices with index set $P^{++}$, then we define addition to be matrix addition, multiplication to be matrix multiplication, and involution to be matrix transpose. More precisely, for all $f, g \in \mathbb{Z}_{\geq 0}[P^{++} \times P^{++}]$, we set

$$(f+g)(\lambda,\mu) := f(\lambda,\mu) + g(\lambda,\mu) \text{ for all } \lambda,\mu \in P^{++}$$
$$(f \cdot g)(\lambda,\mu) := \sum_{\nu \in P^{++}} f(\lambda,\nu) g(\nu,\mu) \text{ for all } \lambda,\mu \in P^{++}$$
$$f^*(\lambda,\mu) := f(\mu,\lambda) \text{ for all } \lambda,\mu \in P^{++}.$$

Let 0 denote the zero map of $P^{++} \times P^{++}$ to $\mathbb{Z}_{\geq 0}$. Then

$$\mathbb{Z}_{\geq 0}[P^{++} \times P^{++}] \equiv (\mathbb{Z}_{\geq 0}[P^{++} \times P^{++}], +, 0, \cdot, \mathcal{T}_\rho, *)$$

is an involutive-semiring.

**Theorem 3.3**: The set $\mathrm{TCh}(\mathfrak{g})$ is a subinvolutive-semiring of $\mathbb{Z}_{\geq 0}[P^{++} \times P^{++}]$. Moreover, the map

$$\pi_T : \mathrm{Rep}(\mathfrak{g}) \to \mathrm{TCh}(\mathfrak{g})$$
$$\pi_T(R) := \mathcal{T}_R \text{ for all } R \in \mathrm{Rep}(\mathfrak{g})$$

is an involutive-semiring isomorphism.

This is a nontrivial result. Essentially the theorem says that the tensor characters of $\mathfrak{g}$ encode the involutive-semiring structure of $\mathrm{Rep}(\mathfrak{g})$.

As an aside, we mention that tensor characters also exist for a finite group. In fact, if $G$ is a finite group, then one can form in an analogous manner the tensor character involutive-semiring $\mathrm{TCh}(G)$. Moreover, if $\mathrm{Rep}(G)$ denotes the representation involutive-semiring of $G$, then there is a natural isomorphism of $\mathrm{TCh}(G)$ to $\mathrm{Rep}(G)$. The usual trace characters are the simultaneous eigenvalues of these tensor characters.

For example, in the case $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$, we have

$$\mathcal{T}_\lambda(\mu,\nu) = \left\{ \begin{matrix} 1 & \begin{matrix} \text{if } \lambda+\mu-\nu \in 2\mathbb{Z}+1 \\ \quad \text{and } |\mu-\lambda|+1 \leq \nu \leq \mu+\lambda-1 \end{matrix} \\ 0 & \text{otherwise.} \end{matrix} \right\} \text{ for all } \lambda,\mu,\nu \in \mathbb{Z}_{\geq 1}.$$

From this one may derive the product formula

$$R_\lambda \cdot R_\mu = R_{|\mu-\lambda|+1} + R_{|\mu-\lambda|+3} + \cdots + R_{\mu+\lambda-3} + R_{\mu+\lambda-1} \text{ for all } \lambda,\mu \in \mathbb{Z}_{\geq 1}.$$

## 3.3 Multiplicity Characters

Multiplicity characters are the second type of character that we consider. Let $R \in \text{Rep}(\mathfrak{g})$. Any representation $V \in R$ has a weight space decomposition

$$V = \bigoplus_{\lambda \in P} V(\lambda).$$

The dimensions of these weight spaces are independent of the choice of $V \in R$. Define a map

$$\begin{aligned}&\mathcal{M}_R : P \to \mathbb{Z}_{\geq 0}\\&\mathcal{M}_R(\lambda) := \dim(V(\lambda)) \text{ for all } \lambda \in P.\end{aligned}$$

We refer to $\mathcal{M}_R$ as the **multiplicity character** of $R$. We say that $\mathcal{M}_R$ is irreducible if $R$ is irreducible and we use the shorthand notation

$$\mathcal{M}_\lambda := \mathcal{M}_{R_\lambda} \text{ for all } \lambda \in P^{++}.$$

Some properties of the multiplicity characters of $\mathfrak{g}$ are listed below.

**Proposition 3.4**: The multiplicity characters of $\mathfrak{g}$ satisfy

(i) $\mathcal{M}_R(w\lambda) = \mathcal{M}_R(\lambda)$ for all $R \in \text{Rep}(\mathfrak{g}), w \in W, \lambda \in P$,

(ii) $\mathcal{M}_{R^*}(\lambda) = \mathcal{M}_R(-\lambda)$ for all $R \in \text{Rep}(\mathfrak{g}), \lambda \in P$,

(iii) $\mathcal{M}_{\rho+\lambda}(\lambda) = 1$ for all $\lambda \in P^+$,

(iv) $\mathcal{M}_{\rho+\lambda}(\mu) = 0$ for all $\lambda \in P^+, \mu \in P$ such that $\lambda - \mu \notin Q^+$,

(v) $\mathcal{M}_\rho(\lambda) = \left\{\begin{matrix} 1 & \text{if } \lambda = 0 \\ 0 & \text{otherwise} \end{matrix}\right\}$ for all $\lambda \in P$.

Part (i) of the above proposition shows that the map $\mathcal{M}_R$ is uniquely defined by its restriction to the set $P^+$. Let $\text{MCh}(\mathfrak{g})$ denote the set of all multiplicity characters of $\mathfrak{g}$. It turns out that we can define addition, multiplication, and involution of multiplicity characters so that the set $\text{MCh}(\mathfrak{g})$ has the structure of an involutive-semiring. We refer to $\text{MCh}(\mathfrak{g})$ with this structure as the **multiplicity character involutive-semiring** of $\mathfrak{g}$.

As with tensor characters, we derive the involutive-semiring structure of $\text{MCh}(\mathfrak{g})$ from a larger involutive-semiring. Let $\mathbb{Z}[P]$ denote the set of all maps of $P$ to $\mathbb{Z}$

with finite support. Define addition in $\mathbb{Z}[P]$ to be pointwise and multiplication to be convolution. More precisely, for all $f, g \in \mathbb{Z}[P]$, set

$$(f+g)(\lambda) := f(\lambda) + g(\lambda) \text{ for all } \lambda \in P$$
$$(f \cdot g)(\lambda) := \sum_{\mu \in P} f(\lambda - \mu) g(\mu) \text{ for all } \lambda \in P$$

Let 0 denote the zero map of $P$ to $\mathbb{Z}$. Define involution so that

$$f^*(\lambda) := f(-\lambda) \text{ for all } \lambda \in P.$$

Then $\mathbb{Z}[P] \equiv (\mathbb{Z}[P], +, 0, \cdot, \mathcal{M}_\rho, *)$ is an involutive-semiring.

**Theorem 3.5**: The set $\mathrm{MCh}(\mathfrak{g})$ is a subinvolutive-semiring of $\mathbb{Z}[P]$. Moreover, the map

$$\pi_M : \mathrm{Rep}(\mathfrak{g}) \to \mathrm{MCh}(\mathfrak{g})$$
$$\pi_M(R) := \mathcal{M}_R \text{ for all } R \in \mathrm{Rep}(\mathfrak{g})$$

is an involutive-semiring isomorphism.

Thus we see that the multiplicity characters of $\mathfrak{g}$ also encode the involutive-semiring structure of $\mathrm{Rep}(\mathfrak{g})$. In fact, the multiplicity characters and tensor characters of $\mathfrak{g}$ are related by the following formula.

**Theorem 3.6** (Racah-Speiser Formula): The multiplicity and tensor characters of $\mathfrak{g}$ are related by the formula

$$\mathcal{T}_R(\lambda, \mu) = \sum_{w \in W} \mathrm{sgn}(w) \mathcal{M}_R(\mu - w\lambda) \text{ for all } \lambda, \mu \in P^{++}.$$

For example, in the case $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$, we have

$$\mathcal{M}_\lambda(\mu) = \left\{ \begin{matrix} 1 & \begin{matrix} \text{if } \lambda - \mu \in 2\mathbb{Z}+1 \\ \text{and } -\lambda + 1 \leq \mu \leq \lambda - 1 \end{matrix} \\ 0 & \text{otherwise.} \end{matrix} \right\} \text{ for all } \lambda \in \mathbb{Z}_{\geq 1}, \mu \in \mathbb{Z}.$$

## 3.4 Virtual Isomorphism Classes

We can express the Racah-Speiser formula more concisely if we introduce virtual isomorphism classes and virtual multiplicity characters. Let $\lambda \in P$. Define the notations

$$\begin{aligned} R_\lambda &:= \eta(\lambda) R_{\zeta(\lambda)} \in \mathrm{Rep}(\mathfrak{g})^{\mathbb{Z}}, \\ \mathcal{M}_\lambda &:= \eta(\lambda) \mathcal{M}_{\zeta(\lambda)} \in \mathrm{MCh}(\mathfrak{g})^{\mathbb{Z}} \subseteq \mathbb{Z}[P]. \end{aligned}$$

We refer to $R_\lambda$ (respectively $\mathcal{M}_\lambda$) as the **virtual isomorphism class** (respectively **virtual multiplicity character**) of $\mathfrak{g}$ of weight $\lambda$. The $\mathbb{Z}$-algebra isomorphism ${\pi_M}^{\mathbb{Z}}$ maps $R_\lambda$ to $\mathcal{M}_\lambda$.

**Theorem 3.7**: Multiplication in $\mathrm{Rep}(\mathfrak{g})^{\mathbb{Z}}$ may be expressed in terms of virtual multiplicity characters through the formula

$$R_\lambda \cdot R_\mu = \sum_{\nu \in P} \mathcal{M}_\lambda(\nu) R_{\mu+\nu} \text{ for all } \lambda, \mu \in P.$$

*Proof.* Suppose $\lambda, \mu \in P^{++}$. Then

$$\begin{aligned} R_\lambda \cdot R_\mu &= \sum_{\nu \in P^{++}} \mathcal{T}_\lambda(\mu, \nu) R_\nu \\ &= \sum_{\nu \in P^{++}} \sum_{w \in W} \mathrm{sgn}(w) \mathcal{M}_\lambda(\nu - w\mu) R_\nu \\ &= \sum_{\nu \in P^{++}} \sum_{w \in W} \mathrm{sgn}(w) \mathcal{M}_\lambda(w\nu - \mu) R_\nu \\ &= \sum_{\nu \in P^{++}} \sum_{\substack{\xi \in P \\ \zeta(\xi) = \nu}} \eta(\xi) \mathcal{M}_\lambda(\xi - \mu) R_\nu \\ &= \sum_{\xi \in P} \mathcal{M}_\lambda(\xi - \mu) \eta(\xi) R_{\zeta(\xi)} \\ &= \sum_{\xi \in P} \mathcal{M}_\lambda(\xi - \mu) R_\xi \\ &= \sum_{\xi \in P} \mathcal{M}_\lambda(\xi) R_{\mu+\xi}. \end{aligned}$$

For arbitrary $\lambda \in P$, we have

$$\begin{aligned} R_\lambda \cdot R_\mu &= \eta(\lambda) R_{\zeta(\lambda)} \cdot R_\mu \\ &= \eta(\lambda) \sum_{\xi \in P} \mathcal{M}_{\zeta(\lambda)}(\xi) R_{\mu+\xi} \\ &= \sum_{\xi \in P} \eta(\lambda) \mathcal{M}_{\zeta(\lambda)}(\xi) R_{\mu+\xi} \end{aligned}$$

$$= \sum_{\xi \in P} \mathcal{M}_\lambda(\xi) R_{\mu+\xi}.$$

By commutativity, the result also holds when $\mu \in P$ is arbitrary and $\lambda$ is restricted to $P^{++}$. However, we can apply the same argument as above to extend this to arbitrary $\lambda \in P$. □

We can also use virtual isomorphism classes to concisely express the Weyl character formula. Let $\lambda \in P$. Define a map

$$\delta_\lambda : P \to \mathbb{Z}$$
$$\delta_\lambda(\mu) = \left.\begin{cases} \eta(\lambda)\eta(\mu) & \text{if } \zeta(\lambda) = \zeta(\mu) \\ 0 & \text{otherwise} \end{cases}\right\} \text{ for all } \mu \in P.$$

We refer to $\delta_\lambda$ as the **delta map** of $\mathfrak{g}$ of weight $\lambda$.

**Proposition 3.8**: The delta maps of $\mathfrak{g}$ satisfy

(i) $\delta_\lambda(w\mu) = \text{sgn}(w)\delta_\lambda(\mu)$ for all $w \in W; \lambda, \mu \in P$,

(ii) $\delta_\lambda(\mu) = \delta_\mu(\lambda)$ for all $\lambda, \mu \in P$.

The following is a well-known consequence of the Racah-Speiser formula.

**Theorem 3.9** (Weyl Character Formula): The virtual multiplicity characters of $\mathfrak{g}$ are related to the delta maps by the formula

$$\mathcal{M}_\lambda \cdot \delta_\rho = \delta_\lambda \text{ for all } \lambda \in P.$$

*Proof.* First suppose $\lambda, \mu \in P^{++}$. Then

$$\begin{aligned}
(\mathcal{M}_\lambda \cdot \delta_\rho)(\mu) &= \sum_{\nu \in P} \mathcal{M}_\lambda(\mu - \nu)\delta_\rho(\nu) \\
&= \sum_{w \in W} \text{sgn}(w)\mathcal{M}_\lambda(\mu - w\rho) \\
&= \mathcal{T}_\lambda(\rho, \mu) \\
&= \mathcal{T}_\rho(\lambda, \mu) \\
&= \begin{cases} 1 & \text{if } \lambda = \mu \\ 0 & \text{otherwise} \end{cases} \\
&= \delta_\lambda(\mu).
\end{aligned}$$

For arbitrary $\lambda \in P$, we have

$$\begin{aligned}(\mathcal{M}_\lambda \cdot \delta_\rho)(\mu) &= \left((\eta(\lambda)\mathcal{M}_{\zeta(\lambda)}) \cdot \delta_\rho\right)(\mu) \\ &= \eta(\lambda)(\mathcal{M}_{\zeta(\lambda)} \cdot \delta_\rho)(\mu) \\ &= \eta(\lambda)\delta_{\zeta(\lambda)}(\mu) \\ &= \delta_\lambda(\mu).\end{aligned}$$

Since $\delta_\lambda(\mu) = \delta_\mu(\lambda)$, the result also holds when $\mu \in P$ is arbitrary but $\lambda$ is restricted to $P^{++}$. However, we can apply the same argument as above to extend this to arbitrary $\lambda \in P$. □

For example, in the case $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$, we have

$$\begin{aligned}R_{-\lambda} &= -R_\lambda \in \mathrm{Rep}(A_1)^{\mathbb{Z}} \text{ for all } \lambda \in \mathbb{Z}_{\geq 1} \\ R_0 &= 0 \in \mathrm{Rep}(A_1)^{\mathbb{Z}}.\end{aligned}$$

With this notation, Theorem 3.7 says that

$$R_\lambda \cdot R_\mu = R_{\lambda-\mu+1} + R_{\lambda-\mu+3} + \cdots + R_{\lambda+\mu-3} + R_{\lambda+\mu-1} \quad \text{for all } \lambda \in \mathbb{Z}_{\geq 1}, \mu \in \mathbb{Z}.$$

By extending the isomorphism $\pi_M$ to the integerizations, we also get

$$\mathcal{M}_\lambda \cdot \mathcal{M}_\mu = \mathcal{M}_{\lambda-\mu+1} + \mathcal{M}_{\lambda-\mu+3} + \cdots + \mathcal{M}_{\lambda+\mu-3} + \mathcal{M}_{\lambda+\mu-1} \quad \text{for all } \lambda \in \mathbb{Z}_{\geq 1}, \mu \in \mathbb{Z}.$$

The Weyl character formula says that

$$\mathcal{M}_\lambda(\mu+1) - \mathcal{M}_\lambda(\mu-1) = \delta_\lambda(\mu) \text{ for all } \lambda, \mu \in \mathbb{Z}.$$

## 3.5 Tilde Isomorphism Classes

A central part of this thesis is the introduction of tilde isomorphism classes. Let $\lambda \in P$. Define the shorthand notations

$$\begin{aligned}\widetilde{R}_\lambda &:= \sum_{w \in W} R_{\rho + w\lambda} \in \mathrm{Rep}(\mathfrak{g})^{\mathbb{Z}}, \\ \widetilde{\mathcal{M}}_\lambda &:= \sum_{w \in W} \mathcal{M}_{\rho + w\lambda} \in \mathrm{MCh}(\mathfrak{g})^{\mathbb{Z}} \subseteq \mathbb{Z}[P].\end{aligned}$$

We refer to $\widetilde{R}_\lambda$ (respectively $\widetilde{\mathcal{M}}_\lambda$) as the **tilde isomorphism class** (respectively **tilde multiplicity character**) of $\mathfrak{g}$ of weight $\lambda$. The $\mathbb{Z}$-algebra isomorphism ${\pi_M}^{\mathbb{Z}}$ maps $\widetilde{R}_\lambda$ to $\widetilde{\mathcal{M}}_\lambda$.

Some properties of tilde isomorphism classes and tilde multiplicity characters are listed below.

**Proposition 3.10**: The tilde isomorphism classes and tilde multiplicity characters of $\mathfrak{g}$ satisfy

(i) $\widetilde{R}_{w\lambda} = \widetilde{R}_\lambda$ for all $w \in W, \lambda \in P$,

(ii) $\widetilde{\mathcal{M}}_{w\lambda} = \widetilde{\mathcal{M}}_\lambda$ for all $w \in W, \lambda \in P$,

(iii) $\widetilde{\mathcal{M}}_\lambda(w\mu) = \widetilde{\mathcal{M}}_\lambda(\mu)$ for all $w \in W; \lambda, \mu \in P$,

(iv) $\widetilde{\mathcal{M}}_\lambda(\lambda) = |W_\lambda|$ for all $\lambda \in P^+$,

(v) $\widetilde{\mathcal{M}}_\lambda(\mu) = 0$ for all $\lambda \in P^+, \mu \in P$ such that $\lambda - \mu \notin Q^+$,

(vi) $\widetilde{\mathcal{M}}_0(\lambda) = \left\{\begin{matrix} |W| & \text{if } \lambda = 0 \\ 0 & \text{otherwise} \end{matrix}\right\}$ for all $\lambda \in P$.

*Proof.* We will only prove Parts (iv) and (v). Let $\lambda \in P^+$. Then

$$\lambda - w\lambda \in Q^+ \text{ for all } w \in W.$$

In particular, we have

$$w\lambda - w \notin Q^+ \text{ for all } w \in W \text{ such that } w\lambda \neq \lambda.$$

Thus

$$\begin{aligned}
\widetilde{\mathcal{M}}_\lambda(\lambda) &= \sum_{w \in W} \mathcal{M}_{\rho + w\lambda}(\lambda) \\
&= \sum_{\substack{w \in W \\ w\lambda = \lambda}} \mathcal{M}_{\rho+\lambda}(\lambda) \\
&= |W_\lambda| \mathcal{M}_{\rho+\lambda}(\lambda) \\
&= |W_\lambda|.
\end{aligned}$$

For Part (v), let $\lambda \in P^+, \mu \in P$ and suppose $\lambda - \mu \notin Q^+$. Let $w \in W$. Then

$$\lambda - w\lambda = \alpha \in Q^+.$$

Thus

$$\begin{aligned}\lambda - \mu \notin Q^+ &\Rightarrow \alpha - w\lambda - \mu \notin Q^+ \\ &\Rightarrow w\lambda - \mu \notin Q^+ \\ &\Rightarrow \mathcal{M}_{\rho+w\lambda}(\mu) = 0 \\ &\Rightarrow \sum_{w\in W} \mathcal{M}_{\rho+w\lambda}(\mu) = 0 \\ &\Rightarrow \widetilde{\mathcal{M}}_\lambda(\mu) = 0.\end{aligned}$$

□

The following conjecture argues that tilde multiplicity characters have a particular simple form.

**Conjecture 3.11**: The tilde multiplicity characters of $\mathfrak{g}$ satisfy

$$\widetilde{\mathcal{M}}_\lambda(\mu) = \left\{\begin{matrix} |W_\lambda| & \text{if } \zeta(\lambda) = \zeta(\mu) \\ 0 & \text{otherwise} \end{matrix}\right\} \text{ for all } \lambda, \mu \in P.$$

The following proposition shows that this conjecture is at least reasonable.

**Proposition 3.12**: Let $\lambda \in P$. Then

$$\widetilde{R}_\lambda = \sum_{\mu\in P} \widetilde{\mathcal{M}}_\lambda(\mu) R_{\rho+\mu}.$$

*Proof.* First note that, for arbitrary $\xi \in P$, we have

$$\begin{aligned}R_\xi &= R_\xi \cdot R_\rho \\ &= \sum_{\mu\in P} \mathcal{M}_\xi(\mu) R_{\rho+\mu}.\end{aligned}$$

Inserting this into the definition of $\widetilde{R}_\lambda$, we get

$$\begin{aligned}\widetilde{R}_\lambda &= \sum_{w\in W} R_{\rho+w\lambda} \\ &= \sum_{w\in W}\sum_{\mu\in P} \mathcal{M}_{\rho+w\lambda}(\mu) R_{\rho+\mu} \\ &= \sum_{\mu\in P}\sum_{w\in W} \mathcal{M}_{\rho+w\lambda}(\mu) R_{\rho+\mu}\end{aligned}$$

$$= \sum_{\mu \in P} \widetilde{\mathcal{M}}_\lambda(\mu) R_{\rho+\mu}.$$

□

If Conjecture 3.11 is true, then we have nice expressions for the products of tilde isomorphism classes and tilde multiplicity characters.

**Theorem 3.13**: We have the product formula

$$\widetilde{R}_\lambda \cdot R_\mu = \sum_{\nu \in P} \widetilde{\mathcal{M}}_\lambda(\nu) R_{\mu+\nu} \text{ for all } \lambda, \mu \in P.$$

If Conjecture 3.11 holds, then this simplifies to

$$\widetilde{R}_\lambda \cdot R_\mu = \sum_{w \in W} R_{\mu+w\lambda} \text{ for all } \lambda, \mu \in P.$$

*Proof.* Let $\lambda, \mu \in P$. Consider

$$\begin{aligned}
\widetilde{R}_\lambda \cdot R_\mu &= \sum_{w \in W} R_{\rho+w\lambda} \cdot R_\mu \\
&= \sum_{w \in W} \sum_{\nu \in P} \mathcal{M}_{\rho+w\lambda}(\nu) R_{\mu+\nu} \\
&= \sum_{\nu \in P} \sum_{w \in W} \mathcal{M}_{\rho+w\lambda}(\nu) R_{\mu+\nu} \\
&= \sum_{\nu \in P} \widetilde{\mathcal{M}}_\lambda(\nu) R_{\mu+\nu}.
\end{aligned}$$

□

**Theorem 3.14**: We have the product formula

$$\widetilde{R}_\lambda \cdot \widetilde{R}_\mu = \sum_{\nu \in P} \widetilde{\mathcal{M}}_\lambda(\mu) \widetilde{R}_{\mu+\nu} \text{ for all } \lambda, \mu \in P.$$

If Conjecture 3.11 holds, then this simplifies to

$$\widetilde{R}_\lambda \cdot \widetilde{R}_\mu = \sum_{w \in W} \widetilde{R}_{\mu+w\lambda} \text{ for all } \lambda, \mu \in P.$$

*Proof.* Let $\lambda, \mu \in P$. Consider

$$\begin{aligned}
\widetilde{R}_\lambda \cdot R_\mu &= \widetilde{R}_\lambda \cdot \sum_{w \in W} R_{\rho + w\mu} \\
&= \sum_{w \in W} \widetilde{R}_\lambda \cdot R_{\rho + w\mu} \\
&= \sum_{w \in W} \sum_{\nu \in P} \widetilde{\mathcal{M}}_\lambda(\nu) R_{\rho + w\mu + \nu} \\
&= \sum_{w \in W} \sum_{\nu \in P} \widetilde{\mathcal{M}}_\lambda(w\nu) R_{\rho + w\mu + w\nu} \\
&= \sum_{w \in W} \sum_{\nu \in P} \widetilde{\mathcal{M}}_\lambda(\nu) R_{\rho + w(\mu + \nu)} \\
&= \sum_{\nu \in P} \widetilde{\mathcal{M}}_\lambda(\nu) \sum_{w \in W} \widetilde{R}_{\rho + w(\mu + \nu)} \\
&= \sum_{\nu \in P} \widetilde{\mathcal{M}}_\lambda(\nu) \widetilde{R}_{\mu + \nu}.
\end{aligned}$$

□

In the case of $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$, we can explicitly verify Conjecture 3.11:

$$\widetilde{M}_\lambda(\mu) = \left\{ \begin{matrix} 2 & \text{if } \mu = \lambda = 0 \\ 1 & \text{if } \mu = \pm\lambda \neq 0 \\ 0 & \text{otherwise} \end{matrix} \right\} \text{ for all } \lambda, \mu \in \mathbb{Z}.$$

This leads to the simple formulas

$$\begin{aligned}
\widetilde{R}_\lambda &= R_{\lambda+1} - R_{\lambda-1} \in \mathrm{Rep}(A_1)^{\mathbb{Z}} \text{ for all } \lambda \in \mathbb{Z} \\
\widetilde{R}_\lambda \cdot R_\mu &= R_{\mu-\lambda} + R_{\mu+\lambda} \text{ for all } \lambda, \mu \in \mathbb{Z} \\
\widetilde{R}_\lambda \cdot \widetilde{R}_\mu &= \widetilde{R}_{\mu-\lambda} + \widetilde{R}_{\mu+\lambda} \text{ for all } \lambda, \mu \in \mathbb{Z}.
\end{aligned}$$

## 3.6 Exponent Characters

Exponent characters are the third type of character that we consider. Exponent characters are essentially complex versions of the multiplicity characters. They are important because, as we will see in the next chapter, slices of the exponents characters at some fixed input form the one-dimensional representations of $\mathrm{Rep}(\mathfrak{g})^{\mathbb{C}}$.

Let $P^{\mathbb{C}}$ denote the complexification of the weight lattice of $\mathfrak{g}$. Let $R \in \mathrm{Rep}(\mathfrak{g})$. Define a map

$$\begin{aligned}&\mathcal{E}_R : P^{\mathbb{C}} \to \mathbb{C}\\&\mathcal{E}_R(\lambda) := \sum_{\mu \in P} \mathcal{M}_R(\mu) \exp(\langle \lambda, \mu \rangle) \text{ for all } \lambda \in P^{\mathbb{C}}.\end{aligned}$$

We refer to $\mathcal{E}_R$ as the **exponent character** of $R$. We say that $\mathcal{E}_R$ is irreducible if $R$ is irreducible we use the shorthand notation

$$\mathcal{E}_\lambda := \mathcal{E}_{R_\lambda} \text{ for all } \lambda \in P^{++}.$$

Some properties of the exponent characters of $\mathfrak{g}$ are listed below.

**Proposition 3.15**: The exponent characters of $\mathfrak{g}$ satisfy

(i) $\mathcal{E}_R(w\lambda) = \mathcal{E}_R(\lambda)$ for all $R \in \mathrm{Rep}(\mathfrak{g}), w \in W, \lambda \in P^{\mathbb{C}}$,

(ii) $\mathcal{E}_{R^*}(\lambda) = \mathcal{E}_R(-\lambda) = \overline{\mathcal{E}_R(\lambda)}$ for all $R \in \mathrm{Rep}(\mathfrak{g}), \lambda \in P^{\mathbb{C}}$,

(iii) $\mathcal{E}_\rho(\lambda) = 1$ for all $\lambda \in P^{\mathbb{C}}$,

(iv) $\mathcal{E}_\lambda(0) = \dim(V_\lambda)$ for all $\lambda \in P$.

Let $\mathrm{ECh}(\mathfrak{g})$ denote the set of all exponent characters of $\mathfrak{g}$. It turns out that we can define addition, multiplication, and involution of exponent characters so that the set $\mathrm{ECh}(\mathfrak{g})$ has the structure of an involutive-semiring. We refer to $\mathrm{ECh}(\mathfrak{g})$ with this structure as the **exponent character involutive-semiring** of $\mathfrak{g}$.

Let $\mathbb{C}[P^{\mathbb{C}}]$ denote the set of all maps of $P^{\mathbb{C}}$ to $\mathbb{C}$. Here we view the set $\mathbb{C}$ as an involutive-semiring with involution as complex conjugation. Define addition, multiplication, and involution in $\mathbb{C}[P^{\mathbb{C}}]$ to be pointwise. Let 0 denote the zero map of $P^{\mathbb{C}}$ to $\mathbb{C}$. Then $\mathbb{C}[P^{\mathbb{C}}] \equiv (\mathbb{C}[P^{\mathbb{C}}], +, 0, \cdot, \mathcal{E}_\rho, *)$ is an involutive-semiring.

**Theorem 3.16**: The set $\mathrm{ECh}(\mathfrak{g})$ is a subinvolutive-semiring of $\mathbb{C}[P^{\mathbb{C}}]$. Moreover the map

$$\begin{aligned}&\pi_E : \mathrm{Rep}(\mathfrak{g}) \to \mathrm{ECh}(\mathfrak{g})\\&\pi_E(R) := \mathcal{E}_R \text{ for all } R \in \mathrm{Rep}(\mathfrak{g})\end{aligned}$$

is an involutive-semiring isomorphism.

# Chapter 4

# The Involutive-Semiring $\mathrm{Rep}_k(\mathfrak{g})$

Let $\mathfrak{g}$ be a finite-dimensional semisimple complex Lie algebra. In the previous chapter, we studied the free commutative involutive-semiring $\mathrm{Rep}(\mathfrak{g})$. In this chapter, we consider certain subinvolutive-monoids of $\mathrm{Rep}(\mathfrak{g})$ with multiplication defined by a projection of the product in $\mathrm{Rep}(\mathfrak{g})$. This projection depends on an affine Weyl group of $\mathfrak{g}$.

## 4.1 Affine Weyl Group

Let $k$ be a positive integer and let $Q$ denote the root lattice of $\mathfrak{g}$. The **affine Weyl group** of $\mathfrak{g}$ at level $k$ is the semidirect product $W_k := kQ \rtimes W$ with multiplication defined so that

$$(\alpha, w)(\beta, x) := (w\alpha + \beta, wx) \text{ for all } (\alpha, w), (\beta, x) \in W_k.$$

The group $W_k$ acts on the weight lattice $P$ by the affine transformation

$$(\alpha, w)\lambda := w\lambda + \alpha \text{ for all } \lambda \in P, (\alpha, w) \in W_k.$$

We always have the subgroup inclusion

$$\begin{aligned} &W \hookrightarrow W_k \\ &w \mapsto (0, w) \text{ for all } w \in W. \end{aligned}$$

We can extend the notion of sign to $W_k$ if we set

$$\mathrm{sgn}(\alpha, w) := \mathrm{sgn}(w) \text{ for all } (\alpha, w) \in W_k.$$

Recall that the finite Weyl group of $\mathfrak{g}$ acts freely and transitively on the finite set of infinite-volume Weyl chambers. By contrast, the affine Weyl group of $\mathfrak{g}$ at level $k$ acts freely and transitively on the infinite set of finite-volume affine Weyl chambers.

There is exactly one affine Weyl chamber that lies inside the dominant Weyl chamber and intersects zero on its boundary. Label this affine Weyl chamber as dominant. Let ${P_k}^+$ (respectively ${P_k}^{++}$) denote the intersection of $P$ with the dominant affine Weyl chamber union its boundary (respectively the dominant affine Weyl chamber). Then ${P_k}^+$ is a set of representatives for the orbits of the weight lattice under the action of $W_k$ and ${P_k}^{++}$ is the subset of ${P_k}^+$ consisting of weights with trivial stabilizer.

We can describe the sets ${P_k}^+$ and ${P_k}^{++}$ more explicitly in terms of fundamental weights. Recall the direct sum decomposition

$$P \cong \bigoplus_{i=1}^{r} \mathbb{Z}\Lambda_i.$$

Let $a_1, \ldots, a_r$ denote the Coxeter labels of $\mathfrak{g}$. For any

$$\lambda = \sum_{i=1}^{r} \lambda^i \Lambda_i \in P,$$

we introduce the zeroth coordinate

$$\lambda^0 := k - \sum_{i=1}^{r} a_i \lambda^i \in \mathbb{Z}.$$

Then we have

$$\begin{aligned} {P_k}^{++} &= \left\{ \sum_{i=1}^{r} \lambda^i \Lambda_i \in P : \lambda^0 \geq 1, \lambda^1 \geq 1, \ldots, \lambda^r \geq 1 \right\} \\ {P_k}^{+} &= \left\{ \sum_{i=1}^{r} \lambda^i \Lambda_i \in P : \lambda^0 \geq 0, \lambda^1 \geq 0, \ldots, \lambda^r \geq 0 \right\}. \end{aligned}$$

Note that ${P_k}^{++}$ contains $\rho$ and is nonempty if and only if $k \geq h$, where $h$ is the Coxeter number of $\mathfrak{g}$. In terms of the Coxeter labels, we have

$$h = 1 + \sum_{i=1}^{r} a_i.$$

We will always assume that $k \geq h$ so that $\mathrm{Rep}_k(\mathfrak{g})$ is nonzero.

As with the case for finite Weyl groups, the action of the affine Weyl group on the weight lattice can be encoded using two special maps. We use the notations

$$\zeta_k : P \to P_k{}^+$$
$$\eta_k : P \to \{-1, 0, 1\}$$

and define these maps as follows. Let $\lambda \in P$. Since $P_k{}^+$ is a set of representatives for the orbits of $P$ under the action of $W_k$, there is exactly one weight in the intersection of $P_k{}^+$ and the orbit of $\lambda$. Define $\zeta_k(\lambda)$ to be this weight. If there is a unique $(\alpha, w) \in W_k$ such that $(\alpha, w)\lambda = \mu$, then set $\eta_k(\lambda) = \mathrm{sgn}(w)$. Otherwise set $\eta_k(\lambda) = 0$.

We refer to $\zeta_k$ (respectively $\eta_k$) as the **affine zeta map** (respectively **affine eta map**) of $\mathfrak{g}$ at level $k$. Some properties of the affine zeta and eta maps of $\mathfrak{g}$ are listed below.

**Proposition 4.1**: The affine zeta and eta maps of $\mathfrak{g}$ at level $k$ satisfy

(i) $\zeta_k((\alpha, w)\lambda) = \zeta_k(\lambda)$ for all $(\alpha, w) \in W_k, \lambda \in P$,

(ii) $\eta_k((\alpha, w)\lambda) = \mathrm{sgn}(w)\eta_k(\lambda)$ for all $(\alpha, w) \in W_k, \lambda \in P$,

(iii) $\eta_k(\lambda) = 0 \Leftrightarrow \zeta_k(\lambda) \in P_k{}^+ \setminus P_k{}^{++}$ for all $\lambda \in P$.

Like their finite counterparts, the affine zeta and eta maps of $\mathfrak{g}$ at level $k$ have efficient computer implementations. In fact, the same algorithm that we described for the zeta and eta maps of $\mathfrak{g}$ works for the affine variants if we include the affine reflection $s_0$ about the hyperplane $\lambda^0 = 0$.

One can verify that the set $P_k{}^{++}$ is closed under the involution map. Thus

$$\lambda^* = \zeta(-\lambda) = \zeta_k(-\lambda) \text{ for all } \lambda \in P_k{}^{++}.$$

For a simple example, consider the case $g = \mathfrak{sl}_2(\mathbb{C})$:

$$\begin{aligned} Q &= 2\mathbb{Z} \\ W_k &= 2k\mathbb{Z} \rtimes \{\pm 1\} \\ P_k{}^+ &= \{0, 1, \ldots, k\} \\ P_k{}^{++} &= \{1, 2, \ldots, k-1\} \end{aligned}$$

$$\zeta_k(\lambda) = \begin{cases} \mu & \text{if } \lambda \in \mu + 2k\mathbb{Z} \text{ for some } \mu \in {P_k}^{++} \\ 0 & \text{if } \lambda \in k\mathbb{Z} \\ -\mu & \text{if } \lambda \in -\mu + 2k\mathbb{Z} \text{ for some } \mu \in {P_k}^{++} \end{cases} \text{ for all } \lambda \in P$$

$$\eta_k(\lambda) = \begin{cases} 1 & \text{if } \lambda \in \mu + 2k\mathbb{Z} \text{ for some } \mu \in {P_k}^{++} \\ 0 & \text{if } \lambda \in k\mathbb{Z} \\ -1 & \text{if } \lambda \in -\mu + 2k\mathbb{Z} \text{ for some } \mu \in {P_k}^{++} \end{cases} \text{ for all } \lambda \in P.$$

In the case $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$, we have

$$\begin{aligned} Q &= \{\lambda = (\lambda^1, \lambda^2) \in \mathbb{Z}^2 : \lambda^1 - \lambda^2 \in 3\mathbb{Z}\} \\ W_k &= kQ \rtimes W \\ \lambda^0 &= k - \lambda^1 - \lambda^2 \text{ for all } \lambda = (\lambda^1, \lambda^2) \in \mathbb{Z}^2 \\ {P_k}^{+} &= \{\lambda = (\lambda^1, \lambda^2) \in \mathbb{Z}^2 : \lambda^0 \geq 0, \lambda^1 \geq 0, \lambda^2 \geq 0\} \\ {P_k}^{++} &= \{\lambda = (\lambda^1, \lambda^2) \in \mathbb{Z}^2 : \lambda^0 \geq 1, \lambda^1 \geq 1, \lambda^2 \geq 1\}. \end{aligned}$$

## 4.2 Involutive-Semiring Structure

We are now ready to define what we mean by the affine truncation of $\text{Rep}(\mathfrak{g})$ at level $k$. Let $\text{Rep}_k(\mathfrak{g})$ denote the free commutative subinvolutive-monoid of $\text{Rep}(\mathfrak{g})$ with basis $\{R_\lambda : \lambda \in {P_k}^{++}\}$. As a direct sum, we have

$$\text{Rep}_k(\mathfrak{g}) = \bigoplus_{\lambda \in {P_k}^{++}} \mathbb{Z}_{\geq 0} R_\lambda.$$

We have the inclusion

$$\iota : \text{Rep}_k(\mathfrak{g}) \to \text{Rep}(\mathfrak{g}).$$

Define the map

$$\begin{aligned} &\psi_k : \text{Rep}(\mathfrak{g}) \to \text{Rep}_k(\mathfrak{g})^{\mathbb{Z}} \\ &\psi_k(R_\lambda) := \eta_k(\lambda) R_{\zeta_k(\lambda)} \text{ for all } \lambda \in P^{++}. \end{aligned}$$

Define multiplication in $\text{Rep}_k(\mathfrak{g})$ so that

$$R_\lambda \cdot R_\mu := \psi_k(\iota(R_\lambda) \cdot \iota(R_\mu)) \text{ for all } \lambda, \mu \in {P_k}^{++}.$$

Of course, this multiplication is not well-defined unless it just so happens that

$$R_\lambda \cdot R_\mu \in \text{Rep}_k(\mathfrak{g}) \subseteq \text{Rep}_k(\mathfrak{g})^{\mathbb{Z}} \text{ for all } \lambda, \mu \in {P_k}^{++}.$$

This is certainly not obvious. We will not go into details, but one way to show this is to construct $\mathrm{Rep}_k(\mathfrak{g})$ as the set of representations of a quantum universal enveloping algebra of $\mathfrak{g}$ at a root of unity, together with the operations of direct sum, truncated tensor product, and dual. In any case, the involutive-monoid $\mathrm{Rep}_k(\mathfrak{g})$ with the above multiplication is an involutive-semiring, which we refer to as the **affine truncation** of $\mathrm{Rep}(\mathfrak{g})$ at level $k$.

## 4.3 Affine Tensor Characters

Let $R \in \mathrm{Rep}_k(\mathfrak{g})$ and let $\lambda \in {P_k}^{++}$. Since $\mathrm{Rep}_k(\mathfrak{g})$ is a free involutive-semiring, we know that the product $R \cdot R_\lambda$ can be uniquely written (up to reordering) as a finite sum of basis elements. In other words, there is a map

$$(\mathcal{T}_k)_R : {P_k}^{++} \times {P_k}^{++} \to \mathbb{Z}_{\geq 0}$$

such that

$$R \cdot R_\lambda = \sum_{\mu \in {P_k}^{++}} (\mathcal{T}_k)_R(\lambda, \mu) R_\mu \text{ for all } \lambda \in {P_k}^{++}.$$

We refer to the map $(\mathcal{T}_k)_R$ as the **affine tensor character** of $R$ at level $k$. We use the shorthand notation

$$(\mathcal{T}_k)_\lambda := (\mathcal{T}_k)_{R_\lambda} \text{ for all } \lambda \in {P_k}^{++}.$$

Some properties of the affine tensor characters of $\mathfrak{g}$ are listed below.

**Proposition 4.2**: The affine tensor characters of $\mathfrak{g}$ at level $k$ satisfy

(i) $(\mathcal{T}_k)_{R^*}(\lambda, \mu) = (\mathcal{T}_k)_R(\lambda^*, \mu^*)$ for all $R \in \mathrm{Rep}_k(\mathfrak{g}); \lambda, \mu \in {P_k}^{++}$,

(ii) $(\mathcal{T}_k)_{R^*}(\lambda, \mu) = (\mathcal{T}_k)_R(\mu, \lambda)$ for all $R \in \mathrm{Rep}_k(\mathfrak{g}); \lambda, \mu \in {P_k}^{++}$,

(iii) $(\mathcal{T}_k)_R(\lambda, \mu) \leq (\mathcal{T}_{k+1})_R(\lambda, \mu)$ for all $R \in \mathrm{Rep}_k(\mathfrak{g}); \lambda, \mu \in {P_k}^{++}$,

(iv) $\lim_{k \to \infty} (\mathcal{T}_k)_R(\lambda, \mu) = \mathcal{T}_R(\lambda, \mu)$ for all $R \in \mathrm{Rep}(\mathfrak{g}); \lambda, \mu \in P^{++}$,

(v) $(\mathcal{T}_k)_\lambda(\mu, \nu) = (\mathcal{T}_k)_\mu(\lambda, \nu)$ for all $\lambda, \mu, \nu \in {P_k}^{++}$,

(vi) $(\mathcal{T}_k)_\rho(\lambda, \mu) = \begin{Bmatrix} 1 & \text{if } \lambda = \mu \\ 0 & \text{otherwise} \end{Bmatrix}$ for all $\lambda, \mu \in {P_k}^{++}$.

Let $\mathrm{TCh}_k(\mathfrak{g})$ denote the set of all affine tensor characters of $\mathfrak{g}$ at level $k$. It turns out that we can define addition, multiplication, and involution of affine tensor characters so that the set $\mathrm{TCh}_k(\mathfrak{g})$ has the structure of an involutive-semiring. We refer to $\mathrm{TCh}_k(\mathfrak{g})$ with this structure as the **affine tensor character involutive-semiring** of $\mathfrak{g}$ at level $k$.

Let $\mathbb{Z}_{\geq 0}[{P_k}^{++} \times {P_k}^{++}]$ denote the set of all maps of ${P_k}^{++} \times {P_k}^{++}$ to $\mathbb{Z}_{\geq 0}$. If we think of maps in this set as finite nonnegative integer matrices, then we define addition to be matrix addition, multiplication to be matrix multiplication, and involution to be matrix transpose. Let 0 denote the zero map. Then

$$\mathbb{Z}_{\geq 0}[{P_k}^{++} \times {P_k}^{++}] \equiv (\mathbb{Z}_{\geq 0}[{P_k}^{++} \times {P_k}^{++}], +, 0, \cdot, (\mathcal{T}_k)_\rho, *)$$

is an involutive-semiring.

**Theorem 4.3**: The set $\mathrm{TCh}_k(\mathfrak{g})$ is a subinvolutive-semiring of

$$\mathbb{Z}_{\geq 0}[{P_k}^{++} \times {P_k}^{++}].$$

Moreover, the map

$$\begin{aligned} &(\pi_k)_T : \mathrm{Rep}_k(\mathfrak{g}) \to \mathrm{TCh}_k(\mathfrak{g}) \\ &(\pi_k)_T(R) := (\mathcal{T}_k)_R \text{ for all } R \in \mathrm{Rep}_k(\mathfrak{g}) \end{aligned}$$

is an involutive-semiring isomorphism.

The affine tensor characters of $\mathfrak{g}$ at level $k$ may be computed directly from the tensor characters of $\mathfrak{g}$ using the following formula.

**Theorem 4.4** (Kac-Walton Formula): For any $R \in \mathrm{Rep}_k(\mathfrak{g})$, the affine tensor character of $R$ at level $k$ is related to the tensor character of $\iota(R) \equiv R$ by the formula

$$(\mathcal{T}_k)_R(\lambda, \mu) = \sum_{(\alpha,w)\in W_k} \mathrm{sgn}(w)\mathcal{T}_R(\lambda, (\alpha, w)\mu) \text{ for all } \lambda, \mu \in {P_k}^{++}.$$

By combining the Kac-Walton formula with the Racah-Speiser formula, we can also write the affine tensor characters of $\mathfrak{g}$ at level $k$ in terms of the multiplicity characters of $\mathfrak{g}$.

**Theorem 4.5**: For any $R \in \mathrm{Rep}_k(\mathfrak{g})$, the affine tensor character of $R$ at level $k$ is related to the multiplicity character of $\iota(R) \equiv R$ by the formula

$$(\mathcal{T}_k)_R(\lambda, \mu) = \sum_{(\alpha,w)\in W_k} \mathrm{sgn}(w)\mathcal{M}_R(\mu - (\alpha, w)\lambda) \text{ for all } \lambda, \mu \in (P_k)^{++}.$$

For example, in the case of $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$, we have

$$(\mathcal{T}_k)_\lambda(\mu,\nu) = \begin{cases} 1 & \begin{array}{l}\text{if } \lambda+\mu-\nu \in 2\mathbb{Z}+1 \\ \quad\text{and } |\mu-\lambda|+1 \le \nu \le \mu+\lambda-1 \\ \quad\text{and } \nu \le 2k-\mu-\lambda-2\end{array} \\ 0 & \text{otherwise} \end{cases}$$

$$\text{for all } \lambda,\mu,\nu \in \{1,\ldots,k-1\}.$$

From this one may deduce that the product of any two irreducible isomorphism classes has the form

$$R_\lambda \cdot R_\mu = R_{|\mu-\lambda|+1} + R_{|\mu-\lambda|+3} + \cdots + R_{m-3} + R_{m-1}$$
$$m = \min\{\mu+\lambda, 2k-\mu-\lambda-1\} \text{ for all } \lambda,\mu \in \{1,\ldots,k-1\}.$$

## 4.4 Affine Virtual Isomorphism Classes

We can express multiplication in $\mathrm{Rep}_k(\mathfrak{g})$ more concisely in terms of multiplicity characters if we introduce affine virtual isomorphism classes. Let $\lambda \in P$. Define the notation

$$(R_k)_\lambda := \eta_k(\lambda) R_{\zeta_k(\lambda)} \in \mathrm{Rep}_k(\mathfrak{g})^{\mathbb{Z}}.$$

We refer to $(R_k)_\lambda$ as the **affine virtual isomorphism class** of $\mathfrak{g}$ at level $k$ of weight $\lambda$.

**Theorem 4.6**: We have the product formula

$$(R_k)_\lambda \cdot (R_k)_\mu = \sum_{\nu\in P} \mathcal{M}_\lambda(\nu)(R_k)_{\mu+\nu} \text{ for all } \lambda,\mu \in P.$$

There is also an affine variant of the Weyl character formula which may be expressed using an affine variant of the delta map. Let $\lambda \in P$. Define the map

$$(\delta_k)_\lambda : P \to \mathbb{Z}$$
$$(\delta_k)_\lambda(\mu) = \left\{ \begin{array}{ll} \eta_k(\lambda)\eta_k(\mu) & \text{if } \zeta_k(\lambda) = \zeta_k(\mu) \\ 0 & \text{otherwise} \end{array} \right\} \text{ for all } \mu \in P.$$

We refer to $(\delta_k)_\lambda$ as the **affine delta map** of $\mathfrak{g}$ at level $k$.

**Theorem 4.7** (Weyl-Kac Character Formula): The virtual multiplicity characters of $\mathfrak{g}$ are related to the affine delta maps of $\mathfrak{g}$ at level $k$ by the formula

$$\mathcal{M}_\lambda \cdot (\delta_k)_\rho = (\delta_k)_\lambda \text{ for all } \lambda \in P.$$

The proof is similar to the proof of the Weyl character formula.

For example, in the case $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$, we have

$$(R_k)_\lambda \cdot (R_k)_\mu = (R_k)_{\mu-\lambda+1} + (R_k)_{\mu-\lambda+3} + \cdots + (R_k)_{\mu+\lambda-3} + (R_k)_{\mu+\lambda-1} \quad \text{for all } \lambda \in \mathbb{Z}_{\geq 1}, \mu \in \mathbb{Z}.$$

## 4.5 Affine Tilde Isomorphism Classes

Let $\lambda \in P$. Define the notation

$$(\widetilde{R}_k)_\lambda := \sum_{w \in W} (R_k)_{\rho + w\lambda} \in \mathrm{Rep}_k(\mathfrak{g})^{\mathbb{Z}}.$$

We refer to $(\widetilde{R}_k)_\lambda$ as the **affine tilde isomorphism class** of $\mathfrak{g}$ at level $k$ of weight $\lambda$. These affine tilde isomorphism classes have many of the same properties as their finite counterparts.

**Proposition 4.8**: Let $\lambda \in P$. Then

$$(\widetilde{R}_k)_\lambda = \sum_{\mu \in P} \widetilde{\mathcal{M}}_\lambda(\mu)(R_k)_{\rho+\mu}.$$

**Theorem 4.9**: We have the product formula

$$(\widetilde{R}_k)_\lambda \cdot (R_k)_\mu = \sum_{\nu \in P} \widetilde{\mathcal{M}}_\lambda(\nu)(R_k)_{\mu+\nu} \text{ for all } \lambda, \mu \in P.$$

If Conjecture 3.11 holds, then we also have

$$(\widetilde{R}_k)_\lambda \cdot (R_k)_\mu = \sum_{w \in W} (R_k)_{\mu + w\lambda} \text{ for all } \lambda, \mu \in P.$$

**Theorem 4.10**: We have the product formula

$$(\widetilde{R}_k)_\lambda \cdot (\widetilde{R}_k)_\mu = \sum_{\nu \in P} \widetilde{\mathcal{M}}_\lambda(\nu)(\widetilde{R}_k)_{\mu+\nu} \text{ for all } \lambda, \mu \in P.$$

If Conjecture 3.11 holds, then we also have

$$(\widetilde{R}_k)_\lambda \cdot (\widetilde{R}_k)_\mu = \sum_{w\in W} (\widetilde{R}_k)_{\mu+w\lambda} \text{ for all } \lambda, \mu \in P.$$

The proofs of the above propositions and theorems mirror the finite case.

## 4.6 Affine Exponent Characters

We also consider affine exponent characters. These are essentially a special subclass of the exponent characters of $\mathfrak{g}$. Let $R \in \mathrm{Rep}_k(\mathfrak{g})$. Define a map

$$(\mathcal{E}_k)_R : P \to \mathbb{C}$$
$$(\mathcal{E}_k)_R(\lambda) := \sum_{\mu\in P} \mathcal{M}_R(\mu) \exp\left(\frac{2\pi i\langle\lambda,\mu\rangle}{k}\right) \text{ for all } \lambda \in P.$$

We refer to $(\mathcal{E}_k)_R$ as the **affine exponent character** of $R$. We say that $(\mathcal{E}_k)_R$ is irreducible if $R$ is irreducible and we use the shorthand notation

$$(\mathcal{E}_k)_\lambda := (\mathcal{E}_k)_{R_\lambda} \text{ for all } \lambda \in {P_k}^{++}.$$

Some properties of affine exponent characters are listed below.

**Proposition 4.11**: The affine exponent characters of $\mathfrak{g}$ at level $k$ satisfy

(i) $(\mathcal{E}_k)_R((\alpha, w)\lambda) = (\mathcal{E}_k)_R(\lambda)$ for all $R \in \mathrm{Rep}_k(\mathfrak{g}), (\alpha, w) \in W_k, \lambda \in P$,

(ii) $(\mathcal{E}_k)_{R^*}(\lambda) = (\mathcal{E}_k)_R(-\lambda) = \overline{(\mathcal{E}_k)_R(\lambda)}$ for all $R \in \mathrm{Rep}_k(\mathfrak{g}), \lambda \in P$,

(iii) $(\mathcal{E}_k)_\rho(\lambda) = 1$ for all $\lambda \in P$.

Let $\mathrm{ECh}_k(\mathfrak{g})$ denote the set of all affine exponent characters of $\mathfrak{g}$.

**Theorem 4.12**: The set $\mathrm{ECh}_k(\mathfrak{g})$ is a subinvolutive-semiring of $\mathbb{C}[P]$. Moreover, the map

$$(\pi_k)_E : \mathrm{Rep}_k(\mathfrak{g}) \to \mathrm{ECh}_k(\mathfrak{g})$$
$$(\pi_k)_E(R) := (\mathcal{E}_k)_R \text{ for all } R \in \mathrm{Rep}_k(\mathfrak{g})$$

is an involutive-semiring isomorphism.

We refer to $\mathrm{ECh}_k(\mathfrak{g})$ as the **exponent character involutive-semiring** of $\mathfrak{g}$.

# Chapter 5

# Representations of $\mathrm{Rep}_k(\mathfrak{g})$

Let $S$ be an involutive-semiring. In Chapter 2, we defined a representation of $S$ to be an involutive-semiring homomorphism

$$\pi : S \to \mathrm{M}_r(\mathbb{Z}_{\geq 0}).$$

However, with this definition, only the root systems having all roots of the same length correspond to representations of $\mathrm{Rep}_k(A_1)$. To get all root systems, we need to weaken the definition of a representation slightly. Thus we define an **involutionizable-representation** of $S$ to be a semiring homomorphism

$$\pi : S \to \mathrm{M}_r(\mathbb{Z}_{\geq 0})$$

for which there is a diagonal matrix $D \in \mathrm{M}_r(\mathbb{Z}_{\geq 0})$ with positive diagonal entries so that

$$D \cdot \pi(s^*) = \pi(s)^\mathsf{T} \cdot D \text{ for all } s \in S.$$

We define an involutionizable-representation of an involutive-algebra in a similar way. Note that, if $D$ is the identity matrix, then we just have the usual definition of a representation of $S$. Thus an involutionizable-representation is really an extension of the definition of a representation. Since any positive multiple of $D$ also satisfies the above condition, we will always assume that the smallest diagonal entry of $D$ is a 1.

In Chapter 8, we will find it convenient to restrict the definition of a representation slightly in the following manner. We declare a representation $\pi$ of $S$ to be **symmetric** if

$$\pi(s^*) = \pi(s) \text{ for all } s \in S.$$

In other words, a representation of $S$ is symmetric if it assigns a symmetric nonnegative integer matrix to every element of $S$. If involution in $S$ is just the identity, then all representations of $S$ are symmetric. Note that we could also define a "symmetrizable-representation" in an analogous manner to "involutionizable-representation", but we do not consider examples of these in this thesis.

## 5.1 Basic Theory and Notation

Let $\pi$ be an involutionizable-representation of $\mathrm{Rep}_k(\mathfrak{g})$. Since $\mathrm{Rep}_k(\mathfrak{g})$ is free, the semiring homomorphism $\pi$ is completely determined by its image on the basis of $\mathrm{Rep}_k(\mathfrak{g})$. Thus we are led to introduce the shorthand notation

$$N_\lambda := \pi(R_\lambda) \in \mathrm{M}_r(\mathbb{Z}_{\geq 0}) \text{ for all } \lambda \in {P_k}^{++}.$$

In practice, we only need to specify a handful of these matrices (for example, the ones that correspond to the fundamental weights, assuming $k \neq h$) since the rest can be generated recursively using the multiplication and involution formulas of $\mathrm{Rep}_k(\mathfrak{g})$.

Some immediate properties of the matrices $N_\lambda$ are listed below.

**Proposition 5.1**: The nonnegative integer matrices $N_\lambda$ satisfy

(i) $N_\lambda \cdot N_\mu = \sum_{\nu \in {P_k}^{++}} (\mathcal{T}_k)_\lambda(\mu, \nu) N_\nu$ for all $\lambda, \mu \in {P_k}^{++}$,

(ii) $N_\rho = I$,

(iii) $D \cdot N_{\lambda^*} = (N_\lambda)^\intercal \cdot D$ for all $\lambda \in {P_k}^{++}$.

In fact, the above conditions are precisely the ones needed so that the nonnegative integer matrices $N_\lambda$ define an involutionizable-representation of $\mathrm{Rep}_k(\mathfrak{g})$.

Extend the representation $\pi$ in the standard way to a $\mathbb{Z}$-involutive-algebra homomorphism

$$\pi^{\mathbb{Z}} : \mathrm{Rep}_k(\mathfrak{g})^{\mathbb{Z}} \to \mathrm{M}_r(\mathbb{Z}_{\geq 0})^{\mathbb{Z}} \cong \mathrm{M}_r(\mathbb{Z}).$$

In terms of virtual isomorphism classes, we define

$$(N_k)_\lambda := \pi^{\mathbb{Z}}((R_k)_\lambda) = \eta_k(\lambda) N_{\zeta_k(\lambda)} \text{ for all } \lambda \in P.$$

Some immediate properties of these matrices are listed below.

**Proposition 5.2**: The integer matrices $(N_k)_\lambda$ satisfy

(i) $(N_k)_{(\alpha,w)\lambda} = \operatorname{sgn}(w)(N_k)_\lambda$ for all $(\alpha, w) \in W_k, \lambda \in P$,

(ii) $(N_k)_\lambda \cdot N_\mu = \sum_{\nu \in P} \mathcal{M}_\lambda(\nu)(N_k)_{\mu+\nu}$ for all $\lambda, \mu \in P$,

(iii) $D \cdot (N_k)_{-\lambda} = ((N_k)_\lambda)^\mathsf{T} \cdot D$ for all $\lambda \in P$.

Using the above proposition, it is clear that the integer matrices $(N_k)_\lambda$ define an involutionizable-representation of $\operatorname{Rep}(\mathfrak{g})^\mathbb{Z}$. Essentially this means that we can lift any involutionizable-representation of $\operatorname{Rep}_k(\mathfrak{g})$ to an involutionizable-representation of $\operatorname{Rep}(\mathfrak{g})^\mathbb{Z}$.

Conversely, the following theorem says when we can restrict an involutionizable-representation of $\operatorname{Rep}(\mathfrak{g})^\mathbb{Z}$ to an involutionizable-representation of $\operatorname{Rep}_k(\mathfrak{g})$.

**Theorem 5.3**: Let $\sigma$ be an involutionizable-representation of $\operatorname{Rep}(\mathfrak{g})^\mathbb{Z}$ which satisfies the following two conditions:

(i) $\sigma(R_{(\alpha,w)\lambda}) = \sigma(R_\lambda)$ for all $(\alpha, w) \in W_k, \lambda \in P$,

(ii) $\sigma(R_\lambda) \in \mathrm{M}_r(\mathbb{Z}_{\geq 0})$ for all $\lambda \in {P_k}^{++}$.

Then $\sigma$ restricts to an involutionizable-representation of $\operatorname{Rep}_k(\mathfrak{g})$.

This notion of lifting allows us to uniformly consider representations of $\operatorname{Rep}_k(\mathfrak{g})$ and affine representation of $\operatorname{Rep}(\mathfrak{g})$ as representations of $\operatorname{Rep}(\mathfrak{g})^\mathbb{Z}$.

## 5.2 Affine Tilde Matrices

As one might expect, we define the affine tilde matrices

$$(\widetilde{N}_k)_\lambda := \pi^\mathbb{Z}((\widetilde{R}_k)_\lambda) = \sum_{w \in W} N_{\rho + w\lambda} \in \mathrm{M}_r(\mathbb{Z}) \text{ for all } \lambda \in P.$$

Some properties of these affine tilde matrices are listed below.

**Proposition 5.4**: The affine tilde matrices $(\widetilde{N}_k)_\lambda$ satisfy

(i) $(\widetilde{N}_k)_{(\alpha,w)\lambda} = (\widetilde{N}_k)_\lambda$ for all $(\alpha, w) \in W_k, \lambda \in P$,

(ii) $(\widetilde{N}_k)_0 = |W|I$,

(iii) $D \cdot (\widetilde{N}_k)_{-\lambda} = ((\widetilde{N}_k)_\lambda)^\mathsf{T} \cdot D$ for all $\lambda \in P$.

Since $\pi$ is a representation of $\mathrm{Rep}_k(\mathfrak{g})$, we have the following theorems.

**Theorem 5.5**: We have the product formula

$$(\widetilde{N}_k)_\lambda \cdot (N_k)_\mu = \sum_{\nu \in P} \widetilde{\mathcal{M}}_\lambda(\nu)(N_k)_{\mu+\nu} \text{ for all } \lambda, \mu \in P.$$

If Conjecture 3.11 holds, then we also have

$$(\widetilde{N}_k)_\lambda \cdot (N_k)_\mu = \sum_{w \in W} (N_k)_{\mu+w\lambda} \text{ for all } \lambda, \mu \in P.$$

**Theorem 5.6**: We have the product formula

$$(\widetilde{N}_k)_\lambda \cdot (\widetilde{N}_k)_\mu = \sum_{\nu \in P} \widetilde{\mathcal{M}}_\lambda(\nu)(\widetilde{N}_k)_{\mu+\nu} \text{ for all } \lambda, \mu \in P.$$

If Conjecture 3.11 holds, then we also have

$$(\widetilde{N}_k)_\lambda \cdot (\widetilde{N}_k)_\mu = \sum_{w \in W} (\widetilde{N}_k)_{\mu+w\lambda} \text{ for all } \lambda, \mu \in P.$$

## 5.3 Exponents

In this section, we use the inclusion

$$\mathrm{M}_r(\mathbb{Z}) \hookrightarrow \mathrm{M}_r(\mathbb{C})$$

to view the integer matrices $N_\lambda$ and the diagonal matrix $D$ as complex matrices. Since $\mathrm{Rep}_k(\mathfrak{g})$ is commutative, the matrices $N_\lambda$ must all commute. Since $D$ is diagonal, it commutes as well. Finally, using the condition

$$D \cdot N_{\lambda^*} = (N_\lambda)^\mathsf{T} \cdot D \text{ for all } \lambda \in {P_k}^{++},$$

we see that $N_\lambda$ commutes with its transpose and thus is normal. From linear algebra, we know that a set of normal matrices that commute are simultaneously diagonalizable by a unitary matrix. An alternate way of stating this is that, if we extend $\pi$ to a $\mathbb{C}$-involutive-algebra homomorphism

$$\pi^{\mathbb{C}} : \mathrm{Rep}_k(\mathfrak{g})^{\mathbb{C}} \to \mathrm{M}_r(\mathbb{C}),$$

then $\pi^{\mathbb{C}}$ decomposes into a direct sum of one-dimensional representations. It turns out that all one-dimensional representations of $\mathrm{Rep}_k(\mathfrak{g})^{\mathbb{C}}$ have nice descriptions as "slices" of the affine exponent characters of $\mathfrak{g}$ at level $k$.

**Theorem 5.7**: Let $\mu \in {P_k}^{++}$. Define a map

$$(\mathcal{E}_k)^{\mu} : \mathrm{Rep}_k(\mathfrak{g})^{\mathbb{C}} \to \mathrm{M}_1(\mathbb{C})$$
$$(\mathcal{E}_k)^{\mu}(R) := [(\mathcal{E}_k)_R(\mu)] \text{ for all } R \in \mathrm{Rep}_k(\mathfrak{g})^{\mathbb{C}}.$$

Then $(\mathcal{E}_k)^{\mu}$ is a representation of $\mathrm{Rep}_k(\mathfrak{g})^{\mathbb{C}}$. Moreover, all (nonzero) one-dimensional representations of $\mathrm{Rep}_k(\mathfrak{g})^{\mathbb{C}}$ are of this form. Finally, for all $\lambda, \mu \in {P_k}^{++}$, if there is an isomorphism of $(\mathcal{E}_k)^{\lambda}$ to $(\mathcal{E}_k)^{\mu}$, then $\lambda = \mu$.

This leads to the following theorem.

**Theorem 5.8**: Let $\pi$ be a representation of $\mathrm{Rep}_k(\mathfrak{g})$ of rank $r$. Then there are weights $\mu_1, \ldots, \mu_r \in {P_k}^{++}$ such that

$$\pi^{\mathbb{C}} \cong (\mathcal{E}_k)^{\mu_1} \oplus \cdots \oplus (\mathcal{E}_k)^{\mu_r}.$$

Moreover, if $\pi$ is indecomposable, then the weight $\rho$ occurs exactly once in the multiset $\{\mu_1, \ldots, \mu_r\}$.

The last part of the this theorem follows from Perron-Frobenius theory: the decomposition factor $(\mathcal{E}_k)^{\rho}$ corresponds to a sort of Perron-Frobenius eigenvalue. For more details, see [8].

We remark here that this theorem should probably be corrected to allow for the eigenvalue zero.

We refer to the weights $\mu_1, \ldots, \mu_r$ in the above theorem as the **exponents** of $\pi$. The multiplicity of an exponent is the number of times it appears in the multiset $\{\mu_1, \ldots, \mu_r\}$. For example, we know that the weight $\rho$ of an indecomposable representation of $\mathrm{Rep}_k(\mathfrak{g})$ has multiplicity one.

It is easy to check that isomorphic representations of $\mathrm{Rep}_k(\mathfrak{g})$ have the same exponents. However, the converse is not true; it is possible to have two nonisomorphic representations of $\mathrm{Rep}_k(\mathfrak{g})$ with the same exponents.

The following conjecture suggests that representations of $\mathrm{Rep}_k(\mathfrak{g})$ may be verified completely in terms of their exponents.

**Conjecture 5.9**: Let $\sigma$ be an indecomposable representation of $\mathrm{Rep}(\mathfrak{g})^{\mathbb{Z}}$ of rank $r$ which sends the fundamental isomorphism classes (that is, the isomorphism classes of representations whose highest weights are fundamental weights) to nonnegative integer matrices. Then $\sigma$ restricts to a representation of $\mathrm{Rep}_k(\mathfrak{g})$ if and only if there are weights $\mu_1, \ldots, \mu_r \in {P_k}^{++}$ such that

$$\sigma^{\mathbb{C}} \cong (\mathcal{E}_k)^{\mu_1} \oplus \cdots \oplus (\mathcal{E}_k)^{\mu_r},$$

and such that $\rho$ occurs exactly once in the multiset $\{\mu_1, \ldots, \mu_r\}$.

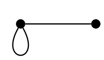
Again this conjecture should probably be corrected to allow for the eigenvalue zero. It is easy to show that, if $\sigma^{\mathbb{C}}$ has the required direct sum decomposition, then $\sigma$ restricts to a representation of $\mathrm{Rep}_k(\mathfrak{g})^{\mathbb{Z}}$; the difficult part of the conjecture is to show that the condition on the multiplicity of $\rho$ is enough to get nonnegativity.

## 5.4 Examples

For a specific example, consider the involutive-semiring $\mathrm{Rep}_5(\mathfrak{sl}_2(\mathbb{C})) \equiv \mathrm{Rep}_5(A_1)$ and the representation $\pi$ of $\mathrm{Rep}_5(A_1)$ defined so that

$$N_2 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

We can describe this $2 \times 2$ matrix more visually in the form of a graph:



This representation has the name $T_2$. We can use the various product formulas for $\mathrm{Rep}_5(A_1)$ to obtain Table 5.1.

By computing the eigenvalues of $N_2$, one can verify that the two exponents of $T_2$ are 1 and 3. However, from now on, we will use the convention that all exponents include the component $\mu^0$; that is, we will write

$$\mu = \sum_{i=1}^{r} \mu^i \Lambda_i \equiv (\mu^0, \mu^1, \ldots, \mu^r) \in \mathbb{Z}^{r+1},$$

| $\lambda$ | $(N_k)_\lambda$ | $(\widetilde{N}_k)_\lambda$ |
|---|---|---|
| 0 | $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ | $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ |
| 1 | $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ | $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ |
| 2 | $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ | $\begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$ |
| 3 | $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ | $\begin{bmatrix} 0 & -1 \\ -1 & 1 \end{bmatrix}$ |
| 4 | $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ | $\begin{bmatrix} -1 & -1 \\ -1 & 0 \end{bmatrix}$ |
| 5 | $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ | $\begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$ |
| 6 | $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ | $\begin{bmatrix} -1 & -1 \\ -1 & 0 \end{bmatrix}$ |
| 7 | $\begin{bmatrix} -1 & -1 \\ -1 & 0 \end{bmatrix}$ | $\begin{bmatrix} 0 & -1 \\ -1 & 1 \end{bmatrix}$ |
| 8 | $\begin{bmatrix} -1 & -1 \\ -1 & 0 \end{bmatrix}$ | $\begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$ |
| 9 | $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ | $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ |
| 10 | $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ | $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ |

Table 5.1: The representation $T_2$ of $\mathrm{Rep}_5(A_1)$.

where, as previously defined, we set

$$\mu^0 = k - \sum_{i=1}^{r} a_i \mu^i \in \mathbb{Z}.$$

With this convention, the exponents of $T_2$ are $(4, 1)$ and $(2, 3)$.

The classification of involutionizable-representations of $\mathrm{Rep}_k(A_1)$ is well-known and comes from the classification of finite-dimensional semisimple Lie algebras.

**Theorem 5.10**: The only indecomposable involutionizable-representations of $\mathrm{Rep}_k(A_1)$ are, up to isomorphism, those listed in Table 5.2.

The exponents of the representations in Table 5.2 are listed in Table 5.3.

Table 5.2: List of involutionizable-representations of $\mathrm{Rep}_k(A_1)$.

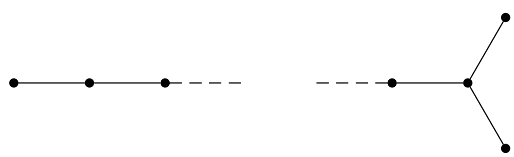
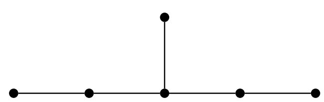
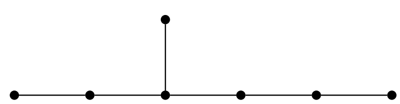
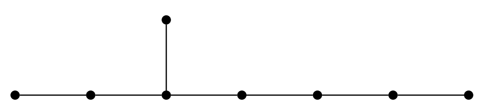
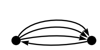
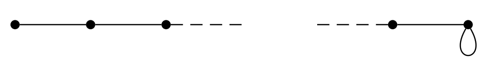
| **Name** | **Rank** | **Graph** |
|---|---|---|
| $A_r$ | $r \geq 1$ |  |
| $B_r$ | $r \geq 2$ |  |
| $C_r$ | $r \geq 3$ |  |
| $D_r$ | $r \geq 4$ |  |
| $E_6$ | 6 |  |
| $E_7$ | 7 |  |
| $E_8$ | 8 |  |
| $F_4$ | 4 |  |
| $G_2$ | 2 |  |
| $T_r$ | $r \geq 1$ |  |

Table 5.3: Exponents of involutionizable-representations of $\mathrm{Rep}_k(A_1)$.

| **Name** | **Level** | **Exponents** |
|---|---|---|
| $A_r$ | $r+1$ | $(r,1)$ $(r-1,2)$ $\cdots$ $(1,r)$ |
| $B_r$ | $2r$ | $(2r-1,1)$ $(2r-3,3)$ $\cdots$ $(1,2r-1)$ |
| $C_r$ | $2r$ | $(2r-1,1)$ $(2r-3,3)$ $\cdots$ $(1,2r-1)$ |
| $D_r$ | $2r-2$ | $(2r-3,1)$ $(2r-1,3)$ $\cdots$ $(1,2n-3)$ $(n-1,n-1)$ |
| $E_6$ | 12 | $(11,1)$ $(8,4)$ $(7,5)$ $(5,7)$ $(4,8)$ $(1,11)$ |
| $E_7$ | 18 | $(17,1)$ $(13,5)$ $(11,7)$ $(9,9)$ $(7,11)$ $(5,13)$ $(1,17)$ |
| $E_8$ | 30 | $(29,1)$ $(23,7)$ $(19,11)$ $(17,13)$ $(13,17)$ $(11,19)$ $(7,23)$ $(1,29)$ |
| $F_4$ | 12 | $(11,1)$ $(7,5)$ $(5,7)$ $(1,11)$ |
| $G_2$ | 6 | $(5,1)$ $(1,5)$ |
| $T_r$ | $2r+1$ | $(2r,1)$ $(2r-2,3)$ $\cdots$ $(2,2r-1)$ |

# Chapter 6

# Affine Representations of $\mathrm{Rep}(\mathfrak{g})$

There are way too many representations of $\mathrm{Rep}(\mathfrak{g})$ in the sense that there are representations of $\mathrm{Rep}(\mathfrak{g})$ whose nonnegative integer matrices have arbitrarily large entries. However, there is a subclass of representations of $\mathrm{Rep}(\mathfrak{g})$ which behaves like the representations of $\mathrm{Rep}_k(\mathfrak{g})$. We refer to the representations in this subclass as affine representations.

## 6.1 Basic Theory and Notation

Let $\pi$ be an involutionizable-representation of $\mathrm{Rep}(\mathfrak{g})$. Since $\mathrm{Rep}(\mathfrak{g})$ is a free, it suffices to define $\pi$ on the basis of $\mathrm{Rep}(\mathfrak{g})$. Thus we introduce the notation

$$N_\lambda := \pi(R_\lambda) \in \mathrm{M}_r(\mathbb{Z}_{\geq 0}) \text{ for all } \lambda \in P^{++}.$$

In practice, we only need to specify a small number of these matrices (for example, the ones that correspond to the fundamental weights) since the rest can be generated recursively using multiplication and involution. Some immediate properties of the matrices $N_\lambda$ are listed below.

**Proposition 6.1**: The nonnegative integer matrices $N_\lambda$ satisfy

(i) $N_\lambda \cdot N_\mu = \sum_{\nu \in P^{++}} \mathcal{T}_\lambda(\mu, \nu) N_\nu$ for all $\lambda, \mu \in P^{++}$,

(ii) $N_\rho = I$,

(iii) $D \cdot N_{\lambda^*} = (N_\lambda)^\mathsf{T} \cdot D$ for all $\lambda \in P^{++}$.

In fact, the above conditions are precisely the ones needed so that the nonnegative integer matrices $N_\lambda$ define an involutionizable-representation of $\mathrm{Rep}(\mathfrak{g})$.

Now we extend $\pi$ to a $\mathbb{Z}$-involutive-algebra homomorphism

$$\pi^{\mathbb{Z}} : \mathrm{Rep}(\mathfrak{g})^{\mathbb{Z}} \to \mathrm{M}_r(\mathbb{Z}_{\geq 0})^{\mathbb{Z}} \cong \mathrm{M}_r(\mathbb{Z}).$$

In terms of virtual isomorphism classes, we define

$$N_\lambda := \pi^{\mathbb{Z}}(R_\lambda) = \eta(\lambda) N_{\zeta(\lambda)} \in \mathrm{M}_r(\mathbb{Z}) \text{ for all } \lambda \in P.$$

Some properties of these integer matrices are listed below.

**Proposition 6.2**: The integer matrices $N_\lambda$ satisfy

(i) $N_{w\lambda} = \mathrm{sgn}(w) N_\lambda$ for all $w \in W, \lambda in P$,

(ii) $N_\lambda \cdot N_\mu = \sum_{\mu \in P} \mathcal{M}_\lambda(\nu) N_{\mu+\nu}$ for all $\lambda, \mu \in P$,

(iii) $D \cdot N_{-\lambda} = (N_\lambda)^{\mathsf{T}} \cdot D$ for all $\lambda \in P$.

Note that involutionizable-representations of $\mathrm{Rep}_k(\mathfrak{g})$ (with some work) and involutionizable-representations of $\mathrm{Rep}(\mathfrak{g})$ (trivially) both extend to involutionizable-representation of $\mathrm{Rep}(\mathfrak{g})^{\mathbb{Z}}$.

## 6.2 Tilde Matrices

Define the tilde matrices

$$\widetilde{N}_\lambda := \pi^{\mathbb{Z}}(\widetilde{R}_\lambda) = \sum_{w \in W} N_{\rho + w\lambda} \in \mathrm{M}_r(\mathbb{Z}) \text{ for all } \lambda \in P.$$

Some properties of these tilde matrices are listed below.

**Proposition 6.3**: The tilde matrices $\widetilde{N}_\lambda$ satisfy

(i) $\widetilde{N}_{w\lambda} = \widetilde{N}_\lambda$ for all $w \in W, \lambda in P$,

(ii) $\widetilde{N}_0 = |W| I$,

(iii) $D \cdot \widetilde{N}_{-\lambda} = (\widetilde{N}_\lambda)^{\mathsf{T}} \cdot D$ for all $\lambda \in P$.

We also have the usual product theorems.

**Theorem 6.4**: We have the product formula

$$\widetilde{N}_\lambda \cdot N_\mu = \sum_{\nu \in P} \widetilde{\mathcal{M}}_\lambda(\nu) N_{\mu+\nu} \text{ for all } \lambda, \mu \in P.$$

If Conjecture 3.11 holds, then we also have

$$\widetilde{N}_\lambda \cdot N_\mu = \sum_{w \in W} N_{\mu+w\lambda} \text{ for all } \lambda, \mu \in P.$$

**Theorem 6.5**: We have the product formula

$$\widetilde{N}_\lambda \cdot \widetilde{N}_\mu = \sum_{\nu \in P} \widetilde{\mathcal{M}}_\lambda(\nu) \widetilde{N}_{\mu+\nu} \text{ for all } \lambda, \mu \in P.$$

If Conjecture 3.11 holds, then we also have

$$\widetilde{N}_\lambda \cdot \widetilde{N}_\mu = \sum_{w \in W} \widetilde{N}_{\mu+w\lambda} \text{ for all } \lambda, \mu \in P.$$

Recall that the affine tilde matrices of a representation of $\mathrm{Rep}_k(\mathfrak{g})$ satisfy the condition

$$(\widetilde{N}_k)_{(\alpha,w)\lambda} = (\widetilde{N}_k)_\lambda \text{ for all } (\alpha, w) \in W_k, \lambda \in P,$$

With this in mind, we make the following definition. For any positive integer $k$, we say that a representation $\pi$ of $\mathrm{Rep}(\mathfrak{g})$ is **affine** at level $k$ if

$$\widetilde{N}_{(\alpha,w)\lambda} = \widetilde{N}_\lambda \text{ for all } (\alpha, w) \in W_k, \lambda \in P.$$

We define affine representations of $\mathrm{Rep}(\mathfrak{g})^{\mathbb{Z}}$ and $\mathrm{Rep}(\mathfrak{g})^{\mathbb{C}}$ in an analogous fashion. Note that, if $\pi$ is affine at level $k$, then $\pi$ is also affine at any positive multiple of $k$. Thus, we will usually assume that $k$ is as small as possible. In the case of $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$, the affine representations correspond to affine Lie algebras.

## 6.3 Exponents

As with the case of $\mathrm{Rep}_k(\mathfrak{g})$, we can always simultaneously diagonalize the matrices $N_\lambda$ by a unitary matrix. Alternately, if we extend $\pi$ to an involutive-algebra homomorphism

$$\pi^{\mathbb{C}} : \mathrm{Rep}(\mathfrak{g})^{\mathbb{C}} \to \mathrm{M}_r(\mathbb{C}),$$

then $\pi^{\mathbb{C}}$ decomposes into a direct sum of one-dimensional representations of $\mathrm{Rep}(\mathfrak{g})^{\mathbb{C}}$.

The following conjectures argue that affine representations of $\mathrm{Rep}(\mathfrak{g})$ at level $k$ have a notion of exponent that closely matches the notion of exponent for representations of $\mathrm{Rep}_k(\mathfrak{g})$.

**Conjecture 6.6**: Let $\mu \in {P_k}^+$. Define a map

$$(\mathcal{E}_k)^\mu : \mathrm{Rep}_k(\mathfrak{g})^{\mathbb{C}} \to \mathrm{M}_1(\mathbb{C})$$
$$(\mathcal{E}_k)^\mu(R) := [(\mathcal{E}_k)_R(\mu)] \text{ for all } R \in \mathrm{Rep}_k(\mathfrak{g})^{\mathbb{C}}.$$

Then $(\mathcal{E}_k)^\mu$ is an affine representation of $\mathrm{Rep}(\mathfrak{g})^{\mathbb{C}}$. Moreover, all (nonzero) one-dimensional affine representations of $\mathrm{Rep}_k(\mathfrak{g})^{\mathbb{C}}$ are of this form. Finally, for all $\lambda, \mu \in {P_k}^+$, if there is an isomorphism of $(\mathcal{E}_k)^\lambda$ to $(\mathcal{E}_k)^\mu$, then $\lambda = \mu$.

This leads to the following conjecture.

**Conjecture 6.7**: Let $\pi$ be an affine representation of $\mathrm{Rep}(\mathfrak{g})$ at level $k$ of rank $r$. Then there are weights $\mu_1, \ldots, \mu_r \in {P_k}^+$ such that

$$P^{\mathbb{C}} \cong (\mathcal{E}_k)^{\mu_1} \oplus \cdots \oplus (\mathcal{E}_k)^{\mu_r}.$$

Moreover, if $\pi$ is indecomposable, then the weight 0 occurs exactly once in the multiset $\{\mu_1, \ldots, \mu_r\}$.

This conjecture does not need to be corrected to allow for the eigenvalue zero because we can always choose a multiple of $k$ so that some weight $\mu \in {P_k}^+$ has $(\mathcal{E}_k)^\mu = 0$. We refer to the weights $\mu_1, \ldots, \mu_r$ in the above conjecture as the **exponents** of $\pi$. The multiplicity of an exponent is the number of times it appears in the multiset $\{\mu_1, \ldots, \mu_r\}$. For example, if $\pi$ is indecomposable, then we know that the weight 0 has multiplicity one.

This conjecture should probably be proved by viewing $\mathrm{Rep}(\mathfrak{g})$ as the limit of $\mathrm{Rep}_k(\mathfrak{g})$ as $k$ approaches infinity.

The following conjecture suggests that representations of $\mathrm{Rep}_k(\mathfrak{g})$ may be verified completely in terms of their exponents.

**Conjecture 6.8**: Let $\sigma$ be an affine indecomposable representation of $\mathrm{Rep}(\mathfrak{g})^{\mathbb{Z}}$ of rank $r$ which sends the fundamental isomorphism classes to nonnegative integer matrices. Then $\sigma$ restricts to an affine representation of $\mathrm{Rep}(\mathfrak{g})$ if and only if there are weights $\mu_1, \ldots, \mu_r \in {P_k}^+$ such that

$$\sigma^{\mathbb{C}} \cong (\mathcal{E}_k)^{\mu_1} \oplus \cdots \oplus (\mathcal{E}_k)^{\mu_r},$$

and such that 0 occurs exactly once in the multiset $\{\mu_1, \ldots, \mu_r\}$.

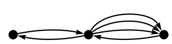
This conjecture essentially says that the condition on the multiplicity of 0 is enough to get nonnegativity.

## 6.4 Examples

For a specific example, consider the involutive-semiring $\mathrm{Rep}(\mathfrak{sl}_c(\mathbb{C})) \equiv \mathrm{Rep}(A_1)$ and the representation $\pi$ of $\mathrm{Rep}(A_1)$ defined so that

$$N_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 3 \\ 0 & 1 & 0 \end{bmatrix}.$$

We can describe this $3 \times 3$ matrix more visually in the form of a graph:



This representation has the name ${G_2}^{(1)}$. We can use the various product formulas for $\mathrm{Rep}(A_1)$ to obtain Table 5.1.

The components of the $N_\lambda$ matrices grow arbitrarily large as $\lambda$ approaches infinite. However, the $\widetilde{N}_\lambda$ matrices are invariant under the action of $W_2$ (on the subscript), and so ${G_2}^{(1)}$ is an affine involutionizable-representation of $\mathrm{Rep}(A_1)$ at level 2.

By computing the eigenvalues of $N_2$, one can verify that the three exponents of ${G_2}^{(1)}$ are 0, 1, and 2. However, as with the affine case, we will use the convention that all exponents of affine representations of $\mathrm{Rep}(\mathfrak{g})$ will be written to include the component $\mu^0$; that is, we will write

$$\mu = \sum_{i=1}^{r} \mu^i \Lambda_i \equiv (\mu^0, \mu^1, \ldots, \mu^r) \in \mathbb{Z}^{r+1}.$$

With this convention, the exponents of ${G_2}^{(1)}$ are $(2,0)$, $(1,1)$, and $(0,2)$.

| $\lambda$ | $N_\lambda$ | $\widetilde{N}_\lambda$ | $D \cdot \widetilde{N}_\lambda$ |
|---|---|---|---|
| 0 | $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ | $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ | $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{bmatrix}$ |
| 1 | $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ | $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 3 \\ 0 & 1 & 0 \end{bmatrix}$ | $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 3 \\ 0 & 3 & 0 \end{bmatrix}$ |
| 2 | $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 3 \\ 0 & 1 & 0 \end{bmatrix}$ | $\begin{bmatrix} -1 & 0 & 3 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ | $\begin{bmatrix} -1 & 0 & 3 \\ 0 & 2 & 0 \\ 3 & 0 & 3 \end{bmatrix}$ |
| 3 | $\begin{bmatrix} 0 & 0 & 3 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{bmatrix}$ | $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 3 \\ 0 & 1 & 0 \end{bmatrix}$ | $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 3 \\ 0 & 3 & 0 \end{bmatrix}$ |
| 4 | $\begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 6 \\ 0 & 2 & 0 \end{bmatrix}$ | $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ | $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{bmatrix}$ |

Table 6.1: The affine representation ${G_2}^{(1)}$ of $\mathrm{Rep}(A_1)$.

The classification of affine involutionizable-representations of $\mathrm{Rep}(A_1)$ is well-known: it comes from the classification of affine Lie algebras.

**Theorem 6.9**: The only affine indecomposable involutionizable-representations of $\mathrm{Rep}(A_1)$ are, up to isomorphism, those listed in Table 6.2.

The exponents of the representations in Table 6.2 are listed in Table 6.3.

Table 6.2: List of affine involutionizable-representations of $\mathrm{Rep}(A_1)$.

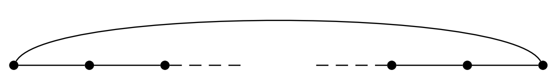
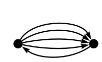
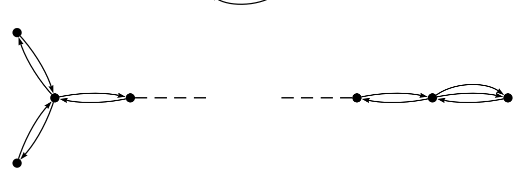
| **Name** | **Rank** | **Graph** |
|---|---|---|
| ${A_{r-1}}^{(1)}$ | $r \geq 2$ |  |
| $\frac{1}{2}{A_{2r-1}}^{(1)}$ | $r \geq 1$ |  |
| ${A_1}^{(2)}$ | 2 |  |
| ${B_{r-1}}^{(1)}$ | $r \geq 4$ |  |

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Table 6.2: List of affine involutionizable-representations of $\mathrm{Rep}(A_1)$ (continued).

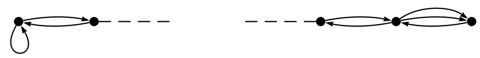
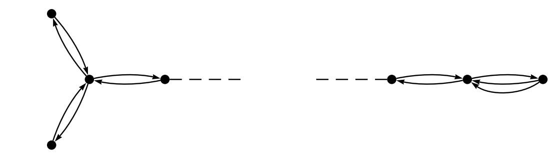
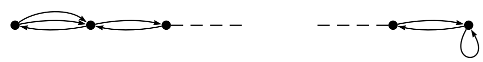
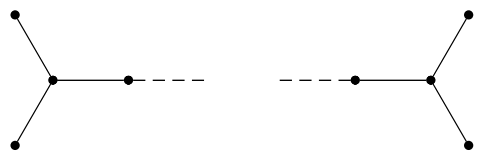
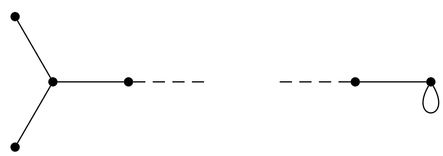
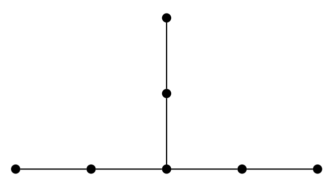
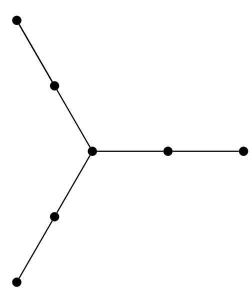
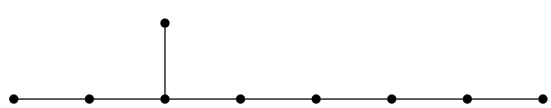
| **Name** | **Rank** | **Graph** |
|---|---|---|
| ${B_{r-1}}^{(2)}$ | $r \geq 3$ |  |
| $\frac{1}{2}{B_{2r-1}}^{(2)}$ | $r \geq 2$ |  |
| ${BC_{r-1}}^{(2)}$ | $r \geq 2$ |  |
| ${C_{r-1}}^{(1)}$ | $r \geq 4$ |  |
| ${C_{r-1}}^{(2)}$ | $r \geq 2$ |  |
| $\frac{1}{2}{C_{2r-1}}^{(2)}$ | $r \geq 2$ |  |
| ${D_{r-1}}^{(1)}$ | $r \geq 5$ |  |
| $\frac{1}{2}{D_{2r-1}}^{(1)}$ | $r \geq 3$ |  |
| ${E_6}^{(1)}$ | 7 |  |
| ${E_7}^{(1)}$ | 8 |  |
| ${E_8}^{(1)}$ | 9 |  |

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Table 6.2: List of affine involutionizable-representations of Rep($A_1$) (continued).

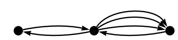
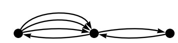
| **Name** | **Rank** | **Graph** |
|---|---|---|
| ${F_4}^{(1)}$ | 5 |  |
| ${F_4}^{(2)}$ | 5 |  |
| ${G_2}^{(1)}$ | 3 |  |
| ${G_2}^{(3)}$ | 3 |  |

Table 6.3: Exponents of affine involutionizable-representations of Rep($A_1$).

| **Name** | **Level** | **Exponents** |
|---|---|---|
| ${A_{r-1}}^{(1)}$<br>$r$ odd | $r-1$ | $(r-1,0)$ $(r-2,1)$ $\cdots$ $(0,r-1)$<br>$(r-2,1)$ $(r-3,2)$ $\cdots$ $(1,r-2)$ |
| ${A_{r-1}}^{(1)}$<br>$r$ even | $2r-1$ | $(r,0)$ $(r-2,2)$ $\cdots$ $(1,r-1)$<br>$(r-2,2)$ $(r-4,4)$ $\cdots$ $(1,r-1)$ |
| $\frac{1}{2}{A_{2r-1}}^{(1)}$ | $r$ | $(r,0)$ $(r-1,1)$ $\cdots$ $(1,r-1)$ |
| ${A_1}^{(2)}$ | 1 | $(1,0)$ $(0,1)$ |
| ${B_{r-1}}^{(1)}$ | $2r-4$ | $(2r-4,0)$ $(2r-6,2)$ $\cdots$ $(0,2r-4)$<br>$(r-2,r-2)$ |
| ${B_{r-1}}^{(2)}$ | $2r-4$ | $(2r-4,0)$ $(2r-6,2)$ $\cdots$ $(0,2r-4)$<br>$(r-2,r-2)$ |
| $\frac{1}{2}{B_{2r-1}}^{(2)}$ | $2r-1$ | $(2r-1,0)$ $(2r-3,2)$ $\cdots$ $(1,2r-2)$ |
| ${BC_{r-1}}^{(2)}$ | $r-1$ | $(r-1,0)$ $(r-2,1)$ $\cdots$ $(0,r-1)$ |
| ${C_{r-1}}^{(1)}$ | $2r-4$ | $(2r-4,0)$ $(2r-6,2)$ $\cdots$ $(0,2r-4)$<br>$(r-2,r-2)$ |
| ${C_{r-1}}^{(2)}$ | $r-1$ | $(r-1,0)$ $(r-2,1)$ $\cdots$ $(0,r-1)$ |
| $\frac{1}{2}{C_{2r-1}}^{(2)}$ | $2r-1$ | $(2r-1,0)$ $(2r-3,2)$ $\cdots$ $(1,2r-2)$ |
| ${D_{r-1}}^{(1)}$ | $2r-6$ | $(2r-6,0)$ $(2r-4,2)$ $\cdots$ $(0,2r-6)$<br>$(r-3,r-3)$ $(r-3,r-3)$ |

Continued on next page . . .

Table 6.3: Exponents of affine involutionizable-representations of $\mathrm{Rep}(A_1)$ (continued).

| **Name** | **Level** | **Exponents** |
|---|---|---|
| $\frac{1}{2}D_{2r-1}{}^{(1)}$ | $4r-6$ | $(4r-6,0)$ $(4r-2,4)$ $\cdots$ $(2,4r-8)$ $(2r-3,2r-3)$ |
| $E_6{}^{(1)}$ | 6 | $(6,0)$ $(4,2)$ $(4,2)$ $(3,3)$ $(2,4)$ $(2,4)$ $(0,6)$ |
| $E_7{}^{(1)}$ | 12 | $(12,0)$ $(9,3)$ $(8,4)$ $(6,6)$ $(6,6)$ $(4,8)$ $(3,9)$ $(0,12)$ |
| $E_8{}^{(1)}$ | 30 | $(30,0)$ $(24,6)$ $(20,10)$ $(18,12)$ $(15,15)$ $(12,18)$ $(10,20)$ $(6,24)$ $(0,30)$ |
| $F_4{}^{(1)}$ | 6 | $(0,6)$ $(2,4)$ $(3,3)$ $(4,2)$ $(6,0)$ |
| $F_4{}^{(2)}$ | 6 | $(0,6)$ $(2,4)$ $(3,3)$ $(4,2)$ $(6,0)$ |
| $G_2{}^{(1)}$ | 2 | $(0,2)$ $(1,1)$ $(2,0)$ |
| $G_2{}^{(3)}$ | 2 | $(0,2)$ $(1,1)$ $(2,0)$ |

# Chapter 7

# Root Systems of Higher Type

In the introduction, we saw how a root system defines an involutionizable-representation of $\mathrm{Rep}_k(A_1)$. In this chapter, we look at the reverse procedure: how to associate a root system to an involutionizable-representation of $\mathrm{Rep}_k(A_1)$. As observed by Ocneanu [11], this same procedure works in general, allowing us to define root systems of higher type.

## 7.1 Basic Theory and Notation

Let $\pi$ be an involutionizable-representation of $\mathrm{Rep}_k(\mathfrak{g})$ or an affine involutionizable-representation of $\mathrm{Rep}(\mathfrak{g})$. We will only explicitly consider the first case: the other can be handled similarly. Then $\pi$ is a semiring homomorphism

$$\pi : \mathrm{Rep}_k(\mathfrak{g}) \to \mathrm{M}_r(\mathbb{Z}_{\geq 0})$$

for which there is a diagonal nonnegative integer matrix $D \in \mathrm{M}_r(\mathbb{Z}_{\geq 0})$ with positive diagonal entries so that

$$D \cdot \pi(R^*) = \pi(R)^{\mathsf{T}} \cdot D \text{ for all } R \in \mathrm{Rep}_k(\mathfrak{g}).$$

We assume that the smallest diagonal entry of $D$ is 1. If $k = h$, then the only representations of $\mathrm{Rep}_k(\mathfrak{g})$ are the zero representations. Otherwise, we have $k > h$ and $D$ is unique.

As usual, we define the integer matrices

$$(N_k)_\lambda := \pi^{\mathbb{Z}}((R_k)_\lambda) \in \mathrm{M}_r(\mathbb{Z}) \text{ for all } \lambda \in P$$
$$(\widetilde{N}_k)_\lambda := \pi^{\mathbb{Z}}((\widetilde{R}_k)_\lambda) \in \mathrm{M}_r(\mathbb{Z}) \text{ for all } \lambda \in P.$$

Let $J$ be the index set of the matrices in $\mathrm{M}_r(\mathbb{Z})$. We will usually take

$$J = \{1, \ldots, r\}.$$

Let $Y = J \times P$. Define a map

$$\langle \cdot, \cdot \rangle : Y \times Y \to \mathbb{Z}$$
$$\langle (a, \lambda), (b, \mu) \rangle := [D \cdot (\widetilde{N}_k)_{\mu-\lambda}]_{a,b} \text{ for all } (a, \lambda), (b, \mu) \in Y.$$

Some obvious properties of this map are listed below.

**Proposition 7.1**: The map $\langle \cdot, \cdot \rangle$ satisfies

(i) $\langle (a, \lambda), (b, \lambda) \rangle = |W|[D]_{a,b}$ for all $a, b \in J; \lambda \in P$,

(ii) $\langle (a, \lambda), (b, \mu) \rangle = \langle (b, \mu), (a, \lambda) \rangle$ for all $(a, \lambda), (b, \mu) \in Y$,

(iii) $\langle (a, (\alpha, w)\lambda), (b, (\alpha, w)\mu) \rangle = \langle (a, \lambda), (b, \mu) \rangle$
for all $(a, \lambda), (b, \mu) \in J; (\alpha, w) \in W_k$.

*Proof.* For Part (i), let $a, b \in J$ and let $\lambda \in P$. Then

$$\begin{aligned} \langle (a, \lambda), (b, \lambda) \rangle &= [D \cdot (\widetilde{N}_k)_{\lambda-\lambda}]_{a,b} \\ &= |W|[D \cdot I]_{a,b} \\ &= |W|[D]_{a,b}. \end{aligned}$$

For Part (ii), let also $\mu \in P$. Then

$$\begin{aligned} \langle (a, \lambda), (b, \mu) \rangle &= [D \cdot (\widetilde{N}_k)_{\lambda-\mu}]_{a,b} \\ &= [((\widetilde{N}_k)_{-\lambda+\mu})^{\mathsf{T}} \cdot D]_{a,b} \\ &= [(D \cdot (\widetilde{N}_k)_{\mu-\lambda})^{\mathsf{T}}]_{a,b} \\ &= [D \cdot (\widetilde{N}_k)_{\mu-\lambda}]_{b,a} \\ &= \langle (b, \mu), (a, \lambda) \rangle. \end{aligned}$$

For Part (iii), let also $(w, \alpha) \in W_k$. Then

$$\begin{aligned}\langle (a, (\alpha, w)\lambda), (b, (\alpha, w)\mu) \rangle &= [D \cdot (\widetilde{N}_k)_{(\alpha,w)\lambda-(\alpha,w)\mu}]_{a,b} \\ &= [D \cdot (\widetilde{N}_k)_{(\alpha,w)(\lambda-\mu)}]_{a,b} \\ &= [D \cdot (\widetilde{N}_k)_{\lambda-\mu}]_{a,b} \\ &= \langle (a, \lambda), (b, \mu) \rangle.\end{aligned}$$

□

The following conjecture argues that the map $\langle \cdot, \cdot \rangle$ coincides with the inner product on the image of $Y$ in some Euclidean space.

**Conjecture 7.2**: Let $\mathbb{R}^n$ denote the $n$-dimensional real Euclidean space with orthonormal basis $e_1, \ldots, e_n$. Take $n = r|W|$. Then there is map

$$\varphi : Y \to \mathbb{R}^n$$

such that the following conditions hold.

(i) $|\varphi(Y)| < \infty$,

(ii) $0 \notin \varphi(Y)$,

(iii) $\operatorname{span}(\varphi(Y)) = \mathbb{R}^n$,

(iv) $\langle \varphi(a, \lambda), \varphi(b, \mu) \rangle = \langle (a, \lambda), (b, \mu) \rangle$ for all $(a, \lambda), (b, \mu) \in Y$.

If $\pi$ is an affine representation of $\operatorname{Rep}(\mathfrak{g})$, then we either remove the spanning condition or allow $n \leq r|W|$. If the map $\varphi$ exists, then we set $\Phi = \varphi(Y)$ and we refer to $\Phi$ as a **root system** of type $[\mathfrak{g}]$. If $\pi$ is an affine representation of $\operatorname{Rep}(\mathfrak{g})$, then we refer to $\Phi$ as an **affine root system** of type $[\mathfrak{g}]$.

The following conjecture argues that $\Phi$ has a natural basis of simple-like roots.

**Conjecture 7.3**: There is a subset $X \subseteq P$ with $|X| = |W|$ such that, for any representation of $\operatorname{Rep}_k(\mathfrak{g})$, the $r|W| \times r|W|$ integer matrix

$$[A]_{\alpha,\beta} = \langle \alpha, \beta \rangle \text{ for all } \alpha, \beta \in \varphi(J \times X)$$

is positive definite and all elements of $\Phi$ may be written as integer linear combinations of elements of $\varphi(J \times X)$ using the formulas

$$\sum_{a \in J} [\widetilde{N}_\lambda]_{a,b} \varphi(a, \mu) = \sum_{w \in W} \varphi(b, \mu + w\lambda) \text{ for all } b \in J; \lambda, \mu \in P.$$

If $\pi$ is an affine representation of $\text{Rep}(\mathfrak{g})$, then we only require that the matrix $A$ be positive semidefinite with determinant zero. If the set $X$ exists, then we refer to $A$ as the **Cartan matrix** of $\pi$ and to elements of $\varphi(J \times X)$ as **simple roots**.

In practice, there seem to be several choices for the set $X$, but all lead to the same Cartan matrix up to a permutation of the indices. In the case $\mathfrak{g} \in A_1$, the only choices that work are those that contain exactly two adjacent weights. For example, one choice is

$$X = \{0, 1\}.$$

In the case $\mathfrak{g} \in A_2$, the only choices that work are those that contain exactly six weights in a dense triangular pattern. For example, one choice is

$$X = \{(0,0), (0,1), (0,2), (1,0), (1,1), (2,0)\}.$$

In the case $\mathfrak{g} \in C_2$ with the long fundamental weight assigned to the second component, one choice is

$$X = \{(0,0), (1,0), (2,0), (1,0), (1,1), (2,-1), (2,0), (2,1)\}.$$

In the case $\mathfrak{g} \in G_2$ with the long fundamental weight assigned to the second component, one choice is

$$X = \{(-1,1), (-1,2), (0,0), (0,1), (1,0), (1,1), (2,0), (2,1), (3,0), (3,1), (4,-1), (4,0)\}.$$

The details here are admittedly scarce, but we leave a more careful working of these ideas to future work. Instead, we end this chapter by looking at a large number of examples of root systems and Cartan matrices for the cases $g \in A_1$ and $\mathfrak{g} \in A_2$.

## 7.2 Root Systems of Type $A_1$

The first example we consider is the one-dimensional representation $T_1$ of $\mathrm{Rep}_6(A_1)$ which sends the fundamental weight to the identity:

| $\lambda$ | $N_\lambda$ | $D \cdot \widetilde{N}_\lambda$ | $\varphi(1, \lambda)$ |
|---|---|---|---|
| 0 | [0] | [2] | $e_1 - e_2$ |
| 1 | [1] | [1] | $e_1 - e_3$ |
| 2 | [1] | [−1] | $e_2 - e_3$ |
| 3 | [0] | [−2] | $e_2 - e_1$ |
| 4 | [−1] | [−1] | $e_3 - e_1$ |
| 5 | [−1] | [1] | $e_3 - e_2$ |
| 6 | [0] | [2] | $e_1 - e_2$ |

The index $\lambda$ is understood to extend to $\pm\infty$. We have included only one period of the table. The roots listed in the right-most column demonstrate one possible embedding of $Y$ into a Euclidean space of dimension 2.

Using $X = \{0, 1\}$, we can work out the Cartan matrix of this representation:

$$A(T_1) = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

The second example we consider is the two-dimensional representation $A_2$ of $\mathrm{Rep}_6(A_1)$ which sends the fundamental weight to the reverse identity:

| $\lambda$ | $N_\lambda$ | $D \cdot \widetilde{N}_\lambda$ | $\varphi(1,\lambda)$ | $\varphi(2,\lambda)$ |
|---|---|---|---|---|
| 0 | $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ | $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ | $e_1 - e_2$ | $e_6 - e_5$ |
| 1 | $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ | $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ | $e_4 - e_5$ | $e_1 - e_3$ |
| 2 | $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ | $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ | $e_2 - e_3$ | $e_4 - e_6$ |
| 3 | $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ | $\begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix}$ | $e_5 - e_6$ | $e_2 - e_1$ |
| 4 | $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ | $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ | $e_3 - e_1$ | $e_5 - e_4$ |
| 5 | $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ | $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ | $e_6 - e_4$ | $e_3 - e_2$ |
| 6 | $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ | $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ | $e_1 - e_2$ | $e_6 - e_5$ |

The Cartan matrix of this representation is

$$A(A_2) = \begin{bmatrix} 2 & 0 & 0 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}.$$

Note that the matrix $A(A_2)$ decomposes into a direct sum of $A(T_1)$ with itself. Similarly, the root system of the representation $A_2$ consists of two interwoven copies of the root system of the representation $T_1$. This is no coincidence: it is not too hard to show that, whenever the matrix assigned to the fundamental weight by an arbitrary representation of $\mathrm{Rep}_k(A_1)$ has the block form

$$N_2 = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix},$$

the Cartan matrix decomposes into a direct sum of a matrix with itself and the root system consists of two interwoven copies of the same root system. Moreover, if $B = C$, then the matrix $B$ also defines a representation of $\mathrm{Rep}_k(A_1)$ with half the rank.

From now on, whenever we get two interwoven copies of the same root system, we will omit one copy from the table.

Consider the representation $A_3$ of $\mathrm{Rep}_4(A_1)$:

| $\lambda$ | $N_\lambda$ | $D \cdot \widetilde{N}_\lambda$ | $\varphi(1,\lambda)$ | $\varphi(2,\lambda)$ | $\varphi(3,\lambda)$ |
|---|---|---|---|---|---|
| 0 | $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ | $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ | $e_1 - e_2$ | | $e_4 - e_3$ |
| 1 | $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ | $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ | | $e_1 - e_3$ | |
| 2 | $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ | $\begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix}$ | $e_2 - e_3$ | | $e_1 - e_4$ |
| 3 | $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ | $\begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$ | | $e_2 - e_4$ | |
| 4 | $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ | $\begin{bmatrix} 0 & 0 & -2 \\ 0 & -2 & 0 \\ -2 & 0 & 0 \end{bmatrix}$ | $e_3 - e_4$ | | $e_2 - e_1$ |
| 5 | $\begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$ | $\begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$ | | $e_3 - e_1$ | |
| 6 | $\begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$ | $\begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix}$ | $e_4 - e_1$ | | $e_3 - e_2$ |
| 7 | $\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ | $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ | | $e_4 - e_2$ | |
| 8 | $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ | $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ | $e_1 - e_2$ | | $e_4 - e_3$ |

The Cartan matrix of this representation is

$$A(A_3) = \begin{bmatrix} 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 & 0 \\ 1 & 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 0 & 2 \end{bmatrix} \cong \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \oplus \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}.$$

Consider the representation $G_2$ of $\mathrm{Rep}_6(A_1)$:

| $\lambda$ | $N_\lambda$ | $D \cdot \widetilde{N}_\lambda$ | $\varphi(1,\lambda)$ | $\varphi(2,\lambda)$ |
|---|---|---|---|---|
| 0 | $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ | $\begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix}$ | $e_1 - e_2$ | |
| 1 | $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ | $\begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix}$ | | $e_1 - 2e_2 + e_3$ |
| 2 | $\begin{bmatrix} 0 & 3 \\ 1 & 0 \end{bmatrix}$ | $\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$ | $-e_2 + e_3$ | |
| 3 | $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ | $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ | | $-e_1 - e_2 + 2e_3$ |
| 4 | $\begin{bmatrix} 0 & 3 \\ 1 & 0 \end{bmatrix}$ | $\begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}$ | $e_3 - e_1$ | |
| 5 | $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ | $\begin{bmatrix} 0 & -3 \\ -3 & 0 \end{bmatrix}$ | | $-2e_1 + e_2 + e_3$ |
| 6 | $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ | $\begin{bmatrix} -2 & 0 \\ 0 & -6 \end{bmatrix}$ | $-e_1 + e_2$ | |
| 7 | $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ | $\begin{bmatrix} 0 & -3 \\ -3 & 0 \end{bmatrix}$ | | $-e_1 + 2e_2 - e_3$ |
| 8 | $\begin{bmatrix} 0 & -3 \\ -1 & 0 \end{bmatrix}$ | $\begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}$ | $e_2 - e_3$ | |
| 9 | $\begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$ | $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ | | $e_1 + e_2 - 2e_3$ |
| 10 | $\begin{bmatrix} 0 & -3 \\ -1 & 0 \end{bmatrix}$ | $\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$ | $-e_3 + e_1$ | |
| 11 | $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ | $\begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix}$ | | $2e_1 - e_2 - e_3$ |
| 12 | $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ | $\begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix}$ | $e_1 - e_2$ | |

The Cartan matrix of this representation is

$$A(G_2) = \begin{bmatrix} 2 & 0 & 0 & 3 \\ 0 & 6 & 3 & 0 \\ 0 & 3 & 2 & 0 \\ 3 & 0 & 0 & 6 \end{bmatrix} \cong \begin{bmatrix} 2 & 3 \\ 3 & 6 \end{bmatrix} . \oplus \begin{bmatrix} 2 & 3 \\ 3 & 6 \end{bmatrix} .$$

Consider the representation $B_3$ of $\mathrm{Rep}_6(A_1)$:

| $\lambda$ | $N_\lambda$ | $D \cdot \widetilde{N}_\lambda$ | $\varphi(1,\lambda)$ | $\varphi(2,\lambda)$ | $\varphi(3,\lambda)$ |
|---|---|---|---|---|---|
| 0 | $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ | $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ | $e_1 - e_2$ | | $e_1 + e_2 + e_3 - e_4$ |
| 1 | $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ | $\begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}$ | | $e_1 - e_2 + e_3 - e_4$ | |
| 2 | $\begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ | $\begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & -2 \end{bmatrix}$ | $e_3 - e_4$ | | $-2e_2$ |
| 3 | $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ | $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ | | $-e_1 - e_2 + e_3 - e_4$ | |
| 4 | $\begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ | $\begin{bmatrix} 0 & 0 & -2 \\ 0 & -2 & 0 \\ -2 & 0 & 2 \end{bmatrix}$ | $-e_1 - e_2$ | | $-e_1 + e_2 + e_3 - e_4$ |
| 5 | $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ | $\begin{bmatrix} 0 & -2 & 0 \\ -2 & 0 & -2 \\ 0 & -2 & 0 \end{bmatrix}$ | | $-2e_1$ | |
| 6 | $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ | $\begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$ | $-e_1 + e_2$ | | $-e_1 - e_2 - e_3 + e_4$ |
| 7 | $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ | $\begin{bmatrix} 0 & -2 & 0 \\ -2 & 0 & -2 \\ 0 & -2 & 0 \end{bmatrix}$ | | $-e_1 + e_2 - e_3 + e_4$ | |
| 8 | $\begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ | $\begin{bmatrix} 0 & 0 & -2 \\ 0 & -2 & 0 \\ -2 & 0 & 2 \end{bmatrix}$ | $-e_3 + e_4$ | | $2e_2$ |
| 9 | $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ | $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ | | $e_1 + e_2 - e_3 + e_4$ | |

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| | | | | | |
|---|---|---|---|---|---|
| 10 | $\begin{bmatrix}0 & 2 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{bmatrix}$ | $\begin{bmatrix}0 & 0 & 2\\ 0 & 2 & 0\\ 2 & 0 & -2\end{bmatrix}$ | $e_1 + e_2$ | | $e_1 - e_2 - e_3 + e_4$ |
| 11 | $\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$ | $\begin{bmatrix}0 & 2 & 0\\ 2 & 0 & 2\\ 0 & 2 & 0\end{bmatrix}$ | | $2e_1$ | |
| 12 | $\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$ | $\begin{bmatrix}2 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 4\end{bmatrix}$ | $e_1 - e_2$ | | $e_1 + e_2 + e_3 - e_4$ |

The Cartan matrix of this representation is

$$A(B_3) = \begin{bmatrix}2 & 0 & 0 & 0 & 2 & 0\\ 0 & 4 & 0 & 2 & 0 & 2\\ 0 & 0 & 4 & 0 & 2 & 0\\ 0 & 2 & 0 & 2 & 0 & 0\\ 2 & 0 & 2 & 0 & 4 & 0\\ 0 & 2 & 0 & 0 & 0 & 4\end{bmatrix} \cong \begin{bmatrix}2 & 2 & 0\\ 2 & 4 & 2\\ 0 & 2 & 4\end{bmatrix} \oplus \begin{bmatrix}2 & 2 & 0\\ 2 & 4 & 2\\ 0 & 2 & 4\end{bmatrix}.$$

## 7.3 Affine Root Systems of Type $A_1$

Consider the affine representation ${A_1}^{(1)}$ of Rep($A_1$) at level 2:

| $\lambda$ | $N_\lambda$ | $D \cdot \widetilde{N}_\lambda$ | $\varphi(1,\lambda)$ | $\varphi(2,\lambda)$ |
|---|---|---|---|---|
| 0 | $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ | $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ | $e_1 - e_2$ | $e_3 - e_4$ |
| 1 | $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ | $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ | $e_1 - e_4$ | $e_3 - e_2$ |
| 2 | $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ | $\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$ | $e_3 - e_4$ | $e_1 - e_2$ |
| 3 | $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ | $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ | $e_3 - e_2$ | $e_1 - e_4$ |
| 4 | $\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$ | $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ | $e_1 - e_2$ | $e_3 - e_4$ |

The Cartan matrix of this representation is

$$A({A_1}^{(1)}) = \begin{bmatrix} 2 & 0 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 1 & 1 & 2 & 0 \\ 1 & 1 & 0 & 2 \end{bmatrix}.$$

Consider the affine representation ${A_2}^{(1)}$ of $\mathrm{Rep}(A_1)$ at level 3:

| $\lambda$ | $N_\lambda$ | $D \cdot \widetilde{N}_\lambda$ | $\varphi(1,\lambda)$ | $\varphi(2,\lambda)$ | $\varphi(3,\lambda)$ |
|---|---|---|---|---|---|
| 0 | $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ | $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ | $e_1 - e_2$ | $e_3 - e_4$ | $e_5 - e_6$ |
| 1 | $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ | $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ | $e_5 - e_4$ | $e_1 - e_6$ | $e_3 - e_2$ |
| 2 | $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ | $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ | $e_3 - e_6$ | $e_5 - e_2$ | $e_1 - e_4$ |
| 3 | $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ | $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ | $e_1 - e_2$ | $e_3 - e_4$ | $e_5 - e_6$ |

The Cartan matrix of this representation is

$$A({A_2}^{(1)}) = \begin{bmatrix} 2 & 0 & 0 & 0 & 1 & 1 \\ 0 & 2 & 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 2 & 0 & 0 \\ 1 & 0 & 1 & 0 & 2 & 0 \\ 1 & 1 & 0 & 0 & 0 & 2 \end{bmatrix}.$$

Consider the affine representation ${C_2}^{(2)}$ of $\text{Rep}(A_1)$ at level 2:

| $\lambda$ | $N_\lambda$ | $D \cdot \widetilde{N}_\lambda$ | $\varphi(1,\lambda)$ | $\varphi(2,\lambda)$ | $\varphi(3,\lambda)$ |
|---|---|---|---|---|---|
| 0 | $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ | $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ | $e_1 + e_2$ | | $e_1 - e_2$ |
| 1 | $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ | $\begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}$ | | $2e_1$ | |
| 2 | $\begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix}$ | $\begin{bmatrix} 0 & 0 & 2 \\ 0 & 4 & 0 \\ 2 & 0 & 0 \end{bmatrix}$ | $e_1 - e_2$ | | $e_1 + e_2$ |
| 3 | $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 0 \\ 2 & 0 & 1 \end{bmatrix}$ | $\begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}$ | | $2e_1$ | |
| 4 | $\begin{bmatrix} 0 & 4 & 0 \\ 2 & 0 & 2 \\ 0 & 4 & 0 \end{bmatrix}$ | $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ | $e_1 + e_2$ | | $e_1 - e_2$ |

The Cartan matrix of this representation is

$$A({C_2}^{(2)}) = \begin{bmatrix} 2 & 0 & 0 & 0 & 2 & 0 \\ 0 & 4 & 0 & 2 & 0 & 2 \\ 0 & 0 & 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 & 0 & 0 \\ 2 & 0 & 2 & 0 & 4 & 0 \\ 0 & 2 & 0 & 0 & 0 & 2 \end{bmatrix} \cong \begin{bmatrix} 2 & 2 & 0 \\ 2 & 4 & 2 \\ 0 & 2 & 2 \end{bmatrix} \oplus \begin{bmatrix} 2 & 2 & 0 \\ 2 & 4 & 2 \\ 0 & 2 & 2 \end{bmatrix}.$$

Consider the affine representation ${G_2}^{(1)}$ of $\mathrm{Rep}(A_1)$ at level 2:

| $\lambda$ | $N_\lambda$ | $D \cdot \widetilde{N}_\lambda$ | $\varphi(1,\lambda)$ | $\varphi(2,\lambda)$ | $\varphi(3,\lambda)$ |
|---|---|---|---|---|---|
| 0 | $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ | $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{bmatrix}$ | $e_1 - e_2$ | | $e_1 + e_2 - 2e_3$ |
| 1 | $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ | $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 3 \\ 0 & 3 & 0 \end{bmatrix}$ | | $e_1 - e_3$ | |
| 2 | $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 3 \\ 0 & 1 & 0 \end{bmatrix}$ | $\begin{bmatrix} -1 & 0 & 3 \\ 0 & 2 & 0 \\ 3 & 0 & 3 \end{bmatrix}$ | $e_2 - e_3$ | | $2e_1 - e_2 - e_3$ |
| 3 | $\begin{bmatrix} 0 & 0 & 3 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{bmatrix}$ | $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 3 \\ 0 & 3 & 0 \end{bmatrix}$ | | $e_1 - e_3$ | |
| 4 | $\begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 6 \\ 0 & 2 & 0 \end{bmatrix}$ | $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{bmatrix}$ | $e_1 - e_2$ | | $e_1 + e_2 - 2e_3$ |

The Cartan matrix of this representation is

$$A({G_2}^{(1)}) = \begin{bmatrix} 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 & 3 \\ 0 & 0 & 6 & 0 & 3 & 0 \\ 0 & 1 & 0 & 2 & 0 & 0 \\ 1 & 0 & 3 & 0 & 2 & 0 \\ 0 & 3 & 0 & 0 & 0 & 6 \end{bmatrix} \cong \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 3 \\ 0 & 3 & 6 \end{bmatrix} \oplus \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 3 \\ 0 & 3 & 6 \end{bmatrix}.$$

Consider the affine representation ${G_2}^{(3)}$ of $\mathrm{Rep}(A_1)$ at level 2:

| $\lambda$ | $N_\lambda$ | $D \cdot \widetilde{N}_\lambda$ | $\varphi(1,\lambda)$ | $\varphi(2,\lambda)$ | $\varphi(3,\lambda)$ |
|---|---|---|---|---|---|
| 0 | $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ | $\begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ | $e_1 + e_2 - 2e_3$ | | $e_1 - e_2$ |
| 1 | $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ | $\begin{bmatrix} 0 & 3 & 0 \\ 3 & 0 & 3 \\ 0 & 3 & 0 \end{bmatrix}$ | | $2e_1 - e_2 - e_3$ | |
| 2 | $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 3 & 0 \end{bmatrix}$ | $\begin{bmatrix} -3 & 0 & 3 \\ 0 & 6 & 0 \\ 3 & 0 & 1 \end{bmatrix}$ | $e_1 - 2e_2 + e_3$ | | $e_1 - e_3$ |
| 3 | $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 3 & 0 \\ 3 & 0 & 2 \end{bmatrix}$ | $\begin{bmatrix} 0 & 3 & 0 \\ 3 & 0 & 3 \\ 0 & 3 & 0 \end{bmatrix}$ | | $2e_1 - e_2 - e_3$ | |
| 4 | $\begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 2 \\ 0 & 6 & 0 \end{bmatrix}$ | $\begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ | $e_1 + e_2 - 2e_3$ | | $e_1 - e_2$ |

The Cartan matrix of this representation is

$$A({G_2}^{(3)}) = \begin{bmatrix} 6 & 0 & 0 & 0 & 3 & 0 \\ 0 & 6 & 0 & 3 & 0 & 3 \\ 0 & 0 & 2 & 0 & 3 & 0 \\ 0 & 3 & 0 & 6 & 0 & 0 \\ 3 & 0 & 3 & 0 & 6 & 0 \\ 0 & 3 & 0 & 0 & 0 & 2 \end{bmatrix} \cong \begin{bmatrix} 6 & 3 & 0 \\ 3 & 6 & 3 \\ 0 & 3 & 2 \end{bmatrix} \oplus \begin{bmatrix} 6 & 3 & 0 \\ 3 & 6 & 3 \\ 0 & 3 & 2 \end{bmatrix}.$$

## 7.4 Root Systems of Type $A_2$

Consider the representation $AB_1$ of $\mathrm{Rep}_3(A_2)$:

| $\lambda$ | $N_\lambda$ | $D\widetilde{N}_\lambda$ | $\varphi(1,\lambda)$ |
|---|---|---|---|
| $(0,0)$ | $[0]$ | $[6]$ | $2e_1 - e_2 - e_3$ |
| $(0,1)$ | $[0]$ | $[0]$ | |
| $(0,2)$ | $[0]$ | $[0]$ | |
| $(1,0)$ | $[0]$ | $[0]$ | |
| $(1,1)$ | $[1]$ | $[-3]$ | $-e_1 + 2e_2 - e_3$ |
| $(1,2)$ | $[0]$ | $[0]$ | |
| $(2,0)$ | $[0]$ | $[0]$ | |
| $(2,1)$ | $[0]$ | $[0]$ | |
| $(2,2)$ | $[-1]$ | $[-3]$ | $-e_1 - e_2 + 2e_3$ |

The Cartan matrix of this root system is

$$A = \begin{bmatrix} 6 & 0 & 0 & 0 & -3 & 0 \\ 0 & 6 & 0 & 0 & 0 & -3 \\ 0 & 0 & 6 & -3 & 0 & 0 \\ 0 & 0 & -3 & 6 & 0 & 0 \\ -3 & 0 & 0 & 0 & 6 & 0 \\ 0 & -3 & 0 & 0 & 0 & 6 \end{bmatrix} \cong \begin{bmatrix} 6 & -3 \\ -3 & 6 \end{bmatrix} \oplus \begin{bmatrix} 6 & -3 \\ -3 & 6 \end{bmatrix} \oplus \begin{bmatrix} 6 & -3 \\ -3 & 6 \end{bmatrix}$$

Consider the representation $BB_1$ of $\mathrm{Rep}_3(A_2)$:

| $\lambda$ | $N_\lambda$ | $D\widetilde{N}_\lambda$ | $\varphi(1,\lambda)$ |
|---|---|---|---|
| $(0,0)$ | $[0]$ | $[6]$ | $e_1+e_2+e_3+e_4+e_5+e_6$ |
| $(0,1)$ | $[0]$ | $[2]$ | $e_1+e_2+e_3+e_4-e_5-e_6$ |
| $(0,2)$ | $[0]$ | $[-2]$ | $e_1-e_2-e_3+e_4-e_5-e_6$ |
| $(0,3)$ | $[0]$ | $[2]$ | $e_1-e_2-e_3+e_4+e_5+e_6$ |
| $(1,0)$ | $[0]$ | $[2]$ | $e_1+e_2+e_3-e_4+e_5-e_6$ |
| $(1,1)$ | $[1]$ | $[-2]$ | $e_1+e_2-e_3-e_4-e_5-e_6$ |
| $(1,2)$ | $[1]$ | $[-2]$ | $e_1-e_2-e_3-e_4-e_5+e_6$ |
| $(1,3)$ | $[0]$ | $[2]$ | $e_1-e_2+e_3-e_4+e_5+e_6$ |
| $(2,0)$ | $[0]$ | $[-2]$ | $-e_1+e_2-e_3-e_4+e_5-e_6$ |
| $(2,1)$ | $[1]$ | $[-2]$ | $-e_1+e_2-e_3-e_4-e_5+e_6$ |
| $(2,2)$ | $[0]$ | $[-2]$ | $-e_1-e_2+e_3-e_4-e_5+e_6$ |
| $(2,3)$ | $[-1]$ | $[-2]$ | $-e_1-e_2+e_3-e_4+e_5-e_6$ |
| $(3,0)$ | $[0]$ | $[2]$ | $-e_1+e_2-e_3+e_4+e_5+e_6$ |
| $(3,1)$ | $[0]$ | $[2]$ | $-e_1+e_2+e_3+e_4-e_5+e_6$ |
| $(3,2)$ | $[-1]$ | $[-2]$ | $-e_1-e_2+e_3+e_4-e_5-e_6$ |
| $(3,3)$ | $[-1]$ | $[-2]$ | $-e_1-e_2-e_3+e_4+e_5-e_6$ |

This root system was identified by Ocneanu in [11], but without an explicit construction. The Cartan matrix is

$$A = \begin{bmatrix} 6 & 2 & -2 & 2 & -2 & -2 \\ 2 & 6 & 2 & 2 & 2 & -2 \\ -2 & 2 & 6 & -2 & 2 & -2 \\ 2 & 2 & -2 & 6 & 2 & 2 \\ -2 & 2 & 2 & 2 & 6 & 2 \\ -2 & -2 & -2 & 2 & 2 & 6 \end{bmatrix}.$$

# Chapter 8

# Classifications, Constructions, and More Examples

In this chapter, we provide the rest of our original results. In particular, we present our classification of the symmetric representations of $\text{Rep}(A_2)$ and the symmetric affine representations of $\text{Rep}(A_2)$. Next, we present a graph construction that uses these symmetric representations to produce many new and unusual (nonsymmetric) representations of $\text{Rep}_k(A_2)$ and affine representations of $\text{Rep}(A_2)$. Finally, we list the output of computer programs written by this author which generate representations of $\text{Rep}_k(A_2)$ and affine representations of $\text{Rep}(A_2)$ up to some maximum number of vertices.

## 8.1 Symmetric Classification

The involutive-semiring $\text{Rep}_k(A_2)$ is generated over multiplication and involution by the isomorphism class of weight $(2,1) \in P$. Thus any representation of $\text{Rep}_k(A_2)$ is completely determined by its image on $R_{(2,1)}$. In other words, we can think of a representation of $\text{Rep}_k(A_2)$ as a single nonnegative integer matrix $G = \pi(R_{(2,1)})$. We refer to $G$ as the graph of $\pi$.

As with the case of $\text{Rep}_k(A_1)$, we can safely identify a representation of $\text{Rep}_k(A_2)$ with its graph. In particular, two representations of $\text{Rep}_k(A_2)$ are isomorphic if and only if their corresponding graphs are isomorphic. A representation of $\text{Rep}_k(A_2)$ is indecomposable if and only if its graph is connected. A representation of $\text{Rep}_k(A_2)$ is symmetric if and only if its graph is symmetric.

The same argument above also applies to affine representations of $\mathrm{Rep}(A_2)$.

**Theorem 8.1**: The only indecomposable symmetric representations of $\mathrm{Rep}_k(A_2)$ are, up to isomorphism, those listed in Table 8.1. The only indecomposable symmetric affine representations of $\mathrm{Rep}(A_2)$ are, up to isomorphism, those listed in Table 8.3.

We prove this theorem in the appendix.

Table 8.1: List of symmetric representations of $\mathrm{Rep}_k(A_2)$.

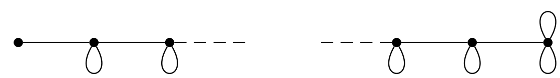
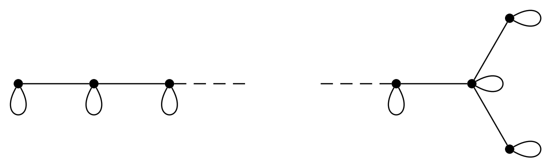
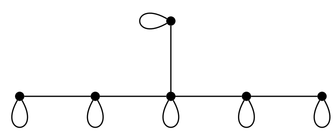
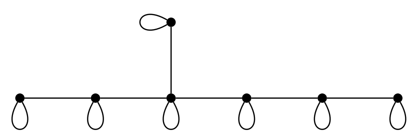
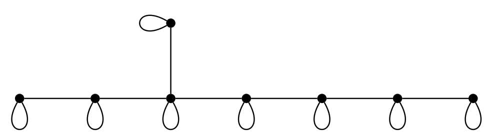
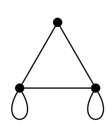
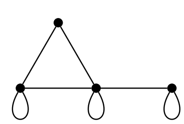
| **Name** | **Rank** | **Graph** |
|---|---|---|
| $AB_r$ | $r \geq 1$ |  |
| $AT_r$ | $r \geq 2$ |  |
| $BB_r$ | $r \geq 1$ |  |
| $BD_r$ | $r \geq 4$ |  |
| $BT_r$ | $r \geq 1$ |  |
| $E_6$ | 6 |  |
| $E_7$ | 7 |  |
| $E_8$ | 8 |  |
| $H_3$ | 3 |  |
| $H_4$ | 4 |  |

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Table 8.1: List of symmetric representations of $\mathrm{Rep}_k(A_2)$ (continued).

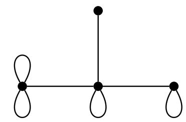
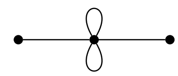
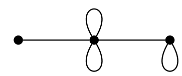
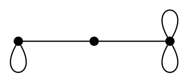
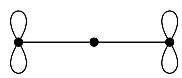
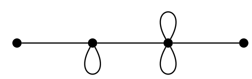
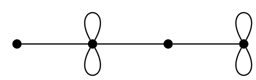
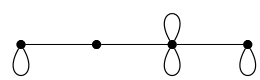
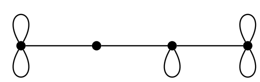
| **Name** | **Rank** | **Graph** |
|---|---|---|
| $Q_4$ | 4 |  |
| $S_{0,2,0}$ | 3 |  |
| $S_{0,2,1}$ | 3 |  |
| $S_{1,0,2}$ | 3 |  |
| $S_{2,0,2}$ | 3 |  |
| $S_{0,1,2,0}$ | 4 |  |
| $S_{0,2,0,2}$ | 4 |  |
| $S_{1,0,2,1}$ | 4 |  |
| $S_{2,0,1,2}$ | 4 |  |

Table 8.2: Exponents of symmetric representations of $\mathrm{Rep}_k(A_2)$.

| **Name** | **Level** | **Exponents** |
|---|---|---|
| $AB_r$ | $2r+1$ | $(2r-1,1,1)$ $(2r-3,2,2)$ $\cdots$ $(1,r,r)$ |
| $AT_r$ | $4r$ | $(4r-2,1,1)$ $(4r-6,3,3)$ $\cdots$ $(2,2r-1,2r-1)$ |
| $BB_r$ | $2r+2$ | $(2r,1,1)$ $(2r-2,2,2)$ $\cdots$ $(2,r,r)$ |
| $BD_r$ | $4r-4$ | $(4r-6,1,1)$ $(4r-10,3,3)$ $\cdots$ $(2,2r-3,2r-3)$ $(2r-2,r-1,r-1)$ |
| $BT_r$ | $4r+2$ | $(4r,1,1)$ $(4r-4,3,3)$ $\cdots$ $(8,2r-3,2r-3)$ |

Continued on next page . . .

Table 8.2: Exponents of symmetric representations of $\mathrm{Rep}_k(A_2)$ (continued).

| **Name** | **Level** | **Exponents** |
|---|---|---|
| $E_6$ | 24 | $(22,1,1)$ $(16,4,4)$ $(14,5,5)$ $(10,7,7)$ $(8,8,8)$ $(2,11,11)$ |
| $E_7$ | 36 | $(34,1,1)$ $(26,5,5)$ $(22,7,7)$ $(18,9,9)$ $(14,11,11)$ $(10,13,13)$ $(2,17,17)$ |
| $E_8$ | 60 | $(58,1,1)$ $(46,7,7)$ $(38,11,11)$ $(34,13,13)$ $(26,17,17)$ $(22,19,19)$ $(14,23,23)$ $(2,29,29)$ |
| $H_3$ | 12 | $(10,1,1)$ $(4,4,4)$ $(2,5,5)$ |
| $H_4$ | 30 | $(28,1,1)$ $(16,7,7)$ $(8,11,11)$ $(4,13,13)$ |
| $Q_4$ | 24 | $(22,1,1)$ $(14,5,5)$ $(10,7,7)$ $(2,11,11)$ |
| $S_{0,2,0}$ | 12 | $(10,1,1)$ $(4,4,4)$ $(2,5,5)$ |
| $S_{0,2,1}$ | 18 | $(16,1,1)$ $(8,5,5)$ $(4,7,7)$ |
| $S_{1,0,2}$ | 9 | $(7,1,1)$ $(5,2,2)$ $(1,4,4)$ |
| $S_{2,0,2}$ | 12 | $(10,1,1)$ $(8,2,2)$ $(2,5,5)$ |
| $S_{0,1,2,0}$ | 30 | $(28,1,1)$ $(16,7,7)$ $(8,11,11)$ $(4,13,13)$ |
| $S_{0,2,0,2}$ | 20 | $(18,1,1)$ $(14,3,3)$ $(6,7,7)$ $(2,9,9)$ |
| $S_{1,0,2,1}$ | 24 | $(22,1,1)$ $(14,5,5)$ $(10,7,7)$ $(2,11,11)$ |
| $S_{2,0,1,2}$ | 15 | $(13,1,1)$ $(11,2,2)$ $(7,4,4)$ $(1,7,7)$ |

Table 8.3: List of symmetric affine representations of $\mathrm{Rep}(A_2)$.

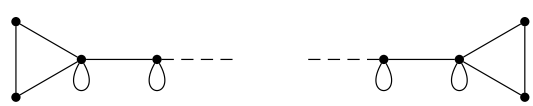
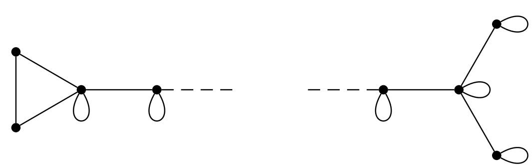
| **Name** | **Rank** | **Graph** |
|---|---|---|
| $CC_r$ | $r \geq 5$ |  |
| $CD_r$ | $r \geq 5$ |  |

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Table 8.3: List of symmetric affine representations of $\mathrm{Rep}(A_2)$ (continued).

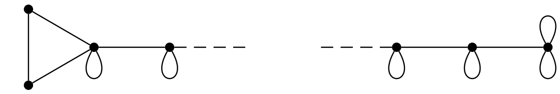
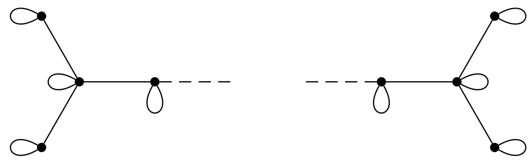
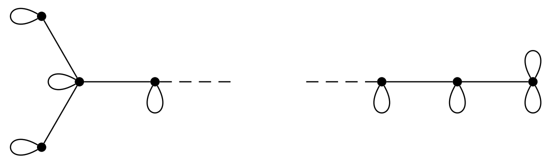
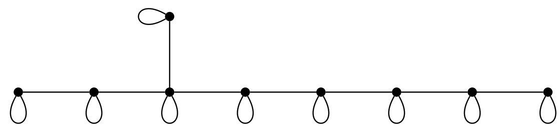
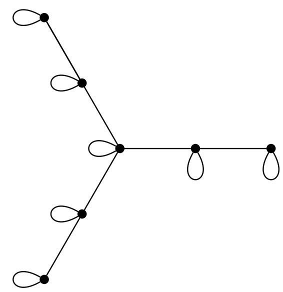
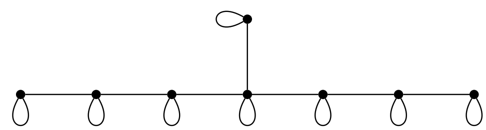
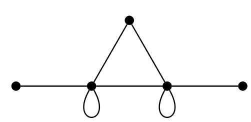
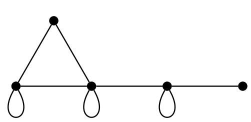
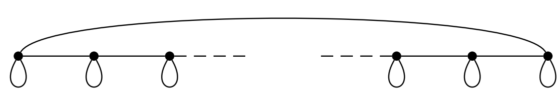
| **Name** | **Rank** | **Graph** |
|---|---|---|
| $CT_r$ | $r \geq 3$ |  |
| $DD_r$ | $r \geq 5$ |  |
| $DT_r$ | $r \geq 3$ |  |
| $E_9$ | 9 |  |
| $F_7$ | 7 |  |
| $G_8$ | 8 |  |
| $I_5$ | 5 |  |
| $J_5$ | 5 |  |
| $K_r$ | $r \geq 1$ |  |

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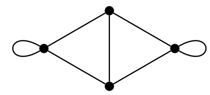
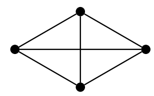
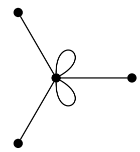
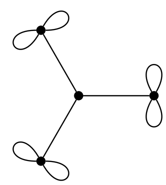
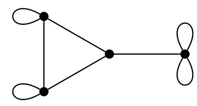
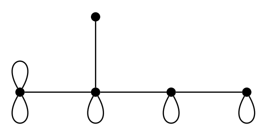
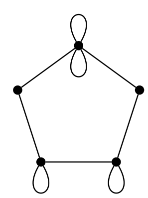
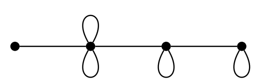
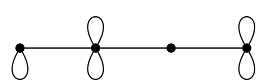
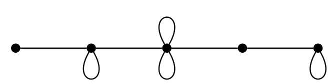
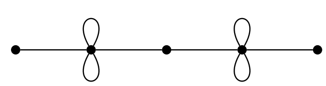
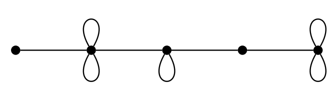
Table 8.3: List of symmetric affine representations of $\mathrm{Rep}(A_2)$ (continued).

| **Name** | **Rank** | **Graph** |
|---|---|---|
| $L_4$ | 4 |  |
| $M_4$ | 4 |  |
| $N_4$ | 4 |  |
| $O_4$ | 4 |  |
| $P_4$ | 4 |  |
| $Q_5$ | 5 |  |
| $R_5$ | 5 |  |
| $S_{0,2,1,1}$ | 4 |  |
| $S_{1,2,0,2}$ | 4 |  |
| $S_{0,1,2,0,1}$ | 5 |  |
| $S_{0,2,0,2,0}$ | 5 |  |
| $S_{0,2,1,0,2}$ | 5 |  |

Continued on next page . . .

Table 8.3: List of symmetric affine representations of Rep($A_2$) (continued).

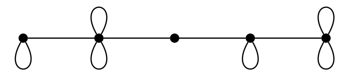
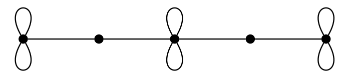
| Name | Rank | Graph |
|---|---|---|
| $S_{1,2,0,1,2}$ | 5 |  |
| $S_{2,0,2,0,2}$ | 5 |  |
| $TT_r$ | $r \geq 2$ |  |

Table 8.4: Exponents of symmetric affine representations of Rep($A_2$).

| Name | Level | Exponents |
|---|---|---|
| $CC_r$ | $2r-6$ | $(2r-6,0,0)$ $(2r-8,1,1)$ $\cdots$ $(0,r-3,r-3)$ $(0,n-3,n-3)$ $(0,r-3,r-3)$ |
| $CD_r$ | $4r-12$ | $(4r-12,0,0)$ $(4r-16,2,2)$ $\cdots$ $(0,2r-6,2r-6)$ $(2r-6,r-3,r-3)$ $(0,2r-6,2r-6)$ |
| $CT_r$ | $4r-6$ | $(4r-6,0,0)$ $(4r-10,2,2)$ $\cdots$ $(2,2r-4,2r-4)$ $(0,2r-3,2r-3)$ |
| $DD_r$ | $4r-12$ | $(4r-12,0,0)$ $(4r-16,2,2)$ $\cdots$ $(0,2r-6,2r-6)$ $(2r-6,r-3,r-3)$ $(2r-6,r-3,r-3)$ |
| $DT_r$ | $8r-12$ | $(8r-12,0,0)$ $(8r-20,4,4)$ $\cdots$ $(4,4r-8,4r-8)$ $(4r-6,2r-3,2r-3)$ |
| $E_9$ | 60 | $(60,0,0)$ $(48,6,6)$ $(40,10,10)$ $(36,12,12)$ $(30,15,15)$ $(24,18,18)$ $(20,20,20)$ $(12,24,24)$ $(0,30,30)$ |
| $F_7$ | 12 | $(12,0,0)$ $(8,2,2)$ $(8,2,2)$ $(6,3,3)$ $(4,4,4)$ $(4,4,4)$ $(0,6,6)$ |
| $G_8$ | 24 | $(24,0,0)$ $(18,3,3)$ $(16,4,4)$ $(12,6,6)$ $(12,6,6)$ $(8,8,8)$ $(6,9,9)$ $(0,12,12)$ |
| $I_5$ | 12 | $(12,0,0)$ $(6,3,3)$ $(4,4,4)$ $(0,6,6)$ $(0,6,6)$ |
| $J_5$ | 30 | $(30,0,0)$ $(18,6,6)$ $(10,10,10)$ $(6,12,12)$ $(0,15,15)$ |

Continued on next page . . .

Table 8.4: Exponents of symmetric affine representations of Rep($A_2$) (continued).

| **Name** | **Level** | **Exponents** |
|---|---|---|
| $K_r$ | $r$ | $r$ odd $\begin{cases} (r,0,0)\ (r-2,1,1)\ \cdots\ (0,\frac{r-1}{2},\frac{r-1}{2}) \\ (r-2,1,1)\ (r-4,2,2)\ \cdots\ (0,\frac{r-1}{2},\frac{r-1}{2}) \end{cases}$ |
| | | $r$ even $\begin{cases} (r,0,0)\ (r-2,1,1)\ \cdots\ (0,\frac{r-2}{2},\frac{r-2}{2}) \\ (r-2,1,1)\ (r-4,2,2)\ \cdots\ (0,\frac{r}{2},\frac{r}{2}) \end{cases}$ |
| $L_4$ | 4 | $(4,0,0)\ (2,1,1)\ (0,2,2)\ (0,2,2)$ |
| $M_4$ | 2 | $(2,0,0)\ (0,1,1)\ (0,1,1)\ (0,1,1)$ |
| $N_4$ | 6 | $(6,0,0)\ (2,2,2)\ (2,2,2)\ (0,3,3)$ |
| $O_4$ | 6 | $(6,0,0)\ (4,1,1)\ (4,1,1)\ (0,3,3)$ |
| $P_4$ | 6 | $(6,0,0)\ (4,1,1)\ (2,2,2)\ (0,3,3)$ |
| $Q_5$ | 12 | $(12,0,0)\ (8,2,2)\ (6,3,3)\ (4,4,4)\ (0,6,6)$ |
| $R_5$ | 12 | $(12,0,0)\ (8,2,2)\ (6,3,3)\ (0,6,6)\ (0,6,6)$ |
| $S_{0,2,1,1}$ | 15 | $(15,0,0)\ (9,3,3)\ (5,5,5)\ (3,6,6)$ |
| $S_{1,2,0,2}$ | 7 | $(7,0,0)\ (5,1,1)\ (3,2,2)\ (1,3,3)$ |
| $S_{0,1,2,0,1}$ | 20 | $(20,0,0)\ (12,4,4)\ (10,5,5)\ (4,8,8)\ (0,10,10)$ |
| $S_{0,2,0,2,0}$ | 24 | $(24,0,0)\ (18,3,3)\ (8,8,8)\ (6,9,9)\ (0,12,12)$ |
| $S_{0,2,1,0,2}$ | 8 | $(8,0,0)\ (6,1,1)\ (4,2,2)\ (2,3,3)\ (0,4,4)$ |
| $S_{1,2,0,1,2}$ | 20 | $(20,0,0)\ (16,2,2)\ (10,5,5)\ (8,6,6)\ (0,10,10)$ |
| $S_{2,0,2,0,2}$ | 24 | $(24,0,0)\ (18,3,3)\ (16,4,4)\ (6,9,9)\ (0,12,12)$ |
| $TT_r$ | $2r$ | $(2r,0,0)\ (2r-2,1,1)\ \cdots\ (2,r-1,r-1)$ |

## 8.2 Tensor Construction

Let $N$ denote the $3 \times 3$ matrix

$$N := \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Then we have the following theorem.

**Theorem 8.2**: Let $G$ be a representation of $\mathrm{Rep}_k(A_2)$ (respectively an affine representation of $\mathrm{Rep}(A_2)$ at level $k$). Then the Kronecker product

$$G \otimes N = \begin{bmatrix} 0 & G & 0 \\ 0 & 0 & G \\ G & 0 & 0 \end{bmatrix}.$$

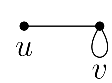
is also a representation of $\mathrm{Rep}_k(A_2)$ (respectively an affine representation of $\mathrm{Rep}_k(A_2)$ at level $k$).

We refer to the graph $G \otimes N$ as the **tensor construction** of $G$. Using the tensor construction, we can, without loss of generality, safely consider only graphs of the block form

$$G = \begin{bmatrix} 0 & A & 0 \\ 0 & 0 & B \\ C & 0 & 0 \end{bmatrix}.$$

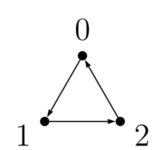
Any graph with this block form is said to be $N$-colourable. The matrices $A$, $B$, and $C$ need not be square.

For an example of the tensor construction, consider the graph

$AB_2 =$ 

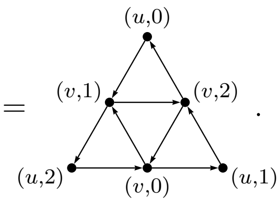
.

Label the vertices of $N$ so that

$N =$ 

.

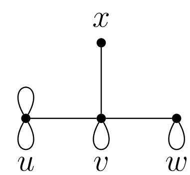
The tensor construction of $AB_2$ is then

$AB_2 \otimes N =$  .



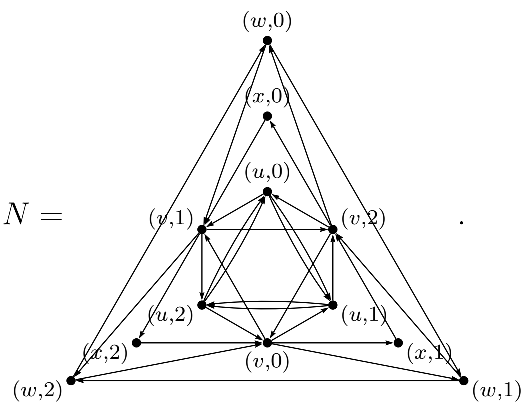
This is a well-known representation [1, 11] of $\mathrm{Rep}_5(A_2)$ with exponents $(3,1,1)$ $(2,2,1)$ $(2,1,2)$ $(1,3,1)$ $(1,2,2)$ $(1,1,3)$.

For another example of the tensor construction, consider the graph

$Q_4 =$  .



Working out the tensor construction, we get

$Q_4 \otimes N =$  .



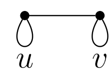
This is a previously unknown representation of $\mathrm{Rep}_{24}(A_2)$ with exponents $(22,1,1)$ $(14,5,5)$ $(11,11,2)$ $(11,2,11)$ $(10,7,7)$ $(7,10,7)$ $(7,7,10)$ $(5,14,5)$ $(5,5,14)$ $(2,11,11)$ $(1,22,1)$ $(1,1,22)$.

## 8.3 Automorphism Construction

In this section, we describe a construction that generalizes the tensor construction. The input to this new construction is a graph, an automorphism of the graph, and a map that assigns integers to the edges of the graph. The output is an $N$-colourable graph that, surprisingly often, turns out to be a representation of $\mathrm{Rep}_k(A_2)$ or an affine representation of $\mathrm{Rep}(A_2)$.

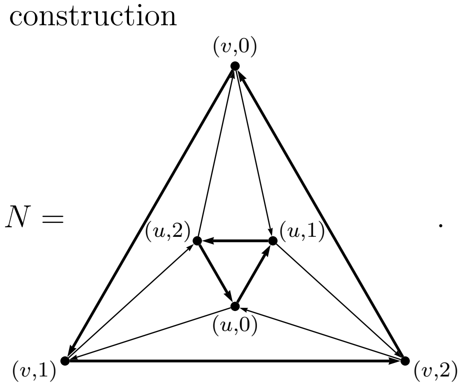
The definition of this new construction, which we label "the automorphism construction", is somewhat ad hoc: the definition comes from noticing patterns in the output of computer programs written by this author that generate representations of $\mathrm{Rep}_k(A_2)$ and affine representations of $\mathrm{Rep}(A_2)$ up to some maximum number of vertices. More specifically, about 60 percent of small (fewer than 15 vertices) $N$-colourable representations of $\mathrm{Rep}_k(A_2)$ and affine representations of $\mathrm{Rep}(A_2)$ appear to be tensor constructions. As the number of vertices increases, this percentage stays roughly the same, but more and more of the graphs in the remaining 40 percent begin to have structural properties that come from some "larger version" of the tensor construction.

To explain what we mean by "larger version", suppose every vertex of $G$ has exactly one loop. Then, in the tensor construction of $G$, a vertex of $G$ corresponds to a 3-cycle in $G \otimes N$, and an edge in $G$ corresponds to a 6-cycle in $G \otimes N$ which zig-zags between the two 3-cycles of its endpoints. For a concrete example, take the graph

$BB_2 =$  .



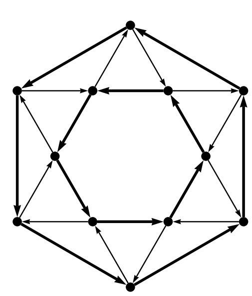
This graph has the tensor construction

$BB_2 \otimes N =$  .



The bold inner 3-cycle corresponds to the vertex $u$, the bold outer 3-cycle corresponds to the vertex $v$, and the unbold 6-cycle that zig-zags between these two 3-cycles corresponds to the edge $uv$.

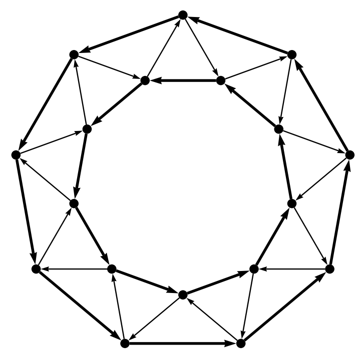
Now suppose we replace the bold 3-cycles with 6-cycles, and the 6-cycle zig-zag with a 12-cycle zig-zag. Then we get

$2BB_2 \otimes_{(2)} N =$  .

We will explain the notation $2BB_2 \otimes_{(2)} N$ when we formally define the automorphism construction. Essentially it means that the graph above comes from two copies of $BB_2$, along with an automorphism of (functional) order two that swaps the two copies.

The graph $2BB_2 \otimes_{(2)} N$ is a representation of $\mathrm{Rep}_{12}(A_2)^{\mathbb{Z}}$ with exponents $(8,2,2)$ $(7,4,1)$ $(7,1,4)$ $(4,7,1)$ $(4,4,4)$ $(4,4,4)$ $(4,4,4)$ $(4,1,7)$ $(2,8,2)$ $(2,2,8)$ $(1,7,4)$ $(1,4,7)$. Since the exponent $\rho = (10,1,1)$ does not appear in the above list, we refer to $2BB_2 \otimes_{(2)} N$ as a **pseudo-representation**.

In a similar manner, we can replace the bold 3-cycles in the tensor construction $BB_2 \otimes N$ with 9-cycles, and the unbold 6-cycle zig-zag with an 18-cycle zig-zag. With this modification, we get

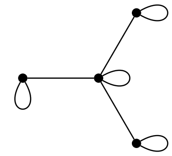
$3BB_2 \otimes_{(3)} N =$  .

As before, the notation $3BB_2 \otimes_{(3)} N$ essentially means that the above graph comes from three copies of $BB_2$, along with an automorphism of (functional) order three that cyclically permutes the three copies.

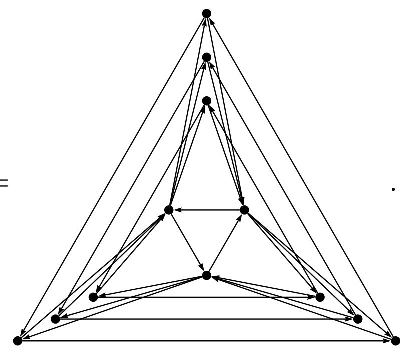
The graph $3BB_2 \otimes_{(3)} N$ is an affine representation of $\mathrm{Rep}(A_2)^{\mathbb{Z}}$ at level 6 with exponents $(4,2,0)$ $(4,1,1)$ $(4,0,2)$ $(3,2,1)$ $(3,1,2)$ $(2,4,0)$ $(2,3,1)$ $(2,2,2)$ $(2,2,2)$ $(2,2,2)$ $(2,1,3)$ $(2,0,4)$ $(1,4,1)$ $(1,3,2)$ $(1,2,3)$ $(1,1,4)$ $(0,4,2)$ $(0,2,4)$. Since the exponent $0 = (0,0,0)$ does not appear in the above list, we refer to $3BB_2 \otimes_{(3)} N$ as an **affine pseudo-representation**.

In general, for any positive integer $m$, we can replace the bold 3-cycles in $BB_2 \otimes N$ with $3m$-cycles, and the unbold 6-cycle zig-zag with a $6m$-cycle zig-zag. We denote the resulting graph using $mBB_2 \otimes_{(m)} N$. Unfortunately, computer tests seem to indicate that the graphs $mBB_2 \otimes_{(m)} N$ are all pseudo-representations except for $m = 1$. However, when we take graphs other than

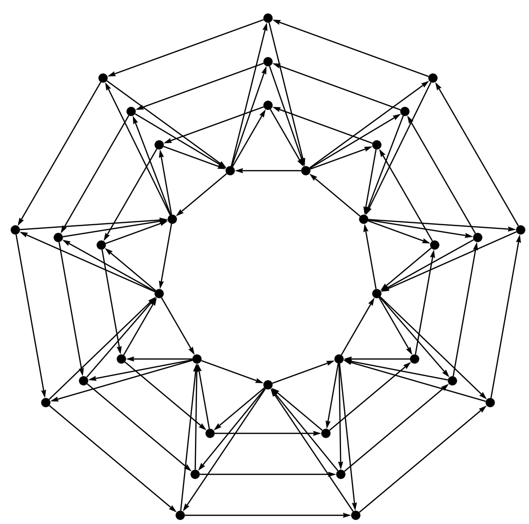
$BB_2$, more interesting things happen. For example, consider the graph

$BD_4 =$  .

Working out the tensor construction, we get

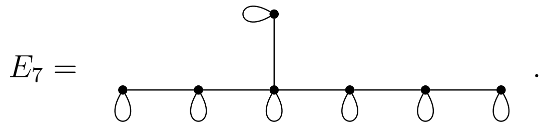
$BD_4 \otimes N =$  .

Taking $m = 3$, we have

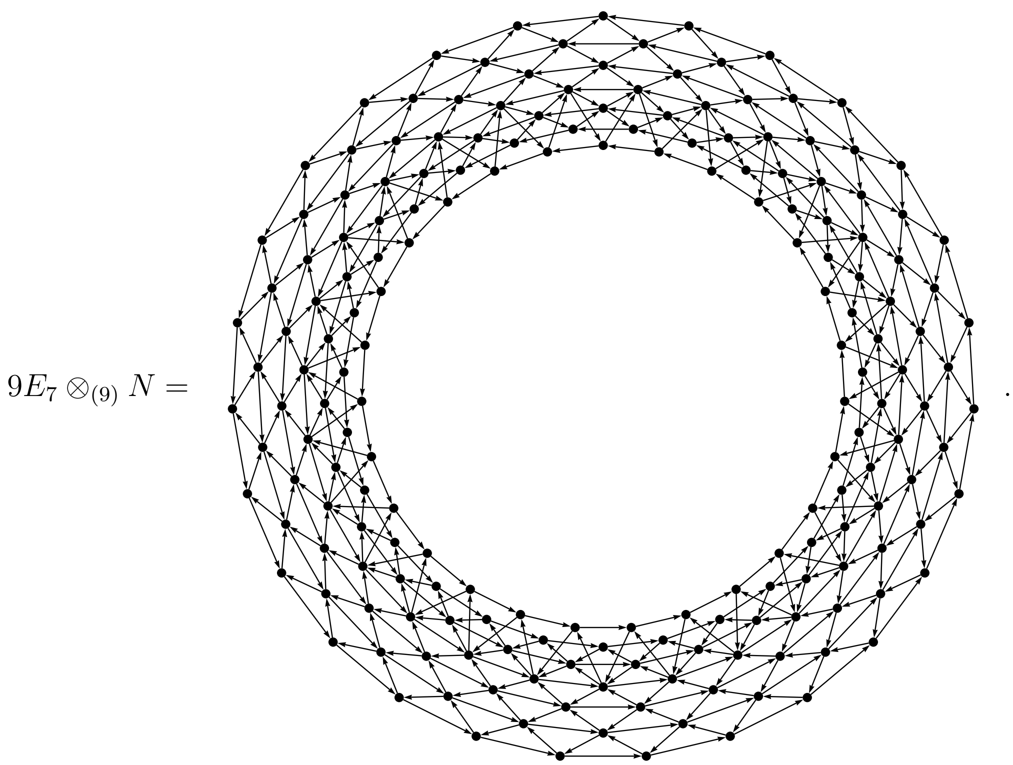
$3BD_4 \otimes_3 N =$  .

It turns out that $3BD_4 \otimes_{(3)} N$ is a representation of $\text{Rep}_{12}(A_2)$ with primary indices $(10, 1, 1)$ $(9, 2, 1)$ $(9, 1, 2)$ $(7, 3, 2)$ $(7, 2, 3)$ $(6, 5, 1)$ $(6, 5, 1)$ $(6, 3, 3)$ $(6, 3, 3)$ $(6, 1, 5)$ $(6, 1, 5)$ $(5, 6, 1)$ $(5, 6, 1)$ $(5, 5, 2)$ $(5, 2, 5)$ $(5, 1, 6)$ $(5, 1, 6)$ $(3, 7, 2)$ $(3, 6, 3)$ $(3, 6, 3)$ $(3, 3, 6)$ $(3, 3, 6)$ $(3, 2, 7)$ $(2, 9, 1)$ $(2, 7, 3)$ $(2, 5, 5)$ $(2, 3, 7)$ $(2, 1, 9)$ $(1, 10, 1)$ $(1, 9, 2)$ $(1, 6, 5)$ $(1, 6, 5)$ $(1, 5, 6)$ $(1, 5, 6)$ $(1, 2, 9)$ $(1, 1, 10)$. Computer tests seem to indicate that the graphs $mBD_4 \otimes_{(m)} N$ are all pseudo-representations except for $m \in \{1, 3\}$.

Even more interesting things happen when we try other graphs. For example, consider the graph



Taking $m = 9$, we get



It turns out that $9E_7 \otimes_{(7)} N_3$ is a representation of $\mathrm{Rep}_{36}(A_2)$ with exponents $(34,1,1)$ $(33,2,1)$ $(33,1,2)$ $(31,3,2)$ $(31,2,3)$ $(29,5,2)$ $(29,2,5)$ $(27,7,2)$ $(27,2,7)$ $(26,9,1)$ $(26,7,3)$ $(26,5,5)$ $(26,3,7)$ $(26,1,9)$ $(25,10,1)$ $(25,9,2)$ $(25,2,9)$ $(25,1,10)$ $(23,11,2)$ $(23,10,3)$ $(23,3,10)$ $(23,2,11)$ $(22,13,1)$ $(22,11,3)$ $(22,9,5)$ $(22,7,7)$ $(22,5,9)$ $(22,3,11)$ $(22,1,13)$ $(21,14,1)$ $(21,13,2)$ $(21,10,5)$ $(21,5,10)$ $(21,2,13)$ $(21,1,14)$ $(19,15,2)$ $(19,14,3)$ $(19,10,7)$ $(19,7,10)$ $(19,3,14)$ $(19,2,15)$ $(18,17,1)$ $(18,15,3)$ $(18,13,5)$ $(18,11,7)$ $(18,9,9)$ $(18,7,11)$ $(18,5,13)$ $(18,3,15)$ $(18,1,17)$ $(17,18,1)$ $(17,17,2)$ $(17,14,5)$ $(17,10,9)$ $(17,9,10)$ $(17,5,14)$ $(17,2,17)$ $(17,1,18)$ $(15,19,2)$ $(15,18,3)$ $(15,14,7)$ $(15,11,10)$ $(15,10,11)$ $(15,7,14)$ $(15,3,18)$ $(15,2,19)$ $(14,21,1)$ $(14,19,3)$ $(14,17,5)$ $(14,15,7)$ $(14,13,9)$ $(14,11,11)$ $(14,9,13)$ $(14,7,15)$ $(14,5,17)$ $(14,3,19)$ $(14,1,21)$ $(13,22,1)$ $(13,21,2)$ $(13,18,5)$ $(13,14,9)$ $(13,13,10)$ $(13,10,13)$ $(13,9,14)$ $(13,5,18)$ $(13,2,21)$ $(13,1,22)$ $(11,23,2)$ $(11,22,3)$ $(11,18,7)$ $(11,15,10)$ $(11,14,11)$ $(11,11,14)$ $(11,10,15)$ $(11,7,18)$ $(11,3,22)$ $(11,2,23)$ $(10,25,1)$ $(10,23,3)$ $(10,21,5)$ $(10,19,7)$ $(10,17,9)$ $(10,15,11)$ $(10,13,13)$

$(10,11,15)$ $(10,9,17)$ $(10,7,19)$ $(10,5,21)$ $(10,3,23)$ $(10,1,25)$ $(9,26,1)$ $(9,25,2)$ $(9,22,5)$ $(9,18,9)$ $(9,17,10)$ $(9,14,13)$ $(9,13,14)$ $(9,10,17)$ $(9,9,18)$ $(9,5,22)$ $(9,2,25)$ $(9,1,26)$ $(7,27,2)$ $(7,26,3)$ $(7,22,7)$ $(7,19,10)$ $(7,18,11)$ $(7,15,14)$ $(7,14,15)$ $(7,11,18)$ $(7,10,19)$ $(7,7,22)$ $(7,3,26)$ $(7,2,27)$ $(5,29,2)$ $(5,26,5)$ $(5,22,9)$ $(5,21,10)$ $(5,18,13)$ $(5,17,14)$ $(5,14,17)$ $(5,13,18)$ $(5,10,21)$ $(5,9,22)$ $(5,5,26)$ $(5,2,29)$ $(3,31,2)$ $(3,26,7)$ $(3,23,10)$ $(3,22,11)$ $(3,19,14)$ $(3,18,15)$ $(3,15,18)$ $(3,14,19)$ $(3,11,22)$ $(3,10,23)$ $(3,7,26)$ $(3,2,31)$ $(2,33,1)$ $(2,31,3)$ $(2,29,5)$ $(2,27,7)$ $(2,25,9)$ $(2,23,11)$ $(2,21,13)$ $(2,19,15)$ $(2,17,17)$ $(2,15,19)$ $(2,13,21)$ $(2,11,23)$ $(2,9,25)$ $(2,7,27)$ $(2,5,29)$ $(2,3,31)$ $(2,1,33)$ $(1,34,1)$ $(1,33,2)$ $(1,26,9)$ $(1,25,10)$ $(1,22,13)$ $(1,21,14)$ $(1,18,17)$ $(1,17,18)$ $(1,14,21)$ $(1,13,22)$ $(1,10,25)$ $(1,9,26)$ $(1,2,33)$ $(1,1,34)$. Computer tests seem to indicate that the graphs $mE_7 \otimes_{(m)} N_3$ are all pseudo-representations except for $m \in \{1,3,9\}$.

We may repeat the above procedure unambiguously on all graphs that satisfy the following conditions:

1. All vertices have exactly one loop.
2. Any two vertices are adjacent by at most one edge.
3. There are no $n$-cycles for all $n \geq 3$.

Let $G$ be such a graph. Table 8.5 lists the values of $m$ for which computer tests indicate that the graph $mG \otimes_{(m)} N$ is a representation of $\mathrm{Rep}_k(A_2)$.

Graphs of the form $mG \otimes_{(m)} N$ do not appear to exhibit these strange patterns when $G$ is a symmetric affine representation of $\mathrm{Rep}(A_2)$. In fact, computer tests indicate that $mG \otimes_{(m)} N$ is always an affine representations of $\mathrm{Rep}(A_2)$ for any value of $m$ when $G \in \{DD_n, E_9, F_7, G_8\}$.

The procedure of replacing, in the tensor construction, the 3-cycles that correspond to vertices with one loop by $3k$-cycles, and replacing the 6-cycle zig-zags that correspond to edges by $6k$-cycles, has a more general (and more precise) form that we call the automorphism construction.

Let $G$ be a graph, let $\pi$ be an automorphism of $G$, let $R$ be an index set for the edges and loops of $G$, and let $\rho$ be the map of $R$ to $V(G) \times V(G)$ which assigns elements of $R$ to edges and loops of $G$. More precisely, let $\rho$ be a map which satisfies

$$E(G)(u,v) = |\{r \in R : (u,v) = \rho(r)\}| \text{ for all } u,v \in V(G).$$

| Graph | $m$ |
|---|---|
| $mBB_n \otimes_{(m)} N$ | 1 |
| $mBD_4 \otimes_{(m)} N$ | 1, 3 |
| $mBD_5 \otimes_{(m)} N$ | 1 |
| $mBD_6 \otimes_{(m)} N$ | 1, 5 |
| $mBD_7 \otimes_{(m)} N$ | 1, 3 |
| $mBD_8 \otimes_{(m)} N$ | 1, 7 |
| $mBD_9 \otimes_{(m)} N$ | 1 |
| $mBD_{10} \otimes_{(m)} N$ | 1, 3, 9 |
| $mBD_{11} \otimes_{(m)} N$ | 1, 5 |
| $mBD_{12} \otimes_{(m)} N$ | 1, 11 |
| $mE_6 \otimes_{(m)} N$ | 1, 2, 4 |
| $mE_7 \otimes_{(m)} N$ | 1, 3, 9 |
| $mE_8 \otimes_{(m)} N$ | 1, 2, 3, 5, 6, 10, 15 |

Table 8.5: Examples of representations of $\mathrm{Rep}_k(A_2)$ of the form $mG \otimes_{(m)} N$.

Let $\eta$ be a map of $R$ to $\mathbb{Z}$. We define the **automorphism construction** of $G$ with respect to $\pi$ and $\eta$, denoted $G \otimes_{\pi,\eta} N$, to be the graph with vertex set

$$V(G \otimes_{\pi,\eta} N) = V(G) \times \{0, 1, 2\}$$

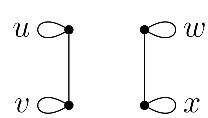
and with edge map defined by the rule

$$E(G \otimes_{\pi,\eta} N) : V(G \otimes_{\pi,\eta} N) \times V(G \otimes_{\pi,\eta} N) \to \mathbb{Z}_{\geq 0}$$

$$E(G \otimes_{\pi,\eta} N)((u,i),(v,j)) = \begin{cases} |\{r \in R : (u, \pi^{\eta(r)}(v)) = \rho(r)\}| & \text{if } j = i+1 \\ |\{r \in R : (u, \pi^{\eta(r)-1}(v)) = \rho(r)\}| & \text{if } j = i-2 \\ 0 & \text{otherwise} \end{cases}$$

for all $u, v \in V(G); i, j \in \{0, 1, 2\}$.

Using this definition, we show how to recover the graph $2BB_2 \otimes_{(2)} N$. Consider the disconnected graph

$2BB_2 =$ 

.

Choose $\pi$ to be the automorphism

$$\pi(u) = w \qquad \pi(w) = u$$
$$\pi(v) = x \qquad \pi(x) = v.$$

Since $2BB_2$ has 4 edges and 4 loops, choose $R$ to be the set

$$R = \{1, 2, 3, 4, 5, 6, 7, 8\}.$$

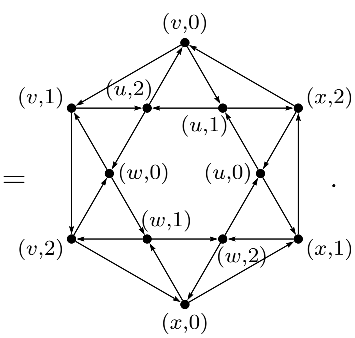
Choose $\rho$ to be the map of $R$ to $V(2BB_2) \times V(2BB_2)$ defined by

$$\begin{aligned} \rho(1) &= (u, u) & \rho(5) &= (w, w) \\ \rho(2) &= (u, v) & \rho(6) &= (w, x) \\ \rho(3) &= (v, u) & \rho(7) &= (x, w) \\ \rho(4) &= (v, v) & \rho(8) &= (x, x). \end{aligned}$$

Choose $\eta$ to be the map of $R$ to $\mathbb{Z}$ defined by

$$\eta(r) = \begin{cases} 1 & \text{if } r \in \{2, 6\} \\ 0 & \text{otherwise.} \end{cases}$$

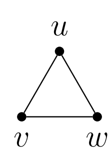
Then the automorphism construction of $2BB_2$ with respect to $\pi$ and $\eta$ produces the graph

$2BB_2 \otimes_{(2)} N =$ 

.

Of course, we already know that the above graph is a pseudo-representation. (See the appendix.) We will now look at examples of the automorphism construction which produce true representations of $\mathrm{Rep}_k(A_2)$.

Consider the graph

$AC_3 =$ 

.

This graph is an affine pseudo-representation at level 6 with exponents $(4, 1, 1)$ $(0, 3, 3)$ $(0, 3, 3)$. Let $\pi$ be the automorphism of $AC_3$ defined by

$$\pi(u) = v \qquad \pi(v) = w \qquad \pi(w) = u.$$

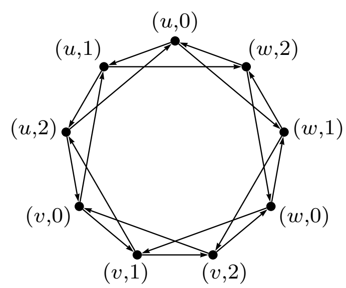
Since $AC_3$ has 6 edges, choose $R$ so that

$$R = \{1, 2, 3, 4, 5, 6\}.$$

Let $\rho$ be the map of $R$ to $V(AC_3) \times V(AC_3)$ defined by

$$\begin{aligned} \rho(1) &= (u, v) & \rho(4) &= (v, u) \\ \rho(2) &= (v, w) & \rho(5) &= (w, v) \\ \rho(3) &= (w, u) & \rho(6) &= (u, w). \end{aligned}$$

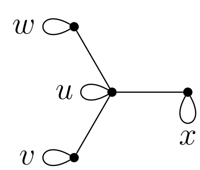
Let $\eta$ be the map of $R$ to $\mathbb{Z}$ that sends every element of $R$ to zero. (In this case, it was unnecessary to define $\rho$ because $\eta$ assigns the same integer to every edge.) Then the automorphism construction of $AC_3$ with respect to $\pi$ and $\eta$ produces the graph

$AC_3 \otimes_{\pi,\eta} N =$  .



The above graph is a representation of $\text{Rep}_6(A_2)$ with exponents $(4, 1, 1)$ $(3, 2, 1)$ $(3, 1, 2)$ $(2, 3, 1)$ $(2, 1, 3)$ $(1, 4, 1)$ $(1, 3, 2)$ $(1, 2, 3)$ $(1, 1, 4)$.

Consider the graph

$BD_4 =$  .



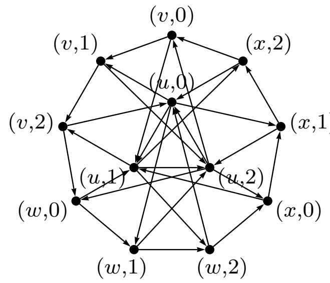
This graph is an representation of $\text{Rep}_{12}(A_2)$ with exponents $(10, 1, 1)$ $(6, 3, 3)$ $(6, 3, 3)$ $(2, 5, 5)$. Let $\pi$ be the automorphism of $BD_4$ defined by

$$\begin{aligned} \pi(u) &= u & \pi(w) &= x \\ \pi(v) &= w & \pi(x) &= v. \end{aligned}$$

Since $BD_4$ has 6 edges and 4 loops, choose $R$ so that

$$R = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}.$$

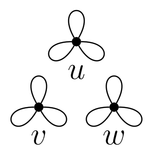
Let $\eta$ be the map of $R$ to $\mathbb{Z}$ that sends every element of $R$ to zero. Then the automorphism construction of $BD_4$ with respect to $\pi$ and $\eta$ produces the graph

$$BD_4 \otimes_{\pi,\eta} N =$$  .



The above graph is a representation of $\mathrm{Rep}_{12}(A_2)$ with exponents $(10, 1, 1)$ $(6, 5, 1)$ $(6, 1, 5)$ $(5, 6, 1)$ $(5, 5, 2)$ $(5, 2, 5)$ $(5, 1, 6)$ $(2, 5, 5)$ $(1, 10, 1)$ $(1, 6, 5)$ $(1, 5, 6)$ $(1, 1, 10)$.

Consider the graph

$$3K_1 =$$  .



Let $\pi$ be the automorphism of $3K_1$ defined by

$$\pi(u) = v \qquad\qquad \pi(v) = w \qquad\qquad \pi(w) = u.$$

Since $3K_1$ has 9 loops, choose $R$ so that

$$R = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}.$$

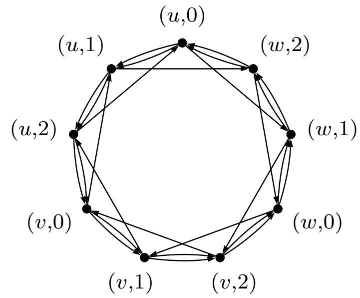
Let $\rho$ be the map of $R$ to $V(3K_1) \times V(3K_1)$ defined by

$$\rho(1) = \rho(2) = \rho(3) = (u, u)$$
$$\rho(4) = \rho(5) = \rho(6) = (v, v)$$
$$\rho(7) = \rho(8) = \rho(9) = (w, w).$$

Let $\eta$ be the map of $R$ to $\mathbb{Z}$ defined by

$$\eta(r) = \begin{cases} 2 & \text{if } r \in \{1, 4, 7\} \\ 0 & \text{otherwise} \end{cases} \qquad \text{for all } r \in R.$$

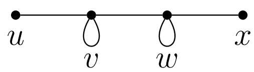
Then the automorphism construction of $3K_1$ with respect to $\pi$ and $\eta$ produces the graph

$$3K_1 \otimes_{\pi,\eta} N =$$ 

.

This graph is an affine representation of $\mathrm{Rep}(A_2)$ at level 3 with exponents $(3,0,0)$ $(2,1,0)$ $(2,0,1)$ $(1,2,0)$ $(1,0,2)$ $(0,3,0)$ $(0,2,1)$ $(0,1,2)$ $(0,0,3)$.

Consider the graph

$$AA_4 =$$ 

.

This graph is an affine pseudo-representation at level 8 with exponents $(6,1,1)$ $(4,2,2)$ $(2,3,3)$ $(0,4,4)$. Let $\pi$ be the automorphism of $AA_4$ defined by

$$\pi(u) = x \qquad \pi(x) = u$$
$$\pi(v) = w \qquad \pi(w) = v.$$

Since $AA_4$ has 6 edges and 2 loops, choose $R$ so that

$$R = \{1,2,3,4,5,6,7,8\}.$$

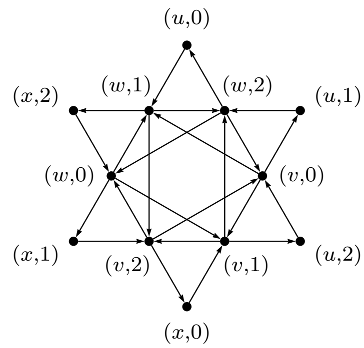
Let $\rho$ be the map of $R$ to $V(AA_4) \times V(AA_4)$ defined by

$$\rho(1) = (u,v) \qquad \rho(5) = (w,v)$$
$$\rho(2) = (v,u) \qquad \rho(6) = (w,w)$$
$$\rho(3) = (v,v) \qquad \rho(7) = (w,x)$$
$$\rho(4) = (v,w) \qquad \rho(8) = (x,v).$$

Let $\eta$ be the map of $R$ to $\mathbb{Z}$ defined by

$$\eta(r) = \begin{cases} 1 & \text{if } r \in \{1,8\} \\ 0 & \text{otherwise} \end{cases} \qquad \text{for all } r \in R.$$

Then the automorphism construction of $AA_4$ with respect to $\pi$ and $\eta$ produces the graph

$AA_4 \otimes_{\pi,\eta} N =$ 

.

This graph is a representation of $\mathrm{Rep}_8(A_2)$ with exponents $(6,1,1)$ $(4,3,1)$ $(4,1,3)$ $(3,4,1)$ $(3,3,2)$ $(3,2,3)$ $(3,1,4)$ $(2,3,3)$ $(1,6,1)$ $(1,4,3)$ $(1,3,4)$ $(1,1,6)$. The graph $AA_4 \otimes_{\pi,\eta} N$ is well-known in connection with subfactors and modules of SU(3) and goes by the various names of $\mathcal{E}_5$, $E_8$, and so on.

In the automorphism construction, the map $\eta$ is allowed to be very general. However, there appear to be strong restrictions on the maps that actually produce true representations. Almost certainly the map $\eta$ has to be invariant under the automorphism $\pi$ in the following sense: for all $r, s \in R$, we require that $\eta(r) = \eta(s)$ whenever $\rho(r) = (u, v)$ and $\rho(s) = (\pi(u), \pi(v))$.

Furthermore, if all vertices of $G$ have exactly one loop, if any two vertices of $G$ are adjacent by at most one edge, and if $G$ has no $n$-cycles for all $n \geq 3$, then very likely we require the following two additional conditions:

1. For all $r \in R$, if $\rho(r) = (u, u)$ for some $u \in V(G)$, then $\eta(r) = 0$.
2. For all $r, s \in R$, if $\rho(r) = (u, v)$ and $\rho(s) = (v, u)$ for some $u, v \in V(G)$, then one of $\eta(r)$ and $\eta(s)$ is zero while the other is one.

If both $G$ and $\eta$ satisfy all of the above conditions, then, in most practical cases, the automorphism construction of $G$ only depends (up to isomorphism) on the automorphism $\pi$. Thus, under this circumstance, we will drop the $\eta$ from the notation $G \otimes_{\pi,\eta} N$ and just write $G \otimes_\pi N$.

Finally, the automorphisms of $G$ that actually produce a connected automorphism construction are usually quite limited, and, if the order of the automorphism is known, can be chosen without affecting the automorphism construction (again, up to isomorphism). In such cases, we will just write $G \otimes_{(m)} N$ to mean the automorphism construction of $G$ with respect to an automorphism of order $m$.

The details here are admittedly somewhat vague, and should be worked out more exactly. The purpose of this condensed notation is just to give some quick examples of more large automorphism constructions. These examples are listed in Table 8.6.

| Graph | $m$ |
|---|---|
| $mBB_3 \otimes_{(2m)} N$ | 2 |
| $mBB_5 \otimes_{(2m)} N$ | 1, 3 |
| $mBB_7 \otimes_{(2m)} N$ | 4 |
| $mBB_9 \otimes_{(2m)} N$ | 1, 5 |
| $mBB_{11} \otimes_{(2m)} N$ | 2, 6 |
| $mBD_4 \otimes_{(3m)} N$ | 1, 2, 8 |
| $mBD_5 \otimes_{(2m)} N$ | 2, 4 |
| $mBD_7 \otimes_{(2m)} N$ | 2, 6 |
| $mBD_9 \otimes_{(2m)} N$ | 2, 4, 8 |
| $mBD_{11} \otimes_{(2m)} N$ | 2, 10 |
| $mE_6 \otimes_{(2m)} N$ | 1, 2, 3, 6 |

Table 8.6: Examples of representations of $\mathrm{Rep}_k(A_2)$ of the form $mG \otimes_{(nm)} N$.

Graphs of the form $mG \otimes_{(nm)} N$ do not appear to exhibit these strange patterns when $G$ is a symmetric affine representation of $\mathrm{Rep}(A_2)$. In particular, computer tests indicate that the graphs

$$mDD_5 \otimes_{(4m)} N$$
$$mF_7 \otimes_{(2m)} N$$
$$mF_7 \otimes_{(3m)} N$$
$$mG_8 \otimes_{(2m)} N$$

are affine representations of $\mathrm{Rep}(A_2)$ for any value of $m$.

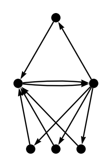
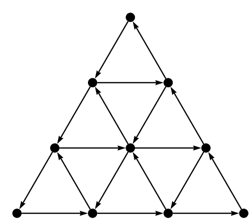
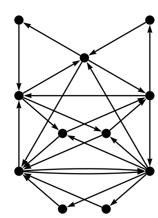
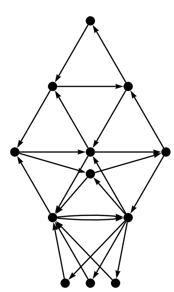
The list of all representation of $\mathrm{Rep}_k(A_2)$ and affine representations of $\mathrm{Rep}(A_2)$ that may be obtained through the automorphism construction is quite large, and we will not try to make a complete list here. In the next section, we will look at representations that are not obtained through the automorphism construction.

## 8.4 Irregular Representations

A large portion of the work for this thesis involved writing computer programs to generate all representations of $\mathrm{Rep}_k(A_2)$ and affine representations of $\mathrm{Rep}(A_2)$ up to some maximum number of vertices. We list the output of these programs here.

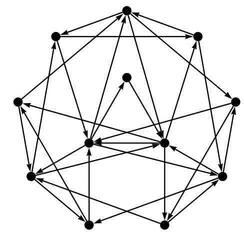
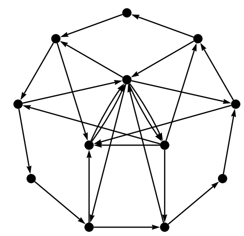
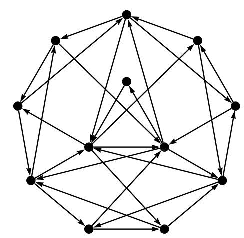
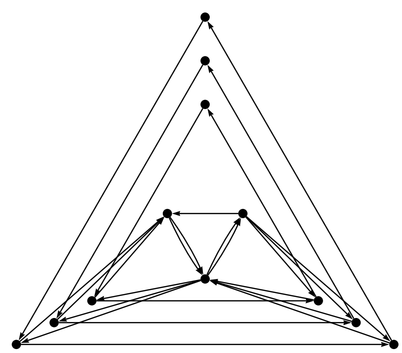
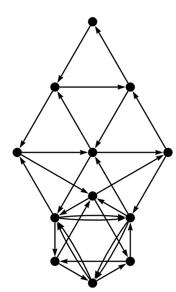
We refer to $N$-colourable representations of $\mathrm{Rep}_k(A_2)$ and $N$-colourable affine representations of $\mathrm{Rep}(A_2)$ that do not come from the automorphism construction as **irregular**. Table 8.7 lists the computer-generated output of irregular representations of $\mathrm{Rep}_k(A_2)$. Table 8.9 lists the computer-generated output of irregular affine representations of $\mathrm{Rep}(A_2)$.

Table 8.7: Some irregular representations of $\mathrm{Rep}_k(A_2)$.

| **Name** | **Level** | **Graph** |
|---|---|---|
| 6.5 | 6 |  |
| 10.3 | 6 |  |
| 11.2 | 12 |  |
| 12.15 | 9 |  |

Continued on next page . . .

Table 8.7: Some irregular representations of $\mathrm{Rep}_k(A_2)$ (continued).

| **Name** | **Level** | **Graph** |
|---|---|---|
| 12.23 | 12 |  |
| 12.25 | 12 |  |
| 12.26 | 12 |  |
| 12.28 | 12 |  |
| 12.40 | 24 |  |

Continued on next page . . .

Table 8.7: Some irregular representations of $\mathrm{Rep}_k(A_2)$ (continued).

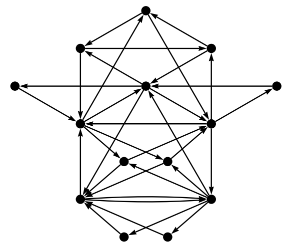
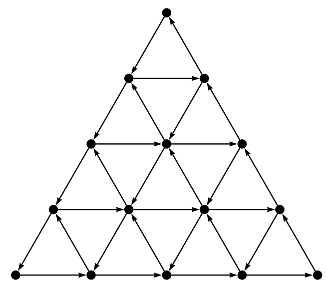
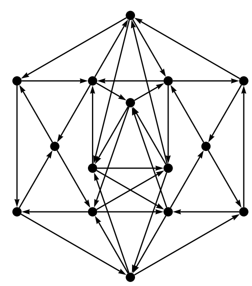
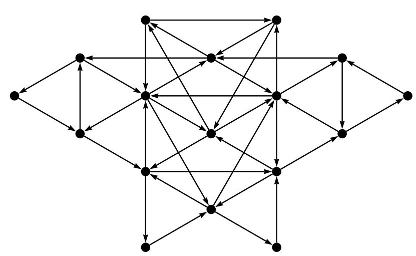
| **Name** | **Level** | **Graph** |
|---|---|---|
| 14.6 | 24 |  |
| 15.9 | 7 |  |
| 15.20 | 12 |  |
| 17.3 | 12 |  |

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Table 8.7: Some irregular representations of $\mathrm{Rep}_k(A_2)$ (continued).

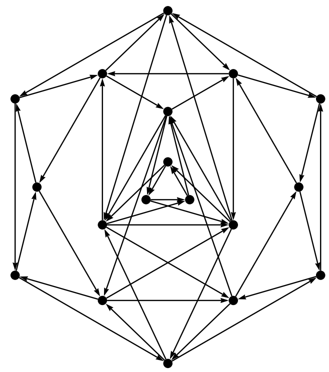
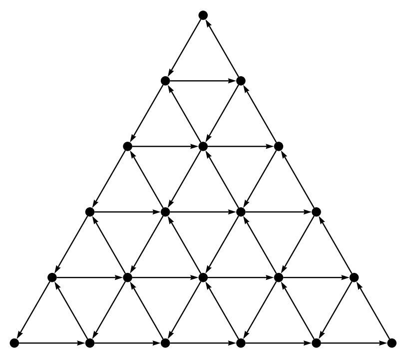
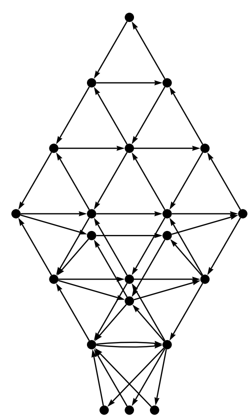
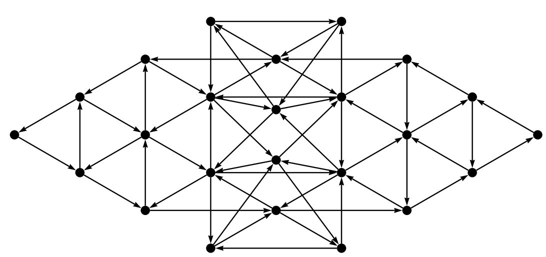
| **Name** | **Level** | **Graph** |
|---|---|---|
| 18.26 | 24 |  |
| 21.10 | 8 |  |
| 21.17 | 12 |  |
| 24.23 | 24 |  |

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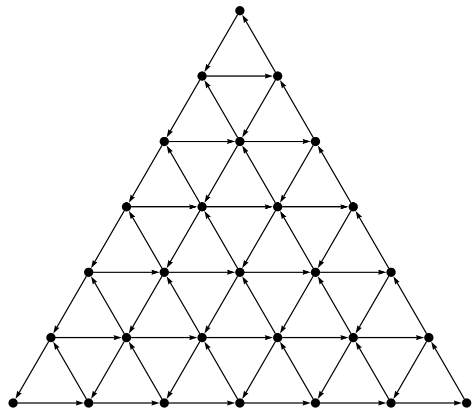
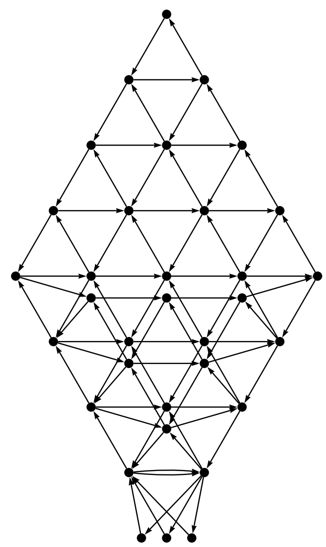
Table 8.7: Some irregular representations of $\mathrm{Rep}_k(A_2)$ (continued).

| Name | Level | Graph |
|---|---|---|
| 28.1 | 9 |  |
| 33.5 | 15 |  |

Continued on next page . . .

Table 8.7: Some irregular representations of $\mathrm{Rep}_k(A_2)$ (continued).

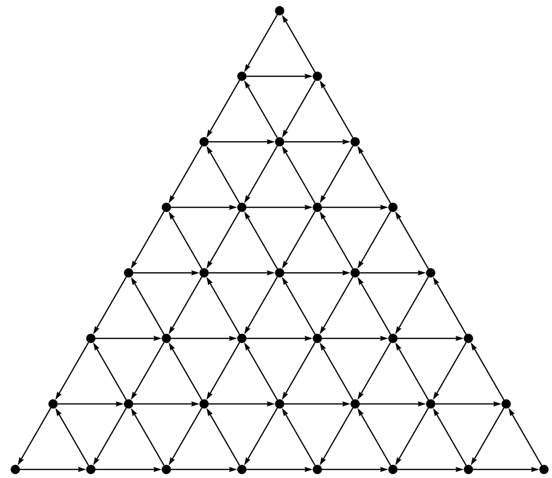
| Name | Level | Graph |
|---|---|---|
| 36.4 | 10 |  |

Table 8.8: Exponents of irregular representations of $\mathrm{Rep}_k(A_2)$.

| Name | Level | Exponents |
|---|---|---|
| 6.5 | 6 | $(4,1,1)$ $(2,2,2)$ $(2,2,2)$ $(2,2,2)$ $(1,4,1)$ $(1,1,4)$ |
| 10.3 | 6 | $(4,1,1)$ $(3,2,1)$ $(3,1,2)$ $(2,3,1)$ $(2,2,2)$ $(2,1,3)$ $(1,4,1)$ $(1,3,2)$ $(1,2,3)$ $(1,1,4)$ |
| 11.2 | 12 | $(10,1,1)$ $(6,3,3)$ $(5,5,2)$ $(5,2,5)$ $(4,4,4)$ $(4,4,4)$ $(3,6,3)$ $(3,3,6)$ $(2,5,5)$ $(1,10,1)$ $(1,1,10)$ |
| 12.15 | 9 | $(7,1,1)$ $(5,2,2)$ $(4,4,1)$ $(4,1,4)$ $(3,3,3)$ $(3,3,3)$ $(3,3,3)$ $(2,5,2)$ $(2,2,5)$ $(1,7,1)$ $(1,4,4)$ $(1,1,7)$ |
| 12.23 | 12 | $(10,1,1)$ $(6,5,1)$ $(6,1,5)$ $(5,6,1)$ $(5,5,2)$ $(5,2,5)$ $(5,1,6)$ $(2,5,5)$ $(1,10,1)$ $(1,6,5)$ $(1,5,6)$ $(1,1,10)$ |
| 12.25 | 12 | $(10,1,1)$ $(6,1,5)$ $(6,5,1)$ $(5,1,6)$ $(5,2,5)$ $(5,5,2)$ $(5,6,1)$ $(2,5,5)$ $(1,1,10)$ $(1,5,6)$ $(1,6,5)$ $(1,10,1)$ |
| 12.26 | 12 | $(10,1,1)$ $(6,3,3)$ $(6,3,3)$ $(5,5,2)$ $(5,2,5)$ $(3,6,3)$ $(3,6,3)$ $(3,3,6)$ $(3,3,6)$ $(2,5,5)$ $(1,10,1)$ $(1,1,10)$ |
| 12.28 | 12 | $(10,1,1)$ $(3,3,6)$ $(3,3,6)$ $(2,5,5)$ $(5,2,5)$ $(3,6,3)$ $(3,6,3)$ $(6,3,3)$ $(6,3,3)$ $(5,5,2)$ $(1,10,1)$ $(10,1,1)$ |

Continued on next page . . .

Table 8.8: Exponents of irregular representations of $\mathrm{Rep}_k(A_2)$ (continued).

| **Name** | **Level** | **Exponents** |
|---|---|---|
| 12.40 | 24 | $(22,1,1)$ $(14,5,5)$ $(11,11,2)$ $(11,2,11)$ $(10,7,7)$ $(7,10,7)$ $(7,7,10)$ $(5,14,5)$ $(5,5,14)$ $(2,11,11)$ $(1,22,1)$ $(1,1,22)$ |
| 14.6 | 24 | $(22,1,1)$ $(14,5,5)$ $(11,11,2)$ $(11,2,11)$ $(10,7,7)$ $(8,8,8)$ $(8,8,8)$ $(7,10,7)$ $(7,7,10)$ $(5,14,5)$ $(5,5,14)$ $(2,11,11)$ $(1,22,1)$ $(1,1,22)$ |
| 15.9 | 7 | $(5,1,1)$ $(4,1,2)$ $(4,2,1)$ $(3,1,3)$ $(3,2,2)$ $(3,3,1)$ $(2,1,4)$ $(2,2,3)$ $(2,3,2)$ $(2,4,1)$ $(1,1,5)$ $(1,2,4)$ $(1,3,3)$ $(1,4,2)$ $(1,5,1)$ |
| 15.20 | 12 | $(10,1,1)$ $(7,4,1)$ $(7,1,4)$ $(6,3,3)$ $(5,5,2)$ $(5,2,5)$ $(4,7,1)$ $(4,1,7)$ $(3,6,3)$ $(3,3,6)$ $(2,5,5)$ $(1,10,1)$ $(1,7,4)$ $(1,4,7)$ $(1,1,10)$ |
| 17.3 | 12 | $(10,1,1)$ $(7,1,4)$ $(7,4,1)$ $(6,3,3)$ $(5,2,5)$ $(5,5,2)$ $(4,1,7)$ $(4,4,4)$ $(4,4,4)$ $(4,7,1)$ $(3,3,6)$ $(3,6,3)$ $(2,5,5)$ $(1,1,10)$ $(1,4,7)$ $(1,7,4)$ $(1,10,1)$ |
| 18.26 | 24 | $(22,1,1)$ $(14,8,2)$ $(14,5,5)$ $(14,2,8)$ $(11,11,2)$ $(11,2,11)$ $(10,7,7)$ $(8,14,2)$ $(8,2,14)$ $(7,10,7)$ $(7,7,10)$ $(5,14,5)$ $(5,5,14)$ $(2,14,8)$ $(2,11,11)$ $(2,8,14)$ $(1,22,1)$ $(1,1,22)$ |
| 21.10 | 8 | $(6,1,1)$ $(5,2,1)$ $(5,1,2)$ $(4,3,1)$ $(4,2,2)$ $(4,1,3)$ $(3,4,1)$ $(3,3,2)$ $(3,2,3)$ $(3,1,4)$ $(2,5,1)$ $(2,4,2)$ $(2,3,3)$ $(2,2,4)$ $(2,1,5)$ $(1,6,1)$ $(1,5,2)$ $(1,4,3)$ $(1,3,4)$ $(1,2,5)$ $(1,1,6)$ |
| 21.17 | 12 | $(10,1,1)$ $(8,2,2)$ $(7,4,1)$ $(7,1,4)$ $(6,3,3)$ $(5,5,2)$ $(5,2,5)$ $(4,7,1)$ $(4,4,4)$ $(4,4,4)$ $(4,4,4)$ $(4,1,7)$ $(3,6,3)$ $(3,3,6)$ $(2,8,2)$ $(2,5,5)$ $(2,2,8)$ $(1,10,1)$ $(1,7,4)$ $(1,4,7)$ $(1,1,10)$ |
| 24.23 | 24 | $(22,1,1)$ $(16,7,1)$ $(16,1,7)$ $(14,5,5)$ $(11,11,2)$ $(11,8,5)$ $(11,5,8)$ $(11,2,11)$ $(10,7,7)$ $(8,11,5)$ $(8,5,11)$ $(7,16,1)$ $(7,10,7)$ $(7,7,10)$ $(7,1,16)$ $(5,14,5)$ $(5,11,8)$ $(5,8,11)$ $(5,5,14)$ $(2,11,11)$ $(1,22,1)$ $(1,16,7)$ $(1,7,16)$ $(1,1,22)$ |

Continued on next page . . .

Table 8.8: Exponents of irregular representations of $\mathrm{Rep}_k(A_2)$ (continued).

| **Name** | **Level** | **Exponents** |
|---|---|---|
| 28.1 | 9 | $(7,1,1)$ $(6,2,1)$ $(6,1,2)$ $(5,3,1)$ $(5,2,2)$ $(5,1,3)$ $(4,4,1)$ $(4,3,2)$ $(4,2,3)$ $(4,1,4)$ $(3,5,1)$ $(3,4,2)$ $(3,3,3)$ $(3,2,4)$ $(3,1,5)$ $(2,6,1)$ $(2,5,2)$ $(2,4,3)$ $(2,3,4)$ $(2,2,5)$ $(2,1,6)$ $(1,7,1)$ $(1,6,2)$ $(1,5,3)$ $(1,4,4)$ $(1,3,5)$ $(1,2,6)$ $(1,1,7)$ |
| 33.5 | 15 | $(13,1,1)$ $(11,2,2)$ $(10,4,1)$ $(10,1,4)$ $(9,3,3)$ $(8,5,2)$ $(8,2,5)$ $(7,7,1)$ $(7,4,4)$ $(7,1,7)$ $(6,6,3)$ $(6,3,6)$ $(5,8,2)$ $(5,5,5)$ $(5,5,5)$ $(5,5,5)$ $(5,2,8)$ $(4,10,1)$ $(4,7,4)$ $(4,4,7)$ $(4,1,10)$ $(3,3,9)$ $(3,6,6)$ $(3,9,3)$ $(2,11,2)$ $(2,8,5)$ $(2,5,8)$ $(2,2,11)$ $(1,13,1)$ $(1,10,4)$ $(1,7,7)$ $(1,4,10)$ $(1,1,13)$ |
| 36.4 | 10 | $(8,1,1)$ $(7,2,1)$ $(7,1,2)$ $(6,3,1)$ $(6,2,2)$ $(6,1,3)$ $(5,4,1)$ $(5,3,2)$ $(5,2,3)$ $(5,1,4)$ $(4,5,1)$ $(4,4,2)$ $(4,3,3)$ $(4,2,4)$ $(4,1,5)$ $(3,6,1)$ $(3,5,2)$ $(3,4,3)$ $(3,3,4)$ $(3,2,5)$ $(3,1,6)$ $(2,7,1)$ $(2,6,2)$ $(2,5,3)$ $(2,4,4)$ $(2,3,5)$ $(2,2,6)$ $(2,1,7)$ $(1,8,1)$ $(1,7,2)$ $(1,6,3)$ $(1,5,4)$ $(1,4,5)$ $(1,3,6)$ $(1,2,7)$ $(1,1,8)$ |

Table 8.9: Some irregular affine representations of $\mathrm{Rep}(A_2)$.

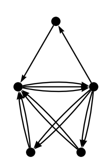
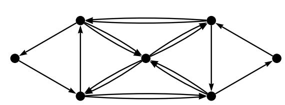
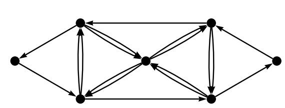
| **Name** | **Level** | **Graph** |
|---|---|---|
| 5.1 | 3 |  |
| 7.1 | 6 |  |
| 7.2 | 12 |  |

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Table 8.9: Some irregular affine representations of $\mathrm{Rep}(A_2)$ (continued).

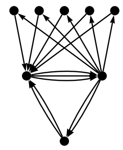
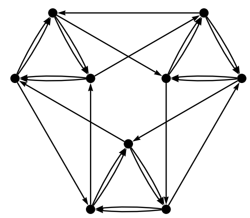
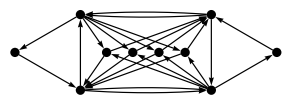
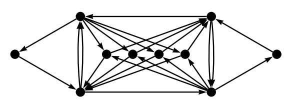
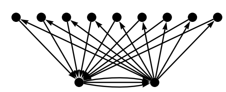
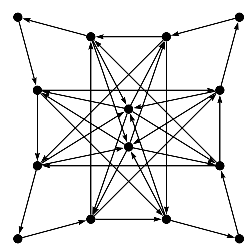
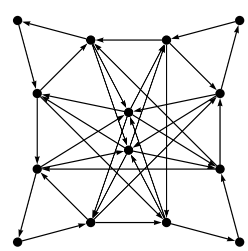
| **Name** | **Level** | **Graph** |
|---|---|---|
| 8.1 | 3 |  |
| 9.2 | 3 |  |
| 10.1 | 6 |  |
| 10.4 | 12 |  |
| 11.1 | 3 |  |
| 14.1 | 6 |  |
| 14.2 | 6 |  |

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Table 8.9: Some irregular affine representations of $\mathrm{Rep}(A_2)$ (continued).

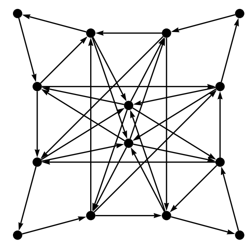
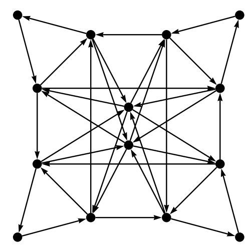
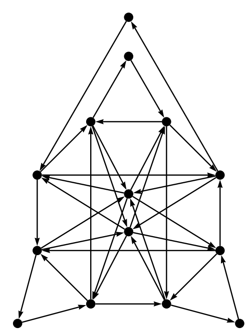
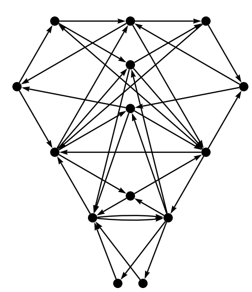
| **Name** | **Level** | **Graph** |
|---|---|---|
| 14.3 | 12 |  |
| 14.4 | 12 |  |
| 14.5 | 24 |  |
| 14.7 | 30 |  |

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Table 8.9: Some irregular affine representations of $\mathrm{Rep}(A_2)$ (continued).

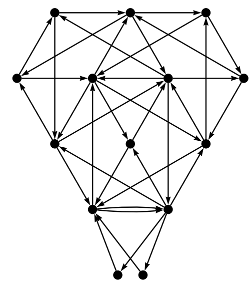
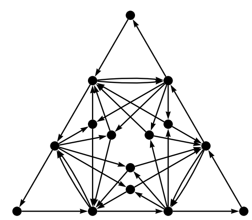
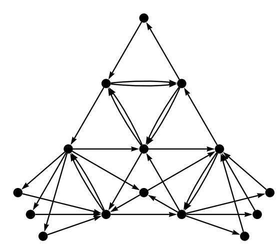
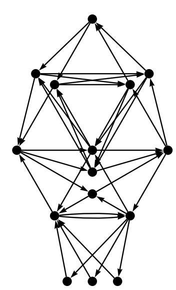
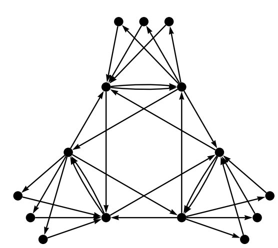
| **Name** | **Level** | **Graph** |
|---|---|---|
| 14.8 | 60 |  |
| 15.1 | 3 |  |
| 15.2 | 3 |  |
| 15.8 | 6 |  |
| 15.30 | 21 |  |

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Table 8.9: Some irregular affine representations of Rep($A_2$) (continued).

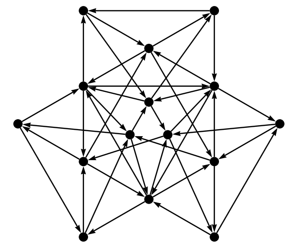
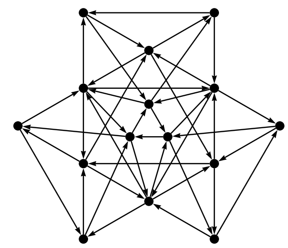
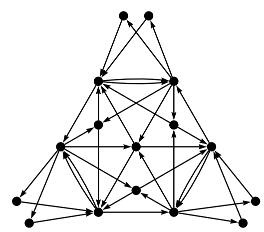
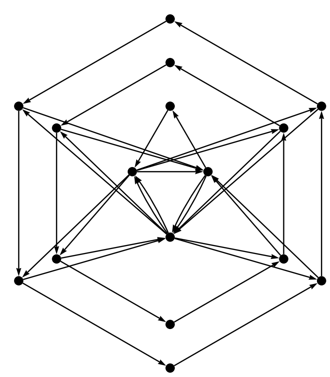
| Name | Level | Graph |
|---|---|---|
| 15.40 | 42 |  |
| 15.41 | 84 |  |
| 16.1 | 3 |  |
| 16.2 | 12 |  |

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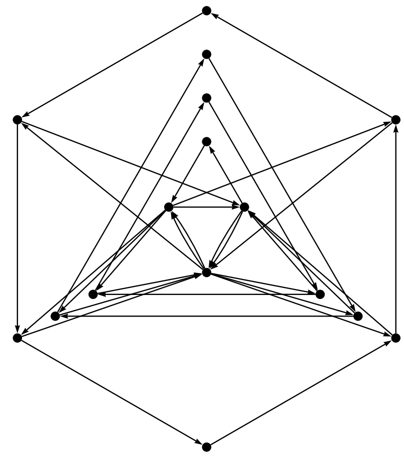
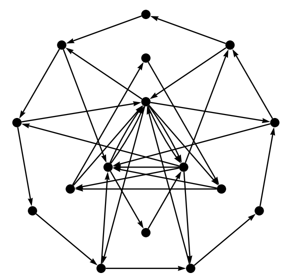
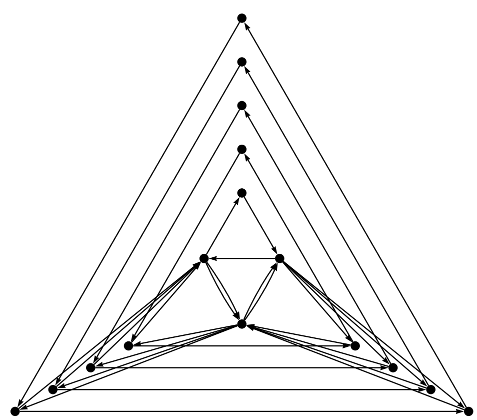
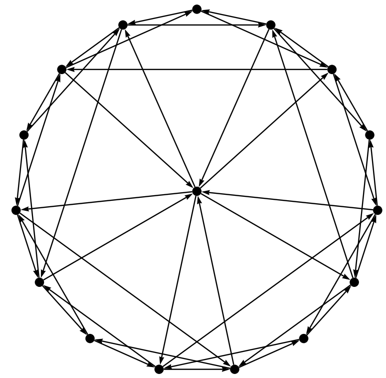
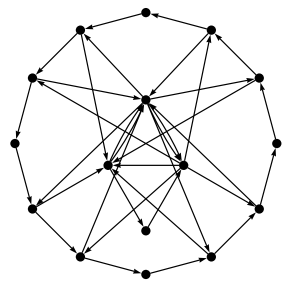
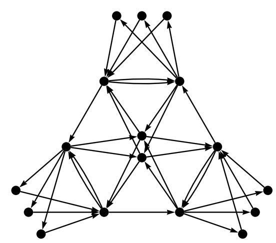
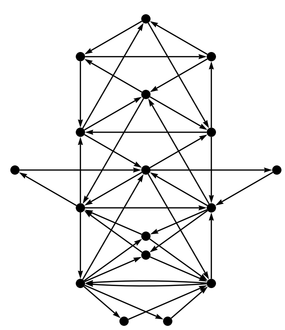
Table 8.9: Some irregular affine representations of Rep($A_2$) (continued).

| **Name** | **Level** | **Graph** |
| --- | --- | --- |
| 16.3 | 12 |  |
| 16.4 | 12 |  |
| 16.5 | 12 |  |

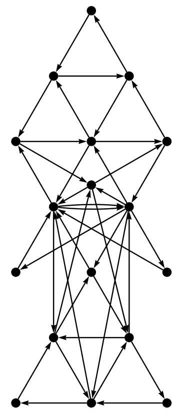
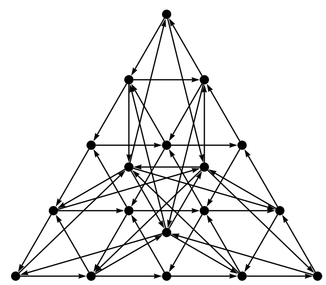
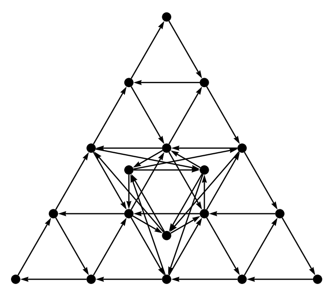
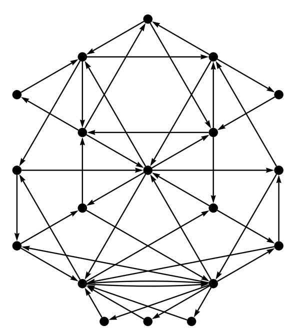
Continued on next page . . .

Table 8.9: Some irregular affine representations of Rep($A_2$) (continued).

| Name | Level | Graph |
|---|---|---|
| 16.6 | 15 |  |
| 16.7 | 24 |  |
| 17.1 | 9 |  |
| 17.2 | 12 |  |

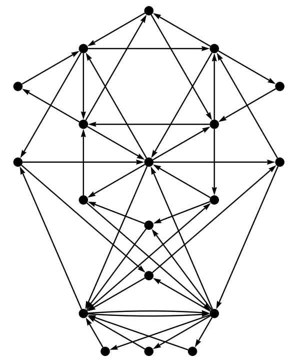
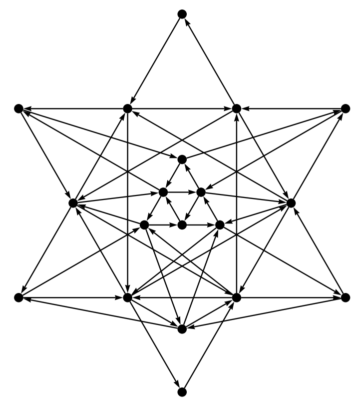
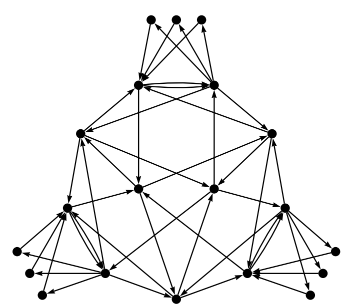
Continued on next page . . .

Table 8.9: Some irregular affine representations of $\mathrm{Rep}(A_2)$ (continued).

| **Name** | **Level** | **Graph** |
|---|---|---|
| 17.4 | 60 |  |
| 18.32 | 70 |  |
| 18.33 | 84 |  |
| 19.1 | 12 |  |

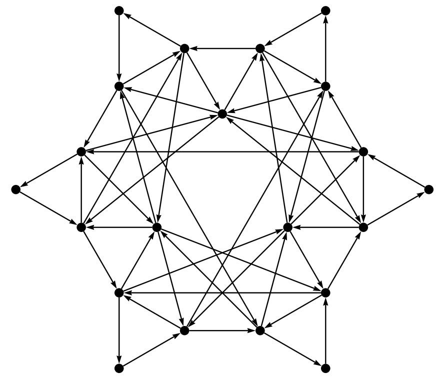
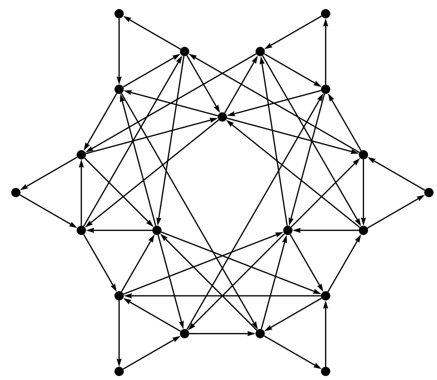
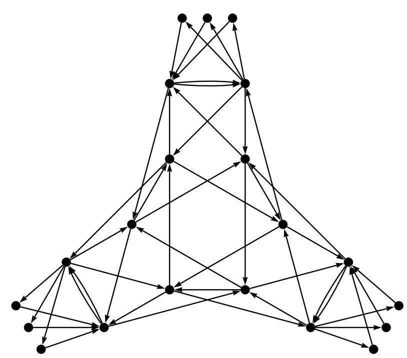
Continued on next page . . .

Table 8.9: Some irregular affine representations of $\mathrm{Rep}(A_2)$ (continued).

| **Name** | **Level** | **Graph** |
|---|---|---|
| 19.2 | 24 |  |
| 19.3 | 210 |  |
| 20.1 | 6 |  |

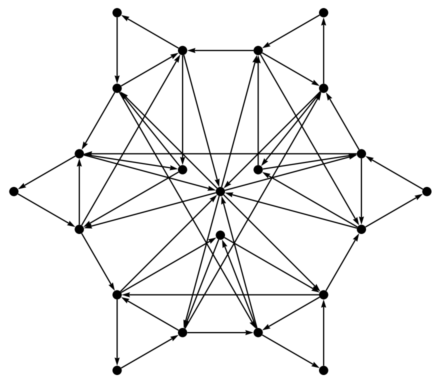
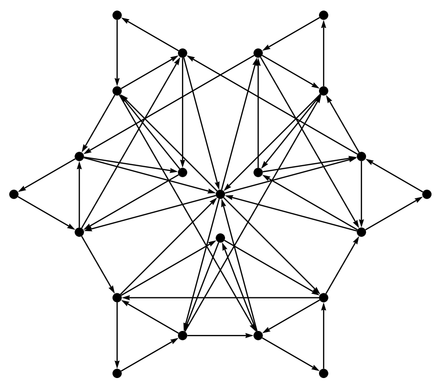
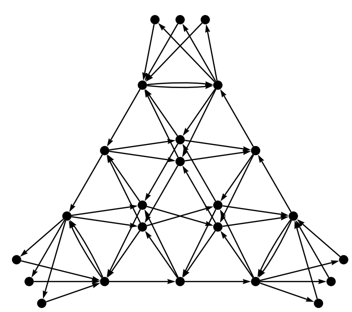
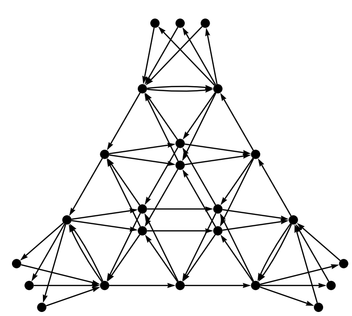
Continued on next page . . .

Table 8.9: Some irregular affine representations of $\mathrm{Rep}(A_2)$ (continued).

| Name | Level | Graph |
|---|---|---|
| 21.1 | 6 |  |
| 21.13 | 12 |  |
| 21.30 | 39 |  |

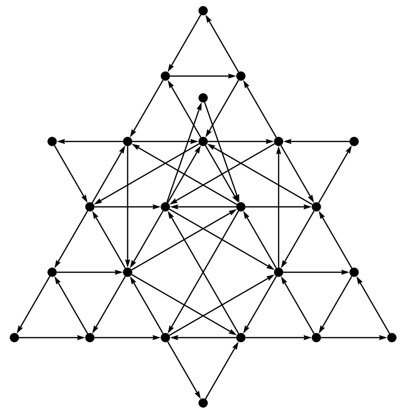
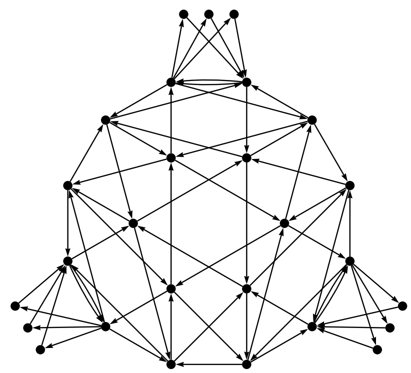
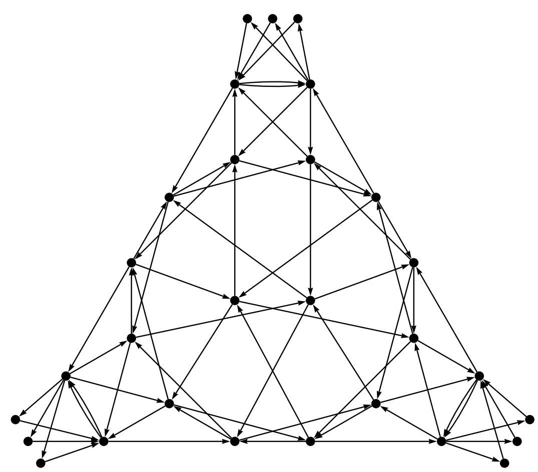
Continued on next page . . .

Table 8.9: Some irregular affine representations of Rep($A_2$) (continued).

| **Name** | **Level** | **Graph** |
|---|---|---|
| 22.1 | 6 |  |
| 22.2 | 12 |  |
| 24.9 | 12 |  |
| 24.10 | 12 |  |

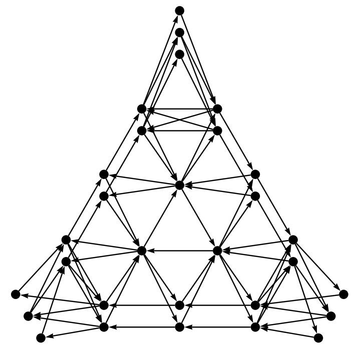
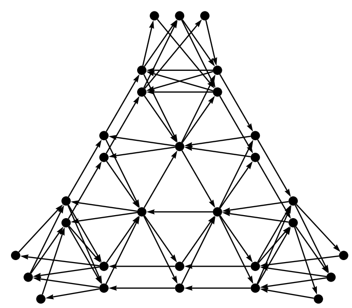
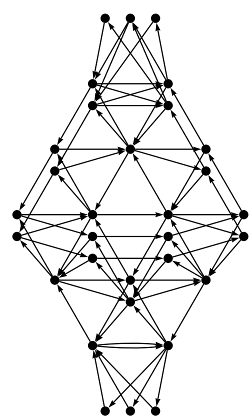
Continued on next page . . .

Table 8.9: Some irregular affine representations of $\mathrm{Rep}(A_2)$ (continued).

| **Name** | **Level** | **Graph** |
|---|---|---|
| 24.11 | 12 |  |
| 27.28 | 57 |  |
| 29.1 | 21 |  |

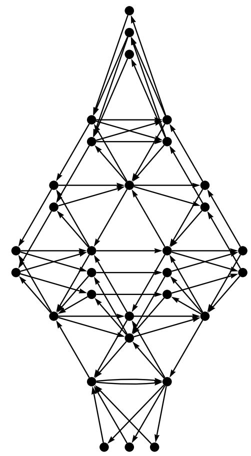
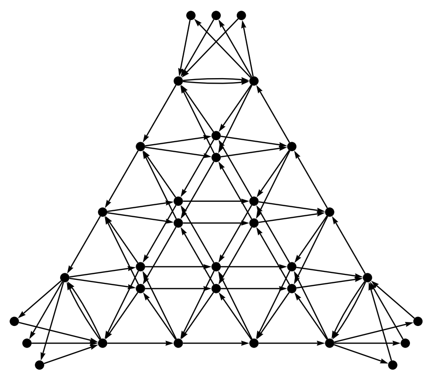
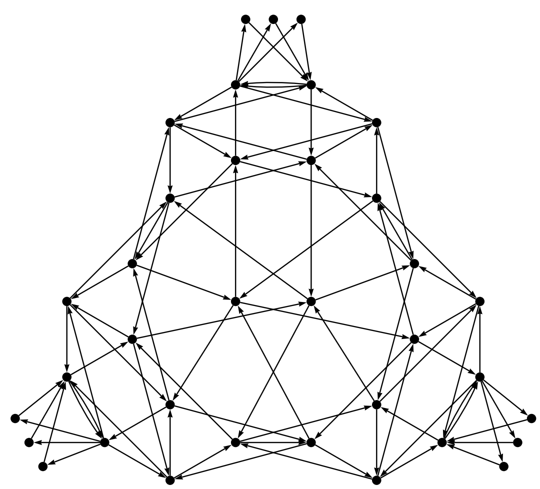
Continued on next page . . .

Table 8.9: Some irregular affine representations of $\mathrm{Rep}(A_2)$ (continued).

| **Name** | **Level** | **Graph** |
|---|---|---|
| 30.21 | 24 |  |
| 30.32 | 48 |  |
| 31.1 | 18 |  |

Continued on next page . . .

Table 8.9: Some irregular affine representations of Rep($A_2$) (continued).

| Name | Level | Graph |
|---|---|---|
| 31.2 | 36 |  |
| 33.4 | 15 |  |
| 35.1 | 9 |  |

Continued on next page . . .

Table 8.9: Some irregular affine representations of Rep($A_2$) (continued).

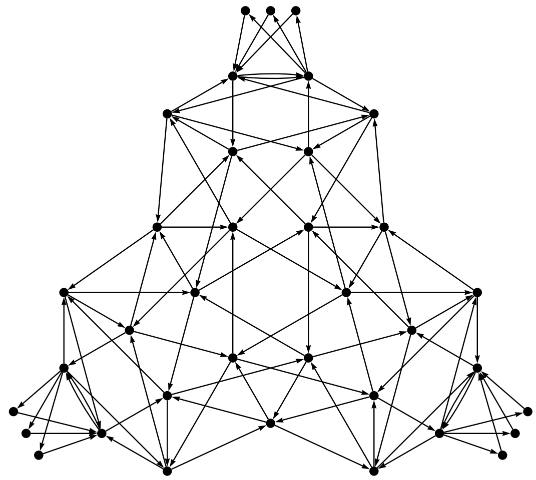
| **Name** | **Level** | **Graph** |
|---|---|---|
| 36.39 | 42 |  |

Table 8.10: Exponents of irregular affine representations of Rep($A_2$).

| **Name** | **Level** | **Exponents** |
|---|---|---|
| 5.1 | 3 | $(3,0,0)$ $(1,1,1)$ $(1,1,1)$ $(0,3,0)$ $(0,0,3)$ |
| 7.1 | 6 | $(6,0,0)$ $(3,3,0)$ $(3,0,3)$ $(2,2,2)$ $(0,6,0)$ $(0,3,3)$ $(0,0,6)$ |
| 7.2 | 12 | $(12,0,0)$ $(6,3,3)$ $(4,4,4)$ $(3,6,3)$ $(3,3,6)$ $(0,12,0)$ $(0,0,12)$ |
| 8.1 | 3 | $(3,0,0)$ $(1,1,1)$ $(1,1,1)$ $(1,1,1)$ $(1,1,1)$ $(1,1,1)$ $(0,3,0)$ $(0,0,3)$ |
| 9.2 | 3 | $(3,0,0)$ $(2,1,0)$ $(2,0,1)$ $(1,2,0)$ $(1,0,2)$ $(0,3,0)$ $(0,2,1)$ $(0,1,2)$ $(0,0,3)$ |
| 10.1 | 6 | $(6,0,0)$ $(3,3,0)$ $(3,0,3)$ $(2,2,2)$ $(2,2,2)$ $(2,2,2)$ $(2,2,2)$ $(0,6,0)$ $(0,3,3)$ $(0,0,6)$ |
| 10.4 | 12 | $(12,0,0)$ $(6,3,3)$ $(4,4,4)$ $(4,4,4)$ $(4,4,4)$ $(4,4,4)$ $(3,6,3)$ $(3,3,6)$ $(0,12,0)$ $(0,0,12)$ |
| 11.1 | 3 | $(3,0,0)$ $(1,1,1)$ $(1,1,1)$ $(1,1,1)$ $(1,1,1)$ $(1,1,1)$ $(1,1,1)$ $(1,1,1)$ $(1,1,1)$ $(0,3,0)$ $(0,0,3)$ |

Continued on next page . . .

Table 8.10: Exponents of irregular affine representations of $\mathrm{Rep}(A_2)$ (continued).

| **Name** | **Level** | **Exponents** |
|---|---|---|
| 14.1 | 6 | $(6,0,0)$ $(3,3,0)$ $(3,3,0)$ $(3,3,0)$ $(3,0,3)$ $(3,0,3)$ $(3,0,3)$ $(2,2,2)$ $(2,2,2)$ $(0,6,0)$ $(0,3,3)$ $(0,3,3)$ $(0,3,3)$ $(0,0,6)$ |
| 14.2 | 6 | $(6,0,0)$ $(3,3,0)$ $(3,2,1)$ $(3,1,2)$ $(3,0,3)$ $(2,3,1)$ $(2,2,2)$ $(2,2,2)$ $(2,1,3)$ $(1,3,2)$ $(1,2,3)$ $(0,6,0)$ $(0,3,3)$ $(0,0,6)$ |
| 14.3 | 12 | $(12,0,0)$ $(6,6,0)$ $(6,6,0)$ $(6,3,3)$ $(6,0,6)$ $(6,0,6)$ $(4,4,4)$ $(4,4,4)$ $(3,6,3)$ $(3,3,6)$ $(0,12,0)$ $(0,6,6)$ $(0,6,6)$ $(0,0,12)$ |
| 14.4 | 12 | $(12,0,0)$ $(6,6,0)$ $(6,3,3)$ $(6,3,3)$ $(6,0,6)$ $(4,4,4)$ $(4,4,4)$ $(3,6,3)$ $(3,6,3)$ $(3,3,6)$ $(3,3,6)$ $(0,12,0)$ $(0,6,6)$ $(0,0,12)$ |
| 14.5 | 24 | $(24,0,0)$ $(12,9,3)$ $(12,6,6)$ $(12,3,9)$ $(9,12,3)$ $(9,3,12)$ $(8,8,8)$ $(8,8,8)$ $(6,12,6)$ $(6,6,12)$ $(3,12,9)$ $(3,9,12)$ $(0,24,0)$ $(0,0,24)$ |
| 14.7 | 30 | $(30,0,0)$ $(18,6,6)$ $(15,15,0)$ $(15,0,15)$ $(12,12,6)$ $(12,6,12)$ $(10,10,10)$ $(10,10,10)$ $(6,18,6)$ $(6,12,12)$ $(6,6,18)$ $(0,30,0)$ $(0,15,15)$ $(0,0,30)$ |
| 14.8 | 60 | $(60,0,0)$ $(36,12,12)$ $(30,15,15)$ $(24,12,24)$ $(24,24,12)$ $(20,20,20)$ $(20,20,20)$ $(15,15,30)$ $(15,30,15)$ $(12,12,36)$ $(12,24,24)$ $(12,36,12)$ $(0,0,60)$ $(0,60,0)$ |
| 15.1 | 3 | $(3,0,0)$ $(2,1,0)$ $(2,0,1)$ $(1,2,0)$ $(1,1,1)$ $(1,1,1)$ $(1,1,1)$ $(1,1,1)$ $(1,1,1)$ $(1,1,1)$ $(1,0,2)$ $(0,3,0)$ $(0,2,1)$ $(0,1,2)$ $(0,0,3)$ |
| 15.2 | 3 | $(3,0,0)$ $(2,1,0)$ $(2,0,1)$ $(1,2,0)$ $(1,1,1)$ $(1,1,1)$ $(1,1,1)$ $(1,1,1)$ $(1,1,1)$ $(1,1,1)$ $(1,0,2)$ $(0,3,0)$ $(0,2,1)$ $(0,1,2)$ $(0,0,3)$ |
| 15.8 | 6 | $(6,0,0)$ $(4,1,1)$ $(3,3,0)$ $(3,0,3)$ $(2,2,2)$ $(2,2,2)$ $(2,2,2)$ $(2,2,2)$ $(2,2,2)$ $(2,2,2)$ $(1,4,1)$ $(1,1,4)$ $(0,6,0)$ $(0,3,3)$ $(0,0,6)$ |
| 15.30 | 21 | $(21,0,0)$ $(12,6,3)$ $(12,3,6)$ $(7,7,7)$ $(7,7,7)$ $(7,7,7)$ $(7,7,7)$$(7,7,7)$ $(7,7,7)$ $(6,12,3)$ $(6,3,12)$ $(3,12,6)$ $(3,6,12)$ $(0,21,0)$ $(0,0,21)$ |

Continued on next page . . .

Table 8.10: Exponents of irregular affine representations of Rep($A_2$) (continued).

| **Name** | **Level** | **Exponents** |
|---|---|---|
| 15.40 | 42 | (42, 0, 0) (24, 12, 6) (24, 6, 12) (21, 21, 0) (21, 0, 21) (14, 14, 14) (14, 14, 14) (14, 14, 14) (12, 24, 6) (12, 6, 24) (6, 24, 12) (6, 12, 24) (0, 42, 0) (0, 21, 21) (0, 0, 42) |
| 15.41 | 84 | (84, 0, 0) (48, 24, 12) (48, 12, 24) (42, 21, 21) (28, 28, 28) (28, 28, 28) (28, 28, 28) (24, 48, 12) (24, 12, 48) (21, 42, 21) (21, 21, 42) (12, 48, 24) (12, 24, 48) (0, 84, 0) (0, 0, 84) |
| 16.1 | 3 | (3, 0, 0) (2, 1, 0) (2, 0, 1) (1, 2, 0) (1, 1, 1) (1, 1, 1) (1, 1, 1) (1, 1, 1) (1, 1, 1) (1, 1, 1) (1, 1, 1) (1, 0, 2) (0, 3, 0) (0, 2, 1) (0, 1, 2) (0, 0, 3) |
| 16.2 | 12 | (12, 0, 0) (6, 6, 0) (6, 6, 0) (6, 6, 0) (6, 3, 3) (6, 0, 6) (6, 0, 6) (6, 0, 6) (4, 4, 4) (3, 6, 3) (3, 3, 6) (0, 12, 0)(0, 6, 6) (0, 6, 6) (0, 6, 6) (0, 0, 12) |
| 16.3 | 12 | (12, 0, 0) (6, 6, 0) (6, 6, 0) (6, 3, 3) (6, 3, 3) (6, 0, 6) (6, 0, 6) (4, 4, 4) (3, 6, 3) (3, 6, 3) (3, 3, 6) (3, 3, 6) (0, 12, 0) (0, 6, 6) (0, 6, 6) (0, 0, 12) |
| 16.4 | 12 | (12, 0, 0) (6, 6, 0) (6, 5, 1) (6, 3, 3) (6, 1, 5) (6, 0, 6) (5, 6, 1) (5, 1, 6) (4, 4, 4) (3, 6, 3) (3, 3, 6) (1, 6, 5) (1, 5, 6) (0, 12, 0) (0, 6, 6) (0, 0, 12) |
| 16.5 | 12 | (12, 0, 0) (6, 6, 0) (6, 3, 3) (6, 3, 3) (6, 3, 3) (6, 0, 6) (4, 4, 4) (3, 6, 3) (3, 6, 3) (3, 6, 3) (3, 3, 6) (3, 3, 6) (3, 3, 6) (0, 12, 0) (0, 6, 6) (0, 0, 12) |
| 16.6 | 15 | (15, 0, 0) (9, 3, 3) (9, 3, 3) (6, 6, 3) (6, 6, 3) (6, 3, 6) (6, 3, 6) (5, 5, 5) (3, 9, 3) (3, 9, 3) (3, 6, 6) (3, 6, 6) (3, 3, 9) (3, 3, 9) (0, 15, 0) (0, 0, 15) |
| 16.7 | 24 | (24, 0, 0) (12, 12, 0) (12, 12, 0) (12, 9, 3) (12, 3, 9) (12, 0, 12) (12, 0, 12) (9, 12, 3) (9, 3, 12) (8, 8, 8) (3, 12, 9) (3, 9, 12) (0, 24, 0) (0, 12, 12) (0, 12, 12) (0, 0, 24) |
| 17.1 | 9 | (9, 0, 0) (6, 3, 0) (6, 0, 3) (3, 6, 0) (3, 3, 3) (3, 3, 3) (3, 3, 3) (3, 3, 3) (3, 3, 3) (3, 3, 3) (3, 3, 3) (3, 3, 3) (3, 0, 6) (0, 9, 0) (0, 6, 3) (0, 3, 6) (0, 0, 9) |

Continued on next page . . .

Table 8.10: Exponents of irregular affine representations of Rep($A_2$) (continued).

| **Name** | **Level** | **Exponents** |
|---|---|---|
| 17.2 | 12 | (12, 0, 0) (8, 2, 2) (6, 6, 0) (6, 3, 3) (6, 0, 6) (4, 4, 4) (4, 4, 4) (4, 4, 4) (4, 4, 4) (4, 4, 4) (3, 6, 3) (3, 3, 6) (2, 8, 2) (2, 2, 8) (0, 12, 0) (0, 6, 6) (0, 0, 12) |
| 17.4 | 60 | (60, 0, 0) (36, 12, 12) (30, 30, 0) (30, 15, 15) (30, 0, 30) (24, 24, 12) (24, 12, 24) (20, 20, 20) (20, 20, 20) (15, 30, 15) (15, 15, 30) (12, 36, 12) (12, 24, 24) (12, 12, 36) (0, 60, 0) (0, 30, 30) (0, 0, 60) |
| 18.32 | 70 | (70, 0, 0) (42, 14, 14) (40, 20, 10) (40, 10, 20) (35, 35, 0) (35, 0, 35) (30, 40, 10) (30, 10, 40) (28, 28, 14) (28, 14, 28) (14, 42, 14) (14, 28, 28) (14, 14, 42) (10, 40, 20) (10, 20, 40) (0, 70, 0) (0, 35, 35) (0, 0, 70) |
| 18.33 | 84 | (84, 0, 0) (48, 24, 12) (48, 12, 24) (42, 42, 0) (42, 21, 21) (42, 0, 42) (28, 28, 28) (28, 28, 28) (28, 28, 28) (24, 48, 12) (24, 42, 21) (24, 21, 42) (24, 12, 48) (12, 48, 24) (12, 24, 48) (0, 84, 0) (0, 42, 42) (0, 0, 84) |
| 19.1 | 12 | (12, 0, 0) (8, 2, 2) (6, 6, 0) (6, 6, 0) (6, 3, 3) (6, 0, 6) (6, 0, 6) (4, 4, 4) (4, 4, 4) (4, 4, 4) (4, 4, 4) (3, 6, 3) (3, 3, 6) (2, 8, 2) (2, 2, 8) (0, 12, 0) (0, 6, 6) (0, 6, 6) (0, 0, 12) |
| 19.2 | 24 | (24, 0, 0) (16, 4, 4) (12, 12, 0) (12, 9, 3) (12, 3, 9) (12, 0, 12) (9, 12, 3) (9, 3, 12) (8, 8, 8) (8, 8, 8) (8, 8, 8) (8, 8, 8) (4, 16, 4) (4, 4, 16) (3, 12, 9) (3, 9, 12) (0, 24, 0) (0, 12, 12) (0, 0, 24) |
| 19.3 | 210 | (210, 0, 0) (126, 42, 42) (120, 60, 30) (120, 30, 60) (105, 105, 0) (105, 0, 105) (84, 84, 42) (84, 42, 84) (70, 70, 70) (60, 120, 30) (60, 30, 120) (42, 126, 42) (42, 84, 84) (42, 42, 126) (30, 120, 60) (30, 60, 120) (0, 210, 0) (0, 105, 105) (0, 0, 210) |
| 20.1 | 6 | (6, 0, 0) (4, 1, 1) (4, 1, 1) (3, 3, 0) (3, 0, 3) (2, 2, 2) (2, 2, 2) (2, 2, 2) (2, 2, 2) (2, 2, 2) (2, 2, 2) (2, 2, 2) (2, 2, 2) (1, 4, 1) (1, 4, 1) (1, 1, 4) (1, 1, 4) (0, 6, 0) (0, 3, 3) (0, 0, 6) |

Continued on next page . . .

Table 8.10: Exponents of irregular affine representations of Rep($A_2$) (continued).

| **Name** | **Level** | **Exponents** |
|---|---|---|
| 21.1 | 6 | (6, 0, 0) (4, 2, 0) (4, 0, 2) (3, 3, 0) (3, 2, 1) (3, 1, 2) (3, 0, 3) (2, 4, 0) (2, 3, 1) (2, 2, 2) (2, 2, 2) (2, 2, 2) (2, 1, 3) (2, 0, 4) (1, 3, 2) (1, 2, 3) (0, 6, 0) (0, 4, 2) (0, 3, 3) (0, 2, 4) (0, 0, 6) |
| 21.13 | 12 | (12, 0, 0) (8, 4, 0) (8, 0, 4) (6, 5, 1) (6, 3, 3) (6, 1, 5) (5, 6, 1) (5, 1, 6) (4, 8, 0) (4, 4, 4) (4, 4, 4) (4, 4, 4) (4, 0, 8) (3, 6, 3) (3, 3, 6) (1, 6, 5) (1, 5, 6) (0, 12, 0) (0, 8, 4) (0, 4, 8) (0, 0, 12) |
| 21.30 | 39 | (39, 0, 0) (27, 9, 3) (27, 3, 9) (18, 15, 6) (18, 6, 15) (15, 18, 6) (15, 6, 18) (13, 13, 13) (13, 13, 13) (13, 13, 13) (13, 13, 13) (13, 13, 13) (13, 13, 13) (9, 27, 3) (9, 3, 27) (6, 18, 15) (6, 15, 18) (3, 27, 9) (3, 9, 27) (0, 39, 0) (0, 0, 39) |
| 22.1 | 6 | (6, 0, 0) (4, 2, 0) (4, 0, 2) (3, 3, 0) (3, 2, 1) (3, 1, 2) (3, 0, 3) (2, 4, 0) (2, 3, 1) (2, 2, 2) (2, 2, 2) (2, 2, 2) (2, 2, 2) (2, 1, 3) (2, 0, 4) (1, 3, 2) (1, 2, 3) (0, 6, 0) (0, 4, 2) (0, 3, 3) (0, 2, 4) (0, 0, 6) |
| 22.2 | 12 | (12, 0, 0) (8, 4, 0) (8, 0, 4) (6, 5, 1) (6, 3, 3) (6, 1, 5) (5, 6, 1) (5, 1, 6) (4, 8, 0) (4, 4, 4) (4, 4, 4) (4, 4, 4) (4, 4, 4) (4, 0, 8) (3, 6, 3) (3, 3, 6) (1, 6, 5) (1, 5, 6) (0, 12, 0) (0, 8, 4) (0, 4, 8) (0, 0, 12) |
| 24.9 | 12 | (12, 0, 0) (9, 3, 0) (9, 0, 3) (6, 6, 0) (6, 6, 0) (6, 3, 3) (6, 0, 6) (6, 0, 6) (4, 4, 4) (4, 4, 4) (4, 4, 4) (4, 4, 4) (4, 4, 4) (4, 4, 4) (3, 9, 0) (3, 6, 3) (3, 3, 6) (3, 0, 9) (0, 12, 0) (0, 9, 3) (0, 6, 6) (0, 6, 6) (0, 3, 9) (0, 0, 12) |
| 24.10 | 12 | (12, 0, 0) (9, 3, 0) (9, 0, 3) (6, 6, 0) (6, 3, 3) (6, 3, 3) (6, 0, 6) (4, 4, 4) (4, 4, 4) (4, 4, 4) (4, 4, 4) (4, 4, 4) (4, 4, 4) (3, 9, 0) (3, 6, 3) (3, 6, 3) (3, 3, 6) (3, 3, 6) (3, 0, 9) (0, 12, 0) (0, 9, 3) (0, 6, 6) (0, 3, 9) (0, 0, 12) |
| 24.11 | 12 | (12, 0, 0) (8, 4, 0) (8, 0, 4) (6, 6, 0) (6, 4, 2) (6, 3, 3) (6, 2, 4) (6, 0, 6) (4, 8, 0) (4, 6, 2) (4, 4, 4) (4, 4, 4) (4, 4, 4) (4, 2, 6) (4, 0, 8) (3, 6, 3) (3, 3, 6) (2, 6, 4) (2, 4, 6) (0, 12, 0) (0, 8, 4) (0, 6, 6) (0, 4, 8) (0, 0, 12) |

Continued on next page . . .

Table 8.10: Exponents of irregular affine representations of $\mathrm{Rep}(A_2)$ (continued).

| **Name** | **Level** | **Exponents** |
|---|---|---|
| 27.28 | 57 | $(57,0,0)$ $(42,9,6)$ $(42,6,9)$ $(33,21,3)$ $(33,3,21)$ $(27,18,12)$ $(27,12,18)$ $(21,33,3)$ $(21,3,33)$ $(19,19,19)$ $(19,19,19)$ $(19,19,19)$ $(19,19,19)$ $(19,19,19)$ $(19,19,19)$ $(18,27,12)$ $(18,12,27)$ $(12,27,18)$ $(12,18,27)$ $(9,42,6)$ $(9,6,42)$ $(6,42,9)$ $(6,9,42)$ $(3,33,21)$ $(3,21,33)$ $(0,57,0)$ $(0,0,57)$ |
| 29.1 | 21 | $(21,0,0)$ $(16,4,1)$ $(16,1,4)$ $(12,6,3)$ $(12,3,6)$ $(11,8,2)$ $(11,2,8)$ $(8,11,2)$ $(8,2,11)$ $(7,7,7)$ $(7,7,7)$ $(7,7,7)$ $(7,7,7)$ $(7,7,7)$ $(7,7,7)$ $(7,7,7)$ $(7,7,7)$ $(6,12,3)$ $(6,3,12)$ $(4,16,1)$ $(4,1,16)$ $(3,12,6)$ $(3,6,12)$ $(2,11,8)$ $(2,8,11)$ $(1,16,4)$ $(1,4,16)$ $(0,21,0)$ $(0,0,21)$ |
| 30.21 | 24 | $(24,0,0)$ $(18,6,0)$ $(18,0,6)$ $(12,12,0)$ $(12,12,0)$ $(12,9,3)$ $(12,6,6)$ $(12,6,6)$ $(12,3,9)$ $(12,0,12)$ $(12,0,12)$ $(9,12,3)$ $(9,3,12)$ $(8,8,8)$ $(8,8,8)$ $(8,8,8)$ $(6,18,0)$ $(6,12,6)$ $(6,12,6)$ $(6,6,12)$ $(6,6,12)$ $(6,0,18)$ $(3,12,9)$ $(3,9,12)$ $(0,24,0)$ $(0,18,6)$ $(0,12,12)$ $(0,12,12)$ $(0,6,18)$ $(0,0,24)$ |
| 30.32 | 48 | $(48,0,0)$ $(36,12,0)$ $(36,0,12)$ $(24,24,0)$ $(24,21,3)$ $(24,15,9)$ $(24,12,12)$ $(24,9,15)$ $(24,3,21)$ $(24,0,24)$ $(21,24,3)$ $(21,3,24)$ $(16,16,16)$ $(16,16,16)$ $(16,16,16)$ $(15,24,9)$ $(15,9,24)$ $(12,36,0)$ $(12,24,12)$ $(12,12,24)$ $(12,0,36)$ $(9,24,15)$ $(9,15,24)$ $(3,24,21)$ $(3,21,24)$ $(0,48,0)$ $(0,36,12)$ $(0,24,24)$ $(0,12,36)$ $(0,0,48)$ |
| 31.1 | 18 | $(18,0,0)$ $(14,2,2)$ $(12,6,0)$ $(12,0,6)$ $(10,4,4)$ $(9,9,0)$ $(9,6,3)$ $(9,3,6)$ $(9,0,9)$ $(8,8,2)$ $(8,2,8)$ $(6,12,0)$ $(6,9,3)$ $(6,6,6)$ $(6,6,6)$ $(6,6,6)$ $(6,6,6)$ $(6,3,9)$ $(6,0,12)$ $(4,10,4)$ $(4,4,10)$ $(3,9,6)$ $(3,6,9)$ $(2,14,2)$ $(2,8,8)$ $(2,2,14)$ $(0,18,0)$ $(0,12,6)$ $(0,9,9)$ $(0,6,12)$ $(0,0,18)$ |

Continued on next page . . .

Table 8.10: Exponents of irregular affine representations of Rep($A_2$) (continued).

| **Name** | **Level** | **Exponents** |
|---|---|---|
| 31.2 | 36 | $(36,0,0)$ $(28,4,4)$ $(24,12,0)$ $(24,0,12)$ $(20,8,8)$ $(18,15,3)$ $(18,9,9)$ $(18,3,15)$ $(16,16,4)$ $(16,4,16)$ $(15,18,3)$ $(15,3,18)$ $(12,24,0)$ $(12,12,12)$ $(12,12,12)$ $(12,12,12)$ $(12,12,12)$ $(12,0,24)$ $(9,18,9)$ $(9,9,18)$ $(8,20,8)$ $(8,8,20)$ $(4,28,4)$ $(4,16,16)$ $(4,4,28)$ $(3,18,15)$ $(3,15,18)$ $(0,36,0)$ $(0,24,12)$ $(0,12,24)$ $(0,0,36)$ |
| 33.4 | 15 | $(15,0,0)$ $(12,3,0)$ $(12,0,3)$ $(9,6,0)$ $(9,3,3)$ $(9,3,3)$ $(9,0,6)$ $(6,9,0)$ $(6,6,3)$ $(6,6,3)$ $(6,3,6)$ $(6,3,6)$ $(6,0,9)$ $(5,5,5)$ $(5,5,5)$ $(5,5,5)$ $(5,5,5)$ $(5,5,5)$ $(5,5,5)$ $(3,12,0)$ $(3,9,3)$ $(3,9,3)$ $(3,6,6)$ $(3,6,6)$ $(3,3,9)$ $(3,3,9)$ $(3,0,12)$ $(0,15,0)$ $(0,12,3)$ $(0,9,6)$ $(0,6,9)$ $(0,3,12)$ $(0,0,15)$ |
| 35.1 | 9 | $(9,0,0)$ $(7,1,1)$ $(7,1,1)$ $(6,3,0)$ $(6,0,3)$ $(5,2,2)$ $(5,2,2)$ $(4,4,1)$ $(4,4,1)$ $(4,1,4)$ $(4,1,4)$ $(3,6,0)$ $(3,3,3)$ $(3,3,3)$ $(3,3,3)$ $(3,3,3)$ $(3,3,3)$ $(3,3,3)$ $(3,3,3)$ $(3,3,3)$ $(3,0,6)$ $(2,5,2)$ $(2,5,2)$ $(2,2,5)$ $(2,2,5)$ $(1,7,1)$ $(1,7,1)$ $(1,4,4)$ $(1,4,4)$ $(1,1,7)$ $(1,1,7)$ $(0,9,0)$ $(0,6,3)$ $(0,3,6)$ $(0,0,9)$ |
| 36.39 | 42 | $(42,0,0)$ $(33,6,3)$ $(33,3,6)$ $(27,12,3)$ $(27,3,12)$ $(24,12,6)$ $(24,6,12)$ $(21,21,0)$ $(21,0,21)$ $(18,15,9)$ $(18,9,15)$ $(15,18,9)$ $(15,9,18)$ $(14,14,14)$ $(14,14,14)$ $(14,14,14)$ $(14,14,14)$ $(14,14,14)$ $(14,14,14)$ $(12,27,3)$ $(12,24,6)$ $(12,6,24)$ $(12,3,27)$ $(9,18,15)$ $(9,15,18)$ $(6,33,3)$ $(6,24,12)$ $(6,12,24)$ $(6,3,33)$$(3,33,6)$ $(3,27,12)$ $(3,12,27)$ $(3,6,33)$ $(0,42,0)$ $(0,21,21)$ $(0,0,42)$ |

Table 8.11 lists the graphs in Tables 8.7 and 8.9 which are previously known in connection with representations of $\mathrm{Rep}_k(A_2)$ and affine representations of $\mathrm{Rep}(A_2)$. Alternate names for the graphs are listed under the main citation(s) where they appear. The graph 12.40 is omitted from the list since it appears in [16] without a name.

| | **Citation** | | |
|---|---|---|---|
| **Name** | [1], [3] | [6] | [11] |
| 6.5 | $\mathcal{D}^{(6)}$ | | $A_3/3$ |
| 10.3 | $\mathcal{A}^{(6)}$ | | $\mathbf{A_3}$ |
| 11.2 | $\mathcal{E}_4^{(12)}$ | | $(A_9/3)^{tc}$ |
| 12.15 | $\mathcal{D}^{(9)}$ | | $A_6/3$ |
| 12.23 | $\mathcal{E}_2^{(12)}$ | | $E_9/3 = (E_9)^c$ |
| 12.28 | $\mathcal{E}_1^{(12)}$ | | $\mathbf{E_9}$ |
| 14.4 | $\Sigma(36 \times 3)$ | $E$ | |
| 15.9 | $\mathcal{A}^{(7)}$ | | $\mathbf{A_4}$ |
| 16.5 | $\Sigma(72 \times 3)$ | $F$ | |
| 17.3 | $\mathcal{E}_5^{(12)}$ | | $(A_9/3)^t$ |
| 17.4 | $\Sigma(360 \times 3)$ | $L$ | |
| 18.33 | | $K$ | |
| 21.10 | $\mathcal{A}^{(8)}$ | | $\mathbf{A_5}$ |
| 21.17 | $\mathcal{D}^{(12)}$ | | $A_9/3$ |
| 24.11 | $\Sigma(216 \times 3)$ | $G$ | |
| 24.13 | $\mathcal{E}^{(24)}$ | | $\mathbf{E_{21}}$ |
| 28.1 | $\mathcal{A}^{(9)}$ | | $\mathbf{A_6}$ |
| 33.5 | $\mathcal{D}^{(15)}$ | | $A_{12}/3$ |
| 36.4 | $\mathcal{A}^{(10)}$ | | $\mathbf{A_7}$ |

Table 8.11: Previously-known graphs appearing in Tables 8.7 and 8.9.

# Chapter 9

# Conclusion

Here is a brief summary of what we accomplished in this thesis:

- We formulated a basic mathematical theory of representations of $\mathrm{Rep}_k(\mathfrak{g})$ and affine representations of $\mathrm{Rep}(\mathfrak{g})$.
- Borrowing an idea of Ocneanu, we showed how to associate root systems and Cartan matrices of higher type to involutionizable-representations of $\mathrm{Rep}_k(\mathfrak{g})$, and also to affine involutionizable-representations of $\mathrm{Rep}(\mathfrak{g})$.
- We classified the symmetric representations of $\mathrm{Rep}_k(A_2)$ and the symmetric affine representations of $\mathrm{Rep}(A_2)$.
- We defined a construction, labelled "the automorphism construction", which produces many new and interesting (nonsymmetric) representations of $\mathrm{Rep}_k(A_2)$ and affine representations of $\mathrm{Rep}(A_2)$.
- We provided a computer-generated list, up to about 36 vertices, of those representations of $\mathrm{Rep}_k(A_2)$ and those affine representations of $\mathrm{Rep}(A_2)$ which do not appear to arise from the automorphism construction.

Here are some questions which are open for future research:

- We showed how to assign root systems of higher type to representations of $\mathrm{Rep}_k(\mathfrak{g})$. Do these root systems have Weyl groups? Are there Lie algebras of higher type? More specifically, can we define a Lie algebra of higher type using an analogue of the Cartan-Weyl basis or the Chevalley-Serre basis?

- What accounts for the discrepancy between the root systems that assign to affine representations of $\text{Rep}(A_1)$ and the usual notion of an affine root system? For example, the root systems that we assign to affine representations of $\text{Rep}(A_1)$ are not closed under negation.

- What accounts for the interesting patterns the arise from the automorphism construction? For example, computer tests indicate that the graph $mE_8 \otimes_{(m)} N$ is a representation of $\text{Rep}_k(A_2)$ precisely when

  $$m \in \{1, 2, 3, 5, 6, 10, 15\}.$$

  Where do these numbers come from?

- Which of the new examples of representations of $\text{Rep}_k(A_2)$ in this thesis correspond to modular invariants?

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# Appendix A

# Appendix

We provide here a proof of the classification of symmetric representations of $\mathrm{Rep}_k(A_2)$ and symmetric affine representations of $\mathrm{Rep}(A_2)$ stated in Theorem 8.1. For the proof, we will use graph theory notation and terminology.

Let $\pi$ be a symmetric representation of $\mathrm{Rep}_k(A_2)$. Since $\mathrm{Rep}_k(A_2)$ is generated over multiplication and involution by $R_{(2,1)}$, we know that $\pi$ is completely determined by the nonnegative integer matrix

$$N_{(2,1)} := \pi(R_{(2,1)}) \in \mathrm{M}_r(\mathbb{Z}_{\geq 0}).$$

Moreover, this matrix is symmetric. Let $G$ be the symmetric graph whose adjacency matrix is $N_{(2,1)}$. Then the vertex set of $G$, denoted $V(G)$, can be identified with the index set of $\mathrm{M}_r(\mathbb{Z}_{\geq 0})$, and for any two vertices $a, b \in V(G)$, we say that $a$ and $b$ are connected by

$$E(G)(a,b) := [N_{(2,1)}]_{a,b} \in \mathbb{Z}_{\geq 0}$$

number of edges. The representation $\pi$ is irreducible if and only if $G$ is connected. We refer to $G$ as the graph of $\pi$.

Let $\pi$ and $\sigma$ be two symmetric representations of $\mathrm{Rep}_k(A_2)$ and let $G$ and $H$ be their respective graphs. Then there is a representation isomorphism of $\pi$ to $\sigma$ if and only if there is a graph isomorphism of $G$ to $H$. This allows us to identify a representation of $\mathrm{Rep}_k(A_2)$ with its graph.

The above argument also applies to symmetric affine representations of $\mathrm{Rep}(A_2)$.

**Theorem A.1**: Let $G$ be a symmetric graph. If $G$ is a representation of $\mathrm{Rep}_k(A_2)$, then the eigenvalues of $G$ lie in the real interval $(-1, 3)$. Conversely, if $G$ is an affine representation of $\mathrm{Rep}(A_2)$, then the eigenvalues of $G$ lie in the real interval $[-1, 3]$.

Technically, the second part of this theorem relies on Conjecture 6.7.

We proceed to classify all symmetric graphs whose eigenvalues lie in the real interval $[-1, 3]$. It turns out that nearly of the symmetric graphs whose eigenvalues lie in the real interval $[-1, 3]$ are either symmetric representations of $\mathrm{Rep}_k(A_2)$ or symmetric affine representations of $\mathrm{Rep}(A_2)$. The leftover symmetric graphs are listed in Table A.1. All of these leftover graphs happen to be symmetric affine representations of $\mathrm{Rep}(A_2)^{\mathbb{Z}}$. We refer to them as symmetric pseudo-representations. Their exponents are listed in Table A.2.

Table A.1: List of symmetric pseudo-representations.

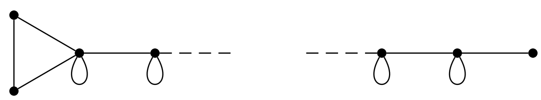
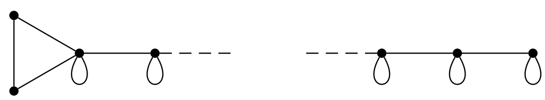
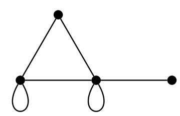
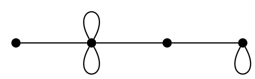
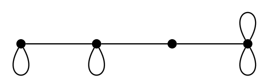
| **Name** | **Rank** | **Graph** |
|---|---|---|
| $AA_r$ | $r \geq 2$ |  |
| $AC_r$ | $r \geq 3$ |  |
| $BC_r$ | $r \geq 3$ |  |
| $I_4$ | 4 |  |
| $S_{0,2,0,1}$ | 4 |  |
| $S_{1,1,0,2}$ | 4 |  |

Table A.2: Exponents of symmetric pseudo-representations.

| **Name** | **Level** | **Exponents** |
|---|---|---|
| $AA_n$ | $2n$ | $(2n-2,1,1)$ $(2n-4,2,2)$ $\cdots$ $(0,n,n)$ |
| $AC_n$ | $4n-6$ | $(4n-8,1,1)$ $(4n-12,3,3)$ $\cdots$ $(0,2n-3,2n-3)$ $(0,2n-3,2n-3)$ |
| $BC_n$ | $4n-4$ | $(4n-6,1,1)$ $(4n-10,3,3)$ $\cdots$ $(2,2n-3,2n-3)$ $(0,2n-2,2n-2)$ |

Continued on next page . . .

Table A.2: Exponents of symmetric pseudo-representations (continued).

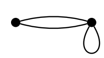
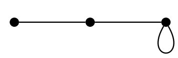
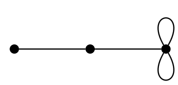
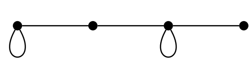
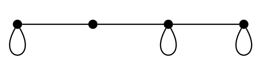
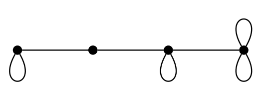
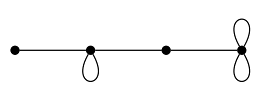
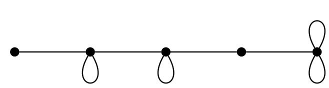
| **Name** | **Level** | **Exponents** |
|---|---|---|
| $I_4$ | 18 | $(16,1,1)$ $(8,5,5)$ $(4,7,7)$ $(0,9,9)$ |
| $S_{0,2,0,1}$ | 14 | $(12,1,1)$ $(8,3,3)$ $(4,5,5)$ $(0,7,7)$ |
| $S_{1,1,0,2}$ | 30 | $(24,3,3)$ $(20,5,5)$ $(12,9,9)$ $(0,15,15)$ |

For any symmetric graph $G$, let $\mu_{\max}(G)$ and $\mu_{\min}(G)$ denote its maximum and minimum eigenvalue, respectively. Then we have the following lemma.

**Lemma A.2**: Let $G$ be a connected symmetric graph whose eigenvalues lie in the real interval $[-1,3]$ and let $H$ be a symmetric graph with $\mu_{\min}(H) < -1$. Then $H$ is not an induced subgraph of $G$.

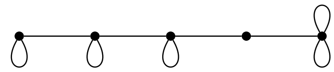
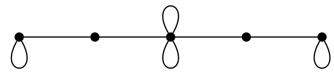
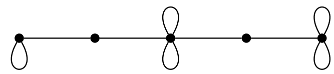
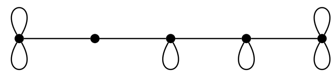
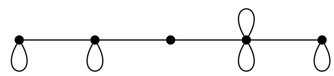
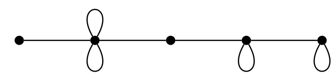
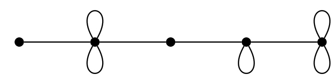
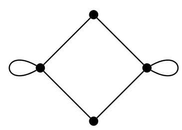
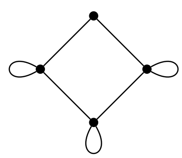
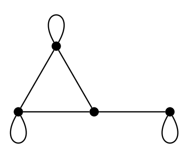
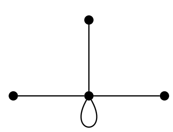
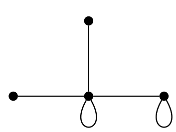
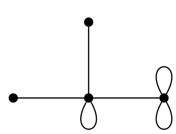
This lemma is well-known in algebraic graph theory: induced subgraphs have greater minimum eigenvalues. We refer to the $H$ in the above lemma as a forbidden induced subgraph. Table A.3 lists the forbidden induced subgraphs that we need for our classification.

Table A.3: List of forbidden induced subgraphs.

| **Name** | **Graph** | $\boldsymbol{\mu_{\max}}$ | $\boldsymbol{\mu_{\min}}$ |
|---|---|---|---|
| $Y_1$ |  | 2.0 | −2.0 |
| $Y_2$ |  | 2.5615 . . . | −1.5615 . . . |
| $Y_3$ |  | 1.4142 . . . | −1.4142 . . . |
| $Y_4$ |  | 1.8019 . . . | −1.2469 . . . |
| $Y_5$ |  | 2.4811 . . . | −1.1700 . . . |
| $Y_6$ |  | 2.1935 . . . | −1.1935 . . . |
| $Y_7$ |  | 2.3556 . . . | −1.0952 . . . |
| $Y_8$ |  | 2.7640 . . . | −1.0614 . . . |
| $Y_9$ |  | 2.5764 . . . | −1.1267 . . . |
| $Y_{10}$ |  | 2.6581 . . . | −1.1063 . . . |

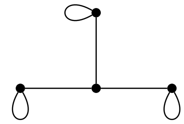
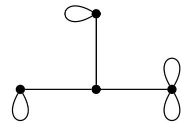
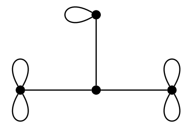
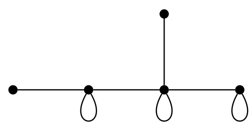
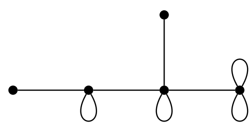
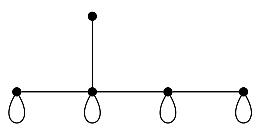
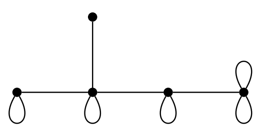
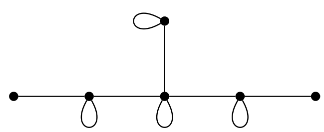
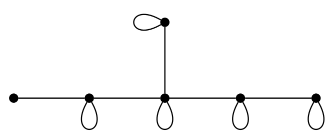
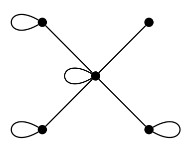
Continued on next page . . .

Table A.3: List of forbidden induced subgraphs (continued).

| **Name** | **Graph** | $\mu_{\mathbf{max}}$ | $\mu_{\mathbf{min}}$ |
|---|---|---|---|
| $Y_{11}$ |  | $2.6964\ldots$ | $-1.0431\ldots$ |
| $Y_{12}$ |  | $2.8608\ldots$ | $-1.1149\ldots$ |
| $Y_{13}$ |  | $2.9429\ldots$ | $-1.0665\ldots$ |
| $Y_{14}$ |  | $2.8866\ldots$ | $-1.0264\ldots$ |
| $Y_{15}$ |  | $2.9540\ldots$ | $-1.0384\ldots$ |
| $Y_{16}$ |  | $2.8370\ldots$ | $-1.0850\ldots$ |
| $Y_{17}$ |  | $2.9215\ldots$ | $-1.0541\ldots$ |
| $Y_{18}$ |  | $2.5615\ldots$ | $-1.5615\ldots$ |
| $Y_{19}$ |  | $2.8136\ldots$ | $-1.3429\ldots$ |
| $Y_{20}$ |  | $2.8608\ldots$ | $-1.1149\ldots$ |
| $Y_{21}$ |  | $2.3027\ldots$ | $-1.3027\ldots$ |
| $Y_{22}$ |  | $2.4811\ldots$ | $-1.1700\ldots$ |
| $Y_{23}$ |  | $1.2541\ldots$ | $-1.1149\ldots$ |

Continued on next page . . .

Table A.3: List of forbidden induced subgraphs (continued).

| **Name** | **Graph** | $\mu_{\mathbf{max}}$ | $\mu_{\mathbf{min}}$ |
|---|---|---|---|
| $Y_{24}$ |  | $2.3027\ldots$ | $-1.3027\ldots$ |
| $Y_{25}$ |  | $2.6751\ldots$ | $-1.2143\ldots$ |
| $Y_{26}$ |  | $2.8608\ldots$ | $-1.1149\ldots$ |
| $Y_{27}$ |  | $2.7050\ldots$ | $-1.1388\ldots$ |
| $Y_{28}$ |  | $2.9738\ldots$ | $-1.0911\ldots$ |
| $Y_{29}$ |  | $2.7640\ldots$ | $-1.0614\ldots$ |
| $Y_{30}$ |  | $2.9540\ldots$ | $-1.0384\ldots$ |
| $Y_{31}$ |  | $2.8608\ldots$ | $-1.1149\ldots$ |
| $Y_{32}$ |  | $2.8988\ldots$ | $-1.0467\ldots$ |
| $Y_{33}$ |  | $2.9122\ldots$ | $-1.1986\ldots$ |

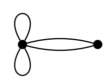
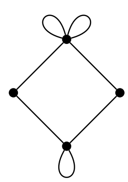
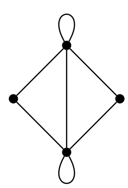
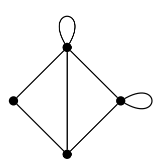
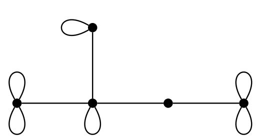
**Lemma A.3**: Let $G$ be a symmetric graph whose eigenvalues lie in the real interval $[-1, 3]$ and let $H$ be a symmetric graph with $\mu_{\max}(H) \geq 3$. Then $H$ is not a proper subgraph of $G$.

This lemma is also well-known in algebraic graph theory: proper subgraphs have strictly lesser maximum eigenvalues. We refer to $H$ in the above lemma as a forbidden proper subgraph. It turns out that all affine representations of $\mathrm{Rep}(A_2)$ are forbidden proper subgraphs. This is implied by Conjecture 6.7, but it can also be verified case-by-case.

**Lemma A.4**: Let $G$ be a connected symmetric graph whose eigenvalues lie in the real interval $[-1, 3]$ and let $H$ be a symmetric graph with $\mu_{\max}(H) > 3$, or with $\mu_{\max}(H) = 3$ and $\mu_{\min}(H) < -1$. Then $H$ is not a subgraph of $G$.

This lemma follows by combining Lemmas A.2 and A.3. We refer to the $H$ in the above lemma as a forbidden subgraph. Table A.4 lists the forbidden subgraphs that we need for our classification.

Table A.4: List of forbidden subgraphs.

| **Name** | **Graph** | $\boldsymbol{\mu_{\max}}$ | $\boldsymbol{\mu_{\min}}$ |
|---|---|---|---|
| $X_1$ |  | $3.2360\ldots$ | $-1.2360\ldots$ |
| $X_2$ |  | $3.0$ | $-3.0$ |
| $X_3$ |  | $3.0$ | $-1.4142\ldots$ |
| $X_4$ |  | $3.2360\ldots$ | $-1.2360\ldots$ |
| $X_5$ |  | $3.1401\ldots$ | $-1.2734\ldots$ |
| $X_6$ |  | $3.0$ | $-1.0952\ldots$ |

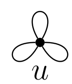
**Lemma A.5**: Let $G$ be a connected symmetric graph whose eigenvalues lie in the real interval $[-1, 3]$ and let $H$ be an induced subgraph of $G$. Then the eigenvalues of $H$ lie in the real interval $[-1, 3]$.

The lemma also follows by combining Lemmas A.2 and A.3.

We now prove that the only symmetric graphs whose eigenvalues lie in the real interval $[-1, 3]$ are those listed in Tables 8.1, 8.3, and A.1. The proof is somewhat lengthy, so we divide the various cases into separate propositions. For the remainder of this appendix, let $G$ be a connected symmetric graph whose eigenvalues lie in the real interval $[-1, 3]$.

**Proposition A.6**: Suppose at least one vertex of $G$ has three or more loops. Then $G = K_1$.

*Proof.* Let $u$ be a vertex with three or more loops. Then we have

$G \supseteq$  $= K_1$.



Since $K_1$ is a forbidden proper subgraph, we must have $G = K_1$. □

In all of the remaining propositions, we will assume that every vertex of $G$ has at most two loops.

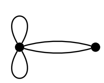
**Proposition A.7**: Suppose some two vertices of $G$ are adjacent by two or more edges. Then $G = K_2$.

*Proof.* Let $u$ and $v$ be two vertices that are adjacent by two or more edges. Then we have

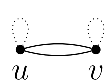
$G \supseteq$  .



The forbidden subgraph

 $= X_1$

requires that $u$ and $v$ have at most one loop:

$G \supseteq$  .



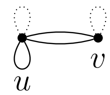
Here the dotted loops indicate that $u$ and $v$ have only one loop available. The forbidden subgraph

 $= X_2$

requires that $u$ and $v$ are adjacent by at most two edges. Thus there are no more edges to consider. The forbidden induced subgraph


$= Y_1$

requires that at least one of the vertices $u$ and $v$ has one loop. Without loss of generality, assume $u$ has one loop:

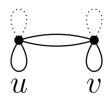
$G \supseteq$ 

.

The forbidden induced subgraph


$= Y_2$

requires that $v$ also has one loop:

$G \supseteq$ 

$= K_2$.

Since $K_2$ is a forbidden proper subgraph, we must have $G = K_2$. □

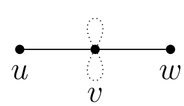
In all of the remaining propositions, we will assume that any two vertices of $G$ are adjacent by at most one edge. Thus for any two vertices $u, v \in V(G)$, we will use the expression "$uv$ is an edge" to mean that $u$ and $v$ are adjacent by one edge.

**Proposition A.8**: Suppose $G$ contains three vertices $u$, $v$, and $w$ such that the following conditions hold:

1. The vertex $v$ has no loops.
2. The pairs $uv$ and $wv$ are edges while $uw$ is not.

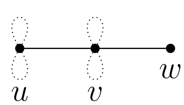
Then $u$ and $w$ both have at least one loop.

*Proof.* We have the following situation:

$G \supseteq$ 

.

Here the dotted pair of loops indicates that $v$ has no loops. Suppose to the contrary that at least one of the vertices $u$ and $w$ has no loops. Without loss of generality, assume $u$ has no loops. Then we have

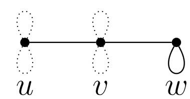
$G \supseteq$ 

.

The forbidden induced subgraph

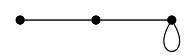

$= Y_3$

requires that $w$ has at least one loop:

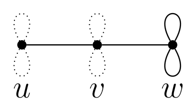
$G \supseteq$ 

.

The forbidden induced subgraph


$= Y_4$

requires that $w$ has at least two loops:

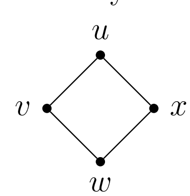
$G \supseteq$ 

$= Y_5$.

Since $Y_5$ is a forbidden induced subgraph, we have a contradiction. We conclude that $u$ and $w$ both have at least one loop. □

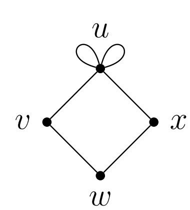
**Proposition A.9**: Suppose $G$ has at least one 4-cycle. Then $G \in \{K_4, L_4, M_4\}$.

*Proof.* Label the vertices of one of the 4-cycles using $u$, $v$, $w$, and $x$. Then we have

$G \supseteq$ 

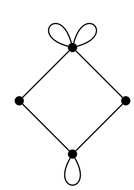
.

First we show that the vertices $u$, $v$, $w$, and $x$ have at most one loop. Suppose to the contrary that at least one of the vertices $u$, $v$, $w$, and $x$ has two loops. Without loss of generality, assume $u$ has two loops. Then we have

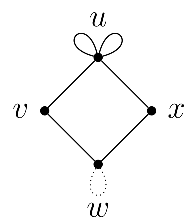
$G \supseteq$ 

.

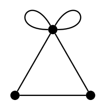
The forbidden subgraph


$= X_3$

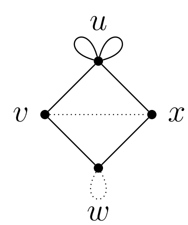
requires that $w$ has no loops:

$G \supseteq$  .

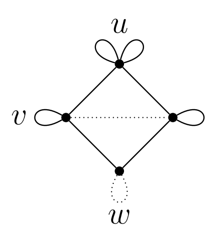
The forbidden proper subgraph

 $= CT_3$

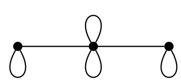
requires that $vx$ is not an edge:

$G \supseteq$  .

By Proposition A.8, the vertices $v$ and $x$ both have at least one loop. Thus

$G \supseteq$  .

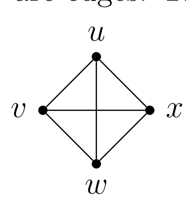
This contradicts the forbidden proper subgraph

 $= DT_3$.

Thus $u$, $v$, $w$, and $x$ all have at most one loop.

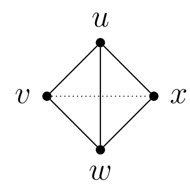
We proceed to consider the various cases in which combinations of the pairs $uw$ and $vx$ are edges.

**Case 1:** Suppose $uw$ and $vx$ are edges. Then we have

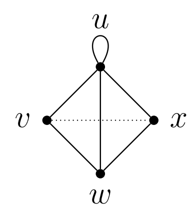
$G \supseteq$  $= L_4$.

Since $L_4$ is a forbidden proper subgraph, we have $G = L_4$.

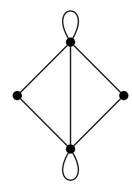
**Case 2:** Suppose $uw$ is an edge while $vx$ is not. Then we have

$G \supseteq$  .

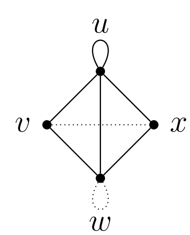

Suppose to the contrary that at least one of $u$ and $w$ has exactly one loop. Without loss of generality, assume $u$ has exactly one loop. Then we have

$G \supseteq$  .

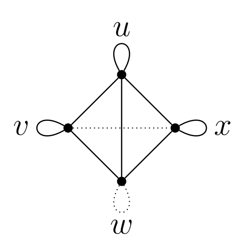

The forbidden subgraph

 $= X_4$

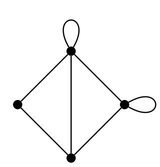
requires that $w$ has no loops:

$G \supseteq$  .

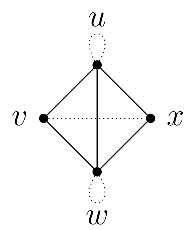

By Proposition A.8, the vertices $v$ and $x$ both have at least one loop:

$G \supseteq$  .


This contradicts the forbidden subgraph

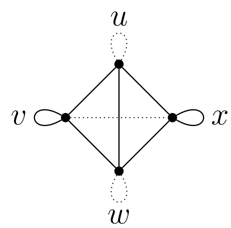
 $= X_5$.

Thus $u$ and $w$ have no loops:

$G \supseteq$ 

.

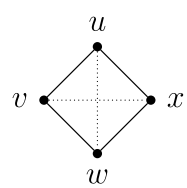
By Proposition A.8, the vertices $v$ and $x$ both have at least one loop:

$G \supseteq$ 

$= M_4$.

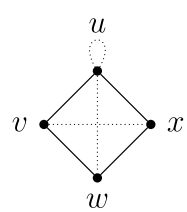
Since $M_4$ is a forbidden proper subgraph, we have $G = M_4$.

**Case 3:** Suppose $uw$ and $vx$ are not edges. Then we have

$G \supseteq$ 

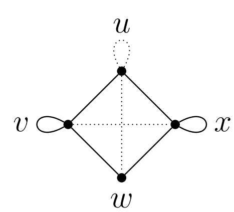
.

Suppose to the contrary that one of the four vertices $u$, $v$, $w$, and $x$ has no loops. Without loss of generality, assume $u$ has no loops. Then we have

$G \supseteq$ 

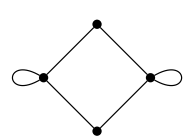
.

By Proposition A.8, the vertices $v$ and $x$ both have at least one loop:

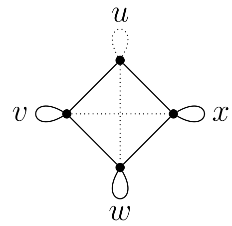
$G \supseteq$ 

.

The forbidden induced subgraph

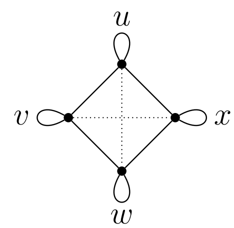
 $= Y_{18}$

requires that $w$ has at least one loop:

$G \supseteq$  $= Y_{19}$.



Since $Y_{19}$ if a forbidden induced subgraph, we have a contradictions. Thus $u$, $v$, $w$, and $x$ all have at least one loop:

$G \supseteq$  $= K_4$.



Since $K_4$ is a forbidden proper subgraph, we have $G = K_4$.

In all cases, we have $G \in \{K_4, L_4, M_4\}$. □

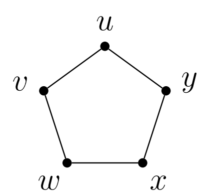
**Proposition A.10**: Suppose $G$ satisfies the following conditions:

1. There are no 4-cycles.

2. There is at least one 5-cycle having a vertex with two loops.

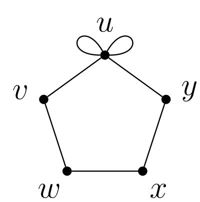
Then $G = R_5$.

*Proof.* Label the vertices of one of the 5-cycles having a vertex with two loops using $u$, $v$, $w$, $x$, and $y$. Then we have

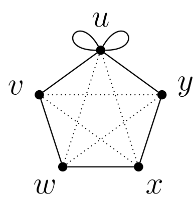
$G \supseteq$  .



Without loss of generality, assume $u$ has two loops:

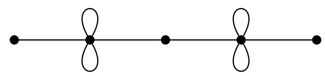
$G \supseteq$  .

Since $G$ has no 4-cycles, we know that $uw$, $vx$, $wy$, $xu$, and $yv$ are not edges:

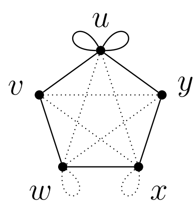
$G \supseteq$ 

.

The forbidden proper subgraph

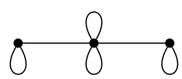
 $= S_{0,2,0,2,0}$

requires that $w$ and $x$ have at most one loop:

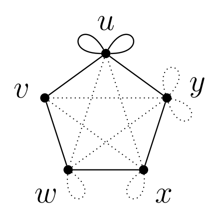
$G \supseteq$ 

.

The forbidden proper subgraph

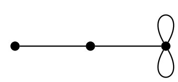
 $= DT_3$

requires that at least one of the vertices $v$ and $y$ has no loops. Without loss of generality, assume $y$ has no loops:

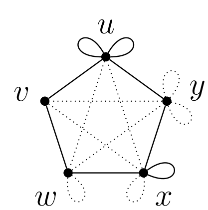
$G \supseteq$ 

.

The forbidden induced subgraph

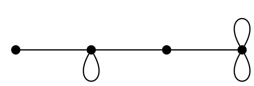
 $= Y_5$

requires that $x$ has at least one loop:

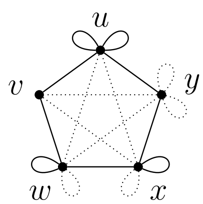
$G \supseteq$ 

.

The forbidden induced subgraph

 $= Y_9$

requires that $w$ has at least one loop:

$G \supseteq$ 

$= R_5.$

Since $R_5$ is a forbidden proper subgraph, we have $G = R_5$. □

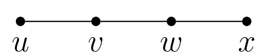
**Proposition A.11**: Suppose $G$ contains four vertices $u$, $v$, $w$, and $x$ such that the following conditions hold:

1. All four vertices have at most one loop.

2. The pairs $uv$, $vw$, and $wx$ are edges while $uw$, $ux$, and $vx$ are not.

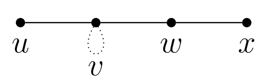
Then $v$ and $w$ both have exactly one loop.

*Proof.* We have the following situation:

$G \supseteq$ 

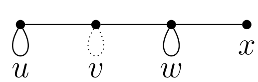
.

Suppose to the contrary that at least one of $v$ and $w$ has no loops. Without loss of generality, assume $v$ has no loops. Then we have

$G \supseteq$ 

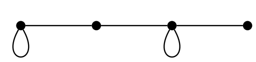
.

By Proposition A.8, the vertices $u$ and $w$ both have at least one loop:

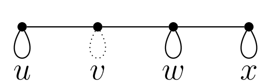
$G \supseteq$ 

.

The forbidden induced subgraph


$= Y_6$

requires that $u$ has at least one loop:

$G \supseteq$ 

$= Y_7.$

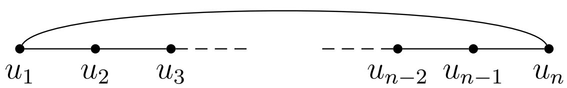
Since $Y_7$ is a forbidden induced subgraph, we have a contradiction. We conclude that $v$ and $x$ both have one loop. □

**Proposition A.12**: Suppose $G$ satifies the following conditions:

1. There are no 4-cycles.
2. There is at least one $n$-cycle for some $n \geq 5$ whose vertices all have at most one loop.

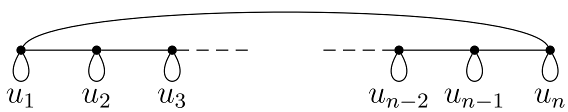
Then $G = K_n$.

*Proof.* Label the vertices of one of the $n$-cycles whose vertices all have at most one loop using $u_1, \dots, u_n$. Then we have

$G \supseteq$ 

.

Since $n \geq 5$, by Proposition A.11, the vertices $u_1, \dots, u_n$ all have one loop:

$G \supseteq$ 

$= K_n$.

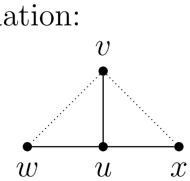
Since $K_n$ is a forbidden proper subgraph, we have $G = K_n$. □

**Proposition A.13**: Suppose that $G$ contains four vertices $u$, $v$, $w$, and $x$ such that following conditions hold:

1. All four vertices have at most one loop.
2. The pairs $uv$, $uw$, and $ux$ are edges while $vw$ and $vx$ are not.

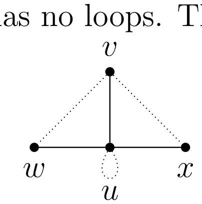
Then $u$ has exactly one loop.

*Proof.* We have the following situation:

$G \supseteq$ 

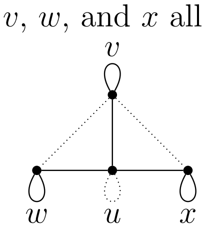
.

Suppose to the contrary that $u$ has no loops. Then we have

$G \supseteq$ 

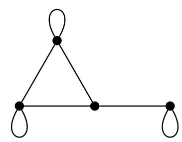
.

By Proposition A.8, the vertices $v$, $w$, and $x$ all have exactly one loop:

$G \supseteq$ 

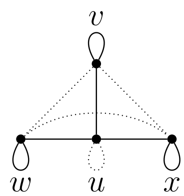
.

The forbidden induced subgraph



$= Y_{20}$

requires that $wx$ is not an edge:

$G \supseteq$  $= Y_{24}$.



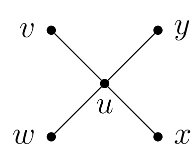
Since $Y_{24}$ is a forbidden induced subgraph, we have a contradiction. Therefore $u$ has one loop. □

**Proposition A.14**: Suppose $G$ satisfies the following conditions:

1. All vertices have at most one loop.
2. There are no $n$-cycles for all $n \geq 4$.
3. There is at least one vertex that is adjacent to four or more other vertices.

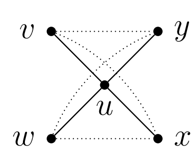
Then $G \in \{CC_5, CD_5, DD_5\}$.

*Proof.* Let $u$ be the vertex that is adjacent to four or more other vertices, and label four of these other vertices using $v$, $w$, $x$, and $y$. Then we have

$G \supseteq$  .

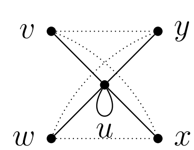


Since $G$ has no 4-cycles, only disjoint pairs of $v$, $w$, $x$, and $y$ can be edges. Without loss of generality, we can assume that the only possible edges are $vw$ and $xy$. With this arrangement, the pairs $vx$, $vy$, $wx$, and $wy$ are not edges:

$G \supseteq$  .



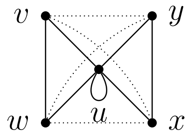
By Proposition A.13, the vertex $u$ has one loop:

$G \supseteq$  .



There are three cases to consider.

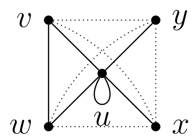
**Case 1:** Suppose $vw$ and $xy$ are edges. Then we have

$G \supseteq$ 

$= CC_5.$

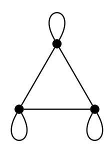
Since $CC_5$ is a forbidden proper subgraph, we have $G = CC_5$.

**Case 2:** Suppose exactly one of the pairs $vw$ and $xy$ is an edge. Without loss of generality, assume $vw$ is an edge while $xy$ is not. Then we have

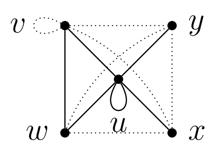
$G \supseteq$ 

.

The forbidden proper subgraph

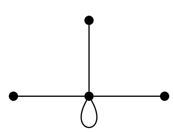

$= K_3$

requires that $v$ or $w$ has no loops. Without loss of generality, assume $v$ has no loops:

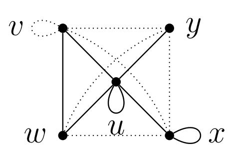
$G \supseteq$ 

.

The forbidden induced subgraph

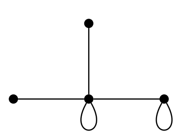

$= Y_{21}$

requires that $x$ or $y$ has one loop. Without loss of generality, assume $x$ has one loop:

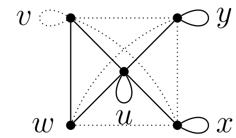
$G \supseteq$ 

.

The forbidden induced subgraph


$= Y_{22}$

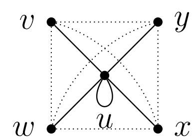
requires that $y$ has one loop:

$G \supseteq$ 

$= CD_5$.

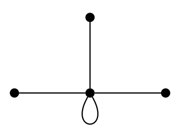
Since $CD_5$ is a forbidden proper subgraph, we have $G = CD_5$.

**Case 3:** Suppose $vw$ and $xy$ are not edges. Then we have

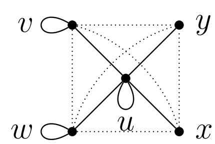
$G \supseteq$ 

.

The forbidden induced subgraph

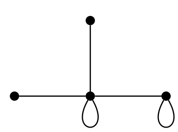

$= Y_{21}$

requires that at least two of the vertices $v$, $w$, $x$, and $y$ have one loop. Without loss of generality, assume $v$ and $w$ have one loop:

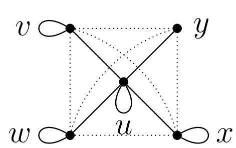
$G \supseteq$ 

.

The forbidden induced subgraph

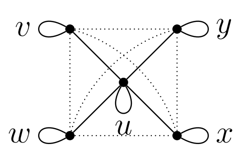

$= Y_{22}$

requires that $x$ or $y$ has one loop. Without loss of generality, assume $x$ has one loop:

$G \supseteq$ 

$= Y_{33}$.

Since $Y_{33}$ is a forbidden induced subgraph, the vertex $y$ must have one loop:

$G \supseteq$ 

$= DD_5$.

Since $DD_5$ is a forbidden proper subgraph, we have $G = DD_5$.

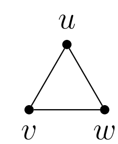
In all cases, we have $G \in \{CC_5, CD_5, DD_5\}$. □

**Proposition A.15**: Suppose $G$ satisfies the following conditions:

1. All vertices have at most one loop.
2. There are no $n$-cycles for all $n \geq 4$.
3. There is exactly one 3-cycle.
4. All vertices are adjacent to at most three other vertices.
5. All vertices except for the vertices of the 3-cycle are adjacent to at most two other vertices.

Then $G \in \{AC_n, BC_n : n \geq 3\} \cup \{H_3, H_4, I_4, I_5, J_5, K_3\}$.

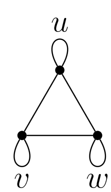
*Proof.* Label the vertices of the one 3-cycle using $u$, $v$, and $w$. Then we have

$G \supseteq$ 

.

There are four cases to consider.

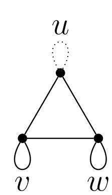
**Case 1:** Suppose the vertices $u$, $v$, and $w$ all have one loop. Then we have

$G \supseteq$ 

$= K_3$.

Since $K_3$ is a forbidden proper subgraph, we have $G = K_3$.

**Case 2:** Suppose exactly two of the vertices $u$, $v$, and $w$ have one loop. Without loss of generality, assume $u$ has no loops while $v$ and $w$ have one loop. Then we have

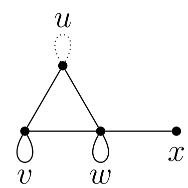
$G \supseteq$ 

$= H_3$.

Combining Proposition A.13 with the fact that there are no 4-cycles, we see that the vertex $u$ is not adjacent to any additional vertices. There are three cases to consider.

**Case 2.1:** Suppose $v$ and $w$ are not adjacent to any additional vertices. Then there are no more vertices, edges, or loops to consider. Thus $G = H_3$.

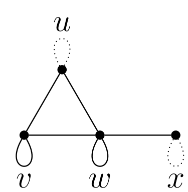
**Case 2.2:** Suppose exactly one of the vertices $v$ and $w$ is adjacent to an additional vertex. Without loss of generality, assume $w$ is adjacent to an additional vertex $x$ while $v$ is not adjacent to any additional vertices. Then we have

$G \supseteq$  .



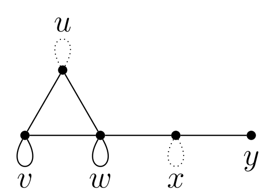
Since all vertices of $G$ are adjacent to at most three other vertices, we see that $w$ is not adjacent to any additional vertices. Since $G$ has no 4-cycles, the pairs $ux$ and $vx$ are not edges. There are two cases to consider.

**Case 2.2.1:** Suppose $x$ has no loops. Then we have

$G \supseteq$  $= I_4$.



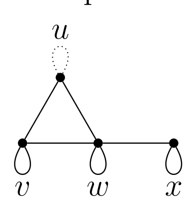
We show that $x$ is not adjacent to any additional vertices. Suppose to the contrary that $x$ is adjacent to an additional vertex $y$. Then we have

$G \supseteq$  .



By Proposition A.11, the vertex $x$ has one loop. This contradicts the assumption that $x$ has no loops. Thus $x$ is not adjacent to an additional vertex. Since there are no more vertices, edges, or loops to consider, we must have $G = I_4$.

**Case 2.2.2:** Suppose $x$ has one loop. Then we have

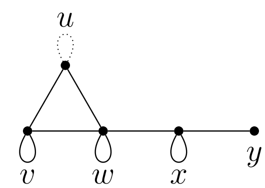
$G \supseteq$  $= H_4$.



There are two cases to consider.

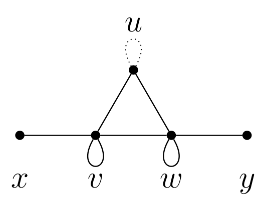
**Case 2.2.2.1:** Suppose $x$ is not adjacent to any additional vertuces. Then there are no more vertices, edges, or loops to consider. Thus $G = H_4$.

**Case 2.2.2.2:** Suppose $x$ is adjacent to an additional vertex $y$. Then we have

$G \supseteq$  $= J_5.$

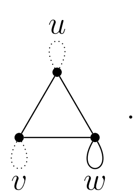

Since $J_5$ is a forbidden proper subgraph, we have $G = J_5$.

**Case 2.3:** Suppose both $v$ and $w$ are adjacent to additional vertices. Since $G$ has no 4-cycles, these additional vertices must be distinct. Thus assume $v$ is adjacent to an additional vertex $x$, and $w$ to an additional vertex $y$:

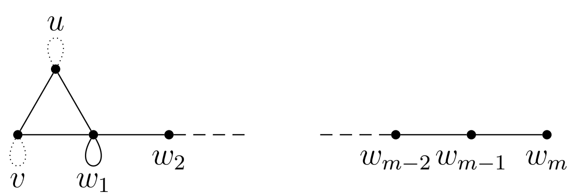
$G \supseteq$  $= I_5.$


Since $I_5$ is a forbidden proper subgraph, we have $G = I_5$.

**Case 3:** Suppose exactly two of the vertices $u$, $v$, and $w$ have no loops. Without loss of generality, assume $u$ and $v$ have no loops while $w$ has exactly one loop. Then we have

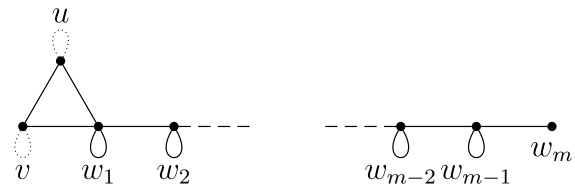
$G \supseteq$  .


By Proposition A.13, the vertices $u$ and $v$ are not adjacent to any additional vertices. Since there are no $n$-cycles for all $n \geq 4$, since all vertices, except possibly for $w$, are adjacent to at most two other vertices, and since $w$ is adjacent to at most three other vertices, the graph $G$ can only consist of a path attached to the 3-cycle at the vertex $w$. Label the vertices of this path using $w = w_1, \ldots, w_m$. Then we have

$G \supseteq$  .

By Proposition A.11, all of the vertices $w_2, \ldots, w_{m-1}$ have one loop:

$G \supseteq$




.

There are two cases to consider.

**Case 3.1:** Suppose $w_m$ has no loops. Then $m \geq 2$ and we have

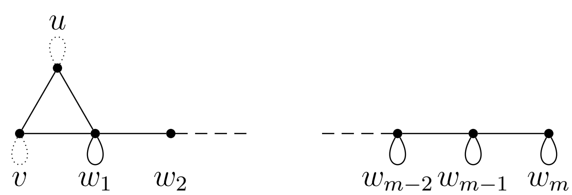
$G =$


$= AC_{m+2}$.

Thus $G = AC_{m+2}$. Since $m \geq 2$, we have $G \in \{AC_n : n \geq 4\}$.

**Case 3.2:** Suppose $w_m$ has one loop. Then we have

$G =$

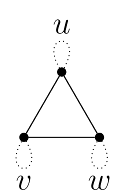



$= BC_{m+2}$.

Thus $G = BC_{m+2}$. Since $m \geq 1$, we have $G \in \{BC_n : n \geq 3\}$.

**Case 4:** Suppose all of the vertices $u$, $v$, and $w$ have no loops. Then we have

$G \supseteq$




$= AC_3$.

By Proposition A.13, none of the vertices $u$, $v$, and $w$ are adjacent to additional vertices. Thus $G = AC_3$.

In all cases, we have $G \in \{AC_n, BC_n : n \geq 3\} \cup \{H_3, H_4, I_4, I_5, J_5, K_3\}$. □

**Proposition A.16**: Suppose $G$ satisfies the following conditions:

1. All vertices have at most one loop.

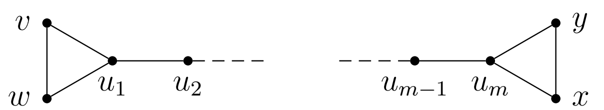
2. There are no $n$-cycles for all $n \geq 4$.

3. There are at least two 3-cycles.

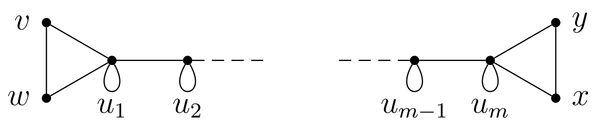
4. All vertices are adjacent to at most three other vertices.

Then $G \in \{CC_n : n \geq 6\}$.

*Proof.* Let $u_1, \ldots, u_m$ be the shortest path between two of the 3-cycles. Let $u_1$, $v$, and $w$ be the vertices of the first 3-cycle, and let $u_m$, $x$, and $y$ be the vertices of the second. Since $G$ has no 4-cycles, and since all vertices of $G$ are adjacent to at most three other vertices, these two 3-cycles do not share any vertices. Thus $m \geq 2$ and we have

$G \supseteq$ 


By Proposition A.11, the vertices $u_1, \ldots, u_m$ all have one loop:

$G \supseteq$ 

$= CC_{m+4}$.

Since $CC_{m+4}$ is a forbidden proper subgraph, we have $G = CC_{m+4}$. Since $m \geq 2$, it follows that $G \in \{CC_n : n \geq 6\}$. □

**Proposition A.17**: Suppose $G$ satifies the following conditions:

1. All vertices have at most one loop.

2. There are no $n$-cycles for all $n \geq 4$.

3. There is at least one 3-cycle.

4. All vertices of $G$ are adjacent to at most three other vertices.

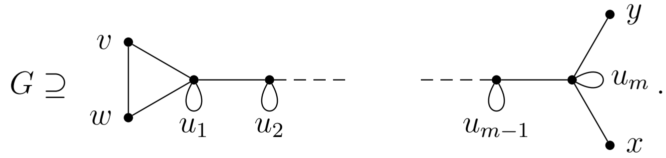
5. At least one vertex outside of a 3-cycle is adjacent to three other vertices.

Then $G \in \{CD_n : n \geq 6\}$.

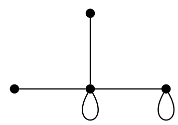
*Proof.* Let $u_1, \ldots, u_m$ be the shortest path between a 3-cycle and a vertex outside of a 3-cycle that is adjacent to three other vertices. Let $u_1$, $v$, and $w$ be the vertices of the 3-cycle, and let $x$ and $y$ be the other two vertices to which $u_m$ is adjacent. Since there are no $n$-cycles for all $n \geq 4$, the vertices $x$ and $y$ are distinct from $v$ and $w$. Thus we have

$G \supseteq$ 

By Proposition A.11, the vertices $u_1, \ldots, u_m$ all have one loop:

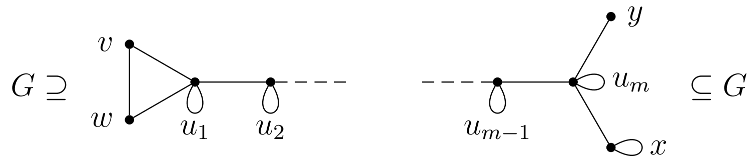



The forbidden induced subgraph

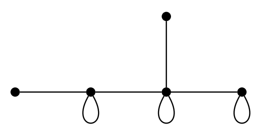


$= Y_{22}$

requires that $x$ or $y$ has one loop. Without loss of generality, assume $x$ has one loop:

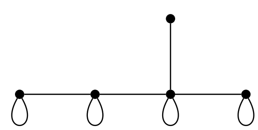



If $m = 2$ and $v$ has no loops, then the forbidden induced subgraph



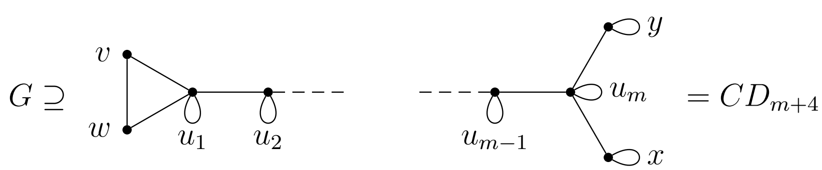
$= Y_{27}$

requires that $y$ has one loop. Conversely, if $m \geq 3$ or if $v$ has one loop, then the forbidden induced subgraph



$= Y_{29}$

requires that $y$ has one loop. In all cases, the vertex $y$ has one loop:




Since $CD_{m+4}$ is a forbidden proper subgraph, we have $G = CD_{m+4}$. Since $m \geq 2$, it follows that $G \in \{C_n : n \geq 6\}$. □

**Proposition A.18**: Suppose $G$ satisfies the following conditions:

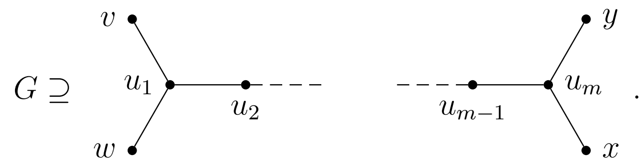
1. All vertices have at most one loop.

2. There are no $n$-cycles for all $n \geq 3$.

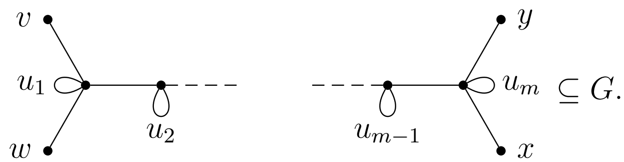
3. At least two vertices are adjacent to three or more other vertices.

Then $G = \{DD_n : n \geq 6\}$.

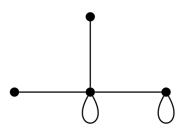
*Proof.* Let $u_1, \ldots, u_m$ be the shortest path between two of the vertices that are adjacent to three or more other vertices. Let $v$ and $w$ be two of the other vertices to which $u_1$ is adjacent, and let $x$ and $y$ be two of the other vertices to which $u_m$ is adjacent. Since $G$ has no $n$-cycles for all $n \geq 3$, these vertices are all distinct. Thus we have




By Proposition A.11, the vertices $u_1, \ldots, u_m$ all have one loop:

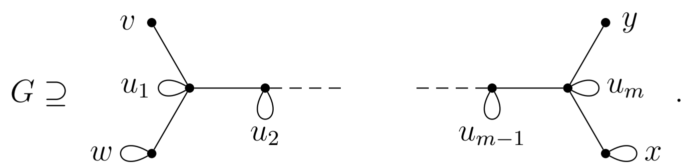



The forbidden induced subgraph

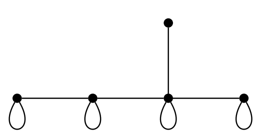


$= Y_{22}$

requires that $v$ or $w$ has one loop, and that $x$ or $y$ has one loop. Without loss of generality, assume $w$ and $x$ have one loop:

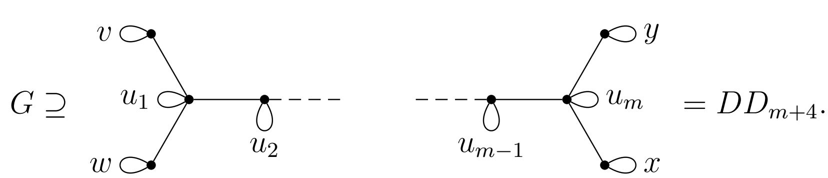



The forbidden induced subgraph



$= Y_{29}$

requires that $v$ and $y$ have one loop:



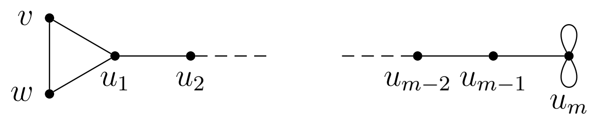
Since $DD_{m+4}$ is a forbidden proper subgraph, we have $G = DD_{m+4}$. Since $m \geq 2$, it follows that $G \in \{DD_n : n \geq 6\}$. □

**Proposition A.19**: Suppose $G$ satifies the following conditions:

1. There are no $n$-cycles for all $n \geq 4$.
2. There is at least one 3-cycle.
3. At least one vertex has two loops.

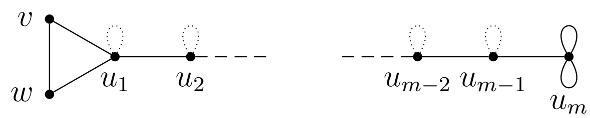
Then $G = \{CT_n : n \geq 3\} \cup \{P_4\}$.

*Proof.* Let $u_1, \ldots, u_m$ be the shortest path between a vertex of a 3-cycle and a vertex that has two loops. Let $u_1$, $v$, and $w$ be the vertices of the 3-cycle. Then we have

$G \supseteq$ 

.

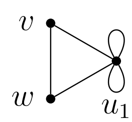
Since the path $u_1, \ldots, u_m$ is the shortest path between a vertex of a 3-cycle with a vertex that has two loops, the vertices $u_1, \ldots, u_{m-1}$ all have at most one loop:

$G \supseteq$ 

.

There are three cases to consider.

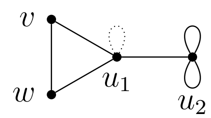
**Case 1:** Suppose $m = 1$. Then we have

$G \supseteq$ 

$= CT_3$.

Since $CT_3$ is a forbidden proper subgraph, we have $G = CT_3$.

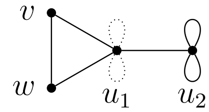
**Case 2:** Suppose $m = 2$. Then we have

$G \supseteq$ 

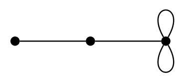
.

There are two cases to consider.

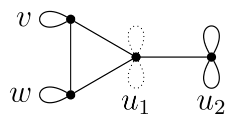
**Case 2.1:** Suppose $u_1$ has no loops. Then we have

$G \supseteq$  .


The forbidden induced subgraph

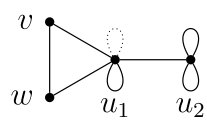
 $= Y_5$

requires that $v$ and $w$ have at least one loop:

$G \supseteq$  $= P_4$.

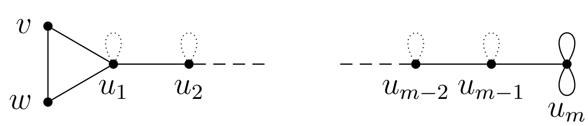

Since $P_4$ is a forbidden proper subgraph, we have $G = P_4$.

**Case 2.2:** Suppose $u_1$ has one loop. Then we have

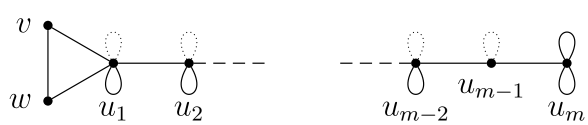
$G \supseteq$  $= CT_4$.


Since $CT_4$ is a forbidden proper subgraph, we have $G = CT_4$.

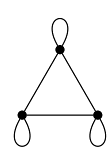
**Case 3:** Suppose $m \geq 3$. Then we have

$G \supseteq$  .

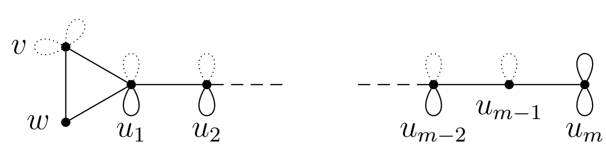

By Proposition A.11, the vertices $u_1, \ldots, u_{m-2}$ all have one loop:

$G \supseteq$  .

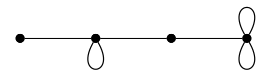

The forbidden proper subgraph

 $= K_3$

requires that $v$ or $w$ has no loops. Without loss of generality, assume $v$ has no loops:

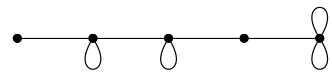
$G \supseteq$  .

If $m = 3$, then the forbidden induced subgraph



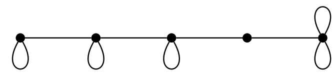
$= Y_9$

requires that $u_{m-1}$ has one loop. If $m = 4$, then the forbidden induced subgraph



$= Y_{10}$

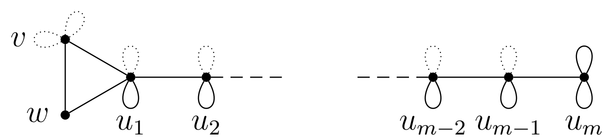
requires that $u_{m-1}$ has one loop. If $m \geq 5$, then the forbidden induced subgraph



$= Y_{11}$

requires that $u_{m-1}$ has one loop. In all cases, the vertex $u_{m-1}$ has one loop:

$G \supseteq$




$= CT_{m+2}$.

Since $CT_{m+2}$ is a forbidden proper subgraph, we have $G = CT_{m+2}$. Since $m \geq 3$, it follows that $G \in \{CT_n : n \geq 5\}$.

In all cases, we have $G = \{CT_n : n \geq 3\} \cup \{P_4\}$. □

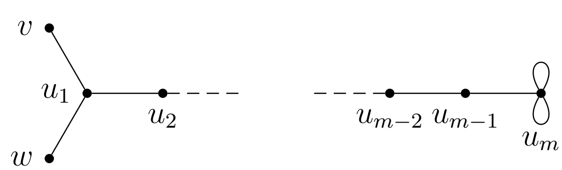
**Proposition A.20**: Suppose $G$ satifies the following conditions:

1. There are no $n$-cycles for all $n \geq 3$.
2. At least one vertex has two loops.
3. At least one vertex that is adjacent to three or more other vertices.

Then $G \in \{DT_n : n \geq 3\} \cup \{N_4, O_4, Q_4, Q_5\}$.

*Proof.* Let $u_1, \ldots, u_m$ be the shortest path between a vertex that is adjacent to three or more other vertices and a vertex that has two loops. Let $v$ and $w$ be two of the other vertices to which $u_1$ is adjacent. (If $m = 1$, there will be one more vertex.) Then we have

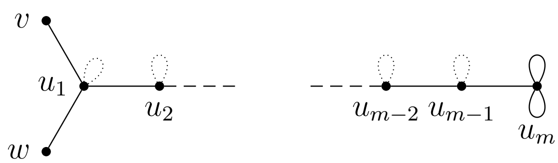
$G \supseteq$




.

Since $u_1, \ldots, u_m$ is the shortest path connecting a vertex that is adjacent to three or more other vertices with a vertex that has two loops, the vertices

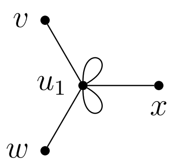
$u_1, \ldots, u_{m-1}$ all have at most one loop:

$G \supseteq$ 

.

There are three cases to consider.

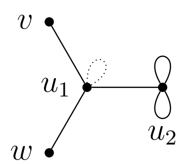
**Case 1:** Suppose $m = 1$. Then $u_1$ is adjacent to at least one more vertex $x$ and we have

$G \supseteq$ 

$= N_4$.

Since $N_4$ is a forbidden proper subgraph, we have $G = N_4$.

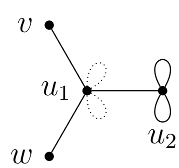
**Case 2:** Suppose $m = 2$. Then we have

$G \supseteq$ 

.

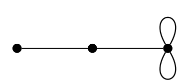
There are two cases to consider.

**Case 2.1:** Suppose $u_1$ has no loops. Then we have

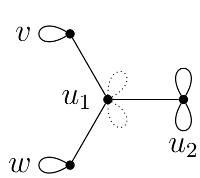
$G \supseteq$ 

.

The forbidden induced subgraph

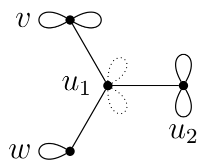

$= Y_5$

requires that $v$ and $w$ have at least one loop:

$G \supseteq$ 

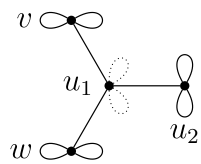
$= Y_{25}$.

The forbidden induced subgraph $Y_{25}$ requires that $v$ or $w$ has two loops. Without loss of generality, assume $v$ has two loops. Then we have

$G \supseteq$ 

$= Y_{26}$.

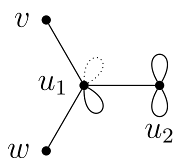
The forbidden induced subgraph $Y_{26}$ requires that $w$ has two loops:

$G \supseteq$ 

$= O_4$.

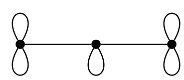
Since $O_4$ is a forbidden proper subgraph, we have $G = O_4$.

**Case 2.2:** Suppose $u_1$ has one loop. Then we have

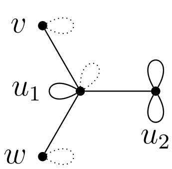
$G \supseteq$ 

.

The forbidden proper subgraph

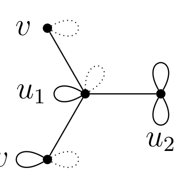

$= TT_3$

requires that $v$ and $w$ have at most one loop:

$G \supseteq$ 

$= Y_{23}$.

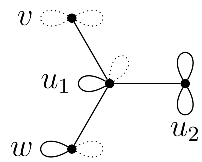
The forbidden induced subgraph $Y_{23}$ requires that $v$ or $w$ has at least one loop. Without loss of generality, assume $w$ has at least one loop:

$G \supseteq$ 

$= Q_4$.

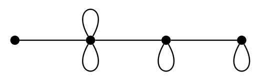
If there are no additional vertices, edges, or loops, then we have $G = Q_4$. Therefore, suppose $G$ properly contains $Q_4$. There are two cases to consider.

**Case 2.2.1:** Suppose $v$ has no loops. Then we have

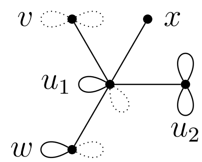
$G \supseteq$ 

.

The forbidden proper subgraph

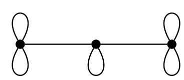

$= S_{0,2,1,1}$

requires that $u_2$ is not adjacent to any additional vertices. We show that $u_1$ is not adjacent to any additional vertices. Suppose to the contrary that $u_1$ is adjacent to an additional vertex $x$:

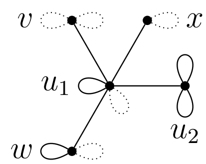
$G \supseteq$ 

.

The forbidden proper subgraph

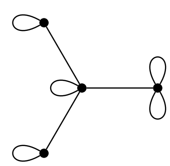

$= TT_3$

requires that $x$ has at most one loop:

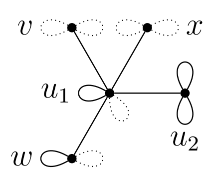
$G \supseteq$ 

.

The forbidden proper subgraph

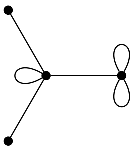

$= DT_4$

requires that $x$ has no loops:

$G \supseteq$ 

.

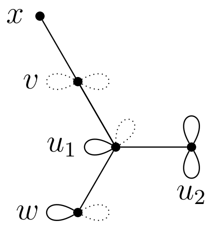
This contradicts the forbidden induced subgraph



$= Y_{23}.$

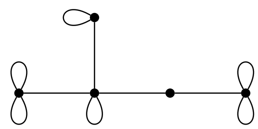
Thus $u_1$ is not adjacent to any additional vertices. Next we show that $v$ is not adjacent to any additional vertices. Suppose to the contrary that $v$ is adjacent to an additional vertex $x$. Then we have

$G \supseteq$




.

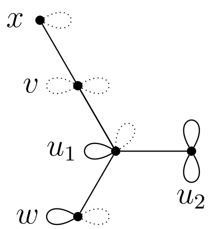
The forbidden subgraph



$= X_6$

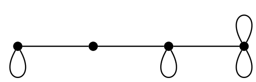
requires that $x$ has at most one loop:

$G \supseteq$




.

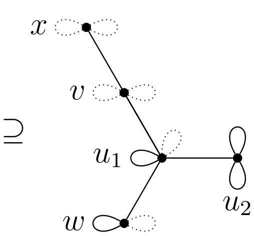
The forbidden induced subgraph



$= Y_8$

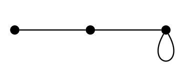
requires that $x$ has no loops:

$G \supseteq$



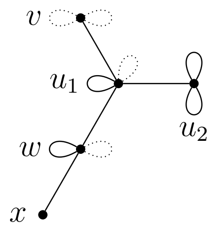

.

But this contradicts the forbidden induced subgraph



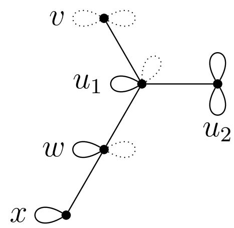
$= Y_4.$

Thus $v$ is not adjacent to any additional vertcies. The only remaining possibility is for $w$ to be adjacent to an additional vertex $x$:

$G \supseteq$ 

$= Y_{28}.$

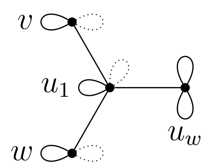
The forbidden induced subgraph $Y_{28}$ requires that $x$ has at least one loop:

$G \supseteq$ 

$= Q_5.$

Since $Q_5$ is a forbidden proper subgraph, we have $G = Q_5$.

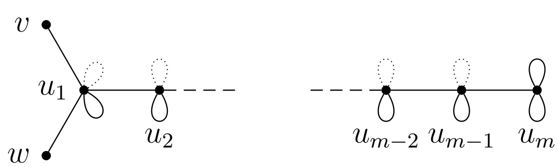
**Case 2.2.2:** Suppose $v$ has exactly one loop. Then we have

$G \supseteq$ 

$= DT_4.$

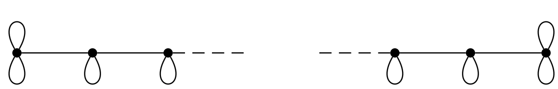
Since $DT_4$ is a forbidden proper subgraph, we have $G = DT_4$.

**Case 3:** Suppose $m \geq 3$. By Proposition A.11, the vertices $u_1, \ldots, u_{m-1}$ all have exactly one loop:

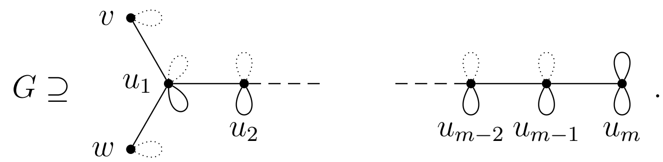
$G \supseteq$ 

.

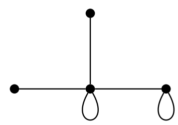
The forbidden proper subgraph


$= TT_{m+1}$

requires that $v$ and $w$ have at most one loop:

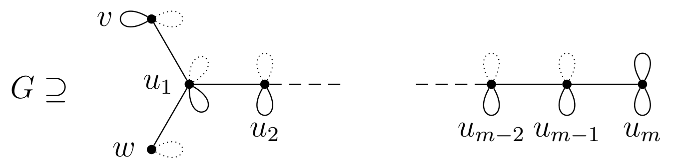



The forbidden induced subgraph

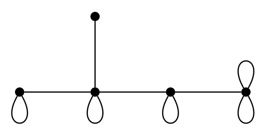


$= Y_{22}$

requires that at least one of $v$ and $w$ has one loop. Without loss of generality, assume $v$ has one loop:

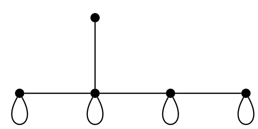



If $m = 3$, then the forbidden induced subgraph



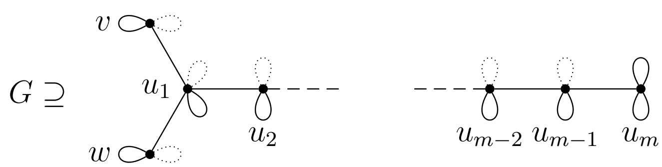
$= Y_{30}$

requires that $w$ has one loop. If $m \geq 4$, then the forbidden induced subgraph



$= Y_{29}$

requires that $w$ has one loop. In all cases, the vertex $w$ has one loop:




$= DT_{m+2}$.

Since $DT_{m+2}$ is a forbidden proper subgraph, we have $G = DT_{m+2}$. Since $m \geq 3$, it follows that $G \in \{DT_n : n \geq 5\}$.

In all cases, we have $G \in \{DT_n : n \geq 3\} \cup \{N_4, O_4, Q_4, Q_5\}$. □

**Proposition A.21**: Suppose $G$ satisfies the following conditions:

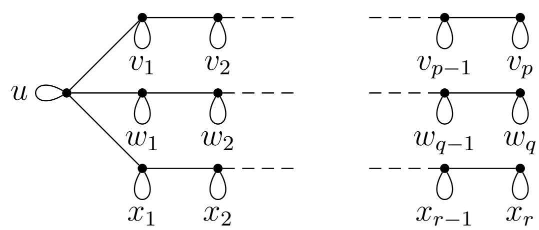
1. All vertices have exactly one loop.
2. There are no $n$-cycles for all $n \geq 3$.

3. All vertices are adjacent to at most three other vertices.

4. There is exactly one vertex that is adjacent to three other vertices.

Then $G \in \{BD_n : n \geq 4\} \cup \{E_6, E_7, F_7, E_8, G_8, E_9\}$.

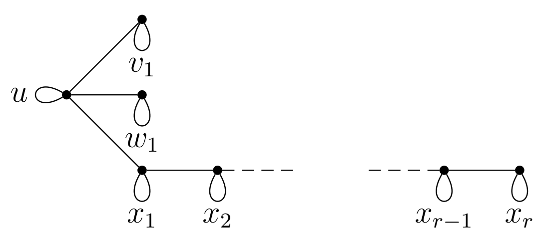
*Proof.* Let $u$ be the one vertex that is adjacent to three other vertices. Since all other vertices are adjacent to at most two other vertices, and since there are no $n$-cycles for all $n \geq 3$, we see that $G$ consists of three paths that attach at the vertex $u$. Label the vertices of these three paths using $v_1, \ldots, v_p$; $w_1, \ldots, w_q$; and $x_1, \ldots, x_r$. Then we have

$G =$  .



There are four cases to consider.

**Case 1:** Suppose at least two of the conditions $p = 1$, $q = 1$, and $r = 1$ hold. Without loss of generality, assume $p = 1$ and $q = 1$. Then we have

$G =$  $= BD_{r+3}$.



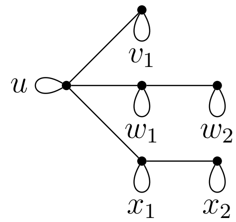
Thus $G = BD_{r+3}$. Since $r \geq 1$, we have $G \in \{BD_n : n \geq 4\}$.

**Case 2:** Suppose exactly one of the conditions $p = 1$, $q = 1$, and $r = 1$ hold. Without loss of generality, assume $p = 1$, $q \geq 2$, and $r \geq 2$. There are two cases to consider.

**Case 2.1:** Suppose at least one of the conditions $q = 2$ and $r = 2$ hold. Without loss of generality, assume $q = 2$. There are four cases to consider.

**Case 2.1.1:** Suppose $r = 2$. Then we have

$G =$

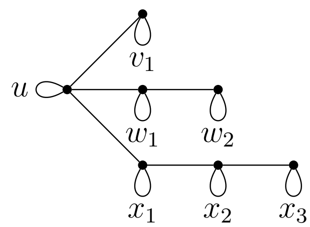



$= E_6.$

Thus $G = E_6$.

**Case 2.1.2:** Suppose $r = 3$. Then we have

$G =$

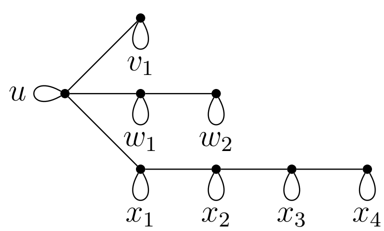



$= E_7.$

Thus $G = E_7$.

**Case 2.1.3:** Suppose $r = 4$. Then we have

$G =$

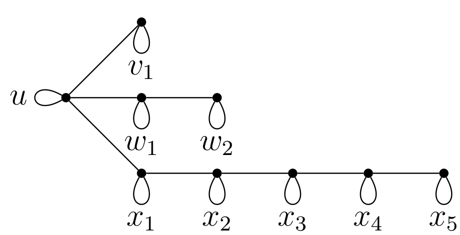



$= E_8.$

Thus $G = E_8$.

**Case 2.1.4:** Suppose $r \geq 5$. Then we have

$G \supseteq$

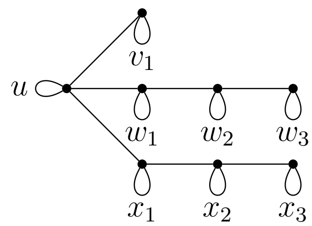



$= E_9.$

Since $E_9$ is a forbidden proper subgraph, we have $G = E_9$.

**Case 2.2:** Suppose none of the conditions $q = 2$ and $r = 2$ hold. Then we have $q \geq 3$ and $r \geq 3$. Thus

$G \supseteq$

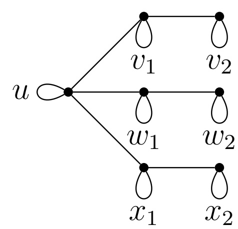



$= G_8$.

Since $G_8$ is a forbidden proper subgraph, we have $G = G_8$.

**Case 3:** Suppose none one of the conditions $p = 1$, $q = 1$, and $r = 1$ hold. Then we have $p \geq 2$, $q \geq 2$, and $r \geq 2$. Thus

$G \supseteq$




$= F_7$.

Since $F_7$ is a forbidden proper subgaph, we have $G = F_7$.

In all cases, we have $G \in \{BD_n : n \geq 4\} \cup \{E_6, E_7, F_7, E_8, G_8, E_9\}$. □

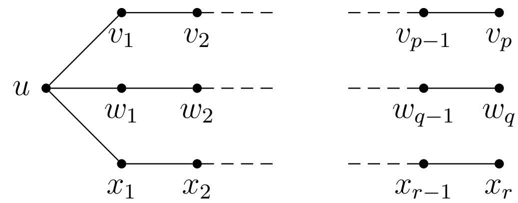
**Proposition A.22**: Suppose $G$ satisfies the following conditions:

1. All vertices have at most one loop.
2. There are no $n$-cycles for all $n \geq 3$.
3. All vertices are adjacent to at most three other vertices.
4. There is exactly one vertex that is adjacent to three other vertices.
5. At least one vertex has no loops.

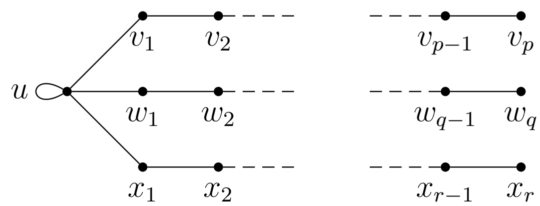
Then $G \in \{AD_n : n \geq 4\}$.

*Proof.* Since all other vertices are adjacent to at most two other vertices, and since there are no $n$-cycles for all $n \geq 3$, we see that $G$ consists of three paths that attach at the vertex $u$. Label the vertices of these three paths using

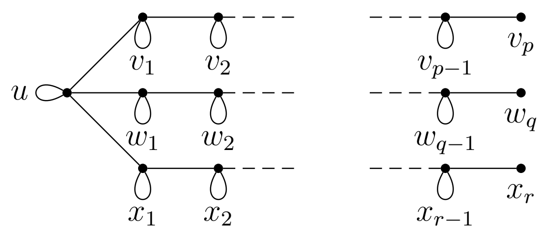
$v_1, \ldots, v_p$; $w_1, \ldots, w_q$; and $x_1, \ldots, x_r$. Then we have

$G \supseteq$  .

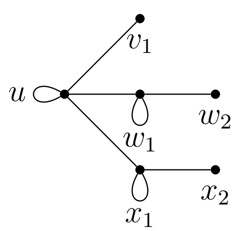
By Proposition A.13, the vertex $u$ has one loop:

$G \supseteq$  .

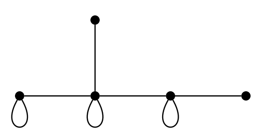
By Proposition A.11, the vertices $v_1, \ldots, v_{p-1}$; $w_1, \ldots, w_{q-1}$; and $x_1, \ldots, x_{r-1}$ all have one loop:

$G \supseteq$  .

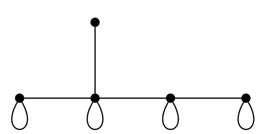
We show that at least two of the conditions $p = 1$, $q = 1$, and $r = 1$ hold. Suppose to the contrary that at most one of the conditions $p = 1$, $q = 1$, and $r = 1$ holds. Without loss of generality, assume $q \geq 2$ and $r \geq 2$. Then we have

$G \supseteq$  .

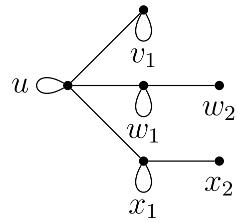
The forbidden induced subgraph

 $= Y_{27}$

requires that $v_1$ or $x_2$ has one loop. Thus if $x_2$ has no loops, then $v_1$ has one loop. The forbidden induced subgraph

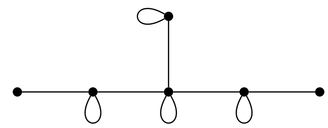
 $= Y_{29}$

requires that $v_1$ has one loop if $x_2$ has one loop. In all cases, $v_1$ has one loop:

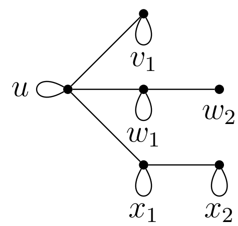
$G \supseteq$  .



The forbidden induced subgraph

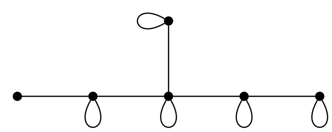
 $= Y_{31}$

requires that $w_2$ or $x_2$ has one loop. Without loss of generality, assume $x_2$ has one loop:

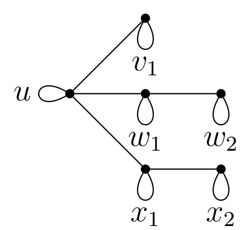
$G \supseteq$  .



The forbidden induced subgraph

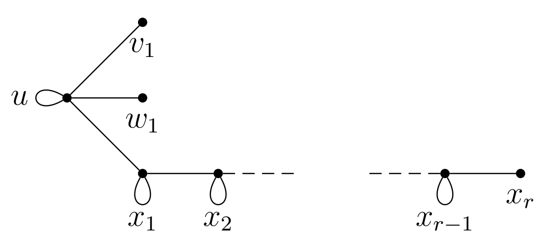
 $= Y_{32}$

requires that $w_2$ has one loop:

$G \supseteq$  $= E_6$.

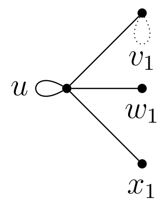


Since $E_6$ is a forbidden proper subgraph, we have $G = E_6$. But all vertices of $E_6$ have one loop, which contradicts the assumption that at least one vertex of $G$ has no loops. Thus at least two of the conditions $p = 1$, $q = 1$, and $r = 1$ hold. Without loss of generality, assume $p = 1$ and $q = 1$. Then we have

$G \supseteq$  .

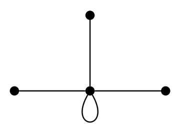
We consider two cases.

**Case 1:** Suppose at least one of $v_1$, $w_1$, and $x_1$ has no loops. Without loss of generality, assume $v_1$ has no loops:

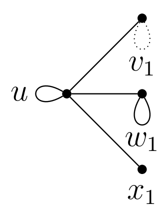
$G \supseteq$ 

.

The forbidden induced subgraph

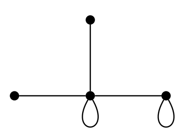

$= Y_{21}$

requires that $w_1$ or $x_1$ has one loop. Without loss of generality, assume $w_1$ has one loop:

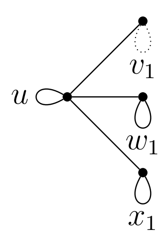
$G \supseteq$ 

.

The forbidden induced subgraph


$= Y_{22}$

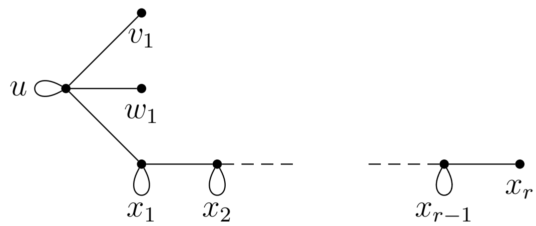
requires that $w_1$ has one loop:

$G =$ 

$= AD_4$.

Thus $G = AD_4$.

**Case 2:** Suppose at all of $v_1$, $w_1$, and $x_1$ have one loop. Since at least one vertex of $G$ has no loops, we must have $r \geq 2$ and $x_r$ has no loops. Thus

$G \supseteq$  $= AD_{r+3}$.


Thus $G = AD_{r+3}$. Since $r \geq 2$, we have $G \in \{AD_n : n \geq 5\}$.

In all cases, we have $G \in \{AD_n : n \geq 4\}$. □

**Proposition A.23**: Suppose $G$ satifies the following conditions:

1. There are no $n$-cycles for all $n \geq 3$.
2. All vertices are adjacent to at most two other vertices.
3. At least one vertex that is adjacent to two other vertices does not have exactly one loop.

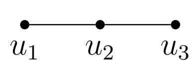
Then $G \in \{AD_3, DT_3, S_{0,2,0}, S_{0,2,1}, S_{1,0,2}, S_{2,0,2}, S_{0,1,2,0}, S_{0,2,0,1}, S_{0,2,0,2}, S_{0,2,1,1}, S_{1,0,2,1}, S_{1,1,0,2}, S_{1,2,0,2}, S_{2,0,1,2}, S_{0,1,2,0,1}, S_{0,2,0,2,0}, S_{0,2,1,0,2}, S_{1,2,0,1,2}, S_{2,0,2,0,2}\}$.

*Proof.* Since there are no $n$-cycles for all $n \geq 3$, and since all vertices are adjacent to at most two other vertices, then graph $G$ must be a path. Label the vertices of this path using $u_1, \ldots, u_m$. Then we have

$G \supseteq$  .

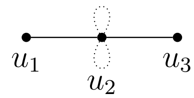

Since there is at least one vertex that is adjacent to two other vertices (the vertex that does not have exactly one loop), we must have $m \geq 3$. There are three cases to consider.

**Case 1:** Suppose $m = 3$. Then we have

$G \supseteq$  .


Since $u_2$ is the only vertex that is adjacent to exactly two other vertices, the vertex $u_2$ does not have exactly one loop. There are two cases to consider.

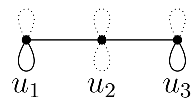
**Case 1.1:** Suppose $u_2$ has no loops:

$G \supseteq$ 

.

By Proposition A.8, the vertices $u_1$ and $u_3$ have at least one loop. There are three cases to consider.

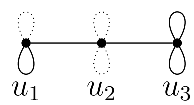
**Case 1.1.1:** Suppose $u_1$ and $u_3$ have exactly one loop:

$G =$ 

$= AD_3$.

Thus $G = AD_3$.

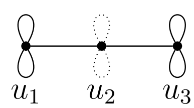
**Case 1.1.2:** Suppose one of $u_1$ and $u_3$ has exactly one loop while the other has two. Without loss of generality, assume $u_1$ has exactly one loop while $u_3$ has exactly two loops:

$G \supseteq$ 

$= S_{1,0,2}$.

Thus $G = S_{1,0,2}$.

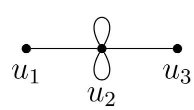
**Case 1.1.3:** Suppose $u_1$ and $u_3$ have two loops:

$G \supseteq$ 

$= S_{2,0,2}$.

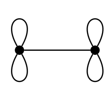
Thus $G = S_{2,0,2}$.

**Case 1.2:** Suppose $u_2$ has two loops:

$G \supseteq$ 

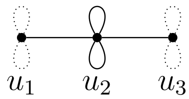
.

The forbidden proper subgraph


$= TT_2$

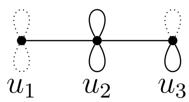
requires that $u_1$ and $u_3$ have at most one loop. There are three cases to consider.

**Case 1.2.1:** Suppose $u_1$ and $u_3$ have no loops:

$G =$  $= S_{0,2,0}$.

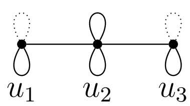
Thus $G = S_{0,2,0}$.

**Case 1.2.2:** Suppose one of $u_1$ and $u_3$ has no loops while the other has exactly one loop. Without loss of generality, assume $u_1$ has no loops while $u_3$ has exactly one loop:

$G =$  $= S_{0,2,1}$.

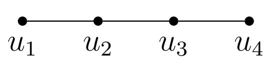
Thus $G = S_{0,2,1}$.

**Case 1.2.3:** Suppose $u_1$ and $u_3$ have exactly one loop:

$G =$  $= DT_3$.

Thus $G = DT_3$.

**Case 2:** Suppose $m = 4$. Then we have

$G \supseteq$  .

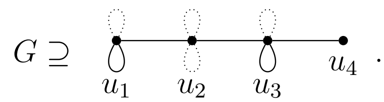
Since $u_2$ and $u_3$ are the only vertices that are adjacent to two other vertices, at least one of the vertices $u_2$ and $u_3$ does not have exactly one loop. Without loss of generality, assume $u_2$ does not have exactly one loop.

Let $H$ be the subgraph of $G$ induced by the vertices $u_1$, $u_2$, and $u_3$. By Lemma A.5, we know that the eigenvalues of $H$ lie in the real interval $[-1, 3]$. Thus the graph $H$ satisfies the conditions of this proposition. Thus $H$ must be one of the six graphs identified in Case 1. Of these six possibilities, the graph $DT_3$ is a forbidden proper subgraph and may be excluded. This leaves five possibilities:

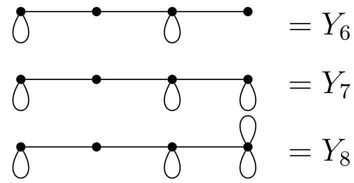
$$H \in \{AD_3, S_{0,2,0}, S_{0,2,1}, S_{1,0,2}, S_{2,0,2}\}.$$

Thus there are five cases to consider.

**Case 2.1:** Suppose $H = AD_3$. Then we have

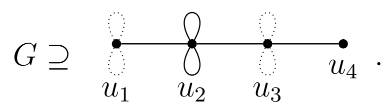



However, the forbidden induced subgraphs

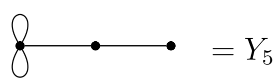



eliminate all possible choices for the number of loops to assign to $u_4$ and so this case is impossible.

**Case 2.2:** Suppose $H = S_{0,2,0}$. Then we have

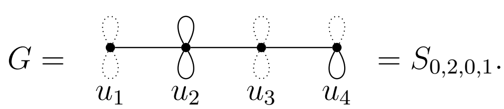



The forbidden induced subgraph



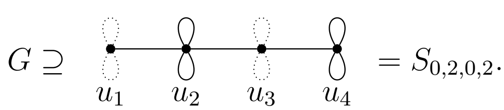

requires that $u_4$ has at least one loop. There are two cases to consider.

**Case 2.2.1:** Suppose $u_4$ has exactly one loop:




Thus $G = S_{0,2,0,1}$.

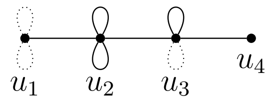
**Case 2.2.2:** Suppose $u_4$ has two loops:




Thus $G = S_{0,2,0,2}$.

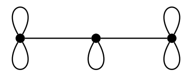
**Case 2.3:** Suppose $H = S_{0,2,1}$. There are two cases to consider depending on how we map the vertices of $S_{0,2,1}$ to $u_1$, $u_2$, and $u_3$.

**Case 2.3.1:** Suppose the vertex of $S_{0,2,1}$ with no loops maps to $u_1$:

$G \supseteq$ 

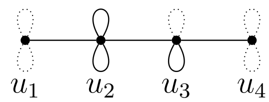
.

The forbidden proper subgraph

 $= TT_3$

requires that $u_4$ has at most one loop. There are two cases to consider.

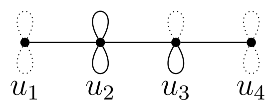
**Case 2.3.1.1:** Suppose $u_4$ has no loops:

$G \supseteq$ 

$= S_{0,1,2,0}$.

Thus $G = S_{0,1,2,0}$.

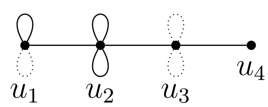
**Case 2.3.1.2:** Suppose $u_4$ has exactly one loop:

$G =$ 

$= S_{0,2,1,1}$.

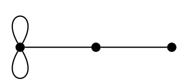
Thus $G = S_{0,2,1,1}$.

**Case 2.3.2:** Suppose the vertex of $S_{0,2,1}$ with no loops maps to $u_3$:

$G \supseteq$ 

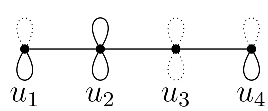
.

The forbidden induced subgraph

 $= Y_5$

requires that $u_4$ has at least one loop. There are two cases to consider.

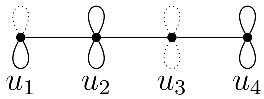
**Case 2.3.2.1:** Suppose $u_4$ has exactly one loop:

$G =$ 

$= S_{1,0,2,1}$.

Thus $G = S_{1,0,2,1}$.

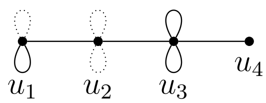
**Case 2.3.2.2:** Suppose $u_4$ has exactly two loops:

$G =$  $= S_{1,2,0,2}$.

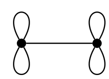

Thus $G = S_{1,2,0,2}$.

**Case 2.4:** Suppose $H = S_{1,0,2}$. There are two possibilities, depending on how we map the vertices of $S_{1,0,2}$ to $u_1$, $u_2$, and $u_3$.

**Case 2.4.1:** Suppose the vertex of $S_{1,0,2}$ with exactly one loop maps to $u_1$:

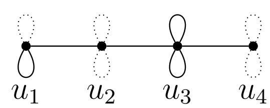
$G \supseteq$  .


The forbidden proper subgraph

 $= TT_2$

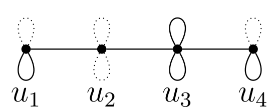
requires that $u_4$ has at most one loop. There are two cases to consider.

**Case 2.4.1.1:** Suppose $u_4$ has no loops:

$G =$  $= S_{0,2,0,1}$.

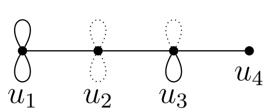

Thus $G = S_{0,2,0,1}$.

**Case 2.4.1.2:** Suppose $u_4$ has exactly one loop:

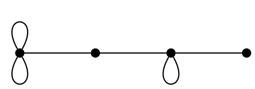
$G =$  $= S_{1,0,2,1}$.


Thus $G = S_{1,0,2,1}$.

**Case 2.4.2:** Suppose the vertex of $S_{1,0,2}$ with exactly one loop maps to $u_3$:

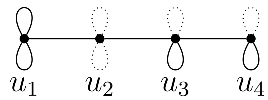
$G \supseteq$  .


The forbidden induced subgraph

 $= Y_9$

requires that $u_4$ has at least one loop. There are two cases to consider.

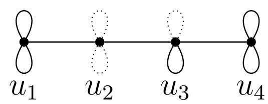
**Case 2.4.2.1:** Suppose $u_4$ has exactly one loop:

$G =$ 

$= S_{1,1,0,2}$.

Thus $G = S_{1,1,0,2}$.

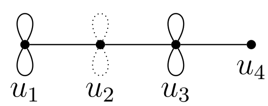
**Case 2.4.2.2:** Suppose $u_4$ has two loops:

$G =$ 

$= S_{2,0,1,2}$.

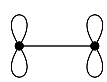
Thus $G = S_{2,0,1,2}$.

**Case 2.5:** Suppose $H = S_{2,0,2}$. Then we have

$G \supseteq$ 

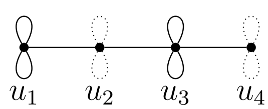
.

The forbidden proper subgraph


$= TT_2$

requires that $u_4$ has at most one loop. There are two cases to consider.

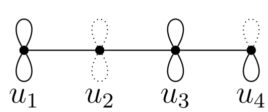
**Case 2.5.1:** Suppose $u_4$ has no loops:

$G =$ 

$= S_{0,2,0,2}$.

Thus $G = S_{0,2,0,2}$.

**Case 2.5.2:** Suppose $u_4$ has exactly one loop:

$G =$ 

$= S_{1,2,0,2}$.

Thus $G = S_{1,2,0,2}$.

**Case 3:** Suppose $n = 5$. Then the only vertices that are adjacent to two other vertices are $u_2$, $u_3$, and $u_4$. Thus at least one of the three vertices does not have exactly one loop. Without loss of generality, assume either (1) the vertex $u_2$ does not have exactly one loop, or (2) both $u_2$ and $u_4$ have exactly one loop while $u_3$ does not. Let $H$ be the subgraph of $G$ induced by the four vertices $u_1$, $u_2$, $u_3$, and $u_4$. Then $H$ satisfies the conditions of this proposition and so must be one of the eight graphs identified in Case 2. Two of these eight graphs, namely $S_{1,2,0,2}$ and $S_{0,2,1,1}$, are forbidden proper subgraphs, and so may be excluded. This leaves six possibilities:

$$H \in \{S_{0,1,2,0}, S_{0,2,0,1}, S_{0,2,0,2}, S_{1,0,2,1}, S_{1,1,0,2}, S_{2,0,1,2}\}.$$

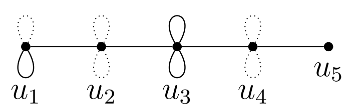
Note that none of the six choices for $H$ have both the first and third vertices with exactly one loop, or both the second and fourth vertices with exactly one loop. Thus we can exclude possibility (2) and assume that $u_2$ does not have exactly one loop. There are two cases to consider.

**Case 3.1:** Suppose $u_2$ has no loops. Then

$$H \in \{S_{0,2,0,1}, S_{0,2,0,2}, S_{1,0,2,1}, S_{1,1,0,2}, S_{2,0,1,2}\}.$$

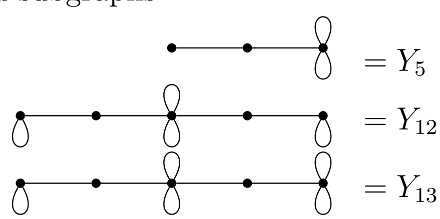
The graph $S_{0,1,2,0}$ does not appear in this list because neither of its two internal vertices has no loops. There are five cases to consider.

**Case 3.1.1:** Suppose $H = S_{0,2,0,1}$. Then we have

$G \supseteq$ 

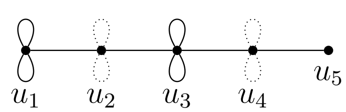
.

The forbidden induced subgraphs



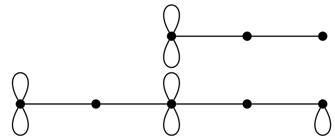

leave no possibilities for the number of loops to assign to $u_5$.

**Case 3.1.2:** Suppose $H = S_{0,2,0,2}$. Then we have

$G \supseteq$ 

.

The forbidden induced subgraphs

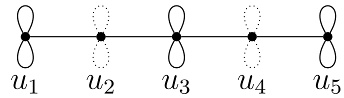


$= Y_5$

$= Y_{13}$

require that $u_5$ has two loops:

$$G = \quad$$

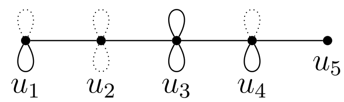



$$= S_{2,0,2,0,2}.$$

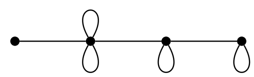
Thus $G = S_{2,0,2,0,2}$.

**Case 3.1.3:** Suppose $H = S_{1,0,2,1}$. Then we have

$$G \supseteq$$



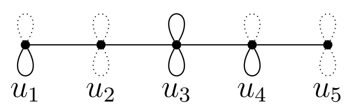

The forbidden proper subgraph



$$= S_{0,2,1,1}$$

reqiures that $u_5$ has no loops:

$$G =$$

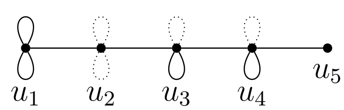



$$= S_{0,1,2,0,1}.$$

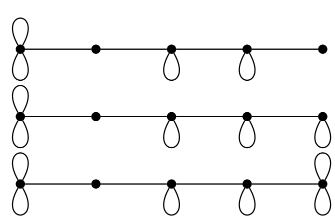
Thus $G = S_{0,1,2,0,1}$.

**Case 3.1.4:** Suppose $H = S_{1,1,0,2}$. Then we have

$$G \supseteq$$




The forbidden induced subgraphs



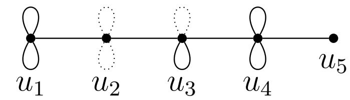
$= Y_{10}$

$= Y_{11}$

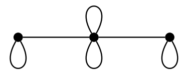
$= Y_{14}$

leave no possibilities for the number of loops to assign to $u_5$.

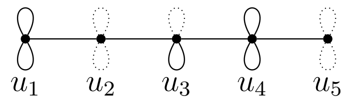
**Case 3.1.5:** Suppose $H = S_{2,0,1,2}$. Then we have

$G \supseteq$  .

The forbidden proper subgraph

 $= DT_3$

requires that $u_5$ has no loops:

$G =$  $= S_{0,2,1,0,2}$.

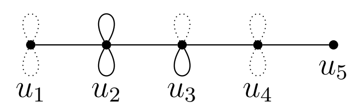
Thus $G = S_{0,2,1,0,2}$.

**Case 3.2:** Suppose $u_2$ has two loops. Then we have

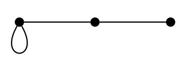
$$H \in \{S_{0,1,2,0}, S_{0,2,0,1}, S_{0,2,0,2}, S_{1,0,2,1}\}.$$

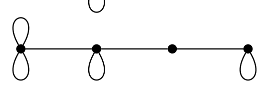
The graphs $S_{1,1,0,2}$ and $S_{2,0,1,2}$ are not included in this list because neither of their two interval vertices has two loops. Thus there are four cases to consider.

**Case 3.2.1:** Suppose $H = S_{0,1,2,0}$. Then we have

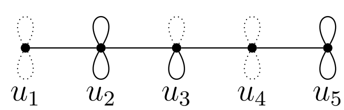
$G \supseteq$  .

The forbidden induced subgraphs

 $= Y_4$

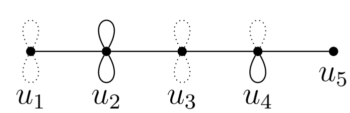
 $= Y_8$

require that $u_5$ has two loops:

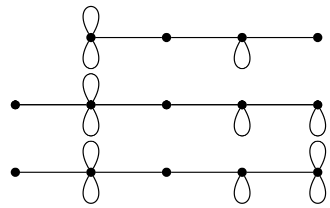
$G =$  $= S_{0,2,1,0,2}$.

Thus $G = S_{0,2,1,0,2}$.

**Case 3.2.2:** Suppose $H = S_{0,2,0,1}$. Then we have

$G \supseteq$  .

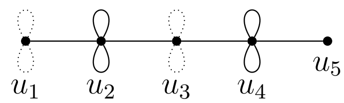
The forbidden induced subgraphs



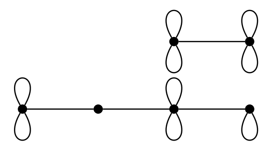
$= Y_9$
$= Y_{16}$
$= Y_{17}$

leave no possibilities for the number of loops to assign to $u_5$.

**Case 3.2.3:** Suppose $H = S_{0,2,0,2}$. Then we have

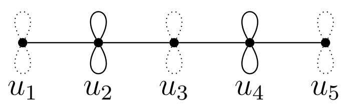
$G \supseteq$ 
 .

The forbidden proper subgraphs



$= TT_2$
$= S_{1,2,0,2}$

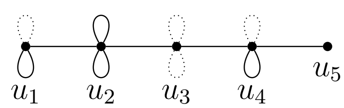
require that $u_5$ has no loops:

$G \supseteq$ 

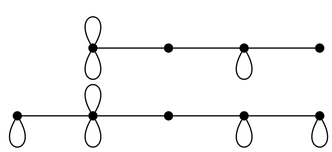
$= S_{0,2,0,2,0}$.

Thus $G = S_{0,2,0,2,0}$.

**Case 3.2.4:** Suppose $H = S_{1,0,2,1}$. Then we have

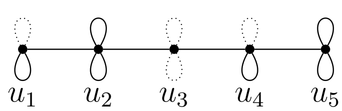
$G \supseteq$ 
 .

The forbidden induced subgraphs



$= Y_9$
$= Y_{15}$

requires that $u_5$ has two loops:

$G \supseteq$ 

$= S_{1,2,0,1,2}$.

Thus $G = S_{1,2,0,1,2}$.

**Case 4:** Suppose $n \geq 6$. Then the only vertices that are adjacent to two other vertices are $u_2$, $u_3$, $u_4$, and $u_5$. Thus one of these four vertices does not have exactly one loop. Without loss of generality, assume that $u_2$ or $u_3$ does not have exactly one loop. Let $H$ be the subgraph of $G$ induced by the five vertices $u_1$, $u_2$, $u_3$, $u_4$, and $u_5$. Then $H$ satisfies the conditions of this proposition and so is one of the five graphs identified in Case 3:

$$H \in \{S_{1,0,2,1,0}, S_{2,0,1,2,0}, S_{2,0,2,0,2}, S_{1,2,0,1,2}, S_{0,2,0,2,0}\}.$$

All five of these graphs are forbidden proper subgraphs. Thus there is nothing to consider in this case.

In all cases, we have $G \in \{AD_3, DT_3, S_{0,2,0}, S_{0,2,1}, S_{1,0,2}, S_{2,0,2}, S_{0,1,2,0}, S_{0,2,0,1}, S_{0,2,0,2}, S_{0,2,1,1}, S_{1,0,2,1}, S_{1,1,0,2}, S_{1,2,0,2}, S_{2,0,1,2}, S_{0,1,2,0,1}, S_{0,2,0,2,0}, S_{0,2,1,0,2}, S_{1,2,0,1,2}, S_{2,0,2,0,2}\}$. □

**Proposition A.24**: Suppose $G$ has an $n$-cycle for some $n \geq 6$. Then all vertices of this $n$-cycle have at most one loop.

*Proof.* Suppose to the contrary that $G$ has an $n$-cycle for some $n \geq 6$, and that at least one vertex of this $n$-cycles has two loops. Label the vertices of this $n$-cycle using $u_1, \ldots, u_n$, and assume without loss of generality that $u_2$ has two loops. Let $H$ be the subgraph of $G$ induced by the vertices $u_1$, $u_2$, $u_3$, $u_4$, and $u_5$. Then $H$ is a proper subgraph of $G$ and satisfies the conditions of Proposition A.23. Since $H$ has five vertices, the possibilities are:

$$H \in \{S_{0,1,2,0,1}, S_{0,2,0,2,0}, S_{0,2,1,0,2}, S_{1,2,0,1,2}, S_{2,0,2,0,2}\}.$$

Since all five possibilities are forbidden proper subgraphs, we have a contradiction. □

**Proposition A.25**: Suppose $G$ satisfies the following conditions:

1. There are no $n$-cycles for all $n \geq 3$.
2. All vertices are adjacent to at most two other vertices.
3. All vertices that are adjacent to exactly two other vertices have exactly one loop.

Then $G \in \{AB_1, BB_1, BT_1\} \cup \{AA_n, AB_n, AT_n, BB_n, BT_n, TT_n : n \geq 2\}$.

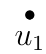
*Proof.* Since there are no $n$-cycles for all $n \geq 3$, and since all vertices are adjacent to at most two other vertices, the graph $G$ must be a path. Label the vertices of this path using $u_1$, ..., $u_m$. Then we have

$\subseteq G.$

We divide into cases based on the value of $m$.

**Case 1:** Suppose $m = 1$. There are three cases to consider.

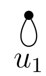
**Case 1.1:** Suppose $u_1$ has no loops:

$G =$ 

$= AB_1$.

Thus $G = AB_1$.

**Case 1.2:** Suppose $u_1$ has exactly one loop:

$G =$ 

$= BB_1$.

Thus $G = BB_1$.

**Case 1.3:** Suppose $u_2$ has two loops:

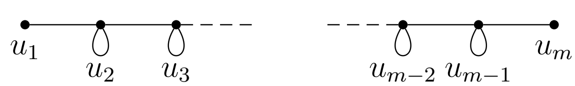
$G =$ 

$= BT_1$.

Thus $G = BT_1$.

**Case 2:** Suppose $m \geq 2$. By assumption, all the vertices that are adjacent to exactly two other vertices have exactly one loop. Thus $u_2$, ..., $u_{m-1}$ all have exactly one loop. Without loss of generality, there are six cases to consider.

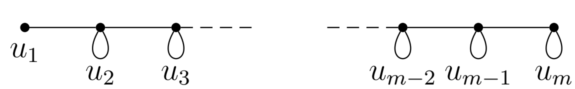
**Case 2.1:** Suppose $u_1$ and $u_m$ have no loops:

$G =$ 

$= AA_m$.

Thus $G = AA_m$. Since $m \geq 2$, we have $G \in \{AA_n : n \geq 2\}$.

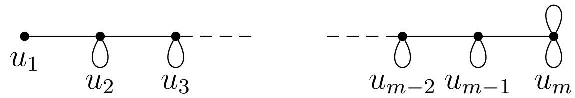
**Case 2.2:** Suppose $u_1$ has no loops while $u_m$ has exactly one loop:

$G =$ 

$= AB_m$.

Thus $G = AB_m$. Since $m \geq 2$, we have $G \in \{AB_n : n \geq 2\}$.

**Case 2.3:** Suppose $u_1$ has no loops while $u_m$ has two loops:

$G \supseteq$  $= AT_m$.

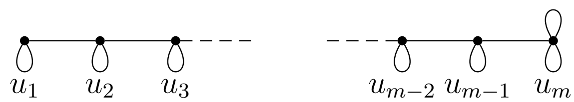

Thus $G = AT_m$. Since $m \geq 2$, we have $G \in \{AT_n : n \geq 2\}$.

**Case 2.4:** Suppose $u_1$ and $u_m$ have exactly one loop:

$G =$  $= BB_m$.


Thus $G = BB_m$. Since $m \geq 2$, we have $G \in \{BB_n : n \geq 2\}$.

**Case 2.5:** Suppose $u_1$ has exactly one loop while $u_m$ has two loops:

$G =$  $= BT_m$.


Thus $G = BT_m$. Since $m \geq 2$, we have $G \in \{BT_n : n \geq 2\}$.

**Case 2.6:** Suppose $u_1$ and $u_m$ have two loops:

$G =$  $= TT_m$.


Thus $G = TT_m$. Since $m \geq 2$, we have $G \in \{TT_n : n \geq 2\}$.

In all cases, we have

$$G \in \{AB_1, BB_1, BT_1\} \cup \{AA_n, AB_n, AT_n, BB_n, BT_n, TT_n : n \geq 2\}.$$

□

**Theorem A.26**: Let $G$ be a connected symmetric graph whose eigenvalues lie in the real interval $[-1, 3]$. Then $G$ is one of the graphs listed in Tables 8.1, 8.3, and A.1.

*Proof.* We consider various cases concerning the graph $G$.

**Case 1:** Suppose at least one vertex has three or more loops. Then Proposition A.6 implies that $G = K_1$.

**Case 2:** Suppose all vertices have at most two loops.

**Case 2.1:** Suppose some two vertices are adjacent by two or more edges. Then Proposition A.7 implies that $G = K_2$.

**Case 2.2:** Suppose any two vertices are adjacent by at most one edge.

**Case 2.2.1:** Suppose there is at least one 4-cycle. Then Proposition A.9 implies that $G \in \{K_4, L_4, M_4\}$.

**Case 2.2.2:** Suppose there are no 4-cycles.

**Case 2.2.2.1:** Suppose there is at least one 5-cycle.

**Case 2.2.2.1.1:** Suppose at least one of the 5-cycles has a vertex with two loops. Then Proposition A.10 implies that $G = R_5$.

**Case 2.2.2.1.2:** Suppose every vertex of a 5-cycle has at most one loop. Then Proposition A.12 implies that $G = K_5$.

**Case 2.2.2.2:** Suppose there are no 5-cycles.

**Case 2.2.2.2.1:** Suppose there is an $n$-cycle for some $n \geq 6$. By Proposition A.24, all vertices of this $n$-cycle have at most one loop. By Proposition A.12, we have $G = K_n$.

**Case 2.2.2.2.2:** Suppose there are no $n$-cycles for all $n \geq 6$.

**Case 2.2.2.2.2.1:** Suppose there is at least one 3-cycle.

**Case 2.2.2.2.2.1.1:** Suppose at least one vertex has two loops. Then Proposition A.19 implies that $G \in \{P_4\} \cup \{CT_n : n \geq 3\}$.

**Case 2.2.2.2.2.1.2:** Suppose all vertices have at most one loop.

**Case 2.2.2.2.2.1.2.1:** Suppose at least one vertex is adjacent to four or more other vertices. Then Proposition A.14 implies that $G \in \{CC_5, CD_5, DD_5\}$. However, since $DD_5$ has no 3-cycles, we more precisely have $G \in \{CC_5, CD_5\}$.

**Case 2.2.2.2.2.1.2.2:** Suppose all vertices are adjacent to at most three other vertices.

**Case 2.2.2.2.2.1.2.2.1:** Suppose there is at least two 3-cycles. Then Proposition A.16 implies that $G \in \{CC_n : n \geq 6\}$.

**Case 2.2.2.2.2.1.2.2.2:** Suppose there is only one 3-cycle.

**Case 2.2.2.2.2.1.2.2.2.1:** Suppose there is at least one vertex disjoint from the 3-cycle that is adjacent to three or more other vertices. Then Proposition A.17 implies that $G \in \{CD_n : n \geq 6\}$.

**Case 2.2.2.2.2.1.2.2.2.2:** Suppose all vertices, except possibly for the vertices of the 3-cycle, are adjacent to at most two other vertices. Then Proposition A.15 implies that $G \in \{AC_n, BC_n : n \geq 3\} \cup \{H_3, H_4, I_4, I_5, J_5, K_3\}$.

**Case 2.2.2.2.2.2:** Suppose there are no 3-cycles.

**Case 2.2.2.2.2.2.1:** Suppose at least one vertex is adjacent to four or more other vertices. Then Proposition A.14 implies that $G \in \{CC_5, CD_5, DD_5\}$. However, since $CC_5$ and $CD_5$ have 3-cycles, we more precisely have $G = DD_5$.

**Case 2.2.2.2.2.2.2:** Suppose all vertices are adjacent to at most three other vertices.

**Case 2.2.2.2.2.2.2.1:** Suppose at least one vertex is adjacent to three or more other vertices.

**Case 2.2.2.2.2.2.2.1.1:** Suppose at least one vertex has two loops. Then Proposition A.20 implies that $G \in \{DT_n : n \geq 3\} \cup \{N_4, O_4, Q_4, Q_5\}$.

**Case 2.2.2.2.2.2.2.2.1.2:** Suppose all vertices have at most one loop.

**Case 2.2.2.2.2.2.2.2.1.2.1:** Suppose at least two vertices are adjacent to three or more other vertices. Then Proposition A.18 implies that $G = \{DD_n : n \geq 6\}$.

**Case 2.2.2.2.2.2.2.2.1.2.2:** Suppose at most one vertex is adjacent to three or more other vertices.

**Case 2.2.2.2.2.2.2.2.1.2.2.1:** Suppose exactly one vertex is adjacent to three or more other vertices.

**Case 2.2.2.2.2.2.2.2.1.2.2.1.1:** Suppose at least one vertex has no loops. Then Proposition A.22 implies that $G \in \{AD_n : n \geq 4\}$.

**Case 2.2.2.2.2.2.2.2.1.2.2.1.2:** Suppose all vertices have exactly one loop. Then Proposition A.21 implies that

$$G \in \{BD_n : n \geq 4\} \cup \{E_6, E_7, E_8, E_9, F_7, G_8\}.$$

**Case 2.2.2.2.2.2.2.2.1.2.2.2:** Suppose all vertices are adjacent to at most two other vertices.

**Case 2.2.2.2.2.2.2.2.1.2.2.2.1:** Suppose all vertices that are adjacent to exactly two other vertices have exactly one loop. Then Proposition A.25 implies that $G \in \{AB_1, BB_1, BT_1\} \cup \{AA_n, AB_n, AT_n, BB_n, BT_n, TT_n : n \geq 2\}$.

**Case 2.2.2.2.2.2.2.2.1.2.2.2.2:** Suppose at least one vertex that is adjacent to two other vertices does not have exactly one loop. Then Proposition A.23 implies that

$$G \in \{AD_3, S_{1,0,2}, S_{2,0,2}, S_{0,2,0}, S_{0,2,1}, DT_3, S_{1,0,2,0}, S_{1,0,2,1}, S_{2,0,1,1}, S_{2,0,1,2}, S_{2,0,2,0}, S_{2,0,2,1}, S_{0,2,1,0}, S_{0,2,1,1}, S_{1,0,2,1,0}, S_{2,0,1,2,0}, S_{2,0,2,0,2}, S_{1,2,0,1,2}, S_{0,2,0,2,0}\}.$$

In all cases, the graph $G$ appears in one of Tables 8.1, 8.3, and A.1. □